Sudan University of Science and Technology

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Commutants and Abelian Toeplitz Algebra and Operators with Sarason Toeplitz Product on Bergman and Fock Spaces

المبدلات الابيلية لمؤثرات وجبر تبوليتز مع ضرب تبوليتز ساراسون علي فضاءات بيرجمان وفوك

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by:

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Dedication

To my Family

Acknowledgements

First I would like to Thank our greatest ALLAH. Then I would like to thank my supervisor Prof. Dr. Shawgy Hussein Abdalla who has given me great advice and help in this research . My thanks are due to every one who assisted by a way or another to bring this study to the light.

Abstract

We study by the function theory on Cartan domains, the Berezin-Toeplitz symbol calculus, the essential commutant of analytic Toeplitz operators and algebras associated with spherical isometries and on the Bergman space. The classification of reducing subspaces of a class of multiplication operators on the Bergman space, the Hardy space of the bidisk with the Totally Abelian Toeplitz operators and geometric invariants associated with their symbol curves are considered. The Hankel operators on Fock spaces and related Bergman kernel estimates and asymptotics for generalized Fock spaces are introduced . We obtain the localization, compactness and Sarason's Toeplitz product problem in Bergman and Fock spaces .

الخلاصة

درسنا بواسطة نظرية الدالة علي مجالات كارتان وحسبان رمز بيرزن-تبوليتز والمبدل الأساسي لمؤثرات تبوليتز التحليلية والجبريات المشاركة مع الايزوميتريس الدائرية وعلي فضاء بيرجمان. قمنا بإعتبار تصنيف الفضاءات الجزئية المختزلة لعائلة المؤثرات الضربية علي فضاء بيرجمان وفضاء هاردي للقرص الثنائي مع مؤثرات تبوليتز الابيلية الكلية واللامتغيرات الهندسية المشاركة مع منحنياتها الرمزية. تم إدخال مؤثرات هانكل علي فضاءات فوك وتقديرات نواة بيرجمان ذات العلاقه والمقاربات لفضاءات فوك المعممة. تم الحصول علي الموضعية والتراص ومسالة ضرب ساراسون تبوليتز في فضاءات بيرجمان وفوك.

Introduction

For $T \in B(H)^n$ be an essentially normal spherical isometry with empty point spectrum on a separable complex Hilbert space H , and let $\mathcal{A}_T \subset B(H)$ be the unital dual operator algebra generated by T . In this note we show that every operator $S \in B(H)$ in the essential commutant of $\mathcal{A}_T$ has the form $S = X + K$ with a T -Toeplitz operator X and a compact operator K. Our proof actually covers a larger class of subnormal op-erator tuples, called A-isometries, which includes for example the tuple $T = (M_{z_1}, \dots, M_{z_n}) \in B(H^2(\sigma))^n$ consisting of the multiplication operators with the coordinate functions on the Hardy space $H^2(\sigma)$ asso-ciated with the normalized surface measure σ on the boundary ∂D of a strictly pseudoconvex domain $D \subset \mathbb{C}^n$.

We study the commutant of an analytic Toeplitz operator. For $\phi \in H^\infty$, let $\phi = x$F be its innerouter factorization. Our main result is that if there exists $\lambda \in C$ such that x factors as $x = x_1 x_2 \dots x_n$ each x_i an inner function, and if $F - \lambda$ is divisible by each x_i, then $\{T_\phi\}' = \{T_x\}' \cap \{T_F\}'$. We develop a machinery to study multiplication operators on the Bergman space via the Hardy space of the bidisk. Using the machinery we study the structure of reducing subspaces of a multiplication operator on the Bergman space.

We show that the key step in the proof, which is a curious result about nilpotent operators. One corollary of our main result is that if $x(z) = z^n, n \geq 1$, then $\{T_\phi\}' = \{T_x\}' \cap \{T_F\}'$, another is that if $\phi \in H^\infty$ is univalent then $\{T_\phi\}' = \{T_z\}'$. We are also able to prove that if the inner factor of f ϕ is $x(z) = z^n, n \geq 1$, then $\{T_\phi\}' = \{T_{z^s}\}'$ where s is a positive integer maximal with respect to the property that z^n and $F(z)$ are both functions of z^s. We conclude by raising six questions. We completely classify reducing subspaces of the multiplication operator by a Blaschke product ϕ with order three on the Bergman space to solve a conjecture of Zhu.

We study the compactness of operators on the Bergman space of the unit ball and on very generally weighted Bargmann-Fock spaces in terms of the behavior of their Berezin transforms and the norms of the operators acting on reproducing kernels. In particular, in the Bergman space setting we show how a vanishing Berezin transform combined with certain (integral) growth conditions on an operator T are sufficient to imply that the operator is compact.

We studies totally Abelian operators in the context of analytic Toeplitz operators on both the Hardy and Bergman space. When the symbol is a meromorphic function on $\mathbb{C}$, we establish the connection between the totally Abelian property of these operators and geometric properties of their symbol curves. It is found that winding numbers and multiplicities of self-intersection of symbol curves play an important role. Techniques of group theory, complex analysis, geometry and operator theory are intrinsic.

We introduce a function space integrable mean oscillation (IMO) on C^n. With *IMO*, for all possible $1 \leq p, q < \infty$ we characterize those symbols *f* on C^n for which the Hankel operators H_f and H_f are simultaneously bounded (or compact) from Fock space F_α^p to Lebesgue space L_α^q. Sarason's Toeplitz product problem asks when the operator $T_u T_v$ is bounded on various Hilbert spaces of analytic functions, where u and v are analytic. The problem is highly nontrivial for Toeplitz operators on the Hardy space and the Bergman space (even in the case of the unit disk).

The Contents

Chapter 1
Function Theory on Cartan Domains and Essential Commutant

We determine the essential commutant of the set of all analytic Toeplitz operators on $H^2(\sigma)$ and thus extend results proved by Davidson (1977) for the unit disc and Ding and Sun (1997) for the unit ball.

Section (1.1): The Berezin-Toeplitz Symbol Calculus:

For Ω be a bounded symmetric (Cartan) domain in C^n with $dv(z)$ normalized Lebesgue measure on Ω. We let $L = L^2(\Omega, dv)$ be the usual space of Lebesgue square-integrable complex-valued functions on Ω. By $H^2 = H^2(\Omega, dv)$ we denote the Bergman subspace of L^2 consisting of holomorphic functions. The orthogonal projection operator from L^2 onto H^2 is denoted by P. Forf in $L^\infty(\Omega)$, the space of essentially bounded, measurable functions on Ω, we will consider the multiplication operator

$$M_f g = fg,$$

onL^2 the Hankel operator on L^2

$$H_f = (I - P)M_f P,$$

and the Toeplitz operator on H^2

$$T_f g = PM_f g.$$

The main problem we consider is to determine the maximal conjugate-closed subalgebraΩ of L^∞ for which

$$T_f T_g - T_{fg}$$

is a compact operator for all f, g in Q. The algebra Q is the homolog of the quasi-continuous functions for the corresponding classical problem with Toeplitz operators on the Hardy space of the unit circle [2].

Our results can be viewed as a major part of Berezin's program to "quantize" curved space [4]. As is clear in the case of C_n with Gaussian measure [6, 12, 3], the $map\ f \to T_f$ is a good candidate for a "quantization modulo the compact operators" on any domain in C_n.

We develop a theory of functions of "vanishing mean oscillation at the boundary" of the classical domains. This theory is of independent interest and is evidence that some of the modern results in one complex variable [3] remain valid for domains in C_n when formulated in terms of the Bergman metric.

It is elementary algebra that (Proposition (1.1.20),)

$$Q = \{f \in L^\infty : H_f \text{ and } H_{\bar{f}} I \text{ are compact}\}.$$

To characterize Q in function-theoretic terms, we need to introduce the Bergman reproducing kernel $K(z, a)$. For a in $\Omega, K(\cdot, a)$ is in H^2 and for any f in H^2

$$f(a) = \langle f, K(\cdot, a)\rangle$$

in the L^2 inner product $\langle\ ,\ \rangle$.The function $K(\cdot,\ \cdot)$ is actually defined and continuous on $\Omega\ X\ \overline{\Omega}$(where $\overline{\Omega}$ is the closure of Ω in C^n). The normalized (in H^2) reproducing kernel is denoted by

$$k_a(Z) = K(z,a)(K(a,a))^{-1/2}.$$

The functions $K(\cdot,\ \cdot)$ are well understood on bounded symmetric domains and have many useful properties. We shall make essential use of the *Berezin transform off* in L^{∞}, which is given by

$$\tilde{f}(a) = \langle f k_a, k_a \rangle .$$

For typographical reasons, we will write $(|f|^2)^{\sim}$ for the Berezin transform of $|f|^2$.

We will denote the topological boundary of Ω in $C^n$by $\partial\Omega$. Since $\partial\Omega$ is compact, by $z \rightarrow \partial\Omega$ for $z$in Ω, we will simply mean that the usual distance function $d(z,\partial\Omega)$ has the property that

$$d(z,\partial\Omega) \rightarrow 0$$

For f bounded and continuous on Ω we write $f \in BC(\Omega)$ and we define

$$Osc_z(f) = sup\{|f(z) - f\ (w)| : \beta(z,w) \leq 1\}$$

where $\beta(\cdot,\ \cdot)$ is the Bergman metric on Ω. The closed Bergman metric ball centered at z (with radius r) is denoted by

$$E(z,r) = \{w \in Q\Omega: \beta(z,w) \leq r\}.$$

We say that f in $BC(\Omega)$ is in VO_∂ (*vanishing oscillation at the boundary*) if

$$\lim_{z\rightarrow\partial\Omega} Osc_z(f) = 0.$$

It is not hard to check that VO_∂ is a norm-closed, conjugate closed subal-gebra of the sup norm algebra $BC(\Omega)$.

We also define the algebra $\mathcal{J}$ by

$$\mathcal{J} = \{f \in L^{\infty}: (|f|^2)^{\sim}(z) \rightarrow 0\ as\ z \rightarrow \partial\Omega\}$$

and the algebra $\tilde{Q}$by

$$\tilde{Q} = \{f \in L^{\infty}: (|f|^2)^{\sim}(z) - \tilde{f}(Z)^2 \rightarrow 0\ as\ z \rightarrow \partial\Omega\}.$$

Finally, we define the algebra $VMO_\partial(r)$ by writing

$$\hat{f}(z,r) = |E(z,r)|^{-1}\int_{E(z,r)} f\ (w) dv(w),$$

Where $|A|$ is the *v*-measure of any measurable subset A, and setting

$$VMO_\partial(r) = \{f \in L^{\infty}: |E(z,r)|^{-1}\int_{E(z,r)} \left|f\ (w) - \hat{f}(z,r)\right| dv(w) \rightarrow 0\ as\ z \rightarrow \partial\Omega\}$$

Our main result is that, for a class of Ω-including the ball and the polydisc-described later,

Theorem (1.1.1)[1]: *For f in $L^{\infty}(\Omega)$ the following are equivalen*t:

(a) $f \in Q$

(b) $f \in \tilde{Q}$

(c) $f \in VMO_\partial(r)$

(d) $f \in VO_\partial + \mathcal{J}$.

Moreover, for f in $Q, \hat{f}$ and $\hat{f}(\cdot, r)$ are in VO_∂ with $f - \tilde{f}$ and $f - \hat{f}(\cdot, r)$in $\mathcal{J}$. The decomposition $Q = VO_\partial + \mathcal{J}$ is almost unique:

$$VO_\partial \cap \mathcal{J} = C_\partial \equiv \{f \in \mathrm{BC}(\Omega): f(\mathrm{z}) \to 0 \text{ as } \mathrm{z} \to \partial\Omega\}$$

and

$$\tilde{f} - \hat{f}(\cdot, r)$$

is in $C_\partial\Omega$ for $f$in Q and $r > 0$ arbitrary.

Corollary (1.1.2)[1]: $VMO_\partial(r)$ is independentof $r > 0$.

In applications of Theorem (1.1.1), the metric geometry of $(Q, f)$is very significant. For general, $\Omega, \beta$is unbounded and the balls $E(z, r)$ are compact inC^n. As $a \to \partial\Omega, E(a, r) \to \partial\Omega$ in the sense that

$$\sup_{w \in E(a.r)} d(w, \partial\Omega) \to 0 \quad .$$

For $\Omega = \boldsymbol{B}$, the open unit ball in C^n, more is true: as $a \to \partial\boldsymbol{B}, E(a, r) \to a$. in the sense that

$$\sup_{w \in E(a,r)} |W - a| \to O$$

It follows easily that

$$C(\overline{\boldsymbol{B}}) \subset VO_\partial(\boldsymbol{B}) \qquad (*)$$

and we recover an old result of [2]. On the other hand, $(*)$ fails for higher rank domains.

For arbitrary bounded symmetric $\Omega, VO_\partial(\Omega)$and $\mathcal{J}(\Omega)$ are nontrivial (bigger than scalars + C_∂ and C_∂ respectively). In fact, it is easy to check that

$$e^{i\sqrt{\beta(O,z)}}$$

is always in VO_∂.

Using Theorem (1.1.1), Fredholm and index theory for the C*-algebras $\tau(Q)$, generated by all T_f with f in Q, becomes accessible. The algebras $VO_\partial(Q)$ are interesting in their own right. For SC the full ideal of compact operators and Ω as in Theorem (1.1.1), we have

Theorem (1.1.3)[1]: *For f in Q, $T_f$is compact if and only if $f \in \mathcal{J}$. There are *-isomorphisms*

$$\tau(Q)/\mathcal{K} \simeq Q/\tau \simeq VO_\partial/C_\partial.$$

The prototype for Theorems (1.1.1) and (1.1.3) is the earlier work [4, 8] for the domain $\Omega = C_n$ with dv replaced by normalized Gaussian measure. The "point at infinity" plays the role of the boundary $\partial\Omega$. Working with "Carleson rectangles" instead of Bergman metric balls, [3] then established versions of Theorems (1.1.1) and (1.1.3) for $\Omega = \boldsymbol{D}$, the open unit disk in **C**.

For $\Omega = \boldsymbol{D}$, [2] showed that

$$Q(\boldsymbol{D}) \cap H^\infty(\boldsymbol{D}) = B_0(\boldsymbol{D}) \cap L^\infty(\boldsymbol{D}),$$

whereH^{∞} is the usual space of bounded analytic functions and $B_0(\boldsymbol{D})$ is the "little Bloch space" of analytic functions f on **D** such that

$$\lim_{z\to\partial \boldsymbol{D}}(1 - |Z|^2)|f'(z)| = 0.$$

His method extends to the ball **B** in C^n to yield

$$Q(\boldsymbol{B}) \cap H^{\infty}(\boldsymbol{B}) = B_0(\boldsymbol{B}) \cap L^{\infty}(\boldsymbol{B}),$$

Where $B_0(\boldsymbol{B})$ consists of all analytic functions on **B** with

$$\lim_{z\to\partial \boldsymbol{B}}(1 - |Z|^2)|\nabla f(z)| = 0.$$

This sort of result fails spectacularly for higher rank domains. Using [22],we have

Theorem (1.1.4)[1]: *for* $\operatorname{rank}(\Omega) > 1, Q(\Omega) \cap H^{\infty}(\Omega)$.*consists of just the constant functions*

Although our methods of proof in Theorem (1.1.1) use many properties of bounded symmetric domains, the results are likely to hold more generally. In particular, for $\Omega = C^n$ we have the related results of [5,8]. We conjecture that Theorem (1.1.1) holds for all strictly pseudoconvex domains.

Here we collect the properties of bounded symmetric domains which will be used in the main results. [4] will provide essentially all of our requirements.

Any bounded symmetric domain Ω in C^n is, by definition, a Hermitian symmetric space [5] with complete Riemannian metric the Bergman metric $\beta(\cdot\,,\ \cdot)$. We always assume that Ω is in its Harish-Chandra realization so that 0 ∈ Ω. It follows directly from [6] that for each a in Ω there is a biholomorphic automorphism of Ω, φ_a, with the properties

(a) $\varphi_a(a) = 0$
(b) $\varphi_a \circ \varphi_a =$ identity map.

For $\Omega = \boldsymbol{D}$, the open unit disk in $\boldsymbol{C}, \varphi_a(z) = (a - z)(1 - \bar{a}z)^{-1}$. For $\Omega = \boldsymbol{B}$, the open unit ball in C^n, the φ_a's are explicitly described in [7]. We denote by Aut(Ω) the group of all biholomorphic automorphisms of Ω.

For the Bergman reproducing kernel mentioned, we have the well-known transformation law

$$K\big(\varphi(z), \varphi(w)\big)(J_c\varphi)(z)\overline{(J_c\varphi)(w)} = K(z,w), \tag{1}$$

hereφ is a biholomorphic map from Ω to Ω′ and $(J_c\varphi)(z)$ denotes the determinant of the complex Jacobian of φ. Note also that $|(J_c\varphi)(z)|^2$ is the determinant of the corresponding real Jacobian [8].

Using normalized volume measure dv on Ω (instead of the more customary unnormalized volume measure), it is clear from [14, 13] that we can assume

$$K(z,0) = 1$$

for all z in Ω.

Proposition (1.1.5)[1]: *For* φ_a*described above*,

$$|(J_c\varphi_a)(z)|^2 = |k_a(z)|^2 .$$

Proof. Using the transformation law (1), we see that

$$K(z,a) = K(\varphi_a(z),0)(J_c\varphi_a)(z)\overline{(J_c\varphi_a)(a)}$$
$$K(a,a) = K(0,0)|(J_c\varphi_a)(a)|^2.$$

It follows that

$$|K(z,a)|^2 = |(J_c\varphi_a)(z)|^2 K(a,a)$$

whence

$$|(J_c\varphi_a)(z)|^2 = |k_a(z)|^2$$

and, in fact, for $|\lambda_a| = 1$

$$(J_c\varphi_a)(z) = \lambda_a k_a(z).$$

It follows from [9] that for bounded symmetric domains Ω in the standard representation, with normalized volume measure, the kernel functions $K(\cdot\,,\ \cdot)$ have the special properties:

(a) $K(0,a) = 1 = K(a,0)$,
(b) $K(z,a) \neq 0, z \in \Omega, a \in \overline{\Omega}$,
(c) $\lim_{a-\partial\Omega} K(a,\ a) = +\infty$,
(d) $K(z,a)^{-1}$is a smooth function $on\ \boldsymbol{C}^n \times \boldsymbol{C}^n$.

Of course, $K(z,a) = \overline{K(a,z)}$. Here $\overline{\Omega}$ denotes the closure of Ω in C^n and $\partial\Omega$ is the topological boundary. The complex conjugate of f is denoted

One useful consequence of (1)-(3) is

Proposition (1.1.6)[1]: $On\ \Omega, k_a(\varphi_a(z))k_a(z) \equiv 1$

Proof. From Proposition (1.1.5) and the fact that φ_a is an involution, it follows that

$$|k_a(\varphi_a(z))k_a(z)| = 1.$$

Using analyticity, $k_a(\varphi_a(z))k_a(z)$ is constant on Ω and we can evaluate this constant at $z = 0$.

Another consequence of (a)-(c) is that, for f any polynomial in $z = (z_1, \dots, z_n)$, we have

$$\langle f, k_a\rangle = K(a,a)^{-1/2} f(a)$$

so that

$$\lim_{a\to\partial\Omega} \langle f, k_a\rangle = 0.$$

Using density of polynomials in $H^2(\Omega)$, it follows that the net $\{k_a\}$ con-verges weakly to 0 as $a \to \partial\Omega$.

For a in Ω, we consider the operator

$$(U_a f)(\mathrm{w}) = k_a(\mathrm{w}) f(\varphi_a(w))$$

on $L^2(\Omega)$. Using Proposition (1.1.6), it is easy to check that

(a) $U_a^2 = I$
(b) $U_a = U_a^*$ is unitary
(c) $U_a P = P U_a$.

Now we turn to the properties of the Bergman metric $\beta(\cdot\,,\cdot)$ on Ω. As noted earlier, $\beta(\cdot\,,\cdot)$ is a complete Riemannian metric [10] and it follows [12] that:

(a) the β metric topology on Ω is the usual topology,

(b) the closed β -metric balls

$$\mathrm{E(a,r)} = \{\mathrm{w} \in \Omega : \beta(\mathrm{a}, w) \leq r\}$$

are compact.

It follows easily that for a in Ω

$$\lim_{z \to \partial\Omega} \beta(a, z) = +\infty.$$

We will need to know that the $E(a,r)$ concentrate at $\partial\Omega$as $a \to \partial\Omega$.

We have

Lemma (1.1.7)[1]: *For r fixed,* $a \to \partial\Omega$ *if and only if*

$$\sup_{z \in E(a,r)} d(z, \partial\Omega) \to 0.$$

Proof. It is enough to show that, as $a \to \partial\Omega$, $E(a,r)$ is eventually in the complement of any compact subset of Ω. For C such a compact subset and $z_o \in C$, we let

$$\sup_{z \in C} \beta(z_o, z) = R.$$

Now, $\beta(a, z_o) \to \infty as\ a \to \partial\Omega$, and for $\beta(a, z_o) > R + r$, we see that $E(a,r) \cap C$ is empty.

Lemma (1.1.8)[1]:*For* $\Omega = \boldsymbol{B}$, *the open unit ball in* C^n,*as* $d(a, \partial\boldsymbol{B}) \to O$,*we have*

$$\sup_{z \in E(a,r)} |z - a| \to 0.$$

Moreover, if $a \to a_0$ in $\partial\boldsymbol{B}$ then

$$\sup_{z \in E(a,r)} |z - a_0| \to 0.$$

Proof. For z in $E(a,r)$, we must have $\varphi_a(z)$ in $E(0,r)$ so, by com-pactness of $E(0,r)$, there is a $\mu = \mu, (r)$ with

$$|\varphi_a(z)| \leq \mu < 1.$$

It follows from [11] that

$$|z - a| \leq \mu \left(\frac{1 - |a|^2}{1 - \mu^2|a|^2}\right)^{\frac{1}{2}} + |a| \frac{\mu^2(1 - |a|^2)}{1 - \mu^2|a|^2}$$

Hence, as $d(a, \partial\boldsymbol{B}) \to 0(|a| - 1)$ we have

$$\sup_{z \in E(a,r)} |z - a| \to 0.$$

Moreover, if $a \to a_0$ in $\partial\boldsymbol{B}$ then the triangle inequality implies that

$$\sup_{z \in E(a,r)} |z - a_0| \to 0.$$

The last geometric fact we will use is another consequence of the fact that (Ω, β) is a complete Riemannian manifold. For $\beta(z,w) \leq r$, consider the geodesic $arc\ \gamma$ from z to w of length $\beta(z,w)$. For any $r^{|} > 0$, we can find $m = m(r, r^{|})$ points $\{z_j\}$ on $\gamma\ with\ z_1 = z, z_m = w$ and

$$\beta(z, z_j) \leq r$$

$$\beta(z_j, z_j + 1) \leq r^{|}.$$

We will require some information about the dv measure of the metric balls $E(a, r)$. We write

$$|E(a,r)| = \int_{E(a\,,r)} 1 dv(z)\,.$$

By Proposition (1.1.5) and the invariance of $\beta(\cdot\,,\ \cdot)$ under biholomorphic transformations [12], we see that $\varphi_a E(a,r) =\ E(0,r)$ and

$$|E(a,r)| = \int_{E(0\,,r)} |k_a(w)|^2 dv(w)\,.$$

This fact allows us to compare volumes of different metric balls.

Lemma (1.1.9)[1]: *For a, b in* Ω *with* $\beta(a,b) \leq\ R and\, r, s > 0$*, we have*

$$0 < m(R,r,s) \leq \frac{|E(a,r)|}{|E(b,s)|} \leq\ M(R,r,s) < \infty.$$

Proof. *It is sufficient to check the upper bound separately on*

$$\frac{|E(a,r)|}{|E(b,r)|}, \qquad \frac{|E(b,r)|}{|E(b,s)|}.$$

We have

$$|E(c,t)| = \quad K(c,c)^{-1} \int_{E(0,t)} |K(z,c)|^2 I dv(z)$$

and by the compactness of $E(0,t)$ and the fact that $K(z,c)$ is continuous and nonvanishing on $E(0,t)\ \times\ \overline{\Omega}$, we see that

$$0 < m, \leq |E(c,t)|\, K(c,c) \leq M_t < \infty$$

for all *c* in Ω. It follows immediately that

$$\frac{|E(a,r)|}{|E(b,s)|} \leq \frac{M_r}{m_s}$$

Moreover,

$$\frac{|E(a,r)|}{|E(b,r)|} \frac{|K(a,a)|}{|K(b,b)|} \leq \frac{M_r}{m_s}$$

so that

$$\frac{|E(a,r)|}{|E(b,r)|} \leq \frac{M_r}{m_s} \frac{|K(b,b)|}{|K(a,a)|}$$

Using the transformation law for $K(\cdot\,,\cdot)$ and Propositions (1.1.5) and (1.1.6), we have

$$\frac{|K(b,b)|}{|K(a,a)|} = \frac{|K(\varphi_a(b), \varphi_a(b))|}{|K(\varphi_a(b), a)|^2}$$

Since $\beta(a,b) \leq R, \varphi_a(b) \in E(0,R)$. Again using the fact that $K(z,c)$ is continuous and nonvanishing on the compact set $E(0,R) \times \overline{\Omega}$, we see that there is a constant K_R with

$$\frac{|K(b,b)|}{|K(a,a)|} \leq K_R$$

and

$$\frac{|E(a,r)|}{|E(b,r)|} \leq \frac{M_r}{m_s} K_R$$

This finishes the proof.

The map $(a,r) \to |E(a,r)|$ will be of considerable interest to us. We will use

Lemma(1.1.10)[1]: *The map $(a,r) \to |E(a,r)|$ is continuous in each variable separately.*

Proof. For fixed a, we can easily check continuity of $r \to |E(a,r)|$ provided we know

$$S(a,r) = \{z \in \Omega: \beta(a,z) = r\}$$

has $S(a,r) = 0$. This follows directly from the fact that (Ω, β) is a complete Riemannian manifold. The exponential map sends the sphere of radius r in the tangent space at a onto $S(a,r)$. Differentiability of the exponential map [9] provides the appropriate estimate to show that $S(a,r) = 0$.

For fixed r, we recall that

$$|E(a,r)| = \int_{E(0,r)} \frac{|K(z,a)|^2}{K(a,a)} dv(w).$$

Continuity of $K(z,a)$, compactness of $E(0,r)$, and the fact that $K(a,a) \neq 0$ now show that $a \to |E(a,r)|$is continuous.

We also need a somewhat different kind of estimate.

Lemma (1.1.11)[1]: *For r > 0, there are constants $E(r), \epsilon(r)$ so that*

$$\infty > E(r) \geq |k_a(z)|^2 |E(a,r)| \geq \epsilon(r) > 0$$

for*a, z* in Ω with $\beta(a,z) \leq r$.

Proof. We have

$$|k_a(z)|^2 |E(a,r)| = \frac{|K(z,a)|^2}{K(a,a)^2} \int_{E(0,r)} |K(w,a)|^2 dv(w)$$

so, using previously discussed properties of $K(\cdot,\cdot)$,

$$0 < m_r \leq \int_{E(0,r)} |K(w,a)|^2 dv(w) \leq M_r < \infty,$$

and it suffices to consider

$$\frac{|K(z,a)|}{K(a,a)}$$

Again using the transformation rules and Proposition (1.1.6), we obtain

$$\frac{|K(z,a)|}{K(a,a)} = \frac{1}{|K(\varphi_a(z),a)|}$$

Since $(\varphi_a(z)$ is in $E(0,r)$ and $K(\cdot\,,\cdot)$ is continuous and nonvanishing on the compact set $E(0,r) \times \overline{\Omega}$, the desired estimate follows.

There is one important analytic requirement which remains. We need a version of a result in [12] for $\Omega = \boldsymbol{B}$, the open unit ball in C^n. This estimate has been somewhat extended by [13].

Lemma (1.1.12)[1]: *For* Ω *a finite product of balls or the irreducible rank two domain of* 2×2 *contractive symmetric matrices (in* C^3*), there are* $\epsilon_0 > 0$ *and* $q > 1$ *with*

$$\infty > M_\Omega^{\epsilon_0,q} = \sup_{z\in\Omega} \int_\Omega |K(z,w)|^{(1-2\epsilon_0)q} K(w,w)^{q\epsilon_0} dv(w)$$

Proof. Direct calculation using [17, p. 17] [10]. *Hence forth, we will assume when needed that Lemma (1.1.12) holds for* Ω.

For Ω a bounded symmetric domain in C^n and $\beta(\cdot\,,\ \cdot)$ the Bergman metric, we now derive some function-theoretic results which will be needed later. We have, first, for *f*in $BC(\Omega)$:

Lemma (11.13)[1]: *The function f is in* VO_a *if and only if for any fixed* $r > O$

$$\lim_{a\to\partial\Omega} sup\{\, |f(a) - f(w)|: \beta(a,w) \le r\} = 0 \ .$$

Proof. One direction is trivial. Suppose *f* is in VO_∂ and $r > 0$ is given. Recall that, by Lemma (1.1.7),

$$\sup_{z\in E(a,r)} d(z,\partial\Omega) \to 0$$

as a $\to \partial\Omega$. Recall also that there are $m = m(r,1)$ points z_j for any *w* in $E(a,r)$, so that $\{z_j\}$ are in $E(a,r)$ and $\beta(z_j,z_{j+1}) \le 1\, with\, z_1 = a$ and $Z_m = w$. By compactness of $E(a,r)$,

$$sup\{\, |f(a) - f(z)|: \beta(a,z) \le r\} = |f(a) - f(w)|$$

for some *w* in $E(a,r)$. Thus,

$$|f(a) - f(w)| \le \sum_{j=1}^{m-1} \left|f(z_j) - f(z_{j+1})\right|$$

for $\{z_j\}$ as above. As $a \to \partial\Omega$, the $z_j \to \partial\Omega$ and, by the definition of VO_∂, for any $\epsilon > 0$, if $d(a,\partial\Omega)$ is small enough then

$$\left|f(z_j) - f(z_{j+1})\right| < \epsilon/m$$

so that

$$|f(a) - f(w)| < \epsilon.$$

We can now prove

Theorem (1.1.14)[1]: *If f is in* VO_∂ *then*

$$\|f(a) - f\circ\varphi_a(\cdot)\|_p \to 0$$

As $a \to \partial\Omega$ for all $p > 1$.

Proof. This follows easily from Lemma (1.1.13), the invariance of $\beta(\cdot\,,\ \cdot)$ under biholomorphic maps, and the Lebesgue dominated convergence theorem.

we will establish some relations between the function spaces $\bar{Q}, VMO_\partial(r)$, and $VO_\partial + \mathcal{J}$ described in the introduction.

These relations are purely function-theoretic and are part of the proof of Theorem (1.1.1). We begin with

Lemma (1.1.15) [1]: *The following are equivalent*:

(a) $f \in VMO_\partial(r)$

(b) $\lim\limits_{a\to\partial\Omega} |E(a,r)|^{-2} \int_{E(a,r)} \int_{E(a,r)} |f(w) - f(z)|^2 dv(w)dv(z) = o.$

Proof. By direct calculation

$$|E(a,r)|^{-2} \int_{E(a,r)} \int_{E(a,r)} |f(w) - f(z)|^2 dv(w)dv(z)$$

$$= 2|E(a,r)|^{-1} \left|f(w) - \hat{f}(a,r)\right|^2 dv(w).$$

The desired result follows, using the fact that f is in L^∞ and standard estimates.

Lemma (1.1.16)[1]: For $r^| > r$, we have $VMO_\partial(r^|) \subset VMO_\partial(r)$.

Proof. Using Lemma (1.1.15), it suffices to check that

$$\left|E(a,r^|)\right|^{-1} |E(a,r)| \geq m(0,r,r^|) > 0.$$

This is exactly the content of Lemma (1.1.9).

We can now check

Theorem (1.1.17)[1]: *The space $\tilde{Q}$ is contained in $VMO_\partial(r)$.*

Proof. Direct calculation shows that

$$2(|f|^{2\sim}(a) - \left|\tilde{f}(a)\right|^2) = \int_\Omega \int_\Omega |f(z) - f(w)|^2 |k_a(z)|^2 |k_a(w)|^2 dv(z)dv(w)$$

$$\geq \int_{E(a,r)} \int_{E(a,r)} |f(z) - f(w)|^2 |k_a(z)|^2 |k_a(w)|^2 dv(z)dv(w)\ .$$

Now using Lemma (1.1.11), we see that

$$2[\epsilon(r)]^{-2} |f|^{2\sim}(a) - |\tilde{f}(a)|^2)$$

$$\geq |E(a,r)|^{-2} \int_{E(a,r)} \int_{E(a,r)} |f(z) - f(w)|^2\, dv(z)dv(w)$$

The desired result follows from Lemma (1.1.15).

Next we consider the function $\hat{f}(\cdot,r)$.

Lemma(1.1.18)[1]: *For f in L^∞ (Ω) and $r > 0, \hat{f}\ (\cdot,r)$ is in $BC(\Omega)$.*

Proof. Boundedness is clear. Using Lemma (1.1.10), it suffices to check that the map

$$a \to \int_{E(a,r)} f(w)dv(w) \equiv F(a)$$
is continuous. We note that, for $\beta(a, a_0)$ small,
$$E(a_0, r - \beta(a, a_0)) \subset E(a, r).$$
Thus,
$$F(a) = \int_{E(a,r)} f(w)dv(w)$$
$$= \int_{E(a_0, r - \beta(a,a_0))} f(w)dv(w) + \int_{E(a,r)\setminus E(a_0, r-\beta(a,a_0))} f(w)dv(w),$$
$$F(a_0) = \int_{E(a_0, r - \beta(a,a_0))} f(w)dv(w) + \int_{E(a,r)\setminus E(a_0, r-\beta(a,a_0))} f(w)dv(w),$$
and it follows that
$$|F(a) - F(a_0)| \le \|f\|_\infty \{|E(a,r)| + |E(a_0,r)| - 2|E(a_0, r - \beta(a,a_0))|\}.$$
Again using Lemma (1.1.10), it follows that $|F(a) - F(a_0)| \to 0 \ as \ a \to a_0$.

Finally, we can prove

Theorem (1.1.19)[1]: *For* $r > O$ *and fin* $VMO_\partial(r), \hat{f}(\cdot, r/2)$ *is in* VMO_∂ *and* $f - \hat{f}(., r/2)$ *is in* $\mathfrak{I}$*. Thus,* $VMO_\partial(r)$ *is contained in*$VO_\partial + \mathfrak{I}$.

Proof. For f in $VMO_\partial(r)$, we know that $\hat{f}(\cdot, r/2)$ is in $BC(\Omega)$ by Lemma (1.1.18). By Lemma (1.1.13), to check that$\hat{f}(\cdot, r/2)$ is in VO_∂ it will suffice to check that
$$\lim_{a - \partial\Omega} sup\{|\hat{f}(a, r/2) - \hat{f}(z, r/2)| : \beta(a,z) \le r/2\} = 0.$$
We have for $\beta(a,z) \le r/2$
$$|\hat{f}(a, r/2) - \hat{f}(z, r/2)|$$
$$\le |E(a, r/2)|^{-1}|E(z, r/2)|^{-1}$$
$$\times \int_{E\left(a,\frac{r}{2}\right)} \int_{E\left(z,\frac{r}{2}\right)} |f(u) - f(w)|dv(u)dv(w)$$
$$\le \frac{|E(a,r)|^2}{|E(a,r/2)||E(z,r/2)|} \frac{1}{|E(a,r)|^2}$$
$$\times \int_{E(a,r)} \int_{E(a,r)} |f(u) - f(w)|dv(u)dv(w).$$
Thus, by Lemma (1.1.9) and the Cauchy-Schwarz inequality, we have
$$|\hat{f}(a, r/2) - \hat{f}(z, r/2)|$$
$$\le M(0, r, r/2)M(r/2, r, r/2)$$
$$\times \left\{|E(a,r)|^2 \int_{E(a,r)} \int_{E(a,r)} |f(u) - f(w)|^2 dv(u)dv(w)\right\}^{1/2}$$
and Lemma (1.1.15) implies that $\hat{f}(\cdot, r/2)$ is in VO_∂.

Next, we check that $g = f - \hat{f}(\cdot, r/2)$is in $\mathcal{I}$.We need to show that

$$\int_{\Omega} |g(w)||k_a(W)|^2 dv(w) \to 0$$

As $a \to \partial\Omega$.

Given $\epsilon > 0$, using Proposition (1.1.5) and

$$\Omega = \bigcup_{k=1}^{\infty} E(O, k)$$

there is an $R = R(\epsilon)$ with

$$\begin{aligned}\int_{\Omega} |g(w)||k_a(w)|^2 dv(w) &= \int_{\Omega} \left|g \circ \varphi_a\big(\varphi_a(w)\big)\right||k_a(w)|^2 dv(w)\\ &= \int_{\Omega} |g \circ \varphi_a(z)| dv(z)\\ &\leq \int_{E(0,R)} |g \circ \varphi_a(z)| dv(z) + \|g\|_{\infty} |\Omega - E(0,R)|\\ &\leq \int_{E(0,R)} |g \circ \varphi_a(z)| dv(z) + \epsilon \leq \int_{E(a,R)} |g(w)||k_a(w)|^2 dv(w) + \epsilon\end{aligned}$$

Moreover,

$$\infty > E(R) \geq |E(a,R)||k_a(w)|^2$$

for all w in $E(a,R)$ by Lemma (1.1.11). It follows that it will suffice to check that

$$|E(a,R)|^{-1} \int_{E(a,R)} |g(w)|\, dv(w) \to O$$

as $a \to \partial\Omega$

Using the fact that $\varphi_a E(0,R) = E(a,R)$, it is easy to find $m = m(R,r)$ points a_j in $E(a,R)$ so that

$$E(a,R) \subset \bigcup_{j=1}^{m} E(a_j, r/2) .$$

Thus,

$$|E(a,R)|^{-1} \int_{E(a,R)} |g(w)|\, dv(w) \leq \sum_{j=1}^{m} |E(a,R)|^{-1} \int_{E(a_j,r/2)} |g(w)|\, dv(w) .$$

Moreover, since $\beta(a_j, a) \leq R$ by Lemma (1.1.9),

$$|E(a,R)|^{-1} \leq E\big(a_j, r/2\big)^{-1} M(R, r/2, R)$$

and so

$$|E(a,R)|^{-1} \int_{E(a,R)} |g(w)|\, dv(w) \leq M(R, r/2, R) \sum_{j=1}^{m} |E(a,R)|^{-1} \int_{E(a_j,r/2)} |g(w)|\, dv(w) .$$

Using Lemma (1.1.7), it will now suffice to check that

$$|E(a, r / 2)|^{-1} \int_{E(a,r/2)} |g(w)|\, dv(w) \to O$$

as $a \to \partial\Omega$

To finish the proof, we note that

$$\begin{aligned} |E(a, r / 2)|^{-1} \int_{E(a,r/2)} |g(w)|\, dv(w) \\ \leq |E(a, r / 2)|^{-1} \int_{E(a,r/2)} \left|f(w) - \hat{f}(a, r / 2)\right| dv(w) \\ + |E(a, r / 2)|^{-1} \int_{E(a,r/2)} \left|\hat{f}(a, r / 2) - \hat{f}(w, r / 2)\right| dv(w) \end{aligned}$$

As $a \to \partial\Omega$, the first term on the right goes to zero since f is in $VMO_\partial(r/2)$ by Lemma (1.1.16). Since $\hat{f}(\cdot\,, r / 2)$ is in VO_∂, Lemma (1.1.13) shows that the second term on the right goes to zero as $a \to \partial\Omega$.

We now turn to the operator-theoretic estimates which are required. Let $\mathcal{K}$ denote the algebra of all compact op-erators on the appropriate Hilbert space. Note that A is in $\mathcal{K}$ if and only if $A * A$ is in $\mathcal{K}$. For completeness we check

Proposition (1.1.20)[1]: $Q = \{f \in L^\infty : H_f, H_{\bar{f}} \in \mathcal{K}\}$.

Proof. It is easy to check that $H_f, H_{\bar{f}}$ are in $\mathcal{K}$ if and only if

$$[P, M_f] = PM_f - M_f P$$

is in $\mathcal{K}$. Since

$$[P, M_{gf}] = [P, M_g]M_f + M_g[P, M_f]$$

we see that $\{f \in L^\infty : H_f, H_{\bar{f}} \in \mathcal{K}\}$ is a closed (since $\mathcal{K}$ is norm closed) conjugate-closed subalgebra of L^∞.

Using

$$PM_{gf}P - (PM_fP)(PM_gP) = PM_f(I - P)M_gP,$$

we see that for $H_{\bar{f}}$ or H_g in $\mathcal{K}$, $T_{fg} - T_fT_g$ is in $\mathcal{K}$. Choosing $f = \bar{g}$, we also see that

$$T_{|g|^2} - T_{\bar{g}}T_g = H_g^* H_g$$

so that H_g is in $\mathcal{K}$ if and only if $T_{|g|^2} - T_{\bar{g}}T_g$ is in $\mathcal{K}$. the desired conclusion follows. A natural ideal in Q is described as follows:

$$\mathcal{J}_{\mathcal{K}} = \{f \in Q : T_f \in \mathcal{K}\}.$$

Lemma (1.1.21) [1]: *$\mathcal{J}_{\mathcal{K}}$ is a closed ideal in Q. f is in $\mathcal{J}_{\mathcal{K}}$ if and only if $f \in L^\infty$ and $T_{|f|^2} \in \mathcal{K}$.*

Proof. It is easy to check that f is in $\mathcal{J}_{\mathcal{K}}$ if and only if M_fP is in $\mathcal{K}$ (equivalently, $T_{|f|^2} \in \mathcal{K}$)and $f \in L^\infty$. It is now easy to check that $\mathcal{J}_{\mathcal{K}}$ is an ideal in L^∞ and that $\mathcal{J}_{\mathcal{K}}$ is closed.

For f in $L^\infty(\Omega)$, we define

$$J_1(z,f) = \int_\Omega |f(z) - f(w)|\,|K(z,w)|\,K(w,w)^{\epsilon_o} dv(w)$$

$$J_2(z,f) = \int_\Omega |f(w)|\,|K(z,w)|\,K(w,w)^{\epsilon_o} dv(w)$$

where ϵ_o is chosen as in Lemma (1.1.12). We have

Lemma(1.1.22)[1]: *For $\epsilon_o > 0, q > 1$ as in Lemma (1.1.12) and $p^{-1} + q^{-1} = 1$, there is a positive constant M, independent off,such that*

$$J_1(z,f) \le MK(z, z)^{\epsilon_o} \|f(z) - f \circ \varphi_z\|_p$$

$$J_2(z,f) \le MK(z, z)^{\epsilon_o} \| f \circ \varphi_z\|_p.$$

Proof. The argument for J_2exactly parallels the argument for J_1. By change of variables for $J_1(z, f)$, we have

$$J_1(z,f) \int_\Omega f(z) - f \circ \varphi_z(w)$$

$$\times |K(z, \varphi_z(w))| K(\varphi_z(w), \varphi_z(w))^{\epsilon_o} |k_z(w)|^2 dv(w).$$

Using Proposition (1.1.6) and the transformation laws for $K(\cdot,\cdot)$ we find that

$$|K(z, \varphi_z(w))| = \frac{K(z,z)}{|K(z,w)|}$$

and

$$K(\varphi_z(w), \varphi_z(w))\,|k_z(w)|^2 = K(w,w).$$

Thus, by the Jordan-Hölder inequality and Lemma (1.1.12)

$$J_1(z,f) = K(z,z)^{\epsilon_o} |f(z) - f \circ \varphi_z(w)|\,|K(z,w)|^{1-2\epsilon_o} K(w,w)^{\epsilon_o} dv(w)$$

$$\le \left(M_\Omega^{\epsilon_o,q}\right)^{1/q} K(z,z)^{\epsilon_o} \|f(z) - f \circ \varphi_z\|_p.$$

This completes the proof.

We also need

Lemma (1.1.23)[1]: *For $\chi = \chi c$ the characteristic function of any compact subset C of $\Omega, M_\chi H_f$ and $M_\chi P M_f P$ are compact operators on $L^2(\Omega)$.*

Proof. For g in the range of P,

$$(M_\chi H_f) g(z) = \int_\Omega \chi(z)(f(z) - f(w)) K(z,w) g(w) dv(w)$$

$$(M_\chi P M_f P) g(z) = \int_\Omega \chi(z) f(w) K(z,w) g(w) dv(w).$$

The integral kernels are bounded so that the operators are Hilbert-Schmidt.

We can now establish the next link in the main result:

Theorem (1.1.24)[1]: $VO_\partial + \mathcal{I}$ is contained in $\mathcal{Q}$.

Proof. Since $\mathcal{Q}$ is an algebra, we need only check that VO_∂ and $\mathcal{I}$ are separately contained in $\mathcal{Q}$. Suppose that f is in VO_∂. By Theorem (1.1.14), $\|f(a) - f \circ \varphi_a\|_p \to 0$ as $d(a, \partial\Omega) \to 0$. Given $\epsilon > 0$, choose $\delta = \delta(\epsilon, f)$ small enough so that

$$\|f(a)- f\circ\varphi_z\|_p<\epsilon$$
When ever
$$d(a,\partial\Omega)<\delta.$$
Let $\chi=\chi c$ be the characteristic function of the compact set
$$C=\{a\in\Omega: d(a,\partial\Omega)\geq\delta\}.$$
By Lemma (1.1.23), $M_\chi H_f$ is in $\mathcal{K}$.Moreover, for g in $H^2(\Omega)$ and $\epsilon_o>0$ as in Lemma (1.1.12),
$$(H_f g)(z)=\int_\Omega (f(z)-f(w))K(z,w)g(w)dv(w).$$
so that
$$\left|(H_f g)(Z)\right|^2 \leq \int_\Omega |f(z)-f(w)||K(z,w)|K(w,w)^{\epsilon_o}\,dv(w) \times \int_\Omega |f(z)-f(w)||K(z,w)|K(w,w)^{-\epsilon_o}|g(w)|^2\,dv(w)$$
by the Cauchy-Schwarz Lemma. Using Lemma (1.1.22), we now have
$$\left|(H_f g)(Z)\right|^2\leq 2\|f\|_\infty\, M_\epsilon K(z,z)^{\epsilon_o}\int_\Omega |K(z,w)|K(w,w)^{-\epsilon_o}|g(w)|^2\,dv(w)$$
for z in $\Omega\backslash C$. Thus by Fubini's Theorem and another application of Lemma (1.1.22), we have
$$\left\|\left(H_f-M_\chi H_f\right)g\right\|_2^2=\int_{\Omega\backslash C}\left|(H_f g)(Z)\right|^2 dv(z) \leq 2\|f\|_\infty\, M_\epsilon\int_\Omega K(w,w)^{-\epsilon_o}|g(w)|^2\,dv(w) \times\int_\Omega |K(z,w)|K(z,z)^{\epsilon_o}\,dv(z)\leq 2\|f\|_\infty M^2\epsilon\|f\|_2^2$$
Since $\epsilon_o>0$ is arbitrary, it follows that H_f is in $\mathcal{K}$. The same argument shows that $H_{\bar f}$ is in $\mathcal{K}$ and so f is in $\mathcal{Q}$ by Proposition (1.1.20).

To show that $\mathcal{I}$ is contained in $\mathcal{Q}$, we actually check that $\mathcal{I}$ is contained in $\mathcal{I}_\mathcal{K}$. For f in $\mathcal{I}$, it is easy to see that $(|f|^P)^\sim$ is in $C_\partial(\Omega)$for $p>1$. To show that f is in $\mathcal{I}_\mathcal{K}$, it suffices to show that $T_{|f|^2}$ is in $\mathcal{K}$. The proof is completed by noting that
$$\|f\circ\varphi_a\|_P^P=(|f|^P)^\sim(a)$$
and estimating
$$\left\|PM_{|f|^2}P-M_\chi PM_{|f|^2}P\right\|$$
using the method above.

Corollary (1.1.25)[1]: $\mathcal{I}=\mathcal{I}_\mathcal{K}$

Proof. We need only check that $\mathcal{I}_{\mathcal{K}}$ is contained in $\mathcal{I}$. We recall that the net $\{k_a\}$ converges weakly to zero as $a \to \partial\Omega$. Hence, for f in $\mathcal{I}_{\mathcal{K}}, T_{|f|^2}$ is in $\mathcal{K}$ and so

$$\langle T_{|f|^2} k_a, k_a \rangle = (|f|^P)\tilde{}(a) \to 0$$

as $a \to \partial\Omega$. An application of the Cauchy-Schwarz inequality now implies that f is in $\mathcal{I}$.

Corollary (1.1.26)[1]: $C_\partial(\Omega)$ is contained in $\mathcal{I}$.

Proof. Using Corollary (1.1.25) and Lemma (1.1.21), it suffices to check that $T_{|f|^2}$ is in $\mathcal{K}$ for $|f|^2$ in $C_\partial(\Omega)$. This is an easy exercise.

To establish that Q is contained in $\tilde{Q}$, we require several preliminary results. The next Lemma appears in [14]. We sketch the proof.

Lemma (1.1.27) [1]: $P|_{L^\infty(\Omega)} = PE$ *is a compact operator from the Banach space* $L^\infty(\Omega)$ *to* $H^2(\Omega)$.

Proof. [15]. Let E be the injection of $L^\infty(\Omega)$ into $L^2(\Omega)$. Then $P|_{L^\infty(\Omega)} =$ PE and for $M_{\chi\sigma}$, the operator of multiplication by the character-istic function of the compact set $\sigma, \sigma \subset \Omega$, we have

$$PE = PM_{\chi_\sigma}E + PM_{\chi_{\Omega\setminus\sigma}}E.$$

Note that $PM_{\chi\sigma}$ is a compact operator since P is an integral operator with smooth kernel away from the boundary $\partial\Omega$. Choose. σ so that $|\Omega \setminus \sigma| < \epsilon$. Then, for $\|f\|_\infty \leq 1$, we have

$$\left\|PM_{\chi_{\Omega\setminus\sigma}}Ef\right\|_2 = \|PM_{f\chi\Omega\setminus\sigma}\|_2 \leq \|\chi\Omega \setminus \sigma\|_2 < \sqrt{\epsilon}$$

so that

$$\left\|PM_{\chi_{\Omega\setminus\sigma}}E\right\| < \sqrt{\epsilon}.$$

Hence, PE is a norm limit of compact operators.

We also need

Lemma (1.1.28) [1]: If $\{f_n\}$ *is a sequence of real-valued functions in* $L^2(\Omega)$ with $\| f_n - h\|_2 \to 0$ *for* h *in* $H^2(\Omega)$, *then* h *is a constantfunction.*

Proof. Write $h = u + iv$ with u, v real-valued. Then

$$|f_n(z) - h(z)|^2 = |f_n(z) - u(z)|^2 + |v(z)|^2.$$

It follows that

$$\| f_n - h\|_2 \geq \| v\|_2$$

so that $v = 0$ and h is a real-valued holomorphic function. It is now elementary that h must be a constant function.

We can establish

Theorem (1.1.29)[1]: Q is contained in $\tilde{Q}$.

Proof. Since Q is conjugate-closed, it is enough to show that for f real-valued in Q, f must be in $\tilde{Q}$. For f real-valued, f in Q, we know

$$(|f|^2)^{\sim}(a) - \left|\tilde{f}(a)\right|^2 = \left\|\tilde{f}(a) - f \circ \varphi_a\right\|_2^2 \geq 0.$$

Suppose that

$$\overline{\lim_{a \to \partial\Omega}} \left\|\tilde{f}(a) - f \circ \varphi_a\right\|_2^2 \geq 0$$

so there are $a_n \to \partial\Omega$ with

$$\left\|\tilde{f}(a_n) - f \circ \varphi_n\right\|_2^2 \geq \rho > 0$$

for $= 1, 2, \ldots.$. Since f is in $Q, H_f = (I - P)M_f P$ is a compact operator by Proposition (1.1.20). Since $\{k_{a_n}\}$ converges weakly to zero as $a_n \to \partial\Omega$, we must have

$$\lim_{n \to \infty} \left\|(I - P) f k_{a_n}\right\|_2 = 0$$

We now use the unitary operators

$$(U_a f)(w) = k_a(w) f \circ \varphi_a(w)$$

discussed earlier and recall that $[P, U_a] = 0$ and

$$U_a(f \circ \varphi_a) = f k_a.$$

It is easy to check that

$$\lim_{n \to \infty} \left\|(I - P) f \circ \varphi_{a_n}\right\|_2 = 0 \qquad (2)$$

Noting that $\{f \circ \varphi_{a_n}\}$ is a bounded subset of L^∞ (Ω), Lemma (1.1.27) implies that there is an h in $H^2(\Omega)$ and a subsequence $\{a_{n_k}\}$ so that

$$\lim_{k \to \infty} \left\|P(f \circ \varphi_{a_{n_k}}) - h\right\|_2 = 0\,. \qquad (3)$$

Combining (2) and (3), we see that

$$\lim_{k \to \infty} \left\|(f \circ \varphi_{a_{n_k}}) - h\right\|_2 = 0\,.$$

It follows from Lemma (1.1.28) that h is a constant function. Thus

$$\tilde{f}(a_{n_k}) = \langle f \circ \varphi_{a_{n_k}}, 1\rangle \to \langle h, 1\rangle = h$$

and it follows from the estimate

$$\left\|\tilde{f}(a_{n_k}) - f \circ \varphi_{a_{n_k}}\right\|_2 \leq \left\|(f \circ \varphi_{a_{n_k}}) - h\right\|_2 + \left\|\tilde{f}(a_{n_k}) - h\right\|_2$$

that

$$\lim_{k \to \infty} \left\|\tilde{f}(a_{n_k}) - f \circ \varphi_{a_{n_k}}\right\|_2 = 0\,.$$

This contradicts (a).

It follows that

$$\lim_{a \to \partial\Omega} \left\|\tilde{f}(a) - f \circ \varphi_n\right\|_2^2 = 0$$

and so f is in $\tilde{Q}$.

We can now assemble the results of earlier sections. We need

Lemma (1.1.30)[1]: *For f in $\tilde{Q}$, wehave $\tilde{f} - \hat{f}(\cdot, r)$ in $C_\partial(\Omega)$ for any $r > O$ .It Follows that f is in VO_∂.*

Proof. It is easy to see that $\tilde{f}$ is in $BC(\Omega)$ because of the continuity of $K(\cdot,\cdot)$ and the definition of $\tilde{f}$. Moreover, $\hat{f}(\cdot,r)$ is in VO_∂ by Theorems (1.1.17)and (1.1.19). It remains to estimate $\tilde{f}-\hat{f}(\cdot,r)$. Using Lemma (1.1.11) and direct calculation, we have

$$\epsilon(r)^{-1}[(|f|^2)^{\sim}(a)-|\tilde{f}(a)|^2] = \epsilon(r)^{-1}\int_\Omega |f(w)-\tilde{f}(a)|^2|k_a(w)|^2dv(w)$$

$$\geq |E(a,r)|^{-1}\int_{E(a,r)} |f(w)-\tilde{f}(a)|^2dv(w) \geq \hat{f}(a,r)-\tilde{f}(a)^2.$$

The desired result follows.

We also require a characterization of VO_∂ which is of intrinsic interest.

Theorem (1.1.31)[1]: $VO_\partial = \tilde{Q}\cap\{f: f-\tilde{f}\in C_\partial(\Omega)\}$.

Proof. By Theorem (1.1.14), if f is in VO_∂ then

$$\lim_{a\to\partial\Omega}\|f(a)-f\circ\varphi_a\|_2^2 = 0$$

Direct calculation shows that

$$\|f(a)-f\circ\varphi_a\|_2^2 = (|f|^2)^{\sim}(a)-|\tilde{f}(a)|^2+|\tilde{f}(a)-f(a)|^2.$$

Note that

$$(|f|^2)^{\sim}(a)\geq|\tilde{f}(a)|^2$$

by the Cauchy-Schwarz inequality. It follows that if (1) holds and f is in $BC(\Omega)$ then f is in $\tilde{Q}$ and $\tilde{f}-f$ is in $C_\partial(\Omega)$. By Lemma (1.1.30), we see that f is in VO_∂. The desired result follows from (b).

For f in $L^\infty(\Omega)$ the following are equivalent:

(a) $f\in Q$
(b) $f\in\tilde{Q}$
(c) $f\in VMO_\partial(r)$
(d) $f\in VO_\partial+\mathcal{I}$.

Moreover, for f in Q, f and $\hat{f}(\cdot,r)$ are in VO_∂ with $f-\tilde{f}$ and $f-\hat{f}(\cdot,r)$ in $\mathcal{I}$ for $r>0$. We have

$$VO_\partial\cap\mathcal{I}=C_\partial(\Omega) \text{ and } \tilde{f}-\hat{f}(\cdot,r) \text{ is in } C_\partial(\Omega) \text{ for } f \text{ in } Q \text{ and } r>0.$$

Proof. $(b)\Rightarrow(c)$ is Theorem (1.1.19). $(c)\Rightarrow(d)$ is Theorem (1.1.19). $(d)\Rightarrow(a)$ is Theorem (1.1.24). $(a)\Rightarrow(b)$ is Theorem (1.1.29). Theorem (1.1.19) also shows that $\hat{f}(\cdot,r)$ is in VO_∂ with $f-\tilde{f}(\cdot,r)$ in $\mathcal{I}$. By Lemma (1.1.30), $\tilde{f}-\hat{f}(\cdot,r)$ is in $C_\partial(\Omega)$. By Corollary(1.1.26) of Theorem (1.1.24), $C_\partial(\Omega)$ is contained in $\mathcal{I}$ and it follows that $f-\tilde{f}$ is in $\mathcal{I}$.

Finally, using Theorem (1.1.31) we see that for f in $VO_\partial\cap\mathcal{I}$, $f-\tilde{f}$ and $(|f|)^{\sim}$ are in $C_\partial(\Omega)$.It followsthat $\tilde{f}$ is in $C_\partial(\Omega)$ so that f is also in $C_\partial(\Omega)$. The reverse inclusion is elementary in view of Corollary (1.1.26) of Theorem (1.1.24).

$VMO_\partial(r)$ is independent of $r>0$.

Proof. $VMO_\partial(r)=Q$ for all $r>0$.

*For f in Q, T_f is compact if and only iff is in $\mathcal{I}$. We have *isomorphisms*

$$\tau(Q)/\mathcal{K} \simeq Q/\mathcal{I} - VOad/Ca(Q).$$

Proof. We need to check, first, that $\mathcal{K}$ is contained in $\tau(Q)$. A standard argument shows it is enough to check irreducibility of $\tau(Q)$. Using Theorem (1.1.24) and noting that $\mathcal{I}$ contains the set L_C^∞ of L^∞ functions with compact support, it suffices to check the irreducibility of the set $\{T_f: f \in L_C^\infty\}$. Cutting down the coordinate functions z_j to compact subsets of Ω and an application of the Lebesgue dominated convergence theorem as in [15] shows that the irreducibility of $\{T_f: f \in L_C^\infty\}$ follows from the easily checked irreducibility of $\{T_{zj}: j = 1, 2, \ldots, n\}$ (cf. [12]).

By Corollary(1.1.25) of Theorem (1.1.24), for f in Q, T_f is in $\mathcal{K}$ if and only if f is in $\mathcal{I}$. Moreover, for all g, h in Q

$$T_g T_h - T_{gh}$$

is in $\mathcal{K}$. It follows easily from standard *-algebra facts that $\tau(Q)/\mathcal{K} \simeq Q/\mathcal{I}$. The last isomorphism is clear from Theorem (1.1.1).

Theorem (1.1.32) [1]: [14]. For **B** the open unit ball in $\boldsymbol{C}^n$, the algebra $C(\overline{\boldsymbol{B}})$ is contained in $Q(\boldsymbol{B})$.

Proof. If f is in $C(\overline{\boldsymbol{B}})$, then for arbitrary $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ for which $d(z, a) < \delta(\epsilon)$ implies that $|f(a) - f(z)| < \epsilon$. By Lemma (1.1.8), there is a $\delta' = \delta'(r, \delta(\epsilon)) > 0$ so that $d(a, \partial\boldsymbol{B}) < \delta'$ implies $d(z, a) < \delta(\epsilon)$ for all z in $E(a, r)$. This shows that $C(\overline{\boldsymbol{B}})$ is contained in $VO_\partial(\boldsymbol{B})$. An application of Theorem (1.1.24) completes the proof.

For rank $(\Omega) > 1, Q(\Omega) \cap H^\infty(\Omega)$ *consists of just the constant functions.*

Proof. Let

$$G_z = 1/2\left(\frac{\partial^2}{\partial_{z_i}\partial_{\bar{z}_i}} log\, K(z,z)\right)$$

be the (infinitesimal) Bergman metric on Ω. For f holomorphic on $\Omega, z = (z_1, \ldots, z_n), \bar{w} = (\bar{w}_1, \ldots, \bar{w}_n)(z_i,\ w_i\ in\ \boldsymbol{C})$, define [12]

$$Q_f(z) = \sup \frac{|\langle \nabla f(z), \bar{w}\rangle|}{\sqrt{\langle G_2 w, w\rangle}} : |w| = 1$$

where $\langle\ ,\ \rangle$ is the usual inner product on $\boldsymbol{C}^n$ and

$$\nabla f(z) = \frac{\partial f}{\partial_{z_1}}, \ldots, \frac{\partial f}{\partial_{z_n}}$$

is the analytic gradient of f. It is known [12] that

$$Q_f(\varphi(z)) = Q_{f\circ\varphi}(z)$$

for all φ in Aut(Ω). It is also known [12] for rank $(\Omega) > 1$ that if

$$\lim_{z\to\partial\Omega} Q_f(z) = 0, \qquad (*)$$

then f is a constant function. Thus, it suffices to show that (*) holds for f in $Q \cap H^\infty$.

Since f is in H^∞, it is clear that $f = \tilde{f}$ and $H_f = 0$. It follows from Theorem (1.1.29) and Theorem (1.1.31) that, for $f H^\infty \cap \mathcal{Q}, f$ must be in VO_∂ and so

$$\lim_{a\to\partial\Omega} \mathcal{Q}_f(z) \left\| f(a) - f \circ \varphi_a \right\|_2^2 = 0 \qquad (**)$$

Next, we show that (*) follows from (**).

By [12], there is a smooth function $V(z,w)$ on $\boldsymbol{C}^n \times \boldsymbol{C}^n$ so that $K(z,w) = V(z,w)^{-1}$. Thus, if g is analytic on Ω, then

$$\mathrm{g}(z) = \int_\Omega g(w) V(z,w)^{-1} dv(w)$$

$$\frac{\partial g}{\partial \mathrm{z_i}}(z) = -\int_\Omega g(w) V(z,w)^{-2} \frac{\partial V}{\partial \mathrm{z_i}}(z,w) dv(w).$$

Note that $K(0,w) \equiv K(z,0) \equiv 1$. Thus, $V(0,w) \equiv 1$. Since $V(z,w)$ is smooth on $\boldsymbol{C}^n \times \boldsymbol{C}^n$, and Ω is bounded in $\boldsymbol{C}^n$, there is a constant $M > 0$ such that

$$\left| \frac{\partial V}{\partial \mathrm{z_i}}(0,w) \right| \le M$$

for all $1 \le i \le n$ and w in Ω. Therefore, we have

$$\left| \frac{\partial g}{\partial \mathrm{z_i}}(0) \right| \le M \int_\Omega |g(w)| dv(w).$$

Replacing g by $g - g(0)$ yields

$$|\nabla g(0)|^2 \le nM^2 \int_\Omega |g(w) - g(0)|^2 dv(w).$$

Finally, for $g = f \circ \varphi_a$ pawe have

$$|\nabla (f \circ \varphi_a)(0)|^2 \le nM^2 \left\| f(a) - f \circ \varphi_a \right\|_2^2$$

and so

$$\lim_{a\to\partial\Omega} |\nabla (f \circ \varphi_a)(0)| = 0$$

Since

$$\alpha^2 = \inf_{\|w\|=1} \langle G_0 w, w \rangle > 0$$

$[Tim_1]$ it follows from the definition of $\mathcal{Q}_f(z)$ that

$$\mathcal{Q}_{f\circ\varphi_a}(0) \le \alpha^{-1} |\nabla (f \circ \varphi_a)(0)|$$

and so

$$\mathcal{Q}_f(a) = \mathcal{Q}_f(\varphi_a(0)) = \mathcal{Q}_{f\circ\varphi_a}(0)$$

tends to 0 as $a \to \partial\Omega$. This completes the proof.

We conclude with a remark about the extension of Theorem (1.1.1) and some open problems.

Conjecture: Theorem (1.1.1) holds for Ω any strictly pseudoconvex domain.

Of course, a different method of proof will be needed because of the sparsity of analytic automorphisms.

Section (1.2): Analytic Toeplitz Operators Associated with Spherical Isometries

For m denote the linear Lebesgue measure on the unit circle $\partial\mathbb{D}$. A classical theorem of Davidson from 1977 (Theorem 1 in [26]) asserts that an operator S on the Hardy space $H^2(m)$ commutes modulo compact operators with all analytic Toeplitz operators if and only if S is a compact perturbation of a Toeplitz operator $\mathbb{B}_n$ with symbol $f \in H^\infty(m) + C(\partial\mathbb{D})$, where $H^\infty(m)$ refers to the space of all bounded holomorphic functions on $\mathbb{D}$ regarded as a subspace of $L^\infty(m)$ by passing to non-tangential boundary values.

[24] by Ding and Sun from 1997 an analogue of this result is obtained for the Hardy space on the open Euclidean unit ball $\mathbb{B}_n \subset \mathbb{C}^n$. If σ denotes the normalized surface measure on $\partial\mathbb{B}_n$, then by Theorem 2 in [24], an operator $S \in B(H^2(\sigma\ ))$ essentially commutes with all analytic Toeplitz operators if and only if $S\ = \mathbb{B}_n + K$, where K is compact and $f \in L^\infty(\sigma\ )$ has the property that the associated Hankel operator $H_f\ =\ P_{H^2(\sigma)^\perp}\ M_f\ |H^2(\sigma)$ is compact. For $n > 1$, this class of symbols strictly contains the space $H^\infty(\sigma) + C(\partial\mathbb{B}_n)$ $(see$ [27]), while equality holds in the case $n = 1$.

We establish variants of the cited results for Toeplitz operators associated with spherical isometries or, more general, with A-isometries. Recall that a spherical isometry on a complex Hilbert space H is a commuting tuple $T \in B(H)^n$ satisfying

$$\sum_{i=1}^{n} T_i * T_i = 1_H\ .$$

Given a spherical isometry T, there is an abstract theory of T-Toeplitz operators $X \in B(H)$ defined by Prunaru [25] as the solutions of the operator equation $\sum_{i=1}^{n} T_i * XT_i = X$. From this point of view, the result of Ding and Sun cited above describes the essential commutant of the dual algebra

$$\mathcal{A}_T\overline{\mathbb{C}[T_1, \ldots, T_n]}^{w^*} \subset B(H)$$

generated by the special spherical isometry $T = (T_1, \ldots, T_n) \in B(H)^n$ consisting of the multiplication operators $T_i = M_{z_i}$ with the coordinate functions on the Hardy space $H = H^2\ (\sigma)$. Formulated in the setting of general spherical isometries, the main result is the following (cf. Theorem (1.2.24)):

If $T \in B(H)^n$ is an essentially normal spherical isometry with empty point spectrum, then every operator $S \in B(H\ )$ in the essential commutant of $\mathcal{A}_T$ has the form $S = X + K$ with a T-Toeplitz operator X and a compact operator K on H.

As an application we deduce concrete analogues of the above-mentioned results of Davidson and Ding–Sun for multiplication tuples on Hardy-type function spaces. To be more specific, let μ denote a regular Borel probability measure on $\partial\mathbb{B}_n$ with the property that all one-point sets have μ-measure zero. Then the multiplication tuple $T_z = \left(M_{z_1}, \ldots, M_{z_n}\right) \in B(H^2(\mu))^n$ on the associated Hardy space

$$H^2(\mu) = \overline{\mathbb{C}[z_1, \dots, z_n]}^{\|\cdot\|_{2,\mu}} \subset L^2(\mu)$$
is a spherical isometry whose T_z-Toeplitz operators are precisely the compressions
$$T_f = P_{H^2(\mu)} M_f \setminus H^2(\mu)$$
of multiplication operators $M_f: L^2(\mu) \to L^2(\mu)$ with symbols $f \in L^\infty(\mu)$. The analytic Toeplitz operators are those with a symbol belonging to the space
$$H^\infty(\mu) = \overline{\mathbb{C}[z_1, \dots, z_n]}^{w^*} \subset L^\infty(\mu).$$
We show the following form (see Corollary (1.2.25)):
If $T_z \in B(H^2(\mu))^n$ *is essentially normal, then an operator essentially commutes with all analytic Toeplitz operators if and only if it has the form* $S = T_f + K$ *with a compact operator* K *and a symbol* $f \in L^\infty(\mu)$ *for which the associated Hankel operator* $H_f = P_{H^2(\mu)^\perp} M_f \setminus H^2(\mu)$ *is compact.*

We actually prove stronger versions of the above results for so-called regular A-isometries. The precise definition will be given. Let us just mention at the moment that this class is general enough to cover multiplication tuples with the coordinate functions on strictly pseudo-convex domains. For example, we obtain the following exact analogue of the above-mentioned theorem of Ding and Sun in the strictly pseudoconvex situation.

If σ *denotes the normalized surface measure on the boundary* ∂D *of a strictly pseudoconvex domain* $D \subset \mathbb{C}^n$ *with* C^2*-boundary, then an operator* S *in* $B(H^2(\sigma))$ *essentially commutes with all analytic Toeplitz operators on* $H^2(\sigma)$ *if and only if it has the form* $S = T_f + K$ *with a compact operator* K *and a symbol* $f \in L^\infty(\sigma)$ *for which the associated Hankel operator* H_f *is compact.*

We extend Prunaru's theory [26] on the existence of short exact Toeplitz sequences from the case of spherical isometries to the class of A-isometries and refine his results in the essentially normal case. To illustrate this for a spherical isometry $T \in B(H)^n$, let us write $\mathcal{T}_C(T) = C^*(T_f : f \in C(\partial\mathbb{B}_n))$ for the C*-algebra generated by all T-Toeplitz operators with continuous symbols (for the definition of T_f ,). Then Proposition (1.2.18) says the following:

Let $T \in B(H)^n$ *be an essentially normal, non-normal spherical isometry. If* $\mathcal{T}_C(T)$ *is irreducible, then there is a short exact sequence of* C***-algebras*
$$0 \longrightarrow \mathcal{K}(H) \overset{\subset}{\rightarrow} \mathcal{T}_C(T) \overset{\sigma}{\rightarrow} C(\sigma_n(T)) \longrightarrow 0,$$
where σ *maps the Toeplitz operator* T_f *to* $f\,|\sigma n(T)$ *for every* $f \in C(\partial\mathbb{B}_n)$.

As the above examples show (see also Theorem 3.5 in [27]), many interesting aspects of the theory of Toeplitz operators on classical Hardy spaces can be rediscovered of multi-variable subnormal isometries. The role of the surface measure in the classical theory will then be played by a scalar spectral measure of the minimal normal extension for the underlying subnormal tuple. In general, this measure is far from being explicitly known. So one cannot hope to find as detailed results as in the

-

classical case. It seems worth wile to pursue this connection further. An interesting question arises from a recent result of Xia (Theorem 1 in [26]) who answered a longstanding problem for Toeplitz operators on the unit disc. By the cited theorem, the condition that $T_{\overline{\theta}} X T_\theta - X$ is compact for every inner function $\theta \in H^\infty(m)$ implies that $X \in B(H^2(m))$ is a compact perturbation of a Toeplitz operator. In spherical isometries $T \in B(H)^n$, the T-Toeplitz operators with inner symbols naturally correspond to isometries in the dual operator algebra $\mathcal{A}_T$. So we may ask:

If $T \in B(H)^n$ is an essentially normal spherical isometry with empty point spectrum, and $X \in B(H)$ has the property that $J * XJ - X$ is compact for every isometry $\in \mathcal{A}_T$. Must X then necessarily be a compact perturbation of a T -Toeplitz operator?

Xia's proof depends on a special sequence of inner functions $(\theta_k)_{k\geq 0}$ consisting of finite Blaschke products, for which a multi-variable substitute is out of sight at the moment. So it seems that more sophisticated methods are needed to solve this problem.

Let H be a separable complex Hilbert space. A commuting tuple $T \in B(H)^n$ is called a spherical isometry if it satisfies the relation

$$\sum_{i=1}^{n} T_i * T_i = 1_H .$$

A result of Athavale [28] from 1990 saying that each spherical isometry is subnormal marks the starting point of the structure theory for this class of multi-operators. Since our approach to spherical isometries and their generalizations is based on the property of subnormality, we briefly recall some central facts about subnormal operator tuples. By definition, a subnormal tuple $T \in B(H)^n$ possesses an extension to a tuple $U \in B(\widehat{H})^n$ consisting of commuting normal operators on some Hilbert space $\widehat{H}$ containing . If the only reducing subspace for U that contains H is the space $\widehat{H}$ itself, then the tuple $U \in B(\widehat{H})^n$ is called *a minimal normal extension of T* . Given any normal extension U of T, one can always obtain a minimal one by restricting U to the space $\bigvee_{\alpha\in\mathbb{N}_0^n}(U^*)^\alpha H$. It is well known that any two minimal normal extensions of T are unitarily equivalent. In particular, the normal spectrum of T, which is defined by $\sigma_n(T) = \sigma(U)$ for some minimal normal extension U of T, does not depend on the choice of U. A result of Putinar [29] guarantees that $\sigma_n(T)$ is always contained in $\sigma(T)$.

Now, fix a subnormal tuple $T \in B(H)^n$ together with a minimal normal extension $U \in B(\widehat{H})^n$. Then one can choose a separating vector $z \in H$ for U , which means that the projection-valued spectral measure $E(\cdot)$ for U and the scalar-valued measure $\mu = \langle E(\cdot)z, z\rangle$ are mutually absolutely continuous. The measure μ obtained in this way is a finite regular positive Borel measure supported by $\sigma_n(T) = \sigma\ (U)$, and will be called a *scalar spectral measure* for U . From the identity $\mu(\sigma_n(T)) = \|z\|^2$ it follows that μ is a probability measure if the underlying separating vector $z \in H$ is a unit vector. Since, up to mutual absolute continuity, the measure μ does not depend on the special choice of U, we may speak of μ as a *scalar spectral measure associated with T*. By the spectral theorem for normal tuples, there exists an isomorphism of von Neumann algebras

$$\Psi_U : L^\infty(\mu) \to W^*(U) \subset B(\widehat{H}),$$

mapping the coordinate functions to the corresponding components of U . Defining

$$\mathcal{R}_T = \{f \in L^\infty(\mu): \Psi_U(f)H \subset H\}$$

one obtains a weak* closed subalgebra of $L^\infty(\mu)$ called the *restriction algebra*. The induced mapping

$$\gamma_T : \mathcal{R}_T \to B(H), \qquad f \longmapsto \Psi_U(f)|H$$

is known to be isometric again (see Conway [25]). Thus γ_T defines a weak* continuous isometric algebra homomorphism mapping z_i to T_i for $i = 1, \dots, n$. It should be mentioned that the restriction algebra $\mathcal{R}_T$ is independent of the choice of the minimal normal extension U and the concrete spectral measure μ.

From these general considerations about subnormal tuples we now return to the special case of a spherical isometry $T \in B(H)^n$. According to Athavale [29], T is subnormal and the spectral inclusion $\sigma_n(T) \subset \partial\mathbb{B}_n$ holds. An obvious density argument for the polynomials implies that the restriction algebra always contains the ball algebra $A(\mathbb{B}_n) = \{f \in C(\overline{\mathbb{B}}_n): f\,|\mathbb{B}_n \text{ is holomorphic}\}$. The rich function-theoretic structure of $A(\mathbb{B}_n)$ and suitable weak* closures then leads to interesting structure theorems for spherical isometries such as the reflexivity [29] of the dual operator algebra generated by T or factorization properties of type $\mathbb{A}_1$ and $\mathbb{A}_{1,\aleph_0}$ (see [25]). Replacing $A(\mathbb{B}_n)$ by an arbitrary function algebra A containing the polynomials one obtains the following very general notion of an isometric operator tuple introduced by [26].

Definition(1.2.1)[23]: Let $K \subset \mathbb{C}^n$ be a compact set and let $A \subset C(K)$ be a closed subalgebra containing the restrictions of the polynomials $C[z]$ in *n* complex variables $z = (z_1, \dots, z_n)$. A subnormal tuple $T \in B(H)^n$ is called an *A-isometry* if $\sigma_n(T)$ is contained in the Shilov boundary ∂_A of A and $\subset \mathcal{R}_T$.

By definition the Shilov boundary $\partial_A \subset K$ is the smallest closed set such that $\|f\|_{\infty,K} = \|f\|_{\infty,\partial_A}$ holds for every $f \in A$. Since the Shilov boundary of $A(\mathbb{B}_n)$ coincides with the topological boundary $\partial\mathbb{B}_n$, the remarks preceding the definition show that spherical isometries are precisely the $A(\mathbb{B}_n)$-isometries.

Other natural examples of A-isometries can be found of generalized Hardy spaces. Fix a compact set $K \subset \mathbb{C}^n$, a closed subalgebra $A \subset C(K)$ containing the polynomials $\mathbb{C}[z]|K$ and a positive measure $\mu \in M^+(\partial_A)$. The multiplication tuple

$M_z = (M_{z_1}, \ldots, M_{z_n}) \in B(L^2(\mu))^n$ is normal with scalar spectral measure μ and Taylor spectrum $\sigma(M_z) = supp(\mu) \subset \partial_A$. The associated functional calculus is given by the map $\Psi_{M_z}: L^\infty(\mu) \to B(L^2(\mu)), f \longmapsto M_f$. A Stone–Weiertrass argument shows that the restriction T_z of M_z to the invariant sub-space

$$H_A^2(\mu) = \bar{A}^{\|\cdot\|_{2,\mu}} \subset L^2(\mu)$$

has M_z as minimal normal extension. Since $H_A^2(\mu)$ is invariant under each multiplication operator M_f with symbol $f \in A$, it follows that $\mathcal{R}_{T_z} \supset A$. Thus

$$T_z = (M_{z_1}, \ldots, M_{z_n}) \in B(H_A^2(\mu))^n$$

is an A-isometry. Note that the multiplication tuples with the coordinate functions on the classical Hardy spaces over strictly pseudoconvex or bounded symmetric domains in $\mathbb{C}^n$.

Definition(1.2.2)[23]: A multiplication tuple of the form $T_z \in B(H_A^2(\mu))^n$ described above will be called a Hardy-space A-isometry.

Let us now return from these concrete examples to the study of a general A-isometry $T \in B(H)^n$. Fix a minimal normal extension $U \in B(\widehat{H})^n$ and a scalar spectral measure μ of T. we may consider μ as an element of $M^+(\partial_A)$ in the sequel.

Since the restriction algebra is weak* closed and contains A, it also contains the dual algebra

$$H_A^\infty(\mu) = \bar{A}^{w^*} \subset L^\infty(\mu).$$

If we denote the image of $H_A^\infty(\mu)$ under the canonical map γ_T introduced above by

$$\mathcal{H}_T = \gamma_T(H_A^\infty(\mu)) \subset B(H),$$

which is a weak⋆ closed subalgebra of B(H) , then we obtain a dual algebra isomorphism, that is, a weak* homeomorphism and isometric isomorphism

$$\gamma_T : H_A^\infty(\mu) \to \mathcal{H}_T , \qquad f \mapsto \Psi_U(f)|H,$$

extending the polynomial functional calculus of T . This map will be referred to as the canonical functional calculus for T. Via γ_T one can analyze the operator algebra $\mathcal{H}_T$ by studying the function algebra $H_A^\infty(\mu)$. A special role is played by the family

$$I_\mu = \{\theta \in H_A^\infty(\mu): |\theta| = 1\, \mu - \text{almost everywhere on } \partial_A\},$$

whose elements are called *μ-inner functions*. As in the case of spherical isometries, there is a one-to-one correspondence between I_μ and the operator family

$$\mathcal{T}_T = \{J \in \mathcal{H}_T : J \text{ is isometric}\}$$

A word-by-word repetition of the proof of Lemma 1.1 in [27] yields the following result.

Lemma(1.2.3)[23]: *Let $T \in B(H)^n$ be an A-isometry with associated scalar spectral meassure μ in $M^+(\partial_A)$. Then $\mathcal{T}_T = \gamma_T(I_\mu)$, where γ_T is the canonical functional calculus of T .*

In [25], Aleksandrov gives a sufficient condition ensuring $H_A^\infty(\mu)$ to have a rich supply of inner functions. More explicitly, a triple (A, K, μ) consisting of a compact set $K \subset \mathbb{C}^n$, a closed subalgebra $A \subset C(K)$ and a measure $\mu \in M^+(K)$, is called *regular* in the sense of Aleksandrov if the following approximation problem is solvable: *For every $\varphi \in C(K)$ with $\varphi > 0$, there exists a sequence of functions (φk) in A with $|\varphi_k| < \varphi$ on K and* $\log_{k\to\infty} |\varphi k| = \varphi$ *μ-almost everywhere on K.* One of the main results in [24] says that, if the measure μ in a regular triple is *continuous* in the sense that one-point sets have μ-measure zero, then the set of all μ-inner functions is rich in the following sense (see Corollary 29 in [25]).

Theorem (1.2.4)[23]: *(Aleksandrov). Let (A, K, μ) be a regular triple with a continuous measure μ in $M^+(K)$. Then the weak* sequential closure of the set I_μ contains all $L^\infty(\mu)$-equivalence classes of functions $f \in A$ with $|f| \le 1$ on K.*

In [30, Proposition 2.4 and Corollary 2.5] it was observed that the following weaker version of this density assertion is valid without any continuity assumption on the measure.

Proposition (1.2.5)[23]: *For every regular triple (A, K, μ), we have*

$$H_A^\infty(\mu) = \overline{LH}^{w^*}(I_\mu) \text{ and } L^\infty(\mu) = \overline{LH}^{w^*}\{\bar{\eta} \cdot \theta : \eta, \theta \in I_\mu\}.$$

Now we introduce a regularity criterion for A-isometries which guarantees that the above density results hold for the associated scalar spectral measures.

Definition (1.2.6)[23]: An A-isometry $T \in B(H)^n$ is called regular if, for some or equivalently every scalar spectral measure $\mu \in M^+(\partial_A)$ associated with T, the triple $(A|\partial_A, \partial_A, \mu)$ is regular.

In general, the regularity condition is hard to check. Nevertheless there are examples of function algebras A for which every A-isometry is regular. For example, if $D \subset \mathbb{C}^n$ is a relatively compact strictly pseudoconvex open set and

$$A(D) = \{f \in C(\overline{D}) : f\,|D \text{ is holomorphic}\}$$

is the generalized ball-algebra, then $\partial_A(D) = \partial D$ and the triple $(A(D)|\partial D, \partial D, \mu)$ is regular for every measure $\mu \in M^+(\partial D)$ (see Aleksandrov [24] or, for a more detailed explanation, [28]).

Proposition (1.2.7)[23]: *Every $A(D)$-isometry on a relatively compact strictly pseudoconvex open set $D \subset \mathbb{C}^n$ (in particular, every spherical isometry) is regular.*

As another example, take $A\, C(K)$. Then $\partial_A = K$ and $(C(K), K, \mu)$ is regular for every measure $\mu \in M^+(K)$. Now, a look at Definition (1.2.1) shows that the regular *C(K)*-isometries are precisely the normal tuples $T \in B(H)^n$ with Taylor spectrum contained in K.

The regularity of an A-isometry $T \in B(H)^n$ has immediate and far-reaching consequences for the structure of the dual operator algebras associated with *T* and its minimal normal extension U. For later reference, we collect some of them in the following proposition. Recall from Lemma (1.2.3) that the family of all isometries in $\mathcal{H}_T$ is $\mathcal{J}_T = \gamma_T(I_\mu) \subset B(H)$. Considering the normal tuple $U \in B(\widehat{H})^n$ also as an A-isometry, the corresponding set of all isometries contained in $\mathcal{H}_U$ is $\mathcal{J}_U = \Psi_U(I_\mu)B(\widehat{H})$. Having in mind that the point spectrum

$$\sigma_p(T) = \zeta \in \mathbb{C}^n : \bigcap_{i=1}^{n} ker(\zeta_i - T_i) \neq \emptyset$$

coincides with the set $\Delta_\mu = \{\zeta \in \partial_A : \mu(\{\zeta\}) > 0\}$ of all one-point atoms of one (equivalently any) scalar spectral measure μ (cf. the remarks following Proposition 3.1 in [24] for the case of spherical isometries), the following approximation results are immediate consequences of Lemma (1.2.3), Theorem (1.2.4) and Proposition (1.2.5).

Proposition (1.2.8)[23]: *Let $T \in B(H)^n$ be a regular A-isometry with minimal normal extension $U \in B(\widehat{H})^n$. Then the following assertions hold*:

(a) *The families of isometries $\mathcal{J}_T$ and $\mathcal{J}_U$ defined above satisfy*

$$\mathcal{H}_T = \overline{LH}^{w^*}(\mathcal{J}_T) \text{ and } W^*(U) = \overline{LH}^{w^*}(\{J_1^* J_2 : J_1, J_2 \in \mathcal{J}_U\}).$$

(b) *If T has empty point spectrum, then the dual operator algebra $\mathcal{H}_T$ contains a weak* zero sequence of isometries $J_k = \gamma_T(\theta_k)$ with $\theta_k \in I_\mu$ for $k \geq 1$.*

It seems that a profound theory of Toeplitz operators for A-isometries can only be established under the assumption that the associated families of isometries $\mathcal{J}_T$ *and* $\mathcal{J}_U$ are sufficiently rich (in the sense of part (a) above). This is the reason why we mostly consider regular A-isometries from now on.

Recall that Toeplitz operators associated with a spherical isometry $T \in B(H)^n$ have been introduced by Prunaru in [29] as the solutions $X \in B(H)$ of the operator equation $\sum_{i=1}^{n} T_i^* X T_i = X$. This relation is modelled after the classical Brown–Halmos condition characterizing Hardy-space Toeplitz operators on the unit disc [30]. A recent result of the authors (Proposition 3.1 in [31]) shows that the following definition for general A-isometries is consistent with Prunaru's definition for spherical isometries.

Definition (1.2.9)[23]: Let $T \in B(H)^n$ be an A-isometry. Then an operator $X \in B(H)$ is called a T- Toeplitz operator if

$$J^* X J = X \quad for\ every\ isometry\ J \in \mathcal{H}_T.$$

We write $\mathcal{T}(T)$ for the set of all T-Toeplitz operators on H.

To give an alternative characterization of T-Toeplitz operators, fix a minimal normal extension $U \in B(\widehat{H})^n$ and write $(U)'$ for the commutant of U in $B(\widehat{H})$, $P_H \in B(\widehat{H})$ for the orthogonal projection onto H. Then every operator $X \in B(H)$ of the form

$$X = P_H A|H \quad with\ A \in (U)'$$

belongs to $\mathcal{T}(T)$. Indeed, if $J = \gamma_T(\theta)$ is an isometry in $\mathcal{H}_T$ and h, k are arbitrary elements of H, then the fact that $\theta \in H_A^\infty(\mu)$ is inner immediately implies that $\langle J^*XJ\,h,k\rangle = \langle A\Psi_U(\theta)h, \Psi_U(\theta)k\rangle = \langle Ah, \Psi_U(|\theta|^2)k\rangle = \langle Ah,k\rangle = \langle Xh,k\rangle$. In particular, for every function $f \in L^\infty(\mu)$, we obtain an element $T_f \in \mathcal{T}(T)$ by setting

$$T_f = P_H\, \Psi_U\,(f)|H \in B(H),$$

called the T *-Toeplitz operator with symbol* f. The corresponding *Hankel operator with symbol* f is defined to be

$$H_f = (1 - P_H)\Psi_U\,(f\,)|H \in B\; H, H^\perp\,.$$

In case of a *regular* A-isometry the different types of Toeplitz operators considered above are related as follows.

Proposition (1.2.10)[23]: *Given a regular A-isometry* $T \in B(H)^n$ *with minimal normal extension* $U \in B(\widehat{H})^n$*, the following assertions hold*:

(a) *The* T *-Toeplitz operators possess the representation* $\mathcal{T}(T) = P_H(U)'|H$.

(b) *If* $W^*(U)$ *is a maximal abelian* W^**-algebra, then* $\mathcal{T}(T\,) = \{T_\varphi : \varphi \in L^\infty(\mu)\}$.

Proof. Note that the T -Toeplitz operators in the sense of Definition (1.2.9) are just the operators $X \in B(H)$ that are T -Toeplitz with respect to the commuting family of isometries $(\gamma_T(\theta))_{\theta\in I_\mu}$ in the sense of Prunaru (Definition 1.1 in [28]). The representation

$$W^*(U) = \overline{LH}^{w^*}(\{J_1^*J_2 : J_1, J_2 \in \mathcal{T}_U\})$$

obtained in Proposition (1.2.8) shows that the commutant of the family $(\Psi_U(\theta\,))_{\theta\in I_\mu}$ coincides with $(W^*(U\,))' = (U)'$. But then the minimality of U as a normal extension of T implies that $(\Psi_U(\theta\,))_{\theta\in I_\mu}$ is the minimal normal extension of the commuting family $(\gamma_T(\theta))_{\theta\in I_\mu}$ of isometries. Using Theorem 1.2 in Prunaru [32] for commuting families of isometries, we obtain that

$$\mathcal{T}(T) = P_H\,(\Psi_U(\theta))'_{\theta\in I_\mu}\,|H = P_H(U)'|H.$$

To prove part (b), observe that if $W^*(U\,)$ is a maximal abelian W^*-algebra, then $W^*(U\,) = (W^*(U))' = (U)'$ by Proposition 4.62 in Douglas [32]. Therefore $\mathcal{T}(T\,) = P_H(U)'|H = P_H\,\Psi_U(L^\infty(\mu))|H$, as desired.

Corollary (1.2.11)[23]: *For every regular Hardy-space A-isometry* $T = T_z \in B(H_A^2(\mu))^n$ *associated with a probability measure* $\mu \in M_1^+(\partial_A)$*, we obtain the identity*

$$\mathcal{T}(T) = \left\{X \in B\left(H_A^2(\mu)\right) : T_{\overline{\theta}}XT_\theta = X \; for\; every\; \theta \in I_\mu\right\} = T_\varphi : \varphi\, \in L^\infty(\mu)\,.$$

Proof. Remember that the minimal normal extension of T is $U = M_z \in B(L^2(\mu))^n$. Propo-sition 4.50 in Douglas [34] says that $W^*(U) = \{M_\varphi : \varphi \in L^\infty(\mu)\} \subset B(L^2(\mu))$ is a maximal abelian W^*-algebra. Hence the assertion follows from Lemma (1.2.3) and part (b) of the above proposition.

Let us add a simple lemma with two elementary properties of Toeplitz operators that will often be used without a comment throughout. For abbreviation, we say that a map $\Gamma : L^{\infty}(\mu) \to B(H)$ is *pointwise boundedly SOT-continuous* if, for every bounded sequence $(f_k)_{k\geq 1}$ in $L^{\infty}(\mu)$ converging pointwise μ-almost everywhere to some $f \in L^{\infty}(\mu)$ (at the level of representatives), we have $\Gamma(f) = SOT - \lim_{k\to\infty} \Gamma(f_k)$.

Lemma (1.2.12)[23]: *Let $T \in B(H)^n$ be an A-isometry with minimal normal extension $U \in B(\widehat{H})^n$. Then the following assertions hold:*

(a) For $Y \in B(\widehat{H})$, the maps

$\Gamma : L^{\infty}(\mu) \to B(H), f \mapsto P_H(\Psi_U(f)Y)|H$, *and* $\Gamma^*: L^{\infty}(\mu) \to B(H), \Gamma^*(f) = \Gamma(f)^*$ *are pointwise boundedly SOT-continuous.*

(b) *Given $Y \in (U)'$, $f \in L^{\infty}(\mu)$ and $g, h \in H_A^{\infty}(\mu)$, we have*

$$P_H(\Psi_U(\bar{g}f\, h)Y)|H = T_{\bar{g}}\,(P_H(\Psi_U(f)Y)|H)T_h$$

and in particular $T_{\bar{g}f\,h} = T_{\bar{g}}\, T_f\, T_h$.

Proof. Fix an arbitrary vector $x \in H$ and set $y = Yx$. Then the desired continuity property for Γ follows from the dominated convergence theorem and the estimate

$$\|\Gamma(f)x\|^2 \leq \|\Psi_U(f)y\|^2 = \int_{\partial_A} |f|^2 d\,\langle E(\cdot)y, y\rangle \quad (f \in L^{\infty}(\mu)).$$

An analogous argument applies to $\|\Gamma(f)^*x\|^2 \leq \|Y^*\|^2\, \|\Psi_U(f)x\|^2$. This proves part (a). In order to verify part (b), note that, for $x, y \in H$, the scalar product $\langle P_H\, \Psi_U(\bar{g}f\, h)Y\, x, y\rangle$ can be rewritten as

$$\langle \Psi_U(g)^*\Psi_U(f)Y\, \Psi_U(h)x, y\rangle = \langle \Psi_U(f)Y\, T_h x, T_g\, y\rangle = \langle T_{\bar{g}}\, P_H\, \Psi_U(f)Y\ T_h x, y\rangle,$$

as desired.

By applying the main result of the of Prunaru (Theorem 1.2 in [30]) to the setting explained in the proof of Proposition (1.2.10), we obtain the following version of Prunaru's result for -Toeplitz operators associated with regular A-isometries.

Proposition (1.2.13)[23]: (Prunaru). For a regular A-isometry $T \in B(H)^n$ with minimal normal extension $U \in B(H)^n$, the following assertions hold:

a) *The compression map* $: (U)' \to B(H), Y \mapsto P_H Y|H$, *is a complete isometry with range $ran(\rho) = \mathcal{T}(T)$.*

b) There is a surjective unital *-representation $\pi: C^*(\mathcal{T}(T)) \to (U)' \subset B(\widehat{H})$ satisfying the identity $\pi(\rho\,(Y)) = Y$ *for every* $Y \in (U)'$.

c) *There exists a completely positive and unital projection $\Phi: B(H) \to B(H)$ onto $ran(\Phi) = \mathcal{T}(T)$ such that $\Phi(X) = P_H\, \pi(X)|H$ holds for every $X \in C^*(\mathcal{T}(T))$.*

d) *The kernels $ker(\Phi|C^*(\mathcal{T}(T)))$ and $ker(\pi)$ are equal and coincide with the two-sided closed* ideal in $C^*(\mathcal{T}(T))$ generated by all operators of the form $XY - \Phi(XY)$ with *X, Y* in $\mathcal{T}(T)$.

Let $\mu \in M^+(\partial_A)$ be a scalar spectral measure associated with a regular A-isometry T in $B(H)^n$. For $f \in L^{\infty}(\mu)$, let us denote by $\mathcal{R}(f)$ the essential range of f, that is, the set of all $w \in \mathbb{C}$ such that $\mu(\{z \in \partial_A : |f(z) - w| < \epsilon\}) > 0$ for every $\epsilon > 0$. It follows from part (a) of Theorem (1.2.4) that

$$\|T_f\| = \|\Psi_U(f)\| = \|f\|_{L^\infty(\mu)}$$

for every $f \in L^\infty(\mu)$, and it is well known (see e.g. Satz 3.3.5 in [32]) that this property is equivalent to the condition that the spectral inclusion

$$\mathcal{R}(f) \subset \sigma(T_f)$$

holds for every function $f \in L^\infty(\mu)$. If the point spectrum of T is empty, then exactly as in the proof of Proposition 3.3 from [35] it follows that the norm and essential norm of every T-Toeplitz operator coincide.

Corollary (1.2.14)[23]: *Let $T \in B(H)^n$ be a regular A-isometry and let $\mu \in M^+(\partial_A)$ be a scalar spectral measure associated with T.*

a) For every $f \in L^\infty(\mu)$, we have $\|T_f\| = \|f\|_{L^\infty(\mu)}$ and $\mathcal{R}(f) \subset \sigma(T_f)$.

b) If T has empty point spectrum, then for every T-Toeplitz operator $X \in B(H)$, the equality $\|X\| = inf\{\|X - K\| : K \in \kappa(H)\}$ holds.

As another immediate consequence , we obtain the existence of a generalized Toeplitz sequence which, in some sense, justifies the definition of Toeplitz operators via the condition $J^*XJ = X$ for every $J \in \mathfrak{T}_T$.

Corollary (1.2.15)[23]: *For every regular A-isometry $T \in B(H)^n$, there is a short exact sequence*

$$0 \longrightarrow SC(T) \overset{\subset}{\rightarrow} C^*(\mathcal{T}(T)) \overset{\pi}{\rightarrow} (U)' \longrightarrow 0,$$

where $SC(T)$ stands for the two-sided closed ideal in $C^*(\mathcal{T}(T))$ generated by all operators of the form $XY - \Phi(XY)$ with $X, Y \in \mathcal{T}(T)$.

Restricting the map π from the full Toeplitz C^*-algebra $C^*(\mathcal{T}(T))$ to the C^*-algebra

$$\mathcal{T}_C(T) = C^*(\{T_f : f \in C(\partial_A)\}) \subset B(H)$$

generated by all Toeplitz operators with continuous symbols, we obtain the next result. Let $SC_C(T) \subset \mathcal{T}_C(T)$ be the closed two-sided ideal generated by all semi-commutators $T_f T_g - T_{fg}$ with $f, g \in C(\partial_A)$, and let $C_C(T) \subset \mathcal{T}_C(T)$ be the closed two-sided ideal generated by all com-mutators $T_f T_g - T_g T_f$ with $f, g \in C(\partial_A)$. It is elementary to see that $C_C(T) \subset S\,C_C(T)$.

Corollary(1.2.16)[23]: *For every regular A-isometry $T \in B(H)^n$, there is a short exact sequence*

$$0 \longrightarrow S\,C_C(T) = C_C(T) \overset{\subset}{\rightarrow} \mathcal{T}_C(T) \overset{\sigma}{\rightarrow} C(\sigma_n(T)) \longrightarrow 0$$

with a $*$-homomorphism σ satisfying $\sigma(T_f) = f|\sigma_n(T)$ for every $f \in C(\partial_A)$.

Proof. With the notations from Proposition (1.2.13), we have $\rho(\Psi_U(f)) = P_H \Psi_U(f)|H = T_f$ for $f \in C(\partial_A)$. Hence by part (b) of Proposition (1.2.13), the restriction of the map $\pi : C^*(\mathcal{T}(T)) \to (U)'$ to $\mathcal{T}_C(T)$ yields a surjective C^*-algebra homomorphism $\tilde{\pi}: \mathcal{T}_C(T) \to C^*(U)$ $with$ $\tilde{\pi}(T_f) = \Psi_U(f)$ for all $f \in C(\partial_A)$. We want to determine the kernel of $\tilde{\pi}$, which is a closed two-sided ideal in $\mathcal{T}_C(T)$. First observe that part (c) of Proposition (1.2.13) yields the identity

$$\Phi(T_{f_1} \cdots T_{f_k}) = P_{H^\pi}(T_{f_1} \cdots T_{f_k})|H = P_{H^{\tilde{\pi}}}(T_{f_1}) \cdots \tilde{\pi}(T_{f_k})|H$$
$$= P_H \Psi_U(f_1 \cdots f_k)|H = T_{f_1 \cdots f_k},$$

valid for all $k \geq 1$ and all $f_1, \cdots, f_k \in C(\partial_A)$.

Let us denote by $\mathcal{T}_A(T) \subset B(H)$ the C^*-subalgebra generated by all operators T_f with $f \in A$. Then $\mathcal{T}_A(T)$ is the closed linear span of all finite products $X = T_{f_1} \ldots T_{f_k}$ such that f_i or $\bar{f_i}$ belongs to A for all i. Set $T_i = T_{f_i}$ for all i. Since $\subset \mathcal{R}_T$, we obtain that

$$\Phi(X) = T_{f_1 \cdots f_k} = \prod_{i \in I} T_i \prod_{j \in J} T_j \in \mathcal{T}_A(T),$$

where J is the set of all indices $j \in \{1, \ldots, k\}$ with $f_j \in A$ and I consists of the remaining indices. Hence $\mathcal{T}_A(T)$ is invariant under Φ. If $1 \in I$, then

$$X - \Phi(X) = T_1 (T_2 \cdots T_k - \Phi(T_2 \cdots T_k))$$

and if $1 \in J$, then

$$X - \Phi(X) = T_1(T_2 \cdots T_k) - \Phi(T_2 \cdots T_k)T_1$$
$$= T_1 (T_2 \cdots T_k - \Phi(T_2 \cdots T_k)) + T_1\Phi(T_2 \cdots T_k) - \Phi(T_2 \cdots T_k)T_1.$$

Hence by induction on k we find that $X - \Phi(X) \in C_C(T)$ for every finite product X as above.

For $f, g \in C(\partial_A)$ with $T_f, T_g \in \mathcal{T}_A(T)$, we obtain that $T_{fg} = \Phi(T_f T_g) \in \mathcal{T}_A(T)$. By Stone –Weierstrass, $T_f \in \mathcal{T}_{\mathcal{T}A}(T)$ for all $f \in C(\partial_A)$ and therefore $\mathcal{T}_A(T) = \mathcal{T}_C(T)$. Hence $\tilde{\Phi} = \Phi|\mathcal{T}_C(T)$ is a continuous linear map with $\tilde{\Phi}^2 = \tilde{\Phi}$. By part (d) of Proposition (1.2.13),

$$ran(1 - \tilde{\Phi}) = ker(\tilde{\Phi}) = ker(\Phi) \cap \mathcal{T}_C(T) = ker(\pi) \cap \mathcal{T}_C(T) = ker(\tilde{\pi})$$

By the above inductive argument, it follows that $ker(\tilde{\pi}) = ran(1 - \tilde{\Phi}) \subset C_C(T))$

On the other hand, for all $f, g \in C(\partial_A)$,

$$T_f T_g - T_{fg} = T_f T_g - \Phi(T_f T_g) \in ker(\tilde{\pi}) = \mathrm{ran}(1 - \tilde{\Phi}).$$

Hence $C_C(T) = SC_C(T) = ker(\tilde{\pi})$. To complete the proof, we define the symbol map σ as the composition $\sigma = \Gamma \circ \tilde{\pi}$

In the classical theory of Hardy space Toeplitz tuples on the unit ball or, more general, on strictly pseudoconvex domains in C^n (see [24]), the first space in the above short exact sequence coincides with the ideal $\mathcal{K}(H) \subset B(H)$ of all compact operators on H. While this fails to be true for arbitrary A-isometries, it holds under some natural additional assumptions on T including essential normality. Recall that a commuting tuple $T \in B(H)^n$ is said to be *essentially normal* if its self-commutators are compact, that is, if

$$[T_i, T_i^*] = T_iT_i^* - T_i^*T_i \in \mathcal{K}(H) \qquad (i = 1,\dots,n).$$

In other words, the images $\pi(T_i)$ of the components of T under the canonical map

$$\pi : B(H) \to C(H) = B(H)/\mathcal{K}(H), \qquad X \mapsto X + \mathcal{K}(H)$$

into the Calkin-algebra form a commuting tuple $\pi(T) = (\pi(T_1),\dots,\pi(T_n))$ of normal elements in $C(H)$. Some useful characterizations of essentially normal A-isometries are collected in the following lemma.

Lemma (1.2.17)[23]: *For an A-isometry $T \in B(H)^n$, the following assertions are equivalent*:

(a) *The tuple T is essentially normal.*

(b) *All Hankel operators H_f with continuous symbol $f \in C(\partial_A)$ are compact.*

(c) *For every $f \in C(\partial_A)$ and every $Y \in B(H)$, the semi-commutators*

$$(P_HY|H)T_f - P_H(Y\Psi_U(f))|H \quad and \quad T_f(P_HY|H) - P_H(\Psi_U(f)Y|H)$$

are compact.

(d) *The semi-commutators $T_fT_g - T_{fg}$ are compact whenever $f \in C(\partial_A)$ and $g \in L^\infty(\mu)$ (or, equivalently, whenever $f, g \in C(\partial_A)$).*

Proof. It is well known that a subnormal tuple $T \in B(H)^n$ with minimal normal extension $U \in B(\widehat{H})^n$ is essentially normal if and only if

$$[U_i, P_H] \in \mathcal{K}(\widehat{H}) \quad (i = 1,\dots,n),$$

or equivalently, if $\pi(P_H) \in C(\widehat{H})$ belongs to the commutant of the $C*$-algebra generated by the commuting normal elements $\pi(U_i)$ $(i = 1,\dots,n)$. In the setting of the lemma, this immediately implies the compactness of all commutators $[\Psi_U(f), P_H]$ with $f \in C(\partial_A)$, and thus of all Hankel operators

$$H_f = (1 - P_H)\Psi_U(f)P_H|H = (1 - P_H)[\Psi_U(f)P_H]|H \quad (f \in C(\partial_A)).$$

This settles the implication $(a) \Rightarrow (b)$. Now, fix arbitrary elements $Y \in B(\widehat{H})$ and $f \in C(\partial_A)$.

A look at the algebraic identities

$$(P_HY|H)T_f - P_H(Y\Psi_U(f))|H = P_HY(P_H\Psi_U(f) - \Psi_U(f))|H$$
$$= P_HY(P_H - 1)\Psi_U(f)|H = -P_HYH_f$$

and $\qquad T_f(P_HY|H) - P_H(\Psi_U(f)Y)|H = ((P_HY^*|H)T_{\bar{f}} - P_H(Y^*\Psi_U(\bar{f}))|H))^*$

shows that (b) implies (c). Setting $Y = \Psi_U(g)$ with $g \in L^\infty(\mu)$ in the last part, we obtain (d) as special case. Using the decomposition

$$T_i, T_i^* = T_{z_i}T_{\bar{z}_i} - T_{\bar{z}_i}T_{z_i} = (T_{z_i}T_{\bar{z}_i} - T_{z_i\bar{z}_i}) + (T_{\bar{z}_iz_i} - T_{\bar{z}_i}T_{z_i}) \quad (i = 1,\dots,n)$$

we get back to condition (a), as desired.

Part (d) of the preceding lemma can be used to calculate the *commutator ideal* of the Toeplitz algebra $\mathcal{T}_C(T)$, that is, the closed two-sided ideal of $\mathcal{T}_C(T)$ generated by all commutators $[A, B] = AB - BA$ of operators $A, B \in \mathcal{T}_C(T)$. Recall that a subset $S \subset B(H)$ is called irreducible if there is no non-zero proper closed subspace $M \subset H$ which is reducing for H. It is well known that the classical Toeplitz tuples T_z on the Hardy space $H^2(\sigma)$ with respect to the surface measure of the unit sphere or the boundary of a strictly pseudoconvex domain in $\mathbb{C}^n$ are essentially normal and generate an irreducible Toeplitz algebra $\mathcal{T}_C(T_z)$ (see Upmeier [24]).

Proposition (1.2.18)[23]: *Let $T \in B(H)^n$ be an essentially normal, non-normal regular A-isometry. If the Toeplitz C*-algebra $\mathcal{T}_C(T)$ is irreducible, then the commutator ideal of $\mathcal{T}_C(T)$ is $\mathcal{K}(H)$, and there is a short exact sequence of C*-algebras*

$$0 \longrightarrow \mathcal{K}(H) \overset{\subset}{\to} \mathcal{T}_C(T) \overset{\sigma}{\to} C(\sigma_n(T)) \longrightarrow 0,$$

where the symbol homomorphism σ satisfies $\sigma(T_f) = f\,|\sigma_n(T)$ for every $f \in C(\partial_A)$.

Proof. Let $\mathcal{C} \subset \mathcal{T}_C(T)$ denote the commutator ideal. By the assumption on *T* not to be normal, and part (d) of the previous lemma, we conclude that

$$0 \neq \mathcal{C} \subset \mathcal{SC}_C(T) \subset \mathcal{K}(H).$$

In particular, it follows that $\mathcal{T}_C(T) \cap \mathcal{K}(H) \neq 0$. Hence $\mathcal{T}_C(T) \supset \mathcal{K}(H)$ by the assumed irre-ducibility (Theorem 5.39 in [12]). So both $\mathcal{C}$ and $\mathcal{SC}_C(T)$ are non-zero closed ideals of $\mathcal{K}(H)$. Since $\mathcal{K}(H)$ is known to contain no proper closed ideals, we conclude that $C = \mathcal{SC}_C(T) = \mathcal{K}(H)$. Hence the asserted short exact sequence is just the one established in Corollary (1.2.16).

The *essential commutant* of an arbitrary subset $\mathcal{F} \subset B(H)$ is defined as

$$EssCom(\mathcal{F}) = \{C \in B(H) : CY - YC \in \mathcal{K}(H) \; for\ all\ Y \in \mathcal{F}\}.$$

In other words, an operator *C* belongs to $EssCom(\mathcal{F})$ if and only if its image $\pi(C)$ in the Calkin algebra belongs to the commutant $(\pi(F))'$. Obviously, $EssCom(\mathcal{F})$ is always a norm-closed subalgebra of $B(H)$. This is devoted to a detailed study of the essential commutant of the dual algebra $\mathcal{H}_T$ associated with a regular essentially normal *A*-isometry. The following two simple observations show how the assumption on *T* to be essentially normal influences the structure of $EssCom(T)$.

Lemma (1.2.19)[23]: *If $T \in B(H)^n$ is an essentially normal regular A-isometry, then we have $EssCom(T) = EssCom(\mathcal{T}_C(T))$, and this is a C* -algebra.*

Proof. To prove the non-trivial inclusion, fix an element $R \in EssCom(T)$. $Since\ \pi(R)$ com-mutes with the commuting normal elements $\pi(T_i)$ $(i = 1, \ldots, n)$, it commutes with $C^*(\pi(T))$. By Lemma (1.2.17) the map $C(\partial_A) \to C(H), f \mapsto \pi(T_f)$, is a *C**-algebra homomorphism. The theorem of Stone–Weierstrass implies that $\pi(T_f) \in C^*(\pi(T))$ for all $f \in C(\partial_A)$ and hence that $\pi(\mathcal{T}_C(T)) \subset C^*(\pi(T))$.

Therefore $R \in EssCom(\mathcal{T}_C(T))$ .Since $\mathcal{T}_C(T)) \subset B(H)$ is a self-adjoint subset, its essential commutant is a C^* -algebra.

For an arbitrary element $f \in L^\infty(\mu)$, we define the support $supp(f)$ to be the support of the measure μ_f induced by f via the formula $\mu_f(\omega) = \int_\omega |f|\, d\mu$ for every Borel subset $\omega \subset \partial_A$. By definition, $supp(f) \subset \partial_A$ is closed and $supp(f)^c$ is the largest open set $G \subset \partial_A$ with the property that $f = 0$ μ-almost everywhere on G. Moreover, if $g \in C(\partial_A)$ is a function with $g = 1$ on $supp(f)$, then $(1-g) \cdot f = 0$ and $gf = f$ μ-almost everywhere on ∂_A.

Lemma (1.2.20)[23]: *Suppose that $T \in B(H)^n$ is an essentially normal A-isometry and that $R \in EssCom(T)$. Then, for every choice of operators $Y_1, Y_2 \in (U)'$ and every pair of elements $f_1, f_2 \in L^\infty(\mu)$ with disjoint supports, we have*

$$(P_H(\Psi_U(f_1)Y_1)|H)\, R\, (P_H\, (\Psi_U(f_2)Y_2)|H) \in \mathcal{K}(H).$$

Proof. Let us abbreviate the factors on both sides of R by $X_1 = P_H(\Psi_U(f_1)Y_1)|H$ and $X_2 = P_H\, (\Psi_U(f_2)Y_2)|H$. By Urysohn's lemma, we can choose a continuous function $h : \partial_A \to [0,1]$ with $h = 1$ on $supp(f_1)$ and $h = 0$ on $supp(f_2)$. With this choice of h, an application of Lemma (1.2.17)(c) guarantees that

$$\pi(X_1) = \pi(X_1 T_h) \quad and \quad \pi(T_h X_2) = 0.$$

Since $R \in EssCom(T) = EssCom(\mathcal{T}_C(T))$ (see Lemma (1.2.19)), we obtain that

$$\pi(X_1 R X_2) = \pi(X_1 T_h R X_2) = \pi(X_1 R T_h X_2) = 0,$$

as desired.

As most ideas occurring, the previous lemma goes back in its original form to Davidson [26]. Our study of $EssCom(\mathcal{H}_T)$ has been inspired by corresponding results of Le [28] and Ding and Sun [27] (see also [28] and [29]) who developed Davidson's technique further in the several-variable case.

For the remainder, we fix a regular A-isometry $T \in B(H)^n$ with $\sigma_p(T) = \emptyset$ and denote its minimal normal extension by $U \in B(H)^n$.

Lemma (1.2.21)[23]: *For every element $S \in EssCom(\mathcal{H}_T)$, there are a weak* zero sequence of isometries $(J_k)_{k\geq 1}$ in $\mathcal{H}_T$ and an operator $Y_S \in (U)'$ such that the limit*

$$X_S = w^* - \lim_{k\to\infty} J_k^*\, S J_k$$

exists and satisfies $X_S = P_H Y_S|H$.

Proof. Let $S \in EssCom(\mathcal{H}_T)$ be given. According to Proposition (1.2.8) there is a weak* zero sequence $(J_k)_{k\geq 1}$ of isometries in $\mathcal{H}_T$. By passing to a subsequence we can achieve that the limit $X_S = w^* - \lim_{k\to\infty} J_k^*\, S J_k \in B(H)$ exists. For every isometry $V \in \mathcal{H}_T$, we obtain that

$$V^* X_S V = w^* - \lim_{k\to\infty} J_k^*\, V^* S V J_k = w^* - \lim_{k\to\infty} J_k^*\, S J_k = X_S.$$

Here we have used that $[S, V] \in \mathcal{K}(H)$ and that $w^* - \lim_{k\to\infty} J_k^* K J_k = 0$ for every compact operator K on H. Thus X_S is a T-Toeplitz operator. By Proposition (1.2.10) there is an operator $Y_S \in (U)$ with $X_S = P_H Y_S|H$.

Before we continue, we need an elementary topological lemma ensuring the existence of suitable open covers of compact sets $Q \subset \mathbb{C}^n$. Since the real dimension is involved, we formulate it for compact sets in $\mathbb{R}^m$. Given a subset $F \subset \mathbb{R}^m$, we denote its diameter with respect to the Euclidean norm by $|F| = \sup_{x,y\in F} |x - y|$.

Lemma (1.2.22)[23]: *Let $Q \subset \mathbb{R}^m$ be compact and let $\varepsilon > 0$ be given. Then there exists a finite open cover $Q = \bigcup_{j\in J} U_j$ consisting of relatively open sets $U_j \subset Q$ with $|U_j| < \varepsilon$ and such that the index set J admits a decomposition $J = J_1 \cup \cdots \cup J_{2^m}$ with the property that each of the families $(U_i)_{i\in J_l}$ $(l = 1, \ldots, 2^m)$ consists of pairwise disjoint sets.*

Proof. For the convenience of the reader, we indicate the elementary ideas. Clearly it suffices to prove the assertion for every compact rectangle $Q \subset \mathbb{R}^m$. For $m = 1$, the result obviously holds. Suppose that the assertion is true for some $m \geq 1$, and let $Q = Q^1 \times Q^2$ be a compact rectangle with $Q^1 \subset \mathbb{R}, Q^2 \subset \mathbb{R}^m$. Choose open covers $(U_j^1)_{j\in J^1}$ for Q^1 and $(U_k^2)_{j\in J^2}$ for Q^2 as in the assertion. Let

$$J^1 = J_1^1 \cup J_2^1 \quad \textit{and} \quad J^2 = J_1^2 \cup \cdots \cup J_{2^m}^2$$

be the corresponding decompositions of the index sets. Define open sets

$$U_{(j,k)} = U_j^1 \times U_k^2 \subset Q \qquad (j \in J^1, k \in J^2)$$

and index sets

$$J = J^1 \times J^2 \ \text{ and } \ J_{(a,b)} = J_a^1 \times J_b^2$$

Then $\left(U_{(j,k)}\right)_{(j,k)\in J}$ is a cover of Q by open sets of diameter $|U_{(j,k)}| \leq |U_j| + |U_k| < 2\varepsilon, J$ is the disjoint union of all $J_{(a,b)}$ and the families $\left(U_{(j,k)}\right)_{(j,k)\in J_{(a,b)}}$ consist of pairwise disjoint sets.

Let $Y \in (U)'$ and $S \in EssCom(T)$ be given operators. By Lemma (1.2.12) the map

$$F: L^\infty(\mu) \to B(H) \quad \textit{by} \quad F(f) = T_f S - P_H(\Psi_U(f)Y|H).$$

is pointwise boundedly SOT-continuous. A straightforward application of Lemma (1.2.20) (and Lemma (1.2.19)) yields that, for any pair of functions $f, g \in L^\infty(\mu)$ with disjoint supports, each of the products

$$F(f)F(g), F(f)^*F(g), F(f)F(g)^* \in B(H)$$

is compact.

Our main result will follow by applying the following general observation to functions of the above type.

Proposition (1.2.23)[23]: *Let $F : L^\infty(\mu) \to B(H)$ be a linear map such that*

(P1) *F is pointwise boundedly SOT-continuous*;
(P2) $F(\chi)$ *is not compact for a characteristic function* χ *of some Borel set in* ∂_A;
(P3) *if* $f, g \in L^{\infty}(\mu)$ *have disjoint supports, then each of the products* $F(f)F(g), F(f)F(g)^*, F(f)^*F(g)$ *is compact.*

Then there are a positive real number $\rho > 0$ *and a sequence* $(f_k)_{k\geq 1}$ *of continuous functions* $f_k : \partial_A \to [0,1]$ *with disjoint supports satisfying* $\|F(f_k)\| > \rho$ *for all* $k \geq 1$.

Proof. Let $\alpha = \|\pi(F(\chi))\|/2 > 0$ and define

$$\varepsilon = \{f \in C(\partial_A): 0 \leq f \leq 1 \ \ and \ \ \|F(f\chi)\| > \alpha/(2N)\}$$

with $\varepsilon = 2^{2n}$. We obtain a decreasing sequence $(E_k)_{k\geq 1}$ of closed subsets of ∂_A by defining each E_k as the closure of the set

$$\bigcup supp(f): f \in \varepsilon \ \ with \ |supp(f)| \leq \frac{1}{k}$$

We first prove that $E = \bigcap_{k\geq 1} E_k$ is non-empty. Let us assume the converse. Then $\|F(f\chi)\| \leq \alpha/(2N)$ for every $f \in C(\partial_A)$ with $0 \leq f \leq 1$ and $|supp(f)| \leq \frac{1}{k}$.

According to Lemma (1.2.22) we can choose an open cover $\partial_A = U_1 \cup \cdots \cup U_r$ such that $|U_j| \leq 1/k$ $(j = 1, \ldots, r)$ and such that the set $\{1, \ldots, r\}$ is the disjoint union of sets $J_1, \ldots, J_N$ with the property that each of the families $(U_j)_{i\in J_l}$ $(l = 1, \ldots, N)$ consists of pairwise disjoint sets. Let $(h_j)_{j=1,\ldots,r}$ be a continuous partition of unity relative to the open cover $(U_j)_{j=1,\ldots,r}$. In view of the decomposition

$$\pi(F(\chi)) = \frac{\pi(F(\chi) + F(\chi)^*)}{2} + i\,\frac{\pi(F(\chi) - F(\chi)^*)}{2i},$$

we can choose an $\varepsilon \in \{-1, +1\}$ such that $\pi\|(F(\chi) + \varepsilon F(\chi)^*)\| > \alpha$. Then

$$A_j = \pi\, F(h_j\chi) + \varepsilon F(h_j\chi)^* \qquad (j = 1, \ldots, r)$$

defines a family $(A_j)_{j=1,\ldots,r}$ of normal elements in the Calkin algebra such that $A_\mu A_\nu = 0$ whenever μ, ν are different indices in one of the sets $J_l (l = 1, \ldots, N)$.
A simple spectral radius argument then yields the estimates

$$\left\|\sum_{j\in J_l} A_j\right\| \leq \max_{j\in J_l}\|A_j\| \leq 2 \cdot \max_{j\in J_l}\|F(h_j\chi)\| \leq \alpha/N \qquad (l = 1, \ldots, N)$$

which leads to the contradiction

$$\alpha < \left\|\sum_{j=1}^{r} A_j\right\| \leq \sum_{l=1}^{N}\left\|\sum_{j\in J_l} A_j\right\| \leq \alpha.$$

Thus we have shown that $E = \bigcap_{k\geq 1} E_k \neq \emptyset$.

Define $\rho = \alpha/(2N)$. In the second step we prove the existence of a sequence $(g_k)_{k\geq 1}$ in $C(\partial_A)$ with $0 \leq g_k \leq 1$ and pairwise disjoint supports such that $\|F(g_k\chi)\| > \rho$ for all $k \geq 1$.

To this end, let us fix a point $z_0 \in E$. Suppose that $g_1, \ldots g_k \in E$ are functions with pairwise disjoint supports such that

$$d = dist\ z_0, \bigcup_{j=1}^{k} supp(g_j) > 0.$$

Since $z_0 \in E$, there is a function $f \in \varepsilon$ with $|\,supp(f)| < d/3$ and $dist(z_0, supp(f)) < d/3$. If $z_0 \notin supp(f)$, we define $g_{k+1} = f$. Otherwise we choose a sequence of functions $(\kappa_j)_{j\geq 1}$ in $C(\partial_A)$ with $0 \leq \kappa_j \leq 1, z_0 \notin supp(\kappa_j)$ for all $j \geq 1$ and

$$\kappa_j(z) \overset{j}{\rightarrow} 1 \quad (z \in \partial_A \setminus \{z_0\}).$$

By hypothesis $\sigma_p(T) = \emptyset$ and hence μ has no one-point atoms. Therefore $(\kappa_j f\chi)_j$ is a bounded sequence in $L^\infty(\mu)$ which converges pointwise μ-almost everywhere to the function $f\chi$. Using condition (P1) we find that

$$F(f\chi) = SOT - \lim_{j\to\infty} F(\kappa_j f\chi)$$

Since $f \in \varepsilon$, we can choose a natural number $j \geq 1$ with $F(\kappa_j f\chi) > \alpha/(2N)$. In this case we define $g_{k+1} = \kappa_j f$. In both cases we obtain a family $(g_j)_{j=1,\ldots,k+1}$ of functions in ε with pairwise disjoint supports not containing z_0.

Inductively, one finds a sequence $(g_k)_{k\geq 1}$ in ε with pairwise disjoint supports and $\|F(g_k\chi)\| > \rho$ for all $k \geq 1$.

A standard application of Lusin's theorem (Theorem 7.4.3 and Proposition 3.1.2 in [24]) shows that there is a sequence of continuous functions $h_j: \partial_A \to [0,1]$ such that $(h_j) \overset{j}{\rightarrow} \chi$ μ-almost everywhere. Again using hypothesis (P1) we find that

$$F(g_k\chi) = SOT - \lim_{j\to\infty} F(g_k h_j)$$

for every $k \geq 1$. Hence, for every $k \geq 1$, there is a natural number j_k such that $F(g_k h_{jk}) > \rho$. The observation that the resulting functions $f_k = g_k h_{jk}$ have all required properties completes the proof.

Now we are able to prove the main theorem . Recall that, by Proposition (1.2.7), every spherical isometry is a regular $A(\mathbb{B}_n)$-isometry.

Theorem(1.2.24)[23]: *Let $T \in B(H)^n$ be an essentially normal regular A-isometry with $\sigma_p(T) = \emptyset$, and let $S \in B(H)$ be an operator that essentially commutes with $\mathcal{H}_T$. Then there are a T-Toeplitz operator $X \in B(H)$ and a compact operator $K \in \mathcal{K}(H)$ with*

$$S = X + K.$$

Proof. According to Lemma (1.2.21) there is a sequence $(J_k)_{k\geq 1}$ of isometries in $\mathcal{H}_T$ in such that the limit

$$X_S = w^* - \lim_{j\to\infty} J_k^* S J_k$$

defines a T-Toeplitz operator. By Proposition (1.2.10) there is an operator $Y_S \in (U)'$ with $X_S = P_H Y_S|H$. By Lemma (1.2.3) we can choose a sequence $(\theta_i)_{i\geq 1}$ of bounded measurable functions $\theta_i : \partial_A \to \mathbb{C}$ with $|\theta_i| = 1$ on ∂_A such that θ_i, or better its equivalence class in $L^\infty(\mu)$, belongs to $H_A^\infty(\mu)$ and satisfies $J_i = \gamma_T(\theta i)$ for every $i \geq 1$. As seen before, the continuous mapping $F: L^\infty(\mu) \to B(H)$ defined by

$$F(f) = T_f S - P_H\,(\Psi_U(f)Y_S)|H$$

satisfies the hypotheses (P1) and (P3) of Proposition (1.2.23). To complete the proof it suffices to show that $F\,(1)$ is a compact operator. We even show that

$$FL^{\infty}(\mu) \subset \mathcal{K}(H\,).$$

we assume that the inclusion does not hold. Since every bounded measurable function can be approximated uniformly by finite linear combinations of characteristic functions of Borel sets, there is a characteristic function χ of some Borel set in ∂_A such that $F(\chi)$ is not compact. As an application of Proposition (1.2.23) we find that there are a real number $\rho > 0$ and a sequence $(f_k)_{k\geq 1}$ of continuous functions $f_k : \partial_A \to [0,1]$ with pairwise disjoint supports $A_k = supp(f_k)$ and $\|F(f_k)\| > \rho$ for all $k \geq 1$.

Let us fix an index $k \geq 1$. Choose a real number t with $0 < t < 4^{-k}$ and $t \cdot \|F(1)\| < \rho/2$. Then the function $\phi = f_k + t \in C(\partial_A)$ is strictly positive on ∂_A and satisfies the estimates

$$\|\varphi\|_{\infty,\partial_A} \leq 2, \qquad \|\varphi\|_{\infty,\partial_A\setminus Ak} < 4^{-k} \qquad , \|F(\varphi)\| > \rho/2.$$

Since $(\partial_A|\,A,\partial_A,\mu)$ is regular, there is a sequence $(\varphi_j)_{j\geq 1}$ in A with $|\varphi_j| < \sqrt{\varphi}$ on ∂_A and $|\varphi_j| \overset{j}{\rightarrow} \sqrt{\varphi}$ μ-almost everywhere on ∂_A. Property (P1) implies that

$$F(\varphi) = SOT - \lim_{j\to\infty} F|\varphi_j|^2.$$

Choose a natural number j with $\|F(|\varphi_j|^2)\| > \rho/2$ and set $g_k = \varphi_j$ .Then $g_k \in A$ satisfies the estimates $\|g_k\|_{\infty,\partial_A} \leq 2$ and $\|g_k\|_{\infty,\partial_A\setminus A_k} < 2^{-k}$. The identity

$$F|g_k|^2 = T_{\bar{g}kgk}\,S - P_H\,(\Psi_U(\bar{g}kgk)Y_S)|H = T_{\bar{g}k}\,T_{gk}\,S - P_H(\Psi_U(gk\,)Y_S)|H$$
$$= T_{\bar{g}k}F(gk)$$

implies that $\|F(gk)\| > \rho/4$. The observation that

$$w^* \lim_{i\to\infty} J_i^*(T_{gk}J_iS - ST_{gk}J_i) = w^* - \lim_{i\to\infty}\big(T_{gk}S - J_i^*SJ_iT_{gk}\big) = T_{gk}\,S - (P_HY_S|H\,)_{T_{gk}}$$
$$= T_{gk}\,S - P_H(\Psi_U(gk\,)Y_S)|H = F(gk)$$

allows us to choose a natural number i such that

$$\|T_{gk}J_iS - ST_{gk}J_i\| > \rho/4.$$

The functions $h_k = {}_{gk}\theta_i$, where for every given $k \geq 1$ the index i is chosen as explained above, satisfy the estimates

$$\|h_k\|_{\infty,\partial A} \leq 2 \;\; and \quad \|h_k\|_{\infty,\partial_A\setminus A_k} < 2^{-k}.$$

Furthermore by construction the functions h_k, or better their equivalence classes in $L^{\infty}(\mu)$,belong to $H_A^{\infty}(\mu)$ and the commutators $B_k = [T_{h_k}, S] \in B(H)$ are compact operators with $B_k \geq \rho/4$.

By passing to a subsequence, we can achieve that the limit

$$c = \lim_{k\to\infty}\|B_k\| \in [\rho/4,\infty)$$

exists. Since the sequence $(h_k)_{k\geq 1}$ is uniformly bounded on ∂_A and converges to zero pointwise on ∂_A, it follows that both sequences $(B_k)_{k\geq 1}$ and $(B_k^*)_{k\geq 1}$ converge to zero in the strong op-erator topology (see Lemma (1.2.12)). A result proved by Muhly and Xia in [29, Lemma 2.1] shows that, by passing to a subsequence again, we can achieve that the series

$$B = \sum_{k=0}^{\infty} B_k$$

converges in the strong operator topology and satisfies $\|\pi(B)\| = c \geq \rho/4$. Since every point $z \in \partial_A$ belongs to at most one of the sets A_k, the partial sums of the series $\sum_{k=0}^{\infty} h_k$are uniformly bounded on ∂_A and converge pointwise to a function $h : \partial_A \to \mathbb{C}$ Clearly, (the equivalence class of). h belongs to $H_A^\infty(\mu)$ and the identities

$$T_h = \sum_{k=1}^{\infty} Th_k \qquad and\ [T_h, S] = \sum_{k=1}^{\infty} [Th_k, S] = B$$

hold in the strong operator topology. But then $T_h \in \mathcal{H}_T$ would be an operator with non-compact commutator $[S, T_h]$. This contradicts the hypothesis and thus completes the proof.

Corollary (1.2.25)[23]: *Let $T \in B(H)^n$ be an essentially normal regular A-isometry with $\sigma_p(T) = \emptyset$. Denote by $U \in B(\widehat{H})^n$ the minimal normal extension of T. Suppose that $W^*(U) \subset B(\widehat{H})$ is a maximal abelian von Neumann algebra. Then a given operator $S \in B(H)$ essentially commuteswith $\mathcal{H}_T$ if and only if S has the form $S = T_f + K$ with a compact operator $K \in \mathcal{K}(H)$ and a symbol $f \in L^\infty(\mu)$ having the property that the associated Hankel operator H_f is compact.*

Proof. Suppose that $S \in B(H)$ essentially commutes with $\mathcal{H}_T$. Fix a weak* zero sequence of isometries $(J_k)_{k\geq 1}$ in $\mathcal{H}_T$ such that the weak* limit

$$X = w^* - \lim_{k\to\infty} J_k^* S J_k \in B(H)$$

defines a T-Toeplitz operator (see Lemma (1.2.21)). By Proposition (1.2.10) (b) there is a function $f \in L^\infty(\mu)$ such that $= P_H\, \Psi_U(f)|H = T_f$. The proof of the preceding theorem shows that the image of the map

$$F : L^\infty(\mu) \to B(H), \qquad F(h) = T_h S - P_H\, \Psi_U(h_f)|H$$

is contained in $\mathcal{K}(H)$. In particular, the operator $K = F(1) = S - T_f$ is compact.

Because of

$$\begin{aligned} F(\bar{f}) = \Psi_U S - T_{|f|^2} &= T_{\bar{f}}\, T_f - T_{|f|^2} + T_{\bar{f}} K \\ &= P_H \Psi_U(\bar{f}) P_H \Psi_U(f)|H - P_H \Psi_U(\bar{f})\Psi_U(f)|H + \Psi_U K \\ &= -P_H \Psi_U(\bar{f}) P_{H^\perp} \psi_U(f)|H + T_{\bar{f}} K = -H_f^* H_f + T_{\bar{f}} K \end{aligned}$$

we find that $H_f^* H_f$ and hence also H_f is compact.

Conversely, suppose that $f \in L^\infty(\mu)$ is a function such that H_f is compact. Then, for every $g \in H_A^\infty(\mu)$, it follows that

$$\begin{aligned} T_f\, T_g = P_H\, \Psi_U(f)\Psi_U(g)|H &= P_H\, \Psi_U(g) P_H\, \Psi_U(f)|H + P_H\, \Psi_U(g) H_f \\ &= T_g T_f + P_H\, \Psi_U(g) H_f. \end{aligned}$$

Thus T_f essentially commutes with $\mathcal{H}_T$.

The preceding corollary in particular applies to each essentially normal regular Hardy-space A-isometry $T = T_z \in B(H_A^2(\mu))^n$ (Definition (1.2.2)) with empty point spectrum. To give a concrete example, let $D \subset \mathbb{C}^n$ be a relatively compact strictly pseudoconvex open set with C^2-boundary ∂D. The normalized surface measure σ on the boundary ∂D has no one-point atoms. The associated Toeplitz tuple $T_z = (T_{z_1}, \ldots, T_{z_n}) \in B(H_{A(D)}^2(\sigma))^n$ is a regular Hardy-space $A(D)$-isometry. The space $H_{A(D)}^2(\sigma)$ coincides with the classical Hardy space $H^2(\sigma) \subset L^2(\sigma)$ on the boundary ∂D (see Section 7 in [28]). To identify $H_{A(D)}^\infty(\sigma)$ we use the map $r: H^\infty(D) \to L^\infty(\sigma), f \mapsto f^*$, associating with each function $f \in H^\infty(D)$ its non-tangential boundary value f^* (see Theorem 8.4.1 in [27]). Fix $f \in D(D)$. Since $f \in H^2(D)$, there exists a function $\tilde{f} \in L^2(\sigma)$ such that

$$f(z) = (\mathcal{P}\tilde{f})(z) = \int_{\partial D} P(z,w)\tilde{f}(w)d\sigma(w) \quad (z \in D),$$

where P denotes the Poisson kernel of D (Theorem 8.3.6 in [27]). In the proof of Theorem 8.4.1 in [27] it is shown that $r(f) = \tilde{f}$.Hence $\tilde{f} \in L^\infty(\sigma)$ and $\|\tilde{f}\|_{\infty,\sigma} = \|f\|_{\infty,D}$. Thus the map $r: H^\infty(D) \to L^\infty(\sigma)$ is isometric. As usual we denote its range by $H^\infty(\sigma)$. Since $\mathcal{P}(r(f)) = f$ for every $f \in H^\infty(D)$, the inverse of r is the Poisson transformation $\mathcal{P} : H^\infty(\sigma) \to H^\infty(D)$.

Standard arguments show that $H^\infty(\sigma) \subset L^\infty(\sigma)$ is weak* closed. We briefly indicate a possible proof. Let (f_k) be a sequence in the closed unit ball of $H^\infty(D)$ such that $g = w^* - \lim_k r(f_k)$ exists in $L^\infty(\sigma)$. By Krein–Smulian's theorem and the separability of $L^1(\sigma)$ it suffices to show that $g \in H^\infty(\sigma)$. By Montel's theorem we may suppose that (f_k) converges to some function $f \in H^\infty(D)$ uniformly on every compact subset of D. Since $r(f)$ and g are functions in $L^\infty(\sigma)$ such that

$$\mathcal{P}(r(f))(z) = f(z) = \lim_k f_k(z) = \lim_k \int_{\partial D} P(z,w)(rf_k)(w)d\sigma(w) = \mathcal{P}(g)(z)$$

for all $z \in D$, it follows that $g = r(f)$ (cf. the proof of Theorem 8.4.1 in [17]).

As an application one obtains the weak* continuity of the map $r : H^\infty(D) \to H^\infty(\sigma)$. Since $H^\infty(D) = (L^1(D)/{}^\perp H^\infty(D))'$ has a separable predual, it suffices to show that $(r(f_k))$ is a weak* zero sequence in $L^\infty(\sigma)$ for each weak* zero sequence (f_k) in $H^\infty(D)$. But this follows from the observation that

$$\langle [P(z,\cdot)], r(f_k)\rangle = \int_{\partial D} P(z,w)r(f_k)(w)\, d\sigma(w) = f_k(z) \xrightarrow{k} 0$$

for all $z \in D$ and the fact that the predual $L^1(\sigma)/{}^\perp H^\infty(\sigma)$ of $H^\infty(\sigma)$ is the closed linear span of all equivalence classes $[P(z,\cdot)]$ $(z \in D)$.

Since $H^\infty(\sigma) \subset L^\infty(\sigma)$ is weak* closed, the inclusion $H^\infty_{A(D)}(\sigma) = \overline{(A(D)|\partial D)}^{w^*} \subset r(H^\infty(D)) = H^\infty(\sigma)$ holds. The reverse inclusion $H^\infty(\sigma) \subset H^\infty_{A(D)}(\sigma)$ follows from the weak* continuity of r and the fact that there is an open neighbourhood U of $\overline{D}$ in $\mathbb{C}^n$ such that $\mathcal{O}(U)|D$ is sequentially weak* dense in $H^\infty(D)$ (Proposition 2.1.6 in [28]).

Since $\gamma_{T_z}(f) = \Psi_U(f)|H^2(\sigma) = T_f$ for $f \in H^\infty(\sigma)$, the dual algebra $\mathcal{H}_{T_z}$ coincides with the set of all analytic Toeplitz operators, that is, Toeplitz operators T_φ with symbol φ in $H^\infty(\sigma)$. By Theorem 4.2.17 in Upmeier [24] the tuple T_z is essentially normal. So the last corollary applies to this case and yields a description of the essential commutant of the set of all analytic Toeplitz operators, which extends Theorem 2 of Ding and Sun [29].

Corollary (1.2.26)[23]: *Let σ be the normalized surface measure on the boundary ∂D of a strictly pseudoconvex domain $D \subset \mathbb{C}^n$ with C^2-boundary. Then an operator $S \in B(H^2(\sigma))$ essentially commutes with all analytic Toeplitz operators on $H^2(\sigma)$ if and only* *if it has the form* $S = T_f + K$ *with a compact operator* K *and a symbol* $f \in L^{\infty}(\sigma)$*, for which the associated Hankel operator* H_f *is compact.*

Since in the setting of Corollary (1.2.26) there are no non-zero compact Toeplitz operators, it follows that a Toeplitz operator T_f with symbol $f \in L^{\infty}(\sigma)$ essentially commutes with all analytic Toeplitz operators if and only if H_f is compact. A more concrete description of this class of operators for the case of the unit ball can be found in [25].

Chapter 2
The Commutant and Multiplication Operators

We show results about nilpotent operators. One corollary of result is that if $x(z) = z^n, n \geq 1$, then $\{T_\phi\}' = \{T_x\}' \cap \{T_F\}'$, another is that if $\phi \in H^\infty$ is univalent then $\{T_\phi\}' = \{T_z\}'$. We are also able to show that if the inner factor of f ϕ is $x(z) = z^n, n \geq 1$, then $\{T_\phi\}' = \{T_{z^s}\}'$ where s is a positive integer maximal with respect to the property that z^n and $F(z)$ are both functions of z^s. We conclude by raising six questions. We completely classify reducing subspaces of the multiplication operator by a Blaschke product ϕ with order three on the Bergman space to solve a conjecture of Zhu.

Section (2.1): Analytic Toeplitz Operators

For H^2 denote the Hilbert space of functions f analytic in the open unit disc **D** for which the functions $f_r(\theta) = f(re^{i\theta})$ are uniformly bounded in L^2-norm for r < 1, and let H^∞ denote the linear manifold of bounded functions in H^2. For $\phi \in H^\infty, T_\phi (or) T_{\phi(z)}$ is the analytic Toeplitz operator on H^2 defined by the relation $(T_\phi f)(z) = \phi(z) f(z)$. These operators have received a great deal of attention recently and many of their properties are well known ([54], [55]). The operator T_z is often called the unilateral shift and is the canonical example of a completely nonunitary isometry of defect one. Every analytic Toeplitz operator commutes with T_z, in fact, every operator that commutes with T_z is an analytic Toeplitz operator. We study the commutant of an arbitrary analytic Toeplitz operator. We obtain some partial results characterizing the commutant of an analytic Toeplitz operator as well as some partial results characterizing those analytic functions whose associated Toeplitz operators have commutant equal to that of T_z. The main result stated in terms of pure isometries, the rest contain numerous results on the commutant of analytic Toeplitz operators.

It is well known that if $f \in H^2$ then there is a function $f^* \in L^2(\boldsymbol{T})$ such that $f(re^{i\theta})$ converges almost everywhere to $f^*(e^{i\theta})$. $\chi \in H^\infty$ is said to be an inner function if $|\chi^*(e^{i\theta})| = 1$ almost everywhere (or equivalently, if T_x is an isometry). Every inner function χ has a factorization $\chi(Z) = e^{i\gamma} B(z) S(z)$ with $|e^{i\gamma}| = 1$ where $B(z)$ is a Blaschke product of the form

$$B(z) = z^n \prod_{k=1}^{\infty} \frac{|\alpha_k|}{\alpha_k} \frac{\alpha_k - z}{1 - \alpha_k z}, \qquad 0 < |\alpha_k| < 1,$$

And $S(z)$ is a singular inner function of the form

$$S(z) = exp\left\{-\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t)\right\}$$

with μ a singular measure. $F \in H^\infty$ is said to be an outer function if F is of the form

$$F(z) = exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} k(t) d(t)\right\}$$

where k is a real-valued integrable function (or equivalently,if T_F has dense range). Every nonconstant function $\phi \in H^\infty$ has a unique factorization of the form $\phi(z) = \chi(z) F(z)$ where $\chi \in H^\infty$ is an inner function and $F \in H^\infty$ is an outer function ([56]). Results show that this factorization plays a key role in determining the commutant of T_ϕ. Although we are primarily interested in analytic Toeplitzoperators it will be convenient to state some of our results more generally. An isometry Von a Hilbert space $\mathcal{H}$ is called a pure isometry ([52], [53]) if $\bigcap_{n=0}^{\infty} V^n\mathcal{H} = \{0\}$. The dimension of the defect space $K_V = \mathcal{H} \ominus V\mathcal{H}$ is called the defect or multiplicity of V, and one easily obtains the decomposition $\mathcal{H} = \sum_{n=0}^{\infty}\oplus V^n K_V$. Any two pure isometries of the same multiplicity are unitarily equivalent. Thus V is unitarily equivalent to the unilateral shift U_+ on $l_+^2(K_V) = \sum_{n=0}^{\infty}\oplus K_V$ defined by

$$U_+(x_0, x_1, x_2, \dots) = (0, x_0, x_1, x_2, \dots).$$

It is easily verified that the commutant of U_+ consists of those bounded linear operators on $l_+^2(K_V)$ of the form $\sum_{n=0}^{\infty} \widehat{A}_n U_+^n$ where, for A a bounded linear operator on $K_{V'}, \widehat{A}$ (the inflation of A) is defined on $l_+^2(K_V)$ by

$$\widehat{A}(x_0, x_1, x_2, \dots) = (Ax_0, Ax_1, Ax_2, \dots).$$

Thus the commutant of a pure isometry can be characterized.

If $\chi \in H^\infty$ is a nonconstant inner function then T_x is a pure isometry (this follows since no $0 \neq f \in H^\infty$ is infinitely divisible by χ), which has finite defect if and only if χ is a finite Blaschke product. Hence the commutant of T_x for χ an inner function is well known, and we attempt to characterize the commutant of an arbitrary analytic Toeplitz operator in terms of these objects. For a Hilbert space $\mathcal{H}, \mathcal{B}(\mathcal{H})$ will denote the algebra of all bounded linear operators on $\mathcal{H}$. If $A \in \mathcal{B}(\mathcal{H})$ then $\{A\}'$ will denote the commutant of A, that is, $\{A\}' = \{B \in \mathcal{B}(\mathcal{H}): AB = BA\}$.

Throughout K will denote a Hilbert space, $\mathcal{H}$ will denote the Hilbert space $l_+^2(K)$, and U_+ will denote the unilateral shift on $\mathcal{H}$.

Lemma (2.1.1)[50]: *Suppose $S \in \mathcal{B}(\mathcal{H})$ has dense range and commutes with U_+. If $T \in \mathcal{B}(\mathcal{H})$ commutes with SU_+, then T has a lower triangular operator-valued matrix on $\mathcal{H}$.*

Proof. Since T lower triangular is equivalent to T^* upper triangular, which in turn, is equivalent to the subspaces $M_n = \sum_{k=0}^{n}\oplus K$ invariant for T^*, it suffices to prove that T^* leaves M_n invariant. Since T commutes with SU_+, T^* commutes with $S^*U_+^*$ and hence with $S^{*n+I}\, U_+^{*n+I}$ Thus T^* leaves invariant the null space of $S^{*n+I}\, U_+^{*n+I}$. Because S has dense range, $S^{*n+I}$is one-to-one, and

$$null\ (S^{*n+I}\, U_+^{*n+I}) = null\ (U_+^{*n+I}) = M_n$$

Hence T is lower triangular.

The following lemma is essential.

Lemma (2.1.2)[50]: *Let N be a nilpotent operator on K and let $\chi_0 = \lambda I + N$ where $0 \neq \lambda \in C$. If $B, A_0, A_1, A_2, \dots \in \mathcal{B}(K)$ satisfy*

a. $\|A_k\| \le M, k = 0,1,2, \dots$, and

b. $A_k\chi_0 = \chi_0 A_{k-1} + B, k = 1,2,3, \dots$,

then $A_0 = A_1 = A_2$

Proof. By Theorem 1 in [8], K decomposes into $\sum_{i=1}^{n}\dot{\oplus} K_i$ and χ_0 has a lower triangular operator-valued matrix with diagonal elements λI_i. We show that $A_0 = A_1 = A_2 = \cdots$ by showing that they must have the same

$$(1,n), (1,n-1), \dots, (1,1), (2,n), \dots, \quad (2,1), \dots, (n,n), \dots, (n,1)$$

operatorentries with respect to o this decomposition. We repeatedly use the obvious fact that the lemma is true if $N = 0$. More precisely, if $D, C_0, C_1, C_2, \dots \in \mathcal{B}(K', K'')$ satisfying $\|C_k\| \le M$ and $\lambda C_k = \lambda C_{k-I} + D$ for k = 1, 2, 3, ... then $C_0 = C_1 = C_2 = \cdots$. To see that this is the case merely observe that $\lambda C_k = \lambda C_{k-I} + D$ implies $\lambda C_k = \lambda C_0 + kD$. In order that $\|C_k\| \le M$ we must have that $D = 0$ and hence $C_0 = C_1 = C_2 = \cdots$.

Now the $(1, n)$ entry of (b) is

$$\lambda(A_k)_{1,n} = \lambda(A_{k-1} + (B)_{1,n}, \quad k = 0, 1, 2, \ldots,$$

and so by the above remark and by the fact that (a) implies $\|(A_k)_{i,j}\| \leq M$ for $i, j = 0, 1, 2, \ldots, n$, $k = 0, 1, 2, \ldots$ one concludes that $(A_0)_{1,n} = (A_1)_{1,n} = (A_2)_{1,n} = \cdots$ .The $(1, n-1)$ entry of (b) is

$$\lambda(A_k)_{1,n-1} + (\chi_0)_{n,n-1}(A_k)_{1,n} = \lambda(A_{k-1})_{1,n-1} + (B)_{1,n-1}$$

or since $(A_0)_{1,n} = (A_1)_{1,n} = \cdots$, that

$$\lambda(A_k)_{1,n-1} = \lambda(A_{k-1})_{1,n-1} + [(B)_{1,n-1} - (\chi_0)_{n,n-1}(A_0)_{1,n}]$$

and again we conclude that $(A_0)_{1,n-1} = (A_1)_{1,n-1} = \cdots$.

Now the (i, j) entry of (b) is

$$\lambda(A_k)_{i,j} + (A_k)_{i,j+1} + (\chi_0)_{j+1,j} + \cdots + (A_k)_{i,n}(\chi_0)_{n,j}$$
$$= \lambda(A_{k-1})_{i,j} + (\chi_0)_{i,j-1}(A_{k-1})_{i-1,j} + \cdots + (\chi_0)_{i,1}(A_{k-1})_{1,j} + (B.)_{i.j}$$

Let us now assume that $(A_0)_{p,q} = (A_1)_{p,q} = (A_2)_{p,q} =\ldots$ for all $p = i, j < q \leq n$ and $1 \leq p < i, q = j$. Then

$$\lambda(A_k)_{i,j} = \lambda(A_{k-1})_{i,j} + [(\chi_0)_{i,i-1}(A_0)_{i-1} +\ldots+(\chi_0)_{i,1}(A_0)_{i,j} + (B)_{i,j} - (A_0)_{i,j+1}$$
$$+ (\chi_0)_{j+1,j} - \ldots - (A_0)_{i,n}(\chi_0)_{n,j}\,]$$

and we conclude that $(A_0)_{i,j} = (A_1)_{i,j} = \cdots$. Inductively we obtain that

$$(A_0)_{i,j} = (A_1)_{i,j} = \cdots \quad for\ all\ i, j = 1, 2, \ldots, n$$

and so $A_0, A_1, A_2, \ldots$.

Corollary (2.1.3)[50]: If N is a nilpotent operator on K and $\chi_0 = \lambda I + N$ where $0 \neq \lambda \in \boldsymbol{C}$, then

$$\|\chi_0^n A \chi_0^{-n}\| \leq M \quad for\ n = 0, 1, 2, \ldots \quad implies \quad A_k\chi_0 = \chi_0 A.$$

If K is finite dimensional, then the converse is also true.

Proof. Let $A_k = \chi_0^k A \chi_0^{-k}$ for $k = 0, 1, 2, \ldots$. Then $A_k\chi_0 = \chi_0 A_{k-1}$ and the result follows from Lemma (2.1.2) by setting $B = 0$.

In order to see that the converse is true if $K$is finite dimensional, first observe that if χ_0 satisfies the conclusion of the corollary, that is, $\chi_0 = \lambda I + N$ for $0 \neq \lambda$ and N nilpotent, then so does any operator $S\chi_0 S^{-1}$ similar to χ_0.

Now using Jordan canonical forms it is easy to see that for any invertible operator χ_0 with two or more distinct eigen values there is another operator A satisfying $\|\chi_0^n A \chi_0^{-n}\| \leq M$ but $A\chi_0 \neq \chi_0 A$. For example, if $\mu, \lambda \neq 0$ and if $\chi_0 = (\lambda I_1 + N_1) \oplus (\mu I_2 + N_2)$ on $[e_n^{(1)}]_{n=1}^{n_1} \oplus [e_n^{(2)}]_{n=1}^{n_2}$ where $N_i\, e_n^{(i)}$ equals $e_{n+1}^{(i)}$ if $n < n_i$ and equals 0 if $n = n_i$ if A is defined by $Ae_0^{(2)} = (\lambda/\mu)e_n^{(1)}$ and 0 otherwise, then $\chi_0 A \chi_0^{-1} = (\lambda/\mu)A$. So that if $\lambda \neq \mu$ and $|\mu| \geq |\lambda|$ then $\|\chi_0^n A \chi_0^{-n}\| \leq |\lambda/\mu|$ and $A\chi_0 \neq \chi_0 A$.

Lemma ((2.1.4)[50]: Suppose $T \in \mathcal{B}(\mathcal{H})$ has a lower triangular operator valued matrix on $\mathcal{H}$. If T commutes with $\chi = (\sum_{n=0}^{\infty} \hat{\chi}_n U_+^n)U_+$ where $\chi_0 = \lambda I + N$ with $\chi_0 = \lambda \in \boldsymbol{C}$ and N nilpotent, then T commutes with U_+.

Proof. We will show that T commutes with U_+ by inductively proving $T_{k,o} = T_{k+1,1} = T_{k+2,2} = \ldots$ for $k = 0, 1, 2, \ldots$ Notice that $\|T_{k+j,j}\| \leq \|T\|$ for all $, j = 0, 1, 2, \ \ldots$.

If $1 \leq j < i$ then the (i, j) entry of $T\chi = \chi T$ is

$$T_{i,j+1}\chi_0 + T_{i,j+2}\chi_1 + \cdots + T_{i,i}\chi_{i-j-1} = \chi_{i-j-1}T_{i,i} + \chi_{i-j}T_{j+1,j} + \ldots + \chi_0 T_{i-1,j} \quad (1)$$

If $i = j + 1$ we obtain $T_{j+1,j+1}\chi_0 = \chi_0 T_{ij}$ and Lemma (2.1.2) implies that $T_{0,0} = T_{1,1} = T_{2,2} = \cdots$. Let us now assume that $T_{l,0} = T_{l+1,1} = T_{l+2,2} = \cdots$ for all $l \leq k$. Setting $i = j + k + 2$ in (1) we obtain

$$T_{k+1+j+1,j+1}\chi_0 = \chi_0 T_{k+1+j,j} + \left[\chi_1 T_{k,0} + \cdots + \chi_{k+1}T_{0,0} - T_{k,0}\chi_1 - \cdots - T_{0,0}\chi_{k+1}\right]$$

Applying Lemma (2.1.2) we obtain that $T_{k+I,0} = T_{k+2,1} = \cdots$ and hence by induction $T_{k,0} = T_{k+1,1} = \ldots$ for all $k = 0,1,2,\ldots$ Thus Tcommutes with U_+.

Theorem (2.1.5)[50]: Let V be a pure isometry on a Hilbert space $\mathcal{H}$, and $S \in \mathcal{B}(\mathcal{H})$ have dense range and commute with V. Suppose there exists a $\lambda \in \boldsymbol{C}$ such that V factors as a product of pure isometries $V_1, V_2, \ldots, V_n$ and such that $S - \lambda I = V_i S_i$ for each $i = 1, 2, \ldots, n$, where each V_i commutes with each S_i. Then $\{SV\}' - \{S\}' \cap \{V\}'$.

Proof. It clearly suffices to prove the result for $V = U_+$ on $\mathcal{H} = l_+^2(K)$. If T commutes with S and U_+ then it obviously commutes with SU_+. So assume that T commutes with SU_+. Lemma (2.1.1) implies that T has a lower triangular operator- valued matrix on $\mathcal{H}$. Since S has dense range and commutes with U_+, it follows that $\lambda \neq 0$ and that $S = \sum_{n=0}^{\infty} \hat{\chi}_n U_+^n$. We need only show that $\chi_0 - \lambda I$ is nilpotent for then Lemma (2.1.4) will imply that T commutes with U_+. Since U_+ is one-to-one and commutes with S, it then follows that T also commutes with S.

We will in fact show that χ_0 has a decomposition as described in the proof of Lemma (2.1.2). By hypothesis $U_+ = V_1 V_2 \ldots V_n$ and S commutes with each V_i. Since χ_0^* is the restriction of S^* to

$$K = \mathcal{H} \ominus U_+\mathcal{H} = (\mathcal{H} \ominus V_1\mathcal{H})V_1(\mathcal{H} \ominus V_2\mathcal{H}) \oplus \ldots \oplus V_1 V_2 \ldots V_{n-1}(\mathcal{H} \ni V_n\mathcal{H})$$

it follows that χ_0^* is upper triangular and hence that χ_0 is lower riangulaL.

Let $(\chi_0)_{ii}$ be the compression of χ_0 to $V_1 V_2 \ldots V_{i-1}(\mathcal{H} \ominus V_i\mathcal{H})$. If $f, g \in (\mathcal{H} \ominus V_i\mathcal{H})$ then, since $S = \lambda I + V_i S_i)$ we obtain

$$S(V_1 V_2 \ldots V_{i-1} f) = (V_1 V_2 \ldots V_{i-1})Sf = \lambda V_1 V_2 \ldots V_{i-1} f + V_1 V_2 \ldots V_{i-1} V_i S_i f.$$

But $(V_i S_i V_1 V_2 \ldots V_{i-1} f, V_1 V_2 \ldots V_{i-1} g) = 0$, hence $(\chi_0)_{ii} = \lambda I_i$.

we reformulate Theorem (2.1.5) in terms of analytic Toeplitz operators and obtain numerous consequences.

Theorem (2.1.6)[50]: *Let $\phi \in H^\infty$ and $\phi \in \chi F$ be its inner-outer factorization. If for some $\lambda \in \boldsymbol{C}, \chi$ factors as $\chi = \chi_1 \chi_2 \ldots \chi_n$ with each χ_i an inner function and $F - \lambda$ divisible by each χ_i, then $\{T_\phi\}' = \{T_\chi\}' \cap \{T_F\}'$.*

Proof. If χ is constant, then the result is obvious since $\phi = F$. If χ is nonconstant, then as remarked earlier T_χ is a pure isometry and T_F has dense range and commutes with T_χ. By hypothesis there exist $g_i \in H^\infty$ such that $F(z) - \lambda = \chi_i(z) g_i(z)$. Hence $T_F - \lambda I = T_{\chi_i} T_{g_i}$, and of course T_{χ_i} commutes with T_{g_i}. Thus Theorem (2.1.1) implies $\{T_\phi\}' = \{T_\chi\}' \cap \{T_F\}'$.

Corollary (2.1.7)[50]: *Let $\phi \in H^\infty$ and $\phi \in \chi F$ be its inner-outer factorization. If*

$$\chi(z) = z^k \prod_{i=1}^{\infty} \left[\frac{|\alpha_i|}{\alpha_i} \frac{\alpha_i - z}{1 - \bar{\alpha}_i z}\right]^{n_i},$$

$\alpha_i \in D$ distinct, $n_i \leq N$, and $F(0) = F(\alpha_i)$ for all i, then $\{T_\phi\}' = \{T_\chi\}' \cap \{T_F\}'$.

Proof. Factor $\chi(z) = \chi_1(z)\chi_2(z) \ldots \chi_N(z)$ where each $\chi_i(z)$ is a Blaschke product in which distinct α_k appear at most once. Since $F(0) = F(\alpha_i)$ for all i, $F - F(0)$ is divisible by each χ_i and Theorem (2.1.6) implies the conclusion.

Corollary (2.1.8)[50]: *Let* $\phi \in H^\infty$ *and* $\phi \in \chi F$ *be its inner-outer factorization. If* $\chi(z) = ((\alpha - z)/(1 - \bar{\alpha} z))^n, n > 0, \alpha \in \boldsymbol{D}$ *then* $\{T_\phi\}' = \{T_\chi\}' \cap \{T_F\}'$.
Proof. Obvious by Corollary (2.1.7).

Proposition (2.1.9)[50]: *Suppose* $\phi \in H^\infty$ *is such that* $\phi - \phi(\alpha)$ *has a zero of order* $n > 1$ *at* $\alpha \in \boldsymbol{D}$ *and that there exists* $\epsilon > 0$ *such that*

$$(\phi(z) - \phi(\alpha))/(z - \alpha)^n \geq \epsilon > 0 \quad for\ all\ z \in \boldsymbol{D}, \qquad z \neq \alpha.$$

Then the inner factor of $\phi - \phi(\alpha)$ *is* $((\alpha - z)/(1 - \bar{\alpha} z))^n$.
Proof. The hypotheses imply that $(z - \alpha)^n/(\phi(z) - \phi(\alpha)) \in H^\infty$. Writing

$$\left(\frac{z - \alpha}{1 - \bar{\alpha} z}\right)^n = \frac{(z - \alpha)^n}{\phi(z) - \phi(\alpha)} (\phi(z) - \phi(\alpha)) \left(\frac{1}{1 - \bar{\alpha} z}\right)^n$$

and noting that the inner-outer factorization of any function is unique, one easily concludes that the inner factor of $\phi - \phi(\alpha)$ is$((z - \alpha)/(1 - \bar{\alpha} z))^n$ (seep. 51 in[6]).
Corollary (2.1.10)[50]: *If* $\phi \in H^\infty$ *is such that* $\phi - \phi(\alpha)$ *has a simple zero for some* $\alpha \in \boldsymbol{D}$ *and* $|(\phi(z) - \phi(\alpha))/(z - \alpha)| \geq \epsilon > 0$ *for all* $z \in \boldsymbol{D}$, $z \neq \alpha$, *then* $\{T_\phi\}' = \{T_z\}'$.
Proof. Proposition (2.1.9) implies the inner factor of $\phi - \phi(\alpha)$ is $(\alpha - z)/(1 - \bar{\alpha} z)$. Since $\{T_{(\alpha - z)/(1 - \bar{\alpha} z)}\}' = \{T_z\}'$, Corollary (2.1.8) implies $\{T_\phi\}' = \{T_z\}'$.
Corollary (2.1.11)[50]: *If* $\phi \in H^\infty$ *is univalent then* $\{T_\phi\}' = \{T_z\}'$.
Proof. A univalent function satisfies the hypothesis of Corollary(2.1.10) for every $\alpha \in \boldsymbol{D}$.
Proposition(2.1.12)[50]: *The Fredholm spectrum of an analytic Toeplitz operator* T_ϕ *isexactly the cluster set* C_ϕ.
Proof. Recall that λ is not in the Fredholm spectrum of an operator T if and only if $T - \lambda$ is Fredholm, that is $T - \lambda$ has closed range and the null spaces of $T - \lambda$ and $(T - \lambda)^*$ are both finite dimensional. For convenience we consider $\lambda = 0$. Note that $T_\phi - \lambda = T_{\phi - \lambda}$ is always oneto-one unless $\phi(z) \equiv \lambda$.

Suppose $0 \notin C(\phi)$ and write $\phi = BSF$ where B is a Blaschke product, S is a singular inner function, and F is an outer function. If S were nonconstant or if B were an infinite Blaschke product, then one could easily find $z_n \in \boldsymbol{D}, |z_n| \to 1$, such that $S(z_n)B(z_n) \to 0$ contradicting $0 \notin C(\phi)$ since $F \in H^\infty$. Hence $S(z) = 1$ and B is a finite Blaschke product. Since $0 \notin \overline{\phi(\boldsymbol{D})}$ implies $|\phi(e^{i\theta})| = |F(e^{i\theta})| \geq \epsilon > 0$ almost everywhere, it follows that $F^{-1} \in H^\infty$ and that T_F is invertible. But T_B is clearly Fredholm, hence $T_\phi = T_{\dot{B}} T_F$ is also Fredholm.

Conversely suppose that T_ϕ is Fredholm and write $\phi = BSF$. Since T_ϕ has closed range, ϕ is bounded below [55, and so T_F is invertible. If S were non-constant or if B were an infinite Blaschke product then $dim(null\ T_B^* T_S^*) = +\infty$. Since $null\ (T_\phi^*) = null\ (T_B^* T_S^*)$ we must again have that B is a finite Blaschke product. Thus the inner factor B of ϕ is continuous on **T** with $|B(e^{i\theta})| = 1$ and so $0 \notin C(\phi)$.

We will completely characterize the commutant of T_ϕ for any function $\phi \in H^\infty$ whose inner factor is $z^n, n \geq 1$. Analogous results also hold for any function $\phi \in H^\infty$ whose inner factor is $((\alpha - z)/(1 - \alpha z))^n, n \geq 1,\ \alpha \in \boldsymbol{D}$.

Lemma (2.1.13)[50]: *Suppose T commutes with* $T_{z^n}, n > 1$, *and with* T_f *where* $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots \in \mathrm{H}^\infty$. *Let* $p \geq 1$ *be the smallest integer for which* $a_p \neq 0$, *and let* $p = qn + r$ *where* $0 \leq q$ *and* $0 \leq r < n$. *Then T commutes with* T_g *where*

$$g(z) = (f(z) - a_0)/z^{qn} = a_p z^r + \dots .$$

Proof. Since T_f, commutes with $U_+ = T_{z^n}$ we have that $T_f = \sum_{\mathrm{k}=0}^{\infty} \hat{A}_k U_+^{\mathrm{k}}$ If $q = 0$ then the result follows. So assume $q \geq 1$. By the definition of p we have that $A_o = a_0 I, A_1 = 0, \dots, A_{q-1} = 0$. Since T commutes with T_f we obtain

$$T\left(\sum_{\mathrm{k}=0}^{\infty} \hat{A}_k U_+^{\mathrm{k}}\right) = \left(\sum_{\mathrm{k}=0}^{\infty} \hat{A}_k U_+^{\mathrm{k}}\right) T \text{ or } T\left(\sum_{\mathrm{k}=\mathrm{q}}^{\infty} \hat{A}_k U_+^{\mathrm{k}}\right) = \left(\sum_{\mathrm{k}=\mathrm{q}}^{\infty} \hat{A}_k U_+^{\mathrm{k}}\right) T.$$

Hence

$$T\left(\sum_{\mathrm{k}=0}^{\infty} \hat{A}_{k+q} U_+^{\mathrm{k}}\right) = \left(\sum_{\mathrm{k}=0}^{\infty} \hat{A}_{k+q} U_+^{\mathrm{k}}\right) T .$$

since T commutes with U_+, and since U_+ is isometric. But by definition of g we have that $T_g = \sum_{\mathrm{k}=0}^{\infty} \hat{A}_{k+q} U_+^{\mathrm{k}}$.

Lemma (2.1.14)[50]: *Suppose T commutes with* T_{z^n}, $n > 1$, *and with* T_f , *where* $f(z) = a_0 + a_r z^r + \dots + \in H^\infty, a_r \neq 0,\ 1 \leq r < n$. *If r divides n then T commutes with* T_{z^r}.

Proof. Since T_f commutes with $U_+ = T_{z^n}$, we have that $T_f = \sum_{\mathrm{k}=0}^{\infty} \hat{A}_k U_+^{\mathrm{k}}$ Since r divides n, say $n = qr$, we have that A_0 is unitarily equivalent to

$$\begin{pmatrix} a_0 I & & & 0 \\ Y & a_0 I & & \\ & Y & & \\ & & \ddots & \\ & & Y & a_0 I \end{pmatrix}$$

on $\sum_{l=1}^{q} \oplus \boldsymbol{C}^r$, where $Y = a_r\, I + N$ on $\boldsymbol{C}^r$, $a_r \neq 0$, N nilpotent. Since T commutes with $U_+ = T_{z^n}$ we also have that $\sum_{\mathrm{k}=0}^{\infty} \hat{B}_k U_+^{\mathrm{k}}$. In order that T commute with T_f it is necessary tha B_o commute with A_o. But one easily checks that since $Y = a_r\, I + N, a_r \neq 0$, N nilpotent this implies that B_o is lower triangular on $\sum_{l=1}^{q} \oplus \boldsymbol{C}^r$, which in turn implies that T is lower triangular with respect to the decomposition of H^2 for $V_+ = T_{z^r}$. Notice that

$$T_f - a_0\, I = \left(\sum_{\mathrm{k}=0}^{\infty} \hat{X}_k U_+^{\mathrm{k}}\right) V_+$$

and $X_0 = Y = a_r\, I + N$ with N a nilpotent operator. Lemma (2.1.4) now applies and we obtain that T commutes with $V_+ = T_{z^r}$.

Lemma (2.1.15)[50]: Suppose T commutes with $T_{z^n}, n \geq 1$, and with T_f where $f(z) = a_0 + a_p z^P + \cdots \in H^\infty, a_p \neq 0$ If $p = qn + r$ with $0 \leq q$ and $0 < r < n$ then T commutes with T_{z^s} where $s = g.c.d.(\mathrm{r}, \mathrm{n})$.

Proof. By Lemma (2.1.13), T commutes with T_g where

$$g(z) = a_p z^r + \cdots$$

Now if $s = r$ then r divides n and Lemma (2.1.14) implies that T commutes with T_{z^s}. Otherwise there exist integers $t, k \geq 0$ such that $s = tr - kn$. Hence $tr = kn + s, 0 \leq k, 0 < s < n$. Since T commutes with T_g, T also commutes with $T_g^t = T_{g^t}$ where $g(z)^t = a_p^t z^{tr} + \cdots$ By Lemma (2.1.13),T then commutes with T_b where $b(z) = a_p^t z^s + \ldots, 0 < s < n$. Since s divides n, Lemma (2.1.14) now implies that T commutes with T_{z^s},

Theorem (2.1.16)[50]: If $\phi \in H^\infty$ has inner-outer factorization $\phi = \chi F$ where $\chi(z) = z^n, n \geq 1$, then $\{T_\phi\}' = \{T_{z^s}\}'$ where $s > 1$ is the positive integer which is maximal with respect to the property that both z^n and $F(z)$ are functions of z^s (equivalently: that ϕ is a functionof z^s).

Proof. Clearly if χ and F are both functions of z^s and if T commutes with T_{z^s}, then T commutes with T_χ and T_F, and hence with T_ϕ.

Now suppose that T commutes with T_ϕ. Corollary (2.1.8) then implies that T commutes with T_{z^n} and T_F where $F(z) = a_0 + a_1 z + a_2 z^2 + \cdots \in H^\infty$. Let $s > 1$ be maximal with respect to the property that both z^n and $F(z)$ are functions of z^s. Denote the sequence of integers $p \geq 1$ such that $a_p \neq 0$ by $p_1 < p_2 < p_3 < \cdots$. Then $s = g.c.d.(n, p_1, p_2, \ldots)$. Let $s_k = g.c.d.(n, p_1, p_2, \ldots, p_k)$. Now suppose $p_1 = q_1 n + r_1$ with $0 \leq q_1$, and $0 \leq r_1 < n$. If $r_1 = 0$ then $s_1 = n$ and T commutes with $T_{z^{s_1}}$ while if $0 < s_1$ then Lemma (2.1.15) implies that T commutes with $T_{z^{s_1}}$ since $g.c.d.(n, r_1) = g.c.d.(n, p_1) = s_1$. In either case T commutes with $T_{z^{s_1}}$. From this one concludes that T also commutes with T_{f_1} where $f_1(z) = f(z) - a_0 - a_{p_1} z^{p_1} = a_{p_2} z^{p_2} + \cdots$. Now suppose $p_2 = q_2 s_1 + r_2$ with $0 \leq$ and $0 \leq r_2 < s_1$. If $r_2 = 0$ then $s_2 = s_1$ and T commutes with $T_{z^{s_2}}$, while if $0 < r_2$ then Lemma (2.1.15) impliesthat T commutes with $T_{z^{s_2}}$ since $g.c.d.(s_1, r_2) = g.c.d.(s_1, p_2) = g.c.d.(n, p_1, p_2) = s_2$ From this one concludes that T commutes with T_{f_2} where $f_2(z) = f(z) - a_0 - a_{p_1} z^{p_1} - a_{p_2} z^{p_2} = a_{p_3} z^{p_3} + \cdots$. Now suppose $p_3 = q_3 s_2 + r_3$ with $0 \leq q_3$ and $0 \leq r_3 < s_2$. If $r_3 = 0$ then $s_3 = s_2$ and T commutes with $T_{z^{s_3}}$, while if $0 < r_3$ then Lemma (2.1.15) implies that T commutes with $T_{z^{s_3}}$ since $g.c.d.(s_2, r_3) = g.c.d.(s_2, p_3) = g.c.d.(n, p_1, p_2, p_3) = s_3$. Continuing in this manner we obtain that T commutes with $T_{z^{s_k}}$ for every k. Hence T commutes with T_{z^s}.

Corollary (2.1.17)[50]: *Let $\phi \in H^\infty$ and $\phi = \chi F$ be its inner-outer factorization with $\chi(z) = z^n, n \geq 1$, and $F(z) = a_0 + a_1 z + a_2 z^2 + \cdots$. If there exists an integer $p \geq 1$ such that $a_p \neq 0$ and $g.c.d.(n, p) = 1$ then $\{T_\phi\}' = \{T_z\}'$.*

Proof. Theorem (2.1.16) applies and the hypotheses imply that $s = 1$.

Corollary (2.1.18)[50]: *If $n, k \geq I$ are positive integers, then $\{T_{z^n}\}' \cap \{T_{z^k}\}' = \{T_{z^s}\}'$ where $s = g.c.d.(n, k)$.*

Proof. Observe that if T commutes with T_{z^n} and T_{z^k} then it also commutes with $T_{z^n(1+z^k)}$ Since $1 + z^k$ is outer, Theorem (2.1.16) implies that T also commutes with T_{z^s} where s can be described as in the corollary.

Corollary (2.1.19)[50]: *If $n, k \geq 1$ are positive integers and $0 < |a| < 1$, then $\{T_{z^n}\}' \cap \left\{T_{((\alpha - z)/(1 - \bar{\alpha} z))^k}\right\}' = \{T_z\}'$.*

Proof. If T commuteswith T_{z^n} and $T_{((\alpha-z)/(1-\bar{\alpha}z))^k}$ then T also commutes with T_ϕ where $\phi(z)=z^n(1+((\alpha-z)\,/\,(1-\bar{\alpha}z))^k)$.Since the coefficient of z in the outer part of ϕ is nonzero, Corollary(2.1.17) implies the result [3.55].

Section (2.2): Bergman Space by the Hardy Space of the Bidisk

For $\mathbb{D}$ be the open unit disk in $\mathbb{C}$. Let dA denote the Lebesgue area measure on the unit disk $\mathbb{D}$, normalized so that the measure of $\mathbb{D}$ equals 1. The Bergman space L_a^2 is the Hilbert space consisting of the analytic functions on $\mathbb{D}$ that are also in the space $L_a^2(\mathbb{D},dA)$ of square integrable functions on $\mathbb{D}$.

We study multiplication operators on L_a^2 by bounded analytic functions on the unit disk $\mathbb{D}$ via the Hardy space of bidisk. The theme is to use the theory of multivariable operators to study a single operator. The main idea is to lift the Bergman shift up as the compression of a commuting pair of isometries on a nice subspace of the Hardy space of bidisk. This idea was used in studying the Hilbert modules by R. Douglas and V. Paulsen [65], operator theory in the Hardy space over the bidisk by R. Douglas and R. Yang [66], [67], [68] and [69], the higher-order Hankel forms by S. Ferguson and R. Rochberg [70] and [71], and the lattice of the invariant subspaces of the Bergman shift by S. Richter [72].

For a bounded analytic function ϕ on the unit disk, the multiplication operator M_ϕ is defined on the Bergman space L_a^2 given by $M_\phi h=\phi h$ for $h\in L_a^2$. Let $e_n=\sqrt{n+1z^n}$. Then $\{e_n\}_0^\infty$ form an orthonormal basis of the Bergman space L_a^2 . On the basis $\{e_n\}$, the multiplication operator M_z by z is a weighted shift operator, called the Bergman shift: $M_ze_n=\sqrt{\frac{n+1}{n+2}}e_{n+1}$.

The multiplication operators on the Bergman space possess a very rich structure theory. Even the lattice of the invariant subspaces of the Bergman shift M_z is huge [73]. The Bergman shift M_z has a universal property [74]: for any strict contraction S on a Hilbert space H, there always exists a pair of invariant subspaces of M_z, say $\mathcal{M}$ and $\mathcal{N}$ in Lat M_z (the invariant subspace lattice of M_z is the set of subspaces $\mathcal{M}$ of L_a^2 with $M_z\mathcal{M}\subset\mathcal{M}$), such that $S\cong P_{\mathcal{M}\ominus\mathcal{N}}\{M_z|_{\mathcal{M}\ominus\mathcal{N}}\}$, where $P_{\mathcal{M}\ominus\mathcal{N}}$ denotes the orthogonal projection of $L_a^2(\mathbb{D})$ onto $\mathcal{M}\ominus\mathcal{N}$. This indicates that existence of the invariant subspace problem for Hilbert space operator is equivalent to whether Lat M_z is saturated, i.e., for any $\mathcal{M},\mathcal{N}\in$ Lat M_z, with $\mathcal{M}\supset\mathcal{N}$ and $dim(\mathcal{M}\ominus\mathcal{N})=\infty$, whether there always exists some $\Omega\in$ $Lat\ M_z$ such that

$$\mathcal{N}\subsetneqq\Omega\subsetneqq\mathcal{M}.$$

Let $\mathbb{T}$ denote the unit circle. The torus $\mathbb{T}^2$ is the Cartesian product $\mathbb{T}\times\mathbb{T}$. The Hardy space $H^2(\mathbb{T}^2)$ over the bidisk is $H^2(\mathbb{T})\otimes H^2(\mathbb{T})$. Let P be the orthogonal projection from the space $L^2(\mathbb{T}^2)$ of the Lebesgue square integrable functions on the torus $\mathbb{T}^2$ onto $H^2(\mathbb{T}^2)$. The Toeplitz operator on $H^2(\mathbb{T}^2)$ with symbol f in $L^\infty(\mathbb{T}^2)$ق is defined by $T_f(h)=P(fh)$, for $h\in H^2(\mathbb{T}^2)$. Clearly, T_z and T_w are a pair of doubly commuting pure isometries on $H^2(\mathbb{T}^2)$. For each integer $n\geq 0$, let

$$P_n(z,w)=\sum_{i=0}^{n}z^iw^{n-i}=\frac{z^{n+1}-w^{n+1}}{z-w}$$

Let $\mathcal{H}$ be the subspace of $H^2(\mathbb{T}^2)$ spanned by functions $\{P_n\}_{n=0}^{\infty}$. The orthogonal complement of $\mathcal{H}$ in $H^2(\mathbb{T}^2)$ is

$$[z-w] = closure_{H^2(\mathbb{T}^2)}\{(z-w)H^2(\mathbb{T}^2)\}.$$

Then $[z-w]$ is an invariant subspace of analytic Toeplitz operators T_f for $f \in H^\infty(\mathbb{T}^2)$. Let $P_{\mathcal{H}}$ be the orthogonal projection from $L^2(\mathbb{T}^2)$ onto $\mathcal{H}$. It is easy to check that

$$P_{\mathcal{H}} T_z|_{\mathcal{H}} = P_{\mathcal{H}} T_w|_{\mathcal{H}}.$$

Let $\mathcal{B}$ denote the operator above. It was shown explicitly in [75] and implicitly in [76] that $\mathcal{B}$ is unitarily equivalent to the Bergman shift M_z on the Bergman space L_a^2 via the following unitary operator $U: L_a^2(\mathbb{D}) \to \mathcal{H}$,

$$Uz^n = \frac{P_n(z,w)}{n+1}.$$

Clearly, for each $f(z,w) \in \mathcal{H}$,

$$(U^*f)(z) = f(z,z).$$

for $z \in D$. The simple observation that $P_n(z,w) = \frac{z^{n+1}-w^{n+1}}{z-w}$ gives that for each $f(z,w) \in \mathcal{H}$, there is a function $\tilde{f}(z)$ in the Dirichlet space $\mathcal{D}$ such that

$$f(z,w) = \frac{\tilde{f}(z)-\tilde{f}(w)}{z-w}$$

Thus, for each Blaschke product $\phi(z)$ with finite order, the multiplication operator M_ϕ on the Bergman space is unitarily equivalent to $\phi(\mathcal{B})$ on $\mathcal{H}$.

We will study the operator $\phi(\mathcal{B})$ on the Hardy space of the bidisk to shed light on properties of the multiplication operator M_ϕ. This method seems to be effective in dealing with the multiplication operators on the Bergman space because functions in the Hardy space of the bidisk behave slightly better than the functions in the Bergman space.
The difficulty to study $\mathcal{B}$ on $\mathcal{H}$ is to get better understanding the projection $P_{\mathcal{H}}$. We will get a lot of mileage from developing a ‘‘heavy’’ machinery on the Hardy space of the bidisk how to get rid of $P_{\mathcal{H}}$ in the expression

$$\phi(\mathcal{B})^n f = \frac{1}{n+1} P_{\mathcal{H}}\big(P_n(\phi(z),\phi(w))f\big),$$

for $f \in \mathcal{H}$. To do this, letting L_0 denote the space $ker\, T^*_{\phi(z)} \cap ker\, T^*_{\phi(w)} \cap \mathcal{H}$, for each $e \in L_0$, we construct functions $\{d_e^k\}$ such that for each $l \geqq 1$,

$$p_l\,(\phi(z),\phi(w))e + \sum_{k=0}^{l-1} p_k(\phi(z),\phi(w))\, d_e^{l-k} \in \mathcal{H}$$

and

$$p_l\,(\phi(z),\phi(w))e + p_{l-1}\,(\phi(z),\phi(w))d_e^0 \in \mathcal{H}.$$

On one hand, we have a precise formula of d_e^0. On the other hand, d_e^k is orthogonal to L_0. These constructions are useful in studying the reducing subspaces of $\phi(\mathcal{B})$. A reducing sub-space $\mathcal{M}$ for an operator T on a Hilbert space H is a subspace $\mathcal{M}$ of H such that $T\mathcal{M} \subset \mathcal{M}$ and $T^*\mathcal{M} \subset \mathcal{M}$. A reducing subspace $\mathcal{M}$ of T is called minimal if only reducing subspaces contained in $\mathcal{M}$ are $\mathcal{M}$ and $\{0\}$. As in [77], a subspace $\mathcal{N}$ of H is a wandering subspace of T if $\mathcal{N}$ is orthogonal to $T^n\mathcal{N}$ for each $= 1,2,\ldots$. If $\mathcal{M}$ is an invariant subspace of T, it is clear that $\mathcal{M} \ominus T\mathcal{M}$ is a wandering subspace of T, and we will refer this subspace as the wandering subspace of $\mathcal{M}$.

For a reducing subspace $\mathcal{M}$ of $\phi(\mathcal{B})$, and e in the wandering subspace of $\mathcal{M}$,

$$p_l\big(\phi(z),\phi(w)\big)e+\sum_{k=0}^{l-1}p_k\big(\phi(z),\phi(w)\big)\,d_e^{l-k}\ \in\mathcal{M}.$$

Although for a Blaschke product ϕ of finite order, M_ϕ is not an isometry, using the machinery on the Hardy space of bidisk we will show that there exists a unique reducing subspace $\mathcal{M}_0$, the so called distinguished subspace, on which the restriction of M_ϕ is unitarily equivalent to the Bergman shift, which will play an important role in classifying reducing subspaces of M_ϕ. The functions d_e^1 and d_e^0 have the following relation.

Theorem (2.2.1)[64]: *If $\mathcal{M}$ is a reducing subspace of $\phi(\mathcal{B})$ orthogonal to the distinguished reducing subspace $\mathcal{M}_0$, for each e in the wandering subspace for $\mathcal{M}$, then there is an element $\tilde{e}$ in the wandering subspace for $\mathcal{M}$ and a number λ such that*

$$d_e^1=d_e^0+\tilde{e}+\lambda\,e_0 \tag{2}$$

To understand the structure of minimal reducing subspaces of $\phi(\mathcal{B})$ we lift a reducing subspace of $\phi(\mathcal{B})$ as a reducing subspace of the pair of doubly commuting isometries $T_{\phi(z)}$ and $T_{\phi(w)}$. For a given reducing subspace $\mathcal{M}$ of $(\mathcal{B})$, define the lifting $\widetilde{\mathcal{M}}$ of $\mathcal{M}$:

$$\widetilde{\mathcal{M}}=span\{\phi(z)^l,\phi(w)^k\mathcal{M};\ l,k\geqq 0\}.$$

Since $\mathcal{M}$ is a reducing subspace of $\phi(\mathcal{B})$ and $\big|\widetilde{\mathcal{M}}$ is a reducing subspace of the pair of doubly commuting isometries $T_{\phi(z)}$and $T_{\phi(w)}$, by the Wold decomposition of the pair of isometries on $\mathcal{M}$, we have

$$\widetilde{\mathcal{M}}=\bigoplus_{l,k\geqq 0}\phi(z)^l,\phi(w)^kL_{\widetilde{\mathcal{M}}},$$

where $L_{\widetilde{\mathcal{M}}}$ is the wandering subspace

$$L_{\widetilde{\mathcal{M}}}=ker\,T_{\phi(z)}^*\cap ker\,T_{\phi(w)}^*\cap\widetilde{\mathcal{M}}$$

The following theorem gives a complete description of the wandering subspace $L_{\widetilde{\mathcal{M}}}$.

Theorem (2.2.2)[64]: Suppose that $\mathcal{M}$ is a reducing subspace of $\phi(\mathcal{B})$ orthogonal to $\mathcal{M}_0$. If $\left\{e_1^{(M)},\dots,e_{q_M}^{(M)}\right\}$ is a basis of the wandering subspace of $\mathcal{M}$, then

$$L_{\widetilde{\mathcal{M}}}=spane\left\{e_1^{(M)},\dots,e_{q_M}^{(M)};d^1_{e_1^{(M)}},\dots,d^1_{e_{q_M}^{(M)}}\right\}$$

and

$$dim\,L_{\widetilde{\mathcal{M}}}=2q_M\,.$$

To prove Theorem (2.2.2), first we use the Wold decomposition of the pair of doubly commuting isometries $T_{\phi(z)}$and $T_{\phi(w)}$on the lifting $\mathcal{K}_\phi\big(=\widetilde{\mathcal{H}}\big)$ of $\mathcal{H}$ to get the dimension of the wandering subspace $\mathcal{L}_\phi(=L_{\widetilde{\mathcal{H}}})$. By means of the Fredholm theory in [8], we are able to show that the dimension of $\mathcal{L}_\phi$ equals $2N-1$, where N is the order of the Blaschke product ϕ.

Then by means of the finite dimension of the wandering subspace of these isometries on the reducing subspace we will be able to obtain some structure theorems on reducing subspaces of the multiplication operators by finite Blaschke products on the Bergman space.

Theorem (2.2.3)[64]: *Suppose that $\Omega,\mathcal{M}$ and $\mathcal{N}$ are three distinct nontrivial minimal reducing subspaces contained in $\mathcal{M}_0^\perp$ for $\phi(\mathcal{B})$. If*

$$\Omega\subset\mathcal{M}\oplus\mathcal{N},$$

then there is a unitary operator $U:\mathcal{M}\to\mathcal{N}$ such that U commutes with $\phi(\mathcal{B})$ and $\phi(\mathcal{B})^*$.

The machinery on the Hardy space of the bidisk is not only useful in classifying the reducing subspaces of multiplication operators on the Bergman space, but also it is helpful in understanding the lattice of invariant subspaces of the Bergman shift as in [79] and hence the invariant subspace problem. We develop the Bergman function theory [80], [81] via the Hardy space of the bidisk.

Applying the machinery developed , we will be able to disprove Zhu's conjecture in the following theorem. For a holomorphic function ϕ on the unit disk and a point c in the unit disk, we say that c is a critical point of ϕ if its derivative vanishes at c.

Theorem (2.2.4)[64]: *Let ϕ be a Blaschke product with three zeros. If $\phi(z)$ has a multiple critical point in the unit disk, then M_ϕ has three nontrivial minimal reducing subspaces. If ϕ does not have any multiple critical point in the unit disk, then M_ϕ has only two nontrivial minimal reducing subspaces.*

The multiplication operator on the Bergman space is completely di¤erent from that in the Hardy space. By the famous Beurling Theorem [82], the lattice of the invariant subspaces of the multiplication operator by z on the Hardy space is completely determined by inner functions. A Beurling's theorem was recently obtained for the Bergman space [83]. On one hand, on the Hardy space, for an inner function ϕ, the multiplication operator by ϕ is a pure isometry and hence unilateral shift (with arbitrary multiplicity). So its reducing sub-spaces are in one-to-one correspondence with the closed subspaces of $H^2 \ominus \phi H^2$ [84], [85]. Therefore, it has infinitely many reducing subspaces provided that ϕ is any inner function other than a Möbius function. Many people have studied the problem of determining reducing subspaces of a multiplication operator on the Hardy space of the unit circle [86], [87] and [88]. For an inner function ϕ, the multiplication operator by ϕ on the Bergman space is a contraction but not an isometry. On the other hand, surprisingly, on the Bergman space, it was shown in [89] and [90] that for a Blaschke product ϕ with two zeros, the multiplication operator M_ϕ has only two nontrivial reducing subspaces. Zhu [66] conjectured that for a Blaschke product ϕ with N zeros, the lattice of reducing subspaces of the operator M_ϕ is generated by N elements. In other words, M_ϕ has exactly N nontrivial minimal reducing subspaces.

Bochner's theorem [65], [66] says that every Blaschke product with N zeros has exactly $N-1$ critical points in the unit disk $\mathbb{D}$. Theorem (2.2.4) gives a classifcation of reducing subspaces for M_ϕ for a Blaschke product ϕ with three zeros.

Critical points of ϕ have important geometric meaning about the self-mapping of the unit disk. The work of Stephenson [70], [75], [76] suggests that the geometric version of the above theorem should be in terms of the Riemann surfaces. A finite Blaschke product ϕ with N zeros is an N to 1 conformal map of $\overline{\mathbb{D}}$ onto $\overline{\mathbb{D}}$. Bochner's theorem [75], [76] says that ϕ has exactly $N-1$ critical points in the unit disk $\mathbb{D}$ and none on the unit circle. Let $\mathcal{C}$ denote the set of the critical points of ϕ in $\mathbb{D}$ and $\mathcal{F} = \phi^{-1} \circ \phi(\mathcal{C})$. Then $\mathcal{F}$ is a finite set, and $\phi^{-1} \circ \phi$ is an N-branched analytic function defined and arbitrarily continuable in $\mathbb{D} / \mathcal{F}$. Not all of the branches of $\phi^{-1} \circ \phi$ can be continued to a different branch, for example z is a single valued branch of $\phi^{-1} \circ \phi$. The Riemann surface for $\phi^{-1} \circ \phi$ over $\mathbb{D}$ is an N-sheeted cover of $\mathbb{D}$ at most $N(N-1)$ branch points, and it is not connected. The geometric version of Theorem (2.2.4) is the following theorem.

Theorem (2.2.5)[64]: *Let ϕ be a Blaschke product with three zeros. Then the number of non-trivial minimal reducing subspaces of M_ϕ equals the number of connected components of the Riemann surface of $\phi^{-1} \circ \phi$ over $\mathbb{D}$.*

We would like to point out that there are many essential differences in analysis and geometry between Blaschke products with order three and Blaschke products with order two. On one hand, for Blaschke products ϕ with order two, $\phi^{-1} \circ \phi$ contains two analytic functions on the unit disk and hence the Riemann surface for $\phi^{-1} \circ \phi$ over $\mathbb{D}$ is just two copies of the unit disk. On the other hand, for the most Blaschke products with order three, $\phi^{-1} \circ \phi$ has three multivalue functions on the unit disk and the Riemann surface for $\phi^{-1} \circ \phi$ over $\mathbb{D}$ has two connected components. This phenomenon makes it difficult for us to classify the reducing subspaces of a multiplication operator with the Blaschke product of order higher than two. It seems that the machinery developed is inevitable in classifying the reducing subspaces of the multiplication operator by a Blaschke product of higher order.

The problem of determining reducing subspaces of a multiplication operator is equi-valent to finding projections in the commutant of the operator which is the set of operators commuting with the multiplication operators. Everyvon Neumann algebra is generated by its projections. Theorem (2.2.4) says that everyvon Neumann algebra contained in the commutant of mulitplication operator by the Blaschke product with third order is commutative. A lot of nice and deep work on the commutant of a multiplication operator has been done on the Hardy space [66], [63], [64] while Blaschke products with finite zeros play an important role. Indeed Cowen proved that for $f \in H^\infty$, if the inner factor of $f - f\,(\alpha)$ is a Blaschke product ϕ with finite order for some $\alpha \in \mathbb{D}$, then the commutant of the multiplication operator by f equals the commutant of the multiplication operator by the finite Blaschke product ϕ [66]. Thus the structure of lattice of reducing subspaces of the multiplication operator by a Blaschke product with finite order is useful in studying the general multiplication operators on the Bergman space.

One application of the machinery on the Hardy space of the bidsk is that it was proved in [62] that the multiplication operator on the Bergman space is unitarily equivalent to a weighted unilateral shift operator of finite multiplicity if and only if its symbol is a constant multiple of the *N*-th power of a Mo¨bius transform. Another one is that we have obtained a complete description of nontrivial minimal reducing subspaces of the multipli-cation operator by ϕ on the Bergman space of the unit disk for the fourth order Blaschke product ϕ [61].

Using Theorems (2.2.1) and (2.2.3), for a finite Blaschke product ϕ, we are able to show that for two distinct nontrivial minimal reducing subspaces of $\phi(\mathcal{B})$, either they are orthogonal or $\phi(\mathcal{B})$ has two distinct unitarily equivalent reducing subspaces and has also infinitely many minimal reducing subspaces. Thus either $\phi(\mathcal{B})$ has infinitely many minimal reducing subspaces or the number of nontrivial minimal reducing subspaces of $\phi(\mathcal{B})$ is less than or equal to the order of ϕ. We say that two reducing subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\phi(\mathcal{B})$ are unitarily equivalent if there is a unitary operator $U : \mathcal{M} \to \mathcal{N}$ such that U commutes with $\phi(\mathcal{B})$ and $(\mathcal{B})$.

The adjoint of the multiplication operator by a finite Blaschke product is in a Cowen-Douglas class [67]. The theory of Cowen-Douglas classes will be useful in studying the multiplication operators on the Bergman space. On the other hand, we would like to see some applications of the results obtained to the study of the Cowen-Douglas classes.

First we introduce notations and show some properties of functions in $\mathcal{H}$. Then we compute the dimension of the wandering space for the lifting $\widetilde{\mathcal{H}}$ of $\mathcal{H}$. The dimension is useful for us to find the wandering space for the lifting $\mathcal{M}$ of a reducing subspace $\mathcal{M}$ of $\phi(\mathcal{B})$.

For $\alpha \in \mathcal{D}$, let k_α be the reproducing kernel of the Hardy space $H^2(\mathbb{T})$ at α. That is, for each function f in $H^2(\mathbb{T})$,

$$f(\alpha) = \langle f, k_\alpha \rangle$$

In fact, $k_\alpha = 1 / (1 - \bar{\alpha} z)$. For ϕ in $H^\infty(\mathbb{T})$, let $\hat{T}_\phi$ denote the analytic Toeplitz operator with symbol ϕ on $H^2(\mathbb{T})$, given by

$$\hat{T}_\phi h = \phi h.$$

Thus it is easy to check that

$$\hat{T}_\phi^* k_\alpha = \overline{\phi(\alpha)} k_\alpha. \tag{3}$$

For an integer $s \geqq 0$, let

$$k_\alpha^s(z) = \frac{s!\, z^s}{(1 - \bar{\alpha} z)^{s+1}}.$$

Lemma (2.2.6)[64]: For each $f \in H^\infty(\mathbb{T})$,

$$\hat{T}_f^* k_\alpha^s = \sum_{l=0}^{s} \frac{s!}{l!\,(s-l)!} \overline{f^{(l)}(\alpha)} k_\alpha^{s-l}.$$

Lemma (2.2.6) gives that the kernel of the Toeplitz operator $\hat{T}_\phi^*$ on the Hardy space of the unit circle is spanned by $\left\{ \{ k_{\alpha_k}^{s_k} \}_{s_k = 0, \dots, n_k} \right\}_{k=0,\dots,K}$.

Recall that $\mathcal{H}$ is the subspace of $H^2(\mathbb{T}^2)$ spanned by functions $\{p_n\}_{n=0}^\infty$. The following two lemmas give some properties for functions in $\mathcal{H}$ or $\mathcal{H}^\perp$.

Lemma (2.2.7)[64]: *If f is in $H^2(\mathbb{T}^2)$ and continuous on the closed bidisk and e is in $\mathcal{H}$, then*

$$\langle f(z,w), e(z,w) \rangle = \langle f(z,z), e(z,0) \rangle = \langle f(w,w), e(0,w) \rangle$$

Lemma(2.2.8) [64]: *For $h(z,w) \in H^2(\mathbb{T}^2)$, h is in $\mathcal{H}^\perp$ if $h(z,z) = 0$, for $z \in \mathbb{D}$.*

Proof. As pointed out before,

$$\mathcal{H}^\perp = c1\{(z-w) H^2(\mathbb{T}^2)\}$$

Let z be in $\mathbb{D}$. For each function $f(z,w) \in (z-w)H^2(\mathbb{T}^2)$, $f(z,z) = 0$ for each $h \in \mathcal{H}^\perp$.

Conversely, assume that for a function $h \in \mathcal{H}^\perp$, $h(z,z) = 0$, for $z \in \mathbb{D}$. For $0 < r < 1$, define

$$h_r(z,w) = h(rz, rw).$$

Then for each fixed $0 < r < 1$, $h_r(z,z) = 0$, and $h_r(z,w)$ is continuous on the closed bidisk and in $H^2(\mathbb{T}^2)$.

By Lemma (2.2.7), for each $e(z,w)$ in $\mathcal{H}$,

$$\langle h_r(z,w), e(z,w) \rangle = \langle h_r(z,z), e(z,0) \rangle = 0.$$

On the other hand, by [73], Theorem 3.4.3,

$$\langle h_r(z,w), e(z,w) \rangle = \lim_{r \to 1^-} \langle h_r(z,w), e(z,w) \rangle = 0.$$

Hence we conclude that h is in $\mathcal{H}^\perp$.

The Dirichlet space $\mathcal{D}$ consists of analytic functions on the unit disk whose derivative is in the Bergman space L^2_α.

Theorem (2.2.9)[64]: *For each $f(z,w)$ in $H^2(T^2)$, f is in $\mathcal{H}$ if and only if there is a function $\tilde{f}(z)$ in $\mathcal{D}$ such that*

$$f(z,w) = \frac{\tilde{f}(z) - \tilde{f}(w)}{\mathrm{z} - \mathrm{w}}$$

for two distinct points z, w in the unit disk.

This immediately gives the following three lemmas.

Lemma (2.2.10)[64]: *Suppose that $e(z,w)$ is in $\mathcal{H}$. If $e(z,z) = 0$ for each z in the unit disk, then $e(z,w) = 0$ for (z,w) on the torus.*

Lemma (2.2.11)[64]: If $e(z,w)$ is in $\mathcal{H}$, then

$$e(z,w) = e(w,z).$$

Lemma (2.2.12)[64]: Suppose $f(z,w)$ is in $\mathcal{H}$. Let $F(z) = f(z,0)$. Then

$$f\,(\lambda,\lambda) = \lambda F'(\lambda) + F(\lambda),$$

for each $\lambda \in \mathbb{D}$.

For an operator *T* on a Hilbert space, let $ker\ T$ denote the kernel of *T*. Then

$$ker\ T^* = H \ominus TH.$$

Given a pure isometry *U* on a Hilbert space *H*, the classical Wold decomposition theorem [79] states that

$$H = \bigoplus_{n \geqq 0} U^n E\,,$$

where $E = H \ominus UH$ is the *wandering subspace* for *U* and equals $ker\ T^*$.

For a function ϕ in $H^\infty(\mathbb{D})$, we can view $\phi(z)$ and $\phi(w)$ as functions on the torus $\mathbb{T}^2$. While M_ϕ is not an isometry on the Bergman space of the unit disk, the analytic Toeplitz operators $T_{\phi(z)}$ and $T_{\phi(w)}$ are a pair of doubly commuting pure isometries on the Hardy space $H^2(\mathbb{T}^2)$ of torus. Since

$$T_z^* p_n = T_w^* p_n = p_{n-1}$$

for $n \geqq 1$ and

$$T_z^* p_0 = T_w^* p_0 = 0,$$

$\mathcal{H}$ is an invariant subspace for both T_z^* and T_w^*. So $\mathcal{H}$ is also an invariant subspace for both $T_{\phi(z)}^*$ and $T_{\phi(w)}^*$. Recall the lifting $\mathcal{K}_\phi$ of $\mathcal{H}$:

$$\mathcal{K}_\phi = span\{\phi^l(z)\phi^k(w)\mathcal{H};\ l,k\ \geqq 0\}.$$

Then $\mathcal{K}_\phi$ is a reducing subspace for both $T_{\phi(z)}$ and $T_{\phi(w)}$, and so $T_{\phi(z)}$ and $T_{\phi(w)}$ are also a pair of doubly commuting isometries on $\mathcal{K}_\phi$.

We consider the Wold decompositions for the pair on both $\mathcal{K}_\phi$ and $\mathcal{K}_\phi^\perp\big(H^2(\mathbb{T}^2) \ominus \mathcal{K}_\phi\big)$. Introduce wandering subspaces

$$\mathcal{L}_\phi = kerT_{\phi(z)}^* \cap kerT_{\phi(w)}^* \cap \mathcal{K}_\phi,$$

and

$$\widehat{\mathcal{L}_\phi} = kerT_{\phi(z)}^* \cap kerT_{\phi(w)}^* \cap \mathcal{K}_\phi^\perp.$$

To get the dimension of the wandering subspaces $\mathcal{L}_\phi$ and $\widehat{\mathcal{L}_\phi}$, we will dentify the wandering subspace $\widehat{\mathcal{L}_\phi}$ for the Blaschke product ϕ with distinct zeros. The following lemma follows from the remark after Lemma (2.2.6) about ker $\hat{T}_\phi^*$.

Lemma (2.2.13)[64]: *If $\phi(z)$ is a Blaschke product with distinct zeros $\{\alpha_i\}_{i=1}^N$s, then intersection of the kernel of $T_{\phi(z)}^*$ and $T_{\phi(w)}^*$ is spanned by $\{k_{\alpha_i}(z)k_{\alpha_j}(w)\}_{i,j=1}^N$.*

The following lemma is implicit in the proof of Theorem 3 [79]. But we give a complete proof of the lemma.

Lemma (2.2.14)[64]: *Suppose that $\phi(z)$ is a Blaschke product with distinct zeros $\{\alpha_i\}_{i=1}^{N}$. Then the wandering space $\widehat{\mathcal{L}_\phi}$ is equal to the space spanned by*

$$\left\{k_{\alpha_i}(z)k_{\alpha_j}(w) - k_{\alpha_j}(z)k_{\alpha_i}(w);\ 1 \leqq i < j \leqq N\right\} \text{ and}$$

$$T^*_{z-w}\left[k_{\alpha_l}(z)k_{\alpha_{l+1}}(w) + k_{\alpha_{l+1}}(z)k_{\alpha_q}(w) + k_{\alpha_q}(z)k_{\alpha_l}\right]; 2 \leqq l+1 < q \leqq N.$$

Moreover, the dimension of $\widehat{\mathcal{L}_\phi}$ *equals* $(N-1)^2$.

Proof. First we show

$$\widehat{\mathcal{L}_\phi} = kerT^*_{\phi(z)} \cap kerT^*_{\phi(w)} \cap \mathcal{H}^\perp.$$

Since $\mathcal{H} \subset \mathcal{K}_\phi$,

$$\widehat{\mathcal{L}_\phi} \subset kerT^*_{\phi(z)} \cap kerT^*_{\phi(w)} \cap \mathcal{H}^\perp.$$

Conversely, if f is in $kerT^*_{\phi(z)} \cap kerT^*_{\phi(w)} \cap \mathcal{H}^\perp$ then f is in $kerT^*_{\phi(z)} \cap kerT^*_{\phi(w)}$ and orthogonal to $\mathcal{H}$. Thus for each $g(z,w) = \sum_{l,k\geqq 0} \phi(z)^l \phi(w)^k h_{kl} \in \mathcal{H}$ where $h_{kl} \in \mathcal{H}$ we have

$$\langle f, g\rangle = \sum_{k,l\geqq 0} \langle f, \phi(z)^l \phi(w)^k h_{kl}\rangle = \sum_{k,l\geqq 0} \langle \left[T^*_{\phi(z)}\right]^l \left[T^*_{\phi(w)}\right]^k f, h_{lk}\rangle = 0$$

So f is also in $\widehat{\mathcal{L}_\phi}$. Hence we have

$$\widehat{\mathcal{L}_\phi} = kerT^*_{\phi(z)} \cap kerT^*_{\phi(w)} \cap \mathcal{H}^\perp.$$

We want to prove that the dimension of $\widehat{\mathcal{L}_\phi}$ is $(N-1)^2$. Without loss of generality, we assume that $a_I = 0$. By Lemma (2.2.13), the N^2 dimensional space $kerT^*_{\phi(z)} \cap kerT^*_{\phi(w)}$ is spanned by $\{k_{\alpha_i}(z)k_{\alpha_j}(w)\}_{i,j=1}^{N}$. So it follows from Lemma (2.2.8) that $\widehat{\mathcal{L}_\phi}$ consists of the elements h in $kerT^*_{\phi(z)} \cap kerT^*_{\phi(w)}$ which satisfy $h(z,z) = 0$. That is,

$$\widehat{\mathcal{L}_\phi} = \left\{ h = \sum_{i=1}^{N}\sum_{j=1}^{N} c_{ij} k_{\alpha_i}(z) k_{\alpha_j}(w); h(z,z) = \sum_{i=1}^{N}\sum_{j=1}^{N} c_{ij} k_{\alpha_i}(z) k_{\alpha_j}(z) = 0 \right\}$$

For any $h \in \widehat{\mathcal{L}_\phi}$, taking the limit at infinity and testing the multiplicity at its poles $1 / \overline{a_J}$ of the function $h(z,z)$, we immediately have that $h(z,z) = 0$ implies $c_{jj} = 0, j = 1,2,\dots,N$. That is,

$$\widehat{\mathcal{L}_\phi} = \left\{ h = \sum_{i\neq j,i=1}^{N}\sum_{j=1}^{N} c_{ij} k_{\alpha_i}(z) k_{\alpha_j}(w); h(z,z) = \sum_{i\neq j,i=1}^{N}\sum_{j=1}^{N} c_{ij} k_{\alpha_i}(z) k_{\alpha_j}(z) = 0 \right\}$$

Observe that $k_{a_i}(z)k_{a_j}(w) = a_{ij}k_{a_i}(z) + b_{ij}k_{a_i}(z)$ where $a_{ij} = \frac{\overline{a_\iota}}{\overline{a_\iota} - \overline{a_J}}$ and $b_{ij} = \frac{-\overline{a_J}}{\overline{a_\iota} - \overline{a_J}}$, and $k_{\alpha_2}(z), \dots, k_{\alpha_N}(z)$ are linearly independent. Write $h(z,z)$ as linear combination of $k_{\alpha_j}(z), j = 2,\dots,N$, then all the coe‰cients of $k_{\alpha_j}(z)$ must be zero. So we have a system of another $N-1$ linear equations governing $c_{ij}, i \neq j, i,j = 1,\dots,N$. It is easy to check that the rank of the coe‰cient matrix of the system is $N-1$. Hence the dimension of $\widehat{\mathcal{L}_\phi}$ (as the solution space of $N^2 - N$ unknown variables governed by $N-1$ linearly independent equations) equals $N^2 - N - (N-1)$. The proof is finished.

We show the main result.

Theorem (2.2.15)[64]: *Let ϕ be a Blaschke product with N zeros in the unit disk. Then*

$$\mathcal{K}_\phi = \bigoplus_{l,k\geqq 0} \phi^l(z)\phi^k(w)\ \mathcal{L}_\phi,$$

and

$$H^2(\mathbb{T}^2) \ominus \mathcal{K}_\phi = \bigoplus_{l,k\geqq 0} \phi^l(z)\phi^k(w)\ \widehat{\mathcal{L}_\phi}$$

The dimension of $\widehat{\mathcal{L}_\phi}$ *equal* $(N-1)^2$ *and The dimension of* $\mathcal{L}_\phi$ *equals* $2N-1$

Proof. As pointed out early, $T_{\phi(z)}$ and $T_{\phi(w)}$ are a pair of doubly commuting isometries on both $\mathcal{K}_\phi$ and $H^2(\mathbb{T}^2) \ominus \mathcal{K}_\phi$. Consider the Wold decomposition of $T_{\phi(z)}$ on $\mathcal{K}_\phi$ to get

$$\mathcal{K}_\phi = \bigoplus_{l\geqq 0} \phi(z)^l E$$

where E is the wandering subspace for $T_{\phi(z)}$ given by

$$E = \mathcal{K}_\phi \ominus \left[T_{\phi(z)}\mathcal{K}_\phi\right] = ker\left[T^*_{\phi(z)}\big|_{\mathcal{K}_\phi}\right] = kerT^*_{\phi(z)} \cap \mathcal{K}_\phi.$$

Since $T_{\phi(z)}$ and $T_{\phi(w)}$ are doubly commuting, E is a reducing subspace of $T_{\phi(w)}$. Thus $T_{\phi(w)}\big|_E$ is still an isometry. The Wold decomposition theorem again gives

$$E = \bigoplus_{k\geqq 0} \phi(w)^k E_1$$

where E_1 is the wandering subspace for $T_{\phi(w)}\big|_E$ given by

$$E_1 = E \ominus T_{\phi(w)}E = kerT^*_{\phi(w)} \cap E = ker\, T^*_{\phi(z)} \cap T^*_{\phi(w)} \cap \mathcal{K}_\phi$$

This gives

$$\mathcal{K}_\phi = \bigoplus_{l,k\geqq 0} \phi^l(z)\phi^k(w)\ \mathcal{L}_\phi.$$

Considering the Wold decompositions of $T_{\phi(z)}$ and $T_{\phi(w)}$ on $H^2(\mathbb{T}^2) \ominus \mathcal{K}_\phi$, similarly we obtain

$$H^2(\mathbb{T}^2) \ominus \mathcal{K}_\phi = \bigoplus_{l,k\geqq 0} \phi^l(z)\phi^k(w)\ \widehat{\mathcal{L}_\phi}.$$

Noting

$$ker\, T^*_{\phi(z)} \cap kerT^*_{\phi(w)} = \mathcal{L}_\phi \oplus \widehat{\mathcal{L}_\phi}$$

we have

$$\dim\left[ker\, T^*_{\phi(z)} \cap kerT^*_{\phi(w)}\right] = \dim\left[\mathcal{L}_\phi\right] + \dim\left[\widehat{\mathcal{L}_\phi}\right].$$

By Lemma (2.2.13), the dimension of $ker\, T^*_{\phi(z)} \cap kerT^*_{\phi(w)}$equals N^2. Hence

$$\dim\left[\mathcal{L}_\phi\right] = N^2 - \dim\left[\widehat{\mathcal{L}_\phi}\right].$$

To finish the proof, it suffices to show that the dimension of $\left[\widehat{\mathcal{L}_\phi}\right]$ equals $(N-1)^2$. By Lemma(2.2.14) , for a Blaschke product $\phi(z)$ with distinct zeros, the dimension of $\widehat{\mathcal{L}_\phi}$ equals $(N-1)^2$. We need to show that this is still true for a Blaschke product B with N zeros which perhaps contains some repeated zeros. To do so, for a given 1 $\lambda \in \mathbb{D}$, let $\phi_\lambda(z)$ be the Möbius transform $\frac{z-\lambda}{1-\bar{\lambda}z}$. Then $\phi_\lambda \circ \phi(z)$ is a Blaschke product with N zeros in the unit disk and

$$T_{\phi_\lambda\circ\phi(z)} = \left(T_{\phi(z)} - \lambda I\right)\left(I - \lambda T_{\phi(z)}\right)^{-1}.$$

Thus $\mathcal{K}_\phi = \mathcal{K}_{\phi_\lambda\circ\phi}$, and so

$$\widehat{\mathcal{L}_{\phi_\lambda\circ\phi}} = ker\, T^*_{\phi_\lambda\circ\phi(z)} \cap kerT^*_{\phi_\lambda\circ\phi(w)} \cap \left[H^2(T^2) \ominus \mathcal{K}_{\phi_\lambda\circ\phi}\right]$$
$$= ker\, T^*_{\phi(z)-\lambda} \cap kerT^*_{\phi(w)-\lambda} \cap \left[H^2(T^2) \ominus \mathcal{K}_\phi\right].$$

The last equality follows from that

$$ker\, T^*_{\phi(z)-\lambda} = ker\, T^*_{\phi_\lambda\circ\phi(z)}$$

and

$$ker\ T^*_{\phi(w)-\lambda} = kerT^*_{\phi_\lambda\circ\phi(w)}$$

We have the fact that

$$dim\widehat{\mathcal{L}_{\phi_\lambda\circ\phi}} = -\text{index}\left(T^*_{\phi(z)-\lambda}, T^*_{\phi(w)-\lambda}\right),$$

where $\text{index}\left(T^*_{\phi(z)-\lambda}, T^*_{\phi(w)-\lambda}\right)$ is the Fredholm index of the pair $\text{index}\left(T^*_{\phi(z)-\lambda}, T^*_{\phi(w)-\lambda}\right)$, which was first introduced in [78]. It was shown in [78] that the Fredholm index of the pair $\text{index}\left(T^*_{\phi(z)-\lambda}, T^*_{\phi(w)-\lambda}\right)$ is a continuous mapping from the set of the Fredholm tuples to the set of integers. Thus for a sufficiently small λ,

$$\text{index}\left(T^*_{\phi(z)-\lambda}, T^*_{\phi(w)-\lambda}\right) = \text{index}\left(T^*_{\phi(z)}, T^*_{\phi(w)}\right).$$

If λ is not in the critical values set $\mathcal{C} = \{\mu \in \mathbb{D}; \mu = \phi(z)\ and\ \phi'^{(z)} = 0\ for\ some\ z \in \mathbb{D}\}$ of ϕ, then $\phi_\lambda \circ \phi(z)$ is a Blaschke product with N distinct zeros in $\mathbb{D}$. In fact, Bochner's theorem implies that there are at most $N-1$ points in $\mathcal{C}$. In this case, by Lemma (2.2.14),

$$dim\widehat{\mathcal{L}_{\phi_\lambda\circ\phi}} = (N-1)^2$$

Since the set $\mathcal{C}$ has zero area, we conclude that the dimension of $\widehat{\mathcal{L}_\phi}$ equals $(N-1)^2$.
we will construct a family $\{d_e^k\}$ of functions and a function d_e^0 in $\mathcal{L}_\phi$ for each $e \in ker\ T^*_{\phi(z)} \cap kerT^*_{\phi(w)} \cap \mathcal{H}$, which have properties in Theorem (2.2.1) to present the proof of Theorem (2.2.1) that gives a relation between d_e^1 and d_e^0. The relation is very useful for us to understand the structure of the minimal reducing subspaces in the rest .

Let L_0 be $ker\ T^*_{\phi(z)} \cap kerT^*_{\phi(w)} \cap \mathcal{H}$. It is easy to check that the dimension of L_0 equals the order of the Blaschke product.

First we will show that for a given reducing subspace $\mathcal{M}$ for $\phi(\mathcal{B})$, for each $e \in \mathcal{M} \cap L_0$ and each integer $l \geqq 1$, there is a family of functions $\{d_e^k\}_{k=1}^l$ such that

$$p_l\left(\phi(z), \phi(w)\right)e + \sum_{k=0}^{l-1} p_k\left(\phi(z), \phi(w)\right)d_e^{l-k} \in \mathcal{M}.$$

These functions are useful in studying the structure of the multiplication operator M_ϕ on the Bergman space.
The following lemma shows that for each reducing subspace $\mathcal{M}$ of $\phi(\mathcal{B})$, the intersection of $\mathcal{M}$ and L_0 is nontrivial.
Lemma (2.2.16)[64]: If $\mathcal{M}$ is a nontrivial reducing subspace for $\phi(\mathcal{B})$, then the wandering sub-space of $\mathcal{M}$ is contained in L_0.
Proof. Let $\mathcal{M}$ be a nontrivial reducing subspace for $\phi(\mathcal{B})$. For each f in $\mathcal{H}$, $\mathcal{P}_\mathcal{M} f$ is in $\mathcal{M}$. Thus for each e in the wandering subspace $\mathcal{M} \ominus \phi(\mathcal{B})\mathcal{M} of\ \mathcal{M}$,

$$0 = \langle e, \phi(\mathcal{B})\mathcal{P}_\mathcal{M} f\rangle = \langle e, \mathcal{P}_\mathcal{M}\phi(\mathcal{B})f\rangle$$
$$= \langle e, \phi(\mathcal{B})f\rangle = \langle T^*_{\phi(z)}e, f\rangle.$$

The second equality follows from that $\mathcal{M}$ is a reducing subspace and the last equality follows from the fact that for each $f \in \mathcal{H}$,

$$\phi(\mathcal{B})^* f = T^*_{\phi(z)} f = T^*_{\phi(w)} f .$$

So $T^*_{\phi(z)}e = 0$. Similarily, we also have that $T^*_{\phi(w)}e = 0$. This gives that e is in L_0 to complete the proof.
Lemma (2.2.17)[64]: If $\mathcal{M}$ is a reducing subspace for $\phi(\mathcal{B})$, then $\phi(\mathcal{B})^*\mathcal{M} = \mathcal{M}$.
Proof. First note that for a Blaschke product $\phi(z)$ with finite order, $\phi(\mathcal{B})$ is Fredholm and the kernel of $\phi(\mathcal{B})$ contains only zero. Thus

$$\phi(\mathcal{B})^*\mathcal{H} = \mathcal{H}$$

Suppose that $\mathcal{M}$ is a reducing subspace for $\phi(\mathcal{B})$. Let $\mathcal{N} = \mathcal{M}^{\perp}$. Then

$$\phi(\mathcal{B})^* = \phi(\mathcal{B})^*|_{\mathcal{M}} \oplus \phi(\mathcal{B})^*|_{\mathcal{N}}$$

under the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$. Since $\phi(\mathcal{B})^*$ is subjective,

$$\phi(\mathcal{B})^*|_{\mathcal{M}}\mathcal{M} = \mathcal{M}.$$

This completes the proof.

Theorem (2.2.18)[64]: *Suppose that $\mathcal{M}$ is a reducing subspace for $\phi(\mathcal{B})$. For a given e in the wandering subspace of $\mathcal{M}$, there is a unique family of functions $\{d_e^k\} \subset \mathcal{L}_\phi \ominus L_0$ such that*:

$$p_l\left(\phi(z), \phi(w)\right)e + \textstyle\sum_{k=0}^{l-1} p_k\left(\phi(z), \phi(w)\right)d_e^{l-k} \text{ is in } \mathcal{M}, \text{ for each } l \geqq 1.$$

$$p_{\mathcal{H}}\left[p_l\left(\phi(z), \phi(w)\right)d_e^k\right] \text{ is in } \mathcal{M} \text{ for each } k \geqq 1., \text{ and } l \geqq 0.$$

Proof. For a given e in the wandering subspace of $\mathcal{M}$, first we will use mathematical induction to construct a family of functions $\{d_e^k\}$.

By Lemma (2.2.16), e is in L_0. A simple calculation gives $T_{\phi(z)}^*\left[\left(\phi(z) + \phi(w)\right)e\right] = e$, and $T_{\phi(w)}^*\left[\left(\phi(z) + \phi(w)\right)e\right] = e$. By Lemma (2.2.17), there is a unique function $\tilde{e} \in \ominus L_0$ such that

$$T_{\phi(z)}^*\tilde{e} = T_{\phi(w)}^*\tilde{e} = e.$$

This gives

$$T_{\phi(z)}^*\left[\tilde{e} - \left(\phi(z) + \phi(w)\right)e\right] = e - e = 0,$$

and

$$T_{\phi(w)}^*\left[\tilde{e} - \left(\phi(z) + \phi(w)\right)e\right] = e - e = 0,$$

to get that letting $d_e^1 = \tilde{e} - \left(\phi(z) + \phi(w)\right)e$, d_e^1 is in $ker\, T_{\phi(z)}^* \cap ker\, T_{\phi(w)}^*$, and

$$p_1\left(\phi(z), \phi(w)\right)e + d_e^1 = \left(\phi(z) + \phi(w)\right)e + d_e^1 \in \mathcal{M}.$$

Because both $\tilde{e}$ and e are in $\mathcal{M}$, we have that d_e^1 is in $\mathcal{K}_\phi$, and hence d_e^1 is in $\mathcal{L}_\phi$.

Next we show that d_e^1 is orthogonal to L_0. To do so, let $f \in L_0$. A simple calculation gives

$$\langle d_e^1, f\rangle = \langle \tilde{e} - \left(\phi(z) + \phi(w)\right)e, f\rangle = \langle \tilde{e}, f\rangle - \langle\left(\phi(z) + \phi(w)\right)e, f\rangle$$
$$= 0 - \langle e, T_{\phi(z)}^* f + T_{\phi(w)}^* f\rangle = 0.$$

The third equality follows from that $\tilde{e}$ is in $\mathcal{M} \ominus L_0$. This gives that d_e^1 is in $\mathcal{L}_\phi \ominus L_0$.

Assume that for $n < l$ there is a family of functions $\{d_e^k\}_{k=1}^n \subset \mathcal{L}_\phi \ominus L_0$ such that

$$p_n\left(\phi(z), \phi(w)\right)e + \sum_{k=0}^{n-1} p_k\left(\phi(z), \phi(w)\right)d_e^{n-k} \in \mathcal{M}.$$

Let $E = p_n\left(\phi(z), \phi(w)\right)e + \sum_{k=0}^{n-1} p_k\left(\phi(z), \phi(w)\right)d_e^{n-k}$. By Lemma (2.2.17) again, there is a unique function $\tilde{E}$ in $\mathcal{M} \ominus L_0$ such that

$$T_{\phi(z)}^*\tilde{E} = T_{\phi(w)}^*\tilde{E} = E$$

Let $F = p_{n+1}\left(\phi(z), \phi(w)\right)e + \sum_{k=1}^{n} p_k\left(\phi(z), \phi(w)\right)d_e^{n+1-k}$ Since

$$T_{\phi(z)}^*\left[p_k\left(\phi(z), \phi(w)\right)f\right] = T_{\phi(w)}^*\left[p_k\left(\phi(z), \phi(w)\right)f\right] = p_{k-1}\left(\phi(z), \phi(w)\right)f,$$

for each f in $\mathcal{L}_\phi$ and $k \geqq 1$, simple calculations give

$$T_{\phi(z)}^*F = T_{\phi(w)}^*F = E.$$

Thus

$$T_{\phi(z)}^*\left(\tilde{\mathrm{E}} - \mathrm{F}\right) = T_{\phi(w)}^*\left(\tilde{\mathrm{E}} - \mathrm{F}\right) = \mathrm{E} - \mathrm{E} = 0.$$

So letting $d_e^{n+1} = \widetilde{\mathrm{E}} - \mathrm{F}$, d_e^{n+1} is in $ker\ T^*_{\phi(z)} \cap ker\ T^*_{\phi(w)}$. Noting $\widetilde{\mathrm{E}}$ is orthogonal to L_0 , we have that for each $f \in L_0$,

$$\langle d_e^{n+1}, f\rangle = \langle \widetilde{\mathrm{E}}, f\rangle - \langle \mathrm{F}, f\rangle$$

$$= -\left[\langle p_{n+1}\big(\phi(z),\phi(w)\big)e, f\rangle + \sum_{k=1}^{n} \langle p_k\ \big(\phi(z),\phi(w)\big)d_e^{n+1-k}, f\rangle\right] = 0.$$

to get that d_e^{n+1} is in $\mathcal{L}_\phi \ominus L_0$. Hence

$$p_{n+1}\big(\phi(z),\phi(w)\big)e + \sum_{k=1}^{n} p_k\ \big(\phi(z),\phi(w)\big)d_e^{n+1-k} + d_e^{n+1} \in \mathcal{M}$$

This gives a family of functions $\{d_e^k\} \subset \mathcal{L}_\phi \ominus L_0$ satifying Property (2).

To finish the proof we only need to show that Property (3) holds. A simple calculation gives

$$2\phi(\mathcal{B})e\ = P_{\mathcal{H}}\big(p_1\big(\phi(z),\phi(w)\big)e\big) = P_{\mathcal{H}}\big(p_1\big(\phi(z),\phi(w)\big)e + d_e^1\big) - P_{\mathcal{H}}(d_e^1)$$

$$= p_1\big(\phi(z),\phi(w)\big)e + d_e^1 - P_{\mathcal{H}}(d_e^1).$$

This implies

$$P_{\mathcal{H}}(d_e^1) = \big[p_1\big(\phi(z),\phi(w)\big)e + d_e^1\big] - 2\phi(\mathcal{B})e \in \mathcal{M}.$$

Noting that $d_e^1 - P_{\mathcal{H}}d_e^1$ is in $\mathcal{H}^\perp = [z-w]$ and $[z-w]$ is an invariant subspace for analytic Toeplitz operators, we have that

$$\big[\mathrm{p}_{\mathrm{l}-1}\big(\phi(z),\phi(w)\big)\ (d_e^1 - P_{\mathcal{H}}d_e^1)\big] \in \mathcal{H}^\perp,$$

and so

$$P_{\mathcal{H}}\big[\mathrm{p}_{\mathrm{l}-1}\big(\phi(z),\phi(w)\big)\ (d_e^1 - P_{\mathcal{H}}d_e^1)\big] = 0,$$

to get

$$P_{\mathcal{H}}\big[p_1\big(\phi(z),\phi(w)\big)d_e^{n+1}\big]\ = P_{\mathcal{H}}\left\{\mathrm{p}_{\mathrm{l}-1}\big(\phi(z),\phi(w)\big)[P_{\mathcal{H}}d_e^1]\right\} \in \mathcal{M}.$$

Assume that $P_{\mathcal{H}}\big[p_1\big(\phi(z),\phi(w)\big)d_e^k\big] \in \mathcal{M}$ for $k \leqq n$ and any $l \geqq 0$. To finish the proof by induction we only need to show that

$$P_{\mathcal{H}}\big[p_1\big(\phi(z),\phi(w)\big)d_e^{n+1}\big] \in \mathcal{M}$$

for any $l \geqq 0$.

A simple calculation gives

$$(n+2)\phi(\mathcal{B})^{n+1}e$$

$$= P_{\mathcal{H}}\left[p_{n+1}\big(\phi(z),\phi(w)\big)e + \sum_{k=0}^{n} p_k\ \big(\phi(z),\phi(w)\big)d_e^{n+1-k}\right]$$

$$-\left\{P_{\mathcal{H}}[d_e^{n+1}] + P_{\mathcal{H}}\left[\sum_{k=1}^{n} p_k\ \big(\phi(z),\phi(w)\big)d_e^{n+1-k}\right]\right\}.$$

Thus

$$P_{\mathcal{H}}[d_e^{n+1}] = P_{\mathcal{H}}\left[p_{n+1}\big(\phi(z),\phi(w)\big)e + \sum_{k=0}^{n} p_k\ \big(\phi(z),\phi(w)\big)d_e^{n+1-k}\right]$$

$$-\left\{(n+2)\phi(\mathcal{B})^{n+1}e + P_{\mathcal{H}}\left[\sum_{k=1}^{n} p_k\ \big(\phi(z),\phi(w)\big)d_e^{n+1-k}\right]\right\}.$$

Property (2) gives that the first term in the last equality is $\mathcal{M}$, the induction hypothesis gives that the last term is in $\mathcal{M}$ and the second term belongs to $\mathcal{M}$ since $e \in \mathcal{M}$ and $\mathcal{M}$ is a reducing subspace for $\phi(\mathcal{B})$. So $P_{\mathcal{H}}[d_e^{n+1}]$ is in M. Therefore we conclude

$$P_{\mathcal{H}}\left[p_1\big(\phi(z),\phi(w)\big)d_e^{n+1}\right] = P_{\mathcal{H}}\left[p_1\big(\phi(z),\phi(w)\big)(P_{\mathcal{H}}d_e^{k+1})\right] \in \mathcal{M}$$

to complete the proof.

In the special case for $\mathcal{H}$, as $\mathcal{H}$ is a reducing subspace for $\phi(\mathcal{B})$, Theorem (2.2.18) immediately gives the following theorem.

Theorem (2.2.19) [64]: *For a given $e \in L_0$ there is a unique family of functions $\{d_e^k\} \subset \mathcal{L}_\phi \ominus L_0$ such that*

$$p_l\big(\phi(z),\phi(w)\big)e + \sum_{k=0}^{l-1} p_k\left(\phi(z),\phi(w)\right)d_e^{l-k} \in \mathcal{H}$$

for each $l \geqq 1$.

Next for a given $e \in L_0$, we will show that the function $d_e^0(z,w)$ given by

$$d_e^0(z,w) = we(0,w)e_0(z,w) - w\phi_0(w)e(z,w)$$

is in $\mathcal{L}_\phi$ and satisfies

$$p_l\big(\phi(z),\phi(w)\big)e + p_{l-1}\big(\phi(z),\phi(w)\big)d_e^0 \in \mathcal{H}$$

for each $l \geqq 1$.

Recall that ϕ is a Blaschke product with zeros $\{\alpha_k\}_0^K$ and α_k repeats $n_k + 1$ times, and $\phi(z) = z\phi_0(z)$ where $\phi_0$is a Blaschke product with $N-1$ zeros. Let $e_0 = \frac{\phi(z)-\phi(w)}{z-w}$. Theorem (2.2.9) gives that e_0 is in $\mathcal{H}$ since ϕ is a Blaschke product with finite order. This also gives that $e_0(z,0) = \phi_0(z)$.

Theorem (2.2.20)[64]: *Let ϕ be a nonzero function f in $\mathcal{H}$. $p_l\big(\phi(z),\phi(w)\big) f \in \mathcal{H}$, for some $l \geqq 1$ if and only if $f = \lambda e_0$ for some constant λ.*

The proof of Theorem (2.2.20) is left to the readers.

Theorem (2.2.20) gives that

$$\mathcal{M}_0 = \underset{l\geqq 1}{span}\{p_l\big(\phi(z),\phi(w)\big)e_0\}$$

is a reducing subspace of $\phi(\mathcal{B})$. We will study the space.

For each $e(z,w)$ in L_0, let

$$d_e^0(z,w) = we(0,w)e_0(z,w) - w\phi_0(w)e(z,w)$$

Theorem (2.2.21)[64]: *For each $e(z,w)$ in L_0, $d_e^0(z,w)$ is a function in $\mathcal{L}_\phi$ such that*

$$p_l\big(\phi(z),\phi(w)\big)e + p_{l-1}\big(\phi(z),\phi(w)\big)d_e^0 \in \mathcal{H}, \tag{4}$$

for $l \geqq 1$.

Proof. First we show that the function $d_e^0(z,w)$ is in $ker\ T_{\phi(z)}^* \cap ker\ T_{\phi(w)}^*$. To do this, by Theorem (2.2.9), write

$$e(z,w) = \frac{\phi_e(z) - \phi_e(w)}{z-w} \tag{5}$$

for some function ϕ_e in the Dirichlet space $\mathcal{D}$ with $\phi_e(0) = 0$. Letting $w = 0$ in the above equality gives that $e(z,0) = e(0,z) = \bar{z}\phi_e(z)$.

$$d_e^0(z,w) = we(0,w)e_0(z,w) - w\phi_0(w)e(z,w)$$
$$= \phi_e(w)\left[\frac{\phi(z)-\phi(w)}{z-w}\right] - \phi(w)\left[\frac{\phi_e(z)-\phi_e(w)}{z-w}\right]$$
$$= \frac{\phi_e(w)\phi(z)-\phi(w)\phi_e(z)}{z-w}.$$

This gives that $d_e^0(z,w)$ is a symmetric function of z and w. Since $e_0(z,w)$ and $e(z,w)$ are symmetric functions of z and w in L_0, we have

$$T^*_{\phi(z)}[d_e^0(z,w)] = T^*_{\phi(z)}[\mathrm{w}e(0,w)e_0(z,w) - w\phi_0(w)e(z,w)]$$
$$= \mathrm{w}e(0,w)T^*_{\phi(z)}[e_0(z,w)] - w\phi_0(w)T^*_{\phi(z)}e(z,w) = 0$$
$$T^*_{\phi(w)}[d_e^0(z,w)] = T^*_{\phi(w)}[d_e^0(w,z)] = T^*_{\phi(w)}[\mathrm{z}e(0,z)e_0(w,z) - z\phi_0(z)e(w,z)]$$
$$= \mathrm{z}e(0,z)T^*_{\phi(w)}[e_0(z,w)] - z\phi_0(z)T^*_{\phi(w)}e(z,w) = 0$$

to get that $d_e^0(z,w)$ is in $ker\ T^*_{\phi(z)} \cap ker\ T^*_{\phi(w)}$.

Next we show that $d_e^0(z,w)$ satisfies (4). To do this, let

$$E_l = p_l(\phi(z),\phi(w))e + p_{l-1}(\phi(z),\phi(w))d_e^0.$$

We show that

$$E_l = \frac{\phi_e(z)\phi^l(z) - \phi_e(w)\phi^l(w)}{z-w}.$$

By Theorem (2.2.9), this gives that E_l is in $\mathcal{H}$. Simple calculations give

$$p_l(\phi(z),\phi(w))e = \left[\frac{\phi^{l+1}(z)-\phi^{l+1}(w)}{\phi(z)-\phi(w)}\right]\left[\frac{\phi_e(z)-\phi_e(w)}{z-w}\right]$$
$$= \frac{\phi^{l+1}(z)\phi_e(z) - \phi^{l+1}(z)\phi_e(w) - \phi^{l+1}(w)\phi_e(z) + \phi^{l+1}(z)\phi_e(w)}{(\phi(z)-\phi(w))(z-w)}$$

and

$$p_{l-1}(\phi(z),\phi(w))d_e^0 = \left[\frac{\phi^l(z)-\phi^l(w)}{\phi(z)-\phi(w)}\right]\left[\frac{\phi_e(w)\phi(z)-\phi(w)\phi_e(z)}{z-w}\right]$$
$$= \frac{\phi^{l+1}(z)\phi_e(w) - \phi^l(z)\phi(w)\phi_e(z) - \phi^l(w)\phi(z)\phi_e(w) + \phi^{l+1}(w)\phi_e(z)}{(\phi(z)-\phi(w))(z-w)}$$

Thus

$$E_l = p_l(\phi(z),\phi(w))e + p_{l-1}(\phi(z),\phi(w))d_e^0$$
$$= \frac{\phi^l(z)\phi_e(z)(\phi(z)-\phi(w)) - \phi^l(w)\phi_e(w)(\phi(z)-\phi(w))}{(\phi(z)-\phi(w))(z-w)}$$
$$= \frac{\phi_e(z)\phi^l(z) - \phi_e(w)\phi^l(w)}{z-w}.$$

Since $p_1(\phi(z),\phi(w))e + d_e^0$ is in $\mathcal{H}$ and $p_1(\phi(z),\phi(w))e$ is in $\mathcal{K}_\phi$, we conclude that d_e^0 is in $\mathcal{K}_\phi$. Hence d_e^0 is in $\mathcal{L}_\phi$. This completes the proof.

Now we are ready to prove Theorem (2.2.1).

Since $\mathcal{M}$ is orthogonal to $\mathcal{M}_0$, we have

$$\mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_0^\perp = \mathcal{M}_0 \oplus \mathcal{M} \oplus [\mathcal{M}_0^\perp \cap \mathcal{M}^\perp].$$

Thus

$$L_0 = Ce_0 \oplus [\mathcal{M} \cap L_0] \oplus [\mathcal{M}_0^\perp \cap \mathcal{M}^\perp \cap L_0].$$

So e is orthogonal to e_0, and

$$L_0 \ominus e_0 = [\mathcal{M} \cap (L_0 \ominus e_0)] \oplus [\mathcal{M}_0^\perp \cap \mathcal{M}^\perp \cap (L_0 \ominus e_0)].$$

Let P_0 denote the orthogonal projection from $H^2(\mathbb{T}^2)$ onto the space Ce_0. Let $d_e = d_e^0 - P_0 d_e^0$. Then d_e is orthogonal to e_0. Theorems (2.2.20) and (2.2.21) give

$$p_l\big(\phi(z),\phi(w)\big)e + p_{l-1}\big(\phi(z),\phi(w)\big)d_e \in \mathcal{H}, \tag{6}$$

for $l \geqq 1$. By Theorem (2.2.19), there is a function $d_e^k \in \mathcal{L}_\phi \ominus L_0$ such that

$$p_l\big(\phi(z),\phi(w)\big)e + \sum_{k=0}^{l-1} p_k\left(\phi(z),\phi(w)\right)d_e^{l-k} \in \mathcal{M} ,$$

for each $l \geqq 1$. Thus

$$d_e - d_e^1 = p_1\big(\phi(z),\phi(w)\big)e + d_e - \big(p_1\big(\phi(z),\phi(w)\big)e + d_e^1\big) \in \mathcal{H}.$$

So $d_e - d_e^1$ is in $L_0 \ominus e_0$. Write

$$d_e - d_e^1 = e' + e''$$

for $e' \in \mathcal{M} \cap (L_0 \ominus e_0)$ and $e'' \in \mathcal{M}^\perp \cap (L_0 \ominus e_0)$. Thus (6) gives that the following function is in $\mathcal{H}$:

$$\begin{aligned} p_2\big(\phi(z),\phi(w)\big)e + p_1\big(\phi(z),\phi(w)\big)d_e = \\ \big[p_2\big(\phi(z),\phi(w)\big)e + p_1\big(\phi(z),\phi(w)\big)d_e^1 + d_e^2\big] + \big[p_1\big(\phi(z),\phi(w)\big)e' + d_{e'}^1\big] \\ + \big[p_1\big(\phi(z),\phi(w)\big)e'' + d_{e''}^1\big] - \big(d_e^2 + d_{e'}^1 + d_{e''}^1\big). \end{aligned}$$

Theorem (2.2.18) gives that the first term and the second term in the right-hand side are in $\mathcal{M}$ and the third term is in $\mathcal{M}^\perp$. Thus the last term must be in $\mathcal{H}$ and hence

$$d_e^2 + d_{e'}^1 + d_{e''}^1 \in \mathcal{H} \cap ker\, T_{\phi(z)}^* \cap T_{\phi(w)}^* = L_0$$

By Theorem (2.2.18) again, we have

$$d_e^2 + d_{e'}^1 + d_{e''}^1 \in \mathcal{L}_\phi \ominus L_0,$$

to get

$$d_e^2 + d_{e'}^1 + d_{e''}^1 = 0$$

This gives

$$P_{\mathcal{H}} d_{e''}^1 = -\left(P_{\mathcal{H}} d_{e'}^1 + P_{\mathcal{H}} d_e^2\right)$$

On the other hand, Theorem (2.2.18) gives $P_{\mathcal{H}} d_{e'}^1 + P_{\mathcal{H}} d_e^2$ is in $\mathcal{M}$ and $P_{\mathcal{H}} d_{e''}^1$ is in $\mathcal{M}^\perp$. Thus $P_{\mathcal{H}} d_{e''}^1 = 0$, and so simple calculations give

$$\begin{aligned} \left\|d_{e''}^1\right\|^2 &= \langle d_{e''}^1, d_{e''}^1\rangle = \langle d_{e''}^1, p_1\big(\phi(z),\phi(w)\big)e'' + d_{e''}^1\rangle \\ &= \langle d_{e''}^1, p_{\mathcal{H}}\big[p_1\big(\phi(z),\phi(w)\big)e'' + d_{e''}^1\big]\rangle \\ &= \langle p_{\mathcal{H}}\big(d_{e''}^1\big), p_1\big(\phi(z),\phi(w)\big)e'' + d_{e''}^1\rangle = 0. \end{aligned}$$

Hence we have that $d_{e''}^1 = 0$, to get

$$p_1\big(\phi(z),\phi(w)\big)e'' \in \mathcal{H}.$$

Theorem (2.2.20) gives that $e'' = \lambda e_0$, for some constant λ. Since $e'' \in \mathcal{M}^\perp \cap (L_0 \ominus e_0)$ we conclude that $e'' = 0$ to get $d_e = d_e^1 + e'$. Letting $\tilde{e} = -e' \in \mathcal{M}$, we obtain

$$d_e^1 = d_e + \tilde{e} = d_e^0 - P_0 d_e^0 + \tilde{e} = d_e^0 + \tilde{e} + \lambda e_0$$

as desired. The last equality follows from that $P_0 d_e^0 = \lambda e_0$ for some constant. This completes the proof.

Theorems (2.2.1) and (2.2.19) are useful in studying reducing subspaces of $\phi(\mathcal{B})$. we will use them to show that there always exists a unique reducing subspace $\mathcal{M}_0$ for $\phi(\mathcal{B})$ such that the restriction of $\phi(\mathcal{B})$ on $\mathcal{M}_0$ is unitarily equivalent to the Bergman shift. The existence of such a reducing subspace is the main result in [68]. Moreover, we will show that such reducing space is unique. We call $\mathcal{M}_0$ to be the distinguished reducing subspace for $\phi(\mathcal{B})$. In fact, $\mathcal{M}_0$ is unitarily equivalent the subspace $span\{\phi'\phi^m; m =$

$0,\ldots,n,\ldots\}$ of the Bergman space [27] if $\phi(0) = 0$. Furthermore we will show that only the multiplication operator by a finite Blaschke product has such nice property.

Assume that ϕ be a Blaschke product of order N with $\phi(0) = 0$. Recall $e_0(z,w) = \frac{\phi(z)-\phi(w)}{z-w}$. The following lemmas will be used in the proofs of Theorems (2.2.25) and 26. The proofs of those lemmas are left to the readers.

Lemma (2.2.22) [64]: *Let f be a function in $H^2(\mathbb{T}^2)$. Then*

$$p_{\mathcal{H}}\big[\phi(z)p_n\big(\phi(z),\phi(w)\big)f\big] = \frac{n+1}{n+2}p_{\mathcal{H}}\big[p_{n+1}\big(\phi(z),\phi(w)\big)f\big].$$

Lemma (2.2.23) [64]: *Let $\phi(z)$ be an inner function satisfying $\frac{\phi(z)-\phi(w)}{z-w} \in H^2(\mathbb{T}^2)$, then*

$$\frac{\phi(z)-\phi(w)}{z-w} \perp \phi(z)H^2(\mathbb{T}^2).$$

Lemma (2.2.24) [64]: *For an inner function $\phi(z)$, $\frac{\phi(z)-\phi(w)}{z-w}$ is in $H^2(\mathbb{T}^2)$ iff $\phi(z)$ is a finite Blaschke product. Moreover, for a Blaschke product f of order N,*

$$\|e_0\|^2 = N.$$

We show the first main result.

Theorem (2.2.25) [64]: *Let ϕ be a Blaschke product of order N. There is a unique reducing subspace $\mathcal{M}_0$ for $\phi(\mathcal{B})$ such that $\phi(\mathcal{B})|_{\mathcal{M}_0}$ is unitarily equivalent to the Bergman shift. In fact,*

$$\mathcal{M}_0 = \operatorname{span}_{l\geqq 0}\{p_l\big(\phi(z),\phi(w)\big)e_0\}, \qquad \textit{and} \quad \left\{\frac{p_l\big(\phi(z),\phi(w)\big)e_0}{\sqrt{l+1}\sqrt{N}}\right\}_0^\infty$$

form an orthonormal basis of $\mathcal{M}_0$.

Proof. First we show that there exists a reducing subspace $\mathcal{M}_0$ of $\phi(\mathcal{B})$ such that $\phi(\mathcal{B})|_{\mathcal{M}_0}$ is unitarily equivalent to the Bergman shift.

Let

$$\mathcal{M}_0 = \operatorname{span}_{l\geqq 0}\{p_l\big(\phi(z),\phi(w)\big)e_0\}$$

As pointed out before, Theorem (2.2.20) gives that $\mathcal{M}_0$ is a reducing subspace of $\phi(\mathcal{B})$. Here $e_0(z,w) = \frac{\phi(z)-\phi(w)}{z-w}$.

A simple calculation gives

$$\big\|p_l\big(\phi(z),\phi(w)\big)e_0\big\|_2^2 = (l+1)\|e_0\|_2^2,$$

and

$$\langle p_l\big(\phi(z),\phi(w)\big)e_0, p_n\big(\phi(z),\phi(w)\big)e_0\rangle = 0,$$

for $n \neq l$. Let $E_n = \frac{p_n(\phi(z),\phi(w))e_0}{\sqrt{n+1}\|e_0\|_2}$. Thus $\{E_n\}$ is an orthonormal basis of $\mathcal{M}_0$. By Lemma (2.2.22) we have

$$\begin{aligned}\phi(\mathcal{B})\big[p_n\big(\phi(z),\phi(w)\big)e_0\big] &= P_{\mathcal{H}}\big[\phi(z)p_n\big(\phi(z),\phi(w)\big)e_0\big]\\ &= P_{\mathcal{H}}\left[\frac{n+1}{n+2}p_{n+1}\big(\phi(z),\phi(w)\big)e_0\right] = \frac{n+1}{n+2}p_{n+1}\big(\phi(z),\phi(w)\big)e_0,\end{aligned}$$

to obtain

$$\phi(\mathcal{B})E_n = \frac{\phi(\mathcal{B})\big[p_n\big(\phi(z),\phi(w)\big)e_0\big]}{\sqrt{n+1}\|e_0\|_2} = \frac{n+1}{n+2}\frac{p_{n+1}\big(\phi(z),\phi(w)\big)e_0}{\sqrt{n+1}\|e_0\|_2} = \sqrt{\frac{n+1}{n+2}}E_{n+1}.$$

Clearly, $\phi(\mathcal{B})^* E_0 = 0$. This implies that $\phi(\mathcal{B})|_{\mathcal{M}_0}$ is unitarily equivalent to the Bergman shift.

Suppose that $\mathcal{M}_1$ is a reducing subspace of $\phi(\mathcal{B})$ and $\phi(\mathcal{M})|_{\mathcal{M}_1}$ to the Bergman shift, i.e., there is an orthonormal basis $\{F_n\}$ of $\mathcal{M}_1$ such that

$$\phi(\mathcal{B})F_n = \sqrt{\frac{n+1}{n+2}}E_{n+1}$$

Next we will show that $\mathcal{M}_1 = \mathcal{M}_0$. Observe

$$P_{\mathcal{H}}\big[\big(\phi(z)+\phi(w)\big)F_0\big] = 2\phi(\mathcal{B})F_0 = \frac{2}{\sqrt{2}}F_1.$$

Thus

$$\big\|P_{\mathcal{H}}\big[\big(\phi(z)+\phi(w)\big)F_0\big]\big\|^2 = 2.$$

Since

$$T^*_{\phi(z)}F_0 = \phi(\mathcal{B})^*F_0 = 0,$$

a simple calculation gives

$$\begin{aligned}\big\|\big(\phi(z)+\phi(w)\big)F_0\big\|^2 &= \langle\big(\phi(z)+\phi(w)\big)F_0, \big(\phi(z)+\phi(w)\big)F_0\rangle\\ &= \langle\phi(z)F_0,\phi(z)F_0\rangle + \langle\phi(w)F_0,\phi(w)F_0\rangle + \langle\phi(z)F_0,\phi(w)F_0\rangle\\ &\quad + \langle\phi(w)F_0,\phi(z)F_0\rangle = 2\langle F_0,F_0\rangle = 2.\end{aligned}$$

Thus we obtain

$$P_{\mathcal{H}^\perp}\big[\big(\phi(z)+\phi(w)\big)F_0\big] = 0$$

because

$$\big\|\big(\phi(z)+\phi(w)\big)F_0\big\|^2 = \big\|P_{\mathcal{H}}\big[\big(\phi(z)+\phi(w)\big)F_0\big]\big\|^2 + \big\|P_{\mathcal{H}^\perp}\big[\big(\phi(z)+\phi(w)\big)F_0\big]\big\|^2.$$

So $p_1\big(\phi(z),\phi(w)\big)F_0 = \big(\phi(z),\phi(w)\big)F_0$ is in $\mathcal{H}$. Theorem (2.2.20) gives that $F_0 = \lambda e_0$ for some constant λ. Thus $\mathcal{M}_0$ is a subspace of $\mathcal{M}_1$ but $\mathcal{M}_0$ is a reducing subspace of $\phi(\mathcal{B})|_{\mathcal{M}_1}$, which is unitarily equivalent to the Bergman shift. But the Bergman shift is irreducible. So we conclude that $\mathcal{M}_1 = \mathcal{M}_0$, to complete the proof.

For $\phi(z) \in H^\infty(\mathbb{D})$, let S_ϕ denote $P_{\mathcal{H}}M_\phi\big|_{\mathcal{H}}$. Then

$$U^*S_\phi U = M_\phi,$$

where M_ϕ is the multiplication operator on $L^2_a(\mathbb{D})$. Indeed, for each $g \in \mathcal{H}$ and any $z \in \mathbb{D}$, we have

$$\begin{aligned}\big(U^*S_\phi g\big)(z) &= \big(S_\phi g\big)(z,z) = (P_{\mathcal{H}}\phi g)(z,z) = (\phi g - P_{\mathcal{H}^\perp}\phi g)(z,z) = \phi(z)g(z,z)\\ &= \big(M_\phi U^* g\big)(z).\end{aligned}$$

The last equality follows from Lemma (2.2.8). This gives that $U^*S_\phi = M_\phi U$. Thus $U^*S_\phi U = M_\phi$.

Theorem (2.2.25) tells us that for each finite Blaschke product ϕ, M_ϕ has a unique the distinguished reducing subspace. The following theorem shows that only a multiplication operator by a finite Blaschke product has such property.

Theorem (2.2.26) [64]: *Let $\phi \in H^\infty(\mathbb{D})$. Then M_ϕ acting on $L^2_a(\mathbb{D})$ has the distinguished reducing subspace iff ϕ is a finite Blaschke product.*

Proof. We only need to prove that if M_ϕ has the distinguished reducing subspace, then ϕ is a finite Blaschke product. Now, assume M_ϕ has the distinguished reducing subspace $\mathcal{M}$ such that $M_\phi\big|_{\mathcal{M}}$ is unitarily equivalent to the Bergman shift M_z, that is, there exists a unitary operator $U: M \to L_a^2(\mathbb{D})$ such that $U^*M_zU = M_\phi\big|_{\mathcal{M}}$. Let $K_\lambda^{\mathcal{M}}$ be the reproducing kernel of $\mathcal{M}$ for $\lambda \in \mathbb{D}$. Clearly, $K_\lambda^{\mathcal{M}} \neq 0$, except for at most a countable set. Thus we have

$$\left|\langle \mathcal{M}_\phi K_\lambda^{\mathcal{M}}, K_\lambda^{\mathcal{M}}\rangle\right| = |\phi(\lambda)|\left\|K_\lambda^{\mathcal{M}}\right\|^2 = \left|\langle M_z U K_\lambda^{\mathcal{M}}, U K_\lambda^{\mathcal{M}}\rangle\right| \leqq \|M_z\| \left\|UK_\lambda^{\mathcal{M}}\right\|^2 \leqq \left\|K_\lambda^{\mathcal{M}}\right\|^2,$$

to get that $|\phi(\lambda)| \leqq 1$ except for at most a countable set. So $\|\phi\|_\infty \leqq 1$ Since S_ϕ acting on $\mathcal{H} = H^2(\mathbb{T}^2) \ominus [z-w]$ is unitarily equivalent to M_ϕ acting on $L_a^2(\mathbb{D})$ this means that S_ϕ, restricted on its corresponding reducing subspace $\mathcal{N}$, is unitarily equivalent to M_z acting on $L_a^2(\mathbb{D})$, that is, there exists a unitary operator $V : \mathcal{N} \to L_a^2(\mathbb{D})$ such that $V^*M_zV = S_\phi\big|_{\mathcal{N}}$. Set $e_n = V^*e_n'$, where $e_n' = \sqrt{n+1}z^n$ for $n = 0,1,\ldots$. Then $S_\phi^* e_0 = 0$, and hence $M_{\phi(z)}^* e_0 = 0$ and $M_{\phi(w)}^* e_0 = 0$, where $M_{\phi(z)}$ and $M_{\phi(w)}$ are the operators acting on $H^2(\mathbb{T}^2)$. Noticing $S_{\phi(z)} = S_{\phi(w)}$, we have

$$\left\|VS_{(\phi(z)+\phi(w))}e_0\right\|^2 = \|z+z\|^2 = 2,$$

to obtain

$$\langle \phi(z)e_0, \phi(w)e_0 \rangle = \langle M_{\phi(w)}^* e_0, M_{\phi(z)}^* e_0\rangle = 0.$$

Thus

$$\left\|\big(\phi(z) + \phi(w)\big)e_0\right\|^2 = \|\phi(z)e_0\|^2 + \|\phi(w)e_0\|^2 \leqq 2.$$

Since

$$2 = \left\|VS_{(\phi(z)+\phi(w))}e_0\right\|^2 = \left\|VP_{\mathcal{H}}\left(\phi(z) + \phi(w)\right)e_0\right\|^2 = \left\|P_{\mathcal{H}}\left(\phi(z) + \phi(w)\right)e_0\right\|^2,$$

we have

$$\big(\phi(z) + \phi(w)\big)e_0 \in \mathcal{H},$$

to obtain

$$e_0 = c\,\frac{\phi(z) - \phi(w)}{z - w}$$

for some constant c. This follows from Theorem (2.2.20).
On the other hand,

$$\left\|\big(\phi(z) + \phi(w)\big)e_0\right\|^2 = \|\phi(z)e_0\|^2 + \|\phi(w)e_0\|^2 \leqq 2.$$

As showed above, $\|\phi\|_\infty \leqq 1$. We have that $\|\phi(z)e_0\|^2 = 1$ to get

$$\int_{\mathbb{T}^2} (|\phi(z)|^2 - 1)|e_0|^2 dm_2 = 0.$$

Thus $|\phi(z)| = 1$ almost all on the unit circle and so ϕ is an inner function. Lemma (2.2.24) gives that ϕ is a finite Blaschke product. This completes the proof.
we will first show that every nontrivial minimal reducing subspace of $\phi(\mathcal{B})$ is orthogonal to the distinguished subspace $\mathcal{M}_0$ if it is other than $\mathcal{M}_0$. We will show the proof of Theorem (2.2.3).

Theorem (2.2.27) [64]: *Suppose that Ω is a nontrivial minimal reducing subspace for $\phi(\mathcal{B})$. If Ω does not equal $\mathcal{M}_0$ then Ω is a subspace of $\mathcal{M}_0^\perp$.*

Proof. By Lemma (2.2.16), there is a function e in $\Omega \cap L_0$ such that $e = \lambda e_0 + e_1$ for some constant λ and a function e_1 in $\mathcal{M}_0^\perp \cap L_0$. By Theorem (2.2.18),

$$p_1\big(\phi(z), \phi(w)\big)e + d_e^1 \in \Omega.$$

Here d_e^1 is the function constructed in Theorem (2.2.18). Let

$$\mathrm{E} = \phi(\mathcal{B})^*[\phi(\mathcal{B})\mathrm{e}] - \frac{1}{2}\mathrm{e}.$$

Since $p_1(\phi(z), \phi(w))e_0$ is in $\mathcal{H}$, we obtain

$$\phi(\mathcal{B})e_0 = \frac{p_1(\phi(z), \phi(w))e_0}{2}$$

Simple calculations give

$$E = \phi(\mathcal{B})^*\{\phi(\mathcal{B})[\lambda e_0 + e_1]\} - \frac{1}{2}[\lambda e_0 + e_1] = -\frac{1}{2}\phi(\mathcal{B})^* P_{\mathcal{H}} d_{e_1}^1.$$

The sixth equality holds because $p_1(\phi(z) + \phi(w))e_1 + d_{e_1}^1 \in \mathcal{H}$. The eighth equality follows from that $d_{e_1}^1$ is in $\mathcal{L}_\phi$. We claim that $E \neq 0$. If this is not true, we would have

$$\frac{1}{2}\phi(\mathcal{B})^* P_{\mathcal{H}} d_{e_1}^1 = 0$$

This gives that $P_{\mathcal{H}} d_{e_1}^1$ is in L_0. And hence

$$0 = \langle P_{\mathcal{H}} d_{e_1}^1, d_{e_1}^1 \rangle = \left\| d_{e_1}^1 \right\|^2.$$

This gives that $d_{e_1}^1 = 0$. Thus we obtain that $p_1(\phi(z) + \phi(w))e_1 \in \mathcal{H}$. By Theorem (2.2.20), we get that e_1 is linearly dependent on e_0. This contradicts that $e_1 \in \mathcal{M}_0^\perp$. By Theorem (2.2.18), $P_{\mathcal{H}} d_{e_1}^1$ is in $\mathcal{M}$ and so is $E = -\frac{1}{2}\phi(\mathcal{B})^* P_{\mathcal{H}} d_{e_1}^1$. This implies that E is in $\Omega \cap \mathcal{M}_0^\perp$. We conclude that $\Omega \cap \mathcal{M}_0^\perp = \Omega$ since Ω is minimal to complete the proof.

Lemma (2.2.28) [64]: *If $\mathcal{M}$ and $\mathcal{N}$ are two mutually orthogonal reducing subspaces of $\phi(\mathcal{B})$, then $\widetilde{\mathcal{M}}$ is orthogonal to $\widetilde{\mathcal{N}}$.*

Proof. Let $f = \sum_{l,k \geqq 0} \phi(z)^l \phi(w)^k m_{lk}$ and $g = \sum_{l,k \geqq 0} \phi(z)^l \phi(w)^k \mathrm{n}_{lk}$ for finite numbers of elements $\mathrm{m}_{\mathrm{lk}} \in \mathcal{M}$ and $\mathrm{n}_{\mathrm{lk}} \in \mathcal{N}$. Then

$$\langle f, g \rangle = \langle \sum_{l,k \geqq 0} \phi(z)^l \phi(w)^k m_{lk}, \sum_{l,k \geqq 0} \phi(z)^l \phi(w)^k n_{lk} \rangle = \sum_{l,k \geqq 0} \sum_{l_1,k_1 \geqq 0} \langle \phi(z)^{l-l_1} \phi(w)^{k-k_1} m_{lk}, n_{lk} \rangle.$$

Since $\mathcal{M}$ is orthogonal to $\mathcal{N}$ and both $\mathcal{M}$ and $\mathcal{N}$ are invariant subspaces of $T_{\phi(z)}^*$ and $T_{\phi(w)}^*$, the above inner product $\langle f, g \rangle$ must be zero. Thus we conclude that $\widetilde{\mathcal{M}}$ is orthogonal to $\widetilde{\mathcal{N}}$ to complete the proof.

Suppose that $\{e_1^{(M)}, \dots, e_{qM}^{(M)}\}$ forms a basis of $\mathcal{M} \cap L_0$. First we show

$$span\left\{e_1^{(M)}, \dots, e_{qM}^{(M)}; d_{e_1^{(M)}}^1, \dots, d_{e_{qM}^{(M)}}^1\right\} \subset L_{\mathcal{M}}.$$

Note that $\{e_1^{(M)}, \dots, e_{qM}^{(M)}; d_{e_1^{(M)}}^1, \dots, d_{e_{qM}^{(M)}}^1\}$ are contained in $\mathcal{L}_\phi$. It suffices to show

$$\left\{e_1^{(M)}, \dots, e_{qM}^{(M)}; d_{e_1^{(M)}}^1, \dots, d_{e_{qM}^{(M)}}^1\right\} \subset \widetilde{\mathcal{M}}.$$

Since $\mathcal{M} \cap L_0$ contains $\{e_1^{(M)}, \dots, e_{qM}^{(M)}\}$ for each $l, k \geqq 0, \phi(z)^l \phi^k(w) e_i^{(M)}$ is in $\widetilde{\mathcal{M}}$. By Theorem (2.2.18), we have

$$p_1(\phi(z), \phi(w))e_i^{(M)} + d_{e_i^{(M)}}^1 \in \mathcal{M}.$$

So we have that $d_{e_i^{(M)}}^1 \in \widetilde{\mathcal{M}}$, to obtain

$$span\left\{e_1^{(M)},\ldots,e_{qM}^{(M)};d^1_{e_1^{(M)}},\ldots,d^1_{e_{qM}^{(M)}}\right\}\subset L_{\widetilde{\mathcal{M}}}.$$

Next we will show that $\{e_1^{(M)},\ldots,e_{qM}^{(M)};d^1_{e_1^{(M)}},\ldots,d^1_{e_{qM}^{(M)}}\}$ are linearly independent. Suppose that for some constants λ_i and μ_i,

$$\sum_{i=1}^{q}\lambda_i e_i^{(M)}+\sum_{i=1}^{q}\mu_i d^1_{e_i^{(M)}}=0.$$

Thus

$$\sum_{i=1}^{q}\lambda_i e_i^{(M)}=-\sum_{i=1}^{q}\mu_i d^1_{e_i^{(M)}}.$$

The right-hand side of the above equality is in L_0 but the left-hand side of the equality is orthogonal to L_0. So we have

$$\sum_{i=1}^{q}\lambda_i e_i^{(M)}=0$$

and

$$\sum_{i=1}^{q}\mu_i d^1_{e_i^{(M)}}=0$$

The first equality gives that $\lambda_i=0$ and the second equality gives

$$d^1_{\sum_{i=1}^{q}\mu_i e_i^{(M)}}=0.$$

Because $\mathcal{M}$ is orthogonal to $\mathcal{M}_0$, by Theorem (2.2.20), we have

$$\sum_{i=1}^{q}\mu_i e_i^{(M)}=0$$

This gives that $\mu_i=0$. Hence $\{e_1^{(M)},\ldots,e_{qM}^{(M)};d^1_{e_1^{(M)}},\ldots,d^1_{e_{qM}^{(M)}}\}$ are linearly independent. So far, we have obtained

$$dim\,L_{\widetilde{\mathcal{M}}}\geqq 2qM.$$

To finish the proof, we only need to show that

$$dim\,L_{\widetilde{\mathcal{M}}}\geqq 2qM.$$

To do so, we consider the decomposition of $\mathcal{H}$,

$$\mathcal{H}=\mathcal{M}_0\oplus\mathcal{M}\oplus[\mathcal{M}_0^{\perp}\cap\mathcal{M}^{\perp}],$$

and

$$L_0=[\mathcal{M}_0\cap L_0]\oplus[\mathcal{M}\cap L_0]\oplus\{[\mathcal{M}_0^{\perp}\cap\mathcal{M}^{\perp}]\cap L_0\}.$$

Then

$$dim\{[\mathcal{M}_0^{\perp}\cap\mathcal{M}^{\perp}]\cap L_0\}=dimL_0-dim[\mathcal{M}_0\cap\ L_0]-\ [\mathcal{M}\cap\ L_0]=N-1-qM.$$

Letting $=[\mathcal{M}_0^{\perp}\cap\mathcal{M}^{\perp}]$, Lemma (2.2.28) gives

$$\mathcal{K}=\widetilde{\mathcal{M}}_0\oplus\widetilde{\mathcal{M}}\oplus\widetilde{\mathcal{N}},$$

and

$$\mathcal{L}_\phi=L_{\widetilde{\mathcal{M}}_0}\oplus L_{\widetilde{\mathcal{M}}}\oplus L_{\widetilde{\mathcal{N}}},$$

Replacing $\mathcal{M}$ by $\mathcal{N}$ in the above argument gives

$$dim\ L_{\widetilde{\mathcal{N}}} \geqq 2(N-1-qM).$$

By Theorem (2.2.15), so we have

$$2N-1 = 1 + dim[L_{\widetilde{\mathcal{M}}}] + dim[L_{\widetilde{\mathcal{N}}}].$$

Hence

$$\dim[L_{\widetilde{\mathcal{M}}}] = 2\mathrm{N} - 2 - \dim[L_{\widetilde{\mathcal{N}}}] \leqq 2\mathrm{N} - 2 - 2(N-1-qM) = 2\mathrm{qM}$$

This completes the proof.

Lemma (2.2.29) [64]: *Suppose that $\mathcal{M}$, $\mathcal{N}$, and Ω are three distinct nontrivial minimal reducing subspaces of $\phi(\mathcal{B})$ such that*

$$\Omega \subset \mathcal{M} \oplus \mathcal{N}$$

If $\mathcal{M}$, $\mathcal{N}$, and Ω are orthogonal to $\mathcal{M}_0$, then

$$\widetilde{\mathcal{M}} \cap \widetilde{\Omega} = \widetilde{\mathcal{N}} \cap \widetilde{\Omega} = \{0\}.$$

Proof. Since the intersection $\widetilde{\mathcal{M}} \cap \widetilde{\Omega}$ is also a reducing subspace of the pair of isometries $T^*_{\phi(z)}$ and $T^*_{\phi(w)}$, the Wold decomposition of the pair of the pair of isometries on $\widetilde{\mathcal{M}} \cap \widetilde{\Omega}$ gives

$$\widetilde{\mathcal{M}} \cap \widetilde{\Omega} = \bigoplus_{l,k \geqq 0} \phi(z)^l \phi(w)^k L_{\widetilde{\mathcal{M}} \cap \widetilde{\Omega}},$$

where $L_{\widetilde{\mathcal{M}} \cap \widetilde{\Omega}}$ is the wandering space given by

$$\begin{aligned} \mathrm{L}_{\widetilde{\mathcal{M}} \cap \widetilde{\Omega}} &= \ker T^*_{\phi(z)} \cap T^*_{\phi(w)} \cap \widetilde{\mathcal{M}} \cap \widetilde{\Omega} \\ &= \left[\ker T^*_{\phi(z)} \cap T^*_{\phi(w)} \cap \widetilde{\mathcal{M}}\right] \cap \left[\ker T^*_{\phi(z)} \cap T^*_{\phi(w)} \cap \widetilde{\Omega}\right] = L_{\widetilde{\mathcal{M}}} \cap L_{\widetilde{\Omega}}. \end{aligned}$$

To prove that $\widetilde{\mathcal{M}} \cap \widetilde{\Omega} = \{0\}$, it suffices to show

$$L_{\widetilde{\mathcal{M}}} \cap L_{\widetilde{\Omega}} = \{0\}.$$

To do this, let $q \in L_{\widetilde{\mathcal{M}}} \cap L_{\widetilde{\Omega}}$. By Theorem (2.2.2), there are functions $e_M, \tilde{e}_M \in \mathcal{M} \cap L_0$ and $e_\Omega, \tilde{e}_\Omega \in \Omega \cap L_0$ such that

$$q = e_M + d^1_{\tilde{e}_M} = e_\Omega + d^1_{\tilde{e}_\Omega}$$

The above two equalities give

$$e_M - e_\Omega = d^1_{\tilde{e}_M - \tilde{e}_\Omega}.$$

On the other hand, $d^1_{\tilde{e}_M - \tilde{e}_\Omega}$ is orthogonal to L_0. Thus

$$d^1_{\tilde{e}_M - \tilde{e}_\Omega} = e_M - e_\Omega = 0.$$

This gives

$$e_M = e_\Omega.$$

But e_M is in $\mathcal{M}$ and e_Ω is in Ω and hence both e_M and e_Ω are zero. Since $d^1_{\tilde{e}_M - \tilde{e}_\Omega} = 0$, Theorem (2.2.20) implies that $\tilde{e}_M - \tilde{e}_\Omega$ linearly depends on e_0. Since both $\mathcal{M}$ and Ω are orthogonal to $\mathcal{M}_0$, we have that $\tilde{e}_M = \tilde{e}_\Omega$. Thus we obtain $\tilde{e}_M = 0$ to conclude that $q = 0$, as desired. So

$$\widetilde{\mathcal{M}} \cap \widetilde{\Omega} = \{0\}$$

Similarly we obtain

$$\widetilde{\mathcal{N}} \cap \widetilde{\Omega} = \{0\}$$

Lemma (2.2.30) [64]: *Suppose that $\mathcal{M}$, $\mathcal{N}$, and Ω are three distinct nontrivial minimal reducing subspaces of $\phi(\mathcal{B})$ such that*

$$\Omega \subset \mathcal{M} \oplus \mathcal{N}$$

If $\mathcal{M}$, $\mathcal{N}$, and Ω are orthogonal to $\mathcal{M}_0$, then

$$P_{\widetilde{\mathcal{M}}} L_{\widetilde{\Omega}} = L_{\widetilde{M}},$$

and

$$P_{\widetilde{\mathcal{N}}} L_{\widetilde{\Omega}} = L_{\widetilde{N}},$$

where $P_{\widetilde{\mathcal{M}}}$ denotes the orthogonal projection from $H^2(\mathbb{T}^2)$ onto $\widetilde{\mathcal{M}}$.

Proof. Since $\mathcal{M}$ is orthogonal to $\mathcal{N}$, Lemma (2.2.28) gives that $\widetilde{\mathcal{M}}$ is orthogonal to $\widetilde{\mathcal{N}}$ and

$$\widetilde{\Omega} \subset \widetilde{\mathcal{M}} \oplus \widetilde{\mathcal{N}}.$$

We will show that $P_{\widetilde{\mathcal{M}}} L_{\widetilde{\Omega}} = L_{\widetilde{M}}$.

Since $\Omega \subset \mathcal{M} \oplus \mathcal{N}$, we have

$$\Omega \cap L_0 \subset [\mathcal{M} \cap L_0] \oplus [\mathcal{N} \cap L_0].$$

For each $e^{(\Omega)} \in \Omega \cap L_0$, there are two functions $e^{(\mathrm{M})} \in \mathcal{M} \cap L_0$ and $e^{(\mathrm{N})} \in \mathcal{N} \cap L_0$ such that

$$e^{(\Omega)} = e^{(\mathrm{M})} + e^{(\mathrm{N})},$$

$$d^1_{e^{(\Omega)}} = d^1_{e^{(\mathrm{M})}} + d^1_{e^{(\mathrm{N})}}.$$

By Theorem (2.2.2), $d^1_{e^{(\mathrm{M})}}$ is in $\widetilde{\mathcal{M}}$ and $d^1_{e^{(\mathrm{N})}}$ is in $\widetilde{\mathcal{N}}$. Since $\mathcal{M}$, $\mathcal{N}$, and Ω are orthogonal to $\mathcal{M}_0$, the above decompositions are unique. Thus

$$\mathrm{P}_{\widetilde{\mathcal{M}}} e^{(\Omega)} = e^{(\mathrm{M})},$$

and

$$P_{\widetilde{\mathcal{M}}} d^1_{e^{(\Omega)}} = d^1_{e^{(\mathrm{M})}}.$$

So for each $\mathrm{f} = e^{(\Omega)} + d^1_{\tilde{e}^{(\Omega)}} \in L_{\widetilde{\Omega}}$, where $e^{(\Omega)}$ and $\tilde{e}^{(\Omega)}$, we have

$$P_{\widetilde{\mathcal{M}}} f = e^{(\mathrm{M})} + d^1_{e^{(\mathrm{M})}}.$$

is in $L_{\widetilde{\mathrm{M}}}$ to obtain

$$P_{\widetilde{\mathcal{M}}} L_{\widetilde{\Omega}} \subset L_{\widetilde{\mathcal{M}}}.$$

To prove that $P_{\widetilde{\mathcal{M}}} L_{\widetilde{\Omega}} \subset L_{\widetilde{\mathcal{M}}}$, it suffices to show that

$$P_{\widetilde{\mathcal{M}}} : L_{\widetilde{\Omega}} \to L_{\widetilde{\mathcal{M}}}$$

is subjective. If this is not so, by Theorem (2.2.2), there are two functions $e, \tilde{e} \in \mathcal{M} \cap L_0$ such that $0 \neq e + d^1_{\tilde{e}}$ is orthogonal to $P_{\widetilde{\mathcal{M}}} L_{\widetilde{\Omega}}$.

Assume that $\{e_1, \dots, e_{q\Omega}\}$ is a basis of $\Omega \cap L_0$. Then

$$P_{\widetilde{\mathcal{M}}} L_{\widetilde{\Omega}} = span\, \{e_1^{(M)}, \dots, e_{q\Omega}^{(M)}; d^1_{e_1^{(M)}}, \dots, d^1_{e_{q\Omega}^{(M)}}\}$$

If $e \neq 0$, then $\langle e, e_{\mathrm{i}}^{(M)} \rangle = 0$, for $1 \leqq i \leqq q\Omega$. Thus

$$0 = \langle e, e_i^{(M)} \rangle = \langle e, e_i^{(M)} + e_i^{(N)} \rangle = \langle e, e_i \rangle,$$

and

$$\langle e, d^1_{e_i} \rangle = 0,$$

For each $1 \leqq \mathrm{i} \leqq \mathrm{q}\Omega$. So e is orthogonal to $\mathrm{L}_{\widetilde{\Omega}} = \mathrm{span}\{\mathrm{e}_1, \dots, \mathrm{e}_{\mathrm{q}\Omega}; \mathrm{d}^1_{\mathrm{e}_1}, \dots, \mathrm{d}^1_{\mathrm{e}_{\mathrm{q}\Omega}}\}$. Noting e is in L_0, we see that e is orthogonal to $\phi(z)^l \phi(w)^k L_{\widetilde{\Omega}}$, for each $l > 0$ or $k > 0$. This gives that e is orthogonal to $\widetilde{\Omega}$ and hence orthogonal to Ω. Since e is in $\mathcal{M}$, e must be orthogonal to the closure of $P_{\mathcal{M}}\Omega \in \mathcal{M}$, which is also a reducing subspace of $\phi(\mathcal{B})$. Therefore e is orthogonal to $\mathcal{M}$, which is a contradiction.

If $e = 0$, then $d^1_{\tilde{e}} \neq 0$ and

$$0 = \langle d^1_{\tilde{e}}, d^1_{e_i^{(M)}} \rangle = \langle d^1_{\tilde{e}}, P_{\widetilde{\mathcal{M}}} d^1_{e_i} \rangle.$$

and

$$\langle d^1_{\tilde{e}}, e_i \rangle = 0$$

for each $1 \leqq i \leqq q\Omega$. This gives that $d^1_{\tilde{e}}$ is orthogonal to $L_{\widetilde{\Omega}}$. But $d^1_{\tilde{e}}$ is also in $\mathcal{L}_\phi$. We have that for any $\in L_{\widetilde{\Omega}}$,

$$\langle d^1_{\tilde{e}}, \phi(z)^l \phi(w)^k f \rangle = 0,$$

for $l > 0$ or $k > 0$. We have that $d^1_{\tilde{e}}$ is orthogonal to $\widetilde{\Omega}$ and hence orthogonal to Ω to obtain that $P_{\mathcal{H}} d^1_{\tilde{e}}$ is orthogonal to Ω. On the other hand, by Theorem (2.2.18), $P_{\mathcal{H}} d^1_{\tilde{e}}$ is in $\mathcal{M}$. Thus $P_{\mathcal{H}} d^1_{\tilde{e}}$ is orthogonal to the closure of $P_{\mathcal{M}}\Omega$ and so $P_{\mathcal{H}} d^1_{\tilde{e}}$ must be zero because the closure of $P_{\mathcal{M}}\Omega$ equals $\mathcal{M}$. Therefore,

$$0 = \langle P_{\mathcal{H}} d^1_{\tilde{e}}, p_1(\phi(z), \phi(w))\tilde{e} + d^1_{\tilde{e}} \rangle = \langle d^1_{\tilde{e}}, p_1(\phi(z), \phi(w))\tilde{e} + d^1_{\tilde{e}} \rangle = \langle d^1_{\tilde{e}}, d^1_{\tilde{e}} \rangle = \left\| d^1_{\tilde{e}} \right\|^2.$$

The second equality follows from that $p_1(\phi(z), \phi(w))\tilde{e} + d^1_{\tilde{e}}$ is in $\mathcal{H}$ and the third equality follows from that $d^1_{\tilde{e}}$ is orthogonal to $p_1(\phi(z), \phi(w))\tilde{e}$. This gives that $d^1_{\tilde{e}} = 0$, which is a contradiction. We have obtained that $P_{\widetilde{\mathcal{M}}}: L_{\widetilde{\Omega}} \to L_{\widetilde{\mathcal{M}}}$ is subjective and hence

$$P_{\widetilde{\mathcal{M}}} L_{\widetilde{\Omega}} = L_{\widetilde{\mathcal{M}}}.$$

Similarly we obtain

$$P_{\widetilde{\mathcal{N}}} L_{\widetilde{\Omega}} = L_{\widetilde{\mathcal{N}}}.$$

This completes the proof.

First we will show

$$P_{\mathcal{M}} = P_{\mathcal{H}} P_{\widetilde{\mathcal{M}}}.$$

Let $\mathcal{N}_1$ denote the orthogonal complementary of $\mathcal{M} \oplus \mathcal{N}$ in $\mathcal{H}$. Write

$$\mathcal{H} = \mathcal{M} \oplus \mathcal{N} \oplus \mathcal{N}_1$$

Lemma (2.2.28) gives

$$\widetilde{\mathcal{H}} = \widetilde{\mathcal{M}} \oplus \widetilde{\mathcal{N}} \oplus \widetilde{\mathcal{N}}_1.$$

For each function f in $H^2(\mathbb{T}^2)$, write

$$f = f_{\widetilde{\mathcal{H}}} \oplus f_2 = f_{\widetilde{\mathcal{M}}} \oplus f_{\widetilde{\mathcal{N}}} \oplus f_{\widetilde{\mathcal{N}}_1} \oplus f_2,$$

where f_2 is orthogonal to $\widetilde{\mathcal{H}}, f_{\widetilde{\mathcal{H}}} \in \widetilde{\mathcal{H}}, f_{\widetilde{\mathcal{M}}} \in \widetilde{\mathcal{M}}, f_{\widetilde{\mathcal{N}}} \in \widetilde{\mathcal{N}}$, and $f_{\widetilde{\mathcal{N}}_1} \in \widetilde{\mathcal{N}}_1$. Since $\widetilde{\mathcal{M}}$ contains $\mathcal{M}$, we write

$$f_{\widetilde{\mathcal{M}}} = f_{\mathcal{M}} \oplus f_3,$$

for two functions $f_{\mathcal{M}} \in \mathcal{M}$ and $f_3 \in \widetilde{\mathcal{M}} \ominus \mathcal{M}$. Thus f_3 is orthogonal to both $\widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{N}}_1$ and hence orthogonal to both $\mathcal{N}$ and $\mathcal{N}_1$. So f_3 is orthogonal to

$$\mathcal{H} = \mathcal{M} \oplus \mathcal{N} \oplus \mathcal{N}_1.$$

This gives that $P_{\mathcal{H}} f_3 = 0$. We have

$$P_{\mathcal{H}} P_{\widetilde{\mathcal{M}}} f = P_{\mathcal{H}} f_{\widetilde{\mathcal{M}}} = P_{\mathcal{H}} f_{\mathcal{M}} + P_{\mathcal{H}} f_3 = P_{\mathcal{H}} f_{\mathcal{M}} = f_{\mathcal{M}},$$

and

$$P_{\mathcal{M}} f = f_{\mathcal{M}},$$

to get

$$P_{\mathcal{M}} = P_{\mathcal{H}} P_{\widetilde{\mathcal{M}}}.$$

Next we will show that $P_{\mathcal{M}}$ is subjective from Ω onto $\mathcal{M}$. For each $q \in \mathcal{M}$, by Lemma (2.2.30), there are functions $q_{lk} \in L_{\widetilde{\Omega}}$ such that

$$q = \sum_{l,k \geqq 0} \phi(z)^l \phi(w)^k m_{lk}\,,$$

and

$$\|q\|^2 = \sum_{l,k \geqq 0} \|m_{lk}\|^2 \ < \infty,$$

where $m_{lk} = P_{\widetilde{\mathcal{M}}} q_{lk}$. Since $L_{\widetilde{\Omega}}$ and $L_{\widetilde{\mathrm{M}}}$ are finite dimension spaces, there are two positive constants c_1 and c_2 such that

$$c_1\|q_{lk}\| \leqq \|m_{lk}\| \leqq c_2\|q_{lk}\|.$$

Define

$$\tilde{q} = \sum_{l,k \geqq 0} \phi(z)^k \phi(w)^l q_{lk}\,.$$

Thus

$$\|\tilde{q}\|^2 = \sum_{l,k \geqq 0} \|q_{lk}\|^2 \leqq c_2 \sum_{l,k \geqq 0} \|m_{lk}\|^2 < \infty.$$

So we obtain that $\tilde{q}$ is in $\widetilde{\Omega}$, and

$$\tilde{q} = \sum_{l,k \geqq 0} \phi(z)^l \phi(w)^k q_{lk} = \sum_{l,k \geqq 0} \phi(z)^l \phi(w)^k \left[P_{\widetilde{\mathcal{M}}} q_{lk} + P_{\widetilde{\mathcal{N}}} q_{lk}\right]$$

$$= \sum_{l,k \geqq 0} \phi(z)^l \phi(w)^k\, m_{lk} + \sum_{l,k \geqq 0} \phi(z)^l \phi(w)^k \left[P_{\widetilde{\mathcal{N}}} q_{lk}\right] = q + q_N,$$

where $q_N = \sum_{l,k \geqq 0} \phi(z)^k \phi(w)^l \left[P_{\widetilde{\mathcal{N}}} q_{lk}\right]$ is in $\widetilde{\mathcal{N}}$. Hence $P_{\widetilde{\mathcal{M}}}\tilde{q} = q$. We have

$$P_{\mathcal{H}} P_{\widetilde{\mathcal{M}}} \tilde{q} = P_{\mathcal{H}} q = q,$$

to obtain

$$P_{\mathcal{M}} \tilde{q} = P_{\mathcal{H}} P_{\widetilde{\mathcal{M}}} \tilde{q} = \mathrm{q},$$

Since $\mathcal{M}$ is a subspace of $\mathcal{H}$, $P_{\mathcal{M}} = P_{\mathcal{M}} P_{\mathcal{H}}$. Thus

$$P_{\mathcal{M}} P_{\mathcal{H}} \tilde{q} \ = P_{\mathcal{M}} \tilde{q} = \mathrm{q}.$$

Writing $q_{lk} = \ \mathrm{e}_{\mathrm{kl}}^{(\Omega)} + \mathrm{d}^1_{\tilde{\mathrm{e}}_{\mathrm{kl}}^{(\Omega)}}$. for functions $\mathrm{e}_{\mathrm{kl}}^{(\Omega)}, \tilde{\mathrm{e}}_{\mathrm{kl}}^{(\Omega)} \in \Omega \cap \mathrm{L}_0$, we have

$$P_{\mathcal{H}} \tilde{q} \ = \sum_{l,k \geqq 0} P_{\mathcal{H}}(\phi(z)^l \phi(w)^k q_{lk}) = \sum_{l,k \geqq 0} P_{\mathcal{H}} \phi(z)^l \phi(w)^k \left(\mathrm{e}_{\mathrm{kl}}^{(\Omega)} + \mathrm{d}^1_{\tilde{\mathrm{e}}_{\mathrm{kl}}^{(\Omega)}}\right)$$

$$= \sum_{l,k \geqq 0} \left(P_{\mathcal{H}} \phi(z)^l \phi(w)^k \mathrm{e}_{\mathrm{kl}}^{(\Omega)}\right) + \sum_{l,k \geqq 0} \left(P_{\mathcal{H}} \phi(z)^l \phi(w)^k \mathrm{d}^1_{\tilde{\mathrm{e}}_{\mathrm{kl}}^{(\Omega)}}\right)$$

$$= \sum_{l,k \geqq 0} \left(P_{\mathcal{H}} \phi(z)^l \phi(w)^k \mathrm{e}_{\mathrm{kl}}^{(\Omega)}\right) + \sum_{l,k \geqq 0} \left[P_{\mathcal{H}} \phi(z)^l \phi(w)^k \left(P_{\mathcal{H}} \mathrm{d}^1_{\tilde{\mathrm{e}}_{\mathrm{kl}}^{(\Omega)}}\right)\right].$$

The last equality follows from that $(z)^l \phi(w)^k (1 - P_{\mathcal{H}}) \mathrm{d}^1_{\tilde{\mathrm{e}}_{\mathrm{kl}}^{(\Omega)}}$ is orthogonal to $\mathcal{H}$. The first sum in the last equality is in Ω and Theorem (2.2.18) gives that the second sum in the equality is in Ω also. Letting $\omega = P_{\mathcal{H}} \tilde{q}$, we have proved that $P_{\mathcal{M}} \omega = q$ to get that

$$P_{\mathcal{M}} \Omega = \mathcal{M}.$$

On the other hand, $ker[P_{\mathcal{M}}|_{\Omega}\,] \subset \Omega$ is a reducing subspace of $\phi(\mathcal{B})$. Since Ω is a non-trivial minimal reducing space of $\phi(\mathcal{B})$, we see that $ker[P_{\mathcal{M}}|_{\Omega}\,] = \{0\}$. This implies that $P_{\mathcal{M}} : \Omega \rightarrow$

$\mathcal{M}$ is bijective and bounded. By the closed graph theorem we conclude that $P_{\mathcal{M}}|_{\Omega}$ is invertible.
Similarly we can show that $P_{\mathcal{N}}|_{\Omega}$ is invertible. Define

$$S = [P_{\mathcal{N}}|_{\Omega}][P_{\mathcal{M}}|_{\Omega}]^{-1}.$$

Then S is an invertible operator from $\mathcal{M}$ onto $\mathcal{N}$. Both S and S^* commute with $\phi(\mathcal{B})$ because Ω, $\mathcal{M}$ and $\mathcal{N}$ are three distinct nontrivial minimal reducing subspaces for $\phi(\mathcal{B})$. Thus S^*S commutes with $\phi(\mathcal{B})$. Making the polar decomposition of S, we write

$$S = U|S|,$$

for some unitary operator U from $\mathcal{M}$ onto $\mathcal{N}$, where $|S| = [S^*S]^{1/2}$. So U commutes with both $\phi(\mathcal{B})$ and $\phi(\mathcal{B})^*$. This completes the proof.

Theorem (2.2.31)[64]: *Let $\mathcal{M}$ and $\mathcal{N}$ be two distinct nontrivial minimal reducing subspaces of $\phi(\mathcal{B})$. Then either they are orthogonal or $\phi(\mathcal{B})$ has two distinct unitarily equivalent reducing subspaces and has also infinitely many minimal reducing subspaces.*

Proof. Let $\mathcal{M}$ and $\mathcal{N}$ be two distinct nontrivial minimal reducing subspaces of $\phi(\mathcal{B})$. Consider

$$\mathcal{W} = [closure(\mathcal{M} + \mathcal{N})] \ominus \mathcal{M}.$$

Then $\mathcal{W}$ is a reducing subspace of $\phi(\mathcal{B})$. For each $y \in closure(\mathcal{M} + \mathcal{N})$, we have

$$y = P_{\mathcal{M}}y + P_{\mathcal{M}^{\perp}}y.$$

Thus

$$closure(\mathcal{M} + \mathcal{N}) = \mathcal{M} \oplus \mathcal{W}.$$

If $\mathcal{M}$ and $\mathcal{N}$ are not orthogonal, by Theorem (2.2.27), $\mathcal{M}$, $\mathcal{N}$ are orthogonal to the distiguished minimal reducing subspace $\mathcal{M}_0$, and then $\mathcal{N}$ does not equal $\mathcal{W}$ and

$$\mathcal{N} \subset \mathcal{M} \oplus \mathcal{W}. \quad (7)$$

Now we show that $\mathcal{W}$ is a minimal reducing subspace of $\phi(\mathcal{B})$. Since $\mathcal{M}$ and $\mathcal{N}$ are distinct and they are minimal reducing subspaces, we have that the intersection of $\mathcal{M}$ and $\mathcal{N}$ equals 0. Noting that $\mathcal{N} \cap \mathcal{M}^{\perp}$ is a reducing subspace and contained in $\mathcal{N}$, we see that $\mathcal{N} \cap \mathcal{M}^{\perp}$ equals 0 to get that

$$ker(P_{\mathcal{M}}|_{\mathcal{N}}) = \{0\}.$$

This gives that for each $q \in \mathcal{N}, P_{\mathcal{M}} \neq 0$. Since $\mathcal{N}$ is a minimal reducing subspace, for each $0 \neq q_0$ in $\mathcal{N}$, the closure of $\{q - P_{\mathcal{M}}q; q \in \mathcal{N}\}$ is the reducing subspace generated by $q_0 - P_{\mathcal{M}}q_0$ which equals $\mathcal{W}$. Thus $\mathcal{W}$ is a minimal reducing subspace. By (7), we observe that $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{W}$ satisfy the conditions in Theorem (2.2.3). So $\mathcal{M}$ is unitarily equivalent to $\mathcal{W}$.
Now for each α in $[0,1]$, define

$$\mathcal{N}_{\alpha} = closure\{q - \alpha P_{\mathcal{M}}q; q \in \mathcal{N}\}.$$

As $\mathcal{N}$ is a minimal reducing subspace, each $\mathcal{N}_{\alpha}$ is a minimal reducing subspace. For α and β in $[0,1]$, and q_1 and q_2 in $\mathcal{N}$, if

$$q_1 - \alpha P_{\mathcal{M}}q_1 = q_2 - \beta P_{\mathcal{M}}q_2,$$

then

$$q_1 - q_2 = \alpha P_{\mathcal{M}}q_1 - \beta P_{\mathcal{M}}q_2.$$

The right-hand side of the above equality is in $\mathcal{N}$ but the left-hand side is in $\mathcal{M}$. Thus q_1 equals q_2 and α equals β. So $\mathcal{N}_{\alpha}$ does not equal $\mathcal{N}_{\beta}$ provided β does not equal α. Hence we get infinitely many minimal reducing subspaces to complete the proof.

Theorem (2.2.32)[64]: *Let ϕ be a Blaschke product of finite order N. Then either $\phi(\mathcal{B})$ has infinitely many minimal reducing subspaces or the number of nontrivial reducing subspaces of $\phi(\mathcal{B})$ is less than or equal to N.*

Proof. If $\phi(\mathcal{B})$ does not have infinitely many nontrivial reducing subspaces, by Theorem (2.2.31), any two distinct reducing subspaces must be orthogonal. Let $\{\mathcal{M}_j\}_{j=1}^{N_1}$ be the set of distinct minimal reducing subspaces of $\phi(\mathcal{B})$. Thus

$$\oplus_{j=1}^{N_1} \mathcal{M}_j \subset \mathcal{H}.$$

Lemma (2.2.16) gives that the dimension of $\mathcal{M}_j \cap L_0$ is at least one. So

$$dim\, L_0 \geqq dim\left\{\left[\oplus_{j=1}^{N_1} \mathcal{M}_j\right] \cap L_0\right\} \geqq N_1.$$

On the other hand,

$$L_0 = ker\, T^*_{\phi(z)} \cap ker\, T^*_{\phi(w)} \cap \mathcal{H}.$$

As pointed out before, the dimension of L_0 equals N. Thus

$$N \geqq N_1.$$

So the number of nontrivial minimal reducing subspaces of $\phi(\mathcal{B})$ is less than or equal to the order N of ϕ. The proof is completed.

We will prove Theorems (2.2.4) and (2.2.5). For the Blaschke product $\phi(z) = z^2 \frac{z-\alpha}{1-\bar{\alpha}z}$ with a nonzero point α in $\mathbb{D}$, for each e in the wandering subspace of a reducing subspace of $\phi(\mathcal{B})$ we will be able to calculate d_e^0 in Theorem (2.2.21) and L_0 precisely. By Theorem (2.2.1), the fact that d_e^1 is orthogonal to L_0 leads to some algebraic equations. By solving the algebraic equations, we will show that $\phi(\mathcal{B})$ has only two nontrivial minimal reducing subspaces.

Theorem (2.2.33)[64]: *For the Blaschke product $\phi(z) = z^2 \frac{z-\alpha}{1-\bar{\alpha}z}$ with a nonzero point α in $\mathbb{D}$, $\phi(\mathcal{B})$ has only two minimal reducing subspaces $\{\mathcal{M}_0, M_0^{\perp}\}$.*

Proof. For a given nonzero point α in the unit disk, let $\phi_0(z) = z\frac{z-\alpha}{1-\bar{\alpha}z}$ .The Mittag-Leffler expansion of the finite Blaschke product ϕ_0 is

$$\phi_0(z) = -\frac{1-|\alpha|^2}{\bar{\alpha}^2} - \frac{1}{\bar{\alpha}}z + \frac{1-|\alpha|^2}{\bar{\alpha}^2} k_\alpha(z).$$

So

$$\phi(z) = z\phi_0(z) = -\frac{1-|\alpha|^2}{\bar{\alpha}^2}z - \frac{1}{\bar{\alpha}}z^2 + \frac{1-|\alpha|^2}{\bar{\alpha}^2} k_\alpha(z).$$

Hence

$$e_0(z,w) = -\frac{1-|\alpha|^2}{\bar{\alpha}^2} - \frac{1}{\bar{\alpha}}p_1(z,w) + \frac{1-|\alpha|^2}{\bar{\alpha}^2} k_\alpha(z)k_\alpha(w).$$

It is easy to see that

$$L_0 = span\{1, p_1(z,w), k_\alpha(z)k_\alpha(w)\}.$$

Theorem (2.2.25) gives

$$\mathcal{M}_0 = span_{l\geqq 0}\{p_l(\phi(z),\phi(w))e_0\}.$$

By Theorem (2.2.27), for each minimal reducing subspace Ω not equal to $\mathcal{M}_0$, Ω is a subspace of $M_0^{\perp}$. So we only need to show that $M_0^{\perp}$ is a minimal reducing subspace for $\phi(\mathcal{B})$.

Assume that $M_0^\perp$ is not a minimal reducing subspace for $\phi(\mathcal{B})$. Then $\mathcal{H}$ is the direct sum of three reducing subspaces of $\phi(\mathcal{B})$. We may assume

$$\mathcal{H} = \bigoplus_{i=0}^{2} \mathcal{M}_i.$$

Now choose a nonzero vector e_i in the wandering subspace for $\mathcal{M}_i$, which is contained in L_0. Since $\{e_i\}_{i=0}^{2}$ are mutually orthogonal to each other, they form a basis for L_0. On the other hand, those functions $1, p_1(z,w)$ and $k_\alpha(z)k_\alpha(w)$ are a basis for L_0. Thus there are some constants c_{ij} such that

$$e_i = c_{i0} + c_{i1}\, p_1(z,w) + c_{i2} k_\alpha(z) k_\alpha(w).$$

First we show that neither c_{12} nor c_{22} equals zero. Since

$$\langle e_0, 1\rangle = e_0(0,0) = \phi_0(0) = 0$$

and

$$\langle e_0, p_1\rangle = \langle e_0, p_1(z,z)\rangle = \langle e_0(z,0), 2z\rangle = 2\langle \phi_0, z\rangle = -2\alpha$$

we have that 1 is in $M_0^\perp$ but $p_1(z,w)$ is not in $M_0^\perp$, to get that $c_{i2} \neq 0$ for $i = 1,2$.
Next we show that $e_i(0,\alpha)$ equals 0 for $i = 0,1,2$. To do this, note that the dimension of the wandering subspace for each $\mathcal{M}_i$ equals 1. By Theorem (2.2.1), there are constants β_i and λ_i such that

$$d_{e_i}^1 = d_{e_i}^0 + \beta_i e_i + \lambda_i e_0.$$

Thus for $i, j \geqq 1$ and $i \neq j$,

$$\langle d_{e_i}^0, e_j\rangle = \langle d_{e_i}^1 - \beta_i e_i - \lambda_i e_0, e_j\rangle = \langle d_{e_i}^1, e_j\rangle - \beta_i\langle e_i, e_j\rangle - \lambda_i\langle e_0, e_j\rangle = 0.$$

The last equality follows from the fact that $d_{e_i}^1$ is orthogonal to L_0 and $\{e_i\}$ is an orthogonal basis for L_0 .
By Theorem (2.2.21), we have

$$\langle d_{e_i}^0, e_j\rangle = \langle w e_i(0,w) e_0(z,w) - \phi(w) e_i(z,w), e_j\rangle = \langle w e_i(0,w) e_0(z,w), e_j(z,w)\rangle.$$

Simple calculations give

$$\begin{aligned}
&\langle w e_i(0,w) e_0(z,w), p_1(z,w)\rangle \\
&= \langle w e_i(0,w) e_0(w,w), p_1(z,w)\rangle \quad \text{(by Lemma (2.2.7))} \\
&= \langle w e_i(0,w) \phi'(w), p_1(0,w)\rangle \quad \text{(by Lemma (2.2.9))} \\
&= \langle w e_i(0,w) \phi'(w), w\rangle = e_i(0,0)\phi'(0) = 0.
\end{aligned}$$

It is easy to see that

$$\langle w e_i(0,w) e_0(z,w), 1\rangle = 0.$$

These give

$$\begin{aligned}
\langle d_{e_i}^0, e_j\rangle &= c_{j2}\langle w e_i(0,w) e_0(z,w), k_\alpha(z)k_\alpha(w)\rangle = c_{j2}\langle w e_i(0,w) e_0(w,w), k_\alpha(z)k_\alpha(w)\rangle \\
&= c_{j2}\langle w e_i(0,w) e_0(w,w), k_\alpha(w)\rangle = c_{j2}\alpha e_i(0,\alpha) e_0(\alpha,\alpha).
\end{aligned}$$

Noting that $e_0(\alpha,\alpha) = \phi'(0) \neq 0$, we have $c_{j2} e_i(0,\alpha) = 0$, to get that $e_i(0,\alpha) = 0$ for $i = 1,2$. Also we have $e_0(0,\alpha) = \phi_0(\alpha) \neq 0$. Since $\{e_0, e_1, e_2\}$ forms a basis for L_0 and $p_1(z,w)$ is in L_0, $p_1(z,w)$ is a linear combination of functions e_0, e_1 and e_2. Thus $p(0,\alpha)$ must be zero. But $p_1(0,\alpha) = \alpha \neq 0$. This leads to a contradiction. So $M_0^\perp$ is a minimal reducing subspace of $\phi(\mathcal{B})$ to complete the proof.
Now we are ready to prove Theorems (2.2.4) and (2.2.5).

Suppose that ϕ is a Blaschke product with three zeros. As pointed out M_ϕ is unitarily equivalent to $\phi(\mathcal{B})$, in the rest proof we will concern only $\phi(\mathcal{B})$.

First observe that for $\lambda \in \mathbb{D}$ and a subspace $\mathcal{M}$ of $\mathcal{H}$, $\mathcal{M}$ is a reducing subspace of $\phi(\mathcal{B})$ if and only if $\mathcal{M}$ is a reducing subspace of $\phi_\lambda \circ \phi(\mathcal{B})$.
If $\phi(z)$ has a multiple critical point in the unit disk, then

$$\phi = \phi_\lambda \circ z^3 \circ \phi_\mu$$

for two numbers $\lambda, \mu \in \mathbb{D}$. Thus every reducing subspace of $\phi(\mathcal{B})$ is also a reducing subspace of $\phi_\mu(\mathcal{B})^3$. But $\phi_\mu(\mathcal{B})^3$ is unitarily equivalent to the direct sum of three weighted shifts and hence it has only three minimal reducing subspaces.
If ϕ does not have any multiple critical point in the unit disk, by Bochner's theorem [75], $\phi(z)$ always has a critical point c in the unit disk. Let $\lambda = \phi(c)$. Then

$$\phi_\lambda \circ \phi \circ \phi_c(z) = z^2 \frac{z-a}{z-\bar{a}z},$$

for some nonzero point $a \in \mathbb{D}$. Let $\psi(z) = \phi_\lambda \circ \phi \circ \phi_c(z)$. By Theorem (2.2.33), we conclude that $\psi(\mathcal{B})$ has only two minimal reducing subspaces. Hence $\phi(\mathcal{B})$ has only two minimal reducing subspaces. This completes the proof.
Let ϕ be a Blaschke product with three zeros.
As in the proof of Theorem (2.2.4), if $\phi(z)$ has a multiple critical point in the unit disk, then

$$\phi = \phi_\lambda \circ z^3 \circ \phi_\mu$$

for two numbers $\lambda, \mu \in \mathbb{D}$. In this case, the Riemann surface of $\phi^{-1} \circ \phi$ over $\mathbb{D}$ has the same number of connected components as the one of $z^{-3} \circ z^3$ over $\mathbb{D}$ does. But the latter one has three connected components and M_ϕ has the only three nontrivial minimal reducing subspaces. Thus the number of nontrivial minimal reducing subspaces of M_ϕ equals the number of connected components of the Riemann surface of $\phi^{-1} \circ \phi$ over $\mathbb{D}$.
If ϕ does not have any multiple critical point in the unit disk, as in the proof of Theorem (2.2.4), $\phi(z)$ always has a critical point c in the unit disk. Let $\lambda = \phi(c)$. Then

$$\phi_\lambda \circ \phi \circ \phi_c(z) = \psi(z),$$

where $\psi(z) = z^2 \frac{z-a}{z-\bar{a}z}$ for some nonzero point $a \in \mathbb{D}$. By the example in [24], except for the trivial branch z, nontrivial branches of $\psi^{-1} \circ \psi$ are all continuations of one another. Thus the Riemann surface of $\psi^{-1} \circ \psi$ over $\mathbb{D}$ has only two connected components. So does the Riemann surface of $\phi^{-1} \circ \phi$ over $\mathbb{D}$. By Theorem (2.2.33), the number of nontrivial minimal reducing subspaces of M_ϕ equals the number of connected components of the Riemann surface of $\phi^{-1} \circ \phi$ over $\mathbb{D}$. This completes the proof.

Chapter 3
Bergman Kernel Asymptotics and Hankel Operators

We show that for a large class of measures, we find that these quantities satisfy asymptotic relations similar to the simple exact relations which hold in the model case $m(t) = e^{-t}$. We show that the main result says that a Hankel operator on such a Fock space is bounded if and only if the symbol belongs to a certain BMOA space, defined via the Berezin transform. The latter space coincides with a corresponding Bloch space which is defined by means of the Bergman metric. This characterization of boundedness relies on certain precise estimates for the Bergman kernel and the Bergman metric. Characterizations of compact Hankel operators and Schatten class Hankel operators are also given. In the latter case, results on Carleson measures and Toeplitz operators along with Hörmander's L^2 estimates for the $\bar{\partial}$operator are key ingredients in the proof.

Section (3.1): Generalized Fock Spaces

Given a positive measure $m(t)dt$ on $\boldsymbol{R}^+$ and its moment sequence $\gamma_n = \int_0^\infty t^n m\,(t)dt\,, n = 0, 1, 2\,, \ldots,$ we form the associated Bergrnan kernel function, $K_m(x) = \sum \gamma_n^{-1} x^n$ .We also form the new measure $(K_m(t))^{-1} m\,(t)dt$ and its kernel function, $K_{(K_m)^{-1}m}$. If we start with $m(t) = e^{-t}$ and do the computations, we find three striking facts: for all $t \in \boldsymbol{R}^+$ and all $a \in \mathbb{C}$,

(A) $$m(t)K_m(t) = 1,$$

(B) $$K_{(K_m)^{-1}m}(t) = 2K_m^2(t),$$

and

(C) $$\int\int_{\mathbb{C}} \left|K_{(K_m)^{-1}m}(\bar{a}z)\right| m(|z|^2)\ \frac{dxdy}{\pi} = 2K_m(|a|^2)$$

Our interest in these identities arose w h e n we were doing operator theory on the Fock space, the Hilbert space of entire functions square integrable with respect to the Gaussian density. We wanted to know if similar relations or useful substitutes held in Bergrnau spaces of entire functions square integrable with respect to other radial measures, $\pi^{-1} m(|z|^2)dxdy$. However, although operator theoretic issues influence our discussion of the consequences of our main results, neither our results here nor our methods involve operator theory. We hope to pursue the operator theoretic implications of these results else where. See [107].

We collect background information about conjugate functions of convex functions (in the sense of Fenchel, Legendre, and Young) which arises both as we pass from the density m to its moments γ_n, and as we pass from the coefficients of K_m to its values. has informal statements and proof outlines for our two main technical results: Theorem (3.1.2) — which shows how the growth of the density function controls the asymptotic growth of the moment sequence — and Theorem (3.1.3) — which shows how the growth of the coefficient sequence controls the growth *of* $K_m\left(re^{i\theta}\right)$ for large *r*. The next two have the statements and proofs of Theorems (3.1.2) and (3.1.3). The basic approach for Theorem (3.1.2) is Laplace's method for asymptotic estimation of integrals which depend on a parameter. To prove Theorem (3.1.3), we join Laplace's method with Poisson summation, we combine Theorem (3.1.2) and Theorem (3.1.3) to give Theorem (3.1.12), our estimates for the Bergman kernel functions. A consequence of that theorem is Corollary (3.1.13), which includes the result that, as $r \to \infty$,

$$m(r)K(r)\sim\frac{-\left(r\frac{d}{dr}\right)^2\log m(r)}{r}.$$

In particular, if $m(r)\sim ar^be^{-cr^d}$, with $a,b,c,d>0$, we have

$$m(r)K(r)\sim cd^2r^{d-1},$$

which is a version of (A). If we take the estimates for K in terms of m and then use Theorem (3.1.2) and Theorem (3.1.3) again to estimate $K_{(K_m)^{-1}m}$, we find that the two expressions in (B) are asymptotically equal. In fact, as is suggested by the example o f the exponential density, we see in Theorem (3.1.14) that

$$K_{(K_m)^{-\alpha}m}\sim(1-\alpha)(K_m)^{1+\alpha} \tag{1}$$

for $\alpha>0$. We also show that the Berezin transform for these Bergrnan spaces is given asymptotically by integration against a Gaussian density. This and (1) are then used to give an asymptotic version of (C) in Corollary (3.1.16).

Asummary of these and related results along with some discussion of the operator theory is in [107].

Suppose A(s) is a convex function defined on an $I\subset R$. (When convenient, we set $(s)=+\infty for\ s\notin I$.)We recall the definition o f the *conjugate function* of A.

$$A^*(x)=\sup_{s\in\mathbf{R}}\{xs-A(s)\}. \tag{2}$$

This transformation occurs in various contexts, at times associated with the names Fenchel, Legendre, or Young.

Lemma (3.1.1)[106]: *Suppose A is smooth and $A,A',A''>0$. Set $s(x)=A'^{-1}(x)$ and $x(s)=A'(s)$. Then $A^*,A^{*\prime},A^{*\prime\prime}>0$ and we have, for all s, x,*

(a) $s(x(s))=s\,,x(s(x))=x$,

(b) $sx\le A(s)+A^*(x)$,

(c) $A^*(x)=xA'^{-1}(x)-A(A'^{-1}(x))=xs(x)-A(s(x))$,

(d) $s(x)=A^{*\prime}(x)=A'^{-1}(x)$,

(e) $A^{**}(s)=A(s)$,

(f) $A^{*\prime\prime}(x)=A''(s(z))^{-1}$,

(g) $A^{*(3)}(x)A^{*\prime\prime}(x)^{-3/2}=-A^{(3)}(s(x))A''(s(x))^{-3/2}$,

(h) $A^{*(4)}(x)A^{*\prime\prime}(x)^{-2}=-A^{(4)}(s(x))A''(s(x))^{-2}+3A^{(3)}(s(x))^2A''(s(x))^{-3}$.

Note: We are only interested in asymptotic behavior for large *s* and large *x*. Hence, if necessary to insure that the hypotheses are satisfied, we can first restrict A to an interval (M,∞) and then set $A=+\infty$ on $(-\infty,M]$. In that case, the conclusions of the lemma hold for all sufficiently large *x, s*.

Proof. The proof of related results under minimal smoothness assumptions requires care, but here there is no problem. The first statement follows from the definitions, as does the second, which is often called Young's conjugate function inequality. Our assumptions insure that the supremum in (b) is attained at the unique critical point *of* $xs-A(s)$. This gives the formula for A^*. The first equality in (d) follows from differentiating (c). The relation (e) comes from (c) and (d). Formula (f) follows from differentiating (e). Equality (g) follow s from differentiating (f) and noting that $s'(x)=A^{*\prime\prime}(x)=A''(s(x))^{-1}$. Formuala 8follows from differentiating (g), using $s'(x)=A^{*\prime\prime}(x)=A''(s(x))^{-1}$, and then using (g).

The model pair for what we do later is

$$A(s) = e^s - s\,,$$
$$A^*(x) = (x+1)\,log(x+1) - (x+1),$$

which corresponds to $m(t) = exp(-t)$. More generally, for $(t) = exp(-t^\beta)$, we have

$$A(s) = e^{\beta s} - s\,, \tag{3}$$
$$A^*(x) = \left(\frac{x+1}{\beta}\right) log\left(\frac{x+1}{\beta}\right) - \left(\frac{x+1}{\beta}\right),$$

The theorems and proofs have substantial technical details. However, the basic ideas are quite straightforward. we present the ideas.

Given a positive function $a(s)$ defined on $\boldsymbol{R}^+$, set

$$A(s) = -\log a(e^s) - s. \tag{4}$$

We suppose that for all large s

$$A(s), A'(s), A''(s), A^{(3)}(s), A^{(4)}(s) > 0. \tag{5}$$

Set $s_x = s(x) = A'^{-1}(x)$. Suppose b is a positive function which varies slowly compared to a and set $B(s) = \log b(e^s)$. Let γ_n be the moments of the measure $a(t)b(t)\ at; \gamma_n = \int_0^\infty t^n a(t)b(t)\,dt$.

Theorem (3.1.2) [106]: *As $n \to \infty$, we have*

$$\gamma_n \sim e^{A^*(n)} \frac{\sqrt{2\pi}}{\sqrt{A''(s_n)}} e^{B(s_n)}.$$

In the simplest case, when $a(t) = e^{-t}$ and $b(t) = 1$, this is Stirling's formula. Now suppose *c(x)* is a positive function on $\boldsymbol{R}^+$. Set

$$\Gamma(x) = \log c(x)\,. \tag{6}$$

Suppose that for all large x

$$\Gamma(x), \Gamma'(x)\,, \Gamma''(x) > 0. \tag{7}$$

However, in contrast to the previous theorem, we now require that as $x \to \infty$

$$\Gamma'(\mathrm{x}) \to \infty\,, \qquad \Gamma''(\mathrm{x})\,, \Gamma^{(3)}(\mathrm{x}), \Gamma^{(4)}(\mathrm{x}) \to 0. \tag{8}$$

Let Γ^* be the conjugate function of Γ and set $x_s = x(s) = \Gamma'^{-1}(s)$. Suppose that d is a positive function which varies slowly compared to c. Let f be the holomorphic function

$$f(z) = \sum_0^\infty d(n)c(n)^{-1}\,z^n.$$

Theorem (3.1.3) [106]: *f is entire. Forsmall θ we have as $s \to \infty$*

$$f(e^{s+i\theta}) \sim e^{\Gamma^*(s)} \frac{\sqrt{2\pi}}{\sqrt{\Gamma''(x_s)}} \mathrm{d}(x_s)\boldsymbol{e}^{ix_s\theta}\ \boldsymbol{e}^{-\theta^2}/(2\Gamma''(x_s)).$$

Our main kernel estimate, Theorem (3.1.12), follows quickly from these two results. First we apply Theorem (3.1.2) with the choices $a(t) = m(t^2), b(t) = I$ and then Theorem (3.1.3) with the choices $(x) = A^*(x), d(x) = c(x)/\gamma_x$. Because we are able to put some o f the behavior o f the mon~ents into the correction term d, we obtain kernel estimates whose main term involves A^{**}. We then use the fact that $A^{**} = A$. To get estimates for $K_{K_{\overline{m}}^{\alpha} m}$, we repeat the cycle, using as our new starting choice for a the square of the function used the first time.

This forces a nonconstant choice for b. However, b turns out to be slowly varying, so again the main term o f the estimate involves $A^{**} = A$.
To prove Theorem (3.1.2), we use Laplace's method for asymptotic evaluation of integrals as it adapts to our situation. We want to estimate

$$\begin{aligned}\gamma_n &= \int_0^\infty t^n a(t) b(t)\, dt \\ &= \int_0^\infty e^{n \log t + \log a(t)} b(t)\, dt \\ &= \int_{-\infty}^\infty e^{ns + \log t + \log a(e^s) + s} b(e^s)\, ds \\ &= \int_{-\infty}^\infty e^{ns - A(s)} e^{B(s)}\, ds. \qquad (9)\end{aligned}$$

The hypotheses insure that, for fixed large n, the function $ns - A(s)$ has a maximum value at the point $s_n = A'^{-1}(n)$. The value is $A^*(n) = ns_n - A(s_n)$. We now expand $ns - A(s)$ in a Taylor series about its critical point s_n:

$$ns - A(s) = A^*(n) - \frac{1}{2} A''(s_n)(s - s_n)^2 + \mathcal{R}.$$

Here $\mathcal{R}$ is the remainder term. I f we could drop $\mathcal{R}$ and replace $B(s)$, which is built from a slowly varying function, by $B(s_n)$ then w e could evaluate the integral and would have γ_n *equal* to the desired estimate. The technical details o f the proof involve estimating the errors that result from dropping $\mathcal{R}$ and replacing $B(s)$ by $B(s_n)$.
Introduce the new integration variable $u = s - s_n$. Using $A'(s_n) = n$, we have

$$ns - A(s) = A^*(n) - [A(u + s_n) - A(s_n) - A'(s_n)u].$$

We need to estimate

$$\gamma_n = e^{A^*(n)} \int_{-\infty}^\infty e^{-[A(s_n+u) - A(s_n) - A'(s_n)u]} e^{B(s_n+u)}\, du. \qquad (10)$$

To do this we select a positive function $\delta = \delta(n)$ and split the integral as

$$\int_{-\infty}^\infty \dots \mathrm{du} = \int_{\mathrm{u}<-\delta} \dots \mathrm{du} + \int_{|\mathrm{u}|<\delta} \dots \mathrm{du} + \int_{\mathrm{u}>\delta} \dots \mathrm{du} = \mathrm{L} + \mathrm{C} + \mathrm{R}.$$

To estimate C, we want to kno w that, uniformly in $\{u: |\mathrm{u}| < \delta\}$, we have for some appropriate small $\mathcal{K}$

$$A(s_n + u) = A(s_n) + A'(s_n)u + A''(s_n)\, u^2/2 + O(\mathcal{K}),$$
$$B(s_n + u) = B(s_n) + O(\mathcal{K}).$$

Those estimates follow from the hypotheses on a and b and Taylor's theorem. Using them, we have

$$e^{A^*(n)} C = e^{A^*(n)} \int_{|\mathrm{u}|<\delta} e^{-A''(s_n)u^2/2} e^{B(s_n)} [1 + O(\mathcal{K})] \mathrm{du}.$$

Introducing the new variable $v = u\sqrt{A''(s_n)}$,we find that

$$e^{A^*(n)} C = \frac{e^{A^*(n)} e^{B(s_n)}}{\sqrt{A''(s_n)}} \int_{|\mathrm{u}|<\delta\sqrt{A''(s_n)}} \mathrm{e}^{-v^2/2} [1 + \mathrm{O}(\mathcal{K})]\, dv.$$

If we know $\delta^2(n)A"(s_n) \to \infty$, we can conclude that

$$e^{A^*(n)}C = e^{A^*(n)}\frac{\sqrt{2\pi}}{\sqrt{A"(s_n)}}e^{B(s_n)}[1 + O(\mathcal{K})].$$

The tails, L and R, can be estimated b y tails o f Gaussian integrals and are seen to be $O(e^{-\delta^2(n)A"(s_n)/10})$. Combining the estimates for L, C, and R gives Theorem (3.1.2).
In the second theorem, we can pass from the sum to the corresponding integral and use a similar argument to get the estimates on the positive real axis. However, that approach doesn't capture the cancellation which occurs off the axis. Hence w e split the sum into three terms and estimate the main term, the central one, using Poisson summation.

In Theorem (3.1.3), we show that if we are given the moments $\{\gamma_n\}$ o f a density, then the asymptotic growth of the kernel function is given by

$$f(e^s) \sim \sqrt{2\pi}e^{(\log\gamma)^*(s)}, \qquad s \to \infty.$$

Rewriting this in terms of the Taylor coefficients a_n $(= \gamma_n^{-1})$ o f f, we have

$$a_n \sim \frac{1}{\sqrt{2\pi}}e^{-(\log f(e^s))^*(n)}, \quad n \to \infty.$$

In the other direction, one can ask whether, given an entire function which satisfies appropriate conditions, we can conclude this sort o f asymptotic growth for the coefficients. That such estimates do, in fact, hold for a large class o f entire functions is a result of Hayman [108].

Suppose $f(z) = \sum_0^\infty a_n z^n$ is an entire function with positive coefficients. Set

$$F(s) = \log f(e^s).$$

We say that f is *admissible* if $F"(e^s) \to \infty$ as $s \to \infty$ and there is a positive function $\delta(r)$, defined for all sufficiently large r, such that $0 < \delta(r) < \pi$,

$$f(re^{i\theta}) \sim f(r)e^{iF'(\log r)^\theta}e^{-\frac{1}{2}F''(\log r)^{\theta^2}} \quad as\ r \to \infty$$

uniformly for $|\theta| \le \delta(r)$, and

$$f(re^{i\theta}) = o(1)\frac{f(r)}{\sqrt{F"(\log r)}},$$

uniformly for $\delta(r) \le |\theta| < \pi$.
Corollary II of [109] is
Theorem (3.1.4) [106]: *If $f(x)$ is admissible, then as* $n \to \infty$

$$a_{n+1} \sim \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{F"(F'^{-1}(n))}}e^{-F^*(n)}.$$

In fact, this follows quite easily from the admissibility of f. Most of the work in [110] is in establishing that a substantial number of functions are admissible, in showing that the class o f admissible functions has interesting closure properties, and in deriving further consequences of admissibility. Our Theorem (3.1.3) insures that the kernel functions we construct are admissible.

In Hayman's theorem as well as Theorems (3.1.2) and (3.1.3), we see that the transforms have asymptotics described to leading order using the conjugate function. That is in keeping with the heuristic "principle o f duality o f phases", for describing the asymptotic behavior of Fourier (and related) transforms ([104], p. 358). The principle has a long tradition in this context. Hayman's results are related to earlier results of Wiener and Martin [105], [106] and still earlier results of Hardy and Fejor, both of whom attribute the basic insight to Riemann. (For this see the discussion in [111].) More recent related results in this tradition are given by Evgrafov [107] and Bemdtsson [112].
Related questions have been considered for measures and kernel functions defined on the unit disk. See [112], with more recent results by Kriete and MacCluer [113] and Kriete [114]. Here are two o f the results of [113].
Suppose that we are working on the Bergman space o f the disk with radial weight $(2\pi)^{-1}w(r)dx\,dy$. Thus $K(x)=\sum\gamma_n^{-1}x^n$ with $\gamma_n=\int_0^1 r^{2n+1}w(r)\,dr$. Set $A(s)=log\,2-log\,w(e^{-s/2})+s$. Under appropriate conditions on *w*, a result analogous to Theorem (3.1.2) is obtained.
Theorem (3.1.5) [106]: *As* $n\to\infty, \gamma_n \sim \sqrt{\pi}\sqrt{A^{*\prime\prime}(-n)}e^{A^*(-n)}$.
This theorem is used in the proof o f the following quantitative alternative to (A), which plays a major technical role in [113]:
Theorem (3.1.6) [106]: *As* $r\to 1^-,\ m\,(r)K(r)\nearrow\ \infty$.
We learned that Kriete has taken his work further and obtained rather comprehensive results on the unit disk [116]. Although the detailed formalism of [114] and [115] differ from ours, there is certainly a similarity between those methods and ours.
Related questions have been studied for nonradial weights using a variety of function theoretic techniques. For instance, it is shown in [118] that under some regularity conditions on the function $w(z)\geq 0$, *and* with the assumption that $-log\,w$ is subharmonic, the Bergman kernel $K(z,\zeta)$ for the space $L^2(\mathbb{D},w(z)\,dx\,dy)\cap Hol$ satisfies
Proposition (3.1.7) [106]: *There are positive constants* C_1 *and* C_2 *so that*

$$C_1<\frac{K(z,z)w(z)}{-\Delta\,log\,w(z)}<C_2.$$

Similar techniques produce an analogous result for Bergman spaces on the plane. These should be compared with Theorem (3.1.12), which deals with smooth *radial* weights (on the plane). That result states that, as $z\to\infty$,

$$\frac{K(z,z)w(z)}{-\Delta\,log\,w(z)}=1+o(1).$$

Christ, Berndtsson, Ortega-Cerdà and Seip, Delin, and others have obtained refined estimates on Bergman kernel functions, including estimates off the diagonal, using (9 techniques. It is a theorem of Miles and Williamson [110], which proved a conjecture of

Renyi and Vincze, that $m(t) = e^{-t}$ is essentially the only function which satisfies (A). It would be interesting to know if there were analogous uniqueness results related to (C). As mentioned, we shall prove that for fixed α, as $\alpha \to \infty$

$$K_{K_m^{-\alpha} m}(t) \sim (1+\alpha) K_m^{1+\alpha}(t). \tag{11}$$

In his interesting study of Berezin quantization, Englis [112] shows that in certain cases, for fixed *t,* (11) holds as $\to \infty$. His methods and viewpoint are quite different from ours. we discuss briefly the possibility of obtaining asymptotics as $\alpha \to \infty$ by our methods. Suppose $m(t)dt$ is a positive measure on $[0, \infty)$. For $x > 0$, set $\gamma_x = \gamma(x) = \int_0^\infty t^x m(t)\, dt$. We assume that m does not have compact support and that $\gamma(x)$ is finite for all *x*. We write $m(t) = a(t)b(t)$ with a as the main term and b as a slowly varying correction. Although $m = ab$ is the object of interest, most of our computations, and hence also the hypotheses, are in terms of auxiliary functions, *A* and *B*, defined by

$$A(s) = -\log a(e^s) - s, \tag{12}$$

$$B(s) = \log b(e^s). \tag{13}$$

Set $sx = s(x) = A'^{-1}(x)$. Fix $\varepsilon, 1/4 < \varepsilon < 1/2$. We suppose that for all sufficiently large x

$$A^{(i)}(x) > 0, \qquad i = 0, \dots, 4, \tag{14}$$

$$\mathrm{A}'''(x) = \mathrm{O}\left(\mathrm{A}''^{3/2-\varepsilon(\mathrm{z})}(x)\right), \tag{15}$$

$$\mathrm{A}^{(4)}(x) = O\left(\mathrm{A}''^{2-2\varepsilon(\mathrm{z})}(x)\right). \tag{16}$$

The core hypothesis for the proof of Theorem (3.1.2) is that we can find an auxiliary positive function δ such that $\delta^2(x)\mathrm{A}''(s_x) \to \infty$ and $\delta^3(x)\mathrm{A}'''(s_x) \to 0$ as $x \to \infty$. We surrendered a slight a m o u n t o f generality b y assuming (1 5), but that lets us mak e a simple choice for δ. Select α with $0 < \alpha < \varepsilon/2 - 1/8$ and set

$$\delta(x) = A''(s_x)^{-1/2+\alpha} \, . \tag{17}$$

The model case for the hypotheses is $A(s) = e^{\beta s} - s$. In that case, (15) and (16) hold with any $\varepsilon < 1/2$. The same is true for $A(s) = e^{\,h(s)}$ with any function *h* of regular and modest growth. Hence , for the examples we have in mind, we coul d restrict attention to A which satisfy (15) and (16) for all ε up to 1/2. In fact, suppose *A* were to fail (15) for a fixed ε because there is some $\varphi < 1/2 - \varepsilon$ so that $\mathrm{A}''' \geq C\mathrm{A}''^{1+\varphi}$. *In* such a case, we could compare A'' with the exact solution of $f' = Cf^{1+\varphi}$ and conclude that $A''(s)$ cannot be finite for all $s > 0$. Such *A* are not of interest here. However, we carry the extra generality o f allowing (15) and (16) to fail for *some* $\varepsilon < 1/2$ because it may be useful in some other context. That being said, w e should note that in the following discussion it m a y be convenient to think of the model case $\varepsilon = (1/2)^-, \alpha = 0^+$.

Our pointwise estimates on the derivatives of *A* imply interval estimates.

Lemma (3.1.8) [106]: *I f we have (15), (16), and (17), then we also have*

$$\sup_{|t|<\delta} A''(s_x + t) = (1 + o(1))\, A''(s_x)\,, \tag{18}$$

$$\sup_{|t|<\delta} A'''(s_x + t) = O(A''(s_x)^{3/2-\varepsilon}), \tag{19}$$

$$\sup_{|t|<\delta} A^{(4)}(s_x + t) = O(A''(s_x)^{2-2\varepsilon}). \tag{20}$$

Proof. Set $g(t) = A''(s_x + t)$. B y (15), $g' = O(g^{3/2-\varepsilon})$ and hence $g^{-3/2+\varepsilon}g' = O(1)$. Pick and fix some $t_0, |t_0| < \delta$. Integrating, we find

$$|g^{-1/2+\varepsilon}(t_0) - g^{-1/2+\varepsilon}(0)| = 0(\delta).$$

Hence, recalling the definitions of g and δ, we have

$$\frac{g^{-1/2+\varepsilon}(t_0)}{g^{-1/2+\varepsilon}(0)} = 1 + |g^{1/2-\varepsilon}(0)|O(g(0)^{-1/2+\alpha}) = 1 + O(g(0)^{\alpha-\varepsilon}),$$

as required for (18). For (19), note that b y (15) we have

$$A'''(s_x + t) = O(A''^{3/2-\varepsilon}(s_x + t)),$$

and by (18) w e can replace $A''^{3/2-\varepsilon}(s_x + t)$ b y $A''^{3/2-\varepsilon}(s_x)$. We obtain (20) by the same reasoning.

We say that a positive function b is slowl y varying in the first sense with respect to a, ε and α and write $b \in SVI(a, \varepsilon, \alpha)$ if, for B given b y (13),

$$B' = O(\delta^2 A''') = O\left(A''^{\frac{1}{2}-\varepsilon+2\alpha}\right), \tag{21}$$

$$B'' = O((A''^{1/2-\varepsilon+2\alpha})^2). \tag{22}$$

Note that $0 < 1/2 - \varepsilon + 2\alpha < 1/4$. We know from the previous lemma that A'' and A''' satisfy interval estimates. Hence so do B' and " .

We use the following notation for a class of error terms which will be negligible for our purposes. Write $X = X(x) = O(\varepsilon)$ if there is a positive c such that $=$ $O(exp(-A''(s_x)^c))$.

Suppose a and b are positivefunctions on $\mathbf{R}^+$, *A and B are defined by (12) and (13), and* δ *is given by (17). Suppose A satisfies (14), (15), (16), and hence also (18), (19), and (20). Suppose* $b \in SVI(a, \varepsilon, \alpha)$. *Let*

$$\gamma(x) = \int_0^\infty t^x a(t) b(t)\, dt.$$

As $x \to \infty$, we have

$$\gamma(x) = e^{A^*(x)} \frac{\sqrt{2\pi}}{\sqrt{A''(s_x)}} e^{B(s_x)}\left(1 + O(A''(s_x)^{6\alpha-2\varepsilon}) + O(\varepsilon)\right). \tag{23}$$

Furthermore, as $x \to \infty$

$$(\log \gamma)'(x) \to \infty, \tag{24}$$

$$A^{*'}(x) - (\log \gamma)'(x) = O\left(A''^{-\frac{1}{2}+2\alpha-\varepsilon}(x)\right), \tag{25}$$

$$A^{*''}(x) - (\log \gamma)''(x) = O\left(A''^{-1+6\alpha-2\varepsilon}(x)\right). \tag{26}$$

Notes: 1.

$$6\alpha - 2\varepsilon < -1/4, \quad -1/2 + 2\alpha - \varepsilon < -3/4, \quad -1 + 6\alpha - 2\varepsilon < -5/4.$$

(c) The formulation of (23) is redundant, as $O(\varepsilon)$ is smaller than $A''^{6\alpha-2\varepsilon}$. We include it separately because, while the error term $O(A''^{6\alpha-2\varepsilon})$ can obviously be refined by straightforward (but lengthy) analysis, the exponential error term appears to be intrinsic to the method.

(d) (24), (25), and (26) are technical estimates we shall use when we use the output from this theorem as input for Theorem (3.1.3).

Proof. Fix *x* large. We write δ for $\delta(x)$. Set

$$\alpha_i = A^{(i)}(s_x), \qquad i = 0, 1, \ldots,$$
$$\beta_i = B^{(i)}(s_x), \qquad i = 0, 1, \ldots.$$

We saw at (10) in the proof outline that

$$\gamma(x) = e^{A^*(x)} e^{\beta_0} \int_{-\infty}^{\infty} e^{-[A(s_x+u)-\alpha_0-\alpha_1 u]} e^{B(s_x+u)-\beta_0}\, du.$$

We need to estimate

$$I_j = \int_{-\infty}^{\infty} u^j e^{-[A(s_x+u)-\alpha_0-\alpha_1 u]} e^{B(s_x+u)-\beta_0}\, du$$

for $j = 0, 1, 2$. The analysis of the tails, *L* and *R,* is the same for $j = 0, 1,$ and 2,; we present the discussion only for $j = 0$. We have

$$I_0 = \int_{u<-\delta} \ldots \mathrm{du} + \int_{|u|<\delta} \ldots \mathrm{du} + \int_{u>\delta} \ldots \mathrm{du} = \mathrm{L} + \mathrm{C} + \mathrm{R}.$$

We first estimate *L*. Integration by parts gives

$$A(s_x + u) - \alpha_0 - \alpha_1 u = \int_u^0 (r-u)\, A''(s_x + r)\, dr.$$

In the integral defining $L, u < -\delta < 0$; thus the integrand in the previous integral is positive. Using this and the monotonicity of A'', we continue with

$$A(s_x + u) - \alpha_0 - \alpha_1 u \geq \int_{-\delta}^{0} (\mathrm{r}-\mathrm{u})\, \mathrm{A}''(\mathrm{s_x}+\mathrm{r})\, \mathrm{dr} \geq \mathrm{A}''(\mathrm{s_x}-\delta) \int_{-\delta}^{0} (\mathrm{r}-\mathrm{u})\ \mathrm{dr}$$
$$= -\frac{1}{2}(2u+\delta)\delta \mathrm{A}''(\mathrm{s_x}-\delta).$$

Thus

$$\mathrm{L} \leq \int_{-\infty}^{-\delta} e^{\frac{1}{2}(2u+\delta)\delta \mathrm{A}''(\mathrm{s_x}-\delta)+B(s_x+u)-\beta_0}\, du$$
$$= e^{\frac{1}{2}\delta^2 \mathrm{A}''(s_x-\delta)} \int_{-\infty}^{-\delta} e^{uA''(s_x-\delta)+B(s_x+u)-\beta_0}\, du.$$

Now

$$e^{B(s_x+u)-\beta_0} = exp \int_0^u B'(s_x+u)\, dt.$$

Usin g (21) and recalling that A'' is monotone, we find

$$e^{B(s_x+u)-\beta_0} = \; exp(O(1)|u|\alpha_2^{\theta})\,,$$

where $\theta = \; 1/2 - \varepsilon + 2\,\alpha$ is between 0 and 1/4. Hence

$$L \le e^{\frac{1}{2}\delta^2 A''(s_x-\delta)} \int_{-\infty}^{-\delta} exp(u\delta A''(s_x - \delta) + O(1)|u|\alpha_2^{\theta})\, du\,.$$

Taking into account (17), and recalling that u is negative in the region of integration, we have

$$\begin{aligned} u\delta A''(s_x - \delta) + O(1)|u|\alpha_2^{\theta} &= u\delta\big(\alpha_2\big(1 + o(1)\big) + O(1)\delta^{-1}\alpha_2^{\theta}\big) \\ &= u\delta\big(\alpha_2\big(1 + o(1)\big) + O(1)\alpha_2^{\frac{1}{2}+\alpha-\varepsilon}\big) \; = u\delta\alpha_2(1 + o(1))\,. \end{aligned}$$

Thus we can continue with

$$\begin{aligned} L &\le e^{\frac{1}{2}\delta^2 A''(s_x-\delta)} \int_{-\infty}^{-\delta} exp(u\delta\alpha_2(1 + o(1)))\, du \\ &= \frac{1}{\big(1 - \mathrm{o}(1)\big)\delta\alpha_2} e^{\frac{1}{2}\delta^2\alpha_2(1+o(1))} e^{-\delta^2\alpha_2(1-o(1))} \\ &\le \frac{(1 + o(1))}{\alpha_2} e^{-(\frac{1}{2}-o(1))\delta^2\alpha_2} = O(\varepsilon). \end{aligned}$$

Hence also $\alpha_2^{1/2}\, L = O(\varepsilon)$, which is what we require.

We now look at R. We need to estimate

$$\int_{-\infty}^{\infty} e^{-[A(s_x+u)-\alpha_0-\alpha_1 u]} e^{B(s_x+u)-\beta_0}\, du.$$

By Taylor's theorem,

$$A(s_x + u) - \alpha_0 - \alpha_1 u\,[B(s_x + u) - \beta_0] = -\beta_0 u \; + \frac{1}{2}(A''(s_x + \xi)) - B''(s_x + \xi)u^2$$

for some $\xi \in (0, u)$. Taking into account (21), (22), and the motonicity of A'', we continue with

$$\begin{aligned} A(s_x + u) - \alpha_0 - \alpha_1 u\,[B(s_x + u) - \beta_0] &= o(\alpha_2^{1/2})u \\ &+ \frac{1}{2}(A''(s_x + \xi) + o(A''(s_x + \xi)))u^2 \\ &\ge o(\alpha_2^{1/2})u + (\frac{1}{2} + o(1))\alpha_2 u^2. \end{aligned}$$

Henc e

$$\begin{aligned} \int_{\delta}^{\infty} e^{-[A(s_x+u)-\alpha_0-\alpha_1 u]} e^{B(s_x+u)-\beta_0}\, du &\le \int_{\delta}^{\infty} e^{-\left[o(\alpha_2^{1/2})u+(1/2+o(1))\alpha_2 u^2\right]} du \\ &\le \alpha_2^{-1/2} \int_{\delta\alpha_2^{1/2}}^{\infty} e^{-[o(1)v+(1/2+o(1))u^2]} dv = O(\varepsilon). \end{aligned}$$

Hence $\alpha_2^{1/2} R \; = O(\varepsilon)$, which is what we needed.

We now estimate C. For $j = 0, 1, 2$, we need to estimate

$$C^j = \int_{|u|<\delta} u^j e^{-[A(s_x+u)-\alpha_0-\alpha_1 u]} e^{B(s_x+u)-\beta_0} du \ .$$

We no w need to take the Taylor series analysis given in the pro f outline one step further. By Taylor's theorem, we have

$$e^{-[A(s_x+u)-\alpha_0-\alpha_1 u]} e^{B(s_x+u)-\beta_0} = -\frac{1}{2}\alpha_2 u^2 - \frac{1}{6}\alpha_3 u^3 - \frac{1}{24}\hat{\alpha}_4 u^4 + \beta_1 u + \frac{1}{2}\hat{\beta}_2 u^2.$$

Here we use decoration to indicate terms which must be evaluated away from s_x: $\hat{\alpha}_4 = A^{(4)}(w)$ for some $w, |s_x - w| < \delta, \hat{\beta}_2 = B''(w')$ for some $w', |s_x - w'| < \delta$. We separate the main quadratic term, the odd powers, and the error term. We set $D = -\frac{1}{6}\alpha_3 u^3 + \beta_1 u$ and $E = \frac{1}{24}\hat{\alpha}_4 u^4 + \frac{1}{2}\hat{\beta}_2 u^2$, Set $A = \alpha_2^{3\alpha-\varepsilon}$ and note that $3\alpha - \varepsilon < -1/8$. Noting (15), (16), (21), and (22) and Lemma (3.1.8), we have $D = O(A), E = O(A^2)$, and $D^2 = O(A^2)$. Taking into account that $A \to 0$, we have

$$\begin{aligned} C^j &= \int_{|u|<\delta} u^j e^{-\alpha_2 u^2/2} e^D e^E du = \int_{|u|<\delta} u^j e^{-\alpha_2 u^2/2}(1 + D + O(D^2))(1 + O(E))du \\ &= \int_{|u|<\delta} u^j e^{-\alpha_2 u^2/2}(1 + D + O(A^2))du \\ &= \int_{|u|<\delta} u^j e^{-\alpha_2 u^2/2}(1 + D)du + O(A^2)\int_{|u|<\delta} u^j e^{-\alpha_2 u^2/2} du \\ &= \int_{|u|<\delta} u^j e^{-\alpha_2 u^2/2}(1 + D)du + O(A^2)\alpha_2^{-(j+1)/2}. \end{aligned}$$

For $j = 0,2, \int_{|u|<\delta} u^j D e^{-\alpha_2 u^2/2} du$ is the integral of an odd function over a symmetric interval and hence we can drop *D* from the expressions for c_0 and c_2. We next pass to integrals over the entire line. This introduces an error of $O(\varepsilon)$, which we absorb into the larger error terms. We have

$$C_0 = \int_{-\infty}^{\infty} e^{-\alpha_2 u^2/2} du + O(A^2)\alpha_2^{-1/2} = \alpha_2^{-1/2}(\sqrt{2\pi} + O(A^2));$$

$$C_2 = \int_{-\infty}^{\infty} u^2 e^{-\alpha_2 u^2/2} du + O(A^2)\alpha_2^{-3/2} = \alpha_2^{-3/2}\left(\sqrt{2\pi} + O(A^2)\right).$$

For $j = 1$ the integral involving $u^j\ e^{-\alpha_2 u^2/2}$ vanishes; and we have, using (15), (17), and (21),

$$\begin{aligned} C_1 &= \int_{-\infty}^{\infty} u e^{-\alpha_2 u^2/2} D du + O(A^2)\alpha_2^{-1} = O(\alpha_2^{-3/2}\ \beta_1) + O(\alpha_2^{-5/2}\alpha_3) + O(A^2)\alpha_2^{-1} \\ &= \alpha_2^{-1}(\alpha_2^{-\varepsilon+2\alpha} + \alpha_2^{-\varepsilon} + A^2) = \alpha_2^{-1}(\alpha_2^{-\varepsilon+2\alpha}). \end{aligned}$$

Hence

$$e^{A^*(x)} e^{\beta_0} I_0 = e^{A^*(x)} e^{\beta_0} \alpha_2^{-1/2}\left(\sqrt{2\pi} + O(A^2)\right), \tag{27}$$

$$e^{A^*(x)} e^{\beta_0} I_1 = e^{A^*(x)} e^{\beta_0} \alpha_2^{-1} O(\alpha_2^{-\varepsilon+2\alpha}) \quad (28),$$

$$e^{A^*(x)}e^{\beta_0}I_2 = e^{A^*(x)}e^{\beta_0}\alpha_2^{-3/2}\left(\sqrt{2\pi} + O(A^2)\right). \tag{29}$$

This gives us (23).

We now proceed to verify (24), (25), and (26). For appropriate K, we have

$$\gamma(x) = e^{A^*(x)}e^{\beta_0}\int_{-\infty}^{\infty} K(s_x + u)\, du.$$

If we differentiate (9) and then follow the same pattern of analysis we find

$$\gamma^{(j)}(x) = e^{A^*(x)}e^{\beta_0}\int_{-\infty}^{\infty} (s_x + u)^j K(s_x + u)\, du$$

for $j = 1, 2$. If we set

$$J_j = \int_{-\infty}^{\infty} u^j K(s_x + u)\, du\,, \quad j = 0\,, 1\,, \ldots,$$

then we can write

$$\begin{aligned} \gamma &= e^{A^*}e^{\beta_0}J_0, \\ \gamma' &= e^{A^*}e^{\beta_0}(s_x J_0 + J_1)\,, \\ \gamma'' &= e^{A^*}e^{\beta_0}(s_x^2 J_0 + 2s_x J_1 + J_2)\,. \end{aligned} \tag{30}$$

From Lemma(3.1.1), we know that $A^{*\prime}(x) = s_x$ *and* $A^{*\prime\prime}(x) = A''(x)^{-1}$. Hence the quantities we want to estimate are

$$(\log\gamma)' = s_x + \frac{J_1}{J_0}, \qquad A^{*\prime} - (\log\gamma)' = -\frac{J_1}{J_0},$$

$$A^{*\prime} - (\log\gamma)'' = A''^{-1} - \frac{J_2}{J_0} + \left(\frac{J_1}{J_0}\right)^2.$$

From (27) and (28), we know that

$$-\frac{J_1}{J_0} = \frac{e^{A^*(x)}e^{\beta_0}\alpha_2^{-1}\, O(\alpha_2^{2\alpha-\varepsilon})}{e^{A^*(x)}e^{\beta_0}\alpha_2^{-1/2}\left(\sqrt{2\pi} + O(\alpha_2^{6\alpha-2\varepsilon})\right)} = \alpha_2^{-1}\, O(\alpha_2^{2\alpha-\varepsilon}) = O(\alpha_2^{-1+6\alpha-2\varepsilon}),$$

This gives (25). Also, noting that $s_x \to \infty$, we have (24). This also gives the required estimate for $(J_1/J_0)^2$ in (25). We complete (25) by noting

$$\begin{aligned} A''^{-1}(s_x) - \frac{J_2}{J_0} &= \alpha_2^{-1} - \frac{e^{A^*(x)}e^{\beta_0}\alpha_2^{-3/2}\left(\sqrt{2\pi} + O(\alpha_2^{6\alpha-2\varepsilon})\right)}{e^{A^*(x)}e^{\beta_0}\alpha_2^{-1/2}\left(\sqrt{2\pi} + O(\alpha_2^{6\alpha-2\varepsilon})\right)} \\ &= \alpha_2^{-1} - \alpha_2^{-1}(\,1 + O(\alpha_2^{6\alpha-2\varepsilon})) = O(\alpha_2^{-1+6\alpha-2\varepsilon})), \end{aligned}$$

as required.

We start with positive functions c and d defined on $\boldsymbol{R}^+$. We want to estimate $f(z) = \sum c(n)^{-1}d(n)\, z^n$. Here *c* will be our main term with *d* a slowly varying correction. We will do our computational work with the auxiliary functions

$$\Gamma(x) = log\ c(x), \tag{31}$$
$$\Delta(x) = log\ d(x). \tag{32}$$
Let Γ^* be the conjugate function to Γ and set $x_s = x(s) = \Gamma'^{-1}(s) = \Gamma^{*\prime}(s)$. We suppose that $1/4 < \varepsilon < 1/2$ is such that as $x \to \infty$
$$\Gamma(x), \Gamma'(x), \Gamma''(x), -\Gamma^{(3)}(x) > 0, \tag{33}$$
$$\Gamma(x), \Gamma'(x) \to \infty\ , \tag{34}$$
$$\Gamma''(x), \Gamma^{(3)}(x), \Gamma^{(4)}(x) \to 0, \tag{35}$$
$$\Gamma'''(x) = O\left(\Gamma''^{\frac{3}{2}+\varepsilon}(x)\right), \tag{36}$$
$$\Gamma^{(4)}(x) = O(\Gamma''^{2+2\varepsilon}(x)). \tag{37}$$
Note that if this holds for e, then it also holds for any ε' such that $1/4 < \varepsilon' < \varepsilon$.
In analogy to the previous theorem, the core hypothesis for the proof is no w that we can find an auxiliary function λ such that $\lambda^2\Gamma'' \to \infty$ and $\lambda^3\Gamma''' \to 0$. Assumption (36) allows us to use
$$\lambda(s) = \Gamma''(x_s)^{-\frac{1}{2}-\beta} \tag{38}$$
for some $\beta, 0 < \beta < \varepsilon/\ 3 - 1/12$, which we now regard as selected and fixed. As before, a convenient choice to keep in mind is $\varepsilon = (1/2)^-, \beta = 0^+$.
Lemma (3.1.9) [106]: *If we have (36), (37), and (38), then we also have*
$$\sup_{|\theta|<\lambda} |\Gamma''(x_s + \theta)| = \big(1 + o(1)\big)|\Gamma''(x_s)|\ , \tag{39}$$
$$\sup_{|\theta|<\lambda} |\Gamma'''(x_s + \theta)| = O\left(\Gamma'''(x_s)^{\frac{3}{2}+\varepsilon}\right), \tag{40}$$
$$\sup_{|\theta|<\lambda} \left|\Gamma^{(4)}(x_s + \theta)\right| = O\big(\Gamma^{(4)}(x_s)^{2+2\varepsilon}\big). \tag{41}$$
Proof. The proof is the natural modification o f the proof of Lemma (3.1.6).
Suppose that d is a positive C^2 function. We say that d is slowly varying in the second sense with respect to c, ε, and β, and write $d \in SV\ I\ I\ (c, \varepsilon, \beta)$, if, as $s \to \infty$,
$$\Gamma'(s) - \Delta'(s) \to \infty, \tag{42}$$
$$\Delta' = O\left(\Gamma^{''\frac{1}{2}+\varepsilon-\beta}\right), \tag{43}$$
$$\Delta'' = O\big((\Gamma''^{1/2+\varepsilon-\beta})\big)^2. \tag{44}$$
Note that the assumptions imply
$$\frac{3}{4} < \frac{1}{2} + \varepsilon - \beta < 1.$$
In analogy with A and B, these estimates on F imply interval estimates for Δ.
For these hypotheses the model case is
$$\Gamma(x) = \frac{x+1}{\gamma}\log\frac{x+1}{\gamma} - \frac{x+1}{\gamma}, \qquad \Gamma'(x) = \frac{1}{\gamma}\log\frac{x+1}{\gamma}, \qquad \Gamma''(x) = \frac{1}{\gamma(x+1)}$$
and ε can be chosen as close as desired to 1/2.

Set

$$c_i = \Gamma^{(i)}(x_s), \qquad i = 0, 1, 2, \ldots$$
$$d_i = \Delta^{(i)}(x_s), \qquad i = 0, 1, 2, \ldots$$
$$\sigma = x_s - [x_s].$$

Define the scaled parameters

$$C_3 = c_3 c_2^{-3/2},$$
$$E = d_1 c_2^{-1/2} - \frac{1}{2} c_3 c_2^{-3/2},$$
$$\Theta = \Theta(\theta, 0) = \theta c_2^{-1/2},$$
$$\Theta(n) = \Theta(\theta, n) = (\theta + 2\pi n) c_2^{-1/2}.$$

Note that the previous assumptions insure that $C_3, E = o(1)$.
Again we want notation for a class o f small error terms. We write $Y = Y(x_s) = O(\mathcal{F})$ i f for some $> 0, Y = O(exp(-\Gamma''(x_s)^{-c}))$.
Set

$$\tau = 2\varepsilon - 6\beta - \frac{1}{2}. \tag{45}$$

The assumptions on ε and β insure that $0 < \tau < 1/2$.

Theorem (3.1.3) [106]: *Suppose Γ satisfies (33), (34),(35), (36), (37), and hence also(39), (40), and (41) and that $d \in SV\,I\,I\,(c, \varepsilon, \beta)$. Set $f(z) = \sum_0^\infty d(n) c(n)^{-1} z^n$. Then f is entire and, as $s \to \infty$, has the asymptotic growth*

$$f\left(e^{s+i\theta}\right) = e^{\Gamma^*(s)} \frac{\sqrt{2\pi}}{\sqrt{\Gamma''(x)}} d(x_s) e^{i\theta x_s} \left[e^{-\frac{1}{2}\Theta^2}(1 + i\Theta E + i\Theta^3 C_3) + O\left(c_2^{\frac{1}{2}+\tau}\right) + O(\mathcal{F}) \right]. \tag{46}$$

In particular, fo $c_2^{1/2-\beta} < |\theta| < \pi$

$$\left|f\left(e^{s+i\theta}\right)\right| = f(e^s) O\left(c_2^{\frac{1}{2}+\tau}\right). \tag{47}$$

Note: As with the previous theorem, the formulation in (46) is redundant. The $O(\mathcal{F})$ error term, which is smaller, is intrinsic to the method; the other could be mechanically refined. Hence we present both.
we use the results of this theorem as input for Theorem (3.1.2). To do that, we require certain estimates on the derivatives of *f* on the axis. We present those estimates as a lemma now because it is convenient to include their proof along with the proof of Theorem (3.1.3).Lemma: *.In the situation of Theorem (3.1.3), we have the following additional estimates:*

$$z \frac{d}{dz} f(e^s) = x_s L_o + L_1, \tag{48}$$

$$\left(z\frac{d}{dz}\right)^2 f(e^s) = x_s^2 L_o + 2x_s L_1 + L_2, \tag{49}$$

where for $j = 0,1,2$

$$L_j = e^{\Gamma^*(s)+\Delta(x_s)}\sqrt{\frac{2\pi}{c_2}}\left[\sqrt{\frac{c_2}{2\pi}}J_j + O(c_2^{1/2+\tau}) + O(\mathcal{F})\right] \tag{50}$$

with

$$J_0 = 2\pi\frac{1}{\sqrt{c_2}}, \qquad J_1 = \sqrt{2\pi}\,\frac{-c_3+2d_1c_2}{2(\sqrt{c_2})^5}, \qquad J_2 = \sqrt{2\pi}\frac{1}{(\sqrt{c_2})^3}. \tag{51}$$

Proof. The hypothesis (42) insures that $lim(d(n)c(n)^{-1})^{1/n} = 0$. Thus f is entire. We need to estimate

$$\left(z\frac{d}{dz}\right)^2 f(z) = \sum_0^\infty n^i d(n)c(n)^{-1}z^n, \qquad i = 0,1,2\,.$$

We split the sum into a central part and tails. The tails will be estimated by the corresponding integrals, using analysis similar to that in the previous proof. In order to capture the cancellation when $\theta \neq 0$, we treat the central part differently.

The analysis of the tail terms is not essentially changed by the factors n^i; hence we present the estimates only for $i = 0$. We have

$$f(x) = \sum d(n)c(n)^{-1}x^n = \sum e^{n\log x - \Gamma(n)+\Delta(n)}.$$

Writing $x = e^s$, we have $f\left(e^{s+i\theta}\right) = \sum \exp(ns - \Gamma(n) + \Delta(n) + i\theta)$. Fix *s* large. For this $s, xs - \Gamma(x)$ has its maximum at x_s. Set $u = n - x_s$. Thus $e^{in\theta} = e^{ix_s\theta}e^{iu\theta}$. For typographic convenience, we set

$$\Omega = -(\Gamma\,(x_s+u) - c_2 - c_1u\,) + \Delta(x_s+u) - d_0.$$

Bringing a factor of $e^{\Gamma^*(n)}e^{ix_s\theta}$ outside the sum, recalling that $c_1 = \Gamma'(x_s) = s$, and doing a bit o f rearranging, we find

$$f(e^{s+i\theta}) = e^{\Gamma^*(s+\Delta(s)+i\theta x_s)}\sum_{-x_s}^{\infty} e^{\Omega+iu\theta}\,.$$

We need to show

$$\sqrt{\frac{c_2}{2\pi}}\sum_{-x_s}^{\infty} e^{\Omega+iu\theta} = e^{-\frac{1}{2}\Theta^2}(1 + i\Theta E + i\Theta^3 C_3) + O\left(c_2^{\frac{1}{2}+\tau}\right) + O(\mathcal{F}). \tag{52}$$

We use $\lambda = \lambda(s)$ as given by (38) to split the range of summation into three parts, again *L, C, and R.* We start the analysis with *L*. We drop the unimodular factor and dominate the sum by the corresponding integral. That is,

$$\sqrt{\frac{c_2}{2\pi}}L = O(1)\sqrt{\frac{c_2}{2\pi}}\int_{-x_s}^{-\lambda} e^{\Omega}du\,. \tag{53}$$

We now estimate the integrand. We have

$$\Gamma(x_s+u)-c_0-c_1u=\int_u^0 (r-u)\Gamma''(x_s+r)\,dr.$$

By the monotonicity of Γ'' we see that if $u<-\lambda$ then

$$\Gamma(x_s+u)-c_0-c_1u \geq \int_{-\lambda}^0 (r-\&u)\Gamma''(x_s+r)dr \geq \Gamma''(x_s)\int_{-\lambda}^0 (r-\&u)dr$$

$$=\frac{1}{2}\Gamma''(x_s)\lambda(2u+\lambda).$$

Thus, we continue (53) with

$$\sqrt{\frac{c_2}{2\pi}}L=O(1)\sqrt{\frac{c_2}{2\pi}}e^{\lambda^2 c_2/2}\int_{-x_s}^{-\lambda} e^{c_2 u\lambda+\Delta(x_s+u)-d_0}\,du$$

$$=O(1)e^{\lambda^2 c_2/2}\int_{-\infty}^{-\lambda\sqrt{c_2}} e^{w\lambda\sqrt{c_2}+[\Delta(x_s+w/\sqrt{c_2})-d_0]}\,dw.$$

We need to estimate the integral. Using (43) to estimate $[\Delta(x_s+w/\sqrt{c_2})-d_0]$in the integral, we find that, for some positive K,

$$\sqrt{\frac{c_2}{2\pi}}\int_{-\infty}^{-\lambda\sqrt{c_2}} \ldots\,dw=O(1)e^{\lambda^2 c_2/2}\int_{-\infty}^{-\lambda\sqrt{c_2}} e^{w\lambda\sqrt{c_2}}e^{-Kw}\,dw$$

$$=O(1)\frac{e^{\lambda^2 c_2/2}e^{(-\lambda^2 c_2+K\lambda\sqrt{c_2})}}{\lambda\sqrt{c_2}-K}=O(1)\frac{e^{-\lambda^2 c_2/2+K\lambda\sqrt{c_2})}}{\lambda\sqrt{c_2}-K}$$

$$=O(1)O\left(exp\left(-\frac{\lambda^2 c_2}{2}+\lambda\sqrt{c_2}K\right)\right)=O(\mathcal{F}).$$

as required. We now look at R. I f $u\geq\lambda$, then

$$\Gamma(x_s+u)-c_0-c_1u=\int_0^u (u-r)\Gamma''(x_s+r)dr \geq \int_0^\lambda (u-r)\Gamma''(x_s+r)dr$$

$$\geq \Gamma''(x_s+\lambda)\int_0^\lambda (u-r)dr=\frac{1}{2}\lambda\Gamma''(x_s+\lambda)(2u-\lambda).$$

Set $\tilde{c}_2=\Gamma''(x_s+\lambda)$. Then

$$R=O(1)\int_\lambda^\infty e^{-\frac{1}{2}\lambda\tilde{c}_2(2u-\lambda)+\Delta(x_s+u)-d_0}\,du.$$

Thus

$$\sqrt{\frac{c_2}{2\pi}}R=O(1)\sqrt{c_2}\,e^{\frac{1}{2}\lambda^2\tilde{c}_2}\int_\lambda^\infty e^{-\lambda\tilde{c}_2 u+\Delta(x_s+u)-d_0}\,du.$$

Lemma (3.1.9) insures $\tilde{c}_2 \sim c_2$. The hypothesis (43) and the monotonicity of Γ'' insure $\Delta(x_s+u)-d_0=O(1)c_2^{1/2}u$. Thus we need to estimate

$$I = O(1)\sqrt{c_2}exp((1+o(1))c_2\lambda^2/2)\int_\lambda^\infty exp(-\lambda(1+0(I))c_2 u + O(I)c_2^{1/2}u)du$$
$$= O(1)\sqrt{c_2}exp((1 + o(1))c_2\lambda^2/2)\int_\lambda^\infty exp\left(\left[-(1+o(I))\lambda^2 c_2 + O(I)c_2^{1/2}\right]\frac{u}{\lambda}\right)du\,.$$

We know $\lambda c_2^{1/2} \to \infty$. Hence, for large s,

$$\left[-(1+o(1))\lambda^2 c_2 + O(1)\lambda c_2^{1/2}\right]\frac{u}{\lambda} \le \left[-\frac{2}{3}\lambda^2 c_2\right]\frac{u}{\lambda} = -\frac{2}{3}\lambda c_2 u.$$

Thus

$$I = O(1)\sqrt{c_2}exp((1+o(1))c_2\lambda^2/2)\int_\lambda^\infty exp(-2\lambda c_2 u/3)\,du$$
$$= O(1)\sqrt{c_2}exp((1+o(1))c_2\lambda^2/2)(c_2\lambda)^{-1}\,exp(-2\lambda^2 c_2/3)$$
$$= O(1)\frac{1}{\sqrt{c_2}}\lambda^{-1}exp((\frac{1}{2}-\frac{2}{3}+o(1))c_2\lambda^2) = O(\mathcal{F}),$$

as required.

We now need to estimate the center term,

$$\sum\nolimits_c = e^{\Gamma^*(s)+\Delta(x_s)+i\theta x_s}\sum_{|[u]|<\lambda} e^{\Omega+iu\theta}\,.$$

By Taylor's theorem, we have, for $|u| < \lambda$,

$$\Omega = -\frac{1}{2}c_2u^2 - \frac{1}{6}c_3u^3 + d_1u - \frac{1}{24}\hat{c}_4u^4 + \frac{1}{2}\hat{d}_2u^2.$$

Here $\hat{c}_4 = \Gamma^{(4)}(w)$ and $\hat{d}_2 = \Delta''(w')$, with $w, w' \in (x_s - \lambda, x_s + \lambda)$. Using (36), (37), (43), and (44), we find

$$|\hat{c}_4u^4| + |\hat{d}_2u^2| = O(c_2^{1/2+r}),$$
$$(|c_3u^3| + |d_1u|)^2 = O(c_2^{1/2+r}).$$

We have

$$\sum_{|[u]|<\lambda} e^{\Omega+iu\theta} = \sum\left(1 + d_1u - \frac{c_3u^3}{6} + O(c_2^{1/2+r})\right)exp\left(-\frac{c_2u^2}{2} + iu\theta\right)$$
$$= \sum\left(1 + d_1u - \frac{c_3u^3}{6}\right)exp\left(-\frac{c_2u^2}{2} + iu\theta\right)$$
$$+ \sum O(c_2^{1/2+r})exp\left(-\frac{c_2u^2}{2} + iu\theta\right)$$
$$= \sum\nolimits_1 + \sum\nolimits_2\,. \tag{54}$$

We estimate $\sum_2$ by passing to absolute values, estimating the truncated Gaussian sum by the corresponding Gaussian integral over the entire line, and then evaluating the integral. This gives $\sum_2 = O(c_2^{1/2+r})O(c_2^{-1/2}) = O(c_2^r)$, which is what we needed. (Recall from (52) that we pick up an additional factor of $O(c_2^{1/2})$ outside o f the sum $\sum_{|[u]|<\lambda}$.) Write $u = k - \sigma$ with $k \in \boldsymbol{Z}$ and $\sigma = x_s - [x_s]$. Now $\sum_1$ is a sum with the range $|k| < \lambda$. However, the natural estimates show that we change things only by $O(\mathcal{F})$ if we replace that with the sum over all integers. We do that and thus no w need to estimate

$$\sum_{-\infty}^{\infty}\left(1 + \mathrm{d}_1(\mathrm{k}-\sigma) - \frac{c_3(\mathrm{k}-\sigma)^3}{6}\right) exp\left(-\frac{c_2(\mathrm{k}-\sigma)^2}{2} + i(\mathrm{k}-\sigma)\theta\right).$$

By the Poisson summation formula ([114], p. 8), this equals $\sum_{-\infty}^{\infty} h(n)$, where

$$h(n) = \int_{-\infty}^{\infty}\left(1 + d_1(x-\sigma) - \frac{c_3(x-\sigma)^3}{6}\right)$$
$$\times\, exp\left(-\frac{c_2(x-\sigma)^2}{2} + i(x-\sigma)\theta\right)e^{2\pi inx}\ dx$$
$$= e^{2\pi in\sigma}\int_{-\infty}^{\infty}(1 + d_1y - c_3y^3\ /\ 6)exp(-c_2y^2\ /\ (2 + 2\pi n + \theta)iy)\,dy\,.$$

Starting with the formula for $\int_{-\infty}^{\infty} e^{-ty^2+2sy}dy$ ([115], p. 8) and differentiating with respect to *s, t,* and then both, we find

$$\int_{-\infty}^{\infty} e^{-ty^2+2sy}dy = \frac{\sqrt{\pi}}{\sqrt{t}}\ e^{s^2/t}, \tag{55}$$

$$\int_{-\infty}^{\infty} ye^{-ty^2+2sy}dy = \frac{s}{t}\frac{\sqrt{\pi}}{\sqrt{t}}\ e^{s^2/t}, \tag{56}$$

$$\int_{-\infty}^{\infty} y^2e^{-ty^2+2sy}dy = \left[\frac{1}{2t} + \left(\frac{s}{t}\right)^2\right]\frac{\sqrt{\pi}}{\sqrt{t}}\ e^{s^2/t}, \tag{57}$$

$$\int_{-\infty}^{\infty} y^3e^{-ty^2+2sy}dy = \left[\frac{3s}{2t^2} + \left(\frac{s}{t}\right)^2\right]\frac{\sqrt{\pi}}{\sqrt{t}}\ e^{s^2/t}. \tag{58}$$

For $n = 0$, direct computation gives

$$h(0) = e^{-\Theta^2/2c_2}\left(1 + \left(i\Theta\frac{d_1}{d_2} - \frac{1}{2}\left(\frac{c_3}{c_2^2}\right)\right) + i\frac{\Theta^3C_3}{c_2^3}\right)\frac{\sqrt{2\pi}}{\sqrt{c_2}}$$

or, in terms of the scaled parameters,

$$h(0) = e^{-\frac{1}{2}\Theta^2}(1 + i\Theta E + i\Theta^3C_3)\frac{\sqrt{2\pi}}{\sqrt{c_2}}.$$

In general,

$$h(n) = e^{-\frac{1}{2}\Theta(n)^2}(1 + i\Theta(n)E + i\Theta(n)^3C_3)\frac{\sqrt{2\pi}}{\sqrt{c_2}}e^{2\pi in\sigma}.$$

In general, $\sum_{n\neq 0}|h(n)|$ is dominated by a geometric series which is dominated by $h(0)O(\mathcal{F})$. However, this fails to be uniform in θ; in fact, $\Theta(-\pi,1)=\Theta(\pi,0)$. However, this is only an issue if $n=\pm 1$ and $e^{i\theta}$ is near the negative real axis. In that case, however, both terms are $O(e^{-c_2^{-1}})=O(\mathcal{F})$. Hence all the $h(n)$, for $n>0$, can be absorbed into the various error terms. Finally, notice that when $c_2^{-1/2-\beta}<|\theta|, h(0)=O(\mathcal{F})$. Hence the main term is the contribution associated to $\sum_2$ in (54), which we saw was $O(c_2^{-1/2+r})$. Thus (46) and (47) are done.

We now proceed to the proof of the lemma. For $j=1,2$ we want to estimate

$$\left(z\frac{d}{d(z)}\right)^j f(e^s)=e^{\Gamma^*(s)}d(x_s)\sum_{-x_s}^{\infty}(x_s+u)^j e^{\Omega}.$$

Straightforward manipulation shows that (48) and (49) hold with

$$L_j=e^{\Gamma^*(s)}d(x_s)\sum_{-x_s}^{\infty}u^j e^{\Omega}$$

for j = 0, 1, 2. We estimate L_1 and L_2 using the same type o f analysis as in the proof of the theorem (which, in fact, treated L_0). That is, the tails contribute an error that is $O(\mathcal{F})$, and the central part of the sum is analyzed using Poisson summation. The situation here is slightly easier because we only want estimates on the positive real axis. Hence the terms in the Poisson summation corresponding to $n\neq 0$ contribute a total error which is $O(\mathcal{F})$. This gives, up to an error term of $O(c_2^r)+O(\mathcal{F}), L_j=J_j$ This gives us (50) with

$$J_j=\int_{-\infty}^{\infty}y^j(1+d_1y-c_3y^3/6)exp(-c_2y^2/2)\,dy$$

For $j=0,1,2$. Evaluating those integrals using (55) - (58) then produces the statements in the lemma.

We shall use the output of Theorem (3.1.2) as input for Theorem and then use the output from Theorem (3.1.3) as input for Theorem (3.1.2). Here we collect the bookkeepin g lemmas which show that the functions which arise in this process satisfy the required hypotheses.

First, suppose that we have *a* and *b* which satisfy the hypotheses of Theorem (3.1.2), that *A* and *B* are given by (12) and (13), and that $\{\gamma(n)\}$ are the associated moments . We want to use $\{\gamma(n)^{-1}\}$ as power series coefficients in a way which keeps the focus on a as the primary term. To do this, we define e,d by $c(x)=exp(A^*(x)), d(x)=c(x)\gamma(x)^{-1}$ and define Γ and Δ b y (31) and (32).

Lemma (3.1.10) [106]: *Suppose a,ε, and α satisfy the hypothese s of Theorem (3.1.2) and $b\in SV\,I\,(a,\varepsilon,\alpha/3)$. Then, with the same ε, with $\beta=\alpha$, and with $\lambda=\Gamma''^{-1/2-\beta}$, the data ε,β,Γ, and Δ satisfy the hypotheses of Theorem (3.1.3). That is, with the same ε, Γ satisfies (33), (34), (35), (36), a n d 0 7) , and $d\in SV\,I\,I\,(c,\varepsilon,\beta)$.*

Proof. The statements about Γ follow from the hypotheses on *A*, the fact that $\Gamma=A^*$, and Lemma (3.1.1).

To see that $d \in SV\,I\,I\,(c\,,\varepsilon,\beta)$, note that

$$\begin{aligned}\Delta &= log\ c\ - \log\gamma \\ &= \Gamma - \log\gamma \\ &= A^* - \log\gamma\end{aligned}$$

Hence , by (24), (25), and (26), Δ satisfies (43) and (44).

Suppose, now, that we had a, that $b = 1$, that we had a choice of α, and that we then invoked Theorem (3.1.2) with the choice $\alpha^* = \alpha/3$. Of course, for b constant function, $d \in SV\,I(\alpha\,,\varepsilon,\alpha^*)$. Noting the previous lemma, we can then apply Theorem (3.1.3) to the functions c and d just described. That will produce an entire function f . Suppose we have a fixed > -1. We want to apply Theorem (3.1.2) to the functions a_σ and b_σ, selected so that $a_\sigma b_\sigma = af^{-\sigma}$. We set $a_\sigma = e^{-\sigma A}, b_\sigma = e^{\sigma A} f^{-\sigma}$ *and* $A_\sigma(s) = -\,log\ a_\sigma(e^s) - s, B_\sigma(s) = log\ b_\sigma(e^s)$.

Lemma (3.1.11) [106]: *Using a new smaller* α*, we can apply Theorem (3.1.2) to the functions* a_σ *and* b_σ*. That is, for a smaller* α*,* A_σ *satisfies (14),(15), (16), (19), and (20). Furthermore,* $b_\sigma \in SV\,I(a_\sigma,\varepsilon,\alpha)$.

Proof. Our choice $c(x) = exp(A^*(x))$ in Theorem (3.1.3) gives $\Gamma = A^*$ and hence $= \Gamma^* = A^{**} = A$. Thus

$$\begin{aligned}A_\sigma(s) &= -log a_\sigma(e^s) - s \\ &= -\,log\ a(e^s) - s + \sigma A(s) \\ &= (1+\sigma)A(s),\end{aligned}$$

and the conclusions for A_σ are immediate. We have $B_\sigma(s) = \text{In}\, b_\sigma(e^s) = \sigma(\Gamma^*(s) - log\ f(e^s))$. Hence we need estimates for $(\Gamma^* - log\ f)'$ and $(\Gamma^* - log\ f)''$. Set $= zd/dz$. Direct computation yields

$$(log\ f(e^s))'(e^s) = \frac{\mathcal{D}f(e^s)}{f(e^s)},$$

$$(log\ f(e^s))'(e^s) = \frac{\mathcal{D}^2 f(e^s)}{f(e^s)} - \left(\frac{\mathcal{D}f(e^s)}{f(e^s)}\right)^2$$

(Recall that $\Gamma^{*\prime}(s) = x_s = A'(s)$ and, by Lemma (3.1.1), $\Gamma^{*\prime\prime} = \Gamma^{\prime\prime -1} = c_2^{-1}$. Thus, using (48) and (49), we have

$$\begin{aligned}(\Gamma^* - log\ f)' &= \Gamma^{*\prime}(s) - \frac{\mathcal{D}f\ (e^s)}{f\ (e^s)}, \\ &= x_s - \left(\frac{x_s L_0 + L_1}{L_0}\right) = -\frac{L_1}{L_0}. \\ &= -\frac{\sqrt{c_2/\pi}J_1 + O(c_2^{1/2+r})}{\sqrt{c_2/\pi}J_0 + O(c_2^{1/2+r})}\end{aligned}$$

The last equality follows by using (50) and absorbing $O(\mathcal{F})$ into the other, larger, error term. Using the values of J_0 and J_1, we continue with

$$(\Gamma^* - log\ f)' = \frac{(-c_3 + 2d_1c_2)/2c_2^2 + O(c_2^{1/2+r})}{1 + O(c_2^{1/2+r})}$$

$$= -\left(\frac{-c_3 + 2d_1 c_2)}{2c_2^2} + O(c_2^{1/2+r})\right)\left(1 + O(c_2^{1/2+r})\right)$$
$$= \left(O(c_3 c_2^{-2}) + O(d_1 c_2^{-1}) + O(c_2^{1/2+r})\right)\left(1 + O(c_2^{1/2+r})\right)$$
$$= O\left(c_2^{-1/2-\varepsilon+\beta}\right)$$
$$= O(A''^{1/2-\varepsilon+\beta}).$$
Using this estimate for L_1/L_0, we analyze the second derivative by
$$(\Gamma^* - \log f)'' = \Gamma^{*\prime\prime}(s) - \frac{\mathcal{D}^2 f(e^s)}{f(e^s)} + \left(\frac{\mathcal{D} f(e^s)}{f(e^s)}\right)^2$$
$$= \Gamma''^{-1}(x_s) - \left(\frac{x_s^2 L_0 + 2x_s L_1 + L_2}{L_0}\right) + \left(\frac{x_s L_0 + L_1}{L_0}\right)^2$$
$$= c_2^{-1} - \frac{L_2}{L_0} + \left(\frac{L_1}{L_0}\right)^2$$
$$= c_2^{-1} - \frac{\sqrt{c_2/2\pi} J_2 + O(c_2^{1/2+r})}{\sqrt{c_2/2\pi} J_0 + O(c_2^{1/2+r})} + \left(O(A''^{1/2-\varepsilon+\beta})\right)^2$$
$$= c_2^{-1} - \frac{c_2^{-1} + O(c_2^{1/2+r})}{1 + O(c_2^{1/2+r})} + \left(O(A''^{1/2-\varepsilon+\beta})\right)^2$$
$$= c_2^{-1} - c_2^{-1} + c_2^{-1} O(c_2^{1/2+r}) + \left(O(A''^{1/2-\varepsilon+\beta})\right)^2$$
$$= O(c_2^{1/2+r}) + \left(O(A''^{1/2-\varepsilon+\beta})\right)^2$$
$$= O(A''^{1/2\varepsilon+\beta}).$$
This gives the required estimates for B' and B''.
we suppose that m is given and fixed and that $A(s) = -\log m(e^s) - s$ satisfies the hypotheses of Theorem (3.1.2) for some selected ε, α. We use the notation of Theorem (3.1.2) and its proof and of Theorem and its proof with the choice $\Gamma = A^*$. In particular, we denote the derivatives of A by α's and of Γ by c's.
Many of our estimates will be in terms of the function A''. We would like to be able to relate those estimates both to the starting function m and to the function φ defined by $m(|z|^2) = \exp(-2\varphi(z))$, which is often used as a parameterization in this context. B y straightforward calculation, we have
$$A''(\log x^2) = -\left(x\frac{d}{dx}\right)^2 (\log m)(x^2) = x^2(\Delta\varphi)(x).$$
Let H_m be the weighted Bergman space,
$$H_m = L^2\left(\mathbb{C}, m\,(r^2)\frac{r\,dr\,d\theta}{\pi}\right) \cap Hol.$$
For each $w \in \mathbb{C}$, there is a Bergman kernel function $k_m = k_{m,w}$ which is characterized as that element of H_m which satisfies $f(w) = \langle f, k_w \rangle$ for all f in H_m. Because the monomials are an orthogonal basis *of* $H_m, k_w(z) = \sum_{n=0}^{\infty} \|z^n\|^{-2} (\bar{w} z)^n$. Thus, setting

$\gamma_n = \int_0^\infty x^n m(x)\, dx$ and $K(z) = k_m(z) = \sum_{n=0}^\infty \gamma_n^{-1} z^n$, we have$k_w(z) = K(\overline{w}z)$. We are interested in estimating k_w and related objects. We start with *m*, use Theorem (3.1.2) to estimate the γ's in terms of *m*, and then use those estimates in Theorem (3.1.3) to estimate *K* . There is no loss of generality in assuming that *w* is real and positive, and we make that assumption for the rest.

In describing various small quantities, we use the shorthand

$$S(x) = A''(log\ x)^{-1} .$$

Here is our main estimate for the Bergma n kernel.

Theorem (3.1.12) [106]: *As* $\to \infty$, *for* $|\theta| \le S(wr)^{1/2-6\alpha}$,

$$k_w\left(re^{i\theta}\right) = e^{A(\log wr)}\ A''(log\ wr)e^{i\theta A'(\log wr)}\left(e^{-A''(log\ wr)\theta^2\backslash 2} + O\left(S(wr)^{1\backslash 2+r}\right)\right). \quad (59)$$

and thus

$$k_w(re^{i\theta})m(wr\ )\sim \frac{A''(log\ wr)}{wr}\ e^{-A''(log\ wr)\theta^2\backslash 2}. \quad (60)$$

On the diagonal,

$$k_w(w) = e^{A(2\log\ wr)}A''(2log\ w)(1 + O(S(w^2)^{1\backslash 2+r})). \quad (61)$$

Far from the axis, $|\theta| \le S(wr)^\gamma$ *for (any) fixed* $\gamma > 0$,

$$k_w\left(re^{i\theta}\right) = k_w(r)O(S\left(w^2\right)^{1\backslash 2+r})). \quad (62)$$

Note: Recall from Theorem (3.1.3) that $\tau = 2\varepsilon - 6\beta - 1/2$ and $0 < \tau < 1/2$.

Proof. We apply Theorem (3.1.2) with the choice $A(s) = -log\ m(e^s) - s$, B =0. Let γ be the moment function we obtain. Lemma (3.1.10) insures that we can then use Theorem (3.1.3) with the choices $c = exp(A^*), d\ = exp(A^*)\gamma^{-1}$ (and thus $c^{-1}d = \gamma^{-1}$). Theorem (3.1.3) shows that on the positive axis

$$f(e^s)\sim e^{\Gamma^*(s)}\ \frac{\sqrt{2\pi}}{\sqrt{\Gamma''(x_s)}}\ d(x_s)(1 + O(S(w^2)^{1\backslash 2+r}).$$

We have $\Gamma = A^*$ and hence $\Gamma^* = A^{**} = A$, the last by Lemma (3.1.1). We also know, from that lemma, that $\Gamma'' = A'' = A''^{-1}$ and hence $c_2 = A''^{-1}$. Finally, $=$ $\sqrt{A''}Texp(B)/\sqrt{2\pi}$. In this case, $B = 0$ and hence $f(e^s)\sim e^A A''$. From the definitions, we have $e^{A(\log t)} = 1/(tm(t))$. Recalling that $K = f$ gives (59). The other estimates follow b y restricting to appropriate θ.

From this theorem, we get an asymptotic version of (A).

Corollary (3.1.13) [106]:

$$m(r^2)k_r(r)\sim\frac{A''(\log r^2)}{r^2}.$$

In particular, if $m(r)\sim ar^b e^{-cr^d}s(r)$, *where* $a, b, c, d > 0$, *and* $s \in SVI(ar^b e^{-cr^d}, \varepsilon, \alpha)$ *for* ε, α *allowed in Theorem (3.1.2), then*

$$m(r^2)k_r(r) \sim\ cd^2 r^{2d-2}.$$

Theorem (3.1.12) is not enough to give a version of (B). It shows that $log\, k_m = (-\, log m)(1 + o(1))$; but to get to a version of (B), we need to know that a similar estimate holds after we apply $(xd/dx)^2$ to each side. For that reason, we need to invoke (9) and Theorem (3.1.2) again.

Theorem (3.1.14) [106]: Fix $\sigma > 0$ and set $K_{\sigma,w} = k_{mK_m^{-\sigma},w}$. As $r \to \infty$, for $|\theta| \leq S(r)^{1\backslash 2-6\alpha}$

$$K_{\sigma,w}(re^{i\theta}) \sim (1+\sigma)K_{m,w}(re^{i\theta})^{1+\sigma}.$$

For $\pi \geq |\theta| > S(r)^{1\backslash 2-6\alpha}$,

$$K_{\sigma,w}(re^{i\theta}) \sim K_{\sigma,w}(r)O(\,S(r)^{1\backslash 2+r})\,.$$

Proof. We have $A(s) = -log\, m(e^s) - s$. We want to apply Theorem (3.1.2) with the choices $a_\sigma = \; m\, e^{-\sigma A}$ *and* $b_\sigma = e^{\sigma A}K_m(x\,)^{-\sigma}$. The associated function A_σ is

$$A_\sigma(s) = -\log a_\sigma\,(e^s)\; - s = -\log m\,(e^s) - s - \sigma A(s) = (1+\sigma)A(s)\,.$$

We saw in the proof of Theorem (3.1.12) that $K_m(e^s) \sim e^A A''b = e^A A''$. *The* same argument applied to a_σ and b_σ gives

$$\begin{aligned} K_{\sigma,w}(x) &\sim e^{A_\sigma}A''_\sigma b_\sigma \\ &= e^{(1+\sigma)A}(1+\sigma)A''b_\sigma \\ &= e^{(1+\sigma)A}(1+\sigma)A''\left(e^A K_{m,w}(x\,)^{-1}\right)^\sigma \\ &\sim e^{(1+\sigma)A}(1+\sigma)A''(A'')^\sigma \\ &= (1+\sigma)(e^A A'')^{1+\sigma} \\ &\sim (1+\sigma)K_{m,w}(x)^{1+\sigma}. \end{aligned}$$

For small θ, the proof of Theorem (3.1.3) goes through with $\Theta(n)^2$ increased by a factor of $1+\sigma$. For large θ, the argument in that proof gives the required estimates.

Rather than integrate these estimates to get an asymptotic version of (C), we do a slightly more general computation in the following.

The Berezin transform B_m is a valuable tool for studying Toeplitz operators on H_m. For a smooth function $F, B_m(F)$ is defined by

$$\begin{aligned} B_m, F(w) &= \; \langle F\frac{k_w}{\|k_w\|}, \frac{k_w}{\|k_w\|}\rangle \\ &= \int\int_{\mathbb{C}} F(z)\frac{|K(\overline{w}z)|^2}{K(|w|^2)}m(|z|^2)\frac{dxdy}{\pi}\,. \end{aligned} \tag{63}$$

If we look at the Fock spaces, $m_\sigma(|z|^2) = exp(-(1+\sigma)\,|z|^2)$, then we have

$$\begin{aligned} B_{m_\sigma}\, F(w) &= \int\int_{\mathbb{C}} F(z)e^{-(1+\sigma)|z-w|^2}\frac{dxdy}{\pi} \\ &= F(w) + \frac{1}{4}\frac{1}{(1+\sigma)}\Delta F(w) + O\left(\frac{1}{(1+\sigma)^2}\right)\,. \end{aligned}$$

We would like analogues of these formulas for our more general weights. The general theory of reproducing kernels insures that the Berezin measure

$$d\mu = \frac{|K(\overline{w}z)|^2}{K(|w|^2)}m(|z|^2)\frac{dxdy}{\pi}$$

is always a probability measure. We now want to study $d\mu$ using our asymptotic estimates on the kernel function. First, however, we introduce a further restriction on a, which we formulate in terms o f the auxiliary function A o f (4). We require $A''(s)$ to be dominated by $exp(s^2)$ in a controlled way. Suppose, therefore, that there exists constants $C > 0, \alpha_0 > 0$ such that for $\alpha > \alpha_0$ and $t > A''(\alpha_0)\ 1/2$ we have

$$\log \frac{A''(\alpha + \mathrm{t})}{A''(\alpha)} \leq \mathrm{Ct}^2. \tag{64}$$

For context, note that in the model case $A(t) = e^{\beta t} - t$ the left-hand side equals βt.

For smooth functions F defined on $\mathbb{C}$, set $\|F\| = \sum_{n\leq 3} sup\,|\nabla^n F|$. In addition to rectangular coordinates on $\mathbb{C}$ we will use coordinates (s,θ) where $w = e^{\omega}$ and $z = e^{\omega+s+i\theta}$ and also use the scaled coordinates (S,Θ) where $S = \sqrt{A''(2\omega)}s$ and $\Theta = \sqrt{A''(2\omega)}\theta$.

In particular, we still have the hypotheses and conclusions of Theorem (3.1.12) and Theorem (3.1.14).

Theorem (3.1.15) [106]: *In addition to the hypotheses of the previous, suppose that (64) holds. Given F with* $\|F\| < \infty$, *we have*

$$B_{m'} F(w) = \int\int_{\mathbb{C}} F(z) e^{-(S^2+\Theta^2)} \frac{dSd\Theta}{\pi} + O(1)\|F\| S(w^2)^{e-\alpha}.$$

Proof. We start from (63). We first estimate the integral over the unit disk. On $\mathbb{D}$ we can bound F by $\|F\|$ and $|K(wz)|$ by $K(w)$. Thus we have

$$\left| \int\int_{\mathbb{D}} F(z) \frac{|K(wz)|^2}{K(|w|^2)} m(|z|^2) \frac{dxdy}{\pi} \right| \leq \|F\|(wz) l\ 2 \frac{K(w)^2}{K(w^2)}.$$

To show that this can be absorbed into the error term, we need to control $K(w)^2/K(w^2)$ for large *w*. Recall that $\omega = \log w$. By Theorem (3.1.12), it is enough to show that

$$\frac{e^{2A(\omega)} A''(\omega)^2}{e^{A(2\omega)} A''(2\omega)} = O(A''(2\omega)^{-1}) \,.$$

Hence it suffices to show that

$$j(\omega) = 2A(\omega) - A(2\omega) + 2\ log\ A''(\omega)$$

is bounded above. We compute

$$j'(\omega) = 2A'(\omega) - 2A'(2\omega) + 2\ \frac{A''(\omega)}{A''(\omega)}$$

and use the intermediate value theorem on the first pair of terms and the hypothesis (14) on the third. It follows that for some $\widetilde{\omega} \in (\omega, 2\omega)$

$$j'(\omega) \leq -\ 2\omega A''(\widetilde{\omega})\ + 2A''(\omega)^{1/2}\,.$$

Recalling that A'' is monotone increasing and unbounded, we see that j' is negative for all large w, which gives what we need.

We now pass to coordinates (s,θ), where $w = e^{\omega}$ and $z = e^{\omega+s+i\theta}$ and so $dx\,dy = e^{2(\omega+s)}\,ds\,d\theta$. By definition, $m(|z|^2) = m(e^{2(\omega+s)}) = e^{-2(\omega+s)} e^{-A(2\omega+2s)}$.

Hence $\pi^{-1} m(|z|^2) dx\,dy = \pi^{-1} e^{-A(2\omega+2s)}\,ds\,d\theta$.

Set $R = \{(s,\theta) : |\theta| < O(S(wr)^{1/2-6\alpha})\}$. *In R* we use the asymptotic estimates for *K* given in Theorem (3.1.12). This lets us estimate the integrand by

$$F(z)\,\frac{\left|e^{A(2\omega+s)}A''(2\omega+s)e^{-A''(2\omega+s)\theta^2/2}(1+\mathrm{O}(\mathrm{S}(\mathrm{wr})^{1/2+\mathrm{r}}))\right|^2}{e^{A(2\omega)}\,A''(2\omega)(1+\mathrm{O}(\mathrm{S}(\mathrm{wr})^{1/2+\mathrm{r}}))}\;e^{-A(2\omega+2s)}.$$

Note that $(1+\mathrm{O}(\mathrm{S}(\mathrm{wr})^{1/2+\mathrm{r}}))^2/(1+\mathrm{O}(\mathrm{S}(\mathrm{wr})^{1/2+\mathrm{r}}))=(1+\mathrm{O}(\mathrm{S}(\mathrm{wr})^{1/2+\mathrm{r}}))$. We shall see that our approximations to the Berezin measure converge to a probability measure; in the course of that analysis, it will be clear that the norms of the approximations are uniformly bounded. Hence the error made by dropping the factors $(1+\mathrm{O}(\mathrm{S}(\mathrm{wr})^{1/2+\mathrm{r}}))$ in the integral can be safely absorbed into the error term. Thus, in *R*, the integrand can be estimated by

$$F(z)e^{2A(2\omega+s)-A(2\omega)-A(2\omega+2s)}\;\frac{A''(2\omega+s)^2}{A''(2\omega)}\;e^{-A''(2\omega+s)\theta^2}. \qquad (65)$$

We have estimated the integral over the unit disk, i.e., $<-\,\omega$. We now consider the region where $-\,\omega<s<-\delta$. Set

$$h(s)=2A(2\omega+s)-A(2\omega)-A(2\omega+2s)$$

and put $\delta=\delta(2\omega)$. We dominate $F(z)$ by $\|F\|$ and first do the integral in θ. Near the axis, we use the Gaussian estimate of (65). Integrating that gives $(\sqrt{\pi}\;+\;o(1))A''(2\omega+s)^{-1/2}$. Using the estimate in (62) away from the axis, w e get a further contribution of $(A"(2\omega+s)^{-1/2})$. Thus, integrating in θ contributes a factor of $O(1)A''(2\omega+s)^{-1/2}$. Hence we must estimate

$$\int_{-\omega}^{-\delta} e^{h(s)}\,\frac{A''(2\omega+s)^2}{A''(2\omega)A''(2\omega+s)^{-1/2}}\,ds\,.$$

Now A'' is increasing; hence the fraction in the integrand is at most 1. We need to estimate $\int_{-\omega}^{-\delta} e^{h(s)}\,ds$. We have $h'(s)=2A'(2\omega+s)-2A'(2\omega+2s)$. Since A' is increasing and *s* is negative, h' is positive and thus h is increasing. Thus the integral is dominated by $we^{h(-\delta)}$. To estimate $h(-\delta)$, we compute $h''(s)=2A''(2\omega+s)-4A''(2\omega+2s)$ and take note o f (18). We find that, on $(-\,\delta,0)$, $h''(s)=-2A''(2\omega)(1+o(1))$. Noting that $h(0)=h'(0)=0$ and integrating twice gives $h(-\delta)=-A''(2\omega)\delta^2(1+o(1))$. Recalling that $A''^{1/2}\delta=A''^{k}$ for some $k>0$, we conclude that the integral is dominated by any negative power of A'', a better estimate than needed.

Now we consider the integral over the region where $s>\delta$. Note that $h(0)=h'(O)=0$ *and* $h''(O)=-2A''(2\omega)$. Hence, by Taylor's theorem,

$$h(s)=-\,A''(2\omega)s^2+\frac{1}{6}h'''(s^*)s^3\,,$$

with s^* between 0 and *s*. Again we dominate $F(z)$ by $\|F\|$ and first do the integral in θ, making the same estimates as in the previous case. We are reduced to estimating

$$\int_{\delta}^{\infty} e^{-\,A''(2\omega)s^2+\frac{1}{6}h'''(s^*)s^3}\,\frac{A''(2\omega+s)^2}{A''(2\omega)A''(2\omega+s)^{1/2}}\,ds\,.$$

We make the change of variables $s\;=\;S/\sqrt{A''(2\omega)}$ and introduce the shorthand φ. We then need to estimate

$$\int_{\delta\sqrt{A''(2\omega)}}^{\infty} e^{-S^2+\varphi S^3}\,\frac{A''(2\omega+SA''^{-1/2}(2\omega))^{3/2}}{A''(2\omega)^{3/2}}\,dS\,.$$

In the region of integration, s is positive and hence s^* is positive. We compute $h'''(t) = 2A'''(2\omega + t) - 8A'''(2\omega + 2t)$. Recalling that A''' is positive and increasing, we conclude that φ is negative. Hence we make the integral larger by dropping φS^3. We thus need to estimate

$$\int_{\delta\sqrt{A''(2\omega)}}^{\infty} e^{-S^2} \frac{A''(2\omega + sA''^{-1/2}(2\omega))^{3/2}}{A''(2\omega)^{3/2}}\, dS.$$

The estimate (64) insures that the fraction in the integral is dominated by $exp(C'S^2A''^{-1}(2\omega))$. This insures that the integral is $O(e\ -A''^{\theta})$ for some $\theta > 0$, which is more than we need.

Now we look at the range $|s| < \delta$ First we consider the part of that region outside of R. Using (62), we see that, for fixed s, the integration in θ (outside of R) yields an integrand of the form

$$O(1)\|F\|e^{A''(2\omega)s^2+\frac{1}{6}h'''(s^*)s^3} \frac{A''(2\omega)^2}{A''(2\omega)^{3/2+r}}.$$

In $|s| < \delta$ the hypotheses on A insure that the quotient in this expression is $O(A''(2\omega)^{1/2-r})$ and that $h'''(s^*)s^3$ is $o(1)$. Thus we must estimate the integral of $O(1)\|F\|e^{-A''(2\omega)(1+o(1))s^2}A''(2\omega)^{1/2-r}$. Doing the s integration gives $O(1)\ \|F\|A''(2\omega)^{-r}$, which is an acceptable error term.

What remains is the main contribution, the integral over the region where s and θ are both small. In that region, we first note that the hypotheses on A and standard Taylor estimates insure that

$$e^{-A''(2\omega)s^2+h'''(s^*)s^3/6}\, e^{-A''(2\omega+s)\theta^2} = e^{-A''(2\omega)(s^2+\theta^2)}(1 + O(S(w^2)^{\varepsilon})).$$

Hence, making the change o f variable $(S,\Theta) = (\sqrt{A''(2\omega)}s, \sqrt{A''(2\omega)}\theta)$, we obtain, up to a term which can be safely absorbed into the error term,

$$\int_{|\Theta|<A''(2\omega)^{\Theta\omega}} \int_{|S|<\delta A''(2\omega)^{\frac{1}{2}}} F(z)e^{-(S^2+\Theta^2\&)} \frac{A''(2\omega + s)^2}{A''(2\omega)^2} \frac{dS\, d\Theta}{\pi}.$$

Using the Taylor expansion of $A''(2\omega +\ s)$ about $s = 0$, (17), and (19), we see that $A''(2\omega + s)^2/A''(2\omega)^2 = 1 + O(A''(2\omega))^{\alpha-\varepsilon}$. It remains only to note that the passage from $\int_{|\Theta|<A''(2\omega)^{\Theta\omega}} \int_{|S|<\delta A''(2\omega)^{\frac{1}{2}}}$ to $\int \int_{\mathbb{C}}$ introduces an error which, is $O(\mathcal{F})$.

This estimate gives our asymptotic version of (C):

Corollary (3.1.16) [106]: *As* $w \to \infty$,

$$\int \int_{\mathbb{C}} |K_{K_m^{-1}m}(\widetilde{w}z)|m(|z|^2)\ \frac{dx\, dy}{\pi} \sim 2K_m(|w|^2). \qquad (66)$$

Proof. We use the notation of Theorem (3.1.14), that is, $K_0 = K_m$ and $K_1 = K_{K_m^{-1}m}$. We want to estimate

$$I = \int \int_{\mathbb{C}} \frac{|K_1(\widetilde{w}z)|}{K_0(|w|^2)} m(|z|^2)\ \frac{dx\, dy}{\pi}.$$

The same arguments as those in the proof of Theorem (3.1.14) insure that

$$I \sim \int \int_{R} \frac{|K_1(\widetilde{w}z)|}{K_0(|w|^2)} m(|z|^2)\ \frac{dx\, dy}{\pi},$$

where

$$R = \{(s, \theta) : |s| < \delta, |\theta| < S(2w + r)^{1/2-6\alpha}/10\}.$$

We rewrite this as

$$I \sim \int\int_R \frac{|K_1(\widetilde{w}z)|}{|K_0(\widetilde{w}z)|^2} \frac{|K_0(\widetilde{w}z)|^2}{K_0(|w|^2)} m(|z|^2) \frac{dx\,dy}{\pi}$$

$$\sim \int\int_R F_w(z) \frac{|K_0(\widetilde{w}z)|^2}{K_0(|w|^2)} m(|z|^2) \frac{dx\,dy}{\pi},$$

where $F_w = |K_1(\widetilde{w}z)|/|K_0(\widetilde{w}z)|^2$ on *R*. Theorem (3.1.14) insures that $F_w \sim 2$ on *R*. There is no problem extending F_w to the entire plane with $\|F_w\|$ bounded independently of *w*. We now apply the previous theorem with $F = F_w$ and find that

$$I = \int\int_{\mathbb{C}} F_w(z)e - (S^2 + \Theta^2)\frac{dS\,d\Theta}{\pi} + o(1).$$

Recalling that $F_w \sim 2$, we obtain $I \sim 2$, which is the desired conclusion.
the results are estimates in a fixed Bergman space which are asymptotic as $|z| \to \infty$. However, instead of a fixed density m, we could look at the family of densities $m_\sigma = K_{K_m^{-1-\sigma} m}$ and investigate the asymptotic behavior of the kernel function and Berezin transform for fixed z and as $\sigma \to \infty$. Such questions are of interest in quantization, with $(1+\sigma)^{-1}$ playing the role of Planck's constant. See [115] and [116] for instances o f such estimates as well as further discussion. Here we discuss briefly the type of results that could perhaps be obtained by the methods, and why we have not yet obtained them.
First, consider Theorem (3.1.14). We have $K_\sigma(re^{i\theta}) \sim (1+\sigma)K_m(re^{i\theta})^{1+\sigma}$. There may be a more refined result such as

$$K_\sigma = (1+\sigma)K_m^{1+\sigma} + (something) + \frac{1}{(\sigma+1)}(something) + O\left(\frac{1}{\sigma^2}\right).$$

However, the proof which we give fails to produce such a result. That proof gives

$$K_\sigma = (1+\sigma)K_m^{1+\sigma}\left(1 + O\left(\frac{1}{A''\beta}\right)\right)^\sigma$$

for some positive β. This is fine for fixed σ and large *r*, but not for fixed *r* and large σ. Th e fact that the right-hand side involves a factor $(1 + smatl)^\sigma$ seems to be intrinsic to the structure of our proof.
It also seems plausible that more is true in Theorem (3.1.15). We can estimate the Gaussian integral by writing *F* near $z = 2\omega$ as a Taylor polynomial of degree 2 in the variables *s* and θ. The integral of the Taylor remainde r gives a contribution smaller than the error term. The polynomial - times - Gaussian can be integrated explicitly, and we obtain

$$B_m\,F(w) = F(w) + \frac{w^2}{4A''(2\omega)}\Delta F(w) + O(1)\|F\|A''(2\omega)^{\alpha-\varepsilon}.$$

However, this presentation is misleading. We do not know that the third term on the right is smaller than the middle one. The difficulty is not in the estimation of the Gaussian integral, which produces an error that is $O(A''^{-2})$. The problem is the error terms on the estimates whic h led to the Gaussian integral. If it were known that the error terms resulting from that analysis were $O(A''^{-2})$, then we would in fact have

$$B_m\, F(w) = F(w) + \frac{w^2}{4A''(2\omega)}\, \Delta F(w) + O(A''^{-2})\|F\|.$$

We can carry the speculation a step further. If, instead of fixed m, we n o w look at the family of densities $m_\sigma = K_{K_m^{-1-\sigma}m}$ and write B_σ for the corresponding Berezin transforms, we would have

$$B_\sigma F(w) \sim F(w) + \frac{w^2}{4A''(2\omega)}\, \Delta F(w) + O(A''^{-2})\|F\|$$

Now recall from the proof of Theorem (3.1.14) that $A_\sigma = (1+\sigma)A_0$. We could next regard m, F, and w as fixed and let σ grow. That would give, as $\sigma \to \infty$,

$$B_\sigma F(w) \sim F(w) + \frac{1}{1+\sigma}\frac{w^2}{4A_0''(2\omega)}\, \Delta F(w) + O\left(\frac{1}{\sigma^2}\right).$$

Estimates such as this, even with an error term $(1\backslash\sigma)$, woul d be sufficient to give correspondence principle for Berezin quantization schemes; see the Introduction of [117] . It may be that the methods here can be developed to obtain such estimates for large σ. However, it appears that doing this by direct estimation would be quite awkward.

Section (3.2): Fock Spaces and Related Bergman Kernel Estimates

We presents Hankel operators with anti-holomorphic symbols for a large class of weighted Fock spaces. Thus certain natural analogues of BMOA, the Bloch space, the little Bloch space, and the Besov spaces are identified and shown to play similar roles as their classical counterparts do. We will see that these spaces con-tain all holomorphic polynomials and are infinite-dimensional whenever the weight decays so fast that there exist functions of infinite order belonging to the Fock space.

Consider a C^3-function Ψ: $[0,+\infty[\to[0,+\infty[$ such that

$$\Psi'(x) > 0, \qquad \Psi''(x) \geq 0, \qquad and \qquad \Psi'''(x) \geq 0. \tag{67}$$

We will refer to such a function as a logarithmic growth function. Note that (67) effectively says that Ψ should grow at least as a linear function. Set

$$d\mu_\Psi(z) := e^{-\Psi(|z|^2)} d\,V(z),$$

where dV denotes Lebesgue measure on $\mathbb{C}^n$, and let $\mathcal{A}^2(\Psi)$ be the Fock space defined as the closure of the set of holomorphic polynomials in $L^2(\mu_\Psi)$. We observe that $\mathcal{A}^2(\Psi)$ coincides with the classical Fock space when Ψ is a suitably normalized linear function.

It is immediate that

$$s_d := \int_0^{+\infty} x^d\, e^{-\Psi(x)} dx < +\infty$$

for all nonnegative integers d . Moreover, as shown in [130], the series

$$F_s(\zeta) := \sum_{d=0}^{+\infty} \frac{\zeta^d}{s_d}\,, \qquad \zeta \in \mathbb{C}$$

has an infinite radius of convergence and $\mathcal{A}^2(\Psi)$is a reproducing kernel Hilbert space with reproducing kernel

$$K_\Psi(z,w) = \frac{1}{(n-1)!} F_s^{(n-1)}(\langle z,w\rangle), \qquad z,w \in \mathbb{C}^n.$$

This implies that the orthogonal projection P from $L^2(\mu_\Psi)$ onto $\mathcal{A}^2(\Psi)$ can be expressed as

$$(P_\Psi g)(z) = \int_{\mathbb{C}^n} K_\Psi(z,w) g(w) d\mu_\Psi(w), \qquad z \in \mathbb{C}^n,$$

for every function g in $L^2(\mu_\Psi)$. The domain of this integral operator can be extended to include functions g that satisfy $K_\Psi(z,\cdot)g \in L^1(\mu_\Psi)$ for every z in $\mathbb{C}^n$. This extension allows us to define (big) Hankel operators. To do so, denote by $\mathcal{T}(\Psi)$ the class of all f in $L^2(\mu_\Psi)$ such that $f_\varphi K_\Psi(z,\cdot) \in L^1(\mu_\Psi)$ for all holomorphic polynomials φ and z in $\mathbb{C}^n$ and the function

$$H_f(\varphi)(z) := \int_{\mathbb{C}^n} K_\Psi(z,w)\varphi(w)[f(z) - f(w)]\, d\mu_\Psi(w), \quad z \in \mathbb{C}^n$$

is in $L^2(\mu_\Psi)$. This is a densely defined operator from $\mathcal{A}^2(\Psi)$ into $L^2(\mu_\Psi)$ which will be called the Hankel operator H_f with symbol f. It can be written in the form

$$H_f(\varphi) = (I - P_\Psi)(f_\varphi)$$

for all holomorphic polynomials φ. It is clear that the class $\mathcal{T}(\Psi)$ contains all holomorphic polynomials.

The main theorem involves the analogues in our setting of the space BMOA and the Bloch space. The analogue of BMOA is most conveniently defined via the Berezin transform, which for a linear operator T on $\mathcal{A}^2(\Psi)$ is the function $\tilde{T}$ defined on $\mathbb{C}^n$ by

$$\tilde{T}(z) := \frac{\langle T K_\Psi(\cdot,z), K_\Psi(\cdot,z)\rangle}{K_\Psi(z,z)}.$$

If $T = M_f$ is the operator of multiplication by the function f, then we just set $\widetilde{M_f} = \tilde{f}$. We set

$$\|f\|_{BMO} := \sup_{z\in\mathbb{C}^n} (MO\ f)(z),$$

where

$$(\mathrm{MO\ f})(z) := \sqrt{\widetilde{|f|^2}(z) - \left|\tilde{f}(z)\right|^2},$$

and define $BMO(\Psi)$ as the set of functions f on $\mathbb{C}^n$ for which $\widetilde{|f|^2}(z)$ is finite for every z and $\|f\|_{BMO} < \infty$. It is plain that $BMO(\Psi)$ is a subset of $\mathcal{T}(\Psi)$. The space $BMO(\Psi)$ is the subspace of $BMO(\Psi)$ consisting of analytic elements; this space is in turn a subset of $\mathcal{T}(\Psi) \cap \mathcal{A}^2(\Psi)$.

We introduce the Bergman metric associated with Ψ. Set $\Lambda_\Psi(z) = log\ K_\Psi(z,z)$ and

$$\beta^2(z,\xi) := \sum_{j,k=1}^{n} \frac{\partial^2 \Lambda_\Psi(z)}{\partial z \partial \bar{z}_k} \xi_j \bar{\xi}_k$$

for arbitrary vectors $z = (z_1, \ldots, z_n)$ and $\xi = (\xi_1, \ldots, \xi_n)$ in $\mathbb{C}^n$. The corresponding distance ϱ is given by

$$\varrho(z,w) := \inf_\gamma \int_0^1 \beta(\gamma(t), \gamma'(t)) dt \ , \qquad (68)$$

where the infimum is taken over all piecewise C^1-smooth curves $\gamma : [0,1] \to \mathbb{C}^n$ such that $\gamma(0) = z$ and $\gamma(1) = w$. We define the Bloch space ß(Ψ) to be the space of all entire functions *f* such that

$$\|f\|_{ß(\Psi)} := \sup_{z\in\mathbb{C}^n}\left[\sup_{\xi\in\mathbb{C}^n\setminus\{0\}} \frac{\left|\langle(\nabla f)(z), \bar{\xi}\rangle\right|}{\beta(z,\xi\,)}\right] < +\infty. \quad (69)$$

In what follows, the function

$$\Phi(x) := x\Psi'(x)$$

will play a central role. By (67), we have that both $\Phi'(x) > 0$ and $\Phi''(x) > 0$, and it may be checked that $\Phi'(|z|^2)$ coincides with the Laplacian of $\Psi'(|z|^2)$ when $n = 1$ and in general is bounded below and above by positive constants times this Laplacian for arbitrary $n > 1$.

We prepared to state the main result.

Theorem (3.2.1)[129]: *Let* Ψ *be a logarithmic growth function, and suppose that there exists a real number* $\eta < 1/2$ *such that*

$$\Phi''(t) = O\left(t^{-\frac{1}{2}}\,[\Phi'(t)]^{1+\eta}\right) \text{ when } t \to \infty. \quad (70)$$

If f is an entire function on $\mathbb{C}^n$, *then the following statements are equivalent*:

(i) *The function f belongs to* $\mathcal{T}(\Psi)$ *and the Hankel operator* $H_{\bar{f}}$ *on* $\mathcal{A}^2(\Psi)$ *is bounded*;

(i) *The function f belongs to* $BMOA(\Psi)$;

(iii) *The function f belongs to* ß(Ψ).

Note that the additional assumption (70) is just a mild smoothness condition, which holds whenever Ψ is a nontrivial polynomial or a reasonably well-behaved function of super-polynomial growth.

As part of the proof of Theorem (3.2.1), we will perform a precise computation of the asymptotic behavior of $\beta(z,\xi\,)$ when $|z| \to \infty$. We state this result as a separate theorem.

Theorem (3.2.2) [129]: *Let* Ψ *be a logarithmic growth function, and suppose that there exists a real number* $\eta < 1/2$ *such that* (70) *holds. Then we have, that* $\beta^2(z,\xi) = (1+o(1))|\xi|^2\Psi'(|z|^2) + |z,\xi|^2\Psi''(|z|^2)$ *when* $|z| \to \infty$.

We observe that for the classical Fock space (Ψ a linear function) we have $\Psi''(x) \equiv 0$, and so the "directional" term in $\beta(z,\xi\,)$ is not present. Note also that ß(Ψ) contains all polynomials and is infinite-dimensional whenever the growth of $\Psi'(x)$ is super-polynomial. In the language of entire functions, this means that $\mathcal{A}^2(\Psi)$ contains functions of infinite order. When $n = 1, \beta^2(z,\xi)$ can be replaced by $\Phi'(|z|^2)|\xi|^2$. The same is also true when Ψ is a polynomial, because then Ψ' and Φ' have the same asymptotic behavior. In the latter case, our two theorems give the following precise result: If Ψ is a polynomial of degree *d* , then ß(Ψ) consists of all holomorphic polynomials of degree at most *d* ; cf. Theorem A in [130].

The implication $(i) \Rightarrow (ii)$ in Theorem (3.2.1) is standard; it follows from general arguments for reproducing kernels. Likewise, the implication $(ii) \Rightarrow (iii)$ can be established by a well-known argument concerning the Bergman metric. Our proof of Theorem (3.2.1) therefore deals mainly with the implica-tion $(iii) \Rightarrow (i)$. The crucial technical ingredients in the proof of this result are certain estimates for the Bergman kernel $K_\Psi(z,w)$. Such estimates have previously been obtained by F. Holland and R. Rochberg in [131]. The results of [131] are not directly applicable because we need more precise off-diagonal estimates for the kernel than those given. Our method of proof is similar to that of [131], but our approach highlights more explicitly the interplay between the smoothness of and the off-diagonal decay of the Bergman kernel. This is where the additional smoothness condition (70) comes into play; many of our estimates can be performed with sufficient precision without the assumption that (70) holds, but some condition of this kind seems to be needed for our off-diagonal estimates.

The fact that the Bergman metric is the notion used to define the Bloch space ß(Ψ) suggests that Theorem (3.2.1) should be extendable beyond the case of radial weights. To obtain such an extension, one would need a replacement of our Fourier-analytic approach, which relies crucially on the representation of the Bergman kernel as a power series.

The machinery developed to prove Theorem (3.2.1) leads with little extra effort to a characterization of compact Hankel operators in terms of the obvious counterparts to VMOA and the little Bloch space; for details. In the study of Schatten class Hankel operators, however, some additional techniques will be used. We will need local information about the Bergman metric, namely that balls of fixed radius in the Bergman metric are effectively certain ellipsoids in the Euclidean metric of $\mathbb{C}^n$. These results appear to be of independent interest; they lead to a characterization of Carleson measures and in turn to a characterization of the spectral properties of Toeplitz operators. Building on these results and using L^2 estimates for the ∂ operator, we obtain a characterization of Schatten class Hankel operators. Boundedness and compactness of Hankel operators with arbitrary symbols have previously been considered only for the classical Fock space (Ψ a linear function); see, [133]. The methods, relying on the transitive self-action of the group $\mathbb{C}^n$, cannot be extended beyond this special case. Hankel operators with anti-holomorphic symbols defined on more general weighted Fock spaces were studied recently in [139] and [138], where it was shown that anti-holomorphic polynomials do not automatically induce bounded Hankel operators. For Bergman kernel estimates in similar settings, see [134] and [135]. We finally mention [133] and [134]; the first of these focuses on small Hankel operators and the Heisenberg group action, while the second deals with Hankel operators for the Bergman projection on smoothly bounded pseudoconvex domains in $\mathbb{C}^n$.

Throughout, $U(z) \lesssim V(z)$ (or equivalently $V(z) \gtrsim U(z)$) means that there is a constant C such that $U(z) \le CV(z)$ holds for all z in the set in question, which may be a space of functions or a set of numbers. If both $U(z) \lesssim V(z)$ and $V(z) \gtrsim U(z)$, then we write $U(z) \simeq V(z)$.

The following standard argument shows that (i) implies (ii) in Theorem (3.2.1). To begin note that if f is in $\mathcal{A}^2(\psi)$, then $\tilde{f} = f$. Moreover, by the definition of the reproducing kernel, a computation shows that

$$\widetilde{|f|^2}(z) - |f(z)|^2 = \int_{\mathbb{C}^n} |f(\xi) - f(z)|^2 \frac{|K(\xi, z)|^2}{K_\Psi(z,z)} d\mu(\xi) = \frac{\left\|H_{\bar f} K_\Psi(\cdot, z)\right\|^2}{K_\Psi(z,z)}. \quad (71)$$

Hence, if $H_{\bar f}$ is bounded, then $\|f\|_{BMO} < +\infty$.

The implication (ii) ⇒ (iii) is a consequence of the following lemma, the proof of which is exactly as the proof of Corollary 1 in [134] .

Lemma (3.2.3) [129]: *Suppose that f is in $MOA(\Psi)$. Then for every piecewise C^1-smooth curve $\gamma: [0,1] \to \mathbb{C}^n$ we have*

$$\left|\frac{d}{dt}(f\circ\gamma)(t)\right| \le 2\sqrt{2}\beta(\gamma(t),\gamma(t))(MO\, f)(\gamma(t)).$$

If we choose $(t) = z + t\,\xi$, then we obtain

$$\frac{|\langle(\nabla f)(z),\bar{\xi}\rangle|}{\beta(z,\xi)} \le 2\sqrt{2}(MO\, f)(z) \quad (72)$$

for all z in $\mathbb{C}^n$ and ξ in $\mathbb{C}^n\setminus\{0\}$.

This is a somewhat elaborate preparation for the proof of Theorem (3.2.2) and also the proof of the implication (iii) ⇒ (i) in Theorem (3.2.1).

Set

$$\theta_0(r) := [r\Phi'(r)]^{-1/2}.$$

The key estimates for the Bergman kernel are the following.

Lemma (3.2.4) [129]: *Suppose that* (70) *holds. Let z and w be arbitrary points in $\mathbb{C}^n$ such that $z, w \neq 0$, and write $\langle z,w\rangle = r\,e^{i\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$. Then we have*

$$\frac{1}{[\Psi'(r)]^{n-1}}\frac{|K_\Psi(z,w)|}{e^{\Psi(r)}} \lesssim \begin{cases} \Phi'(r), & |\theta| \le \theta_0(r), \\ r^{-3/2}[\Phi'(r)]^{-1/2}|\theta|^{-3}, & |\theta| > \theta_0(r). \end{cases}$$

Moreover, there exists a positive constant c such that if $\theta < c\theta_0(r)$, then

$$|K_\Psi(z,w)| \gtrsim (r)[\Phi'(r)]^{n-1}e^{\Psi(r)}.$$

We collect a few results.

Lemma (3.2.5) [129]: *Let η be as in Theorem (3.2.1). Then, for any fixed $\alpha > \eta$, we have*

$$\sup_{|\tau| \le t^{1/2}[\Phi'(t)]^{-\alpha}} \Phi'(t+\tau) = (1+o(1))\Phi'(t)$$

when $t \to \infty$.

Proof. The proof is similar to the proof of Lemma 6 in [131]. By (70), $[\Phi'(x)]^{-1-\eta} \times \Phi''(x) = O(x^{-1/2})$ when $x \to \infty$, which implies that

$$|[\Phi'(t+\tau)]^{-\eta} - [\Phi'(t)]^{-\eta}| = |\tau|O(t^{-1/2}\tau)$$

when $t \to \infty$. The result follows from this relation.

In order to estimate $|K_\Psi(z,w)|$, we need precise information about the moments s_d . To this end, note that the integrand of

$$\int_0^\infty x^t\, e^{-\Psi(x)}dx$$

attains its maximum at $x = \Phi^{-1}(t)$. Set

$$h_t(x) = -t\log x + \Psi x) - (-t\log\Phi^{-1}(t) + \Psi(\Phi^{-1}(t)))$$

and

$$I(t) = \int_0^\infty e^{-h_t(x)}dx;$$

we may then write

$$s_d = e^{d\log\Phi^{-1}(d) - \Psi(\Phi^{-1}(d))}I(d).$$

We have the following precise estimate for $I(t)$.

Lemma (3.2.6) [129]: *For the function* $I(t)$ *we have*

$$I(t) = (\sqrt{2\pi} + o(1)) \left[\frac{\Phi^{-1}(t)}{\Phi'(\Phi^{-1}(t))} \right]^{1/2}$$

when $t \to \infty$.

Proof. Set $(x) = \sqrt{x}[\Phi'(x)]^{-\alpha}$, where $\eta < \alpha < 1/2$. Since

$$h_t''(x) = \frac{\Phi'(x)}{x} + \frac{t}{x^2} - \frac{\Phi(x)}{x^2} = \frac{\Phi'(x)}{x} + \frac{1}{x^2}[\Phi(\Phi^{-1}(t)) - \Phi(x)],$$

we have, by Lemma (3.2.5),

$$h_t''(x) = h_t''(\Phi^{-1}(t))(1 + o(1))$$

when $|x - \Phi^{-1}(t)| \le \tau(\Phi^{-1}(t))$. On the other hand, by the convexity of h_t , we then have

$$|h_t(x)| \ge \frac{1}{2}(h_t''(\Phi^{-1}(t)) + o(1))\tau(\Phi^{-1}(t))|x - \Phi^{-1}(t)|$$

for $|x - \Phi^{-1}(t)| \ge \tau(\Phi^{-1}(t))$. Setting for simplicity

$$c = h_t''(\Phi^{-1}(t)) = \frac{\Phi'(\Phi^{-1}(t))}{\Phi^{-1}(t)},$$

we then get

$$I(t) = \int_{|x| \le \tau(\Phi^{-1}(t))} e^{-\frac{1}{2}(c+o(1))x^2}\, dx + E(t), \tag{73}$$

where

$$|E(t)| \le 2 \int_{x \ge \tau(\Phi^{-1}(t))} e^{-\frac{1}{2}(c+o(1))\tau(\Phi^{-1}(t))x}\, dx.$$

Thus the result follows, since the integral in (73) can be estimated by the corresponding Gaussian integral from $-\infty$ to ∞.

We will estimate a number of integrals in a similar fashion, using Lemma (3.2.5) to split the domain of integration. The integrands will be of the type $e^{-g_t(x)}S_t(x)$ and satisfy the following:

g_t attains its minimum at a point $x_0 = x_0(t) \to \infty$ with $x_0(x) = (1 + o(1))c$ for $|x - x_0| \le \tau$ and $1/\tau = o(c)$ when $t \to \infty$.

K For $|x - x_0| \le \tau$, $S_t(x)$ can be estimated by a constant *C* times $|x - x_0|^m$ for some positive integer *m*.

When $|x - x_0| \ge \tau$ and $|x - x_0|$ grows, the function $e^{-g_t(x)}S_t(x)$ decays so fast that

$$\int_0^\infty e^{-g_t(x)}|S_t(x)|dx = (1 + o(1)) \int_{|x-x_0| \ge \tau} e^{-g_t(x)}|S_t(x)|d\,x.$$

Taking into account the formula

$$\int_0^\infty x^m e^{-\frac{1}{2}cx^2}\, dx = (c/2)^{-(m+1)/2} \int_0^\infty x^m e^{-x^2} dx\,, \tag{74}$$

we then arrive at the estimate

$$\int_0^\infty e^{-h_t\,(x) S_t(x) dx} = O(Cc^{-(m+1)/2}) \tag{75}$$

when $t \to \infty$.

We will at one point encounter a slightly different variant of this scheme, obtained by replacing (II) by the following:

(II') For $|x - x_0| \geq \tau$, we have $S(x) = (1 + o(1))(x - x_0)$ when $t \to \infty$.

In this case, because of the symmetry around the point x_0, we get the slightly better estimate

$$\int_0^\infty e^{-h_t(x)} S(x) d\,x \;=\; o(c^{-1}) \tag{76}$$

when $t \to \infty$.

To avoid tedious repetitions, in the following we will omit most of the details of such calculus arguments. We will briefly state that conditions (I), (II), (III) (or, respectively, (I), (II'), (III)) are satisfied and conclude that this leads to the estimate (75) (or, respectively, (76)).

In the proof of the next lemma, we will use this scheme three times.

Lemma (3.2.7) [129]: *We have*

$$I'(t) = O\big([\Phi^{-1}(t)\Phi'(\Phi^{-1}(t))]^{-1/2} I(t)\big)\,;$$
$$I''(t) = O([\Phi^{-1}(t)\Phi'(\Phi^{-1}(t))]^{-1} I(t))\,;$$
$$I(t) = O\big([\Phi^{-1}(t)\Phi'(\,\Phi^{-1}(t))]^{-3/2} I(t)\big)$$

when $t \to \infty$.

Proof. We begin by noting that I' can be computed in the following painless way:

$$I'(t) = \int_0^\infty log \frac{x}{\Phi^{-1}(t)} e^{-h_t(x)} dx\,; \tag{77}$$

this holds because $h_t'\,(\Phi^{-1}(t)) = 0$. For the same reason, we get

$$I''(t) = \int_0^\infty \left[-\frac{\left(\Phi^{(-1)}\right)'(t)}{\Phi^{-1}(t)} + \left(log \frac{x}{\Phi^{-1}(t)} \right)^2 \right] e^{-h_t(x)} dx \tag{78}$$

and

$$I'''(t) = \int_0^\infty \left[-\left[\frac{\left(\Phi^{(-1)}\right)'(t)}{\Phi^{-1}(t)} \right]' - 3 \frac{\left(\Phi^{(-1)}\right)'(t)}{\Phi^{-1}(t)} log \frac{x}{\Phi^{-1}(t)} + \left(log \frac{x}{\Phi^{-1}(t)} \right)^3 \right] e^{-h_t(x)} dx \tag{79}$$

We use that $[\Phi^{-1}]'(t) = 1/(\Phi^{-1}(t))$, and then in (79) we also use the fact that

$$\left[\frac{1}{\Phi'(\Phi^{(-1)}(t))\Phi^{-1}(t)} \right]' = -\frac{\Phi''(\Phi^{-1}(t))}{[\Phi'(\Phi^{-1}(t))]^3 \Phi^{-1}(t)} - \frac{1}{[\Phi'(\Phi^{-1}(t))\Phi^{-1}(t)]^2}\,; \tag{80}$$

we apply condition (70) to the first term on the right-hand side. When we estimate the integrals in (77), (78), and (79), we use that

$$\left| log \frac{x}{\Phi^{-1}(t)} \right| \leq\; e \frac{|x - \Phi^{-1}(t)|}{\Phi^{-1}(t)}$$

for $x \geq e^{-1}\Phi^{-1}(t)$ and that, say,

$$\left|log\frac{x}{\Phi^{-1}(t)}\right| \le log\frac{1}{\Phi^{-1}(t)}$$

when $1 \le x < e^{-1}\Phi^{-1}(t)$. In each case, the integrand satisfies conditions (I), (II), (III) with $g_t = h_t$, so that we may use (75). The desired results for I', I'', I''' now follow from (75). We will need similar estimates for the function

$$L_r(t) = exp\; t\; log\; r - t\; log\Phi^{-1}(t) + \Phi^{-1}(t),$$

where *r* is a positive parameter.

Lemma (3.2.8) [129]: *We have*

$$L'_r(t) = \left(-log\frac{\Phi^{-1}(t)}{r}\right)L_r(t);$$

$$L''_r(t) = \left[\left(log\frac{\Phi^{-1}(t)}{r}\right)^2 - \frac{1}{\Phi'(\Phi^{-1}(t))\Phi^{-1}(t)}\right]L_r(t);$$

$$L'''_r(t) = \left[\left(-log\frac{\Phi^{-1}(t)}{r}\right)^3 + \frac{3log\frac{\Phi^{-1}(t)}{r}}{\Phi'(\Phi^{-1}(t))\Phi^{-1}(t)} + O\left([\Phi'(\Phi^{-1}(t))\Phi^{-1}(t)]^{-3/2}\right)\right]L_r(t)$$

when $t \to \infty$.

Proof. The first and the second of these formulas are obtained by direct computation. We arrive at the estimate for the third derivative by again using (80) and then applying condition (70).

We begin by recalling that

$$K_\Psi(z,w) = k(z,w),$$

where

$$k(\zeta) := \frac{1}{(n-1)!}\sum_{d=n-1}^{\infty}\frac{d(d-1)\cdots(d-n+2)}{s_d}\zeta^{d-n+1}.$$

We set $\langle z,w\rangle = re^{i\theta}$ and assume that $r > 0$ and $|\theta| \le \pi$. We may then write

$$\frac{\langle z,w\rangle^d}{s_d} = \frac{L_r(d)}{I(d)}exp(id\,\theta)$$

and hence

$$\langle z,w\rangle^{n-1}K_\Psi(z,w) = r^{n-1}exp(i(n-1)\theta)k\,re^{i\theta}$$
$$= \frac{1}{(n-1)!}\sum_{d=n-1}^{\infty}d(d-1)\cdots(d-n+2)\frac{L_r(d)}{I(d)}exp(id\,\theta).$$

Let $\Omega(t)$ be a function in $C^3(R)$ so that

$$\Omega(t) = \frac{1}{(n-1)!}\frac{t(t-1)\cdots(t-n+2)L_r(t)}{I(t)}$$

for $t \geq n-1$ and $\Omega(t) = 0$ for $t \leq n-2$. Then the Poisson summation formula gives

$$r^{n-1}\, exp(i(n-1)\theta)k\left(r\, e^{i\theta}\right) = \sum_{-\infty}^{\infty} \widetilde{\Omega}(j),$$

where

$$\widetilde{\Omega}(j) = \int_{-\infty}^{\infty} \Omega(t) e^{i(2\pi j+\theta)t}\, dt\ .$$

Integrating by parts, we obtain

$$r^{n-1}|k(r\, e^{i\theta})| \leq |\widetilde{\Omega}(0)| + \|\Omega'''\|_1 \sum_{j=1}^{\infty} \frac{2}{(2\pi)^3 (j-1/2)^3}.$$

Since

$$|\widetilde{\Omega}(0)| \leq\ min(\|\Omega\|_1, |\theta|^{-3}\|\Omega'''\|_1),$$

the proof of the first part of the lemma is complete if we can prove that

$$\|\Omega\|_1 \lesssim \left(\Phi(r)\right)^{n-1} \Phi'(r) e^{\Psi(r)} \tag{81}$$

and

$$\|\Omega'''\|_1 \lesssim \left(\Phi(r)\right)^{n-1} \frac{e^{\Psi(r)}}{r^{\frac{3}{2}} e\sqrt{\Phi'(r)}}. \tag{82}$$

We first estimate $\|\Omega\|_1$. We write $L_r(t) = exp(-gr(t))$ and claim that conditions (I), (II), (III) above hold. To see this, we observe that, by the first formula of Lemma (3.2.8), L_r attains its maximum at $= \Phi(r)$. Moreover, g_r is a convex function and

$$g_r''(t) = \frac{1}{\Phi'\left(\Phi^{-1}(t)\right)\Phi^{-1}(t)}.$$

Lemma (3.2.5) implies that

$$g_r''(t) = (1+o(1))g_r''((r))$$

when $|t-\Phi(r)| \leq \sqrt{r}[\Phi'(r)]^{1-2\alpha}$. The remaining details are carried out as in the proof of Lemma (3.2.6). Using (75) with $m = 0$ and Lemma (3.2.6), we therefore get

$$\|\Omega\|_1 = |\Phi(r)(\Phi(r) - 1)\cdots(\Phi(r) - n + 2)| \frac{L_r\left(\Phi(r)\right)}{I\left(\Phi(r)\right)}\left(\sqrt{2\pi} + o(1)\right)[\Phi'(r)r]^{\frac{1}{2}}$$

$$= \left(1+o(1)\right)\left(\Phi(r)\right)^{n-1}\Phi'(r)e^{\Psi(r)},$$

which shows that (81) holds.

To arrive at (82), we need a pointwise estimate for Ω''' . To simplify the writing, we set

$a = \left|log\frac{\Phi^{-1}(t)}{r}\right|$ *and* $b = [\Phi'(\Phi^{-1}(t))\Phi^{-1}(t)]^{-1/2}$Then using the Leibniz rule along with Lemmas (3.2.7) and (3.2.8), we get

$$|\Omega'''(t)| \lesssim (a^3 + a^2 b + ab^2 + b^3)\Omega(t).$$

By a straightforward calculus argument, we verify that each of the terms in this expression satisfies (I), (II), and (III) above, again with $x_0 = \Phi(r)$ $\tau = \sqrt{r}[\Phi'(r)]^{1-2\alpha}$. We now use (75) to achieve the desired estimate for each of the terms $a^m b^{3-m}\Omega(t)$.

The previous proof also gives the second estimate when $\theta = 0$, because then $\widetilde{\Omega}(0) = \|\Omega\|_1$. To prove it in general, we need to check that $k(r) \simeq |k(re^{i\theta})|$ when $|\theta| \leq c[r\Phi'(r)]^{-1/2}$. To this end, note that

$$\widetilde{\Omega}(0) = e^{i\theta\Phi(r)} \int_{-\infty}^{\infty} \Omega(t) e^{i\theta(t-\Phi(r))} dt ,$$

which implies that

$$|\widetilde{\Omega}(0)| \geq \|\Omega\|_1 \int_{-\infty}^{\infty} \Omega(t)|\theta||t - \Phi(r)|dt.$$

The integral on the right is computed using (75) with $m = 1$, and so we get

$$|\widetilde{\Omega}(0)| \geq \|\Omega\|_1 \big(1 - C|\theta|[r\Phi'(r)]^{1/2}\big) .$$

Thus the second estimate in Lemma (3.2.4) holds for c sufficiently small.

We close by proving some estimates for another function that will be important later. Set

$$Q_x(r) = \frac{1}{2}(\Psi(r^2) + \Psi(x^2)) - xr). \tag{83}$$

Lemma (3.2.9) [129]: *Let α be a positive number such that $\eta < \alpha < 1/2$, let x_1 and x_2 be the two points such that $x_1 < x < x_2$ and*

$$|x - x_1| = |x - x_2| = [(x)] - \alpha ,$$

and set $c = (0)$. When $r \to \infty$, we have

$$Q''_x(r) = \big(1 + o(1)\big)\Phi'(x^2), \;\; x_1 \leq r \leq x_2; \tag{84}$$

$$Q_x(r) \geq \frac{c}{4}(x - r)^2 + \left(\frac{1}{4} + o(1)\right)[\Phi'(x^2)]^{1-2\alpha} , r < x_1; \tag{85}$$

$$Q_x(r) \geq \frac{c}{4}(x - r)^2 + \left(\frac{1}{4} + o(1)\right)[\Phi'(x^2)]^{1-2\alpha} , r > x_2. \tag{86}$$

Proof. We begin by noting that

$$Q'_x(r) = r\Psi'(r^2) - x\Psi'(xr)$$

and

$$Q''_x(r) = \Psi'(r^2) + 2r^2\Psi''(r^2) - x^2\Psi''(xr).$$

We observe that for $x_1 \leq r \leq x_2$ Lemma (3.2.5) applies:

$$Q''_x(r) = \Phi'(r^2) + r^2\Psi''(r^2) - x2\Psi''(xr) = (1 + o(1))\Phi'(x^2),$$

and so we have established (84). For $r < x_1$, we use the following estimate:

$$Q_x(r) \geq \frac{1}{2}\int_r^x \Psi'(s^2)(s - x)ds + \frac{1}{2}\int_{x-[\Phi'(x^2)]^{-\alpha}}^{x} \int_{x-[\Phi'(x^2)]^{-\alpha}}^{t} Q''_x(u)dudt$$
$$\geq \frac{c}{4}(x - r)^2 + \left(\frac{1}{4} + o(1)\right)[\Phi'(x^2)]^{1-2\alpha} ,$$

where we used again Lemma (3.2.5) in the last step. Now observe that since $\Psi''(y)$ is a nondecreasing function, we have

$$Q''_x(r) \geq \Phi'(r^2)$$

for $r \geq x$. We therefore obtain for $x > x_2$:

$$Q_x(r) \geq \frac{1}{2}\int_x^r \Psi'(s^2)(s - x)ds + \frac{1}{2}\int_{r-[\Phi'(x^2)]^{-\alpha}}^{r} \int_{r-[\Phi'(x^2)]^{-\alpha}}^{t} Q''_x(u)dudt$$
$$\geq \frac{c}{4}(x - r)^2 + \left(\frac{1}{4} + o(1)\right)[\Phi'(r^2)]^{1-2\alpha} ,$$

where Lemma (3.2.5) is applied once more. Hence (86) also holds.
We begin by recalling that

$$K_\Psi(z,z) = k(r^2),$$

where

$$k(r) = \sum_{n=0}^{\infty} c_d\, r^d\,,$$

and

$$c_d := \frac{(d\ +\ 1)\ \cdots\ (d\ +\ n\ -\ 1)}{(n\ -\ 1)!\ sd + n - 1}.$$

A computation shows that

$$\beta^2(z,\xi) := |\xi|^2 \frac{k'(|z|^2)}{k(|z|^2)} + \ |\langle z,\xi\rangle|^2\ \left[\frac{k''((|z|^2))}{k((|z|^2))} - \left(\frac{k'(|z|^2)}{k(|z|^2)}\right)^2\right].$$

Thus Theorem (3.2.2) is a consequence of the following lemma.

Lemma (3.2.10) [129]: *Suppose that* (70) *holds. Then we have*

$$\frac{k'\,(r)}{k(r)} = (1 + o(1))\Psi'(r),$$

$$\left(\frac{k'\,(r)}{k(r)}\right)' = (1 + o(1))\Psi''(r) + o(1)\frac{\Psi'(r)}{r}$$

when $r\ \to\ \infty$.

The proof of this lemma relies on the following estimates.

Lemma (3.2.11) [129]: *Suppose that* (70) *holds and let the coefficients* c_d *be as defined above. Then we have*

$$\sum_{d=1}^{\infty} c_d\,(d - \Phi(r))r^d = o([r\Phi'(r)]^{1/2}k(r)), \tag{87}$$

$$\sum_{d=1}^{\infty} c_d\,(d - \Phi(r))^2 r^d =\ \big(1 + o(1)\big)r\Phi'(r)k(r) \tag{88}$$

when $r \to \infty$.

Proof. The proof is essentially the same as the proof for the diagonal estimates in Lemma (3.2.4). The only difference is that we replace the function $\Omega(t)$ by $\big(t - \Phi(r)\big)\Omega(t)$ and $(t - \Phi(r))^2\Omega(t)$, respectively. In the first case, we have a functionthat satisfies condition (II′). This means that we may use (76) to arrive at (87). To establish (88), we may apply (74) with $m = 2$ and take into account that 2 we have the explicit factor $(t - \Phi(r))^2$ front of $\Omega(t)$.

We write

$$k'(r) = \frac{\Phi(r)}{r}(k(r) + O(1)) + \frac{1}{r}\sum_{d=1}^{\infty} c_d(d\ -\ \Phi(r))r^d\,;$$

using Lemma (3.2.11), we obtain

$$\frac{k'(r)}{k(r)} = (1+o(1))\Psi'(r) + o\left(\left[\frac{\Phi'(r)}{r}\right]^{1/2}\right).$$

The desired estimate for k / k follows because, in view of Lemma (3.2.5,) we have
$\Phi(r) \geq \int_{r-r^{1/2}[\Phi'(r)]^{-\alpha}}^{r} \Phi'(t)dt = (1+o(1))r^{1/2}[\Phi'(r)]^{1-\alpha}$
for some $\alpha < 1/2$.
To arrive at the second estimate, we first observe that

$$k''(r) = \frac{\Phi(r)-1}{r}(k'(r)+O(1)) + \frac{1}{r}\sum_{d=2}^{\infty} c_d d\left(d\&-\&\Phi(r)\right)r^{d-1}$$
$$= \frac{\Phi(r)-1}{r}(k'(r)+O(1)) + \frac{\Phi(r)}{r^2}\sum_{d=2}^{\infty} c_d(d-\Phi(r))r^d$$
$$+\frac{1}{r^2}\sum_{d=2}^{\infty} c_d(d-\Phi(r))^2 r^d.$$

Combining our expressions for k' and k'', we find that

$$k''(r)k(r) - \left(k'(r)\right)^2 = \frac{k(r)}{r^2}\sum_{d=2}^{\infty} c_d\left(d-\Phi(r)\right)^2 r^d - \frac{1}{r^2}\left[\sum_{d=2}^{\infty} c_d\left(d-\Phi(r)\right)r^d\right]^2 -$$
$$\frac{k(r)k'(r)}{r} + \Psi'(r)O(k(r)+k'(r)).$$

Using again Lemma (3.2.11) and the estimate already obtained for k'/k, we get

$$\left(\frac{k'(r)}{k(r)}\right)' = (1+o(1))\frac{\Phi'(r)}{r} - (1+o(1))\frac{\Phi(r)}{r^2}$$

from which the second estimate in Lemma (3.2.10) follows.
We finally turn to the proof that (iii) implies (i) in Theorem (3.2.1). A different proof, using L^2 estimates for the $\bar{\partial}$ operator, will be given, subject to an additional mild smoothness condition. The proof gives a more informative norm estimate, which will be crucial in our study of Schatten class Hankel operators. The proof to be given below has the advantage that it does not require f to be holomorphic.
Using the reproducing formula, we find that

$$H_{\bar{f}}g(z) = \int_{\mathbb{C}^n}\left(\overline{f(z)} - \overline{f(w)}\right)K_\Psi(z,w)g(w)d\mu_\Psi(w).$$

Therefore, by the definition of ß(Ψ), we have

$$|H_{\bar{f}}g(z)| \leq \|f\|_{ß(\Psi)}\int_{\mathbb{C}^n}\varrho(z,w)K_\Psi(z,w)g(w)d\mu_\Psi(w).$$

Thus it suffices to prove that the operator A defined as

$$Ag(z) = \int_{\mathbb{C}^n}\varrho(z,w)K_\Psi(z,w)g(w)d\mu_\Psi(w)$$

is bounded on $L^2(\mu_\Psi)$.
We shall use a standard technique known as Schur's test [18, p. 42]. Set

$$H(z,w) = \varrho(z,w)|K_\Psi(z,w)|e^{-\frac{1}{2}(\Psi(|z|^2)+\Psi|w|^2))}.$$

By the Cauchy–Schwarz inequality, we obtain

$$|(Ag)(z)|^2 e^{-\Psi(|z|^2)} \lesssim \int_{\mathbb{C}^n} H(z,\zeta)dV(\zeta) \int_{\mathbb{C}^n} H(z,w)|g(w)|^2 e^{-\Psi|w|^2)dV(w)}.$$

This means that the operator *A* is bounded on $L^2(\mu_\Psi)$ if

$$\sup_z \int_{\mathbb{C}^n} H(z,\zeta)dV(\zeta) < \infty. \tag{89}$$

Our task is therefore to establish (89).

We may assume that $z = (x, 0, \dots, 0)$ with $x > 0$. We begin by estimating $\varrho(z,w)$. To this end, write $w = (w_1, \xi)$ with ξ a vector in $\mathbb{C}^{n-1}$ and $w_1 = re^{i\theta}$ when $n > 1$. Set $e_1 = (1, 0, \dots, 0)$ and consider the three curves

$$\gamma_1(t) = xe^{it}\, e_1, \qquad 0 \le t \le \theta\,,$$

$$\gamma_2(t) = \big(x + t(r-x)\big)e^{i\theta}\, e_1,\ 0 \le t \le 1,$$

$$\gamma_3(t) = \big(re^{i\theta}\,, t\,\xi\big), 0\ \le t \le 1,$$

which together constitute a piecewise smooth curve from z to *w*. (When $n = 1, \gamma_3$ does not appear and can be neglected.) Note that

$$|\langle\gamma_1(t\,), \gamma_1'(t)\rangle| = |\gamma_1(t)||\gamma_1'(t)\ = x^2,$$

$$|\langle\gamma_2(t), \gamma_2'(t)\rangle| = |\gamma_2(t\,)||\gamma_2'(t\,)| = (x + t(r-x))|x-r|,$$

$$|\gamma_3(t\,), \gamma_3'(t) = t|\xi|^2.$$

By these observations and Theorem (3.2.2), we get the following estimate:

$$\varrho(z,w) \lesssim x|\theta|[\Phi'(x^2)]^{1/2} + [\Phi'(max(x^2,r^2))]^{1/2}|x-r|$$

$$+ |\xi|\,[\Psi'(r^2+|\xi|^2)]^{\frac{1}{2}} + |\xi|^2[\Psi''(r^2+|\xi|^2)]^{1/2}.$$

When estimating the last term on the right-hand side of this inequality, we will use that

$$[\Psi'(y)]^2 \gtrsim \Psi''(y), \tag{90}$$

which is a consequence of our assumptions (67) and (70). Indeed, assuming $\Psi'' > 0$, we have $y\Psi''(y) \simeq \Phi'(y)$ sinceΨ'' is a nondecreasing function. Thus (90) is equivalent to the following:

$$\Phi(t) \gtrsim t^{1/2}\,[\Phi'(t)]^{1/2}.$$

We arrive at this estimate because

$$\Phi(t) = \Phi(0) + \int_0^t \Phi'(\tau)d\tau \ge \Phi(0) + (1+o(1))t^{1/2}\,[\Phi'(t)]^{1/2},$$

where in the second step we used Lemma (3.2.5) with $\alpha = 1/2$.

For $\zeta = |\zeta|e^{i\theta}$, we set

$$h(\zeta\,) = \begin{cases} \Phi'(|\zeta|), & |\theta|\ \le\ \theta_0(|\zeta|), \\ |\zeta|^{-3/2}[\Phi'(|\zeta|)]^{-1/2}|\theta|^{-3}, & |\theta|\ >\ \theta_0(|\zeta\,|). \end{cases}$$

Using this notation and Lemma (3.2.4), we then obtain

$H(z,w) \lesssim \varrho(x,w)h(xre^{i\theta})[\Psi'(xr)]^{n-1}e^{-\frac{1}{2}(\Psi(x^2)+\Psi(r^2+|\xi|^2))-\Psi(xr)}$.By Fubini's theorem, we may compute the integral in (89) by first integrating with respect to the vector ξ over $\mathbb{C}^{n-1}$ and then taking an area integral with respect to the complex variable w_1 over $\mathbb{C}$. Since $y \mapsto \Psi(r^2+y^2)$ attains its maximum at $y = 0$ and has a second derivative larger than

$2\Psi'(r^2)$, we have that $\Psi(r^2+y^2)-\Psi(r^2)\geq\Psi'(r^2)y^2$. Using spherical coordinates along with this fact, we find that

$$\int_{\mathbb{C}^{n-1}} e^{-\Psi(r^2+|\xi|^2)}dV_{n-1}(\xi)\lesssim e^{-\Psi(r^2)}[\Psi'(r2)]^{-n+1}.$$

Similarly, again using spherical coordinates, we get

$$\int_{\mathbb{C}^{n-1}}\Theta(r,|\xi|)e^{-\Psi(r^2+|\xi|^2)}dV_{n-1}(\xi)=C\int_0^\infty\Theta(r,y)y^{2n-2}e^{-\Psi(r^2+y^2)}dy,$$

where C is the surface area of the unit sphere in $\mathbb{C}^{n-1}$ and is any suitable function of two variables. From the estimate for $\varrho(z,w)$ and (90) we see that we are interested in the following two choices: (1) $\Theta(r,y)=y[\Psi'(r^2+y^2)]^{1/2}$ and (2) $\Theta(r,y)=y^2\Psi(r^2+y^2)$. In case (1), we use the Cauchy–Schwarz inequality, so that we get

$$\int_{\mathbb{C}^{n-1}}|\xi|[\Psi'(r^2+|\xi|^2)]^{1/2}e^{-\Psi(r^2+|\xi|^2)}dV_{n-1}(\xi)$$

$$\lesssim e^{-\Psi(r^2)}\left[\int_0^\infty y^{4n-3}e^{-(\Psi(r^2+y^2)-\Psi(r^2))}dy\right]^{1/2}.$$

Estimating $\Psi(r^2+y^2)-\Psi(r^2)$ as above, we therefore get

$$\int_{\mathbb{C}^{n-1}}|\xi|[\Psi'(r^2+|\xi|^2)]^{1/2}e^{-\Psi(r^2+|\xi|^2)}dV_{n-1}(\xi)\lesssim e^{-\Psi(r^2)}[\Psi'(r^2)]^{-n+1}.$$

In case (2), we integrate by parts and get

$$\int_{\mathbb{C}^{n-1}}|\xi|^2\Psi'(r^2+|\xi|^2)e^{-\Psi(r^2+|\xi|^2)}dV_{n-1}(\xi)\lesssim\int_0^\infty y^{2n-1}e^{-\Psi(r^2+y^2)}dy.$$

We proceed as above and obtain

$$\int_{\mathbb{C}^{n-1}}|\xi|^2\Psi'(r^2+|\xi|^2)e^{-\Psi(r^2+|\xi|^2)}dV_{n-1}(\xi)\lesssim e^{-\Psi(r^2)}[\Psi'(r^2)]^{-n+1}.$$

With σ denoting Lebesgue measure on $\mathbb{C}$, we therefore get

$$\int_{\mathbb{C}^n}H(z,w)dV(w)\lesssim\int_{\mathbb{C}}G(x,r,\theta)\left[\frac{\Psi'(r\,x)}{\Psi'(r^2)}\right]^{n-1}h(xre^{i\theta})e^{-Q_x(r)}d\sigma(re^{i\theta}),$$

where

$$G(x,r,\theta)=x|\theta|[\Phi'(x^2)]^{1/2}+[\Phi'(max(x^2,r^2))]^{1/2}|x-r|+1$$

and Q_x is as defined by (83).

We now resort to polar coordinates; simple calculations show that

$$\int_{-\pi}^{\pi}h(xre^{i\theta})d\theta\lesssim\left[\frac{\Phi'(xr)}{xr}\right]^{\frac{1}{2}}and\int_{-\pi}^{\pi}|\theta|h(xre^{i\theta})d\theta\lesssim\frac{1}{xr}$$

so that

$$\int_{\mathbb{C}^n}H(z,w)dV(w)\lesssim\int_0^\infty\big(S_x(r)+T_x(r)\big)e^{-Q_x(r)}rdr,$$

where

$$S_x(r)=\left(\frac{[\Phi'(x^2)]^{1/2}}{r}+\left[\frac{\Phi'(xr)}{xr}\right]^{\frac{1}{2}}\right)\left[\frac{\Psi'(rx)}{\Psi'(r^2)}\right]^{n-1}$$

and

$$T_x(r) = \varphi(\max(x^2, r^2))|x - r| \left[\frac{\Phi'(xr)}{xr}\right]^{\frac{1}{2}} \left[\frac{\Psi'(rx)}{\Psi'(r^2)}\right]^{n-1}$$

By Lemma (3.2.9) and a straightforward argument, we find that both $S_x\, e^{-Q_x}$ and $T_x\, e^{-Q_x}$ satisfy conditions (I), (II), (III) of Sect. 3 (with $x = t, Q_x = g_t$, $x_0 = x$, and $\tau = [\Phi'(x)]^{-\alpha}$). Hence (75) applies with *m* = 0 and *m* = 1 for the respective integrands, so that we get

$$\sup_{x>0} \int_0^\infty S_x(r) e^{-Q_x(r)} r dr < \infty \text{ and } \sup_{x>0} \int_0^\infty T_x(r) e^{-Q_x(r)} r dr < \infty.$$

We may therefore conclude that (89) holds.

We now turn to a study of the relation between the spectral properties of Hankel operators and the asymptotic behavior of their symbols. We begin with the case of compact Hankel operators.

An entire function is said to be of vanishing mean oscillation with respect to
if $(MO\ f\ )(z) = o(1)$ as $|z| \to +\infty$. Entire functions of vanishing mean oscillation form a closed subspace of $BMOA(\Psi)$ which we will denote by $MOA(\Psi)$. In ac-cordance with our preceding discussion, we define the little Bloch space $ß_0(\Psi)$ as the collection of functions *f* in $ß(\Psi)$ for which

$$\sup_{\xi \in \mathbb{C}^n \setminus \{0\}} \frac{\left|\langle \nabla f\ (z), \bar{\xi} \rangle\right|}{\beta(z, \xi)} = o(1) \quad when\ |z| \to +\infty.$$

The main result reads as follows.

Theorem (3.2.12) [129]: *Let Ψ be a logarithmic growth function, and suppose that there exists a real number $\eta < 1/2$ such that* (70) *holds. If f is an entire function on $\mathbb{C}^n$, then the following statements are equivalent*:

(i) *The function f belongs to $T\ (\Psi)$ and the Hankel operator $H_{\bar{f}}$ on $\mathcal{A}^2(\Psi)$ is compact*;

(b) *The function f belongs to $VMOA(\Psi)$* ;

(iii) *The function f belongs to* $ß_0(\Psi)$.

Our proof of Theorem (3.2.12) requires the following two lemmas.

Lemma (3.2.13) [129]: *The normalized Bergman kernels $K_\Psi(\cdot, z)/\sqrt{K_\Psi(z, z)}$ converge weakly to* 0 *in $\mathcal{A}^2(\Psi)$ when $|z| \to +\infty$.*

Proof. Since the holomorphic polynomials are dense in $\mathcal{A}^2(\Psi)$, it suffices to show that for any non-negative integer *m*, we have

$$\frac{|z|^m}{\sqrt{K_\Psi(z, z)}} \to 0$$

as$|z| \to +\infty$. But this holds trivially because $K_\Psi(z, z)$ is an infinite power series in $|z|^2$with positive coefficients.

Lemma (3.2.14) [129]: *Let $f: \mathbb{C}^n \to \mathbb{C}$ be a function for which there exist positive numbers **R** and ε such that*

$$|f(z) - f(w)| \leq \varepsilon \varrho(z, w)$$

whenever $|z| \geq \boldsymbol{R}$. Then there exists a function $f_0: \mathbb{C}^n \to \mathbb{C}$ such that $f(z) = f_0(z)$ for $|z| \geq \boldsymbol{R}$ and

$$|f_0(z) - f_0(w)| \leq \varepsilon\varrho(z,w)$$

for all points z and w in $\mathbb{C}^n$.

Proof. We argue as in the proof of Lemma 5.1 in [133]. We assume without loss of generality that f is real-valued and set

$$f_0(z) := \inf_{w\in\mathbb{C}^n} \{f(w) + \varepsilon\varrho(z,w)\}.$$

Then a straightforward argument using the triangle inequality for the Bergman metric shows that f_0 has the desired properties.

We first prove the implication (i) ⇒ (ii). Assuming that $H_{\bar f}$ is compact, we obtain, using Lemma (3.2.13), that

$$[(MO\ f)(z)]^2 = \frac{\left\|H_{\bar f}K_\Psi(\cdot,z)\right\|^2}{K_\Psi(z,z)} \to 0$$

when $|z| \to +\infty$. This gives the desired conclusion.

We next note that the implication (ii) ⇒ (iii) is immediate from (72).

Finally, to prove that (iii) implies (i), in view of Theorem (3.2.1), we only need to prove that the bounded Hankel operator $H_{\bar f}$ is compact whenever (iii) is satisfied. To see that this holds, we choose an arbitrary positive ε. Assuming (iii), we may find a positive R_0 such that

$$|(\nabla f)(z), \bar\xi| \leq \frac{\varepsilon}{2}\beta(z,\xi)$$

whenever $|z| \geq R_0$ and ξ is in $\mathbb{C}^n\backslash\{0\}$. Then for some $R > R_0$ we have

$$|f(z) - f(w)| \leq \varepsilon\varrho(z,w)$$

as long as $|z| \geq R$. Indeed, this follows because $\beta(z,\xi)/|\xi\ | \to \infty$ when $|z| \to \infty$ so that, whenever $|z|$ is sufficiently large, $\varrho(z,w)$ is "essentially" determined by the contribution to the integral in (68) from the points that lie outside the ball of radius R_0 centered at 0. Now let f_0 be the function obtained from Lemma (3.2.14). We write

$$H_{\bar f} = H_{\bar f - \bar f_0} + H_{\bar f_0}$$

and observe that $\bar f - \bar f_0$ is a compactly supported continuous function on $\mathbb{C}^n$. Hence $H_{\bar f - \bar f_0}$ is compact. On the other hand, if g is a holomorphic polynomial, then

$$|H_{\bar f_0}\ g(z)| \lesssim \int_{\mathbb{C}^n} \left|\bar f_0(w) - \bar f_0(z)\right|\ |K_\Psi(z,w)g(w)|\, d\mu_\Psi\,(w)$$

$$\leq\ \varepsilon\int_{\mathbb{C}^n} \beta(z,\xi)\ |K_\Psi(z,w)g(w)|\, d\mu_\Psi\,(w)$$

so that, by the proof of Theorem (3.2.1), we see that $\left\|H_{\bar f_0}\right\| \lesssim \varepsilon$. The implication (iii) ⇒ (i) follows because ε can be chosen arbitrarily small.

In what follows, we will need the analogue of Lemma (3.2.5) for the function Ψ when $n > 1$. We will therefore assume that

$$\Psi''(t) = O\left(t^{-\frac{1}{2}}\,[\Psi'(t)]^{1+\eta}\right) when\ t\ \to\ \infty \tag{91}$$

for some $\eta < 1/2$ whenever $n > 1$. This is again a mild smoothness condition on Ψ.

Lemma (3.2.15) [129]: *Assume that* (91) *holds for some $\eta < 1/2$. Then, for any fixed $\alpha > \eta$, we have*

$$\sup_{|\tau|\le t^{1/2}\,[\Psi'(t)]^{-\alpha}} \Psi'(t+\tau) = (1+o(1))\Psi'(t)$$

when $t \to \infty$.

We are interested in describing geometrically the Bergman ball

$$B(z,a) = \{w : \varrho(z,w) < a\}.$$

Let P_z denote the orthogonal projection in $\mathbb{C}^n$ onto the complex line $\{\zeta z: \zeta \in \mathbb{C}\}$, where z is an arbitrary point in $\mathbb{C}^n\backslash\{0\}$. It will be convenient to let P_0 denote the identity map. We use the notation

$$D(z,a) = \{w: |z-P_z w| \le a[\Phi'(|z|^2)]^{-1/2}, |w-P_z w| \le a[\Psi'(|z|^2)]^{-1/2}\}.$$

Then we have the following result.

Lemma (3.2.16) [129]: *Suppose that there exists a real number* $\eta < 1/2$ *such that* (70) *holds and that* (91) *holds if* $n > 1$. *Then, for every positive number* a, *there exist two positive numbers m and M such that*

$$D(z,m) \subset B(z,a) \subset D(z,M)$$

for every z *in* $\mathbb{C}^n$.

$$\varrho(z,w) \simeq |z-P_z w|[\Phi'(|z|^2]^{1/2} + |w-P_z w|[\Psi'(|z|^2)]^{1/2} \quad (92)$$

for w in $D(z,M)$ for any fixed positive number M. (The latter term vanishes and can be disregarded when n = 1.) To begin with, we note that Theorem (3.2.2) gives that

$$\varrho(z,w) \simeq \inf_{\gamma} \int_0^1 \left(|\gamma'(t)|[\Psi'(|z|^2)]^{\frac{1}{2}} + |\langle\gamma(t),\gamma'(t)\rangle|[\Psi''(|\gamma(t)|^2)]^{\frac{1}{2}}\right) dt \;, (93)$$

where the infimum is taken over all piecewise smooth curves $\gamma: [0,1] \to \mathbb{C}^n$ such that $\gamma(0) = z$ and $\gamma(1) = w$. If we choose γ to be the line segment from z to $P_z w$ followed by the line segment from $P_z w$ to w and use that $\Psi''(x) = o([\Psi'(x)]^{1/2})$ on the latter part of , we get from (93) that

$$\varrho(z,w) \simeq |z-P_z w|[\Phi'(|z|^2]^{1/2} + |P_z w - w|[\Psi'(|z|^2)]^{1/2} + |P_z w - w|^2 o(\Psi'(|z|^2)).$$

This gives the desired bound from above because, by assumption, $|P_z w - w| \le M[\Psi'(|z|^2)]^{-1/2}$.

To prove the bound from below, we argue in the following way. Let $\ell(\gamma)$ denote the Euclidean length of γ . Set

$$\varrho_\gamma^*(z,w) = \int_0^1 \left(|\gamma'(t)|[\Psi'(|\gamma(t)|^2)]^{\frac{1}{2}} + |\langle\gamma(t),\gamma'(t)\rangle|[\Psi''(|\gamma(t)|^2]^{\frac{1}{2}}\right) dt$$

and $\varrho_\gamma^*(z,w) = inf_\gamma\, \varrho_\gamma^*(z,w)$. We observe that (93) implies that

$$\varrho_\gamma^*(z,w) \gtrsim \inf_t [\Psi'(|\gamma(t)|^2)]^{1/2}\ell(\gamma) \quad (94)$$

whenever, say, $\varrho_\gamma^*(z,w) \le 2\varrho^*(z,w)$. Since we know by the first part of the proof that $\varrho(z,w) \lesssim 1$, this implies that

$$\ell(\gamma) \lesssim \inf_t [\Psi'(|\gamma(t)|^2)]^{-1/2}.$$

By Lemma (3.2.15), we therefore have

$$\ell(\gamma) \lesssim [\Psi'(|\gamma(t)|^2)]^{-1/2},$$

which, in view of (94), in turn gives

$$\ell(\gamma) \lesssim [\Psi'(|\gamma(t)|^2)]^{-1/2}\varrho(z,w). \quad (95)$$

Now let γ be any curve such that $\varrho_\gamma^*(z,w) \le 2\varrho^*(z,w)$. We then get from (93) that

$$\varrho(z,w) \gtrsim |z-w|[\Psi'(|\gamma(t)|^2)]^{1/2} + \int_0^1 |\langle\gamma(t),\gamma'(t)\rangle|[\Psi''(|\gamma(t)|^2)]^{1/2}dt \ . (96)$$

Set $\gamma_0(t) = P_z(\gamma(t))$ and $\gamma_1(t) = \gamma(t) - \gamma_0(t)$. Note that $\gamma_1(0) = 0$ and that $\ell(\gamma_1) \le \ell(\gamma)$. By orthogonality and the triangle inequality, we get

$$\int_0^1 |\langle\gamma(t),\gamma'(t)\rangle|[\Psi''(|\gamma(t)|^2)]^{1/2}dt$$
$$\ge \int_0^1 |\gamma_0(t),\gamma_0'(t)|[\Psi''(|\gamma_0(t)|^2)]^{1/2}dt$$
$$- \int_0^1 |\gamma_1(t),\gamma_1'(t)|[\Psi''(|\gamma(t)|^2)]^{1/2}dt.$$

Let t_1 be the smallest *t* such that $|z-\gamma_0(t)| = |z - P_z w|$. Using that $\Psi''(x) = o([\Psi'(x)]^2)$ and (95), we then get

$$\int_0^1 |\langle\gamma(t),\gamma'(t)\rangle|[\Psi''(|\gamma(t)|^2)]^{1/2}dt \ \ge (1$$
$$+ o(1)) \int_0^{t_1} |z||\gamma_0'(t)|[\Psi''(|z|^2)]^{1/2}dt - [\ell(\gamma)]^2 o(\Psi'(|z|^2))$$
$$\gtrsim |z - P_z w||z|[\Psi''(|z|^2)]^{1/2} - o(1)(z,w)$$

when $|z| \to \infty$. Plugging this estimate into (96), we obtain the desired bound from below.
It follows from the previous lemma that the Euclidean volume of $B(z,r)$ can be estimated as

$$|B(z,r)| \simeq [\Phi'|z|^2)]^{-\frac{1}{2}}[\Psi'(|z|^2)]^{\frac{n-1}{2}} \qquad (97)$$

when *r* is a fixed positive number. We will now use this fact to establish two covering lemmas.

Lemma (3.2.17) [129]: *Suppose that there exists a real number* $\eta < 1/2$ *such that* (70) *holds and that* (91) *holds if* $n > 1$. *Let R be a positive number and m a positive integer. Then there exists a positive integer N such that every Bergman ball* $B(a,r)$ *with* $r \le R$ *can be covered by N Bergman balls* $B(a_k, \frac{r}{m})$.

Proof. Fix a ball $B(a,r)$. Choose $a_0 := a$ and let a_1 be a point in $\mathbb{C}^n$ such that $\varrho(a,a_1) = r/m$. Now iterate so that in the k-th step a_k is chosen as a point in the complement of $\bigcup_{j=1}^{k-1} B(a_j, r/m)$ minimizing the distance from a, and let J be the smallest k such that $\varrho(a,a_k) \ge r$. Then the balls $B(a_0,r/m),\dots,B(a_{J-1},r/m)$ constitute a covering of $B(a,r)$. By the triangle inequality, we see that the sets $B(a_j, r/(2m))$ are mutually disjoint, and they are all contained in $B(a, r + r/(2m))$ when $j < J$. Hence

$$\sum_{j=0}^{J-1} |B(a_j, r/(2m))| \le |B(a, r + r/(2m))| \ .$$

On the other hand, by (97), it follows that there is a positive number *C* depending on and m but not on a such that

$$\frac{1}{C}|B(a,r\ +\ r/(2m))| \leq |B(aj\ ,r/(2m)|)$$

for every j . We observe that it suffices to take N to be the smallest positive integer larger than or equal to C.

Inspired by the construction in the previous lemma, we introduce the following notion. We say that a sequence of distinct points (a_k) in C^n is a -lattice if there exists a positive number r such that the balls $B(a_k, r)$ constitute a covering of $\mathbb{C}^n$ and the balls $B(a_k\ , r/2)$ are mutually disjoint. Replacing a by, say, 0, and r/m by r in the previous proof, we have a straightforward way of constructing a Ψ-lattice. Note that since the balls $B(a_k\ , r/2)$ are mutually disjoint, we must have $\varrho(a_k\ , a_j) \geq r$ when $\neq j$. The number , which may fail to be unique, is called a covering radius for the Ψ -lattice (a_k). The supremum of all the covering radii is again a covering radius; it will be called the maximal covering radius for (a_k).

Lemma (3.2.18) [129]: *Suppose that there exists a real number* $\eta < 1/2$ *such that* (70) *holds and that* (91) *holds if* $n > 1$, *and let R be a positive number. Then there exists a positive integer N such that if* (a_k) *is a* Ψ*-lattice with maximal covering* $radius\ r \leq\ R/2$, *then every point z in* $\mathbb{C}^n$ *belongs to at most N of the sets* $B(a_k, 2r\ )$.

Proof. Let N be the integer obtained from Lemma (3.2.17) for the given R when $m = 4$ and assume that $z \in \bigcap_{j=1}^{N+1} B(a_{k_j}, 2r)$. Then a_{k_j} is in $B(z, 2r\ )$ for every $j = 1, \dots, N + 1$. If the sets $B(z_1, r/2), \dots, B(z_1\ , r/2)$ constitute a covering of $B(z, 2r\ )$, the existence of which is guaranteed by Lemma (3.2.17), then at least one of the sets $B(z_k\ , r/2)$ must contain two of the points $a_{k_j}\ , j\ = 1, \dots, N\ +\ 1$. On the other hand, by the triangle inequality, we have reached a contradiction because the minimal distance between any two points in the sequence (a_k) cannot be smaller than .

For a nonnegative Borel measure ν on $\mathbb{C}^n$, we set

$$d\nu_\Psi\ (z) = e^{-\Psi(|z|^2)} d\nu(z).$$

Such a measure ν is called a Carleson measure for $\mathcal{A}^2(\Psi)$ if there is a positive constant C such that

$$\int_{\mathbb{C}^n} |f(z)|^2 d\nu_\Psi(z) \leq C \int_{\mathbb{C}^n} |f(z)|^2 d\mu_\Psi(z)$$

for every function f in $\mathcal{A}^2(\Psi)$. Thus ν is a Carleson measure for $\mathcal{A}^2(\Psi)$ if and only if the embedding E_ν of $\mathcal{A}^2(\Psi)$ into the space $L^2(\nu_\Psi)$ is bounded.

Theorem (3.2.19) [129]: *Let* Ψ *be a logarithmic growth function, and suppose that there exists a real number* $\eta < 1/2$ *such that* (70) *holds and that* (91) *holds if* $n > 1$. *If* ν *is a nonnegative Borel measure on* $\mathbb{C}^n$, *then the following statements are equivalent*:

(i) ν *is a Carleson measure for* $\mathcal{A}^2(\Psi)$;

There is a constant $C > 0$ *such that*

$$\int_{\mathbb{C}^n} \frac{|K_\Psi(w, z)|^2}{K\ (z, z)} d\nu_\Psi(w) \leq C$$

for every z *in* $\mathbb{C}^n$;

For every positive number r , *there is a positive number C such that*

$$\nu(B(z, r)) \leq C|B(z, r)|$$

for every z in $\mathbb{C}^n$;

There exist a Ψ-lattice (a_k) and a positive number C such that

$$\nu(B(a_k\,,r)) \le C|B(a_k\,,r)|$$

for every point k, where r is the maximal covering radius for (a_k).

We prepare for the proof of Theorem (3.2.19) by establishing the following two lemmas.

Lemma (3.2.20) [129]: *Suppose that there exists a real number $\eta < 1/2$ such that* (70) *holds and that* (70) *holds if $n > 1$. Then there exists a positive number r_0 such that*

$$|K_\Psi(w,z)|^2 \simeq K\,(z,z)K(w,w)$$

holds for z and w whenever $\varrho(z,w) \le r_0$.

Proof. The lemma follows from Lemma (3.2.4) along with Lemma (3.2.16).

Lemma (3.2.21) [129]: *Suppose that there exists a real number $\eta < 1/2$ such that* (70) *holds and that* (91) *holds if $n > 1$, and let r_0 be the constant from Lemma* (3.2.20). *Then there is a constant C such that*

$$|f(z)|^2 e^{-\Psi(|z|^2)} \le \frac{C}{|B(z,r)|}\int_{B(z,r\,)} |f(w)|^2 d\mu_\Psi\,(w)$$

for every entire function f on $\mathbb{C}^n$ and every z in $\mathbb{C}^n$.

Proof. By Lemma (3.2.20), the holomorphic function $w \mapsto\; K(z,w)$ does not vanish at any point in $B(z,r)$. Thus the function $w \mapsto\; |f(w)|^2|K_\Psi(z,w)|^{-2}$ is subharmonic in $B(z,r)$. Choosing m as in Lemma (3.2.16), we therefore get

$$\begin{aligned}|f(z)|^2|K(z,z)|^{-2} &\lesssim \frac{1}{|D(z,m)|}\int_{D(z,m)} |f(w)|^2|K_\Psi(z,w)|^{-2}d\,V(w)\\ &\lesssim \frac{1}{|B(z,r\,)|}\int_{B(z,r\,)} |f(w)|^2|K_\Psi(z,w)|^{-2}d\,V(w)\,.\end{aligned}$$

Applying Lemma (3.2.20) to the integrand to the left and then Lemma (3.2.4) to each side, we arrive at the desired estimate.

Note that, by (97)the lemma is valid for all positive r , with the additional proviso that C depends on r .

We begin by noting that the implication (i) ⇒ (ii) is trivial because it is just the statement that the Carleson measure condition holds for the functions $K\,(\cdot,z)$. To prove that (ii) implies (iii), we assume that (ii) holds and consider a ball $B(z,r)$ where r is a fixed positive number. Then, by Lemma (3.2.20) and (97), we have

$$\frac{1}{|B(z,r\,)|} \lesssim \frac{|K_\Psi(z,w)|^2}{K(z,z)} e^{-\Psi(|z|^2)}$$

when $\varrho(z,w) \le r_0$, and therefore we obtain

$$\frac{\nu(B(z,r))}{|B(z,r)|} \lesssim \int_{\mathbb{C}^n} \frac{|K_\Psi(z,w)|^2}{K_\Psi(z,z)} e^{-\Psi(|w|^2)} d\nu(w) \le C\,.$$

The implication (iii) ⇒ (iv) is trivial (modulo the existence of Ψ-lattices), and we are therefore done if we can prove that (iv) implies (i). To this end, assume that (iv) holds, and let (a_k) be a Ψ-lattice with maximal covering radius . By Lemma (3.2.21), we see that

$$\sup_{z\in B(a_k,r)} |f(z)|^2 e^{-\Psi(|z|^2)}| \lesssim \frac{1}{|B(a_k,2r)|}\int_{|B(a_k,2r)|} |f(w)|^2 d\mu_\Psi(z)$$

for every k. We therefore get

$$\int_{\mathbb{C}^n} |f(z)|^2 d\mu_\Psi(z) \lesssim \sum_k \int_{B(a_k,2r)} |f(w)|^2 d\mu_\Psi(w) \lesssim \int_{\mathbb{C}^n} |f(w)|^2 d\mu_\Psi(z),$$

where the latter inequality holds by Lemma (3.2.18).

For ν a nonnegative Borel measure on $\mathbb{C}^n$, we define the Toeplitz operator T_ν on $\mathcal{A}^2(\Psi)$in the following way:

$$(T_\nu f)(z) := \int_{\mathbb{C}^n} f(w) K_\Psi(z,w) e^{-\Psi(|w|^2)} d\nu(w).$$

computation shows that $E_\nu^* E_\nu = T_\nu$. Thus Theorem (3.2.19) characterizes bounded Toeplitz operators. Compact Toeplitz operators can likewise be characterized via so-called vanishing Carleson measures; an obvious and straightforward modification of Theorem (3.2.19) gives a description of such measures. Toeplitz operators belonging to the Schatten classes S_p are characterized by the following theorem.

Theorem (3.2.22) [129]: *Let Ψ be a logarithmic growth function, and suppose that there exists a real number $\eta < 1/2$ such that* (70) *holds and that* (91) *holds if $n > 1$. If ν is a nonnegative Borel measure on $\mathbb{C}^n$ and $p \geq 1$, then the following statements are equivalent*:

(i) *The Toeplitz operator T_ν on $\mathcal{A}^2(\Psi)$ belongs to the Schatten class S_p* ;

(ii) *There exists a Ψ-lattice (a_k) such that*

$$\sum_{k=1}^{\infty} \left(\frac{\nu(B(a_k,r))}{|B(a_k,r)|}\right)^p < +\infty,$$

where r is the maximal covering radius for (a_k).

For the proof of this theorem, we require the following two lemmas.

Lemma (3.2.23) [129]: *Suppose that (e_j) is an orthonormal basis for$\mathcal{A}^2(\Psi)$ and that (a) is a Ψ-lattice. Then the operator J on $\mathcal{A}^2(\Psi)$ defined by*

$$J\, e_j(z) := \frac{K_\Psi(z,a_j)}{\sqrt{K_\Psi(a_j,a_j)}}$$

is bounded.

Proof. For two arbitrary functions $f = \sum_j c_j e_j$ and g in $\mathcal{A}^2(\Psi)$, the reproducing formula and the Cauchy–Schwarz inequality give

$$|\langle Jf, g\rangle|^2 = \left|\sum_j c_j \frac{\overline{g(a_j)}}{\sqrt{K_\Psi(a_j,a_j)}}\right|^2 \leq \left(\sum_j |c_j|^2\right)\left(\sum_k \frac{|g(ak)|^2}{K_\Psi(a_k,a_k)}\right).$$

If we set

$$\nu := \sum_k \frac{e^{\Psi(|a_j|^2)}}{K_\Psi(a_j,a_j)} \delta_{a_j}$$

then we may write this estimate as

$$|\langle Jf,g\rangle|^2 \le \ \|f\|^2_{\mathcal{A}^2(\Psi)} \int_{\mathbb{C}^n} |g(z)|^2 d\nu_\Psi(z).$$

By Theorem (3.2.19), we see that ν is a Carleson measure, which implies that J is a bounded operator on $\mathcal{A}^2(\Psi)$.

Lemma (3.2.24) [129]: *Suppose that T is a positive operator on $\mathcal{A}^2(\Psi)$. Then the trace of T can be computed as*

$$Tr(T) = \int_{\mathbb{C}^n} \tilde{T}(z) K_\Psi(z,z) d\mu_\Psi\,(z).$$

Proof. We write $K_\Psi(z,w) = \sum_{k=0}^{\infty} e_k(z)\overline{e_k(w)}$, where (e_k) is an orthonormal basis for $\mathcal{A}^2(\Psi)$. The lemma is then proved by means of the following computation:

$$Tr(T\,) = \sum_{k=0}^{\infty} \langle Tf_k, f_k\rangle_{\mathcal{A}^2(\Psi)} \ = \int_{\mathbb{C}^n} \langle TK_\Psi(.\,,z), K_\Psi(.\,,z)\rangle_{\mathcal{A}^2(\Psi)}\ d\mu_\Psi(z).$$

We begin by assuming that T_ν is in S_p . Pick a Ψ-lattice (a_j) and let r be its maximal covering radius. By (97) and Lemma (3.2.20), we have

$$\sum_k \left(\frac{\nu(B(a_k\,,r))}{|B(a_k\,,r)|}\right)^p \simeq \sum_k \left(\int_{B(a_k\,,r\,)} K_\Psi(w,w) d\nu_\Psi(w)\right)^p$$

$$\simeq \sum_k \left(\int_{B(a_k\,,r\,)} \frac{|K_\Psi(a_k,w)|^2}{K_\Psi(a_k,a_k)} d\nu_\Psi(w)\right)^p .$$

By Lemma (3.2.18) and our assumption on ν, this gives

$$\sum_k \left(\frac{\nu(B(a_k,r))}{|B(a_k\,,r)|}\right)^p \lesssim \sum_k \left(\int_{\mathbb{C}^n} \frac{|K_\Psi(a_k,w)|^2}{K_\Psi(a_k,a_k)} d\mu_\Psi(w)\right)^p$$

If we construct J as in Lemma (3.2.23), then the right-hand side equals $\sum_k |\langle J^*\, T_\nu\, Je_k\,, e_k\rangle|^p$. Since J is a bounded operator, $J^*\, T_\nu\, J$ also belongs to S_p , and so the latter sum converges. We conclude that (i) implies (ii).

We will use an interpolation argument to prove that (ii) implies (i). We already know from Theorem (3.2.19) that T_ν is in the Schatten class S_∞ whenever $\nu(B(a_k,r)) \le C|B(a_k,r)|$ for some positive constant C. Suppose now that

$$\sum_k \frac{\nu(B(a_k,r))}{|B(a_k\,,r)|} < +\infty,$$

and let (e_j) be an orthonormal basis for $\mathcal{A}^2(\Psi)$. By the reproducing formula, we have

$$\langle T_\nu\, e_j, e_j\rangle = \int_{\mathbb{C}^n} |e_j(w)|^2 d\,\nu_\Psi(w),$$

which implies that

$$\sum_j |\langle T_\nu\, e_j, e_j\rangle| = \int_{\mathbb{C}^n} K_\Psi(w,w) d\nu_\Psi(w) \le \sum_k \int_{B(a_k\,,r\,)} K_\Psi(w,w) d\nu_\Psi(w)\,.$$

Again using Lemma (3.2.4), we then get

$$\sum_j |\langle T_\nu\, e_j, e_j\rangle| \lesssim \sum_k \frac{\nu(B(a_k, r))}{|B(a_k\,, r)|} < +\infty,$$

which means that T_ν belongs to S_1. By interpolation, we conclude that (ii) implies (i).
We remark that the theorems proved generalize results for the classical Fock space when $n = 1$ obtained recently in [140]. It may be noted that Theorem (3.2.19) above could be elaborated to include two additional conditions for membership in S_p, in accordance with Theorem 4.4 in [141]. The proof would be essentially the same as the proof of the latter theorem. Note that [142] also treats Schatten class membership of Toeplitz operators for $p < 1$.
We suggest two possible definitions of Besov spaces, in accordance with our respective definitions of $BMOA(\Psi)$ and ß(Ψ) . We let ß$_m^p(\Psi)$ denote the set of entire functions f such that

$$\int_{\mathbb{C}^n} [MO\ f(z)]^p\ K_\Psi(z, z) d\mu_\Psi(z) < \infty;$$

for a function $h: \mathbb{C}^n \to \mathbb{C}^n$, we set

$$|h(z)|_\beta = \sup_{\xi\in\mathbb{C}^n\setminus\{0\}} \frac{|\langle h(z), \bar{\xi}\rangle|}{\beta(z, \xi\ )},$$

and we let ß$_d^p(\Psi)$be the set of entire functions f for which

$$\int_{\mathbb{C}^n} |\nabla f\ z)|_\beta^p\ K_\Psi(z, z) d\mu_\Psi(z) < \infty.$$

These definitions are in line with those of K. Zhu for Hankel operators on the Bergman space of the unit ball in $\mathbb{C}^n$ [143].
It is immediate from (72) that ß$_m^p(\Psi)$ $\subset$ ß$_d^p(\Psi)$. The basic question is whether these spaces coincide and in fact characterize Schatten class Hankel operators with anti-holomorphic symbols. The following theorem gives an affirmative answer to this question.
Theorem (3.2.25) [129]: *Let Ψ be a logarithmic growth function, and suppose that there exists a real number $\eta < 1/2$ such that* (70) *holds and that* (91) *holds if $n > 1$. If f is an entire function on $\mathbb{C}^n$ and $p \geq 2$, then the following statements are equivalent*:
(i) *The function f belongs to $\mathcal{T}(\Psi)$and the Hankel operator $H_{\bar{f}}$ on $\mathcal{A}^2(\Psi)$ is in the Schatten class S_p* ;
(ii) The function f belongs to ß$_m^p(\Psi)$;
(iii) *The function f belongs to* ß$_d^p(\Psi)$.
Proof. We have already observed that the implication (ii) ⇒ (iii) is an immediate consequence of (72). The implication (i) ⇒ (ii) relies on the following general Hilbert space argument. If (i) holds, then the operator $[H_{\bar{f}}^*\ H_{\bar{f}}]^{\frac{p}{2}}$ is in the trace class S_1. Applying Lemma (3.2.24) and using the spectral theorem along with Hölder's inequality, we obtain

$$Tr\left([H_{\bar f}^* H_{\bar f}]^{\frac{p}{2}}\right) = \int_{\mathbb{C}^n} \langle [H_{\bar f}^* H_{\bar f}]^{\frac{p}{2}} K_\Psi(.,z), K_\Psi(.,z)\rangle \, d\mu_\Psi(z)$$
$$\gtrsim \int_{\mathbb{C}^n} \left[\frac{\|H_{\bar f} K_\Psi(z,z)\|^2}{K_\Psi(z,z)}\right]^{\frac{p}{2}} K_\Psi(z,z) d\mu_\Psi(z).$$

Recalling the computation made in (71), we arrive at (ii).

Our proof of the implication (iii) ⇒ (i) will use a version of L. Hörmander's L^2 estimates for the $\bar\partial$ operator. To this end, write $\Delta_\Psi(z) = \Psi(|z|^2)$ and observe that

$$\alpha^2(z,\xi) := \sum_{j,k=1}^{n} \frac{\partial^2 \Delta_\Psi(z)}{\partial_{z_j}\partial_{\bar z_k}} \xi_j \, \bar\xi_k = |\xi|^2 \Psi'(|z|^2) + |\langle z,\xi\rangle|^2 \Psi''(|z|)$$

for arbitrary vectors $z = (z_1, \dots, z_n)$ and $\xi = (\xi_1, \dots, \xi_n)$ in $\mathbb{C}^n$. By Theorem (3.2.2), we therefore have $\alpha(z,\xi) \simeq \beta(z,\xi)$. Now let $L^2_\beta(\mu)$ be the space of vector-valued functions $h = (h_1, \dots, h_n)$, identified with the corresponding $(0,1)$-forms $h_1 d\bar z_1 + \cdots + h_n d\bar z_n$ such that

$$\|h\|^2_{L^2_\beta(\mu_\Psi)} := \int_{\mathbb{C}^n} |h(z)|^2_\beta \, d\mu_\Psi(z) < \infty.$$

It follows from Theorem 2.2 in [144] (a special case of a theorem proved by J.-P. Demailly in [145]) that the operator S giving the canonical solution to the $\bar\partial$-problem is bounded from $L^2_\beta(\mu_\Psi)$ into $L^2(\mu_\Psi)$.

Since f is holomorphic, we have

$$\partial(H_{\bar f} g) = \overline{\nabla f} g$$

when g is in $\mathcal{A}^2(\Psi)$, whence $H_{\bar f} g = S(\overline{\nabla f} g)$. Thus it follows that

$$\|H_{\bar f} g\|_{L^2(\mu_\Psi)} \lesssim \int_{\mathbb{C}^n} |\nabla f(z)|^2_\beta |g(z)|^2 d\,\mu_\Psi(z). \qquad (98)$$

If we set $d\nu(z) = |\nabla f(z)|^2_\beta \, dV(z)$, this may be written as

$$H_{\bar f}^* H_{\bar f} \lesssim M^*_{|\nabla f|_\beta} M_{|\nabla f|_\beta} = T_\nu ,$$

where as before $M_h$denotes the operator of multiplication by h from $\mathcal{A}^2(\Psi)$ into $L^2(\mu_\Psi)$.

By Theorem E, it remains to verify that (iii) implies that for some Ψ –lattice (a_k) we have

$$\sum_{k=1}^{\infty} \left(\frac{\nu(B(a_k,r))}{|B(a_k,r)|}\right)^{p/2} < +\infty, (99)$$

where r is the maximal covering radius for (a_k). To this end, we first observe that Hölder's inequality gives that

$$\left(\frac{\nu(B(z,r))}{|B(z,r)|}\right)^{p/2} \lesssim \frac{1}{|B(z,r)|} \int_{B(z,r)} |\nabla f(z)|^p_\beta \, dV(w) .$$

Hence, using (97) and Lemma (3.2.4), we obtain

$$\left(\frac{\nu(B(z,r))}{|B(z,r)|}\right)^{p/2} \lesssim \int_{B(z,r)} |\nabla f(z)|^p_\beta \, K(z,z) dV(w) .$$

Now choosing any Ψ-lattice (a_k) and using Lemma (3.2.18), we arrive at (99).

Several remarks are in order. First, note that (98) gives another proof of the implication (iii) ⇒ (i) in Theorem (3.2.1), subject to the additional smoothness condition (91). Second, as shown in [146], there are nontrivial Hankel operators in S_p only when $p > 2n$. This fact is easy to see from Theorem (3.2.25) when $n = 1$, because then

$$|\nabla f(z)|_\beta \simeq f'(z)|[\Phi'(|z|^2)]^{-1/2},$$

whence f is in $ß_d^p(\Psi)$ if and only if

$$\int_{\mathbb{C}} |f'(z)|^p[\Phi'(|z|^2)]^{1-p/2}dV(z) < \infty. \tag{100}$$

When $n > 1$, the computation of $|\nabla f(z)|_\beta$ is less straightforward, but we always have

$$|\nabla f(z)|[\Phi'(|z|^2)]^{-1/2} \lesssim |\nabla f(z)|_\beta \lesssim |\nabla f(z)|[\Psi'(|z|^2)]^{-1/2}.$$

The estimate from above shows that the condition

$$\int_{\mathbb{C}^n} |\nabla f(z)|^p\Phi'(|z|^2[\Psi'(|z|^2)]^{n-1-\frac{p}{2}}dV(z) < \infty \tag{101}$$

is sufficient for f to belong to $ß_d^p(\Psi)$, and the estimate from below shows that this is also necessary when $\Phi'/\Psi'$is a bounded function. We conclude from (100) and (101) that if the growth of Ψ' is super-polynomial, then $ß_d^p(\Psi)$ is infinite-dimensional and contains all polynomials if and only if $p > 2n$. This is immediate when $n = 1$, and it follows also when $n > 1$ because

$$\int_0^\infty \frac{\Psi''(t)}{[\Psi'(t)]^{1+\delta}}dt \le \frac{1}{\delta[\Psi'(0)]^\delta} < \infty$$

for every $\delta > 0$. If, on the other hand, is a polynomial, then Φ'/Ψ' is a bounded function, and one may use (101) and Theorem (3.2.25) to deduce Theorem B in [147]. It is not hard to check that if f is a monomial and $n > 1$, then

$$|\nabla f(z)|_\beta \simeq |\nabla f(z)||[\Psi'(|z|^2)]^{-1/2}$$

for z belonging to a set of infinite volume measure. By Lemma 2.12 in [148] and Theorem (3.2.25) above, one may therefore conclude as in [149] that $ß_d^p(\Psi)$ is nontrivial only if $p > 2n$.

Chapter 4
Localization and Compactness with Essential Commutant

We show the weighted Bargmann-Fock space setting we show that the reproducing kernel thesis for compactness holds for operators satisfying similar growth conditions. The main results extend the results of Xia and Zheng to the case of the Bergman space when $1 < p < \infty$, and in the weighted Bargmann-Fock space setting, our results provide new, more general conditions that imply the work of Xia and Zheng via a more familiar approach that can also handle the $1 < p < \infty$ case. For $\mathcal{T}$ be the C^*-algebra generated by the Toeplitz operators $\{T_f : f \in L^\infty(\boldsymbol{B}, dv)\}$on the Bergman space of the unitball. We show that the essential commutant of T equals $\{T_g : g \in VO_{bdd}\} + \mathcal{K}$, where VO_{bdd} is the collection of bounded functions of vanishing oscillation on **B** and $\mathcal{K}$ denotes the collection of compact operators on $L_a^2(\boldsymbol{B}, dv)$.

Section (4.1): Bergman and Fock Spaces

The Bargmann-Fock space $\mathcal{F}^p := \mathcal{F}^p(\mathbb{C}^n)$ is the collection of entire functions f on $\mathbb{C}^n$ such that $f(\cdot)e^{-\frac{|\cdot|}{2}} \in L^p(\mathbb{C}^n, dv)$. It is well known that $\mathcal{F}^2$ is a reproducing kernel Hilbert space with reproducing kernel given by $K_z(w) = e^{\bar{z}w}$. As usual, we denote by k_z the normalized reproducing kernel at z. For a bounded operator T on $\mathcal{F}^p$, the Berezin transform of T is the function defined by

$$T(z) = \langle Tk_z, k_z\rangle_{\mathcal{F}^2}.$$

It was proved recently by Bauer that the vanishing of the Berezin transform is sufficient for compactness whenever the operator is in the Toeplitz algebra [150]. However, it is generally very difficult to check whether a given operator T is in the Toeplitz algebra, unless T is itself a Toeplitz operator or a combination of a few Toeplitz operators, and as such one would like a "simpler" sufficient condition to guarantee this.

In the recent and interesting [151], Xia and Zheng introduced a class of "sufficiently localized" operators on $\mathcal{F}^2$ which includes the algebraic closure of the Toeplitz operators. These are the operators T acting on $\mathcal{F}^2$ such that there exist constants $2n < \beta < \infty$ and $0 < C < \infty$ with

$$|\langle Tk_z, k_w\rangle_{\mathcal{F}^2}| \leq \frac{C}{(1 + |z - w|)^\beta}. \tag{1}$$

It was proved by Xia and Zheng that every bounded operator T from the C^* algebra generated by sufficiently localized operators whose Berezin transform vanishes at infinity, i.e.,

$$\lim_{|z|\to\infty} \langle Tk_z, k_z\rangle_{\mathcal{F}^2} = 0 \tag{2}$$

is compact on $\mathcal{F}^2$. One of their main innovations is providing an easily checkable condition (1) which is general enough to imply compactness from the seemingly much weaker condition (2).

We extend the Xia-Zheng notion of suffi-ciently localized operators to both a much wider class of weighted Fock spaces (in particular, the class of so-called "generalized Bargmann-Fock spaces" considered in [152]) and to a larger class of operators. Note that (1) easily implies

$$\sup_{z\in\mathbb{C}^n}\int_{\mathbb{C}^n} |\langle Tk_z\,,k_w\rangle_{\mathcal{F}^2}|\; dv(w)\; <\; \infty\,;$$

and consequently one should look at generalizations of sufficiently localized operators that allow for weaker integral conditions. Also, note that the ideas in [150] are essentially frame theoretic (see [151] for a discussion of the ideas in [150] from this point of view) and therefore one cannot easily extend these ideas to the non-Hilbert space setting. We will provide a simpler, more direct proof of the main result in [150] which follows a more traditional route and which can be extended to other (not necessarily Hilbert) spaces of analytic functions. In particular, we show that our main result, in an appropriately modified form, holds for the classical Bergman space A^p on the ball we will discuss the possibility of extending our results to a very wide class of weighted Bergman spaces.)

The extension of the main results in [150] to a larger class of operators and to a wider class of weighted Fock spaces is as follows. Let $d^c = \frac{i}{4}\,(\bar{\partial}\, -\, \partial)$ and let *d* be the usual exterior derivative. For the rest let $\phi \in C^2(\mathbb{C}^n)$ be a real valued function on $\mathbb{C}^n$ such that

$$c\omega_0 <\, dd^c\phi\, < C\omega_0$$

holds uniformly pointwise on $\mathbb{C}^n$ for some positive constants *c* and *C* (in thesense of positive (1, 1) forms) where $\omega_0 = dd^c|\cdot|^2$ is the standard Euclidean Kähler form. Furthermore, for $0 < p \le \infty$, define the generalized Bargmann-Fock space F_ϕ^p to be the space of entire functions *f* on $\mathbb{C}^n$ such that $fe^{-\phi} \in L^p(\mathbb{C}^n, dv)$ (for a detailed study of the linear space properties of $\mathcal{F}_\phi^p$ see [153]). For operators *T* acting on the reproducing kernels $(z,w)\; of\; \mathcal{F}_\phi^2$, we impose the following conditions. We first assume that

$$\sup_{z\in\mathbb{C}^n}\int_{\mathbb{C}^n}\left|\langle Tk_z\,,k_w\rangle_{\mathcal{F}_\phi^2}\right|\, dv(w) < \infty\,, \sup_{z\in\mathbb{C}^n}\int_{\mathbb{C}^n}\left|\langle T^*k_z\,,k_w\rangle_{\mathcal{F}_\phi^2}\right|\, dv(w) < \infty, \quad (3)$$

which is enough to conclude that the operator *T* initially defined on the linear span of the reproducing kernels extends to a bounded operator on $\mathcal{F}_\phi^p$ for $1 \le p \le \infty$. To show that the operator is compact, we impose the following additional assumptions on *T*:

$$\lim_{r\to\infty}\sup_{z\in\mathbb{C}^n}\int_{D(z,r)^c}\left|\langle Tk_z\,,k_w\rangle_{\mathcal{F}_\phi^2}\right|\, dv(w) = 0,\quad \lim_{r\to\infty}\sup_{z\in\mathbb{C}^n}\int_{D(z,r)^c}\left|\langle T^*k_z\,,k_w\rangle_{\mathcal{F}_\phi^2}\right|\, dv(w)\; =\; 0. \quad (4)$$

Definition (4.1.1)[149]: We will say that a linear operator *T* on $\mathcal{F}_\phi^p$ is weakly localized (and for convenience write $T \in A_\phi(\mathbb{C}^n)$) if it satisfies the conditions (3) and (4).

Note that every sufficiently localized operator on $\mathcal{F}^2$ in the sense of Xia and Zheng obviously satisfies (3) and (4) and is therefore weakly localized in our sense too. Now if $D(z,r)$ is the Euclidean ball with center z and radius *r*, and if $\|T\|_e$ denotes the essential norm of a bounded operator T on $\mathcal{F}_\phi^p$ then the following theorem is one of the main results:

Theorem (4.1.2) [149]: *Let* $1 < p < \infty$ *and let T be an operator on* $\mathcal{F}_\phi^p$ *which belongs to the norm closure of* $A_\phi(\mathbb{C}^n)$). *Then there exists* $r, C > 0$ *(both depending on T) such that*

$$\|T\|_e \leq C \limsup_{|z|\to\infty} \sup_{w\in D(z,r)} |\langle Tk_z, k_w\rangle| .$$

In particular, if

$$\lim_{|z|\to\infty} \|Tk_z\|_{\mathcal{F}^p_\phi} = 0$$

then T is compact on F^p_ϕ.

Now if $A(\mathbb{C}^n)$ is the class of sufficiently localized operators on $\mathcal{F}^2$ then note that an application of Proposition 1.4 in [155] in conjunction with Theorem (4.1.2) immediately proves the following theorem, which provides the previously mentioned generalization of the results in [150].

Theorem (4.1.3) [149]: *Let* $1 < p < \infty$ *and let* T *be an operator on* $\mathcal{F}^p$ *which belongs to the norm closure of* $A(\mathbb{C}^n)$. *If* $\lim_{|z|\to\infty} |\langle Tk_z, k_z\rangle_{\mathcal{F}^2}| = 0$ *then T is compact.*

We write the so called "Fock-Sobolev spaces" from [154] as generalized Bargmann-Fock spaces, so that in particular Theorem (4.1.2) immediately applies to these spaces (see [155]).

To state the main result in the Bergman space. Let $\mathbb{B}_n$ denote the unit ball in $\mathbb{C}^n$ and let the space $A^p := A^p(\mathbb{B}_n)$ denote the classical Bergman space, i.e., the collection of all holomorphic functions on $\mathbb{B}_n$ such that

$$\|f\|^p_{A^p} := \int_{\mathbb{B}_n} |f(z)|^p \, dv(z) < \infty.$$

The function $K_z(w) := (1 - \bar{z}w)^{-(n+1)}$ is the reproducing kernel for A^2 and

$$K_z(w) := \frac{(1-|z|^2)^{\frac{n+1}{2}}}{(1 - \bar{z}w)^{(n+1)}}$$

is the normalized reproducing kernel at the point z. We also will let $d\lambda$ denote the invariant measure on $\mathbb{B}_n$, i.e.,

$$d\lambda(z) = \frac{dv(z)}{(1-|z|^2)^{n+1}}$$

Now let $1 < p < \infty$ and let $\frac{1}{p} + \frac{1}{p'} = 1$. We are interested in operators T acting on the reproducing kernels of A^2 that satisfy the following conditions. First, we assume that there exists $0 < \delta < min\{p, p'\}$ such that

$$\sup_{z\in\mathbb{B}_n} \int_{\mathbb{B}_n} |\langle Tk_z, k_w\rangle_{A^2}| \frac{\|K_z\|_{A^2}^{1-\frac{2\delta}{p'(n+1)}}}{\|K_w\|_{A^2}^{1-\frac{2\delta}{p'(n+1)}}} d\lambda(w) < \infty,$$

$$\sup_{z\in\mathbb{B}_n} \int_{\mathbb{B}_n} |\langle T^*k_z, k_w\rangle_{A^2}| \frac{\|K_z\|_{A^2}^{1-\frac{2\delta}{p(n+1)}}}{\|K_w\|_{A^2}^{1-\frac{2\delta}{p(n+1)}}} d\lambda(w) < \infty \quad (5)$$

These are enough to conclude that the operator T initially defined on the linear span of the reproducing kernels extends to a bounded operator on A^p (see the comments following the proof of Proposition (4.1.10)). To treat compactness we assumptions on T: there exists $0 < \delta < min\{p, p'\}$ such that

$$\sup_{z\in\mathbb{B}_n} \int_{\mathrm{D}(z,\mathrm{r})^{\mathrm{c}}} |\langle Tk_z, k_w\rangle_{A^2}| \frac{\|K_z\|_{A^2}^{1-\frac{2\delta}{p'(n+1)}}}{\|K_w\|_{A^2}^{1-\frac{2\delta}{p'(n+1)}}} d\lambda(w) \to 0,$$

$$\sup_{z\in\mathbb{B}_n} \int_{\mathrm{D}(z,\mathrm{r})^{\mathrm{c}}} |\langle T^*k_z, k_w\rangle_{A^2}| \frac{\|K_z\|_{A^2}^{1-\frac{2\delta}{p(n+1)}}}{\|K_w\|_{A^2}^{1-\frac{2\delta}{p(n+1)}}} d\lambda(w) \to 0 \quad (6)$$

as $r \to \infty$.

Definition (4.1.4) [149]: We say that a linear operator T on A^p is p weakly localized (which we denote by $T \in A_p(\mathbb{B}_n)$) if it satisfies conditions (5) and (6).

Note that the condition $0 < \delta < min\{p, p'\}$ implies that both $1 - \frac{2\delta}{p(n+1)}$ and $1 - \frac{2\delta}{p'(n+1)}$ are strictly between $\frac{n-1}{\mathrm{n}+1}$ and 1. Furthermore, note that when $p = p' = 2$, we have that $\frac{n-1}{\mathrm{n}+1} < 1 - \frac{\delta}{(n+1)} < 1$ precisely when $0 < \delta < 2$. Thus, in this case we can rewrite condition (5) in the following simpler way: there exists $\frac{n-1}{\mathrm{n}+1} < a < 1$ where

$$\sup_{z\in\mathbb{B}_n} \int_{\mathbb{B}_n} |\langle Tk_z, k_w\rangle_{A^2}| \frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a} d\lambda(w) < \infty, \sup_{z\in\mathbb{B}_n} \int_{\mathbb{B}_n} |\langle T^*k_z, k_w\rangle_{A^2}| \frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a} d\lambda(w) < \infty$$

We can similarly rewrite condition (6) when $p = 2$.

We prove the following result.

Theorem (4.1.5) [149]: *Let $1 < p < \infty$ and let* T *be an operator on A^p which belongs to the norm closure of $A^p(\mathbb{B}_n)$. If*

$$\lim_{|z|\to 1} \langle Tk_z, k_z\rangle_{A^2} = 0$$

then T is compact.

It will be clear that the method of proof also will work for the weighted Bergman space A^p_α.

Note that this result is known through deep work of Suàrez, [159] in the case of A^p when the operator T belongs to the Toeplitz algebra generated by L^∞ symbols (see also [157] for the case of weighted Bergman spaces.) We will prove below that the Toeplitz algebra on A^p generated by L^∞ symbols is a subalgebra of the norm closure of $\mathcal{A}_p(\mathbb{B}_n)$. In particular, the results provide a considerably simpler proof of the main results in [158] for the $p \neq 2$ situation (though it should be noted that a similar simplification when $p = 2$ was provided in [156]).

We provide the extension of the the Xia and Zheng result to the Bergman space on the unit ball $\mathbb{B}_n$, and in particular we prove Theorem (4.1.5). we prove Theorems (4.1.2) and (4.1.3) which provides an extension of the Xia and Zheng result in the case of the generalized Bargmann-Fock spaces. Finally we will briefly discuss some interesting open problems related to these results.

Let φ_z be the Möbius map of $\mathbb{B}_n$ that interchanges 0 and z. It is well known that

$$1-|\varphi_z(w)|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-\bar{z}w|^2},$$

and as a consequence we have that

$$|\langle k_z, k_w\rangle_{A^2}| = \frac{1}{\left\|K_{\varphi_z(w)}\right\|_{A^2}}. \tag{7}$$

Using the automorphism φ_z , the pseudohyperbolic and Bergman metrics on $\mathbb{B}_n$ are defined by

$$\rho(z,w) := |\varphi_z(w)| \ and \ \ \beta(z,w) := \frac{1}{2} log \frac{1+\rho(z,w)}{1-\rho(z,w)}.$$

Recall that these metrics are connected by $\rho = \frac{e^{2\beta}-1}{e^{2\beta}+1} = \ tanh\, \beta$ and it is well-known that these metrics are invariant under the automorphism group of $\mathbb{B}_n$. We let

$$D(z,r) := \{w \in \mathbb{B}_n : \beta(z,w) \le r\} = \{w \in \mathbb{B}_n : \rho(z,w) \le s = tanh\, r\},$$

denote the hyperbolic disc centered at z of radius r. Recall also that the orthogonal (Bergman) projection of $L^2(\mathbb{B}_n, dv)$ onto A^2 is given by the integral operator

$$P(f)(z) := \int_{\mathbb{B}_n} \langle K_w, K_z \rangle_{A^2} f(w) dv(w).$$

Therefore, for all $f \in A^2$ we have

$$f(z) = \int_{\mathbb{B}_n} \langle f, k_w\rangle_{A^2}\, k_w(z)\, d\lambda(w). \tag{8}$$

As usual an important ingredient in our treatment will be the Rudin-Forelli estimates, see [151] or [156]. Recall the standard Rudin-Forelli estimates:

$$\int_{\mathbb{B}_n} \frac{|\langle K_z, K_w\rangle_{A^2}|^{\frac{r+s}{2}}}{\|K_z\|_{A^2}^{s} \|K_w\|_{A^2}^{r}} d\lambda(w) \le C = C(r,s) < \infty, \forall_z \in \mathbb{B}_n \tag{9}$$

for all $r > \kappa > s > 0$, where $\kappa = \kappa_n := \frac{2n}{n+1}$. We will use these in the following form: For all $\frac{n-1}{n+1} < a < 1$ we have that

$$\int_{\mathbb{B}_n} |\langle k_z, k_w\rangle_{A^2}| \frac{\|K_z\|_{A^2}^{a}}{\|K_w\|_{A^2}^{a}} d\lambda(w) \le C = C(a) < \infty, \forall z \in \mathbb{B}_n. \tag{10}$$

To see that this is true in the classical Bergman space setting, for a given $\frac{n-1}{n+1} < a < 1$ set $r = 1 + a$ and $s = 1 - a > 0$. Then $r + s = 2$, and since $a > \frac{n-1}{n+1}$ we have that $r = 1 + a > \frac{2n}{n+1}$. Furthermore since $0 < a < 1$ we have that $0 < s < 1 \le \frac{2n}{n+1}$. By plugging this in (9) we obtain (10).

We will also need the following uniform version of the Rudin-Forelli estimates.

Lemma (4.1.6) [149]: *Let* $\frac{n-1}{n+1} < a < 1$. *Then*

$$\lim_{R\to\infty}\sup_{z\in\mathbb{B}_n}\int_{\mathrm{D}(z,\mathrm{R})^c}|\langle k_z,k_w\rangle_{A^2}|\frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a}d\lambda(w) = 0 . \tag{11}$$

Proof. Notice first that

$$\int_{\mathrm{D}(z,\mathrm{R})^c}|\langle k_z,k_w\rangle_{A^2}|\frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a}d\lambda(w) = \int_{\mathrm{D}(0,\mathrm{R})^c}\left|\langle k_z,k_{\varphi_z(w)}\rangle_{A^2}\right|\frac{\|K_z\|_{A^2}^a}{\left\|K_{\varphi_z(w)}\right\|_{A^2}^a}d\lambda(w)$$

$$= \int_{\mathrm{D}(0,\mathrm{R})^c}\frac{|\langle k_z,k_w\rangle_{A^2}|}{\|K_w\|_{A^2}^{1+a}}d\lambda(w)$$

$$= \int_{\mathrm{D}(0,\mathrm{R})^c}\frac{dv(w)}{|1-\bar{w}z|^{(n+1)a}(1-|w|^2)^{\frac{n+1}{2}(1-a)}}$$

$$= \int_{R'}^1\int_{\mathbb{S}_n}\frac{r^{2n-1}d\xi dr}{\left|1-zr\bar{\xi}\right|^{(n+1)^a}(1-r^2)^{\frac{n+1}{2}(1-a)}}$$

where in the last integral $R = log\,\frac{1+R'}{1-R'}$. Notice that $R' \to 1$ when $R\to\infty$ and note that the last integral can be written as

$$\int_{R'}^1 I_{(n+1)a-n}(rz)\,\frac{r^{2n-1}dr}{(1-r^2)^{\frac{n+1}{2}(1-a)}}\ ,$$

where

$$I_c(z) := \int_{\mathbb{S}_n}\frac{d\xi}{\left|1-zr\bar{\xi}\ \right|^{c+n}}$$

By standard estimates (see [11, p. 15] for example), we have that

$$I_{(n+1)a-n}(rz)\lesssim\begin{cases}1,\ if\ (n+1)a-n<0\\ \log\dfrac{1}{1-|rz|^2},if(n+1)a-n=0\\ \dfrac{1}{(1-|rz|^2)^{(n+1)a-n}},if(n+1)a-n>0,\end{cases}$$

which gives us that

$$\int_{\mathrm{D}(z,\mathrm{R})^c}|\langle k_z,k_w\rangle_{A^2}|\frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a}d\lambda(w)$$

$$\lesssim\begin{cases}\displaystyle\int_{R'}^1\frac{r^{2n-1}}{(1-r^2)^{\frac{n+1}{2}(1-a)}}dr,if(n+1)a-n<0\\ \displaystyle\int_{R'}^1 log\,\frac{1}{1-r^2}\frac{r^{2n-1}}{(1-r^2)^{\frac{1}{2}}}dr,if(n+1)a-n=0\\ \displaystyle\int_{R'}^1\frac{r^{2n-1}}{(1-r^2)^{(n+1)a-n+\frac{n+1}{2}(1-a)}}dr,if(n+1)a-n>0\end{cases}$$

Since $a < 1$, it is easy to see that all the functions appearing on the right hand side are integrable on $(0, 1)$. Therefore, we obtain the desired conclusion by taking the limit as $R \to \infty$ (which is the same as $R' \to 1$).
First, we want to make sure that the class of weakly localized operators is large enough to contain some interesting operators. This is indeed true since every Toeplitz operator with a bounded symbol belongs to this class.
Proposition (4.1.7) [149]: *Each Toeplitz operator* T_u *on* A^p *with a bounded symbol* $u(z)$ *is in* $\mathcal{A}_p(\mathbb{B}_n)$ *for any* $1 < p < \infty$.
Proof. Clearly it is enough to show that

$$\sup_{z\in\mathbb{B}_n} \int_{\mathrm{D}(z,\mathrm{r})^{\mathrm{c}}} |\langle T_u\, k_z, k_w\rangle_{A^2}| \frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a} d\lambda(w) \to 0\,, \sup_{z\in\mathbb{B}_n} \int_{\mathrm{D}(z,\mathrm{r})^{\mathrm{c}}} |\langle T_{\bar{u}}\, k_z, k_w\rangle_{A^2}\,| \frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a} d\lambda(w) \to 0$$

as $r \to \infty$ for all $n - 1 < a < \infty$By definition

$$T_u\, k_z(w) = P(uk_z)(w) = \int_{\mathbb{B}_n} \langle K_x, K_w\rangle_{A^2} u(x) k_z(x)\, dv(x).$$

Therefore,

$$|\langle T_u\, k_z, k_w\rangle_{A^2}| \le \int_{\mathbb{B}_n} |\langle k_w, k_x\rangle_{A^2}|\, |u(x)|\, |\langle k_z, k_x\rangle_{A^2}|\, d\lambda(x)$$

$$\le \|u\|_\infty \int_{\mathbb{B}_n} |\langle k_w, k_x\rangle_{A2}\, \langle k_x, k_z\,\rangle_{A2}|\, d\lambda(x).$$

Now for $z, \mathrm{x} \in \mathbb{B}_\mathrm{n}$, set

$$I_z(x) := |\langle k_x, k_z\rangle_{A2}| \int_{\mathrm{D}(z,\mathrm{r})^{\mathrm{c}}} |\langle k_w, k_x\rangle_{A^2}|\, \frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a}\, d\lambda(w)$$

First note that

$$\int_{\mathrm{D}(z,\mathrm{r})^{\mathrm{c}}} |\langle T_u\, k_z, k_w\rangle_{A^2}| \frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a} d\lambda(w)$$

$$\le \|u\|_\infty \int_{\mathrm{D}(z,\mathrm{r})^{\mathrm{c}}} \int_{\mathbb{B}_n} |\langle k_w, k_x\rangle_{A2}\, \langle k_x, k_z\,\rangle_{A2}|\, d\lambda(x) \frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a} d\lambda(w)$$

$$= \|u\|_\infty \int_{\mathbb{B}_n} \int_{\mathrm{D}(z,\mathrm{r})^{\mathrm{c}}} |\langle k_w, k_x\rangle_{A2}\, \frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a} d\lambda(w) \langle k_x, k_z\,\rangle_{A2}|\, d\lambda(x)$$

$$= \|u\|_\infty \int_{\mathbb{B}_n} I_z(x)\, d\lambda(x) = \|u\|_\infty \int_{\mathbb{B}_n} \int_{\mathrm{D}(z,\frac{\mathrm{r}}{2})} + \int_{\mathrm{D}(z,\frac{\mathrm{r}}{2})^{\mathrm{c}}} I_z(x)\, d\lambda(x)$$

To estimate the first integral notice that for $\mathrm{x} \in \mathrm{D}(z, \frac{\mathrm{r}}{2})$we have $\mathrm{D}(z, \mathrm{r})^{\mathrm{c}} \subset \mathrm{D}\left(\mathrm{x}, \frac{\mathrm{r}}{2}\right)^{\mathrm{c}}$.
Therefore, the first integral is no greater than

$$\int_{\mathrm{D}(z,\frac{\mathrm{r}}{2})} \int_{\mathrm{D}(\mathrm{x},\frac{\mathrm{r}}{2})^{\mathrm{c}}} |\langle \mathrm{k_w}, \mathrm{k_x}\rangle_{\mathrm{A2}}| \frac{\|\mathrm{K}_z\|_{\mathrm{A}^2}^{\mathrm{a}}}{\|\mathrm{K_w}\|_{\mathrm{A}^2}^{\mathrm{a}}} \mathrm{d}\lambda(\mathrm{w})\, |\langle \mathrm{k_x}, \mathrm{k}_z\,\rangle_{\mathrm{A2}}| \mathrm{d}\lambda(\mathrm{x}).$$

It is easy to see that the last expression is no greater than $C(a)A\left(\frac{r}{2}\right)$, where

$$A(r) = \sup_{z\in\mathbb{B}_n}\int_{D(z,r)^c} |\langle k_z, k_w\rangle_{A2}| \frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a} d\lambda(w) .$$

and $C(a)$ is just the bound from the standard Rudin-Forelli estimates (10).

Estimating the second integral is simpler. The second integral is clearly no greater than

$$\int_{D(z,\frac{r}{2})}\int_{\mathbb{B}_n} |\langle k_w, k_x\rangle_{A2}| \frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a} d\lambda(w)\ |\langle k_x, k_z\rangle_{A2}| d\lambda(x).$$

By the standard Rudin-Forelli estimates (10) the inner integral is no greater than

$$C(a)\frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a}\ ,$$

where the constant $C(a)$ is independent of z and x. So, the whole integral is bounded by $(a)A\left(\frac{r}{2}\right)$. Therefore

$$\sup_{z\in\mathbb{B}_n}\int_{D(z,r)^c} |\langle T_u\, k_z, k_w\rangle_{A^2}| \frac{\|K_z\|^a}{\|K_w\|^a} d\lambda(w) \le \|u\|_\infty\left(C(a)A\left(\frac{r}{2}\right) + C(a)A\left(\frac{r}{2}\right)\right).$$

Applying the uniform Rudin-Forelli estimates (11) in Lemma (4.1.6) completes the proof since $2C(a)\|u\|_\infty A\left(\frac{r}{2}\right) \to 0\ as\ r \to \infty.$

We next show that the class of weakly localized operators forms a $*$-algebra.

Proposition (4.1.8) [149]: *If* $1 < p < \infty$ *then* $\mathcal{A}_p(\mathbb{B}_n)$ *is an algebra. Furthermore,* $\mathcal{A}_2(\mathbb{B}_n)$ *is a* $*$*−algebra.*

Proof. It is trivial that $T \in \mathcal{A}_2(\mathbb{B}_n)$ implies $T^* \in \mathcal{A}_2(\mathbb{B}_n)$. It is also easy to see that any linear combination of two operators in $\mathcal{A}_p(\mathbb{B}_n)$ must be also in $\mathcal{A}_p(\mathbb{B}_n)$. It remains to prove that if $T, S \in \mathcal{A}_p(\mathbb{B}_n)$, then $T\,S \in \mathcal{A}_p(\mathbb{B}_n)$. To that end, we have that

$$\int_{D(z,r)^c} |\langle TSk_z, k_w\rangle_{A2}| \frac{\|K_z\|_{A^2}^{1-\frac{2\delta}{p\prime(n+1)}}}{\|K_w\|_{A^2}^{1-\frac{2\delta}{p\prime(n+1)}}} d\lambda(w)$$

$$= \int_{D(z,r)^c} |\langle Sk_z, T^*k_w\rangle_{A2}| \frac{\|K_z\|_{A^2}^{1-\frac{2\delta}{p\prime(n+1)}}}{\|K_w\|_{A^2}^{1-\frac{2\delta}{p\prime(n+1)}}} d\lambda(w)$$

$$= \int_{D(z,r)^c} \left|\int_{\mathbb{B}_n} \langle Sk_z, k_x\rangle_{A2}\langle k_x, T^*k_w\rangle_{A2} d\lambda(x)\right| \frac{\|K_z\|_{A^2}^{1-\frac{2\delta}{p\prime(n+1)}}}{\|K_w\|_{A^2}^{1-\frac{2\delta}{p\prime(n+1)}}} d\lambda(w)$$

$$\le \int_{\mathbb{B}_n}\int_{D(z,r)^c} |\langle k_z, T^*k_w\rangle_{A2}| \frac{d\lambda(w)}{\|K_w\|_{A^2}^{1-\frac{2\delta}{p\prime(n+1)}}} |\langle Sk_z, k_x\rangle_{A2}| \|K_z\|_{A^2}^{1-\frac{2\delta}{p\prime(n+1)}} d\lambda(x)$$

Proceeding exactly as in the proof of the previous Proposition and using the conditions following from $T, S \in \mathcal{A}_p(\mathbb{B}_n)$ in the place of the local Rudin-Forelli estimates (11) (and replacing a with $1-\frac{2\delta}{p\prime(n+1)}$ we obtain that

$$\lim_{r\to\infty}\int_{\mathrm{D}(z,\mathrm{r})^{\mathrm{c}}} |\langle TSk_z, k_w\rangle_{A^2}| \frac{\|K_z\|_{A^2}^{1-\frac{2\delta}{p\prime(n+1)}}}{\|K_w\|_{A^2}^{1-\frac{2\delta}{p\prime(n+1)}}} d\lambda(w)\, d\lambda(w) \;=\; 0.$$

The corresponding condition for $(TS)^*$ is proved in exactly the same way.

We next show that every weakly localized operator can be approximated by infinite sums of well localized pieces. To state this property we need to recall the following proposition proved in [156].

Proposition (4.1.9) [149]: *There exists an integer $N > 0$ such that for any $r > 0$ there is a covering $\mathcal{F}_r = \{\mathcal{F}_j\}$ of $\mathbb{B}_n$ by disjoint Borel sets satisfying*

(a) *every point of $\mathbb{B}_n$ belongs to at most N of the sets $G_j := \{z \in \mathbb{B}_n : d(z, \mathcal{F}_j) \le r\}$,*

(b) $diam_d\, \mathcal{F}_j \le 2r$ *for every* j.

We use this to prove the following proposition, which is similar to what appears in [6], but exploits condition (6).

Proposition (4.1.10) [149]: *Let $1 < p < \infty$ and let T be in the norm closure of $\mathcal{A}_p(\mathbb{B}_n)$. Then for every $\epsilon > 0$ there exists $r > 0$ such that for the covering $\mathcal{F}_r = \{\mathcal{F}_j\}$ (associated to r) from Proposition (4.1.9), we have:*

$$\left\| TP - \sum_j M_{1_{\mathcal{F}_j}} TPM1G_{1_{G_j}} \right\|_{A^p \to L^p(\mathbb{B}_n, dv)} < \epsilon.$$

Proof. By Proposition (4.1.8) in conjunction with Proposition (4.1.9) and a simple approximation argument, we may assume that $T \in \mathcal{A}_p(\mathbb{B}_n)$. Now define

$$S = TP - \textstyle\sum_j M_{1_{\mathcal{F}_j}} TPM1G_{1_{G_j}}.$$

Given ϵ choose *r* large enough so that

$$\sup_{z\in\mathbb{B}_n}\int_{\mathrm{D}(z,\mathrm{r})^{\mathrm{c}}} |\langle Tk_z, k_w\rangle_{A^2}| \frac{\|K_z\|_{A^2}^{1-\frac{2\delta}{p\prime(n+1)}}}{\|K_w\|_{A^2}^{1-\frac{2\delta}{p\prime(n+1)}}} d\lambda(w) < \epsilon$$

and 1− 2δ

$$\sup_{z\in\mathbb{B}_n}\int_{\mathrm{D}(z,\mathrm{r})^{\mathrm{c}}} |\langle T^*k_z, k_w\rangle_{A^2}| \frac{\|K_z\|_{A^2}^{1-\frac{2\delta}{p\prime(n+1)}}}{\|K_w\|_{A^2}^{1-\frac{2\delta}{p\prime(n+1)}}} d\lambda(w) < \epsilon$$

Now for any $z \in \mathbb{B}_n$ let $\in F_{j_0}$, so that

$$|Sf(z)| \leq \int_{\mathbb{B}_n} \sum_j 1_{F_{j(z)}}\, 1_{G_j^c}(w) |\langle T^* K_z , K_w\rangle_{A^2}|\, |f(w)|\, dv(w)$$
$$= \int_{G_{j_0}^c} |\langle T^* K_z , K_w\rangle_{A^2}|\, |f(w)|\, dv(w)$$
$$\leq \int_{\mathrm{D}(z,\mathrm{r})^{\mathrm{c}}} |\langle T^* K_z , K_w\rangle_{A^2}|\, |f(w)|\, dv(w)\,.$$

To finish the proof, we will estimate the operator norm of the integral operator on $L^p(\mathbb{B}_n, dv)$ with kernel $1_{D(z,r)^c}(w)|\langle T^* K_z , K_w\rangle_{A^2}|$ by using the classical Schur test.

To that end, let $h(w) = \|\mathrm{K_w}\|_{A^2}^{\frac{2\delta}{pp'(n+1)}}$ so that

$$\int_{\mathbb{B}_n} 1_{D(z,r)^c}(w)|\langle T^* K_z , K_w\rangle_{A^2}| h(w)^{p'} dv(w)$$
$$= \int_{D(z,r)^c} |\langle T^* K_z , K_w\rangle_{A^2}| \|\mathrm{K_w}\|_{A^2}^{\frac{2\delta}{p(n+1)}} dv(w)$$
$$= \int_{D(z,r)^c} |\langle T^* k_z , k_w\rangle_{A^2}| \|\mathrm{K}_z\| \|\mathrm{K_w}\|_{A^2}^{\frac{2\delta}{p(n+1)}-1} d\lambda(w) \leq \epsilon \|\mathrm{K}_z\|_{A^2}^{\frac{2\delta}{p(n+1)}}$$
$$= \epsilon h(z)^{p'}$$

Similarly, we have that

$$\int_{\mathbb{B}_n} 1_{D(z,r)^c}(w)|\langle T^* K_z , K_w\rangle_{A^2}| h(z)^p dv(z) \leq \epsilon h(w)^p$$

which completes the proof.

It should be noted that a very similar Schur test argument actually proves that condition (5) implies that T is bounded on A^p.

We can now prove one of our main results whose proof uses the ideas in [6, Theorem 4.3] and [155, Lemma 5.3]. First, for any $w \in \mathbb{B}_n$ and $1 < p < \infty$, let $k_w^{(p)}$ be the "p - normalized reproducing kernel" defined by

$$k_w^{(p)} = \frac{K(z,w)}{\|\mathrm{K}_w\|^{\frac{2}{\mathrm{p}'}}}.$$

Clearly we have that $k_w^{(2)} = k_w$ and an easy computation tells us that $\left\|k_w^{(p)}\right\|_{A^p} \approx 1$ (where obviously we have equality when $p = 2$).

Theorem (4.1.11) [149]: *Let* $1 < p < \infty$ *and let T be in the norm closure of* $\mathcal{A}_p(\mathbb{B}_n)$. *Then there exists* $r, C > 0$ *(both depending on T) such that*

$$\|T\|_e \leq C \limsup_{|z|\to 1-} \sup_{w\in D(z,r)} |\langle T k_z^{(p)}, k_w^{(p')}\rangle_{A^2}|$$

Where $\|T\|_e$ *is the essential norm of T as a bounded operator on* A^p.

Proof. Since $P: L^p(\mathbb{B}_n, dv) \to A^p$ is a bounded projection, it is enough to estimate the essential norm of $T = TP$ as an operator on from A^p to $L^p(\mathbb{B}_n, dv)$.

Clearly if $\|TP\|_e = 0$ then there is nothing to prove, so assume that $\|TP\|_e > 0$. By Proposition (4.1.10) there exists $r > 0$ such that for the covering $\mathcal{F}_r = \{F_j\}$ associated to r (from Proposition (4.1.9))

$$\left\| TP - \sum_j M_{1_{F_j}} T\, P\, M_{1_{G_j}} \right\|_{A^p \to L^p(\mathbb{B}_n, dv)} < \frac{1}{2} \|TP\|_e.$$

Since $\sum_{j<m} M_{1_{F_j}} T\, P\, M_{1_{G_j}}$ is compact for every $m \in \mathbb{N}$ we have that $\|TP\|_e$ (as an operator from A^p to $L^p(\mathbb{B}_n, dv)$) can be estimated in the following way:

$$\begin{aligned}\|TP\|_e &\le \left\| TP - \sum_{j<m} M_{1_{F_j}} T\, P\, M_{1_{G_j}} \right\|_{A^p \to L^p(\mathbb{B}_n, dv)} \\ &\le \left\| TP - \sum_{j} M_{1_{F_j}} T\, P\, M_{1_{G_j}} \right\|_{A^p \to L^p(\mathbb{B}_n, dv)} + \|T_m\|_{A^p \to L^p(\mathbb{B}_n, dv)} \\ &\le \frac{1}{2} \|TP\|_e + \|T_m\|_{A^p \to L^p(\mathbb{B}_n, dv)},\end{aligned}$$

Where

$$T_m = \sum_{j \ge m} M_{1_{F_j}} T\, P\, M_{1_{G_j}}.$$

We will complete the proof by showing that there exists $C > 0$ where

$$\limsup_{m\to\infty} \|T_m\|_{A^p \to L^p(\mathbb{B}_n, dv)} \lesssim C \limsup_{|z|\to 1^-} \sup_{w \in D(z,r)} |\langle T k_z^{(p)}, k_w^{(p')} \rangle_{A^2}| + \frac{1}{4} \|TP\|_e.$$

If $f \in A^p$ is arbitrary of norm no greater than 1, then

$$\begin{aligned}\|T_m f\|_{A^p}^p &= \sum_{j \ge m} \left\| M_{1_{F_j}} T\, P\, M_{1_{G_j}} f \right\|_{A^p}^p \\ &= \sum_{j\ge m} \frac{\left\| M_{1_{F_j}} T\, P\, M_{1_{G_j}} f \right\|_{A^p}^p}{\left\| M_{1_{G_j}} f \right\|_{A^p}^p} \left\| M_{1_{F_j}} T\, P\, M_{1_{G_j}} f \right\|_{A^p}^p \le N \sup_{j \ge m} \left\| M_{1_{F_j}} T\, l_j \right\|_{A^p}^p\end{aligned}$$

where

$$l_j := \frac{P\, M_{1_{G_j}} f}{\left\| M_{1_{G_j}} f \right\|_{A^p}}.$$

Therefore, we have that

$$\|T_m\|_{A^p \to L^p(\mathbb{B}_n, dv)} \lesssim \sup_{j \ge m} \sup_{\|f\|_{A^p} \le 1} \left\{ \left\| M_{1_{F_j}} T\, l_j \right\|_{A^p} : l_j = \frac{P\, M_{1_{G_j}} f}{\left\| M_{1_{G_j}} f \right\|_{A^p}} \right\}$$

and hence

$$\limsup_{m\to\infty}\|T_m\|_{A^p\to L^p(\mathbb{B}_n,dv)} \lesssim \limsup_{j\to\infty}\sup_{\|f\|_{A^p}\le 1}\left\{\left\|M_{1_{F_j}}T\,l_j\right\|_{A^p} : l_j=\frac{P\,M_{1_{G_j}}f}{\left\|M_{1_{G_j}}f\right\|_{A^p}}\right\}$$

Now pick a sequence $\{f_j\}$ in A^p with $\|f\|_{A^p}\le 1$ such that

$$\limsup_{j\to\infty}\sup_{\|f\|\le 1}\left\{\left\|M_{1_{F_j}}T\,g\right\|_{A^p} : g=\frac{P\,M_{1_{G_j}}f}{\left\|M_{1_{G_j}}f\right\|_{A^p}}\right\}-\frac{1}{4}\|TP\|_e \le \limsup_{j\to\infty}\left\|M_{1_{F_j}}T\,g_j\right\|_{A^p},$$

where

$$g_j=\frac{P\,M_{1_{G_j}}f_j}{\left\|M_{1_{G_j}}f_j\right\|_{A^p}}=\frac{\int_{G_j}\langle f,k_w^{(p\prime)}\rangle_{A^2}\,k_w^{(p)}\,d\lambda(w)}{\left(\int_{G_j}\left|\langle f,k_u^{(p\prime)}\rangle_{A^2}\right|^p d\lambda(w)\right)^{\frac{1}{p}}}=\int_{G_j}\tilde{a}_j(w)k_w^{(p)}\,d\lambda(w)$$

where

$$\tilde{a}_j(w)=\frac{\langle f,k_w^{(p')}\rangle_{A^2}}{\left(\int_{G_j}\left|\langle f,k_u^{(p\prime)}\rangle_{A^2}\right|^p d\lambda(w)\right)^{\frac{1}{p}}}.$$

Finally, by the reproducing property and H¨older's inequality, we have that

$$\begin{aligned}
\limsup_{j\to\infty}\left\|M_{1_{F_j}}T\,g_i\right\|_{A^p}^p &\le \limsup_{j\to\infty}\int_{F_j}\left(\int_{G_j}|\tilde{a}_j(w)|\left|T\,k_w^{(p)}(z)\right|d\lambda(w)\right)^p dv(z)\\
&= \limsup_{j\to\infty}\int_{F_j}\left(\int_{G_j}|\tilde{a}_j(w)|\left|\langle T\,k_w^{(p)},k_z^{(p')}\rangle_{A^2}\right|d\lambda(w)\right)^p d\lambda(z)\\
&\le \limsup_{|z|\to 1^-}\sup_{w\in D(z,3r)}\left|\langle Tk_z^{(p)},k_w^{(p')}\rangle_{A^2}\right|^p\left(\sup_j\lambda(G_j)^p\int_{G_j}|\tilde{a}_j(w)|^p\,d\lambda(w)\right)\\
&\le C(r)\limsup_{|z|\to 1^-}\sup_{w\in D(z,3r)}\left|\langle Tk_z^{(p)},k_w^{(p')}\rangle_{A^2}\right|^p
\end{aligned}$$

since by Proposition (4.1.9) we have that $z\in F_j$ and $w\in G_j$ implies that $d(z,w)\le 3r$ and $\lambda(G_j)\le C(r)$ where $C(r)$ is independent of j.

We will finish off with a proof of Theorem (4.1.5). First, for $z\in\mathbb{B}_n$, define

$$U_z^{(p)}f(w) := f(\varphi_z(w))(k_z(w))^{\frac{2}{p}}$$

which via a simple change of variables argument is clearly an isometry on A^p. As was shown in [9], an easy computation tells us that there exists a unimodular function $\Phi(\cdot,\cdot)$ on $\mathbb{B}_n\times\mathbb{B}_n$ where

$$\left(U_z^{(p)}\right)^* k_w^{(p\prime)}=\Phi(z,w)k_{\phi_z(w)}^{(p')}. \tag{12}$$

With the help of the operators $U_z^{(p)}$, we will prove the following general result which in conjunction with Theorem (4.1.11) proves Theorem (4.1.5). Note that proof is similar to the proof of [155, Proposition 1.4].

Proposition (4.1.12) [149]: *If T is any bounded operator on A^p for $1 < p < \infty$ then the following are equivalent*

(a) $\lim_{|z|\to 1^-} \sup_{w\in D(z,r)} \left|\langle T\, k_z^{(p)}, k_w^{(p')}\rangle_{A^2}\right| = 0 \quad for\ all\ r > 0,$

(b) $\lim_{|z|\to 1^-} \sup_{w\in D(z,r)} \left|\langle T\, k_z^{(p)}, k_w^{(p')}\rangle_{A^2}\right| = 0 \quad for\ some\ r > 0,$

(c) $\lim_{|z|\to 1^-} \sup_{w\in D(z,r)} |\langle T\, k_z, k_z\rangle_{A^2}| = 0.$

Proof. Trivially we have that $(a) \Rightarrow (b)$, and the fact that $(b) \Rightarrow (c)$ follows by definition and setting $z = w$. We will complete the proof by showing that $(c) \Rightarrow (a)$.
Assume to the contrary that $|\langle \mathrm{T}\, \mathrm{k}_z, \mathrm{k}_z\rangle_{\mathrm{A}^2}|$ vanishes as $|z| \to 1^-$ but that

$$\lim_{|z|\to 1^-} \sup_{\mathrm{w}\in \mathrm{D}(z,\mathrm{r})} \left|\langle \mathrm{T}\, \mathrm{k}_z^{(\mathrm{p})}, \mathrm{k}_\mathrm{w}^{(\mathrm{p}')}\rangle_{\mathrm{A}^2}\right| \neq 0$$

for some fixed $r > 0$. Thus, there exists sequences $\{z_m\}, \{w_m\}$ and some $0 < r_0 < 1$ where $\lim_{m\to\infty} |z_m| = 1$ and $|w_m| \leq r_0$ for any $m \in \mathbb{N}$, and where

$$\limsup_{\mathrm{m}\to\infty} \left|\langle \mathrm{T}\, \mathrm{k}_{z_m}^{(\mathrm{p})}, \mathrm{k}_{\varphi_{z_m(w_m)}}^{(\mathrm{p}')}\rangle_{\mathrm{A}^2}\right| > \epsilon \tag{13}$$

for some $\epsilon > 0$. Furthermore, passing to a subsequence if necessary, we may assume that $\lim_{m\to\infty} w_m = w \in \mathbb{B}_n$. Note that since $|w_m| \leq r_0 < 1$ for all m, we trivially have $\lim_{\mathrm{m}\to\infty} k_{w_m}^{(p')} = k_\mathrm{w}^{(p')}$ where the convergence is in the $A^{p'}$ norm.
Let $\mathcal{B}(A^p)$ be the space of bounded operators on A^p. Since the unit ball in $\mathcal{B}(A^p)$ is WOT compact, we can (passing to another subsequence if necessary) assume that

$$\hat{T} = WOT - \lim_{m\to\infty} U_{z_m}^{(p)} T \left(U_{z_m}^{(p')}\right)^* .$$

Thus, we have that

$$\begin{aligned}
\limsup_{\mathrm{m}\to\infty} \left|\langle \mathrm{T}\, \mathrm{k}_{z_m}^{(\mathrm{p})}, \mathrm{k}_{\varphi_{z_m(w_m)}}^{(\mathrm{p}')}\rangle_{\mathrm{A}^2}\right| &= \limsup_{\mathrm{m}\to\infty} \left|\langle U_{z_m}^{(p)} T \left(U_{z_m}^{(p')}\right)^* \mathrm{k}_0^{(\mathrm{p})}, \mathrm{k}_{w_m}^{(\mathrm{p}')}\rangle_{\mathrm{A}^2}\right| \\
&= \limsup_{\mathrm{m}\to\infty} \left|\langle U_{z_m}^{(p)} T \left(U_{z_m}^{(p')}\right)^* \mathrm{k}_0^{(\mathrm{p})}, \mathrm{k}_{w}^{(\mathrm{p}')}\rangle_{\mathrm{A}^2}\right| \\
&= \left|\langle \hat{T}\mathrm{k}_0, \mathrm{k}_\mathrm{w}\rangle_{\mathrm{A}^2}\right|.
\end{aligned}$$

However, for any $z \in \mathbb{B}_n$

$$\left|\langle \hat{T} k_0^{(p)}, k_z^{(p')}\rangle\right| = \lim_{m\to\infty} \left|\langle U_{z_m}^{(p)} T \left(U_{z_m}^{(p')}\right)^* \mathrm{k}_z^{(\mathrm{p})}, \mathrm{k}_z^{(\mathrm{p}')}\rangle\right| \approx \lim_{m\to\infty} \left|\langle T\, k_{\varphi_{z_m(z)}}^{(p)}, k_{\varphi_{z_m(z)}}^{(p')}\rangle\right| = 0$$

since by assumption $|\langle \mathrm{T}\, \mathrm{k}_z , \mathrm{k}_z\rangle|$ vanishes as $|z| \to 1^-$. Thus, since the Berezin transform is injective on A^p, we get that $T = 0$, which contradicts (13) and completes the proof.

we will prove Theorems (4.1.2) and (4.1.3). Some parts of the proofs are essentially identical to proof of Theorem (4.1.5) and so we will we only outline the necessary modifications.

$$D(z,r) := \{w \in \mathbb{C}^n : |w - z| < r\}$$

denote the standard Euclidean disc centered at the point z of radius $r > 0$. For $z \in \mathbb{C}^n$, we define

$$U_z f\ (w) := \ f\ (z - \ w) k_z(w),$$

which via a simple change of variables argument is clearly an isometry on $\mathcal{F}^p$ (though note in general that it is not clear whether U_z even maps $\mathcal{F}_\phi^p$ into itself). Recall also that the orthogonal projection of $L^2(\mathbb{C}^n, e^{-2\phi}dv)$ onto $\mathcal{F}_\phi^2$ is given by the integral operator

$$P\ (f\ )(z) := \int_{\mathbb{C}^n} \langle K_w, K_z\rangle_{\mathcal{F}_\phi^2} f\ (w)\ e^{-2\phi(w)} dv\ .$$

Therefore, for all $f \in \mathcal{F}_\phi^p$ we have

$$f(z) = \int_{\mathbb{C}^n} \langle f, \widetilde{k_w}\rangle_{\mathcal{F}_\phi^2} \widetilde{k_w}(z) dv(w) \tag{14}$$

where $\widetilde{k_w}(z) := K_w(z) e^{-\phi(w)}$. Note that $|K(z,z)| \approx \ e^{2\phi(z)}$ (see [158]) so that

$$\left|\widetilde{k_w}(z)\right| \approx |k_w(z)|. \tag{15}$$

The following analog of Lemma (4.1.6) is simpler to prove in this case.

Lemma (4.1.13) [149]:

$$\lim_{R\to\infty} \sup_{z\in\mathbb{C}^n} \int_{D(z,R)^c} \left|\langle k_z\ , k_w\rangle_{\mathcal{F}_\phi^2}\right|\ dv(w) = 0. \tag{16}$$

To prove this, simply note that there exists $\epsilon > 0$ such that $\left|\langle k_z\ , k_w\rangle_{\mathcal{F}_\phi^2}\right| \leq \ e^{-\epsilon|z-w|}$ for all $z, w \in \mathbb{C}^n$. The proof of this is then immediate since

$$\int_{D(z,R)^c} \left|\langle k_z\ , k_w\rangle_{\mathcal{F}_\phi^2}\right|\ dv(w) \leq \int_{D(0,R)^c} e^{-\epsilon|w|dv(w)}$$

which clearly goes to zero as $R \to \infty$.

As in the Bergman case, $\mathcal{A}_\phi(\mathbb{C}^n)$ contains all Toeplitz operators with bounded symbols. Also, as was stated in the introduction, any $T \in \mathcal{A}_\phi(\mathbb{C}^n)$ is automatically bounded on $\mathcal{F}_\phi^p$ for all $1 \leq p \leq \infty$. To prove this, note that it is enough to prove that T is bounded on $\mathcal{F}_\phi^1$ and $\mathcal{F}_\phi^\infty$ by complex interpolation (see [155]). To that end, we only prove that T is bounded on $\mathcal{F}_\phi^1$ since the proof that T is bounded on $\mathcal{F}_\phi^\infty$ is similar. If $T \in \mathcal{A}_\phi(\mathbb{C}^n)$ and $f \in \mathcal{F}_\phi^1$, then the reproducing property gives us that

$$|\mathrm{T}f(z)|\ \mathrm{e}^{-\phi(z)} \approx \ \left|\langle \mathrm{f}, \mathrm{T}^*\mathrm{k}_z\rangle_{\mathcal{F}_\phi^2}\right| \lesssim \int_{\mathbb{C}^n} |\mathrm{f}(\mathrm{u})| \left|\langle \mathrm{T}^*\mathrm{k}_z, \mathrm{k}_\mathrm{u}\rangle_{\mathcal{F}_\phi^2}\right| \mathrm{e}^{-\phi(\mathrm{u})}\ \mathrm{dv}(\mathrm{u}).$$

Thus, by Fubini's theorem, we have that

$$\|Tf\|_{\mathcal{F}_\phi^1} \leq \int_{\mathbb{C}^n} |\mathrm{f}(\mathrm{u})| \left(\int_{\mathbb{C}^n} \left|\langle T^*k_z\ , k_u\rangle_{\mathcal{F}_\phi^2}\right| dv(z)\right)\ \mathrm{e}^{-\phi(\mathrm{u})}\ dv(u)\ \lesssim\ \|f\|_{\mathcal{F}_\phi^1}\ .$$

In addition, $\mathcal{A}_\phi(\mathbb{C}^n)$ satisfies the following two properties:

Proposition (4.1.14) [149]: *Each Toeplitz operator T_u on $\mathcal{F}_\phi^p$ with a bounded symbol $u(z)$ is weakly localized.*

Proof. Since $\left|\langle k_z\ , k_w\rangle_{\mathcal{F}_\phi^2}\right| \leq \ e^{-\epsilon|z-w|}$ for some $\epsilon > 0$ we have that

$$\left|\langle T_u k_z\,, k_w\rangle_{\mathcal{F}^2_\phi}\right| \lesssim \|u\|_{L^\infty}\int_{\mathbb{C}^n}\left|\langle k_z, k_x\rangle_{\mathcal{F}^2_\phi}\right|\left|\langle k_x, k_w\rangle_{\mathcal{F}^2_\phi}\right| dx$$
$$\lesssim \|u\|_{L^\infty}\int_{\mathbb{C}^n} e^{-\epsilon|z-x|}e^{-\epsilon|x-w|}\; dx.$$
Now if $|z - w| \geq r$ then by the triangle inequality we have that either $|z - x| \geq r/2$ or $|x - w| \geq r/2$ so that
$$\int_{D(z,r)^c}\left|\langle T_u k_z\,, k_w\rangle_{\mathcal{F}^2_\phi}\right| dw \lesssim e^{-\frac{\epsilon r}{2}}\|u\|_{L^\infty}\int_{D(z,r)^c}\int_{\mathbb{C}^n} e^{-\frac{\epsilon}{2}|z-x|e-\frac{\epsilon}{2}|x-w|}\; dx\, dw$$
$$\lesssim e^{-\frac{\epsilon r}{2}}\|u\|_{L^\infty}$$
Note that T_u is suffciently localized even in the sense of Xia and Zheng by [10, Proposition 4.1]. Also note that a slight variation of the above argument shows that the Toeplitz operator $T_\mu \in \mathcal{A}_\phi(\mathbb{C}^n)$ if μ is a positive Fock-Carleson measure on $\mathbb{C}^n$ (see [158]).

Proposition (4.1.15) [149]: $\mathcal{A}_\phi(\mathbb{C}^n)$ *forms a* $*$*-algebra.*

We will omit the proof of this proposition since it is proved in exactly the same way as it is in the Bergman space case (where the only difference is that one uses (14) in conjunction with (15) instead of (8)).

We next prove that operators in the norm closure of $\mathcal{A}_\phi(\mathbb{C}^n)$ can also be approximated by infinite sums of well localized pieces. To state this property we need to recall the following proposition proved in [156]

Proposition (4.1.16) [149]: *There exists an integer* $N > 0$ *such that for any* $r > 0$ *there is a covering* $\mathcal{F}_r = \{F_j\}$ *of* $\mathbb{C}^n$ *by disjoint Borel sets satisfying*

(a) *every point of* $\mathbb{C}^n$ *belongs to at most N of the sets* $G_j := \{z \in \mathbb{C}^n : d(z, F_j) \leq r\}$,

(b) $diam_d\, F_j \leq 2r$ *for every j.*

We use this to prove the following proposition, which is similar to what appears in [156], but exploits condition (4) (and is proved in a manner that is similar to the proof of [155, Lemma 5.2]). Note that for the rest, L^p_ϕ will refer to the space of measurable functions f on $\mathbb{C}^n$ such that $f\, e^{-\phi} \in L^p(\mathbb{C}^n, dv)$.

Proposition (4.1.17) [149]: *Let* $1 < p < \infty$ *and let T be in the norm closure of* $\mathcal{A}_\phi(\mathbb{C}^n)$. *Then for every* $\epsilon > 0$ *there exists* $r > 0$ *such that for the covering* $\mathcal{F}_r = \{F_j\}$ *(associated to* r*) from Proposition (4.1.16)*
$$\left\|TP - \sum_j M_{1_{F_j}} TPM_{1_{G_j}}\right\|_{\mathcal{F}^p_\phi \to L^p_\phi} < \epsilon.$$
Proof. Again by an easy approximation argument we can assume that $T \in \mathcal{A}_\phi(\mathbb{C}^n)$. Furthermore, we first prove the theorem for $p = 2$.

Define
$$S = TP - \sum_j M_{1_{F_j}} TPM_{1_{G_j}}.$$
Given ϵ choose r large enough so that

$$\sup_{z\in\mathbb{C}^n}\int_{D(z,r)^c}\left|\langle T^*k_z,k_w\rangle_{\mathcal{F}_\phi^2}\right|dv(w)<\epsilon \quad and \quad \sup_{z\in\mathbb{C}^n}\int_{D(z,r)^c}\left|\langle Tk_z,k_w\rangle_{\mathcal{F}_\phi^2}\right|dv(w)<\epsilon.$$

Now for any $z\in\mathcal{A}_\phi(\mathbb{C}^n)$, pick j_0 such that $z\in F_{j_0}$. Then we have that

$$\begin{aligned}|Sf(z)| &\le \int_{\mathbb{C}^n}\sum_j 1_{F_j}(z)1_{G_j^c}(w)\ \left|\langle T^*K_z,K_w\rangle_{\mathcal{F}_\phi^2}\right||f(w)|\,e^{-2\phi(w)}\,dv(w)\\ &= \int_{G_{j_0}^c}\left|\langle T^*K_z,K_w\rangle_{\mathcal{F}_\phi^2}\right||f(w)|\,e^{-2\phi(w)}\,dv(w)\\ &\le \int_{D(z,r)^c}\left|\langle T^*K_z,K_w\rangle_{\mathcal{F}_\phi^2}\right||f(w)|\,e^{-2\phi(w)}\,dv(w).\end{aligned}$$

To finish the proof when $p=2$, we will estimate the operator norm of the integral operator on L_ϕ^2 with kernel $1_{D(z,r)^c}(w)\left|\langle T^*K_z,K_w\rangle_{\mathcal{F}_\phi^2}\right|$ using the classical Schur test. To that end, let $h(z)=e^{\frac{1}{2}\phi(z)}$ so that

$$\begin{aligned}&\int_{\mathbb{C}^n}1_{D(z,r)^c}(w)\ \left|\langle T^*K_z,K_w\rangle_{\mathcal{F}_\phi^2}\right|h(w)^2e^{-2\phi(w)}\,dv(w)\\ &\approx h(z)^2\int_{D(z,r)^c}\left|\langle T^*k_z,k_w\rangle_{\mathcal{F}_\phi^2}\right|\,dv(w)\ \lesssim\ \epsilon h(z)^2.\end{aligned}$$

Similarly, we have that

$$\int_{\mathbb{C}^n}1_{D(z,r)^c}(w)\ \left|\langle T^*K_z,K_w\rangle_{\mathcal{F}_\phi^2}\right|h(w)^2e^{-2\phi(w)}\,dv(z)\ \lesssim\ \epsilon h(w)^2$$

which finishes the proof when $p=2$.

Now assume that $1<p<2$. Since T is bounded on $\mathcal{F}_\phi^1$, we easily get that

$$\left\|\sum_j M_{1_{F_j}}TPM_{1_{G_j}}\right\|_{\mathcal{F}_\phi^1\to L_\phi^1}<\infty$$

which by complex interpolation proves the proposition when $1<p<2$. Finally when $2<p<\infty$, one can similarly get a trivial $L_\phi^1\to\mathcal{F}_\phi^1$ operator norm bound on

$$\left(\sum_j M_{1_{F_j}}TPM_{1_{G_j}}\right)^*=\sum_j P\,M_{1_{G_j}}T^*PM_{1_{F_j}}$$

since T^* is bounded on $\mathcal{F}_\phi^1$. Since $(\mathcal{F}_\phi^p)^*=\mathcal{F}_\phi^q$ when $1<p<\infty$ where q is the conjugate exponent of p (see [158]), duality and complex interpolation now proves the proposition when $2<p<\infty$.

Because of (15), the proof of the next result is basically the same as the proof of Theorem (4.1.11) .

Theorem (4.1.18) [149]: *Let* $1 < p < \infty$ *and let T be in the norm closure of* $\mathcal{A}_\phi(\mathbb{C}^n)$*. Then there exists r,* $C > 0$ *(both depending on T) such that*

$$\|T\|_e \leq C \limsup_{|z|\to\infty} \sup_{w\in D(z,r)} \left|\langle T\, k_z, k_w\rangle_{\mathcal{F}^2_\phi}\right|$$

Where $\|T\|_e$ *is the essential norm of T as a bounded operator on* $\mathcal{F}^p_\phi$.

As was stated in the beginning, the operator U_z for $z \in \mathbb{C}^n$ is an isometry on $\mathcal{F}^p$. Furthermore, since a direct calculation shows that

$$|U_z\, k_w(u)| \approx |k_{z-w}(u)|,$$

the proof of Theorem (4.1.3) now follows immediately by combining Theorem (4.1.18) with [155, Proposition 1.4].

The reader should clearly notice that the proof of Theorem (4.1.11) did not in any way use the existence of a family of "translation" operators $\left\{U_z^{(p)}\right\}_{z\in\mathbb{C}^n}$ on A^p that satisfies

$$\left|\left(U_z^{(p)}\right)^* k_w^{(p')}\right| \approx \left|k_{\phi_z}^{(p')}\right| \tag{17}$$

(and moreover, one can make a similar remark regarding Theorem (4.1.18)). In particular, a trivial application of Hölder's inequality in conjunction with the above remark implies that one can prove the so called "reproducing kernel thesis" for operators in the norm closure of $\mathcal{A}_\phi(\mathbb{B}^n)$ (respectively, $\mathcal{A}_\phi(\mathbb{C}^n)$ *without* the use of any "translation" operators. It would therefore be interesting to know if our results can be proved for the weighted Bergman spaces on the ball that were considered in [153] for example. Moreover, it would be interesting to know whether one can use the ideas to modify the results in [156] to include spaces where condition A.5 on the space of holomorphic functions at hand is not necessarily true (note that it is precisely this condition that allows one to easily cook up "translation operators").

It would also be very interesting to know whether "translation" operators are in fact crucial for proving Proposition (4.1.12) and its generalized Bargmann-Fock space analog (again see [155, Proposition 1.4]). More generally, it would be fascinating to know precisely how these translation operators fit into the "Berezin transform implies compactness".

As was noted earlier, the techniques in [153] are essentially frame theoretic, and therefore are rather different than the techniques usedin. In particular, a crucial aspect of [154] involves a localization result somewhat similar in spirit to Proposition (4.1.17) and which essentially involves treating a "sufficiently localized" operator T as a sort of matrix with respect to the frame $\{k_\sigma\}_{\sigma\in\mathbb{Z}^{2n}}$ for $\mathcal{F}^2$. Also, note that the techniques in [155] were extended in [156] to the generalized Bargmann-Fock space setting to obtain results for $\mathcal{F}^2_\phi$ that are similar to (but slightly weaker than) the results obtained. Because of these considerable differences in localization schemes, it would be interesting to know if one can combine the localization ideas from with that of [157] to obtain new or sharper results on $\mathcal{F}^2_\phi$ (or even just new or sharper results on $\mathcal{F}^2$).

Corollary (4.1.18) [264]: *Let* $\frac{n-1}{n+1} < a < 1$*. Then*

$$\lim_{R\to\infty} \sup_{z^2-1\in\mathbb{B}_n} \int_{D(z^2-1,R)^c} \left|\langle k_{z^2-1}, k_{w^2-1}\rangle_{A^2}\right| \frac{\left\|K_{z^2-1}\right\|^a_{A^2}}{\left\|K_{w^2-1}\right\|^a_{A^2}}\, d\lambda(w^2-1) = 0.$$

Proof. Notice first that $\int_{D(z^2-1,R)^c} |\langle k_{z^2-1}, k_{w^2-1}\rangle_{A^2}| \frac{\|K_{z^2-1}\|_{A^2}^a}{\|K_{w^2-1}\|_{A^2}^a} d\lambda(w^2-1) =$
$\int_{D(0,R)^c} \left|\langle k_{z^2-1}, k_{\varphi_{z^2-1}(w^2-1)}\rangle_{A^2}\right| \frac{\|K_{z^2-1}\|_{A^2}^a}{\left\|K_{\varphi_{z^2-1}(w^2-1)}\right\|_{A^2}^a} d\lambda(w^2-1) =$
$\int_{D(0,R)^c} \frac{|\langle k_{z^2-1}, k_{w^2-1}\rangle_{A^2}|}{\|K_{w^2-1}\|_{A^2}^{1+a}} d\lambda(w^2-1) = \int_{D(0,R)^c} \frac{dv(w^2-1)}{|1-\overline{(w^2-1)}(z^2-1)|^{(n+1)a}(1-|w^2-1|^2)^{\frac{n+1}{2}(1-a)}} =$
$\int_{R'}^1 \int_{\mathbb{S}_n} \frac{(1+3\epsilon)^{2n-1} d\xi d(1+3\epsilon)}{|1-(z^2-1)(1+3\epsilon)\bar{\xi}|^{(n+1)^a}(-3\epsilon(2+3\epsilon))^{\frac{n+1}{2}(1-a)}}$
where in the last integral $R = \log\frac{1+R'}{1-R'}$. Notice that $R' \to 1$ when $R \to \infty$ and note that the last integral can be written as

$$\int_{R'}^1 I_{(n+1)a-n}((1+3\epsilon)(z^2-1)) \frac{(1+3\epsilon)^{2n-1} d(1+3\epsilon)}{(-3\epsilon(2+3\epsilon))^{\frac{n+1}{2}(1-a)}},$$

where

$$I_c(z^2-1) := \int_{\mathbb{S}_n} \frac{d\xi}{|1-(z^2-1)(1+3\epsilon)\bar{\xi}|^{c+n}}$$

By standard estimates for example), we have that

$$I_{(n+1)a-n}((1+3\epsilon)(z^2-1)) \lesssim \begin{cases} 1, & \text{if } (n+1)a-n<0 \\ \log\frac{1}{1-|(1+3\epsilon)(z^2-1)|^2}, & \text{if} (n+1)a-n=0 \\ \frac{1}{(1-|(1+3\epsilon)(z^2-1)|^2)^{(n+1)a-n}}, & \text{if} (n+1)a-n>0, \end{cases}$$

which gives us that

$$\int_{D(z^2-1,R)^c} |\langle k_{z^2-1}, k_{w^2-1}\rangle_{A^2}| \frac{\|K_{z^2-1}\|_{A^2}^a}{\|K_{w^2-1}\|_{A^2}^a} d\lambda(w^2-1)$$
$$\lesssim \begin{cases} \int_{R'}^1 \frac{(1+3\epsilon)^{2n-1}}{(-3\epsilon(2+3\epsilon))^{\frac{n+1}{2}(1-a)}} d(1+3\epsilon), & \text{if} (n+1)a-n<0 \\ \int_{R'}^1 \log\frac{1}{-3\epsilon(2+3\epsilon)} \frac{(1+3\epsilon)^{2n-1}}{(1-(1+3\epsilon)^2)^{\frac{1}{2}}} d(1+3\epsilon), & \text{if} (n+1)a-n=0 \\ \int_{R'}^1 \frac{(1+3\epsilon)^{2n-1}}{(-3\epsilon(2+3\epsilon))^{(n+1)a-n+\frac{n+1}{2}(1-a)}} d(1+3\epsilon), & \text{if} (n+1)a-n>0 \end{cases}$$

Since $a<1$, it is easy to see that all the functions appearing on the right hand side are integrable on $(0,1)$. Therefore, we obtain the desired conclusion by taking the limit as $R \to \infty$ (which is the same as $R' \to 1$).

Corollary (4.1.19)[264]: *Each sequence of Toeplitz operators* $(T_j)_u$ *on* $A^{1+\epsilon}$ *with a bounded symbol* $u(z^2-1)$ *is in* $\mathcal{A}_{1+\epsilon}(\mathbb{B}_n)$ *for any* $0<\epsilon<\infty$.

Proof. Clearly it is enough to show that

$$\sup_{z^2-1\in\mathbb{B}_n}\int_{\mathrm{D}(z^2-1,1+3\epsilon)^{\mathrm{c}}}\sum_j |\langle (T_j)_u k_{z^2-1}, k_{w^2-1}\rangle_{A^2}| \frac{\|K_{z^2-1}\|_{A^2}^a}{\|K_{w^2-1}\|_{A^2}^a} d\lambda(w^2-1) \to 0,$$

$$\sup_{z^2-1\in\mathbb{B}_n}\int_{\mathrm{D}(z^2-1,1+3\epsilon)^{\mathrm{c}}}\sum_j |\langle (T_j)_{\bar{u}} k_{z^2-1}, k_{w^2-1}\rangle_{A^2}| \frac{\|K_{z^2-1}\|_{A^2}^a}{\|K_{w^2-1}\|_{A^2}^a} d\lambda(w^2-1) \to 0$$

as $\epsilon \to \infty$ for all $n-1 < a < \infty$.

By definition

$$(T_j)_u k_{z^2-1}(w^2-1) = P(uk_{z^2-1z^2-1})(w^2-1)$$
$$= \int_{\mathbb{B}_n}\langle K_{x^2-1}, K_{w^2-1}\rangle_{A^2} u(x^2-1)k_{z^2-1}(x^2-1)\, dv(x^2-1).$$

Therefore,

$$|\sum_j \langle (T_j)_u k_{z^2-1}, k_{w^2-1}\rangle_{A^2}|$$
$$\leq \int_{\mathbb{B}_n}|\langle k_{w^2-1}, k_{x^2-1}\rangle_{A^2}|\, |u(x^2-1)|\, |\langle k_{z^2-1}, k_{x^2-1}\rangle_{A^2}|\, d\lambda(x^2-1)$$
$$\leq \|u\|_\infty \int_{\mathbb{B}_n}|\langle k_{w^2-1}, k_{x^2-1}\rangle_{A^2} \langle k_{x^2-1}, k_{z^2-1}\rangle_{A^2}|\, d\lambda(x^2-1).$$

Now for $z^2-1, x^2-1 \in \mathbb{B}_\mathrm{n}$, set

$$I_{z^2-1}(x^2-1) := |\langle k_{x^2-1}, k_{z^2-1}\rangle_{A^2}| \int_{\mathrm{D}(z^2-1,1+3\epsilon)^{\mathrm{c}}} |\langle k_{w^2-1}, k_{x^2-1}\rangle_{A^2}| \frac{\|K_{z^2-1}\|_{A^2}^a}{\|K_{w^2-1}\|_{A^2}^a} d\lambda(w^2-1)$$

First note that $\int_{D(z^2-1,1+3\epsilon)^c}\sum_j |\langle (T_j)_u k_{z^2-1}, k_{w^2-1}\rangle_{A^2}| \frac{\|K_{z^2-1}\|_{A^2}^a}{\|K_{w^2-1}\|_{A^2}^a} d\lambda(w^2-1) \leq \|u\|_\infty \int_{D(z^2-1,1+3\epsilon)^c}\int_{\mathbb{B}_n} |\langle k_{w^2-1}, k_{x^2-1}\rangle_{A^2} \langle k_{x^2-1}, k_{z^2-1}\rangle_{A^2}|\, d\lambda(x^2-1) \frac{\|K_{z^2-1}\|_{A^2}^a}{\|K_{w^2-1}\|_{A^2}^a} d\lambda(w^2-1) = \|u\|_\infty \int_{\mathbb{B}_n}\int_{D(z^2-1,1+3\epsilon)^c} |\langle k_{w^2-1}, k_{x^2-1}\rangle_{A^2} \frac{\|K_{z^2-1}\|_{A^2}^a}{\|K_{w^2-1}\|_{A^2}^a} d\lambda(w^2-1)\langle k_{x^2-1}, k_{z^2-1}\rangle_{A^2}|\, d\lambda(x^2-1) = \|u\|_\infty \int_{\mathbb{B}_n} I_{z^2-1}(x^2-1)\, d\lambda(x^2-1) = \|u\|_\infty \int_{\mathbb{B}_n}\int_{D(z^2-1,\frac{1+3\epsilon}{2})} + \int_{D(z^2-1,\frac{1+3\epsilon}{2})^c} I_{z^2-1}(x^2-1)\, d\lambda(x^2-1)$

To estimate the first integral notice that for $x^2-1 \in \mathrm{D}(z^2-1,\frac{1+3\epsilon}{2})$ we have $\mathrm{D}(z^2-1, 1+3\epsilon)^{\mathrm{c}} \subset D\left(x^2-1,\frac{1+3\epsilon}{2}\right)^c$. Therefore, the first integral is no greater than

$$\int_{D(z^2-1,\frac{1+3\epsilon}{2})}\int_{D(x^2-1,\frac{1+3\epsilon}{2})^c} |\langle k_{w^2-1}, k_{x^2-1}\rangle_{A^2}| \frac{\|K_{z^2-1}\|_{A^2}^a}{\|K_{w^2-1}\|_{A^2}^a} d\lambda(w^2-1)\, |\langle k_{x^2-1}, k_{z^2-1}\rangle_{A^2}| d\lambda(x^2-1).$$

It is easy to see that the last expression is no greater than $\mathrm{C(a)A}\left(\frac{1+3\epsilon}{2}\right)$, where

$$A(1+3\epsilon) = \sup_{z^2-1\in\mathbb{B}_n}\int_{D(z^2-1,1+3\epsilon)^c} |\langle k_{z^2-1}, k_{w^2-1}\rangle_{A^2}| \frac{\|K_{z^2-1}\|_{A^2}^a}{\|K_{w^2-1}\|_{A^2}^a} d\lambda(w^2-1).$$

and C(a) is just the bound from the standard Rudin-Forelli estimates

Estimating the second integral is simpler. The second integral is clearly no greater than

$$\int_{D(z^2-1,\frac{r}{2})}\int_{\mathbb{B}_n} |\langle k_{w^2-1}, k_{x^2-1}\rangle_{A^2}| \frac{\|K_{z^2-1}\|_{A^2}^a}{\|K_{w^2-1}\|_{A^2}^a} d\lambda(w^2-1) \left|\langle k_{x^2-1}, k_{z^2-1}\rangle_{A^2}\right| d\lambda(x^2-1).$$

By the standard Rudin-Forelli estimates the inner integral is no greater than

$$C(a)\frac{\|K_{z^2-1}\|_{A^2}^a}{\|K_{w^2-1}\|_{A^2}^a} \ ,$$

where the constant $C(a)$ is independent of z^2-1 and x^2-1. So, the whole integral is bounded by $(a)A\left(\frac{1+3\epsilon}{2}\right)$. Therefore

$$\sup_{z^2-1\in\mathbb{B}_n}\int_{D(z^2-1,1+3\epsilon)^c} |\langle (T_j)_u\, k_{z^2-1}, k_{w^2-1}\rangle_{A^2}| \frac{\|K_{z^2-1}\|^a}{\|K_{w^2-1}\|^a} d\lambda(w^2-1)$$
$$\leq \|u\|_\infty\left(C(a)A\left(\frac{1+3\epsilon}{2}\right)+\ C(a)A\left(\frac{1+3\epsilon}{2}\right)\right).$$

Applying the uniform Rudin-Forelli estimates completes the proof since

$$2C(a)\|u\|_\infty A\left(\frac{1+3\epsilon}{2}\right) \to\ 0 \text{ as } \epsilon\ \to\ \infty.$$

Corollary (4.1.20)[264]:*If* $0<\epsilon<\infty$ *then* $\mathcal{A}_{1+\epsilon}(\mathbb{B}_n)$ *is an algebra. Furthermore,* $\mathcal{A}_2(\mathbb{B}_n)$ *is a* $*-$*algebra.*

Proof. It is trivial that $T_j\in\mathcal{A}_2(\mathbb{B}_n)$ implies $T_j^*\in\mathcal{A}_2(\mathbb{B}_n)$. It is also easy to see that any linear combination of two operators in $\mathcal{A}_{1+\epsilon}(\mathbb{B}_n)$ must be also in $\mathcal{A}_{1+\epsilon}(\mathbb{B}_n)$. It remains to prove that if $T_j, S\in\mathcal{A}_{1+\epsilon}(\mathbb{B}_n)$, then $T_j\, S\in\mathcal{A}_{1+\epsilon}(\mathbb{B}_n)$. To that end, we have that

$$\int_{D(z^2-1,1+3\epsilon)^c}\sum_j\ |\langle T_jSk_{z^2-1}, k_{w^2-1}\rangle_{A^2}| \frac{\|K_{z^2-1}\|_{A^2}^{1-\frac{2\delta}{\left(\frac{1+\epsilon}{\epsilon}\right)(n+1)}}}{\|K_{w^2-1}\|_{A^2}^{1-\frac{2\delta}{\left(\frac{1+\epsilon}{\epsilon}\right)(n+1)}}} d\lambda(w^2-1) =$$

$$\int_{D(z^2-1,1+3\epsilon)^c}\sum_j\ |\langle Sk_{z^2-1}, T_j^*k_{w^2-1}\rangle_{A^2}| \frac{\|K_{z^2-1}\|_{A^2}^{1-\frac{2\delta}{\left(\frac{1+\epsilon}{\epsilon}\right)(n+1)}}}{\|K_{w^2-1}\|_{A^2}^{1-\frac{2\delta}{\left(\frac{1+\epsilon}{\epsilon}\right)(n+1)}}} d\lambda(w^2-1) =$$

$$\int_{D(z^2-1,1+3\epsilon)^c}\left|\int_{\mathbb{B}_n}\sum_j\ \langle Sk_{z^2-1}, k_{x^2-1}\rangle_{A^2}\langle\, k_{x^2-1}, T_j^*k_{w^2-1}\rangle_{A^2} d\lambda(x^2-1)\right| \frac{\|K_{z^2-1}\|_{A^2}^{1-\frac{2\delta}{\left(\frac{1+\epsilon}{\epsilon}\right)(n+1)}}}{\|K_{w^2-1}\|_{A^2}^{1-\frac{2\delta}{\left(\frac{1+\epsilon}{\epsilon}\right)(n+1)}}} d\lambda(w^2-1) \leq$$

$$\int_{\mathbb{B}_n}\int_{D(z^2-1,1+3\epsilon)^c}\sum_j\ |\langle k_{z^2-1}, T_j^*k_{w^2-1}\rangle_{A^2}| \frac{d\lambda(w^2-1)}{\|K_{w^2-1}\|_{A^2}^{1-\frac{2\delta}{\left(\frac{1+\epsilon}{\epsilon}\right)(n+1)}}} |\langle Sk_{z^2-1}, k_{x^2-1}\rangle_{A^2}| \|K_{z^2-1}\|_{A^2}^{1-\frac{2\delta}{\left(\frac{1+\epsilon}{\epsilon}\right)(n+1)}} d\lambda(x^2-1)$$

Proceeding exactly as in the proof of the previous Proposition and using the conditions following from $T_j, S\in\mathcal{A}_{1+\epsilon}(\mathbb{B}_n)$ in the place of the local Rudin-Forelli estimates (and replacing a with $1-\frac{2\delta}{\left(\frac{1+\epsilon}{\epsilon}\right)(n+1)}$ we obtain that

$$\lim_{\epsilon\to\infty}\int_{D(z^2-1,1+3\epsilon)^c}\sum_j |\langle T_jSk_{z^2-1},k_{w^2-1}\rangle_{A^2}|\frac{\|K_{z^2-1}\|_{A^2}^{1-\frac{2\delta}{\left(\frac{1+\epsilon}{\epsilon}\right)(n+1)}}}{\|K_{w^2-1}\|_{A^2}^{1-\frac{2\delta}{\left(\frac{1+\epsilon}{\epsilon}\right)(n+1)}}}d\lambda(w^2-1)\,d\lambda(w^2-1) = 0.$$

The corresponding condition for $(T_jS)^*$ is proved in exactly the same way.

Corollary (4.1.21)[264]:*Let* $0<\epsilon<\infty$ *and let the* sequence T_j *be in the norm closure of* $\mathcal{A}_{1+\epsilon}(\mathbb{B}_n)$. *Then for every* $\epsilon>0$ *there exists* $\epsilon\geq 0$ *such that for the covering* $\mathcal{F}_{1+\epsilon}=\{\mathcal{F}_{j_0}\}$ *(associated to* $1+\epsilon$*) we have:*

$$\left\|\sum_j T_jP-\sum_{j_0}\sum_j (M_1)_{F_{j_0}}T_jPM_1(G_1)_{G_{j_0}}\right\|_{A^{1+\epsilon}\to L^{1+\epsilon}(\mathbb{B}_n,dv)}<\epsilon.$$

Proof. a simple approximation argument, we may assume that $T_j\in\mathcal{A}_{1+\epsilon}(\mathbb{B}_n)$. Now define

$$S=\sum_j T_jP-\sum_{j_0}\sum_j (M_1)_{F_{j_0}}T_jPM_1(G_1)_{G_{j_0}}.$$

Given ϵ choose $1+\epsilon$ large enough so that

$$\sup_{z^2-1\in\mathbb{B}_n}\int_{D(z^2-1,1+\epsilon)^c}\sum_j |\langle T_jk_{z^2-1},k_{w^2-1}\rangle_{A^2}|\frac{\|K_{z^2-1}\|_{A^2}^{1-\frac{2\delta}{\left(\frac{1+\epsilon}{\epsilon}\right)(n+1)}}}{\|K_{w^2-1}\|_{A^2}^{1-\frac{2\delta}{\left(\frac{1+\epsilon}{\epsilon}\right)(n+1)}}}d\lambda(w^2-1)<\epsilon$$

and

$$\sup_{z^2-1\in\mathbb{B}_n}\int_{D(z^2-1,1+\epsilon)^c}\sum_j |\langle {T_j}^*k_{z^2-1},k_{w^2-1}\rangle_{A^2}|\frac{\|K_{z^2-1}\|_{A^2}^{1-\frac{2\delta}{\left(\frac{1+\epsilon}{\epsilon}\right)(n+1)}}}{\|K_{w^2-1}\|_{A^2}^{1-\frac{2\delta}{\left(\frac{1+\epsilon}{\epsilon}\right)(n+1)}}}d\lambda(w^2-1)<\epsilon$$

Now for any $z^2-1\in\mathbb{B}_n$ let $z^2-1\in F_{j_0}$, so that

$$\begin{aligned}|\sum_j Sf_j(z^2-1)| &\leq \int_{\mathbb{B}_n}\sum_{j_0}\sum_j 1_{F_{j_0}(z^2-1)}\,1_{G_{j_0}^c}(w^2-1)|\langle {T_j}^*K_{z^2-1},K_{w^2-1}\rangle_{A^2}|\,|f_j(w^2-1)|\,dv(w^2-1)\\ &= \int_{G_{j_0}^c}\sum_j |\langle {T_j}^*K_{z^2-1},K_{w^2-1}\rangle_{A^2}|\,|f_j(w^2-1)|\,dv(w^2-1)\\ &\leq \int_{D(z^2-1,1+\epsilon)^c}\sum_j |\langle {T_j}^*K_{z^2-1},K_{w^2-1}\rangle_{A^2}|\,|f_j(w^2-1)|\,dv(w^2-1).\end{aligned}$$

To finish the proof, we will estimate the operator norm of the integral operator on $L^{1+\epsilon}(\mathbb{B}_n, dv)$ with kernel $1_{D(z^2-1,1+\epsilon)^c}(w^2-1)|\langle T_j^* K_{z^2-1}, K_{w^2-1}\rangle_{A^2}|$ by using the classical Schur test. To that end, let $h(w^2-1) = \|\mathrm{K}_{w^2-1}\|_{A^2}^{\frac{2\delta}{(1+\epsilon)\left(\frac{1+\epsilon}{\epsilon}\right)(n+1)}}$ so that

$$\int_{\mathbb{B}_n} \sum_j 1_{D(z^2-1,1+\epsilon)^c}(w^2-1)|\langle T_j^* K_{z^2-1}, K_{w^2-1}\rangle_{A^2}| h(w^2-1)^{\frac{1+\epsilon}{\epsilon}} dv(w^2-1)$$

$$= \int_{D(z^2-1,1+\epsilon)^c} \sum_j |\langle T_j^* K_{z^2-1}, K_{w^2-1}\rangle_{A^2}| \|\mathrm{K}_{w^2-1}\|_{A^2}^{\frac{2\delta}{(1+\epsilon)(n+1)}} dv(w^2-1)$$

$$= \int_{D(z^2-1,1+\epsilon)^c} \sum_j |\langle T_j^* k_{z^2-1}, k_{w^2-1}\rangle_{A^2}| \|\mathrm{K}_{z^2-1}\| \|\mathrm{K}_{w^2-1}\|_{A^2}^{\frac{2\delta}{(1+\epsilon)(n+1)}-1} d\lambda(w^2-1)$$

$$\leq \epsilon \|\mathrm{K}_{z^2-1}\|_{A^2}^{\frac{2\delta}{(1+\epsilon)(n+1)}} = \epsilon h(z^2-1)^{\frac{1+\epsilon}{\epsilon}}$$

Similarly, we have that

$$\int_{\mathbb{B}_n} \sum_j 1_{D(z^2-1,1+\epsilon)^c}(w^2-1)|\langle T_j^* K_{z^2-1}, K_{w^2-1}\rangle_{A^2}| h(z^2-1)^{1+\epsilon} dv(z^2-1)$$
$$\leq \epsilon h(w^2-1)^{1+\epsilon}$$

which completes the proof.

Corollary (4.1.22)[264]:*Let $0 < \epsilon < \infty$ and let the sequence T_j be in the norm closure of $\mathcal{A}_{1+\epsilon}(\mathbb{B}_n)$. Then there exists $\epsilon \geq 0$ (both depending on T_j) such that*

$$\left\|\sum_j T_j\right\|_e \leq (1+\epsilon) \limsup_{|z^2-1|\to 1} \sup_{w^2-1\in D(z^2-1,1+2\epsilon)} \sum_j |\langle T_j k_{z^2-1}^{(1+\epsilon)}, k_{w^2-1}^{\left(\frac{1+\epsilon}{\epsilon}\right)}\rangle_{A^2}|$$

where $\|T_j\|_e$ is the essential norm of T_j as a bounded operator on $A^{1+\epsilon}$.

Proof. Since $P: L^{1+\epsilon}(\mathbb{B}_n, dv) \to A^{1+\epsilon}$ is a bounded projection, it is enough to estimate the essential norm of $T_j = T_j P$ as an operator on from $A^{1+\epsilon}$ to $L^{1+\epsilon}(\mathbb{B}_n, dv)$.

Clearly if $\sum_j \|T_j P\|_e = 0$ then there is nothing to prove, so assume that $\sum_j \|T_j P\|_e > 0$. there exists $\epsilon \geq 0$ such that for the covering $\mathcal{F}_{1+\epsilon} = \{F_{j_0}\}$ associated to $1+\epsilon$

$$\left\|\sum_j \left(T_j P - \sum_{j_0} (M_1)_{F_{j_0}} T_j P (M_1)_{G_{j_0}}\right)\right\|_{A^{1+\epsilon}\to L^{1+\epsilon}(\mathbb{B}_n,dv)} < \frac{1}{2}\sum_j \|T_j P\|_e.$$

Since $\sum_{j_0<m}\sum_j (M_1)_{F_{j_0}} T_j P (M_1)_{G_{j_0}}$ is compact for every $m \in \mathbb{N}$ we have that $\sum_j \|T_j P\|_e$ (as an operator from $A^{1+\epsilon}$ to $L^{1+\epsilon}(\mathbb{B}_n, dv)$) can be estimated in the following way:

$$\left\|\sum_j T_j P\right\|_e \leq \left\|\sum_j T_j P - \sum_{j_0<m}\sum_j (M_1)_{F_{j_0}} T_j P (M_1)_{G_{j_0}}\right\|_{A^{1+\epsilon}\to L^{1+\epsilon}(\mathbb{B}_n,dv)}$$
$$\leq \left\|\sum_j T_j P - \sum_{j_0}\sum_j (M_1)_{F_{j_0}} T_j P (M_1)_{G_{j_0}}\right\|_{A^{1+\epsilon}\to L^{1+\epsilon}(\mathbb{B}_n,dv)}$$

$$+\sum_j \left\|\left(T_j\right)_m\right\|_{A^{1+\epsilon}\to L^{1+\epsilon}(\mathbb{B}_n,dv)} \le \frac{1}{2}\sum_j \left\|T_jP\right\|_e + \sum_j \left\|\left(T_j\right)_m\right\|_{A^{1+\epsilon}\to L^{1+\epsilon}(\mathbb{B}_n,dv)},$$

Where

$$\sum_j \left(T_j\right)_m = \sum_{j_0\ge m}\sum_j (M_1)_{F_{j_0}} T_j\, P\, (M_1)_{G_{j_0}}.$$

We will complete the proof by showing that there exists $\epsilon \ge 0$ where

$$\limsup_{m\to\infty}\sum_j \left\|(T_j)_m\right\|_{A^{1+\epsilon}\to L^{1+\epsilon}(\mathbb{B}_n,dv)}$$
$$\lesssim (1+\epsilon)\ \limsup_{|z^2-1|\to 1^-}\ \sup_{w^2-1\in D(z^2-1,1+\epsilon)}\sum_j \left|\langle T_j k_{z^2-1}^{(1+\epsilon)}, k_{w^2-1}^{\left(\frac{1+\epsilon}{\epsilon}\right)}\rangle_{A^2}\right|$$
$$+\frac{1}{4}\sum_j \left\|T_jP\right\|_e.$$

If $f_j \in A^{1+\epsilon}$ is arbitrary of norm no greater than 1, then

$$\sum_j \left\|(T_j)_m f_j\right\|_{A^{1+\epsilon}}^{1+\epsilon} = \sum_{j_0\ge m}\sum_j \left\|(M_1)_{F_{j_0}} T_j\, P\, (M_1)_{G_{j_0}} f_j\right\|_{A^{1+\epsilon}}^{1+\epsilon}$$
$$= \sum_{j_0\ge m}\sum_j \frac{\left\|(M_1)_{F_{j_0}} T_j\, P\, (M_1)_{G_{j_0}} f_j\right\|_{A^{1+\epsilon}}^{1+\epsilon}}{\left\|(M_1)_{G_{j_0}} f_j\right\|_{A^{1+\epsilon}}^{1+\epsilon}}\left\|(M_1)_{F_{j_0}} T_j\, P\, (M_1)_{G_{j_0}} f_j\right\|_{A^{1+\epsilon}}^{1+\epsilon}$$
$$\le N \sup_{j_0\ge m}\sum_j \left\|(M_1)_{F_{j_0}} T_j\, l_{j_0}\right\|_{A^{1+\epsilon}}^{1+\epsilon}$$

where

$$l_{j_0} := \sum_j \frac{P\,(M_1)_{G_{j_0}} f_j}{\left\|(M_1)_{G_{j_0}} f_j\right\|_{A^{1+\epsilon}}}.$$

Therefore, we have that

$$\left\|\sum_j (T_j)_m\right\|_{A^{1+\epsilon}\to L^{1+\epsilon}(\mathbb{B}_n,dv)}$$
$$\lesssim \sup_{j\ge m}\ \sup_{\|f_j\|_{A^{1+\epsilon}}\le 1}\sum_j \left\{\left\|(M_1)_{F_{j_0}} T_j\, l_{j_0}\right\|_{A^{1+\epsilon}} : l_{j_0} = \frac{P\,(M_1)_{G_{j_0}} f_j}{\left\|(M_1)_{G_{j_0}} f_j\right\|_{A^{1+\epsilon}}}\right\}$$

and hence

$$\limsup_{m\to\infty}\sum_j \left\|(T_j)_m\right\|_{A^{1+\epsilon}\to L^{1+\epsilon}(\mathbb{B}_n,dv)}$$
$$\lesssim \limsup_{j\to\infty}\ \sup_{\|f_j\|_{A^{1+\epsilon}}\le 1}\sum_j \left\{\left\|(M_1)_{F_{j_0}} T_j\, l_{j_0}\right\|_{A^{1+\epsilon}} : l_{j_0} = \frac{P\,(M_1)_{G_{j_0}} f_j}{\left\|(M_1)_{G_{j_0}} f_j\right\|_{A^{1+\epsilon}}}\right\}$$

Now pick a sequence $\{(f_{j_0})_j\}$ in $A^{1+\epsilon}$ with $\|f_j\|_{A^{1+\epsilon}} \leq 1$ such that

$$\limsup_{j_0\to\infty} \sup_{\|f_j\|\leq 1} \sum_j \left\{ \left\|(M_1)_{F_{j_0}} T_j\, g\right\|_{A^{1+\epsilon}} : g = \frac{P\,(M_1)_{G_{j_0}} f_j}{\left\|(M_1)_{G_{j_0}} f_j\right\|_{A^{1+\epsilon}}} \right\} - \frac{1}{4}\sum_j \|T_j P\|_e$$

$$\leq \limsup_{j_0\to\infty} \sum_j \left\|(M_1)_{F_{j_0}} T_j\, g_j\right\|_{A^{1+\epsilon}},$$

where

$$g_{j_0} = \sum_j \frac{P\,(M_1)_{G_{j_0}} f_j}{\left\|(M_1)_{G_{j_0}} f_j\right\|_{A^{1+\epsilon}}} = \frac{\int_{G_{j_0}} \sum_j \langle f_j, k_{w^2-1}^{(\frac{1+\epsilon}{\epsilon})}\rangle_{A^2}\, k_{w^2-1}^{(1+\epsilon)} d\lambda(w^2-1)}{\left(\int_{G_{j_0}} \sum_j \left|\langle f_j, k_u^{(\frac{1+\epsilon}{\epsilon})}\rangle_{A^2}\right|^{1+\epsilon} d\lambda(w^2-1)\right)^{\frac{1}{1+\epsilon}}}$$

$$= \int_{G_{j_0}} \tilde{a}_j(w^2-1) k_{w^2-1}^{(1+\epsilon)} d\lambda(w^2-1)$$

where

$$\tilde{a}_j(w^2-1) = \frac{\sum_j \langle f_j, k_{w^2-1}^{(\frac{1+\epsilon}{\epsilon})}\rangle_{A^2}}{\left(\int_{G_{j_0}} \sum_j \left|\langle f_j, k_u^{(\frac{1+\epsilon}{\epsilon})}\rangle_{A^2}\right|^{1+\epsilon} d\lambda(w^2-1)\right)^{\frac{1}{1+\epsilon}}}.$$

Finally, by the reproducing property and Hölder's inequality, we have that

$$\limsup_{j_0\to\infty} \sum_j \left\|(M_1)_{F_{j_0}} T_j\, g_i\right\|_{A^{1+\epsilon}}^{1+\epsilon}$$

$$\leq \limsup_{j\to\infty} \int_{F_{j_0}} \sum_j \left(\int_{G_{j_0}} \left|\tilde{a}_j(w^2-1)\right| \left|T_j\, k_{w^2-1}^{(1+\epsilon)}(z^2-1)\right| d\lambda(w^2-1)\right)^{1+\epsilon} dv(z^2-1)$$

$$= \limsup_{j\to\infty} \int_{F_{j_0}} \sum_j \left(\int_{G_{j_0}} \left|\tilde{a}_j(w^2-1)\right| \left|\langle T_j\, k_{w^2-1}^{(1+\epsilon)}, k_{z^2-1}^{\left(\frac{1+\epsilon}{\epsilon}\right)}\rangle_{A^2}\right| d\lambda(w^2-1)\right)^{1+\epsilon} d\lambda(z^2-1)$$

$$\leq \limsup_{|z^2-1|\to 1^-} \sup_{w^2-1\in D(z^2-1,3(1+\epsilon))} \sum_j \left|\langle T_j k_{z^2-1}^{(1+\epsilon)}, k_{w^2-1}^{\left(\frac{1+\epsilon}{\epsilon}\right)}\rangle_{A^2}\right|^{1+\epsilon} \left(\sup_j \lambda(G_{j_0})^{1+\epsilon} \int_{G_{j_0}} |\tilde{a}_j(w^2-1)|^{1+\epsilon} d\lambda(w^2-1)\right)$$

$$\leq C(1+\epsilon) \lim_{|z^2-1|\to 1^-} sup \sup_{w^2-1\in D(z^2-1,3(1+\epsilon))} \sum_j \left| \langle T_j k_{z^2-1}^{(1+\epsilon)}, k_{w^2-1}^{\left(\frac{1+\epsilon}{\epsilon}\right)} \rangle_{A^2} \right|^{1+\epsilon}$$

since we have that $z^2 - 1 \in F_{j_0}$ and $w^2 - 1 \in G_{j_0}$ implies that $d(z^2 - 1, w^2 - 1) \leq 3(1+\epsilon)$ and $\lambda(G_{j_0}) \leq C(1+\epsilon)$ where $C(1+\epsilon)$ is independent of j.

We will finish off with a. First, for $z^2 - 1 \in \mathbb{B}_n$, define

$$U_{z^2-1}^{(1+\epsilon)} f_j(w^2-1) := f_j(\varphi_{z^2-1}(w^2-1))(k_{z^2-1}(w^2-1))^{\frac{2}{1+\epsilon}}$$

which via a simple change of variables argument is clearly an isometry on $A^{1+\epsilon}$.

Corollary (4.1.23)[264]:: *If T_j are any bounded operators on $A^{1+\epsilon}$ for $0 < \epsilon < \infty$ then the following are equivalent*

(a) $\lim_{|z^2-1|\to 1^-} \sup_{w^2-1\in D(z^2-1,1+\epsilon)} \sum_j \left| \langle T_j k_{z^2-1}^{(1+\epsilon)}, k_{w^2-1}^{\left(\frac{1+\epsilon}{\epsilon}\right)} \rangle_{A^2} \right| = 0$ for all $\epsilon \geq 0$,

(b) $\lim_{|z^2-1|\to 1^-} \sup_{w^2-1\in D(z^2-1,1+\epsilon)} \sum_j \left| \langle T_j k_{z^2-1}^{(1+\epsilon)}, k_{w^2-1}^{\left(\frac{1+\epsilon}{\epsilon}\right)} \rangle_{A^2} \right| = 0$ for some $\epsilon \geq 0$,

(c) $\lim_{|z^2-1|\to 1^-} \sup_{w^2-1\in D(z^2-1,1+\epsilon)} \sum_j \left| \langle T_j k_{z^2-1}, k_{z^2-1} \rangle_{A^2} \right| = 0.$

Proof. Trivially we have that $(a) \Rightarrow (b)$, and the fact that $(b) \Rightarrow (c)$ follows by definition and setting $z^2 = w^2$. We will complete the proof by showing that $(c) \Rightarrow (a)$.

Assume to the contrary that $\left|\langle T_j \mathrm{k}_{z^2-1}, \mathrm{k}_{z^2-1} \rangle_{\mathrm{A}^2}\right|$ vanishes as $|z^2-1| \to 1^-$ but that

$$\lim_{|z^2-1|\to 1^-} \sup_{w^2-1\in D(z^2-1,1+\epsilon)} \sum_j \left| \langle T_j k_{z^2-1}^{(1+\epsilon)}, k_{w^2-1}^{\left(\frac{1+\epsilon}{\epsilon}\right)} \rangle_{A^2} \right| \neq 0$$

for some fixed $\epsilon \geq 0$. Thus, there exists sequences $\{(z^2-1)_m\}, \{(w^2-1)_m\}$ and some $0 < \epsilon < 1$ where $\lim_{m\to\infty} |(z^2-1)_m| = 1$ and $|(w^2-1)_m| \leq 1-\epsilon$ for any $m \in \mathbb{N}$, and where

$$\limsup_{\mathrm{m}\to\infty} \sum_j \left| \langle T_j \mathrm{k}_{(z^2-1)_m}^{(1+\epsilon)}, \mathrm{k}_{\varphi_{(z^2-1)_m((w^2-1)_m)}}^{\left(\frac{1+\epsilon}{\epsilon}\right)} \rangle_{\mathrm{A}^2} \right| > \epsilon$$

for some $\epsilon > 0$. Furthermore, passing to a subsequence if necessary, we may assume that $\lim_{m\to\infty} (w^2-1)_m = w^2 - 1 \in \mathbb{B}_n$. Note that since $|(w^2-1)_m| \leq 1-\epsilon < 1$ for all m, we trivially have $\lim_{\mathrm{m}\to\infty} k_{(w^2-1)_m}^{\left(\frac{1+\epsilon}{\epsilon}\right)} = k_{w^2-1}^{\left(\frac{1+\epsilon}{\epsilon}\right)}$ where the convergence is in the $A^{\frac{1+\epsilon}{\epsilon}}$ norm.

Let $\mathcal{B}(A^{1+\epsilon})$ be the space of bounded operators on $A^{1+\epsilon}$. Since the unit ball in $\mathcal{B}(A^{1+\epsilon})$ is WOT compact, we can (passing to another subsequence if necessary) assume that

$$\hat{T}_j = WOT - \lim_{m\to\infty} \sum_j U_{(z^2-1)_m}^{(1+\epsilon)} T_j \left(U_{(z^2-1)_m}^{\left(\frac{1+\epsilon}{\epsilon}\right)} \right)^* .$$

Thus, we have that

$$\limsup_{\mathrm{m}\to\infty} \sum_j \left| \langle T_j \mathrm{k}_{(z^2-1)_m}^{(1+\epsilon)}, \mathrm{k}_{\varphi_{(z^2-1)_m((w^2-1)_m)}}^{\left(\frac{1+\epsilon}{\epsilon}\right)} \rangle_{\mathrm{A}^2} \right|$$

$$= \limsup_{\mathrm{m}\to\infty} \left| \sum_j \langle U_{(z^2-1)_m}^{(1+\epsilon)} T_j \left(U_{(z^2-1)_m}^{\left(\frac{1+\epsilon}{\epsilon}\right)} \right)^* k_0^{(1+\epsilon)}, k_{(w^2-1)_m}^{\left(\frac{1+\epsilon}{\epsilon}\right)} \rangle_{A^2} \right|$$

$$= \limsup_{m\to\infty} \sum_j \left| \langle U^{(1+\epsilon)}_{(z^2-1)_m} T_j \left(U^{(\frac{1+\epsilon}{\epsilon})}_{(z^2-1)_m} \right)^* k_0^{(1+\epsilon)}, k_{w^2-1}^{\left(\frac{1+\epsilon}{\epsilon}\right)} \rangle_{A^2} \right|$$
$$= \sum_j \left| \langle \hat{T}_j k_0, k_{w^2-1} \rangle_{A^2} \right|.$$

However, for any $z^2 - 1 \in \mathbb{B}_n$

$$\sum_j \left| \langle \hat{T}_j k_0^{(1+\epsilon)}, k_{z^2-1}^{\left(\frac{1+\epsilon}{\epsilon}\right)} \rangle \right| = \lim_{m\to\infty} \sum_j \left| \langle U^{(1+\epsilon)}_{(z^2-1)_m} T_j \left(U^{\left(\frac{1+\epsilon}{\epsilon}\right)}_{(z^2-1)_m} \right)^* k_{z^2-1}^{(1+\epsilon)}, k_{z^2-1}^{(\frac{1+\epsilon}{\epsilon})} \rangle \right|$$
$$\approx \lim_{m\to\infty} \sum_j \left| \langle T_j \, k^{(1+\epsilon)}_{\varphi_{(z^2-1)_m}(z^2-1)}, k^{\left(\frac{1+\epsilon}{\epsilon}\right)}_{\varphi_{(z^2-1)_m}(z^2-1)} \rangle \right| = 0$$

since by assumption $\left| \langle T_j \, k_{z^2-1} , k_{z^2-1} \rangle \right|$ vanishes as $|z^2 - 1| \to 1^-$. Thus, since the Berezin transform is injective on $A^{1+\epsilon}$, we get that $T_j = 0$, which contradicts and completes the proof.

Corollary (4.1.24)[264]:: *Each sequence of Toeplitz operators* $(T_j)_u$ *on* $\mathcal{F}_\phi^{1+\epsilon}$ *with a bounded symbol* $u(z^2 - 1)$ *is weakly localized.*

Proof. Since $\left| \langle k_{z^2-1} , k_{w^2-1} \rangle_{\mathcal{F}_\phi^2} \right| \le e^{-\epsilon |z^2 - w^2|}$ for some $\epsilon > 0$ we have that

$$\left| \sum_j \langle (T_j)_u k_{z^2-1} , k_{w^2-1} \rangle_{\mathcal{F}_\phi^2} \right|$$
$$\lesssim \|u\|_{L^\infty} \int_{\mathbb{C}^n} \left| \langle k_{z^2-1}, k_{x^2-1} \rangle_{\mathcal{F}_\phi^2} \right| \left| \langle k_{x^2-1}, k_{w^2-1} \rangle_{\mathcal{F}_\phi^2} \right| d(x^2 - 1)$$
$$\lesssim \|u\|_{L^\infty} \int_{\mathbb{C}^n} e^{-\epsilon |z^2 - x^2|} e^{-\epsilon |x^2 - w^2|} \, d(x^2 - 1).$$

Now if $|z^2 - w^2| \ge 1 + \epsilon$ then by the triangle inequality we have that either $|z^2 - x^2| \ge (1 + \epsilon)/2$ or $|x^2 - w^2| \ge (1 + \epsilon)/2$ so that

$$\int_{D(z^2-1,1+\epsilon)^c} \sum_j \left| \langle (T_j)_u k_{z^2-1} , k_{w^2-1} \rangle_{\mathcal{F}_\phi^2} \right| d(w^2 - 1)$$
$$\lesssim e^{-\frac{\epsilon(1+\epsilon)}{2}} \|u\|_{L^\infty} \int_{D(z^2-1,1+\epsilon)^c} \int_{\mathbb{C}^n} e^{-\frac{\epsilon}{2}|z^2 - x^2|} e^{-\frac{\epsilon}{2}|x^2 - w^2|} \, d(x^2 - 1) \, d(w^2 - 1) \lesssim e^{-\frac{\epsilon(1+\epsilon)}{2}} \|u\|_{L^\infty}$$

Note that $(T_j)_u$ is suffciently localized even in the sense of Xia and Zheng by [170, Proposition 4.1]. Also note that a slight variation of the above argument shows that the sequence of Toeplitz operators $(T_j)_\mu \in \mathcal{A}_\phi(\mathbb{C}^n)$ if μ is a positive Fock-Carleson measure on $\mathbb{C}^n$ (see [178] for precise definitions).

Corollary (4.1.25)[264]:: *Let* $0 < \epsilon < \infty$ *and let sequence* T_j *be in the norm closure of* $\mathcal{A}_\phi(\mathbb{C}^n)$. *Then for every* $\epsilon > 0$ *there exists* $\epsilon \ge 0$ *such that for the covering* $\mathcal{F}_{1+\epsilon} = \{F_{j_0}\}$ *(associated to* $1 + \epsilon$*)*

$$\left\| \sum_j T_j P - \sum_{j_0} \sum_j (M_1)_{F_{j_0}} T_j P (M_1)_{G_{j_0}} \right\|_{\mathcal{F}_\phi^{1+\epsilon} \to L_\phi^{1+\epsilon}} < \epsilon.$$

Proof. Again by an easy approximation argument we can assume that $T_j \in \mathcal{A}_\phi(\mathbb{C}^n)$. Furthermore, we first prove the theorem for $\epsilon = 1$.
Define

$$S = \sum_j T_j P - \sum_{j_0} \sum_j (M_1)_{F_{j_0}} T_j P (M_1)_{G_{j_0}}.$$

Given ϵ choose $1 + \epsilon$ large enough so that

$$\sup_{z^2-1 \in \mathbb{C}^n} \int_{D(z^2-1,1+\epsilon)^c} \sum_j \left| \langle T_j{}^* k_{z^2-1}, k_{w^2-1} \rangle_{\mathcal{F}_\phi^2} \right| dv(w^2-1) < \epsilon$$

and

$$\sup_{z^2-1 \in \mathbb{C}^n} \int_{D(z^2-1,1+\epsilon)^c} \sum_j \left| \langle T_j k_{z^2-1}, k_{w^2-1} \rangle_{\mathcal{F}_\phi^2} \right| dv(w^2-1) < \epsilon.$$

Now for any $z^2 - 1 \in \mathcal{A}_\phi(\mathbb{C}^n)$, pick j_0 such that $z^2 - 1 \in F_{j_0}$. Then we have that

$$\begin{aligned}
&\left| \sum_j S f_j (z^2-1) \right| \\
&\leq \int_{\mathbb{C}^n} \sum_{j_0} \sum_j 1_{F_{j_0}}(z^2-1) 1_{G_{j_0}^c}(w^2-1) \left| \langle T_j{}^* K_{z^2-1}, K_{w^2-1} \rangle_{\mathcal{F}_\phi^2} \right| \left| f_j(w^2-1) \right| e^{-2\phi(w^2-1)} dv(w^2-1) \\
&= \int_{G_{j_0}^c} \sum_j \left| \langle T_j{}^* K_{z^2-1}, K_{w^2-1} \rangle_{\mathcal{F}_\phi^2} \right| \left| f_j(w^2-1) \right| e^{-2\phi(w^2-1)} dv(w^2-1) \\
&\leq \int_{D(z^2-1,1+\epsilon)^c} \sum_j \left| \langle T_j{}^* K_{z^2-1}, K_{w^2-1} \rangle_{\mathcal{F}_\phi^2} \right| \left| f_j(w^2-1) \right| e^{-2\phi(w^2-1)} dv(w^2-1).
\end{aligned}$$

To finish the proof when $\epsilon = 1$, we will estimate the operator norm of the integral operator on L_ϕ^2 with kernel $1_{D(z^2-1,1+\epsilon)^c}(w^2-1) \left| \langle T_j{}^* K_{z^2-1}, K_{w^2-1} \rangle_{\mathcal{F}_\phi^2} \right|$ using the classical Schur test. To that end, let $h(z^2-1) = e^{\frac{1}{2}\phi(z^2-1)}$ so that

$$\begin{aligned}
&\int_{\mathbb{C}^n} \sum_j 1_{D(z^2-1,1+\epsilon)^c}(w^2-1) \left| \langle T_j{}^* K_{z^2-1}, K_{w^2-1} \rangle_{\mathcal{F}_\phi^2} \right| h(w^2-1)^2 e^{-2\phi(w^2-1)} dv(w^2-1) \\
&\approx h(z^2-1)^2 \int_{D(z^2-1,1+\epsilon)^c} \sum_j \left| \langle T_j{}^* k_{z^2-1}, k_{w^2-1} \rangle_{\mathcal{F}_\phi^2} \right| dv(w^2-1) \\
&\lesssim \epsilon h(z^2-1)^2.
\end{aligned}$$

Similarly, we have that

$$\int_{\mathbb{C}^n} \sum_j 1_{D(z^2-1,1+\epsilon)^c}(w^2-1) \left| \langle T_j{}^* K_{z^2-1}, K_{w^2-1} \rangle_{\mathcal{F}_\phi^2} \right| h(w^2-1)^2 e^{-2\phi(w^2-1)} dv(z^2-1) \lesssim \epsilon h(w^2-1)^2$$

which finishes the proof when $\epsilon = 1$.

Now assume that $0 < \epsilon < 1$. Since sequence T_j is bounded on $\mathcal{F}_{\phi}^{1}$, we easily get that

$$\left\| \sum_{j_0} \sum_{j} (M_1)_{F_{j_0}} T_j P (M_1)_{G_{j_0}} \right\|_{\mathcal{F}_{\phi}^{1} \to L_{\phi}^{1}} < \infty$$

which by complex interpolation proves the proposition when $1 < \epsilon < 2$. Finally when $-2 < \epsilon < \infty$, one can similarly get a trivial $L_{\phi}^{1} \to \mathcal{F}_{\phi}^{1}$ operator norm bound on

$$\left(\sum_{j_0} \sum_{j} (M_1)_{F_{j_0}} T_j P (M_1)_{G_{j_0}} \right)^{*} = \sum_{j_0} \sum_{j} P\,(M_1)_{G_{j_0}} {T_j}^{*} P (M_1)_{F_{j_0}}$$

since T* is bounded on $\mathcal{F}_{\phi}^{1}$. Since $(\mathcal{F}_{\phi}^{1+\epsilon})^{*} = \mathcal{F}_{\phi}^{\frac{1+\epsilon}{\epsilon}}$ when $0 < \epsilon < \infty$ where $\frac{1+\epsilon}{\epsilon}$ is the conjugate exponent of $1 + \epsilon$.

Section (4.2): Toeplitz Algebra on the Bergman Space

Suppose that $\mathcal{Z}$ is a collection of bounded operators on a Hilbert space $\mathcal{H}$. Recall that the *essential commutant* of $\mathcal{Z}$ is defined to be

$$EssCom(\mathcal{Z}) = \{A \in ß(\mathcal{H}) : [A, T\,] is\ compact\ for\ every\ T \in \mathcal{Z}\}.$$

Obviously, EssCom($\mathcal{Z}$) is always a norm-closed unital operator algebra that contains $\mathcal{K}(\mathcal{H})$, the collection of compact operators on$\mathcal{H}$. If$\mathcal{Z}$is closed under the*-operation,then EssCom($\mathcal{Z}$) is a C^{*}-algebra.

In [163], Johnson, Parrott and Popa characterized the essential commutant of every von Neumann algebra. Ever since, essential-commutant problems have always attracted attention. We determine the essential commutant of the Toeplitz algebra on the *Bergman space* of the unit ball. Before stating our result, let us first explain the historical background of this problem.

Recall that the essential-commutant problem for the Toeplitz algebra on the *Hardy space* was solved long ago. We write $\mathcal{T}^{Hardy}$ for the Toeplitz algebra on the Hardy space H^2. Also, write $T_f^{Hardy}$for Toeplitz operators on H^2. In [164], Davidson showed that

$$EssCom(\mathcal{T}^{Hardy}) = \{T_f^{Hardy} : f \in QC\} + \mathcal{K}^{Hardy}, \tag{18}$$

where $QC = VMO \cap L^{\infty}$ and $\mathcal{K}^{Hardy}$ is the collection of compact operators on the Hardy space. Later, this result was generalized in [165] to the setting of the Hardy space $H^2(S)$of the unit sphere in $\boldsymbol{C}^n$. (see [166].) We now even know that the essential commutant of $\{T_f^{Hardy} : f \in QC\}$ is strictly larger than $\mathcal{T}^{Hardy}$ [167]. In other words, the image of $\mathcal{T}^{Hardy}$in the Calkin algebra does not satisfy the double-commutant relation.

In view of these Hardy-space results, it may appear surprising that in the decades since [168], no progress has been made on the corresponding essential-commutant problems on the Bergman space. This will fundamentally change the situation by proving the Bergman-space analogue of (18). At the same time, the material helps explain why it took so long for progress to be made on the Bergman space: the Bergman-space case deals with a different kind of structure and requires ideas and techniques that were developed only in the last few years.

For **B** denote the open unit ball $\{z \in \boldsymbol{C}^n : |z| < 1\}$in $\boldsymbol{C}^n$. Letdvbe the volume measure on **B** with the normalization $v(\boldsymbol{B}) = 1$. The Bergman space $L_a^2(\boldsymbol{B}, dv)$ is the subspace

$$\{h \in L^2(\boldsymbol{B}, dv) :\ h\ is\ analytic\ on\ \boldsymbol{B}\}$$

of $L^2(\boldsymbol{B}, dv)$. Write P for the orthogonal projection from $L^2(\boldsymbol{B}, dv)$ onto $L_a^2(\boldsymbol{B}, dv)$. For each $f \in L^\infty(\boldsymbol{B}, dv)$, we have the Toeplitz operator T_f defined by the formula

$$T_f h = P(fh), \qquad h \in L_a^2(\boldsymbol{B}, dv).$$

The *Toeplitz algebra* $\mathcal{T}$ on the Bergman space $L_a^2(\boldsymbol{B}, dv)$ is the C^*-algebra generated by the collection of Toeplitz operators

$$\{T_f : f \in L^\infty(\boldsymbol{B}, dv)\}.$$

In a recent [169], the structure of the Toeplitz algebra $\mathcal{T}$ was explored in some depth. It is the knowledge gained there that enables us to determine EssCom($\mathcal{T}$).

The natural description of EssCom($\mathcal{T}$) involves functions of *vanishing oscillation* on **B**, which were first introduced by Berger, Coburn and Zhu in [170]. These functions are defined in terms of the *Bergman metric* on **B**. For each $z \in \boldsymbol{B}\backslash\{0\}$, we have the Möbiu stransform φ_z given by the formula

$$\varphi_z(\zeta) = \frac{1}{1 - \langle \zeta, z\rangle}\left\{z - \frac{\langle \zeta, z\rangle}{|z|^2} z - (1 - |z|^2)^{\frac{1}{2}}\left(\zeta - \frac{\langle \zeta, z\rangle}{|z|^2} z\right)\right\}, \qquad \zeta \in \boldsymbol{B},$$

[14, p. 25]. In the case $z = 0$, we define $\varphi_0(\zeta) = -\zeta$. Then the formula

$$\beta(z, w) = \frac{1}{2} log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}, \qquad z, w \in \boldsymbol{B},$$

gives us the Bergman metric on **B**. Recall that a function g on **B** is said to have vanishing oscillation if it satisfies the following two conditions: (i) g is continuous on **B**; (ii) the limit

$$\lim_{|z|\uparrow 1} \sup_{\beta(z,w)\le 1} |g(z) - g(w)| = 0$$

holds. We will write VO for the collection of functions of vanishing oscillation on **B**. Moreover, we write

$$VO_{bdd} = VO \cap L^\infty(\boldsymbol{B}, dv).$$

In other words, VO_{bdd} denotes the collection of functions of vanishing oscillation that are also bounded (hence the subscript "bdd") on **B**.

Let us write $\mathcal{K}$ for the collection of compact operators on $L_a^2(\boldsymbol{B}, dv)$. It is well known that $\mathcal{T} \supset \mathcal{K}$. The following is the main result:

Theorem (4.2.1): [162] *The essential commutant of the Toeplitz algebra* $\mathcal{T}$ *equals*

$$\{T_g : g \in VO_{bdd}\} + \mathcal{K}.$$

It was already known in [169] that if $g \in VO_{bdd}$, then the operators $P\, M_g\, (1 - P)$ and $(1 - P)M_g P$ are compact on $L^2(\boldsymbol{B}, dv)$. Therefore it follows that

$$EssCom(\mathcal{T}) \supset \{T_g : g \in VO_{bdd}\} + \mathcal{K}. \tag{19}$$

Our task is to prove the inclusion

$$\text{EssCom}(\mathcal{T}) \subset \{T_g : g \in VO_{bdd}\} + \mathcal{K}, \tag{20}$$

which will take quite a few steps. We conclude by giving an outline of the proof of (20), which also serves to explain the organization.

The proof of (20) involves a "reverse bound" for certain matrix norms. While the bound itself is elementary, we will prove it as our first step.

An ingredient that is essential to the proof is the modified kernel $\psi_{z,i}$, $z \in \boldsymbol{B}$ and $i \in \boldsymbol{Z}_+$. Modified kernels were previously used in various spaces [169]. We recall these functions and other relevant material. The ends with Proposition (4.2.11), which is a step in the proof of (20). This proposition allows us to "harvest" a specific piece of a non-compact operator for analysis: if A is a non-compact operator on $L_a^2(\boldsymbol{B}, dv)$, then there is an operator of the form

$$F = \sum_{z\in\Gamma} \psi_{z,i} \otimes e_z \ , \tag{21}$$

where Γ is a set that is separated with respect to the Bergman metric and $\{e_z : z \in \Gamma\}$ is an orthonormal set, such that AF is not compact.

Our proof requires a special class of operators to test the membership $A \in EssCom(\mathcal{T})$. In fact, our test operators have the form

$$T = \sum_{z\in\Gamma} c_z \psi_{z,i} \otimes \psi_\gamma(z), i,$$

where the set Γ is separated, the map $\gamma : \Gamma \to \boldsymbol{B}$ satisfies the condition $\beta\big(z, \gamma(z)\big) \le C$for every $z \in \Gamma$, and the set of coefficients $\{c_z : z \in \Gamma\}$ is bounded. But to use such a T as a test operator, we must know that$\in \mathcal{T}$. we show that such a T is *weakly localized*. Therefore by a result from [169], we have $T \in \mathcal{T}$. Then, using the fact that $T \in \mathcal{T}$, we prove Lemma (4.2.18), which provides conditions for excluding an operator from EssCom($\mathcal{T}$), another necessary step in the proof of Theorem (4.2.1).

Using the modified kernel $\psi_{z,i}$, we introduced themodified Berezin trans-forms $ß_i(X)$ of any operator $X, i \in \boldsymbol{Z}_+$. When $i = 0, ß_0(X)$ is just the usual Berezin transform of X. But our proof uses $ß_i(X)$ for an $i \ge 8n + 1$, which necessitates the introduction of the modified Berezin transforms. Using the fact that $T \in \mathcal{T}$, we show that if $X \in EssCom(\mathcal{T})$, then $ß_i(X) \in VO_{bdd}$ for every $i \in \boldsymbol{Z}_+$, which is also a step in the proof of Theorem (4.2.1).

contains some specific estimates required in the proof, which involve the condition $i \ge 8n + 1$.

We prove Theorem (4.2.1); more specifically, we prove inclusion (20). To do that, fix an $i \ge 8n + 1$. Let $X \in EssCom(\mathcal{T})$ be given. Since we know that $ß_i(X) \in VO_{bdd}$, it suffices to show that $X - T_{ß_i}(X)$ is compact. If$X - T_{ß_i}(X)$were not compact, then there would be an F of the form (21) such that$(X - T_{ß_i}(X))F$ is not compact. Then, by a lengthy deduction process that involves thebounds provided, we show that the non-compactness of $(X - T_{ß_i}(X))F$ implies that

$$A = (X - T_{ß_i}(X))^* \ (X - T_{ß_i}(X))$$

satisfies the conditions in Lemma (4.2.18). By that lemma, we would have to conclude that $A \notin EssCom(\mathcal{T})$, which is obviously a contradiction.

For each $k \in \boldsymbol{N}$, let M_k denote the collection of $k \times k$ matrices. Each $A \in M_k$ is naturally identified with the corresponding operator on (the column Hilbert space) $\boldsymbol{C}^k$. We write $\mathcal{D}_k$ for the collection of $k{\times}k$ diagonal matrices. Let DP_k denote the collection of $k \times k$diagonal matrices whose diagonal entries are either 1 or 0. That is, each $E \in DP_k$ is a diagonal matrix that is also a projection. For every $A \in M_k$, $k \in \boldsymbol{N}$, we define

$$C_k(A) = max\{\|[A, E]\|: E \in DP_k\}.$$

Lemma (4.2.2) [162]: *For*$D \in \mathcal{D}_k$*and*$A \in M_k$,$k \in \boldsymbol{N}$, *we have*$\|[A, D]\| \le \ 4\|D\|C_k(A)$.

Proof. It suffices to show that $\|[A, D]\| \le 2\|D\|C_k(A)$ for $D \in \mathcal{D}_k$ with real entries. Given such a D, we list its diagonal entriesin the ascending order as

$$d_1 \le \cdot \cdot \cdot \le d_k \ .$$

Then there is a permutation $\sigma(1), \dots, \sigma(k)$ of $1, \dots, k$ such that for every $i \in \{1, \dots, k\}, di$is in the intersection of the $\sigma(i)$-th row and the $\sigma(i)$-th column of D. Note that $d_1 \ge -\|D\|$ and $d_k \le D$. For each$i \in \{1, \dots, k\}$, let$E_i$be the$k \times k$diagonal matrixwhose entry in the intersection of the $\sigma(j)$-th row and the $\sigma(j)$-th column equals 1 for every $i \leq j \leq k$, and whose other entries are all 0. Then E_1 is the $k \times k$ identity matrix, and we have

$$D = d_1E_1 + (d_2 - d_1)E_2 + \cdots + (d_k - d_{k-1})E_k .$$

Accordingly, for any $A \in M_k$, we have

$$\begin{aligned}\|[A, D]\| &\leq |d_1| \ \|[A, E_1]\| + (d_2 - d_1)\|[A, E_2]\| + \cdots + (d_k - d_{k-1})\|[A, E_k]\| \\ &\leq (d_2 - d_1)C_k(A) + \cdots + (d_k - d_{k-1})C_k(A) = (d_k - d_1)C_k(A) \\ &\leq 2\|D\|C_k(A)\end{aligned}$$

as promised.

For each $k \in \boldsymbol{N}$, let VD_k be the collection of $A \in M_k$ whose diagonal entries are all zero. That is, VD stands for *vanishing diagonal.*

Lemma (4.2.3) [162]: *If* $A \in VD_k, k \in \boldsymbol{N}$, *then* $\|A\| \leq sup\{\|[A, D]\|: D \in \mathcal{D}_k, D \leq 1\}$.

Proof. Let $k \in \boldsymbol{N}$ be given. For each $\theta \in \boldsymbol{R}$, let V_θ be the $k \times k$ diagonal matrix whose diagonal entries are, in the natural order, $e^{i\theta}, e^{i2\theta}, \ldots, e^{ik\theta}$. Let $A \in VD_k$. Since the diagonal entries of A are all zero, elementary calculation shows that

$$\int_0^{2\pi} V_\theta^* AV_\theta \, d\theta = 0.$$

Hence

$$A = \frac{1}{2\pi}\int_0^{2\pi} (A - V_\theta^* AV_\theta)d\theta = \frac{1}{2\pi}\int_0^{2\pi} [A, V_\theta^*]V_\theta d\theta .$$

Thus for each $A \in VD_k$, there is a $\theta(A) \in [0, 2\pi]$ such that $\left\|[A, V_{\theta(A)}^*]V_\theta(A)\right\| \geq \|A\|$.Since $\left\|[A, V_{\theta(A)}^*]\right\| = \left\|[A, V_{\theta(A)}^*]V_\theta(A)\right\|$, the lemma follows.

The following bound is a step in the proof of (20):

Proposition (4.2.4) [162]:*If*$\in VD_k, k \in \boldsymbol{N}$, *then* $\|A\| \leq 4C_k(A)$.

Proof. The conclusion follows immediately from Lemmas (4.2.3) and (4.2.2).

As usual, we write $H^\infty(\boldsymbol{B})$ for the collection of bounded analytic functions on **B**. Also, we write $\|h\|_\infty = \sup_{\zeta \in \boldsymbol{B}} |h(\zeta)|$ for $h \in H^\infty(\boldsymbol{B})$. Naturally, we consider $H^\infty(\boldsymbol{B})$ assubset of the Bergman space $L_a^2(\boldsymbol{B}, dv)$. Recall that the formula

$$k_z(\zeta) = \frac{(1 - |z|^2)^{(n+1)/2}}{(1 - \langle \zeta , z\rangle)^{n+1}}, \ z, \zeta \in \boldsymbol{B},$$

gives us the normalized reproducing kernel for $L_a^2(\boldsymbol{B}, dv)$. For each integer $i \geq 0$, we define the modified kernel function

$$\psi_{z,i}(\zeta) = \frac{(1 - |z|^2)^{\left\{\frac{n+1}{2}\right\}+i}}{(1 - \langle \zeta, z\rangle)^{n+1+i}}, \qquad z, \zeta \in \boldsymbol{B} .$$

If we introduce the multiplier

$$m_z(\zeta) = \frac{1 - |z|^2}{1 - \langle \zeta , z\rangle}$$

for each $z \in \boldsymbol{B}$, then we have the relation $\psi_{z,i} = m_z^i k_z$. As we have seen previously [7,8,18], this modification gives$\psi_{z,i}$ a faster "decaying rate" than k_z, which is what makes the estimate in Lemma (4.2.29) possible.

Obviously, $\|m_z\|_\infty \leq 1 + |z| < 2$ for every $z \in \boldsymbol{B}$. Therefore for every $i \in \boldsymbol{Z}_+$ we have$\psi_{z,i} \leq 2^i$. On the other hand,$\psi_{z,i}, k_z = 1$. Hence the inequality

$$1 \leq \left\|\psi_{z,i}\right\| \leq 2^i \tag{22}$$
holds for all $i \in \boldsymbol{Z}_+$ and $z \in \boldsymbol{B}$.

Definition (4.2.5) [162]: (a) For $z \in \boldsymbol{B}$ and $r > 0$, denote $D(z,r) = \{\zeta \in \boldsymbol{B} : \beta(z,\zeta) < r\}$.

(b) Let $a > 0$. A subset Γ of **B** is said to be a-separated if $D(z,a) \cap D(w,a) = \emptyset$ for all distinct elements z, w in Γ.

(c) A subset $\Gamma$of **B** is simply said to be separated if it is a-separated for some $a > 0$.

Lemma (4.2.6) [162]: [169, Lemma 2.2] *Let Γbe a separated set in* **B**.

(a)*For each $0 < R < \infty$, there is a natural number $N = N(\Gamma, R)$ such that* card$\{v \in \Gamma : \beta(u,v) \leq R\} \leq N$for every$u \in \Gamma$.

(b) *For every pair of$z \in \boldsymbol{B}$and$\rho > 0$, there is a finite partition$\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_m$such that for every$i \in \{1, \ldots, m\}$, the conditions$u, v \in \Gamma_iandu \neq v$imply$\beta(\varphi_u(z), \varphi_u(z)) > \rho$.*

Recall that for each $z \in \boldsymbol{B}$, the formula
$$(U_z h)(\zeta) = k_z(\zeta) h(\varphi_z(\zeta)), \qquad \zeta \in \boldsymbol{B} \text{ and } h \in L_a^2(\boldsymbol{B}, dv),$$
defines a unitary operator.

Lemma (4.2.7) [162]:[169, Lemma 2.6] *Given any separated set Γ in* **B**, *there exists a constant $0 < B(\Gamma) < \infty$ such that the following estimate holds*: *Let $\{h_u : u \in \Gamma\}$ be functions in $H^\infty(\boldsymbol{B})$ such that $\sup_{u \in \Gamma} \|h_u\|_\infty < \infty$, and let $\{e_u : u \in \Gamma\}$ be any orthonormal set. Then*
$$\left\| \sum_{u \in \Gamma} (U_u h_u) \otimes e_u \right\| \leq \sup_{u \in \Gamma} \|h_u\|_\infty .$$
Suppose that Γ is a separated set in **B**, $i \in \boldsymbol{Z}_+$ and $z \in \boldsymbol{B}$. For each such triple $\{\Gamma, i, z\}$, we define the operator
$$E_{\Gamma;z;i} = \sum_{u \in \Gamma} \psi_{\varphi_u(z),i} \otimes \psi_{\varphi_u(z),i}.$$
Corollary (4.2.8) [162]:*LetΓbe a separated set in* **B** *and let $i \in \boldsymbol{Z}_+$. Given any $R > 0$, there is a constant $C_{3.4} = C_{3.4}(R)$ such that the inequality $\left\|E_{\Gamma;z;i}\right\| \leq C_{3.4}$ holds for every $z \in \boldsymbol{B}$ satisfying the condition $\beta(z,0) \leq R$.*

Proof. Let $u \in \Gamma$ and $z \in \boldsymbol{B}$. By [164, Theorem 2.2.2], we have $\psi_{\varphi_u(z),i} = U_u h_{u,z;i}$, where
$$h_{u,z;i} = \left(\frac{1 - \langle u, z\rangle}{|1 - \langle u, z\rangle|}\right)^{n+1} (m_{\varphi_u}(z) \circ \varphi_u)^i k_z.$$
Obviously,$\left\|h_{u,z;i}\right\|_\infty \leq 2^i \|k_z\|_\infty \leq C(1 - |z|)^{-(n+1)/2}$, where $C = 2i + (1/2)(n+1)$. Com-bining this with the fact that$|z| = (e^{2\beta(z,0)} - 1)/(e^{2\beta(z,0)} + 1)$, we obtain
$$\left\|h_{u,z;i}\right\|_\infty \leq C(e^{2\beta(z,0)} + 1)^{(n+1)/2}.$$
Applying Lemma (4.2.7), we see that the constant $C_{3.4} = C_{3.4}(R) = B^2(\Gamma) C^2 (e^{2R} + 1)^{n+1}$ suffices for the given $R > 0$.

Let $d\lambda$ denote the standard Möbius-invariant measure on **B**. That is,
$$d\lambda(z) = \frac{dv(z)}{(1 - |z|^2)^{n+1}}.$$
Proposition (4.2.9) [162]:[168, Proposition 4.1]*For each integer $i \geq 0$, there exist scalars$0 < c \leq C < \infty$ which are determined by i and n such that the self-adjoint operator*

$$R_i = \int \psi_{z,i} \otimes \psi_{z,i} d\lambda(z)$$

satisfies the operator inequality $cP \leq R_i \leq C\ P$ *on the Hilbert space* $L^2(\boldsymbol{B}, dv)$.

Lemma (4.2.10) [162]:[166, Lemma 4.2]*Let*$\{X, M, \mu\}$*be a (finite or infinite) measure space. Let H be a separable Hilbert space and let* $\mathcal{K}(H)$ *denote the col lection of compact operators on H . Suppose that* $F: X \to \mathcal{K}(H)$ *is a weakly* $\mathcal{M}$*-measurable map. If*

$$\int_X \|F(x)\| d\mu(x) < \infty,$$

then

$$K = \int_X F(x) d\mu(x)$$

is a compact operator on the Hilbert space H.

Using the above facts, we can "decompose" non-compactness on the Bergman space:

Proposition (4.2.11) [162]: *Let*A*be a bounded, non-compact operator on*$L_a^2(\boldsymbol{B}, dv)$*. Then for every*$i \in \boldsymbol{Z}_+$*, there is a* 1*-separated set* Γ *in* **B** *such that the operator*

$$A \sum_{z \in \Gamma} \psi_{z,i} \otimes e_z$$

is not compact, where $\{e_z : z \in \Gamma\}$ *is any orthonormal set.*

Proof. Let $\mathcal{L}$ be a subset of **B** which is maximal with respect to the property that

$$D(u,1) \cap D(v,1) = \emptyset \text{ for all } u \neq v \text{ in } \mathcal{L}. \tag{23}$$

The maximality of $\mathcal{L}$ implies that

$$\bigcup_{u \in \mathcal{L}} D(u,2) = \boldsymbol{B}. \tag{24}$$

Now define the function

$$\Phi = \sum_{u \in \mathcal{L}} \chi_{D(u,2)}$$

on **B**. By (23) and Lemma (4.2.6)(a), there is a natural number $N \in \boldsymbol{N}$ such that

$$card\{v \in \mathcal{L} : D(u,2) \cap D(v,2) \neq \emptyset\} \leq N$$

for every $u \in \mathcal{L}$. This and (24) together tell us that the inequality

$$1 \leq \Phi \leq N \tag{25}$$

holds on the unit ball **B**.

Given any integer $i \in \boldsymbol{Z}_+$, we define the operator

$$R_i' = \int \Phi(z) \psi_{z,i} \otimes \psi_{z,i} d\lambda(z) = \sum_{u \in \mathcal{L}} \int_{D(u,2)} \psi_{z,i} \otimes \psi_{z,i} d\lambda(z).$$

Then (25) implies that $R_i \leq R_i' \leq N R_i$. Applying Proposition (4.2.9), we see that the operator inequality $c \leq R_i' \leq N\ C$ holds on the Bergman space $L_a^2(\boldsymbol{B}, dv)$. That is, R_i is both bounded and invertible on $L_a^2(\boldsymbol{B}, dv)$. By the Möbius invariance of both β and $d\lambda$, we have

$$R_i' = \sum_{u \in \mathcal{L}} \int_{D(0,2)} \psi_{\varphi_u(z),i} \otimes \psi_{\varphi_u(z),i} d\lambda(z) = \int_{D(0,2)} E_{\mathcal{L};z;i}\, d\lambda(z).$$

Let A be a bounded, non-compact operator on $L_a^2(\boldsymbol{B}, dv)$. Since R_i' is invertible, the operator

$$AR_i' = \int\limits_{D(0,2)} AE_{\mathcal{L};z;i}\, d\lambda(z)$$

is not compact. By Corollary (4.2.8), there is a finite bound for $\|E_{\mathcal{L};z;i}\|$, $z \in D(0,2)$. ThusLemma (4.2.10) tells us that there is a$w \in D(0,2)$ such that$AE_{\mathcal{L};z;i}$is not compact, i.e.,

$$A \sum_{u\in\mathcal{L}} \psi_{\varphi_u(w),i} \otimes \psi_{\varphi_u(w),i} \tag{26}$$

is not compact. Since $\mathcal{L}$ is 1-separated, Lemma (4.2.6)(b) provides a partition $\mathcal{L} = \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_m$such that for each$j \in \{1, \ldots, m\}$, we have$\beta(\varphi_u(w), \varphi_v(w)) > 2$ for all$u \neq v$in $\mathcal{L}_j$. That is, each set $\Gamma_j = \{\varphi_u(w) : u \in \mathcal{L}_j\}$ is also 1-separated, $j \in \{1, \ldots, m\}$. The non-compactness of (26) implies that there is a $j_0 \in \{1, \ldots, m\}$ such that the operator

$$A \sum_{z\in\Gamma_{j_0}} \psi_{z,i} \otimes \psi_{z,i} \tag{27}$$

is not compact. Finally, let $\{e_z : z \in \Gamma_{j_0}\}$ be any orthonormal set and define the operator

$$F = \sum_{z\in\Gamma_{j_0}} \psi_{z,i} \otimes e_z \,.$$

By Corollary (4.2.8), F is a bounded operator. Since (27) equals AFF^*, we conclude that AF is not compact. This completes the proof.

To prove Theorem (4.2.1), we obviously need plenty of operators to test the membership $A \in EssCom(\mathcal{T})$. In view ofProposition (4.2.11), it is easy to understand that the most suitable "test operators" are discrete sums constructed from the modified kernel $\psi_{z,i}$. But then a problem immediate arises: how do we know that these operators belong to ?

It was first discovered in [170] that *localization* is a powerful tool for analyzing operators on reproducing-kernel Hilbert spaces. Recently, Isralowitz, Mitkovski and Wick further explored this idea in [171] by introducing the notion of *weakly localized operators* on the Bergman space. This in turn led to [169], which settles the membership problem mentioned in the preceding paragraph.

Definition (4.2.12) [162]: Let a positive number $(n-1)/(n+1) < s < 1$ be given.

(c) A bounded operator B on the Bergman space $L_a^2(\boldsymbol{B}, dv)$ is said to be *s*-weakly localized if it satisfies the conditions

$$\sup_{z\in\boldsymbol{B}} \int |\langle Bk_z,\, k_w\rangle| \left(\frac{1-|w|^2}{1-|z|^2}\right)^{\frac{s(n+1)}{2}} d\lambda(w) < \infty,$$

$$\sup_{z\in\boldsymbol{B}} |\langle B^*k_z,\, k_w\rangle| \left(\frac{1-|w|^2}{1-|z|^2}\right)^{\frac{s(n+1)}{2}} d\lambda(w) < \infty,$$

$$\lim_{r\to\infty} \sup_{z\in\boldsymbol{B}} \int\limits_{\boldsymbol{B}\setminus D(z,r)} |\langle Bk_z,\, k_w\rangle| \left(\frac{1-|w|^2}{1-|z|^2}\right)^{\frac{s(n+1)}{2}} d\lambda(w) = 0 \quad and$$

$$\lim_{r\to\infty} \sup_{z\in \boldsymbol{B}} \int_{\boldsymbol{B}\setminus D(z,r)} |\langle B^*Bk_z,\ k_w\rangle| \left(\frac{1-|w|^2}{1-|z|^2}\right)^{s(n+1)/2} d\lambda(w) = 0.$$

(c) Let $\mathcal{A}_s$ denote the collection of s-weakly localized operators defined as above.

(d) Let $C^*(\mathcal{A}_s)$ denote the C^*-algebra generated by $\mathcal{A}_s$.

Theorem (4.2.13) [162]:[169, Theorem 1.3]*For every* $(n-1)/(n+1) < s < 1$ *we have* $C^*(\mathcal{A}_s) = \mathcal{T}$.

Lemma (4.2.14) [162]:[169, Lemma 2.3]*For all* $u, v, x, y \in \boldsymbol{B}$ *we have*

$$\frac{(1-|\varphi_u(x)|^2)^{1/2}(1-|\varphi_v(y)|^2)^{1/2}}{|1-\langle\varphi_u(x),\varphi_v(y)\rangle|} \le\ 2e^{\beta(x,0)+\beta(y,0)}\frac{(1-|u|^2)^{1/2}(1-|v|^2)^{1/2}}{|1-\langle u,v\rangle|}.$$

Corollary (4.2.15) [162]:*For every triple of* $z, z', \zeta \in \boldsymbol{B}$ *we have* $|k_z(\zeta)| \le (2e^{\beta(z,z')})^{n+1}|k_{z'}(\zeta)|$.

Proof. Given any triple of $z, z', \zeta \in \boldsymbol{B}$, we apply Lemma (4.2.14) to the case where $u = z', v = \zeta, x = \varphi_{z'}(z)$ and $y = 0$, which gives us

$$\frac{(1-|z|^2)^{1/2}(1-|\zeta|^2)^{1/2}}{|1-\langle z,\zeta\rangle|} \le\ 2e^{\beta(\varphi_{z'}(z),0)}\frac{(1-|z'|^2)^{1/2}(1-|\zeta|^2)^{1/2}}{|1-\langle z',\zeta\rangle|}.$$

Since $\beta(\varphi_{z'}(z),0) =\ \beta(z,z')$, this implies the conclusion of the corollary.

We now present the "test operators" mentioned earlier.

Proposition (4.2.16) [162]: *Let* Γ *be a separated set in* **B**. *Suppose that* $\gamma\colon \Gamma \to \boldsymbol{B}$ *is a map for which there is a* $0 < C < \infty$ *such that*

$$\beta\big(u,\gamma(u)\big) \le C \tag{28}$$

for every $u \in \Gamma$. *Then for every* $i \in \boldsymbol{Z}_+$ *and every bounded set of complex coefficients* $\{c_u : u \in \Gamma\}$, *the operator*

$$T = \sum_{u\in\Gamma} c_u\psi_{u,i} \otimes \psi_{\gamma(u),i} \tag{29}$$

belongs to the Toeplitz algebra $\mathcal{T}$.

Proof. We need the Forelli–Rudin estimates in [10]. Fix an $(n-1)/(n+1) < s < 1$ and set

$$A = \sup_{x\in\boldsymbol{B}} \int |\langle k_x, k_w\rangle| \left(\frac{1-|w|^2}{1-|x|^2}\right)^{\frac{s(n+1)}{2}} d\lambda(w)\ \ and$$

$$B(R) = \sup_{x\in\boldsymbol{B}} \int_{\beta(w,x)\ge R} |\langle k_x, k_w\rangle| \left(\frac{1-|w|^2}{1-|x|^2}\right)^{\frac{s(n+1)}{2}} d\lambda(w)$$

for $R > 0$. From [170, p. 1558] we know that $A < \infty$ and $B(R) \to 0$ as $R \to \infty$. To show that $\in \mathcal{T}$, by Theorem (4.2.13), it suffices to show that $T \in \mathcal{A}_s$.

Thus we need to verify that

$$\lim_{r\to\infty} \sup_{z\in \boldsymbol{B}} \int_{\boldsymbol{B}\setminus D(z,r)} |\langle Tk_z, k_w\rangle| \left(\frac{1-|w|^2}{1-|z|^2}\right)^{s(n+1)/2} d\lambda(w) = 0. \tag{30}$$

To prove this, let us write $C_1 = sup\{|c_u| : u \in \Gamma\}$, which is assumed to be finite. For every pair of $z, w \in \boldsymbol{B}$ we have

$$|\langle Tk_z, k_w\rangle| \le C_1 \sum_{u\in\Gamma} \left|\langle k_z, \psi_{\gamma(u),i}\rangle\langle \psi_{u,i}, k_w\rangle\right|$$
$$\le 2^{2i} C_1 (1 - |z|^2)^{\frac{n+1}{2}} (1-|w|^2)^{\frac{n+1}{2}} \sum_{u\in\Gamma} |k_{\gamma(u)}(z) k_u(w)|. \qquad (31)$$

By the assumption on Γ, there is an $a > 0$ such that $D(u,a) \cap D(v,a) = \emptyset$ for all= v in Γ. By Corollary (4.2.15), we have $|k_u(w)| \le (2e^a)^{n+1}|k_x(w)|$ for every $x \in D(u,a)$. Similarly, by (28) and Corollary (4.2.15), we have $|k_\gamma(u)(z)| \le (2e^a)^{n+1}|k_x(z)|$ for every $x \in D(u,a)$. Substituting these in(31), we find that if we set$C_2 = 2^{2i}C_1(4e^{2a+C})^{n+1}$,then

$$|\langle Tk_z, k_w\rangle| \le C_2 \sum_{u\in\Gamma} \left|\langle k_z, k_{x_u}\rangle\langle k_{x_u}, k_w\rangle\right|,$$

where $x_u \in D(u,a)$ for every $u \in \Gamma$. Thus for any $z \in \boldsymbol{B}$ and $r > 0$, we have

$$\int_{\boldsymbol{B}\setminus D(z,r)} |\langle Tk_z, k_w\rangle| \left(\frac{1-|w|^2}{1-|z|^2}\right)^{\frac{s(n+1)}{2}} d\lambda(w)$$
$$\le \int_{\beta(z,w)\ge r} C_2 \sum_{u\in\Gamma} \int_{D(u,a)} |\langle k_z, k_x\rangle\langle k_x, k_w\rangle| \frac{d\lambda(x)}{\lambda(D(u,a))} \left(\frac{1-|w|^2}{1-|z|^2}\right)^{\frac{s(n+1)}{2}} d\lambda(w)$$
$$\le \frac{C}{\lambda(D(0,a))} \int \int_{\beta(z,w)\ge r} |\langle k_z, k_x\rangle\langle k_x, k_w\rangle| \left(\frac{1-|w|^2}{1-|z|^2}\right)^{\frac{s(n+1)}{2}} d\lambda(w) d\lambda(x).$$

The rest of the proof resembles the proof of [10, Proposition 2.2]: Write the last integral in the form of $I_1 + I_2$.

If $\beta(z,x) < r/2$ and $\beta(z,w) \ge r$, then $\beta(x,w) \ge r/2$. Hence

$$I_1 \le |\langle k_z, k_x\rangle| \left(\frac{1-|x|^2}{1-|z|^2}\right)^{\frac{s(n+1)}{2}} \int_{\beta(w,x)\ge r/2} |\langle k_z, k_w\rangle| \left(\frac{1-|w|^2}{1-|x|^2}\right)^{\frac{s(n+1)}{2}} d\lambda(w) d\lambda(x)$$

Since the inner integral is at most $B(r/2)$, we have $I_1 \le AB(r/2)$. On the other hand,

$$\le \int_{\beta(z,x)\ge r/2} |\langle k_z, k_x\rangle| \left(\frac{1-|x|^2}{1-|z|^2}\right)^{\frac{s(n+1)}{2}} \int |\langle k_z, k_w\rangle| \left(\frac{1-|w|^2}{1-|x|^2}\right)^{\frac{s(n+1)}{2}} d\lambda(w) d\lambda(x)$$

Since the inner integral does not exceed A, we have $I_2 \le AB(r/2)$. Hence

$$\int_{\boldsymbol{B}\setminus D(z,r)} |\langle Tk_z, k_w\rangle| \left(\frac{1-|w|^2}{1-|z|^2}\right)^{\frac{s(n+1)}{2}} d\lambda(w) \le \frac{2C_2AB(r/2)}{\lambda(D(0,a))}$$

for all $z \in \boldsymbol{B}$and $r > 0$, which proves (30). By the same argument, if we replace T by T^*in(30), the limit also holds. This completes the verification that$T \in \mathcal{A}_s$.

For an operator A on a Hilbert space $\mathcal{H}$, we write $\|A\|_Q$forits *essential norm*, i.e.,

$$\|A\|_Q = inf\,\{\|A - K\|: K\ is\ any\ compact\ operator\ on\ \mathcal{H}\}.$$

Next we use operators of the form (29) to test membership in $EssCom(\mathcal{T})$. To do this, we also need a familiar lemma:

Lemma (4.2.17) [162]:[172, Lemma 2.1]*Let*$\{B_i\}$*be a sequence of compact operators on a Hilbert space* $\mathcal{H}$ *satisfying the following conditions*:

(a) *Both sequences* $\{B_i\}$ *and* $\{B_i^*\}$ *converge to* 0 *in the strong operator topology.*

(b) *Thelimit*$\lim_{i\to\infty}\|B_i\|$*exists.*

Then there exist natural numbers $i(1) < i(2) < \cdots < i(m) < \cdots$*such that the sum*

$$\sum_{m=1}^{\infty} B_{i(m)} = \lim_{N\to\infty} \sum_{m=1}^{N} B_{i(m)}$$

exists in the strong operator topology and we have

$$\left\|\sum_{m=1}^{\infty} B_{i(m)}\right\|_Q = \lim_{i\to\infty}\|B_i\|.$$

Lemma (4.2.18) [162]:*LetAbe a bounded operator on*$L_a^2(\boldsymbol{B}, dv)$. *Suppose that there exist an*$i \in \boldsymbol{Z}_+$, *a separated set* Γ *in* $\boldsymbol{B}$ *and a* $c > 0$ *such that the fol lowing two conditions hold*:

(a) There is a sequence $E_1, \dots, E_j, \dots$ *of finite subsets of* Γ *such that*

$$\left\|\left\|\left[A, \sum_{z\in E_j} \psi_{z,i} \otimes \psi_{z,i}\right]\right\|\right\| \geq c \quad for\ every\ j \geq 1.$$

(b) $inf\{|z| : z \in E_j\} \to 1 as j \to \infty$.

Then A does not belong to the essential commutant of Γ .

Proof. For convenience, for each subset E of Γ we denote

$$S_E = \sum_{z\in E} \psi_{z,i} \otimes \psi_{z,i}.$$

By Corollary (4.2.8), S_Γ is a bounded operator. Therefore $\|S_E\| \leq \|S_\Gamma\| < \infty$ for every $E \subset \Gamma$.

Since each E_j is a finite set and since (2) holds, passing to a subsequence if necessary, we may assume that $E_j \cap E_k = \emptyset$ for all $j = k$ in **N**. Denote $= \cup_{j=1}^{\infty} E_j$. Then

$$\langle S_\varepsilon h, h\rangle = \sum_{j=1}^{\infty} \langle S_{E_j} h, h\rangle$$

or every $h \in L_a^2(\boldsymbol{B}, dv)$. Obviously, this implies that the sequence of operators $\{S_{E_j}\}$ converges to 0 weakly. But since $S_{E_j} \geq 0$ for every j, from this weak convergence we deduce that the operator sequence $\{S_{E_j}\}$ converges to 0 strongly. Define

$$B_j = [A, S_{E_j}]$$

for every $j \in \boldsymbol{N}$. Then we have the strong convergence $B_j \to 0$ and $B_j^* \to 0$ as $j \to \infty$, i.e., condition (a) in Lemma (4.2.17) is satisfied by these operators. Since $\|B_j\| \leq 2\|A\|\|S_\Gamma\|$ for every j, there is a subsequence $\{j_\nu\}$ of the natural numbers such that the limit

$$d = \lim_{\nu\to\infty}\|B_{j_\nu}\|$$

exists. That is, the subsequence $\{B_{j_\nu}\}$ satisfies both condition (a) and condition (b) in Lemma (4.2.17). By that lemma, there are $\nu(1) < \nu(2) <\cdots< \nu(m) <\cdots$ such that thesum

$$B = \sum_{m=1}^{\infty} B_{j_{\nu(m)}} = \lim_{N\to\infty} \sum_{m=1}^{N} B_{j_{\nu(m)}}$$

converges in the strong operator topology with $\|B\|_Q = d$. By condition (1), $d \geq c > 0$.Thus B is not a compact operator.

For each $N \in \boldsymbol{N}$, define

$$T_N = \sum_{m=1}^{N} S_{E_{j_{\nu(m)}}}.$$

If we set $\mathcal{F} = \cup \bigcup_{=1}^{\infty} E_{j_{\nu(m)}}$, then we obviously have the weak convergence $T_N \to S_{\mathcal{F}}$ as $N \to \infty$. Thus, taking weak limit, we obtain

$$B = \lim_{N\to\infty} \sum_{m=1}^{N} [A, S_{E_{j_{\nu(m)}}}] = \lim_{N\to\infty} [A, T_N] = [A, S_{\mathcal{F}}].$$

This shows that the commutator $[A, S_{\mathcal{F}}]$is not compact. Since Proposition (4.2.16) tells us that $S_{\mathcal{F}} \in \mathcal{T}$, we conclude that $A \notin EssCom(\mathcal{T})$.

We begin with some general elementary facts.

Lemma (4.2.19) [162]:*LetTbe a bounded, self-adjoint operator on a Hilbert space$\mathcal{H}$. Then foreach unit vector $x \in H$ we have*

$$\|[T, x\otimes x]\| = \|(T - \langle Tx, x\rangle)x\|.$$

Proof. Let $x \in H$. By the self-adjointness of T, we have $T\langle Tx, x\rangle \in \boldsymbol{R}$. Therefore

$$[T, x\otimes x] = \{(T - \langle Tx, x\rangle)x\} \otimes x - x \otimes \{(T - \langle Tx, x\rangle)x\} = h\otimes x - x\otimes h,$$

where we write $h = (T - \langle Tx, x\rangle)x$. In the case $\|x\| = 1$, we have $\langle h, x\rangle = 0$. Hence

$$\|[T, x\otimes x]\| = \|h\otimes x - x\otimes h\| = \|h\|\|x\| = \|h\| = \|(T - \langle Tx, x\rangle)x\|$$

for every unit vector $x \in \mathcal{H}$.

Lemma (4.2.20) [162]:*LetTbe a bounded, self-adjoint operator on a Hilbert space $\mathcal{H}$. Then forevery pair of unit vectors $x, y \in \mathcal{H}$ we have*

$$|\langle Tx, x\rangle - \langle Ty, y\rangle \leq \|[T, x\otimes y]\| + \|[T, x\otimes x]\| + \|[T, y\otimes y]\|. \quad (32)$$

Proof. By the self-adjointness of T, we have

$$[T, x\otimes y] = (Tx)\otimes y - x\otimes (Ty)$$
$$= \{(T - \langle Tx, x\rangle)x\} \otimes y - x\otimes \{(T - \langle Ty, y\rangle)y\} + (\langle Tx, x\rangle - \langle Ty, y\rangle)x\otimes y.$$

Since x and y are unit vectors, we have $\|x\otimes y\| = \|x\|\|y\| = 1$. Therefore

$$|\langle Tx, x\rangle\langle Ty, y\rangle| = \|(\langle Tx, x\rangle - \&\langle Ty, y\rangle)x\otimes y\|$$
$$\leq \|[T, x\otimes y]\| + \|\{(T - \langle Tx, x\rangle x\} \otimes y\| + \|x\otimes \{(T - \langle Ty, y\rangle)y\}\|$$
$$= \|[T, x\otimes y]\| + \|(T - \langle Tx, x\rangle)x\|\} + \|(T - \langle Ty, y\rangle)y\|.$$

Applying Lemma (4.2.19) to the last two terms above, we obtain (32).

Lemma (4.2.21) [162]:*Let$\{z_j\}$be a sequence in***B***such that*

$$\lim_{j\to\infty} sup\, |z_j| = 1. \quad (33)$$

Then there is a sequence $j_1 < j_2 < \cdots < j_i < \cdots$ *of natural numbers such that* $|z_{j_i}| < |z_{j_{i+1}}|$ *for every* $i \in \boldsymbol{N}$ *and such that the set* $\{z_{j_i}: i \in \boldsymbol{N}\}$ *is separated.*

Proof. For $z \in \boldsymbol{B}$, we have $\beta(z, 0) = (1/2)\, log\{(1 + |z|)/(1 - |z|)\}$. Therefore (33) implies

$$\lim_{j\to\infty} sup\, \beta(z_j, 0) = \infty.$$

Using the triangle inequality for β, the conclusion of the lemma follows from an easy inductive selection of $j_1 < j_2 < \cdots < j_i < \cdots$.

Proposition (4.2.22) [162]:*Suppose that*$\{z_j\}$*is a sequence in* **B** *such that*

$$\lim_{j\to\infty} |z_j| = 1. \tag{34}$$

Furthermore, suppose that $\{w_j\}$ *is a sequence in* **B** *for which there is a constant* $0 < C < \infty$ *such that*

$$\beta(z_j, w_j) \le C \tag{35}$$

for every $j \in \boldsymbol{N}$. *Then for every* $i \in \boldsymbol{Z}_+$ *and every* $X \in EssCom(\mathcal{T})$ *we have*

$$\lim_{j\to\infty} \left\| [X, \psi_{z_j,i} \otimes \psi_{w_j,i}] \right\| = 0. \tag{36}$$

Proof. For the given $\{z_j\}, \{w_j\}, i$ and X, suppose that (36) did not hold. Then, replacing $\{z_j\}, \{w_j\}$by subsequences if necessary, we may assume that there is a$c > 0$ such that

$$\lim_{j\to\infty} \left\| [X, \psi_{z_j,i} \otimes \psi_{w_j,i}] \right\| = c. \tag{37}$$

We will show that this leads to a contradiction.

By (34) and Lemma (4.2.21), there is a sequence $j_1 < j_2 < \cdots < j_\nu <\cdots$of natural numbers such that $|z_{j_\nu}| < |z_{j_{\nu+1}}|$ for every $\nu \in \boldsymbol{N}$ and such that the set $\{z_{j_\nu}: \nu \in \boldsymbol{N}\}$is separated. For each$\nu \in \boldsymbol{N}$, we now define the operator

$$B_\nu = [X, \psi_{z_{j_\nu},i} \otimes \psi_{w_{j_\nu},i}],$$

whose rank is at most 2. Since $\beta(w_j, 0) = (1/2)\, log\{(1 + |w_j|)/(1 - |w_j|)\}$, (34) and (35) together imply that $|w_j| \uparrow 1$ as $j \to \infty$. Thus both sequences of vectors$\{\psi_{z_j,i}\}$and$\{\psi_{w_j,i}\}$converge to 0 weakly in$L_a^2(\boldsymbol{B}, dv)$. Consequently we have the convergence

$$\lim_{\nu\to\infty} B_\nu = 0 \ \ and \qquad \lim_{\nu\to\infty} B_\nu^* = 0$$

in the strong operator topology. Thus by (37) and Lemma (4.2.17), there is a subsequence $(1) < \nu(2) < \cdots < \nu(m) < \cdots$of natural numbers such that the sum

$$B = \sum_{m=1}^{\infty} B_{\nu(m)}$$

converges strongly with $\|B\|_Q = c > 0$. Thus$B$ is not compact. Now define the operator

$$A = \sum_{m=1}^{\infty} \psi_{z_{j_{\nu(m)}},i} \otimes \psi_{w_{j_{\nu(m)}},i}.$$

Since the set $\{z_{j_\nu} : \nu \in \boldsymbol{N}\}$ is separated and since (35) holds, by Proposition (4.2.16) we have $A \in \mathcal{T}$. Since $X \in EssCom(\mathcal{T})$, the commutator $[X, A]$ *compact.* On the other

hand,we clearly have $[X,A]=B$, which, according to Lemma (4.2.17), is a non-compact operator. This is the contradiction promised earlier.
Next we introduce modified Berezin transforms. To do this, we first need to normalize $\psi_{z,i}$. For all$i\in \boldsymbol{Z}_+$and$z\in \boldsymbol{B}$, we define

$$\tilde{\psi}_{z,i}=\frac{\psi_{z,i}}{\|\psi_{z,i}\|}.$$

Keep in mind that $1\le \|\psi_{z,i}\|\le 2^i$ (see (22)). Suppose that A is a bounded operator on $L_a^2(\boldsymbol{B},dv)$. Then for each $i\in \boldsymbol{Z}_+$we define the function

$$ß_i(A)(z)=\langle A\,\tilde{\psi}_{z,i},\tilde{\psi}_{z,i}\rangle, z\in \boldsymbol{B},$$

on the unit ball. Of course, $ß_0(A)$ is just the usual Berezin transform (also called Berezin *symbol*) of A. For each$i>0$, we consider$ß_i(A)$ as a modified Berezin transform of A.
Proposition(4.2.23) [162]:*If$X\in EssCom(\mathcal{T})$, then*$ß_i(X)\in VO_{bdd}$*for every*$i\in \boldsymbol{Z}_+$.
Proof. Let $i\in \boldsymbol{Z}_+$ be given. Since $\mathcal{T}$ is closed under the *-operation, so is $EssCom(\mathcal{T})$. Hence it suffices to consider a self-adjoint $X\in EssCom(\mathcal{T})$. Obviously, $ß_i(X)$ is both bounded and continuous on **B**. If it were true that $ß_i(X)\notin VO$, then there would be a $c>0$ and sequences$\{z_j\},\{w_j\}$in **B** with

$$\lim_{j\to\infty}|z_j|=1 \tag{38}$$

such that for every $j\in \boldsymbol{N}$, we have $\beta(z_j,w_j)\le 1$ and

$$\left|\langle X\tilde{\psi}_{z_j,i},\tilde{\psi}_{z_j,i}\rangle-\langle X\tilde{\psi}_{w_j,i},\tilde{\psi}_{w_j,i}\rangle\right|=\left|ß_i(X)(z_j)-ß_i(X)(w_j)\right|\ge c. \tag{39}$$

But on the other hand, it follows from Lemma (4.2.20) that

$$\begin{aligned}|\langle X\tilde{\psi}_{z_j,i},\tilde{\psi}_{z_j,i}\rangle-\langle X\tilde{\psi}_{w_j,i},\tilde{\psi}_{w_j,i}\rangle|\\ &\le \left\|[X,\tilde{\psi}_{z_j,i}\otimes\tilde{\psi}_{w_j,i}]\right\|+\left\|[X,\tilde{\psi}_{z_j,i}\otimes\tilde{\psi}_{z_j,i}]\right\|\\ &+\left\|[X,\tilde{\psi}_{w_j,i}\otimes\tilde{\psi}_{w_j,i}]\right\|. \end{aligned}\tag{40}$$

By (38) and the condition $\beta(z_j,w_j)\le 1, j\in \boldsymbol{N}$, we can apply Proposition (4.2.22) to obtain

$$\lim_{j\to\infty}\left\|\left[X,\psi_{z_j,i}\otimes\psi_{w_j,i}\right]\right\|=0\ \ and\ \ \lim_{j\to\infty}\left\|[X,\psi_{z_j,i}\otimes\psi_{z_j,i}]\right\|=0. \tag{41}$$

bviously, conditions (38) and $\beta(z_j,w_j)\le 1, j\in \boldsymbol{N}$, also imply $\lim_{j\to\infty}|w_j|=1$. Thus Proposition (4.2.22) also provides that

$$\lim_{j\to\infty}\left\|[X,\psi_{w_j,i}\otimes\psi_{w_j,i}]\right\|\ =0. \tag{42}$$

By (22), we have $\|[X,\tilde{\psi}_{z,i}\otimes\tilde{\psi}_{w,i}]\|\le\ \ [X,\psi_{z,i}\otimes\psi_{w,i}]$for all $z,w\in \boldsymbol{B}$. Thus (40),(41) and (42) together contradict (39).
Lemma (4.2.24) [162]:[163, Theorem 11]*If*$g\in VO_{bdd}$, *then*

$$\lim_{|z|\uparrow 1}\|(g-g(z))k_z\|=0.$$

Proposition (4.2.25) [162]:*If*$X\in EssCom(\mathcal{T})$, *then for every*$i\in \boldsymbol{Z}_+$*we have*

$$\lim_{|z|\uparrow 1}(X-T_{ß_i(X)})\psi_{z,i}\ =\ 0.$$

Proof. Again, it suffices to consider any self-adjoint $X\in EssCom(\mathcal{T})$. Let $i\in \boldsymbol{Z}_+$ be given. Then from Lemma (4.2.19), Proposition (4.2.22) and (22) we deduce that

$$\lim_{|z|\uparrow 1}\left\|(X-ß_i(X)(z))\psi_{z,i}\right\| \le 2^i \lim_{|z|\uparrow 1}(X-ß_i(X)(z))\psi_{z,i} \le 2^i \lim_{|z|\uparrow 1}\left\|[X,\psi_{z,i}\otimes\psi_{z,i}]\right\| = 0.$$

Therefore the proposition will follow if we can show that

$$\lim_{|z|\uparrow 1}\left\|(T_{ß_i(X)}-ß_i(X)(z))\psi_{z,i}\right\| = 0. \tag{43}$$

But

$$\left\|(T_{ß_i(X)}-ß_i(X)(z))\psi_{z,i}\right\| \le \left\|\left(ß_i(X)-ß_i(X)(z)\right)\psi_{z,i}\right\| = \left\|\left(ß_i(X)-ß_i(X)(z)\right)m_z^i k_z\right\| \le 2^i\left\|\left(ß_i(X)-ß_i(X)(z)\right)k_z\right\|. \tag{44}$$

Proposition (4.2.23) tells us that$ß_i(X) \in VO_{bdd}$, which enables us to apply Lemma (4.2.24) in the case $g = ß_i(X)$. Thus (43) follows from (44) and Lemma (4.2.24).

Here we present a number of estimates that will be needed in the proof of Theorem (4.2.1).First of all, we need a more precise version of Lemma (4.2.6)(a).

Lemma (4.2.26) [162]: *There is a constant$C_{6.1}$such that if$\Gamma$is any 1-separated set in***B***and if*$1 \le R < \infty$*, then for every*$u \in \Gamma$*we have*

$$card\{v \in \Gamma: \beta(u,v) \le R\} \le C_{6.1}e^{2nR}. \tag{45}$$

Proof. Since Γ is 1-separated, for every pair of $x = y$ in $\{v \in \Gamma : \beta(u,v) \le R\}$, we have $D(x,1) \cap D(y,1) = \emptyset$. Also, if $\beta(u,v) \le R$, then $D(v,1) \subset D(u,R+1)$. By the Möbiusinvariance of both β and $d\lambda$, we have $\lambda(D(z,t)) = \lambda(D(0,t))$ for all $z \in \boldsymbol{B}$ and $t > 0$. Hence for every $u \in \Gamma$ we have

$$card\{v \in \Gamma: \beta(u,v) \le R\} \le \frac{\lambda D(u,R+1))}{\lambda(D(0,1))} = \frac{\lambda(D(0,R+1))}{\lambda(D(0,1))} \tag{46}.$$

On the other hand, $\beta(0,z) = (1/2)\, log\{(1+|z|)/(1-|z|)\}, z \in \boldsymbol{B}$. Therefore

$$D(0,R+1) = \{z \in \boldsymbol{B}: |z| < \rho\}, \qquad where\ \rho = (e^{2R+2}-1)/(e^{2R+2}+1).$$

By the radial-spherical decomposition of the volume measure *dv*, we have

$$\lambda(D(0,R+1)) = \int_{|z|<\rho}\frac{dv(z)}{(1-|z|^2)^{n+1}} = \int_0^\rho \frac{2nr^{2n-1}dr}{(1-r^2)^{n+1}} \le \int_0^\rho \frac{2ndr}{(1-r)^{n+1}} \le \frac{2}{(1-\rho)^n}.$$

Obviously, $1-\rho \ge e^{-2R-2}$. Therefore$\lambda(D(0,R+1)) \le 2e^{2n}e^{2nR}$. Substituting this in (46), we see that (45) holds for the constant $C_{6.1} = 2e^{2n}/\lambda(D(0,1))$.

Lemma (4.2.27) [162]: [168, Lemma 4.2] *Given any integer* $i \ge 1$, *there is a constant*$C_{6.2}$*such that*

$$\left|\langle\psi_{z,i},\psi_{w,i}\rangle\right| \le C_{6.2}e^{-i\beta(z,w)} \tag{47}$$

for all $z, w \in \boldsymbol{B}$.

Lemma (4.2.28) [162]: [166, Lemma 4.1] *Let X be a set and let E be a subset of* $X \times X$. *Supposethat m is a natural number such that*

$$card\{y \in X : (x,y) \in E\} \le m \ and\ card\{y \in X : (y,x) \in E\} \le m$$

for every $x \in X$. *Then there exist pairwise disjoint subsets* $E_1, E_2, \ldots, E_{2m}$ *of E such that*

$$E = E_1 \cup E_2 \cup \ldots \cup E_{2m}$$

and such that for each $1 \leq j \leq 2m$, *the conditions* $(x,y),(x',y') \in E_j$ *and* $(x,y) \neq (x',y')$ *imply both* $x \neq x'$ *and* $y = y'$.

Lemma (4.2.29) [162]:*Let an integer* $i \geq 8n+1$ *be given. Then there is a constant*$C_{6.4}$*such that the following estimate holds*: *Let* Γ *be a* 1*-separated set in* **B** *and let* $\{e_u: u \in \Gamma\}$ *be any orthonormal set. Let* $1 \leq R < \infty$. *Then*

$$\left\| \sum_{(u,v)\in F} \langle \psi_{v,i}, \psi_{u,i} \rangle e_u \otimes e_v \right\| \leq C_{6.4} e^{-(4n+1)R}$$

for every $F \subset \{(u,v) \in \Gamma \times \Gamma : \beta(u,v) \geq R\}$.

Proof. We partition such an F in the form

$$F = E^{(1)} \cup E^{(2)} \cup \cdots \cup E^{(k)} \cdots , \ where$$

$$E^{(k)} = \{(u,v) \in F: kR \leq \beta(u,v) < (k+1)R\}, \qquad k \in \boldsymbol{N}.$$

Accordingly,

$$\sum_{(u,v)\in F} \langle \psi_{v,i}, \psi_{u,i} \rangle e_u \otimes e_v = T^{(1)} + T^{(2)} + \cdots + T^{(k)} \cdots , \ where \tag{48}$$

$$T^{(k)} = \sum_{(u,v)\in E^{(k)}} \langle \psi_{v,i}, \psi_{u,i} \rangle e_u \otimes e_v , \ k \in \boldsymbol{N}.$$

By Lemma (4.2.26), for each $u \in \Gamma$ we have

$$card\{v \in \Gamma: (u,v) \in E^{(k)}\} \leq C_{6.1} e^{2n(k+1)R} \quad and$$

$$card\{v \in \Gamma: (v,u) \in E^{(k)}\} \leq C_{6.1} e^{2n(k+1)R}.$$

Thus, by Lemma (4.2.28), each $E^{(k)}$ admits a partition

$$E^{(k)} = E_1^{(k)} \cup \cdots \cup E_{2m_k}^{(k)} \quad with \ m_k \leq C_{6.1} e^{2n(k+1)R}$$

such that for every $j \in \{1,\ldots,2m_k\}$, the conditions $(u,v),(u',v') \in E_j^{(k)}$ and $(u,v) \neq (u',v')$ imply both $u \neq u'$ and $v = v'$.

Accordingly, we decompose each $T^{(k)}$ in the form

$$T^{(k)} = T_1^{(k)} + \cdots + T_{2m_k}^{(k)}, \qquad where \tag{49}$$

$$T_j^{(k)} = \sum_{(u,v)\in E_j^{(k)}} \langle \psi_{v,i}, \psi_{u,i} \rangle e_u \otimes e_v , j \ \in \{1,\ldots,2m_k\}.$$

The above-mentioned property of $E_j^{(k)}$ meansthat both projections $(u,v) \mapsto u$ and$(u,v) \mapsto v$ are injective on $E_j^{(k)}$. Therefore

$$\left\| T_j^{(k)} \right\| = \sup_{(u,v)\in E_j^{(k)}} |\langle \psi_{v,i}, \psi_{u,i} \rangle| \,.$$

Applying Lemma (4.2.27), this gives us $\left\| T_j^{(k)} \right\| \leq C_{6.2} e^{-ikR}$. By (49), *we* now have

$$\left\| T^{(k)} \right\| \leq 2m_k C_{6.2} e^{-ikR} \leq \ 2C_{6.1} C_{6.2} e^{2n(k+1)R} e^{-ikR} \ = C_1 e^{-\{ik-2n(k+1)\}R},$$

where $C_1 = \ 2C_{6.1}C_{6.2}$. Since $i \geq 8n+1$ and $k \geq \ 1$, we have

$$ik - 2n(k+1) \geq (8n+1)k - 2n \cdot 2k = (4n+1)k.$$

Hence $\left\| T^{(k)} \right\| \leq \ C_1 e^{-(4n+1)kR}$. Combining this with (49), we obtain

$$\left\|\sum_{(u,v)\in F}\langle\psi_{v,i},\psi_{u,i}\rangle e_u\otimes e_v\right\|\le\sum_{k=1}^{\infty}\|T^{(k)}\|\le C_1\sum_{k=1}^{\infty}e^{-(4n+1)kR}.$$

Recall that we assume $R\ge 1$. Thus, factoring out $e^{-(4n+1)R}$ on the right, we see thatthe lemma holds for the constant$C_{6.4}=C_1\sum_{k=1}^{\infty}e^{-(4n+1)(k-1)}$.

Lemma (4.2.30) [162]:*Given any* $i\ge 8n+1$, *there is a positive number* $2\le R(i)<\infty$ *such that the following holds true for every* $R\ge R(i)$: *Let* Γ *be a subset of* **B** *with the property that* $\beta(u,v)\ge R$ *for* $u\ne v$ *in* Γ, *and let* $\{e_u: u\in\Gamma\}$ *be an orthonormal set. Then the operator*

$$\Psi=\sum_{u,v\in\Gamma}\langle\psi_{v,i},\psi_{u,i}\rangle e_u\otimes e_v$$

satisfies the condition $\|\Psi x\|\ge(1/2)\|x\|$ *for every vector x of the form*

$$x=\sum_{u\in\Gamma}c_u e_v,\qquad\sum_{u\in\Gamma}|c_u|^2<\infty. \tag{50}$$

Proof. Given any $i\ge 8n+1$, let $2\le R(i)<\infty$ besuch that $C_{6.4}e^{-(4n+1)R(i)}\le 1/2$, where$C_{6.4}$ is the constant provided by Lemma (4.2.29). Let $R\ge R(i)$, and suppose that Γ has the property that $\beta(u,v)\ge R$ for $u\ne v$ in Γ. We have $=D+Y$, where

$$D=\sum_{u\in\Gamma}\|\psi_{u,i}\|^2 e_u\otimes e_u\ \ and\ Y=\sum_{\substack{u,v\in\Gamma\\u\ne v}}\langle\psi_{v,i},\psi_{u,i}\rangle e_u\otimes e_u.$$

By (22), we have $\|Dx\|\ge\|x\|$for every vector *x* of the form (50). By the propertyof Γ, we can apply Lemma (4.2.29) to obtain $Y\le C_{6.4}e^{-(4n+1)R}\le C_{6.4}e^{-(4n+1)R(i)}\le 1/2$Clearly, the conclusion of the lemma follows from these two inequalities.

As we have already mentioned, (19) is known and we only need to prove (20). To do this, we first fix an integer $i\ge 8n+1$. Let *X* be any operator in $EssCom(\mathcal{T})$. Then Proposition (4.2.23) tells us that $ß_{i(X)}\in VO_{bdd}$. Thus $T_{ß_{i(X)}}\in EssCom(\mathcal{T})$ by (19). Toprove (20), it suffices to show that the operator $X-T_{ß_{i(X)}}$ is compact. Assume the contrary, i.e., $X-T_{ß_{i(X)}}$ is not compact. We will show that this non-compactness leads to the conclusion that the operator

$$A=\left(X-T_{ß_{i(X)}}\right)^*\left(X-T_{ß_{i(X)}}\right) \tag{51}$$

does not belong to $EssCom(\mathcal{T})$, which is a contradiction.

Since $X-T_{ß_{i(X)}}$is assumed not to be compact, Proposition (4.2.11) provides a 1-separated set Γ in **B** such that the operator

$$Y=\left(X-T_{ß_{i(X)}}\right)\sum_{u\in\Gamma}\psi_{u,i}\otimes e_u \tag{52}$$

is also not compact, where $\{e_u: u\in\Gamma\}$ is an orthonormal set, which will be fixed for the rest of the proof. Our next step is to fix certain constants.

First of all, the non-compactness of *Y*means that

$$\|Y\|_Q=d>0. \tag{53}$$

Since $\psi_{u,i} = m_u^i k_u = U_u m_u^i \circ \varphi_u$ and since $\|m_u\|_\infty \le 2, u \in \Gamma$, by Lemma (4.2.7) we have

$$\sum_{u \in G} \psi_{u,i} \otimes e_u \le 2^i B(\Gamma) \quad \mathit{for\ every} \quad G \subset \Gamma. \tag{54}$$

Let $R(i) \ge 2$ be the positive number provided by Lemma (4.2.30) for the selected integer i. We then pick a positive number $R > R(i)$ such that

$$4\|X\|^2 C_{6.4} e^{-R} \le \frac{d^2}{2^{4i+6} B^4(\Gamma) C_{6.1}^2}, \tag{55}$$

where $C_{6.1}$ and $C_{6.4}$ are the constants provided by Lemmas (4.2.26) and (4.2.29) respectively. By Lemma (4.2.26), there is a natural number$N \le C_{6.1} e^{2nR}$such that

$$card\{v \in \Gamma : \beta(u,v) \le R\} \le N$$

for every $u \in \Gamma$. By a standard maximality argument, there is a partition $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_N$ such that for every $\nu \in \{1, \ldots, N\}$,

$$\mathit{the\ conditions} \quad u, v \in \Gamma_\nu \ \mathit{and} \ u \ne v \ \mathit{imply}\ \beta(u,v) > R. \tag{56}$$

For each $\nu \in \{1, \ldots, N\}$, define

$$Y_\nu = (X - T_{ß_{i(X)}}) \sum_{u \in \Gamma_\nu} \psi_{u,i} \otimes e_u .$$

By (52) and (53), there is a $\mu \in \{1, \ldots, N\}$ such that $\|Y_\mu\|_Q \ge d/N$.

By Lemma (4.2.18), to obtain the promised contradiction $A \notin EssCom(\mathcal{T})$, it suffices to produce, for each $j \in N$, a finite subset $E_j \subset \Gamma_\mu \cap \{z \in \boldsymbol{B} : |z| \ge 1 - (1/j)\}$ such that

$$\left|\left|\left|\left[A, \sum_{u \in E_j} \psi_{u,i} \otimes \psi_{u,i}\right]\right|\right|\right| \ge \frac{d^2}{2^{4i+6} B^4(\Gamma) C_{6.1}^2 e^{4nR}} . \tag{57}$$

Let $j \in N$ be given. To find the $E_j$described above, we set $G_j = \Gamma_\mu \cap \{z \in \boldsymbol{B} : |z| \ge 1 - (1/j)\}$. Note that $\Gamma_\mu \backslash G_j$ is a finite set. Thus if we define

$$Z_j = (X - T_{ß_{i(X)}}) \sum_{u \in G_j} \psi_{u,i} \otimes e_u ,$$

then $Y_\mu - Z_j$ is a finite-rank operator, and consequently $\|Z_j\|_Q = \|Y_\mu\|_Q \ge d/N$. Hence

$$\|Z_j^* Z_j\|_Q = \|Z_j\|_Q^2 \ge (d/N)^2 \ge \frac{d^2}{C_{6.1}^2 e^{4nR}}.$$

Obviously, we have $Z_j^* Z_j = D + W$, where

$$D = \sum_{u \in G_j} \left\|(X - T_{ß_{i(X)}})\psi_{u,i}\right\|^2 e_u \otimes e_u \quad \mathit{and} \quad W = \sum_{\substack{u,v \in G_j \\ u \ne v}} \langle A\psi_{v,i}, \psi_{u,i}\rangle e_u \otimes e_v .$$

proposition (4.2.25) implies that D is a compact operator. Hence

$$\|W\| \ge \|W\|_Q = \|Z_j^* Z_j\|_Q \ge \frac{d^2}{C_{6.1}^2 e^{4nR}} .$$

For each $k \in \boldsymbol{N}$, define the orthogonal projection

$$F_k = \sum_{\substack{u \in G_j \\ |u| \le 1-(1/k)}} e_u \otimes e_u \,.$$

Then obviously we have the strong convergence $F_k W F_k \to W$ as $k \to \infty$. Therefore there is a $k(j) \in \boldsymbol{N}$ such that if we set $G_j = \{u \in Gj : |u| \le 1 - (1/k(j))\}$ and

$$\mathrm{W}' = \sum_{\substack{u,v \in G_j' \\ u \ne v}} \langle A\psi_{v,i}, \psi_{u,i}\rangle e_u \otimes e_v, \tag{58}$$

then

$$\|W'\| \ge (1/2)W \ge \frac{d^2}{2C_{6.1}^2 e^{4nR}} \,. \qquad (59).$$

Obviously, G_j' is a finite set and the diagonal of W' vanishes.

We now apply Proposition (4.2.4) to the finite-rank operator W'. By that proposition, there is an $E_j \subset G_j'$ such that the orthogonal projection

$$Q = \sum_{u \in E_j} e_u \otimes e_u$$

has the property $4\|[\mathrm{W}', \mathrm{Q}]\| \ge \|W'\|$. If we define

$$J = \sum_{u \in G_j' \setminus E_j} e_u \otimes e_u \,,$$

then $[W', Q] = JW'Q - QW'J$. Since W' is self-adjoint, this gives us $\|[W', Q]\| = \|JW'Q\|$. Combining these facts with (59), we obtain

$$\|JW'Q\| \ge \frac{d^2}{8C_{6.1}^2 e^{4nR}}.$$

On the other hand, since $\{G_j' \setminus E_j\} \cap E_j = \emptyset$, from (58) we see that

$$JW'Q = \sum_{u \in G_j' \setminus E_j} \sum_{v \in E_j} A\psi_{v,i}, \psi_{u,i}\, e_u \otimes e_v = S^* AT \,,$$

where

$$S = \sum_{u \in G_j' \setminus E_j} \psi_{u,i} \otimes e_u \quad \text{and} \quad T = \sum_{u \in E_j} \psi_{u,i} \otimes e_u \,.$$

By the finite dimensionalities involved here, there are unit vectors $\xi \in span\{e_u : u \in E_j\}$ and $\eta \in span\{e_u : u \in G_j' \setminus E_j\}$ such that $|\langle S^*AT\,\xi, \eta\rangle| = \|S^*AT\|$. Hence

$$|\langle S^*AT\,\xi, \eta\rangle \ge \frac{d^2}{8C_{6.1}^2 e^{4nR}}.$$

Since $R > R(i)$ and $E_j \subset G_j \subset \Gamma_\mu$, by (56) and Lemma (4.2.30), we have $\|T^*T\,x\| \ge (1/2)\|x\|$ for every $x \in span\{e_u : u \in E_j\}$. This implies that T^*T is surjective on $span\{e_u : u \in E_j\}$. Hence there is an $x_0 \in span\{e_u : u \in E_j\}$ with $\|x_0\| \le 2$ such that $\xi = T^*Tx_0$. Similarly, there is a $y_0 \in span\{e_u : u \in G_j' \setminus E_j\}$ with $\|y_0\| \le 2$ such that $\eta = S^*Sy_0$. Therefore

$$|\langle S^*SS^*AT\,T^*T\,x_0, y_0\rangle| = |\langle S^*AT\,T^*T\,x_0, S^*Sy_0\rangle| = \ |S^*AT\xi, \eta| \geq \frac{d^2}{8C_{6.1}^2 e^{4nR}}$$

Since $\|x_0\| \leq 2$ and $\|y_0\| \leq 2$, this implies

$$\|S^*SS^*AT\,T^*T\| \geq \frac{d^2}{32C_{6.1}^2 e^{4nR}}.$$

By (54), we have $\|T\| \leq 2^i B(\Gamma)$ and $\|S\| \leq 2^i B(\Gamma)$. Hence

$$\|SS^*ATT^*\| \leq \frac{d^2}{2^{2i+5}B^2(\Gamma)C_{6.1}^2 e^{4nR}}. \tag{60}$$

On the other hand,

$$\begin{aligned}\|SS^*ATT^*\| &\leq \|SS^*[A, TT^*]\| \ + \|SS^*T\,T^*A\| \\ &\leq \|SS^*\|\|[\,A, TT^*]\,\| + \|S\|\ \|S^*T\|\|T^*\|\|A\| \\ &\leq \ 2^{2i}\ B^2(\Gamma)\|[\&A, T\,T^*\&]\| \\ &+ 2^{2i}\ B^2(\Gamma)\|A\|\|S^*T\|\,.\end{aligned} \tag{61}$$

Recalling (51), we clearly have $\|A\| \leq 4\|X\|^2$. Thus from (60) and (61) we deduce

$$\|[\,A, TT^*]\| + 4\|X\|^2\|S^*T\| \geq \frac{d^2}{2^{4i+5}B^4(\Gamma)C_{6.1}^2 e^{4nR}}. \tag{62}$$

To estimate $\|S^*T\|$, note that

$$S^*T = \sum_{u\in G_j'\backslash E_j}\sum_{u\in E_j}\langle\psi_{v,i}\,, \psi_{u,i}\rangle e_u \otimes e_v \quad .$$

Obviously, $\{G_j'\backslash E_j\} \times E_j \subset \{(u, v) \in \Gamma_\mu \times \Gamma_\mu : u \neq v\}$. By (75), we can apply Lemma (4.2.29) to obtain $\|S^*T\| \leq C_{6.4}e^{-(4n+1)R}$. Substituting this in (62), we have

$$\|[\,A, TT^*]\| + 4\|X\|^2 C_{6.4}e^{-(4n+1)R} \geq \frac{d^2}{2^{4i+5}B^4(\Gamma)C_{6.1}^2 e^{4nR}}.$$

We now apply condition (55) in the above and then simplify. The result of this is

$$\|[\,A, TT^*]\| \geq \frac{d^2}{2^{4i+6}B^4(\Gamma)C_{6.1}^2 e^{4nR}}.$$

this proves (57) and completes the proof of Theorem (4.2.1).

Chapter 5
Classification of Reducing Subspaces of a Class of Multiplication Operators and Totally Abelian Toeplitz Operators

We obtain a complete description of nontrivial minimal reducing subspaces ofthe multiplication operator by a Blaschke product with four zeros on the Bergman space of the unit disk via the Hardy space of the bidisk. As a byproduct, under a mild condition we provides an affirmative answer to a question raised , and also construct some examples to show that the answer is negative if the associated conditions are weakened.

Section (5.1): Bergman Space by the Hardy Space of the Bidisk

For $\mathbb{D}$ be the open unit disk in $\mathbb{C}$. Let dA denote Lebesgue area measure on the unit disk $\mathbb{D}$, normalized so that the measure of $\mathbb{D}$ equals 1. The Bergman space L_a^2 is the Hilbert space consisting of the analytic functions on $\mathbb{D}$ that are also in the space $L^2(\mathbb{D}, dA)$ of square integrable functions on$\mathbb{D}$. For a bounded analytic functionϕon the unit disk, the multiplication operator $M_\phi$with symbol ϕ is defined on theBergman space L^2 given by $M_\phi h = \phi h$ for $h \in L^2$. On the basis $\{e_n\}_{n=0}^{\infty}$, where e_n isequal to $\sqrt{n+1}z^n$, the multiplication operator M_z by z is a weighted shift operator, said to be the Bergman shift

$$M_z e_n = \sqrt{\frac{n+1}{n+\ 2}}\, e_{n+1}\,.$$

A reducing subspace M for an operator T on a Hilbert space H is a subspace M of H such that$TM \subset M$and$T^*M \subset M$. A reducing subspace M of T is called minimalif M does not have any nontrivial subspaces which are reducing subspaces. We classify reducing subspaces of M_ϕ for the Blaschke product ϕ with four zeros by identifying its minimal reducing subspaces. We lift the Bergman shift up as a compression of a commuting pair of isometries on a nice subspace of the Hardy space of the bidisk. This idea was used in studying the Hilbert modules by R. Douglas and V. Paulsen [185], operator theory in the Hardy space over the bidisk by R. Douglas and R. Yang [183–184], the higher order Hankel forms by S. Ferguson and R. Rochberg [182], and the lattice of the invariant subspaces of the Bergman shift by S. Richter [183].

On the Hardy space of the unit disk, for an inner function ϕ, the multiplication operator by ϕ is a pure isometry. So its reducing subspaces are in one-to-one correspondence with the closed subspaces of $H^2 \ominus \phi H^2$ [183]. Therefore, it has infinitelymany reducing subspaces provided that ϕ is any inner function other than a Möbius function. Many people have studied the problem of determining reducing subspaces of a multiplication operator on the Hardy space of the unit circle [184].

The multiplication operators on the Bergman space possess a very rich structure theory. Even the lattice of the invariant subspaces of the Bergman shift M_z is huge[183]. But the lattice of reducing subspaces of the multiplication operator by a finite Blaschke on the Bergman space seems to be simple. On the Bergman space, Zhu [185] showed that for a Blaschke product ϕ with two zeros, the multiplication operator M_ϕ has exactly two nontrivial reducing subspaces $\mathcal{M}_0$ and $\mathcal{M}_0^\perp$. In fact, the restriction of the multiplication operator on $\mathcal{M}_0$ is unitarily equivalent to the Bergman shift. Usingthe Hardy space of the bidisk in [186], we show that the multiplication operator with a finite Blaschke product ϕ has a unique reducing subspace $\mathcal{M}_0(\phi)$, on which the restriction of M_ϕ is unitarily equivalent to the Bergman shift and if a multiplication operator has such a reducing subspace, then its symbol must be a finite Blaschke product. The space $\mathcal{M}_0(\phi)$ is called the distinguished reducing subspace of M_ϕ and is equal to

$$\bigvee \boldsymbol{W}\{\phi'\phi^n : n = 0, 1, \dots, m, \dots\}$$

if ϕ vanishes at 0 in [185], *i.e,*

$$\phi(z) = cz \prod_{k=1}^{n} \frac{z - \alpha_k}{1 - \overline{\alpha_k} z},$$

for some points $\{\alpha_k\}$ in the unit disk and a unimodular constant *c*. The space has played an important role in classifying reducing subspaces of M_ϕ . In [183], we have shown that for a Blaschke product $\phi$of the third order, except for a scalar multiple of the third power of a Mobius transform, M_ϕ has exactly two nontrivial minimal reducing subspaces $\mathcal{M}_0$ and $\mathcal{M}_0^\perp$. This continues our study on reducing subspaces of the multiplication operators $M_\phi$on the Bergman space in [184] by using the Hardy space of the bidisk. We will obtain a complete description of nontrivial minimal reducing subspaces of $M_\phi$for the fourth order Blaschke product ϕ.

We introduce some notation to lift the Bergman shift as the compression of some isometry on a subspace of the Hardy space of the bidisk and state some theorems in [188] which will be used later. We state the main result and present its proof. Since the proof is long, two difficult cases in the proof are considered.

Let $\mathbb{T}$ denote the unit circle. The torus $\mathbb{T}^2$ is the Cartesian product $\mathbb{T} \times \mathbb{T}$. Let $d\sigma$ be the rotation invariant Lebesgue measure on $\mathbb{T}^2$. The Hardy space $H^2(\mathbb{T}^2)$ is the sub-space of $L^2(\mathbb{T}^2, d\sigma)$, where functions in $H^2(\mathbb{T}^2)$ can be identified with the boundary value of the function holomorphic in the bidisc $\mathbb{D}^2$ with the square summable Fourier coefficients. The Toeplitz operator on $H^2(\mathbb{T}^2)$ with symbol f in $L^\infty(\mathbb{T}^2, d\sigma)$ is defined by $T_f(h) = P(fh)$, for $h \in H^2(\mathbb{T}^2)$, where P is the orthogonal projection from $L^2(\mathbb{T}^2, d\sigma)$ onto $H^2(\mathbb{T}^2)$,.

For each integer $n \geq 0$, let $p_n(z, w) = \sum_{i=0}^{n} z^i w^{n-i}$. Let $\mathcal{H}$ be the subspace of$H^2(\mathbb{T}^2)$spanned by functions$\{p_n\}_{n=0}^{\infty}$ .Thus

$$H^2(\mathbb{T}^2) = \mathcal{H} \oplus c1\{(z - w)H^2(\mathbb{T}^2)\}.$$

Let $\mathcal{B} = P_{\mathcal{H}} T_z|_{\mathcal{H}} = P_{\mathcal{H}} T_w|_{\mathcal{H}}$, where $P_{\mathcal{H}}$ is the orthogonal projection from $L^2(\mathbb{T}^2, d\sigma)$ onto$\mathcal{H}$. So $\mathcal{B}$is unitarily equivalent to the *Bergman shift* $M_z$on theBergman space L_a^2 via the following unitary operator $U: L_a^2(\mathbb{D}) \to \mathcal{H}$,

$$Uz^n = \frac{p_n(z, w)}{n + 1}.$$

This implies that the Bergman shift is lifted up as the compression of an isometry on a nice subspace of $H^2(\mathbb{T}^2)$. Indeed, for each finite Blaschke product $\phi(z)$, the multiplication operator M_ϕ on the Bergman space is unitarily equivalent to $\phi(\mathcal{B})$ on $\mathcal{H}$.

Let L_0 be $ker\ T^*_{\phi(z)} \cap\ ker\ T^*_{\phi(\mathrm{w})} \cap \mathcal{H}$. In [189], for each $e\ \in L_0$, we construct functions $\{d_e^k\}$ and d_e^0 such that for each $l \geq 1$,

$$p_l(\phi(z), \phi(\mathrm{w}))e + \sum_{k=0}^{l-1} p_k(\phi(z), \phi(\mathrm{w}))d_e^{l-k} \in \mathcal{H}$$

and

$$p_l(\phi(z), \phi(\mathrm{w}))e\ + p_{l-1}(\phi(z), \phi(\mathrm{w}))d_e^0 \in \mathcal{H}.$$

On one hand, we have a precise formula of d_e^0:

$$d_e^0(z, w) = we(0, w)e_0(z, w) - w\phi_0(w)e(z, w), \tag{1}$$

where e_0 is the function$z - w$. On the other hand, deis orthogonal to

$$ker\ T^*_{\phi(z)} \cap\ ker\ T^*_{\phi(\mathrm{w})} \cap \mathcal{H},$$

and for a reducing subspace M and $e \in \mathcal{M}$,

$$p_l(\phi(z), \phi(\mathrm{w}))e\ + \sum_{k=0}^{l-1} p_k(\phi(z), \phi(\mathrm{w}))d_e^{l-k} \in \mathcal{M}.$$

Moreover, the relation between d_e^1 and d_e^0 is given by [9, Theorem 1] as follows.

Theorem(5.1.1)[181]*: If $\mathcal{M}$ is a reducing subspace of $\phi(\mathcal{B})$ orthogonal to the distinguished reducing subspace $\mathcal{M}_0$, for each $e \in \mathcal{M} \cap L_0$,then there is an element $\tilde{e} \in \mathcal{M} \cap L_0$ and a number λ such that*

$$d_e^1\ = d_e^0 + \tilde{e} +\ \lambda e_0. \tag{2}$$

Since it is not difficult to calculate $\tilde{e}$ and λ precisely for Blaschke products with smaller order, we are able to classify minimal reducing subspaces of a multiplication operator by a Blaschke product of the fourth order. The main ideas in the proofs of Theorems (5.1.7) and (5.1.8) are that by complicated computations we use (2) to derive conditions on zeros of the Blaschke product of the fourth order.

we often use Theorem (5.1.1) and Theorems 1 and 25 in [189] stated as follows.

Theorem (5.1.2) [181]*: There is a unique reducing subspace $\mathcal{M}_0$ for $\phi(\mathcal{B})$ such that $\phi(\mathcal{B})|_{\mathcal{M}_0}$isunitarily equivalent to the Bergman shift. In fact,*

$$\mathcal{M}_0 = \bigvee_{l\geq 0} \{p_l(\phi(z), \phi(\mathrm{w}))e_0 \quad and \left\{\frac{p_l(\phi(z), \phi(w))e_0}{\sqrt{l+1}\|e_0\|}\right\}_0^\infty$$

form an orthonormal basis of $\mathcal{M}_0$.

Let $\mathcal{M}_0$ be the distinguished reducing subspace for $\phi(\mathcal{B})$. Then $\mathcal{M}_0$ is unitarily equivalent to a reducing subspace of M_ϕ contained in the Bergman space, denoted by $\mathcal{M}_0(\phi)$. The space plays an important role in classifying the minimal reducing subspaces of M_φ in Theorem (5.1.4).

In [189] we showed that for a nontrivial minimal reducing subspace Ω for $\phi(\mathcal{B})$, either Ω equals $\mathcal{M}_0$ or Ω is a subspace of $\mathcal{M}_0^\perp$. The condition in the following theorem is natural.

Theorem(5.1.3) [181]*:Suppose that$\Omega, \mathcal{M}$, and$\mathcal{N}$ are three distinct nontrivial minimal reducing subspaces for $\phi(\mathcal{B})$ and $\Omega \subset \mathcal{M} \oplus\ \mathcal{N}$. If they are contained in $\mathcal{M}_0^\perp$, then there is a unitary operator $U: \mathcal{M} \to \mathcal{N}$ such that U commutes with $\phi(\mathcal{B})$ and $\phi(\mathcal{B})^*$.*

Let ϕ be a Blaschke product with four zeros. we will obtain a complete description of minimal reducing subspaces of the multiplication operator M_{ϕ} . First observe that the multiplication operator M_{z^4} is a weighted shift with multiplicity 4:

$$M_{z^4}e_n = \sqrt{\frac{n+1}{n+5}}e_{n+4},$$

where e_n equals $\sqrt{n+1}z^n$. By [184, Theorem B], M_{z^4}has exactly four nontrivial minimal reducing subspaces:

$$\mathcal{M}_j = \bigvee\{z^n : n \equiv j \bmod 4\}, \qquad j = 1,2,3,4.$$

It is not difficult to see that the set of finite Blaschke products forms a semigroup under composition of two functions. For a finite Blaschke product ϕ we say that ϕ is decomposable if there are two Blaschke products ψ_1 and ψ_2 with orders greater than 1 such that

$$\phi(z) = \psi_1 \circ \psi_2(z).$$

For each λ in $\mathbb{D}$, let ϕ_λ denote the Möbius transform:

$$\phi_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda}z}.$$

Define the operator $U_\lambda$on the Bergman space as follows: $U_\lambda f = f \circ \phi_\lambda k_\lambda$for $f$in $L_a^2$where k_λ is the normalized reproducing kernel$\frac{(1-|\lambda|^2)}{(1-\lambda z)^2}$. Clearly, $U_\lambda$is a self-adjoint unitary operator on the Bergman space. Using the unitary operator U_λ wehave$\mathcal{M}_0(\phi) = U_\lambda \mathcal{M}_0(\phi \circ \phi_\lambda)$, where λ is a zero ofthe finite Blaschke product ϕ. This easily follows from that $\phi \circ \phi_\lambda$ vanishes at 0 and $U_\lambda^* M_\phi U_\lambda = M_{\phi\circ\phi_\lambda}$.

We say that two Blaschke products ϕ_1 and ϕ_2 are equivalent if there is a complexnumber λ in $\mathbb{D}$ such that $\phi_1 = \phi_\lambda \circ \phi_2$. For two equivalent Blaschke products $\phi_1$and ϕ_2, M_{ϕ_1} and M_{ϕ_2} are mutually analytic function calculi of each other and hence share reducing subspaces. The following main result gives a complete description of minimal reducing subspaces.

Theorem*(5.1.4)* [181]*:* *Let ϕ be a Blaschke product with four zeros. One of the following holds.*

(i) If ϕ is equivalent to z^4, i.e.,*ϕ is a scalar multiple of the fourth power ϕ_c^4 of the Möbiustransform ϕ_c for some complex number c in the unit disk, M_ϕ has exactly four nontrivial minimal reducing subspaces*

$$\{U_c\mathcal{M}_1, U_c\mathcal{M}_2, U_c\mathcal{M}_3, U_c\mathcal{M}_4\}.$$

(ii) If ϕ is is decomposable but not equivalent to z^4,i.e,$\phi = \psi_1 \circ \psi_2$ for two Blaschke products ψ_1 and ψ_2 with orders 2 but not both of ψ_1 and ψ_2 are a scalar multiple of z^2, then M_ϕ has exactly three nontrivial minimal reducing subspaces

$$\{\mathcal{M}_0(\phi), \mathcal{M}_0(\psi_2) \ominus \mathcal{M}_0(\varphi), \mathcal{M}_0(\psi_2)^\perp\}.$$

(iii) If ϕ is not decomposable, then M_ϕ has exactly two nontrivial minimal reducing subspaces

$$\{\mathcal{M}_0(\phi), \mathcal{M}_0(\phi)^\perp\}.$$

To prove the above theorem we need the following two lemmas, which tell us when a Blaschke product with order 4 is decomposable.

Lemma (5.1.5) [181]: *If a Blaschke product ϕ with order four is decomposable, then the numerator of the rational function $\phi(z)-\phi(w)$ has at least three irreducible factors.*

Proof. Suppose that ϕ is the Blaschke product with order four. Let $f(z,w)$ be the numerator of the rational function $\phi(z)-\phi(w)$. If ϕ is decomposable, then $\phi = \psi_1 \circ \psi_2$ for two Blaschke products ψ_1 and ψ_2 with order two. Let $g(z,w)$ be the numerator of the rational function $\psi_1(z) - \psi_1(w)$. Clearly, $z-w$ is a factor of $g(z,w)$. Thus we can write $g(z,w)=(z- w)\,p(z,w)$ for some polynomial $p(z,w)$ of z and w to get

$$g(\psi_2(z),\psi_2(w)) = (\psi_2(z)- \psi_2(w))\,p(\psi_2(z),\psi_2(w)).$$

On the other hand, we also have

$$\psi_2(z)-\psi_2(w) = \frac{(z- w)\,p_2(z,w)}{q_2(z,w)}$$

for two polynomials $p_2(z,w)$ and $q_2(z,w)$ with no common factor. In fact, $q_2(z,w)$ and the numerator of the rational function

$$p(\psi_2(z),\psi_2(w))$$

do not have a common factor also. So we obtain

$$g(\psi_2(z),\psi_2(w)) = \frac{(z-w)\,p_2\,(z,w)}{q_2(z,w)}p(\psi_2(z),\psi_2(w)).$$

Since $f(z,w)$ is the numerator of the rational function $g(\psi_2(z),\psi_2(w))$, this gives that $f(z,w)$ has at least three factors.

For $\alpha,\beta \in \mathbb{D}$, define

$$f_{\alpha,\beta}\,(w,z) = w^2(w-\alpha)(w-\beta)(1-\bar{\alpha}z)(1-\bar{\beta}z)-z^2(z-\alpha)(z-\beta)(1-\bar{\alpha}w)(1-\bar{\beta}w).$$

It is easy to see that $f_{\alpha,\beta}\,(w,z)$ is the numerator of $z^2\phi_\alpha(z)\phi_\beta(z)- w^2\phi_\alpha(w)\phi_\beta(w)$. The following lemma gives a criterion for when the Blaschke product $z^2\phi_\alpha(z)\phi_\beta(z)$ is decomposable.

Lemma (5.1.6) **[181]:** *For α and β in $\mathbb{D}$, one of the following holds.*

(i) If both α and β equal zero, then

$$f_{\alpha,\beta}\,(w,z) = (w-z)(w+z)(w-iz)(w+iz).$$

(ii) If α does not equal either β or $-\beta$, then $f_{\alpha,\beta}\,(w,z) = (w-z)\,p(w,z)$ for some irreducible polynomial $p(w,z)$.

(iii) If α equals either β or $-\beta$, but does not equal zero, then

$$f_{\alpha,\beta}\,(w,z) = (w-z)\,p(w,z)q(w,z)$$

for two irreducible distinct polynomials $p(w,z)$ and $q(w,z)$.

Proof. Clearly, (i) holds.

To prove (ii), by the example in [183, p. 6] we may assume that neither α nor β equals 0. First observe that $(w-z)$ is a factor of the polynomial $f_{\alpha,\beta}\,(w,z)$. Taking a long division gives $f_{\alpha,\beta}\,(w,z) = (w-z)g_{\alpha,\beta}(w,z)$, where

$$g_{\alpha,\beta}(w,z) = (1-\bar{\alpha}z)(1-\bar{\beta}z)w^3 + (z-(\alpha+\beta))(1-\bar{\alpha}z)(1-\bar{\beta}z)w^2$$
$$+ (z-\alpha)(z-\beta)(1-(\bar{\alpha}+\bar{\beta})z)w + z(z-\alpha)(z-\beta)$$

Next we will show that $g_{\alpha,\beta}(w,z)$ is irreducible. To do this, we assume that $g_{\alpha,\beta}(w,z)$ is reducible to derive a contradiction.

Assuming that $g_{\alpha,\beta}(w,z)$ is reducible, we can factor $g_{\alpha,\beta}(w,z)$ as the product of two polynomials $p(w,z)$ *and* $q(w,z)$ of z and w with degree of w greater than or equal to one. Write

$$p(w,z) = a_1(z)w + a_0(z), \qquad q(w,z) = b_2(z)w^2 + b_1(z)w + b_0(z),$$

where $a_j(z)$ and $b_j(z)$ are polynomials of z .Since $g_{\alpha,\beta}(w,z)$ equals the product of $p(w,z)$ *and* $q(w,z)$, taking the product and comparing coefficients ofw^kgive

$$a_1(z)b_2(z) = (1-\bar{\alpha}z)(1-\bar{\beta}z), \tag{3}$$
$$a_1(z)b_1(z) + a_0(z)b_2(z) = (z-(\alpha+\beta))(1-\bar{\alpha}z)(1-\bar{\beta}z), \tag{4}$$
$$a_1(z)b_0(z) + a_0(z)b_1(z) = (z-\alpha)(z-\beta)(1-(\bar{\alpha}+\bar{\beta})z), \tag{5}$$
$$a_0(z)b_0(z) = z(z-\alpha)(z-\beta). \tag{6}$$

Equation (3) gives that one of

$$a_1(z) = (1-\bar{\alpha}z),$$
$$a_1(z) = (1-\bar{\alpha}z)(1-\bar{\beta}z),$$
$$a_1(z)=1.$$

In the first case that $a_1(z) = (1-\bar{\alpha}z)$, (3) gives $b_2(z) = (1-\bar{\beta}z)$. Thus by equation (4) we have

$$a_0(z)(1-\bar{\beta}z) = (1-\bar{\alpha}z)[(z-(\alpha+\beta))(1-\bar{\beta}z) - b_1(z)],$$

to get that $(1-\bar{\alpha}z)$ is a factor of $a_0(z)$, and hence is also a factor of a factor $z(z-\alpha)(z-\beta)$ by (6). This implies that α must equal 0. It is a contradiction.

In the second case that $a_1(z) = (1-\bar{\alpha}z)(1-\bar{\beta}z)$, we have that $b_2(z) = 1$ to getthat either the degree of $b_1(z)$ or the degree of $b_0(z)$ must be one while the degrees of $b_1(z)$ and$b_0(z)$ are at most one. So the degree of$a_0(z)$ is at most two. Also $a_0(z)$ doesnot equal zero. Equation (4) gives

$$(1-\bar{\alpha}z)(1-\bar{\beta}z)b_1(z) + a_0(z) = (z-(\alpha+\beta))(1-\bar{\alpha}z)(1-\bar{\beta}z).$$

Thus $a_0(z) = c_1(1-\bar{\alpha}z)(1-\bar{\beta}z)$ for some constant c_1But equation (6) gives

$$c_1(1-\bar{\alpha}z)(1-\bar{\beta}z)b_0(z) = z(z-\alpha)(z-\beta).$$

Either $c_1 = 0$ or $(1-\bar{\alpha}z)(1-\bar{\beta}z)$ is a factor of $z(z-\alpha)(z-\beta)$. This is impossible.

In the third case that $a_1(z) = 1$, then $b_2(z) = (1-\bar{\alpha}z)(1-\bar{\beta}z)$. Since the root $w$of $f_{\alpha,\beta}(w,z)$ is a nonconstant function of z, the degree of $a_0(z)$ must be one. Thus the degrees of $b_1(z)$ and $b_0(z)$ are at most two. By equation (4) we have

$$(1-\bar{\alpha}z)(1-\bar{\beta}z)a_0(z) + b_1(z) = (z-(\alpha+\beta))(1-\bar{\alpha}z)(1-\bar{\beta}z),$$

to get $b_1(z) = (1-\bar{\alpha}z)(1-\bar{\beta}z)[(z-(\alpha+\beta)) - a_0(z)]$. Since the degree of $b_1(z)$isat most two, we have

$$a_0(z) = (z-(\alpha+\beta)) - c_0, \qquad b_1(z) = c_0(1-\bar{\alpha}z)(1-\bar{\beta}z).$$

Equations (6) and (5) give

$$[(z-(\alpha+\beta)) - c_0]b_0(z) = z(z-\alpha)(z-\beta)$$

and

$$b_1(z)[(z-(\alpha+\beta))-c_0]+b_0(z) = (z-\alpha)(z-\beta)(1-(\bar{\alpha}+\bar{\beta})z)\,.$$

Multiplying both sides of the last equality by $[(z-(\alpha+\beta))-c_0]$ gives

$$b_1(z)[(z-(\alpha+\beta))-c_0]^2+z(z-\alpha)(z-\beta)$$
$$= \left[\left(z(\alpha+\beta)\right)-c_0\right](z-\alpha)(z-\beta)\left(1\left(\bar{\alpha}+\bar{\beta}\right)z\right).$$

This leads to

$$[c_0(1-\bar{\alpha}z)(1-\bar{\beta}z)[(z-(\alpha+\beta))-c_0]^2+z(z-\alpha)(z-\beta)$$
$$= [(z-(\alpha+\beta))-c_0](z-\alpha)(z-\beta)(1-(\bar{\alpha}+\bar{\beta})z).$$

If $c_0 = 0$, then the above equality gives that $(z-\alpha)(z-\beta)$ is a factor of $[(z-(\alpha+\beta))-c_0]^2$. This is impossible.

If $c_0 = 0$, then we have

$$z(z-\alpha)(z-\beta) = [(z-(\alpha+\beta))](z-\alpha)(z-\beta)(1\left(\bar{\alpha}+\bar{\beta}\right),z,$$

to get $\bar{\alpha}+\bar{\beta} = 0$ and hence$\alpha = -\beta$. It is also a contradiction. This completes the proof that $g_{\alpha,\beta}(w,z)$ is irreducible.

To prove (iii), we note that if α equals β, an easy computation gives

$$f_{\alpha,\beta}(w,z) = (w-z)[(1-\bar{\alpha}z)w+(z-\alpha)]$$
$$\times[w(w-\alpha)(1-\bar{\alpha}z)+z(z-\alpha)(1-\bar{\alpha}w)].$$

If $\alpha = -\beta$, we also have

$$f_{\alpha,\beta}(w,z) = (w-z)(w+z)[(1-\bar{\alpha}^2z^2)w^2+(z^2-\alpha^2)].$$

Assume that$\phi$is a Blaschke product with the fourth order.By the Bochner Theorem [187], ϕ has a critical point c in the unit disk. Let $\lambda = \phi(c)$ be the critical value of ϕ. Then there are two points α and β in the unit disk such that

$$\phi_\lambda\circ\phi\circ\phi_c(z) = \eta z^2\phi_\alpha\phi_\beta\,,$$

where η is a unimodule constant. Let ψ be $z^2\phi_\alpha\phi_\beta$. Since $\phi\circ\phi_c$ and ψ are mutually analytic function calculus of each other, both $M_{\phi\circ\phi_c}$and M_ψ share reducing subspaces.

(i) If ϕ is equivalent to z^4, then ψ must equal a scalar multiple of z^4. By [184, Theorem B], M_ψ has exactly four nontrivial minimal reducing subspaces

$$\{\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3,\mathcal{M}_4\}$$

where $\mathcal{M}_j = \vee\{z^n : n\equiv j \bmod 4\}$ for $j = 1,2,3,4$. The four spaces above are also reducing subspaces for $M_{\phi\circ\phi_c}$. Noting $U_c^*M_{\phi\circ\phi_c}U_c = M_\phi$, we have that M_ϕ has exact four nontrivial minimal reducing subspaces

$$\{U_c\mathcal{M}_1,U_c\mathcal{M}_2,U_c\mathcal{M}_3,U_c\mathcal{M}_4\}.$$

(ii) If ϕ is decomposable but not equivalent to z^4, *i.e.,*$\phi = \psi_1\circ\psi_2$ for two Blaschke products ψ_1 and ψ_2 with degree two and not both ψ_1 and ψ_2 are scalar multiples of z^2, by Lemmas (5.1.5) and (5.1.6), then α equals either β or $-\beta$ but does not equal 0. By Theorem (5.1.2), the restriction of M_{ψ_2} on $\mathcal{M}_0(\psi_2)$ is unitarily equivalent to the Bergman shift. Thus $\mathcal{M}_0(\psi_2)$ is also a reducing subspace of M_ϕ and the restriction of $M_\phi = M_{\psi_1\circ\psi_2}$ on $\mathcal{M}_0(\psi_1)$ is unitarily equivalent to M_{ψ_1} on the Bergman space. By Theorem (5.1.2) again, there is a unique reducing subspace $\mathcal{M}_0(\psi_1)$ on which the restriction M_{ψ_1} is unitarily equivalent to the Bergman shift. Thus there is a subspace of $\mathcal{M}_0(\psi_2)$ on which

the restriction of M_ϕ is unitarily equivalent to the Bergman shift. Theorem (5.1.2) implies that $\mathcal{M}_0(\phi)$ is contained in $\mathcal{M}_0(\psi_2)$. Therefore $\mathcal{M}_0(\psi_2) \ominus \mathcal{M}_0(\phi)$ is also a minimal reducing subspace of M_ϕ and

$$L_a^2 = \mathcal{M}_0(\phi) \oplus [\mathcal{M}_0(\psi_2) \ominus \mathcal{M}_0(\phi)] \oplus [\mathcal{M}_0(\psi_2)]^\perp.$$

By [186, Theorem 3.1], $\{\mathcal{M}_0(\phi), [\mathcal{M}_0(\psi_2) \ominus \mathcal{M}_0(\phi)], [\mathcal{M}_0(\psi_2)]^\perp\}$ are nontrivial minimal reducing subspaces of M_ϕ. We will show that they are exact nontrivial minimal reducing subspaces of M_ϕ . If this is not true, then there is another minimal reducing subspace Ω of M_ϕ . By [9, Theorem 38], we have

$$\Omega \subset [\mathcal{M}_0(\psi_2) \ominus \mathcal{M}_0(\phi)] \oplus [\mathcal{M}_0(\psi_2)]^\perp.$$

By Theorem (5.1.3), there is a unitary operator $U : [\mathcal{M}_0(\psi_2) \ominus \mathcal{M}_0(\phi)] \to [\mathcal{M}_0(\psi_2)]^\perp$ which commutes with both M_ϕ and M_ϕ^*. But $dim\ ker\ M_\phi^* \cap [\mathcal{M}_0(\psi_2) \ominus \mathcal{M}_0(\phi)] = 1$ and $dim\ ker\ M\varphi * \cap [\mathcal{M}_0(\psi_2)]^\perp = 2$. This is a contradiction. Thus

$$\{\mathcal{M}_0(\phi), [\mathcal{M}_0(\psi_2) \ominus \mathcal{M}_0(\phi)], [\mathcal{M}_0(\psi_2)]^\perp\}$$

are exact nontrivial minimal reducing subspaces of M_ϕ .

(iii) If ϕ is not decomposable, by Lemma (5.1.6), then ϕ equals $z^3\phi_\alpha$ or $z^3\phi_\alpha\phi_\beta$ for two nonzero points $\alpha\ \beta$ in $\mathbb{D}$ and α does not equal β or $-\beta$. The difficult cases will be dealt with . By Theorems (5.1.7) and (5.1.8), M_ϕ has exactly two nontrivial minimal reducing subspaces $\{\mathcal{M}_0(\phi), \mathcal{M}_0(\phi)^\perp\}$.

we will study reducing subspaces of $M_{z^3\phi_\alpha}$ for a nonzero point $\alpha \in \mathbb{D}$. Recall that $\mathcal{M}_0$ is the distinguished reducing subspace of $\phi(\mathcal{B})$ as in Theorem (5.1.2).

Theorem (5.1.7) [181]: *Let $\phi = z^3\phi_\alpha$ for a nonzero point $\alpha \in \mathbb{D}$. Then $\phi(\mathcal{B})$ has exactly two nontrivial reducing subspaces $\{\mathcal{M}_0, \mathcal{M}_0^\perp\}$.*

Proof. Let $\mathcal{M}_0$ be the distinguished reducing subspace of $\phi(\mathcal{B})$ as in Theorem (5.1.2). By Theorem (5.1.3), we only need to show that $\mathcal{M}_0^\perp$ is a minimal reducing subspace for $\phi(\mathcal{B})$. Assume that $\mathcal{M}_0^\perp$ is not a minimal reducing subspace for $\phi(\mathcal{B})$. Then by [186, Theorem 3.1] we may assume $\mathcal{H} = \oplus_{i=0}^2 M_i$ such that each M_i is a nontrivial reducing subspace for $\phi(\mathcal{B})$, $M_0 = \mathcal{M}_0$ is the distinguished reducing subspace for $\phi(\mathcal{B})$, and $\mathcal{M}_0^\perp = M_1 \oplus M_2$. Recall that

$$\phi_0 = z^2\phi_\alpha, \qquad L_0 = span\{1, p_1, p_2, k_\alpha(z)k_\alpha(w)\},$$
$$L_0 = (L_0 \cap M_0) \oplus (L_0 \cap M_1) \oplus (L_0 \cap M_2).$$

We further assume that

$$dim(M_1 \cap L_0) = 1 \quad and \quad dim(M_2 \cap L_0) = 2.$$

Take $0 \neq e_1 \in M_1 \cap L_0, e_2, e_3 \in M_2 \cap L_0$ such that $\{e_2, e_3\}$ are a basis for $M_2 \cap L_0$.Then $L_0 = span\{e_0, e_1, e_2, e_3\}$.

By (1), we have $d_{e_j}^0 = w_{e_j}(0, w)e_0 - \phi(w)e_j$ and direct computations show that

$$\begin{aligned}\langle d_{e_j}^0, p_k\rangle &= \langle w_{e_j}(0,w)e_0 - \phi(w)e_j, p_k\rangle = \langle w_{e_j}(0,w)e_0, p_k\rangle\ b\ y\ T_{\phi(w)}^* p_k = 0)\\ &= \langle w_{e_j}(0,w)e_0(w,w), p_k(0,w)\rangle = \langle w_{e_j}(0,w)\phi'(w), w^k\rangle\\ &= \langle w^3 e_j(0,w)(w\phi_\alpha'(w) + 3\phi_\alpha(w)), w^k\rangle\\ &= \langle w^{3-k} e_j(0,w)(w\phi_\alpha'(w) + 3\phi_\alpha(w)), 1\rangle = 0\end{aligned}$$

for $0 \leq k \leq 2$, and

$$d_{e_j}^0, k_\alpha(z)k_\alpha(w) = \alpha e_j(0,\alpha)e_0(\alpha,\alpha) \;=\; \alpha e_j(0,\alpha)\frac{\alpha^3}{1-\;|\alpha|^2}.$$

This implies that those functions $d_{e_j}^0$are orthogonal to $\{1, p_1, p_2\}$.

Simple calculations give $\langle e_0, p_k\rangle = \; 0$ for $0 \le k \le 1$,

$$\langle e_0,\, p_2\rangle = \langle e_0(0,w),\, p_2(w,w)\rangle = \frac{3}{2}\phi_0''(0) = -3\alpha \neq 0$$

and

$\langle e_0, k_\alpha(z)k_\alpha(w)\rangle = \; e_0(\alpha,\alpha) \; = \phi'(\alpha) \; = \frac{\alpha^3}{1-|\alpha|^2} \neq 0.$

By Theorem (5.1.1), there are numbers μ, λ_jsuch that

$$d_{e_1}^1 = d_{e_1}^0 + \;\mu e_1 + \;\lambda_1 e_0,$$
$$d_{e_2}^1 = d_{e_2}^0 + \;\tilde{e}_2 + \lambda_2 e_0,$$
$$d_{e_3}^1 = d_{e_3}^0 + \;\tilde{e}_3 + \lambda_3 e_0,$$

where $\tilde{e}_2 , \tilde{e}_3 \in M_2 \cap L_0$.

Now we consider two cases. In each case we will derive a contradiction.

Case 1:$\mu \neq 0$. In this case, we get that e_1 is orthogonal to $\{1,\, p_1\}$. So $\{1,\, p_1, e_0 \,, e_1\}$form an orthogonal basis for L_0.

First we show that $\tilde{e}_2 = 0$. If $\tilde{e}_2 \neq 0$, then we get that $\{1,\, p_1, e_0 \,, \tilde{e}_2\}$ are also anorthogonal basis for L_0. Thus $\tilde{e}_2 = \; ce_1$ for some nonzero number c. However, $\tilde{e}_2$ isorthogonal to e_1, since $\tilde{e}_2 \in M_2$ and $e_1 \in M_1$. This is a contradiction. Thus

$$d_{e_2}^1 = d_{e_2}^0 + \lambda_2 e_0.$$

Since both $d_{e_2}^1$and $d_{e_2}^0$are orthogonal to $p_2$and$\langle e_0 \,, p_2\rangle = 3\alpha \neq \; 0$, we have that$\lambda_2 = 0$to get that $d_{e_2}^0 = d_{e_2}^1$is orthogonal to L_0. On the other hand,

$$\langle d_{e_2}^0 \,, k_\alpha(z)k_\alpha(w)\rangle \;\; = \alpha e_2(0,\alpha)\frac{\alpha^3}{1-|\alpha|^2}.$$

Thus $e_2(0,\alpha) = 0$. Similarly we get that $e_3(0,\alpha) = 0$.

Moreover, since e_2 and e_3 are orthogonal to $\{e_0, e_1\}$, write $e_2 = \; c_{11} + c_{12}p_1$ and $e_3 \; = \; c_{21} + c_{22}p_1$. Thus we have

$$e_2(0,\alpha) = c_{11} + \; c_{12}\alpha \; = \; 0, e_3\,(0,\alpha) \; = c_{21} + c_{22}\alpha \; = \; 0$$

to get that e_2 and e_3 are linearly dependent. This leads to a contradiction in this case.

Case 2: $\mu = 0$. In this case we have$d_{e_1}^1 = d_{e_1}^0 + \lambda_1 e_0$. Similarly to the proof in Case 1,we get that $\lambda_1 = \; 0$,

$$d_{e_1}^1 = d_{e_1}^0 \perp L_0 \tag{7}$$

and

$$e_1(0,\alpha) = 0.$$

Without loss of generality we assume that

$$e_1 = \; -\alpha \; + p_1. \tag{8}$$

Letting e be in $M_2 \cap L_0$ such that e is a nonzero function orthogonal to $\tilde{e}_2$, we have that e is orthogonal to $\{e_0, \tilde{e}_2\}$. Thus e must be in the subspace $span\{1, p_1\}$. So there are two constants $b_1$and b_2 such that $= b_1 + b_2 p_1$. Noting $0 = \; \langle e, e_1\rangle = \; -b_1\bar{\alpha} + 2b_2$, we have $e = b_1/2(2 + \bar{\alpha}p_1)$. Hence we may assume that

$$e = 2 + \bar{\alpha} p_1. \tag{9}$$

By Theorem (5.1.1) we have $d_e^1 = d_e^0 + \tilde{e} + \lambda e_0$ for some number λ and $\tilde{e} \in M_2 \cap L_0$. Thus

$$0 = \langle d_{e_1}^1, d_e^1 \rangle = \langle d_{e_1}^1, d_e^0 + \tilde{e} + \lambda e_0 \rangle = \langle d_{e_1}^1, d_e^0 \rangle = \langle d_{e_1}^0, d_e^0 \rangle (by\ (7)).$$

However, a simple computation gives

$$\langle d_{e_1}^0, d_e^0 \rangle = \langle d_{e_1}^0, we(0,w)e_0 - \phi(w)e \rangle = \langle d_{e_1}^0, we(0,w)e_0 \rangle (by\ T_{\phi(w)}^* d_{e_1}^0 = 0$$
$$= \langle we_1(0,w)e_0 - \phi(w)e_1, we(0,w)e_0 \rangle$$
$$= \langle we_1(0,w)e_0, we(0,w)e_0 \rangle - \langle \phi(w)e_1, we(0,w)e_0 \rangle .$$

We need to calculate two terms in the right-hand side of the above equality. By (8)and (9), the first term becomes

$$\langle we_1(0,w)e_0, we(0,w)e_0 \rangle = \langle w(-\alpha + w)e_0, w(2 + \bar{\alpha}w)e_0 \rangle$$
$$= \langle (-\alpha + w)e_0, (2 + \bar{\alpha}w)e_0 \rangle$$
$$= \langle -\alpha e_0, 2e_0 \rangle + \langle we_0, 2e_0 \rangle + \langle -\alpha e_0, \bar{\alpha}we_0 \rangle + \langle we_0, \bar{\alpha}we_0 \rangle$$
$$= -\alpha \langle e_0, e_0 \rangle + 2 \langle we_0, e_0 \rangle - \alpha^2 \langle e_0, we_0 \rangle .$$

The first term in the right-hand side of the last equality is

$$\langle e_0, e_0 \rangle = \langle e_0(w,w), e_0(0,w) \rangle = \langle w\phi_0' + \phi_0, \phi_0 \rangle$$
$$= \langle w(2w\phi_\alpha + w^2\phi_\alpha'), w^2\phi_\alpha \rangle + \langle \phi_0, \phi_0 \rangle = 2 + \langle w\phi_\alpha', \phi_\alpha \rangle + 1 = 4.$$

The last equality follows from

$$\phi_\alpha = -\frac{1}{\bar{\alpha}} + \frac{\frac{1}{\bar{\alpha}} - \alpha}{1 - \bar{\alpha}w} = -\frac{1}{\bar{\alpha}} + \left(\frac{1}{\bar{\alpha}} - \bar{\alpha}\right) K_\alpha (w).$$

Similarly, we have

$$\langle we_0, e_0 \rangle = \langle we_0 (w,w), e_0(0,w) \rangle = w(w\phi_0' + \phi_0), \phi_0 = \alpha.$$

This gives

$$\langle we_1(0,w)e_0, we(0,w)e_0 \rangle = \langle e_1(0,w)e_0, e(0,w)e_0 \rangle = \langle (-\alpha + w)e_0, (2 + \bar{\alpha}w)e_0 \rangle$$
$$= -2\alpha \langle e_0, e_0 \rangle - \alpha^2 \langle e_0, we_0 \rangle + 2 \langle we_0, e_0 \rangle + \alpha \langle we_0, we_0 \rangle$$
$$= -8\alpha - \alpha|\alpha|^2 + 2\alpha + 4\alpha = -2\alpha - \alpha|\alpha|^2.$$

A simple calculation gives that the second term becomes

$$\langle \phi(w)e_1, we(0,w)e_0 \rangle = \langle \phi_0(w)e_1, (2 + \bar{\alpha}w)e_0 \rangle = \langle \phi_0(w)e_1, 2e_0 \rangle + \langle \phi_0(w)e_1, \bar{\alpha}we_0 \rangle$$
$$= 2\langle \phi_0(w)e_1(w,w), e_0(0,w) \rangle + \alpha \langle \phi_0(w)e_1(w,w), we_0(0,w) \rangle$$
$$= 2\langle e_1(w,w), 1 \rangle + \alpha \langle e_1(w,w), w \rangle = 2\langle -\alpha + 2w, 1 \rangle + \alpha \langle -\alpha + 2w, w \rangle$$
$$= -2\alpha + 2\alpha = 0.$$

Thus we conclude

$$\langle d_{e_1}^0, d_e^0 \rangle = \langle we_1(0,w)e_0, we(0,w)e_0 \rangle - \langle \phi(w)e_1, we(0,w)e_0 \rangle = -2\alpha - \alpha|\alpha|^2$$
$$= -\alpha(2 + |\alpha|^2) \neq 0$$

to get a contradiction in this case. This completes the proof.

we will classify minimal reducing subspaces of $M_{z^2\phi_\alpha\phi_\beta}$ for two nonzero points α and β in $\mathbb{D}$ and with $\alpha \neq \beta$.

Theorem (5.1.8) [181]: *Let ϕ be the Blaschke product $z^2\phi_\alpha\phi_\beta$ for two nonzero pointsαandβin $\mathbb{D}$. If* α *does not equal either* β *or* $-\beta$*, then $\phi(\mathcal{B})$ has exact two nontrivial reducing subspaces* $\{\mathcal{M}_0, \mathcal{M}_0^\perp\}$.

Proof. By [189, Theorem 27], if $\mathcal{N}$ is a nontrivial minimal reducing subspace of $\phi(\mathcal{B})$ which is not equal to $\mathcal{M}_0$, then $\mathcal{N}$ is a subspace of $\mathcal{M}_0^{\perp}$, so we only need to show that $\mathcal{M}_0^{\perp}$ is a minimal reducing subspace for $\phi(\mathcal{B})$ unless $\alpha = -\beta$.
Assume that $\mathcal{M}_0^{\perp}$ is not a minimal reducing subspace for $\phi(\mathcal{B})$. By [186, Theorem 3.1], we may assume $\mathcal{H} = \bigoplus_{i=0}^{2} M_i$ such that each M_i is a reducing subspace for $\phi(\mathcal{B})$, $M_0 = \mathcal{M}_0$ is the distinguished reducing subspace for $\phi(\mathcal{B})$, and $M_1 \oplus M_2 = \mathcal{M}_0^{\perp}$. Recall that $\phi_0 = z\phi_\alpha\phi_\beta$, $L_0 = span\{1, p_1, e_\alpha, e_\beta\}$, with $e_\alpha = k_\alpha(z)k_\alpha(w)$, $e_\beta = k_\beta(z)k_\beta(w)$ and

$$L_0 = (L_0 \cap M_0) \oplus (L_0 \cap M_1) \oplus (L_0 \cap M_2).$$

So we further assume that the dimension $M_1 \cap L_0$ of is one and the dimension of $M_2 \cap L_0$ is two. Take a nonzero element e_1 in $M_1 \cap L_0$. Then by Theorem (5.1.1), there are numbers μ_1, λ_1 such that

$$d_{e_1}^1 = d_{e_1}^0 + \mu_1 e_1 + \lambda_1 e_0. \qquad (10)$$

We only need to consider two possibilities, μ_1 is zero or nonzero. If μ_1 is zero, then (10) becomes

$$d_{e_1}^1 = d_{e_1}^0 + \lambda_1 e_0. \qquad (11)$$

In this case, simple calculations give

$$\begin{aligned}\langle d_{e_1}^0, p_1\rangle &= \langle we_1(0,w)e_0(z,w) - w\phi_0(w)e_1(z,w), p_1(z,w)\rangle \\ &= \langle we_1(0,w)e_0(w,w) - w\phi_0(w)e_1(w,w), p_1(z,w)\rangle \\ &= \langle we_1(0,w)e_0(w,w) - w\phi_0(w)e_1(w,w), p_1(0,w)\rangle \\ &= \langle we_1(0,w)e_0(w,w) - w\phi_0(w)e_1(w,w), w\rangle \\ &= \langle e_1(0,w)e_0(w,w) - \phi_0(w)e_1(w,w), 1\rangle \\ &= e_1(0,0)e_0(0,0) - \phi_0(0)e_1(0,0) = 0,\end{aligned}$$

and

$$\begin{aligned}\langle e_0, p_1\rangle &= \langle e_0(z,w), p_1(z,w)\rangle = \langle e_0(z,w), p_1(w,w)\rangle = \langle e_0(0,w), 2w\rangle = \langle \phi_0(w), 2w\rangle \\ &= 2\langle w\phi_\alpha(w)\phi_\beta(w), w\rangle = 2\phi_\alpha(0)\phi_\beta(0) = 2\phi_\beta \neq 0.\end{aligned}$$

Noting that $d_{e_1}^1$ is orthogonal to L_0, by (11) we have that $\lambda_1 = 0$, and hence

$$d_{e_1}^0 = d_{e_1}^1 \perp L_0.$$

So $\langle d_{e_1}^0, e_\alpha\rangle = 0 = \langle d_{e_1}^0, e_\beta\rangle$. On the other hand,

$$\langle d_{e_1}^0, e_\alpha\rangle = \alpha e_1(0,\alpha)e_0(\alpha,\alpha) - \alpha\phi_0(\alpha)e_1(\alpha,\alpha) = \alpha e_1(0,\alpha)e_0(\alpha,\alpha)$$

and

$$\langle d_{e_1}^0, e_\beta\rangle = \beta e_1(0,\beta)e_0(\beta,\beta) - \beta\phi_0(\beta)e_1(\beta,\beta) = \beta e_1(0,\beta)e_0(\beta,\beta).$$

Consequently,

$$e_1(0,\alpha) = e_1(0,\beta) = 0. \qquad (12)$$

Observe that e_0, e_1, and 1 are linearly independent. If this is not so, then $1 = ae_0 + be_1$ for some numbers *a*, *b*. But $e_1(0,\alpha) = 0$ and $e_0(0,\alpha) = 0$. This forces that $1 = 0$ and leads to a contradiction.
By Theorem (5.1.1), we can take an element $e \in M_2 \cap L_0$ such that $d_e^1 = d_e^0 + e_2 + \mu e_0$ with $e_0 = 0$ and $e_2 \in M_2 \cap L_0$. Thus we have that e_2 is orthogonal to 1 and so e_2 is in $\{1, e_0, e_1\}^{\perp}$ and $\{1, e_0, e_1, e_2\}$ form a basis for L_0. Moreover for any $f \in M_2 \cap L_0$,

$$d_f^1 = d_f^0 + g + \lambda e_0$$

for some number λ and $g \in M_2 \cap L_0$. If g does not equal 0, then g is orthogonal to 1. Thus g is in $\{1, e_0, e_1\}^{\perp}$ and hence $g = ce_2$ for some number c. Therefore taking a nonzero element $e_3 \in M_2 \cap L_0$ which is orthogonal to e_2, we have

$$d^1_{e_2} = d^0_{e_2} + \mu_2 e_2 + \lambda_2 e_0, d^1_{e_3} = d^0_{e_3} + \mu_2 e_3 + \lambda_3 e_0,$$

and $\{e_0, e_1, e_2, e_3\}$ is an orthogonal basis for L_0.

If $\mu_2 = 0$, then by the same reason as before we get

$$\lambda_2 = 0, d^0_{e_2} = d^1_{e_2} \perp L_0 e_2(0,\alpha) = e_2(0,\beta) = 0.$$

So using $p_1 \in L_0 = span\{1, e_0, e_1, e_2\}$, we have $\alpha = p_1(0,\alpha) = p_1(0,\beta) = \beta$, which contradicts our assumption that $\alpha \neq \beta$. Hence $\mu_2 \neq 0$.

Observe that 1 is in $L_0 = span\{e_0, e_1, e_2, e_3\}$ and orthogonal to both e_0 and e_2. Thus $1 = c_1 e_1 + c_3 e_3$ for some numbers c_1 and e_3. So

$$1 = c_1 e_1(0,\alpha) + c_3 e_3(0,\alpha) = c_1 e_1(0,\beta) + c_3 e_3(0,\beta).$$

By (12), we have $1 = c_3 e_3(0,\alpha) = c_3 e_3(0,\beta)$, to obtain that $c_3 \neq 0$ and

$$c_3(0,\alpha) = e_3(0,\beta) = 1/c_3.$$

If $\mu_3 = 0$, then by the same reason as before we get $e_3(0,\alpha) = e_3(0,\beta) = 0$. Hence $\mu_3 \neq 0$.

Now by the linearality of $d^1_{(.)}$ and $d^0_{(.)}$ we have

$$d^1_{\mu_3 e_2 - \mu_2 e_3} = d^0_{\mu_3 e_2 - \mu_2 e_3} + (\mu_3\lambda_2 - \mu_2\lambda_3)e_0.$$

By the same reason as before we get$\mu_3\lambda_2 - \mu_2\lambda_3 = 0$ and$d^0_{\mu_3 e_2 - \mu_2 e_3} = d^1_{\mu_3 e_2 - \mu_2 e_3} \perp L_0$ and therefore

$$\mu_3 e_2(0,\alpha) - \mu_2 e_3(0,\alpha) = \mu_3 e_2(0,\beta) - \mu_2 e_3(0,\beta) = 0.$$

So we get $e_2(0,\alpha) = \mu_2/\mu_3 c_3 = e_2(0,\beta)$. Hence $p_1 \in L_0 = span\{1, e_0, e_1, e_2\}$. This implies that $\alpha = p_1(0,\alpha) = p_1(0,\beta) = \beta$, which again contradicts our as-sumption that $\alpha \neq \beta$.

Another case is that μ_1 is not equal to 0. In this case, (10) can be rewritten as

$$e_1 = \frac{1}{\mu_1} d^1_{e_1} - \frac{1}{\mu_1} d^0_{e_1} - \frac{\lambda_1}{\mu_1} e_0,$$

and we have that e_1 is orthogonal to 1 since $d^1_{e_1}$, $d^0_{e_1}$, and e_0 are orthogonal to 1. Thus 1 is in $M_2 \cap L_0$.

By Theorem (5.1.1), there is an element $e \in M_2 \cap L_0$ and a number λ_0 such that

$$d^1_1 = d^0_1 + e + \lambda_0 e_0. \tag{13}$$

If $e = 0$, then $\lambda = 0$, and hence $d^0_1 \perp L_0$and $1 = 1(0,\alpha) = 1(0,\beta)$. So $e \neq 0$.

Since d^1_1 is in $L_0^{\perp}$, $d^1_1$is orthogonal to 1. Noting that $d^0_1$and e_0 are orthogonal to 1,we have that $e \perp 1$. Hence we get an orthogonal basis $\{e_0, e_1, 1, e\}$ of L_0.

Claim: $e(0,\alpha) - e(0,\beta) = 0$.

Proof. Using Theorem (5.1.1) again, we have that $d^1_e = d^0_e + g + \lambda e_0$ for some $g \in L_0 \cap M_2$.If $g \neq 0$, we have that $g \perp 1$, since d^1_e, d^0_e, and $e_0$are orthogonal to 1. Thus we havethat $g = \mu e$ for some number μ to obtain $d^1_e = d^0_e + \mu e + \lambda e_0$.

Furthermore by the linearality of $d^1_{(.)}$ and $d^0_{(.)}$we have that

$$d^1_{e-\mu 1} = d^0_{e-\mu 1} + (\lambda - \mu\lambda_0)e_0.$$

By the same reason (namely $d^1_{e-\mu 1} \perp L_0, d^0_{e-\mu 1} \perp 1$ and $\langle e_0, 1\rangle \neq 0$) we have that

$$\lambda - \mu\lambda_0 = 0, \quad d^0_{e-\mu 1} = d^1_{e-\mu 1} \perp L_0$$
$$(e - \mu 1)(0,\alpha) = (e - \mu 1)(0,\beta) = 0.$$

Hence we have $e(0,\alpha) - e(0,\beta) = \mu - \mu = 0$ to complete the proof of the claim.

Let us find the value of λ_0 in (13) which will be used to make the coefficients symmetric with respect to α and β. To do this, we first state a technical lemma which will be used in several other places.

Lemma (5.1.9) [181]: *If g is in $H^2(\mathbb{T})$, then* $w\langle g\phi_0', \phi_0\rangle = g(0) + g(\alpha) + g(\beta)$.

Proof. Since ϕ_0 equals $z\phi_\alpha\phi_\beta$, simple calculations give

$$\begin{aligned}\langle wg\phi_0', \phi_0\rangle &= \langle wg(w\phi_\alpha\phi_\beta)', w\phi_\alpha\phi_\beta\rangle = \langle g(w\phi_\alpha\phi_\beta)', \phi_\alpha\phi_\beta\rangle \\ &= \langle g(\phi_\alpha\phi_\beta + w\phi_\alpha'\phi_\beta + w\phi_\alpha\phi_\beta'), \phi_\alpha\phi_\beta\rangle \\ &= \langle g, 1\rangle + \langle wg\phi_\alpha', \phi_\alpha\rangle + \langle wg\phi_\beta', \phi_\beta\rangle \\ &= g(0) + \langle wg\phi_\alpha', \phi_\alpha\rangle + \langle wg\phi_\beta', \phi_\beta\rangle .\end{aligned}$$

Writing ϕ_α as

$$\phi_\alpha = -\frac{1}{\bar\alpha} + \frac{\frac{1}{\bar\alpha} - \alpha}{1 - \bar\alpha w} = -\frac{1}{\bar\alpha} + \frac{1-|\alpha|^2}{\bar\alpha} K_\alpha(w).$$

we have

$$\langle wg\phi_\alpha', \phi_\alpha\rangle = \frac{1-|\alpha|^2}{\alpha}(wg\phi_\alpha')(\alpha) = g(\alpha).$$

The first equality follows from $\langle wg\phi_\alpha', 1\rangle$ equals 0 and the second equality followsfrom

$$\phi_\alpha'(\alpha) = \frac{1}{1-|\alpha|^2}.$$

By the symmetry of α and β, similar computations lead to $\langle wg\phi_\beta', \phi_\beta\rangle = g(\beta)$.

We state the values of λ_0 and $\langle e_0, e_0\rangle$ as a lemma.

Lemma (5.1.10) [181]*:*

$$\lambda_0 = -\frac{\alpha+\beta}{4}, \quad \langle e_0, e_0\rangle = 4,$$

Proof. Since d_1^1 is orthogonal to L_0, e_0 is in L_0, and *e* is orthogonal to e_0, (13) gives

$$0 = \langle d_1^1, e_0\rangle = \langle d_1^0 + e + \lambda_0 e_0, e_0\rangle = \langle d_1^0, e_0\rangle + \lambda_0\langle e_0, e_0\rangle.$$

We need to compute $\langle d_1^0, e_0\rangle$ and $\langle e_0, e_0\rangle$, respectively.

$$\begin{aligned}\langle d_1^0, e_0\rangle &= \langle -\phi(w) + we_0, e_0\rangle = \langle we_0, e_0\rangle = \langle we_0(w,w), e_0(0,w)\rangle = \langle w(w\phi_0' + \phi_0), \phi_0\rangle \\ &= \langle w^2\phi_0', \phi_0\rangle + \langle w\phi_0, \phi_0\rangle = \langle w^2\phi_0', \phi_0\rangle = \alpha + \beta.\end{aligned}$$

The last equality follows from Lemma (5.1.9) with $g = w$.

$$\begin{aligned}\langle e_0, e_0\rangle &= \langle e_0(w,w), e_0(0,w)\rangle = \langle w\phi_0' + \phi_0, \phi_0\rangle = \langle w\phi_0', \phi_0\rangle + \langle \phi_0, \phi_0\rangle \\ &= \langle w\phi_0', \phi_0\rangle + 1 = 4,\end{aligned}$$

where the last equality follows from Lemma (5.1.9) with $g = 1$. Hence $\alpha + \beta + 4\lambda_0 = 0$ and $\lambda_0 = \frac{-\alpha+\beta}{4}$.

Continuing with the proof of Theorem (5.1.8), let P_{L_0} denote the projection of $H^2(\mathbb{T}^2)$ onto L_0. The element $P_{L_0}(k_\alpha(w) - k_\beta(w))$ has the property that for any $g \in L_0$,

$$\langle g, P_{L_0}(k_\alpha(w) - k_\beta(w))\rangle = \langle g, k_\alpha(w) - k_\beta(w)\rangle = g(0,\alpha) - g(0,\beta).$$

Thus $P_{L_0}(k_\alpha(w) - k_\beta(w))$ is orthogonal to g for $g \in L_0$ with $g(0,\alpha) = g(0,\beta)$. So $P_{L_0}(k_\alpha(w) - k_\beta(w))$ is orthogonal to e_0,1, e. On the other hand,

$$\langle p_1, P_{L_0}(k_\alpha(w) - k_\beta(w))\rangle = \alpha - \beta \neq 0.$$

This gives that the element $P_{L_0}(k_\alpha(w) - k_\beta(w))$ is a nonzero element. Therefore there exists a nonzero number b such that $P_{L_0}(k_\alpha(w) - k_\beta(w)) = be1$. Without loss of generality we assume that $e_1 = P_{L_0}(k_\alpha(w) - k_\beta(w))$.

Observe that

$$p_1\big(\phi(z), \phi(w)\big)e_1 + d_{e_1}^1 \in M_1, \quad p_1(\phi(z), \phi(w)) + d_1^1 \in M_2, \qquad M_1 \perp M_2,$$

to get

$$\langle p_1(\phi(z), \phi(w))e_1 + d_{e_1}^1, p_1(\phi(z), \phi(w)) + d_1^1\rangle = 0.$$

Thus we have

$$\begin{aligned} 0 &= \langle p_1\big(\phi(z), \phi(w)\big)e_1 + d_{e_1}^1, p_1\big(\phi(z), \phi(w)\big) + d_1^1\rangle \\ &= \langle\big(\phi(z), \phi(w)\big)e_1, \phi(z) + \phi(w)\rangle + \langle d_{e_1}^1, d_1^1\rangle \\ &= \langle d_{e_1}^1, d_1^1\rangle . \end{aligned} \tag{14}$$

The second equality follows from $d_{e_1}^1, d_1^1 \in ker\, T_{\phi(z)}^* \cap ker T_{\phi(z)}^*$. The last equality follows from $e_1 \perp 1$ and $e_1, 1 \in ker\, T_{\phi(z)}^* \cap ker\, T_{\phi(z)}^*$. Substituting (13) into equation (14), we have

$$\begin{aligned} 0 &= \langle d_{e_1}^1, d_1^0 + e + \lambda_0 e_0\rangle = \langle d_{e_1}^1, d_1^0\rangle = \langle d_{e_1}^1, -\phi(w) + we_0\rangle = \langle d_{e_1}^1, we_0\rangle \\ &= \langle d_{e_1}^0 + \mu_1 e_1 + \lambda_1 e_0, we_0\rangle = \langle d_{e_1}^0, we_0\rangle + \mu_1\langle e_1, we_0\rangle + \lambda_1\langle e_0, we_0\rangle. \end{aligned}$$

The second equation comes from the fact that $d_{e_1}^1$ is orthogonal to L_0 and both e and e_0 are in L_0. The third equation follows from the definition of d_1^0 and the forth equation follows from the fact that $d_{e_1}^1$ is in ker $T_{\phi(z)}^* \cap ker\, T_{\phi(w)}^*$. We need to calculate $\langle d_{e_1}^0, we_0, e_1, we_0\rangle$, and $\langle e_0, we_0\rangle$ separately.

To get $\langle d_{e_1}^0, we_0\rangle$, by the definition of $d_{e_1}^0$we have

$$\begin{aligned} \langle d_{e_1}^0, we_0\rangle &= \langle -\phi(w)e_1 + we_1(0,w)e_0, we_0\rangle \\ &= \langle -\phi(w)e_1, we_0\rangle + \langle we_1(0,w)e_0, we_0\rangle. \end{aligned}$$

Thus we need to compute $\langle -\phi(w)e_1, we_0\rangle$ and $\langle we_1(0,w)e_0, we_0\rangle$ one by one. The equality$\langle -\phi(w)e_1, we_0\rangle = 0$ follows from the following computations.

$$\begin{aligned} \langle -\phi(w)e_1, we_0\rangle &= \langle -w\phi_0(w)e_1, we_0\rangle = -\langle \phi_0(w)e_1, e_0\rangle \\ &= -\langle \phi_0(w)e_1(w,w), e_0(0,w)\rangle = \langle -\phi_0(w)e_1(w,w), \phi_0(w)\rangle \\ &= -\langle e_1(w,w), 1\rangle = -\langle e_1, 1\rangle = 0. \end{aligned}$$

To get$\langle we_1(0,w)e_0, we_0\rangle$, we continue as follows.

$$\begin{aligned}\langle we_1(0,w)e_0, we_0\rangle &= \langle e_1(0,w)e_0, e_0\rangle = \langle e_1(0,w)e_0(w,w), e_0(0,w)\rangle \\ &= \langle e_1(0,w)e_0(w,w), \phi_0(w)\rangle = \langle e_1(0,w)(\phi_0(w) + w\phi_0'(w)), \phi_0(w)\rangle \\ &= \langle e_1(0,w)\phi_0(w), \phi_0(w)\rangle + \langle e_1(0,w)w\phi_0'(w), \phi_0(w)\rangle \\ &= \langle e_1(0,w),1\rangle + \langle e_1(0,w)w\phi_0'(w), \phi_0(w)\rangle \\ &= e_1(0,0) + \langle e_1(0,w)w\phi_0'(w), \phi_0(w)\rangle \\ &= \langle e_1, 1\rangle + \langle e_1(0,w)w\phi_0'(w), \phi_0(w)\rangle = \langle e_1(0,w)w\varphi 0'(w), \phi_0(w)\rangle \\ &= e_1(0,\alpha) + e_1(0,\beta).\end{aligned}$$

The last equality follows from Lemma (5.1.9) and $e_1(0,0) = \langle e_1, 1\rangle = 0$. Hence

$$\langle d^0_{e_1}, we_0\rangle = e_1(0,\alpha) + e_1(0,\beta).$$

Recall that $d_1^1 = d_1^0 + e + \lambda_0 e_0$ is orthogonal to L_0 and e_1 is orthogonal to both e and e_0. Thus

$$0 = \langle e_1, d_1^0 + e + \lambda_0 e_0\rangle = \langle e_1, -\phi(w) + we_0\rangle = \langle e_1, we_0\rangle.$$

From the computation of d_1^0, e_0 in the proof of Lemma (5.1.10) we have showed that $\langle we_0, e_0\rangle = \alpha + \beta$. Therefore we have that

$$e_1(0,\alpha) + e_1(0,\beta) + \lambda_1(\bar{\alpha} + \bar{\beta}) = 0. \qquad (15)$$

On the other hand,

$$0 = \langle d^1_{e_1}, e_0\rangle = d^0_{e_1} + \mu_1 e_1 + \lambda_1 e_0, e_0 = \langle d^0_{e_1}, e_0\rangle + 4\lambda_1$$

and

$$\begin{aligned}\langle d^0_{e_1}, e_0\rangle &= \langle -\phi(w)e_1 + we_1(0,w)e_0, e_0\rangle = \langle we_1(0,w)e_0, e_0\rangle \\ &= \langle we_1(0,w)e_0(w,w), e_0(0,w)\rangle \\ &= \langle we_1(0,w)(\phi_0(w) + w\phi_0'), \phi_0(w)\rangle = \langle w^2 e_1(0,w)\phi_0', \phi_0(w)\rangle \\ &= \alpha e_1(0,\alpha) + \beta e_1(0,\beta).\end{aligned}$$

The last equality follows from Lemma (5.1.9) with $g = we_1(0,w)$. Thus

$$\alpha e_1(0,\alpha) + \beta e_1(0,\beta) + 4\lambda_1 = 0.$$

So

$$\lambda_1 = -\frac{\alpha}{4}e_1(0,\alpha) - \frac{\beta}{4}e_1(0,\beta). \qquad (16)$$

Substituting (16) into (15), we have

$$\left[1 - \frac{\alpha(\bar{\alpha} + \bar{\beta})}{4}\right] e_1(0,\alpha) + \left[1 - \frac{\beta(\bar{\alpha} + \bar{\beta})}{4}\right] e_1(0,\beta) = 0.$$

Recall that $\lambda_0 = -\frac{\alpha+\beta}{4}$, to get

$$(1 + \bar{\lambda}_0\alpha)e_1(0,\alpha) + (1 + \bar{\lambda}_0\beta)e_1(0,\beta) = 0. \qquad (17)$$

We are going to draw another equation about $e_1(0,\alpha)$ and $e_1(0,\beta)$ from the property that $d^1_{e_1}$ is orthogonal to L_0. To do this, recall that

$$e_1 = P_{L_0}(k_\alpha(w) - k_\beta(w)) \in M_1 \cap L_0,$$
$$d^1_{e_1} = d^0_{e_1} + \mu_1 e_1 + \lambda_1 e_0 \perp L_0,$$
$$L_0 = span\{1, p_1, e_\alpha, e_\beta\},$$

$$e_\alpha = k_\alpha(z)k_\alpha(w), e_\beta = k_\beta(z)k_\beta(w).$$

Thus $d_{e_1}^1$ is orthogonal to p_1, e_α and e_β .

Since $d_{e_1}^1$ is orthogonal to p_1 we have $\langle d_{e_1}^0, p_1\rangle + \mu_1\langle e_1, p_1\rangle + \lambda_1\langle e_0, p_1\rangle = 0$.

Noting

$$\langle d_{e_1}^0, p_1\rangle = \langle -\phi(w)e_1 + we_1(0,w)e_0, p1\rangle = \langle we_1(0,w)e_0\, p_1\rangle$$
$$= \langle\, we_1(0,w)e_0(w,w), w\rangle = \langle e_1(0,w)e_0(w,w), 1\rangle = 0,$$
$$\langle e_1, p_1\rangle = \langle P_{L_0}(k_\alpha(w) - k_\beta(w)), p_1\rangle = \langle k_\alpha(w) - k_\beta(w), p_1\rangle = \bar{\alpha} - \bar{\beta},$$
$$\langle e_0, p_1\rangle = \langle e_0(0,w), p_1(w,w)\rangle = \langle \phi_0(w), 2w\rangle = \langle w\phi_\alpha\phi_\beta\,, 2w\rangle = 2\,\langle \phi_\alpha\phi_\beta\,, 1\rangle$$
$$= 2\phi_\alpha(0)\phi_\beta(0) = 2\alpha\beta,$$

we have $(\bar{\alpha} - \bar{\beta})\mu_1 + 2\alpha\beta\lambda_1 = 0$, to obtain

$$\lambda_1 = -\mu_1\frac{\bar{\alpha} - \bar{\beta}}{2\alpha\beta}.$$

Since $d_{e_1}^1 \perp e_\alpha$, we have $\langle d_{e_1}^0, e_\alpha\rangle + \mu_1\langle e_1, e_\alpha\rangle + \lambda_1\langle e_0\,, e_\alpha\rangle = 0$, to get

$$d\langle d_{e_1}^0, e_\alpha\rangle + \mu_1\langle e_1, e_\alpha\rangle - \mu_1\frac{\bar{\alpha} - \bar{\beta}}{2\alpha\beta}, \langle e_0\,, e_\alpha\rangle = 0. \qquad (18)$$

We need to calculate $\langle d_{e_1}^0, e_\alpha\rangle, \langle e_1, e_\alpha\rangle$, and $\langle e_0\,, e_\alpha\rangle$. Simple calculations show that

$$\langle d_{e_1}^0, e_\alpha\rangle = -\phi(w)e_1 + we_1(0,w)e_0, e_\alpha = \langle we_1(0,w)e_0, e_\alpha\rangle$$
$$= \alpha e_1(0,\alpha)e_0(\alpha,\alpha), \qquad (19)$$
$$\langle e_1, e_\alpha\rangle = e_1(\alpha,\alpha) = \langle P_{L_0}\Big(k_\alpha(w) - k_\beta(w)\Big), e_\alpha\rangle = \frac{1}{1-|\alpha|^2} - \frac{1}{1-\alpha\bar{\beta}}$$
$$= \frac{\alpha(\bar{\alpha}-\bar{\beta})}{(1-|\alpha|^2)(1-\alpha\bar{\beta})}, \qquad (20)$$
$$\langle e_0, e_\alpha\rangle = e_0(\alpha,\alpha) = \alpha\phi_0'(\alpha) + \phi_0(\alpha) = \alpha^2\frac{1}{1-|\alpha|^2}\frac{\alpha - \beta}{1-\alpha\bar{\beta}}\,. \qquad (21)$$

Thus (20) and (21) give

$$\frac{e_1(\alpha,\alpha)}{e_0(\alpha,\alpha)} = \frac{\bar{\alpha} - \bar{\beta}}{\alpha(\alpha - \beta)}.$$

Substituting the above equality in equation (18) leads to

$$\alpha e_1(0,\alpha)e_0(\alpha,\alpha) + \mu_1 e_1(\alpha,\alpha) - \mu_1\frac{\bar{\alpha} - \bar{\beta}}{2\alpha\beta}e_0(\alpha,\alpha) = 0.$$

Dividing both sides of the above equality by $e_0(\alpha,\alpha)$ gives

$$\alpha e_1(0,\alpha) + \mu_1\frac{e_1(\alpha,\alpha)}{e_0(\alpha,\alpha)} - \mu_1\frac{\bar{\alpha} - \bar{\beta}}{2\alpha\beta} = 0.$$

Hence we have

$$\alpha e_1(0,\alpha) + \mu_1\frac{\bar{\alpha} - \bar{\beta}}{\alpha(\alpha - \beta)} - \mu_1\frac{\bar{\alpha} - \bar{\beta}}{2\alpha\beta} = 0,$$

to obtain

$$\alpha e_1(0,\alpha) + (\beta + \lambda_0)\frac{2\mu_1(\bar{\alpha}-\bar{\beta})}{\alpha\beta(\alpha-\beta)} = 0. \qquad (22)$$

Similarly, since $d_{e_1}^1$ is orthogonal to e_β , we have

$$\langle d_{e_1}^0, e_\beta\rangle + \mu_1\langle e_1, e_\beta\rangle + \lambda_0 \langle e_0, e_\beta\rangle = 0,$$

to obtain

$$\langle d_{e_1}^0, e_\beta\rangle + \mu_1\langle e_1, e_\beta\rangle - \mu_1 \frac{\bar{\alpha}-\bar{\beta}}{2\alpha\beta}\langle e_0, e_\beta\rangle = 0. \qquad (23)$$

We need to calculate $\langle d_{e_1}^0, e_\beta\rangle$, $\langle e_1, e_\beta\rangle$, and $\langle e_0, e_\beta\rangle$. Simple calculations as above show that

$$\langle d_{e_1}^0, e_\beta\rangle = -\phi(w)e_1 + we_1(0,w)e_0, e_\beta = we_1(0,w)e_0, e_\beta$$
$$= \beta e_1(0,\beta)e_0(\beta,\beta), \quad (24)$$

$$\langle e_1, e_\beta\rangle = e_1(\beta,\beta) = \langle P_{L_0}(k_\alpha(w) - k_\beta(w)), e_\beta\rangle = k\alpha(w) - k\beta(w), e\beta$$
$$= \frac{1}{1-\bar{\alpha}\beta} - \frac{1}{1-|\beta|^2}$$
$$= \frac{\beta(\bar{\alpha}-\bar{\beta})}{(1-\bar{\alpha}\beta)(1-|\beta|^2)}, \qquad (25)$$

$$\langle e_0, e_\beta\rangle = e_0(\beta,\beta) = \beta\phi_0'(\beta) + \phi_0(\beta) = \beta^2\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\frac{1}{1-|\beta|^2} \qquad (26)$$

Combining (25) with (26) gives

$$\frac{e_1(\beta,\beta)}{e_0(\beta,\beta)} = -\frac{\bar{\alpha}-\bar{\beta}}{\beta(\alpha-\beta)}$$
$$\beta e_1(0,\beta)e_0(\beta,\beta) + \mu_1 e_1(\beta,\beta) - \mu_1 2\alpha\beta e_0(\beta,\beta) = 0.$$

Dividing both sides of the above equality by $e_0(\beta,\beta)$ gives

$$\beta e_1(0,\beta) + \mu_1\frac{e_1(\beta,\beta)}{e_0(\beta,\beta)} - \mu_1\frac{\bar{\alpha}-\bar{\beta}}{2\alpha\beta} = 0$$

Hence we have

$$\beta e_1(0,\beta) - \mu_1\frac{\bar{\alpha}-\bar{\beta}}{\beta(\alpha-\beta)} - \mu_1\frac{\bar{\alpha}-\bar{\beta}}{2\alpha\beta} = 0,$$

to get

$$\beta e_1(0,\beta) - (\alpha+\lambda_0)\frac{2\mu_1(\bar{\alpha}-\bar{\beta})}{\alpha\beta(\alpha-\beta)} = 0. \qquad (27)$$

Eliminating $\frac{2\mu_1(\bar{\alpha}-\bar{\beta})}{\alpha\beta(\alpha-\beta)}$ from (22) and (27) gives

$$\alpha(\alpha+\lambda_0)e_1(0,\alpha) + \beta(\beta+\lambda_0)e_1(0,\beta) = 0. \qquad (28)$$

Now combining (27) and (28), we have the following linear system of equations about $e_1(0,\alpha)$and $e_1(0,\beta)$

$$\begin{aligned}(1+\bar{\lambda}_0\alpha)e_1(0,\alpha) + (1+\bar{\lambda}_0\beta)e_1(0,\beta) &= 0\\ \alpha(\alpha+\lambda_0)e_1(0,\alpha) + \beta(\beta+\lambda_0)e_1(0,\beta) &= 0.\end{aligned} \qquad (29)$$

If $e_1(0,\alpha) = e_1(0,\beta) = 0$, then p_1 is in $L_0 = span\{e_0, e_1, 1, e\}$. But noting

$$e_0(0,\alpha) = e_0(0,\beta) \quad and \quad e(0,\alpha) = e(0,\beta)$$

we have $p_1(0,\alpha) = p_1\,(0,\beta)$, which contradicts the assumption that $\alpha = \beta$. So at least one of $e_1(0,\beta)$ and $e_1(0,\beta)$ is nonzero. Then the determinant of the coefficient matrix of system (29) must be zero. This implies

$$\begin{vmatrix} 1+\bar{\lambda}_0\alpha & 1+\bar{\lambda}_0\beta \\ \alpha(\alpha+\lambda_0) & \beta(\beta+\lambda_0) \end{vmatrix} = 0$$

Making elementary row reductions on the above the determinant, we get

$$\begin{vmatrix} (\alpha-\beta)\bar{\lambda}_0 & 1+\bar{\lambda}_0\beta \\ \alpha-\beta)(\alpha+\beta+\lambda_0) & \beta(\beta+\lambda_0) \end{vmatrix} = 0.$$

Since $\alpha+\beta = -4\lambda_0$ and $\alpha-\beta \neq 0$, we have

$$\begin{vmatrix} \bar{\lambda}_0 & 1+\bar{\lambda}_0\beta \\ -3\lambda_0 & \beta(\beta+\lambda_0) \end{vmatrix} = 0.$$

Expanding this determinant we have

$$0 = \bar{\lambda}_0(\beta^2+\beta\lambda_0) + 3\bar{\lambda}_0(1+\bar{\lambda}_0\beta) = \bar{\lambda}_0(\beta^2+\beta\lambda_0+3\beta\lambda_0) + 3\lambda_0$$
$$= \bar{\lambda}_0(\beta^2+4\beta\lambda_0) + 3\lambda_0 = \bar{\lambda}_0(-\alpha\beta) + 3\lambda_0.$$

Taking absolute value on both sides of the above equation, we have

$$0 = |\bar{\lambda}_0(-\alpha\beta) + 3\lambda_0| \geq |\lambda_0|(3-|\alpha\beta|) \geq 2|\lambda_0|,$$

to get $\lambda_0 = 0$. This implies $\alpha+\beta = 0$, to complete the proof of Theorem (5.1.8)

Section (5.2): Geometric Invariants Associated with their Symbol Curves

For $\mathbb{D}$ denotes the unit disk in the complex plane $\mathbb{C}$, and $\mathbb{T}$ denotes the boundary of $\mathbb{D}$. Let $\mathfrak{M}(\overline{\mathbb{D}})$ consist of all meromorphic functions over $\mathbb{C}$ which have no pole on the closed unit disk $\overline{\mathbb{D}}$, and $\mathfrak{R}(\overline{\mathbb{D}})$ denotes the set of all rational functions without poles on $\overline{\mathbb{D}}$. It is clear that $\mathfrak{M}(\overline{\mathbb{D}}) \supseteq \mathfrak{R}(\overline{\mathbb{D}})$. Let $H^\infty(\mathbb{D})$ denote the Banach algebra of all bounded holomorphic functions over $\mathbb{D}$, and $H^\infty(\overline{\mathbb{D}})$, as a subset of $H^\infty(\mathbb{D})$, consists of functions that are holomorphic on $\overline{\mathbb{D}}$. The Hardy space $H^2(\mathbb{D})$ consists of all holomorphic functions on $\mathbb{D}$ whose Taylor coefficients at 0 are square summable. The Bergman space $L_a^2(\mathbb{D})$ consists of all holomorphic functions over $\mathbb{D}$ that are square integrable with respect to the normalized area measure over $\mathbb{D}$. For each function ϕ in $H^\infty(\mathbb{D})$, let T_ϕ denote the Toeplitz operator on the Hardy space $H^2(\mathbb{D})$ or the Bergman space $L_a^2(\mathbb{D})$ according to the context.

Let H be a Hilbert subspace. For an operator T in $B(H)$, $\{T\}'$ denotes the commutant of T; that is,

$$\{T\}' = \{S \in B(H) : ST = TS\},$$

which is a WOT-closed subalgebra of $B(H)$. The operator T is called totally Abelian if $\{T\}'$ is Abelian; equivalently, $\{T\}'$ is a maximal Abelian subalgebra of $B(H)$ [205]. Berkson and Rubel [204] completely characterized totally Abelian operators in $B(H)$ in the case of $\dim H < \infty$, and in this case they proved that T is totally Abelian if and only if T has a cyclic vector. In the case of $dim\ H = \infty$ and H being separable, they also characterized when normal operators (including unitary operators) and non-unitary isometric operators are totally Abelian. Related work on analytic Toeplitz operators on $H^2(\mathbb{D})$ are also initiated by Berkson and Rubel [205], where it is shown that if ϕ is an inner function, then T_ϕ is totally Abelian on $H^2(\mathbb{D})$ if and only if there exist a unimodular constant *c* and a point $\lambda \in \mathbb{D}$ such that $\phi(z) = c\frac{\lambda - z}{1-\bar{\lambda}z}$ [206,Theorem 2.1]. Recall that $\{T_z\}' = \{T_h : h \in H^\infty(\mathbb{D})\}$ is maximal Abelian. It follows that an analytic Toeplitz operator T_ϕ is to-tally Abelian if and only if $\{T_\phi\}' = \{T_z\}'$ (this statement also holds on many function spaces, such as weighted Bergman spaces). But in general, it is hard to judge when $\{T_\phi\}' = \{T_z\}'$ holds. Thus for a generic symbol $\phi \in H^\infty(\mathbb{D})$, it is beyond reach to give a complete characterization for the totally Abelian property of T_ϕ. This leads us to consider the commutants for analytic Toeplitz operators defined on the Hardy space $H^2(\mathbb{D})$. In [207] Deddens and Wong raised several questions on this topic. One of them asks whether for each function $\phi \in H^\infty(\mathbb{D})$, there is an inner function ψ such that $\{T_\phi\}' = \{T_\psi\}'$ and that $\phi = h \circ \psi$ for some $h \in H^\infty(\mathbb{D})$. Baker, Deddens and Ullman [208] proved that for an entire function ϕ, there is a positive integer k such that $\{T_\phi\}' = \{T_{z^k}\}'$ and $\phi = h(z^k)$ for some entire function *h*.

For a function ϕ in $H^\infty(\mathbb{D})$, if there exists a point λ in $\mathbb{D}$ such that the inner part of $\phi - \phi(\lambda)$ is a finite Blaschke product, then ϕ is said to be in Cowen-Thomson's class, denoted by $\phi \in \mathcal{CT}(\mathbb{D})$. It is known that $\mathcal{CT}(\mathbb{D})$ contains all nonconstant functions in $H^\infty(\bar{\mathbb{D}})$. Below, $\mathcal{H}$ denotes the Hardy space $H^2(\mathbb{D})$ or the Bergman space $L_a^2(\mathbb{D})$. As presented below is the remarkable theorem on commutants for analytic Toeplitz operators, due to Thomson and Cowen [205]; also see [206, Chapter 3] for a detailed discussion and see [207].

Theorem (5.2.1)[203]: [Cowen-Thomson] Suppose $\phi \in \mathcal{CT}(\mathbb{D})$. Then there exists a finite Blaschke product B and an H^∞-function ψ such that $\phi = \psi(B)$ and $\{T_\phi\}' = \{T_B\}'$ *holds on* $\mathcal{H}$.

The identity $\phi = \psi(B)$ in Theorem (5.2.1) is called *the Cowen-Thomson representation* of ϕ. Note that this B is of maximal order in the following sense: if there is another finite Blaschke product $\tilde{B}$ and a function $\tilde{\psi}$ in $H^\infty(\mathbb{D})$ satisfying $\phi = \tilde{\psi}(\tilde{B})$, then

$$order\, B \geq order\, \tilde{B}.$$

One defines a quantity $b(\phi)$ to be the maximum of orders of B, for which there is a function ψ in $H^\infty(\mathbb{D})$ such that $\phi = \psi(B)$, and $b(\phi)$ is called *the Cowen-Thomson order* of ϕ. Thus for the finite Blaschke product B in Theorem (5.2.1) we have order $B = b(\phi)$. Once ϕ is fixed, it is not difficult to show that B is uniquely determined in the following sense. If there is another finite Blaschke product B_0 satisfying one of the following:

(i) order $B_0 = b(\phi)$ and there is an $h \in H^\infty$ such that $\phi = h(B_0)$;

(ii) $\{T_\phi\}' = \{T_{B_0}\}'$,

then there is a Moebius map η such that $B_0 = \eta(B)$. This means that *the Cowen-Thomson representation* of ϕ is unique in the sense of modulo Moebius maps.
For a bounded holomorphic function ϕ over $\mathbb{D}$, as we mention $T_\phi$we always assume that T_ϕ is defined on $\mathcal{H}$, the Hardy space $H^2(\mathbb{D})$ or the Bergman space $L_a^2(\mathbb{D})$. The following is an immediate consequence of Theorem (5.2.1), see [208].
Corollary (5.2.2) [203]: *Let ϕ be a nonconstant function in $H^\infty(\overline{\mathbb{D}})$. Then there exist a finite Blaschke product B and a function ψ in $H^\infty(\overline{\mathbb{D}})$ such that $\phi = \psi(B)$ and $\{T_\phi\}' = \{T_B\}'$ holds. If ϕ is entire, then ψ is entire and $B(z) = z^n$ for some positive integer n.*
Suppose ϕ belongs to Cowen-Thomson's class $\mathcal{CT}(\mathbb{D})$. Then T_ϕ is totally Abelian if and only if $b(\phi) = 1$. When ϕ is an entire function, expanding ϕ's Taylor series yields

$$\phi(z) = \sum_{n=0}^{\infty} a_n z^n .$$

Set $N = gcd\{n : a_n \neq 0\}$, and then by Corollary (5.2.2) $\{T_\phi\}' = \{T_{z^N}\}'$. In this case, $b(\phi) = N$. Therefore, for a nonconstant entire function ϕ , T_ϕ is totally Abelian if and only if

$$gcd\{n : a_n = 1.$$

Therefore, for the totally Abelian property of analytic Toeplitz operators T_ϕ it is important to determine the Cowen-Thomson order $b(\phi)$ of ϕ, and it is of interest to determine the exact form of the Blaschke product with order $b(\phi)$. As we will see, there are several ways to study $b(\phi)$. The first attack is made by Baker, Deddens and Ullman [209] in the case of ϕ being an entire function. In what follows, for $c \notin \phi(\mathbb{T})$, let wind$(\phi, c)$ denote the winding number of the curve $\phi(z)$ $(z \in \mathbb{T})$ around the point c. Write $n(\phi)$ for the number

$$min\,\{wind(\phi, \phi(a)) : a \in \mathbb{D}, \phi(a) \notin \phi(\mathbb{T})\}.$$

For a function $\phi \in H^\infty(\overline{\mathbb{D}})$, it is obvious that $b(\phi) \le n(\phi)$. If $b(\phi) = n(\phi)$, ϕ is said to *satisfy the Minimal Winding Number property* (the MWN property). It is shown that a nonconstant entire function ϕ enjoys the MWN property [210]. For functions in $H^\infty(\overline{\mathbb{D}})$, the problem raised in [211] can be reformulated as:
if ϕ is a nonconstant function in $H^\infty(\overline{\mathbb{D}})$, then does ϕ have the MWN Property; that is, $b(\phi) = n(\phi)$?
If the answer is yes, for a large class of analytic Toeplitz operators we can formulate their totally Abelian property in terms of winding number.
For those functions ϕ in $H^\infty(\overline{\mathbb{D}})$ satisfying the MWN property, we can determine the exact form of B appearing in Corollary (5.2.2). Let be a point in $\mathbb{D}$ such that $\phi - \phi(a)$ does not vanish on $\mathbb{T}$ and

$$wind(\phi, \phi(a)) = n(\phi) = b(\phi).$$

Denote the inner factor of $\phi - \phi(a)$ by Ba, and we will show that B_a is the desired finite Blaschke product. For this, let

$$\phi = \psi(B)$$

be the Cowen-Thomson representation of ϕ. By Corollary (5.2.2), ψ is in $H^\infty(\mathbb{D})$. Let

$$\psi - \phi(a) = \eta F$$

be the inner-outer factorization of $\psi - \phi(a)$, where η is inner. We see that

$$\phi - \phi(a) = (\psi - \phi(a)) \circ B = \eta \circ B\, F \circ B.$$

Therefore $B_a = c\eta \circ B$, where c is a constant with $|c| = 1$. Since

$$order\ B_a = wind(\phi, \phi(a)) = b(\phi) = order\ B,$$

this forces η to be a Blaschke factor of order 1. Therefore this B_a is the desired finite Blaschke product in Corollary (5.2.2). In this way, finding B essentially reduces to finding one of these points a (in general, such points a consist of a nonempty open set). In some cases of interest this procedure is feasible (see Theorem (5.2.5)).

The MWN property is quite restricted. We will provide some examples of functions with good smoothness on the unit circle, and they are in Cowen-Thomson's class $\mathcal{CT}(\mathbb{D})$ but do not have the MWN property, see Examples (5.2.32) and (5.2.33). It is known that entire functions have the MWN property [212]. In Theorem (5.2.5) we extend this result to all nonconstant meromorphic functions in $\mathfrak{M}(\overline{\mathbb{D}})$.

Before continuing, we introduce the finite self-intersection property (the FSI property). For a function ϕ in the disk algebra $A(\mathbb{D})$ and $\eta \in \mathbb{T}$, let $N(\phi - \phi(\eta), \mathbb{T})$ denote the cardinality of the set

$$\{w \in \mathbb{T} : \phi(w) - \phi(\eta) = 0\}.$$

called the multiplicity of self-intersection of the curve $\phi(z)$ $(z \in \mathbb{T})$ at the point $\phi(\eta)$. Write

$$N(\phi) = min\, \{N(\phi - \phi(\eta), \mathbb{T}) : \eta \in \mathbb{T}\},$$

called *the multiplicity of self-intersection of the curve* $\phi(z)$ $(z \in \mathbb{T})$. It is not difficult to verify that $b(\phi) \le N(\phi)$. A function ϕ in $A(\mathbb{D})$ is said to have the FSI *property* if there exists a finite subset E of $\mathbb{T}$ such that each point $\xi \in \mathbb{T} \setminus E$ satisfies $N(\phi - \phi(\xi), \mathbb{T}) = 1$ [212].

For meromorphic functions in $\mathfrak{M}(\overline{\mathbb{D}})$, we have the following result.

Theorem (5.2.3) [203]: Suppose ϕ is a nonconstant function in $\mathfrak{M}(\overline{\mathbb{D}})$. The following are equivalent:

(a) the Toeplitz operator T_ϕ is totally Abelian;

(b) ϕ has the FSI property.

(c) $N(\phi) = 1$.

For a nonconstant function ϕ in $H^\infty(\overline{\mathbb{D}})$, let $\phi = \psi(B)$ be the Cowen-Thomson representation of ϕ. If $\psi \in H^\infty(\overline{\mathbb{D}})$ has the FSI property, then is said to have the FSI-*decomposable property*. Quine [205] showed that each nonconstant polynomial has the FSI-decomposable property. In we prove that each nonconstant function in $\mathfrak{M}(\overline{\mathbb{D}})$ also enjoys the same property (see Theorem (5.2.21)).

—

For the characterization of geometric properties of symbol curves, we introduce the semigroup $\mathrm{G}(\phi)$. Precisely, for each continuous function ϕ on $\mathbb{T}$ define $\mathrm{G}(\phi)$ to be the set of all continuous maps ρ from $\mathbb{T}$ to $\mathbb{T}$ satisfying $\phi(\rho) = \phi$.

For a finite Blaschke product ϕ, $\mathrm{G}(\phi)$ is a finite cyclic group, and furthermore $\#(\mathrm{G}(\phi)) = \text{order } \phi$ [204]. For $\phi \in H^\infty(\overline{\mathbb{D}})$, we have the following result.

Theorem (5.2.4) [203]: *Suppose ϕ is a nonconstant function in $H^\infty(\overline{\mathbb{D}})$. Then $\mathrm{G}(\phi)$ is a finite cyclic group.*

Let $o(\phi)$ denote the order of $G(\phi)$; that is, $o(\phi) = \#G(\phi)$. We thus have four integer quantities for a function ϕ: $o(\phi), b(\phi), n(\phi)$ and $N(\phi)$. We will prove that if ϕ is in $H^\infty(\overline{\mathbb{D}})$, then $b(\phi) \le o(\phi) \le n(\phi)$ and $o(\phi) \le N(\phi)$.

For a finite Blaschke product B, $o(B) = order\ B = n(B) = N(B)$ [204]. More generally, we will prove that each nonconstant meromorphic function in $\mathfrak{M}(\overline{\mathbb{D}})$ enjoys this property.

Theorem (5.2.5) [203]: *Suppose ϕ is a nonconstant function in $\mathfrak{M}(\overline{\mathbb{D}})$. Then*

$$n(\phi) = b(\phi) = o(\phi) = N(\phi).$$

In particular, for $\phi \in \mathfrak{M}(\overline{\mathbb{D}})$ the Toeplitz operator T_ϕ is totally Abelian if and only if $o(\phi) = 1$; equivalently, the identity map is the only continuous map $\rho : \mathbb{T} \to \mathbb{T}$ satisfying $\phi(\rho) = \phi$.

This is arranged as follows. first provides some basic properties of the group $G(\phi)$ for ϕ in $H^\infty(\overline{\mathbb{D}})$ and gives the proof of Theorem (5.2.4). focuses on Toeplitz operators with meromorphic symbols, discusses the MWN property, $o(\phi)$, $N(\phi)$ and the Cowen-Thomson order $b(\phi)$ of ϕ for $\phi \in \mathfrak{M}(\overline{\mathbb{D}})$, and gives the proof of Theorem (5.2.5). first presents the proof of Theorem (5.2.3), and then give further results on FSI and FSI-decomposable properties. constructs some examples. On one hand, we give some totally Abelian Toeplitz operators defined by symbols in $\mathfrak{M}(\overline{\mathbb{D}})$. On the other hand, some examples show that conclusion of Theorem (5.2.3) can fail even if the associated functions have good smoothness on $\mathbb{T}$.

This provides some basic properties of $G(\phi)$.

For a function ϕ holomorphic on the closure of a domain Ω, $N(\phi, \Omega)$ or $N(\phi, \overline{\Omega})$ denotes the number of zeros of ϕ on Ω or $\overline{\Omega}$ respectively, counting multiplicity. The winding number of ϕ is defined to be $\mathrm{wind}(\phi, 0)$.

The following shows that each member in $G(\phi)$ has very strong restriction.

Lemma (5.2.6) [203]: *Suppose ϕ is a nonconstant function in $H^\infty(\overline{\mathbb{D}})$. Then every $\rho \in G(\phi)$ is an automorphism of $\mathbb{T}$ with winding number 1.*

Proof. For each $\rho \in G(\phi)$, define

$$\Lambda = \{t \in [0, 2\pi) : \phi'(e^{it})\phi'(\rho(e^{it})) = 0\}.$$

We will show that Λ is a finite set. In fact, let $\mathcal{Z}'$ denote the zero set of ϕ' on $\overline{\mathbb{D}}$ and put

$$\mathcal{F} = \phi^{-1}(\phi(\mathcal{Z})) \cap \mathbb{T}.$$

Since $\phi \in H^\infty(\overline{\mathbb{D}})$, F is a finite set. If $\phi\ (\rho(e^{it})) = 0$, then $\rho(e^{it}) \in \mathcal{F}$. Therefore,

$$e^{it} \in \bigcup_{\varsigma\in\mathcal{F}} \{z \in \mathbb{T} : \phi(\rho(z)) - \phi(\varsigma) = 0\} = \bigcup_{\varsigma\in\mathcal{F}} \{z \in \mathbb{T} : \phi(z) - \phi(\varsigma) = 0\}.$$

Since ϕ is holomorphic on $\overline{\mathbb{D}}$ and nonconstant, the right hand side is a union of finitely many finite sets. Hence $\{t \in [0, 2\pi) : \phi'(e^{it}) = 0\}$ is a finite set, and so is Λ.

Write $(0, 2\pi) \setminus \Lambda = \bigcup_{k=0}^{n-1}(t_k, t_{k+1})$, where $0 = t_0 < t_1 < \cdots t_n = 2\pi$. Since ρ is continuous, there exists a real continuous function θ on $[0, 2\pi]$ such that $\rho(e^{it}) = e^{i\theta(t)}$. Then

$$\phi(e^{it}) = \phi(e^{i\theta(t)}).$$

The Inverse Function Theorem implies that θ is differentiable on $(0, 2\pi)\setminus\Lambda$. Taking derivatives of *t* yields that

$$\theta'(t) = \frac{e^{it}\phi'(e^{it})}{e^{i\theta(t)}\phi'(e^{i\theta(t)})} \neq 0, t \in (0, 2\pi)\setminus \Lambda. \tag{30}$$

Hence for $0 \le k \le n-1$, θ is strictly monotonic on each interval (t_k, t_{k+1}) Since $\phi(\mathbb{T})$ is of zero area measure and $\phi(\mathbb{D})$ is open, one can pick $\lambda \in \mathbb{D}$ such that $\phi(\lambda) \notin \phi(\mathbb{T})$. By the Argument Principle, we have

$$N(\phi - \phi(\lambda), D) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\phi'(\xi)}{\phi(\xi) - \phi(\lambda)} d\xi = \frac{1}{2\pi}\int_0^{2\pi} \frac{\phi'(e^{it})}{\phi(e^{it}) - \phi(\lambda)} e^{it} dt$$
$$= \frac{1}{2\pi}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \frac{\phi'(e^{it})}{\phi(\rho(e^{it})) - \phi(\lambda)} e^{it} dt \overset{(30)}{=} \frac{1}{2\pi}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \frac{\phi'(e^{i\theta(t)})}{\phi(e^{i\theta(t)}) - \phi(\lambda)} e^{i\theta(t)}\theta'(t) dt$$
$$= \frac{1}{2\pi}\sum_{k=0}^{n-1}\int_{\theta(t_k)}^{\theta(t_{k+1})} \frac{\phi'(e^{i\theta})}{\phi(e^{i\theta}) - \phi(\lambda)} e^{i\theta} d\theta = \frac{1}{2\pi}\int_{\theta(0)}^{\theta(2\pi)} \frac{\phi'(e^{i\theta})}{\phi(e^{i\theta}) - \phi(\lambda)} e^{i\theta} d\theta$$
$$= \frac{\theta(2\pi) - \theta(0)}{4\pi^2 i}\int_{\mathbb{T}} \frac{\phi'(\xi)}{\phi(\xi) - \phi(\lambda)} d\xi = \#\rho \cdot N(\phi - \phi(\lambda), \mathbb{D}),$$

where $\#\rho = wind(\rho, 0)$. Since $N(\phi - \phi(\lambda), \mathbb{D})$ is a positive integer, we have $\#\rho = 1$. This implies that ρ is surjective.

It remains to show that ρ is injective. Otherwise, there exist two points ξ_1 and ξ_2 in $\mathbb{T}$ such that $\rho(\xi_1) = \rho(\xi_2) = \eta$. Let A be the set of zeros of $\phi - \phi(\eta)$ in $\mathbb{T}$, and $\rho|_A : A \to A$ is surjective as ρ is surjective. Since *A* is a finite set, $\rho|_A$ is actually a bijection, which is a contradiction to $\rho(\xi_1) = \rho(\xi_2)$. The proof is complete.

Corollary (5.2.7) [203]: *Suppose that ϕ is a nonconstant function in $H^\infty(\overline{\mathbb{D}})$ and both ρ_1 and ρ_2 belong to $G(\phi)$. If $\rho_1(\xi_0) = \rho_2(\xi_0)$ for some point $\xi_0 \in \mathbb{T}$, then $\rho_1 = \rho_2$.*

Proof. Let λ be an arbitrary point in $\mathbb{T} \setminus \{\xi_0\}$. Let A be the zero set of $\phi - \phi(\lambda)$ in $\mathbb{T}$. Arrange the points of $\{\xi_0\} \cup A$ in the counter-clockwise direction:

$$\xi_0, \xi_1, \ldots, \xi_{n_0}\ (n_0 \ge 1).$$

It is clear that $\rho_1(\{\xi_0\} \cup A) = \rho_2(\{\xi_0\} \cup A)$. By Lemma (5.2.6), both ρ_1 and ρ_2 are automorphisms of $\mathbb{T}$ with winding number $\#\rho_1 = \#\rho_2 = 1$. Thus, when moves

along $\mathbb{T}$ in the positive direction, the images $\rho_1(\xi)$ and $\rho_2(\xi)$ run in the same direction. As ξ goes from ξ_0 to ξ_1, ρ_1 and ρ_2 must coincide at the point ξ_1. By induction, we have $\rho_1(\xi_k) = \rho_2(\xi_k)$, $1 \le k \le n_0$. In particular, ρ1(λ) = ρ2(λ). The proof is finished.
In the case of finite Blaschke products, $G(\phi)$ is a finite cyclic group [4].
In what follows we will prove that this result also is true for functions in $H^\infty(\overline{\mathbb{D}})$.
Now we come to the proof of Theorem (5.2.4) (=Theorem (5.2.8)), which is represented as below.
Theorem (5.2.8): **[203]** *Suppose ϕ is a nonconstant function in $H^\infty(\overline{\mathbb{D}})$. Then $G(\phi)$ is a finite cyclic group.*
Proof. By Lemma (5.2.6), $G(\phi)$ is a group. We will show that $G(\phi)$ is a finite cyclic group. Note that for a fixed point $\zeta \in \mathbb{T}, \{z \in \mathbb{T} : \phi(z) = \phi(\zeta)\}$ is a finite set, and by Corollary (5.2.7) $G(\phi)$ is a finite group.
Let ξ_0 be a point on T, and let $\xi_0, \cdots, \xi_{n_0-1}$ be all zeros of $\phi - \phi(\xi_0)$ on $\mathbb{T}$ in the counter-clockwise direction. Then define $\{\xi_j\}_{j=0}^{\infty}$ to be the infinite sequence

$$\xi_0, \cdots, \xi_{n_0-1};\ \xi_0, \cdots, \xi_{n_0-1}; \cdots.$$

That is, for all *j*

$$\xi_j = \xi_{[j]}, \qquad where\ j \equiv [j]\ (mod\ n_0),$$

where $0 \le [j] \le n_0 - 1$. Let d be the minimal positive integer l for which there is a member ρ_0 in $G(\phi)$ satisfying $\rho_0(\xi_0) = \xi_l$. By Lemma (5.2.6) ρ_0 maps each circular arc $\xi_l\widetilde{\xi_l + 1}$ to $\xi_j\widetilde{\xi_j + 1}$ for some j, preserving the orientation. By continuity, if $\rho_0(\xi_0) = \xi_l$, then one has

$$\rho_0(\xi_i) = \xi_{i+l}, 0 \le i \le n_0 - 1. \tag{31}$$

By definition of *d*, there is a member τ in $G(\phi)$ satisfying $\tau(\xi_0) = \xi_d$. To finish the proof of Theorem (5.2.8), it suffices to show that for each member ρ in $G(\phi)$, there is an integer *m* such that $\rho = \tau^m$ (in the sense of composition). Write $\rho(\xi_0) = \xi_l$, and there are two integers $k \ge 0$ and l_0 such that $0 \le l_0 < d$ and

$$l = kd + l_0.$$

Letting $\sigma = \tau^{-k}\rho$, we have $\sigma \in G(\phi)$, and by (31) $\sigma(\xi_0) = \xi_{l_0}$. By definition of *d* we have $l_0 = 0$. By Corollary (5.2.7) $\sigma = id$, forcing $\rho = \tau^k$ to complete the proof.
For two positive integers *m* and *n*, write $m|n$ to denote that *m* divides *n*. By Theorem (5.2.8), if ϕ is a nonconstant function in $H^\infty(\overline{\mathbb{D}})$, $G(\phi)$ is a finite cyclic group. If $\phi = \psi(B)$ is the Cowen-Thomson representation of ϕ, then $G(B)$ is a subgroup of $G(\phi)$, and hence $o(B)|o(\phi)$. Since $o(B) = b(\phi)$, we have

$$b(\phi) \mid o(\phi).$$

The following gives some properties of the order $o(\phi)$ of $G(\phi)$.
Corollary (5.2.9) [203]: S*uppose ϕ is a nonconstant function in $H^\infty(\overline{\mathbb{D}})$. Then for each $\xi \in \mathbb{T}, o(\phi)\ |N(\phi - \phi(\xi), \mathbb{T})$. Besides, for each $a \in \mathbb{D}$, if $\phi(a) \notin \phi(\mathbb{T})$, then*

$$o(\phi)\ |wind(\phi, \phi(a)).$$

In particular, we have

$$o(\phi)\,|N(\phi)\quad and\quad o(\phi)\,|n(\phi),$$

where $N(\phi) = min\,\{N(\phi - \phi(\xi), \mathbb{T}) : \xi \in \mathbb{T}\}$ and

$$n(\phi) = min\,\{wind(\phi, \phi(a)) : a \in \mathbb{D}, \phi(a) \notin \phi(\mathbb{T})\}.$$

Proof. Let us have a close look at the proof of Theorem (5.2.8). Fix $\xi_0 \in \mathbb{T}$, and let $\xi_0, \cdots, \xi_{n_0-1}$ be all zeros of $\phi - \phi(\xi_0)$ on $\mathbb{T}$ in counter-clockwise direction. Let *d* be the minimal positive integer l so that there is a member ρ in $G(\phi)$ satisfying $\rho(\xi_0) = \xi_l$, and τ denotes the generator in $G(\phi)$ satisfying $\tau(\xi_0) = \xi_d$. Then by Lemma (5.2.6) we have

$$\tau(\xi_i) = \xi_{i+d}, 0 \le i \le n_0 - 1,$$

and $d|_{n_0}$. Write

$$n_0 = jd.$$

For $k > 0, \tau^k(\xi_0) = \xi_{kd}$. By Corollary (5.2.7) we have that τ^k is the identity map if and only if $\xi_{kd} = \xi_0$. Therefore j is the minimal positive number k such that τ^k is the identity map, and then

$$j = o(\phi).$$

Since $n_0 = jd, j|_{n_0}$; that is, $o(\phi) \mid N(\phi - \phi(\xi_0), \mathbb{T})$. The first statement is proved. For $0 \le i \le j-1$ let γ_i denote the positively oriented circular arc $\widetilde{\xi_{id}\xi_{(i+1)d}}$, $(\xi_{id} = \xi_0)$. Then $\tau^i(\gamma_0) = \gamma_i$, and $\phi(\tau^i) = \phi$. Also noting that $\phi(\tau^i)$ are closed curves, we have

$$wind(\phi(\gamma_i), \lambda) = wind(\phi(\gamma_0), \lambda), \lambda \in \mathbb{C}\backslash\phi(\mathbb{T}), 0 \le i \le j-1.$$

Since

$$\mathbb{T} = \bigcup_{i=0}^{j-1} \gamma_i \qquad (as \quad curves),$$

$$wind(\phi(\mathbb{T}), \lambda) = j \cdot wind(\phi(\gamma_0), \lambda), \lambda \in \mathbb{C}\backslash\phi(\mathbb{T}).$$

Thus $o(\phi)|wind(\phi, \lambda)$. In particular, we have

$$o(\phi)\,|wind(\phi, \phi(a))$$

for each $a \in \mathbb{D}$ such that $\phi(a) \notin \phi(\mathbb{T})$. The proof is complete.

Recall that a Jordan curve in $\mathbb{C}$ is the image of a continuous injective map from the unit circle $\mathbb{T}$ into $\mathbb{C}$. For $\phi \in H^\infty(\mathbb{D})$, in the case of $\phi(\mathbb{T})$ being a Jordan curve, we have the following.

Proposition (5.2.10) [203]: *Suppose $\phi \in H^\infty(\mathbb{D})$ and its image on $\mathbb{T}$ is a Jordan curve. Then there is a univalent function h on $\mathbb{D}$ and a finite Blaschke product B satisfying* $\phi = h(B)$. *In this case, we have*

$$n(\phi) = b(\phi) = o(\phi) = N(\phi) = order\ B.$$

Proof. Write $\Gamma = \phi(\mathbb{T})$, the image of $\mathbb{T}$ under ϕ. Then Γ is a Jordan curve. We will prove that $\Gamma = \partial\phi(\mathbb{D})$. For this, note that $\partial\phi(\mathbb{D}) \subseteq \Gamma$. Assume conversely that $\partial\phi(\mathbb{D}) \neq \Gamma$. Since Γ is a Jordan curve, $\mathbb{C}\backslash\,\partial\phi(\mathbb{D})$ is connected. A fact from topology states that a domain Ω in $\mathbb{C}$ is a component of $\mathbb{C} \setminus \partial\Omega$. Letting $\Omega = \phi(\mathbb{D})$,

we have $\Omega = \mathbb{C} \setminus \partial\phi(\mathbb{D})$. However, this can not happen since $\mathbb{C} \setminus \partial\phi(\mathbb{D})$ is not bounded. Therefore, $\Gamma = \partial\phi(\mathbb{D})$.
The Jordan curve Γ divides the complex plane $\mathbb{C}$ to an interior region and an exterior region. By $\Gamma = \partial\phi(\mathbb{D})$, we know that ϕ is a proper map and $\phi(\mathbb{D})$ is the interior region of Γ, a simply connected domain. Let h be a conformal map from $\mathbb{D}$ onto $\phi(\mathbb{D})$, and write $\psi = \mathrm{h}^{-1}(\phi)$. Then ψ is a proper holomorphic map from $\mathbb{D}$ to $\mathbb{D}$. Hence ψ is a finite Blaschke product [207, Theorem 7.3.3]. Also, we have $\phi = \mathrm{h}(\psi)$ as desired.
By Carathéodory's Theorem, a conformal map from $\mathbb{D}$ onto a Jordan domain Ω extends to a continuous bijection from $\overline{\mathbb{D}}$ onto $\overline{\Omega}$. Thus h is bijective on $\mathbb{T}$. Rewrite $\psi = \mathrm{B}$. By $\phi = \mathrm{h(B)}$, we get $\mathrm{o}(\phi) = \mathrm{o(B)} = \text{order B}$, $\mathrm{n}(\phi) = \mathrm{n(B)}$, and $\mathrm{N}(\phi) = \mathrm{N(B)}$. Since $\mathrm{B} : \mathbb{T} \to \mathbb{T}$ is a covering map,

$$o(B) = n(B) = N(B) = order\ B,$$

forcing $o(\phi) = n(\phi) = N(\phi) = order\ B$. Since order $\mathrm{B} \leq \mathrm{b}(\phi) \leq \mathrm{o}(\phi)$, we have $\mathrm{o}(\phi) = \mathrm{n}(\phi) = \mathrm{b}(\phi) = \mathrm{N}(\phi) = \text{order B}$. The proof is finished.
focuses on the class $\mathfrak{M}(\overline{\mathbb{D}})$, consisting of all meromorphic functions on $\mathbb{C}$ whose poles are outside $\mathbb{D}$. In this interesting case, we give the proof of Theorem (5.2.5).
We begin with the notion of analytic continuation [209, Chapter 16]. A function element is an ordered pair (f, D), where D is an open disk and f is holomorphic function on D. Two function elements (f_0, D_0) and (f_1, D_1) are called direct continuations of each other if $D_0 \cap D_1$ is not empty and $f_0 = f_1$ holds on $D_0 \cap D_1$. By a curve, we mean a continuous map from [0, 1] into $\mathbb{C}$. Given a function element (f_0, D_0) and a curve γ with $\gamma(0) \in D_0$, if there is a partition of [0, 1]:

$$0 = s_0 < s_1 < \cdots < s_n = 1$$

and function elements (f_j, D_j) $(0 \leq j \leq n)$ such that

(a) (f_j, D_j) and (f_{j+1}, D_{j+1}) are direct continuations for all j with $0 \leq j \leq n - 1$;

(b) $\gamma[s_j, s_{j+1}] \subseteq D_j (0 \leq \mathrm{j} \leq \mathrm{n} - 1)$ and $\gamma(1) \in D_n$, *then* (f_n, D_n) *is called an analytic continuation of (f0, D0) along γ.*

Suppose Ω is a domain satisfying $D_0 \cap \Omega = \emptyset$. A function element (f_0, D_0) is said to admit unrestricted continuation in Ω if for any curve γ in Ω such that $\gamma(0) \in D_0$, (f_0, D_0) admits an analytic continuation along γ. Furthermore, analytic continuation along a curve is essentially unique; that is, if (g, U) is another analytic continuation of (f_0, D_0) along γ, then on $U \cap D_n$ we have $f_n = g$. For $s \in [0, 1]$, we denote by $f_0(\gamma, s)$ the value of analytic continuation of f_0 along γ at the endpoint $\gamma(s)$ of $\gamma_s: t \mapsto \gamma(st), 0 \leq t \leq 1$. In particular, $f_0(\gamma, 1) = f_n(\gamma(1))$.
For example, let $D = \{z \in \mathbb{C}: |z - 1| < 1\}$ and define

$$f(z) = ln\ z, z \in D$$

with $\ln 1 = 0$. Let $\gamma(t) = exp(2t\pi i)$. Then (f, D) admits analytic continua-tion along γ. We have $f(\gamma, 0) = f(1) = 0$, and in general

$$f(\gamma, t) = 2t\pi i, 0 \le t \le 1.$$

Note that $f(\gamma, 1) = 2\pi i \neq f(\gamma, 0)$, but $\gamma(1) = \gamma(0)$.

For a holomorphic function f on a planar domain Ω, if there is a subdo-main V of Ω and a holomorphic function $\rho : V \to \Omega$ such that

$$f(z) = f(\rho(z)), z \in V,$$

then ρ is called *a local inverse of f on V*. The analytic continuation of a local inverse of f is also a local inverse.

For each $z \in \mathbb{C} \setminus \{0\}$, define

$$z^* = 1/\bar{z}.$$

Let A be a subset of the complex plane, and define $A^* = \{z^* : z \in A \setminus \{0\}\}$.

For a meromorphic function f on domain Ω, define f^* by

$$f^*(z) = (f(z^*))^*, z \in \Omega^*.$$

Note that f^* is a meromorphic function if $f \not\equiv 0$.

We need the following lemma.

Lemma (5.2.11) [203]: *Suppose that f is a holomorphic function on a convex domain Ω and $x_0 \in \Omega \cap R$. If there exists a sequence $\{x_k\}$ in $\mathbb{R} \setminus \{x_0\}$ such that $x_k \to x_0 (k \to \infty)$, and $f(x_k) \in \mathbb{R}$, then $f(\Omega \cap \mathbb{R}) \subseteq \mathbb{R}$.*

Proof. Write $U = \Omega \cap \{\bar{z} : z \in \Omega\}$, which is itself a domain. Then define $g(z) = f(z) - \overline{f(\bar{z})}$ on U. For each k, we have

$$g(x_k) = f(x_k) - \overline{f(\overline{x_k})} = f(x_k) - \overline{(f(x_k))} = 0.$$

Since $x_0 \in U$ and x_0 is the accumulation point of $\{x_k\}$, $g \equiv 0$. In particular, for each x in $\Omega \cap \mathbb{R}$, $f(x) = \overline{f(\bar{x})} = \overline{f(x)}$. That is, $f(x) \in \mathbb{R}$ to finish the proof.

For $a \in \mathbb{C}$ and $\delta > 0$, let

$$O(a, \delta) = \{z \in \mathbb{C} : |z - a| < \delta\}.$$

Since there is a Moebius map mapping the real line to the unit circle, Lemma (5.2.11) implies the following.

Corollary (5.2.12) [203]: *Assume f is holomorphic in $O(\zeta_0, \delta)$ where $\zeta_0 \in \mathbb{T}$ and $\delta > 0$. Suppose that there exists a sequence $\{\zeta_k\}$ in $\mathbb{T} \setminus \{\zeta_0\}$ such that $\zeta_k \to \zeta_0 (k \to \infty)$ and $f(\zeta_k) \in \mathbb{T}$. Then $f(O(\zeta_0, \delta) \cap \mathbb{T}) \subseteq \mathbb{T}$.*

For two planar domains Ω_1 and Ω_2, a holomorphic map $f : \Omega_1 \to \Omega_2$ is called proper if $f^{-1}(E)$ is compact for each compact subset E of Ω_2. For a proper holomorphic map $f : \Omega_1 \to \Omega_2$, there exists a positive integer n such that for each $w \in \Omega_2$, $f - w$ has exactly n zeros in Ω_1, counting multiplicity (see [208, Theorem 15.1.9]). The integer n is called *the multiplicity of the holomorphic proper map f.*

An observation is in order. Let f be a nonconstant function holomorphic at a. By complex analysis, there is a holomorphic function ψ over a neighborhood W of a and a positive integer n such that

$$f(z) - f(a) = (z - a)^n \psi(z), z \in W$$

and $\psi(a) \neq 0$. For W sufficiently small, $g(z) = (z - a)\sqrt[n]{\psi(z)}$ is univalent on W, and we have

$$f(z) - f(a) = g(z)^n.$$

Furthermore, we can require W to be a Jordan domain such that $g(W)$ is a disk centered at 0. Therefore, we immediately get the following.

Lemma (5.2.13) [203]: *Suppose f is a nonconstant holomorphic function over a domain containing both a and b, and $f(a) = f(b) = \lambda$. Then for a sufficiently small positive number ε, there are two Jordan neighborhoods W_1 and W_2 of a and b, such that both $f|_{W_1}$ and $f|_{W_2}$ are proper maps onto $\lambda + \varepsilon\mathbb{D}$.*

Furthermore, in this case for each pair (z, w) satisfying $f(z) = f(w), z \in W_1 \setminus\{a\}$, and w $W_2\setminus\{b\}$, there is a local inverse ρ of f such that $\rho(z) = w$ and ρ admits analytic continuation along any curve in $W_1 \setminus\{a\}$, with values in $W_2\setminus\{b\}$.

A problem raised in [208] asks whether each nonconstant function in $H^\infty(\overline{\mathbb{D}})$ has the MWN property. Under a certain condition this is answered by the following result.

Theorem (5.2.14) [203]: Suppose ϕ is a nonconstant function in $\mathfrak{M}(\overline{\mathbb{D}})$. Then

$$n(\phi) = b(\phi) = o(\phi) = N(\phi).$$

The proof of Theorem (5.2.14) is long and thus it is divided into several parts. In what follows we will establish some lemmas and corollaries and then prove Theorem (5.2.14) at the end.

let ϕ be a nonconstant function in $\mathfrak{M}(\overline{\mathbb{D}})$. We write for the set of poles of ϕ in $\mathbb{C}$, and Z' for the set of zeros of ϕ' . Let $X = P \cup \phi^{-1}(\varphi(0)) \cup \phi^{-1}(\phi(Z')), Y = X \cup X^*$, and write

$$Y = \phi^{-1}(\phi(Y)).$$

Note that $\tilde{Y}$ is a countable set containing Y and $\phi^{-1}(\phi(\tilde{Y})) = \tilde{Y}$. Recall that a planar domain minus a countable set is path-connected.

Lemma (5.2.15) [203]: *Suppose that there is a point ξ on $\setminus \tilde{Y}$, a neighborhood U_ξ of ξ, and a local inverse ρ of ϕ satisfying $\rho = \rho^*$ on U_ξ. If γ is a curve in $\mathbb{D}\setminus \tilde{Y}$ such that $\gamma(0) = \xi$, then ρ admits analytic continuation along γ.*

Proof. To reach a contradiction, assume that ρ admits no analytic continuation along γ. Write $\gamma_s(t) = \gamma(st), t \in [0,1]$ and put

$$s_0 = sup\,\{s \in [0,1] : \rho \text{ admits an analytic continuation along } \gamma_s\}.$$

Then it is clear that the map ρ defined on U_ξ admits no analytic continuation along γ_{s_0} ; otherwise there is some $s_1 > s_0$ such that ρ admits an analytic continuation along γ_{s_1} , a contradiction. Recall that $\rho(\gamma, s)$ denotes the value of the analytic continuation of ρ at the endpoint $\gamma(s)$ of γ_s, where $0 \le s < s_0$.

One will show that $\{\rho(\gamma, s) : s \in [0, s_0)\}$ is bounded. Note that $\rho(\gamma, s)$ is continuous in s. If $\{\rho(\gamma, s) : s \in [0, s_0)\}$ is not bounded, then these exists a sequence $\{s_n\} \subseteq [0, s_0)$ such that $\{s_n\}$ tends to s_0, and

$$\lim_{n\to\infty} \rho(\gamma, s_n) = \infty. \tag{32}$$

Since $\gamma \cap \tilde{Y} = \emptyset, \gamma$ has no intersection with $\phi^{-1}(\phi(0))$, and then the local inverse ρ^* of ϕ admits an analytic continuation along γ_s^*, where

$$\gamma_s^*(t) = (\gamma s(t))^*, t \in [0,1].$$

Then by (32)

$$\lim_{n\to\infty} \rho^*(\gamma^*, s_n) \ = \ 0,$$

forcing

$$\lim_{n\to\infty} \phi(\gamma^*(s_n)) = \phi(0).$$

That is, $\phi(\gamma(s_0)^*) \ = \phi(0)$, and hence $\gamma(s_0)^* \in \phi^{-1}(\phi(0))$. But γ has no intersection with the set $\phi^{-1}(\phi(0))^*$, which is a contradiction. Therefore $\{\rho(\gamma, s) \ : \ s \in [0, s_0)\}$ is bounded by a positive number c_0.

Let $\{z_i\}_{i=1}^m$ be all the zeros of $\phi - \phi(\gamma(s_0))$ in $c_0\mathbb{D}$. Since γ has no intersection with $\phi^{-1}(\phi(Z'))$, we have that $\phi'(\gamma(s_0)) \neq 0$ and

$$\phi'(z_i) \neq 0, i \ = \ 1, \ldots, m.$$

Then one can find a connected neighborhood U of $\gamma(s_0)$ and disjoint connected neighborhoods $U_i(i \ = \ 1, \ldots, m)$ of z_i such that $\phi|_U$ and $\phi|_{U_i}$ are univalent. Since $\phi(z_i) \ = \phi(\gamma(s_0))$ for $1 \ \leq \ i \ \leq \ m$, using Lemma (5.2.13) and contracting U and U_i we have that

$$\phi(U_i) \ = \phi(U) \ = \ O(\phi(\gamma(s_0)), \varepsilon), 1 \leq i \leq m,$$

for some $\varepsilon > 0$, and that

$$\phi^{-1}(O(\phi(\gamma(s_0)), \varepsilon)) \cap \ c_0\overline{\mathbb{D}} \subseteq \coprod_{i=1}^{m} U_i. \tag{33}$$

By continuity of ϕ, there exists a positive number $\delta < s_0$ such that

$$\phi(\gamma[s_0 - \ \delta, s_0]) \ \subseteq \ O(\phi(\gamma(s_0)), \varepsilon).$$

By (33) $\{\rho(\gamma, s) : s \in (s_0 - \ \delta, s_0)\}$ is a connected set in $\coprod_{i=1}^{m} U_i$, and thus it is contained in a single U_j for some $1 \leq j \leq m$. Letting

$$\tau = \ (\phi|U_j)^{-1} \circ (\phi|U \ ),$$

we have that τ is a local inverse of ϕ such that $\tau(\gamma(s_0)) = z_j$ and $\tau \ (U) = U_j$.

For each $s \in (s_0 - \delta, s_0)$, let ρ_s be the analytic continuation for ρ along γ_s, and then ρ_s is a direct continuation of τ . Then by combining ρs with , we have that ρ admits analytic continuation along $\gamma_{s_0}$to derive a contradiction. The proof is complete.

Lemma (5.2.16) [203]: *Suppose* $\phi \in \mathfrak{M}(\overline{\mathbb{D}})$ *is not a rational function. Then for each positive number C, there exist two points a and* a' *in* $\mathbb{C}$ *such that* $|\phi(a)| \ > \ C, |\phi(a')| < \frac{1}{C}$ *and* $min\{|a|, |a'|\} \ > \ C$.

Proof. Since $\phi \in \mathfrak{M}(\overline{\mathbb{D}})$, ϕ is a meromorphic function over $\mathbb{C}$. Then either the infinity ∞ is an isolated singularity or ∞ is the limit of poles. If ∞ is an isolated singularity, ∞ is a removable singularity, a pole or an essential singularity. If ∞ were either a removable singularity or a pole, then ϕ would have finitely many singularities (poles), and by complex analysis φ is a rational function. This is a contradiction to our assumption. Therefore, ∞ is an essential singularity of ϕ. By Weierstrass' theorem in complex analysis, for each point $w \in \mathbb{C} \cup \{\infty\}$ there is a sequence $\{z_n\}$ tending to ∞ such that $\{\phi(z_n)\}$ tends to *w*. Hence the conclusion of Lemma (5.2.16) follows.

If ∞ is the limit of poles of φ, then for a fixed number $C > 0$, one can find a point a satisfying $|a| > C$ and $|\phi(a)| > C$. To complete the proof, we will show that there exists a point a' such that $|a'| > C$ and $|\phi(a')| < \frac{1}{C}$. If this were not true, then we would have

$$\frac{1}{|\phi(z)|} \leq C, |z| > C,$$

where $\frac{1}{\phi(z)}$ equals zero if z is one pole of ϕ. Since $\frac{1}{\phi}$ is bounded at a neigh borhood of ∞, ∞ is a removable singularity of $\frac{1}{\phi}$. Then $\frac{1}{\phi}$ has only finitely many poles in $\mathbb{C} \cup \{\infty\}$. Then by complex analysis $\frac{1}{\phi}$ is a rational function, and so is ϕ. This derives a contradiction to finish the proof.

We will use Lemmas (5.2.15) and (5.2.16) to prove the following.

Lemma (5.2.17) [203]: *Suppose $\phi \in \mathfrak{M}(\overline{\mathbb{D}})$ is not a rational function. Then there exists a bounded domain $\Omega \supseteq \overline{\mathbb{D}}$ having the following property: for a point $\xi \in \mathbb{T} \setminus \tilde{Y}$, if there is a neighborhood U_ξ of ξ, and a local inverse ρ of ϕ satisfying $\rho = \rho^*$ on U_ξ, then for each curve γ in $\Omega \setminus \tilde{Y}$ with $\gamma(0) = \xi$, we have $\rho(\gamma, 1) \in \bar{\Omega}$, i.e. the value of the analytic continuation $\tilde{\rho}$ of ρ along γ at endpoint $\gamma(1)$ lies in $\bar{\Omega}$.*

Proof. Suppose $\phi \in \mathfrak{M}(\overline{\mathbb{D}})$ is not a rational function. First we give the construction of Ω. By comments above Lemma (5.2.13) there exists a small neighborhood V of 0 and a biholomorphic function $g : V \to r\mathbb{D} (r > 0)$ such that $\phi(z) - \phi(0) = g(z)^k$ on V for some positive integer k. One can require that V is a Jordan domain and ∂V is contained in $\mathbb{D}$. Put

$$\Gamma = (\partial V)^* = \{1/\bar{z} : z \in \partial V\},$$

a closed Jordan curve outside $\overline{\mathbb{D}}$. Let Ω be the interior of Γ, and then

$$V^* = \mathbb{C} \setminus \bar{\Omega}.$$

Let $\in \mathbb{T} \setminus \tilde{Y}$, and let γ be a curve in $\Omega \setminus \tilde{Y}$ with $\gamma(0) = \xi$. Suppose that there is a neighborhood U_ξ of ξ, and a local inverse ρ of ϕ satisfying $\rho = \rho^*$ on U_ξ. The proof will be finished if we can show that

$$\rho(\gamma, 1) \in \bar{\Omega}.$$

To reach a contradiction, we assume that $\rho(\gamma, 1) \in \mathbb{C} \setminus \bar{\Omega} = V^*$. Let

$$\gamma_s(t) = \gamma(st), t \in [0, 1]$$

and by Lemma (5.2.15) the map ρ defined on U_ξ admits an analytic continuation ρ_s along γ_s. Recall that $\rho(\gamma, s)$ is the value of ρ_s at the endpoint $\gamma_s(1) = \gamma(s)$, and let

$$\sigma(s) = \rho(\gamma, s), s \in [0, 1]. \quad (34)$$

Then σ is a curve in $\mathbb{C} \setminus \tilde{Y}$. Since $\xi \notin \phi^{-1}(\phi(Z')), \rho'(\xi) \neq 0$, and thus there is a sufficiently small neighborhood V_ξ of ξ such that $V_\xi \subseteq U_\xi$,

$$V_\xi = V_\xi^*,$$

and $\rho : V_\xi \to \rho(V_\xi)$ is biholomorphic. Since γ has no intersection with $\phi^{-1}(\phi(Z'))$, the map $\rho^{-1} : \rho(V_\xi) \to V_\xi$ admits an analytic continuation $\widetilde{\rho^{-1}}$ along σ, and by (34) we have

$$\rho^{-1}(\sigma, t) = \gamma(t), t \in [0,1].$$

In particular, we get

$$\widetilde{\rho^{-1}}\big(\sigma(1)\big) = \rho^{-1}(\sigma, 1) = \gamma(1). \tag{35}$$

Let $\{p_i\}_{i=1}^{m}$ be the poles of ϕ on $\bar{\Omega}$. One can construct disjoint connected neighborhoods $U_i (i = 1, \ldots, m)$ of p_i such that

(a) ϕ has no zeros in $\bar{U}_i$ for $1 \le i \le m$;

(b) $\bar{U}_i \cap [\phi^{-1}(\phi(0))]^* \subseteq \{p_i\}$ for $1 \le i \le m$;

(c) For i such that $\phi(p_i^*) = \phi(0)$, there exists a sufficiently small connected neighborhood $V_i \subseteq V$ of 0, such that $\phi|_{U_i^*}, \phi|_{V_i}$ are proper maps satisfying $\phi(U_i^*) = \phi(V_i)$; for other i, let $V_i = V$.

In fact, Condition (a) is easy to fulfill. Since $\phi^{-1}(\phi(0))$ is discrete and $[\phi^{-1}(\phi(0))]^*$ has at most one accumulation point 0, Condition (b) is fulfilled if we let U_i be sufficiently small. By Lemma (5.2.13) we can choose U_i and V_i to satisfy (c) and be as small as possible thus to meet (a) and (b). Therefore, one has (a)-(c) as desired.

Let

$$M = \max_{z \in \bar{\Omega} \setminus \cup_{i=1}^{m} U_i} |\phi(z)|,$$

and define

$$\varepsilon_i = dist(\phi(0), \phi(\overline{U_i^*})), i = 1, \cdots, m.$$

If each ε_i equals zero, set $\varepsilon = +\infty$; otherwise, write

$$\varepsilon = min\,\{\varepsilon_i : \varepsilon_i > 0, 1 \le i \le m\}. \tag{36}$$

Then there exists a number $\delta > 0$ such that

$$\phi(\delta\mathbb{D}) \subseteq O(\phi(0), \varepsilon) \quad and \quad \delta\mathbb{D} \subseteq \bigcap_{i=1}^{m} V_i \quad . \tag{37}$$

By Lemma (5.2.16) we get a point $a \neq \tilde{Y}$ satisfying

$$|a| > \frac{1}{\delta} \qquad and \qquad |\phi(a)| > M.$$

Since $V^* \backslash \tilde{Y}$ is path-connected, we can choose a curve ς in $V^* \backslash \tilde{Y}$ connecting $\rho(\gamma, 1) = \sigma(1)$ with a. By Lemma (5.2.15), $\rho^{-1}: \rho(V_\xi) \to V_\xi$ admits an analytic continuation τ along $\sigma\varsigma$, where $\sigma\varsigma$ is defined by

$$\sigma\varsigma(t) = \begin{cases} \sigma(2t), & 0 \le t \le \frac{1}{2}, \\ \varsigma(2t - 1), & \frac{1}{2} < t \le 1. \end{cases}$$

Since $\rho = \rho^*$ on V_ξ, we have $\rho(V_\xi) = \rho(V_\xi)^*$ and $\rho^{-1} = (\rho^{-1})^* : \rho(V_\xi) \to V_\xi$ admits the analytic continuation τ^* along $\sigma^*\varsigma^*$. Then both τ and τ^* are local inverses of ϕ. Since

$$|\phi(\tau\,(a))| \;=\; |\phi(a)| \;>\; M,$$

by the definition of M we get either $\tau\,(a) \in V^*$ or $\tau(a) \in U_i$ for some i. As follows, to derive contradictions we will distinguish two cases.

Case I: $\tau(a) \in V^*$. For , $z \in V$, let $\phi(w) = \phi(z)$. Recall that on V we have

$$\phi(z)\; - \phi(0) = g(z)^k,$$

and then $g(w)^k = g(z)^k$. Since $g|_V$ is biholomorphic, we get

$$w \;=\; g^{-1} \circ (\lambda g(z)), z \in V,$$

where $\lambda \;=\; exp(\frac{2\pi ji}{k})$ for some integer j in $\{1,\cdot\cdot\cdot,k\}$. Let τ_j denote the map $g^{-1} \circ (\lambda g(z))$, we have $\tau_j(V) = V$ and $\tau_j(0) = 0$.

Note that

$$\tau^*(a^*) = (\tau(a))^* \;\in (V^*)^* \;=\; V.$$

Since $\phi(\tau^*(a^*)) = \phi(a^*)$ and $a^* \in V$, there exists an integer j_0 such that

$$\tau_{j_0}(a^*) = \tau^*(a^*).$$

By noting $\tau = (\tau^*)^*$, we have that τ extends analytically to

$$\tau_{j_0}^* : V^* \to V^* .$$

Recall that $\widetilde{\rho^{-1}}$ and τ are analytic continuations of ρ^{-1} along σ and $\sigma\varsigma$, respectively. Thus $\widetilde{\rho^{-1}}$ also extends analytically to $\tau_{j_0}^*$. Then by (35)

$$\sigma(1) \;= \widetilde{\rho^{-1}}(\sigma(1)) \;=\; \tau_{j_0}^*(\sigma(1)) \;\in\; V^* .$$

This contradicts with the fact that $\gamma \subseteq \Omega$.

Case II: There is some *i* such that $\tau(a) \in U_i$. First we show $\phi(p_i^*) \;= \phi(0)$. In fact, since $a^* \in \delta\mathbb{D}$, by (37) we have

$$|\phi(0)\; - \phi(\tau^*(a^*))| \;=\; |\phi(0)\; - \phi(a^*)| < \varepsilon.$$

Since $\tau^*(a^*) = (\tau\,(a))^* \in U_i^*$,

$$\varepsilon_i \;=\; dist(\phi(0), \phi(\overline{U_i^*})) < \varepsilon,$$

which along with (36) gives $\varepsilon_i = 0$. This shows that $\overline{U_i^*} \cap \phi^{-1}(\phi(0))$ is not empty. By Condition (b) we immediately get $\overline{U_i^*} \cap \phi^{-1}(\phi(0)) \subseteq \{p_i^*\}$, and thus

$$\phi(p_i^*) = \phi(0).$$

Since Condition (a) shows that

$$min\{|\phi(z)| : z \in \overline{U}_\imath, 1 \le i \le m\} \;>\; 0,$$

by Lemma (5.2.16) there is a point $a' \neq \tilde{Y}$ satisfying

$$|\phi(a')| < min\{|\phi(z)| : z \in \overline{U}_\imath, 1 \le i \le m\} \tag{38}$$

And $|a'| > \frac{1}{\delta}$. By (37) $a' \in V_i^*$. Let ζ be a curve in $V_i^* \setminus \tilde{Y}$ joining a with $'$, and let $\tilde{\tau}$ be the analytic continuation of τ along ζ. Since both τ and τ^* are local inverses of ϕ, so are $\tilde{\tau}$ and $\tilde{\tau}^*$, and $\tilde{\tau}^*$ is the analytic continuation of τ^*along ζ^*. By Condition (c) and Lemma (5.2.13), along any curve in $V_i\setminus\{0\}, \tau^*$ admits analytic continuation with values in $U_i^*\setminus\{p_i^*\}$. Thus we have

$$\tilde{\tau}^*(a'^*) \in U_i^*.$$

Since $(\tilde{\tau}(a'))^* = \tilde{\tau}^*(a'^*), \tilde{\tau}(a)$ lies in U_i, and hence $\phi(a') = \phi(\tilde{\tau}(a')) \in \phi(U_i)$. This is a contradiction to (38). In either case, we conclude a contradiction thus finishing the proof of Lemma (5.2.17).

Suppose ϕ is a function in $H^\infty(\overline{\mathbb{D}})$. For $\xi \in \mathbb{T}$, define

$$m(\xi) = \lim_{\delta \to 0^+} \min_{\eta \in O(\xi,\delta) \cap \mathbb{T}} N(\phi - \phi(\eta), \mathbb{T}) \quad .$$

Clearly, $m(\xi) \leq N(\phi - \phi(\xi), \mathbb{T})$. Write

$$S = \{\xi \in \mathbb{T} : m(\xi) < N(\phi - \phi(\xi), \mathbb{T})\}.$$

For $r \in (0,1)$, let A_r denote the annulus

$$\{z \in \mathbb{C} : r < |z| < \frac{1}{r}\}.$$

We need the following lemma.

Lemma (5.2.18) [203]: *Let ϕ be a nonconstant function in $H^\infty(\mathbb{D})$. Then S is count-able.*

Proof. To reach a contradiction, assume S is uncountable. Let $\mathcal{Z}'$ denote the zero set of ϕ' on $\overline{\mathbb{D}}$ and

$$F = \phi^{-1}(\phi(\mathcal{Z}')).$$

For each positive integer j, put

$$S_j = \{\xi \in S : N(\phi - \phi(\xi), \mathbb{T}) = j\}.$$

Then there exists at least a positive integer l such that S_l is uncountable. Recall that an uncountable set in $\mathbb{C}$ has infinitely many accumulation points. One can pick an accumulation point ξ_0 of S_l such that $\xi_0 \in F$.

Since $\phi - \phi(\xi_0)$ has finitely many zeros on $\mathbb{T}$ and $\phi - \phi(\xi_0)$ is holomorphic on $\mathbb{T}$, one can pick an r $(0 < r < 1)$ close to 1 such that all zeros of $\phi - \phi(\xi_0)$ in $\overline{A_r}$ lie on $\mathbb{T}$. By Rouch´e's Theorem, there exists a positive number δ such that for each z in $O(\xi_0, \delta)$

$$N(\phi - \phi(z), A_r) = N(\phi - \phi(\xi_0), A_r) = N(\phi - \phi(\xi_0), \mathbb{T}) = l.$$

On the other hand, there is a sequence $\{\xi_k\}$ in $S_l \cap [O(\xi_0, \delta) \setminus \{\xi_0\}]$, such that $\xi_k \to \xi_0 (k \to \infty)$. Thus,

$$l = N(\phi - \phi(\xi_k), A_r) \geq N(\phi - \phi(\xi_k), \mathbb{T}) = l.$$

This means that each zero of $\phi - \phi(\xi_k)$ in A_r lies on $\mathbb{T}$. Since $\xi_0 \notin F$, there exist l local inverses $\rho_0, \ldots, \rho_{l-1}$ of ϕ defined on $O(\xi_0, \delta)$; that is,

$$\phi(\rho_i) = \phi, i = 0, \ldots, l - 1.$$

Note that $\rho_0(\xi_0), \ldots, \rho_{l-1}(\xi_0)$ are exactly l zeros of $\phi - \phi(\xi_0)$ on $\mathbb{T}$, and for all k we have

$$\rho_i(\xi_k) \in \mathbb{T}, i = 0, \ldots, l - 1.$$

By Corollary (5.2.12), $\rho_i(O(\xi_0, \delta) \cap \mathbb{T}) \subseteq \mathbb{T}, i = 0, \ldots, l - 1$. We can require δ to be small enough such that $\rho_i(O(\xi_0, \delta) \cap \mathbb{T})$ are pairwise disjoint. Hence for each $\xi \in O(\xi_0, \delta) \cap \mathbb{T}, \phi - \phi(\xi)$ has l distinct zeros on $\mathbb{T}$, $\rho_0(\xi), \ldots, \rho_{l-1}(\xi)$. Therefore

$$m(\xi_0) \geq l = N(\phi - \phi(\xi_0), \mathbb{T}),$$

a contradiction to $\xi_0 \in S$, finishing the proof.

By using Lemmas (5.2.17) and (5.2.18), one can prove the following.

Proposition (5.2.19) [203]: Suppose ϕ is a nonconstant function in $\mathfrak{M}(\overline{\mathbb{D}})$. Then

$$o(\phi) = b(\phi).$$

Proof. We will first construct some local inverses of ϕ that maps some arc of $\mathbb{T}$ into T. For this, by Lemma (5.2.18) $\phi(S)$ is countable as well as S, and then $\partial\phi(\mathbb{D}) \setminus \phi(S)$ is uncountable. Then there is a point ξ_0 in $\mathbb{T} \setminus (\tilde{Y} \cup S)$ satisfying

$$\phi(\xi_0) \in \partial\phi(\mathbb{D}).$$

Rewrite $n_0 = N(\phi - \phi(\xi_0), \mathbb{T}) = m(\xi_0)$. Then one can find an $r \in (0,1)$ close to 1 such that

$$n_0 = N(\phi - \phi(\xi_0), \mathbb{T}) = N(\phi - \phi(\xi_0), \overline{A_r}).$$

By application of Rouché's theorem, there exists a positive number $\delta > 0$ satisfying

$$N(\phi - \phi(\xi), \mathbb{T}) \leq N(\phi - \phi(\xi), \overline{A_r}) = N(\phi - \phi(\xi_0), \overline{A_r}) = n_0, \xi \in O(\xi_0, \delta) \cap \mathbb{T}.$$

By definition of $m(\xi_0), N(\varphi - \varphi(\xi), T) \geq n_0$, forcing

$$N(\phi - \phi(\xi), \mathbb{T}) = m(\xi_0) = n_0, \xi \in O(\xi_0, \delta) \cap \mathbb{T}. \quad (39)$$

As done in Lemma (5.2.18) one can find n_0 holomorphic functions $\rho_0, \dots, \rho_{n_0-1}$ on $O(\xi_0, \delta)$ (δ can be decreased if necessary) such that

(a) for $z \in O(\xi_0, \delta)$,

$$N(\phi - \phi(z), A_r) = N(\phi - \phi(\xi_0), A_r) = N(\phi - \phi(\xi_0), \mathbb{T}) = n_0;$$

(b) $\phi(\rho_i) = \phi, 0 \leq i \leq n_0 - 1$;

(c) $\rho_i(O(\xi_0, \delta)) \subseteq A_r, 0 \leq i \leq n_0 - 1$.

In particular, $\rho_0(\xi_0), \dots, \rho_{n_0-1}(\xi_0)$ are exactly those n0 zeros of $\phi - \phi(\xi_0)$ on $\mathbb{T}$. Then by (a)

$$n_0 = N(\phi - \phi(\xi), A_r) \geq N(\phi - \phi(\xi), \mathbb{T}) = n_0, \xi \in O(\xi_0, \delta) \cap \mathbb{T},$$

forcing all zeros of $\phi - \phi(\xi)$ in A_r to fall onto $\mathbb{T}$. Hence by Conditions (b) and (c) we get

$$\rho_i(\xi) \in \mathbb{T}, i = 0, \dots, n_0 - 1.$$

Hence there exists a neighborhood of ξ_0 where we have $\rho_i = \rho_i^*$ for $i = 0, \dots, n_0 - 1$, as they are equal on some arc of $\mathbb{T}$.

By Lemma (5.2.15) for each curve $\wp$ in $\mathbb{C}\setminus\tilde{Y}$ such that $\wp(0) = \xi_0$, each member in $\{\rho_i : i = 0, \dots, n_0 - 1\}$ admits analytic continuation along $\wp$. We will see that the family $\{\rho_i : i = 0, \dots, n_0 - 1\}$ is closed under analytic continuation. For this, assume that γ is a loop in $\mathbb{C} \setminus \tilde{Y}$ with $\gamma(0) = \gamma(1) = \xi_0$. Let $\widetilde{\rho_i}(0 \leq i \leq n_0 - 1)$ be the analytic continuation of ρ_i along γ. Clearly, all these $\widetilde{\rho_i}$ are local inverses of ϕ, i.e. $\phi(\widetilde{\rho_i}) = \phi$. Since $\phi(\xi_0) \in \partial\phi(\mathbb{D})$,

$$\widetilde{\rho_i}(\xi_0) \notin \mathbb{D}.$$

Besides, we have $\rho_i = \rho_i^*$ on some neighborhood of ξ_0, and then

$$\phi(\rho_i) = \phi(\rho_i^*) = \phi.$$

Write $\gamma^*(t) = (\gamma(t))^*$ $(t \in [0,1])$ and define $\tilde{\rho}_i^*$ along γ^*. Hence

$$\phi(\tilde{\rho}_i^*) = \phi, 0 \leq i \leq n_0 - 1.$$

By similar reasoning as above, $\widetilde{\rho_i}^*(\xi_0) \notin \mathbb{D}$. Also noting $\widetilde{\rho_i}(\xi_0) \notin \mathbb{D}$ gives $\widetilde{\rho_i}(\xi_0) \in \mathbb{T}$. Then it follows that $\{\widetilde{\rho_i}(\xi_0) : i = 0, \dots, n_0 - 1\}$ is a permutation of $\{\rho_i(\xi_0) : i =$

$0,\ldots,n_0-1\}$. If two local inverses are equal at one point $\xi_0 \in \phi^{-1}(\phi(Z'))$, by the Implicit Function Theorem they are equal on a neighborhood of this point. Thus we have

$$\widetilde{\rho_i} : i = 0,\ldots,n_0-1\} = \{\rho_i : i = 0,\ldots,n_0-1\}.$$

Give two curves γ_1 and γ_2 with $\gamma_1(0) = \gamma_2(0) = \xi_0$ and $\gamma_1(1) = \gamma_2(1), \gamma_1\gamma_2^{-1}$ is a loop with endpoints ξ_0. Therefore, we have that analytic continuations of the family $\{\rho_i : i = 0,\ldots,n_0-1\}$ along γ_1 are the same as those along γ_2. Thus analytic continuations of the family $\{\rho_i : i = 0,\ldots,n_0-1\}$ does not depend on the choice of the curve. Define

$$B(z) = \prod_{i=0}^{n_0-1} \widetilde{\rho_i}(z), z \in \mathbb{C}\,\tilde{Y}, \tag{40}$$

where we use analytic continuations. In what follows, we will show that B extends analytically to a finite Blaschke product and there are two cases to distinguish:

$$\phi \in \Re(\overline{\mathbb{D}}) \quad or \quad \phi \in \mathfrak{M}(\overline{\mathbb{D}}) \setminus \Re(\overline{\mathbb{D}}).$$

Case I: $\phi \in \Re(\overline{\mathbb{D}})$. Thus ϕ is a rational function, and then $\tilde{Y}$ is a finite set. Assume that the infinity ∞ is a pole of ϕ, without loss of generality. Otherwise, one can compose ϕ with some $\eta \in Aut(\mathbb{D})$ defined by

$$\eta(z) = \frac{\alpha - z}{1 - \bar{\alpha}z},$$

mapping ∞ to a pole $1/\bar{\alpha}$ of ϕ. Replacing ϕ with ϕ reduces to the desired case. Since $\phi \in \Re(\overline{\mathbb{D}})$, there is a constant $c_1 > 1$ such that ϕ is holomorphic on some neighborhood of $c_1\mathbb{D}$. Let

$$M = max\{|\phi(z)| : |z| \le c_1\} < +\infty.$$

Since $\phi(\infty) = \infty$, there exists a constant $c_2 > 0$ satisfying

$$|\phi(z)| > M, |z| > c_2.$$

For $\in c_1\mathbb{D} \setminus \tilde{Y}$, we have

$$|\phi(\widetilde{\rho_i}(z))| = |\phi(z)| \le M, 0 \le i \le n_0 - 1,$$

and then each $\widetilde{\rho_i}(z)$ is bounded by c_2. Hence B is an analytic function bounded by $c_2^{n_0}$. Therefore, B extends analytically to $c_1\mathbb{D}$ since $\tilde{Y}$ is a finite set. Besides, all $\widetilde{\rho_i}(z)$ are unimodular on the circular arc $O(\xi_0,\delta) \cap \mathbb{T}$, and so is B. By Corollary (5.2.12) B is unimodular on $\mathbb{T}$, and hence B is a finite Blaschke product [17].

Case II: $\phi \in \mathfrak{M}(\overline{\mathbb{D}}) \setminus \Re(\overline{\mathbb{D}})$, then ϕ is not a rational function. By Lemma (5.2.17) there exists a bounded domain $\Omega \supseteq \overline{\mathbb{D}}$ such that for $\in \Omega \setminus \tilde{Y}$, we have

$$\widetilde{\rho_i}(z) \in \bar{\Omega}, 0 \le i \le n_0 - 1.$$

For these ρ_i, each analytic continuation along a curve in $\Omega \setminus \tilde{Y}$ is defined by a chain of disks, and hence by (40) B extends naturally to an open set $V (V \subseteq \Omega)$ containing $\Omega \setminus \tilde{Y}$. Since $\tilde{Y}$ is countable, there is a relatively closed countable set Y_0 such that

$$V = \Omega \setminus Y_0.$$

Since $\Omega \setminus \tilde{Y}$ is dense in $\Omega \setminus Y_0$, for $z \in \Omega \setminus Y_0$ we have

$$\widetilde{\rho_i}(z) \in \bar{\Omega}, 0 \le i \le n_0 - 1.$$

Therefore, B is a well-defined bounded analytic function on $\Omega \setminus Y_0$. Since Y_0 is a countable relatively closed set in Ω, Y_0 is H^∞-removable, and thus B extends analytically on Ω. In particular, B is analytic on a neighborhood of $\overline{\mathbb{D}}$. Since each $\widetilde{\rho_\iota}(z)$ is unimodular on the circular arc $O(\xi_0, \delta) \cap \mathbb{T}$, so is B. By Corollary (5.2.12) B is unimodular on $\mathbb{T}$, forcing B to be a finite Blaschke product.

In both cases we have shown that B extends analytically to a finite Blaschke product. All local inverses of B are exactly $\{\widetilde{\rho_\iota} : i = 0, \dots, n_0 - 1\}$, and clearly, order $B = n_0$. By Corollary (5.2.7), each member ρ in $G(\phi)$ is uniquely determined by the value $\rho(\xi_0)$. Thus

$$o(\phi) \le N(\phi - \phi(\xi_0), \mathbb{T}).$$

Note that $\rho_0(\xi_0), \dots, \rho_{n_0-1}(\xi_0)$ are all zeros of $\phi - \phi(\xi_0)$ on $\mathbb{T}$, and thus

$$o(\phi) \le n_0 = order\, B = o(B). \tag{41}$$

On the other hand, $\{\widetilde{\rho_\iota} : i = 0, \dots, n_0 - 1\}$ are local inverses of ϕ, and then $B(z) \mapsto \phi(z)$ is a well-defined analytic function, denoted by h. Since h is bounded on $c_1\mathbb{D}$ minus a finite set where $c_1 > 1$, h extends to a function in $H^\infty(\overline{\mathbb{D}})$, and

$$\phi = h(B).$$

This gives $G(\phi) \supseteq G(B)$. Noting (41), we have $o(\phi) = o(B)$.,

$$o(\phi) \ge b(\phi) \ge order\, B = o(B) = o(\phi).$$

forcing $b(\phi) = o(\phi) = n_0$.

To establish Theorem (5.2.14), we also need the following.

Corollary (5.2.20) [203]: *For a nonconstant function $\phi \in \mathfrak{M}(\overline{\mathbb{D}})$, except for a countable set each point ξ in $\mathbb{T}$ satisfies $N(\phi - \phi(\xi), \mathbb{T}) = b(\phi)$. Furthermore, $N(\phi) = b(\phi)$.*

Proof. By Proposition (5.2.19), we write

$$n_0 = b(\phi) = o(\phi).$$

By Corollary (5.2.9) we have

$$N(\phi - \phi(\xi), \mathbb{T}) \ge n_0, \xi \in \mathbb{T}.$$

Write

$$A = \{\xi \in \mathbb{T}\ N(\phi - \phi(\xi), \mathbb{T}) > n_0\},$$

and it suffices to show that A is countable. Assume conversely that A is uncountable. Since A contains uncountable accumulation points in itself, one can pick an accumulation point η_0 in $\mathbb{T} \setminus (\tilde{Y} \cup S)$. Write

$$l = N(\phi - \phi(\eta_0), \mathbb{T}) > n_0.$$

In the first paragraph of the proof of Proposition (5.2.19), by replacing ξ_0 with η_0 we get l local inverses on some neighborhood of η_0, which maps an arc in $\mathbb{T}$ into $\mathbb{T}$. Let γ be a curve in $\mathbb{C} \setminus (\tilde{Y} \cup S)$ connecting η_0 and ξ_0, and these l local inverses admit analytic continuations along γ, denoted by $\widetilde{\tau_0}, \dots, \widetilde{\tau_{l-1}}$. Also $\widetilde{\tau_0}^*, \dots, \widetilde{\tau_{l-1}}^*$ are exactly analytic continuations along γ^* of the local inverses $\tau_0^*, \dots, \tau_{l-1}^*$ at η_0. For $0 \le i \le l - 1$, neither $\tilde{\tau}_i(\xi_0)$ nor $\tilde{\tau}_i{}^*(\xi_0)$ belongs to $\mathbb{D}$ as $\phi(\xi_0) \in \partial\phi(\mathbb{D})$. Therefore $\{\tilde{\tau}_i(\xi_0) : 0 \le i \le l - 1\}$ are l distinct zeros of $\phi - \phi(\xi_0)$ on $\mathbb{T}$. But by (39),

$$N(\phi - \phi(\xi_0), \mathbb{T}) = n_0 < l,$$

which derives a contradiction. Hence A is countable, as desired.

Now we proceed to present the proof of Theorem (5.2.14).

Suppose that ϕ is a nonconstant meromorphic function in $\mathbb{C}$ without poles on $\overline{\mathbb{D}}$. By Proposition (5.2.19) and Corollary (5.2.20), we have

$$N(\phi) = b(\phi) = o(\phi).$$

It remains to show that

$$n(\phi) = b(\phi).$$

Recall that

$$n(\phi) = \min_{z\in D, \phi(z)\notin\phi(\mathbb{T})} wind(\phi(\mathbb{T}), \phi(z)).$$

By Corollary (5.2.9) $o(\phi) \le n(\phi)$. Since $b(\phi)|o(\phi), b(\phi) \le n(\phi)$. It remains to prove that

$$n(\phi) \le b(\phi).$$

Recall that Z' is the zero of ϕ, and let $F = \phi^{-1}(\phi(Z' \cap \overline{\mathbb{D}}))$. By Corollary (5.2.20) there is a point $w_0 \in \mathbb{T}\backslash F$ such that $\phi(w_0) \in \partial\phi(\overline{\mathbb{D}})$ and

$$(\phi - \phi(w_0), \mathbb{T}) = b(\phi).$$

Since the zeros of ϕ are isolated in $\mathbb{C}$, there exists a positive constant $t > 1$ satisfying

$$N(\phi - \phi(w_0), t\mathbb{D}) = N(\phi - \phi(w_0), \mathbb{D}) = b(\phi).$$

By Rouché's Theorem, there is a positive number δ such that

$$N(\phi - \phi(z), t\mathbb{D}) = N(\phi - \phi(w_0), t\mathbb{D}) = b(\phi), z \in O(w_0, \delta).$$

Let z_0 be a point in $O(w_0, \delta) \cap \mathbb{D}$ such that $\phi(z_0) \notin \phi(\mathbb{T})$, and by the Argument Principle we get

$$wind(\phi(\mathbb{T}), \phi(z_0)) = N(\phi - \phi(z_0), \mathbb{D}) \le N(\phi - \phi(z_0), t\mathbb{D}) = b(\phi).$$

Thus $n(\phi) \le b(\phi)$, forcing $n(\phi) = b(\phi)$. This finishes the proof of Theorem (5.2.14).

It is shown that each nonconstant function in $\mathfrak{M}(\overline{\mathbb{D}})$ has the FSI-decomposable property. Based on this, the proof of Theorem (5.2.3) is furnished.

Recall that $\mathfrak{M}(\overline{\mathbb{D}})$ denotes the class of all meromorphic functions in $\mathbb{C}$ without poles on $\overline{\mathbb{D}}$. We prove the following.

Theorem (5.2.21) [203]: *For a nonconstant function $\phi \in M(D)$, suppose $\phi = \psi(B)$ is the Cowen-Thomson representation of ϕ. Then ψ has the FSI property.*

Later, by using Theorem (5.2.21) one will get Theorem (5.2.3), restated as follows.

Theorem (5.2.22) [203]: S*uppose ϕ is a nonconstant function in $\mathfrak{M}(\overline{\mathbb{D}})$. The following are equivalent:*

(a) the Toeplitz operator T_ϕ is totally Abelian;

(b) ϕ has the FSI property.

(c) $N(\phi) = 1$.

Recall that a point λ in $\mathbb{C}$ is called a point of self-intersection of the curve $\phi(z)$ $(z \in \mathbb{T})$ [214] if there exist two distinct points w_1 and w_2 on $\mathbb{T}$ such that

$$\phi(w_1) = \phi(w_2) = \lambda;$$

equivalently, $N(\phi - \lambda, \mathbb{T}) > 1$. To prove Theorem (5.2.21), we need the following.

Lemma (5.2.23) [203]: Suppose $\phi \in H^{\infty}(\overline{\mathbb{D}})$. Then the cardinality of points of self-intersections of the curve $\phi(z)$ $(z \in \mathbb{T})$ is either finite or $\aleph_1$, the cardinality of continuum.

Proof. Suppose $\phi \in H^{\infty}(\overline{\mathbb{D}})$. Denote the set of all points of self-intersection of $\phi(\mathbb{T})$ by A. If A is finite, the proof is finished.

Assume A is an infinite set. Then $\phi^{-1}(A) \cap \mathbb{T}$ must have an accumulation point ξ_0 on $\mathbb{T}$. By the definition of points of self-intersection, there is a sequence $\{\xi_k\}$ in $\mathbb{T}\setminus\{\xi_0\}$ and a sequence $\{\eta_k\}$ in $\mathbb{T}$ such that $\xi_k \to \xi_0 (k \to \infty)$, and

$$\phi(\xi_k) = \phi(\eta_k), \xi_k \neq \eta_k, \forall k.$$

Without loss of generality, one assumes that $\{\eta_k\}$ itself converges to a point η_0 on $\mathbb{T}$. Thus we have

$$\phi(\xi_0) = \phi(\eta_0) \equiv \lambda_0.$$

Note that ξ_0 may be equal to η_0.

Since ϕ is not constant, by Lemma (5.2.13) there are two simply-connected neighborhoods U of ξ_0, V of η_0 and a positive number ε such that

$$\phi(U) = \phi(V) = \varepsilon\mathbb{D} + \lambda_0,$$

and $\phi|_U$, $\phi|_V$ are proper holomorphic maps, whose multiplicities are equal to the multiplicities of zero of $\phi - \lambda_0$ at ξ_0 and η_0 respectively. Write

$$\widehat{U} = U\setminus\{\xi_0\} \quad and \quad \widehat{V} = V\setminus\{\eta_0\}.$$

Let N be the multiplicity of the zero of $\phi - \lambda_0$ at the point η_0. By Lemma (5.2.13), for each $z \in \widehat{U}$ we have the following:

(a) there exist exactly N distinct local inverses of ϕ on a connected neigh-borhood U_z of z with values in $\widehat{V}$ and $U_z \subseteq \widehat{U}$;

(b) each local inverse in (a) admits analytic continuation along any curve in $\widehat{U}$ starting from the point z.

Note that analytic continuation of a local inverse of ϕ in (a) is also a local inverse, with values in $\widehat{V}$.

We construct a special holomorphic function ω defined on some neighborhood D of 0, which maps infinitely many real numbers to real numbers. The following discussions are based on the upper half plane Π rather than on the unit disk, and this will be more convenient. Let φ be a Moebius transformation mapping $\mathbb{D}$ onto Π, its pole being distinct from ξ_0 and η_0. Rewrite

$$x_k = \varphi(\xi_k) \quad and \quad y_k = \varphi(\eta_k), k \geq 0.$$

We point out that x_k and y_k do not represent the real and imaginary parts of a complex number. Let δ be a positive number such that $\overline{O(x_0, \delta)} \subseteq \varphi(U)$, and we define four simply connected domains:

$$D_0 = \{z \in O(x_0, \delta): Re(z - x_0) > 0\}, D_1 = \{z \in O(x_0, \delta): Im(z - x_0) > 0\};$$
$$D_2 = \{z \in O(x_0, \delta): Re(z - x_0) < 0\}, D_3 = \{z \in O(x_0, \delta): Im(z - x_0) < 0\}.$$

Note that D_1, D_2 and D_3 can be obtained by a rotation of D_0. Since $\varphi^{-1}(D_i)$ is simply connected for $i = 0, 1, 2, 3$, by (a) and (b) we get N local inverses of ϕ with values in $\widehat{V}$;

and by the Monodromy Theorem these local inverses are all analytic on $\varphi^{-1}(D_i)$ for fixed i. Let

$$\tilde{\phi} = \phi \circ \varphi^{-1},$$

and we obtain N local inverses of $\tilde{\phi}$, which are analytic on each domain D_i for $i = 0, 1, 2, 3$. With no loss of generality, assume there are infinitely many points of $\{x_k\}$ lying in D_0. Then there exists at least one local inverse σ_0 of defined on D_0 so that σ0 maps x_k to y_k, for infinitely many k. Define

$$D_{4j+i} = D_i, 0 \leq i \leq 3, j \in \mathbb{Z}_+.$$

Note that

$$D_0 = D_4 = D_8 = \cdots,$$

and there are only finitely many distinct local inverses of $\tilde{\phi}$ on D_0. There must be a minimal positive integer $n_0 \leq N$ satisfying $\sigma_{4n_0} = \sigma_0$. As follows, we will use function elements (σ_i, D_i) $(i = 0, \dots, 4n_0 - 1)$ to construct a holomorphic function on a disk D. Precisely, write $D = O(0, \sqrt[n_0]{\delta})$, and for $z \in D \setminus \{0\}$ define

$$\omega(z) = \begin{cases} \sigma_0(z^{n_0} + x_0), & 0 \leq \arg z < \dfrac{\pi}{2n_0}; \\ \sigma_1(z^{n_0} + x_0), & \dfrac{\pi}{2n_0} \leq \arg z < \dfrac{\pi}{n_0}; \\ \dots & \\ \sigma_{4n_0-1}(z^{n_0} + x_0), & (4n_0 - 1) \leq \arg z < 2\pi. \end{cases}$$

Then ω is well-defined and holomorphic in $D \setminus \{0\}$. Observe that as z tends to x_0 in $D_i (i = 0, \dots, 4n_0 - 1)$, each $\sigma_i(z)$ tends to y_0. Therefore ω is bounded near 0, and hence 0 is a removable singularity of ω. By setting $\omega(0) = y_0$ we get a holomorphic function ω on D.

Since $\omega|_{D\cap\mathbb{R}^+}(x) = \sigma_0(x^{n_0} + x_0)$, and $\sigma_0(x_k) = y_k$ holds for infinitely many k, we have $\omega(\sqrt[n_0]{x_k - x_0}) = y_k \in \mathbb{R}$ as $x_k > x_0$. By Lemma (5.2.11),

$$\omega(D \cap \mathbb{R}) \subseteq \mathbb{R},$$

forcing $\sigma_0(D_0 \cap \mathbb{R}) \subseteq \mathbb{R}$. Letting

$$\gamma = \varphi^{-1}(D_0 \cap \mathbb{R}) \subseteq \mathbb{T},$$

and

$$\tilde{\sigma}_0(w) = \varphi^{-1} \circ \sigma_0 \circ \varphi(w), w \in \varphi^{-1}(D_0) \subseteq U,$$

we have $\tilde{\sigma}_0(\gamma) \subseteq \mathbb{T}$. Clearly σ_0 is not the identity map, and neither is $\tilde{\sigma}_0$. Let

$$W = \{z \in \varphi^{-1}(D_0) \cap \mathbb{T} : \tilde{\sigma}_0(z) = z\},$$

and W is at most countable. Since the cardinality of $\varphi(\gamma)$ is $\aleph_1$, so is $\varphi(\gamma \backslash W)$, finishing the proof of Lemma (5.2.23).

Suppose that ϕ is a nonconstant function in $\mathfrak{M}(\overline{\mathbb{D}})$, and $\phi = \psi(B)$ is the Cowen-Thomson representation. Corollary (5.2.2) says that ψ is in $H^\infty(\mathbb{D})$. By comments below Theorem (5.2.1), B is of maximal order and thus ψ can not be written as a function of a finite Blaschke product of order larger than 1. Again by Corollary (5.2.2) we have $b(\psi) = 1$. Corollary (5.2.20) implies that

$$\{w \in \mathbb{T} : N(\psi - \psi(w), \mathbb{T}) > 1\}$$
is countable, as well as $\{\psi(w) \in \mathbb{T} : N(\psi - \psi(w), \mathbb{T}) > 1\}$. But by Lemma (5.2.23) self-intersections of $\psi(z)$ $(z \in \mathbb{T})$ is finite or has the cardinality of the continuum. Therefore, it must be finite. Thus ψ has the FSI property as desired.

Note (b) ⇒ (c) is trivial. To show (c) ⇒ (a), assume $N(\phi) = 1$. By Corollary (5.2.20) we have $b(\phi) = N(\phi) = 1$. Then Theorem (5.2.1) gives that T_ϕ is totally Abelian.

For (a) ⇒ (b), let $\phi = \psi(B)$ be the Cowen-Thomson representation. Then by Theorem (5.2.21) ψ has the FSI property. Since T_ϕ is totally Abelian,
$$\{T_z\}' = \{T_\phi\}' \supseteq \{T_B\}' \supseteq \{T_z\}' .$$
Then $\{T_\phi\}' = \{T_\phi\}'$, forcing order B = 1. Since $\phi = \psi(B)$, ϕ has the FSI property as desire.

It is straightforward to get equivalent formulations for (a)-(c) in Theorem (5.2.22): (d) there is a point $\xi \in \mathbb{T}$ satisfying $N(\phi - \phi(\xi), \mathbb{T}) = 1$; and (5) except for a countable or finite set every point $\xi \in \mathbb{T}$ satisfies $N(\phi - \phi(\xi), \mathbb{T}) = 1$.

Theorem (5.2.21) shows that each function ϕ in $\mathfrak{M}(\overline{\mathbb{D}})$ has the FSI-decomposable property; that is, for the Cowen-Thomson representation $\phi = \psi(B)$, ψ has the FSI property. In fact, we will see that ψ has quite special form (see Lemma (5.2.24) and Theorem (5.2.26)).

Recall that $\mathfrak{R}(\overline{\mathbb{D}})$ consists of all rational functions which have no pole on D. If P and Q are two co-prime polynomials, order $\frac{P}{Q}$ is defined to be max $\{\deg P, \deg Q\}$. The following is of independent interest and for related work, See Stephenson's [210, Lemma 3.4] [211].

Lemma (5.2.24) [203]: *If f is in $\mathfrak{R}(\overline{\mathbb{D}})$ and there is a function h on $\mathbb{D}$ such that*
$$f = h(B),$$
where B is a finite Blaschke product, then h is $\mathfrak{R}(\overline{\mathbb{D}})$. In this case, we have order $f = order\ h \times order\ B$.

Proof. Suppose f is a function in $\mathfrak{R}(\overline{\mathbb{D}})$ and h is a function on $\mathbb{D}$ satisfying
$$f = h(B),$$
where B is a finite Blaschke product. Let $n = order\ B$, and B has n local inverses, denoted by $\rho_0, \cdots, \rho_{n-1}$. Let Z' denote the zero set of B' in $\mathbb{C}$, and by Bochner's Theorem [216] Z' is a finite subset of $\mathbb{D}$. Write
$$\mathcal{E} = B^{-1}(B(Z')).$$
It is known that each local inverse of B admits unrestricted continuation in $\overline{\mathbb{D}} \setminus \mathcal{E}$. For each j $(0 \leq j \leq n - 1)$, define $\rho_j^*(z) = (\rho_j(z^*))^*$, which admits unrestricted continuation on $C \setminus \overline{\mathbb{D}}$ minus a finite set. Recall that the derivative B' of B does not vanish on $\mathbb{T}$, ρ_j is analytic on $\mathbb{T}$ and $\rho_j^* = \rho_j$ on $\mathbb{T}$. Thus there exists a finite set F_1 such that each ρ_j admits unrestricted continuation on $\mathbb{C} \setminus F_1$.

For each $z \in \mathbb{D}$, by $h(B(z)) = f(z)$ we get
$$f(z) = f\left(\rho_j(z)\right), 0 \leq j \leq n - 1. \tag{42}$$

By analytic continuation, the above also holds for all $z \in \mathbb{C}\setminus F_1$. Let P denote the poles of f, a finite set in $\mathbb{C}$. Then by (42) $B(z) \mapsto f(z)$ extends to $\mathbb{C}\setminus(B(F_1) \cup P)$. Thus on the complex plane minus discrete points, we have

$$f = h(B). \tag{43}$$

If f is a rational function, then its only possible isolated singularities (including ∞) are poles. By (43), h has at most finitely many singularities including ∞, which are either removable singularities or poles. Hence h is a rational function. Since f is holomorphic on $\overline{\mathbb{D}}$, by (43) h is bounded on a neighborhood of $\overline{\mathbb{D}}$ with finitely many singularities possible. Thus h extends analytically on $\overline{\mathbb{D}}$, forcing $h \in \mathfrak{R}(\overline{\mathbb{D}})$.

Suppose f is a rational function. Noting that f can be written as the quotient of two co-prime polynomials, by computations we have that f is a covering map on $\mathbb{C}\setminus(f^{-1}(f(\infty)) \cup \mathcal{E})$, and the multiplicity is exactly the order of f. Since both h and B are rational functions, they can be regarded as covering maps on $\mathbb{C}$ minus some finite set. This leads to the conclusion that

$$order\, f = order\, h \times order\, B,$$

to complete the proof.

By Theorem (5.2.21) and Lemma (5.2.24) we get the following.

Corollary (5.2.25) [203]: *Suppose R is a rational function in $\mathfrak{R}(\overline{\mathbb{D}})$ with prime order. Then either R is a composition of a Moebius transformation and a finite Blaschke product, or R has the FSI property. In the latter case, T_R is totally Abelian.*

Below we come to functions in $\mathfrak{M}(\overline{\mathbb{D}}) \setminus \mathfrak{R}(\overline{\mathbb{D}})$. The main theorem in says that each entire function ϕ has the Cowen-Thomson representation $\phi(z) = \psi(z^n)$ for some entire function ψ and some integer n. The following theorem generalizes the theorem in [210] to functions in $\mathfrak{M}(\overline{\mathbb{D}}) \setminus \mathfrak{R}(\overline{\mathbb{D}})$, and is of independent interest.

Theorem (5.2.26) [203]: *Suppose that $f \in \mathfrak{M}(\overline{\mathbb{D}})$ is not a rational function. Then there is a positive integer n and a function h in $\mathfrak{M}(\overline{\mathbb{D}})$ such that*

$$f(z) = h(z^n)$$

and $\{T_f\}' = \{T_{z^n}\}'$. Furthermore, $n = o(f) = b(f) = n(f) = N(f)$.

Proof. To prove Theorem (5.2.26), we begin with an observation from complex analysis. By Lemma (5.2.13) and comments above it, for a function φ holomorphic on a neighborhood of λ, let $k = order\,(\varphi, \lambda)$, the multiplicity of the zero of $\varphi - \varphi(\lambda)$ at λ. Then there is a Jordan neighborhood W of λ such that $\varphi|_W$ is a $k - to - 1$ proper map onto a neighborhood of $\varphi(\lambda)$. This is right even if $\lambda = \infty$ or λ is a pole of φ (for $\lambda = \infty$, order $(\varphi, \infty) = order\,(\varphi(1/z), 0)$).

Based on this, we will show that if B_0 is a finite Blaschke product of order k, and order (B_0, ∞) equals k, then B_0 is a function of z^k. In fact, either $B_0(\infty) = \infty$ or $B_0(\infty)$ is finite and $|B_0(\infty)|$ is necessarily greater than 1. If $B_0(\infty)$ is finite and $|B_0(\infty)| > 1$, by letting

$$\psi(z) = \frac{1/\overline{B_0(\infty)} - z}{1 - z/B_0(\infty)}$$

we have $\psi \circ B_0(\infty) = \infty$. Then we can assume $B_0(\infty) = \infty$. Note that order $(1/B_0(1/z), 0) = order\ (B_0, \infty) = k$. Write

$$B_0(z) = c\prod_{j=1}^{k}\frac{\alpha_j - z}{1 - \overline{\alpha_j}\, z},$$

where $|c| = 1$ and $\alpha_j \in \mathbb{D}$ for all j, and then

$$1/B_0(1/z) = \bar{c}\prod_{j=1}^{k}\frac{\overline{\alpha_j} - z}{1 - \alpha_j\, z}.$$

Since order $(1/B_0(1/z), 0) = k, \alpha_j = 0$ for all j. Then B_0 is a function of z^k. This fact will be used later.

Suppose that $f \in \mathfrak{M}(\overline{\mathbb{D}})$ is not a rational function. By Theorem (5.2.1) there is a function $h \in H^\infty(\mathbb{D})$ and a finite Blaschke product B such that

$$f(z) = h(B(z)), z \in \mathbb{D},$$

and $\{T_f\}' = \{T_B\}'$. Without loss of generality, assume order $B = n \geq 2$. In the proof of Lemma (5.2.24) we have shown that there is a finite set F_1 such that each local inverse ρ_j of B admits unrestricted continuation on $\mathbb{C}\backslash F_1$. Let P denote the poles of f. By $B(\rho_j) = B$ on $\mathbb{C}\backslash F_1$ and $f(z) = h(B(z))$, we can define a holomorphic function

$$h(z) : B(z) \mapsto f(z)$$

on $\mathbb{C}\backslash(B(F_1) \cup P)$. So h has only isolated singularities. Letting

$$F_0 = \overline{F_1 \cup B^{-1}(P)},$$

we have

$$f(z) = h\big(B(z)\big), z \in \mathbb{C}\backslash F_0 \qquad (44)$$

If order $(B, \infty) = n$, then by the second paragraph of this proof B is a function of z^n, and hence

$$\{T_f\}' = \{T_B\}' = \{T_{z^n}\}' .$$

Reasoning as (44), there is a function $\tilde{h}$ such that $f(z) = \tilde{h}(z^n)$ holds on $\mathbb{C}$ minus a discrete set. In this case, it is straightforward to show $\tilde{h}$ is in $\mathfrak{M}(\overline{\mathbb{D}})$. By Theorem (5.2.14), we have $n = o(f) = b(f) = n(f) = N(f)$ to complete the proof for the case of order $(B, \infty) = n$.

Noting order $(B, \infty) \leq$ order $B = n$, we will prove that order $(B, \infty) < n$ would not happen. For this, assume conversely that order $(B, \infty) < n$ to reach a contradiction. Since B is an n-to-1 map, we have a point $a \in \mathbb{C}$, two neighborhoods $\mathcal{N}_1$ of a and $\mathcal{N}_2$ of ∞ such that $B(a) = B(\infty)$, $B|_{\mathcal{N}_1}$ and $B|_{\mathcal{N}_2}$ are proper maps, and their images are equal. There are two cases to distinguish: either P is a finite set or P is an infinite set.

Case I: P is a finite set. Then F_0 is a finite set. By (44), f has similar behaviors at a and at ∞: since f is a meromorphic function and not a rational function, ∞ is an essential singularity of f, and so is a. But this is a contradiction to the fact that f has no isolated singularities other than poles in $\mathbb{C}$.

Case II: P is an infinite set. Let ∞ be the limit of all poles $\{w_k : k \geq 1\}$ of f. Note that there is an integer k_0 such that $w_k \in \mathcal{N}_2$ for $n \geq k_0$. Then there is a sequence $\{w_k' : k \geq k_0\}$ in $\mathcal{N}_1$ such that

$$B(w_k') = B(w_k) \quad and \quad w_k' \to a\ (k \to \infty).$$

Soon we will see that F_0 contains only finitely many accumulation points, that is, poles of B. In fact, since F_1 is a finite set and $F_0 = F1 \cup B - 1(P)$, the accumulation set of F_0 are exactly that of $B^{-1}(P)$, where $P = \{w_k : k \geq 1\}$. Since ∞ is the limit of $\{w^k\}$ and $B^{-1}(\infty)$ consists of (finitely many) poles of B, it follows that the accumulation points of $B^{-1}(P)$ are poles of B, and hence the accumulation points of F_0 are poles of B.

Since the only accumulation points of F_0 are poles of B, these points w_k' are isolated singularities of f. Noting (44), we have that w_k' are poles of f because w_k are poles of f. But $\{w_k'\}$ tends to the finite point a, and thus a is not an isolated singularity of f. This is a contradiction to $f \in \mathfrak{M}(\overline{\mathbb{D}})$.

Therefore, in both cases we derive a contradiction to finish the proof.

Some comments are in order. Quine [214] showed that each polynomial has the FSI-decomposible property. Precisely, he proved that a nonconstant polynomial can always be written as $p(z^m)$ $(m \geq 1)$ where p is a polynomial of the FSI property. For decomposition of rational functions, See [215].

The following result gives some equivalent conditions for a Toeplitz operator with an entire function as symbol to be totally Abelian, and it follows from Theorems (5.2.21) and (5.2.22). See [215].

Proposition (5.2.27) [203]: *Suppose $\phi(z) = \sum_{k=0}^{\infty} c_k z^k$ is a nonconstant entire function. Then the following are equivalent:*

(a) T_ϕ is totally Abelian;

(b) $n(\phi) = min\,\{wind(\phi, \phi(a)) : a \in \mathbb{D}, \phi(a) \notin f(\mathbb{T})\} = 1$.

(c) there is a point w on $\mathbb{T}$ such that $\phi(w)$ is not a point of self-intersection;

(d) ϕ has only finitely many points of self-intersection on $\mathbb{T}$;

(e) there is a point w in $\mathbb{D}$ such that $\phi - \phi(w)$ has exactly one zero in $\mathbb{D}$, counting multiplicity.

(f) there is a point $\lambda \in \mathbb{C}$ such that $\phi - \lambda$ has exactly one zero in $\mathbb{D}$, counting multiplicity;

(g) $gcd\{c_k : c_k \neq 0\} = 1$.

provides some examples. Some of them are examples of totally Abelian Toeplitz operators, and others will show that the MWN property is quite restricted even for functions of good "smooth" property on $\mathbb{T}$.

We begin with a rational function in $\mathfrak{R}(\overline{\mathbb{D}})$.

Example (5.2.28) [203]: *Let Q be a polynomial without zeros on $\overline{\mathbb{D}}$ and of prime degree q. Let P be a nonconstant polynomial satisfying*

$$deg\,P < q.$$

Suppose that P has at least one zero in $\mathbb{D}$ and let $= \frac{P}{Q}$. We will show that T_R is totally Abelian. For this, assume R has k zeros in $\mathbb{D}$, counting multiplicity. We have

$$1 \le k \le deg\, P < q. \tag{45}$$

If T_R were not totally Abelian, then by Corollary (5.2.25) there would be a finite Blaschke product B and a Moebius map $\tilde{R}$ such that

$$R = \tilde{R} \circ B,$$

and order $B = q$. But by $R = \tilde{R} \circ B$, we have $k \ge order\ B = q$, which is a contradiction to (45). Therefore T_R is totally Abelian.

The following two examples arise from the Riemann-zeta function and the Gamma function. It is shown that under a translation or a dilation of the variable, the corresponding Toeplitz operators are totally Abelian.

Example (5.2.29) [203]: *The Riemann-zeta function $\zeta(z)$ is defined as the analytic continuation of the following:*

$$z \mapsto \sum_{n=1}^{\infty} \frac{1}{n^z}, Re z > 1.$$

It is a meromorphic function in $\mathbb{C}$ and the only pole is $z = 1$. Write $f(z) = \zeta(\frac{z}{2})$, and then $f(z) \in \mathfrak{M}(\overline{\mathbb{D}})$. We claim that T_f is totally Abelian.

For this, by Theorem (5.2.26) it suffices to show that there is no meromorphic function g on $\mathbb{C}$ such that $f(z) = g(z^k)$ for some integer $k \ge 2$. Otherwise, taking $\omega \ne 1$ and $\omega^k = 1$, we have $f(\omega z) = f(z)$. This gives $\zeta(\frac{\omega z}{2}) = \zeta(\frac{z}{2})$, and thus

$$\zeta(\omega z) = \zeta(z).$$

Then ζ has at least two poles 1 and $\overline{\omega}$. This is a contradiction. Therefore, T_f is totally Abelian.

Example (5.2.30) [203]: *The Gamma function $\Gamma(z)$ is a meromorphic function with only poles at non-positive integers*

$$0, -1, -2, \cdots.$$

Let $f(z) = \Gamma(z + 2)$. Then $f(z) \in \mathfrak{M}(\overline{\mathbb{D}})$ and Tf is totally Abelian. Otherwise, by Theorem (5.2.26), there is a function $g \in \mathfrak{M}(\overline{\mathbb{D}})$ such that $f(z) = g(z^k)$ for some integer $k \ge 2$. Let $\omega \ne 1$ and $\omega^k = 1$, and we have $f(z) = f(\omega z)$; that is,

$$\Gamma(z + 2) = \Gamma(\omega z + 2).$$

The poles of $\Gamma(z + 2)$

$$-2, -3, \cdots$$

must be the poles of $\Gamma(\omega z + 2)$

$$-2\overline{\omega}, -3\overline{\omega}, \cdots.$$

This is impossible. Hence T_f is totally Abelian.

Before continuing, recall that a Jordan domain is the interior of a Jordan curve. We need Carathéodory's theorem, which can be found in a standard textbook of complex analysis, see [216] for example.

Lemma (5.2.31) [203]: [*Carath´eodory's theorem] Suppose that Ω is a Jordan domain. Then the inverse Riemann mapping function f from $\mathbb{D}$ onto Ω extends to a 1-to-1 continuous function F from $\bar{\mathbb{D}}$ onto $\bar{\Omega}$. Furthermore, the function F maps T 1-to-1 onto $\partial\Omega$.*

In what follows, we provide some examples to show that in general a function f in the disk algebra $A(\mathbb{D})$ may not satisfy the MWN property, even if f has good smoothness on $\mathbb{T}$.

Example (5.2.32) [203]: *First, we present an easy example of a function in $A(\mathbb{D})$ with good smoothness on $\mathbb{T}$, but not satisfying the MWN property. Put*

$$\Omega_0 = \{z : 0 < |z| < 1, 0 < arg\, z < \pi\},$$

and write $g(z) = z^8, z \in \Omega_0$. Let ϕ_0 be a conformal map from the unit disk $\mathbb{D}$ onto Ω_0. Precisely, write $u(z) = \sqrt{i\frac{z+1}{z-1}}$ with $\sqrt{1} = 1$ and

$$\phi_0(z) = \frac{1 - 2u(z)}{1 + 2u(z)}, z \in D.$$

and put $\phi_1 = g \circ \phi_0$. Note that $\phi_1 - \phi_1(0)$ has finitely many zeros in $\mathbb{D}$ and is away from zero on $\mathbb{T}$. Then the inner factor of $\phi_1 - \phi_1(0)$ contains no singular inner factor; otherwise, there is a sequence $\{\lambda_k\}$ in $\mathbb{D}$ satisfying $|\lambda_k| \to 1$ and $\phi_1(\lambda_k) - \phi_1(0) \to 0$ as $k \to \infty$, a contradiction. Therefore, the inner factor of $\phi_1 - \phi_1(0)$ is a finite Blaschke product, and hence $\phi_1 \in \mathcal{CT}(\mathbb{D})$.

For smoothness of ϕ_1, by Lemma (5.2.31) we have that $\phi_1 \in A(\mathbb{D})$. In addition, by using Schwarz Reflection Principle we see that except for at most three points on $\mathbb{T}$, ϕ_1 extends analytically across $\mathbb{T}$.

However, ϕ_1 does not satisfy the MWN property. In fact, for each point $a \in \mathbb{D}, \phi_1 - \phi_1(a)$ has at least 3 zeros in $\mathbb{D}$. Thus,

$$n(\phi_1) \geq 3.$$

On the other hand, since $N(\phi_1, \mathbb{T}) = 1$, φ1 can not be written as a function of a finite Blaschke product of order larger than 1. Then by Theorem (5.2.1) $\{T_{\phi_1}\}' = \{T_z\}'$. That is, $b(\phi_1) = 1$. But

$$n(\phi_1) > 1,$$

forcing $n(\phi_1) = b(\phi_1)$.

Inspired by this example, put $\Omega_1 = \{z \in \mathbb{C} : |z| < 1, |z - 1| < 1\}$, and let $h : \mathbb{D} \to \Omega$ be a conformal map. Note that by Carath´eodory's theorem, *h* extends continuously to a bijective map from $\bar{\mathbb{D}}$ onto $\bar{\Omega}$. Furthermore, noting that Ω_1 has two cusp points, one can show that except for two cusp points, can be analytically extended across $\mathbb{T}$, as well as h9. Also, $\{T_{h^9}\}' = \{T_z\}'$. The next example shows that Theorems (5.2.3) and (5.2.5) are restricted.

Example(5.2.33) [203]: *By Schwarz-Christoffel formula, one can construct a conformal map f form the upper half-plane onto the rectangle Ω with vertices $\{-\frac{K}{2}, \frac{K}{2}, \frac{K}{2} + iK', -\frac{K}{2} + iK'\}$, where $K, K' > 0$. Precisely, it is defined by*

$$f(z) = C\int_0^z \frac{1}{(\lambda^2 - 1)(\lambda^2 - t^2)} d\lambda, z \in \Omega,$$

where $\sqrt{1} = 1, C > 0$ *and t is a parameter in* $(0,1)$ *[210, Section 2.5]. We can specialize* $K' = 2k\pi$ *for some integer* $k \geq 100$.

Define $h(z) = exp(z - \frac{K}{2}), z \in \mathbb{C}$ *and let* $g(z)$ *be a conformal map from the unit disk onto the upper plane. Write*

$$\phi(z) = h \circ f \circ g(z), z \in \mathbb{D}.$$

It is not difficult to see that

$$n(\phi) \geq k,$$

and for each $\xi \in \mathbb{T}, N(\phi - \phi(\xi), \mathbb{T}) \geq 2$. *Moreover, we have* $N(\phi) = 2$.

Next we show that $o(\phi) = 1$. *For this, note that* $f \circ g$ *maps the unit disk* $\mathbb{D}$ *conformally onto the rectangle* Ω*, and* $f \circ g$ *extends to a continuous bijection from* $\overline{\mathbb{D}}$ *onto* $\bar{\Omega}$*, and* $f \circ g(\mathbb{T}) = \partial\Omega$. *Thus by definition of* $o(\phi)$*, the assertion* $o(\phi) = 1$ *is equivalent to that the only continuous map* $\rho : \partial\Omega \to \partial\Omega$ *satisfying* $h(\rho) = h$ *is the identity map. For this, let*

$$h(\rho(z)) = h(z), z \in \partial\Omega.$$

Then for each z *in* $\partial\Omega, \rho(z) = z + 2k(z)\pi i$ *for some integer* $k(z)$. *But ρ is continuous, forcing* $k(z)$ *to be a constant integer k. Hence*

$$\rho(z) = h(z) + 2k\pi i, z \in \partial\Omega.$$

Since $\rho(\partial\Omega) \subseteq \partial\Omega$*, we have* $k = 0$ *and ρ is the identity map, forcing* $o(\phi) = 1$. *Since* $b(\phi) \mid o(\phi), b(\phi) = 1$. *By Theorem (5.2.1),* T_{φ} *is totally Abelian. But for this function φ we have*

$$b(\phi) = o(\phi) < N(\phi) < n(\phi).$$

We conclude by showing that the function ϕ defined in Example (5.2.33) has good smoothness property on $\mathbb{T}$. Rewrite $h(\mathbb{T}) = v(z)2n$ where $v(z) = exp[\frac{1}{2k}(z - \frac{K}{2})]$. Note that $v \circ f \circ g$ defines a conformal map from $\mathbb{D}$ onto the domain

$$\{z \in \mathbb{C} : exp(\frac{-K}{2k}) < |z| < 1, arg\, z \in (0,\pi)\},$$

whose boundary contains only four "cusp points". By Lemma (5.2.31) we have $v \circ f \circ g \in A(D)$, and by Schwarz Reflection Principle $v \circ f \circ g$ extends analytically across T except for these cusp points. The same is true for ϕ.

Chapter 6
Hankel Operators and Sarason's Toeplitz Product Problem

We obtain all holomorphic functions f for which the Hankel operators H_f are bounded (or compact) from F_α^p to L_α^q. We provide a complete solution to the problem for a class of Fock spaces on the complex plane. In particular, this generalizes an earlier result of Cho, Park, and Zhu.

Section (6.1): Between Fock Spaces

For $\mathbb{C}^n$ be the n-dimensional complex Euclidean space. For $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ in $\mathbb{C}^n$, write $\langle z, w\rangle = z_1\overline{w_1} + \ +z_n\overline{w_n}$, and $|z| = \sqrt{\langle z, z\rangle}$. Given $\alpha > 0$, we consider the Gaussian probability measure

$$dv_\alpha(z) = \left(\frac{\alpha}{\pi}\right)^n e^{-\alpha|z|^2} dv(z)$$

on $\mathbb{C}^n$, where $dv(z)$ is the ordinary Lebesgue volume measure on $\mathbb{C}^n$.

For $1 \le p < \infty$, the space L_α^p consists of all Lebesgue measurable functions f on $\mathbb{C}^n$ for which

$$\|f\|_{p,\alpha}^p = \left(\frac{p\alpha}{2\pi}\right)^n \int_{\mathbb{C}^n} |f(z)e^{-\frac{\alpha}{2}|z|^2}|^p dv(z) < \infty.$$

It is clear that L_α^p is a Banach space with the norm $\|\cdot\|_{p,\alpha}$. We use L^p to stand for the usual p-th Lebesgue space with the norm $\|\cdot\|_{L^p} = \left\{\int_{\mathbb{C}^n} |\cdot|^p dv\right\}^{\frac{1}{p}}$. Let $H(\mathbb{C}^n)$ be the family of all holomorphic functions on $\mathbb{C}^n$. The Fock space F_α^p is defined by

$$F_\alpha^p = L_\alpha^p \cap H(\mathbb{C}^n).$$

It is easy to see that F_α^p is closed in L_α^p. Therefore, F_α^p is a Banach space. The Fock space has been studied by many authors, see [1, 2, 6, 10, 12].

Let $K_\alpha(\cdot)$ be the reproducing kernel of F_α^2, it is well known that $K_\alpha(z, w) = e^{\alpha\langle z,w\rangle}$. The orthogonal projection $P_\alpha : L_\alpha^2 \to F_\alpha^2$ can be represented as

$$P_\alpha f(z) = \int_{\mathbb{C}^n} K_\alpha(z, w)f(w)dv_\alpha(w).$$

With this expression, P_α can be extended to a bounded linear operator from L_α^p to F_α^p for $1 \le p \le \infty$, and for $f \in F_\alpha^p$ there holds $P_\alpha f = f$. Set $k_z(w) = \frac{K_\alpha(w,z)}{\|K_\alpha(\cdot,z)\|_{2,\alpha}}$ to be the normalized F_α^p for $1 \le p < \infty$. Bergman kernel. The set Span$\{k_z : z \in \mathbb{C}^n\}$ Let Γ denote the family of those measurable function f on $\mathbb{C}^n$ satisfying $fk_z \in \bigcup_{p\ge 1} L_\alpha^p$ for each $z \in \mathbb{C}^n$. Given $f \in \Gamma$, the Hankel operator H_f induced by symbol f can be densely defined on F_α^p by

$$H_f g(z) = (I - P_\alpha)(fg)(z)$$

$$= \int_{\mathbb{C}^n} \big(f(z) - f(w)\big) K_\alpha(z, w)g(w)dv_\alpha(w),$$

where I is the identity operator on L_α^p.

During the past decades, a lot of researches have been done on Hankel operators, see [5, 6, 8, 9, 12] for example. More recently in [8], in the case $1 < p \le q < \infty$, Pau, Zhao and Zhu characterized those f on the unit ball $\mathbb{B}^n$ for which both the operators H_f and $H_{\bar{f}}$ are bounded from Bergman space $A_\alpha^p(\mathbb{B}^n)$ to Lebesgue space $L^q(\mathbb{B}^n, dv_\beta)$. In [9, 12] and some others, the behavior of Hankel operators on Fock spaces F_α^p was studied, however the known results are only from F_α^p to L_α^p, with the same exponent p.

We characterize real-valued functions $f \in \Gamma$ so that H_f is bounded (or compact) from F_α^p to L_α^q with $1 \le p, q < \infty$. This is equivalent to characterizing complex-valued functions $f \in \Gamma$ such that both H_f and $H_{\bar{f}}$ are bounded (or compact). The following theorem is the main result. The spaces $BMOq$, VMO^q and $IMO^{s,q}$ will be defined.

Theorem (6.1.1)[230]: *Let $\alpha > 0$ and let $f \in \Gamma$.*

(*a*) *For $1 \le p \le q < \infty, H_f, H_{\bar{f}} : F_\alpha^p \to L_\alpha^q$ are both bounded if and only if $f \in BMO^q$; $H_f, H_{\bar{f}} : F_\alpha^p \to L_\alpha^q$ are both compact if and only if $f \in VMO^q$.*

(*b*) *For $1 \le q < p < \infty, H_f, H_{\bar{f}} : F_\alpha^p \to L_\alpha^q$ are both bounded if and only if $H_f, H_{\bar{f}}:. F_\alpha^p \to L_\alpha^q$ are both compact if and only if $f \in IMO^{s,q}$, where $s = \frac{pq}{p-q}$*

We introduce a family of *IMO* spaces and get some related results. *IMO* spaces are some generalization of the well-known BMO spaces., we define some weighted convolution operators for which we will give some mapping properties from one Lebesgue space to another. This part may have its own interests in operator theory of real variables. We study simultaneous boundedness (and compactness) of H_f and $H_{\bar{f}}$ from F_α^p to L_α^q. Our theory generalizes those of [231] and [132] where only the case $p = q \ge 1$ was considered.

We use C to denote positive constants whose value may change from line to line, but do not depend on functions being considered. For two quantities A and B, we write $A < B \sim$ if there exists some C such that $A \le CB$. We call A and B are equivalent, denoted by $A \simeq B$, if there exists some C such that $C^{-1}A \le B \le CA$. For $1 \le p < \infty$, let L_{loc}^p be the set of all p-th locally Lebesgue integrable functions on $\mathbb{C}^n$. Given some $z \in \mathbb{C}^n$ and $r > 0$, write $B(z,r) = \{w \in \mathbb{C}^n : |w - z| < r\}$. For $f \in L_{loc}^p$, the average function $\widehat{f}_r(z)$ can be defined as

$$\widehat{f}_r(z) = \frac{1}{v\big(B(z,r)\big)} \int_{B(z,r)} f(w) dv(w);$$

and the p-th mean oscillation of f at z is given by

$$MO_{p,r}(f)(z) = \left(\frac{1}{v\big(B(z,r)\big)} \int_{B(z,r)} |f(w) - \widehat{f}_r(z)|^p dv(w) \right)^{\frac{1}{p}}$$

For $1 \le s \le \infty, 1 \le p < \infty$ and $r > 0$, the space $IMO_r^{s,p}$, (s,p)-th integrable mean oscillation, is defined to be the set of all $f \in L_{loc}^p$ such that $MO_{p,r}(f) \in L^s(\mathbb{C}^n)$. For $f \in IMO_r^{s,p}$, write

$$\|f\|_{IMO_r^{s,p}} = \|MO_{p,r}(f)\|_{L^s(\mathbb{C}^n)}.$$

As in [12], the space BMO_r^p consists of those functions $f \in L_{loc}^p$ for which

$$\|f\|_{BMO_r^p} = \sup_{z\in\mathbb{C}^n} \quad MO_{p,r}(f)(z) < \infty,$$

and the space VMO_r^p consists of those $f \in BMO_r^p$ such that

$$\lim_{z\to\infty} M\,O_{p,r}(f)(z) = 0.$$

It is trivial to see that $IMO_r^{\infty,p} = BMO_r^p$. The spaces BMO_r^p and VMO_r^p are independent of r (see Chapter 3 of [232]). Thus, we will write BMO^p for BMO_r^p (and VMO^p for VMO_r^p) for short.

Let $1 \le p < \infty, 1 \le s \le \infty$, and $r > 0$. We denote by $IA^{s,p}$ the space of functions $f \in L_{loc}^p$ with

$$\|f\|_{IA^{s,p}} = \|\left(\widehat{|f|^p}\right)^{\frac{1}{p}}\|_{L^s(\mathbb{C}^n)} < \infty$$

for some $r > 0$. By the knowledge of (p, q) −Fock Carleson measure (see [4]), we know the space $IA^{s,p}$ is independent of r.

Given any $r > 0$, for a continuous function f on $\mathbb{C}^n$, let

$$\omega_r(f)(z) = \sup\{|f(z) - f(w)|: |w - z| < r\}$$

be the oscillation of f at z. We use $\omega(f)(z)$ for $(\omega_1(f)(z)$ for short. Let IO_r^s denote the space of continuous function f on $\mathbb{C}^n$ such that

$$\|f\|_{IO_r^s}=\|(\omega_r(f)\|_{L^s(\mathbb{C}^n)} < \infty.$$

It is easy to see, when $s = \infty$, the space $IA_r^{s,p}$ is consistent with BA_r^p. the space IO_r^s is consistent with BO_r. The definitions of BA_r^p and BO_r can be found in [232].

The next lemma shows that the space IO_r^p is independent of the choice of r. So we write it IO^p.

Lemma (6.1.2) [230]:. *Suppose* $0 < p \le \infty$. *Then for* $r, R \in (0,\infty)$ *and* f *Lebesgue measurable on* $\mathbb{C}^n$, $f \in IO_r^p$ *if and only in* $f \in IO_R^p$. *Furthermore,* $\|f\|_{IO_r^p} \simeq \|f\|_{IO_R^p}$.

$$\|f\|_{IO_R^p} \le C\|f\|_{IO_1^p}$$

for $R > 1$. To see this, notice that $B(0, R) \cap \frac{1}{2}\mathbb{Z}^n$ is finite, say

Proof. We need only to prove,

$$B(0, R) \cap \frac{1}{2}\mathbb{Z}^n = \{\xi_1, \xi_2, \dots, \xi_m\}.$$

Then $\omega_R(f)(z) \le \sum_{j=1,2,\dots,m} \omega(f)(z + \xi_j)$. This gives

$$\|f\|_{IO_R^p} \le C \sum_{j=1,2,.m} \|f(\cdot + \xi_j)\|_{IO_1^p}.$$

But $\|f(\cdot + \xi_j)\|_{IO_1^p} = \|f\|_{IO_1^p}$, the desired estimate follows. This completes the proof.

We now describe the structure of $IMO_r^{s,p}$ via IO^s and $IA^{s,p}$.

Theorem (6.1.3) [230]: *Let* $1 \le p < \infty, r > 0$ *and* $1 < s \le \infty$. *Then* $f \in IMO_r^{s,p}$ *if and only if* f *admits a decomposition* $f = f_1 + f_2$, *where* $f_1 \in IO^s$ *and* $f_2 \in IA^{s,p}$. *Furthermore,* $\|f\|_{IMO_r^{s,p}}$ *is equivalent to*

$$\inf\{\|f_1\|_{IO^s}+\|f_2\|_{IA^{s,p}}:\ f=f_1+f_2, f_1\in IO^s, f_2\in IA^{s,p}\}. \tag{1}$$

Proof. Suppose $f\in IMO_r^{s,p}$. Set $f_1=\hat{f}_{\frac{r}{2}}$ and $f_2=f-f_1$. For $|z-w|\le\frac{r}{2}$, we have

$$|f_1(z)-f_1(w)|\le\left|f_1(z)-\hat{f}_r(z)\right|+\left|\hat{f}_r(z)-f_1(w)\right|$$
$$\le\frac{1}{v\left(B\left(z,\frac{r}{2}\right)\right)}\int_{B\left(z,\frac{r}{2}\right)}\left|f(u)-\hat{f}_r(z)\right|dv(u)+\frac{1}{v(B(w,\frac{r}{2}))}\int_{B\left(z,\frac{r}{2}\right)}|f(u)-\hat{f}_r(z)|dv(u).$$

Since $B\left(z,\frac{r}{2}\right)$ and $B\left(w,\frac{r}{2}\right)$ are both contained in (z,r), it follows from Hölder's inequality that

$$|f_1(z)-f_1(w)|_\sim<\left(\frac{1}{v(B(z,r))}\int_{B(z,r)}|f(u)-\widehat{f}_r(z)|^p dv(u)\right)^{\frac{1}{p}}$$

This and Lemma (6.1.2) tell us $f_1\in IO^s$ and

$$\|f_1\|_{IO^s}\le C\|f\|_{IMO_r^{s,p}}. \tag{2}$$

For f_2,

$$\left(\overline{|f_2|p_{\frac{r}{2}}}\right)^{\frac{1}{p}}(z)\le\left(\frac{1}{v\left(B\left(z,\frac{r}{2}\right)\right)}\int_{B\left(z,\frac{r}{2}\right)}|f(u)-f_1(z)|^p dv(u)\right)^{\frac{1}{p}}$$
$$+\left(\frac{1}{v\left(B\left(z,\frac{r}{2}\right)\right)}\int_{B\left(z,\frac{r}{2}\right)}|f_1(u)-f_1(z)|^p dv(u)\right)^{\frac{1}{p}}$$
$$\le MO_{p,\frac{r}{2}}(f)(z)+\omega_{\frac{r}{2}}(f_1)(z).$$

This and (2) imply $\left(\widehat{|f_2|^p}_{\frac{r}{2}}\right)^{\frac{1}{p}}\in L^s(\mathbb{C}^n)$ with

$$\|f_2\|_{IA^{s,p}}\le C\|f\|_{IMO_r^{s,p}}. \tag{3}$$

Conversely, we show $f\in IMO_r^{s,p}$ whenever $f\in IO^s$ or $f\in IA^{s,p}$ with the desired norm estimates. Suppose $f\in IO^s$. Since

$$MO_{p,r}(f)(z)\le\left(\frac{1}{v(B(z,r))}\int_{B(z,r)}|f(w)-f(z)|^p dv(w)\right)^{\frac{1}{p}}$$
$$+|f(z)-\widehat{f}_r(z)|\le 2\omega_r(f)(z),$$

we have $f\in IMO_r^{s,p}$ with

$$\left\|f\right\|_{IMO_r^{s,p}} \leq C\left\| f_1\right\|_{IO^s}. \tag{4}$$

And for $f \in IA^{s,p}$ we have

$$MO_{p,r}(f)(z) \leq \left(\frac{1}{v\big(B(z,r)\big)}\int_{B(z,r)} |f(w)|^p dv(w)\right)^{\frac{1}{p}} + |\widehat{f_r}(z)|$$

$$\leq 2\big(\widehat{|f_2|^p}\big)^{\frac{1}{p}}.$$

Therefore $f \in IMO_r^{s,p}$ with

$$\|f\|_{IMO_r^{s,p}} \leq C\|f_2\|_{IA^{s,p}}. \tag{5}$$

The estimate (a) comes form (2)-(5) . The proof is finished.

Proposition (6.1.4) [230]: *Let* $1 \leq p < \infty, 1 \leq s \leq \infty, r > 0$. *Then for* $f \in L_{loc}^p$ *there holds*

(*a*) $f \in IMO^{s,p}$ *if and only if there exists a continuous function* $c(z)$ *on* $\mathbb{C}^n$ *such that*

$$\left(\frac{1}{v\big(B(z,r)\big)}\int_{B(z,r)} |f(w) - c(z)|^p dv(w)\right)^{\frac{1}{p}} \in L^s(\mathbb{C}^n)\,. \tag{6}$$

(*b*) $f \in VMO^p$ *if and only if there exists a continuous function* $c(z)$ *on* $\mathbb{C}^n$ *such that*

$$\frac{1}{v\big(B(z,r)\big)}\int_{B(z,r)} |f - c(z)|^p dv \to 0$$

as $Z \to \infty$.

Proof. (a) If $f \in IMO^{s,p}$, then (6) holds with $c(z) = \widehat{f_r}(z)$ which is continuous for $z \in \mathbb{C}^n$. Conversely, suppose (6) holds. By Minkowski's inequality, $MO_{p,r}(f)(z) = \left(\frac{1}{v(B(z,r))}\int_{B(z,r)} |f - \widehat{f_r}(z)|^p dv\right)^{\frac{1}{p}}$

$$\leq \left(\frac{1}{v\big(B(z,r)\big)}\int_{B(z,r)} |f - c(z)|^p dv\right)^{\frac{1}{p}} + |\widehat{f_r}(z) - c(z)|.$$

Meanwhile, Hölder's inequality tells us

$$|\widehat{f_r}(z) - c(z)| = |\frac{1}{v\big(B(z,r)\big)}\int_{B(z,r)} \big(f - c(z)\big)\, dv|$$

$$\leq \left(\frac{1}{v\big(B(z,r)\big)}\int_{B(z,r)} |f - c(z)|^p dv\right)^{\frac{1}{p}} \in L^s(\mathbb{C}^n)\,.$$

Therefore $f \in IMO^{s,p}$.s

The proof of the conclusion (b) is similar and is omitted here. The proof is finished

Proposition (6.1.5) [230]: Suppose $1 \leq p < \infty, 1 < s \leq \infty, r > 0$ and $f \in L_{loc}^p$. If for each $z \in \mathbb{C}^n$, there exist $h_1, h_2 \in H(B(z,r))$ satisfying

$$\left(\frac{1}{v(B(z,r))}\int_{B(z,r)} |f - h_1|^p dv\right)^{\frac{1}{p}} \in L^s(\mathbb{C}^n)$$

and

$$\left(\frac{1}{v(B(z,r))}\int_{B(z,r)} |\bar{f} - h_2|^p dv\right)^{\frac{1}{p}} \in L^s(\mathbb{C}^n),$$

then $f \in IMO^{s,p}$.

Proof. It is well known that, if $v : B(0,1) \to \mathbb{R}^n$ is pluriharmonic, there exists some pluriharmonic function $u : B(0,1) \to \mathbb{R}^n$ such that $u + iv \in H(B(0,1))$. Theorem 1 from [3] tells us there is some constant $C_1 > 0$ such that $||u - u(0)||_{Lp(B(0,1),dv)} \leq C_1||v||_{Lp(B(0,1),dv)}$. Hence, for any $r > 0$, by change of variables we know that, if $v : B(z,r) \to \mathbb{R}^n$ is pluriharmonic, there exists a pluriharmonic function u such that $u + iv \in H(B(z,r))$ and

$$\|u - u(z)\|_{L^p(B(z,r),dv)} \leq C_1\|v\|_{L^p(B(z,r),dv)}. \tag{7}$$

For $f \in L^p_{loc}$, set $\|f\|_{p,B(z,r)} = \left(\frac{1}{v(B(z,r))}\int_{B(z,r)} |f|^p dv\right)^{\frac{1}{p}}$ By triangle inequality we have

$$\|\frac{f+\overline{f}}{2} - \frac{h_1+h_2}{2}\|_{p,B(z,r)}$$
$$\leq \|\frac{f-h_1}{2}\|_{p,B(z,r)} + \|\frac{f-h_2}{2}\|_{p,B(z,r)} \in L^s(\mathbb{C}^n).$$

Since $f + \overline{f}$ is real valued, we get $\| \text{Im } \frac{h_1+h_2}{2}\|_{p,B(z,r)} \in L^s(\mathbb{C}^n)$. Notice that $h_1 + h_2 \in H(B(z,r))$, by (7) we obtain

$$\| \text{Re } \frac{h_1+h_2}{2} - \text{Re } \frac{h_1+h_2}{2}(z)\|_{p,B(z,r)} \leq C\| \text{Im } \frac{h_1+h_2}{2}\|_{p,B(z,r)}$$

Therefore,

$$\|\frac{f+\overline{f}}{2} - \text{Re } \frac{h_1+h_2}{2}(z)\|_{p,B(z,r)}$$
$$\leq \|\frac{f+\overline{f}}{2} - \text{Re } \frac{h_1+h_2}{2}\|_{p,B(z,r)} + \| \text{Re } \frac{h_1+h_2}{2} - \text{Re } \frac{h_1+h_2}{2}(z)\|_{p,B(z,r)}$$
$$\leq \|\frac{f+\overline{f}}{2} - \frac{h_1+h_2}{2}\|_{p,B(z,r)} + \| \text{Re } \frac{h_1+h_2}{2} - \text{Re } \frac{h_1+h_2}{2}(z)\|_{p,B(z,r)}$$

This shows $\|\frac{f+\overline{f}}{2} - \text{Re } \frac{h_1+h_2}{2}(z)\|_{p,B(z,r)} \in L^s(\mathbb{C}^n)$. Similarly, we have

$$\| \text{Im } \frac{f-\overline{f}}{2} - \text{Im } \frac{h_1-h_2}{2}(z)\|_{p,B(z,r)} \in L^s(\mathbb{C}^n).$$

Choose $c(z) = \text{Re } \frac{h_1+h_2}{2}(z) + i \text{ Im } \frac{h_1-h_2}{2}(z)$, then $\|f - c(z)\|_{p,B(z,r)} \in L^s(\mathbb{C}^n)$. From this and Proposition (6.1.4) the desired result follows immediately. The proof is finished.

we consider mapping properties of some weighted convolution operators $T_{f,\varepsilon}$ which will play a key role in our study of Hankel operators from F_α^p to L_α^q for $1 \le q < p < \infty$. Our analysis will be carried out in real analysis.

Given two Lebesgue measurable functions f and g on $\mathbb{R}^m$, if

$$\int_{\mathbb{R}^m} f(x-y)g(y)dv(y)$$

converges, we call the integral above the convolution of f and g.

We use L^p to denote $L^p(\mathbb{R}^m, dv)$ for short. Young' s inequality tells us for $1 \le p \le \infty$ and $\in L^p$, $g \in L^1$, there holds

$$\| \int_{\mathbb{R}^m} f(x-y)g(y)dv(y)\|_{L^p} \le \|g\|_{L^1}\|f\|_{L^p}.$$

More details about convolution can be found in [11].

Given some Lebesgue measurable function f on $\mathbb{R}^m$, set

$$\omega(f)(x) = \sup\{|f(y)-f(x)| : y \in \mathbb{R}'^n, |y-x| < 1\}$$

For $\varepsilon > 0$ fixed, define integral operator $T_{f,\varepsilon}$ as

$$T_{f,\varepsilon}h(x) = \int_{\mathbb{R}^m} \left(\int_0^1 w(f)\big(x+t(y-x)\big)dt\right) e^{-\varepsilon|y-x|}h(y)dv(y).$$

The operator $T_{f,\varepsilon}$ can be considered as a kind of weighted convolution operators. The following result shows the mapping properties of $T_{f,\varepsilon}$ from L^p to L^q.

Theorem (6.1.6) [230]: *Let* $1 \le q < p < \infty$, $\varepsilon > 0$. *Suppose* f *is Lebesgue measurable such that* $f \in IO^{\frac{pq}{p-q}}$.

(*a*) *The integral operator* $T_{f,\varepsilon}$ *is bounded from* L^p *to* L^q. *Moreover,*

$$\|T_{f,\varepsilon}\|_{L^p \to L^q} \le C\|f\|_{IO^{\frac{pq}{p-q}}}.$$

(*b*) *For bounded sequence* $\{h_k\}$ *in* L^p *satisfying* $\lim_{k\to\infty} \sup_{|z|\le R} |h_k(z)| = 0$ *for all* $R > 0$, *there holds* $\lim_{k\to\infty} \| T_{f,\varepsilon}(h_k)\|_{L^q} = 0$.

Proof. (a) Write $s = \frac{pq}{p-q}$, then $s > 1$, and $s' = \frac{pq}{pq-p+q}$. By Hölders inequality we have

$$|T_{f,\varepsilon}h(x)| \le \int_{\mathbb{R}^m} \left(\int_0^1 (\lambda)\,(f)^s\big(x+t(y-x)\big)dt\right)^{\frac{1}{s}} e^{-\varepsilon|y-x|}|h(y)|dv(y)$$

$$\le [\int_{\mathbb{R}^m} (\int_0^1 \omega(f)^s\big(x+t(y-x)\big)dt) e^{-s\cdot\frac{\varepsilon}{2}|y-x|}dv(y)]^{\frac{1}{s}}$$

$$\left[\int_{\mathbb{R}^m} |h(y)|^{s'} e^{-s'\cdot\frac{\varepsilon}{2}|y-x|}dv(y)\right]^{\frac{1}{s}}$$

Write

$$I_1 = \int_{\mathbb{R}^m} \left(\int_0^1 \omega(f)^s\big(x+t(y-x)\big)dt\right) e^{-s\cdot\frac{\varepsilon}{2}|y-x|}dv(y),$$

and

$$I_2 = \int_{\mathbb{R}^m} | h(y)|^{s'} e^{-s' \cdot \frac{\varepsilon}{2}|y-x|} dv(y) .$$

Then Fubini's theorem gives

$$\int_{\mathbb{R}^m} I_1 \, dv(x)$$

$$= \int_{\mathbb{R}^m} d\, v(x) \int_0^1 d\, t \int_{\mathbb{R}^m} w\,(f)^s\big(x + t(y-x)\big) e^{-s \cdot \frac{\varepsilon}{2}|y-x|} dv(y)$$

$$= \int_{\mathbb{R}^m} d\, v(x) \int_0^1 d\, t \int_{\mathbb{R}^m} \omega\,(f)^s(x + tu) e^{-s \cdot \frac{\varepsilon}{2}|u|} dv(u)$$

$$= \int_0^1 d\, t \int_{\mathbb{R}^m} e^{-s \cdot \frac{\varepsilon}{2}|u|}\, dv(u) \int_{\mathbb{R}^m} (\, v(f)^s(x + tu) dv(x)$$

$$= \|f\|_{IO^s}^s \int_0^1 d\, t \int_{\mathbb{R}^m} e^{-s \cdot \frac{\varepsilon}{2}|u|}\, dv(u)$$

$$= C\|f\|_{IO^s}^s.$$

Since $\frac{p}{s} \geq 1$, by Hölder's inequality and Fubini's theorem we get

$$\int_{\mathbb{R}^m} e^{-s' \cdot \frac{\varepsilon}{2}|y-x|}\, |h(y)|^{s'} dv(y)$$

$$\leq \left(\int_{\mathbb{R}^m} e^{-\frac{p\varepsilon}{4}|y-x|}\, |h(y)|^{s' \cdot \frac{p}{s}} dv(y) \right)^{\frac{s'}{p}} \left(\int_{\mathbb{R}^m} e^{-\frac{\varepsilon\ pq}{4pq-p}|y-x|}\, dv(y) \right)^{1-\frac{s'}{p}}$$

Then,

$$\int_{\mathbb{R}^m} (I_2)^{\frac{p}{s}}\, dv(x)$$

$$= \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^m} e^{-s' \cdot \frac{\varepsilon}{2}|y-x|}\, |h(y)|^{s'} dv(y) \right)^{\frac{p}{s}} dv(x)$$

$$\leq C \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^m} e^{-\frac{p\varepsilon}{4}|y-x|}\, |h(y)|^{s' \cdot \frac{p}{s}} dv(y) \right) dv(x)$$

$$= C \int_{\mathbb{R}^m} | h(y)|^p dv(y) .$$

Therefore,

$$\int_{\mathbb{R}^m} \left(T_{f,\varepsilon} h(x)\right)^q dv(x) \leq \int_{\mathbb{R}^m} \left((I_1)^{\frac{q}{s}} (I_2)^{\frac{q}{s}} \right) dv(x)$$

$$\leq \left(\int_{\mathbb{R}^m} I_1\, dv(x)\right)^{\frac{q}{s}} \left(\int_{\mathbb{R}^m} (I_2)^{\frac{p}{s}}\, dv(x)\right)^{\frac{q}{p}}$$

$$\leq C\|f\|_{IO^s}^q \cdot \|h\|_{L^p}^q.$$

This means $T_{f,\varepsilon}$ is bounded from L^p to L^q with

$$\|T_{f,\varepsilon}\|_{L^p \to L^q} \leq C\|f\|_{IO^s},$$

which gives the statement (1).

Now we prove the statement (2). To do this, for $\rho > 0$ we consider $T_{f,\varepsilon}(h)(x) = J_{f,p}(h)(x) + Q_{f,\rho}(h)(x)$ with

$$J_{f,\rho}(h)(x) = \int_{|y-x|\geq\rho} (\int_0^1 \omega(f)\big(x+t(y-x)\big)dt) e^{-\varepsilon|y-x|}|h(y)|dv(y)\,,$$

and

$$Q_{f,p}(h)(x) = \int_{|y-x|<\rho} \left(\int_0^1 \omega(f)\big(x+t(y-x)\big)dt\right) e^{-\varepsilon|y-x|}|h(y)|dv(y)\,.$$

Then,

$$J_{f,p}(h)(x)$$

$$\leq e^{-\frac{\varepsilon}{2}\rho} \int_{|y-x|\geq\rho} \left(\int_0^1 \omega\,(f)\big(x+t(y-x)\big)dt\right) e^{-\frac{\varepsilon}{2}|y-x|}|h(y)|dv(y)$$

$$\leq e^{-\frac{\varepsilon}{2}p} T_{f,\frac{\varepsilon}{2}}(|h|)(x)$$

From (1) we have

$$\|J_{f,\rho}(h)\|_{L^q} \leq Ce^{-\frac{\varepsilon}{2}\rho}\|f\|_{IO^s} \cdot \|h\|_{L^p}. \tag{8}$$

To estimte $\|Q_{f,\rho}(h)\|_{L^q}$, for $R > 0$ let χ_R be the characteristic function of $B(0,R)$. Then

$$\|Q_{f,p}(1)\chi_R\|_{L^q}$$

$$\leq CR^{\frac{m}{p}}\left\{\int_{|x|<R} Q_{f,p}\,(1)^s(x)dv(x)\right\}^{\frac{1}{s}}$$

$$\leq CR^{\frac{m}{p}}\{\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\int_0^1 \omega(f)^s\big(x+t(y-x)\big)e^{-\frac{\varepsilon}{2}|y-x|}dtdv(y)dv(x)\}^{\frac{1}{s}}$$

$$= CR^{\frac{m}{p}}\|f\|_{IO^s}.$$

Therefore,

$$\|Q_{f,\rho}(h)\chi_R\|_{L^q} \leq CR^{\frac{m}{p}}(\sup|h(\xi)|)\|f\|_{IO^s}. \tag{9}$$

On the other hand, it is easy to verify that

$$\sup_{y\in B(x,p),t\in[0,1]} \omega(f)\big(x+t(y-x)\big) \leq \omega_{p+1}(f)(x)\,.$$

By Young's inequality, we know

$$\| \int_{\mathbb{R}^m} e^{-\varepsilon|y-x|}\, |h(y)|dv(y)\|_{L^p} \le C\|h\|_{L^p}.$$

Then, applying Hölder's inequality with the exponents $\frac{p}{p-q}$ and $\frac{p}{q}$ to get

$$\|Q_{f,\rho}(h)(1-\chi_R)\|_{L^q}$$

$$\le \left(\int_{|x|\ge R(} \omega_{\rho+1}\,(f)^s(x) \right)^{\frac{1}{s}} \| \int_{|y-x|<p} e^{-\varepsilon|y-x|}\, |h(y)|dv(y)\|_{L^p}$$

$$\le C(\int_{|x|\ge R} \omega_{\rho+1}(f)^s(x))^{\frac{1}{s}}\, \|h\|_{L^p}. \tag{10}$$

The constants C in (8), (9) and (10) are independent of ρ and R. Now suppose $\{h_k\}_{j=1}^{\infty}$ is a bounded sequence in L^p satisfying $\sup_{|z|\le R}|h_k(z)| \to 0$ as $k \to \infty$ for any fixed $R > 0$. Without loss of generality, we may assume

$\|h_k\|_{L^p} \le 1$. Given any $\varepsilon > 0$, pick some $p > 0$ so that $e^{-\frac{\varepsilon}{2}p} < \varepsilon$. For this ρ, by Lemma (6.1.2) $f \in IO_{p+1}^s$. We have some $R > 0$ such that

$$\left(\int_{|x|\ge R} \omega_{\rho+1}\,(f)^s(x) \right)^{\frac{1}{\varepsilon}} < \varepsilon.$$

From (8), (9) and (10) we have, whenever k is sufficiently large,

$$\|T_{f,\varepsilon}(h_k)\|_{L^q}$$

$$\le \|J_{f,\rho}(h_k)\|_{L^q} + \|Q_{f,\rho}(h_k)\chi_R\|_{L^q} + \|Q_{f,\rho}(h_k)(1-\chi_R)\|_{L^q}$$

$$\le C\left\{ \varepsilon(\|f\|_{IO^s}+1) + R^{\frac{m}{p}} \left(\sup_{|\xi|\le R+\rho} \; |h_k(\xi)| \right) \|f\|_{IO^s} \right\}$$

$$\le K\varepsilon,$$

where the constants K is independent of ε. Therefore,

$$\lim_{k\to\infty} \| T_{f,\varepsilon}(h_k)\|_{L^q} = 0$$

as desired. The proof is finished.

we are going to provide a proof of Theorem (6.1.1). Recall that Γ is the family of those measurable function f on $\mathbb{C}^n$ satisfying $fk_z \in \bigcup_{p\ge 1}\; L_\alpha^p$ for all $z \in \mathbb{C}^n$.

Theorem (6.1.7) [230]: *Suppose* $1 \le p \le q < \infty, \alpha > 0$ *and* $f \in \Gamma$.

(*i*) H_f *and* $H_{\bar{f}}$ *are both bounded from* F_α^p *to* L_α^q *if and only if* $f \in BMO^q$. *Moreover,*

$$\|H_f\|_{F_\alpha^p\to L_\alpha^q} + \|H_{\bar{f}}\|_{F_\alpha^p\to L_\alpha^q} \simeq \|f\|_{BMO^q}. \tag{11}$$

(*ii*) H_f *and* $H_{\bar{f}}$ *are both compact from* F_α^p *to* L_α^q *if and only if* $f \in VMO^q$.

Proof. (i) Suppose $f \in BMO^q$. From Theorem 3.3 in [5] we know $H_f, H_{\bar{f}} :\, F_\alpha^q \to L_\alpha^q$ are bounded. Moreover,

$$\|H_f\|_{F_\alpha^q\to L_\alpha^q} + \|H_{\bar{f}}\|_{F_\alpha^q\to L_{\alpha\sim}^q} < \|f\|_{BMO^q}.$$

Since the Fock spaces have the nest property that the inclusion $F_{CX}^{p} \subset F_{\alpha}^{q}$ is bounded for $0 < p \leq q < \infty$, we have $H_f, H_{\overline{f}} : F_\alpha^p \to L_\alpha^q$ are both bounded, and the left hand side of (11) can be dominated by the right hand side.

Conversely, suppose $H_f, H_{\overline{f}}$ are both bounded from F_α^p to L_α^q. Set

$$M_{\alpha,q}(f)(z) = \|fk_z - \overline{g_z(z)}k_z\|_{q,\alpha},$$

where g_z denotes the holomorphic function on $\mathbb{C}^n$ given by

$$g_z(w) = \frac{P_\alpha\left(\overline{f}k_z\right)(w)}{k_z(w)}, w \in \mathbb{C}^n$$

It is easy to check that $g_z(z)$ is continuous on $\mathbb{C}^n$. Clearly, $\|k_z\|_{p,\alpha} = 1$. By Minkowski's inequality,

$$\begin{aligned} M_{\alpha,q}(f)(z) &\leq \|fk_z - P_\alpha(fk_z)\|_{q,\alpha} + \|P_\alpha(fk_z) - \overline{g_z(z)}k_z\|_{q,\alpha} \\ &= \|H_f(k_z)\|_{q,\alpha} + \|P_\alpha(fk_z) - \overline{g_z(z)}k_z\|_{q,\alpha}. \end{aligned}$$

Notice that

$$\overline{g_z(z)}k_z = P_\alpha(\overline{g_z}k_z)\,. \tag{12}$$

To see this, since $K_z(w) = \overline{K_w(z)}$, by the reproducing formula

$$\begin{aligned} \overline{g_z(z)}k_z(w) &= \|K_z\|_{2,\alpha}^{-1} \cdot \overline{g_z(z)K_w(z)} = \|K_z\|_{2,\alpha}^{-1} \cdot \overline{\langle g_z K_w, K_z\rangle_\alpha} \\ &= \|K_z\|_{2,\alpha}^{-1} \cdot \langle K_z, g_z K_w\rangle_\alpha = \langle \overline{g_z}k_z, K_w\rangle_\alpha \\ &= P_\alpha(\overline{g_z}k_z)(w)\,. \end{aligned}$$

Hence, by (12) and the boundedness of P_α on L_α^q,

$$\begin{aligned} \|P_\alpha(fk_z) - \overline{g_z(z)}k_z\|_{q,\alpha} &= \|P_{(X}(fk_z) - P_\alpha(\overline{g_z}k_z)\|_{q,\alpha} \\ &\leq \|P_\alpha\|_{L_\alpha^q \to L_\alpha^q} \cdot \|(f - \overline{g})k_z\|_{q,\alpha} \\ &= \|P_\alpha\|_{L_\alpha^q \to L_\alpha^q} \cdot \|\overline{f}k_z - g_z k_z\|_{q,\alpha} \\ &= \|P_{(X}\|_{L_\alpha^q \to L_\alpha^q} \cdot \|H_{\overline{f}}(k_z)\|_{q,\alpha}. \end{aligned}$$

Therefore

$$M_{\alpha,q}(f)(z) \leq \left(1 + \|P_\alpha\|_{L_\alpha^q \to L_\alpha^q}\right) \cdot \left(\|H_f(k_z)\|_{q,\alpha} + \|H_{\overline{f}}(k_z)\|_{q,\alpha}\right). \tag{13}$$

Since $H_f, H_{\overline{f}} : F_\alpha^p \to L_\alpha^q$ are both bounded, we get $M_{\alpha,q}(f) \in L^\infty(\mathbb{C}^n)$. Meanwhile, by the fact that $\inf_{w \in B(z,r)} |e^{\alpha\langle w,z\rangle}| \geq Ce^{\frac{\alpha}{2}|z|^2 + \frac{\alpha}{2}|w|^2}$ we have

$$\begin{aligned} &\left(M_{\alpha,q}(f)(z)\right)^q \\ &= \int_{\mathbb{C}^n} |f(w)k_z(w) - \overline{g_z(z)}k_z(w)|^q e^{-\frac{q\alpha}{2}|w|^2} dv(w) \\ &\geq \|K_z\|_{2,\alpha}^{-q} \cdot \int_{B(z,r)} |f(w) - \overline{g_z(z)}|^q |e^{\alpha\langle w,z\rangle}|^q e^{-\frac{q\alpha}{2}|w|^2} dv(w) \\ &\geq C \int_{B(z,r)} |f(w) - \overline{g_z(z)}|^q dv(w)\,. \end{aligned} \tag{14}$$

Therefore,

$$\int_{B(z,r)} | f(w) - \overline{g_z(z)}|^q dv(w) \in L^\infty(\mathbb{C}^n) .$$

It follows from Proposition (6.1.4) that $f \in BMO^q$. And the right-hand side of (11) can be dominated by the left-hand side.

(ii) Suppose $f \in VMO^q s$. By Theorem 3.39 in [232], f admits a decomposition $f = f_1 + f_2$, where f_1 and f_2 satisfies

$$\lim_{z\to\infty} \omega\,(f_1)(z) = 0 \tag{15}$$

and

$$\lim_{z\to\infty} \frac{1}{v\big(B(z,r)\big)} \int_{B(z,r)} | f_2(w)|^q dv(w) = 0. \tag{16}$$

respectively. For f_1, given any $\varepsilon > 0$ we claim that there exists some h with compact support such that

$$\|f_1 - h\|_{BO} < \varepsilon. \tag{17}$$

In fact, by (15) we have some $r > 0$ such that $w(f_1)(z) < \varepsilon$ whenever $|z| \geq r$. Similar to Lemma 3.33 in [232], for $|z| \geq r$ we have

$$|f_1(z) - f_1\left(\frac{r}{|z|} z\right)| < \varepsilon\big(1 + (|z| - r)\big) .$$

Then, $|f_1(z)| < \sup_{|\xi|=r}|f_1(\xi)| + \varepsilon(|z| + 1)$ which implies there is some $R > r + 2$ such that

$$\frac{|f_1(z)|}{|z|} < 2\varepsilon \tag{18}$$

whenever $|z| \geq R - 2$. Set

$$s(z) = \begin{cases} 1 & 0 \leq |z| < R; \\ \dfrac{1}{R}(2R - |z|) & R \leq t < 2R; \\ 0, & |z| \geq 2R. \end{cases}$$

Then, $\omega(s) \leq \frac{1}{R}$. Define $h(z) = f_1(z)s(z)$. For $|z| \leq R - 1, \omega(f_1 - h)(z) = 0$. For $|z| \geq 2R + 1, \omega(f_1 - h)(z) = \omega(f_1)(z) < \varepsilon$. For $R - 1 < |z| < 2R + 1$ and $w \in B(z, 1)$, by (18) we know

$$\begin{aligned} &|\big(f_1(z) - h(z)\big) - \big(f_1(w) - h(w)\big)| \\ &\leq |f_1(w)||s(|w|) - s(|z|)| + \big(1 - s(|z|)\big)|f_1(w) - f_1(z)| \\ &\leq |f_1(w)|\omega(s)(|z|) + (x)(f_1)(z) \\ &\leq |f_1(w)|\frac{1}{R} + \omega(f_1)(z) \\ &= \frac{|f_1(w)|}{|w|}\frac{|w|}{R} + \omega(f_1)(z) \\ &\leq 2\varepsilon\frac{2R + 2}{R} + \varepsilon < 7\varepsilon. \end{aligned}$$

From these (17) follows. Since h has compact support, H_h is compact from F_α^p to L_α^q. Furthermore, (4) and (11), (17) imply

$$\|H_{f_1} - H_h\|_{F_\alpha^p \to L_\alpha^q} \le C\|f_1 - h\|_{BO} < C\varepsilon,$$

where C is independent of εi. Therefore, $H_{f_1} : F_\alpha^p \to L_\alpha^q$ is also compact.

For f_2, set $f_{2,R} = f_2 \cdot \chi_R$, where χ_R is the characteristic function of $B(0,R)$. Since $f_{2,R}$ is compact supported, $H_{f_{2,R}}$ is compact from F_α^p to L_α^q. Since f_2 satisfies (16), from (5) and (11) we have

$$\|H_2 - H_{2,R}\|_{F_\alpha^p \to L_\alpha^q} \le C\|f_2 - f_{2,R}\|_{BA^q} \to 0,$$

as $R \to \infty$. Hence, H_{f_2} is compact from F_α^p to L_α^q. Therefore, H_f is compact from F_α^p to L_α^q. Similarly, $H_{\overline{f}}$ is compact from F_α^p to L_α^q as well.

Conversely, if H_f and $H_{\overline{f}}$ are compact from F_α^p to L_α^q. Notice that k_z converges to 0 weakly in F_α^p as $z \to \infty$, it follows from (13) that

$$\lim_{z\to\infty} M_{\alpha,q}(f)(z) = 0.$$

Now the estimate (14) tells us

$$\lim_{z\to\infty} \int_{B(z,r)} |f(w) - \overline{g_z(z)}|^q dv(w) = 0.$$

From Proposition (6.1.4) we know $f \in VMO^q$. This completes the proof.

Next we characterize the boundedness (and the compactness) of both H_f, $H_{\overline{f}}$ from F_α^p to L_α^q for $1 \le q < p < \infty$. To this end, we first introduce Khinchine's inequality. Let r_k be the Rademacher function defined by

$$r_0(t) = \begin{cases} 1, & if\ 0 \le t - [t] < \dfrac{1}{2} \\ -1, & if\ \dfrac{1}{2} \le t - [t] < 1 \end{cases}$$

and $r_k(t) = r_0(2^k t)$ for $k = 1,2,$, where $[t]$ denotes the largest integer less than or equal to t. For $0 < l < \infty$, we need Khinchine's inequality. That is, there exists some positive constants C_1 and C_2 depending only on l such that

$$C_1\left(\sum_{k=1}^m |b_k|^2\right)^{\frac{l}{2}} \le \int_0^1 |\sum_{k=1}^m b_k\, r_k(t)|^l dt \le C_2\left(\sum_{k=1}^m |b_k|^2\right)^{\frac{l}{2}}$$

for all $m \ge 1$ and complex numbers $b_1, b_2, , b_m$. More details can be found in [7].

As in [232], a sequence $\{a_k\}$ in $\mathbb{C}^n$ is called an r-lattice if the following conditions are satisfied:

(a)$\bigcup_{k=1}^{\infty}\ Ba_k, r) = \mathbb{C}^n$;

(b) $\left\{B\left(a_k, \frac{r}{4}\right)\right\}_{k=1}^{\infty}$ arem utually disjoint.

With these two hypotheses, it is easy to check that

(c) For any $\delta > 0$, there exists a positive integer m depending only on r and δ such that every point in $\mathbb{C}^n$ belongs to at most m of the sets $\{B(a_k,\delta)\}$. Given $r>0$, it is easy to pick $a_k \in \mathbb{C}^n$ such that $\{a_k\}$ is an r-lattice. Given $f \in L^p_{loc}$, we define

$$G_{p,r}(f)(z) = \inf\left\{\left(\frac{1}{v\big(B(z,r)\big)}\int_{B(z,r)} |f-h|^p dv\right)^{\frac{1}{p}} : h \in H\big(B(z,r)\big)\right\}.$$

The atomic decomposition for Fock spaces turns out to be a powerful theorem in the theory of Fock spaces. The following theorem, Theorem (6.1.8), is basically due to [236].

Theorem (6.1.8) [230]: Let $0<p\le\infty$. For r-lattice $\{a_k\}$ and $\{\lambda_k\}\in l^p$,

$$f(z)=\sum_{k=1}^{\infty}\lambda_k\, e^{\alpha\langle z,a_k\rangle-\frac{\alpha}{2}|a_k|^2} \in F^p_\alpha, \tag{19}$$

and $\|f\|_{p,\alpha}\le \|\{\lambda_k\}\|_{l^p}$. Furthermore, there exists some positive constant r_0 such that, for any $0<r<r_0$, the space F^p_α consists exactly of the functions (19), where $\{\lambda_k\}\in l^p$ and $\{a_k\}$ is an r-lattice. And

$$C^{-1}\|f\|_{p,\alpha}\le \inf\|\{\lambda_k\}\|_{lp}\le C\|f\|_{p,\alpha}$$

for all $f\in F^p_\alpha$, where the infimun is taken over all sequences $\{\lambda_k\}$ that give rise to the decomposition in (19).

Lemma (6.1.9)[230]: *For continuous functions f on $\mathbb{C}^n$, there holds*

$$|f(z)-f(w)|\le 4(1+|z-w|)\int_0^1 \omega(f)\big(z+t(w-z)\big)dt$$

for all z and w in $\mathbb{C}^n$.

Proof. Suppose z and w in $\mathbb{C}^n$, if $|z-w|<1$, then for any $t\in[0,1]$, we have

$$\begin{aligned}|f(z)-f(w)| &\le |f(z)-f\big(z+t(w-z)\big)|+|f\big(z+t(w-z)\big)-f(w)|\\ &\le 2\cdot\omega(f)\big(z+t(w-z)\big).\end{aligned}$$

Integrating both sides with respect to t from 0 to 1,

$$|f(z)-f(w)|\le 2\int_0^1 \omega(f)\big(z+t(w-z)\big)dt. \tag{20}$$

Now for $|z-w|\ge 1$, let $N=[|z-w|+1]$, where $[x]$ denotes the largest integer less than or equal to x, and set $z_j=z+\frac{j}{N}(w-z), j=0,1,\dots,N$. Then

$$|z_{j+1}-z_j|=\frac{1}{N}|w-z|<1,$$

Therefore,

$$\begin{aligned}|f(z)-f(w)| &\le \sum_{j=0}^{N-1}|f\big(z_{j+1}\big)-f\big(z_j\big)|\\ &\le 2\sum_{j=0}^{N-1}\int_0^1 \omega(f)\left(z_j+\frac{w-z}{N}\cdot t\right)dt\end{aligned}$$

$$= 2\sum_{j=0}^{N-1} N\int_0^{\frac{1}{m}} \omega(f)\left(z_j + s(w-z)\right)ds$$
$$= 2N\int_0^1 \omega(f)\big(z + t(w-z)\big)dt$$
$$\le 4|z-w|\int_0^1 \omega(f)\big(z+t(w-z)\big)dt.$$
This together with (20) completes the proof of the proposition.

Theorem (6.1.10) [230]: *Let* $1 \le q < p < \infty, \alpha > 0$ *and set* $s = \frac{pq}{p-q}$. *Then for* $f \in \Gamma$, *the following statements are equivalent*:

(a) $H_f, H_{\bar{f}}: F_\alpha^p \to L_\alpha^q$ *are bounded*

(b) $H_f, H_{\bar{f}}: F_\alpha^p \to L_\alpha^q$ *are compact.*

(*c*) $f \in IMO^{s,q}$.

Furthermore,
$$\|H_f\|_{F_\alpha^p \to L_\alpha^q} + \|H_{\bar{f}}\|_{F_\alpha^p \to L_\alpha^q} \simeq \|f\|_{IMO^{s,q}}. \tag{21}$$
Proof. We prove the implications (a) ⇒ (c) and (c) ⇒ (b) . The implication (b) ⇒ (a) is trivial.

(a) ⇒ (c) . Fix $r > 0$, and let $\{a_k\}$ be an r-lattice. According to Theorem (6.1.8), for any $\{\lambda_k\} \in l^p$, we have $g_t(z) = \sum_{k=1}^{\infty} \lambda_k r_k(t) k_{a_k}(z) \in F_\alpha^p$, and $\|g_t\|_{p,\alpha} \le C\|\{\lambda_k\}\|_{l^p}$. It is trivial to check that $|k_{a_j}(z)|^q e^{-\frac{q\alpha}{2}|z|^2} = e^{-\frac{q\alpha}{2}|z-a_j|^2} \simeq 1$ for $\in B(a_j, r)$. By Fubini's theorem and Khinchine's inequality we obtain
$$\int_0^1 \| H_f g_t\|_{q,\alpha}^q dt$$
$$= \int_0^1 \int_{\mathbb{C}^n} |\sum_{k=1}^{\infty} \lambda_k r_k(t) H_f k_{a_k}(z)|^q e^{-\frac{q\alpha}{2}|z|^2} dv(z)dt$$
$$= \int_{\mathbb{C}^n} e^{-\frac{q\alpha}{2}|z|^2} dv(z) \int_0^1 |\sum_{k=1}^{\infty} \lambda_k r_k(t) H_f k_{a_k}(z)|^q dt$$
$$\ge C\int_{\mathbb{C}^n} \left(\sum_{k=1}^{\infty} |\lambda_k|^2 |H_f k_{a_k}(z)|^2\right)^{\frac{q}{2}} e^{-\frac{q\alpha}{2}|z|^2} dv(z)$$
$$\ge C\sum_{j=1}^{\infty} \int_{B(a_j,r)} \left(\sum_{k=1}^{\infty} |\lambda_k|^2 |H_f k_{a_k}(z)|^2\right)^{\frac{q}{2}} e^{-\frac{q\alpha}{2}|z|^2} dv(z)$$
$$\ge C\sum_{j=1}^{\infty} |\lambda_j|^q \int_{B(a_j,r)} |f(z)k_{a_j}(z) - P_\alpha\left(f k_{a_j}\right)(z)|^q e^{-\frac{q\alpha}{2}|z|^2} dv(z)$$

$$\geq C\sum_{j=1}^{\infty}|\lambda_j|^q\int_{B(a_j,r)}|f(z)-\frac{P_\alpha\left(fk_{a_j}\right)(z)}{k_{a_j}(z)}|^q|k_{a_j}(z)|^q e^{-\frac{q\alpha}{2}|z|^2}dv(z)$$

$$\geq C\sum_{j=1}^{\infty}|\lambda_j|^q\left(G_{q,r}(f)(a_j)\right)^q.$$

Meanwhile, the boundedness of $H_f : F_\alpha^p \to L_\alpha^q$ gives

$$\|H_f g_t\|_{q,\alpha}^q \leq C\|H_f\|_{F_\alpha^p\to L_\alpha^q}^q \cdot \|g_t\|_{p,\alpha}^q.$$

Hence,

$$\sum_{j=1}^{\infty}|\lambda_j|^q\left(G_{q,r}(f)(a_j)\right)^q \leq C\|H_f\|_{F_\alpha^p\to L_\alpha^q}^q \cdot \|\{|\lambda_k|^q\}\|_{l^{\frac{p}{q}}}^q.$$

Since the conjugate exponent of $\frac{p}{q}$ is $\frac{p}{p-q}$, a duality argument implies

$$\sum_{j=1}^{\infty}\left(G_{q,r}(f)(a_j)\right)^s \leq C\|H_f\|_{F_\alpha^p\to L_\alpha^q}^s.$$

On the other hand, the fact that

$$G_{q,t}(f)(w) \leq CG_{q,r}(f)(z),$$

when $B(w,t)\subset B(z,r)$ gives

$$\int_{\mathbb{C}^n}\left(G_{q,r}(f)(z)\right)^s dv(z) \leq \sum_{j=1}^{\infty}\int_{B(a_j,r)}\left(G_{q,r}(f)(z)\right)^s dv(z)$$

$$\leq C\sum_{j=1}^{\infty}\left(G_{q,2r}(f)(a_j)\right)^s$$

This shows that , for any $r>0, G_{q,r}(f)\in L^s$. Similarly, the boundedness of $H_{\overline{f}}$ from F_α^p to L_α^q implies $G_{q,r}(\overline{f})\in L^s$ for any $r>0$. By the definition of $G_{q,r}(f)$, for each $z\in\mathbb{C}^n$ we have $h_1,h_2\in H\big(B(z,r)\big)$ such that

$$\left(\frac{1}{v\big(B(z,r)\big)}\int_{B(z,r)}|f-h_1|^q dv\right)^{\frac{1}{q}} \leq 2G_{q,r}(f)(z),$$

and

$$\left(\frac{1}{v\big(B(z,r)\big)}\int_{B(z,r)}|\overline{f}-h_2|^q dv\right)^{\frac{1}{q}} \leq 2G_{q,r}(\overline{f})(z).$$

It follows from Proposition (6.1.5) that $f\in IMO^{s,q}$. Moreover, the right-hand side of (21) can be dominated by the left-hand side.

(c) ⇒(b) . Suppose $\in IMO^{s,q}$. First, we prove that the left-hand can be dominated by the right-hand side. By Theorem (6.1.3), f admits a decomposition $f=f_1+f_2$ with

$$\|f_1\|_{IO^s}+\|f_2\|_{IA^{s,q}} \leq C\|f\|_{IMO^{s,q}}. \tag{22}$$

For $f_2 \in IA^{s,q}$, the boundedness of P_α on L^q_α yields

$$\|H_{f_2}g\|_{q,\alpha} \le \|f_2 g\|_{q,\alpha} + \|P_\alpha(f_2 g)\|_{q,\alpha} \le C\|f_2 g\|_{q,\alpha}.$$

Theorem 3.3 from [234] tells us $d\mu = |f_2|^q dv$ is a(p,q)-Fock Carleson measure. Thus the operator M_{f_2} defined by $M_{f_2}g = f_2 g$ is bounded from F^p_α to L^q_α. Precisely,

$$\|M_{f_2}g\|_{q,\alpha} \le C\|\left(\widehat{|f_2|^q}_r\right)^{\frac{1}{q}}\|_{L^s}\|g\|_{p,\alpha}.$$

Therefore, $H_{f_2} : F^p_\alpha \to L^q_\alpha$ is bounded with

$$\|H_{f_2}\|_{F^p_\alpha \to L^q_\alpha} \le C\|f_2\|_{IA^{s,q}}. \quad (23)$$

Now we treat H_{f_1} with $f_1 \in IO^s$. Notice that there are some $C > 0$ and $\varepsilon > 0$ such that, for $z, w \in \mathbb{C}^n$,

$$|e^{\alpha\langle w,z\rangle}| \le Ce^{\frac{\alpha}{2}|z|^2+\frac{\alpha}{2}|w|^2-2\varepsilon|z-w|}, |z-w| \le Ce^{\varepsilon|z-w|}.$$

For $g \in F^p_\alpha$, set $h(w) = g(w)e^{-\frac{\alpha}{2}|w|^2}$. Then, $h \in L^p(\mathbb{C}^n, dv)$. From Lemma (6.1.9) we have

$$\begin{aligned}
&\|H_{f_1}g\|^q_{q,\alpha} \\
&= \int_{\mathbb{C}^n} |H_{f_1}g(z)|^q e^{-\frac{q\alpha}{2}|z|^2} dv(z) \\
&= \int_{\mathbb{C}^n} \left|\int_{\mathbb{C}^n} \big(f_1(z) - f_1(w)\big)\, g(w)e^{\alpha\langle z,w\rangle} dv_\alpha(w)\right|^q e^{-\frac{q\alpha}{2}|z|^2} dv(z) \\
&\le C\int_{\mathbb{C}^n} \left(\int_{\mathbb{C}^n} |f_1(z) - f_1(w)| \cdot |g(w)e^{-\frac{\alpha}{2}|w|^2}| e^{-2\varepsilon|z-w|} dv(w)\right)^q dv(z) \\
&\le C\int_{\mathbb{C}^n} T_{f_1,\varepsilon}(h)^q(z) dv(z) ,
\end{aligned}$$

Applying Theorem (6.1.6) to obtain $\|H_{f_1}g\|_{q,\alpha} \le C\|f_1\|_{IO^s}\|g\|_{p,\alpha}$. That is,

$$\|H_{f_1}\|_{F^p_\alpha \to L^q_\alpha} \le C\|f_1\|_{IO^s}.$$

This and (22), (23) imply

$$\|H_f\|_{F^p_\alpha \to L^q_\alpha} \le C\|f\|_{IMO^{s,q}}.$$

Since $MO_{q,r}(\overline{f}) = MO_{q,r}(f)$, we have

$$\|H_{\overline{f}}\|_{F^p_\alpha \to L^q_\alpha} \le C\|f\|_{IMO^{s,q}}.$$

So the left-hand side of (21) can be dominated by the right-hand side.

Now we prove the compactness. Write $f = f_1 + f_2$ with (22). Let $\{g_k\}_{k=1}^\infty$ be any weakly null sequence in F^p_α. That is, $\|g_k\|_{p,\alpha} \le C$ and $\{g_k(z)\}_{k=1}^\infty$ tends to 0 uniformly on any compact subset of $\mathbb{C}^n$. Set $h_k(z) = g_k(z)e^{-\frac{\alpha}{2}|z|^2}$. By Theorem (6.1.6), we obtain

$$\|H_{f_1}(g_k)\|_{q,\alpha} \le C\|T_{f_1,\varepsilon}(g_k)\|_{L^q} \to 0$$

as $k \to \infty$. Hence, H_{f_1} is compact. And for f_2, set $f_{2,R} = f_2 \cdot \chi_R$, where χ_R is the characteristic function of $B(0,R)$. $f(2,R)$ is compact supported, so H_{f_2}, R is compact from

F_α^p to L_α^q. By (23),

$$\|H_{f_2}-H_{f_{2,R}}\|_{F_\alpha^p\to L_\alpha^q}\le C\|f_2-f_{2,R}\|_{IA^{s,q}}\to 0$$

as $R\to\infty$. Hence H_{f_2} is compact. The compactness of H_{f_1} and H_{f_2} tells us $H_f=H_{f_1}+H_{f_2}$ is compact from F_α^p to L_α^q. Since $\omega_r(f)(z)=\omega_r(\overline{f})(z)$ and $MO_{q,r}(f)(z)=MO_{q,r}(\overline{f})$, we know $H_{\overline{f}}:\ F_\alpha^p\to L_\alpha^q$ is compact as well. The proof is finished.

As a corollary of Theorems (6.1.7) and (6.1.10), we have the following.

Corollary (6.1.11) [230]: *Let $\alpha>0$ and let $\in H(\mathbb{C}^n)$.*

(i) For $1\le p\le q<\infty$. $H_{\overline{f}}$ is bounded from F_α^p to L_α^q if and only if f is a linear polynomial; $H_{\overline{f}}$ is compact from F_α^p to L_α^q if and only if f is constant.

(ii) For $1\le q<p<\infty$. $H_{\overline{f}}$ is bounded from F_α^p to L_α^q if and only if f is constant.

Proof. (i) We know that $f\in BMO^q\cap H(\mathbb{C}^n)$ if and only if f is a linear polynomial and $f\in VMO^q\cap H(\mathbb{C}^n)$ if and only if f is constant (see Proposition 3.38 and Corollary 3.40 in [232] for the one dimensional cases, and the proof also works in higher dimensions). The desired results follow from Theorem (6.1.7) immediately.

(ii) Fock spaces have the nest property that the inclusion $F_\alpha^q\subset F_\alpha^p$ is bounded for $0<q\le p<\infty$. The compactness of $H_{\overline{f}}$ from F_α^p to L_α^q implies its compactness from F_α^q to L_α^q. Hence f is constant.

Corollary (6.1.12) [264]: *Suppose $0\le\epsilon\le\infty$. Then for $0\le\epsilon<\infty$ and f_j Lebesgue measurable on $\mathbb{C}^n$, $f_j\in IO_{1+\epsilon}^{1+\epsilon}$ if and only in $f_j\in IO_{1+2\epsilon}^{1+\epsilon}$. Furthermore, $\sum_j\ \|f_j\|_{IO_{1+\epsilon}^{1+\epsilon}}\simeq\sum_j\ \|f_j\|_{IO_{1+2\epsilon}^{1+\epsilon}}$.*

$$\|\sum_j\ f_j\|_{IO_{1+2\epsilon}^{1+\epsilon}}\le C\sum_j\ \|f_j\|_{IO_1^{1+\epsilon}}$$

for $\epsilon>0$. To see this, notice that $B(0,1+\epsilon)\cap\frac{1}{2}\mathbb{Z}^n$ is finite, say

Proof. We need only to prove,

$$B(0,1+\epsilon)\cap\frac{1}{2}\mathbb{Z}^n=\{\xi_1,\xi_2,\dots,\xi_m\}.$$

Then $\sum_j\ \omega_{1+\epsilon}(f_j)(z)\le\sum_{j=1,2,\dots,m}\ \omega(f_j)(z+\xi_j)$. This gives

$$\left\|\sum_j\ f_j\right\|_{IO_{1+\epsilon}^{1+\epsilon}}\le C\sum_{j=1,2,.m}\ \|f_j(\cdot+\xi_j)\|_{IO_1^{1+\epsilon}}.$$

But $\sum_j\ \|f_j(\cdot+\xi_j)\|_{IO_1^{1+\epsilon}}=\sum_j\ \|f_j\|_{IO_1^{1+\epsilon}}$, the desired estimate follows. This completes the proof.

We now describe the structure of $IMO_{1+\epsilon}^{1+\epsilon,1+\epsilon}$ via $IO^{1+\epsilon}$ and $IA^{1+\epsilon,1+\epsilon}$.

Corollary (6.1.13) [264]: *Let $0\le\epsilon<\infty$. Then $f_j\in IMO_{1+\epsilon}^{1+\epsilon,1+\epsilon}$ if and only if f_j admits a decomposition $f_j=(f_j)_1+(f_j)_2$, where $(f_j)_1\in IO^{1+\epsilon}$ and $(f_j)_2\in IA^{1+\epsilon,1+\epsilon}$. Furthermore, $\|f_j\|_{IMO_{1+\epsilon}^{1+\epsilon,1+\epsilon}}$ is equivalent to*

$$\inf \sum_j \left\{\|(f_j)_1\|_{IO^{1+\epsilon}} + \|(f_j)_2\|_{IA^{1+\epsilon,1+\epsilon}} : f_j = (f_j)_1 + (f_j)_2, (f_j)_1 \in IO^{1+\epsilon}, (f_j)_2 \in IA^{1+\epsilon,1+\epsilon}\right\}.$$

Proof. Suppose $f_j \in IMO_{1+\epsilon}^{1+\epsilon,1+\epsilon}$. Set $(f_j)_1 = (\hat{f}_j)_{\frac{1+\epsilon}{2}}$ and $(f_j)_2 = f_j - (f_j)_1$. For $|z - w| \le \frac{1+\epsilon}{2}$, we have

$$\left|\sum_j \left((f_j)_1(z) - (f_j)_1(w)\right)\right| \le \sum_j \left|(f_j)_1(z) - (\hat{f}_j)_{1+\epsilon}(z)\right| + \sum_j \left|(\hat{f}_j)_{1+\epsilon}(z) - (f_j)_1(w)\right|$$

$$\le \frac{1}{v\left(B\left(z,\frac{1+\epsilon}{2}\right)\right)} \int_{B\left(z,\frac{1+\epsilon}{2}\right)} \sum_j \left|f_j(u) - (\hat{f}_j)_{1+\epsilon}(z)\right| dv(u)$$

$$+ \frac{1}{v(B(w,\frac{1+\epsilon}{2}))} \int_{B\left(z,\frac{1+\epsilon}{2}\right)} \sum_j |f_j(u) - (\hat{f}_j)_{1+\epsilon}(z)| dv(u).$$

Since $B\left(z,\frac{1+\epsilon}{2}\right)$ and $B\left(w,\frac{1+\epsilon}{2}\right)$ are both contained in $(z, 1+\epsilon)$, it follows from Hölder's inequality that

$$\left|\sum_j \left((f_j)_1(z) - (f_j)_1(w)\right)\right|$$

$$\lesssim \sum_j \left(\frac{1}{v\left(B(z,1+\epsilon)\right)} \int_{B(z,1+\epsilon)} |f_j(u) - (\hat{f}_j)_{1+\epsilon}(z)|^{1+\epsilon} dv(u)\right)^{\frac{1}{1+\epsilon}}$$

This and Lemma (6.1.9) tell us $(f_j)_1 \in IO^{1+\epsilon}$ and

$$\left\|\sum_j (f_j)_1\right\|_{IO^{1+\epsilon}} \le C \sum_j \|f_j\|_{IMO_{1+\epsilon}^{1+\epsilon,1+\epsilon}}.$$

For $(f_j)_2$,

$$\sum_j \left(\overline{\left|(f_j)_2\right|^{1+\epsilon}}_{\frac{1+\epsilon}{2}}\right)^{\frac{1}{1+\epsilon}}(z)$$

$$\leq \sum_j \left(\frac{1}{v\left(B\left(z,\frac{1+\epsilon}{2}\right)\right)}\int_{B\left(z,\frac{1+\epsilon}{2}\right)} |f_j(u)-(f_j)_1(z)|^{1+\epsilon}dv(u)\right)^{\frac{1}{1+\epsilon}}$$

$$+\sum_j \left(\frac{1}{v\left(B\left(z,\frac{1+\epsilon}{2}\right)\right)}\int_{B\left(z,\frac{1+\epsilon}{2}\right)} |(f_j)_1(u)-(f_j)_1(z)|^{1+\epsilon}dv(u)\right)^{\frac{1}{1+\epsilon}}$$

$$\leq \sum_j MO_{1+\epsilon,\frac{1+\epsilon}{2}}(f_j)(z)+\sum_j \omega_{\frac{1+\epsilon}{2}}((f_j)_1)(z) .$$

This and (22) imply $\sum_j \left(\left|\widehat{(f_j)_2}\right|^{1+\epsilon}_{\frac{1+\epsilon}{2}}\right)^{\frac{1}{1+\epsilon}} \in L^{1+\epsilon}(\mathbb{C}^n)$ with

$$\|\sum_j (f_j)_2\|_{IA^{1+\epsilon,1+\epsilon}} \leq C\sum_j \|f_j\|_{IMO^{1+\epsilon,1+\epsilon}_{1+\epsilon}}.$$

Conversely, we show $f_j \in IMO^{1+\epsilon,1+\epsilon}_{1+\epsilon}$ whenever $f_j \in IO^{1+\epsilon}$ or $f_j \in IA^{1+\epsilon,1+\epsilon}$ with the desired norm estimates. Suppose $f_j \in IO^{1+\epsilon}$. Since

$$\sum_j MO_{1+\epsilon,1+\epsilon}(f_j)(z) \leq \sum_j \left(\frac{1}{v(B(z,1+\epsilon))}\int_{B(z,1+\epsilon)} |f_j(w)-f_j(z)|^{1+\epsilon}dv(w)\right)^{\frac{1}{1+\epsilon}}$$

$$+\sum_j |f_j(z)-(\hat{f}_j)_{1+\epsilon}(z)| \leq \sum_j 2\omega_{1+\epsilon}(f_j)(z) ,$$

we have $f_j \in IMO^{1+\epsilon,1+\epsilon}_{1+\epsilon}$ with

$$\left\|\sum_j f_j\right\|_{IMO^{1+\epsilon,1+\epsilon}_{1+\epsilon}} \leq C\sum_j \|(f_j)_1\|_{IO^{1+\epsilon}}.$$

And for $f_j \in IA^{1+\epsilon,1+\epsilon}$ we have

$$\sum_j MO_{1+\epsilon,1+\epsilon}(f_j)(z)$$
$$\leq \sum_j \left(\frac{1}{v(B(z,1+\epsilon))}\int_{B(z,1+\epsilon)} |f_j(w)|^{1+\epsilon} dv(w)\right)^{\frac{1}{1+\epsilon}} + \sum_j |(\hat{f}_j)_{1+\epsilon}(z)|$$
$$\leq 2\sum_j \left(\widehat{|(f_j)_2|^{1+\epsilon}}\right)^{\frac{1}{1+\epsilon}}.$$

Therefore $f_j \in IMO^{1+\epsilon,1+\epsilon}_{1+\epsilon}$ with

$$\|\sum_j f_j\|_{IMO^{1+\epsilon,1+\epsilon}_{1+\epsilon}} \leq C\sum_j \|(f_j)_2\|_{IA^{1+\epsilon,1+\epsilon}}.$$

Corollary (6.1.14) [264]: Suppose $0 \leq \epsilon < \infty$ and $f_j \in L^{1+\epsilon}_{\text{loc}}$. If for each $z \in \mathbb{C}^n$, there exist $(h_j)_1, (h_j)_2 \in H(B(z,1+\epsilon))$ satisfying

$$\sum_j \left(\frac{1}{v(B(z,1+\epsilon))}\int_{B(z,1+\epsilon)} \left|f_j - (h_j)_1\right|^{1+\epsilon} dv\right)^{\frac{1}{1+\epsilon}} \in L^{1+\epsilon}(\mathbb{C}^n)$$

and

$$\sum_j \left(\frac{1}{v(B(z,1+\epsilon))}\int_{B(z,1+\epsilon)} \left|\bar{f}_j - (h_j)_2\right|^{1+\epsilon} dv\right)^{\frac{1}{1+\epsilon}} \in L^{1+\epsilon}(\mathbb{C}^n),$$

then $f_j \in IMO^{1+\epsilon,1+\epsilon}$.

Proof. It is well known that, if $v_j : B(0,1) \to \mathbb{R}^n$ is pluriharmonic sequence, there exists some pluriharmonic sequence of functions $u_j : B(0,1) \to \mathbb{R}^n$ such that $u_j + iv_j \in H(B(0,1))$. Theorem 1 from [3] tells us there is some constant $C_1 > 0$ such that $\left\|\sum_j \left(u_j - u_j(0)\right)\right\|_{L^{1+\epsilon}(B(0,1),dv_j)} \leq C_1 \sum_j \left\|v_j\right\|_{L^{1+\epsilon}(B(0,1),dv_j)}$. Hence, for any $\epsilon \geq 0$, by change of variables we know that, if $v_j : B(z,1+\epsilon) \to \mathbb{R}^n$ is pluriharmonic, there exists a pluriharmonic functions u_j such that $u_j + iv_j \in H\big(B(z,1+\epsilon)\big)$ and

$$\|\sum_j \big(u_j - u_j(z)\big)\|_{L^{1+\epsilon}(B(z,1+\epsilon),dv_j)} \leq C_1 \sum_j \|v_j\|_{L^{1+\epsilon}(B(z,1+\epsilon),dv_j)}.$$

For $f_j \in L^{1+\epsilon}_{\text{loc}}$, set $\sum_j \|f_j\|_{1+\epsilon,B(z,1+\epsilon)} = \sum_j \left(\frac{1}{v(B(z,1+\epsilon))}\int_{B(z,1+\epsilon)} |f_j|^{1+\epsilon} dv\right)^{\frac{1}{1+\epsilon}}$. By triangle inequality we have

$$\left\|\sum_j \left(\frac{f_j + \overline{f}_j}{2} - \frac{(h_j)_1 + (h_j)_2}{2}\right)\right\|_{1+\epsilon,B(z,1+\epsilon)}$$

$$\leq \sum_j \left\| \frac{f_j - (h_j)_1}{2} \right\|_{1+\epsilon,B(z,1+\epsilon)} + \sum_j \left\| \frac{f_j - (h_j)_2}{2} \right\|_{1+\epsilon,B(z,1+\epsilon)} \in L^{1+\epsilon}(\mathbb{C}^n).$$

Since $f_j + \overline{f}_j$ is real valued, we get $\| \operatorname{Im} \frac{(h_j)_1+(h_j)_2}{2} \|_{1+\epsilon,B(z,1+\epsilon)} \in L^{1+\epsilon}(\mathbb{C}^n)$. Notice that $(h_j)_1 + (h_j)_2 \in H\big(B(z,1+\epsilon)\big)$, we obtain

$$\left\| \sum_j \left(\operatorname{Re} \frac{(h_j)_1 + (h_j)_2}{2} - \operatorname{Re} \frac{(h_j)_1 + (h_j)_2}{2}(z) \right) \right\|_{1+\epsilon,B(z,1+\epsilon)}$$

$$\leq C \sum_j \left\| \operatorname{Im} \frac{(h_j)_1 + (h_j)_2}{2} \right\|_{1+\epsilon,B(z,1+\epsilon)}$$

Therefore,

$$\sum_j \left\| \frac{f_j + \overline{f}_j}{2} - \operatorname{Re} \frac{(h_j)_1 + (h_j)_2}{2}(z) \right\|_{1+\epsilon,B(z,1+\epsilon)}$$

$$\leq \sum_j \left\| \frac{f_j + \overline{f}_j}{2} - \operatorname{Re} \frac{(h_j)_1 + (h_j)_2}{2} \right\|_{1+\epsilon,B(z,1+\epsilon)}$$

$$+ \sum_j \left\| \operatorname{Re} \frac{(h_j)_1 + (h_j)_2}{2} - \operatorname{Re} \frac{(h_j)_1 + (h_j)_2}{2}(z) \right\|_{1+\epsilon,B(z,1+\epsilon)}$$

$$\leq \sum_j \left\| \frac{f_j + \overline{f}_j}{2} - \frac{(h_j)_1 + (h_j)_2}{2} \right\|_{1+\epsilon,B(z,1+\epsilon)}$$

$$+ \sum_j \left\| \operatorname{Re} \frac{(h_j)_1 + (h_j)_2}{2} - \operatorname{Re} \frac{(h_j)_1 + (h_j)_2}{2}(z) \right\|_{1+\epsilon,B(z,1+\epsilon)}$$

This shows $\| \frac{f_j+\overline{f}_j}{2} - \operatorname{Re} \frac{(h_j)_1+(h_j)_2}{2}(z) \|_{1+\epsilon,B(z,1+\epsilon)} \in L^{1+\epsilon}(\mathbb{C}^n)$. Similarly, we have

$$\left\| \operatorname{Im} \frac{f_j - \overline{f}_j}{2} - \operatorname{Im} \frac{(h_j)_1 - (h_j)_2}{2}(z) \right\|_{1+\epsilon,B(z,1+\epsilon)} \in L^{1+\epsilon}(\mathbb{C}^n).$$

Choose $c_j(z) = \operatorname{Re} \frac{(h_j)_1+(h_j)_2}{2}(z) + i \operatorname{Im} \frac{(h_j)_1-(h_j)_2}{2}(z)$, then $\|f_j - c_j(z)\|_{1+\epsilon,B(z,1+\epsilon)} \in L^{1+\epsilon}(\mathbb{C}^n)$. From this and the desired result follows immediately. The proof is finished.

Corollary (6.1.15) [264]: *Let* $0 \leq \epsilon < \infty$, $\varepsilon > 0$. *Suppose* f_j *is Lebesgue measurable such that* $f_j \in IO^{\frac{(1+2\epsilon)(1+\epsilon)}{\epsilon}}$.

(*i*) *The integral operator* $T_{\sum_j f_j,\varepsilon}$ *is bounded from* $L^{1+2\epsilon}$ *to* $L^{1+\epsilon}$. *Moreover,*

$$\left\| \sum_j T_{f_j,\varepsilon} \right\|_{L^{1+2\epsilon} \to L^{1+\epsilon}} \leq C \sum_j \|f_j\|_{IO^{\frac{(1+2\epsilon)(1+\epsilon)}{\epsilon}}}.$$

(*ii*) *For bounded sequence* $\{(h_j)_k\}$ *in* $L^{1+2\epsilon}$ *satisfying* $\lim_{k\to\infty} \sup_{|z|\le 1+\epsilon} \sum_j \ |(h_j)_k(z)| = 0$ *for all* $\epsilon \ge 0$, *there holds* $\lim_{k\to\infty} \sum_j \ \left\| T_{f_j,\varepsilon}\left(\left(h_j\right)_k\right)\right\|_{L^{1+\epsilon}} = 0.$

Proof. (i) Then $\epsilon \ge 0$. By Hölder's inequality we have

$$|\sum_j \ T_{f_j,\varepsilon} h_j(x)| \le \int_{\mathbb{R}^m} \sum_j \ \left(\int_0^1 (\lambda)\left(f_j\right)^{1+\epsilon}\left(x + t(\epsilon)\right)dt\right)^{\frac{1}{1+\epsilon}} e^{-\varepsilon|\epsilon|} |h_j(x+\epsilon)| dv(x+\epsilon)$$

$$\le \sum_j \ \left[\int_{\mathbb{R}^m}\left(\int_0^1 \ \omega\left(f_j\right)^{1+\epsilon}\left(x + t(\epsilon)\right)dt\right) e^{-(1+\epsilon)\cdot\frac{\varepsilon}{2}|\epsilon|} dv(x+\epsilon)\right]^{\frac{1}{1+\epsilon}} \left[\int_{\mathbb{R}^m} | \, h_j(x + \epsilon)|^{\frac{1+2\epsilon}{1+\epsilon}} e^{-\frac{1+2\epsilon}{1+\epsilon}\cdot\frac{\varepsilon}{2}|\epsilon|} dv(x+\epsilon)\right]^{\frac{1}{1+\epsilon}}$$

Write

$$I_1 = \int_{\mathbb{R}^m} \sum_j \ \left(\int_0^1 \omega\left(f_j\right)^{1+\epsilon}\left(x + t(\epsilon)\right)dt\right) e^{-(1+\epsilon)\cdot\frac{\varepsilon}{2}|\epsilon|} dv(x+\epsilon)\,,$$

and

$$I_2 = \int_{\mathbb{R}^m} \sum_j \ | \, h_j(x+\epsilon)|^{\frac{1+2\epsilon}{1+\epsilon}} e^{-\frac{1+2\epsilon}{1+\epsilon}\cdot\frac{\varepsilon}{2}|\epsilon|} dv(x+\epsilon)\,.$$

Then Fubini's theorem gives

$$\int_{\mathbb{R}^m} I_1 \, dv(x) = \int_{\mathbb{R}^m} d\,v(x) \int_0^1 d\,t \int_{\mathbb{R}^m} \sum_j \ w\left(f_j\right)^{1+\epsilon}\left(x + t(\epsilon)\right) e^{-(1+\epsilon)\cdot\frac{\varepsilon}{2}|\epsilon|} dv(x+\epsilon)$$

$$= \int_{\mathbb{R}^m} d\,v(x) \int_0^1 d\,t \int_{\mathbb{R}^m} \sum_j \ \omega\left(f_j\right)^{1+\epsilon}(x+tu) e^{-(1+\epsilon)\cdot\frac{\varepsilon}{2}|u|} dv(u)$$

$$= \int_0^1 d\,t \int_{\mathbb{R}^m} e^{-(1+\epsilon)\cdot\frac{\varepsilon}{2}|u|} \, dv(u) \int_{\mathbb{R}^m} \sum_j \ v\left(f_j\right)^{1+\epsilon}(x+tu) dv(x)$$

$$= \sum_j \ \|f_j\|_{IO^{1+\epsilon}}^{1+\epsilon} \int_0^1 d\,t \int_{\mathbb{R}^m} e^{-(1+\epsilon)\cdot\frac{\varepsilon}{2}|u|} \, dv(u)$$

$$= C \sum_j \ \|f_j\|_{IO^{1+\epsilon}}^{1+\epsilon}.$$

Since $\epsilon \ge 0$, by Hölder's inequality and Fubini's theorem we get

$$\int_{\mathbb{R}^m} \sum_j \ e^{-\frac{1+2\epsilon}{1+\epsilon}\cdot\frac{\varepsilon}{2}|\epsilon|} \, |h_j(x+\epsilon)|^{\frac{1+2\epsilon}{1+\epsilon}} dv(x+\epsilon)$$

$$\leq \sum_j \left(\int_{\mathbb{R}^m} e^{-\frac{(1+2\epsilon)\varepsilon}{4}|\epsilon|} |h_j(x+\epsilon)|^{\left(\frac{1+2\epsilon}{1+\epsilon}\right)^2} dv(x+\epsilon)\right)^{\frac{1}{1+\epsilon}} \left(\int_{\mathbb{R}^m} e^{-\frac{1+2\epsilon}{4}|\epsilon|} dv(x+\epsilon)\right)^{\frac{\epsilon}{1+\epsilon}}$$

Then,

$$\int_{\mathbb{R}^m} (I_2)^{\frac{1+2\epsilon}{1+\epsilon}} dv(x)$$

$$= \int_{\mathbb{R}^m} \sum_j \left(\int_{\mathbb{R}^m} e^{-\frac{1+2\epsilon}{1+\epsilon}\cdot\frac{\varepsilon}{2}|\epsilon|} |h_j(x+\epsilon)|^{\frac{1+2\epsilon}{1+\epsilon}} dv(x+\epsilon)\right)^{\frac{1+2\epsilon}{1+\epsilon}} dv(x)$$

$$\leq C \int_{\mathbb{R}^m} \sum_j \left(\int_{\mathbb{R}^m} e^{-\frac{(1+2\epsilon)\varepsilon}{4}|\epsilon|} |h_j(x+\epsilon)|^{\left(\frac{1+2\epsilon}{1+\epsilon}\right)^2} dv(x+\epsilon)\right) dv(x)$$

$$= C \int_{\mathbb{R}^m} \sum_j |h_j(x+\epsilon)|^{1+2\epsilon} dv(x+\epsilon) .$$

Therefore,

$$\int_{\mathbb{R}^m} \sum_j \left(T_{f_j,\varepsilon} h_j(x)\right)^{1+\epsilon} dv(x) \leq \int_{\mathbb{R}^m} \left((I_1)^{\frac{1+\epsilon}{1+\epsilon}} (I_2)^{\frac{1+\epsilon}{1+\epsilon}}\right) dv(x)$$

$$\leq \left(\int_{\mathbb{R}^m} I_1 \, dv(x)\right)^{\frac{1+\epsilon}{1+\epsilon}} \left(\int_{\mathbb{R}^m} (I_2)^{\frac{1+2\epsilon}{1+\epsilon}} dv(x)\right)^{\frac{1+\epsilon}{1+2\epsilon}}$$

$$\leq C \sum_j \|f_j\|_{IO^{1+\epsilon}}^{1+\epsilon} \cdot \|h_j\|_{L^{1+2\epsilon}}^{1+\epsilon}.$$

This means $T_{\sum_j f_j,\varepsilon}$ is bounded from $L^{1+2\epsilon}$ to $L^{1+\epsilon}$ with

$$\|\sum_j T_{f_j,\varepsilon}\|_{L^{1+2\epsilon} \to L^{1+\epsilon}} \leq C \sum_j \|f_j\|_{IO^{1+\epsilon}},$$

which gives the statement (i).

Now we prove the statement (ii). To do this, for $\rho > 0$ we consider $\sum_j T_{f_j,\varepsilon}(h_j)(x) = \sum_j J_{f_j,1+2\epsilon}(h_j)(x) + \sum_j Q_{f_j,\rho}(h_j)(x)$ with

$$\sum_j J_{f_j,\rho}(h_j)(x) = \int_{|\epsilon|\geq\rho} \sum_j \int_0^1 \omega(f_j)(x+t(\epsilon))dt) e^{-\varepsilon|\epsilon|} |h_j(x+\epsilon)| dv(x+\epsilon) ,$$

and

$$\sum_j Q_{f_j,1+2\epsilon}(h_j)(x) = \int_{|\epsilon|<\rho} \sum_j \left(\int_0^1 \omega(f_j)(x+t(\epsilon))dt\right) e^{-\varepsilon|\epsilon|} |h_j(x+\epsilon)| dv(x+\epsilon) .$$

Then,

$$\sum_j J_{f_j,1+2\epsilon}(h_j)(x)$$
$$\leq e^{-\frac{\varepsilon}{2}\rho}\int_{|\epsilon|\geq\rho}\sum_j \left(\int_0^1 \omega\,(f_j)(x+t(\epsilon))dt\right)e^{-\frac{\varepsilon}{2}|\epsilon|}|h_j(x+\epsilon)|dv(x+\epsilon)$$
$$\leq e^{-\frac{\varepsilon}{2}\rho}\sum_j T_{f_j,\frac{\varepsilon}{2}}(|h_j|)(x)$$
From (i) we have
$$\|\sum_j J_{f_j,\rho}(h_j)\|_{L^{1+\epsilon}} \leq Ce^{-\frac{\varepsilon}{2}\rho}\sum_j \|f_j\|_{IO^s}\cdot\|h_j\|_{L^{1+2\epsilon}}.$$
To estimte $\|Q_{f_j,\rho}(h_j)\|_{L^{1+\epsilon}}$, for $\epsilon \geq 0$ let $\chi_{1+\epsilon}$ be the characteristic function of $B(0,1+\epsilon)$. Then
$$\|\sum_j Q_{f_j,1+2\epsilon}(1)\chi_{1+\epsilon}\|_{L^{1+\epsilon}} \leq C(1+\epsilon)^{\frac{m}{1+2\epsilon}}\sum_j \left\{\int_{|x|<1+\epsilon} Q_{f_j,1+2\epsilon}\,(1)^{1+\epsilon}(x)dv(x)\right\}^{\frac{1}{1+\epsilon}}$$
$$\leq C(1+\epsilon)^{\frac{m}{1+2\epsilon}}\sum_j \{\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\int_0^1 \omega(f_j)^{1+\epsilon}(x+t(\epsilon))e^{-\frac{\varepsilon}{2}|\epsilon|}dtdv(x+\epsilon)dv(x)\}^{\frac{1}{1+\epsilon}}$$
$$= C(1+\epsilon)^{\frac{m}{1+2\epsilon}}\sum_j \|f_j\|_{IO^s}.$$
Therefore,
$$\|\sum_j Q_{f_j,\rho}(h_j)\chi_{1+\epsilon}\|_{L^{1+\epsilon}} \leq C(1+\epsilon)^{\frac{m}{1+2\epsilon}}\sum_j \left(\sup|h_j(\xi)|\right)\|f_j\|_{IO^s}.$$
On the other hand, it is easy to verify that
$$\sup_{(x+\epsilon)\in B(x,1+2\epsilon),t\in[0,1]}\sum_j \omega(f_j)(x+t(\epsilon)) \leq \sum_j \omega_{2(1+\epsilon)}(f_j)(x)\,.$$
By Young's inequality, we know
$$\|\int_{\mathbb{R}^m}\sum_j e^{-\varepsilon|\epsilon|}\,|h_j(x+\epsilon)|dv(x+\epsilon)\|_{L^{1+2\epsilon}} \leq C\sum_j \|h_j\|_{L^{1+2\epsilon}}.$$
Then, applying Hölder's inequality with the exponents $\frac{1+2\epsilon}{\epsilon}$ and $\frac{1+2\epsilon}{1+\epsilon}$ to get
$$\|\sum_j Q_{f_j,\rho}(h_j)(1-\chi_{1+\epsilon})\|_{L^{1+\epsilon}}$$
$$\leq \sum_j \left(\int_{|x|\geq 1+\epsilon(}\omega_{\rho+1}\,(f_j)^{1+\epsilon}(x)\right)^{\frac{1}{1+\epsilon}}\|\int_{|\epsilon|<1+2\epsilon} e^{-\varepsilon|\epsilon|}\,|h_j(x+\epsilon)|dv(x$$
$$+\epsilon)\|_{L^{1+2\epsilon}}$$

$$\leq C\sum_j \left(\int_{|x|\geq 1+\epsilon} \omega_{\rho+1}(f_j)^{1+\epsilon}(x)\right)^{\frac{1}{1+\epsilon}} \|h_j\|_{L^{1+2\epsilon}}.$$

The constants C are independent of ρ and $1+\epsilon$. Now suppose $\{(h_j)_k\}_{j=1}^{\infty}$ is a bounded sequence in $L^{1+2\epsilon}$ satisfying $\sup_{|z|\leq 1+\epsilon}|(h_j)_k(z)| \to 0$ as $k\to\infty$ for any fixed $\epsilon \geq 0$. Without loss of generality, we may assume $\sum_j \|(h_j)_k\|_{L^{1+2\epsilon}} \leq 1$. Given any $\varepsilon > 0$, pick some $\epsilon \geq 0$ so that $e^{-\frac{\varepsilon}{2}\rho} < \varepsilon$. For this ρ, by Lemma (6.1.9) $f_j \in IO_{2+\epsilon}^{1+\epsilon}$. We have some $\epsilon \geq 0$ such that

$$\sum_j \left(\int_{|x|\geq 1+\epsilon} \omega_{\rho+1}(f_j)^{1+\epsilon}(x)\right)^{\frac{1}{\varepsilon}} < \varepsilon.$$

we have, whenever k is sufficiently large,

$$\begin{aligned}\|\sum_j T_{f_j,\varepsilon}((h_j)_k)\|_{L^{1+\epsilon}} &\leq \sum_j \|J_{f_j,\rho}((h_j)_k)\|_{L^{1+\epsilon}} + \sum_j \|Q_{f_j,\rho}((h_j)_k)\chi_{1+\epsilon}\|_{L^{1+\epsilon}} \\ &\quad + \sum_j \|Q_{f_j,\rho}((h_j)_k)(1-\chi_{1+\epsilon})\|_{L^{1+\epsilon}} \\ &\leq C\sum_j \left\{\varepsilon(\|f_j\|_{IO^s}+1) + (1+\epsilon)^{\frac{m}{1+\epsilon}}\left(\sup_{|\xi|\leq 1+\epsilon+\rho} |(h_j)_k(\xi)|\right)\|f_j\|_{IO^s}\right\} \\ &\leq K\varepsilon,\end{aligned}$$

where the constants K is independent of ε. Therefore,

$$\lim_{k\to\infty}\sum_j \| T_{f_j,\varepsilon}((h_j)_k)\|_{L^{1+\epsilon}} = 0$$

as desired. The proof is finished.

Corollary (6.1.16) [264]: *For continuous functions f_j on $\mathbb{C}^n$, there holds*

$$\left|\sum_j \left(f_j(z) - f_j(w)\right)\right| \leq 4(1+|z-w|)\int_0^1 \sum_j \omega(f_j)(z+t(w-z))dt$$

for all z and w in $\mathbb{C}^n$.

Proof. Suppose z and w in $\mathbb{C}^n$, if $|z-w|<1$, then for any $t\in[0,1]$, we have

$$\left|\sum_j \left(f_j(z)-f_j(w)\right)\right|$$
$$\leq \sum_j |f_j(z)-f_j(z+t(w-z))| + \sum_j |f_j(z+t(w-z))-f_j(w)|$$
$$\leq 2\cdot\sum_j \omega(f_j)(z+t(w-z)).$$
Integrating both sides with respect to t from 0 to 1,
$$\left|\sum_j \left(f_j(z)-f_j(w)\right)\right| \leq 2\int_0^1 \sum_j \omega(f_j)(z+t(w-z))dt.$$
Now for $|z-w|\geq 1$, let $N=[|z-w|+1]$, where $[x]$ denotes the largest integer less than or equal to x, and set $z_j = z+\frac{j}{N}(w-z), j = 0,1,\dots,N$. Then
$$|z_{j+1}-z_j| = \frac{1}{N}|w-z| < 1,$$
Therefore,
$$\left|\sum_j \left(f_j(z)-f_j(w)\right)\right| \leq \sum_{j=0}^{N-1} |f_j(z_{j+1})-f_j(z_j)| \leq 2\sum_{j=0}^{N-1}\int_0^1 \omega(f_j)\left(z_j+\frac{w-z}{N}\cdot t\right)dt$$
$$= 2\sum_{j=0}^{N-1} N\int_0^{\frac{1}{m}} \omega(f_j)\left(z_j+s(w-z)\right)ds$$
$$= 2N\int_0^1 \sum_j \omega(f_j)(z+t(w-z))dt$$
$$\leq 4|z-w|\int_0^1 \sum_j \omega(f_j)(z+t(w-z))dt.$$
This together completes the proof of the Corollary.

Corollary (6.1.17) [264]: *Let* $0\leq\epsilon<\infty$. *Then for* $f_j\in\Gamma$, *the following statements are equivalent*:

(a) $H_{\sum_j f_j}, H_{\sum_j \bar{f}_j}: F^{1+2\epsilon}_{1+\epsilon} \to L^{1+\epsilon}_{1+\epsilon}$ *are bounded*

(b) $H_{\sum_j f_j}, H_{\sum_j \bar{f}_j}: F^{1+2\epsilon}_{1+\epsilon} \to L^{1+\epsilon}_{1+\epsilon}$ *are compact.*

(*c*) $f_j \in IMO^{\frac{(1+2\epsilon)(1+\epsilon)}{\epsilon},1+\epsilon}$.

Furthermore,
$$\|H_{\sum_j f_j}\|_{F^{1+2\epsilon}_{1+\epsilon}\to L^{1+\epsilon}_{1+\epsilon}} + \|H_{\sum_j \bar{f}_j}\|_{F^{1+2\epsilon}_{1+\epsilon}\to L^{1+\epsilon}_{1+\epsilon}} \simeq \sum_j \|f_j\|_{IMO^{\frac{(1+2\epsilon)(1+\epsilon)}{\epsilon},1+\epsilon}}.$$

Proof. We prove the implications (a) ⇒ (c) and (c) ⇒ (b) . The implication (b) ⇒ (a) is trivial.

(a) ⇒ (c) . Fix $\epsilon \geq 0$, and let $\{a_k\}$ be an $(1+\epsilon)$-lattice. According to Theorem (6.1.1), for any $\{\lambda_k\} \in l^{1+2\epsilon}$, we have $(g_j)_t(z) = \sum_{k=1}^{\infty} \lambda_k\, r_k(t) k_{a_k}(z) \in F_{1+\epsilon}^{1+2\epsilon}$, and $\|(g_j)_t\|_{1+2\epsilon,1+\epsilon} \leq C\|\{\lambda_k\}\|_{l^{1+2\epsilon}}$. It is trivial to check that $|k_{a_j}(z)|^{1+\epsilon} e^{-\frac{(1+\epsilon)^2}{2}|z|^2} = e^{-\frac{(1+\epsilon)^2}{2}|z-a_j|^2} \simeq 1$ for $\in B(a_j, 1+\epsilon)$. By Fubini's theorem and Khinchine's inequality we obtain

$$\int_0^1 \sum_j \| H_{f_j}(g_j)_t\|_{1+\epsilon,1+\epsilon}^{1+\epsilon} dt$$

$$= \int_0^1 \int_{\mathbb{C}^n} \sum_j |\sum_{k=1}^{\infty} \lambda_k\, r_k(t) H_{f_j} k_{a_k}(z)|^{1+\epsilon} e^{-\frac{(1+\epsilon)^2}{2}|z|^2} dv(z) dt$$

$$= \int_{\mathbb{C}^n} e^{-\frac{(1+\epsilon)^2}{2}|z|^2} dv(z) \int_0^1 |\sum_{k=1}^{\infty} \lambda_k\, r_k(t) H_{\sum_j f_j} k_{a_k}(z)|^{1+\epsilon} dt$$

$$\geq C \int_{\mathbb{C}^n} \left(\sum_{k=1}^{\infty} |\lambda_k|^2 |H_{\sum_j f_j} k_{a_k}(z)|^2\right)^{\frac{1+\epsilon}{2}} e^{-\frac{(1+\epsilon)^2}{2}|z|^2} dv(z)$$

$$\geq C \sum_{j=1}^{\infty} \int_{B(a_j,1+\epsilon)} \left(\sum_{k=1}^{\infty} |\lambda_k|^2 |H_{\sum_j f_j} k_{a_k}(z)|^2\right)^{\frac{1+\epsilon}{2}} e^{-\frac{(1+\epsilon)^2}{2}|z|^2} dv(z)$$

$$\geq C \sum_{j=1}^{\infty} |\lambda_j|^{1+\epsilon} \int_{B(a_j,1+\epsilon)} | f_j(z) k_{a_j}(z) - P_{1+\epsilon}\left(f_j k_{a_j}\right)(z)|^{1+\epsilon} e^{-\frac{(1+\epsilon)^2}{2}|z|^2} dv(z)$$

$$\geq C \sum_{j=1}^{\infty} |\lambda_j|^{1+\epsilon} \int_{B(a_j,1+\epsilon)} | f_j(z) - \frac{P_{1+\epsilon}\left(f_j k_{a_j}\right)(z)}{k_{a_j}(z)}|^{1+\epsilon} |k_{a_j}(z)|^{1+\epsilon} e^{-\frac{(1+\epsilon)^2}{2}|z|^2} dv(z)$$

$$\geq C \sum_{j=1}^{\infty} |\lambda_j|^{1+\epsilon} \left(G_{1+\epsilon,1+\epsilon}(f_j)(a_j)\right)^{1+\epsilon}.$$

Meanwhile, the boundedness of $H_{\sum_j f_j} : F_{1+\epsilon}^{1+2\epsilon} \to L_{1+\epsilon}^{1+\epsilon}$ gives

$$\| \sum_j H_{f_j}(g_j)_t\|_{1+\epsilon,1+\epsilon}^{1+\epsilon} \leq C \sum_j \|H_{\sum_j f_j}\|_{F_{1+\epsilon}^{1+2\epsilon} \to L_{1+\epsilon}^{1+\epsilon}}^{1+\epsilon} \cdot \|(g_j)_t\|_{1+2\epsilon,1+\epsilon}^{1+\epsilon}.$$

Hence,

$$\sum_{j=1}^{\infty} |\lambda_j|^{1+\epsilon} \left(G_{1+\epsilon,1+\epsilon}(f_j)(a_j)\right)^{1+\epsilon} \leq C\|H_{\sum_j f_j}\|_{F_{1+\epsilon}^{1+2\epsilon} \to L_{1+\epsilon}^{1+\epsilon}}^{1+\epsilon} \cdot \|\{|\lambda_k|^{1+\epsilon}\}\|_{l^{\frac{1+2\epsilon}{1+\epsilon}}}^{1+\epsilon}.$$

Since the conjugate exponent of $\frac{1+2\epsilon}{1+\epsilon}$ is $\frac{1+2\epsilon}{\epsilon}$, a duality argument implies

$$\sum_{j=1}^{\infty}\left(G_{1+\epsilon,1+\epsilon}(f_j)(a_j)\right)^{\frac{(1+2\epsilon)(1+\epsilon)}{\epsilon}} \le C\sum_j \|H_{f_j}\|_{F_{1+\epsilon}^{1+2\epsilon}\to L_{1+\epsilon}^{1+\epsilon}}^{\frac{(1+2\epsilon)(1+\epsilon)}{\epsilon}}.$$

On the other hand, the fact that

$$G_{1+\epsilon,t}\left(\sum_j f_j\right)(w) \le C\sum_j G_{1+\epsilon,1+\epsilon}(f_j)(z),$$

when $B(w,t)\subset B(z,1+\epsilon)$ gives

$$\int_{\mathbb{C}^n}\sum_j \left(G_{1+\epsilon,1+\epsilon}(f_j)(z)\right)^{\frac{(1+2\epsilon)(1+\epsilon)}{\epsilon}} dv(z)$$

$$\le \sum_{j=1}^{\infty}\int_{B(a_j,1+\epsilon)}\left(G_{1+\epsilon,1+\epsilon}(f_j)(z)\right)^{\frac{(1+2\epsilon)(1+\epsilon)}{\epsilon}} dv(z)$$

$$\le C\sum_{j=1}^{\infty}\left(G_{1+\epsilon,2(1+\epsilon)}(f_j)(a_j)\right)^{\frac{(1+2\epsilon)(1+\epsilon)}{\epsilon}}$$

This shows that , for any $\epsilon \ge 0, G_{1+\epsilon,1+\epsilon}(f_j)\in L^{\frac{(1+2\epsilon)(1+\epsilon)}{\epsilon}}$. Similarly, the boundedness of $H_{\sum_j \bar{f}_j}$ from $F_{1+\epsilon}^{1+2\epsilon}$ to $L_{1+\epsilon}^{1+\epsilon}$ implies $G_{1+\epsilon,1+\epsilon}(\overline{f_j})\in L^{\frac{(1+2\epsilon)(1+\epsilon)}{\epsilon}}$ for any $\epsilon\ge 0$. By the definition of $G_{1+\epsilon,1+\epsilon}(f_j)$, for each $z\in\mathbb{C}^n$ we have $(h_j)_1,(h_j)_2\in H\big(B(z,1+\epsilon)\big)$ such that

$$\sum_j\left(\frac{1}{v(B(z,1+\epsilon))}\int_{B(z,1+\epsilon)}|f_j-(h_j)_1|^{1+\epsilon}dv\right)^{\frac{1}{1+\epsilon}} \le 2\sum_j G_{1+\epsilon,1+\epsilon}(f_j)(z),$$

and

$$\sum_j\left(\frac{1}{v(B(z,1+\epsilon))}\int_{B(z,1+\epsilon)}|\overline{f_j}-(h_j)_2|^{1+\epsilon}dv\right)^{\frac{1}{1+\epsilon}} \le 2\sum_j G_{1+\epsilon,1+\epsilon}(\overline{f_j})(z).$$

It follows that $f_j\in IMO^{\frac{(1+2\epsilon)(1+\epsilon)}{\epsilon},1+\epsilon}$. Moreover, the right-hand side can be dominated by the left-hand side.

(c) ⇒(b) . Suppose $f_j\in IMO^{\frac{(1+2\epsilon)(1+\epsilon)}{\epsilon},1+\epsilon}$. First, we prove that the left-hand side can be dominated by the right-hand side. , f_j admits a decomposition $f_j=(f_j)_1+(f_j)_2$ with

$$\|\sum_j (f_j)_1\|_{IO^{\frac{(1+2\epsilon)(1+\epsilon)}{\epsilon}}}+\|\sum_j (f_j)_2\|_{IA^{\frac{(1+2\epsilon)(1+\epsilon)}{\epsilon},1+\epsilon}}$$

$$\le C\sum_j \|f_j\|_{IMO^{\frac{(1+2\epsilon)(1+\epsilon)}{\epsilon},1+\epsilon}}.$$

For $(f_j)_2\in IA^{\frac{(1+2\epsilon)(1+\epsilon)}{\epsilon},1+\epsilon}$, the boundedness of $P_{1+\epsilon}$ on $L_{1+\epsilon}^{1+\epsilon}$ yields

$$\|\sum_j H_{(f_j)_2} g_j\|_{1+\epsilon,1+\epsilon} \le \sum_j \|(f_j)_2 g_j\|_{1+\epsilon,1+\epsilon} + \sum_j \|P_{1+\epsilon}\big((f_j)_2 g_j\big)\|_{1+\epsilon,1+\epsilon}$$
$$\le C \sum_j \|(f_j)_2 g_j\|_{1+\epsilon,1+\epsilon}.$$
tells us $d\mu = |(f_j)_2|^{1+\epsilon} dv$ is a$(1+2\epsilon, 1+\epsilon)$-Fock Carleson measure. Thus the operator $M_{\sum_j (f_j)_2}$ defined by $\sum_j M_{(f_j)_2} g_j = \sum_j (f_j)_2 g_j$ is bounded from $F^{1+2\epsilon}_{1+\epsilon}$ to $L^{1+\epsilon}_{1+\epsilon}$. Precisely,
$$\|\sum_j M_{(f_j)_2} g_j\|_{1+\epsilon,1+\epsilon} \le C \sum_j \|\left(\widehat{|(f_j)_2|^{1+\epsilon}_{1+\epsilon}}\right)^{\frac{1}{1+\epsilon}}\|_{L^{\frac{(1+2\epsilon)(1+\epsilon)}{\epsilon}}} \|g_j\|_{1+2\epsilon,1+\epsilon}.$$
Therefore, $H_{\sum_j (f_j)_2} : F^{1+2\epsilon}_{1+\epsilon} \to L^{1+\epsilon}_{1+\epsilon}$ is bounded with
$$\|\sum_j H_{(f_j)_2}\|_{F^{1+2\epsilon}_{1+\epsilon} \to L^{1+\epsilon}_{1+\epsilon}} \le C \sum_j \|(f_j)_2\|_{IA^{\frac{(1+2\epsilon)(1+\epsilon)}{\epsilon},1+\epsilon}}.$$
Now we treat $H_{\sum_j (f_j)_1}$ with $(f_j)_1 \in IO^{\frac{(1+2\epsilon)(1+\epsilon)}{\epsilon}}$. Notice that there are some $C > 0$ and $\varepsilon > 0$ such that, for $z, w \in \mathbb{C}^n$,
$$|e^{(1+\epsilon)\langle w,z\rangle}| \le C e^{\frac{1+\epsilon}{2}|z|^2 + \frac{1+\epsilon}{2}|w|^2 - 2\varepsilon|z-w|}, |z-w| \le C e^{\varepsilon|z-w|}.$$
For $g_j \in F^{1+2\epsilon}_{1+\epsilon}$, set $\sum_j h_j(w) = \sum_j g_j(w) e^{-\frac{1+\epsilon}{2}|w|^2}$. Then, $h_j \in L^{1+2\epsilon}(\mathbb{C}^n, dv)$. From Lemma (6.1.9) we have
$$\sum_j \|H_{(f_j)_1} g_j\|^{1+\epsilon}_{1+\epsilon,1+\epsilon} = \int_{\mathbb{C}^n} \sum_j |H_{(f_j)_1} g_j(z)|^{1+\epsilon} e^{-\frac{(1+\epsilon)^2}{2}|z|^2} dv(z)$$
$$= \int_{\mathbb{C}^n} |\int_{\mathbb{C}^n} \sum_j \big((f_j)_1(z) - (f_j)_1(w)\big) g_j(w) e^{(1+\epsilon)\langle z,w\rangle} dv_{1+\epsilon}(w)|^{1+\epsilon} e^{-\frac{(1+\epsilon)^2}{2}|z|^2} dv(z)$$
$$\le C \int_{\mathbb{C}^n} \sum_j \left(\int_{\mathbb{C}^n} |(f_j)_1(z) - (f_j)_1(w)| \cdot |g_j(w) e^{-\frac{1+\epsilon}{2}|w|^2}| e^{-2\varepsilon|z-w|} dv(w)\right)^{1+\epsilon} dv(z)$$
$$\le C \int_{\mathbb{C}^n} \sum_j T_{(f_j)_1,\varepsilon}\left(h_j\right)^{1+\epsilon}(z) dv(z),$$
Applying to obtain $\|\sum_j H_{(f_j)_1} g_j\|_{1+\epsilon,1+\epsilon} \le C \sum_j \|(f_j)_1\|_{IO^{\frac{(1+2\epsilon)(1+\epsilon)}{\epsilon}}} \|g_j\|_{1+2\epsilon,1+\epsilon}$. That is,
$$\|\sum_j H_{(f_j)_1}\|_{F^{1+2\epsilon}_{1+\epsilon} \to L^{1+\epsilon}_{1+\epsilon}} \le C \sum_j \|(f_j)_1\|_{IO^{\frac{(1+2\epsilon)(1+\epsilon)}{\epsilon}}}.$$
$$\|\sum_j H_{f_j}\|_{F^{1+2\epsilon}_{1+\epsilon} \to L^{1+\epsilon}_{1+\epsilon}} \le C \sum_j \|f_j\|_{IMO^{\frac{(1+2\epsilon)(1+\epsilon)}{\epsilon},1+\epsilon}}.$$
Since $\sum_j MO_{1+\epsilon,1+\epsilon}\big(\overline{f_j}\big) = \sum_j MO_{1+\epsilon,1+\epsilon}\big(f_j\big)$, we have

$$\|\sum_j H_{\bar{f}_j}\|_{F^{1+2\epsilon}_{1+\epsilon}\to L^{1+\epsilon}_{1+\epsilon}} \le C\sum_j \|f_j\|_{IMO^{\frac{(1+2\epsilon)(1+\epsilon)}{\epsilon},1+\epsilon}}.$$

So the left-hand side can be dominated by the right-hand side.

Now we prove the compactness. Write $f_j = (f_j)_1 + (f_j)_2$ with. Let $\{(g_j)_k\}_{k=1}^{\infty}$ be any weakly null sequence in $F^{1+2\epsilon}_{1+\epsilon}$. That is, $\|(g_j)_k\|_{1+2\epsilon,1+\epsilon} \le C$ and $\{(g_j)_k(z)\}_{k=1}^{\infty}$ tends to 0 uniformly on any compact subset of $\mathbb{C}^n$. Set $\sum_j (h_j)_k(z) = \sum_j (g_j)_k(z)e^{-\frac{1+\epsilon}{2}|z|^2}$. we obtain

$$\|\sum_j H_{(f_j)_1}((g_j)_k)\|_{1+\epsilon,1+\epsilon} \le C\sum_j \|T_{(f_j)_1,\varepsilon}((g_j)_k)\|_{L^{1+\epsilon}} \to 0$$

as $k\to\infty$. Hence, $H_{\sum_j (f_j)_1}$ is compact. And for $(f_j)_2$, set $(f_j)_{2,3+2\epsilon} = (f_j)_2 \cdot \chi_{3+2\epsilon}$, where $\chi_{3+2\epsilon}$ is the characteristic function of $B(0,3+2\epsilon)$. $f_j(2,3+2\epsilon)$ is compact supported, so $H_{(f_j)_2}, 3+2\epsilon$ is compact from $F^{1+2\epsilon}_{1+\epsilon}$ to $L^{1+\epsilon}_{1+\epsilon}$.,

$$\left\|\sum_j \left(H_{(f_j)_2} - H_{(f_j)_{2,3+2\epsilon}}\right)\right\|_{F^{1+2\epsilon}_{1+\epsilon}\to L^{1+\epsilon}_{1+\epsilon}} \le C\sum_j \|(f_j)_2 - (f_j)_{2,3+2\epsilon}\|_{IA^{\frac{(1+2\epsilon)(1+\epsilon)}{\epsilon},1+\epsilon}} \to 0$$

as $\epsilon\to\infty$. Hence $H_{\sum_j (f_j)_2}$ is compact. The compactness of $H_{\sum_j (f_j)_1}$ and $H_{\sum_j (f_j)_2}$ tells us $H_{\sum_j f_j} = H_{\sum_j (f_j)_1} + H_{\sum_j (f_j)_2}$ is compact from $F^{1+2\epsilon}_{1+\epsilon}$ to $L^{1+\epsilon}_{1+\epsilon}$. Since $\sum_j \omega_{1+\epsilon}(f_j)(z) = \sum_j \omega_{1+\epsilon}(\overline{f_j})(z)$ and $\sum_j MO_{1+\epsilon,1+\epsilon}(f_j)(z) = \sum_j MO_{1+\epsilon,1+\epsilon}(\overline{f_j})$, we know $H_{\sum_j \bar{f}_j}$: $F^{1+2\epsilon}_{1+\epsilon} \to L^{1+\epsilon}_{1+\epsilon}$ is compact as well. The proof is finished.

Corollary (6.1.18) [264]: *Let* $\epsilon \ge 0$ *and let* $f_j \in H(\mathbb{C}^n)$.

(*a*) *For* $0 \le \epsilon < \infty$. $H_{\sum_j \bar{f}_j}$ *is bounded from* $F^{1+\epsilon}_{1+\epsilon}$ *to* $L^{1+2\epsilon}_{1+\epsilon}$ *if and only if* f_j *is a linear polynomial*; $H_{\sum_j \bar{f}_j}$ *is compact from* $F^{1+\epsilon}_{1+\epsilon}$ *to* $L^{1+2\epsilon}_{1+\epsilon}$ *if and only if* f_j *is constant.*

(*b*) *For* $0 \le \epsilon < \infty$. $H_{\sum_j \bar{f}_j}$ *is bounded from* $F^{1+2\epsilon}_{1+\epsilon}$ *to* $L^{1+\epsilon}_{1+\epsilon}$ *if and only if* f_j *is constant.*

Proof. (a) We know that $f_j \in BMO^{1+\epsilon} \cap H(\mathbb{C}^n)$ if and only if f_j is a linear polynomial and $f_j \in VMO^{1+\epsilon} \cap H(\mathbb{C}^n)$ if and only if f_j is constant for the one dimensional cases, and the proof also works in higher dimensions). The desired results follow immediately.

(b) Fock spaces have the nest property that the inclusion $F^{1+\epsilon}_{1+\epsilon} \subset F^{1+2\epsilon}_{1+\epsilon}$ is bounded for $0 \le \epsilon < \infty$. The compactness of $H_{\sum_j \bar{f}_j}$ from $F^{1+2\epsilon}_{1+\epsilon}$ to $L^{1+\epsilon}_{1+\epsilon}$ implies its compactness from $F^{1+\epsilon}_{1+\epsilon}$ to $L^{1+\epsilon}_{1+\epsilon}$. Hence f_j is constant.

Section (6.2): Aclass of Fock Spaces

For $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and let $\mathbb{T} = \partial\mathbb{D}$ denote the unit circle. The Hardy space H^2 consists of functions $f \in L^2(\mathbb{T})$ such that its Fourier coefficients satisfy $\hat{f}_n = 0$ for all $n < 0$. Given a function $\varphi \in L^2(\mathbb{T})$, the Toeplitz operator $T_\varphi : H^2 \to H^2$ is densely defined by $T_\varphi f = P(\varphi f)$, where $P : L^2(\mathbb{T}) \to H^2$ is the Riesz-Szego projection.

The original problem that Sarason proposed in [244] was this: characterize the pairs of outer functions u and v in H^2 such that the operator $T_u T_v$ is bounded on H^2. Inner factors can easily be disposed of, so it was only necessary to consider outer functions in the Hardy space case. It was further observed in [244] that a necessary condition for the boundedness of $T_u T_v$ on H^2 is that

$$\sup_{w \in \mathbb{D}} \; P_w(|u|^2) P_w(|v|^2) < \infty,$$

where $P_w(f)$ means the Poisson transform of f at $w \in \mathbb{D}$. In fact, the arguments in [245] show that

$$\sup_{w \in \mathbb{D}} \; P_w(|u|^2) P_w(|v|^2) \leq 4\|T_u T_v\|^2 . \tag{24}$$

Let A^2 denote the Bergman space consisting of analytic functions in $L^2(\mathbb{D}, dA)$, where dA is ordinary area measure on the unit disk. If $P : L^2(\mathbb{D}, dA) \to A^2$ is the Bergman projection, then Toeplitz operators T_φ on A^2 are defined by $T_\varphi f = P(\varphi f)$. Sarason also posed a similar problem in [246] for the Bergman space: characterize functions u and v in A^2 such that the Toeplitz product $T_u T_v$ is bounded on A^2. It was shown in [247] that

$$\sup_{w \in \mathbb{D}} \; \widetilde{|u|^2}(w) \widetilde{|v|^2}(w) \; \leq \; 16\|T_u T_{\bar{v}}\|^2 \tag{25}$$

for all functions u and v in the Bergman space A^2, where $\widetilde{f}\,(w)$ is the socalled Berezin transform of f at w. This provides a necessary condition for the boundedness of $T_u T_v$ on A^2 in terms of the Berezin transform.

The Berezin transform is well defined. In particular, the classical Poisson transform is the Berezin transform of the Hardy space H^2. So the estimates in (24) and (25) are in exactly the same spirit. Sarason stated in [248] that "it is tempting to conjecture that" $T_u T_v$ is bounded on H^2 or A^2 if and only if $\widetilde{|u|^2}(w)\widetilde{|v|^2}(w)$ is a bounded function on $\mathbb{D}$. It has by now become standard to call this "Sarason's conjecture for Toeplitz products".

It turns out that Sarason's conjecture is false for both the Hardy space and the Bergman space of the unit disk, and the conjecture fails in a big way. See [249] for counter-examples. In these cases, Sarason's problem is naturally connected to certain two-weight norm inequalities in harmonic analysis, and counter-examples for Sarason's conjecture were constructed by means of the dyadic model approach in harmonic analysis.

Another setting where Toeplitz operators have been widely studied is the Fock space. More specifically, we let $\mathcal{F}^2$ be the space of all entire functions f on $\mathbb{C}$ that are square-integrable with respect to the Gaussian measure

$$d\lambda(z) = \frac{1}{\pi} \, e^{-|z|^2} \, dA(z).$$

The function

$$K(z, w) \; = \; e^{z\bar{w}}, z, w \; \in \mathbb{C},$$

is the reproducing kernel of $\mathcal{F}^2$ and the orthogonal projection P from $L^2(\mathbb{C}, d\lambda)$ onto $\mathcal{F}^2$ is the integral operator defined by

$$P\,f(z) = \int_{\mathbb{C}} \; K(z, w) f(w) d\lambda(w), \qquad z \; \in \mathbb{C}.$$

If φ is in $L^2(\mathbb{C}, d\lambda)$ such that the function $z \to \varphi(z)K(z,w)$ belongs to $L^1(\mathbb{C}, d\lambda)$ for any $w \in \mathbb{C}$, we can define the Toeplitz operator T_φ with symbol φ by $T_\varphi f = P(\varphi f)$, or

$$T\varphi f(z) = \int_{\mathbb{C}} K(z,w)\varphi(w)f(w)d\lambda(w), \qquad z \in \mathbb{C},$$

When

$$f(w) = \sum_{k=1}^{N} c_k K(w, c_k)$$

is a finite linear combination of kernel functions. Since the set of all finite linear combinations of kernel functions is dense in $\mathcal{F}^2$, the operator T_φ is densely defined and $T_\varphi f$ is an entire function. See [249] for basic information about the Fock space and Toeplitz operators on it.

In a recent [248], Cho, Park and Zhu solved Sarason's problem for the Fock space. More specifically, they obtained the following simple characterization for $T_u T_v$ to be bounded on $\mathcal{F}^2$: if u and v are functions in $\mathcal{F}^2$, not identically zero, then $T_u T_{\bar{v}}$ is bounded on $\mathcal{F}^2$ if and only if $u = e^q$ and $v = ce^{-q}$, where c is a nonzero constant and q is a complex linear polynomial. As a consequence of this, it can be shown that Sarason's conjecture is actually true for Toeplitz products on $\mathcal{F}^2$;.

we consider the weighted Fock space $\mathcal{F}^2_m$, consisting of all entire functions in $L^2(\mathbb{C}, d\lambda_m)$, where $d\lambda_m$ are the generalized Gaussian measure defined by

$$d\lambda_m(z) = e^{-|z|^{2m}} dA(z), \qquad m \geq 1.$$

Toeplitz operators on $\mathcal{F}^2_m$ are defined exactly the same as the cases above, using the orthogonal projection $P : L^2(\mathbb{C}, d\lambda_m) \to \mathcal{F}^2_m$.

We will solve Sarason's problem and prove Sarason's conjecture for the weighted Fock spaces $\mathcal{F}^2_m$. Our main result can be stated as follows.

Let u and v be in $\mathcal{F}^2_m$, not identically zero. The following conditions are equivalent:

(a) The product $T = T_u T_{\bar{v}}$ is bounded on $\mathcal{F}^2_m$.

(b) There exist a polynomial g of degree at most m and a nonzero complex constant c such that $u(z) = e^{g(z)}$ *and* $v(z) = ce^{-g(z)}$.

(c) The product $\widetilde{|u|^2}(z)\widetilde{|v|^2}(z)$ is a bounded function on $\mathbb{C}$.

Furthermore, in the affirmative case, we have the following estimate of the norm:

$$\|T\| \leq C_1 e^{C_2 \|g\|^2_{H^2}},$$

where $\|g\|_{H^2}$ is the norm in the Hardy space of the unit disc, and C_1 and C_2 are positive constants independent of g.

Let us mention that [250] contains partial results related to Sarason's conjecture on the Fock space. The arguments in [251] depend on the explicit form of the reproducing kernel and the Weyl operators induced by translations of the complex plane. Both of these are no longer available for the spaces $\mathcal{F}^2_m$:

there is no simple formula for the reproducing kernel of $\mathcal{F}^2_m$ and the translations on the complex plane do not induce nice operators on $\mathcal{F}^2_m$.

We recall some properties of the Hilbert space $\mathcal{F}_m^2$. It was shown in [246] that the reproducing kernel of $\mathcal{F}_m^2$ is given by the formula

$$K_m(z,w) = \frac{m}{\pi} \sum_{k=0}^{+\infty} \frac{(z\bar{w})^k}{\Gamma\left(\frac{k+1}{m}\right)}. \tag{26}$$

In terms of the Mittag-Leffler function

$$E_{\gamma,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\gamma k + \beta)}, \qquad \gamma,\beta > 0,$$

we can also write

$$K_m(z,w) = \frac{m}{\pi} E_{\frac{1}{m},\frac{1}{m}}(z\bar{w}). \tag{27}$$

Recall that the asymptotics of the Mittag-Leffler function $E_{\frac{1}{m},\frac{1}{m}}(z)$ as $|z| \to +\infty$ are given by

$$E_{\frac{1}{m},\frac{1}{m}}(z) = \begin{cases} mz^{m-1}e^{z^m}\left(1 + o(1)\right), & |\arg z| \le \frac{\pi}{2m}, \\ O\left(\frac{1}{z}\right), & \frac{\pi}{2m} < |\arg z| \le \pi \end{cases} \tag{28}$$

for $m > \frac{1}{2}$, and by

$$E_{\frac{1}{m},\frac{1}{m}}(z) = m \sum_{j=-N}^{N} z^{m-1}\, e^{2\pi i j(m-1)} e^{z^m e^{2\pi i j m}} + O\left(\frac{1}{z}\right), \qquad -\pi < \arg z \le \pi,$$

for $0 < m \le \frac{1}{2}$, where N is the integer satisfying $N < \frac{1}{2m} \le N + 1$ and the powers z^{m-1} and z^m are the principal branches. See, for example, Bateman and Erdelyi [245], vol. III, 18.1, formulas (21)–(22).

The asymptotic estimates of the Mittag-Leffler function $E_{\frac{1}{m},\frac{1}{m}}$ provide the following estimates for the reproducing kernel $K_m(z,w)$, which is a consequence of the results in [246] and Lemma 3.1 in [247].

Lemma (6.2.1)[243]: For arbitrary points $x, r \in (0,+\infty)$ and $\theta \in (-\pi,\pi)$ we have

$$\left|K_m\left(x, re^{i\theta}\right)\right| \lesssim \begin{cases} (xr)^{m-1} e^{(xr)^m \cos(m\theta)} & |\theta| \le \frac{\pi}{2m} \\ O\left(\frac{1}{xr}\right), & \frac{\pi}{2m} \le |\theta| < \pi \end{cases}$$

as $xr \to +\infty$. Moreover, there is a constant $c > 0$ such that for all $|\theta| \le c\theta_0(xr)$ we have

$$\left|K_m\left(x, re^{i\theta}\right)\right| \gtrsim (xr)^{m-1}\, e^{(xr)m}$$

as $xr \to +\infty$, where $\theta_0(r) = r^{-\frac{m}{2}}/m$.

On several occasions later on we will need to know the maximum order of a function in $\mathcal{F}_m^2$. For example, if we have a non-vanishing function f in $\mathcal{F}_m^2$ and if we know that the order of f is finite, then we can write $f = e^q$ with q being a polynomial. The following estimate allows us to do this.

Lemma (6.2.2) [243]: If $f \in \mathcal{F}_m^2$, there is a constant $C > 0$ such that

$$|f(z)| \leq C\,|z|^{m-1}\, e^{\frac{1}{2}|z|^{2m}}, \qquad z \in \mathbb{C}.$$

Consequently, the order of every function in $\mathcal{F}_m^2$ is at most $2m$.

Proof. By the reproducing property and Cauchy-Schwartz inequality, we have

$$|f(z)| = \left|\int_{\mathbb{C}} f(w)K_m(z,w)d\lambda_m(w)\right| \leq \|f\|\, K_m(z,z)^{1/2}$$

for all $f \in \mathcal{F}_m^2$ and all $z \in \mathbb{C}$. The desired estimate then follows from Lemma (6.2.1). See [4] for more details.

Another consequence of the above lemma is that, for any function $u \in \mathcal{F}_m^2$, the Toeplitz operators T_u and $T_{\bar{u}}$ are both densely defined on $\mathcal{F}_m^2$.

We prove the equivalence of conditions (24) and (25) in the main theorem, which provides a simple and complete solution to Sarason's problem for Toeplitz products on the Fock space $\mathcal{F}_m^2$. We break the proof into several lemmas.

Lemma (6.2.3) [243]: Suppose that u and v are functions in $\mathcal{F}_m^2$, each not identically zero, and that the operator $T = T_u T_{\bar{v}}$ is bounded on $\mathcal{F}_m^2$. Then there exists a polynomial g of degree at most m and a nonzero complex constant c such that $u(z) = e^{g(z)}$ and $v(z) = ce^{-g(z)}$.

Proof. If $T = T_u T_v$ is bounded on $\mathcal{F}_m^2$, then the Berezin transform T is bounded, where

$$T(z) = \langle T_u T_{\bar{v}} k_z, k_z\rangle, \qquad z \in \mathbb{C}.$$

By the reproducing property of the kernel functions, it is easy to see that

$$T(z) = u(z)\overline{v(z)}.$$

Since each k_z is a unit vector, it follows from the Cauchy-Schwarz inequality that

$$|u(z)v(z)| = \left|\tilde{T}(z)\right| \leq \|T\|$$

for all $z \in \mathbb{C}$. This together with Liouville's theorem shows that there exist a constant c such that $uv = c$. Since neither u nor v is identically zero, we have $c \neq 0$. Consequently, both u and v are non-vanishing.

Recall from Lemma (6.2.2) that the order of functions in $\mathcal{F}_m^2$ is at most $2m$, so there is a polynomial of degree d,

$$g(z) = \sum_{k=0}^{d} a_k z^k, \qquad d \leq [2m],$$

such that $u = e^g$ and $v = ce^{-g}$. It remains to show that $d \leq m$.

Since T is bounded on $\mathcal{F}_m^2$, the function

$$F(z,w) = \frac{\langle T\left(K_m(\cdot,w)\right), K_m(\cdot,z)\rangle}{\sqrt{K_m(z,z)}\sqrt{K_m(w,w)}}$$

must be bounded on $\mathbb{C}^2$. On general reproducing Hilbert spaces, we always have

$$\langle T_u T_{\bar{v}} K_w, K_z\rangle = \langle T_{\bar{v}} K_w, T_u K_z\rangle = \langle \bar{v}(w) K_w, u(z) K_z\rangle = u(z)\bar{v}(w)K(z,w).$$

It follows that

$$F(z,w) = \bar{c}e^{g(z)-\overline{g(w)}}\frac{K_m(z,w)}{\sqrt{K_m(z,z)}\sqrt{K_m(w,w)}}.$$

From Lemma (6.2.1) we deduce that

$$|F(z,w)| \gtrsim e^{Re(g(z)-g(w))}e^{-\frac{1}{2}(|z|^m-|w|^m)^2} \tag{29}$$

for all $|\arg(z\overline{w})| \leq c\theta_0(|zw|)$ as $|zw|$ grows to infinity. Choose $x > 0$ sufficiently large and set

$$z(x) = xe^{i\frac{\pi}{2d}}e^{-i\frac{\arg(a_d)}{d}},$$

and

$$w(x) = xe^{i\frac{\pi}{2d}}e^{-i\frac{\arg(a_d)+\frac{c}{2mx^m}}{d}}.$$

Since

$$\theta_0(|z(x)w(x)|) = \frac{1}{mx^m},$$

we can apply (29) to $z(x)$ and $w(x)$ to get

$$e^{Re\left(g(z(x))-g(w(x))\right)} \lesssim \sup_{(z,w)\in\mathbb{C}^2} |F(z,w)| < \infty \tag{30}$$

as x grows to infinity. On the other hand, a few computations show that

$$Re\left(g(z(x)) - g(w(x))\right)$$
$$= \sum_{j=0}^{d} x^j Re\left(a_j e^{ij\frac{\pi}{2d}-i\frac{j}{d}arg[a_d)}\left(1 - e^{-i\frac{cj}{2mdx^m}}\right)\right]$$
$$\left(1 - e^{-i\frac{cj}{2mdx^m}}\right)] = |a_d|\, x^d \sin\left(\frac{c}{2mx^m}\right) + g_{d-1}(x),$$
$$= |a_d|\, x^d \sin\left(\frac{c}{2mx^m}\right) + g_{d-1}(x),\Big)$$

Where

$$g_{d-1}(x) = \sum_{j=0}^{d-1} x^j Re\left(a_j e^{i\frac{j\pi}{2d}-i\frac{j}{d}\arg(a_d)}\left(1 - e - i\frac{cj}{2mdx^m}\right)\right)$$
$$= -\sum_{j=0}^{d-1} |a_j|x^j \sin\left(\frac{j\pi}{2d} + arg\, a_j - \frac{j}{d}\, arg\,(a_d)\right)\sin\frac{cj}{2mdx^m}$$
$$+\sum_{j=0}^{d-1} |a_j|x^j \cos\left[\frac{j\pi}{2d} + arg\, a_j - \frac{j}{d}\, arg\,(a_d)\right]\left[1 - cos\frac{cj}{2mdx^m}\right]$$
$$\lesssim x^{d-1-m}.$$

Therefore, there exist some $x_0 > 0$ and $\delta > 0$ such that

$$Re\left(g(z(x)) - g(w(x))\right) \geq \frac{\delta|a_d|x^d}{x^m}$$

for all $x \geq x_0$. Since $a_d \neq 0$, it follows from (30) that $d \leq m$.

On several occasions later on we will need to estimate the integral

$$I(a) = \int_0^{\infty} e^{-\frac{1}{2}r^{2m}+ar^d}\, r^N\, dr,$$

where $m > 0, 0 \le d \le m, N > -1$, and $a \ge 0$.

First, suppose $a > 1$. By various changes of variables, we have

$$\begin{aligned} I(a) &= \int_0^1 e^{-\frac{1}{2}r^{2m}+ar^d}\, r^N\, dr + \int_1^{\infty} e^{-\frac{1}{2}r^{2m}+ar^d}\, r^N\, dr \\ &\le e^a \int_0^1 r^N dr + \int_1^{\infty} e^{-\frac{1}{2}r^{2m}+ar^m\, r^N}\, dr \\ &= \frac{e^a}{N+1} + e^{\frac{a^2}{2}} \int_1^{\infty} e^{-\frac{1}{2}(r^m-a)^2} r^N\, dr \\ &= \frac{e^a}{N+1} + \frac{e^{\frac{a^2}{2}}}{m} \int_1^{\infty} e\text{^}(-\frac{1}{2}(t-a)^2\, t^{\frac{N+1}{m}-1}\, dt. \end{aligned}$$

If $\frac{N+1}{m} - 1 \le 0$, then

$$I(a) \le \frac{e^a}{N+1} + \frac{\sqrt{2\pi}}{m}\, e^{\frac{a^2}{2}} \le \left(\frac{\sqrt{e}}{N+1} + \frac{\sqrt{2\pi}}{m}\right) e^{\frac{a^2}{2}}.$$

Otherwise, we have $\frac{N+1}{m} - 1 > 0$. Using the fact that $u \mapsto u^{\frac{N+1}{m}-1}$ is increasing, we see that

$$\int_{-\frac{a}{2}}^{\frac{a}{2}} e^{-\frac{t^2}{2}} (t+a)^{\frac{N+1}{m}-1} dt \le \left(\frac{3a}{2}\right)^{\frac{N+1}{m}-1} \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{-\frac{t^2}{2}}\, dt \le \sqrt{2\pi}\, \left(\frac{3a}{2}\right)^{\frac{N+1}{m}-1}.$$

For the same reason we also have

$$\begin{aligned} \int_{\frac{a}{2}}^{+\infty} e^{-\frac{t^2}{2}} (t+a)^{\frac{N+1}{m}-1}\, dt &\le \int_{\frac{a}{2}}^{+\infty} e^{-\frac{t^2}{2}} (3t)^{\frac{N+1}{m}-1} dt \\ &\le 3^{\frac{N+1}{m}-1} \int_0^{+\infty} t^{\frac{N+1}{m}-1} e^{-\frac{t^2}{2}}\, dt = \frac{\sqrt{2}}{2} (3\sqrt{2})^{\frac{N+1}{m}-1} \int_0^{+\infty} u^{\frac{N+1}{m}-1} e^{-u}\, dt \\ &= \frac{\sqrt{2}}{2} (3\sqrt{2})^{\frac{N+1}{m}-1} \Gamma\left(\frac{N+1}{2m}\right). \end{aligned}$$

In the case when $1 - a < -a\, 2$ (or equivalently $a > 2$),

$$\begin{aligned} \int_{1-a}^{-\frac{a}{2}} e^{-\frac{t^2}{2}} (t+a)^{\frac{N+1}{m}-1}\, dt &\le \left(\frac{a}{2}\right)^{\frac{N+1}{m}-1} \int_{1-a}^{-\frac{a}{2}} e^{-\frac{t^2}{2}}\, dt \le \left(\frac{a}{2}\right)^{\frac{N+1}{m}-1} \int_{1-a}^{-\frac{a}{2}} e^{\frac{at}{4}} dt \\ &\le \left(\frac{a}{2}\right)^{\frac{N+1}{m}-1} \frac{4}{a} e^{-\frac{a^2}{8}} \le 2 \left(\frac{a}{2}\right)^{\frac{N+1}{m}-1}. \end{aligned}$$

It follows that there exists a constant $C = C(m, N) > 0$ such that

$$\int_1^{\infty} e^{-\frac{1}{2}(t-a)^2} t^{\frac{N+1}{m}-1} dt = \int_{1-a}^{\infty} e^{-\frac{t^2}{2}} (t+a)^{\frac{N+1}{m}-1} dt \le C\, (1+a)^{\frac{N+1}{m}-1}$$

for $\frac{N+1}{m}-1>0$. It is then easy to find another positive constant $C=C(m,N)$, independent of a, such that

$$I(a)\leq C(1+a)^{\frac{N+1}{m}-1}e^{\frac{a^2}{2}}$$

for all $a\geq 1$ and $\frac{N+1}{m}-1>0$. Therefore,

$$\int_0^{\infty} e^{-\frac{1}{2}r^{2m}+ar^d} r^N\,dr\leq C(1+a)^{\max\left(0,\frac{N+1}{m}-1\right)}e^{\frac{a^2}{2}} \tag{31}$$

for all $a\geq 1$. Since $I(a)$ is increasing in a, the estimate above holds for $0\leq a\leq 1$ as well.

Lemma (6.2.4)[243]: For any $m>0,\delta>0,R\geq 1,N>-1$, and $p\geq 0$, we can find a constant $C>0$ (depending on R,δ,p,N,m but not on a,d,x) such that

$$x^{N+1-p}\int_{\frac{R}{x^2}}^{+\infty} e^{-\frac{x^{2m}}{2}(1+r^{2m})+ax^d(1+\delta r^d)} r^N\,dr\leq C(1+a)^{\max\left(0,\frac{N+p+1}{m}-1\right)}e^{\frac{1+\delta^2}{2}a^2}$$

and

$$x^m\int_{\frac{R}{x^2}}^{+\infty} e^{-\frac{x^{2m}}{2}(1-r^m)^2+ax^d(1-r^d)}r^{\frac{m}{2}}\,dr\leq C(1+a)e^{\frac{a^2}{2}}$$

for all $x>0,a>0$, and $0\leq d\leq m$.

Proof. Let $I=I(m,N,p,R,x,a,d)$ denote the first integral that we are trying to estimate. If $x\geq 1$, we have

$$\begin{aligned}I&=x^{N+1-p}e^{-\frac{x^{2m}}{2}+ax^d}\int_{\frac{R}{x^2}}^{\infty} e^{-\frac{(xr)^{2m}}{2}+a\delta(xr)^d} r^N\,dr\\
&\leq x^{-p}e^{-\frac{x^{2m}}{2}+ax^m}\int_{\frac{R}{x}}^{\infty} e^{-\frac{r^{2m}}{2}+a\delta r^d} r^N dr\\
&\leq e^{-\frac{1}{2}(x^m-a)^2+\frac{a^2}{2}}\frac{\int_{\frac{R}{x}}^{\infty} r^p}{R^p} e^{-\frac{1}{2}r^{2m}+a\delta r^d} r^N\,dr\\
&\leq \frac{e^{\frac{a^2}{2}}}{R^p}\int_{\frac{R}{x}}^{\infty} e^{-\frac{r^{2m}}{2}+a\delta r^d} r^{N+p}dr.\end{aligned}$$

The desired result then follows from (31).

If $0<x<1$, we have

$$I=x^{N+1-p}e^{-\frac{x^{2m}}{2}+ax^d}\int_{\frac{R}{x^2}}^{\infty} e^{-\frac{(xr)^{2m}}{2}+a\delta(xr)^d} r^N\,dr\leq e^a x^{-p}\int_{\frac{R}{x}}^{\infty} e^{-\frac{r^{2m}}{2}+a\delta r^d} r^N dr$$

$$\leq\frac{e^{\frac{a^2}{2}+1}}{R^p}\int_{\frac{R}{x}}^{\infty} e^{-\frac{r^{2m}}{2}+a\delta r^d} r^{N+p}dr.$$

The desired estimate follows from (31) again.

To prove the second part of the lemma, denote by $J = J(m,d,R,x,a)$ the second integral that we are trying to estimate. Then it is clear from a change of variables that for $0 < x < 1$ we have

$$J(m,d,R,x,a) = x^{\frac{m}{2}-1} \int_{\frac{R}{x}}^{+\infty} e^{-\frac{1}{2}(x^m - r^m)^2 + a(x^d - r^d)} r^{\frac{m}{2}}\, dr$$

$$\leq \frac{e^a}{R} x^{\frac{m}{2}} \int_{\frac{R}{x}}^{+\infty} e^{-\frac{1}{2}(x^{2m} - 2(xr)^m + r^{2m})} r^{\frac{m}{2}+1}\, dr$$

$$\leq \frac{e^a}{R} \int_0^{+\infty} e^{-\frac{r^{2m}}{2} + r^m} r^{\frac{m}{2}+1}\, dr = Ce^a \leq C'(1+a)e^{\frac{a^2}{2}},$$

where the constants C and C only depend on R and m.

Next assume that $x \geq 1$. In case $R \leq x^2$ we write $J = J_1 + J_2$, where

$$J_1 = J_1(m,d,R,x,a) = x^m \int_{\frac{R}{x^2}}^{1} e^{-\frac{x^{2m}}{2}(1-r^m)^2 + ax^d(1-r^d)} r^{\frac{m}{2}}\, dr,$$

And

$$J_2 = J_2(m,d,R,x,a) = x^m \int_1^{\infty} e^{-\frac{x^{2m}}{2}(1-r^m)^2 + ax^d(1-r^d)} r^{\frac{m}{2}}\, dr.$$

Otherwise we just use $J \leq J_2$. So it suffices to estimate the two integrals above.

To handle $J_1(m,d,R,x,a)$, we fix $\varepsilon > 0$ and consider two cases. In the case $x^m \leq a(1+\varepsilon)$, we have

$$J_1(m,d,R,x,a) \leq x^m \int_{\frac{R}{x^2}}^{1} e^{-\frac{x^{2m}}{2}(1-r^m)^2 + ax^d(1-r^d)} r^{\frac{m}{2}}\, dr$$

$$\leq a(1+\varepsilon)e^{\frac{a^2}{2}} \int_{\frac{R}{x^2}}^{1} e^{-\frac{1}{2}(x^m(1-r^m)-a)^2} r^{\frac{m}{2}}\, dr \qquad \leq a(1+\varepsilon)e^{\frac{a^2}{2}}.$$

When $x^m \geq a(1+\varepsilon)$, we set $y = x^m$ and $\tau = (y-a)/2$. Then we have

$$\tau \geq \frac{\varepsilon}{2(1+\varepsilon)} y \to +\infty$$

as $y \to +\infty$. By successive changes of variables we see that

$$J_1(m,d,R,x,a) \leq x^m \int_{\frac{R}{x^2}}^{1} e^{-\frac{x^{2m}}{2}(1-r^m)^2 + ax^m(1-r^m)} r^{\frac{m}{2}}\, dr$$

$$= \frac{y}{m} \int_0^{1-\frac{R^m}{y^2}} (1-r)^{\frac{1}{m}-\frac{1}{2}} e^{-\frac{y^2 r^2}{2} + ayr}\, dr$$

$$= \frac{1}{m} \int_0^{y-\frac{R^m}{y}} \left(1 - \frac{r}{y}\right)^{\frac{1}{m}-\frac{1}{2}} e^{-\frac{r^2}{2} + ar}\, dr$$

$$= \frac{e^{\frac{a^2}{2}}}{m} \int_{-a}^{y-a-\frac{R^m}{y}} \left(1 - \frac{a}{y} - \frac{r}{y}\right)^{\frac{1}{m}-\frac{1}{2}} e^{-\frac{r^2}{2}}\, dr.$$

This shows that for $1 \leq m \leq 2$ we have

$$J_1 \leq \frac{e^{\frac{a^2}{2}}}{m} \int_{-a}^{y-a-\frac{R^m}{y}} e^{-\frac{r^2}{2}}\, dr \leq \frac{\sqrt{2\pi}}{m}\, e^{\frac{a^2}{2}}.$$

Thus we suppose that $m > 2$. Then

$$\int_{-\tau}^{\tau} \left(1 - \frac{a}{y} - \frac{r}{y}\right)^{\frac{1}{m}-\frac{1}{2}} e^{-\frac{r^2}{2}}\, dr \leq \left(1 - \frac{a}{y} - \frac{\tau}{y}\right)^{\frac{1}{m}-\frac{1}{2}} \int_{-\tau}^{\tau} e^{-\frac{r^2}{2}}\, dr$$
$$= \left(\frac{\tau}{2y}\right)^{\frac{1}{m}-\frac{1}{2}} \int_{-\tau}^{\tau} e^{-\frac{r^2}{2}}\, dr \leq \sqrt{2\pi} \left(\frac{\varepsilon}{4(1+\varepsilon)}\right)^{\frac{1}{m}-\frac{1}{2}}.$$

Moreover, in case $-a < -\tau$, we have

$$\int_{-a}^{-\tau} \left(1 - \frac{a}{y} - \frac{r}{y}\right)^{\frac{1}{m}-\frac{1}{2}} e^{-\frac{r^2}{2}}\, dv \leq \left(1 - \frac{a}{y} + \frac{\tau}{y}\right)^{\frac{1}{m}-\frac{1}{2}} \int_{-a}^{-\tau} e^{-\frac{\tau|r|}{2}}\, dr$$
$$\leq 2\left(\frac{3\varepsilon}{2(1+\varepsilon)}\right)^{\frac{1}{m}-\frac{1}{2}} \frac{e^{-\frac{\tau^2}{2}}}{\tau} \leq 4\left(\frac{3}{2}\right)^{\frac{1}{m}-\frac{1}{2}} \left(\frac{\varepsilon}{1+\varepsilon}\right)^{\frac{1}{m}-\frac{3}{2}} e^{-\frac{\varepsilon^2}{8(1+\varepsilon)^2}}.$$

Similarly, in case $y - a - \frac{R^m}{y} \geq \tau$, we have

$$\int_{\tau}^{y-a-\frac{R^m}{y}} \left[1 - \frac{a}{y} - \frac{r}{y}\right]^{\frac{1}{m}-\frac{1}{2}} e^{-\frac{r^2}{2}}\, dr \leq \left[\frac{R^m}{y^2}\right]^{\frac{1}{m}-\frac{1}{2}} \int_{\tau}^{y-a-\frac{R^m}{y}} e^{-\frac{\tau r}{2}}\, dr$$
$$\leq 2R^{1-\frac{m}{2}} \left[\frac{\varepsilon}{2(1+\varepsilon)}\right]^{\frac{2}{m}-1} \tau^{-\frac{2}{m}} e^{-\frac{\tau^2}{2}} \left(since\ \tau \geq \frac{\varepsilon}{2(1+\varepsilon)}\right)$$
$$\leq 4R^{1-\frac{m}{2}} \frac{1+\varepsilon}{\varepsilon} e^{-\frac{\varepsilon^2}{8(1+\varepsilon)^2}}.$$

The last three estimates yield

$$J_1 \leq C(1+a)e^{\frac{a^2}{2}}$$

for some C > 0 that is independent of x and a.

To establish the estimate for J_2, we perform a change of variables to obtain

$$J_2 \leq x^m \int_1^{+\infty} e^{-\frac{x^{2m}}{2}(1-r^m)^2}\, r^{\frac{m}{2}}\, dr = \frac{1}{m}\int_0^{+\infty} e^{-\frac{r^2}{2}} \left(\frac{r}{x^m} + 1\right)^{\frac{1}{m}-\frac{1}{2}} dr.$$

If $m \geq 2$, we have

$$J_2 \leq \frac{1}{m}\int_0^{+\infty} e^{-\frac{r^2}{2}}\, dr,$$

and if $1 \leq m < 2$, we have

$$J_2 \leq \frac{1}{m}\int_0^{+\infty} e^{-\frac{r^2}{2}}(r+1)^{\frac{1}{m}-\frac{1}{2}}\, dr.$$

Therefore, $J_2 \le C$ for some $C > 0$ that is independent of x and a. This completes the proof of the lemma.

In the proof of the main theorem, we will have to estimate the following two integrals:

$$I(x,r) = \int_{|\theta|\le\frac{\pi}{2m}} e^{-(xr)^m+2ar^d\sin^2\left(\frac{\theta d}{2}\right)}\left|K_m\left(x,re^{i\theta}\right)\right| d\theta,$$

and

$$J(x,r) = \int_{|\theta|\ge\frac{\pi}{2m}} e^{-(xr)^m+a\left(x^d+r^d\right)}\left|K_m\left(x,re^{i\theta}\right)\right| d\theta,$$

where $x, r, a \in (0,+\infty)$ and $0 \le d \le m$.

Lemma (6.2.5) [243]: For any $m > 0$ there exist positive constants $C = C(m)$ and $R = R(m)$ such that

$$I(x,r) \le C(xr)^{m-1}\int_0^1 e^{-\left((xr)^m-ar^d\right)t^2}\,dt$$

And

$$J(x,r) \le \frac{Ce^{-(xr)^m+a\left(x^d+r^d\right)}}{xr}$$

for all $a > 0, 0 \le d \le m$, and $x > 0$ with $xr > R$.

Proof. It follows from Lemma (6.2.1) that there exist positive constants $C = C(m)$ and $R = R(m)$ such that for all $a > 0$ and $xr > R$ we have

$$I(x,r) \le C(xr)^{m-1}\int_{|\theta|\le\frac{\pi}{2m}} e^{-(xr)^m+(xr)^m\cos(m\theta)+2ar^d\sin^2\left(\frac{\theta d}{2}\right)}\,d\theta$$

$$= 2C(xr)^{m-1}\int_0^{\frac{\pi}{2m}} e^{-2(xr)^m\sin^2\left(\frac{m\theta}{2}\right)+2ar^d\sin^2\left(\frac{\theta d}{2}\right)}\,d\theta$$

$$\le 2C(xr)^{m-1}\int_0^{\frac{\pi}{2m}} e^{-2(xr)^m\sin^2\left(\frac{m\theta}{2}\right)+2ar^d\sin^2\left(\frac{m\theta}{2}\right)}\,d\theta$$

$$\le 2C(xr)^{m-1}\int_0^{\frac{\pi}{2m}} e^{-2\left((xr)^m-ar^d\right)\sin^2\left(\frac{m\theta}{2}\right)}\,d\theta$$

$$= \frac{4C}{m}(xr)^{m-1}\int_0^{\frac{\sqrt{2}}{2}} e^{-2\left((xr)^m-ar^d\right)t^2}\frac{dt}{\sqrt{1-t^2}} \le \frac{4\sqrt{2}C}{m}(xr)^{m-1}\int_0^{\frac{\sqrt{2}}{2}} e^{-2\left((xr)^m-ar^d\right)t^2}\,dt$$

$$\le \frac{4\sqrt{2}C}{m}(xr)^{m-1}\int_0^1 e^{-\left((xr)^m-ar^d\right)t^2}\,dt.$$

The estimate

$$J(x,r) \le \frac{Ce^{-(xr)^m+a\left(x^d+r^d\right)}}{xr}, \qquad xr > R,$$

also follows from Lemma (6.2.1).

Lemma (6.2.6) [243]: For any $m \geq 1$ there exist constants $R = R(m) > 1$ and $C = C(m) > 0$ such that

$$\int_{\frac{R}{x}}^{+\infty} e^{-\frac{1}{2}(x^m - r^m)^2 + a(x^d - r^d)} I(x,r) r\, dr \leq C(1 + a)^{\frac{1}{m}-1} e^{a^2}$$

and

$$\int_{\frac{R}{x}}^{+\infty} e^{-\frac{1}{2}(x^m - r^m)^2} J(x,r) r\, dr \leq C(1 + a)^{\max\left(0,\frac{2}{m}-1\right)} e^{a^2}$$

for all $x > 0, a > 0$, and $0 \leq d \leq m$.

Proof. For convenience we write

$$A_I(x,r) = e^{-\frac{1}{2}(x^m - r^m)^2 + a(x^d - r^d)} I(x,r) r,$$

and

$$A_J(x,r) = e^{-\frac{1}{2}(x^m - r^m)^2} J(x,r) r.$$

Let R and C be the constants from Lemma (6.2.5). In the integrands we have r > R/x, or xr > R, so according to Lemma (6.2.5),

$$I(x,r) \leq C(xr)^{m-1} \int_0^1 e^{-(xr)^m t^2 + ar^d t^2}\, dt.$$

If, in addition, $x \leq 1$, then

$$I(x,r) \leq Cr^{m-1} e^{ar^d},$$

and

$$A_I(x,r) = e^{-\frac{1}{2}(x^m - r^m)^2} e^{ax^d - ar^d} I(x,r) r \leq Cr^m e^a e^{-\frac{1}{2}(x^m - r^m)^2}.$$

It follows that

$$\int_{\frac{R}{x}}^{\infty} A_I(x,r)\, dr \leq Ce^a \int_{\frac{R}{x}}^{\infty} r^m e^{-\frac{1}{2}(x^m - r^m)^2}\, dr \leq Ce^a \int_0^{\infty} r^m e^{-\frac{1}{2}x^{2m} + x^m r^m - \frac{1}{2}r^{2m}}\, dr$$

$$\leq Ce^a \int_0^{\infty} r^m e^{r^{m - \frac{1}{2}r^{2m}}}\, dr \qquad \leq C(1 + a)^{\frac{1}{m}-1} e^{a^2}.$$

for all $a > 0$ and $0 < x \leq 1$.

Similarly, if $x \leq 1$ (and $xr > R$), we deduce from Lemma (6.2.5) and (31) that

$$\int_{\frac{R}{x}}^{\infty} A_J(x,r)\, dr \leq \frac{C}{R} \int_{\frac{R}{x}}^{\infty} e^{-\frac{1}{2}(x^m - r^m)^2} e^{-(x^r)^m + ax^d + ar^d\, r}\, dr$$

$$\leq \frac{Ce^a}{R} \int_{\frac{R}{x}}^{\infty} e^{-\frac{1}{2}r^{2m} + ar^d} r\, dr \quad \leq C'(1 + a) \max\left(0, \frac{2}{m} - 1\right) e^{a^2}.$$

Suppose now that $x \geq 1$ and $rx > R$. By Lemma (6.2.5) again,

$$A_I(x,r) \leq Cr(xr)^{m-1} e^{-\frac{1}{2}(x^m - r^m)^2 + a(x^d - r^d)} \int_0^1 e^{-t^2((xr)^m - ar^d)}\, dt.$$

Fix a sufficiently small $\varepsilon \in (0,1)$. If $(xr)m \geq ar^d(1 + \varepsilon)$, then

$$\int_0^1 e^{-t^2\left((xr)^m - ar^d\right)}\, dt = \frac{1}{\sqrt{(xr)^m - ar^d}} \int_0^{\sqrt{(xr)^m - ar^d}} e^{-s^2}\, ds$$

$$\leq \frac{1}{\sqrt{(xr)^m - ar^d}} \int_0^{\infty} e^{-s^2}\, ds = \frac{\sqrt{\pi}}{2} \frac{(xr)^{-\frac{m}{2}}}{\sqrt{1 - \left(\frac{ar^d}{(xr)^m}\right)}}$$

$$\leq \sqrt{\frac{\pi(1 + \varepsilon)}{4\varepsilon}}\, (xr)^{-\frac{m}{2}},$$

so there exists a constant $C = C(m)$ such that

$$A_I\,(x,r) \leq Cr(xr)^{\frac{m}{2}-1}\, e^{-\frac{1}{2}(x^m - r^m)^2 + a(x^d - r^d)}.$$

If $(xr)^m \leq ar^d(1 + \varepsilon)$, we have

$$A_I(x,r)\, a^{\frac{m-1}{m}}\, r^{\frac{d(m-1)+m}{m}}\quad e^{-\frac{1}{2}(x^{2m} + r^{2m}) + ax^d} \int_0^1 e^{(1-t^2)\left((xr)^m - ar^d\right)}\, dt$$

$$\leq a^{\frac{m-1}{m}}\, r^{\frac{d(m-1)+m}{m}}\, e^{-\frac{1}{2}(x^{2m} + r^{2m}) + a\left(x^d + \varepsilon r^d\right)}.$$

It follows that

$$\int_{\frac{R}{x}}^{+\infty} A_I\,(x,r)\, dr$$

$$\lesssim x^{\frac{m}{2}-1} \int_{\frac{R}{x}}^{+\infty} e^{-\frac{1}{2}(x^m - r^m)^2 + a\left(x^d - r^d\right)} r^{\frac{m}{2}}\, dr$$

$$+ a^{\frac{m-1}{m}} \int_{\frac{R}{x}}^{+\infty} e^{-\frac{1}{2}(x^{2m} + r^{2m}) + a\left(x^d + \varepsilon r^d\right)} dr.$$

The change of variables $r \longmapsto xr$ along with the second part of Lemma (6.2.4) shows that

$$x^{\frac{m}{2}-1} \int_{\frac{R}{x}}^{+\infty} e^{-\frac{1}{2}(x^m - r^m)^2 + a\left(x^d - r^d\right) r^{\frac{m}{2}}}\, dr \leq C(1 + a) e^{\frac{a^2}{2}}.$$

Similarly, the change of variables $r \longmapsto xr$ together with the first part Lemma (6.2.4) shows that

$$\int_{\frac{R}{x}}^{+\infty} r^{\frac{d(m-1)+m}{m}}\, e^{-\frac{1}{2}(x^{2m} + r^{2m}) + a\left(x^d + \varepsilon r^d\right)}\, dr \leq C(1 + a)^{\frac{d(m-1)+m}{m}} e^{\frac{1+\varepsilon^2}{2} a^2}.$$

We may assume that $\varepsilon < 1$. Then we can find a positive constant C such that

$$a^{\frac{m-1}{m}} \int_{\frac{R}{x}}^{+\infty} r^{\frac{d(m-1)+m}{m}}\, e^{-\frac{1}{2}(x^{2m} + r^{2m}) + a\left(x^d + \varepsilon r^d\right)} dr \leq C(1 + a)^{\frac{1}{m}-1}\, e^{a^2}.$$

It follows that

$$\int_{\frac{R}{x}}^{+\infty} A_I(x,r)\,dr \le C(1+a)^{\frac{1}{m}-1} e^{a^2}$$

for some other positive constant C that is independent of a and x. This proves the first estimate of the lemma.

To establish the second estimate of the lemma, we use Lemma (6.2.5) to get

$$xA_J(x,xr) = x^2 re^{-\frac{x^{2m}}{2}(1-r^m)^2} J(x,xr) \le Ce^{-\frac{x^{2m}}{2}(1+r^{2m})+ax^d(1+r^d)}.$$

It follows from this and Lemma (6.2.4) that

$$\int_{\frac{R}{x}}^{+\infty} A_J(x,r)\,dr = x\int_{\frac{R}{x^2}}^{+\infty} A_J(x,xr)\,dr \le C(1+a)^{\max\left(0,\frac{2}{m}-1\right)} e^{a^2}.$$

This completes the proof of the lemma.

Lemma (6.2.7) [243]: If $u(z) = e^{g(z)}$ and $v(z) = e^{-g(z)}$, where g is a polynomial of degree at most m, then the operator $T = T_u T_{\bar{v}}$ is bounded on $\mathcal{F}_m^2$.

Proof. To prove the boundedness of $T = T_u T_{\bar{v}}$, we shall use a standard technique known as Schur's test [18, p.42]. Since

$$T f(z) = \int_{\mathbb{C}} K_m(z,w)e^{g(z)-\overline{g(w)}} f(w)e^{-|w|^{2m}}\,dA(w),$$

we have

$$|T f(z)|e^{-\frac{1}{2}|z|^{2m}} \le \int_{\mathbb{C}} H_g(z,w)|f(w)|e^{-\frac{1}{2}|w|^{2m}}\,dA(w),$$

where

$$H_g(z,w) := |K_m(z,w)|e^{-\frac{1}{2}(|z|^{2m}+|w|^{2m})+Re\left(g(z)-\overline{g(w)}\right)}.$$

Thus T will be bounded on $\mathcal{F}_m^2$ if the integral operator S_g defined by

$$S_g f(z) = \int_{\mathbb{C}} \left(H_g(z,w) + H_g(w,z)\right) f(w)\,dA(w)$$

is bounded on $L^2(\mathbb{C}, dA)$. Let

$$H_g(z) = \int_{\mathbb{C}} H_g(z,w)\,dA(w), \qquad z \in \mathbb{C}.$$

Since

$$H_{-g}(z) = \int_{\backslash calcolus} \mathbb{C}\, H_g(w,z)\,dA(w),$$

for all $z \in \mathbb{C}$, by Schur's test, the operator S_g is bounded on $L^2(\mathbb{C}, dA)$ if we can find a positive constant C such that

$$H_{g(z)} + H_{-g(z)} \le C, \qquad z \in \mathbb{C}.$$

By the Cauchy-Schwarz inequality, we have

$$H_{g_1+g_2}(z) \le \sqrt{H_{2g_1}(z)H_{2g_2}(z)}$$

for all $z \in \mathbb{C}$ and holomorphic polynomials g_1 and g_2. Moreover, if

$$U_\theta(z) = e^{i\theta}z, \qquad z \in \mathbb{C}, \theta \in [-\pi,\pi],$$

Then

$$H_{goU_\theta} = H_g \, o \, U_\theta$$

for all $z \in \mathbb{C}, \theta \in [-\pi, \pi]$, and holomorphic polynomials g. Therefore, we only need prove the theorem for $g(z) = az^d$ with some $a > 0$ and $d \leq m$ and establish that

$$\sup_{x \geq 0} H_g(x) \leq C_1 e^{C_2 a^2}, \tag{32}$$

where C_k are positive constants independent of a and d (but dependent on m). We will see that C_2 can be chosen as any constant greater than 1.

It is also easy to see that we only need to prove (32) for $x \geq 1$. This will allow us to use the inequality $x^d \leq x^m$ for the rest of this proof.

For $R > 0$ sufficiently large (we will specify the requirement on R later) we write

$$H_g(x) = \int_{|xw| \leq R} H_g(x, w)\, dA(w) + \int_{|xw| \geq R} H_g(x, w)\, dA(w).$$

We will show that both integrals are, up to a multiplicative constant, bounded above by $e^{(1+\varepsilon)a^2}$.

By properties of the Mittag-Leffler function, we have

$$|K_m(x, w)| \leq \frac{m}{\pi} E_{\frac{1}{m}, \frac{1}{m}}(R) := C_R, \qquad |xw| \leq R.$$

It follows that the integral

$$I_1 = \int_{x|w| \leq R} H_g(x, w)\, dA(w)$$

Satisfies

$$\begin{aligned}
I_1 &= \int_{x|w| \leq R} |K_m(z, w)| e^{-\frac{1}{2}(|z|^{2m} + |w|^{2m}) + aRe\,(x^d - w^d)}\, dA(w) \\
&\leq C_R \int_{x|w| \leq R} e^{-\frac{1}{2}(x^{2m} + |w|^{2m}) + aRe(x^d - w^d)}\, dA(w) \\
&\leq C_R e^{-\frac{1}{2}x^{2m} + ax^d} \int_{x|w| \leq R} e^{-\frac{|w|^{2m}}{2} + a|w|^d}\, dA(w) \\
&\leq 2\pi C_R e^{-\frac{1}{2}x^{2m} + ax^m} \int_0^{+\infty} e^{-\frac{r^{2m}}{2} + ar^d}\, r\, dr \\
&\leq 2\pi C_R e^{\frac{a^2}{2}} \int_0^{+\infty} e^{-\frac{r^{2m}}{2} + ar^d}\, r\, dr \quad \leq C(1 + a)^{\max\left(0, \frac{2}{m} - 1\right)} e^{a^2},
\end{aligned}$$

where the last inequality follows from (31).

We now focus on the integral

$$I_2 = \int_{x|w| \geq R} H_g(x, w)\, dA(w).$$

Observe that for all x, r, and θ we have

$$\begin{aligned}
Re\left(x^d - r^d e^{id\theta}\right) &= x^d - r^d \cos(d\theta) = x^d - r^d + r^d \left(1 - \cos(d\theta)\right) \\
&= x^d - r^d + 2r^d \sin^2\left(\frac{d\theta}{2}\right).
\end{aligned}$$

It follows from polar coordinates that

$$I_2 = \int_{\frac{R}{x}}^{+\infty} \int_{-\pi}^{\pi} H_g(x, re^{i\theta}) r \, d\theta \, dr$$

$$= \int_{\frac{R}{x}}^{+\infty} \int_{-\pi}^{\pi} e^{-\frac{1}{2}(x^{2m}+r^{2m})+a\left(x^d - r^d \cos(d\theta)\right)} \left|K_m(x, re^{i\theta})\right| r \, d\theta \, dr$$

$$= \int_{\frac{R}{x}}^{\infty} e^{-\frac{1}{2}(x^m - r^m)^2 + a(x^d - r^d) - (xr)^m} r \, dr \int_{-\pi}^{\pi} e^{2ar^d} \sin^2 \left(\frac{d\theta}{2}\right) \left|K_m(x, re^{i\theta})\right| d\theta$$

$$\leq \int_{\frac{R}{x}}^{+\infty} e^{-\frac{1}{2}(x^m - r^m)^2} \left(e^{a(x^d - r^d)} I(x, r) + J(x, r)\right) r \, dr,$$

Where

$$I(x, r) = \int_{|\theta| \leq \frac{\pi}{2m}} e^{-(xr)^m + 2ar^d \sin^2\left(\frac{d\theta}{2}\right)} \left|K_m(x, re^{i\theta})\right| d\theta,$$

and

$$J(x, r) = \int_{|\theta| \geq \frac{\pi}{2m}} e^{-(xr)^m + a\left(x^d + r^d\right)} \left|K_m(x, re^{i\theta})\right| d\theta.$$

By Lemma (6.2.6), there exists another constant $C > 0$ such that

$$I_2 \leq C(1 + a)^{\max\left(0, \frac{2}{m} - 1\right)} e^{a^2}.$$

Therefore,

$$\sup_{z \in \mathbb{C}} \int_{\mathbb{C}} H_g(z, w) \, dA(w) \leq C(1 + a)^{\max\left(0, \frac{2}{m} - 1\right)} e^{a^2}$$

for yet another constant C that is independent of a and d. Similarly, we also have

$$\sup_{z \in \mathbb{C}} \int_{\mathbb{C}} H_{-g}(z, w) \, dA(w) \leq C(1 + a)^{\max\left(0, \frac{2}{m} - 1\right)} e^{a^2}$$

This yields (32) and proves the lemma.

We show that Sarason's conjecture is true for Toeplitz products on the Fock type space $\mathcal{F}_m^2$. We will prove that condition (26) in the main theorem stated is equivalent to conditions (24) and (25). Again we will break the proof down into several lemmas.

Lemma (6.2.8) [243]: Suppose u and v are functions in $\mathcal{F}_m^2$, not identically zero, such that the operator $T = T_u T_{\bar{v}}$ is bounded on $\mathcal{F}_m^2$. Then the function $\widetilde{|u|^2}(z)\widetilde{|v|^2}(z)$ is bounded on the complex plane.

Proof. Since $T_u T_{\bar{v}}$ is bounded on $\mathcal{F}_m^2$, the operator $(T_u T_{\bar{v}})^* = T_u T_{\bar{v}}$ and the products $(T_u T_{\bar{v}})^* T_u T_{\bar{v}}$ and $(T_v T_{\bar{u}})^* T_v T_{\bar{u}}$ are also bounded on $\mathcal{F}_m^2$. Consequently, their Berezin transforms are all bounded functions on $\mathbb{C}$.

For any $z \in \mathbb{C}$ we let k_z denote the normalized reproducing kernel of $\mathcal{F}_m^2$ at z. Then

$$\langle (T_u T_{\bar{v}})^* T_u T_{\bar{v}} k_z, k_z \rangle = \langle T_u T_{\bar{v}} k_z, T_u T_{\bar{v}} k_z \rangle = \langle u\overline{v(z)} k_z, u\overline{v(z)} k_z \rangle = |v(z)|^2 \widetilde{|u|^2}(z)$$

is bounded on $\mathbb{C}$. Similarly $|u(z)|^2 \widetilde{|v|^2}(z)$ is bounded on $\mathbb{C}$. By the proof of Lemma (6.2.3), the product uv is a non-zero complex constant, say, $u(z)v(z) = C$. It follows that the function

$$\widetilde{|v|^2}(z)\widetilde{|u|^2}(z) = |u(z)|^2 \widetilde{|v|^2}(z)|v(z)|^2 \widetilde{|u|^2}(z)\frac{1}{|C|^2}$$

is bounded as well.

To complete the proof of Sarason's conjecture, we will need to find a lower bound for the function

$$\mathcal{B}(z) = \widetilde{|v|^2}(z)|u(z)|^2 ,$$

where $u = e^g, v = e^{-g}$, and g is a polynomial of degree d. We write

$$g(z) = a_d z^d + g_{d-1}(z),$$

where

$$a_d = ae^{i\alpha_d}, \qquad a > 0,$$

and

$$g_{d-1}(z) = \sum_{l=0}^{d-1} a_l z^l .$$

In the remainder we will have to handle several integrals of the form

$$I(x) = \int_J S_x(r)e^{-g_x(r)}\, dr,$$

where S_x and g_x are C^3-functions on the interval J, and the real number x tends to $+\infty$. We will make use of the following variant of the Laplace method (see [250]).

Lemma (6.2.9) [243]: Suppose that

(a) g_x attains its minimum at a point r_x, which tends to $+\infty$ as x tends to $+\infty$, with $c_x = g_x''(r_x) > 0$;

(b) there exists τ_x such that for $|r - r_x| < \tau_x, g_x''(r) = c_x\big(1 + o(1)\big)$ as x ten ds to $+\infty$;

(c) for $|r - r_x| < \tau_x, S_x(r) \sim S_x(r_x)$;

(d) we have

$$\int_J S_x(r)e^{-g_x(r)}\, dr = \big(1 + o(1)\big)\int_{|r-r_x|<\tau_x} S_x(r)e^{-g_x(r)}\, dr$$

Then we have the following estimate

$$I(x) = \left(\sqrt{2\pi} + o(1)\right)[c_x]^{-1/2}\, S_x(r_x)e^{-g_x(r_x)}, \qquad x \to +\infty. \tag{33}$$

The computations in [251] ensure that, under the assumptions on g_x and S_x, we have

$$\int_{|r-r_x|>\tau_x} S_x(r)e^{-g_x(r)}\, dr\, (c_x\tau_x)^{-1}\int_{|t|>\tau_x} e^{-\frac{1}{3}\tau_x c_x t}\, dt. \tag{34}$$

In particular, if one of the two conditions $c_x\tau_x^2 \to +\infty$ and $c_x\tau_x \to +\infty$ is satisfied, then hypothesis (d) in Lemma (6.2.9) holds.

The study of $\mathcal{B}(z)$ will require some additional technical lemmas.

Lemma (6.2.10) [243]: For $z = xe^{i\phi}$, with $x > 0$ and $e^{i(\alpha_d + d\phi)} = 1$, we have

$$\mathcal{B}(z) \gtrsim \int_0^{+\infty} (rx)^{-\frac{m}{2}} r^{2m-1} e^{-h_x(r)} dr$$

as $x \to +\infty$, where

$$h_x(r) = (r^m - x^m)^2 - 2a(x^d - r^d) + C(r^{d-1} + x^{d-1} + 1), \quad (35)$$

for some positive constant C.

Proof. It is easy to see that

$$\mathcal{B}(z) = \int_{\mathbb{C}} |K_m(w,z)|^2 \, e^{2Re(g(z)-g(w))} \, [K_m(z,z)]^{-1} \, e^{-|w|^{2m}} \, dA(w),$$

which, in terms of polar coordinates, can be rewritten as

$$\int_0^{+\infty} \int_{\pi}^{-\pi} \left|K_m(re^{i\theta}, z)\right|^2 \, e^{2Re\left(g(z)-g(re^{i\theta})\right)} \, [K_m(x,x)]^{-1} \, e^{-r^{2m}} r \, dr \, d\theta.$$

By Lemma (6.2.1), B(z) is greater than or equal to

$$\int_0^{+\infty} \int_{|\theta-\phi|\le c\theta_0(rx)} \left|K_m(re^{i\theta}, z)\right|^2 e^{2Re\left(g(z)-g(re^{i\theta})\right)} \, [K_m(x,x)]^{-1} \, e^{-r2m} \, rdr \, d\theta.$$

This together with Lemma (6.2.1) shows that

$$\mathcal{B}(z) \int_0^{+\infty} r^{2(m-1)} e^{-(r^m - x^m)^2} \, I(r,z) r dr,$$

where

$$I(r,z) = \int_{|\theta-\phi|\le c\theta_0(rx)} e^{2Re\left(g(z)-g(re^{i\theta})\right)} \, d\theta.$$

Note that

$$\begin{aligned} I(r,z) &= \int_{|\theta-\phi|\le c\theta_0(rx)} e^{2Re\left[ae^{i\alpha_d}\,(x^d e^{id\phi} - r^d e^{id\theta})\right] + 2Re\left[g_{d-1}(z) - g_{d-1}(re^{i\theta})\right]} d\theta \\ &= \int_{|\theta-\phi|\le c\theta_0(rx)} e^{2Re\left[ae^{i(\alpha_d + d\phi)}(x^d - r^d e^{id(\theta-\phi)})\right] + 2Re\left[g_{d-1}(z) - g_{d-1}(re^{i\theta})\right]} d\theta. \end{aligned}$$

The condition on ϕ yields

$$I(r,z) = \int_{|\theta|\le c\theta_0(rx)} e^{2aRe\left[a(x^d - r^d e^{id\theta})\right] + 2Re\left[g_{d-1}(z) - g_{d-1}(re^{(i\theta+\phi)})\right]} \, d\theta.$$

Since

$$g_{d-1}(z) - g_{d-1}\left(re^{i(\theta+\phi)}\right) = \sum_{l=0}^{d-1} a_l\left(x^l \, e^{il\phi} - r^l \, e^{il(\theta+\phi)}\right),$$

we have

$$Re\left[g_{d-1}(z) - g_{d-1}\left(re^{i(\theta+\phi)}\right)\right] \ge -C(r^{d-1} + x^{d-1} + 1)$$

for some constant C. It follows that

$$I(r,z) \ge e^{-C\left(r^{d-1} + x^{d-1} + 1\right)} \int_{|\theta|\le c\theta_0(rx)} e^{2aRe\left[(x^d - r^d e^{id\theta})\right]} d\theta.$$

For the integral we have

$$J(r,z) := \int_{|\theta|\le c\theta_0(rx)} e^{2aRe[(x^d - r^d e^{id\theta})]} d\theta = \int_{|\theta|\le c\theta_0(rx)} e^{2a(x^d - r^d\cos(d\theta))}\, d\theta$$

$$= \int_{|\theta|\le c\theta_0(rx)} e^{2a(x^d - r^d + (-\cos(d\theta)+1)r^d)} d\theta$$

$$= \int_{|\theta|\le c\theta_0(rx)} e^{2a\left(x^d - r^d + 2\left(\sin\left(\frac{d\theta}{2}\right)^2\right)r^d\right)} d\theta$$

$$\ge e^{2a(x^d - r^d)} \int_{|\theta|\le c\theta_0(rx)} e^{4|a_d|\sin\left(\frac{d\theta}{2}\right)^2 r^d} d\theta$$

$$\ge e^{2a(x^d - r^d)} \int_{|\theta|\le c\theta_0(rx)} d\theta \qquad \gtrsim e^{2a(x^d - r^d)}\,(rx)^{-\frac{m}{2}},$$

which completes the proof of the lemma.

Lemma (6.2.11) [243]: Assume $d = 2m$. For $z = xe^{i\phi}$, where $x > 0$ and $e^{i(\alpha_d + d\phi)} = 1$, we have

$$\mathcal{B}(z)e^{(1+o(1))\frac{2a}{(1+2a)}x^{2m}}, \qquad x \to +\infty.$$

Proof. For x large enough, the function h_x defined in (35) is convex on some interval $[M_x, +\infty)$ and attains its minimum at some point r_x. In order to bound $\mathcal{B}(z)$ from below, we shall use the modified Laplace method from Lemma (6.2.9). Since

$$h'_x(r) = 2mr^{m-1}(r^m - x^m) + 2adr^{d-1} + C(d-1)r^{d-2}, \qquad (36)$$

we have

$$h'_x(r) = 2m(1+2a)r^{2m-1} - 2mx^m r^{m-1} + C(d-1)r^{d-2},$$

and

$$h''_x(r) = 2m(2m-1)(1+2a)r^{2m-2} - 2m(m-1)x^m r^{m-2} + C(d-1)(d-2)r^{d-3}.$$

Writing $h'_x(r_x) = 0$ and letting x tend to $+\infty$, we obtain

$$m(1+2a)(r_x)^{2m-1} \sim mx^m r_x^{m-1},$$

or

$$r_x \sim (1+2a)^{-\frac{1}{m}} x. \qquad (37)$$

Thus there exists ρ_x, which tends to 0 as x tends to $+\infty$, such that

$$r_x = (1+2a)^{-\frac{1}{m}} x(1+\rho_x). \qquad (38)$$

When x tends to $+\infty$, we have

$$h_x(r_x) \sim (r_x^m - x^m)^2 + 2a(r_x^{2m} - x^{2m})$$
$$\sim (r_x^m - x^m)\,[(r_x^m - x^m) + 2a(r_x^m + x^m)]$$
$$\sim x^{2m}[(1+2a)^{-1}(1+\rho_x)^m - 1][(1+2a)^{-1}(1+\rho_x)^m - 1$$
$$+ 2a\,((1+2a)^{-1}(1+\rho_x)^m + 1)] \sim -x^{2m}\frac{2a}{(1+2a)},$$

or

$$-h_x(r_x) \sim x^{2m}\frac{2a}{(1+2a)}. \qquad (39)$$

In order to estimate $c_x := h_x''(r_x)$, we compute that

$$h_x''(r_x) \sim 2m^2 (1 + 2a)^{-1+\frac{2}{m}} x^{2m-2} .$$

Thus we get

$$c_x \approx x^{2m-2} . \tag{40}$$

For r in a neighborhood of r_x we set $r = (1 + \sigma_x) r_x$, where $\sigma_x = \sigma_x(r) \to 0$ as $x \to +\infty$; a little computation shows that

$$h_x''(r) \sim h_x''(r_x)$$

as $x \to +\infty$. Taking $\tau_x = r_x^{1/2}$ and $|r - r_x| < \tau_x$, we have $h_x''(r) = \big(1 + o(1)\big) c_x$, so

$$h_x(r) - h_x(r_x) = \frac{1}{2} c_x (r - r_x)^2 \big(1 + o(1)\big).$$

Thus

$$\int_{|r-r_x|<\tau_x} e^{-\frac{1}{2}c_x(r-r_x)^2(1+o(1))} dr = \int_{|t|<\tau_x} e^{-\frac{1}{2}c_x t^2 (1+o(1))} dt \sim \frac{1}{\sqrt{c_x}} \int_{|y|<\tau_x\sqrt{c_x}} e^{-\frac{1}{2}y^2} dy$$

$$\approx \frac{1}{\sqrt{c_x}},$$

because $c_x \tau_x^2 \approx r_x^{2m-1}$ tends to $+\infty$ as x tends to $+\infty$. Finally, the estimates

$$\begin{aligned}
\mathcal{B}(z) &\gtrsim \int_{|r-r_x|<\tau_x} (r_x)^{-\frac{m}{2}} r^{2m-1} e^{-h_x(r)} dr \\
&= \int_{|r-r_x|<\tau_x} (rx)^{-\frac{m}{2}} r^{2m-1} e^{-h_x(r_x)} e^{-[h_x(r)-h_x(r_x)]} dr \\
&= e^{-h_x(r_x)} \int_{|r-r_x|<\tau_x} (rx)^{-\frac{m}{2}} r^{2m-1} e^{-\frac{1}{2}c_x(r-r_x)^2(1+o(1))} dr \\
&\sim e^{-h_x(r_x)} r_x^{\frac{3}{2}m-1} x^{-\frac{m}{2}} \int_{|r-r_x|<\tau_x} e^{-\frac{1}{2}c_x(r-r_x)^2(1+o(1))} dr \\
&\approx e^{-h_x(r_x)} r_x^{\frac{3}{2}m-1} x^{-\frac{m}{2}} \frac{1}{\sqrt{c_x}}
\end{aligned}$$

along with (37), (39), and (40) give the lemma

Lemma (6.2.12) [243]: Assume $d < 2m$. For $z = xe^{i\phi}$, with $x > 0$ and $e^{i(\alpha_d + d\phi)} = 1$, we have

$$\mathcal{B}(z) \gtrsim e^{(1+o(1))\frac{a^2 d^2}{m^2} x^{2d-2m} - Cx^{d-1-m}}, \qquad x \to +\infty$$

for some positive constant C

Proof. Let $\tau_x = o(x)$ be a positive real number that will be specified later. As in the proof of Lemma (6.2.10) we have

$$\begin{aligned}
\mathcal{B}(z) &\gtrsim \int_0^{+\infty} r^{2(m-1)} e^{-(r^m - x^m)^2} I(r,z) r\, dr \\
&\gtrsim \int_{|r-x|\le\tau_x} r^{2(m-1)} e - (r^m - x^m)^2\, I(r,z) r\, dr,
\end{aligned}$$

where

$$I(r,z) = \int_{|\theta-\phi|\leq c\theta_0(rx)} e^{2Re\left(g(z)-g\left(re^{i\theta}\right)\right)}\, d\theta.$$

There exists $c' > 0$ such that for $|r - x| \leq \tau_x$ we have

$$\begin{aligned} I(r,z) &\geq \int_{|\theta-\phi|\leq c'\theta_0(x^2)} e^{2Re\left(g(z)-g\left(re^{i\theta}\right)\right)}\, d\theta \\ &= \int_{|\theta|\leq c'\theta_0(x^2)} e^{2aRe\left(x^d-r^de^{id\theta}\right)+2Re\left[g_{d-1}(z)-g_{d-1}\left(re^{i\theta}\right)\right]} d\theta \\ &= \int_{|\theta|\leq c'\theta_0(x^2)} e\hat{\ }(2aRe(x^d - r^de^{id\theta}) - 2\sum_{l=0}^{d-1} |a_l|\left|x^l - r^le^{il\theta}\right|\, d\theta. \end{aligned}$$

Now for $|r - x| \leq \tau_x$, we write $r = (1 + \sigma)x$, where σ tends to 0 as $x \to +\infty$. Thus for $0 \leq l \leq d-1$ and $|\theta| \leq c'\theta_0(x^2)$, we obtain

$$\begin{aligned} \left|x^l - r^l e^{il\theta}\right|^2 &= x^{2l}[1 - 2(1 + \sigma)^l \cos(l\theta) + (1 + \sigma)^{2l}] \\ &= x^{2l}\left[1 - 2\left(1 + l\sigma + O(\sigma^2)\right) \cos(l\theta) + 1 + 2l\sigma + O(\sigma^2)\right] \\ &= x^{2l}\left[2\left(1 - \cos(l\theta)\right)(1 + l\sigma) + O(\sigma^2)\right] \\ &\lesssim x^{2l}\left[\sin^2\left(\frac{l\theta}{2}\right) + \sigma^2\right] \qquad \lesssim x^{2l}[\theta^2 + \sigma^2]. \end{aligned}$$

Next choosing $|\sigma| \leq x^{-m}$, we get

$$\left|x^l - r^l e^{il\theta}\right| \lesssim x^{2l}\, x^{-2m} \lesssim x^{2(d-1)-2m}$$

or

$$\left|x^l - r^l e^{il\theta}\right| \lesssim x^{d-1-m}.$$

Thus there exists a positive constant C such that for $|r - x| \leq \tau_x$ and $|\theta| \leq c'\theta_0(x^2)$,

$$2\sum_{l=0}^{d-1} |a_l|\left|x^l - r^l e^{il\theta}\right| \leq Cx^{d-1-m}.$$

It follows that

$$I(r,z) \geq \int_{|\theta|\leq c'\theta_0(x^2)} e^{2aRe\left(x^d-r^de^{id\theta}\right)} - Cx^{d-1-m} d\theta \gtrsim x^{-m}e^{2aRe\left(x^d-r^de^{id\theta}\right)-Cx^{d-1-m}}.$$

Then

$$\begin{aligned} \mathcal{B}(z)\int_{|r-x|\leq\tau_x} & r^{2m-1}e^{-(r^m-x^m)^2}\, x^{-m}e^{2a\left(x^d-r^d\right)-Cx^{d-1-m}}\, dr \\ &= x^{-m}e^{-Cx^{d-1-m}}\int_{|r-x|\leq\tau} r^{2m-1}\, e^{-h_x(r)}dr, \end{aligned}$$

where

$$h_x(r) = (r^m - x^m)^2 - 2a(x^d - r^d).$$

It is easy to see that h_x attains its minimum at r_x with $r_x \sim x$ as $x \to +\infty$. Again we write

$$r_x = x(1 + \rho_x), \tag{41}$$

where ρ_x tends to 0 as $x \to +\infty$. Using the fact that $h'_x(r_x) = 0$, we have $2mx^{2m-1}(1 + \rho_x)^{m-1}[(1 + \rho_x)^m - 1] \sim -2adx^{d-1}(1 + \rho_x)^{d-1}$,
and

$$2mx^{2m-1} m\rho_x \sim -2adx^{d-1}.$$

Therefore,

$$\rho_x \sim -\frac{ad}{m^2} x^{d-2m}. \tag{42}$$

Since

$$h''_x(r) = 2m(2m-1)r^{2m-2} - 2m(m-1)x^m r^{m-2} + 2ad(d-1)r^{d-2}$$

and $d < 2m$, we get

$$h''_x(r_x) \sim 2mx^{2m-2}[(2m-1)(1+\rho_x)^{2m-2} - (m-1)(1+\rho_x)^{m-2}] \sim 2m^2 x^{2m-2}.$$

Also,

$$h_x(r_x) \sim x^{2m}[(1+\rho_x)^m - 1]^2 + 2ax^d[(1+\rho_x)^d - 1] + C(x^{d-1} + r_x^{d-1} + 1) \sim m^2\rho_x^2 x^{2m} + 2ax^d d\rho_x$$

It follows that

$$c_x \sim 2m^2 x^{2m-2}, \tag{43}$$

and

$$-h_x(r_x) \sim \frac{a^2d^2}{m^2} x^{2d-2m}. \tag{44}$$

Reasoning as in the proof of Lemma (6.2.11), we arrive at

$$\mathcal{B}(z) \gtrsim\gtrsim x^{-m} e^{-Cx^{d-1-m}} e^{-h_x(r_x)} x^{2m-1} \frac{1}{\sqrt{c_x}}.$$

The desired estimate then follows from (44), and (43).

Lemma (6.2.13) [243]: Suppose u and v are functions in $\mathcal{F}_m^2$, not identically zero, such that $\widetilde{|u|^2}(z)\widetilde{|v|^2}(z)$ is bounded on the complex plane. Then there exists a nonzero constant C and a polynomial g of degree at most m such that $u(z) = eg(z) \text{ and } v(z) = Ce^{-g(z)}$.

Proof. It is easy to check that for $u \in \mathcal{F}_m^2$ we have

$$u(z) = \int_{\mathbb{C}} u(x)|k_z(x)|^2 \, d\lambda_m(x) = \tilde{u}(z).$$

Also, it follows from the Cauchy-Schwarz inequality that $|u(z)|^2 \le \widetilde{|u|^2}(z)$. So if $\widetilde{|u|^2}(z)\widetilde{|v|^2}(z)$ is bounded on $\mathbb{C}$, then $\mathcal{B}(z)$ and $|u(z)v(z)|^2$ are also bounded. Consequently, uv is a constant, , there is a non-zero constant C and a polynomial g such that $u = e^g$ and $v = Ce^{-g}$. The condition $u \in \mathcal{F}_m^2$ implies that the degree d of g is at most $2m$; see Lemma (6.2.2).

We shall consider the case where $u(z) = e^{g(z)}$ and $v(z) = e^{-g(z)}$. We will show that that the boundedness of $\mathcal{B}(z)$ implies $d \le m$. If $2m$ is an integer, Lemma (6.2.11) shows that we must have $d < 2m$.

Thus, in any case ($2m$ being an integer or not), a necessary condition is $d < 2m$. The desired result now follows from Lemma (6.2.12).

we specialize to the case $m = 1$ and make several additional remarks. Thus for any $\alpha > 0$ we let $\mathcal{F}_\alpha^2$ denote the Fock space of entire functions f on the complex plane $\mathbb{C}$ such that

$$\int_{\mathbb{C}} |f(z)|^2 \, d\lambda_\alpha(z) < \infty,$$

where

$$d\lambda_\alpha(z) = \frac{\alpha}{\pi} e^{-\alpha|z|^2} \, dA(z).$$

Toeplitz operators on $\mathcal{F}_\alpha^2$ are defined exactly the same as before using the orthogonal projection $P_\alpha : L^2(\mathbb{C}, d\lambda_\alpha) \to \mathcal{F}_\alpha^2$.

Suppose u and v are functions in $\mathcal{F}_\alpha^2$, not identically zero. It was proved in [248] that $T_u T_{\bar{v}}$ is bounded on the Fock space $\mathcal{F}_\alpha^2$ if and only if there is a point $a \in \mathbb{C}$ such that

$$u(z) = be^{\alpha \bar{a} z}, \qquad v(z) = ce^{-\alpha \bar{a} z}, \tag{45}$$

where b and c are nonzero constants. This certainly solves Sarason's problem for Toeplitz products on the space $\mathcal{F}_\alpha^2$. But [258] somehow did not address Sarason's conjecture, which now of course follows from our main result.

We want to make two points here. First, the proof of Sarason's conjecture for $\mathcal{F}_\alpha^2$ is relatively simple after Sarason's problem is solved. Second, Sarason's conjecture holds for the Fock space $\mathcal{F}_\alpha^2$ for completely different reasons than was originally thought, namely, the motivation for Sarason's conjecture provided in [252] for the cases of Hardy and Bergman spaces is no longer valid for the Fock space. It is therefore somewhat amusing that Sarason's conjecture turns out to be true for the Fock space but fails for the Hardy and Bergman spaces.

Suppose u and v are given by (45). We have

$$\widetilde{|u|^2}(z) = \|f\, k_z\|^2 = \int_{\mathbb{C}} |f(w) e^{\alpha w \bar{z}} - (\alpha 2)|z|^2 |^2 \, d\lambda_\alpha(w)$$

$$= |b|^2 e^{-\alpha|z|^2} \int_{\mathbb{C}} \left| e^{\alpha w(\bar{a}+\bar{z})} \right|^2 d\lambda_\alpha(w)$$

$$= |b|^2 e^{-\alpha|z|^2 + \alpha|a+z|^2} \qquad = |b|^2 e^{\alpha(|a|^2 + \bar{a}z + a\bar{z})}.$$

Similarly,

$$\widetilde{|v|^2}(z) = |c|^2 e^{\alpha(|a|^2 - \bar{a}z - a\bar{z})}.$$

It follows that

$$\widetilde{|u|^2}(z)\widetilde{|v|^2}(z) = |bc|^2 e^{2\alpha|a|^2}$$

is a constant and hence a bounded function on $\mathbb{C}$.

On the other hand, it follows from Hölder's inequality that we always have

$$|u(z)|^2 \le \widetilde{|u|^2}(z), \qquad u \in \mathcal{F}_\alpha^2, z \in \mathbb{C}.$$

Therefore, if $\widetilde{|u|^2}\widetilde{|v|^2}$ is a bounded function on $\mathbb{C}$, then there exists a positive constant M such that

$$|u(z)v(z)|^2 \le \widetilde{|u|^2}(z)\widetilde{|v|^2}(z) \le M$$

for all $z \in \mathbb{C}$. Thus, as a bounded entire function, uv must be constant, say $u(z)v(z) = \mathbb{C}$ for all $z \in \mathbb{C}$. Since u and v are not identically zero, we must have $C \neq 0$. Since functions in $\mathcal{F}_\alpha^2$ must have order less than or equal to 2, we can write $u(z) = e^{p(z)}$, where

$$p(z) = az^2 + bz + c$$

is a polynomial of degree less than or equal to 2. But $u(z)v(z)$ is constant, so $v(z) = e^{q(z)}$, where

$$q(z) = -az^2 - bz + d$$

is another polynomial of degree less than or equal to 2.

We will show that $a = 0$. To do this, we will estimate the Berezin transform $\widetilde{|u|^2}$ when u is a quadratic exponential function as given above. More specifically, for $C_1 = |e^c|^2$, we have

$$\widetilde{|u|^2}(z) = C_1 \int_{\mathbb{C}} \left|e^{a(z+w)^2+b(z+w)}\right|^2 d\lambda_\alpha(w)$$
$$= C_1 \left|e^{az^2+bz}\right|^2 \int_{\mathbb{C}} \left|e^{aw^2+(b+2az)w}\right|^2 d\lambda_\alpha(w).$$

Write $b + 2az = \alpha\bar{\zeta}$. Then it follows from the inequality $\widetilde{|F|^2} \geq |\widetilde{F}|^2$ for $F \in F_\alpha^2$ again that

$$\widetilde{|u|^2}(z) = C_1 \left|e^{az^2+bz}\right|^2 e^{\alpha|\zeta|^2} \int_{\mathbb{C}} \left|e^{aw^2} k_\zeta(w)\right|^2 d\lambda_\alpha(w)$$
$$\geq C_1\left|e^{az^2+bz}\right|^2 e^{\alpha|\zeta|^2} \left|e^{a\zeta^2}\right|^2.$$

If we do the same estimate for the function v, the result is

$$\widetilde{|v|^2}(z) \geq C_2 \left|e^{-az^2-bz}\right|^2 e^{\alpha|\zeta|^2} \left|e^{-a\zeta^2}\right|^2,$$

where ζ is the same as before and $C_2 = \left|e^d\right|^2$. It follows that

$$\widetilde{|u|^2}(z)\widetilde{|v|^2}(z) \geq C_1C_2e^{2\alpha|\zeta|^2} = C_1C_2e^{2|b+2az|^2/\alpha}.$$

This shows that $\widetilde{|u|^2}\widetilde{|v|^2}$ is unbounded unless $a = 0$. Therefore, the boundedness of $\widetilde{|u|^2}\widetilde{|v|^2}$ implies that

$$u(z) = e^{bz+c}, \qquad v(z) = e^{-bz+d}.$$

By [258], the product T_uT_v is bounded on F_α^2. In fact, T_uT_v is a constant times a unitary operator.

Combining the arguments above and the main result of [8] we have actually proved that the following conditions are equivalent for u and v in F_α^2:

(a) $T_uT_{\bar{v}}$ is bounded on F_α^2.

(b) $T_uT_{\bar{v}}$ is a constant multiple of a unitary operator.

(b) $\widetilde{|u|^2}\widetilde{|v|^2}$ is bounded on $\mathbb{C}$.

(c) $\widetilde{|u|^2}\widetilde{|v|^2}$ is constant on $\mathbb{C}$.

Recall that in the case of Hardy and Bergman spaces, there is actually an absolute constant C (4 for the Hardy space and 16 for the Bergman space) such that

$$\widetilde{|u|^2}(z)\widetilde{|v|^2}(z) \leq C\|T_uT_{\bar{v}}\|^2$$

for all $u, v,$ and z. We now show that such an estimate is not possible for the Fock space. To see this, consider the functions

$$u(z) = e^{\alpha\bar{a}z}, \qquad v(z) = e^{-\alpha\bar{a}z}.$$

By calculations done in [8], we have

$$T_u T_{\bar{v}} = e^{\alpha|a|^2/2} W_a ,$$

where W_a is the Weyl unitary operator defined by $W_a f(z) = f(z-a)k_a(z)$. On the other hand, by calculations done earlier, we have

$$\widetilde{|u|^2}(z)\widetilde{|v|^2}(z) = e^{2\alpha|a|^2} .$$

It is then clear that there is NO constant C such that

$$e^{2\alpha|a|^2} \le C e^{\alpha|a|^2/2}$$

for all $a \in \mathbb{C}$. Therefore, there is NO constant C such that

$$\sup_{z\in\mathbb{C}} \widetilde{|u|^2}(z)\widetilde{|v|^2}(z) \le C\|T_u T_{\bar{v}}\|^2$$

for all u and v. In other words, the easy direction for Sarason's conjecture in the cases of Hardy and Bergman spaces becomes difficult for Fock spaces.

List of Symbols

Symbol		Page
L^1	: Lebesgue space	1
meas	: measure	1
L^∞	: Essential Lebesgue space	1
$W^{1,N}$	: Sobolev space	1
osc	: Oscillation	1
$W_0^{1,p}$	: Sobolev space	1
sup	: Supremum	2
L^1	: Lebesgue integral in the Real line	4
ess	: essential	4
L^r	: Lebesgue space	4
inf	: Infimum	4
Loc	: locally	5
PSR	: Poincaré Sobolev Rearrangement	17
a. e	: Almost Everywhere	20
max	: maximum	20
PDES	: partial differential equations	34
Lip	: Lipschtiz	34
$L^{p(x)}$	: Lebesgue space with variable exponent	37
$AMDS$	: Almost monotone deceasing sequence	37
$AMIS$	: Almost monotone increasing sequence	37
$W^{p(x)}$	: Sobolev with Lebesgue variable exponent	38
$H^{p(.)}$	: Hardy space of variable exponent	50
$W_w^{p(.)}$	: Lebesgue space with variable exponent with a weight	69
$W_{p(.),w}^{\alpha}$	: Sobolev space with variable exponent with a weight	69
supp	: Support	96
L^q	: Dual of Lebesgue space	119
u. s. c.	: upper strictly convex	126
l. s. c.	: lower strictly convex	126
int	: Interior	175
ext	: exterior	175
diag	: diagonal	209
BMO	: Bounded Meau Oscillation	226

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