Sudan University of Science and Technology
College of Graduate Studies

Application of Modified Double Sumudu Transform to Partial Differential Equations and Integro-Partial Differential Equations

تطبيق تحويل سمودو الثنائي المعدل علي المعادلات التفاضلية الجزئية والمعادلات التكاملية التفاضلية الجزئية

A Thesis Submitted in Fulfillment of the Requirements for the Degree of Ph.D in Mathematics

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Dedication

This thesis is dedicated to my beloved father (may ALLAH have mercy on him), my beloved mother, wife, brothers and sisters for their endless support and encouragement.

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Abstract

This research project introduces a new method employed to tackle non-linear partial differential equations, namely Modified Double Sumudu Transform Decomposition Method. This method is a combination of the Modified Double Sumudu Transform and Adomian Decomposition Method. The presented technique is provided and supported with necessary illustrations, together with some attached examples. The results reveal that the new method is very efficient, simple and can be applied to other non-linear problems.

الخلاصة

هذا البحث قدم طريقة جديدة لحل المعادلات التفاضلية الجزئية غير الخطية تم الحصول علي هذه الطريقة بالدمج بين تحويل سمودو الثنائي المعدل وطريقة أدوميان حيث تم تقديم شرح كامل لهذه الطريقة وتدعيمها بعدد من الامثلة ووجد أن هذه الطريقة تنافس الطرق الحديثة فهي طريقة بسيطة وفعالة للغاية ويمكن تطبيقها علي معادلات أخري غير خطية.

Contents

Chapter 1

Introduction

Partial differential equations have become a useful tool for describing most of the natural phenomena of science and engineering models. For example, in physics, the heat flow and the wave propagation, in ecology, most population models are governed by partial differential equations. The dispersion of a chemically reactive material is characterized by partial differential equations. In addition, most physical phenomena of fluid dynamics, quantum mechanics, electricity, plasma physics, propagation of shallow water wave, and many other models are controlled within its domain of validity by partial differential equations. Therefore, it becomes increasingly important to be familiar with all traditional and recently developed methods for solving partial differential equations, and the implementation of these methods.

The non-linear partial differential equations, appear in many applications of mathematics, physics, chemistry and engineering, for this reason the researcher presents a number of methods for solving it, such as Adomian Decomposition Method (ADM) [1], Variation Iteration Method (VIM) [1], Homotopy Perturbation Method (HPM) [1].

A new option appear recently, includes the composition of previous methods with some integral transforms namely Laplace transform, Sumudu transform, or Elzaki transform, these compositions resulted number of methods such as Laplace Decomposition Method (LDM) ([2]-[4]), Laplace Variation Iteration Method (LVIM) [5], Sumudu Decomposition Method (SDM) ([6]-[15]), Sumudu Homotopy Perturbation Method (SHPM) ([16],[17]), Elzaki Variation Iteration

Method (EVIM) [18], Elzaki project Differential Transform Method (EPDTM) [19], Elzaki Homotopy Perturbation Method (EHPM) ([20],[21]), and Elzaki Decomposition Method (EDM) ([22],[23]).

In this thesis the essential motivation of the present study is to extend the application of the Modified Double Sumudu Transform by introduce a new method called Modified Double Sumudu Transform Decomposition Method for solving non-linear partial differential equations.

The significance of this method is its capability of combining easy integral transform Modified Double Sumudu Transform (DET)[24] and an effective method for solving non-linear partial differential equations, namely Adomian Decomposition Method [1].

This method is described and illustrated with some examples in chapter four to explain its effectiveness.

Chapter 2

Sumudu and Modified Sumudu Transform

In the literature, there are several works on the theory and applications of integral transform such as Laplace, Fourier, Mellin, Hankel, Sumudu and Modified Sumudu transform (Elzaki Transform). These transforms use to solve a lot of problems in ordinary differential equation, partial differential equation and integral equations. Now we take a glance for some transforms.

2.1 Sumudu Transform

Sumudu transform it was proposed originally by Watugala (1993) to solve differential equation and control engineering problems.

Definition (2.1.1):Sumudu transform denoted by the operator $S(.)$ defined by the integral equation:

$$F(u) = S[f(t);u] = \frac{1}{u}\int_0^\infty f(t)e^{\frac{-t}{u}}\,dt \quad , \quad u \in (-\tau_1, \tau_2) \qquad (2.1)$$

It appeared like the modification of the well-known Laplace transform.

This transform may be used to solve problems without resorting to a new frequency domain and has many interesting properties which make its visualization easing some of these properties are:

1-The differentiation and integration in the t- domain is equivalent to division and multiplication of the transformed function $F(u)$ by u in the u- domain.

2-The unit-step function in the t - domain is transformed into unity in the u - domain.

3-Degree of the function $f(t)$ in the t - domain is equivalent to degree of $F(u)$ in the u - domain by the same degree factor.

4-The limit of $f(t)$ as t tends to zero is equal to the limit of $F(u)$ as u tends to zero.

5-The limit of $f(t)$ as t tends to infinity is the same as the limit of $F(u)$ as u tends to infinity.

6-The slope of the function $f(t)$ at $t=0$ is the same as the slope of $F(u)$ at $u=0$.

2.1.1 Properties of Sumudu Transform :

1- Linear Property:

$$S[af(t)+bg(t)] = a\,S\,[f(t)] + b\,S[g(t)]$$

$$S[af(t)+bg(t)] = \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}[af(t)+bg(t)]\,dt$$

$$= \frac{a}{u}\int_0^\infty f(t)\,e^{-\frac{t}{u}}\,dt + \frac{b}{u}\int_0^\infty g(t)\,e^{-\frac{t}{u}}\,dt$$

$$= a\,S\,[f(t)] + b\,S[g(t)]$$

2- $\boldsymbol{S[f(at)] = F(au)}$

$$S[f(at)] = \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}[f(at)]\,dt = F(au)$$

2.2 Modified Sumudu Transform:

Modified Sumudu transform was introduced to facilitate the process of solving ordinary and partial differential equations in the time domain.

Definition (2.2.1): Modified Sumudu transform denoted by the operator $E(.)$ defined by the integral equation:

$$E[\ f(t)\ ]=T(v)=\ v\int_{0}^{\infty} f(t)\,e^{\frac{-t}{v}}\ dt \quad ,t\ \geq 0\ ,\ k_1\leq v\leq k_2 \qquad (2.2)$$

2.2.1 Existence of Modified Sumudu Transforms:

The sufficient conditions for the existence of Modified Sumudu transform are that $f\,(t)$ for $t \geq 0$ be Piecewise continuous and of exponential order, Otherwise Modified Sumudu transform may or may not exist. That is means Modified Sumudu transform defined for function in the set A such that the set A defined by:

$$A=\left\{ f(t):\exists M\ ,k_1\ ,k_2\ >0,|\ f(t)|<Me^{\frac{|t|}{k_j}}\ ,if\ \ t\in(-1)^j\ X\ [0,\infty)\right\}$$

The constant M must be finite number, $k_1\ ,k_2$ may be finite or infinite.

Example(2.2.1): If $f(t)=1$ then

$$E(1)=v\int_{0}^{\infty} e^{\frac{-t}{v}}\ dt \quad = \quad v(-v)\ e^{\frac{-t}{v}}\Big|_{0}^{\infty} \ = \ -v^2\left[0-1\right]=v^2$$

$$\Rightarrow E(1) = \ v^2$$

Example (2.2.2): If $f(t)=t$ then

$$E(t)=v\int_{0}^{\infty} t\,e^{\frac{-t}{v}}\ dt \quad = \ v\left[\ t(-v e^{\frac{-t}{v}})\Big|_{0}^{\infty} + v\int_{0}^{\infty} e^{\frac{-t}{v}}\ dt\right]$$

$$=v^2\left[-v\ e^{\frac{-t}{v}}\right]_{0}^{\infty} = v^3$$

$$\Rightarrow E(t) = \ v^3$$

Example (2.2.3): If $f(t)=e^{at}$, then

$$E(e^{at})=v\int_{0}^{\infty} e^{at}\ e^{\frac{-t}{v}}\ dt \quad = \ v\int_{0}^{\infty} e^{-t(\frac{1}{v}-a)}\ \ dt$$

$$= v\left[\frac{v}{1-av}\; e^{-t\left(\frac{1}{v}-a\right)}\right]_0^{\infty} = \frac{v^2}{1-av}$$

$$\Rightarrow E(e^{at}) = \frac{v^2}{1-av}$$

Modified Sumudu transform (Elzaki Transform) of the some functions are listed in the following table

f(t)	E[f(t)]
e^{at}	$\frac{v^2}{1-av}$
$\cos(at)$	$\frac{v^2}{1+a^2v^2}$
$\sin(at)$	$\frac{av^3}{1+a^2v^2}$
$\cosh(at)$	$\frac{v^2}{1-a^2v^2}$
$\sinh(at)$	$\frac{av^3}{1-a^2v^2}$

Theorem (2.2.2): Let $T(v)$ is the Modified Sumudu transform of $f(t)$ [$E(f(t))=T(v)$],then

$$(i)\ E(f'(t)) = \frac{T(v)}{v} - v\, f(0)$$

$$(ii)\ E(f''(t)) = \frac{T(v)}{v^2} - f(0) - v\, f'(0)$$

$$(iii)\ E(f^{(n)}(t)) = \frac{T(v)}{v^n} - \sum_{k=0}^{n-1} v^{2-n+k} f^{(k)}(0)$$

Proof

$$(i)\, E(f'(t)) = v\int_0^\infty f'(t)e^{\frac{-t}{v}}\,dt$$

$$= v\left[f(t)e^{\frac{-t}{v}}\Big|_0^\infty + \frac{1}{v}\int_0^\infty f(t)e^{\frac{-t}{v}}\,dt \right]$$

$$= -v\, f(0) + \int_0^\infty f(t)e^{\frac{-t}{v}}\,dt$$

$$\Rightarrow E(f'(t)) = \frac{T(v)}{v} - v\, f(0)$$

(ii) $Let\ \ g(t) = f'(t)$, then

$$E(g'(t)) = \frac{1}{v}E(g(t)) - v\,g(0) \quad \text{, from (i)}$$

$$\Rightarrow E(f''(t)) = \frac{T(v)}{v^2} - f(0) - v\, f'(0)$$

(iii) By mathematical induction: for $n=1$, hold in (i) ,we assume it hold for n , and proved that it carries to $n+1$

$$\Rightarrow E(f^{(n+1)}(t)) = E(f^{(n)}(t))'$$

$$= \frac{E(f^{(n)}(t))}{v} - v\, f^{(n)}(0)$$

$$= \frac{1}{v}\left[\frac{T(v)}{v^n} - \sum_{k=0}^{n-1} v^{2-n+k}\, f^{(k)}(0)\right] - v\, f^{(n)}(0)$$

$$= \frac{T(v)}{v^{n+1}} - \sum_{k=0}^{n-1} v^{1-n+k}\, f^{(k)}(0) - v\, f^{(n)}(0)$$

$$= \frac{T(v)}{v^{n+1}} - \sum_{k=0}^{n} v^{1-n+k}\, f^{(k)}(0)$$

$$\Rightarrow E(f^{(n)}(t)) = \frac{T(v)}{v^n} - \sum_{k=0}^{n-1} v^{2-n+k}\, f^{(k)}(0)$$

Theorem (2.2.3): Modified Sumudu transform of partial derivatives are:

$$(i)\ E\left[\frac{\partial f(x,t)}{\partial t}\right] = \frac{T(x,v)}{v} - v\, f(x,0)$$

$$(ii)\ E\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right]=\frac{T(x,v)}{v^2}-f(x,0)-v\frac{\partial f(x,0)}{\partial t}$$

$$(iii)\ E\left[\frac{\partial f}{\partial x}\right]=\frac{d}{dx}\left[T(x,v)\right]$$

$$(iv)\ E\left[\frac{\partial^2 f}{\partial x^2}\right]=\frac{d^2}{dx^2}\left[T(x,v)\right]$$

Proof

We use integration by parts as follows:

$$(i)\ E\left[\frac{\partial f(x,t)}{\partial t}\right]=\int_0^\infty v\,\frac{\partial f}{\partial t}e^{-\frac{t}{v}}dt=\lim_{p\to\infty}\int_0^\infty v\,e^{-\frac{t}{v}}\frac{\partial f}{\partial t}dt$$

Let $u=ve^{-\frac{t}{v}}\ \Rightarrow\ du=-e^{-\frac{t}{v}}$ and $dv=\frac{\partial f}{\partial t}\Rightarrow v=f$

$$\Rightarrow E\left[\frac{\partial f(x,t)}{\partial t}\right]=\lim_{p\to\infty}\left[\left[ve^{-\frac{t}{v}}f(x,t)\right]_0^p+\int_0^p e^{-\frac{t}{v}}f(x,t)\,dt\right]$$

$$\Rightarrow E\left[\frac{\partial f(x,t)}{\partial t}\right]=\frac{T(x,v)}{v}-v\,f(x,0)$$

(ii) To find $E\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right]$, let $\frac{\partial f}{\partial t}=g$, then:

$$E\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right]=E\left[\frac{\partial g(x,t)}{\partial t}\right]=\frac{E[g(x,t)]}{v}-v\,g(x,0)$$

$$\Rightarrow E\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right]=\frac{T(x,v)}{v^2}-f(x,0)-v\frac{\partial f(x,0)}{\partial t}$$

$$(iii)\ E\left[\frac{\partial f}{\partial x}\right]=\int_0^\infty v\,e^{-\frac{t}{v}}\frac{\partial f(x,t)}{\partial x}dt=\frac{\partial}{\partial x}\int_0^\infty v\,e^{-\frac{t}{v}}f(x,t)\,dt=\frac{\partial}{\partial x}\left[T(x,v)\right]$$

$$\Rightarrow E\left[\frac{\partial f}{\partial x}\right] = \frac{d}{dx}\left[T(x,v)\right]$$

Also we can find: $E\left[\frac{\partial^2 f}{\partial x^2}\right] = \frac{d^2}{dx^2}\left[T(x,v)\right]$

* We can easily extend this result to the nth partial derivative by using mathematical induction.

2.2.2 Laplace – Modified Sumudu Duality:

Now we showed that modified Sumudu transform is the dual of Laplace transform. Hence, one should be able to reveal it to a great extent in problem solving. Defined for Re(s)> 0, the Laplace transform is given by:

$$F(s) = L(f(t)) = \int_0^\infty f(t)e^{-st}\,dt$$

According to above definition ELzaki and Laplace transforms exhibit a duality , relation expressed as follows:

$$T(v) = vF\left(\frac{1}{v}\right) \quad , \qquad F(s) = sT\left(\frac{1}{s}\right)$$

The (L E D) formula helps us to find inverse Elzaki transform by using contour integral and residues theorems we know:

$$f(t) = L^{-1}[F(s)] = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{st}\,F(s)\,ds$$

Where $s = x + i\,y$ a complex variable

Then from the relation between (L and E) transform we get

$$E^{-1}[T(s)] = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{st}\,sT\left(\frac{1}{s}\right)ds = \sum residues\ of\ \left[e^{st}\,sT\left(\frac{1}{s}\right)\right] \qquad (2.3)$$

We can apply above formula as follows:

Let $T(v) = \frac{v^3}{1+v} \Rightarrow sT\left(\frac{1}{s}\right) = s\left[\frac{\frac{1}{s^3}}{1+\frac{1}{s}}\right] = s\left[\frac{\frac{1}{s^3}}{\frac{s+1}{s}}\right] = s\left[\frac{1}{s^3}\,.\,\frac{s}{s+1}\right] = \frac{1}{s(s+1)}$

Then $E^{-1}[T(v)] = \sum residues\ of \left[\dfrac{e^{st}}{s(s+1)}\right]$, do occur at the poles $s=-1$ *and* $s=0$ with respective values $-e^{-t}$ and 1. Then $f(t) = 1 - e^{-t}$.

2.3 Modified Double Sumudu Transform:

Now we take modified version of double Sumudu transform which is called modified double Sumudu transform. This new transform rivals Sumudu transform and Laplace transform in problem solving.

Definition (2.3.1): Let $f(x,t)$, $t, x \in R^{+}$ be a function which can be expressed as a convergent infinite series, then its Modified Double Sumudu Transform given by:

$$E_2\left[f(x,t),u,v\right] = T(u,v) = uv\int_0^\infty\int_0^\infty f(x,t)e^{-\left(\frac{x}{u}+\frac{t}{v}\right)}dxdt \quad , x,t > 0 . \qquad (2.4)$$

where u, v are complex values.

Example (2.3.1): If $f(x,t) = x$ then

$$E_2(x) = uv\int_0^\infty\int_0^\infty x\, e^{-\left(\frac{x}{u}+\frac{t}{v}\right)}dx\, dt \quad = -uv^2\int_0^\infty x\ e^{-\left(\frac{x}{u}+\frac{t}{v}\right)}dx\Bigg|_0^\infty$$

$$= uv^2\int_0^\infty x\, e^{-\frac{x}{u}}dx$$

Use Integration by Parts gives:

$$E_2(x) = uv^2\left[-xue^{-\frac{x}{u}}\Big|_0^\infty + u\int_0^\infty e^{-\frac{x}{u}}dx\right]$$

$$= uv^2\left[-u^2e^{-\frac{x}{u}}\right]_0^\infty$$

$$= -u^3v^2\left[0-1\right] = u^3v^2$$

Example (2.3.2): If $f(x,t) = 2t$ then

$$E_2(2t) = uv\int_0^\infty\int_0^\infty 2t\, e^{-\left(\frac{x}{u}+\frac{t}{v}\right)}dx\, dt$$

Let $\begin{array}{l} u = 2t \Rightarrow du = 2 \\ dv = e^{-\left(\frac{x}{u}+\frac{t}{v}\right)} \Rightarrow v = -v\, e^{-\left(\frac{x}{u}+\frac{t}{v}\right)} \end{array}$

Use Integration by Parts gives:

$$E_2(2t) = uv\int_0^\infty \left[-2t\,v\, e^{-\left(\frac{x}{u}+\frac{t}{v}\right)}\Big|_0^\infty + 2v\int_0^\infty e^{-\left(\frac{x}{u}+\frac{t}{v}\right)} dt \right] dx$$

$$= uv\int_0^\infty \left[-2v^2 e^{-\left(\frac{x}{u}+\frac{t}{v}\right)}\Big|_0^\infty \right] dx = uv\int_0^\infty -2v^2\left(0 - e^{-\frac{x}{u}}\right) dx$$

$$= uv\int_0^\infty 2v^2 e^{-\frac{x}{u}}\, dx = -2u^2v^3\left[e^{-\frac{x}{u}}\right]_0^\infty$$

$$= -2u^2v^3\left[0-1\right] = 2u^2v^3$$

$$\Rightarrow E_2(2t) = 2u^2v^3$$

Example (2.3.3): If $f(x,t) = e^{x-t}$ then

$$E_2(e^{x-t}) = uv\int_0^\infty\int_0^\infty e^{x-t}\, e^{-\left(\frac{x}{u}+\frac{t}{v}\right)} dx\, dt$$

$$= uv\int_0^\infty\int_0^\infty e^{\left(x-\frac{x}{u}\right)} . e^{-\left(t+\frac{t}{v}\right)} dxdt$$

$$= uv\int_0^\infty\int_0^\infty e^{x\left(1-\frac{1}{u}\right)} . e^{-t\left(1+\frac{1}{v}\right)} dxdt$$

$$= uv\int_0^\infty e^{x\left(1-\frac{1}{u}\right)} . \left[\frac{-v\, e^{-t\left(1+\frac{1}{v}\right)}}{v+1}\right]_0^\infty dx$$

$$= \frac{uv^2}{v+1}\int_0^\infty e^{x\left(1-\frac{1}{u}\right)} dx$$

$$= \frac{uv^2}{v+1}\int_0^\infty e^{-x\left(\frac{1}{u}-1\right)} dx = \frac{uv^2}{v+1} . \frac{-u}{1-u} . e^{-x\left(\frac{1}{u}-1\right)}\Big|_0^\infty$$

$$= \frac{v^2}{v+1} . \frac{u^2}{1-u}$$

Example (2.3.4): If $f(x,t) = \sin t$ then

$$E_2(\sin t) = uv \int_0^\infty \int_0^\infty \sin t \; e^{-\left(\frac{x}{u}+\frac{t}{v}\right)} dx\, dt$$

$$= uv \int_0^\infty \sin t \left[-u\, e^{-\left(\frac{x}{u}+\frac{t}{v}\right)} \right]_0^\infty dt$$

$$= u^2 v \int_0^\infty \sin t \;\; e^{-\frac{t}{v}} \; dt$$

Let
$$u = \sin t \;\Rightarrow du = \cos t$$
$$dv = e^{-\frac{t}{v}} dt \;\;\Rightarrow\;\; v = -v\, e^{-\frac{t}{v}}$$

Use Integration by Parts gives:

$$E_2(\sin t) = u^2 v \int_0^\infty \sin t \;\; e^{-\frac{t}{v}} \; dt \; = u^2 v \left[-v\, e^{-\frac{t}{v}} \sin t \Big|_0^\infty + v \int_0^\infty \cos t \; e^{-\frac{t}{v}} dt \right]$$

$$= u^2 v \left[zero + v \int_0^\infty \cos t \; e^{-\frac{t}{v}} dt \right]$$

$$= u^2 v^2 \int_0^\infty \cos t \; e^{-\frac{t}{v}} dt$$

$$u = \cos t \;\Rightarrow du = -\sin t$$
$$dv = e^{-\frac{t}{v}} dt \;\;\Rightarrow\;\; v = -v\, e^{-\frac{t}{v}}$$

$$= u^2 v^2 \left[-v\, e^{-\frac{t}{v}} \cos t \Big|_0^\infty - v \int_0^\infty \sin t \; e^{-\frac{t}{v}} dt \right]$$

$$= u^2 v^2 \left[v - v \int_0^\infty \sin t \; e^{-\frac{t}{v}} dt \right]$$

$$= u^2 v^3 - u^2 v^3 \int_0^\infty \sin t \; e^{-\frac{t}{v}} dt$$

$$u^2 v \int_0^\infty \sin t \; e^{-\frac{t}{v}} dt \; + \; u^2 v^3 \int_0^\infty \sin t \; e^{-\frac{t}{v}} dt \; = u^2 v^3$$

$$u^2 v \int_0^\infty \sin t \; e^{-\frac{t}{v}} dt \left[1 + v^2 \right] = u^2 v^3$$

$$\Rightarrow E_2(\sin t) = u^2 v \int_0^\infty \sin t \; e^{-\frac{t}{v}} dt \; = \; \frac{u^2 v^3}{1+v^2}$$

Now to obtain Modified double Sumudu transform of partial derivatives we use integration by parts [24], and then we have:

$$E_2\left[\frac{\partial f}{\partial x}\right]=\frac{1}{u}T(u,v)-uT(0,v)$$

$$E_2\left[\frac{\partial^2 f}{\partial x^2}\right]=\frac{1}{u^2}T(u,v)-T(0,v)-u\frac{\partial}{\partial x}T(0,v)$$

$$E_2\left[\frac{\partial f}{\partial t}\right]=\frac{1}{v}T(u,v)-vT(u,0) \qquad (2.5)$$

$$E_2\left[\frac{\partial^2 f}{\partial t^2}\right]=\frac{1}{v^2}T(u,v)-T(u,0)-v\frac{\partial}{\partial t}T(u,0)$$

$$E_2\left[\frac{\partial^2 f}{\partial x\partial t}\right]=\frac{1}{uv}T(u,v)-\frac{v}{u}T(u,0)-\frac{u}{v}T(0,v)+uvT(0,0)$$

Proof:

$$E_2\left[\frac{\partial f}{\partial x}\right]=uv\int_0^\infty\int_0^\infty\frac{\partial}{\partial x}f(x,t)e^{-\left(\frac{x}{u}+\frac{t}{v}\right)}dxdt \quad =v\int_0^\infty e^{-\frac{t}{v}}\left[u\int_0^\infty e^{-\frac{x}{u}}\frac{\partial}{\partial x}f(x,t)dx\right]dt$$

The inner integral gives $\frac{1}{u}T(u,t)-u\,f(0,t)$

$$\Rightarrow E_2\left[\frac{\partial f}{\partial x}\right]=\frac{v}{u}\int_0^\infty e^{-\frac{t}{v}}T(u,t)\,dt-uv\int_0^\infty e^{-\frac{t}{v}}f(0,t)\,dt$$

$$\Rightarrow E_2\left[\frac{\partial f}{\partial x}\right]=\frac{1}{u}T(u,v)-uT(0,v)$$

Also $E_2\left[\frac{\partial f}{\partial t}\right]=\frac{1}{v}T(u,v)-vT(u,0)$

$$E_2\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right]=uv\int_0^\infty\int_0^\infty\frac{\partial^2 f(x,t)}{\partial x^2}e^{-\left(\frac{x}{u}+\frac{t}{v}\right)}dxdt \quad =v\int_0^\infty e^{-\frac{t}{v}}\left[u\int_0^\infty\frac{\partial^2 f(x,t)}{\partial x^2}e^{-\frac{x}{u}}dx\right]dt$$

The inner integral: $u\int_0^\infty\frac{\partial^2 f(x,t)}{\partial x^2}e^{-\frac{x}{u}}dx=\frac{T(u,t)}{u^2}-f(0,t)-u\frac{\partial f(0,t)}{\partial x}$.

By taking Modified Sumudu transform with respect to t for above integral we get:

$$E_2\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right]=\frac{1}{u^2}T(u,v)-T(0,v)-u\frac{\partial}{\partial x}T(0,v)$$

Similarly:

$$E_2\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right]=\frac{1}{v^2}T(u,v)-T(u,0)-v\frac{\partial}{\partial t}T(u,0)$$

Theorem (2.3.2): Consider a function f in the set A defined by:

$$f(x,t)=\left\{f(x,t)\in A:\exists M,k_1,k_2>0\ such\,that\left|f(x,t)\right|\le M\,e^{\frac{x+t}{k_i^2}}\ i=1,2\ and\,(x,t)\in R^2{}_+\right\}$$

With double Laplace transform $F(p,s)$, and Modified double Sumudu transform $T(u,v)$

Then: $T(u,v)=uv\,F\left(\frac{1}{u},\frac{1}{v}\right), where\ M,k_1,k_2\in R^+$

Proof: Let $f(x,t)\in A\ and\ k_1<u,v<k_2, T(u,v)=u^2v^2\int_0^\infty\int_0^\infty f(ux,vt)e^{-(x+t)}dxdt$

Let $\eta=ux\ and\ \lambda=vt$, we have

$$T(u,v)=u^2v^2\int_0^\infty\int_0^\infty f(ux,vt)e^{-(x+t)}dxdt\ =\ uv\int_0^\infty\int_0^\infty f(\eta,\lambda)e^{-\left(\frac{\eta}{u}+\frac{\lambda}{v}\right)}d\eta d\lambda=uv\,F\left(\frac{1}{u},\frac{1}{v}\right)$$

$$\Rightarrow T(u,v)=uv\,F\left(\frac{1}{u},\frac{1}{v}\right)$$

Definition (2.3.3): Let $f(x,t)\ and\ g(x,t)$ be a piecewise continuous function on $[0,\infty)$ and having double Laplace transform $F(p,s)\ and\ G(p,s)$ respectively, then the double
Convolution of the functions $f(x,t)$ and $g(x,t)$ exist and defined by:

$$(f**g)(x,t)=\int_0^t\int_0^x f(\alpha,\beta)\,g(x-\alpha,t-\beta)\,d\alpha\,d\beta$$

$$L_xL_t\left[(f**g)(x,t);(p,s)\right]=F(p,s)G(p,s)$$

Theorem (2.3.4): Let $f(x,t)\ and\ g(x,t)$ be defined in A and having the double Laplace transform $F(p,s)\ and\ G(p,s)$ respectively, and also having Modified double Sumudu transform $M(u,v)\ and\ N(u,v)$ respectively, then the Modified double Sumudu transform of the convolution of $f(x,t)$ *and* $g(x,t)$ is given by:

$$E_2\left[(f**g)(x,t);(u,v)\right]=\frac{1}{uv}M(u,v)N(u,v)$$

Proof: The Laplace transform of $(f**g)(x,t)$ is given by

$$L_xL_t\left[(f**g)(x,t);(p,s)\right]=F(p,s)G(p,s)$$

From theorem (2.3.2) we have:

$$E_2\left[(f**g)(x,t);(u,v)\right]= uvL_xL_t\left[(f**g)(x,t);(p,s)\right]$$

Since $M(u,v)=uvF\left(\frac{1}{u},\frac{1}{v}\right), N(u,v)=uvG\left(\frac{1}{u},\frac{1}{v}\right)$ then

$$E_2\left[(f**g)(x,t);(u,v)\right]= uv\left[F\left(\frac{1}{u},\frac{1}{v}\right)G\left(\frac{1}{u},\frac{1}{v}\right)\right]=uv\left[\frac{M(u,v)}{uv}\cdot\frac{N(u,v)}{uv}\right]=\frac{1}{uv}M(u,v)N(u,v)$$

2.3.1 Convergence of Modified Double Sumudu Transform

Here we need to discuss some theorems of convergence of Modified Double Sumudu Transform

Theorem (2.3.5): Let the function $f(x,t)$ is continuous in the $xt-$ plane, if the integral converges at $u=u_0$, $v=v_0$ then the integral, $uv\int_0^\infty\int_0^\infty f(x,t)e^{-\left(\frac{x}{u}+\frac{t}{v}\right)}dxdt$ is convergence for $u<u_0$, $v<v_0$.

For the proof we will use the following theorems.

Theorem (2.3.6): Suppose that: $v\int_0^\infty f(x,t)e^{-\frac{t}{v}}dt$, converges at $v=v_0$, then the integral converges for $v<v_0$

Proof:

Let $\alpha(x,t)=v_0\int_0^t f(x,s)e^{-\frac{s}{v_0}}ds \quad , \; 0<t<\infty$

Clearly $\alpha(x,0)=0$ and $\lim_{t\to\infty}\alpha(x,t)$ exist.

By fundamental theorem of calculus we have:

$$\alpha_t(x,t)= v_0\ f(x,t)e^{-\frac{t}{v_0}} \tag{2.6}$$

If we choose $\in_1$ and R_1 such that ($0<\in_1<R_1$) and using equation (2.6) we get:

$$v\int_{\in_1}^{R_1} f(x,t)e^{-\frac{t}{v}}dt = v\int_{\in_1}^{R_1}\frac{1}{v_0}e^{\frac{t}{v_0}}\alpha_t(x,t)e^{-\frac{t}{v}}dt = \frac{v}{v_0}\int_{\in_1}^{R_1}\alpha_t(x,t)e^{-\left(\frac{v_0-v}{vv_0}\right)t}dt$$

Integrating the last integral by parts to gives:

$$\frac{v}{v_0}\int_{\in_1}^{R_1}\alpha_t(x,t)e^{-\left(\frac{v_0-v}{vv_0}\right)t}dt = \frac{v}{v_0}\left[\left(\alpha(x,t)e^{-\left(\frac{v_0-v}{vv_0}\right)t}\right)_{\in_1}^{R_1} - \int_{\in_1}^{R_1}\alpha(x,t).e^{-\left(\frac{v_0-v}{vv_0}\right)t}.-\left(\frac{v_0-v}{vv_0}\right)dt\right]$$

$$=\frac{v}{v_0}\left[\alpha(x,R_1)e^{-\left(\frac{v_0-v}{vv_0}\right)R_1} - \alpha(x,\in_1)e^{-\left(\frac{v_0-v}{vv_0}\right)\in_1} + \left(\frac{v_0-v}{vv_0}\right)\int_{\in_1}^{R_1}\alpha(x,t)e^{-\left(\frac{v_0-v}{vv_0}\right)t}dt\right]$$

Now let $\in_1\to 0$, $R_1\to\infty$, if $v<v_0$, then we have

$$v\int_0^{\infty} f(x,t)e^{-\frac{t}{v}}dt=\left(\frac{v_0-v}{v_0^{\ 2}}\right)\int_0^{\infty}\alpha(x,t)e^{-\left(\frac{v_0-v}{vv_0}\right)t}dt$$

Now if the integral on the right converges then the theorem is proved.

By using limit test for convergence we get:

$$\lim_{t\to\infty} t^2\alpha(x,t)e^{-\left(\frac{v_0-v}{vv_0}\right)t} = \lim_{t\to\infty}\left(\frac{t^2}{e^{\left(\frac{v_0-v}{vv_0}\right)t}}\right).\lim_{t\to\infty}(\alpha(x,t))$$

The first limit equal zero at $t\to\infty$ if $v<v_0$ and the second limit exist, then

$$\lim_{t\to\infty} t^2\alpha(x,t)e^{-\left(\frac{v_0-v}{vv_0}\right)t}=0, \text{finite.}$$

Then the integral $v\int_0^\infty f(x,t)e^{-\frac{t}{v}}dt$ is converges at $v < v_0$.

Theorem (2.3.7): Suppose that: $u\int_0^\infty f(x,t)e^{-\frac{x}{u}}dx$, converges at $u = u_0$, then the integral converges for $u < u_0$

Proof

Prove, of this theorem is same as the method in Theorem (2.3.6).

Now the proof of the theorem (2.3.5) is as follows

$$uv\int_0^\infty\int_0^\infty f(x,t)e^{-\left(\frac{x}{u}+\frac{t}{v}\right)}dxdt = u\int_0^\infty e^{-\frac{x}{u}}\left[v\int_0^\infty e^{-\frac{t}{v}} f(x,t)dt\right]dx \qquad (2.7)$$

By using theorem (2.3.6) and theorem (2.3.7) we see the integral in RHS of equation (2.7) is converges for $u < u_0$, $v < v_0$, hence the integral $uv\int_0^\infty\int_0^\infty f(x,t)e^{-\left(\frac{x}{u}+\frac{t}{v}\right)}dxdt$ converges for $u < u_0$, $v < v_0$ ([28] [29]).

2.4 On Some Applications of Modified Double Sumudu Transform to Integro-Partial Differential Equations:

To solve integro-partial differential equations by using Modified double Sumudu transform first convert proposed equation to an algebraic equation, solving this algebraic equation and applying inverse Modified double Sumudu transform we obtain the exact solution of the problem

Example (2.4.1):

Consider the PIDE

$$u_{tt} = u_x + 2\int_0^t (t-s).u(x,s)ds - 2e^x$$

With initial conditions: $u(x,0) = e^x$, $u_t(x,0) = 0$

And boundary conditions: $u(0,t) = \cos t$

Solution:

We take Modified double Sumudu transform for equation:

$$\frac{1}{v^2}T(u,v)-T(u,0)-v\frac{\partial}{\partial t}T(u,0)=\frac{1}{u}T(u,v)-uT(0,v)+2v^2T(0,v)-2\frac{u^2v^2}{1-u}$$

We take single Modified Sumudu transform for conditions

$$T(u,0)=\frac{u^2}{1-u} \quad , \quad \frac{\partial T(u,0)}{\partial t}=0 \quad , \quad T(0,v)=\frac{v^2}{1+v^2}$$

By substituting:

$$\frac{1}{v^2}T(u,v)-\frac{u^2}{1-u}=\frac{1}{u}T(u,v)-\frac{uv^2}{1+v^2}+2v^2T(u,v)-\frac{2u^2v^2}{1-u}$$

$$\left(\frac{1}{u}+2v^2-\frac{1}{v^2}\right)T(u,v)=\frac{uv^2}{1+v^2}-\frac{u^2}{1-u}+\frac{2u^2v^2}{1-u}$$

Multiply both sides by uv^2

$$\Rightarrow\left(v^2+2uv^4-u\right)T(u,v)=\frac{u^2v^4}{1+v^2}-\frac{u^3v^2}{1-u}+\frac{2u^3v^4}{1-u}$$

$$=\frac{u^2v^4-u^3v^4-u^3v^2-u^3v^4+2u^3v^4+2u^3v^6}{(1+v^2)(1-u)}$$

$$\Rightarrow\left(v^2+2uv^4-u\right)T(u,v)=\frac{u^2v^2\left(v^2+2uv^4-u\right)}{(1+v^2)(1-u)}$$

$$\Rightarrow T(u,v)=\frac{u^2v^2}{(1+v^2)(1-u)}$$

Appling inverse Modified double Sumudu transform we get:

$$u(x,t)=e^x\cos t$$

Chapter 3

Adomian Decomposition Method

The Adomian Decomposition method was introduced and developed by George Adomian.

The method has a several advantages it's powerful, effective, and can easily handle a wide class of linear or non-linear, ordinary or partial differential equations, and linear and non-linear integral equations, and it's attacks the problem in a direct way and in a straightforward without any need to restrictive assumptions such as linearization, discretization or perturbation, there is no need is using this method to convert in-homogenous conditions to homogenous conditions as required by other techniques.

3.1 Solving linear ODEs and PDEs by Adomian Decomposition Method:

The adomian decomposition method consist of decomposing the unknown function $u(x,y)$ of any equation into a sum of an infinite number of components defined by the decomposition series

$$u(x,y)=\sum_{n=0}^{\infty}u_n(x,y) \tag{3.1}$$

Where the components $u_n(x,y),\ n\geq 0$ are to be determined in a recursive manner.

We first consider the linear differential equation in an operator from by:

$$Lu+Ru=g \tag{3.2}$$

Where L is mostly the lower order derivative which is assumed to be invertible, R is other linear differential operator, and g is a source term. We next apply the inverse operator L^{-1} to both sides of equation (3.2) and using the given condition to obtain:

$$u=f-L^{-1}(Ru) \tag{3.3}$$

Where the function f represents the terms arising from integration the source term g and from using the given conditions that are assumed to be prescribed. Adomian method defines the solution u by an infinite series of components given by

$$u = \sum_{n=0}^{\infty} u_n \tag{3.4}$$

Where the components $u_0, u_1, u_2, \ldots$ are usually recurrently determined. Substituting (3.4) into both sides of (3.3) we get

$$\sum_{n=0}^{\infty} u_n = f - L^{-1}\left(R\left(\sum_{n=0}^{\infty} u_n \right)\right) \tag{3.5}$$

Equation (3.5) can be rewritten as

$$u_0 + u_1 + u_2 + \ldots = f - L^{-1}\left(R(u_0 + u_1 + u_2 + \cdots)\right) \tag{3.6}$$

We need to find the components $u_0, u_1, u_2, \ldots$

The Adomian method suggests that the zeroth component u_0 is usually defined by the function f described above by all terms that are not included under the inverse operator L^{-1}, which arise from the initial data and from integrating the inhomogeneous term. Accordingly the formal recursive relation is defined by

$$\begin{aligned} u_0 &= f \\ u_{k+1} &= -L^{-1}(R(u_k)), \quad k \geq 0 \end{aligned} \tag{3.7}$$

Or equivalently

$$\begin{aligned} u_0 &= f \\ u_1 &= -L^{-1}(R(u_0)) \\ u_2 &= -L^{-1}(R(u_1)) \\ u_3 &= -L^{-1}(R(u_2)) \\ &\vdots \end{aligned} \tag{3.8}$$

After determined these components, we then substitute them into (3.4) to obtain the solution in a series form.

Now consider the problem**:**

$$u'(x) = u(x), \qquad u(0) = A \tag{3.9}$$

In an operator form, we get

$$Lu = u \tag{3.10}$$

Where

$$L = \frac{d}{dx} \tag{3.11}$$

And

$$L^{-1}(\cdot) = \int_0^x (\cdot)dx \tag{3.12}$$

Applying L^{-1} to both sides of (3.10) and using the initial condition we obtain

$$L^{-1}(Lu) = L^{-1}(u) \tag{3.13}$$

So that

$$u(x) - u(0) = L^{-1}(u) \tag{3.14}$$

$$u(x) = A + L^{-1}(u) \tag{3.15}$$

$$\sum_{n=0}^{\infty} u_n(x) = A + L^{-1}\left(\sum_{n=0}^{\infty} u_n(x)\right) \tag{3.16}$$

Then

$$\begin{aligned} &u_0(x) = A \\ &u_{k+1}(x) = L^{-1}(u_k(x)), \quad k \geq 0 \end{aligned} \tag{3.17}$$

Follows immediately consequently, we obtain,

$$\begin{aligned} u_0(x) &= A \\ u_1(x) &= L^{-1}(u_0(x)) = L^{-1}(A) = Ax \\ u_2(x) &= L^{-1}(u_1(x)) = L^{-1}(Ax) = \frac{Ax^2}{2!} \\ u_3(x) &= L^{-1}(u_2(x)) = L^{-1}(\frac{Ax^2}{2!}) = \frac{Ax^3}{3!} \\ &\vdots \end{aligned} \tag{3.18}$$

Substituting (3.18) into (3.4), we get

$$u(x) = A\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) \tag{3.19}$$

And in a closed form by

$$u(x) = Ae^{x} \tag{3.20}$$

Also consider the Airy's equations

$$u''(x) = xu(x), \qquad u(0) = A, u'(0) = B \tag{3.21}$$

In an operator form we get

$$Lu = xu \tag{3.22}$$

Where

$$L = \frac{d^2}{dx^2} \text{ and } L^{-1}(\cdot) = \int_0^x \int_0^x (\cdot) dx dx$$

Applying L^{-1} to both sides of (3.22) and using the initial conditions

$$L^{-1}(Lu) = L^{-1}(xu) \tag{3.23}$$

$$u(x) - xu'(0) - u(0) = L^{-1}(xu) \tag{3.24}$$

$$u(x) = A + Bx + L^{-1}(xu) \tag{3.25}$$

$$\Rightarrow \sum_{n=0}^{\infty} u_n(x) = A + Bx + L^{-1}\left(x \sum_{n=0}^{\infty} u_n(x) \right) \tag{3.26}$$

$$\begin{aligned} &\Rightarrow u_0(x) = A + Bx \\ &u_{k+1}(x) = L^{-1}\left(xu_k(x)\right), \qquad k \geq 0 \end{aligned} \tag{3.27}$$

Consequently we obtain:

$$\begin{aligned} &u_0(x) = A + Bx \\ &u_1(x) = L^{-1}(x\,u_0(x)) = L^{-1}(Ax + Bx^2) = \frac{Ax^3}{6} + \frac{Bx^4}{12} \\ &u_2(x) = L^{-1}(x\,u_1(x)) = L^{-1}(\frac{Ax^4}{6!} + \frac{Bx^5}{12!}) = \frac{Ax^6}{180} + \frac{Bx^7}{504} \\ &\qquad \vdots \end{aligned} \tag{3.28}$$

Substituting (3.28) into (3.4) we get

$$u(x) = A\left(1 + \frac{x^3}{6} + \frac{x^6}{180} + \cdots\right) + B\left(x + \frac{x^4}{12} + \frac{x^7}{504} + \cdots\right) \tag{3.29}$$

Other components can be easily computed to enhance the accuracy of the approximation.

Next we apply Adomian decomposition method to first-order partial differential equations, consider the inhomogeneous partial differential equations,

$$u_x + u_y = f(x, y), \qquad u(0, y) = g(y), \quad u(x, 0) = h(x) \tag{3.30}$$

In an operator form Equation (3.30) can be written as:

$$L_x u + L_y u = f(x, y) \tag{3.31}$$

Where

$$L_x = \frac{\partial}{\partial x}, \quad L_y = \frac{\partial}{\partial y} \quad \text{and } L^{-1}(\cdot) = \int_0^x (\cdot)dx, \; L_y^{-1}(.) = \int_0^y (.)dy \tag{3.32}$$

This means that

$$L_x^{-1} L_x u(x, y) = u(x, y) - u(0, y) \tag{3.33}$$

Applying L_x^{-1} to both sides of equation (3.31) gives

$$L_x^{-1} L_x u = L_x^{-1}(f(x, y)) - L_x^{-1}(L_y u) \tag{3.34}$$

Or equivalently:

$$u(x, y) = g(y) + L_x^{-1}(f(x, y)) - L_x^{-1}(L_y u) \tag{3.35}$$

We obtained above equation by using equation (3.33) and condition $u(0, y) = g(y)$.

Now the decomposition method sets:

$$u(x, y) = \sum_{n=0}^{\infty} u_n(x, y) \tag{3.36}$$

Substituting (3.36) into both sides of (3.35) we get

$$\sum_{n=0}^{\infty} u_n(x, y) = g(y) + L_x^{-1}(f(x, y)) - L_x^{-1}\left(L_y \left(\sum_{n=0}^{\infty} u_n(x, y) \right)\right) \tag{3.37}$$

This can be re written as

$$u_0 + u_1 + u_2 + \cdots = g(y) + L_x^{-1}(f(x, y)) - L_x^{-1} L_y (u_0 + u_1 + u_2 + \cdots) \tag{3.38}$$

We set

$$u_0(x, y) = g(y) + L_x^{-1}(f(x, y))$$

and

$$u_{k+1}(x,y) = -L_x^{-1}L_y(u_k), \qquad k \geq 0 \tag{3.39}$$

Then we obtain the recursive scheme:

$$\begin{aligned} u_0(x,y) &= g(y) + L_x^{-1}(f(x,y)) \\ u_1(x,y) &= -L_x^{-1}\left(L_y u_0(x,y)\right) \\ u_2(x,y) &= -L_x^{-1}\left(L_y u_1(x,y)\right) \\ u_3(x,y) &= -L_x^{-1}\left(L_y u_2(x,y)\right) \end{aligned} \tag{3.40}$$

And so on, after determined the more components $u_n(x,y)$ substituting in equation (3.36) to find approximate solution.

It is important to note that the solution can also be obtained by finding the y-solution by applying the inverse operator L_y^{-1} to both sides of the equation

$$L_y = f(x,y) - L_x u \tag{3.41}$$

And complete similar to the $x-$solution.

The essential steps of the decomposition method for linear and nonlinear equations, homogenous and inhomogeneous can be outlined as follow:

i- Express the partial differential equations, linear or nonlinear in an operator form.

ii- Apply the inverse operator to both sides of the equation written in an operator form.

iii- Set the unknown function $u(x,y)$ into a decomposition series

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y)$$

We next substitute above series into both sides of the resulting equation.

iv- Identify the zeroth component $u_0(x,y)$ as terms arising from the given conditions and from integration the source term $f(x,y)$.

v- Determine the successive components of the series solution $u_k, k \geq 1$ by applying the recursive scheme (3.39), where each component u_k can be completely determined by using the pervious component u_{k-1}

vi- Substituting the determined components into decomposition series to obtain the solution in a series form.

An exact solution can be easily obtained in many equations if such a closed form solution exists.

The essential steps of the adomian decomposition method will be illustrated by the following examples:

Example (3.1.1): Solve the following partial differential equation

$$u_x + u_y = 2xy^2 + 2x^2y, \quad u(x,0) = 0, \; u(0,y) = 0$$

Solution

$$L_x u + L_y u = 2xy^2 + 2x^2y$$

$$L_x u = 2xy^2 + 2x^2y - L_y u$$

$$L_x^{-1} L_x u = L_x^{-1}\left(2xy^2 + 2x^2y\right) - L_x^{-1} L_y u$$

$$u(x,y) = 0 + L_x^{-1}\left(2xy^2 + 2x^2y - L_y u\right)$$

$$\sum_{n=0}^{\infty} u_n(x,y) = L_x^{-1}\left(2xy^2 + 2x^2y - L_y \sum_{n=0}^{\infty} u_n(x,y)\right)$$

$$u_0 + u_1 + u_2 + \cdots = x^2y^2 + \frac{2}{3}x^3y - L_x^{-1} L_y \left(u_0 + u_1 + \cdots\right)$$

$$u_0 = x^2y^2 + \frac{2}{3}x^3y$$

$$u_{k+1}(x,y) = -L_x^{-1}\left(L_y(u_k)\right)$$

$$u_1(x,y) = -L_x^{-1}\left(L_y\left(x^2y^2 + \frac{2}{3}x^3y\right)\right) = -L_x^{-1}\left(2x^2y^2 + \frac{2}{3}x^3\right)$$

$$= -\frac{2}{3}x^3y - \frac{2}{12}x^4$$

$$u_2(x,y) = -L_x^{-1}\left(L_y\left(-\frac{2}{3}x^3y - \frac{2}{12}x^4\right)\right)$$

$$= -L_x^{-1}\left(-\frac{2}{3}x^3\right) = \frac{2}{12}x^4$$

$$u_3(x,y) = -L_x^{-1}\left(L_y\left(\frac{2}{12}x^4\right)\right) = 0$$

$$\Rightarrow u(x,y) = x^2y^2 + \frac{2}{3}x^3y - \frac{2}{3}x^3y - \frac{1}{6}x^4 + \frac{1}{6}x^4$$

$$\Rightarrow u(x,y) = x^2y^2$$

Example (3.1.2): Solve the following partial differential equation:

$$xu_x + u_y = u, \qquad u(x,0) = 1+x, \quad u(0,y) = e^y$$

Solution

$$xL_xu + L_yu = u$$

$$L_yu = u - xL_xu$$

$$L_y^{-1}L_yu = L_y^{-1}u - L_y^{-1}(xL_xu)$$

$$u(x,y) = 1 + x + L_y^{-1}(u - xL_xu)$$

$$\sum_{n=0}^{\infty} u_n(x,y) = 1 + x + L_y^{-1}\left(\sum_{n=0}^{\infty} u_n(x,y) - xL_x\sum_{n=0}^{\infty} u_n(x,y)\right)$$

$$\Rightarrow u_0 = 1 + x$$

$$u_{k+1}(x,y) = L_y^{-1}\left(u_k - xL_xu_k\right), k \geq 0$$

$$u_1(x,y) = L_y^{-1}\left(1 + x - xL_x(1+x)\right) = L_y^{-1}\left(1 + x - x\right) = y$$

$$u_2(x,y) = L_y^{-1}\left(y - xL_xy\right) = \frac{1}{2}y^2$$

$$u_3(x,y) = L_y^{-1}\left(\frac{1}{2}y^2 - xL_x\frac{1}{2}y^2\right) = \frac{1}{6}y^3$$

$$u_n(x,y) = \frac{y^n}{n!}, \qquad n = 1,2,3,\ldots$$

$$u(x,y) = 1 + x + \sum_{n=0}^{\infty}\frac{y^n}{n!} = x + 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$$

$$\Rightarrow u(x,y) = x + e^y$$

Example (3.1.3): Solve the following partial differential equation

$$u_x + yu_y + zu_z = 3u, \qquad u(0,y,z) = yz, \; u(x,0,z) = u(x,y,0) = 0$$

Solution

$$L_x u + yL_y u + zL_z u = 3u$$
$$L_x u = 3u - yL_y u - zL_z u$$
$$L_x^{-1} L_x u = L_x^{-1}(3u - yL_y u - zL_z u)$$
$$u(x,y,z) = yz + L_x^{-1}\left(3\sum_{n=0}^{\infty} u_n - yL_y \sum_{n=0}^{\infty} u_n - zL_z \sum_{n=0}^{\infty} u_n\right)$$
$$\Rightarrow u_0(x,y,z) = yz$$
$$u_{k+1}(x,y,z) = L_x^{-1}(3u_k - yL_y u_k - zL_z u_k),\ k \geq 0$$
$$u_1(x,y,z) = L_x^{-1}(3u_0 - yL_y u_0 - zL_z u_0)$$
$$= L_x^{-1}(3yz - yL_y yz - zL_z yz)$$
$$= L_x^{-1}(3yz - yz - zy) = xyz$$
$$u_2(x,y,z) = L_x^{-1}(3xyz - yL_y xyz - zL_z xyz)$$
$$= L_x^{-1}\left(3xyz - xyz - xyz\right) = \frac{1}{2}x^2 yz$$
$$u_3(x,y,z) = L_x^{-1}\left(\frac{3}{2}x^2 yz - yL_y \frac{1}{2}x^2 yz - zL_z \frac{1}{2}x^2 yz\right)$$
$$= L_x^{-1}\left(\frac{3}{2}x^2 yz - \frac{1}{2}x^2 yz - \frac{1}{2}x^2 yz\right) = \frac{1}{6}x^3 yz$$
$$u_4(x,y,z) = \frac{1}{24}x^4 yz$$
$$u_n(x,y,z) = \sum_{n=0}^{\infty} \frac{1}{n!}x^n yz$$
$$u(x,y,z) = (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots)yz$$
$$\Rightarrow u(x,y,z) = yze^x$$

3.1.1 The Noise Terms Phenomenon

Now we will present a useful tool that will accelerate the convergence of the Adomian decomposition method.

The noise terms phenomenon provides a major advantage in that it demonstrates a fast convergence of the solution. It is important to note that the noise terms phenomenon that will be introduced, may appear only for inhomogeneous PDEs. In addition, this phenomenon is applicable to all inhomogeneous PDEs of any order. The noise terms, if existed in the components u_0 and u_1 , will provide, in general, the solution in a closed form with only two successive iterations.

In view of these remarks, we now outline the ideas of the noise terms:

1. The noise terms are defined as the identical terms with opposite signs that may appear in the components u_0 and u_1 .
2. The noise terms appear only for specific of inhomogeneous equations whereas noise terms do not appear for homogeneous equations.
3. The noise terms appear if the exact solution is part of zeroth component u_0 .
4. Verification that the remaining non-canceled terms satisfy the equation is necessary and essential.

The phenomenon of the useful noise terms will be explained by the following examples.

Example (3.1.4): Consider the inhomogeneous PDE:

$$u_x + u_y = (1 + x)e^y \ , u(0, y) = 0 \ , u(x,0) = x \ .$$

The inhomogeneous PDE can be rewritten in an operator form by

$$L_x u = (1+x)e^y - L_y u .$$

Applying L_x^{-1} to both sides and using the given condition leads to

$$u(x, y) = \left(x + \frac{x^2}{2!} \right) e^y - L_x^{-1}\left(L_y u \right) \ .$$

Substituting $u(x, y) = \sum_{n=0}^{\infty} u_n(x, y)$ into both sides gives

$$\sum_{n=0}^{\infty} u_n(x,y) = \left(x + \frac{x^2}{2!}\right) e^y - L_x^{-1}\left(L_y\left(\sum_{n=0}^{\infty} u_n(x,y)\right)\right),$$

Proceeding as before, the components $u_0, u_1, u_2, ...$ are determined in a recursive manner by

$$u_0(x,y) = \left(x + \frac{x^2}{2!}\right) e^y ,$$

$$u_1(x,y) = -L_x^{-1}\left(L_y u_0\right) = -\left(\frac{x^2}{2!} + \frac{x^3}{3!}\right) e^y ,$$

$$u_2(x,y) = -L_x^{-1}\left(L_y u_1\right) = \left(\frac{x^3}{3!} + \frac{x^4}{4!}\right) e^y$$

Considering the first two components u_0 and u_1, it is easily observed that the noise terms $\frac{x^2}{2!} e^y$ and $-\frac{x^2}{2!} e^y$ appear in u_0 and u_1 respectively. By canceling the noise terms in u_0, and by verifying that the remaining non-canceled terms of u_0, we find that the exact solution is given by

$$u(x,y) = x e^y .$$

3.1.2 Solution Heat Equation by Adomian Decomposition Method.

The initial boundary value problem that controls the heat conduction in a rod in one, two and three dimensional is given respectively by:

PDE	$u_t = \bar{k} u_{xx} \quad 0 < x < L, t > 0$	
BC	$u(0,t) = 0, \quad t \geq 0$	
	$u(L,t) = 0, \quad t \geq 0$	(3.42)
IC	$u(x,0) = f(x)$	

PDE	$u_t = \bar{k}(u_{xx} + u_{yy}), \quad 0 < x < a, \; 0 < y < b, t > 0$	
BC	$u(0,y,t) = u(a,y,t) = 0$	
	$u(x,0,t) = u(x,b,t) = 0$	(3.43)

IC $\quad u(x,y,0) = f(x,y)$

$$
\begin{aligned}
&\text{PDE} && u_t = \bar{k}(u_{xx} + u_{yy} + u_{zz}),\ 0 < x < a, 0 < y < b, 0 < z < c, t > 0 \\
&\text{BC} && u(0,y,z,t) = u(a,y,z,t) = 0 \\
& && u(x,0,z,t) = u(x,b,z,t) = 0 \\
& && u(x,y,0,t) = u(x,y,c,t) = 0 \\
&\text{IC} && u(x,y,z,0) = f(x,y.z)
\end{aligned}
\tag{3.44}
$$

Where $u \equiv u(x,t)$ represent the temperature of the rod at the position x at time t in one dimensional, $u \equiv u(x,y.t)$ is the temperature of any point located at the position (x,y) of a rectangular plate at any time t in two dimensional and $u \equiv u(x,y,z,t)$ is the temperature of any point located at the position (x,y,z) of a rectangular volume at any time t in three dimensional and $\bar{k}$ is the thermal diffusivity of the material that measures the rod ability to heat conduction.

The boundary conditions **(**BC**)** that describe the temperature u at both ends of the rod and the initial condition (IC) that describe the temperature u at time $t = 0$.

The heat equation in one, two, and three dimensional arises in two different types namely:

1- Homogeneous Heat Equation:

This type of equations is often given respectively by:

$$
\begin{aligned}
&u_t = \bar{k}u_{xx}\ , \quad 0 < x < L, t > 0 \\
&u_t = \bar{k}(u_{xx} + u_{yy}), \quad 0 < x < a, 0 < y < b, t > 0 \\
&u_t = \bar{k}(u_{xx} + u_{yy} + u_{zz}),\ 0 < x < a, 0 < y < b, 0 < z < c, t > 0
\end{aligned}
\tag{3.45}
$$

Further, heat equation with a lateral heat loss is formally derived as a homogeneous partial differential equation respectively of the form:

$$
\begin{aligned}
&u_t = \bar{k}u_{xx} - u,\ \ 0 < x < L, t > 0 \\
&u_t = \bar{k}(u_{xx} + u_{yy}) - u,\ \ 0 < x < a,\ 0 < y < b,\ t > 0 \\
&u_t = \bar{k}(u_{xx} + u_{yy} + u_{zz}) - u,\ 0 < x < a,\ 0 < y < b,\ 0 < z < c, t > 0
\end{aligned}
\tag{3.46}
$$

2- Inhomogeneous Heat Equation:

This type of equation contains one or more terms that do not dependent variable $u(x,t), u(x,y,t)$ and $u(x,y,z,t)$ respectively. It's often given by:

$$
\begin{aligned}
&u_t = \bar{k}u_{xx} + g(x),\ \ 0 < x < L, t > 0 \\
&u_t = \bar{k}(u_{xx} + u_{yy}) + g(x,y),\ \ 0 < x < a, 0 < y < b, t > 0 \\
&u_t = \bar{k}(u_{xx} + u_{yy} + u_{zz}) + g(x,y,z),\ \ 0 < x < a, 0 < y < b, 0 < z < c, t > 0
\end{aligned}
\tag{3.47}
$$

Where $g(x), g(x,y)$ and $g(x,y,z)$ is called the heat source which independent of time.

To solve equation (3.42) we first rewrite in an operator form as follows:

$$L_t u(x,t) = \bar{k}\ L_x u(x,t) \tag{3.48}$$

Where

$$L_t = \frac{\partial}{\partial t},\ \ L_x = \frac{\partial^2}{\partial x^2} \tag{3.49}$$

and

$$L_t^{-1} = \int_0^t (\cdot)dt,\ \ L_x^{-1} = \int_0^x\int_0^x (\cdot)dxdx. \tag{3.50}$$

This means that:

$$L_t^{-1} L_t u(x,t) = u(x,t) - u(x,0) \tag{3.51}$$

Applying L_t^{-1} to both sides of (3.48) and using the initial condition we get

$$u(x,t) = f(x) + \bar{k}\ L_t^{-1}\left(L_x u(x,t)\right) \tag{3.52}$$

From the adomian decomposition method we defined the series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{3.53}$$

Substituting (3.53) into both sides of (3.52) we get:

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x) + \bar{k} L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n(x,t) \right)\right) \tag{3.54}$$

From the adomian decomposition method we defined:

$$\begin{aligned} &u_0(x,t) = f(x) \\ &u_{k+1}(x,t) = \bar{k} L_t^{-1}\left(L_x\left(u_k(x,t) \right)\right), \quad k \geq 0 \end{aligned} \tag{3.55}$$

The components $u_0(x,t), u_1(x,t), u_2(x,t), \ldots$ are determined individually by:

$$\begin{aligned} &u_0(x,t) = f(x) \\ &u_1(x,t) = L_t^{-1} L_x(u_0) = f''(x)t \\ &u_2(x,t) = L_t^{-1} L_x(u_1) = f^{(4)}(x)\frac{t^2}{2!} \\ &u_3(x,t) = L_t^{-1} L_x(u_2) = f^{(6)}(x)\frac{t^3}{3!} \\ &\qquad\vdots \end{aligned} \tag{3.56}$$

Others components can be determined by the similar way.

Finally substituting (3.56) into (3.53) we get the solution $u(x,t)$ of the partial differential equation in a series form as follows:

$$u(x,t) = \sum_{n=0}^{\infty} f^{(2n)}(x)\frac{t^n}{n!} \tag{3.57}$$

To solve equation (3.43) we first rewrite in an operator form as follows:

$$L_t u(x,y,t) = \bar{k}(L_x u + L_y u) \tag{3.58}$$

Where

$$L_t = \frac{\partial}{\partial t},\; L_x = \frac{\partial^2}{\partial x^2},\; L_y = \frac{\partial^2}{\partial y^2} \tag{3.59}$$

and

$$L_t^{-1} = \int_0^t (\cdot)dt,\;\; L_x^{-1} = \int_0^x\int_0^x (\cdot)dxdx,\;\; L_y^{-1} = \int_0^y\int_0^y (\cdot)dydy \tag{3.60}$$

This means that:

$$L_t^{-1} L_t u(x,y,t) = u(x,y,t) - u(x,y,0) \tag{3.61}$$

Applying L_t^{-1} to both sides of (3.58) and using the initial condition we get

$$u(x,y,t) = f(x,y) + \bar{k}\, L_t^{-1}(L_x u + L_y u) \tag{3.62}$$

From the adomian decomposition method we defined the series

$$u(x,y,t)=\sum_{n=0}^{\infty}u_n(x,y,t) \tag{3.63}$$

Substituting (3.63) into both sides of (3.62) we get:

$$\sum_{n=0}^{\infty}u_n(x,y,t)=f(x,y)+\bar{k}L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty}u_n\right)+L_y\left(\sum_{n=0}^{\infty}u_n\right)\right), \tag{3.64}$$

From the adomian decomposition method we defined:

$$\begin{aligned}&u_0(x,y,t)=f(x,y)\\&u_{k+1}(x,y,t)=\bar{k}L_t^{-1}(L_xu_k+L_yu_k),\ k\geq 0\end{aligned} \tag{3.65}$$

Calculate the components $u_0(x,y,t),u_1(x,y,t),u_2(x,y,t),\ldots$ individually and substitute into (3.63) we get the solution $u(x,y,t)$ of the partial differential equation in a series form.

To solve equation (3.44) we first rewrite in an operator form as follows:

$$L_tu(x,y,z,t)=\bar{k}(L_xu+L_yu+L_zu) \tag{3.66}$$

where

$$L_t=\frac{\partial}{\partial t},\ L_x=\frac{\partial^2}{\partial x^2},\ L_y=\frac{\partial^2}{\partial y^2},\ L_z=\frac{\partial^2}{\partial z^2} \tag{3.67}$$

and

$$L_t^{-1}=\int_0^t(\cdot)dt,\ L_x^{-1}=\int_0^x\int_0^x(\cdot)dxdx,\ L_y^{-1}=\int_0^y\int_0^y(\cdot)dydy,\ L_z^{-1}=\int_0^z\int_0^z(\cdot)dzdz \tag{3.68}$$

This means that:

$$L_t^{-1}L_tu(x,y,z,t)=u(x,y,z,t)-u(x,y,z,0) \tag{3.69}$$

Applying L_t^{-1} to both sides of (3.66) and using the initial condition we get

$$u(x,y,z,t)=f(x,y,z)+\bar{k}L_t^{-1}(L_xu+L_yu+L_zu) \tag{3.70}$$

From the adomian decomposition method we defined the series

$$u(x,y,z,t)=\sum_{n=0}^{\infty}u_n(x,y,z,t) \tag{3.71}$$

Substituting (3.71) into both sides of (3.70) we get:

$$\sum_{n=0}^{\infty} u_n(x,y,z,t) = f(x,y,z) + \bar{k} L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n\right) + L_y\left(\sum_{n=0}^{\infty} u_n\right) + L_z\left(\sum_{n=0}^{\infty} u_n\right)\right) \quad (3.72)$$

From the adomian decomposition method we defined:

$$\begin{aligned} u_0(x,y,z,t) &= f(x,y,z) \\ u_{k+1}(x,y,z,t) &= \bar{k} L_t^{-1}(L_x u_k + L_y u_k + L_z u_k),\ k \geq 0 \end{aligned} \quad (3.73)$$

Calculate the components $u_0(x,y,z,t), u_1(x,y,z,t), u_2(x,y,z,t), \ldots$ individually and substitute into (3.71) we get the solution $u(x, y, z, t)$ of the partial differential equation in a series form.

We observe the solution for equation (3.42), (3.43), and (3.44) is obtained by using the initial condition only, but we can show that it's satisfies the given boundary conditions these solutions are obtained by using the inverse operator L_t^{-1}.

The solution for equation (3.42), (3.43), and (3.44) can also be obtained by using the inverse operator L_x^{-1} & L_x^{-1}, L_y^{-1} & $L_x^{-1}, L_y^{-1}, L_z^{-1}$ respectively. In this case we use the boundary conditions and initial condition for this reason the solution of partial differential equation in the t direction reduces the size of computational work compare with the other directions.

Now we have chosen several examples of one dimensional, two dimensional and three dimensional heat equation homogeneous and in homogeneous.

Example (3.1.5): Use the adomian decomposition method to solve the initial-boundary value problems:

PDE $\quad u_t = u_{xx},\ 0 < x < \pi, t > 0$

BC $\quad u(0,t) = 0,\ t \geq 0$

$\quad u(\pi,t) = 0,\ t \geq 0$

IC $\quad u(x,0) = \sin x$

Solution

Rewrite in an operator form:

$$L_t u = L_x u \Rightarrow L_t^{-1} L_t u = L_t^{-1} L_x u$$

Where $L_t = \dfrac{\partial}{\partial t}$ and $L_x = \dfrac{\partial^2}{\partial x^2}$

$$u(x,t) = \sin x + L_t^{-1} L_x u$$

$$\sum_{n=0}^{\infty} u_n = \sin x + L_t^{-1} L_x \sum_{n=0}^{\infty} u_n$$

$$\Rightarrow u_0 = \sin x$$

$$u_{k+1} = L_t^{-1} L_x (u_k),\ k \geq 0$$

$$u_1 = L_t^{-1} L_x (\sin x)$$

$$u_1 = L_t^{-1}(-\sin x) = -t \sin x$$

$$u_2(x,t) = L_t^{-1} L_x \left(-t \sin x\right) = L_t^{-1}\left(t \sin x\right) = \frac{1}{2} t^2 \sin x$$

$$u(x,t) = \sin x \left(1 - t + \frac{t^2}{2!} - \cdots\right) = e^{-t} \sin x$$

$$\Rightarrow u(x,t) = e^{-t} \sin x$$

The solution satisfies the partial differential equation, the boundary condition and the initial condition.

We can also solve above example by using formula (3.57) such that:

$$f(x) = \sin x \Rightarrow f^{(2n)}(x) = (-1)^n \sin x, \quad n = 0,1,2,\cdots$$

$$\Rightarrow u(x,t) = \sin x - \frac{t}{1!} \sin x + \frac{t^2}{2!} \sin x - \frac{t^3}{3!} \sin x + \frac{t^4}{4!} \sin x$$

$$= \sin x \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} \cdots\right) = e^{-t} \sin x$$

$$\Rightarrow u(x,t) = e^{-t} \sin x$$

Example (3.1.6): Use the adomian decomposition method to solve the initial-boundary value problems with lateral heat loss

PDE $\quad u_t = u_{xx} + u_{yy} - u \quad 0 < x, y < \pi, t > 0$

BC $\quad u(0, y, t) = u(\pi, y, t) = 0$

$\quad u(x, 0, t) = -u(x, \pi, t) = e^{-3t} \sin x$

IC $\quad u(x, y, 0) = \sin x \cos y$

Solution

Rewrite in an operator form:

$$L_t u = L_x u + L_y u - u$$
$$L_t^{-1} L_t u = L_t^{-1}(L_x u + L_y u - u)$$
$$u(x,y,t) = \sin x \cos y + L_t^{-1}(L_x u + L_y u - u)$$
$$\Rightarrow u_0(x,y,t) = \sin x \cos y$$
$$u_1 = L_t^{-1}\left(L_x(\sin x \cos y) + L_y(\sin x \cos y) - \sin x \cos y\right)$$
$$u_1 = L_t^{-1}(-3\sin x \cos y) = -3t \sin x \cos y$$
$$u_2 = L_t^{-1}(L_x(-3t \sin x \cos y) + L_y(-3t \sin x \cos y) + 3t \sin x \cos y)$$
$$= L_t^{-1}(9t \sin x \cos y) = \frac{9}{2}t^2 \sin x \cos y = \frac{(3t)^2}{2!} \sin x \cos y$$
$$u_3(x,y,t) = -\frac{(3t)^3}{3!} \sin x \cos y$$
$$u(x,y,t) = \sin x \cos y \left(1 - 3t + \frac{(3t)^2}{2!} - \frac{(3t)^3}{3!} + \cdots\right),$$
$$\Rightarrow u(x,y,t) = e^{-3t} \sin x \cos y$$

Example (3.1.7): Use the adomian decomposition method to solve the inhomogeneous partial differential equation

PDE $\quad u_t = u_{xx} + u_{yy} + u_{zz} + \sin z \quad 0 < x, y, z < \pi, t > 0$

BC $\quad u(0, y, z, t) = \sin z + e^{-2t} \sin y$

$u(\pi y, z, t) = \sin z - e^{-2t} \sin y$

$u(x, 0, z, t) = \sin z + e^{-2t} \sin x$

$u(x, \pi, z, t) = \sin z - e^{-2t} \sin x$

$u(x, y, 0, t) = u(x, y, \pi, t) = e^{-2t} \sin(x + y)$

IC $\quad u(x, y, z, 0) = \sin(x + y) + \sin z$

Solution

Rewrite in an operator form:

$$L_t u = (L_x u + L_y u + L_z u + \sin z)$$
$$L_t^{-1} L_t u = L_t^{-1}(L_x u + L_y u + L_z u + \sin z)$$
$$u(x,y,z,t) = \sin(x+y) + \sin z + t\sin z + L_t^{-1}(L_x u + L_y u + L_z u)$$
$$\Rightarrow u_0 = \sin(x+y) + \sin z + t\sin z$$
$$u_1 = L_t^{-1}\left(L_x\left(\sin(x+y)+\sin z + t\sin z\right) + L_y\left(\sin(x+y)+\sin z + t\sin z\right) + L_z\left(\sin(x+y)+\sin z + t\sin z\right)\right)$$
$$= L_t^{-1}(-\sin(x+y) - \sin(x+y) - \sin z - t\sin z)$$
$$= -2t\sin(x+y) - t\sin z - \frac{t^2}{2!}\sin z$$
$$u_2 = \frac{(2t)^2}{2!}\sin(x+y) + \frac{t^2}{2!}\sin z + \frac{t^3}{3!}\sin z$$
$$u(x,y,z,t) = \sin z + \sin(x+y)\left(1 - 2t + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} + \cdots\right) + \left(t\sin z - t\sin z - \frac{t^2}{2!}\sin z + \frac{t^2}{2!}\sin z + \cdots\right)$$
$$\Rightarrow u(x,y,z,t) = \sin z + e^{-2t}\sin(x+y)$$

3.1.3 Solution of Wave Equation by Adomian Decomposition Method

The wave equation plays a significant role in various physical problems, it’s needed in diverse area of science and engineering. It’s usually describes water wave, the vibrations of a string or a membrane, the propagation of electromagnetic and sound wave, or the transmission of electric signals in a cable.

A: One Dimensional Wave Equation:

A simple wave equation it’s came as the following initial-boundary value problem

PDE $\quad u_{tt} = c^2 u_{xx} \quad 0 < x < l, t > 0$

BC $\quad u(0,t) = 0, u(l,t) = 0, \quad t \geq 0 \qquad (3.74)$

IC $\quad u(x,0) = f(x), u_t(x,0) = g(x)$

Where $u = u(x,t)$ is the displacement of any point of the string at the position x and at time t, and c is a constant related to the elasticity of the material of the string. The term u_{tt} that represents the vertical acceleration. The given boundary conditions indicate the end points of the vibrating string are fixed. Two initial conditions $u(x,0)=f(x)$ *and* $u_t(x,0)=g(x)$ that describe the initial displacement and the initial velocity of any point at the starting time $t=0$ respectively.

To solve above equation we begin by rewriting equation in an operator form

$$L_t u(x,t)=c^2 L_x u(x,t) \tag{3.75}$$

where

$$L_t=\frac{\partial^2}{\partial t^2},\ L_x=\frac{\partial^2}{\partial x^2} \tag{3.76}$$

And

$$L_t^{-1}(\cdot)=\int_0^t\int_0^t(\cdot)dtdt\ ,\ L_x^{-1}(\cdot)=\int_0^x\int_0^x(\cdot)dxdx \tag{3.77}$$

This means that:

$$L_t^{-1}L_t u(x,t)=u(x,t)-tu_t(x,0)-u(x,0) \tag{3.78}$$

$$L_x^{-1}L_x u(x,t)=u(x,t)-xu_x(0,t)-u(0,t) \tag{3.79}$$

Appling L_t^{-1} to both sides of (3.75) and using the initial conditions we obtain:

$$u(x,t)=f(x)+tg(x)+c^2L_t^{-1}(L_x u(x,t)) \tag{3.80}$$

The Adomian's method decomposes the displacement function $u(x,t)$ into a sum of infinite components defined by the infinite series:

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t) \tag{3.81}$$

Substituting (3.81) into both sides of (3.80) gives:

$$\sum_{n=0}^{\infty}u_n(x,t)=f(x)+tg(x)+c^2L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty}u_n(x,t)\right)\right) \tag{3.82}$$

From Adomian's method

$$\begin{aligned}&u_0(x,t)=f(x)+tg(x)\\&u_{k+1}(x,t)=c^2L_t^{-1}\left(L_x\left(u_k(x,t)\right)\right),\ k\geq 0\end{aligned} \tag{3.83}$$

From (3.83) the components $u_0(x,t), u_1(x,t), u_2(x,t),...$ can be determined individually by:

$$\begin{aligned}
u_0(x,t) &= f(x) + tg(x),\\
u_1(x,t) &= c^2 L_t^{-1} L_x(u_0) = c^2\left(\frac{t^2}{2!} f''(x) + \frac{t^3}{3!} g''(x)\right),\\
u_2(x,t) &= c^2 L_t^{-1} L_x(u_1) = c^4\left(\frac{t^4}{4!} f^{(4)}(x) + \frac{t^5}{5!} g^{(4)}(x)\right),\\
u_3(x,t) &= c^2 L_t^{-1} L_x(u_2) = c^6\left(\frac{t^6}{6!} f^{(6)}(x) + \frac{t^7}{7!} g^{(6)}(x)\right),
\end{aligned} \tag{3.84}$$

And so on.

By substituting (3.84) into (3.81) we get the solution of (3.74) in a series form as follows:

$$u(x,t) = \sum_{n=0}^{\infty} c^{2n}\left(\frac{t^{2n}}{(2n)!} f^{(2n)}(x) + \frac{t^{(2n+1)}}{(2n+1)!} g^{(2n)}(x)\right) \tag{3.85}$$

The solution (3.85) can also be obtained by using the inverse operator L_x^{-1} but this solution imposes the use of initial and boundary conditions. For this reason and the reduce the size of calculations, we will apply the decomposition method in the t direction.

The PDE one dimensional wave equation can be came of the form:

$$\text{PDE} \quad u_{tt} = c^2 u_{xx} - au \qquad 0 < x < l,\ t > 0 \tag{3.86}$$

Such that an additional term $-au$ arises when each element of the string is subject to an additional force which is proportional to its displacement.

Also The Inhomogeneous PDE one dimensional wave equation can be came of the form:

$$\text{PDE} \quad u_{tt} = c^2 u_{xx} + h(x,t) \qquad 0 < x < l,\ t > 0 \tag{3.87}$$

Where $h(x,t)$ is the inhomogeneous term

B: Two Dimensional Wave Equation:

The propagation of waves in a two dimensional vibrating membrane of length a and width b is governed by the following initial-boundary value problem

PDE $\quad u_{tt} = c^2(u_{xx} + u_{yy}), \quad 0 < x < a, 0 < y < b, t > 0$

BC $\quad u(0, y, t) = u(a, y, t) = 0, \quad t \geq 0$

$$u(x, 0, t) = u(x, b, t) = 0 \tag{3.88}$$

IC $\quad u(x, y, 0) = f(x, y), u_t(x, y, 0) = g(x, y)$

Where $u = u(x, y, t)$ is the displacement of any point located at the position (x, y) of a vibrating membrane at any time t, and c is a constant related to the elasticity of the material of the rectangular plate.

To solve above equation we begin by rewriting equation in an operator form

$$L_t u(x, y, t) = c^2 \left(L_x u(x, y, t) + L_y u(x, y, t) \right) \tag{3.89}$$

where

$$L_t = \frac{\partial^2}{\partial t^2}, L_x = \frac{\partial^2}{\partial x^2}, \ L_y = \frac{\partial^2}{\partial y^2} \tag{3.90}$$

and

$$L_t^{-1}(\cdot) = \int_0^t \int_0^t (\cdot) dt dt \ \& \ L_x^{-1}(\cdot) = \int_0^x \int_0^x (\cdot) dx dx \ \& \ L_y^{-1}(\cdot) = \int_0^y \int_0^y (\cdot) dy dy \tag{3.91}$$

The solution in the t direction, in the x space, or in the y space will lead to identical results. However the solution in the t direction reduces the size of calculations compared with the other space solutions because it uses the initial conditions only. For this reason we use the solution in the t direction as follows:

$$L_t^{-1} L_t u(x, y, t) = u(x, y, t) - u(x, y, 0) - t u_t(x, y, 0) \tag{3.92}$$

Appling L_t^{-1} to both sides of (3.89) and using the initial conditions we obtain:

$$u(x,y,t) = f(x,y) + tg(x,y) + c^2 L_t^{-1}(L_x u + L_y u) \quad (3.93)$$

The Adomian's method defines the solution $u(x,y,t)$ as an infinite series given by:

$$u(x,y,t) = \sum_{n=0}^{\infty} u_n(x,y,t) \quad (3.94)$$

Substituting (3.94) into both sides of (3.93) gives:

$$\sum_{n=0}^{\infty} u_n = f(x,y) + tg(x,y) + c^2 L_t^{-1}\left(L_x \left(\sum_{n=0}^{\infty} u_n \right) + L_y \left(\sum_{n=0}^{\infty} u_n \right) \right) \quad (3.95)$$

From Adomian's method

$$\begin{aligned} &u_0(x,y,t) = f(x,y) + tg(x,y) \\ &u_{k+1}(x,y,t) = c^2 L_t^{-1}(L_x u_k + L_y u_k),\ k \ge 0 \end{aligned} \quad (3.96)$$

From (3.96) calculate the components $u_0(x,y,t), u_1(x,y,t), u_2(x,y,t), \ldots$ and substitute into (3.94) we get the solution in the series form.

The partial differential equation two dimensional wave equation can be came of the form:

$$\text{PDE}\, u_{tt} = c^2(u_{xx} + u_{yy}) - au, \quad 0 < x < a,\ 0 < y < b,\ t > 0 \quad (3.97)$$

Such that an additional term $-au$ arises when each element of the membrane is subjected to an additional force which is proportional to its displacement $u(x,y,t)$.

Also The Inhomogeneous partial differential equation two dimensional wave equation can be came of the form:

$$\text{PDE} \quad u_{tt} = c^2(u_{xx} + u_{yy}) + h(x,y,t) \quad (3.98)$$

Where $h(x,y,t)$ is the inhomogeneous term.

C: Three Dimensional Wave Equation:

The propagation of waves in a three dimensional volume of length a, width b and height d is governed by the following initial-boundary value problem

$$\text{PDE} \quad u_{tt} = c^2(u_{xx} + u_{yy} + u_{zz}),\ 0 < x < a, 0 < y < b, 0 < z < d,\ t > 0$$

$$\text{BC} \quad u(0,y,z,t) = u(a,y,z,t) = 0, \quad t \geq 0$$

$$u(x,0,z,t) = u(x,b,z,t) = 0 \tag{3.99}$$

$$u(x,y,0,t) = u(x,y,d,t) = 0$$

$$\text{IC} \quad u(x,y,z,0) = f(x,y,z), u_t(x,y,z,0) = g(x,y,z)$$

Where $u = u(x,y,z,t)$ is the displacement of any point located at the position (x,y,z) of a rectangular volume at any time t, and c is the velocity of a propagation wave.

To solve above equation we begin by rewriting equation in an operator form

$$L_t u = c^2\left(L_x u + L_y u + L_z u\right) \tag{3.100}$$

where

$$L_t = \frac{\partial^2}{\partial t^2}, L_x = \frac{\partial^2}{\partial x^2}, L_y = \frac{\partial^2}{\partial y^2}, L_z = \frac{\partial^2}{\partial z^2} \tag{3.101}$$

and

$$L_t^{-1}(\cdot) = \int_0^t\int_0^t (\cdot)dtdt \quad \& L_x^{-1}(\cdot) = \int_0^x\int_0^x (\cdot)dxdx \quad \&$$
$$L_y^{-1}(\cdot) = \int_0^y\int_0^y (\cdot)dydy \quad \& L_z^{-1}(\cdot) = \int_0^z\int_0^z (\cdot)dzdz \tag{3.102}$$

The solution in the t direction, in the x space, in the y space, or in the z space will lead to identical results. However the solution in the t direction reduces the size of calculations compared with the other space solutions because it uses the initial conditions only. For this reason we use the solution in the t direction as follows:

We know that:

$$L_t^{-1}L_t u(x,y,z,t) = u(x,y,z.t) - u(x,y,z,0) - tu_t(x,y,z,0) \tag{3.103}$$

Appling L_t^{-1} to both sides of (3.100) and using the initial conditions we obtain:

$$u(x,y,z,t) = f(x,y,z) + tg(x,y,z) + c^2 L_t^{-1}\left(L_x u + L_y u + L_z u\right) \tag{3.104}$$

The Adomian's method defines the solution $u(x,y,z,t)$ as an infinite series as follows:

$$u(x,y,z,t)=\sum_{n=0}^{\infty}u_n(x,y,z,t) \tag{3.105}$$

Substituting (3.105) into both sides of (3.104) gives:

$$\sum_{n=0}^{\infty}u_n=f(x,y,z)+tg(x,y,z)+c^2L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty}u_n\right)+L_y\left(\sum_{n=0}^{\infty}u_n\right)+L_z\left(\sum_{n=0}^{\infty}u_n\right)\right) \tag{3.106}$$

From Adomian's method

$$\begin{aligned} &u_0(x,y,z,t)=f(x,y,z)+tg(x,y,z)\\ &u_{k+1}(x,y,z,t)=c^2L_t^{-1}(L_xu_k+L_yu_k+L_zu_k),k\geq 0 \end{aligned} \tag{3.107}$$

From (3.107) calculate the components $u_0(x,y,z,t),u_1(x,y,z,t),u_2(x,y,z,t),...$ and substitute into (3.105) we get the solution in the series form.

The PDE three dimensional wave equation can be came of the form:

$$\text{PDE} \quad u_{tt}=c^2(u_{xx}+u_{yy}+u_{zz})-au \tag{3.108}$$

Such that an additional term $-au$ arises when each element of the rectangular volume is subjected to an additional force.

Also The Inhomogeneous partial differential equation three dimensional wave equation can be came of the form:

$$\text{PDE} \quad u_{tt}=c^2(u_{xx}+u_{yy}+u_{zz})+h(x,y,z,t) \tag{3.109}$$

Where $h(x,y,z,t)$ is the inhomogeneous term.

D: Wave Equation in an Infinite Domain:

The initial value problem of the one dimensional wave equation, where the domain of the space variable x is unbounded, it's describes the motion of a very long string that is considered not to have boundaries. It's described by a partial differential equation and initial conditions only as follows:

$$\begin{aligned} &\text{PDE} \quad u_{tt}=c^2u_{xx} \qquad -\infty<x<\infty,\ t>0\\ &\text{IC} \quad u(x,0)=f(x),\ u_t(x,0)=g(x) \end{aligned} \tag{3.110}$$

Such that The solution $u(x,t)$ represents the displacement of the point x at time t, the initial displacement $u(x,0)$ and the initial velocity $u_t(x,0)$ are prescribe by $f(x)$ and $g(x)$ respectively.

To solve above equation we begin by rewriting equation in an operator form

$$L_t u(x,t) = c^2 L_x u(x,t) \tag{3.111}$$

Appling L_t^{-1} to both sides of (3.111) and using the initial conditions we obtain:

$$u(x,t) = f(x) + tg(x) + c^2 L_t^{-1}\left(L_x u(x,t)\right) \tag{3.112}$$

The Adomian's method decomposes the displacement function $u(x,t)$ into a sum of an infinite components defined by the infinite series:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{3.113}$$

Substituting (3.113) into both sides of (3.112) gives:

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x) + tg(x) + c^2 L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right) \tag{3.114}$$

From Adomian's method:

$$\begin{aligned} &u_0(x,t) = f(x) + tg(x) \\ &u_{k+1}(x,t) = c^2 L_t^{-1}\left(L_x\left(u_k(x,t)\right)\right),\ k \geq 0 \end{aligned} \tag{3.115}$$

From (3.115) the components $u_0(x,t), u_1(x,t), u_2(x,t), \ldots$ can be determined individually by:

$$\begin{aligned} u_0(x,t) &= f(x) + tg(x) \\ u_1(x,t) &= c^2 L_t^{-1} L_x(u_0) = f''(x)\frac{(ct)^2}{2!} + c^2 g''(x)\frac{t^3}{3!} \\ u_2(x,t) &= c^2 L_t^{-1} L_x(u_1) = f^{(4)}(x)\frac{(ct)^4}{4!} + c^4 g^{(4)}(x)\frac{t^5}{5!} \\ u_3(x,t) &= c^2 L_t^{-1} L_x(u_2) = f^{(6)}(x)\frac{(ct)^6}{6!} + c^6 g^{(6)}(x)\frac{t^7}{7!} \end{aligned} \tag{3.116}$$

and so on.

By substituting (3.116) into (3.113) we get the solution in a series form as follows:

$$u(x,t)=\left(f(x)+f''(x)\frac{(ct)^2}{2!}+f^{(4)}(x)\frac{(ct)^4}{4!}+f^{6}(x)\frac{(ct)^6}{6!}+\cdots\right)+ \\ +\left(g(x)t+c^2g''(x)\frac{t^3}{3!}+c^4g^{(4)}(x)\frac{t^5}{5!}+c^6g^{(6)}(x)\frac{t^7}{7!}+\cdots\right) \tag{3.117}$$

Or equivalent

$$u(x,t)=\sum_{n=0}^{\infty}\left(\frac{(ct)^{2n}}{(2n)!}f^{(2n)}(x)+c^{2n}\frac{t^{(2n+1)}}{(2n+1)!}g^{(2n)}(x)\right) \tag{3.118}$$

Now we have chosen several examples to illustrate discussion given above

Example (3.1.8): Use the Adomian decomposition method to solve the initial-boundary value problem

PDE $\quad u_{tt}=u_{xx} \quad 0<x<\pi,\ t>0$

BC $\quad u(0,t)=1+\sin t,\ u(\pi,t)=1-\sin t$

IC $\quad u(x,0)=1, u_t(x,0)=\cos x$

Solution

$$L_t u=L_x u$$
$$L_t^{-1}L_t u=L_t^{-1}L_x u$$
$$u(x,t)=t\cos x+1+L_t^{-1}L_x u$$

$$\Rightarrow u(x,t)=t\cos x+1+L_t^{-1}L_x\sum_{n=0}^{\infty}u_n(x,t)$$
$$\Rightarrow u_0=t\cos x+1$$

$$u_1=L_t^{-1}L_x(t\cos x+1)=L_t^{-1}(-t\cos x)=-\frac{t^3}{3!}\cos x$$
$$u_2=L_t^{-1}L_x\left(-\frac{t^3}{3!}\cos x\right)=L_t^{-1}\left(\frac{t^3}{3!}\cos x\right)=\frac{1}{5!}t^5\cos x$$
$$u(x,t)=1+\cos x\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\cdots\right)$$

$$\Rightarrow u(x,t)=1+\cos x\sin t$$

We can also solve above example by using formula (3.85) such that:

$c=1,\ f(x)=1 \quad and \quad g(x)=\cos x,\ g^{(2n)}(x)=(-1)^n\cos x, n=0,1,2,\ldots$

and

$$f^{(2n)}(x)=\begin{cases}1 & n=0\\ 0 & n=1,2,3,\ldots\end{cases}$$

$$\Rightarrow u(x,t)=\left(1+\frac{t}{1!}\cos x\right)-\left(\frac{t^3}{3!}\cos x\right)+\left(\frac{t^5}{5!}\cos x\right)-\left(\frac{t^7}{7!}\cos x\right)\ldots$$

$$=1+\cos x\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}+\ldots\right)$$

$$\Rightarrow u(x,t)=1+\cos x\,\sin t$$

Example (3.1.9): Use the Adomian decomposition method to solve the initial-boundary value problem

PDE $\quad u_{tt}=\frac{1}{2}(u_{xx}+u_{yy})-2,\quad 0<x,y<\pi, t>0$

BC $\quad u(0,y,t)=y^2, u(\pi,y,t)=\pi^2+y^2, u(x,0,t)=x^2, u(x,\pi,t)=\pi^2+x^2$

IC $\quad u(x,y,0)=x^2+y^2, u_t(x,y,0)=\sin x\sin y$

Solution

$$L_t u=\frac{1}{2}\left(L_x u+L_y u\right)-2$$

$$L_t^{-1}L_t u=\frac{1}{2}L_t^{-1}\left(L_x u+L_y u\right)-L_t^{-1}(2)$$

$$\Rightarrow u(x,y,t)=-t^2+t\sin x\,\sin y+x^2+y^2+\frac{1}{2}L_t^{-1}\left(L_x u+L_y u\right)$$

$$\Rightarrow u_0=-t^2+t\sin x\,\sin y+x^2+y^2$$

$$u_1=\frac{1}{2}L_t^{-1}\left[L_x\left(-t^2+t\sin x\,\sin y+x^2+y^2\right)+L_y\left(-t^2+t\sin x\,\sin y+x^2+y^2\right)\right]$$

$$=\frac{1}{2}L_t^{-1}\left(2-t\sin x\,\sin y+2-t\sin x\,\sin y\right)=t^2-\frac{1}{3!}t^3\sin x\,\sin y$$

$$u_2=\frac{1}{5!}t^5\sin x\,\sin y$$

$$\Rightarrow u(x,y,t)=x^2+y^2+\sin x\,\sin y\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\cdots\right)$$

$$\Rightarrow u(x,y,t)=x^2+y^2+\sin x\,\sin y\,\sin t$$

Example (3.1.10): Use the Adomian decomposition method to solve the initial-boundary value problem

PDE $u_{tt} = u_{xx} + u_{yy} + u_{zz} - u,\ 0 < x, y, z < \pi, t > 0$

BC $u(0, y, z, t) = u(\pi, y, z, t) = 0$

$u(x, 0, z, t) = u(x, \pi, z, t) = 0$

IC $u(x, y, z, 0) = 0, u_t(x, y, z, 0) = 2\sin x \sin y \sin z$

Solution

$L_t u = L_x u + L_y u + L_z u - u$

$L_t^{-1} L u = L_t^{-1}(L_x u + L_y u + L_z u - u)$

$\Rightarrow u(x, y, z, t) = 2t \sin x \sin y \sin z + L_t^{-1}\left(L_x u + L_y u + L_z u - u\right)$

$\Rightarrow u_0 = 2t \sin x \sin y \sin z$

$u_1 = L_t^{-1}(L_x(2t \sin x \sin y \sin z) + L_y(2t \sin x \sin y \sin z) + L_z(2t \sin x \sin y \sin z) - 2t \sin x \sin y \sin z)$

$= L_t^{-1}(-8t \sin x \sin y \sin z) = -\frac{(2t)^3}{3!} \sin x \sin y \sin z$

$u_2 = \frac{(2t)^5}{5!} \sin x \sin y \sin z$

$$\Rightarrow u(x, y, z, t) = \sin x \sin y \sin z \left(2t - \frac{(2t)^3}{3!} + \frac{(2t)^5}{5!} - \cdots\right)$$

$$\Rightarrow u(x, y, z, t) = \sin x \sin y \sin z \sin(2t)$$

Example (3.1.11): Use the Adomian decomposition method to solve the initial-boundary value problem

PDE $u_{tt} = 16 u_{xx} \quad -\infty < x < \infty, t > 0$

IC $u(x, 0) = \sin x,\ u_t(x, 0) = 2$

Solution

$$L_t u = 16 L_x u$$
$$L_t^{-1} L_t u = 16 L_t^{-1} L_x u$$
$$\Rightarrow u(x,t) = \sin x + 2t + 16 L_t^{-1} L_x u$$
$$\Rightarrow u_0 = \sin x + 2t$$
$$u_1 = 16 L_t^{-1} L_x (\sin x + 2t) = 16 L_t^{-1}(-\sin x) = -8t^2 \sin x$$
$$u_2 = 16 L_t^{-1} L_x (-8t^2 \sin x) = 16 L_t^{-1}(8t^2 \sin x) = \frac{32}{3} t^4 \sin x$$
$$\Rightarrow u(x,t) = 2t + \sin x - 8t^2 \sin x + \frac{32}{3} t^4 \sin x - \cdots$$
$$= 2t + \sin x \left(1 - \frac{(4t)^2}{2!} + \frac{(4t)^4}{4!} - \cdots\right)$$
$$\Rightarrow u(x,t) = 2t + \sin x \cos(4t)$$

We can solve above example by using formula (3.118) such that:

$$c = 4,\ f(x) = \sin x \ \textit{and}\ g(x) = 2$$
$$f^{(2n)}(x) = (-1)^n \sin x,\ \ n = 0,1,2,\ldots$$

and

$$g^{(2n)}(x) = \begin{cases} 2, & n = 0 \\ 0, & n = 1,2,3,\ldots \end{cases}$$

$$u(x,t) = \sum_{n=0}^{\infty} \frac{(4t)^{2n}}{(2n)!}(-1)^n \sin x + 4^{2n} \frac{t^{2n+1}}{(2n+1)!} g^{(2n)}(x)$$
$$u(x,t) = (\sin x + 2t) + \left(-\frac{(4t)^2}{2!}\sin x\right) + \left(\frac{(4t)^4}{4!}\sin x\right) - \cdots$$
$$= 2t + \sin x \left(1 - \frac{(4t)^2}{2!} + \frac{(4t)^4}{4!} - \cdots\right)$$

$$\Rightarrow u(x,t) = 2t + \sin x \cos(4t)$$

Example (3.1.12): Use the Adomian decomposition method to solve the initial-boundary value problem

PDE $\quad u_{tt} = u_{xx} + 2x + 6t \quad -\infty < x < \infty, t > 0$

IC $\quad u(x,0) = 0, u_t(x,0) = \sin x$

Solution

$$L_t u = L_x u + 2x + 6t$$
$$L_t^{-1} L_t u = L_t^{-1}\left(L_x u + 2x + 6t\right) = L_t^{-1}\left(L_x u\right) + xt^2 + t^3$$
$$\Rightarrow u(x,t) = xt^2 + t^3 + t\sin x + L_t^{-1} L_x u$$
$$\Rightarrow u_0 = xt^2 + t^3 + t\sin t$$
$$u_1 = L_t^{-1} L_x\left(xt^2 + t^3 + t\sin x\right) = L_t^{-1}\left(-t\sin x\right) = -\frac{1}{3!}t^3\sin x$$
$$u_2 = L_t^{-1} L_x\left(-\frac{1}{3!}t^3\sin x\right) = L_t^{-1}\left(\frac{1}{3!}t^3\sin x\right) = \frac{1}{5!}t^5\sin x$$
$$\Rightarrow u(x,t) = xt^2 + t^3 + \sin x\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right)$$
$$\Rightarrow u(x,t) = xt^2 + t^3 + \sin x\ \sin t$$

3.2 Solving Systems of Linear PDEs by Adomian Decomposition Method

We apply the Adomian decomposition method for solving systems of linear PDEs. We write a system in an operator form by

$$L_t u + L_x v = g_1\ ,$$
$$L_t v + L_x u = g_2\ ,$$

With initial data

$$u(x,0) = f_1(x)\ ,\ v(x,0) = f_2(x)\ ,$$

Where L_t and L_x are considered, without less of generality, first order partial differential operators, and g_1 and g_2 are inhomogeneous terms.

Applying L_t^{-1} to the system and using the initial condition yields

$$u(x,t) = f_1(x) + L_t^{-1} g_1 - L_t^{-1} L_x v\ ,$$
$$v(x,t) = f_2(x) + L_t^{-1} g_2 - L_t^{-1} L_x u\ .$$

The Adomian decomposition method suggests that the linear terms $u(x,t)$ and $v(x,t)$ be decomposed by an infinite series of components

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t)\ ,\ \ v(x,t)=\sum_{n=0}^{\infty}v_n(x,t).$$

Where $u_n(x,t)$ and $v_n(x,t)$, $n\geq 0$ are the components of $u(x,t)$ and $v(x,t)$ that will be elegantly determined in a recursive manner.

Then gives

$$\sum_{n=0}^{\infty}u_n(x,t)= f_1(x)+L_t^{-1}g_1 - L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty}v_n(x,t)\right)\right),$$
$$\sum_{n=0}^{\infty}v_n(x,t)= f_2(x)+L_t^{-1}g_2 - L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty}u_n(x,t)\right)\right).$$

Following Adomian analysis, the system is transformed into a set of recursive relation given by

$$u_0(x,t)= f_1(x)+L_t^{-1}g_1\ ,$$
$$u_{k+1}(x,t)=-L_t^{-1}(L_x v_k),\ k\geq 0.$$

And

$$v_0(x,t)= f_2(x)+L_t^{-1}g_2\ ,$$
$$v_{k+1}(x,t)=-L_t^{-1}(L_x u_k),\ k\geq 0.$$

To give a clear overview of the content of this work, several illustrative examples have been selected to demonstrate the efficiency of the method.

Example (3.2.1): Consider the linear system of PDEs

$$L_t u + L_x u + 2v = 0\ ,$$
$$L_t v + L_x v - 2u = 0,$$

With initial condition

$$u(x,0)=\cos x\ ,\ \ v(x,0)=\sin x.$$

Operating with L_t^{-1} and using initial condition we obtain

$$u(x,t) = \cos x - L_t^{-1}(2v + L_x u) \ ,$$
$$v(x,t) = \sin x + L_t^{-1}(2u - L_x v) \ .$$

Using the recursive manner gives

$$u_0(x,t) = \cos x \ ,$$
$$u_{k+1}(x,t) = -L_t^{-1}(2v_k + L_x(u_k)), \ k \geq 0.$$

and

$$v_0(x,t) = \sin x \ ,$$
$$v_{k+1}(x,t) = L_t^{-1}(2u_k - L_x(v_k)), \ k \geq 0.$$

Consequently, the pair of zeroth components is defined by

$$(u_0, v_0) = (\cos x \ , \ \sin x),$$

Using (u_0, v_0) into recursive manner gives

$$(u_1, v_1) = (-t \ \sin x \ , \ t \ \cos x),$$
$$(u_2, v_2) = \left(-\frac{t^2}{2!} \cos x \ , \ -\frac{t^2}{2!} \ \sin x \right),$$
$$(u_3, v_3) = \left(\frac{t^3}{3!} \sin x \ , \ -\frac{t^3}{3!} \cos x \right).$$

Combining the results obtained above we obtain

$$u(x,t) = \cos x \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \ldots \right) - \sin x \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \ldots \right),$$
$$v(x,t) = \sin x \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \ldots \right) + \cos x \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \ldots \right),$$

So that the pair (u, v) is know in a closed form by

$$(u,v)=(\cos(x+t),\ \sin(x+t)) .$$

Example (3.2.2): Consider the linear system

$$L_t\, u + L_x\, v = 0 ,$$
$$L_t\, v + L_x\, u = 0 ,$$

With initial condition

$$u(x,0)=e^{x} \ ,\ v(x,0)=e^{-x} \ .$$

To derive the solution by using the decomposition method, we follow the recursive relation to obtain

$$u_0(x,t)=e^{x} \ ,$$
$$u_{k+1}(x,t)=-L_t^{-1}\left(L_x\, v_k\right),\ k\geq 0.$$

and

$$v_0(x,t)=e^{-x} \ ,$$
$$v_{k+1}(x,t)=-L_t^{-1}\left(L_x\, u_k\right),\ k\geq 0.$$

The remaining components are thus determined by

$$u_1(x,t)=t\,e^{-x} \ ,\ v_1(x,t)=-t\,e^{x} \ ,$$
$$u_2(x,t)=\frac{t^2}{2!}e^{x} \ ,\ v_2(x,t)=\frac{t^2}{2!}e^{-x} ,$$
$$u_3(x,t)=\frac{t^3}{3!}e^{-x} \ ,\ v_3(x,t)=-\frac{t^3}{3!}e^{x} ,$$

and so on. Using above components we obtain

$$u(x,t)=e^{x}\left(1+\frac{t^2}{2!}+\frac{t^4}{4!}+\ldots\right)+e^{-x}\left(t+\frac{t^3}{3!}+\frac{t^5}{5!}+\ldots\right),$$
$$v(x,t)=e^{-x}\left(1+\frac{t^2}{2!}+\frac{t^4}{4!}+\ldots\right)-e^{x}\left(t+\frac{t^3}{3!}+\frac{t^5}{5!}+\ldots\right),$$

This has an exact analytical solution of the form

$$(u,v)=\left(e^{x}\cosh t+e^{-x}\sinh t\ ,\ e^{-x}\cosh t-e^{x}\sinh t\right).$$

3.3 Solving Non-Linear ODEs and PDEs by Adomian Decomposition Method

The method has been applied directly and in a straightforward manner to homogeneous and inhomogeneous problems without any restrictive assumptions or linearization. The method usually decomposes the unknown function u into an infinite sum of components that will be determined recursively through iterations as discussed before.

An important remark should be made here concerning the representation of the nonlinear terms that appear in the equation. Although the linear term u is expressed as an infinite series of components.

The Adomian decomposition method requires a special representation for the nonlinear terms such as $u^2, u^3, u^4, \sin u, e^u, uu_x, u_x^2, etc.$ that appear in the equation. The method introduces a formal algorithm to establish a proper representation for all forms of nonlinear terms. The representation of the nonlinear terms is necessary to handle the nonlinear equations in an effective and successful way.

3.3.1 Calculation of Adomian Polynomials:

It is well known now that Adomian decomposition method suggests that the unknown linear function u may be represented by the decomposition series

$$u=\sum_{n=0}^{\infty}u_n \tag{3.119}$$

The nonlinear term $F(u)$ can be expressed by an infinite series of the so-called Adomian polynomials A_n given in the form

$$F(u)=\sum_{n=0}^{\infty}A_n(u_0,u_1,u_2,...,u_n) \tag{3.120}$$

The Adomian polynomials A_n for the nonlinear term $F(u)$ can be evaluated by using the following expression

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F(\sum_{i=0}^{n}\lambda^i u_i)\right]_{\lambda=0}, \; n = 0,1,2,... \tag{3.121}$$

For example:

$$A_0 = \frac{1}{0!}[F(u_0)] \Rightarrow A_0 = F(u_0)$$

$$A_1 = \frac{1}{1}\frac{d}{d\lambda}[F(u_0+\lambda u_1)] = u_1 F'(u_0+\lambda u_1)\Big|_{\lambda=0} \Rightarrow A_1 = u_1 F'(u_0)$$

$$A_2 = \frac{1}{2}\frac{d^2}{d\lambda^2}[F(u_0+\lambda u_1+\lambda^2 u_2)] = \frac{1}{2}\frac{d}{d\lambda}[(u_1+2\lambda u_2)F'(u_0+\lambda u_1+\lambda^2 u_2)]$$

$$= \frac{1}{2}[(u_1+2\lambda u_2)(F''(u_0+\lambda u_1+\lambda^2 u_2)(u_1+2\lambda u_2)) + (2u_2)F'(u_0+\lambda u_1+\lambda^2 u_2)]_{\lambda=0}$$

$$= \frac{1}{2}[u_1 \cdot F''(u_0)\cdot u_1 + 2u_2 F'(u_0)]$$

$$\Rightarrow A_2 = u_2 F'(u_0) + \frac{1}{2!}u_1^2 F''(u_0)$$

Then Adomian polynomials are given by:

$$\begin{aligned}
A_0 &= F(u_0)\\
A_1 &= u_1 F'(u_0)\\
A_2 &= u_2 F'(u_0) + \frac{1}{2!}u_1^2 F''(u_0)\\
A_3 &= u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!}u_1^3 F'''(u_0)\\
A_4 &= u_4 F'(u_0) + \left(\frac{1}{2!}u_2^2 + u_1 u_3\right)F''(u_0) + \frac{1}{2!}u_1^2 u_2 F'''(u_0) + \frac{1}{4!}u_1^4 F^{(4)}(u_0)
\end{aligned} \tag{3.122}$$

Other polynomials can be generated in a similar manner.

Important observation that the A_0 depends only on u_0, A_1 depends only on u_0 and u_1, A_2 depends only on u_0, u_1 and u_2, and so on.

Calculation of Adomian Polynomials A_n:

I- Nonlinear polynomials

Case 1: $F(u) = u^2$

The polynomials can be obtained as follows:

$$A_0 = F(u_0) = u_0^2,$$
$$A_1 = u_1 F'(u_0) = 2u_0 u_1,$$
$$A_2 = u_2 F'(u_0) + \frac{1}{2!} u_1^2 F''(u_0) = 2u_0 u_2 + u_1^2,$$
$$A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!} u_1^3 F'''(u_0) = 2u_0 u_3 + 2u_1 u_2$$

Case 2: $F(u) = u^3$

The polynomials are given by

$$A_0 = F(u_0) = u_0^3$$
$$A_1 = u_1 F'(u_0) = 2u_0^2 u_1$$
$$A_2 = u_2 F'(u_0) + \frac{1}{2!} u_1^2 F''(u_0) = 3u_0^2 u_2 + 3u_0 u_1^2,$$
$$A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!} u_1^3 F'''(u_0) = 3u_0^2 u_3 + 6u_0 u_1 u_2 + u_1^3$$

Case 3: $F(u) = u^4$

Proceeding as before we find

$$A_0 = u_0^4,$$
$$A_1 = 4u_0^3 u_1,$$
$$A_2 = 4u_0^3 u_2 + 6u_0^2 u_1^2,$$
$$A_3 = 4u_0^3 u_3 + 4u_1^3 u_0 + 12u_0^2 u_1 u_2.$$

In a parallel manner, Adomian polynomials can be calculated for nonlinear polynomials of higher degrees.

II- Nonlinear Derivatives:

Case 1: $F(u) = (u_x)^2$

$$A_0 = F\left(u_0\right) = u_{0_x}^2,$$
$$A_1 = u_1 F'(u_0) = 2u_{0_x} u_{1_x},$$
$$A_2 = u_2 F'(u_0) + \frac{1}{2!} u_1^2 F''(u_0) = u_{2_x} \cdot 2u_{0_x} + \frac{1}{2!} u_{1_x}^2 \cdot 2 = 2u_{0_x} u_{2_x} + u_{1_x}^2,$$
$$A_3 = 2u_{0_x} u_{3_x} + 2u_{1_x} u_{2_x}.$$

Case 2: $F(u) = u_x^3$

The Adomian polynomials are given by

$$A_0 = F(u_0) = u_{0_x}^3,$$
$$A_1 = u_1 F'(u_0) = u_{1_x} \cdot 3u_{0_x}^2 = 3u_{0_x}^2 u_{1_x},$$
$$A_2 = u_2 F'(u_0) + \frac{1}{2!} u_1^2 F''(u_0) = u_{2_x} \cdot 3u_{0_x}^2 + \frac{1}{2!} u_{1_x}^2 \cdot 6u_{0_x} = 3u_{0_x}^2 u_{2_x} + 3u_{0_x} u_{1_x}^2,$$
$$A_3 = 3u_{0_x}^2 u_{3_x} + 6u_{0_x} u_{1_x} u_{2_x} + u_{1_x}^3.$$

Case 3. $F(u) = uu_x = \frac{1}{2} L_x(u^2)$

$$F'(u) = \frac{1}{2} L_x(2u) \,\&\, F''(u) = \frac{1}{2} L_x 2$$

The Adomian polynomials for this linearity are given by

$$A_0 = F(u_0) = u_0 u_{0_x},$$
$$A_1 = u_1 F'(u_0) = u_1 \cdot \frac{1}{2} L_x\, 2u_0 = L_x(u_0 u_1) = u_{0_x} u_1 + u_0 u_{1_x}$$
$$A_2 = u_2 F'(u_0) + \frac{1}{2!} u_1^2 F''(u_0) = u_2 \cdot \frac{1}{2} L_x (2u_0) + \frac{1}{2!} u_1^2 \cdot \frac{1}{2} L_x\, 2$$
$$= \frac{1}{2} L_x (2u_0 u_2 + u_1^2) = u_{0_x} u_2 + u_{1_x} u_1 + u_{2_x} u_0,$$
$$A_3 = u_3 \cdot \frac{1}{2} L_x (2u_0) + u_1 u_2 \frac{1}{2} L_x\, 2 + \frac{1}{3!} u_1^3 \cdot 0 = \frac{1}{2} L_x (2u_0 u_3 + 2u_1 u_2)$$
$$= u_{0_x} u_3 + u_{1_x} u_2 + u_{2_x} u_1 + u_{3_x} u_0.$$

III- Trigonometric Nonlinearity:

Case 1: $F(u) = \sin u$

The Adomian polynomials for this linearity are given by

$$A_0 = \sin u_0$$
$$A_1 = u_1 \cos u_0$$
$$A_2 = u_2 \cos u_0 - \frac{1}{2!} u_1^2 \sin u_0,$$
$$A_3 = u_3 \cos u_0 - u_1 u_2 \sin u_0 - \frac{1}{3!} u_1^3 \cos u_0$$

Case 2: $F(u) = \cos u$

Proceeding as before gives

$$A_0 = \cos u_0$$
$$A_1 = -u_1 \sin u_0,$$
$$A_2 = -u_2 \sin u_0 - \frac{1}{2!} u_1^2 \cos u_0,$$
$$A_3 = -u_3 \sin u_0 - u_1 u_2 \cos u_0 - \frac{1}{3!} u_1^3 \sin u_0$$

IV- Hyperbolic Nonlinearity:

Case 1: $F(u) = \sinh u$

The A_n polynomials for this form of nonlinearity are given by

$$A_0 = \sinh u_0$$
$$A_1 = u_1 \cosh u_0$$
$$A_2 = u_2 \cosh u_0 + \frac{1}{2!} u_1^2 \sinh u_0,$$
$$A_3 = u_3 \cosh u_0 + u_1 u_2 \sinh u_0 - \frac{1}{3!} u_1^3 \cosh u_0$$

Case 2: $F(u) = \cosh u$

The Adomian polynomials are given by

$$A_0 = \cosh u_0$$
$$A_1 = u_1 \sinh u_0$$
$$A_2 = u_2 \sinh u_0 + \frac{1}{2!} u_1^2 \cosh u_0,$$
$$A_3 = u_3 \sinh u_0 + u_1 u_2 \cosh u_0 + \frac{1}{3!} u_1^3 \sinh u_0$$

V- Exponential Nonlinearity:

Case 1: $F(u) = e^u$

The Adomian polynomials for this form of nonlinearity are given by

$$A_0 = e^{u_0}$$
$$A_1 = u_1 e^{u_0},$$
$$A_2 = \left(u_2 + \frac{1}{2!} u_1^2 \right) e^{u_0},$$
$$A_3 = \left(u_3 + u_1 u_2 + \frac{1}{3!} u_1^3 \right) e^{u_0}.$$

Case 2. $F(u) = e^{-u}$

$$A_0 = e^{-u_0}$$
$$A_1 = -u_1 e^{-u_0},$$
$$A_2 = \left(-u_2 + \frac{1}{2!}u_1^2\right)e^{-u_0},$$
$$A_3 = \left(-u_3 + u_1 u_2 - \frac{1}{3!}u_1^3\right)e^{-u_0}.$$

VI- Logarithmic Nonlinearity:

Case 1: $F(u) = \ln u,\ u > 0$

The A_n polynomials for logarithmic nonlinearity are given by

$$A_0 = \ln u_0$$
$$A_1 = \frac{u_1}{u_0}$$
$$A_2 = \frac{u_2}{u_0} - \frac{1}{2}\frac{u_1^2}{u_0^2}$$
$$A_3 = \frac{u_3}{u_0} - \frac{u_1 u_2}{u_0^2} + \frac{1}{3}\frac{u_1^3}{u_0^3}$$

Case 2: $F(u) = \ln(1+u), -1 < u \le 1$

The A_n polynomials are given by

$$A_0 = \ln(1+u_0)$$
$$A_1 = \frac{u_1}{1+u_0}$$
$$A_2 = \frac{u_2}{1+u_0} - \frac{1}{2}\frac{u_1^2}{(1+u_0)^2}$$
$$A_3 = \frac{u_3}{1+u_0} - \frac{u_1 u_2}{(1+u_0)^2} + \frac{1}{3}\frac{u_1^3}{(1+u_0)^3}$$

3.3.2 Solving Nonlinear Ordinary Differential Equations by Adomian Method:

To apply the Adomian decomposition method for solving nonlinear ordinary differential equations, we consider the equation

$$Ly + R(y) + F(y) = g(x) \tag{3.123}$$

where the differential operator L may be considered as the highest order derivative in the equation, R is the remainder of the differential operator, $F(y)$ expresses the nonlinear terms, and $g(x)$ is an inhomogeneous term.

If L is a first order operator defined by:

$$L=\frac{d}{dx} \quad \text{Then } L^{-1} \text{ is given by: } L^{-1}(\cdot)=\int_0^x(\cdot)dx$$

so that:

$$L^{-1}Ly=y(x)-y(0) \tag{3.124}$$

If L is a second order:

$$L=\frac{d^2}{dx^2}\Rightarrow L^{-1}(\cdot)=\int_0^x\int_0^x(\cdot)dxdx$$

$$\Rightarrow L^{-1}Ly=y(x)-y(0)-xy'(0) \tag{3.125}$$

If L is a third order we can easily show that

$$L^{-1}Ly=y(x)-y(0)-xy'(0)-\frac{1}{2!}x^2y''(0) \tag{3.126}$$

And so on for a higher order operators.

Applying L^{-1} to both sides of equation (3.123) gives

$$y(x)=\psi_0-L^{-1}g(x)-L^{-1}Ry-L^{-1}F(y) \tag{3.127}$$

where

$$\psi_0=\begin{cases} y(0) & for\ L=\frac{d}{dx} \\ y(0)+xy'(0) & for\quad \frac{d^2}{dx^2} \\ y(0)+xy'(0)+\frac{1}{2!}x^2y''(0) & for\ L=\frac{d^3}{dx^3} \\ y(0)+xy'(0)+\frac{1}{2!}x^2y''(0)+\frac{1}{3!}x^3y'''(0) & for\ L=\frac{d^4}{dx^4} \\ y(0)+xy'(0)+\frac{1}{2!}x^2y''(0)+\frac{1}{3!}x^3y'''(0)+\frac{1}{4!}x^4y^{(4)}(0) & for\ L=\frac{d^5}{dx^5} \end{cases} \tag{3.128}$$

And substitute $y(x)=\sum_{n=0}^{\infty}y_n$ and $F(y)=\sum_{n=0}^{\infty}A_n$ in equation (3.127)

$$\Rightarrow \sum_{n=0}^{\infty} y_n = \psi_0 - L^{-1}g(x) - L^{-1}R\left(\sum_{n=0}^{\infty} y_n\right) - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right) \tag{3.129}$$

The various components y_n of the solution y can be easily determined by:

$$\begin{aligned} y_0 &= \psi_0 - L^{-1}(g(x)) \\ y_{k+1} &= -L^{-1}\left(Ry_k\right) - L^{-1}(A_k), \quad k \geq 0 \end{aligned} \tag{3.130}$$

The first few components can be written as

$$\begin{aligned} y_0 &= \psi_0 - L^{-1}g(x) \\ y_1 &= -L^{-1}(R(y_0)) - L^{-1}(A_0) \\ y_2 &= -L^{-1}(R(y_1)) - L^{-1}(A_1) \\ y_3 &= -L^{-1}(R(y_2)) - L^{-1}(A_2) \\ y_4 &= -L^{-1}(R(y_3)) - L^{-1}(A_3) \end{aligned} \tag{3.131}$$

We can written $\Phi_n = \sum_{k=0}^{n-1} y_k$ to produce a closed form or may be write the approximate solution as the form:

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \tag{3.132}$$

Example (3.3.1): Solve the first order nonlinear differential equation:

$$y' = \frac{y^2}{1-xy} \qquad y(0) = 1$$

Solution

$$y'(1-xy) = y^2$$

$$y' = xyy' + y^2$$

$$y(x) = 1 + L^{-1}(xyy') + L^{-1}(y^2)$$

$$\sum_{n=0}^{\infty} y_n = 1 + L^{-1}\left(\sum_{n=0}^{\infty} xA_n\right) + L^{-1}\left(\sum_{n=0}^{\infty} B_n\right)$$

$$\Rightarrow y_0(x) = 1$$

$$y_{k+1} = L^{-1}(xA_k) + L^{-1}(B_k), \; k \geq 0$$

$$\Rightarrow y_0 = 1$$

$$y_1 = L^{-1}(xA_0) + L^{-1}(B_0) = L^{-1}(0) + L^{-1}(1) = x$$

$$y_2 = L^{-1}(xA_1) + L^{-1}(B_1) = L^{-1}(x) + L^{-1}(2x) = \frac{3}{2}x^2$$

$$y_3 = L^{-1}(xA_2) + L^{-1}(B_2) = L^{-1}(x(x+3x)) + L^{-1}(3x^2 + x^2) = \frac{8}{3}x^3$$

$$\Rightarrow y(x) = 1 + x + \frac{3}{2}x^2 + \frac{8}{3}x^3 + \frac{125}{24}x^4 + \ldots$$

The exact solution can be expressed in the implicit expression $\Rightarrow y(x) = e^{xy}$

Example (3.3.2): Solve the first order nonlinear differential equation:

$$y' - e^{y} = 0, \quad y(0) = 1$$

Solution

$$L(y) = e^{y}$$
$$L^{-1}L(y) = L^{-1}(e^{y})$$
$$y(x) = 1 + L^{-1}\sum_{n=0}^{\infty} A_n$$
$$\sum_{n=0}^{\infty} y_n = 1 + L^{-1}\sum_{n=0}^{\infty} A_n$$
$$y_0 = 1$$
$$y_{k+1} = L^{-1}(A_k), k \geq 0$$
$$\Rightarrow y_1 = L^{-1}(A_0) = L^{-1}(e) = ex$$
$$y_2 = L^{-1}(A_1) = L^{-1}(xe^{2}) = \frac{1}{2}x^{2}e^{2}$$
$$y_3 = L^{-1}(A_2) = L^{-1}\left[\left(\frac{1}{2}x^{2}e^{2} + \frac{1}{2}x^{2}e^{2}\right)e\right] = L^{-1}\left(x^{2}e^{3}\right) = \frac{1}{3}x^{3}e^{3}$$

And so on, the solution in a series form is given by:

$$y(x) = 1 + ex + \frac{1}{2}(ex)^2 + \frac{1}{3}(ex)^3 + \cdots, -1 \leq ex < 1$$
$$\Rightarrow y(x) = 1 - \ln(1 - ex), \ -1 \leq ex < 1$$

Example (3.3.3): Use the noise term phenomenon to solve the second order nonlinear differential equation

$$y'' + (y')^2 + y^2 = 1 - \sin x, \ y(0) = 0, y'(0) = 1$$

Solution

Rewrite in an operator form and applying L^{-1} to both sides of equation we get

$$y(x) = \sin x + \frac{1}{2}x^{2} - L^{-1}\left(\left(y'\right)^{2} + y^{2}\right).$$

From adomian decomposition method we can write above equation as follows

$$\sum_{n=0}^{\infty} y_n(x) = \sin x + \frac{1}{2}x^2 - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right)$$

$$\Rightarrow y_0 = \sin x + \frac{1}{2}x^2$$

$$y_{k+1} = -L^{-1}(A_k), \quad k \geq 0$$

$$\Rightarrow y_1 = -L^{-1}\left((y')^2 + y^2\right) = -\frac{1}{2!}x^2 + \cdots$$

The zeroth component contains the trigonometric function $\sin x$, therefore it is recommended that the noise terms phenomenon be used here. By canceling the noise terms $\frac{1}{2}x^2$ and $-\frac{1}{2}x^2$ between y_0 and y_1, and justifying that the remaining non-canceled term of y_0 satisfies the differential equation leads to the exact solution given by $y(x) = \sin x$

3.3.3 Solution of Nonlinear Partial Differential Equations by Admian Method:

Nonlinear partial differential equations arise in different areas of physics, engineering, and applied mathematics such as fluid mechanics, condensed matter physics, soliton physics fluid dynamics, plasma physics, solid mechanics and quantum field theory.

Systems of nonlinear partial differential equations have been also noticed to arise in chemical and biological applications.

The first order nonlinear partial differential equation in two independent variables x and y can be generally expressed in the form

$$F(x, y, u, u_x, u_y) = f$$

where f is a function of one or two of the independent variables x and y . Similarly, the second order nonlinear partial differential equation in two independent variables x and y can be expressed by

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = f$$

The nonlinear partial differential equation is called homogeneous if $f = 0$, and inhomogeneous if $f \neq 0$.

A wide variety of physically significant problems modeled by nonlinear partial differential equations, such as the advection problem, the KdV equation, the modified KdV equation, the KP equation, Boussinesq equation.

An important note worth mentioning is that there is no general method that can be employed for obtaining analytical solutions for nonlinear partial differential equations. Several methods are usually used and numerical solutions are often obtained. Further, transformation methods are sometimes used to convert a nonlinear equation to an ordinary equation or to a system of ordinary differential equations. Furthermore, perturbation techniques and discretization methods, that require a massive size of computational work, can be used for some types of equations.

The Adomian decomposition method can be used generally for all types of differential and integral equations. The method can be applied in a straightforward manner and it provides a rapidly convergent series solution. Now we will discuss a general description of the method that will be used for nonlinear partial differential.

We first consider the nonlinear partial differential equation given in an operator form

$$L_x u(x,y) + L_y u(x,y) + R(u(x,y)) + F(u(x,y)) = g(x,y) \tag{3.133}$$

Where L_x and L_y is the highest order differential in x and y respectively , R contains the remaining linear terms of lower derivatives, $F(u(x,y))$ is an analytic nonlinear term, and $g(x,y)$ is an inhomogeneous or forcing term.

The solutions for $u(x,y)$ obtained from the operator equations $L_x u$ and $L_y u$ are equivalent and each converges to the exact solution, the decision as to which operator L_x or L_y should be used to solve the problem depends mainly on two bases:

(i) The operator of lowest order should be selected to minimize the size of computational work.

(ii) The selected operator of lowest order should be of best known conditions to accelerate the evaluation of the components of the solution.

Suppose that the operator L_x meets the two bases of selection, therefore we set

$$L_x u(x,y) = g(x,y) - L_y u(x,y) - R(u(x,y)) - F(u(x,y)) \tag{3.134}$$

Applying L_x^{-1} to both sides we get

$$u(x,y) = \phi_0 - L_x^{-1} g(x,y) - L_x^{-1} L_y u(x,y) - L_x^{-1} R\left(u(x,y)\right) - L_x^{-1} F\left(u(x,y)\right) \tag{3.135}$$

Where

$$\phi_0 = \begin{cases} u(0,y) & for\, L = \dfrac{\partial}{\partial x} \\ u(0,y) + x u_x(0,y) & for\, L = \dfrac{\partial^2}{\partial x^2} \\ u(0,y) + x u_x(0,y) + \dfrac{1}{2!} x^2 u_{xx}(0,y) & for\, L = \dfrac{\partial^3}{\partial x^3} \\ u(0,y) + x u_x(0,y) + \dfrac{1}{2!} x^2 u_{xx}(0,y) + \dfrac{1}{3!} x^3 u_{xxx}(0,y) & for\, L = \dfrac{\partial^4}{\partial x^4} \end{cases} \tag{3.136}$$

And so on.

Then substitute $u(x,y) = \sum_{n=0}^{\infty} u_n(x,y)$ and $F\big(u(x,y)\big) = \sum_{n=0}^{\infty} A_n$ in equation

$$\sum_{n=0}^{\infty} u_n(x,y) = \phi_0 - L_x^{-1} g(x,y) - L_x^{-1} L_y \left(\sum_{n=0}^{\infty} u_n(x,y) \right) - L_x^{-1} R \left(\sum_{n=0}^{\infty} u_n(x,y) \right) - L_x^{-1} \left(\sum_{n=0}^{\infty} A_n \right) \tag{3.137}$$

The components $u_n(x,y),\, n \geq 0$ can be recursively determined by using the relation:

$$\begin{aligned} & u_0(x,y) = \phi_0 - L_x^{-1} g(x,y) \\ & u_{k+1}(x,u) = -L_x^{-1} L_y u_k - L_x^{-1} R(u_k) - L_x^{-1}(A_k), k \geq 0 \\ & \Rightarrow u_0(x,y) = \phi_0 - L_x^{-1} g(x,y) \end{aligned} \tag{3.138}$$

$$\begin{aligned}
u_1(x,y) &= -L_x^{-1}L_y u_0(x,y) - L_x^{-1}R(u_0(x,y)) - L_x^{-1}A_0 \\
u_2(x,y) &= -L_x^{-1}L_y u_1(x,y) - L_x^{-1}R(u_1(x,y)) - L_x^{-1}A_1 \\
u_3(x,y) &= -L_x^{-1}L_y u_2(x,y) - L_x^{-1}R(u_2(x,y)) - L_x^{-1}A_2 \\
u_4(x,y) &= -L_x^{-1}L_y u_3(x,y) - L_x^{-1}R(u_3(x,y)) - L_x^{-1}A_3
\end{aligned} \tag{3.139}$$

Substitute above components to obtain the solution in a series form.

Example (3.3.4):

$$u_t + uu_x = 0, \quad u(x,0) = x, \; t > 0$$

Solution

$$\begin{aligned}
&L_t u = -uu_x \\
&L_t^{-1} L_t u = -L_t^{-1}(uu_x) \\
&u(x,t) = x - L_t^{-1}(uu_x)
\end{aligned}$$

From adomian method

$$\sum_{n=0}^{\infty} u_n(x,t) = x - L_t^{-1}\left(\sum_{n=0}^{\infty} A_n\right)$$

$$\begin{aligned}
&\Rightarrow u_0(x,t) = x \\
&\quad u_{k+1}(x,t) = -L_t^{-1}(A_k), \quad k \geq 0
\end{aligned}$$

The first few components are given by

$$\begin{aligned}
&u_0(x,t) = x \\
&u_1(x,t) = -L_t^{-1}(A_0) = -L_t^{-1}(x) = -xt, \\
&u_2(x,t) = -L_t^{-1}(A_1) = -L_t^{-1}(-2xt) = xt^2, \\
&u_3(x,t) = -L_t^{-1}(A_2) = -L_t^{-1}(3xt^2) = -xt^3, \\
&\Rightarrow u(x,t) = x\left(1 - t + t^2 - t^3 + \cdots\right) \\
&\Rightarrow u(x,t) = \frac{x}{1+t}, \quad |t| < 1
\end{aligned}$$

Chapter 4

Modified Double Sumudu Transform Decomposition Method for Solving Non-linear Partial Differential Equations

We discuss now a new method employed to tackle non-linear partial differential equations, namely Modified Double Sumudu Transform Decomposition Method. This method is a combination of the Modified Double Sumudu Transform and Adomian Decomposition Method. This technique is hereafter provided and supported with necessary illustrations, together with some attached examples.

4.1 General Describe for the Method

To clarify the basic idea of this method, we consider a general inhomogeneous nonlinear partial differential equation with the initial condition of the following form:

$$Lu(x,t)+Ru(x,t)+Nu(x,t)=g(x,t)\ , \tag{4.1}$$

$$u(x,0)=h(x), \qquad u_t(x,0)=f(x). \tag{4.2}$$

Where, L is the second order linear differential operator $L=\frac{\partial^2}{\partial t^2}$, R is the linear differential operator of less order than L, N represents the general nonlinear differential operator and $g(x,t)$ is the source term.

Taking the Modified Double Sumudu Transform on both sides of equation (4.1) and Modified single Sumudu Transform of equation (4.2), we get:

$$E_2(Lu(x,t))+E_2(Ru(x,t))+E_2(Nu(x,t))=E_2(g(x,t))\ , \tag{4.3}$$

$$E(u(x,0))=E(h(x))=T(u,0) \quad and \quad E(u_t(x,0))=E(f(x))=\frac{\partial}{\partial t}T(u,0)\ . \tag{4.4}$$

To substitute Equation (4.4) in (4.3) , after using Equation (2.5), we get:

$$E_2(u(x,t)) = v^2 E_2(g(x,t)) + v^2 E(h(x)) + v^3 E(f(x)) - v^2 E_2(Ru(x,t)) - v^2 E_2(Nu(x,t)). \tag{4.5}$$

Now, with the application of the inverse Double Elzaki Transform on both side of equation (4.5) we get:

$$u(x,t) = G(x,t) - E_2^{-1}\left[v^2 E_2\left[Ru(x,t) + Nu(x,t)\right]\right]. \tag{4.6}$$

Where $G(x,t)$ represents the terms arising from the source term and the prescribed initial conditions.

After that we represent solution as an infinite series given below,

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{4.7}$$

and the nonlinear term can be written as follow,

$$N u(x,t) = \sum_{n=0}^{\infty} A_n(u) \ , \tag{4.8}$$

Where, $A_n(u)$ are Adomian polynomial and it can be calculated by formula given below:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0} \qquad , n = 0,1,2,3,\dots \tag{4.9}$$

To substitute (4.7) and (4.8) in (4.6), we get:

$$\sum_{n=0}^{\infty} u_n(x,t) = G(x,t) - E_2^{-1}\left[v^2 E_2\left(R\sum_{n=0}^{\infty} u_n(x,t) + \sum_{n=0}^{\infty} A_n\right)\right]. \tag{4.10}$$

Then from equation (4.10) we get:

$$u_0(x,t) = G(x,t) \ ,$$

$$u_1(x,t) = -E_2^{-1}\left[v^2 E_2\left[Ru_0(x,t)+A_0\right]\right], \tag{4.11}$$

$$u_2(x,t) = -E_2^{-1}\left[v^2 E_2\left[Ru_1(x,t)+A_1\right]\right].$$

In general, the recursive relation is given by:

$$u_n(x,t) = -E_2^{-1}\left[v^2 E_2\left[Ru_{n-1}(x,t)+A_{n-1}\right]\right], \qquad n \geq 1. \tag{4.12}$$

Finally, we approximate the solution $u(x,t)$ by the series:

$$u(x,t) = \lim_{N\to\infty}\sum_{n=0}^{\infty} u_n(x,t). \tag{4.13}$$

4.2 Application of the Method for Some Nonlinear Equations

Now we choose some different types for Nonlinear Partial differential Equations and solve it by Modified Double Sumudu transform decomposition method

Example (4.2.1): Consider the following nonlinear partial differential equations

$$u_t + uu_x - u_{xx} = 0, \tag{4.14}$$

with initial condition:

$$u(x,0) = x. \tag{4.15}$$

Take the Modified double Sumudu transform to both sides of equation (4.14), we get:

$$\frac{T(u,v)}{v} - vT(u,0) = E_2(u_{xx} - uu_x), \tag{4.16}$$

Take single Modified Sumudu transform to initial condition we get:

$$E(u(x,0)) = T(u,0) = E(x) = u^3, \tag{4.17}$$

Substitute Equation (4.17) in Equation (4.16), we obtain:

$$T(u,v) = v^2 u^3 + vE_2(u_{xx} - uu_x). \tag{4.18}$$

Take the inverse Modified double Sumudu transform to both sides of equation (4.18), we obtain:

$$u(x,t) = x + E_2^{-1}[v E_2(u_{xx} - uu_x)]. \tag{4.19}$$

From the Adomian decomposition method, rewrite (4.19) as follows,

$$\sum_{n=0}^{\infty} u_n(x,t) = x + E_2^{-1}\left[v E_2\left(\sum_{n=0}^{\infty} (u_n)_{xx} - \sum_{n=0}^{\infty} A_n(u) \right)\right]. \tag{4.20}$$

Where, $A_n(u)$ are Adomian polynomials that represent the nonlinear terms. The first few components of $A_n(u)$ are given by:

$$A_0(u) = u_0 (u_0)_x \ ,$$

$$A_1(u) = (u_0)_x u_1 + u_0 (u_1)_x \ , \tag{4.21}$$

$$A_2(u) = (u_0)_x u_2 + (u_1)_x u_1 + (u_2)_x u_0 \ ,$$

$$A_3(u) = (u_0)_x u_3 + (u_1)_x u_2 + (u_2)_x u_1 + (u_3)_x u_0 \ .$$

$$\vdots$$

By comparing both sides of equation (4.20), we get:

$$u_0(x,t) = x, \tag{4.22}$$

$$u_{n+1}(x,t) = E_2^{-1}[v E_2[(u_n)_{xx} - A_n(u)]], \qquad n \geq 0. \tag{4.23}$$

Then:

$$u_1(x,t) = E_2^{-1}\left[v\, E_2\left[(u_0)_{xx} - A_0(u)\right]\right]$$

$$= E_2^{-1}\left[v\,E_2(-x)\right]$$

$$= -E_2^{-1}\left[v^3 u^3\right] = -xt\,, \tag{4.24}$$

$$u_2(x,t) = E_2^{-1}\left[v\,E_2\left[(u_1)_{xx} - A_1(u)\right]\right]$$

$$= E_2^{-1}\left[v\,E_2(2xt)\right]$$

$$= -E_2^{-1}\left[2v^4 u^3\right] = xt^2\,, \tag{4.25}$$

By similar way we get:

$$u_3(x,t) = -xt^3\,. \tag{4.26}$$

And so on. Then the first four terms of the decomposition series for Equation (4.14), is given by:

$$u(x,t) = x - xt + xt^2 - xt^3 + \cdots, \tag{4.27}$$

The solution in a closed form is given by:

$$u(x,t) = \frac{x}{1+t}\quad , \quad |t| < 1\,. \tag{4.28}$$

Example (4.2.2): Consider the following nonlinear partial differential equations:

$$u_{tt} - \frac{2x^2}{t} u u_x = 0\,, \tag{4.29}$$

with initial condition:

$$u(x,0) = 0\,, \qquad u_t(x,0) = x\,. \tag{4.30}$$

Take the Modified double Sumudu transform to both sides of Equation (4.29), we get:

$$\frac{T(u,v)}{v^2} - T(u,0) - v\frac{\partial}{\partial t}T(u,0) = E_2\left(\frac{2x^2}{t}uu_x\right), \tag{4.31}$$

Take single Modified Sumudu transform to initial conditions, we get:

$$E(u(x,0)) = 0 \qquad and \quad E(u_t(x,0)) = \frac{\partial}{\partial t}T(u,0) = E(x) = u^3, \tag{4.32}$$

Substitute Equation (4.32) in Equation (4.31) we obtain:

$$T(u,v) = v^3u^3 + v^2E_2\left[\frac{2x^2}{t}uu_x\right]. \tag{4.33}$$

Take the inverse double Modified Sumudu transform to both sides of Equation (4.33), we obtain:

$$u(x,t) = xt + E_2^{-1}\left[v^2E_2\left[\frac{2x^2}{t}uu_x\right]\right], \tag{4.34}$$

From the Adomian decomposition method, rewrite (4.34) as follows:

$$\sum_{n=0}^{\infty}u_n(x,t) = xt + E_2^{-1}\left[v^2\,E_2\left[\frac{2x^2}{t}\sum_{n=0}^{\infty}A_n(u)\right]\right]. \tag{4.35}$$

Where $A_n(u)$ are Adomian polynomials that represent the nonlinear terms.

The first few components of $A_n(u)$ are given by:

$$A_0(u) = u_0(u_0)_x\ ,$$

$$A_1(u) = (u_0)_x u_1 + u_0(u_1)_x\ , \tag{4.36}$$

$$A_2(u) = (u_0)_x u_2 + (u_1)_x u_1 + (u_2)_x u_0\ ,$$

$$A_3(u) = (u_0)_x u_3 + (u_1)_x u_2 + (u_2)_x u_1 + (u_3)_x u_0\ .$$

$$\vdots$$

By comparing both sides of Equation (4.35), we get:

$$u_0(x,t) = xt, \tag{4.37}$$

$$u_{n+1}(x,t) = E_2^{-1}\left[v^2 E_2\left[\frac{2x^2}{t} A_n\right]\right], \qquad n \geq 0. \tag{4.38}$$

Then:

$$u_1(x,t) = E_2^{-1}\left[v^2 E_2\left[\frac{2x^2}{t} A_0\right]\right]$$

$$= E_2^{-1}\left[v^2 E_2\left[\frac{2x^2}{t}.\ xt^2\right]\right]$$

$$= E_2^{-1}\left[12v^5u^5\right] = \frac{1}{3}x^3t^3, \tag{4.39}$$

By similar way we get:

$$u_2(x,t) = \frac{2}{15}x^5t^5, \tag{4.40}$$

$$u_3(x,t) = \frac{17}{315}x^7t^7, \tag{4.41}$$

And so on. Then the first four terms of the decomposition series for Equation (4.29), is given by:

$$u(x,t) = xt + \frac{1}{3}(xt)^3 + \frac{2}{15}(xt)^5 + \frac{17}{315}(xt)^7 + \cdots, \tag{4.42}$$

The solution in a closed form is given by:

$$u(x,t) = \tan(xt). \tag{4.43}$$

Example (4.2.3): Consider the following KdV equations

$$u_t + 6uu_x + u_{xxx} = 0, \tag{4.44}$$

with initial condition:

$$u(x,0)= x. \tag{4.45}$$

Take the Modified double Sumudu transform to both sides of Equation (4.44), we get:

$$\frac{T(u,v)}{v} - vT(u,0) = -E_2(6uu_x + u_{xxx}), \tag{4.46}$$

Take single Modified Sumudu transform to initial condition we get:

$$E(u(x,0)) = T(u,0) = E(x) = u^3, \tag{4.47}$$

Substitute equation (4.47) in equation (4.46), we obtain:

$$T(u,v) = v^2 u^3 - vE_2(6uu_x + u_{xxx}). \tag{4.48}$$

Take the inverse Modified double Sumudu transform to both sides of Equation(4.48), we obtain:

$$u(x,t) = x - E_2^{-1}[v E_2(6uu_x + u_{xxx})]. \tag{4.49}$$

From the Adomian decomposition method, rewrite equation (4.49) as follows,

$$\sum_{n=0}^{\infty} u_n(x,t) = x - E_2^{-1}\left[v E_2\left(6\sum_{n=0}^{\infty} A_n(u) + \sum_{n=0}^{\infty} (u_n)_{xxx} \right)\right]. \tag{4.50}$$

Where, $A_n(u)$ are Adomian polynomials that represent the nonlinear terms.

The first few components of $A_n(u)$ are given by:

$$A_0(u) = u_0 (u_0)_x ,$$

$$A_1(u) = (u_0)_x u_1 + u_0 (u_1)_x , \tag{4.51}$$

$$A_2(u) = (u_0)_x u_2 + (u_1)_x u_1 + (u_2)_x u_0 ,$$

$$A_3(u)=(u_0)_x u_3+(u_1)_x u_2+(u_2)_x u_1+(u_3)_x u_0 \ .$$

$$\vdots$$

By comparing both sides of equation(4.50), we get:

$$u_0(x,t)=x, \qquad (4.52)$$

$$u_{n+1}(x,t)=-E_2^{-1}\left[v E_2\left[6A_n(u)+(u_n)_{xxx}\right]\right], \qquad n\geq 0. \qquad (4.53)$$

Then:

$$u_1(x,t)=-E_2^{-1}\left[v\, E_2\left[\,6A_0(u)+(u_0)_{xxx}\,\right]\right]$$

$$=-E_2^{-1}\left[v\, E_2(6x)\right]$$

$$=-E_2^{-1}\left[6v^3u^3\right]=-6xt, \qquad (4.54)$$

$$u_2(x,t)=-E_2^{-1}\left[v\, E_2\left[\,6A_1(u)+(u_1)_{xxx}\,\right]\right]$$

$$=-E_2^{-1}\left[v\, E_2(-72xt)\right]$$

$$=E_2^{-1}\left[72v^4u^3\right]=36xt^2, \qquad (4.55)$$

By similar way we get:

$$u_3(x,t)=-216xt^3. \qquad (4.56)$$

And so on, then the first four terms of the decomposition series for equation(4.44) are given by:

$$u(x,t)=x-6xt+36xt^2-216xt^3+\cdots, \qquad (4.57)$$

This can be written as:

$$u(x,t)=x\left[1-6t+(6t)^2-(6t)^3+\cdots\right], \qquad (4.58)$$

The solution in a closed form is given by:

$$u(x,t) \quad = \quad \frac{x}{1+6t} \quad , \qquad |t| < 1 \; . \tag{4.59}$$

Example (4.2.4): Consider the following KdV equations

$$u_t - 6uu_x + u_{xxx} = 0, \tag{4.60}$$

with initial condition:

$$u(x,0) = \frac{1}{6}(x-1). \tag{4.61}$$

Take the Modified double Sumudu transform to both sides of equation(4.60), we get:

$$\frac{T(u,v)}{v} - vT(u,0) = E_2(6uu_x - u_{xxx}), \tag{4.62}$$

Take single Modified Sumudu transform to initial condition we get:

$$E\left(u(x,0)\right) = T\left(u,0\right) = E\left(\frac{1}{6}(x-1)\right) = \frac{1}{6}\left(u^3 - u^2\right), \tag{4.63}$$

Substitute Equation(4.63) in equation (4.62), we obtain:

$$T(u,v) = \frac{1}{6}\left(v^2u^3 - v^2u^2\right) + vE_2(6uu_x - u_{xxx}). \tag{4.64}$$

Take the inverse Modified double Sumudu transform to both sides of equation(4.64), we obtain:

$$u(x,t) = \frac{1}{6}(x-1) + E_2^{-1}\left[vE_2(6\,uu_x - u_{xxx})\right]. \tag{4.65}$$

From the Adomian decomposition method, rewrite equation(4.65) as follows,

$$\sum_{n=0}^{\infty} u_n(x,t) = \frac{1}{6}(x-1) + E_2^{-1}\left[v\, E_2\left(6\sum_{n=0}^{\infty} A_n(u) - \sum_{n=0}^{\infty} (u_n)_{xxx} \right)\right]. \qquad (4.66)$$

Where, $A_n(u)$ are the Adomian polynomials that represent the nonlinear terms.

The first few components of $A_n(u)$ are given by:

$$A_0(u) = u_0 (u_0)_x \ ,$$

$$A_1(u) = (u_0)_x u_1 + u_0 (u_1)_x \ , \qquad (4.67)$$

$$A_2(u) = (u_0)_x u_2 + (u_1)_x u_1 + (u_2)_x u_0 \ ,$$

$$A_3(u) = (u_0)_x u_3 + (u_1)_x u_2 + (u_2)_x u_1 + (u_3)_x u_0 \ .$$

$$\vdots$$

By comparing both sides of equation(4.66), we get:

$$u_0(x,t) = \frac{1}{6}(x-1), \qquad (4.68)$$

$$u_{n+1}(x,t) = E_2^{-1}\left[v E_2 [6 A_n(u) - (u_n)_{xxx}]\right], \qquad n \geq 0. \qquad (4.69)$$

Then:

$$u_1(x,t) = E_2^{-1}\left[v\ E_2 \left[6A_0(u) - (u_0)_{xxx} \right]\right]$$

$$= E_2^{-1}\left[v\, E_2\left(6.\frac{1}{36}(x-1)\right)\right]$$

$$= E_2^{-1}\left[\frac{1}{6}\left(v^3 u^3 - v^3 u^2 \right)\right] = \frac{1}{6}(x-1)t, \qquad (4.70)$$

$$u_2(x,t) = E_2^{-1}\left[v\ E_2 \left[6A_1(u) - (u_1)_{xxx} \right]\right]$$

$$= E_2^{-1}\left[v E_2\left(6.\frac{1}{6}(x-1).\frac{1}{3}t \right)\right] = E_2^{-1}\left[v E_2\left(\frac{1}{3}xt - \frac{1}{3}t \right)\right]$$

$$= E_2^{-1}\left[\frac{1}{3}u^3v^4 - \frac{1}{3}u^2v^4\right] = \frac{1}{6}(x-1)t^2\,, \qquad (4.71)$$

By similar way we get:

$$u_3(x,t) = \frac{1}{6}(x-1)t^3\,. \qquad (4.72)$$

And so on. Then the first four terms of the decomposition series for equation (4.60), is given by:

$$u(x,t) = \frac{1}{6}(x-1)\left(1+t+t^2+t^3+\ldots\right), \qquad (4.73)$$

The solution in a closed form is given by:

$$u(x,t) = \frac{1}{6}\left(\frac{x-1}{1-t}\right) \quad , \quad |t| < 1\,. \qquad (4.74)$$

Conclusion

The combination of Adomian Decomposition Method (ADM) and Modified Double Sumudu Transform Method can produce a very effective method to solve nonlinear partial differential equations. Simply, it can be applied to other nonlinear partial differential equations of higher order.

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