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Absence of Cartan Subalgebra and Generator Masa is q-Deformed with Approximation Property for q-Araki-Woods Algebras

غياب جبر كرتان الجزئي وماسا المولد في تشوه – q مع خاصية تقريب جبريات آراكي – وودز – q

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Dedication

To my Family.

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I would like to thank with all sincerity Allah, and my family for their supports throughout my study. Many thanks are due to my thesis guide, Prof. Dr. Shawgy Hussein AbdAlla Sudan University of Science and Technology.

Abstract

We study some products with mixing subalgebras and non-injectivity with generator Masa of the q-deformed Araki – Woods of von Neumann algebras. We also study a class of II_1 factors with at most one Cartan subalgebra II and structure results for free Araki-Woods and their continuous cores. The Q-Gaussian processes, that is, non- commutative and classical aspects with q-deformed Araki- Woods factors are determined. We characterize the asymptotic matricial models, extension of second quantisation, Haagerup approximation property, absence of Cartan subalgebras, the structure of modular invariant subalgrbras and complete metric approximation property for q- Araki- Woods factors and algebras.

الخلاصة

قمنا بدراسة بعض النواتج مع الجبريات الجزئية المختلطة و غير الاحادية مع ماسا المولد لتشوه $q-$ اراكي-وودز لجبريات فون نيومان. ايضا قمنا بدراسة عائلة العوامل II_1 مع على الاكثر واحد الجبر الجزئي II لكرتان ونتائج التشييد لاراكي-وودز الحرة وجوهرها المستمر. تم تحديد عمليات جاوسيان اي غير التبديلية والنواحي التقليدية مع تشوه-q لعوامل اراكي -وودز. قمنا بتشخيص نماذج مصفوفة المقاربة والتمدد للتكميم الثاني وخاصية تقريب هاقريب والغياب للجبريات الجزئية لكرتان والتشييد للجبريات الجزئية اللامتغيرة بمقياس وخاصية التقريب المترية التامة لعوامل اراكي -وودز q- والجبريات.

Introduction

We show that certain free products of factors of type I and other von Neumann algebras with respect to nontracial, almost periodic states are almost periodic free Araki-Woods factors. In particular, they have the free absorption property and Connes' Sd invariant completely classifies these free products. We studying the structure of Cartan subalgebras of von Neumann factors of type II_1. We provide more examples of II_1 factors having either zero, one or several Cartan subalgebras.

We examine, for $-1 < q < 1$, q-Gaussian processes, i.e. families of operators (non-commutative random variables) $X_t = a_t + a_t^*$ – where the a_t fulfill the q-commutation relations $a_s\, a_t^* - q a_t^*\, a_s = c\,(s,t\,) \cdot 1$ for some covariance function $c(\,\cdot\,,\cdot)$ – equipped with the vacuum expectation state. We show that there is a q- analogue of the Gaussian functor of second quantization behind these processes and that this structure can be used to translate questions on q-Gaussian processes into corresponding (and much simpler) questions in the underlying Hilbert space. We show that the von Neumann algebra generated by q-gaussians is not injective as soon as the dimension of the underlying Hilbert space is greater than 1. The approach is based on a suitable vector valued Khintchine type inequality for Wick products.

Using Speicher central limit Theorem we provide Hiai's q-Araki-Woods von Neumann algebras and the q-deformed Araki-Woods factor with nice asymptotic matricial models.

We show that the normalizer of any diffuse amenable subalgebra of a free group factor $L(F_r)$ generates an amenable von Neumann subalgebra. Moreover, any II_1 factor of the form $Q\,\overline{\bigoplus}\,L(F_r)$, with Q an arbitrary subfactor of a tensor product of free group factors, has no Cartan subalgebras. We show that for any type III_1 free Araki-Woods factor $M = \Gamma(H_R, U_t)''$ associated with an orthogonal representation (U_t) of R on a separable real Hilbert space H_R, the continuous core $M = M \rtimes_\sigma R$ is a semisolid II∞ factor, i.e. for any non-zero finite projection $q \in M$, the II_1 factor qMq is semisolid. If the representation (U_t) is moreover assumed to be mixing, then we show that the core M is solid. We show that all the free Araki-Woods factors $\Gamma(H_R, U_t)''$ have the complete metric approximation property. Using Ozawa-Popa's techniques, we then prove that every nonamenable subfactor $N \subset \Gamma(H_R, U_t)$ which is the range of a normal conditional expectation has no Cartan subalgebra.

Jolissaint and Stalder introduced definitions of mixing and weak mixing for von Neumann subalgebras of finite von Neumann algebras. In this note, we study various algebraic and analytical properties of subalgebras with these mixing properties. We prove some basic results about mixing inclusions of von Neumann algebras and establish a connection between mixing properties and normalizers of von Neumann subalgebras. The special case of mixing subalgebras arising from inclusions of countable discrete groups finds applications to ergodic theory, in particular, a new generalization of a classical theorem of Halmos on the automorphisms of a compact abelian group. To any strongly continuous orthogonal representation of ℝ on a real Hilbert space H_R, Hiai constructed q-deformed Araki– Woods von Neumann algebras for $-1 < q < 1$, which are W∗-algebras arising from non-tracial representations of the q-commutation relations, the latter yielding an interpolation between the Bosonic and Fermionic statistics.

We extend the class of contractions for which the second quantisation on q-Araki-Woods algebras can be defined. By adapting an ultraproduct technique of Junge and Zeng, we prove that radial completely bounded multipliers on q-Gaussian algebras transfer to q-Araki-Woods algebras. As a consequence, we establish the w^* -complete metric approximation property for all qAraki-Woods algebras.

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Chapter 1
Free Products of von Neumann Algebras and a Class of II_1 Factors

We show that for λ, μ $\in]0,1[$, $(M_2(C), \omega_\lambda) * (M_2(C), \omega_\mu)$ is isomorphic to the free Araki-Woods factor whose Sd invariant is the subgroup of R_+^* generated by λ and μ. The proofs are based on algebraic techniques and amalgamated free products. These results give some answers to questions of Dykema and Shlyakhtenko. We also show a rigidity result for some group measure space II_1 factors.

Section (1.1): Free Araki-Woods Factors

In [5] and [7], Dykema investigated free products of finite dimensional and other von Neumann algebras with respect to nontracial faithful states. We are interested in free products of factors of type I. In this respect, we recall Theorem 1 of [5] in the particular case of factors of type I (see Proposition 7.3 of [5]).

Theorem (1.1.1)[1]: (Dykema, [5]). Let

$$(\mathcal{M}, \phi) = (A_1, \phi_1) * (A_2, \phi_2)$$

be the von Neumann algebra free product of factors of type I with respect to faithful states, at least one of which is nontracial. Then $\mathcal{M}$ is a full factor of type III and ϕ is an almost periodic faithful state whose centralizer is isomorphic to the type II_1 factor $L(\mathbf{F}_\infty)$. The point spectrum of the modular operator Δ_ϕ of ϕ, is equal to the subgroup of $\mathbf{R}_+^*$ generated by the union of the point spectra of Δ_{ϕ_1} and of Δ_{ϕ_2}. Thus in Connes' classification, $\mathcal{M}$ is always a factor of type III_λ, with $0 < \lambda \le 1$.

The fact that ϕ is an almost periodic faithful state [3] is an easy consequence of basic results on free products (see [5]). The fact that the centralizer of ϕ is isomorphic to the type II_1 factor $L(\mathbf{F}_\infty)$ is the most difficult part of the theorem. To prove this, Dykema uses sophisticated algebraic techniques on free products that he developed also in [6] and [8]. Finally, the fact that $\mathcal{M}$ is of type III follows from results of [5].

Dykema asked in Question 9.1 [5], whether the type III_λ factors that are obtainable by taking various free products of finite dimensional or hyperfinite algebras are isomorphic to each other, and whether they are isomorphic to the factor of Radulescu [11],

$$(L(Z), \tau_Z) * (M_2(X), \omega_\lambda)$$

where $\omega_\lambda(p_{ij}) = \delta_{ij}\lambda^j/(\lambda+1)$ for $i, j \in \{0,1\}$. Furthermore, he asked in Question 9.3 [5], whether the full factors of type III_1 having the same Sd invariant that are obtainable by taking free products of various finite dimensional or hyperfinite algebras are isomorphic to each other. We will see that we partially give positive answers to these questions.

In [15], Shlyakhtenko introduced a new class of full factors of type III. His idea is to give a version of the CAR functor and of the associated quasi-free states in the framework of Voiculescu's free probability theory [19]. We recall his construction. We can say that to each real Hilbert space $H_\mathbf{R}$ and to each orthogonal representation (U_t) of $\mathbf{R}$ on $H_\mathbf{R}$, he associated a factor $\Gamma(H_\mathbf{R}, U_t)''$ called the free Araki- Woods factor. He proved that $\Gamma(H_\mathbf{R}, U_t)''$ is a type III factor except if $U_t = \mathrm{id}$ for all $t \in \mathbf{R}$. The restriction to $\Gamma(H_\mathbf{R}, U_t)''$ of the vacuum state denoted by φ_U and called the free quasi-free state, is faithful. Moreover, he proved that φ_U is an almost periodic state iff the orthogonal representation (U_t) is almost periodic. Recall in this respect the following definition:

Definition (1.1.2)[1]: (Connes, [3]). Let M be a von Neumann algebra with separable predual which has almost periodic weights. The Sd invariant of M is defined as the intersection over all the almost periodic, faithful, normal, semifinite weights φ of the point spectra of the modular operators Δ_φ.

Connes proved that for a factor of type III, $\mathrm{Sd}(M)$ is a countable subgroup of $\mathbf{R}_+^*$[3]. In the almost periodic case, using a powerful tool called the matricial model, Shlyakhtenko obtains this remarkable result:

Theorem (1.1.3)[1]: (Shlyakhtenko, [12],[15]). Let (U_t) be a nontrivial almost periodic orthogonal representation of $\mathbf{R}$ on the real Hilbert space $H_{\mathbf{R}}$ with $\dim H_{\mathbf{R}} \geq 2$. Let A be the infinitesimal generator of (U_t) on $H_{\mathbf{C}}$, the complexified Hilbert space of $H_{\mathbf{R}}$. Denote $M = \Gamma(H_{\mathbf{R}}, U_t)''$. Let $\Gamma \subset \mathbf{R}_+^*$ be the subgroup generated by the point spectrum of A. Then, M only depends on Γ up to state-preserving isomorphisms.

Conversely, the group Γ coincides with the Sd invariant of the factor M. Consequently, Sd completely classifies the almost periodic free Araki-Woods factors. Moreover, the centralizer of the free quasi-free state φ_U is isomorphic to the type II_1 factor $L(\mathbf{F}_\infty)$.

He proved also that the (unique) free Araki-Woods factor of type III_λ, denoted by $(T_\lambda, \varphi_\lambda)$ is isomorphic to the factor of Rădulescu [11] and "freely absorbs" $L(\mathbf{F}_\infty)$. Since the free Araki-Woods factors satisfy free absorption properties, Shlyakhtenko asked whether the free products of matrix algebras $(A_1, \phi_1) * (A_2, \phi_2)$ are stable by taking free products with $L(\mathbf{Z})$, in other words whether they are free Araki-Woods factors.

We give positive answer to the question of Shlyakhtenko for certain free products of matrix algebras and other von Neumann algebras. Thanks to Theorem 6.6 of [15], it partially gives positive answers to Questions 9.1 and 9.3 of Dykema [5]. For an almost periodic state ϕ, we denote by $\mathrm{Sd}(\phi)$ the subgroup of $\mathbf{R}_+^*$ generated by the point spectrum of the modular operator Δ_ϕ. On $B(\ell^2(\mathbf{N}))$, we denote by ψ_λ the state given by $\psi_\lambda(e_{ij}) = \delta_{ij}\lambda^j(1-\lambda)$ for $i,j \in \mathbf{N}$. For $\beta \in]0,1[$, we denote by $(\mathbf{C}^2, \tau_\beta)$ the algebra generated by a projection q with $\tau_\beta(q) = \beta$. The hyperfinite type II_1 factor together with its trace is denoted by $(\mathcal{R}, \tau)$. At last, we denote by $(T_\Gamma, \varphi_\Gamma)$ the unique (up to state-preserving isomorphism) almost periodic free Araki-Woods factor whose Sd invariant is exactly Γ.

Definition (1.1.4)[1]: Let $\rho: (B, \phi_B) \hookrightarrow (A, \phi_A)$ be an embedding of von Neumann algebras. We shall say that ρ is modular if it is state-preserving and if $\rho(B)$ is globally invariant under the modular group $\left(\sigma_t^{\phi_A}\right)$.

Theorem (1.1.5)[1]: Let $(A_i, \phi_i), i = 1,2$, be two von Neumann algebras endowed with a faithful, normal, almost periodic state ϕ_i, such that for $i = 1,2$

$$(A_i, \phi_i) * (L(\mathbf{Z}), \tau_{\mathbf{Z}}) \cong \left(T_{\mathrm{Sd}(\phi_i)}, \varphi_{\mathrm{Sd}(\phi_i)}\right).$$

Let Γ be the subgroup of $\mathbf{R}_+^*$ generated by $\mathrm{Sd}(\phi_1)$ and $\mathrm{Sd}(\phi_2)$. Assume that for some $,\beta \in]0,1[$, there exist modular embeddings

$$(M_2(\mathbf{C}), \omega_\lambda) \hookrightarrow (A_1, \phi_1)$$
$$\left(\mathbf{C}^2, \tau_\beta\right) \hookrightarrow (A_2, \phi_2),$$

such that $\lambda/(\lambda+1) \leq \min\{\beta, 1-\beta\}$. Then

$$(T_\Gamma, \varphi_\Gamma) \cong (A_1, \phi_1) * (A_2, \phi_2).$$

In particular, for any $\lambda, \mu \in]0,1[$, $(M_2(\mathbf{C}), \omega_\lambda) * \left(M_2(\mathbf{C}), \omega_\mu\right), (M_2(\mathbf{C}), \omega_\lambda) * (\mathcal{R}, \tau)$ and $(B(\ell^2(\mathbf{N})), \psi_\lambda) * (\mathcal{R}, \tau)$ are free Araki-Woods factors.

We devoted to a few reminders on free products and free Araki-Woods factors. We show that $(M_2(\mathbf{C}), \omega_\lambda) * \left(\mathbf{C}^2, \tau_\beta\right)$ is isomorphic in a state-preserving way to $(T_\lambda, \varphi_\lambda)$, whenever $\lambda/(\lambda+1) \leq \min\{\beta, 1-\beta\}$. We prove Theorem (1.1.5) using the "machinery" of amalgamated free products.

We will be working with free products of von Neumann algebras with respect to states. It is useful to remind the following notation:

Notation (1.1.6)[1]: If (M,φ) and (N,ψ) are von Neumann algebras endowed with states φ and ψ, the notation $(M,\varphi) \cong (N,\psi)$ means that there exists a *-isomorphism $\alpha: M \to N$ such that $\psi\alpha = \varphi$.

We remind this well-known proposition concerning free products of von Neumann algebras with respect to states.

Proposition (1.1.7)[1]: ([19]). Let (M_i,φ_i) be a family of von Neumann algebras endowed with faithful normal states. Then, there exists, up to state-preserving isomorphism, a unique von Neumann algebra (M,φ) endowed with a faithful normal state φ such that

1. (M_i,φ_i) embeds into (M,φ) in a state-preserving way,
2. M is generated by the family of subalgebras (M_i) which is a free family in (M,φ).

The free product of (M_i,φ_i) is denoted by $(M,\varphi) = \underset{i\in I}{*}(M_i,\varphi_i)$.

Notation (1.1.8)[1]: ([8]). For von Neumann algebras A and B, with states φ_A and φ_D, the von Neumann algebra

$$\underset{\alpha}{\overset{p}{A}} \oplus \underset{\beta}{\overset{q}{B}}$$

where $\alpha,\beta \geq 0$ and $\alpha+\beta=1$, will denote the algebra $A \oplus B$ whose associated state is $\varphi(a,b) = \alpha\varphi_A(a) + \beta\varphi_B(b)$. Moreover, $p \in A$ and $q \in B$ are projections corresponding to the identity elements of A and B.

Now, we want to remind the construction of the free Araki-Woods factors [15]. Let $H_{\mathbf{R}}$ be a real Hilbert space and let (U_t) be an orthogonal representation of $\mathbf{R}$ on $H_{\mathbf{R}}$. Let $H = H_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$ be the complexified Hilbert space. If A is the infinitesimal generator of (U_t) on H, we remind that $j: H_{\mathbf{R}} \to H$ defined by $j(\zeta) = \left(\frac{2}{A^{-1}+1}\right)^{1/2} \zeta$ is an isometric embedding of $H_{\mathbf{R}}$ into H. Let $K_{\mathbf{R}} = j(H_{\mathbf{R}})$. Introduce the full Fock space of H :

$$\mathcal{F}(H) = \mathbf{C}\Omega \oplus \bigoplus_{n=1}^{\infty} H^{\otimes n}.$$

The unit vector Ω is called vacuum vector. For any $\xi \in H$, we have the left creation operator

$$l(\xi): \mathcal{F}(H) \to \mathcal{F}(H): \begin{cases} l(\xi)\Omega = \xi, \\ l(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n. \end{cases}$$

For any $\xi \in H$, we denote by $s(\xi)$ the real part of $l(\xi)$ given by

$$s(\xi) = \frac{l(\xi) + l(\xi)^*}{2}.$$

The crucial result of Voiculescu [19] claims that the distribution of the operator $s(\xi)$ with respect to the vacuum vector state $\varphi(x) = \langle x\Omega, \Omega\rangle$ is the semicircular law of Wigner supported on the interval $[-\parallel \xi \parallel, \parallel \xi \parallel]$.

Definition (1.1.9)[1]: (Shlyakhtenko, [15]). Let (U_t) be an orthogonal representation of $\mathbf{R}$ on the real Hilbert space $H_{\mathbf{R}} (\dim H_{\mathbf{R}} \geq 2)$. The free Araki-Woods factor associated with $H_{\mathbf{R}}$ and (U_t), denoted by $\Gamma(H_{\mathbf{R}}, U_t)''$, is defined by

$$\Gamma(H_{\mathbf{R}}, U_t)'' = \{s(\xi), \xi \in K_{\mathbf{R}}\}''.$$

The vector state $\varphi_U(x) = \langle x\Omega, \Omega\rangle$ is called the free quasi-free state.

As we said previously, the free Araki-Woods factors provide many new examples of full factors of type III [2],[4],[12]. We can summarize the general properties of free Araki-Woods factors in the following theorem (see also [18]):

Theorem (1.1.9)[1]: (Shlyakhtenko, [12],[13],[14],[15]). Let (U_t) be an orthogonal representation of $\mathbf{R}$ on the real Hilbert space $H_{\mathbf{R}}$ with $\dim H_{\mathbf{R}} \geq 2$. Denote $M = \Gamma(H_{\mathbf{R}}, U_t)''$.

1. M is a full factor.
2. M is of type II_1 iff $U_t = \mathrm{id}$ for every $t \in \mathbf{R}$.
3. M is of type III $_\lambda (0 < \lambda < 1)$ iff (U_t) is periodic of period $\frac{2\pi}{|\log \lambda|}$.
4. M is of type III_1 in the other cases.
5. The factor M has almost periodic states iff (U_t) is almost periodic.

Let $H_{\mathbf{R}} = \mathbf{R}^2$ and $0 < \lambda < 1$. Let

$$U_t = \begin{pmatrix} \cos(\mathrm{t}\log \lambda) & -\sin(\mathrm{t}\log \lambda) \\ \sin(\mathrm{t}\log \lambda) & \cos(\mathrm{t}\log \lambda) \end{pmatrix}. \tag{1}$$

Notation (1.1.10)[1]: ([15]). Denote $(T_\lambda, \varphi_\lambda) := \Gamma(H_{\mathbf{R}}, U_t)''$ where $H_{\mathbf{R}} = \mathbf{R}^2$ and (U_t) is given by Equation (1).

Using a powerful tool called the matricial model, Shlyakhtenko was able to prove the following isomorphism

$$(T_\lambda, \varphi_\lambda) \cong \big(B(\ell^2(\mathbf{N})), \psi_\lambda\big) * (L^\infty[-1,1], \mu),$$

where $\psi_\lambda(e_{ij}) = \delta_{ij}\lambda^j(1-\lambda), i, j \in \mathbf{N}$, and μ is a nonatomic measure on $[-1,1]$. He also proved that $(T_\lambda, \varphi_\lambda)$ is isomorphic to the factor of type III_λ introduced by Rădulescu in [11]. Namely,

$$(T_\lambda, \varphi_\lambda) \cong (M_2(\mathbf{C}), \omega_\lambda) * (L^\infty[-1,1], \mu),$$

where $\omega_\lambda(p_{ij}) = \delta_{ij}\lambda^j/(\lambda+1), i, j \in \{0,1\}$, and μ a nonatomic measure on $[-1,1]$. Moreover, he showed that $(T_\lambda, \varphi_\lambda)$ has a good behaviour when it is compressed by a "right" projection. Denote $(C, \psi) := (B(\ell^2(\mathbf{N})), \psi_\lambda) * (L^\infty[-1,1], \mu)$ and $(D, \omega) := (M_2(\mathbf{C}), \omega_\lambda) * (L^\infty[-1,1], \mu)$. The following proposition is an easy consequence of proofs of Theorems 5.4 and 6.7 of [15]. It will be useful.

Proposition (1.1.11)[1]: Let $(C, \psi), (D, \omega)$ defined as above and $e_{00} \in B(\ell^2(\mathbf{N})) \subset C, p_{00}, p_{11} \in M_2(\mathbf{C}) \subset D$. Then

$$\begin{aligned}(T_\lambda, \varphi_\lambda) &\cong \left(e_{00}Ce_{00}, \frac{1}{\psi(e_{00})}\psi\right) \\ &\cong \left(p_{00}Dp_{00}, \frac{1}{\omega(p_{00})}\omega\right) \\ &\cong \left(p_{11}Dp_{11}, \frac{1}{\omega(p_{11})}\omega\right).\end{aligned}$$

When the representation (U_t) is assumed to be almost periodic, we have seen (Theorem (1.1.3)) that $\Gamma \subset \mathbf{R}_+^*$, the subgroup generated by the point spectrum of A, completely classifies the free Araki-Woods factor $\Gamma(U_t, H_{\mathbf{R}})''$.

Notation (1.1.12)[1]: For any nontrivial countable subgroup $\Gamma \subset \mathbf{R}_+^*$, we shall denote by $(T_\Gamma, \varphi_\Gamma)$ the unique (up to state-preserving isomorphism) almost periodic free Araki-Woods factor whose Sd invariant is exactly Γ. Of course, φ_Γ is its free quasi-free state. If $\Gamma = \lambda^{\mathbf{Z}}$ for $\lambda \in]0,1[$, then $(T_\Gamma, \varphi_\Gamma)$ is of type III_λ; in this case, it will be simply denoted by $(T_\lambda, \varphi_\lambda)$[15], as in Notation (1.1.10). Theorem 6.4 in [15] gives the following formula:

$$(T_\Gamma, \varphi_\Gamma) \cong \mathop{*}_{\gamma \in \Gamma} (T_\gamma, \varphi_\gamma).$$

For any $\beta \in]0,1[$, the von Neumann algebra $\underset{\beta}{\mathbf{C}} \oplus \underset{1-\beta}{\mathbf{C}}$ is simply denoted by $(\mathbf{C}^2, \tau_\beta)$. Let $\lambda \in]0,1[$ and denote $\alpha = \lambda/(\lambda+1)$. We remind that the faithful state ω_λ on $M_2(\mathbf{C})$ is defined as follows: $\omega_\lambda(p_{ij}) = \delta_{ij}\lambda^j/(\lambda+1)$, for $i,j \in \{0,1\}$. We prove the following theorem:

Notation (1.1.13)[1]: The von Neumann algebra of the left-hand side of (2) together with its free product state will be denoted by (M, ω).

We will need the following result due to Voiculescu [21] (see also [6],[8]) which gives a precise picture of the von Neumann algebra generated by two projections p and q free with respect to a faithful trace.

Theorem (1.1.14)[1]: (Voiculescu, [21]). Let $0 < \alpha \le \min\{\beta, 1-\beta\} < 1$. Then

$$\left(\overset{p}{\underset{\alpha}{C}} \oplus \overset{1-p}{\underset{1-\alpha}{C}}\right) * \left(\overset{q}{\underset{\beta}{C}} \oplus \overset{1-q}{\underset{1-\beta}{C}}\right) \cong \overset{(1-p)\wedge q}{\underset{1-\alpha}{C}} \oplus \underbrace{\left(L^\infty\left(\left[0,\frac{\pi}{2}\right],\nu\right) \otimes M_2\right)}_{2\alpha} \oplus \overset{(1-p)\wedge(1-q)}{\underset{1-\alpha-\beta}{C}}, \qquad (2)$$

where ν is a probability measure without atoms on $[0,\pi/2]$, and $L^\infty([0,\pi/2],\nu)$ has trace given by integration against ν. In the picture of the right-hand side of (2), we have

$$p = 0 \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus 0,$$

$$q = 1 \oplus \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} \oplus 0,$$

where $\theta \in [0,\pi/2]$.

Definition (1.1.15)[1]: ([6]). Let $(S_t)_{\iota\in I}$ be a family of subsets of a unital algebra $A \ni 1$. A nontrivial traveling product in $(S_\iota)_{\iota\in I}$ is a product $a_1 \cdots a_n$ such that $a_j \in S_{t_j} (1 \le j \le n)$ and $\iota_1 \ne \iota_2 \ne \cdots \ne \iota_{n-1} \ne \iota_n$. The trivial traveling product is the identity element 1. The set of all traveling products in $(S_t)_{\iota\in I}$, including the trivial one is denoted by $\Lambda((S_t)_{\iota\in I})$. If $|I| = 2$, we will call traveling products alternating products.

We are now ready to prove the following proposition; it gives a precise picture of the compression of the von Neumann algebra (M,ω) by the projection p. The proof is based on algebraic techniques developed in [5],[6],[8], and techniques of computation of $*$-distributions developed in [15] and [20].

Proposition (1.1.16)[1]: Let $(M,\omega) = (M_2(\mathbf{C}), \omega_\lambda) * (\mathbf{C}^2, \tau_\beta)$ and $p = p_{11} \in M_2(\mathbf{C})$. Assume as in Theorem (1.1.14), that $\alpha = \lambda/(\lambda+1) \le \min\{\beta, 1-\beta\}$. Then

$$\left(pMp, \frac{1}{\omega(p)}\omega\right) \cong L(\mathbf{Z}) * \left((\mathbf{C}^2, \tau_\delta) \otimes (B(\ell^2(\mathbf{N})), \psi_\lambda)\right),$$

where $(\mathbf{C}^2, \tau_\delta) = \mathbf{C}_\delta \oplus \underset{1-\delta}{\mathbf{C}}$ with $\delta = \frac{1-\frac{\beta}{1-\lambda}}{1-\lambda}$ and $\psi_\lambda(e_{ij}) = \delta_{ij}\lambda^j(1-\lambda)$, for $i,j \in \mathbf{N}$.

Proof. Let $(M,\omega) = (M_2, \omega_\lambda) * (\mathbf{C}^2, \tau_\beta)$. Let p and q be the projections in M such that $N = W^*(p,q)$ as in Theorem (1.1.14); p and q are free in M with respect to ω and $\omega(p) = \alpha = \lambda/(\lambda+1), \omega(q) = \beta$. Let x and z. We know that $N = W^*(pqp, x, z)$. Denote by $u = p_{01} \in M_2(\mathbf{C})$ the partial isometry from p to $1-p$, i.e. $u^*u = p$ and $uu^* = 1-p$. Then, thanks to Lemma 5.3 from [19]

$$pMp = W^*(pqp, u^*x, u^*zu).$$

Denote $v = u^*x$ and $P = u^*zu$. Since $v^*v = p$ and $vv^* \leq p, v$ is an isometry in pMp. Moreover, since $Pv = u^*zuu^*x = u^*zx = u^*x = v$, we get $vv^* \leq P$. Denote $\omega_p = \frac{1}{\omega(p)}\omega$ the canonical state on pMp. First, we are going to compute the $*$-distributions of the elements v and vP in pMp with respect to ω_p.

Lemma (1.1.17)[1]: Let $\gamma = \beta(\lambda+1) = \beta/(1-\alpha)$. For any $k, l \in \mathbf{N}$,

$$\omega_p(v^k(v^*)^l) \quad = \delta_{kl}\lambda^k, \tag{3}$$

$$\omega_p(v^kP(v^*)^l) \quad = \delta_{kl}\lambda^k\gamma. \tag{4}$$

Proof. Step (0). First, we review the "algebraic trick" of Dykema [6]. Denote $a = p - \omega(p)$ and $b = q - \omega(q)$; we have $N = \overline{\text{span}}^w \Lambda(\{a\},\{b\})$. Let $w \in N$ such that $\omega(w) = \omega(pw) = 0$. By Kaplansky Density Theorem, w is the s.o.-limit of a bounded sequence in span $\Lambda(\{a\},\{b\})$. Note that since a and b are free and $\omega(a) = \omega(b) = 0$, if $y \in$ span $\Lambda(\{a\},\{b\})$, then $\omega(y)$ is equal to the coefficient of 1 in y. Since $\omega(w) = 0$, we may choose that approximating sequence in span $(\Lambda(\{a\},\{b\}) \setminus \{1\})$. Moreover, since $\omega(pw) = 0$, we may also assume that each coefficient of a be zero, i.e. we have a bounded approximating sequence for w of elements of span $(\Lambda(\{a\},\{b\}) \setminus \{1,a\})$.

Step (1). We prove now Equation (3). Assume $k \geq 1$ and $l = 0$, then $v^k = (u^*x)^k$ is a nontrivial alternating product in $\{u^*\}$ and $\{x\}$. Since $\omega(x) = \omega(px) = 0, x$ is a s.o.-limit of a bounded sequence in span $(\Lambda(\{a\},\{b\}) \setminus \{1,a\})$. So to show that $\omega_p(v^k) = 0$, it suffices to show that if s is a nontrivial alternating product in span $(\Lambda(\{a\},\{b\}) \setminus \{1,a\})$ and $\{u^*\}$ then $\omega(s) = 0$. But since $u^*a = -\alpha u^*$ and $au^* = (1-\alpha)u^*$, regrouping gives a nontrivial alternating product in $\{a, u^*\}$ and $\{b\}$, hence by freeness $\omega(s) = 0$. We get also immediatly $\omega_p((v^*)^l) = 0$. Assume at last $k \geq 1$ and $l \geq 1$, then $v^k(v^*)^l = (u^*x)^{k-1}u^*xx^*u(x^*u)^{l-1}$.

Let $y = xx^* - \alpha 1 + \lambda a$. Since $\omega(xx^*) = \omega(x^*x) = \alpha$ and $py = 0, \omega(y) = \omega(py) = 0$, hence y is a s.o.-limit of a bounded sequence in span $(\Lambda(\{a\},\{b\}) \setminus \{1,a\})$. Replacing in $(u^*x)^{k-1}u^*xx^*u(x^*u)^{l-1}$ the term xx^* by $y + \alpha 1 - \lambda a$, and since $u^*au = -\alpha p$, we have

$$\omega_p(v^k(v^*)^l) = \omega_p((u^*x)^{k-1}u^*yu(x^*u)^{l-1}) + \lambda\omega_p((u^*x)^{k-1}p(x^*u)^{l-1}). \tag{5}$$

To prove $\omega_p((u^*x)^{k-1}u^*yu(x^*u)^{l-1}) = 0$, it suffices to show that if r is a nontrivial alternating product in $\text{span}(\Lambda(\{a\},\{b\}) \setminus \{1,a\})$ and $\{u^*, u\}$ then $\omega(s) = 0$. But for the same reasons as above, regrouping gives a nontrivial alternating product in $\{a, u^*, u\}$ and $\{b\}$, hence by freeness $\omega(r) = 0$. If in Equation (3), $k \neq l$, then applying Equation (5) several times we eventually get $\omega_p(u^*x \cdots u^*x)$ or $\omega_p(x^*u \cdots x^*u)$, both of which are zero. If $k = l$, then we eventually get $\lambda^k\omega_p(p) = \lambda^k$. Thus Equation (3) holds.

Step (2). We prove at last Equation (4). Since $\omega(u^*zu) = \lambda\omega(z)$, and $\gamma = \beta(\lambda+1) = \beta/(1-\alpha)$, we get $\omega_p(P) = \gamma$. Assume $k, l \geq 0$, then $v^kP(v^*)^l = (u^*x)^ku^*zu(x^*u)^l$. Since $\omega(z) = \beta$ and $pz = 0, y = z - \beta 1 + \gamma a$ satisfies $\omega(y) = \omega(py) = 0$. Consequently, y is a s.o.-limit of a bounded sequence in $\text{span}(\Lambda(\{a\},\{b\} \setminus \{1,a\})$. Replacing in the product $(u^*x)^ku^*zu(x^*u)^l$ the term z by $y + \beta 1 - \gamma a$, and since $u^*au = -\alpha p$, we have

$$\omega_p(v^kP(v^*)^l) = \omega_p((u^*x)^ku^*yu(x^*u)^l) + \gamma\omega_p((u^*x)^kp(x^*u)^l).$$

For the same reasons, $\omega_p((u^*x)^ku^*yu(x^*u)^l) = 0$ and $\omega_p(v^kP(v^*)^l) = \gamma\omega_p(v^k(v^*)^l)$. Thus Equation (4) holds.

Lemma (1.1.18)[1]: In pMp, the von Neumann subalgebras $W^*(pqp)$ and $W^*(v,P)$ are *-free with respect to ω_p.

Proof. Lemma (1.1.23) in [15] inspired us to prove Lemma (1.1.18). It is slightly more complicated because here, in some sense, the assumptions are weaker and we must additionally deal with the projection P. To overcome these difficulties, we will use the "algebraic trick" [6] mentioned above.

Let $B = W^*(pqp)$ be the von Neumann subalgebra of pMp generated by pqp and $C = W^*(v,P)$ be the von Neumann subalgebra of pMp generated by v and P. Let $g_k = (pqp)^k - \omega_p((pqp)^k)p$ for $k \geq 1$. Let $W_{kl} = v^k(v^*)^l - \delta_{kl}\lambda^k p, W'_{rs} = v^r P(v^*)^s - \gamma\delta_{rs}\lambda^r p$ for $k,l,r,s \in \mathbf{N}$, $k+l>0$. Since

$$\begin{aligned} B &= \overline{\operatorname{span}}^w\{p, g_k \mid k \geq 1\}, \\ C &= \overline{\operatorname{span}}^w\{p, W_{kl}, W'_{rs} \mid k,l,r,s \in \mathbf{N}, k+l>0\}, \end{aligned}$$

it follows that to check freeness of B and C, we must show that

$$\omega_p\left(\underbrace{b_0 w_1 b_1 \cdots w_n b_n}_{W}\right) = 0 \tag{6}$$

where

$$b_j = g_{m_j}, \tag{7}$$

$$w_j = W_{k_j l_j} \tag{8}$$

$$\text{or} \quad w_j = W'_{r_j s_j}, \tag{9}$$

with $k_j, l_j, m_j, r_j, s_j \in \mathbf{N}, k_j + l_j > 0, m_j > 0$ for all j, except possibly b_0 and/or b_n are equal to 1. We shall prove Equation (6) under a weaker assumption, which is that w_j is also allowed to be

$$(u^*x)^{s_j}u^*yu(x^*u)^{t_j}, \tag{10}$$

$s_j, t_j \geq 0$ and $y = xx^* - \alpha 1 + \lambda a$ or $y = z - \beta 1 + \gamma a$ as in proof of Lemma (1.1.17).

We will denote by $W = b_0 w_1 b_1 \cdots w_n b_n$ such a word with w_j as in Equation (8) or (9).

Let w_j be as in Equation (8) with both k_j and l_j nonzero and let $y_1 = xx^* - \alpha 1 + \lambda a$. We will replace this w_j by

$$\begin{aligned} w_j = &\ \left((u^*x)^{k_j-1}u^*y_1u(x^*u)^{l_j-1}\right) \\ &+ \left(\lambda(u^*x)^{k_j-1}(x^*u)^{l_j-1} - \delta_{k_j l_j}\lambda^{k_j}\right) \\ = &\ A_j + B_j. \end{aligned}$$

Let now w_j be as in Equation (9) with both r_j and s_j nonzero and let $y_2 = z - \beta 1 + \gamma a$. We will replace this w_j by

$$\begin{aligned} w_j = &\ ((u^*x)^{r_j}u^*y_2u(x^*u)^{s_j}) \\ &+\gamma\left(\lambda(u^*x)^{r_j}(x^*u)^{s_j} - \delta_{r_j s_j}\lambda^{r_j}\right) \\ = &\ ((u^*x)^{r_j}u^*y_2u(x^*u)^{s_j}) \\ &+\gamma((u^*x)^{r_j-1}u^*y_1u(x^*u)^{s_j-1}) \\ &+\gamma\left(\lambda(u^*x)^{r_j-1}(x^*u)^{s_j-1} - \delta_{r_j s_j}\lambda^{r_j}\right) \\ = &\ A'_j + A''_j + C_j. \end{aligned}$$

After such replacements are done, w can be rewritten as a sum of terms, in which some w_j are replaced by A_j 's, A'_j 's, A''_j 's, some by B_j 's and some by C_j 's. Consider the terms where all replacements are replacements by A_j 's, A'_j 's, A''_j 's. These terms can be written as alternating products in $\Omega = \{x, x^*, y_1, y_2, g_k, xg_k, g_kx^*, xg_kx^*\}$ and $\{u, u^*\}$. But each element $h \in \Omega$ satisfies $\omega(h) = \omega(ph) = 0$, hence h is a s.o.-limit of a bounded sequence in span$(\Lambda(\{a\}, \{b\}) \setminus \{1, a\})$. We use the same argument as before. To prove that ω_p is zero on such terms, it suffices to show that ω is zero on a nontrivial alternating product in span $(\Lambda(\{a\}, \{b\}) \setminus \{1, a\})$ and $\{u, u^*\}$. But regrouping gives a nontrivial alternating product in $\{a, u, u^*\}$ and $\{b\}$. So, by freeness ω is zero on such a product.

In the rest of the terms at least one w_j is replaced by B_j or C_j. Then, since

$$\begin{aligned} B_j &= \lambda\left((u^*x)^{k_j-1}(x^*u)^{l_j-1} - \delta_{k_j-1, l_j-1}\lambda^{k_j-1}\right) \\ C_j &= \gamma\lambda\left((u^*x)^{r_j-1}(x^*u)^{s_j-1} - \delta_{r_j-1, s_j-1}\lambda^{r_j-1}\right), \end{aligned}$$

we see that such a term is once again

$$b_0 w'_1 b_1 \cdots w'_n b_n,$$

so of the same form as W in Equation (6), but now with the total number of symbols u^* and x strictly smaller than the total number of such symbols in W. Thus applying the replacement procedure to each of these terms repeatedly, we finally get $\omega_p(W) = \omega_p(\sum W_i)$, where each W_i has the same form as W in Equation (6), but for which the substrings w_j are either as in Equation (8) with k_j or l_j equal to zero, or w_j is as Equation (10) (so that no further replacements can be performed). But then each W_i can be rewritten as a nontrivial alternating product in Ω and $\{u, u^*\}$, so as before $\omega_p(W_i) = 0$. Thus $\omega_p(W) = 0$.

We finish at last the proof of Proposition (1.1.16). We know that $pMp = W^*(pqp, v, P)$, and thanks to Lemma (1.1.18), $W^*(pqp)$ and $W^*(v, P)$ are $* -$ free in pMp with respect to ω_p. As pqp is with no atoms with respect to ω_p, with the previous notation, we get $W^*(pqp) \cong L(\mathbf{Z})$. Concerning $W^*(v, P)$, let

$$\begin{aligned} e_{ij} &= v^i(p - P)(v^*)^j \\ f_{kl} &= v^k(P - vv^*)(v^*)^l, \end{aligned}$$

for $i, j, k, l \in \mathbf{N}$. With straightforward computations, we see that $\left(e_{ij}\right)_{i,j\in\mathbf{N}}$ and $(f_{kl})_{k,l\in\mathbf{N}}$ are systems of matrix units, for all $i, j, k, l \in \mathbf{N}, e_{ij}f_{kl} = f_{kl}e_{ij} = 0$, and $W^*\left(e_{ij}, f_{kl}\right) = W^*(v, P)$.

Moreover, $\omega_p(e_{ii}) = (1 - \gamma)\lambda^i$ and $\omega_p(f_{kk}) = (\gamma - \lambda)\lambda^k$, with $\gamma = \beta/(1 - \alpha)$. Consequently, with notation of Proposition (1.1.16), we finally get $\left(W^*(v, P), \omega_p\right) \cong (\mathbf{C}^2, \tau_\delta) \otimes (B(\ell^2(\mathbf{N})), \psi_\lambda)$.

The proof is complete.

Notation (1.1.19)[1]: For a von Neumann (A, ϕ_A) endowed with a state ϕ_A, we will denote by A° the kernel of ϕ_A on A.

The next proposition is in some sense a generalization of Theorem 1.2 of [8]. We will write a complete proof.

Proposition (1.1.20)[1]: Let $(A, \phi_A), (B, \phi_B)$ and (C, ϕ_C) be three von Neumann algebras endowed with faithful, normal states such that A is a factor of type I. Let

$$\begin{aligned} (\mathcal{M}, \psi) &= \left((C, \phi_C) \otimes (A, \phi_A)\right) * (B, \phi_B) \\ &\cup \\ (\mathcal{N}, \psi) &= (A, \phi_A) * (B, \phi_B) \end{aligned}$$

and let e be a minimal projection of A. Then in $e\mathcal{M}e$, we have that $e\mathcal{N}e$ and $C \otimes e$ are free with respect to $\psi_e = \frac{1}{\psi(e)}\psi$ and together they generate $e\mathcal{M}e$, so that

$$(e\mathcal{M}e, \psi_e) \cong (C, \phi_C) * (e\mathcal{N}e, \psi_e).$$

Proof. We follow step by step the proof of Theorem 1.2 of [8]. For notational convenience, we identify C with $C \otimes 1 \subset \mathcal{M}$. To see that $e\mathcal{N}e$ and eC generate $e\mathcal{M}e$, note that $\mathcal{N}$ and eC generate $\mathcal{M}$; so span $\Lambda(\mathcal{N}, eC)$ is dense in $\mathcal{M}$ and $e\Lambda(\mathcal{N}, eC)e = \Lambda(e\mathcal{N}e, eC)$.

We shall show that ψ_e is zero on a nontrivial alternating product in $(e\mathcal{N}e)^\circ$ and eC°. Let $a = e - \psi(e)1$. Then $A^\circ = \mathbf{C}a + S$ where

$$S = \{s \in A \mid \psi(s) = 0, \text{ese } = 0\}.$$

Let $x \in (e\mathcal{N}e)^\circ$. Then by Kaplansky Density Theorem, x is a s.o.-limit of a bounded sequence $(R_k)_{k\in\mathbf{N}}$ in span $\Lambda(\{a\} \cup S, B^\circ)$. For $Q \in \operatorname{span}(\Lambda(\{a\} \cup S, B^\circ) \setminus \{1\})$, we see that ψ on eQe is equal to a fixed constant times the coefficient of a in Q. So since $\psi(R_k) \to 0$ and $\psi(eR_ke) \to 0$, we may assume that the coefficients in each R_k of 1 and a are zero. Since $R_k - eR_ke \to 0$ for the s.o. topology, we may also assume that the coefficient of each element of S in R_k is zero, i.e., that each $R_k \in \operatorname{span}(\Lambda(\{a\} \cup S, B^\circ) \setminus (\{1, a\} \cup S))$. To prove the proposition, it suffices to show that ψ is zero on a nontrivial alternating product in $\Lambda(\{a\} \cup S, B^\circ) \setminus (\{1, a\} \cup S)$ and eC°. But regrouping and multiplying some neighboring elements gives (a constant times) a nontrivial alternating product in $\{a\} \cup S \cup (eC^\circ) \cup (SC^\circ)$ and B°. Thus by freeness, ψ is zero on such a product.

Theorem (1.1.21)[1]: If $\alpha = \lambda/(\lambda + 1) \leq \min\{\beta, 1 - \beta\}$, then

$$(M_2(\mathbf{C}), \omega_\lambda) * (\mathbf{C}^2, \tau_\beta) \cong (T_\lambda, \varphi_\lambda)\,. \tag{11}$$

Proof. Apply Proposition (1.1.20) with $(A, \phi_A) = (B(\ell^2(\mathbf{N})), \psi_\lambda), (B, \phi_B) = (L(\mathbf{Z}), \tau), (C, \phi_C) = (\mathbf{C}^2, \tau_\delta)$. Let $e = e_{00} \in B(\ell^2(\mathbf{N}))$, and denote

$$\begin{array}{ll}(\mathcal{M}, \psi) = & \big((\mathbf{C}^2, \tau_\delta) \otimes (B(\ell^2(\mathbf{N})), \psi_\lambda)\big) * (L(\mathbf{Z}), \tau) \\ \quad\cup & \\ (\mathcal{N}, \psi) = & (B(\ell^2(\mathbf{N})), \psi_\lambda) * (L(\mathbf{Z}), \tau).\end{array}$$

We get

$$(e\mathcal{M}e, \psi_e) \cong (\mathbf{C}^2, \tau_\delta) * (e\mathcal{N}e, \psi_e).$$

But with notation , $(\mathcal{N}, \psi) \cong (T_\lambda, \varphi_\lambda)$ is the free Araki-Woods factor of type III_λ. Since $e = e_{00}$, applying Proposition (1.1.11), we get $(e\mathcal{N}e, \psi_e) \cong (T_\lambda, \varphi_\lambda)$. We use now the "free absorption" properties of $(T_\lambda, \varphi_\lambda)$. Denote by $L(\mathbf{F}(s))$ the interpolated free factor with s generators. We know that $(T_\lambda, \varphi_\lambda) * (L(\mathbf{F}_\infty), \tau) \cong (T_\lambda, \varphi_\lambda)$ (Corollary 5.5 in [15]) and $L(\mathbf{Z}) * (\mathbf{C}^2, \tau_\delta) \cong L(\mathbf{F}(1 + 2\delta(1 - \delta)))$(Lemma 1.6 in [8]). Consequently,

$$(e\mathcal{M}e, \psi_e) \cong (\mathbf{C}^2, \tau_\delta) * (T_\lambda, \varphi_\lambda) \cong (T_\lambda, \varphi_\lambda).$$

But, in a canonical way

$$(\mathcal{M}, \psi) \cong (e\mathcal{M}e, \psi_e) \otimes \big(B\big(\ell^2(\mathbf{N})\big), \psi_\lambda\big).$$

Since $(T_\lambda, \varphi_\lambda) \cong (T_\lambda, \varphi_\lambda) \otimes (B(\ell^2(\mathbf{N})), \psi_\lambda)$, we get

$$(\mathcal{M}, \psi) \cong (T_\lambda, \psi_\lambda).$$

We remind that we have proved (Proposition (1.1.16)) that

$$\big(pMp, \omega_p\big) \cong L(\mathbf{Z}) * \Big((\mathbf{C}^2, \tau_\delta) \otimes \big(B\big(\ell^2(\mathbf{N})\big), \psi_\lambda\big)\Big) = (\mathcal{M}, \psi),$$

where $(M, \omega) = (M_2(\mathbf{C}), \omega_\lambda) * \big(\mathbf{C}^2, \tau_\beta\big)$ and $p = p_{11} \in M_2(\mathbf{C})$. Thus,

$$\big(pMp, \omega_p\big) \cong (T_\lambda, \varphi_\lambda).$$

But once again

$$(M, \omega) \cong \big(pMp, \omega_p\big) \otimes (M_2(\mathbf{C}), \omega_\lambda).$$

Since $(T_\lambda,\varphi_\lambda)\cong(T_\lambda,\varphi_\lambda)\otimes(M_2(\mathbf{C}),\omega_\lambda)$, we finally get

$$(M_2(\mathbf{C}),\omega_\lambda)*\left(\mathbf{C}^2,\tau_\beta\right)\cong(T_\lambda,\varphi_\lambda).$$

We prove Theorem (1.1.5). We will be using the "machinery" of amalgamated free products of von Neumann algebras. We introduce some notations and recall some basic facts about free products with amalgamation (see [9],[17],[22]).

Let $(B,\varphi_B),(A_i,\varphi_i),i=1,2$, be three von Neumann algebras endowed with faithful normal states. Assume that there exist modular embeddings $\rho_i:(B,\varphi_B)\hookrightarrow(A_i,\varphi_i)$. Denote by $E_i:A_i\to B$ the unique state-preserving conditional expectation associated with the embedding ρ_i. We shall denote by

$$(M,E):=(A_1,E_1)*_B*(A_2,E_2)$$

the free product with amalgamation over B of A_1 and A_2 w.r.t. the conditional expectations E_1 and E_2.

Let (B,φ_B) and (C,φ_C) bė two von Neumann algebras togetherr with a faithful normal state. Let $(A,\varphi_A)=(B,\varphi_B)*(C,\varphi_C)$ be their free product. We have canonical modular embeddings $\rho_B:(B,\varphi_B)\hookrightarrow(A,\varphi_A),\rho_C:(C,\varphi_C)\hookrightarrow(A,\varphi_A)$ (see [7]).

We shall regard $B,C\subset A$. Define as before $F:A\to B$ to be the (unique) state-preserving conditional expectation. Let $B^\circ=B\cap\ker(\varphi_B),C^\circ=C\cap\ker(\varphi_C)$ and denote as usual $\Omega=\Lambda(B^\circ,C^\circ)$ the set of alternating products in B° and C° including the trivial one. From [5], we know that

$$\forall b\in B,F(b)=b$$
$$\forall z\in\Omega\setminus(B^\circ\cup\{1\}),F(z)=0.$$

The following proposition is well-known from specialists but we will give a proof for the sake of completeness.

Proposition (1.1.22)[1]: We use the same notations as before. Moreover, let (M,φ_M) be a von Neumann algebra such that $B\subset M$ together with $E:(M,\varphi_M)\to(B,\varphi_B)$ a state-preserving conditional expectation. Denote by $(\mathcal{M},G)=(M,E)*_B(B*C,F)$ and denote by $\psi=\varphi_B\circ G$ the canonical state on $\mathcal{M}$. Then,

$$(\mathcal{M},\psi)\cong(M,\varphi_M)*(C,\varphi_C).$$

Proof. We see immediatly that (M,φ_M) and (C,φ_C) embed in $(\mathcal{M},\psi)$ in a state-preserving way and together they generate $\mathcal{M}$. It remains to prove that M and C are free together w.r.t. the state ψ. For notational convenience, we may assume $M,C\subset\mathcal{M}$. Denote $M^\circ=M\cap\ker(\varphi_M),C^\circ=C\cap\ker(\varphi_C)$. Let W be a nontrivial alternating product in M° and C°, so that W can be written

$$W=x_0w_1x_1\cdots w_nx_n,$$

where $x_j\in C^\circ,w_j\in M^\circ$ for all j, except possibly x_0 and/or x_n are equal to 1. Denote $\Omega=\Lambda(B^\circ,C^\circ)$. If $W\in M^\circ$, there is nothing to do. If not, for each j, replace w_j by

$$w_j=w_j'+b_j,$$

where $w_j'\in M\cap\ker E$ and $b_j\in B^\circ$. Applying the replacement procedure and multiplying some neighboring elements, we get $\psi(W)=\psi(\sum W_i)$ where each W_i is a nontrivial alternating product in $M\cap\ker E$ and $\Omega\setminus(B^\circ\cup\{1\})$. But, we saw that $\Omega\setminus(B^\circ\cup\{1\})\subset\ker F$. Thus, by freeness with amalgamation over B, we get $G(W_i)=0$. But $\psi(W_i)=(\psi\circ G)(W_i)=0$, consequently $\psi(W)=0$.

Lemma (1.1.23)[1]: Let (N,φ) be a von Neumann algebra endowed with a faithful normal state such that the centralizer N^φ is a factor. Let $(M_n(\mathbf{C}),\omega)$ be a matrix algebra endowed with a faithful normal state. Let $\rho_i: (M_n(\mathbf{C}), \omega) \hookrightarrow (N, \varphi), i = 1,2$, be two modular embeddings. Then, there exists a unitary $u \in \mathcal{U}(N^\varphi)$ such that $\mathrm{Ad}(u) \circ \rho_1 = \rho_2$.

Proof. Denote by $(e_{kl})_{0 \le k,l \le n-1}$ the matrix unit in $M_n(\mathbf{C})$. Let $i \in \{1,2\}$. Denote $p_i = \rho_i(e_{00})$. Since ρ_i is modular, we have $p_{1,2} \in N^\varphi$ and $\varphi(p_1) = \varphi(p_2) = \omega(e_{00})$. Since N^φ is a factor, there exists a partial isometry $v \in N^\varphi$ such that $p_1 = v^* v$ and $p_2 = vv^*$. Denote $u = \sum_{i=0}^{n-1} \rho_2(e_{i0}) v \rho_1(e_{0i})$. An easy computation shows that u is a unitary and $u \in N^\varphi$, since $\rho_{1,2}$ are modular. Moreover, for any $0 \le k, l \le n-1$,

$$u\rho_1(e_{kl})u^* = \rho_2(e_{kl}).$$

Theorem (1.1.24)[1]: Let (A_1, ϕ_1) and (A_2, ϕ_2) be any von Neumann algebras endowed with faithful, normal states. Assume that for some $\lambda, \beta \in]0,1[$, there exist modular embeddings

$$\begin{aligned} (M_2(\mathbf{C}), \omega_\lambda) &\hookrightarrow (A_1, \phi_1) \\ \left(\mathbf{C}^2, \tau_\beta\right) &\hookrightarrow (A_2, \phi_2), \end{aligned}$$

such that $\lambda/(\lambda+1) \le \min\{\beta, 1-\beta\}$. Then

$$(A_1, \phi_1) * (A_2, \phi_2) \cong (A_1, \phi_1) * (T_\lambda, \varphi_\lambda) * (A_2, \phi_2).$$

Proof. We shall simply denote by M_2 the matrix algebra $M_2(\mathbf{C})$. Denote by $(A, \phi) = (A_1, \phi_1) * (A_2, \phi_2)$ and denote by

$$\begin{aligned} E_1: (A_1, \phi_1) &\to (M_2, \omega_\lambda) \\ E_2: (A_2, \phi_2) &\to \left(\mathbf{C}^2, \tau_\beta\right) \\ \tilde{E}: (A, \phi) &\to \left(\mathbf{C}^2, \tau_\beta\right) \end{aligned}$$

the canonical state-preserving conditional expectations. Since $(M_2, \omega_\lambda) * \left(\mathbf{C}^2, \tau_\beta\right) \cong (T_\lambda, \varphi_\lambda)$ (Theorem (1.1.21)), denote by $F: (T_\lambda, \varphi_\lambda) \to (M_2, \omega_\lambda)$ the associated state-preserving conditional expectation. Let $(N, \psi) = (A_1, \phi_1) * \left(\mathbf{C}^2, \tau_\beta\right)$. Applying Proposition (1.1.22), we get

$$(N, E) \cong (A_1, E_1) \underset{M_2}{*} (M_2 * \mathbf{C}^2, F).$$

Since $(T_\lambda)^{\varphi_\lambda} \cong L(\mathbf{F}_\infty)$ is a factor, applying Lemma (1.1.23) for $n = 2$, we obtain that the modular embedding of (M_2, ω_λ) into $(T_\lambda, \varphi_\lambda)$ is unique up to a conjugation by a unitary in $T_\lambda^{\varphi_\lambda}$, and we have

$$\begin{aligned} (N, E) &\cong (A_1, E_1) * (M_2 * T_\lambda, F) \\ (N, \psi) &\cong (A_1, \phi_1) * (T_\lambda, \varphi_\lambda) (\text{ by Proposition (1.1.22)}) \\ &\cong \left((A_1, \phi_1) * (T_\lambda, \varphi_\lambda)\right) * (T_\lambda, \varphi_\lambda). \end{aligned}$$

From Theorem 11 of [2], we get that the centralizer algebra N^ψ is a factor. If $\rho_i: \left(\mathbf{C}^2, \tau_\beta\right) \hookrightarrow (N, \psi)$ are two modular embeddings, denote $p_i = \rho_i(p) \in N^\psi$ such that $\psi(p_i) = \beta$. Since p_1 and p_2 are equivalent in N^ψ, ρ_1 and ρ_2 are unitarily conjugate. Consequently, using the isomorphism

$$(N, \psi) \cong (A_1, \phi_1) * (T_\lambda, \varphi_\lambda) * \left(\mathbf{C}^2, \tau_\beta\right), \tag{12}$$

we shall denote by $G: (N, \psi) \to \left(\mathbf{C}^2, \tau_\beta\right)$ the associated state-preserving conditional expectation. We finally get

$$(A,\tilde{E}) \cong (A_1 * \mathbf{C}^2, G) \underset{\mathbf{C}^2}{*} (A_2, E_2) \text{ (by Proposition (1.1.22))}$$
$$\cong (A_1 * T_\lambda * \mathbf{C}^2, G) \underset{\mathbf{C}^2}{*} (A_2, E_2) \text{(by (12))}$$
$$(A,\phi) \cong (A_1,\phi_1) * (T_\lambda, \varphi_\lambda) * (A_2,\phi_2). \text{ (by Proposition (1.1.22))}$$
The proof is complete.

Theorem (1.1.5) is a straightforward corollary of Theorem (1.1.24). We end by giving some examples of von Neumann algebras which satisfy assumptions of Theorem (1.1.5). We introduce the class $\mathcal{S}$ of all von Neumann algebras (M,ϕ) with separable predual and endowed with a faithful, normal, almost periodic state ϕ such that
$$(M,\phi) * (L(\mathbf{Z}), \tau_{\mathbf{Z}}) \cong \left(T_{\mathrm{Sd}(\phi)}, \varphi_{\mathrm{Sd}(\phi)}\right).$$
Note that if (M_1,ϕ_1) and (M_2,ϕ_2) are in $\mathcal{S}$, then $(M_1,\phi_1) * (M_2,\phi_2)$ is also in $\mathcal{S}$.

Example (1.1.25)[1]: We give several examples of von Neumann algebras in the class $\mathcal{S}$. This list is not exhaustive and there is nothing really new here: these examples are mere consequences of results in [5],[6],[8],[10],[15],[19], and of Proposition (1.1.20).

1. Type I: All factors of type I endowed with a faithful, normal nontracial state ϕ.
2. Type III: All the almost periodic free Araki-Woods factors $(T_\Gamma, \varphi_\Gamma)$ endowed with their free quasi-free state.
3. Tensor products: All the tensor products $(N,\omega) \otimes (\text{ Type I}, \phi)$, where (N,ω) is:
   4. Any finite-dimensional von Neumann algebra of the form $\underset{\alpha_1}{C} \oplus \cdots \oplus \underset{\alpha_n}{C}$ with $\alpha_i > 0$ for all i and $\sum \alpha_i = 1$.
   5. $(\mathcal{R},\tau)$ the hyperfinite II_1 factor.
   6. Any interpolated free group factor $L(\mathbf{F}(s)), s > 1$
7. Free products: All the free products of the previous examples.

We still do not know whether all the free products of finite dimensional matrix algebras $(A_1,\phi_1) * (A_2,\phi_2)$ are isomorphic to free Araki-Woods factors. Assume that $A_1 = M_n(\mathbf{C})$ with
$$\phi_1 = \mathrm{Tr}\left(\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \cdot\right), \lambda_1 \le \cdots \le \lambda_n.$$
Let $\beta \in]0,1[$ such that $\lambda_1 \le \min\{\beta, 1-\beta\}$. With our techniques, it is not difficult to see that if one can prove that $(A_1,\phi_1) * \left(\mathbf{C}^2, \tau_\beta\right)$ is a free Araki-Woods factor, then all the free products $(A_1,\phi_1) * (A_2,\phi_2)$ are also free Araki-Woods factors. That is exactly what we did for $n = 2$. But one of the crucial ingredients in the proof was the precise picture of Voiculescu in Theorem (1.1.14). This precise description no longer exists for $n \ge 3$ (see [8]).

Section (1.2): Most one Cartan Subalgebra II

A celebrated theorem of Connes ([28]) states that all amenable II_1 factors are isomorphic to the approximately finite dimensional II_1 factor R of Murray and von Neumann. In particular, all group II_1 factors $L(\Gamma)$ associated with ICC (infinite conjugacy class) amenable groups Γ, and all group measure space II_1 factors $L^\infty(X) \rtimes \Gamma$ arising from (essentially) free ergodic probability-measure preserving (abbreviated as p.m.p.) actions $\Gamma \curvearrowright X$ of countable amenable groups Γ on standard probability spaces X, are isomorphic to R.

In contrast to the amenable case, the group measure space II_1 factors $L^\infty(X) \rtimes \Gamma$ of free ergodic p.m.p. actions of non-amenable groups Γ on standard probability spaces X form a rich and particularly important class of II_1 factors. More general crossed product construction provides a wider class. We want to investigate the isomorphy problem of the crossed product II_1 factors. Namely, given the crossed product $M = Q \rtimes \Gamma$ of a finite amenable von Neumann algebra (Q, τ) by a τ-preserving action of a countable group Γ, to what extent can we recover information on the original action $\Gamma \curvearrowright Q$? In particular, does there exist a group measure space II_1 factor $M = L^\infty(X) \rtimes \Gamma$ which remembers completely the group Γ and the action $\Gamma \curvearrowright X$? The first task would be to determine all regular amenable subalgebras of a given II_1 factor M. Recall that a von Neumann subalgebra P of M is said to be regular if the normalizer group of P in M generates M as a von Neumann algebra ([35]). A regular maximal abelian subalgebra A of M is called a Cartan subalgebra ([36]). In the case of a group measure space II_1 factor $M = L^\infty(X) \rtimes \Gamma$, the von Neumann subalgebra $L^\infty(X)$ is a Cartan subalgebra and determining its position amounts to recovering the orbit equivalence relation of the original action $\Gamma \curvearrowright X$ (see [36]). By [29], the approximately finite dimensional II_1 factor R has a unique Cartan subalgebra, up to conjugacy by an automorphism of R. In ([48]), we provided the first class of examples of non-amenable II_1 factors having unique Cartan subalgebra. They are the group measure space II_1 factors $M = L^\infty(X) \rtimes \mathbb{F}_r$ associated with free ergodic p.m.p. profinite actions $\mathbb{F}_r \curvearrowright X$ of free groups $\mathbb{F}_r$. We extend this result from the free groups $\mathbb{F}_r$ to a lager class of countable groups with the property (strong) $(\mathrm{HH})^+$, defined as follows.

Definition (1.2.1)[23]: Let G be a second countable locally compact group. By a 1-cocycle, we mean a continuous map $b: G \to \mathcal{K}$, or a triplet $(b, \pi, \mathcal{K})$ of b and a continuous unitary G-representation π on a Hilbert space $\mathcal{K}$, which satisfies the 1-cocycle identity:

$$\forall g, h \in \Gamma,\ b(gh) = b(g) + \pi_g b(h).$$

The 1-cocycle b is called proper if the set $\{g \in G: \| b(g) \| \le R\}$ is compact for every $R > 0$. Assume that G is non-amenable. We say G has the Haagerup property (see [27],[26]) if it admits a proper 1-cocycle $(b, \pi, \mathcal{K})$. In the case when π can be taken non-amenable (resp. to be weakly contained in the regular representation), we say G has the property (resp. strong) (HH). We say G has the property (strong) $(\mathrm{HH})^+$if Γ has the property (strong) (HH) and the complete metric approximation property (i.e., it is weakly amenable with constant 1).

We will prove that lattices of products of $\mathrm{SO}(n,1) (n \ge 2)$ and $\mathrm{SU}(n,1)$ have the property $(\mathrm{HH})^+$, and that lattices of $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$ have the property strong $(\mathrm{HH})^+$. Building on our previous work ([48]) and Peterson's deformation technology ([49]), we obtain the following.

Since $L^\infty(X) \rtimes \Gamma$ has the complete metric approximation property if Γ has it and the action is profinite, the weak compactness assumption holds automatically.

Corollary (1.2.2)[23]: Let Γ be a countable group with the property $(\mathrm{HH})^+$. Then, $L(\Gamma)$ has no Cartan subalgebra. Moreover, if $\Gamma \curvearrowright X$ is a free ergodic p.m.p. profinite action, then $L^\infty(X)$ is the unique Cartan subalgebra in $L^\infty(X) \rtimes \Gamma$, up to unitary conjugacy.

As in [48], a stronger result holds if Γ has the property strong (HH).

Corollary (1.2.3)[23]: Let Γ be a countable group with the property strong $(\mathrm{HH})^+$.Then, $L(\Gamma)$ is strongly solid, i.e., the normalizer of every amenable diffuse subalgebra generates an amenable von Neumann subalgebra.

Once the Cartan subalgebra $L^\infty(X)$ is determined, the isomorphy problem of $M = L^\infty(X) \rtimes \Gamma$ reduces to that of the orbit equivalence relations. Then, the group Γ and the action $\Gamma \curvearrowright X$ can be recovered if the orbit equivalence cocycle untwists ([59]). Ioana ([42]) proved a cocycle (virtual) super-rigidity result with discrete targets for p.m.p. profinite actions of property (T) groups. Here, we prove a similar result for property (τ) groups, but with some restrictions on the targets. Recall that a (residually finite) group Γ is said to have the property (τ) if the trivial representation is isolated among finite unitary representations. See [43], [44] for more information on this property.

It is plausible that the residual finiteness assumption on Λ is in fact redundant. Since there are groups having both properties $(\mathrm{HH})^+$ and (τ), Theorems A and C together imply a rigidity result for group measure space von Neumann algebras.

Let $\Gamma' \leq \Gamma$ be a finite index subgroup and $\Gamma' \curvearrowright (X', \mu')$ be a m.p. action. Then, the induced action $\mathrm{Ind}_{\Gamma'}^{\Gamma}(\Gamma' \curvearrowright X')$ is the Γ-action on the measure space $\Gamma/\Gamma' \times X'$, given by $g(p, x) = (gp, \sigma(gp)^{-1} g \sigma(p) x)$, where σ is a fixed cross section $\sigma: \Gamma/\Gamma' \to \Gamma$. (The action is unique up to conjugacy.) We say that two p.m.p. actions $\Gamma_i \curvearrowright (X_i, \mu_i), i = 1,2$, are strongly virtually isomorphic if there are a p.m.p. action $\Gamma' \curvearrowright (X', \mu')$ and finite index inclusions $\Gamma' \hookrightarrow \Gamma_i$ such that $\Gamma_i \curvearrowright X_i$ are measure-preservingly conjugate to $\mathrm{Ind}_{\Gamma'}^{\Gamma_i}(\Gamma' \curvearrowright X')$.

Connes and Jones ([30]) gave a first example of II_1 factors having more than one Cartan subalgebra. We present here a new class of examples. To describe it, recall first that if Γ is a discrete group having an infinite normal abelian subgroup H, then $L(H)$ is a Cartan subalgebra of $L(\Gamma)$ if and only if it satisfies the relative ICC condition: for any $g \in \Gamma \setminus H$, the set $\{aga^{-1}: a \in H\}$ is infinite. The group $H \rtimes \Gamma$ acts on H by $(a, g)b = agbg^{-1}$ (cf. Proposition 2.11 in [48]).

We distinguish two Cartan subalgebras by weak compactness. The simplest example is the following. Another example will be presented.

Corollary (1.2.4)[23]: Let $p_1, p_2, \ldots$ be prime numbers. Then the II_1-factor

$$M = L^\infty\left(\varprojlim (\mathbb{Z}/p_1 \cdots p_n \mathbb{Z})^2\right) \rtimes (\mathbb{Z}^2 \rtimes \mathrm{SL}(2, \mathbb{Z}))$$

has more than one Cartan subalgebra.

We observe that in the above, $L(\mathbb{Z}^2)$ is actually an (strong) HT Cartan subalgebra of M, in the sense of [51]. Thus, while an HT factor has unique HT Cartan subalgebra, up to unitary conjugacy, there exist HT factors that have at least two non-conjugate Cartan subalgebras. It is plausible that there is no essentially-free group action which gives rise to the same orbit equivalence relation as $(L(\mathbb{Z}^2) \subset M)$. Such examples were first exhibited by Furman ([37]). See also [46] and [53].

Let G be a locally compact group. We recall that a unitary Γ-representation $(\pi, \mathcal{H})$ is called amenable if there is a state φ on $\mathbb{B}(\mathcal{H})$ which is Ad π-invariant: $\varphi \circ \mathrm{Ad}\, \pi_g = \varphi$ for all $g \in G$. This notion was introduced and studied by Bekka ([24]). Ámong other things, he proved that π is amenable if and only if $\pi \otimes \bar{\pi}$ weakly contains the trivial representation.

Let σ be the conjugate action of G on $L^\infty(G)$: $(\sigma_h f)(g) = f(h^{-1}gh)$ for $f \in L^\infty(G)$ and $g, h \in G$. We say a locally compact group G is inner-amenable if there is a σ-invariant state μ on $L^\infty(G)$ which vanishes on $C_0(G)$. We note that in several literatures it is only required that μ is σ-invariant and $\mu \neq \delta_e$ (in case G is discrete).

Proposition (1.2.5)[23]: A locally compact group G with the property (HH) has the Haagerup property and is not inner-amenable.

Proof. Let $(b, \pi, \mathcal{K})$ be a proper 1-cocycle and suppose that there is a singular σ-invariant state μ on $L^\infty(G)$. For $x \in \mathbb{B}(\mathcal{H})$, we define $f_x \in L^\infty(G)$ by $f_x(g) = \| b(g) \|^{-2} \langle x b(g), b(g) \rangle$. Let $h \in G$ be fixed. Since

$$\|b(h^{-1}gh) - \pi_h^{-1} b(g)\| = \|b(h^{-1}) + \pi_{h^{-1}g} b(h)\| \leq 2 \| b(h) \| ,$$

and $\| b(g) \| \to \infty$ as $g \to \infty$, one has $\sigma_h(f_x) - f_{\pi_h x \pi_h^*} \in C_0(G)$. It follows that the state φ on $\mathbb{B}(\mathcal{H})$ defined by $\varphi(x) = \mu(f_x)$ is Ad π-invariant. This means π is amenable.

We do not know whether the converse is also true. Combined with Proposition 2.11 in [48], the above proposition yields the following.

Corollary (1.2.6)[23]: A discrete group Γ with the property $(\text{HH})^+$ does not have an infinite normal amenable subgroup.

Theorem (1.2.7)[23]: The following are true.

1. Each of the properties (HH), $(\text{HH})^+$, strong (HH) and strong $(\text{HH})^+$ inherits to a lattice of a locally compact group.
2. If G_1 and G_2 have the property (HH) (resp. $(\text{HH})^+$), then so does $G_1 \times G_2$.
3. The groups $\text{SO}(n, 1)$ with $n \geq 2$ and $\text{SU}(n, 1)$ have the property $(\text{HH})^+$.
4. The groups $\text{SL}(2, \mathbb{R})$ and $\text{SL}(2, \mathbb{C})$ have the property strong $(\text{HH})^+$.
5. Suppose Γ is a countable non-amenable group acting properly on a finite dimensional $\text{CAT}(0)$ cube complex. If all hyperplane stabilizer groups are non-co-amenable, then Γ has the property $(\text{HH})^+$. If all hyperplane stabilizer groups are amenable, then Γ has the property strong $(\text{HH})^+$.

Proof. The assertion (a) follows from the fact that the restriction of non-amenable (resp. weakly sub-regular) representation to a lattice is non-amenable (resp. weakly sub-regular). For the assertion (b), just consider the direct sum of 1cocycles. We prove the property $(\text{HH})^+$ for $G = \text{SO}(n, 1) (n \geq 2)$ and $\text{SU}(n, 1)$. It follows from Theorem 3 in [25] that every non-trivial irreducible representation of G is non-amenable. Since G does not have the property (T), by [56], there is a non-trivial irreducible representation with an unbounded 1-cocycle. But, by [57], every unbounded 1-cocycles of G is proper. Thus, G has the property (HH). Weak amenability is proved in [33],[32].

The irreducible representation of $\text{SL}(2, \mathbb{R})$ and $\text{SL}(2, \mathbb{C})$ which have non-trivial 1-cocycles are found in the principal series (see Example 3 in [39]) and hence are weakly equivalent to the regular representation.

If a group Γ acts properly on a $\text{CAT}(0)$ cube complex Σ, then it has a proper 1-cocycle into the $\ell^2(H)$, where H is the set of hyperplanes in Σ. (See [47].) The unitary representation on $\ell^2(H)$ is non-amenable (resp. weakly contained in the regular representation) if and only if all hyperplane stabilizer subgroups are non-co-amenable (resp. amenable). Weak amenability for finite-dimensional $\text{CAT}(0)$ cube complexes is proved in [38],[45].

Note that by a result of [31], the wreath product $(\mathbb{Z}/2\mathbb{Z}), \mathbb{F}_2$ acts properly on an infinite-dimensional $\text{CAT}(0)$ cube complex with all hyperplane stabilizer subgroups amenable (being subgroups of $\bigoplus_{\mathbb{F}_2} \mathbb{Z}/2\mathbb{Z}$). It follows that $(\mathbb{Z}/2\mathbb{Z})\{\mathbb{F}_2$ has the property strong (HH), but not $(\text{HH})^+$.

We use the same conventions and notations as in ([48]). Thus the symbol "Lim" will be used for a state on $\ell^\infty(\mathbb{N})$, or more generally on $\ell^\infty(I)$ with I directed, which extends the ordinary limit, and that the abbreviation "u.c.p." stands for "unital completely positive." We say a map is normal if it is ultraweakly continuous. Whenever a finite von Neumann algebra

M is being considered, it comes equipped with a distinguished faithful normal tracial state, denoted by τ. Any group action on a finite von Neumann algebra is assumed to preserve the tracial state τ. If $M = P \rtimes \Gamma$ is a crossed product von Neumann algebra, then the tracial state τ on M is given by $\tau(au_g) = \delta_{g,e}\tau(a)$ for $a \in P$ and $g \in \Gamma$. A von Neumann subalgebra $P \subset M$ inherits the tracial state τ from M, and the unique τ-preserving conditional expectation from M onto P is denoted by E_P. We denote by $\mathcal{Z}(M)$ the center of M; by $\mathcal{U}(M)$ the group of unitary elements in M; and by

$$\mathcal{N}_M(P) = \{u \in \mathcal{U}(M): (\mathrm{Ad}\, u)(P) = P\}$$

the normalizer group of P in M, where $(\mathrm{Ad}\, u)(x) = uxu^*$. A von Neumann subalgebra $P \subset M$ is called regular if $\mathcal{N}_M(P)'' = M$. A regular maximal abelian von Neumann subalgebra $A \subset M$ is called a Cartan subalgebra. We note that if $\Gamma \curvearrowright X$ is a free ergodic p.m.p. action, then $A = L^\infty(X)$ is a Cartan subalgebra in the crossed product $L^\infty(X) \rtimes \Gamma$. (See [36].)

We recall the definition of weak compactness.

Definition (1.2.8)[23]: Let (P, τ) be a finite von Neumann algebra, and $\Gamma \curvearrowright P$ be a τ preserving action. The action is called weakly compact if there is a net $\eta_n \in L^2(P \mathbin{\bar{\otimes}} \bar{P})_+$ such that

1. $\|\eta_n - (v \otimes \bar{v})\eta_n\|_2 \to 0$ for $v \in \mathcal{U}(P)$;
2. $\|\eta_n - \mathrm{Ad}(u \otimes \bar{u})(\eta_n)\|_2 \to 0$ for $u \in \Gamma$;
3. $\langle (a \otimes 1)\eta_n, \eta_n \rangle = \tau(a)$ for all $a \in P$.

(These conditions force P to be amenable.) A von Neumann subalgebra P of M is called weakly compact if the action $\mathcal{N}_M(P) \curvearrowright P$ is weakly compact.

It is proved in ([48], Proposition 3.4) that if $\Gamma \curvearrowright Q$ is weakly compact, then Q is weakly compact in the crossed product $Q \rtimes \Gamma$

Theorem (1.2.9)[23]: (Theorem 3.5 in [48]). Let M be a finite von Neumann algebra with the complete metric approximation property. Then, every amenable von Neumann subalgebra P is weakly compact in M.

Let $Q \subset M$ be finite von Neumann algebras. Then, the conditional expectation E_Q can be viewed as the orthogonal projection e_Q from $L^2(M)$ onto $L^2(Q) \subset L^2(M)$. It satisfies $e_Q x e_Q = E_Q(x) e_Q$ for every $x \in M$. The basic construction $\langle M, e_Q \rangle$ is the von Neumann subalgebra of $\mathbb{B}(L^2(M))$ generated by M and e_Q. We note that $\langle M, e_Q \rangle$ coincides with the commutant of the right Q-action in $\mathbb{B}(L^2(M))$. The conditional expectation E_Q extends on $\langle M, e_Q \rangle$ by the formula $E_Q(z) e_Q = e_Q z e_Q$ for $z \in \langle M, e_Q \rangle$. The basic construction $\langle M, e_Q \rangle$ comes together with the faithful normal semi-finite trace Tr such that $\mathrm{Tr}(x e_Q y) = \tau(xy)$. We denote

$$C^*(M e_Q M) = \text{the norm-closed linear span of } \{x e_Q y : x, y \in M\}$$

which is an ultraweakly dense C*-subalgebra of $\langle M, e_Q \rangle$. Suppose that θ is a τ-preserving u.c.p. map on M such that $\theta|_Q = \mathrm{id}_Q$. Then, θ can be regarded as a contraction on $L^2(M)$ which commutes the left and right Q-actions on $L^2(M)$. In particular, $\theta \in \langle M, e_Q \rangle$. See Section 1.3 in [51] for more information on the basic construction.

The following is Theorem A.1 in [51] (see also Theorem 2.1 in [52]).

Theorem (1.2.10)[23]: Let $P, Q \subset M$ be finite von Neumann subalgebras. Then, the following are equivalent.

1. There exists a non-zero projection $e \in P' \cap \langle M, e_Q \rangle$ such that $\mathrm{Tr}(e) < \infty$.

2. There exist non-zero projections $p \in P$ and $q \in Q$, a normal $*$-homomorphism $\theta: pPp \to qQq$ and a non-zero partial isometry $v \in M$ such that

$$\forall x \in pPp\; xv = v\theta(x)$$

$$\text{and } v^*v \in \theta(pPp)' \cap qMq, vv^* \in p(P' \cap M)p.$$

Definition (1.2.11)[23]: Let $P, Q \subset M$ be finite von Neumann algebras. Following [52], we say that P embeds into Q inside M, and write $P \preceq_M Q$, if any of the conditions in Theorem (1.2.10) holds.

Let $P \subset \mathcal{N}$ be von Neumann algebras. We say a state φ on $\mathcal{N}$ is P-central if $\varphi(u^*xu) = \varphi(x)$ for all $u \in \mathcal{U}(P)$ and $x \in \mathcal{N}$, or equivalently $\varphi(ax) = \varphi(xa)$ for all $a \in P$ and $x \in \mathcal{N}$.

Lemma (1.2.12)[23]: Let $P, Q \subset M$ be a finite von Neumann algebras, and φ be a P central state on $\langle M, e_Q\rangle$ whose restriction to M is normal. If $P \npreceq_M Q$, then φ vanishes on $C^*(\mathrm{M}e_Q M)$.

Proof. We assume $\varphi\left(C^*(Me_QM)\right) \neq \{0\}$ and prove $P \preceq_M Q$. Since M sits inside the multiplier of $C^*(Me_QM)$, there is an approximate unit f_n of $C^*(Me_QM)$ such that $\|[f_n, u]\| \to 0$ for every $u \in \mathcal{U}(M)$. (That is, (f_n) is a quasi-central approximate unit for the ideal $C^*(Me_QM)$ in the C*-algebra $M + C^*(Me_QM)$.) We may assume that each f_n belongs to the linear span of $\{xe_Qy : x, y \in M\}$ We define positive linear functionals φ_n and ψ on $\langle M, e_Q\rangle$ by $\varphi_n(z) = \varphi(f_nzf_n)$ and $\psi(z) = \operatorname{Lim}\varphi_n(z)$ for $z \in \langle M, e_Q\rangle$. We note that ψ is non-zero and still P-central. We claim that ψ is normal. We observe that the net (φ_n) actually norm converges to ψ (w.r.t. Lim), since $\operatorname{Lim}\psi(f_n) = \operatorname{Lim}\varphi(f_n) = \|\psi\|$. Hence, it suffices to show that each φ_n is normal. Now let n be fixed and $f_n = \sum_{i=1}^{k} x_ie_Qy_i$ Then, for any $z \in \langle M, e_Q\rangle_+$, one has

$$f_n^*zf_n = \sum_{i,j=1}^{k} y_i^*E_Q(x_i^*zx_j)e_Qy_j \leq \sum_{i,j=1}^{k} y_i^*E_Q(x_i^*zx_j)y_j \in M$$

since $\left[E_Q(x_i^*zx_j)\right]_{i,j=1}^{k}$ is a positive element in $\mathbb{M}_k(Q)$ which commutes with $\operatorname{diag}(e_Q, \dots, e_Q)$. Hence, one has

$$\varphi_n(z) = \varphi(f_n^*zf_n) \leq (\varphi|_M)\left(\sum_{i,j=1}^{k} y_i^*E_Q(x_i^*zx_j)y_j\right).$$

This implies that φ_n is normal, and thus so is ψ. It follows that ψ can be regarded as a positive non-zero element in $P' \cap L^1\langle M, e_Q\rangle$ (see Section IX.2 in [58]). Taking a suitable spectral projection, we are done.

We recall that A.1 in [51] shows the following:

Lemma (1.2.13)[23]: Let A and B be Cartan subalgebras of a type II$_1$-factor M. If $A \preceq_M B$, then there exists $u \in \mathcal{U}(M)$ such that $uAu^* = B$.

Finally, we state some elementary lemmas about u.c.p. maps and positive linear functionals.

Lemma (1.2.14)[23]: Let (M, τ) be a finite von Neumann algebra and θ be a τ-symmetric u.c.p. map on M. Then for every $a, x \in M$, one has

$$\|\theta(ax) - \theta(a)\theta(x)\|_2 \leq 2\|x\|_\infty \|a\|_\infty^{1/2} \|a - \theta(a)\|_2^{1/2}.$$

Proof. Let $\theta(x) = V^*\pi(x)V$ be a Stinespring dilation. Then,

$$\begin{aligned}\| \theta(ax) - \theta(a)\theta(x) \|_2 &= \|V^*\pi(x^*)(1 - VV^*)\pi(a^*)V\hat{1}\|_2 \\ &\leq \| x \|_\infty \|(1 - VV^*)^{1/2}\pi(a^*)V\hat{1}\|_2 \\ &= \| x \|_\infty \tau\big(\theta(aa^*) - \theta(a)\theta(a^*)\big)^{1/2}.\end{aligned}$$

Since $\tau \circ \theta = \tau$, this completes the proof.

Lemma (1.2.15)[23]: Let φ and ψ be positive linear functional on a C^*-algebra and $\varepsilon > 0$. Suppose that $\varphi(1) \geq \psi(1)$ and $\varphi(x) - \psi(x) \leq \varepsilon \| x \|$ for all $x \geq 0$. Then, one has $\| \varphi - \psi \| \leq 2\varepsilon$.

Proof. Let $\varphi - \psi = (\varphi - \psi)_+ - (\varphi - \psi)_-$be the Hahn decomposition. Since $(\varphi - \psi)(1) \geq 0$, one has $\|(\varphi - \psi)_-\| \leq \|(\varphi - \psi)_+\| \leq \varepsilon$.

We review the work of Peterson on real closable derivations, in order to give a qualitative version of Lemma 2.3 in [49].

Let (M, τ) be a finite von Neumann algebra. An $M - M$ bimodule is a Hilbert space $\mathcal{H}$ together with normal representations λ of M and ρ of M^{op} such that $\lambda(M) \subset \rho\left(M^{\text{op}}\right)'$. The action of M is referred to as the left M-action and the action of M^{op} is referred to as the right M-action. We write intuitively $a\xi b$ for $\lambda(a)\rho(b^{op})\xi$. By a closable derivation, we mean a map δ from a weakly dense *-subalgebra $\mathcal{D}$ of M into an $M - M$ bimodule $\mathcal{H}$, which is closable as an operator from $L^2(M)$ into $\mathcal{H}$ and satisfies the Leibniz's rule:

$$\delta(xy) = \delta(x)y + x\delta(y)$$

for every $x, y \in \mathcal{D}$. Moreover, a derivation is always assumed to be real: there is a conjugate-linear isometric involution J on $\mathcal{H}$ such that $J(x\delta(y)z) = z^*\delta(y^*)x^*$ for every $x, y, z \in \mathcal{D}$ (which is equivalent to another definition: $\langle\delta(x), \delta(y)z\rangle = \langle z^*\delta(y^*), \delta(x^*)\rangle$ for every $x, y, z \in \mathcal{D}$).

Let $\mathcal{H}$ be an $M - M$ bimodule and $\delta: M \to \mathcal{H}$ be a closable derivation whose closure is denoted by $\bar{\delta}$. Thanks to the important work of [34], [54], $\operatorname{dom}\bar{\delta} \cap M$ is still a weakly dense $*$-subalgebra and $\bar{\delta}$ satisfies the Leibniz's rule there. Hence, for notational simplicity, the closure $\bar{\delta}$ will be written as δ. We recycle some notations from [49]:

$$\Delta = \delta^*\delta, \zeta_\alpha = \sqrt{\frac{\alpha}{\alpha + \Delta}}, \tilde{\delta}_\alpha = \alpha^{-1/2}\delta \circ \zeta_\alpha$$

(note that $\operatorname{ran}\zeta_\alpha \subset \operatorname{dom}\Delta^{1/2} = \operatorname{dom}\delta\big)$ and

$$\tilde{\Delta}_\alpha = \alpha^{-1/2}\Delta^{1/2} \circ \zeta_\alpha = \sqrt{\frac{\Delta}{\alpha + \Delta}}, \theta_\alpha = 1 - \tilde{\Delta}_\alpha.$$

All operators are firstly defined as Hilbert space operators. Since $1 - \sqrt{t} \leq \sqrt{1 - t}$ for all $0 \leq t \leq 1$, one has $\theta_\alpha \leq \zeta_\alpha$ and

$$\|a - \zeta_\alpha(a)\|_2 \leq \|\tilde{\Delta}_\alpha(a)\|_2 = \|\tilde{\delta}_\alpha(a)\|_2 \leq \| a \|_2 \leq \| a \|_\infty$$

for all $a \in M$. By Lemma 2.2 in [49], the operators ζ_α and θ_α map $M \subset L^2(M)$ into M and are τ-symmetric u.c.p. on M.

We recall from [55] the following facts: $\psi_t = \exp\big(-t\Delta^{1/2}\big)$ form a semigroup of u.c.p. maps on M. Let

$$\begin{aligned}\Gamma(b^*, c) &= \Delta^{1/2}(b^*)c + b^*\Delta^{1/2}(c) - \Delta^{1/2}(b^*c) \\ &= \lim_{t \to 0} \frac{\psi_t(b^*c) - \psi_t(b^*)\psi_t(c)}{t} \in L^2(M)\end{aligned}$$

for $b, c \in \operatorname{dom} \Delta^{1/2} \cap M$ and note that

$$\left\langle \sum_i b_i \otimes y_i, \sum_j c_j \otimes z_j \right\rangle_\Gamma = \sum_{i,j} \tau\big(y_i^* \Gamma\big(b_i^*, c_j\big) z_j\big)$$

is a positive semi-definite form on $\big(\operatorname{dom} \Delta^{1/2} \cap M\big) \otimes M$. In particular, one has

$$|\tau(x^*\Gamma(b^*, c)y)| \leq \tau(x^*\Gamma(b^*, b)x)^{1/2}\tau(y^*\Gamma(c^*, c)y)^{1/2}.$$

It follows that

$$\begin{aligned} \|\Gamma(b^*, c)\|_2 &= \sup\ \left\{|\tau(x^*\Gamma(b^*, c)y)|: x, y \in M, \|xx^*\|_2 \leq 1, \|yy^*\|_2 \leq 1\right\} \\ &\leq \sup\ \left\{\tau(x^*\Gamma(b^*, b)x)^{1/2}\tau(y^*\Gamma(c^*, c)y)^{1/2}: -\right\} \\ &\leq \|\Gamma(b^*, b)\|_2^{1/2}\|\Gamma(c^*, c)\|_2^{1/2} \\ &\leq 4 \parallel b \parallel_\infty^{1/2} \parallel \delta(b) \parallel^{1/2} \parallel c \parallel_\infty^{1/2} \parallel \delta(c) \parallel^{1/2}. \end{aligned}$$

Lemma (1.2.16)[23]: (Lemma 2.3 in [49]). For every a, $x \in M$, one has

$$\|\zeta_\alpha(a)\tilde{\delta}_\alpha(x) - \tilde{\delta}_\alpha(ax)\| \leq 10 \parallel x \parallel_\infty \parallel a \parallel_\infty^{1/2} \|\tilde{\delta}_\alpha(a)\|^{1/2}$$

and

$$\|\tilde{\delta}_\alpha(x)\zeta_\alpha(a) - \tilde{\delta}_\alpha(xa)\| \leq 10 \parallel x \parallel_\infty \parallel a \parallel_\infty^{1/2} \|\tilde{\delta}_\alpha(a)\|^{1/2}.$$

Proof. One has

$$\zeta_\alpha(a)\tilde{\delta}_\alpha(x) = \alpha^{-1/2}\delta(\zeta_\alpha(a)\zeta_\alpha(x)) - \tilde{\delta}_\alpha(a)\zeta_\alpha(x) =: A_1 - A_2.$$

We note that $\|A_2\| \leq \parallel x \parallel_\infty \|\tilde{\delta}_\alpha(a)\|$. Let $\delta = V\Delta^{1/2}$ be the polar decomposition. Then, one has

$$\begin{aligned} V^*A_1 &= \zeta_\alpha(a)\tilde{\Delta}_\alpha(x) + \tilde{\Delta}_\alpha(a)\zeta_\alpha(x) - \alpha^{-1/2}\Gamma(\zeta_\alpha(a), \zeta_\alpha(x)) \\ &=: B_1 + B_2 - B_3 \end{aligned}$$

in $L^2(M)$. We note that $\|B_2\| \leq \parallel x \parallel_\infty \|\tilde{\delta}_\alpha(a)\|$; and by the estimate preceding to this lemma that $\|B_3\| \leq 4 \parallel x \parallel_\infty \parallel a \parallel_\infty^{1/2} \|\tilde{\delta}_\alpha(a)\|^{1/2}$. Finally, one has

$$B_1 = \zeta_\alpha(a)\tilde{\Delta}_\alpha(x) = \zeta_\alpha(a)(1 - \theta_\alpha)(x) \approx ax - \theta_\alpha(ax) = \tilde{\Delta}_\alpha(ax).$$

For the above estimates, we used

$$\|\zeta_\alpha(a)x - ax\|_2 \leq \parallel x \parallel_\infty \|a - \zeta_\alpha(a)\|_2 \leq \parallel x \parallel_\infty \|\tilde{\delta}_\alpha(a)\|_2$$

and

$$\begin{aligned} \|\zeta_\alpha(a)\theta_\alpha(x) - \theta_\alpha(ax)\|_2 &\leq \parallel x \parallel_\infty \|(\zeta_\alpha - \theta_\alpha)(a)\|_2 + \|\theta_\alpha(a)\theta_\alpha(x) - \theta_\alpha(ax)\|_2 \\ &\leq \parallel x \parallel_\infty \left(\|\tilde{\delta}_\alpha(a)\|_2 + 2 \parallel a \parallel_\infty^{1/2} \|\tilde{\delta}_\alpha(a)\|^{1/2}\right) \end{aligned}$$

(see Lemma (1.2.14)). Consequently, one has

$$\zeta_\alpha(a)\tilde{\delta}_\alpha(x) \approx A_1 \approx VB_1 \approx \tilde{\delta}_\alpha(ax).$$

This yields the first inequality. Since the derivation is real, one obtains the second as well.

We will need a vector-valued analogue of the above lemma. Let

$$\Omega = \left\{\eta \in L^2(M \bar{\otimes} \bar{M}): (\mathrm{id} \otimes \bar{\tau})(\eta^*\eta) \leq 1 \text{ and } (\mathrm{id} \otimes \bar{\tau})(\eta\eta^*) \leq 1\right\}.$$

We note that if $\{\xi_k\}$ is an orthonormal basis of $L^2(M)$ and $\eta = \sum_{k=1}^\infty x_k \otimes \bar{\xi}_k$, then $(\mathrm{id} \otimes \bar{\tau})(\eta^*\eta) = \sum_k x_k^* x_k$ and $(\mathrm{id} \otimes \bar{\tau})(\eta\eta^*) = \sum_k x_k x_k^*$. (These series converge a priori in $L^1(M)$.) We note that if $\eta \in \Omega$ and $b, c \in M$ with $\parallel b \parallel_\infty \parallel c \parallel_\infty \leq 1$, then $\eta^*, (b \otimes 1)\eta(c \otimes 1) \in \Omega$.

Lemma (1.2.17)[23]: For every $a \in M$ and $\eta \in \Omega$, one has

$$\|(\zeta_\alpha(a) \otimes 1)(\tilde{\delta}_\alpha \otimes 1)(\eta) - (\tilde{\delta}_\alpha \otimes 1)((a \otimes 1)\eta)\|_{\mathcal{H} \bar{\otimes} L^2(\bar{M})}$$

$$\leq 20 \parallel a \parallel_\infty^{1/2} \left\|\tilde{\delta}_\alpha(a)\right\|^{1/2}$$

and

$$\left\|\left(\tilde{\delta}_\alpha \otimes 1\right)(\eta)(\zeta_\alpha(a) \otimes 1) - \left(\tilde{\delta}_\alpha \otimes 1\right)(\eta(a \otimes 1))\right\|_{\mathcal{H}\bar{\otimes}L^2(\bar{M})}$$
$$\leq 20 \parallel a \parallel_\infty^{1/2} \left\|\tilde{\delta}_\alpha(a)\right\|^{1/2}.$$

Proof. Let $a \in M$ be fixed and define a linear map $T: M \to \mathcal{H}$ by

$$T(x) = \zeta_\alpha(a)\tilde{\delta}_\alpha(x) - \tilde{\delta}_\alpha(ax).$$

By Lemma (1.2.16), one has $\parallel T \parallel \leq 10 \parallel a \parallel_\infty^{1/2} \left\|\tilde{\delta}_\alpha(a)\right\|^{1/2}$. By the non-commutative little Grothendieck theorem (Theorem 9.4 in [50]), there are states f and g on M such that

$$\parallel T(x) \parallel^2 \leq \parallel T \parallel^2 \left(f(x^*x) + g(xx^*)\right)$$

for all $x \in M$. It follows that for $\eta = \sum_{k=1}^\infty x_k \otimes \bar{\xi}_k \in \Omega$, one has

$$\left\|(\zeta_\alpha(a) \otimes 1)\left(\tilde{\delta}_\alpha \otimes 1\right)(\eta) - \left(\tilde{\delta}_\alpha \otimes 1\right)((a \otimes 1)\eta)\right\|^2_{\mathcal{H}\bar{\otimes}L^2(\bar{M})}$$
$$= \sum_k \left\|T(x_k)\right\|^2 \leq \sum_k \parallel T \parallel^2 \left(f(x_k^*x_k) + g(x_kx_k^*)\right) \leq 2 \parallel T \parallel^2.$$

The second inequality follows similarly.

Let Γ be a group and $(b, \pi, \mathcal{K})$ be a proper 1-cocycle. Replacing $(b, \pi, \mathcal{K})$ with $(b \oplus \bar{b}, \pi \oplus \bar{\pi}, \mathcal{K} \oplus \bar{\mathcal{K}})$ and considering an operator defined by $J_0(\xi \oplus \bar{\eta}) = \eta \oplus \xi$ if necessary, we may assume that there is a conjugate-linear involution J_0 on $\mathcal{K}$ such that $J_0 b(g) = b(g)$ and $J_0\pi_g J_0 = \pi_g$ for all $g \in \Gamma$. (Note that π is amenable (resp. weakly sub-regular) if and only if so is $\pi \oplus \bar{\pi}$.)

Let $M = Q \rtimes \Gamma$ be the crossed product von Neumann algebra of a finite von Neumann algebra (Q, τ) by a τ-preserving action σ of Γ. We denote by u_g the element in M that corresponds to $g \in \Gamma$. We equip $\mathcal{H} = L^2(Q) \otimes \ell^2(\Gamma) \otimes \mathcal{K}$ with an $M - M$ bimodule structure by the following:

$\mathcal{H}$	$=$	$L^2(Q)$	$\otimes\, \ell^2(\Gamma)$	$\otimes\, \mathcal{K}$
left action by $g \in \Gamma$	:	σ_g	$\otimes\ \lambda_g$	$\otimes\, \pi_g$
left action by $a \in Q$	:	a	$\otimes\ 1$	$\otimes\, 1$
right action by $g \in \Gamma$	:	1	$\otimes\, \rho_g^{-1}$	$\otimes\, 1$
right action by $a \in Q$	:	$\sum_{h\in\Gamma} \sigma_h(a)^{\mathrm{op}}$	$\otimes\, e_h$	$\otimes\, 1$

We define a conjugate-linear involution J on $\mathcal{H}$ by

$$J\left(\hat{a} \otimes \delta_g \otimes \xi\right) = -\widehat{\sigma_{g^{-1}}(a^*)} \otimes \delta_{g^{-1}} \otimes J_0\pi_{g^{-1}}\xi,$$

and the derivation $\delta: M \to \mathcal{H}$ by

$$\delta\left(au_g\right) = \hat{a} \otimes \delta_g \otimes b(g) \in L^2(Q) \otimes \ell^2(\Gamma) \otimes \mathcal{K}$$

for $a \in Q$ and $g \in \Gamma$. It is routine to check that J intertwines the left and the right M-actions, $J\delta\left(au_g\right) = \delta\left(\sigma_{g-1}(a^*)u_{g-1}\right) = \delta\left(\left(au_g\right)^*\right)$ and moreover that δ is a real closable derivation satisfying

$$\Delta\left(au_g\right) = \parallel b(g) \parallel^2 au_g, \zeta_\alpha\left(au_g\right) = \sqrt{\frac{\alpha}{\alpha + \parallel b(g) \parallel^2}}\, au_g$$

and

$$\theta_\alpha(au_g) = \left(1 - \sqrt{\frac{\| b(g) \|^2}{\alpha + \| b(g) \|^2}}\right) au_g$$

for all $a \in Q$ and $g \in \Gamma$. In particular, all θ_α belong to $C^*(Me_Q M)$.

Lemma (1.2.18)[23]: Suppose that π is weakly contained in the regular representation. Then, the $M-M$ bimodule $\mathcal{H}$ is weakly contained in the coarse bimodule $L^2(M) \bar{\otimes} L^2(M)$. In particular, the left M-action on $\mathcal{H}$ extends to a u.c.p. map $\Psi: \mathbb{B}(L^2(M)) \to \mathbb{B}(\mathcal{H})$ whose range commutes with the right M-action.

Proof. It is well-known and not hard to see that if π is weakly contained in the left regular representation λ, then the $M-M$ bimodule $\mathcal{H}$ is weakly contained in $\hat{\mathcal{H}} := L^2(Q) \bar{\otimes} \ell^2(\Gamma) \bar{\otimes} \ell^2(\Gamma)$, where $(\pi, \mathcal{K})$ is replaced with $(\lambda, \ell^2(\Gamma))$ in the definition of $\mathcal{H}$. Let U be the unitary operator on $\hat{\mathcal{H}}$ defined by

$$U\hat{a} \otimes \delta_h \otimes \delta_g = \widehat{\sigma_g(a)} \otimes \delta_{gh} \otimes \delta_g.$$

It is routine to check that $U^*\lambda(M)U \subset \lambda(Q) \bar{\otimes} \mathbb{C}1 \bar{\otimes} \mathbb{B}(\ell^2(\Gamma))$ and $U^*\rho\left(M^{\text{op}}\right)U \subset \rho(Q^{\circ p}) \bar{\otimes} \mathbb{B}(\ell^2(\Gamma)) \bar{\otimes} \mathbb{C}1$, where λ and ρ respectively stand for the left and right actions on $\hat{\mathcal{H}}$. Since the ambient von Neumann algebras are amenable and commuting, $\hat{\mathcal{H}}$ and a fortiori $\mathcal{H}$ is weakly contained in the coarse $M-M$ bimodule, i.e., the binormal representation μ of $M \otimes M^{\text{op}}$ on $\mathcal{H}$ is continuous w.r.t. the minimal tensor norm. Hence, μ extends to a u.c.p. map $\tilde{\mu}$ from $\mathbb{B}(L^2(M)) \bar{\otimes} M^{\text{op}}$ into $\mathbb{B}(\mathcal{H})$. We define $\Psi: \mathbb{B}(L^2(M)) \to \mathbb{B}(\mathcal{H})$ by $\Psi(x) = \tilde{\mu}(x \otimes 1)$. Since M^{op} is in the multiplicative domain of $\tilde{\mu}$, the range of Ψ commutes with the right M-action.

For the following, let $P \subset M$ be an amenable von Neumann subalgebra such that $P \nprec_M Q$, and $\mathcal{G} \subset \mathcal{N}_M(P)$ be a subgroup whose action on P is weakly compact. We may and will assume that $\mathcal{U}(P) \subset \mathcal{G}$. By definition, there exists a sequence $\eta_n \in L^2(M \bar{\otimes} \bar{M})_+$ such that

1. $\|\eta_n - (v \otimes \bar{v})\eta_n\|_2 \to 0$ for $v \in \mathcal{U}(P)$;
2. $\|\eta_n - \text{Ad}(u \otimes \bar{u})(\eta_n)\|_2 \to 0$ for $u \in \mathcal{G}$;
3. $\langle (a \otimes 1)\eta_n, \eta_n \rangle = \tau(a)$ for all $a \in M$.

We note that $\eta_n \in \Omega$.

Lemma (1.2.19)[23]: For every $\alpha > 0$ and $a \in M$, one has

$$\text{Lim}_n \|(\tilde{\delta}_\alpha \otimes 1)((a \otimes 1)\eta_n)\| = \| a \|_2.$$

Proof. Note that $\|(a \otimes 1)\eta_n\|_2 = \| a \|_2$. Define a state on $\langle M, e_Q \rangle$ by

$$\varphi_0(x) = \text{Lim}_n \langle (x \otimes 1)\eta_n, \eta_n \rangle.$$

By construction, φ_0 is a P-central state such that $\varphi_0|_M = \tau$. Since $P \nprec_M Q$ and $\theta_\alpha a \in C^*(Me_Q M)$, Lemma (1.2.14) implies $\varphi_0(a^*\theta_\alpha^*\theta_\alpha a) = 0$. It follows that

$$\begin{aligned} \text{Lim}_n \|(\tilde{\delta}_\alpha \otimes 1)((a \otimes 1)\eta_n)\| &= \text{Lim}_n \|((1-\theta_\alpha)a \otimes 1)\eta_n\|_2 \\ &= \text{Lim}_n \|(a \otimes 1)\eta_n\|_2 \\ &= \| a \|_2. \end{aligned}$$

This completes the proof.

For $\alpha > 0$, a non-zero projection $p \in \mathcal{G}' \cap M$ and n, we denote

$$\eta_n^{p,\alpha} = (\tilde{\delta}_\alpha \otimes 1)((p \otimes 1)\eta_n)$$

and define a state $\varphi_{p,\alpha}$ on $\mathbb{B}(\mathcal{H}) \cap \rho(M^{\text{op}})'$, where $\rho(M^{\text{op}})$ is the right M-action on $\mathcal{H}$, by

$$\varphi_{p,\alpha}(x) = \| p \|_2^{-2} \operatorname{Lim}_n \langle (x \otimes 1)\eta_n^{p,\alpha}, \eta_n^{p,\alpha} \rangle.$$

Lemma (1.2.20)[23]: Let $a \in \mathcal{G}''$. Then, one has

$$\operatorname{Lim}_\alpha \left| \varphi_{p,\alpha}(\zeta_\alpha(a)x) - \varphi_{p,\alpha}(x\zeta_\alpha(a)) \right| = 0$$

uniformly for $x \in \mathbb{B}(\mathcal{H}) \cap \rho\left(M^{\mathrm{op}}\right)'$ with $\| x \|_\infty \leq 1$.

Proof. Let $u \in \mathcal{G}$ and denote $u_\alpha = \zeta_\alpha(u)$. By Lemma (1.2.17), one has

$$\operatorname{Lim}_n \| \eta_n^{p,\alpha} - (u_\alpha \otimes \bar{u})\eta_n^{p,\alpha}(u_\alpha \otimes \bar{u})^* \| \leq 40 \| \tilde{\delta}_\alpha(u) \|^{1/2}.$$

Since $u_\alpha^* u_\alpha \leq 1$, one has for every $x \in (\rho(M^{\circ p})')_+$ that

$$\begin{aligned} \varphi_{p,\alpha}(u_\alpha^* x u_\alpha) \quad &\geq \| p \|_2^{-2} \operatorname{Lim}_n \langle (x \otimes 1)(u_\alpha \otimes \bar{u})\eta_n^{p,\alpha}(u_\alpha \otimes \bar{u})^*, (u_\alpha \otimes \bar{u})\eta_n^{p,\alpha}(u_\alpha \otimes \bar{u})^* \rangle \\ &\geq \varphi_{p,\alpha}(x) - 80 \| p \|_2^{-2} \| \tilde{\delta}_\alpha(u) \|^{1/2} \| x \|_\infty. \end{aligned}$$

By Lemma (1.2.15), one obtains

$$\| \varphi_{p,\alpha}(\cdot) - \varphi_{p,\alpha}(u_\alpha^* \cdot u_\alpha) \| \leq 160 \| p \|_2^{-2} \| \tilde{\delta}_\alpha(u) \|^{1/2}.$$

In particular, $\operatorname{Lim}_\alpha \varphi_{p,\alpha}(1 - u_\alpha^* u_\alpha) = 0$ and

$$\operatorname{Lim}_\alpha \left| \varphi_{p,\alpha}(u_\alpha x) - \varphi_{p,\alpha}(x u_\alpha) \right| = 0$$

uniformly for x with $\| x \|_\infty \leq 1$. This implies that

$$\operatorname{Lim}_\alpha \left| \varphi_{p,\alpha}(\zeta_\alpha(a)x) - \varphi_{p,\alpha}(x\zeta_\alpha(a)) \right| = 0$$

for each $a \in \operatorname{span} \mathcal{G}$ and uniformly for $x \in \mathbb{B}(\mathcal{H}) \cap \rho(M^{\mathrm{op}})'$ with $\| x \|_\infty \leq 1$.
However, by Lemma (1.2.17),

$$\begin{aligned} \left| \varphi_{p,\alpha}(x\zeta_\alpha(a)) \right| \quad &= \| p \|_2^{-2} \left| \operatorname{Lim}_n \langle (x \otimes 1)(\zeta_\alpha(a) \otimes 1)\eta_n^{p,\alpha}, \eta_n^{p,\alpha} \rangle \right| \\ &\leq \| p \|_2^{-1} \| x \|_\infty \left(20 \| a \|_\infty^{1/2} \| a \|_2^{1/2} + \| a \|_2 \right), \end{aligned}$$

and likewise for $\left| \varphi_{p,\alpha}(\zeta_\alpha(a)x) \right|$. Thus, by Kaplansky's Density Theorem, we are done.

Theorem (1.2.21)[23]: Let $M = Q \rtimes \Gamma$ be the crossed product of a finite von Neumann algebra (Q, τ) by a τ-preserving action of a countable group Γ with the property (HH). Let $P \subset M$ be a regular weakly compact von Neumann subalgebra. Then, $P \preceq_M Q$.

Proof. Let $\mathcal{G}'' = M$ and $\varphi_\alpha = \varphi_{1,\alpha}$. By Lemma (1.2.20), one has

$$\operatorname{Lim}_\alpha | \varphi_\alpha(\zeta_\alpha(a)x) - \varphi_\alpha(x\zeta_\alpha(a)) | = 0$$

for every $a \in \mathcal{G}'' = M$ and $x \in \mathbb{B}(\mathcal{H}) \cap \rho(M^{\mathrm{op}})'$. Since

$$\| u_g - \zeta_\alpha(u_g) \| = 1 - \sqrt{\frac{\alpha}{\alpha + \| b(g) \|^2}} \to 0$$

as $\alpha \to \infty$, one has

$$\operatorname{Lim}_\alpha \left| \varphi_\alpha(u_g x u_g^*) - \varphi_\alpha(x) \right| = 0$$

for every $g \in \Gamma$ and $x \in \mathbb{B}(\mathcal{H}) \cap \rho(M^{\mathrm{op}})'$. Hence, the state φ defined by

$$\varphi(x) = \operatorname{Lim}_\alpha \varphi_\alpha(x)$$

on $\mathbb{B}(\mathcal{K}) \subset \mathbb{B}(\mathcal{H}) \cap \rho(M^{\mathrm{op}})'$ is $\operatorname{Ad}\pi$-invariant. Therefore, π is an amenable representation, in contradiction to the property (HH).

Theorem (1.2.22)[23]: Let $M = Q \rtimes \Gamma$ be the crossed product of a finite amenable von Neumann algebra (Q, τ) by a τ-preserving action of a countable group Γ with the property strong (HH). Let $P \subset M$ be an amenable von Neumann subalgebra such that $P \npreceq_M Q$ and $\mathcal{G} \subset \mathcal{N}_M(P)$ be a subgroup whose action on P is weakly compact. Then, the von Neumann subalgebra $\mathcal{G}''$ is amenable.

Proof. We use Haagerup's criterion for amenable von Neumann algebras (Lemma 2.2 in [40]). Let a non-zero projection $p \in \mathcal{G}' \cap M$ and a finite subset $F \subset \mathcal{U}(\mathcal{G}'')$ be given arbitrary. We need to show

$$\left\| \sum_{u \in F} up \otimes \overline{up} \right\|_{M \bar{\otimes} \bar{M}} = |F|.$$

Let $u \in \mathcal{U}(\mathcal{G}'')$. By Lemma (1.2.17) and Lemma (1.2.19), one has

$$\begin{aligned}\varphi_{p,\alpha}(\zeta_\alpha(up)^* \zeta_\alpha(up)) &= \| p \|_2^{-2} \operatorname{Lim}_n \|(\zeta_\alpha(up) \otimes 1)(\tilde{\delta}_\alpha \otimes 1)((p \otimes 1)\eta_n)\|_2^2 \\ &\geq \| p \|_2^{-2} \operatorname{Lim}_n \left(\|(\tilde{\delta}_\alpha \otimes 1)((up \otimes 1)\eta_n)\|_2 - 20 \|\tilde{\delta}_\alpha(up)\|^{1/2} \right)^2 \\ &\geq 1 - 40 \| p \|_2^{-1} \|\tilde{\delta}_\alpha(up)\|^{1/2}.\end{aligned}$$

Hence, by Lemma (1.2.20), one has

$$\operatorname{Lim}_\alpha \left| \varphi_{p,\alpha}(\zeta_\alpha(up)^* x \zeta_\alpha(up)) - \varphi_{p,\alpha}(x) \right| = 0$$

uniformly for $x \in \mathbb{B}(\mathcal{H}) \cap \rho\left(M^{\mathrm{op}}\right)'$ with $\| x \|_\infty \leq 1$. By Lemma (1.2.18), the left M-action on $\mathcal{H}$ extends to a u.c.p. map $\Psi: \mathbb{B}(L^2(M)) \to \mathbb{B}(\mathcal{H}) \cap \rho(M^{\mathrm{op}})'$. The state $\psi_{p,\alpha} = \varphi_{p,\alpha} \circ \Psi$ on $\mathbb{B}(L^2(M))$ satisfies

$$\operatorname{Lim}_\alpha \left| \psi_{p,\alpha}(\zeta_\alpha(up)^* x \zeta_\alpha(up)) - \psi_{p,\alpha}(x) \right| = 0$$

uniformly for $x \in \mathbb{B}(L^2(M))$ with $\| x \|_\infty \leq 1$. By a standard convexity argument in cooperation with the Powers-Størmer inequality, this implies that

$$\operatorname{Lim}_\alpha \left\| \sum_{u \in F} \zeta_\alpha(up) \otimes \overline{\zeta_\alpha(up)} \right\|_{M \bar{\otimes} \bar{M}} = |F|$$

for the finite subset $F \subset \mathcal{U}(\mathcal{G}'')$. Since ζ_α are u.c.p. maps, this yields

$$\left\| \sum_{u \in F} up \otimes \overline{up} \right\|_{M \bar{\otimes} \bar{M}} \geq \operatorname{Lim}_\alpha \left\| \sum_{u \in F} \zeta_\alpha(up) \otimes \overline{\zeta_\alpha(up)} \right\|_{M \bar{\otimes} \bar{M}} = |F|$$

This completes the proof.

The corollaries follow from the corresponding Theorem (1.2.21) and Lemma (1.2.13), because all the von Neumann algebras in consideration have the complete metric approximation property and hence all amenable subalgebras are weakly compact (Theorem (1.2.9)).

We fix a notation for profinite actions. An action $\Gamma \curvearrowright (X, \mu)$ is said to be profinite if (X, μ) is the projective limit of finite-cardinality probability spaces (X_n, μ_n) on which Γ acts consistently. We will identify $L^\infty(X_n, \mu_n)$ as the corresponding Γ-invariant finite-dimensional von Neumann subalgebra of $L^\infty(X, \mu)$. The same thing for L^2. We write $X = \coprod_a X_{a,n}$ for the partition of X corresponding to X_n, i.e., the characteristic functions of $X_{a,n}$'s are the non-zero minimal projections in $L^\infty(X_n)$.

Definition (1.2.23)[23]: Let $\pi: \Gamma \curvearrowright \mathcal{H}$ be a unitary representation. We say π has a spectral gap if there are a finite subset $F \subset \Gamma$ and $\kappa > 0$, called a critical pair, satisfying the following property: denoting by P the orthogonal projection of $\mathcal{H}$ onto the subspace of π-invariant vectors, one has

$$\kappa \| \xi - P\xi \| \leq \max_{g \in F} \|\xi - \pi_g \xi\|$$

for every $\xi \in \mathcal{H}$. (This is equivalent to that the point 1 is isolated (if it exists) in the spectrum of the self-adjoint operator $(2|F|)^{-1} \sum_{g \in F} (\pi_g + \pi_g^*)$ on $\mathcal{H}$.) We say that π has a stable spectral gap if the unitary representation $\pi \otimes \bar{\pi}$ of Γ on $\mathcal{H} \bar{\otimes} \bar{\mathcal{H}}$ has a spectral gap. (Note that we allow rank $P \geq 1$.)

When the unitary representation π arises from a p.m.p. action $\Gamma \curvearrowright X$, we simply say $\Gamma \curvearrowright X$ has a (stable) spectral gap if π has. Assume moreover that the action $\Gamma \curvearrowright X$ is profinite. We say $\Gamma \curvearrowright X$ has a stable spectral gap with growth condition if there are a critical pair (F, κ) such that π^F, the restriction of π to the subgroup of Γ generated by F, does not have a subrepresentation of infinite multiplicity.

Suppose that $\Gamma \curvearrowright \varprojlim X_n$ has a stable spectral gap. Then, π^F has finitely many equivalence classes of irreducible subrepresentations of any given dimension $k \in \mathbb{N}$. (See [41].) It follows that the growth condition is equivalent to that the minimal dimension k_n of a non-zero subrepresentation of $\pi^F|_{L^2(X) \ominus L^2(X_n)}$ tends to infinity.

Lemma (1.2.24)[23]: Let $\Gamma \curvearrowright X$ be a p.m.p. action which is profinite and has a stable spectral gap with growth condition. Let $F \subset \Gamma$ and $\kappa > 0$ be a critical pair. Then, for any $k \in \mathbb{N}$ and unitary elements $\{u_g\}_{g \in F}$ on the k-dimensional Hilbert space ℓ_k^2, one has

$$\frac{\kappa^2}{2}\left(1 - \frac{k}{k_n}\right)\left\|\xi - P_{L^2(X_n) \bar{\otimes} \ell_k^2} \xi\right\|_2 \leq \max_{g \in F}\left\|\xi - (\pi_g \otimes u_g)\xi\right\|_2$$

for every $\xi \in L^2(X) \bar{\otimes} \ell_k^2$ and $n \in \mathbb{N}$.

Proof. We denote $L^2(X_n)^{\perp} = L^2(X) \ominus L^2(X_n)$. It suffices to show

$$\frac{\kappa^2}{2}\left(1 - \frac{k}{k_n}\right) \| \xi \|_2 \leq \max_{g \in F}\left\|\xi - (\pi_g \otimes u_g)\xi\right\|_2$$

for $\xi \in L^2(X_n)^{\perp} \bar{\otimes} \ell_k^2$. We assume $\| \xi \|_2 = 1$ and denote the right hand side of the asserted inequality by ε. We view ξ as a Hilbert-Schmidt operator T_ξ from $\overline{\ell_k^2}$ into $L^2(X_n)^{\perp}$. Note that

$$\left\|T_\xi - \pi_g T_\xi \bar{u}_g^*\right\|_2 = \left\|\xi - (\pi_g \otimes u_g)\xi\right\|_2 \leq \varepsilon.$$

Hence by the Powers-Størmer inequality, the Hilbert-Schmidt operator $S_\xi = \left(T_\xi T_\xi^*\right)^{1/2}$ on $L^2(X_n)^{\perp}$ satisfies

$$\begin{aligned}\left\|S_\xi - \pi_g S_\xi \pi_g^*\right\|_2^2 &\leq \left\|T_\xi T_\xi^* - \pi_g T_\xi T_\xi^* \pi_g^*\right\|_1 \\ &\leq \left\|T_\xi + \pi_g T_\xi \bar{u}_g^*\right\|_2 \left\|T_\xi - \pi_g T_\xi \bar{u}_g^*\right\|_2 \\ &\leq 2\varepsilon.\end{aligned}$$

By the stable spectral gap property, one has

$$\left\|S_\xi - P(S_\xi)\right\|_2^2 \leq 2\varepsilon/\kappa^2.$$

Since $P(S_\xi)$ commutes with π_g for all $g \in F$, growth condition implies that $P(S_\xi) = \sum_i \gamma_i r_i^{-1/2} Q_i$ for some $\gamma_i \geq 0$ and mutually orthogonal projections Q_i with $r_i = \mathrm{Tr}(Q_i) \geq k_n$. Since S_ξ has rank at most k, denoting its range projection by R, one has

$$
\begin{aligned}
\|P(S_\xi)\|_2^2 &= \langle S_\xi, P(S_\xi)\rangle \\
&= \sum_i \gamma_i {r_i}^{-1/2} \mathrm{Tr}(Q_i R S_\xi Q_i) \\
&\le \left(\sum_i \gamma_i^2 r_i^{-1} \mathrm{Tr}(Q_i R Q_i)\right)^{1/2} \left(\textstyle\sum_i \mathrm{Tr}(Q_i S_\xi^* S_\xi Q_i)\right)^{1/2} \\
&\le \left(\sum_i \gamma_i^2 k_n^{-1} k\right)^{1/2} \|S_\xi\|_2 \\
&= (k/k_n)^{1/2} \|P(S_\xi)\|_2 .
\end{aligned}
$$

By combining two inequalities, one obtains

$$1 - (k/k_n) \le 1 - \|P(S_\xi)\|_2^2 = \|S_\xi - P(S_\xi)\|_2^2 \le 2\varepsilon/\kappa^2$$

and hence the desired inequality.

Recall that a cocycle of $\Gamma \curvearrowright X$ with values in a group Λ is a measurable map $\alpha\colon \Gamma \times X \to \Lambda$ satisfying the cocycle identity:

$$\forall g, h \in \Gamma, \mu\text{-a.e. } x \in X,\ \alpha(g, hx)\alpha(h, x) = \alpha(gh, x).$$

A cocycle α which is independent of the x-variable is said to be homomorphism for the obvious reason. Cocycles α and β are said to be equivalent if there is a measurable map $\phi\colon X \to \Lambda$ such that $\beta(g, x) = \phi(gx)\alpha(g, x)\phi(x)^{-1}$ for all $g \in \Gamma$ and μ-a.e. $x \in X$.

Lemma (1.2.25)[23]: Let $\Gamma = \Gamma_1 \times \Gamma_2$ and $\Gamma \curvearrowright X = \lim_{\leftarrow} X_n$ be a p.m.p. profinite action such that $\Gamma_2 \curvearrowright X$ has a stable spectral gap with growth condition. Let (N, τ) be a finite type I von Neumann algebra, and $\alpha\colon \Gamma \times X \to \mathcal{U}(N)$ be a cocycle. Then, for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$\int_X \|\alpha(g, x) - \alpha'_{a(x),n}(g)\|_2^2 dx \le \varepsilon$$

for all $g \in \ker(\Gamma_1 \to \mathrm{Aut}(X_n))$, where $\alpha'_{a,n}(g) = |X_{a,n}|^{-1} \int_{X_{a,n}} \alpha(g, y) dy$ and $a(x)$ is such that $x \in X_{a(x),n}$.

Proof. It suffices to consider each direct summand of N and hence we may assume $N = \mathbb{M}_k(\mathbb{C}) \otimes A$, where A is an abelian von Neumann algebra. For every $g \in \Gamma$, we define $w_g \in L^\infty(X) \mathbin{\bar{\otimes}} N = L^\infty(X, N)$ by $w_g(x) = \alpha(g, g^{-1}x)$. Then, it becomes a unitary 1-cocycle for $\tilde{\sigma} = \sigma \otimes \mathrm{id}_N$:

$$w_{gh} = w_g \tilde{\sigma}_g(w_h).$$

Let $F \subset \Gamma_2$ and $\kappa > 0$ be a critical pair for the stable spectral gap of $\Gamma_2 \curvearrowright X$. Let $\delta = \varepsilon\kappa^2/8$ and take $m \in \mathbb{N}$ and unitary elements $w'_h \in L^\infty(X_m) \mathbin{\bar{\otimes}} N$ such that $\|w_h - w'_h\|_2 < \delta$ for every $h \in F$. For the rest of the proof, we fix $g \in \ker(\Gamma_1 \to \mathrm{Aut}(X_m))$. Since $w'_h = \tilde{\sigma}_g(w'_h)$, one has

$$w_g w'_h \approx w_g \tilde{\sigma}_g(w_h) = w_{gh} = w_{hg} \approx w'_h \tilde{\sigma}_h(w_g),$$

for every $h \in F$. We define trace-preserving *-automorphisms π_h on $L^\infty(X) \mathbin{\bar{\otimes}} N$ by

$$\pi_h(x) = \mathrm{Ad}(w'_h) \circ \tilde{\sigma}_h(x)$$

and note that $\|w_g - \pi_h(w_g)\|_2 \le 2\delta$ for every $h \in F$. We write $\tilde{\pi}_h$ for the restriction of π_h to $L^\infty(X_m) \otimes N$. Note that $\tilde{\pi}_h$ acts as identity on $\mathbb{C}1 \mathbin{\bar{\otimes}} A \subset L^\infty(X_m) \mathbin{\bar{\otimes}} N$.

Let $\{p_a\}$ be the set of non-zero minimal projections in $L^\infty(X_m)$ and define an isometry $V: L^2(X) \to L^2(X) \bar{\otimes} L^2(X_m)$ by $V\xi = |X_m|^{1/2}\sum_a p_a\xi \otimes p_a$. (Here $|X_m|$ stands for the cardinality of the atoms of X_m.) We claim that $(V \otimes 1)\pi_h = (\sigma_h \otimes \tilde{\pi}_h)(V \otimes 1)$. Indeed, if $w_h' = \sum_a \sigma_h(p_a) \otimes y_a$, then

$$\begin{aligned}(\sigma_h \otimes \tilde{\pi}_h)(V \otimes 1)(\xi \otimes c) &= |X_m|^{1/2}(\sigma_h \otimes \tilde{\pi}_h)\sum_a p_a\xi \otimes p_a \otimes c \\ &= |X_m|^{1/2}\sum_a \sigma_h(p_a\xi) \otimes \sigma_h(p_a) \otimes y_a c y_a^* \\ &= V\sum_a \sigma_h(p_a\xi) \otimes y_a c y_a^* \\ &= V\pi_h(\xi \otimes c)\end{aligned}$$

for all $\xi \in L^2(X)$ and $c \in L^2(N)$. Now, it follows that

$$\max_{h\in F}\|(V \otimes 1)w_g - (\sigma_h \otimes \tilde{\pi}_h)(V \otimes 1)w_g\|_2 \leq 2\delta.$$

We observe that if $\tilde{\pi}_h$ is viewed as a unitary operator on $L^2(X_m) \bar{\otimes} L^2(\mathbb{M}_k(\mathbb{C})) \bar{\otimes} L^2(A)$, then it lives in $\mathbb{B}\left(L^2(X_m) \bar{\otimes} L^2(\mathbb{M}_k(\mathbb{C}))\right) \bar{\otimes} A$. Hence Lemma (1.2.24) applies and one obtains

$$\frac{\kappa^2}{2}\left(1 - \frac{mk^2}{k_n}\right)\|(V \otimes 1)w_g - \left(P_{L^2(X_n)} \otimes 1 \otimes 1\right)(V \otimes 1)w_g\|_2 \leq 2\delta$$

for every $n \in \mathbb{N}$. Finally take n to be such that $n \geq m$ and $k_n \geq 2mk^2$. Since $\left(P_{L^2(X_n)} \otimes 1\right)V = VP_{L^2(X_n)}$ for $n \geq m$, one has

$$\begin{aligned}\left(\int_X \|\alpha(g,x) - \alpha'_{a(x),n}(g)\|_2^2 dx\right)^{1/2} &= \|w_g - \left(P_{L^2(X_n)} \otimes 1\right)w_g\|_2 \\ &\leq 4\delta\Big/\left(\kappa^2\left(1 - \left(\frac{mk^2}{k_n}\right)^{1/2}\right)\right) \leq \varepsilon.\end{aligned}$$

We note that $\ker\left(\Gamma_1 \to \mathrm{Aut}(X_n)\right) \subset \ker\left(\Gamma_1 \to \mathrm{Aut}(X_m)\right)$.

We combine the above result with results of Ioana in [42], to obtain the following cocycle rigidity result for profinite actions of product groups.

Theorem (1.2.26)[23]: Let $\Gamma = \Gamma_1 \times \Gamma_2$ and $\Gamma \curvearrowright X$ be an ergodic p.m.p. profinite action such that $\Gamma_i \curvearrowright X$ has a stable spectral gap with growth condition, for each $i = 1,2$. Let Λ be a finite group and $\alpha: \Gamma \times X \to \Lambda$ be a cocycle. Then, there exists a finite index subgroup $\Gamma' \subset \Gamma$ such that for each Γ'-ergodic component $X' \subset X$, the restricted cocycle $\alpha|_{\Gamma' \times X'}$ is equivalent to a homomorphism from Γ' into Λ.

Proof. The proof of this theorem is very similar to that of Theorem B in [42], and hence it will be rather sketchy. Let $Z = X \times X \times \Lambda$ and we will consider the unitary representation $\pi: \Gamma \curvearrowright L^2(Z)$ induced by the m.p. transformation

$$g(x,y,t) = (gx, gy, \alpha(g,x)t\alpha(g,y)^{-1}).$$

Let $\varepsilon > 0$ be arbitrary. Since Λ is discrete, Lemma (1.2.25) implies that there are a normal finite index subgroup Γ' and $n \in \mathbb{N}$ such that $\|\pi(g)\xi_n - \xi_n\|_2 < \varepsilon$ for all $g \in \Gamma'$, where $\xi_n = |X_n|^{1/2}\sum_a \chi_{X_{a,n}\times X_{a,n}\times\{e\}}$. It follows that the circumcenter of $\pi(\Gamma')\xi_n$ is a $\pi(\Gamma')$-invariant vector which is close to ξ_n. Since $\Gamma \curvearrowright X$ is ergodic and Γ' is a normal finite index subgroup in Γ, there are a Γ'-ergodic component $X' \subset X$ and a finite subset $E \subset \Gamma$ such that $X = \bigcup_{s\in E} sX'$. Thus, there are Γ'-ergodic components $X_1', X_2' \subset X$ such that $\xi' =$

$|X'|^{-1}\chi_{X_1'\times X_2'\times\{e\}}$ is close to a $\pi(\Gamma')$-invariant vector. We may assume that $X_1' = X'$. By Corollary (1.2.6) in [42], the cocycle $\alpha|_{\Gamma'\times X'}$ is equivalent to a homomorphism θ via $\phi: X' \to \Lambda$, i.e., $\theta(g) = \phi(gx)\alpha(g,x)\phi(x)^{-1}$. We observe that $\alpha|_{\Gamma'\times sX'}$ is equivalent to $\theta \circ \mathrm{Ad}(s^{-1})$.

Indeed, one has

$$\begin{aligned}\theta(s^{-1}gs) &= \phi(s^{-1}gsx)\alpha(s^{-1}gs,x)\phi(x)^{-1}\\ &= \phi(s^{-1}gsx)\alpha(s^{-1},gsx)\alpha(g,sx)\alpha(s,x)\phi(x)^{-1}\\ &= \psi(gsx)\alpha(g,sx)\psi(sx)^{-1},\end{aligned}$$

where $\psi(sx) = \phi(x)\alpha(s^{-1},sx)$ for $s \in E$ and $x \in X'$.

Theorem (1.2.27)[23]: Let $\Gamma = \Gamma_1 \times \Gamma_2$ be a group with the property (τ) and $\Gamma \curvearrowright X = \lim_{\leftarrow} X_n$ be a p.m.p. profinite action with growth condition such that both $\Gamma_i \curvearrowright X$ are ergodic. Let Λ be a residually-finite group. Then, any cocycle

$$\alpha: \Gamma \times X \to \Lambda$$

virtually untwists, i.e., there exist $n \in \mathbb{N}$ and a cocycle $\beta: \Gamma \times X_n \to \Lambda$ which is equivalent to α.

Proof. By Theorem B and Remark 3.1 in [42], it suffices to show that the unitary representation $\pi: \Gamma \curvearrowright L^2(X \times X \times \Lambda)$ has a spectral gap. Let Λ_j be the finite quotients of Λ. Since Λ is residually finite the unitary representation π is weakly contained in the direct sum $\bigoplus\pi_j$, where π_j is the unitary representation induced by $\Gamma \curvearrowright X \times X \times \Lambda_j$ using the same 1-cocycle composed with the quotient $\Lambda \to \Lambda_j$. Thus, it suffices to show that π_j 's have a uniform spectral gap. We prove this by showing that each π_j is contained in a direct sum of finite representations; then the uniformity follows from property (τ). We may assume that Λ is finite and $\pi_j = \pi$. By Theorem (1.2.26), there is a finite index subgroup Γ' such that for each Γ'-ergodic component $X_k' \subset X$, the restricted cocycle $\alpha|_{\Gamma'\times X_k'}$ is equivalent to a homomorphism $\theta_k: \Gamma' \to \Lambda$ via $\phi_k: X_k' \to \Lambda$, i.e., $\theta_k(g) = \phi_k^k(gx)\alpha(g,x)\phi_k(x)^{-1}$. Let $\sigma_{k,l}$ be the automorphism on $X_k' \times X_l' \times \Lambda$ defined by $\sigma_{k,l}(x,y,t) = (x,y,\phi_k(x)t\phi_l(y)^{-1})$. Then, for $g \in \Gamma'$, one has

$$\begin{aligned}\sigma_{k,l}g\sigma_{k,l}^{-1}(x,y,t) &= (gx,gy,\phi_k(gx)\alpha(g,x)\phi_k(x)^{-1}t\phi_l(y)\alpha(g,y)^{-1}\phi_l(gy)^{-1})\\ &= (gx,gy,\theta_k(g)t\theta_l(g)^{-1}).\end{aligned}$$

Since X' is profinite, this implies that the unitary representation $\pi|_{\Gamma'}$ is contained in a direct sum of finite representations (of the form $\Gamma' \curvearrowright X_n \times X_n \times \Lambda, g(x,y,t) = (gx,gy,\theta_k(g)t\theta_l(g)^{-1})$). Since Γ' has finite index in Γ, the unitary representation $\pi \subset \mathrm{Ind}_{\Gamma'}^{\Gamma}(\pi|_{\Gamma'})$ is contained in a direct sum of finite representations. This completes the proof.

The following two lemmas are well-known.

Lemma (1.2.28)[23]: Let $\Gamma \geq \Delta_1 \geq \Delta_2 \geq \cdots$ be a decreasing sequence of finite index normal subgroups. Then, the left-and-right action $\Gamma \times \Gamma \curvearrowright \lim_{\leftarrow} \Gamma/\Delta_n$ is essentially-free if and only if $\lim_n |Z_n(g)|/|\Gamma/\Delta_n| = 0$ for every $g \in \Gamma$ with $g \neq e$, where $Z_n(g)$ is the centralizer group of g in Γ/Δ_n.

Proof. The 'only if' part is trivial. We prove the 'if' part. Note that the condition implies that $\bigcap\Delta_n = \{e\}$. Let $(g,h) \in \Gamma \times \Gamma$ and observe that

$$\left|\left\{x \in \lim_{\leftarrow}\Gamma/\Delta_n : (g,h)x = x\right\}\right| = \lim_n \frac{|\{x \in \Gamma/\Delta_n : gxh^{-1} = x\}|}{|\Gamma/\Delta_n|}.$$

If $g = e$, then (g, h) acts freely unless $h = e$, too. Thus, let $g \neq e$. If $x, y \in \Gamma/\Delta_n$ are such that $gxh^{-1} = x$ and $gyh^{-1} = y$, then one has $gxy^{-1}g^{-1} = xy^{-1}$, i.e. $xy^{-1} \in Z_n(g)$. It follows that $|\{x \in \Gamma/\Delta_n : gxh^{-1} = x\}| \leq |Z_n(g)|$.

Lemma (1.2.29)[23]: Let F be a finite field. Then, for every $g \in \mathrm{PSL}(2, F)$ with $g \neq e$, one has $|Z(g)|/|\mathrm{PSL}(2, F)| \leq 2/(|F| - 1)$.

Proof. Since the characteristic polynomial of g is quadratic, it can be factorized in some quadratic extension $\tilde{F}$ of F. Thus g is conjugate to a Jordan normal form in $\mathrm{PSL}(2, \tilde{F})$. Now, it is not hard to see that the centralizer of g in $\mathrm{PSL}(2, \tilde{F})$ has cardinality at most $|\tilde{F}| = |F|^2$. On the other hand, it is well-known that $|\mathrm{SL}(2, F)| = |F|(|F|^2 - 1)$.

Corollary (1.2.30)[23]: Let $\Gamma_i = \mathrm{PSL}(2, \mathbb{Z}[\sqrt{2}])$ and $p_1 < p_2 < \cdots$ be prime numbers. Let $\Gamma = \Gamma_1 \times \Gamma_2$ act on $X = \lim_{\leftarrow} \mathrm{PSL}\big(2, (\mathbb{Z}/p_1 \cdots p_n\mathbb{Z})[\sqrt{2}]\big)$ by the left-and-right translation action. Let $\Lambda \curvearrowright Y$ be any free ergodic p.m.p. action of a residuallyfinite group Λ and suppose that $L^\infty(X) \rtimes \Gamma \cong (L^\infty(Y) \rtimes \Lambda)^t$ for some $t > 0$. Then, $t \in \mathbb{Q}$ and the actions $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are strongly virtually isomorphic.

Proof. Since $\mathrm{SL}(2, \mathbb{Z}[\sqrt{2}])$ is an irreducible lattice in $\mathrm{SL}(2, \mathbb{R})^2$, it has property (τ) (see Section 4.3 in [43]) and the property $(\mathrm{HH})^+$(cf. Theorem (1.2.7)). By the above lemmas, the action $\Gamma \curvearrowright X$ is essentially-free. Indeed, consider the homomorphism from $\mathrm{PSL}\big(2, (\mathbb{Z}/p_1 \cdots p_n\mathbb{Z})[\sqrt{2}]\big)$ onto $\mathrm{PSL}(2, F)$, where F is the field either $\mathbb{Z}/p_n\mathbb{Z}$ or $(\mathbb{Z}/p_n\mathbb{Z})[\sqrt{2}]$, depending on whether the equation $x^2 = 2$ is solvable in $\mathbb{Z}/p_n\mathbb{Z}$ or not; and apply Lemma (1.2.29) at $\mathrm{PSL}(2, F)$. Therefore, by Corollary (1.2.2), $L^\infty(X)$ is the unique Cartan subalgebra of $L^\infty(X) \rtimes \Gamma$. It follows that the isomorphism of von Neumann algebras $L^\infty(X) \rtimes \Gamma \cong (L^\infty(Y) \rtimes \Lambda)^t$ gives rise to a stable orbit equivalence between $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$. The growth condition of Theorem (1.2.27) is satisfied because p_k 's are mutually distinct primes and $\mathrm{PSL}\big(2, (\mathbb{Z}/p_1 \cdots p_n\mathbb{Z})[\sqrt{2}]\big) \cong \prod \mathrm{PSL}\big(2, (\mathbb{Z}/p_k\mathbb{Z})[\sqrt{2}]\big)$. Therefore, Theorem (1.2.27) is applicable to the orbit equivalence cocycle $\alpha: \Gamma \times X \to \Lambda$. For the rest of the proof, see [42].

Theorem (1.2.31)[23]: Let $\Gamma \curvearrowright X$ be a free ergodic p.m.p. action of a discrete group Γ having an infinite normal abelian subgroup H satisfying the relative ICC condition. Assume that $H \curvearrowright X$ is ergodic and profinite. Then, both $L^\infty(X)$ and $L(H)$ are Cartan subalgebras of $L^\infty(X) \rtimes \Gamma$. Assume moreover that $\Gamma \curvearrowright X$ is profinite and there is no $H \rtimes \Gamma$-invariant mean on $\ell^\infty(H)$. Then, the Cartan subalgebras $L^\infty(X)$ and $L(H)$ are non-conjugate.

Proof. Since $H \curvearrowright^\sigma X$ is an ergodic and profinite action, one has $X = \lim_{\leftarrow} H/H_n$ for some decreasing sequence $H = H_0 \supset H_1 \supset \cdots$ of finite-index subgroups of H such that $\bigcap H_n = \{e\}$. Recall that a function f is called an eigenfunction of H if there is a character χ on H such that $\sigma_h(f) = \chi(h)f$ for every $h \in H$. We observe that every unitary eigenfunction normalizes $L(H)$ in $L^\infty(X) \rtimes H$, and that $L^\infty(X)$ is spanned by unitary eigenfunctions since $L^\infty(H/H_n)$ is spanned by characters. This proves that $L(H)$ is regular in $L^\infty(X) \rtimes \Gamma$. To prove that $L(H)$ is maximal abelian, let $a \in L(H)' \cap L^\infty(X) \rtimes \Gamma$ be given and $a = \sum_{g \in \Gamma} a_g u_g$ be the Fourier expansion. Then, $[a, u_h] = 0$ implies $\sigma_h(a_g) = a_{hgh^{-1}}$ for all $g \in \Gamma$ and $h \in H$. In particular, one has $\|a_{hgh^{-1}}\|_2 = \|a_g\|_2$. Since $\sum_g \|a_g\|_2^2 = \| a \|_2^2 < \infty$, the relative ICC condition implies that $a_g = 0$ for all $g \notin H$. But for $g \in H$, ergodicity of $H \curvearrowright X$ implies that $a_g \in \mathbb{C}1$. This proves $a \in L(H)$.

For the second assertion, recall that weak compactness is an invariant of a Cartan subalgebra (Proposition 3.4 in [48]). We prove that $L(H)$ is not weakly compact in $L^\infty(X) \rtimes$

Γ. Suppose by contradiction that it is weakly compact. Then, by Proposition 3.2 in [48], there is a state φ on $\mathbb{B}(\ell^2(H))$ which is invariant under the $H \rtimes \Gamma$-action. Restricting φ to $\ell^\infty(H)$, we obtain an $H \rtimes \Gamma$-invariant mean. This contradicts the assumption.

Corollary (1.2.4) is an immediate consequence of Theorem (1.2.31). Here we give another example for which Theorem (1.2.31) applies. Let K be a residually-finite additive group such that $|K| > 1$, and Γ_0 be a residually-finite non-amenable group. The wreath product $\Gamma = K \setminus \Gamma_0$ is defined to be the semidirect product of $H = \bigoplus_{\Gamma_0} K$ by the shift action of Γ_0. Then, there is a decreasing sequence $H_0 \supset H_1 \supset \cdots$ of Γ_0-invariant finite-index subgroups of H such that $\bigcap H_n = \{0\}$. Indeed, let $K_0 \supset K_1 \supset \cdots$ (resp. $\Gamma_{0,0} \supset \Gamma_{0,1} \supset \cdots$) be finite-index subgroups of K (resp. Γ_0) such that $\bigcap K_n = \{0\}$ (resp. $\cap\, \Gamma_{0,n} = \{e\}$). Then, the "augmentation subgroups"

$$H_n = \left\{ \left(a_g\right)_{g\in\Gamma_0} \in H \colon \sum_{h\in\Gamma_{0,n}} a_{gh} \in K_n \text{ for all } g \in \Gamma_0 \right\},$$

which is the kernel of the homomorphism onto $\bigoplus_{\Gamma_0/\Gamma_{0,n}} K/K_n$, satisfy the required conditions. It follows from Theorem (1.2.31) that the II $_1$-factor

$$L^\infty\left(\lim_{\leftarrow} H/H_n\right) \rtimes \Gamma$$

has two non-conjugate Cartan subalgebras, namely $L(H)$ and $L^\infty(\lim_{\leftarrow} H/H_n)$.

Chapter 2
Non-Commutative and Classical Aspects with Non-Injectivity

We show that a large class of q-Gaussian processes possess a non-commutative kind of Markov property, which ensures that there exist classical versions of these non-commutative processes. This answers an old question of Frisch and Bourret [77]. We show that the proof works for the more general setting of a Yang-Baxter deformation. The techniques can also be extended to the so called q-Araki-Woods von Neumann algebras recently introduced by Hiai. We obtain the non injectivity under some asssumption on the spectral set of the positive operator asociated with the deformation.

Section (2.1): q-Gaussian Processes

What we are going to call q-Gaussian processes was essentially introduced by Frisch and Bourret [77]. They considered generalized commutation relations given by operators $A(t)$ and a vacuum vector Ψ_0 with

$$A(t)A^*(t') - qA^*(t')A(t) = \Gamma(t,t')1$$

and

$$A(t)\Psi_0 = 0$$

for some real covariance function Γ (i.e. positive definite function). They study the probabilistic properties of the 'parastochastic' process $M(t) = A(t) + A^*(t)$.

The basic problems arising were the following two types of questions:

1. (realization problem)
   Do there exist operators on some Hilbert space and a corresponding vacuum vector in this Hilbert space which fullfill the above relations, i.e. are there non-commutative realizations of the q-Gaussian processes.
2. (random representation problem)
   Are these non-commutative processes of a classical relevance, i.e. do there exist classical versions of the q-Gaussian processes (in the sense of coinciding time-ordered correlations, see Definition (2.1.33)).

Frisch and Bourret could give the following partial answers to these questions.

1. For $q = \pm 1$ the realization is of course given by the Fock space realization of the bosonic/fermionic relations. The case $q = 0$ was realized by creation and annihilation operators on the full Fock space (note that this was before the introduction of the Cuntz algebras and their extensions [72],[75]). For other values of q the realization problem remained open.
2. The $q = 1$ processes are nothing but the Fock space representations of the classical Gaussian processes. For $q = -1$ a classical realization by a dichotomic Markov process could be given for the special case of exponential covariance $\Gamma(t,t') = \exp(-|t - t'|)$. A classical realization for $q = 0$ could not be found, but they were able to show that there is an interesting representation in terms of Gaussian random matrices.

Starting with [61] there has been another and independent approach to noncommutative probability theory. This wide and quite inhomogenous field - let us just mention as two highlights the quantum stochastic calculus of Hudson-Parthasarathy [82] and the free probability theory of Voiculescu [19] - is now known under the name of 'quantum probability'. At least some of the fundamental motivation for undertaking such investigations can be compared with the two basic questions of Frisch and Bourret:

1. Non-commutative probability theory is meant as a generalization of classical probability theory to the description of quantum systems. Thus first of all their objects are operators on some Hilbert spaces having a meaning as non-commutative anlogues of the probabilistic notions of random variables, stochastic processes, etc.
2. In many investigations in this area one also tries to establish connections between non-commutative and classical concepts. The aim of this is twofold. On one side, one hopes to get a better understanding of classical problems by embedding them into a bigger non-commutative context. Thus, e.g., the Azéma martingale, although classically not distinguished within the class of all martingales, behaves in some respects like a Brownian motion [97]. The non-commutative 'explanation' for this fact comes from the observation of Schürmann [100] that this martingale is one component of a noncommutative process with independent increments. In the other direction, one hopes to get a classical picture (featuring trajectories) of some aspects of quantum problems. A total reduction to classical concepts is in general not possible, but partial aspects may sometimes allow a classical interpretation.

It was of quantum probability where two [66] reintroduced the q-relations - without knowing of, but much in the same spirit as [77]. Around the same time the q-relations were also proposed by Greenberg [79] as an example for particles with 'infinite statistics'.

The main progress in connection with this renewed interest was the solution of the realization problem of Frisch and Bourret. There exist now different proofs for the existence of the Fock representation of the q-relations for all q with $-1 \leq q \leq 1$ [66],[112],[76],[103],[68],[111].

In [96], the idea of Frisch and Bourret to use the q-relations as a model for a generalized noise was pursued further and the Greens function for such dynamical problems could be calculated for one special choice of the covariance function namely for the case of the exponential covariance. We will call this special q process in the following q-Ornstein-Uhlenbeck process. It soon became clear that the special status of the exponential covariance is connected with some kind of (noncommutative) Markovianity - as we will see the q-Ornstein-Uhlenbeck process is the only stationary q-Gaussian Markov process. But using the general theory of Kümmerer on non-commutative stationary Markov processes [87],[88] this readily implies the existence of a classical version (being itself a classical Markov process) of the q-Ornstein-Uhlenbeck process.

Thus we got a positive solution of the random representation problem of Frisch and Bourret in this case. However, the status of the other q-Gaussian processes, in particular q-Brownian motion, remained unclear.

Motivated by our preliminary results, Biane [63] (see also [64],[65]) undertook a deep and beautiful analysis of the free ($q = 0$) case and showed the remarkable result that all processes with free increments are Markovian and thus possess classical versions (with a quite explicit calculation rule for the corresponding transition probabilities). This includes in particular the case of free Brownian motion.

Inspired by this work we could extend our investigations from the case of the q-Ornstein-Uhlenbeck process to all q-Gaussian processes. The results are presented.

Up to now there is only one strategy for establishing the existence of a classical version of a non-commutative process, namely by showing that the process is Markovian. That this implies the existence of a classical version follows by general arguments, the main point is to show that we have this property in the concrete case. Whereas Biane could use the quite developed theory of freeness [19] to prove Markovianity for processes with free increments, there is at the moment (and probably also in the future [104]) no kind of q-freeness for general q. Thus another feature of our considered class of processes is needed to attack the problem of Markovianity. It is the aim that the q-analogue of Gaussianity will do this job.

The essential idea of Gaussianity is that one can pull back all considerations from the measure theoretic (or, in the non-commutative frame, from the operator algebraic) level to an underlying Hilbert space, thus in the end one essentially has to deal with linear problems. The main point is that this transcription between the linear and the algebraic level exists in a consistent way. The best way to see and describe this is by presenting a functor ('second quantization') which translates the Hilbert space properties into operator algebraic properties. Our basic considerations will therefore be on the existence and nice properties of the q-analogue of this functor. Having this functor, the rest is mainly linear theory on Hilbert space level. It turns out that all relevant questions on our q-Gaussian processes can be characterized totally in terms of the corresponding covariance function. In particular, it becomes quite easy to decide whether such a process is Markovian or not.

We remind of some basic facts about the q-Fock space and its relevant operators. Furthermore we collect the needed combinatorial results, in particular on q-Hermite polynomials. We devoted to the presentation of the functor Γ_q of second quantization. The main results (apart from the existence of this object) are the facts that the associated von Neumann algebras are in the infinite dimensional case non-injective II_1-factors and that the functor maps contractions into completely positive maps. Having this q-Gaussian functor the definition and investigation of properties of q Gaussian processes (like Markovianity or martingale property) is quite canonical and parallels the classical case. Thus our presentation of these aspects, we will be quite condensed. We contain the classical interpretation of the q Gaussian Markov processes. As pointed out above general arguments ensure the existence of classical versions for these processes. But we will see that we can also derive quite concrete formulas for the corresponding transition probabilities.

Let $q \in (-1,1)$ be fixed in the following. For a complex Hilbert space $\mathcal{H}$ we define its q-Fock space $\mathcal{F}_q(\mathcal{H})$ as follows:

Let $\mathcal{F}^{\text{finite}}(\mathcal{H})$ be the linar span of vectors of the form $f_1 \otimes \cdots \otimes f_n \in \mathcal{H}^{\otimes n}$ (with varying $n \in \mathbb{N}_0$), where we put $\mathcal{H}^{\otimes 0} \cong \mathbb{C}\Omega$ for some distinguished vector Ω, called vacuum. On $\mathcal{F}^{\text{finite}}(\mathcal{H})$ we consider the sesquilinear form $\langle \cdot,\cdot \rangle_q$ given by sesquilinear extension of

$$\langle f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_m \rangle_q := \delta_{nm} \sum_{\pi \in S_n} q^{i(\pi)} \langle f_1, g_{\pi(1)} \rangle \dots \langle f_n, g_{\pi(n)} \rangle,$$

where S_n denotes the symmetric group of permutations of n elements and $i(\pi)$ is the number of inversions of the permutation $\pi \in S_n$ defined by

$$i(\pi) := \#\{(i,j) \mid 1 \le i < j \le n, \pi(i) > \pi(j)\}.$$

Another way to describe $\langle \cdot,\cdot \rangle_q$ is by introducing the operator P_q on $\mathcal{F}^{\text{finite}}(\mathcal{H})$ by linear extension of

$$P_q \Omega = \Omega$$

$$P_q f_1 \otimes \cdots \otimes f_n = \sum_{\pi \in S_n} q^{i(\pi)} f_{\pi(1)} \otimes \cdots \otimes f_{\pi(n)}.$$

Then we can write

$$\langle \xi, \eta \rangle_q = \langle \xi, P_q \eta \rangle_0 \ \left(\xi, \eta \in \mathcal{F}^{f \text{ inite}}(\mathcal{H})\right),$$

where $\langle\cdot,\cdot\rangle_0$ is the scalar product on the usual full Fock space

$$\mathcal{F}_0(\mathcal{H}) = \bigoplus_{n\geq 0} \mathcal{H}^{\otimes n}.$$

One of the main results of [66] (see also [68],[76],[103],[112]) was the strict positivity of P_q, i.e. $\langle\xi,\xi\rangle_q > 0$ for $0 \neq \xi \in \mathcal{F}^{\text{finite}}(\mathcal{H})$. This allows the following definitions.

Definitions (2.1.1)[60]: a) The q-Fock space $\mathcal{F}_q(\mathcal{H})$ is the completion of $\mathcal{F}^{\text{finite}}(\mathcal{H})$ with respect to $\langle\cdot,\cdot\rangle_q$.

b) Given $f \in \mathcal{H}$, we define the creation operator $a^*(f)$ and the annihilation operator $a(f)$ on $\mathcal{F}_q(\mathcal{H})$ by

$$\begin{aligned} a^*(f)\Omega &= f \\ a^*(f)f_1 \otimes \cdots \otimes f_n &= f \otimes f_1 \otimes \cdots \otimes f_n \end{aligned}$$

and

$$a(f)\Omega = 0$$

$$a(f)f_1 \otimes \cdots \otimes f_n = \sum_{i=1}^{n} q^{i-1}\langle f, f_i\rangle f_1 \otimes \cdots \otimes \check{f_i} \otimes \cdots \otimes f_n,$$

where the symbol $\check{f_i}$ means that f_i has to be deleted in the tensor.

Notation (2.1.2)[60]: For a linear operator $T: \mathcal{H} \to \mathcal{H}'$ between two complex Hilbert spaces we denote by $\mathcal{F}(T): \mathcal{F}^{\text{finite}}(\mathcal{H}) \to \mathcal{F}^{\text{finite}}(\mathcal{H}')$ the linear extension of

$$\begin{aligned} \mathcal{F}(T)\Omega &= \Omega \\ \mathcal{F}(T)f_1 \otimes \cdots \otimes f_n &= (Tf_1) \otimes \cdots \otimes (Tf_n): \end{aligned}$$

In order to keep the notation simple we denote the vacuum for $\mathcal{H}$ and the vacuum for $\mathcal{H}'$ by the same symbol Ω.

It is clear that $\mathcal{F}(T)$ can be extended to a bounded operator $\mathcal{F}_0(T): \mathcal{F}_0(\mathcal{H}) \to \mathcal{F}_0(\mathcal{H}')$ exactly if T is a contraction, i.e. if $\| T \| \leq 1$. The following lemma ensures that the same is true for all other $q \in (-1,1)$, too.

Lemma (2.1.3)[60]: Let $\mathcal{T}: \mathcal{F}^{\text{finite}}(\mathcal{H}) \to \mathcal{F}^{\text{finite}}(\mathcal{H}')$ be a linear operator which fulfills $P_q'\mathcal{T} = \mathcal{T}P_q$, where P_q and P_q' are the operators on $\mathcal{F}^{\text{finite}}(\mathcal{H})$ and $\mathcal{F}^{\text{finite}}(\mathcal{H}')$, respectively, which define the respective scalar product $\langle\cdot,\cdot\rangle_q$. Then one has $\| \mathcal{T} \|_q = \| \mathcal{T} \|_0$. Hence, if $\| \mathcal{T} \|_0 < \infty$ then $\mathcal{T}$ can, for each $q \in (-1,1)$, be extended to a bounded operator from $\mathcal{F}_q(\mathcal{H})$ to $\mathcal{F}_q(\mathcal{H}')$.

Proof. Let $\xi \in \mathcal{F}^{\text{finite}}(\mathcal{H})$. Then

$$\begin{aligned} \| \mathcal{T}\xi \|_q^2 &= \langle \mathcal{T}\xi, \mathcal{T}\xi \rangle_q \\ &= \left\langle \mathcal{T}\xi, P_q'\mathcal{T}\xi \right\rangle_0 \\ &= \left\langle P_q^{1/2}\xi, \mathcal{T}^*\mathcal{T}P_q^{1/2}\xi \right\rangle_0 \\ &\leq \|\mathcal{T}^*\mathcal{T}\|_0 \left\langle P_q^{1/2}\xi, P_q^{1/2}\xi \right\rangle_0 \\ &= \|\mathcal{T}^*\mathcal{T}\|_0 \|\xi\|_q^2, \end{aligned}$$

which implies

$$\| \mathcal{T} \|_q^2 \leq \|\mathcal{T}^*\mathcal{T}\|_0 \leq \|\mathcal{T}^*\|_0 \, \| \mathcal{T} \|_0 = \| \mathcal{T} \|_0^2,$$

and thus $\| \mathcal{T} \|_q \leq \| \mathcal{T} \|_0$. Since we can estimate in the same way, by replacing P_q by P_q^{-1} and P_q' by $P_q' - 1$, also $\| \mathcal{T} \|_0 \leq \| \mathcal{T} \|_q$, we get the assertion.

Notation (2.1.4)[60]: For a contraction $T: \mathcal{H} \to \mathcal{H}'$, we denote the extension of $\mathcal{F}(T)$ from $\mathcal{F}^{\text{finite}}(\mathcal{H}) \to \mathcal{F}^{\text{finite}}(\mathcal{H}')$ to $\mathcal{F}_q(\mathcal{H}) \to \mathcal{F}_q(\mathcal{H}')$ by $\mathcal{F}_q(T)$.

The q-relations one usually encounters some kind of q-combinatorics. Let us just remind of the basic facts.

Notations (2.1.5)[60]: We put for $n \in \mathbb{N}_0$

$$[n]_q := \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1} \left([0]_q := 0\right).$$

Then we have the q-factorial

$$[n]_q! := [1]_q \dots [n]_q, \; [0]_q! := 1$$

and a q-binomial coefficient

$$\binom{n}{k}_q := \frac{[n]_q!}{[k]_q!\,[n-k]_q!} = \prod_{i=1}^{n-k} \frac{1-q^{k+i}}{1-q^i}.$$

Another quite frequently used symbol is the q-analogue of the Pochhammer symbol

$$(a;q)_n := \prod_{j=0}^{n-1} \left(1 - aq^j\right) \text{ in particular } (a;q)_\infty := \prod_{j=0}^{\infty} \left(1 - aq^j\right).$$

The importance of these concepts in connection with the q-relations can be seen from the following q-binomial theorem, which is by now quite standard.

Proposition (2.1.6)[60]: Let x and y be indeterminates which q-commute in the sense $xy = qyx$. Then one has for $n \in \mathbb{N}$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k}_q y^k x^{n-k}.$$

Proof. This is just induction and the easily checked equality

$$\binom{n}{k}_q + q^k \binom{n}{k+1}_q = \binom{n+1}{k+1}_q.$$

In the same way as the usual Hermite polynomials are connected to the bosonic relations, the q-relations are linked to q-analogues of the Hermite polynomials.

Definition (2.1.7)[60]: The polynomials $H_n^{(q)} (n \in \mathbb{N}_0)$, determined by

$$H_0^{(q)}(x) = 1, \; H_1^{(q)}(x) = x$$

and

$$xH_n^{(q)}(x) = H_{n+1}^{(q)}(x) + [n]_q H_{n-1}^{(q)}(x) \; (n \geq 1)$$

are called q-Hermite polynomials.

We recall two basic facts about these polynomials which will be fundamental for our investigations on the classical aspects of q-Gaussian processes.

Theorem (2.1.8)[60]: a) Let ν_q be the measure on the interval $[-2/\sqrt{1-q}, 2/\sqrt{1-q}]$ given by

$$\nu_q(dx) = \frac{1}{\pi}\sqrt{1-q}\sin\theta \prod_{n=1}^{\infty} (1-q^n)\left|1 - q^n e^{2i\theta}\right|^2 dx,$$

where

$$x = \frac{2}{\sqrt{1-q}}\cos\theta \text{ with } \theta \in [0, \pi].$$

Then the q-Hermite polynomials are orthogonal with respect to ν_q, i.e.

$$\int_{-2/\sqrt{1-q}}^{2/\sqrt{1-q}} H_n(x)H_m(x)\nu_q(dx) = \delta_{nm}[n]_q!.$$

b) Let $r > 0$ and $x, y \in [-2/\sqrt{1-q}, 2/\sqrt{1-q}]$. Denote by $p_r^{(q)}(x,y)$ the kernel

$$p_r^{(q)}(x,y) := \sum_{n=0}^{\infty} \frac{r^n}{[n]_q!} H_n^{(q)}(x)H_n^{(q)}(y).$$

Then we have with

$$x = \frac{2}{\sqrt{1-q}}\cos\varphi, \qquad y = \frac{2}{\sqrt{1-q}}\cos\psi$$

the formula

$$p_r^{(q)}(x,y) = \frac{(r^2;q)_\infty}{|(re^{i(\varphi+\psi)};q)_\infty(re^{i(\varphi-\psi)};q)_\infty|^2}.$$

In particular, for $q = 0$, we get

$$p_r^{(0)}(x,y) = \frac{1-r^2}{(1-r^2)^2 - r(1+r^2)xy + r^2(x^2+y^2)}.$$

As usually in q-mathematics these formulas are quite old, namely the orthogonalizing measure ν_q was calculated by Szego [106], whereas the kernel $p_r^{(q)}(x,y)$ goes even back to Rogers [99]. See [70],[83],[78],[91].

An abstract way of dealing with classical Gaussian processes is by using the Gaussian functor Γ. This is a functor from real Hilbert spaces and contractions to commutative von Neumann algebras with specified trace-state and unital trace preserving completely positive maps [94],[95],[80],[101],[102]. A fermionic analogue of this functor is also known, see, e.g., [110],[71].

We will present a q-analogue of the Gaussian functor. Namely, to each real Hilbert space, $\mathcal{H}$, we will associate a von Neumann algebra with specified trace-state, $(\Gamma_q(\mathcal{H}), \mathrm{E})$, and to every contraction $T: \mathcal{H} \to \mathcal{H}'$ a unital completely positive trace preserving map $\Gamma_q(T): \Gamma_q(\mathcal{H}) \to \Gamma_q(\mathcal{H}')$.

Definition (2.1.9)[60]: Let $\mathcal{H}$ be a real Hilbert space and $\mathcal{H}_\mathbb{C}$ its complexification $\mathcal{H}_\mathbb{C} = \mathcal{H} \oplus i\mathcal{H}$. Put, for $f \in \mathcal{H}$,

$$\omega(f) := a(f) + a^*(f) \in B\left(\mathcal{F}_q(\mathcal{H}_\mathbb{C})\right)$$

and denote by $\Gamma_q(\mathcal{H}) \subset B\left(\mathcal{F}_q(\mathcal{H}_\mathbb{C})\right)$ the von Neumann algebra generated by all $\omega(f)$

$$\Gamma_q(\mathcal{H}) := \mathrm{vN}(a(f) + a^*(f) \mid f \in \mathcal{H}).$$

Notation (2.1.10)[60]: We denote by

$$\mathrm{E}: \Gamma_q(\mathcal{H}) \to \mathbb{C}$$

the vacuum expectation state on $\Gamma_q(\mathcal{H})$ given by

$$\mathrm{E}[X] := \langle \Omega, X\Omega \rangle_q \ \left(X \in \Gamma_q(\mathcal{H})\right).$$

We remind of some basic facts about $\Gamma_q(\mathcal{H})$ in the following proposition.

Proposition (2.1.11)[60]: The vacuum Ω is a cyclic and separating trace-vector for $\Gamma_q(\mathcal{H})$, hence the vacuum expectation E is a faithful normal trace on $\Gamma_q(\mathcal{H})$ and $\Gamma_q(\mathcal{H})$ is a finite von Neumann algebra in standard form.

Proof. See Theorems 4.3 and 4.4 in [68].

The first part of the proposition yields in particular that the mapping

$$\Gamma_q(\mathcal{H}) \to \mathcal{F}_q(\mathcal{H}_{\mathbb{C}})$$
$$X \mapsto X\Omega$$
is injective, in this way we can identify each $X \in \Gamma_q(\mathcal{H})$ with some element of the q-Fock space $\mathcal{F}_q(\mathcal{H}_{\mathbb{C}})$.

Notations (2.1.12)[60]: a) Let us denote by
$$L_q^\infty(\mathcal{H}) := \Gamma_q(\mathcal{H})\Omega$$
the image of $\Gamma_q(\mathcal{H})$ under the mapping $X \mapsto X\Omega$.

b) We also put
$$L_q^2(\mathcal{H}) := \mathcal{F}_q(\mathcal{H}_{\mathbb{C}}).$$

Definition (2.1.13)[60]: Let $\Psi: L_q^\infty(\mathcal{H}) \to \Gamma_q(\mathcal{H})$ be the identification of $L_q^\infty(\mathcal{H})$ with $\Gamma_q(\mathcal{H})$ given by the requirement
$$\Psi(\xi)\Omega = \xi \ \text{ for } \ \xi \in L_q^\infty(\mathcal{H}) \subset L_q^2(\mathcal{H}) = \mathcal{F}_q(\mathcal{H}_{\mathbb{C}}).$$

The explicit form of our Wick products is given in the following proposition.

Proposition (2.1.14)[60]: We have for $n \in \mathbb{N}$ and $f_1, \dots, f_n \in \mathcal{H}$ the normal ordered representation

$$\Psi(f_1 \otimes \cdots \otimes f_n) = $$
$$= \sum_{\substack{k,l=0,\dots,n \\ k+l=n}} \sum_{\substack{I_1=\{i(1),\dots,i(k)\} \\ I_2=\{j(1),\dots,j(l)\} \\ \text{with} \\ I_1 \cup I_2 = \{1,\dots,n\} \\ I_1 \cap I_2 = \emptyset}} a^*\left(f_{i(1)}\right) \dots a^*\left(f_{i(k)}\right) a\left(f_{j(1)}\right) \dots a\left(f_{j(l)}\right) \cdot q^{i(I_1,I_2)},$$

where
$$i(I_1, I_2) := \{(p,q) | 1 \le p \le k, 1 \le q \le l, i(p) > j(q)\}.$$

Denote by X the right hand side of the above relation. It is clear that $\Omega = f_1 \otimes \cdots \otimes f_n$, the problem is to see that X can be expressed in terms of the ω 's.

Proof. Note that the formula is true for
$$\Psi(f) = \omega(f) = a(f) + a^*(f)$$
and that the definition of $a^*(f)$ and of $a(f)$ gives

$$\Psi(f \otimes f_1 \otimes \cdots \otimes f_n)$$
$$= \omega(f)\Psi(f_1 \otimes \cdots \otimes f_n) - \sum_{i=1}^{n} q^{i-1} \langle f, f_i \rangle \Psi\left(f_1 \otimes \cdots \otimes \check{f_i} \otimes \cdots \otimes f_n\right).$$

¿ From this the assertion follows by induction.

Note that $\Psi(f_1 \otimes \cdots \otimes f_n)$ is just given by multiplying out $\omega(f_1) \dots \omega(f_n)$ and bring all appearing terms with the help of the relation $aa^* = qa^*a$ into a normal ordered form - i.e. we throw away all normal ordered terms in $\omega(f_1) \dots \omega(f_n)$ which have less than n factors. Thus, for the special case $f_1 = \cdots = f_n$, we are in the realm of the q-binomial theorem and we have the following nice formula.

Corollary (2.1.15)[60]: We have for $n \in \mathbb{N}$ and $f \in \mathcal{H}$
$$\Psi\left(f^{\otimes n}\right) = \sum_{k=0}^{n} \binom{n}{k}_q a^*(f)^k a(f)^{n-k}.$$

Instead of writing $\Psi\left(f^{\otimes n}\right)$ in a normal ordered form we can also express it in terms of $\omega(f)$ with the help of the q-Hermite polynomials.

Proposition (2.1.16)[60]: We have for $n \in \mathbb{N}_0$ and $f \in \mathcal{H}$ with $\| f \| = 1$ the representation

$$\Psi\left(f^{\otimes n}\right)=H_n^{(q)}\left(\omega(f)\right).$$

Proof. This follows by the fact that the $\Psi\left(f^{\otimes n}\right)$ fulfill the same recurrence relation as the $H_n^{(q)}(\omega(f))$, namely

$$\omega(f)\Psi\left(f^{\otimes n}\right)=\Psi\left(f^{\otimes(n+1)}\right)+[n]_q\Psi\left(f^{\otimes(n-1)}\right)$$

and that we have the same initial conditions

$$\Psi\left(f^{\otimes 0}\right)=1,\ \Psi\left(f^{\otimes 1}\right)=\omega(f).$$

We know [22], [19] that for $q=0$ the von Neumann algebra $\Gamma_0(\mathcal{H})$ is isomorphic to the von Neumann algebra of the free group on $\dim\mathcal{H}$ generators - in particular, it is a non-injective II_1-factor for $\dim\mathcal{H}\geq 2$. We conjecture non-injectivity and factoriality in the case $\dim\mathcal{H}\geq 2$ for arbitrary $q\in(-1,1)$, but up to now we can only show the following.

Theorem (2.1.17)[60]: i) For $-1<q<1$ and $\dim\mathcal{H}>16/(1-|q|)^2$ the von Neumann algebra $\Gamma_q(\mathcal{H})$ is not injective.

ii) If $-1<q<1$ and $\dim\mathcal{H}=\infty$ then $\Gamma_q(\mathcal{H})$ is a II_1-factor.

Proof. i) This was shown in a more general context in Theorem 4.2 in [68]. ii) Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$. Fix $n\in\mathbb{N}_0$ and $r(1),\dots,r(n)\in\mathbb{N}$ and consider the operator

$$X:=\Psi\left(e_{r(1)}\otimes\cdots\otimes e_{r(n)}\right).$$

(For $n=0$ this shall be understood as $X=1$.) We put

$$\phi_m(X):=\frac{1}{m}\sum_{i=1}^{m}\omega(e_i)X\omega(e_i)\qquad(m\in\mathbb{N})$$

and claim that $\phi_m(X)$ converges for $m\to\infty$ weakly to $\phi(X):=q^nX$. Because of the m-independent estimate

$$\|\phi_m(X)\|_q\leq\|X\|_q\ \|\omega(e_1)\|_q^2$$

it suffices to show

$$\lim_{m\to\infty}\langle\xi,\phi_m(X)\eta\rangle_q=\langle\xi,\phi(X)\eta\rangle_q$$

for all $\xi,\eta\in\mathcal{F}_q(\mathcal{H}_{\mathrm{C}})$ of the form

$$\xi=e_{a(1)}\otimes\cdots\otimes e_{a(u)},\ \eta=e_{b(1)}\otimes\cdots\otimes e_{b(v)}$$

with $u,v\in\mathbb{N}_0, a(1),\dots,a(u),b(1),\dots,b(v)\in\mathbb{N}$ (for $u=0$ we put $\xi=\Omega$). To see this, put

$$m_0:=\max\ \{a(1),\dots,a(u),b(1),\dots,b(v),r(1),\dots,r(n)\}.$$

Since $\left|\langle\xi,\omega(e_i)X\omega(e_i)\eta\rangle_q\right|\leq M$ for some M (independent of), we have

$$\begin{aligned}\lim_{m\to\infty}\langle\xi,\phi_m(X)\eta\rangle_q &=\lim_{m\to\infty}\frac{1}{m}\sum_{i=m_0+1}^{m}\langle\xi,\omega(e_i)X\omega(e_i)\eta\rangle_q\\ &=\lim_{m\to\infty}\frac{1}{m}\sum_{i=m_0+1}^{m}\left\langle\xi,a(e_i)\Psi\left(e_{r(1)}\otimes\cdots\otimes e_{r(n)}\right)a^*(e_i)\eta\right\rangle_q.\end{aligned}$$

By Proposition (2.1.14),$\Psi\left(e_{r(1)}\otimes\cdots\otimes e_{r(n)}\right)$ is now a linear combination of terms of the form $Y=Y_1Y_2$ with

$$Y_1=a^*\left(e_{r(i(1))}\right)\dots a^*\left(e_{r(i(k))}\right)\ \text{ and }\ Y_2=a\left(e_{r(j(1))}\right)\dots a\left(e_{r(j(l))}\right)$$

with $k+l=n$. Each such term gives, for $i>m_0$, a contribution

$$
\begin{aligned}
\langle \xi, a(e_i) Y a^*(e_i) \eta \rangle_q &= \langle \xi, a(e_i) Y_1 Y_2 a^*(e_i) \eta \rangle_q \\
&= q^{k+l} \langle \xi, Y_1 a(e_i) a^*(e_i) Y_2 \eta \rangle_q \\
&= q^n \big\langle \xi, Y_1 \big(1 + q a^*(e_i) a(e_i)\big) Y_2 \eta \big\rangle_q \\
&= q^n \langle \xi, Y_1 Y_2 \eta \rangle_q \\
&= q^n \langle \xi, Y \eta \rangle_q
\end{aligned}
$$

and hence

$$
\lim_{m\to\infty} \langle \xi, \phi_m(X) \eta \rangle_q = \lim_{m\to\infty} \frac{1}{m} \sum_{i=m_0+1}^{m} q^n \big\langle \xi, \Psi\big(e_{r(1)} \otimes \cdots \otimes e_{r(n)}\big) \eta \big\rangle_q = \langle \xi, q^n X \eta \rangle_q .
$$

Thus we have shown

$$
\mathrm{w}-\lim_{m\to\infty} \ \phi_m(X) = \phi(X).
$$

Let now tr be a normalized normal trace on $\Gamma_q(\mathcal{H})$. Then

$$
\begin{aligned}
\mathrm{tr}[\phi(X)] &= \lim_{m\to\infty} \mathrm{tr}[\phi_m(X)] \\
&= \lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} \ \mathrm{tr}[\omega(e_i) X \omega(e_i)] \\
&= \lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} \ \mathrm{tr}[X \omega(e_i) \omega(e_i)] \\
&= \mathrm{tr}\left[X \cdot \lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} \ \omega(e_i)\omega(e_i)\right] \\
&= \mathrm{tr}[X \phi(1)] \\
&= \mathrm{tr}[X] .
\end{aligned}
$$

Since $\phi^k(X) = q^{kn} X$ converges, for $k \to \infty$, (even in norm) to

$$
\mathrm{E}[X] \cdot 1 = \begin{cases} 0, & n \geq 1 \\ X = 1, & n = 0, \end{cases}
$$

we obtain

$$
\mathrm{tr}[X] = \lim_{k\to\infty} \mathrm{tr}[\phi^k(X)] = \mathrm{tr}\left[\lim_{k\to\infty} \phi^k(X)\right] = \mathrm{E}[X]\,\mathrm{tr}[1] = \mathrm{E}[X].
$$

Thus tr coincides on all operators of the form

$$
X = \Psi\big(e_{r(1)} \otimes \cdots \otimes e_{r(n)}\big) \ (n \in \mathbb{N}_0, r(1), \ldots, r(n) \in \mathbb{N})
$$

with our canonical trace E. Since the set of finite linear combinations of such operators X is weakly dense in $\Gamma_q(\mathcal{H})$, we get the uniqueness of a normalized normal trace on $\Gamma_q(\mathcal{H})$, which implies that $\Gamma_q(\mathcal{H})$ is a factor.

The second part of our q-Gaussian functor Γ_q assigns to each contraction: $\mathcal{H} \to \mathcal{H}'$ a map $\Gamma_q(T): \Gamma_q(\mathcal{H}) \to \Gamma_q(\mathcal{H}')$. The idea is to extend $\Gamma_q(T)\omega(f) = \omega(Tf)$ in a canonical way to all of $\Gamma_q(\mathcal{H})$. In general, the q-relations prohibit the extension as a homomorphism, i.e.

$$
\Gamma_q(T)\omega(f_1) \ldots \omega(f_n) \neq \omega(Tf_1) \ldots \omega(Tf_n) \ \text{ in general} .
$$

But what can be done is to demand the above relation for the normal ordered form, i.e.

$$
\big(\Gamma_q(T) X\big)\Omega = \mathcal{F}_q(T)(X\Omega).
$$

Thus our second quantization $\Gamma_q(T)$ is the restriction of $\mathcal{F}_q(T)$ from $\mathcal{F}_q(\mathcal{H}) = L_q^2(\mathcal{H})$ to $\Gamma_q(\mathcal{H}) \cong L_q^\infty(\mathcal{H})$ and the question on the existence of $\Gamma_q(T)$ amounts to the problem whether $\mathcal{F}_q(T)\left(L_q^\infty(\mathcal{H})\right) \subset L_q^\infty(\mathcal{H}')$. We know that $\mathcal{F}_q(T)$ can be defined for T a contraction and we will see in the next theorem that no extra condition is needed to ensure its nice behaviour with respect to L_q^∞. The case $q = 0$ is due to Voiculescu [22],[19].

Theorem (2.1.18)[60]: a) Let $T: \mathcal{H} \to \mathcal{H}'$ be a contraction between real Hilbert spaces. There exists a unique map $\Gamma_q(T): \Gamma_q(\mathcal{H}) \to \Gamma_q(\mathcal{H}')$ such that

$$\left(\Gamma_q(T)X\right)\Omega = \mathcal{F}_q(T)(X\Omega).$$

The map $\Gamma_q(T)$ is linear, bounded, completely positive, unital and preserves the canonical trace E.

b) If T is isometric, then $\Gamma_q(T)$ is a faithful homomorphism, and if T is the orthogonal projection onto a subspace, then $\Gamma_q(T)$ is a conditional expectation.

Proof. Uniqueness of $\Gamma_q(T)$ follows from the fact that Ω is separating for $\Gamma_q(\mathcal{H}')$. To prove the existence and the properties of $\Gamma_q(T)$ we notice that any contraction T can be factored [81] as $T = POI$ where

1. $I: \mathcal{H} \to \mathcal{K} = \mathcal{H} \oplus \mathcal{H}'$ is an isometric embedding
2. $O: \mathcal{K} \to \mathcal{K}$ is orthogonal
3. $P: \mathcal{K} = \mathcal{H} \oplus \mathcal{H}' \to \mathcal{H}'$ is an orthogonal projection onto a subspace.

Thus if we prove our assertions for each of these three cases then we will also get the general statement for $\Gamma_q(T) = \Gamma_q(P)\Gamma_q(O)\Gamma_q(I)$.

a) Let $I: \mathcal{H} \to \mathcal{K} = \mathcal{H} \oplus \mathcal{H}'$ be an isometric embedding and $Q: \mathcal{K} \to \mathcal{K}$ the orthogonal projection onto $\mathcal{H}$. Then $\mathcal{F}_q(Q)$ is a projection in $\mathcal{F}_q(\mathcal{K}_\mathbb{C})$ and $\mathcal{F}_q(\mathcal{H}_\mathbb{C})$ can be identified with $\mathcal{F}_q(Q)\mathcal{F}_q(\mathcal{K}_\mathbb{C})$. Let us denote by $\omega_\mathcal{K}(f)$ the sum of creation and annihilation operator on $\mathcal{F}_q(\mathcal{K}_\mathbb{C})$. If we put

$$\Gamma_q^\mathcal{K}(\mathcal{H}) := \mathrm{vN}(\omega_\mathcal{K}(f) \mid f \in \mathcal{H}) \subset B\left(\mathcal{F}_q(\mathcal{K}_\mathbb{C})\right),$$

then

$$\Gamma_q^\mathcal{K}(\mathcal{H})\mathcal{F}_q(\mathcal{H}_\mathbb{C}) \subset \mathcal{F}_q(\mathcal{H}_\mathbb{C})$$

and we have the canonical identification

$$\Gamma_q(\mathcal{H}) \cong \Gamma_q^\mathcal{K}(\mathcal{H})\mathcal{F}_q(Q),$$

which gives a homomorphism (and thus a completely positive)

$$\Gamma_q(I): \Gamma_q(\mathcal{H}) \to \Gamma_q(\mathcal{K}).$$

Faithfulness is clear since $\mathcal{F}_q(Q)\Omega = \Omega$ and Ω separating. This yields also that the trace is preserved.

b) Let $P: \mathcal{K} = \mathcal{H} \oplus \mathcal{H}' \to \mathcal{H}'$ be an orthogonal projection, i.e. $PP^* = 1_{\mathcal{H}'}$, where $P^*: \mathcal{H}' \to \mathcal{K}$ is the canonical inclusion. Then

$$\Gamma_q(P)X := \mathcal{F}_q(P)X\mathcal{F}_q(P^*) \qquad \left(X \in \Gamma_q(\mathcal{K})\right)$$

gives the right operator, because we have for $k, l \in \mathbb{N}_0$ and $f_1, \dots, f_k, g_1, \dots, g_l \in \mathcal{K}$

$$\begin{aligned}\mathcal{F}_q(P)a^*(f_1) \dots a^*(f_k)a \quad a(g_1) \dots a(g_l)\mathcal{F}_q(P^*) &= \\ &= a^*(Pf_1) \dots a^*(Pf_k)\mathcal{F}_q(P)\mathcal{F}_q(P^*)a(Pg_1) \dots a(Pg_l) \\ &= a^*(Pf_1) \dots a^*(Pf_k)a(Pg_1) \dots a(Pg_l).\end{aligned}$$

By its concrete form, $\Gamma_q(P)$ is a conditional expectation and

$$\mathrm{E}\left[\mathcal{F}_q(P)X\mathcal{F}_q(P^*)\right] = \left\langle \mathcal{F}_q(P^*)\Omega, X\mathcal{F}_q(P^*)\Omega \right\rangle_q = \langle \Omega, X\Omega \rangle_q = \mathrm{E}[X]$$

shows that it preserves the trace.

c) Let $O: \mathcal{K} \to \mathcal{K}$ be orthogonal, i.e. $OO^* = O^*O = 1_{\mathcal{K}}$. Then, as in b),

$$\Gamma_q(O)X = \mathcal{F}_q(O)X\mathcal{F}_q(O^*),$$

which is, by

$$\mathcal{F}_q(O^*)\mathcal{F}_q(O) = \mathcal{F}_q(1_{\mathcal{K}}) = 1_{\mathcal{F}_q(\mathcal{K}_{\mathbb{C}})}$$

also a faithful homomorphism.

Instead of working on the level of von Neumann algebras we could also consider the C^*-analogues of the above constructions. This would be quite similar. We just indicate the main points.

Definition (2.1.19)[60]: Let $\mathcal{H}$ be a real Hilbert space and $\mathcal{H}_{\mathbb{C}}$ its complexification $\mathcal{H}_{\mathbb{C}} = \mathcal{H} \oplus i\mathcal{H}$. Put, for $f \in \mathcal{H}$,

$$\omega(f) := a(f) + a^*(f) \in B\left(\mathcal{F}_q(\mathcal{H}_{\mathbb{C}})\right)$$

and denote by $\Phi_q(\mathcal{H}) \subset B\left(\mathcal{F}_q(\mathcal{H}_{\mathbb{C}})\right)$ the C^*-algebra generated by all $\omega(f)$,

$$\Phi_q(\mathcal{H}) := C^*(a(f) + a^*(f) \mid f \in \mathcal{H}).$$

Clearly, the vacuum is also a separating trace-vector for $\Phi_q(\mathcal{H})$, it is also cyclic and $\Psi(f_1 \otimes \cdots \otimes f_n) \in \Phi_q(\mathcal{H})$ for all $n \in \mathbb{N}_0$ and all $f_1, \ldots, f_n \in \mathcal{H}$.

The most important fact for our latter considerations is that $\Gamma_q(T)$ can also be restricted to the C^*-level.

Theorem (2.1.20)[60]: a) Let $T: \mathcal{H} \to \mathcal{H}'$ be a contraction between real Hilbert spaces. There exists a unique map $\Phi_q(T): \Phi_q(\mathcal{H}) \to \Phi_q(\mathcal{H}')$ such that

$$\left(\Phi_q(T)X\right)\Omega = \mathcal{F}_q(T)(X\Omega).$$

The map $\Phi_q(T)$ is linear, bounded, completely positive, unital and preserves the canonical trace E.

b) If T is isometric, then $\Phi_q(T)$ is a faithful homomorphism, and if T is the orthogonal projection onto a subspace, then $\Phi_q(T)$ is a conditional expectation.

c) We have $\Phi_q(T) = \Gamma_q(T)/\Phi_q(\mathcal{H})$.

Proof. This is analogous to the proof of Theorem (2.1.18).

We can now also prove the analogue of the second part of Theorem (2.1.17). The analogue of factoriality for C^*-algebras is simplicity.

Theorem (2.1.21)[60]: If $-1 < q < 1$ and $\dim \mathcal{H} = \infty$ then $\Phi_q(\mathcal{H})$ is simple.

Proof. Again, this is similar to the proof of the von Neumann algebra result. We just indicate the main steps.

We use the notations from the proof of Theorem (2.1.17). First, by norm estimates, one can show that the convergence $\lim_{m\to\infty} \phi_m(X) = \phi(X)$ for X of the form $X := \Psi\left(e_{r(1)} \otimes \cdots \otimes e_{r(n)}\right)$ is even a convergence in norm. Since $\phi(X)$ is nothing but $\phi(X) = \Gamma_q(q)X$, where q is regarded as multiplication operator on $\mathcal{H}$, we have, by Theorem (2.1.20), the bound

$$\| \phi(X) \|_q \leq \| X \|_q.$$

This together with the m-independent bound

$$\|\phi_m(X)\|_q \leq \| X \|_q \ \|\omega(e_1)\|_q^2$$

implies that

$$\lim_{m\to\infty} \phi_m(X) = \Gamma_q(q)X \ \text{ uniformly for all } X \in \Phi_q(\mathcal{H}).$$

Now assume we have a non-trivial ideal I in $\Phi_q(\mathcal{H})$ and consider a positive nonvanishing $X \in I$. Then $\phi_m(X) \in I$ for all $m \in \mathbb{N}$ and thus $\Gamma_q(q)X \in I$. Iterating shows $\Gamma_q(q^n)X \in I$ for all $n \in \mathbb{N}$ and because of the uniform convergence $\lim_{n\to\infty} \Gamma_q(q^n)X = \mathrm{E}[X]1$ we obtain $\mathrm{E}[X]1 \in I$. The faithfulness of E implies then $I = \Phi_q(\mathcal{H})$.

Before we define the notion of a q-Gaussian process, we want to present our general frame on non-commutative processes. By T we will denote the range of our time parameter t, typically T will be some interval in $\mathbb{R}$.

Definitions (2.1.22)[60]: a) Let $\mathcal{A}$ be a finite von Neumann algebra and $\varphi: \mathcal{A} \to \mathbb{C}$ a faithful normal trace on $\mathcal{A}$. Then we call the pair $(\mathcal{A}, \varphi)$ a (tracial) probability space.

b) A random variable on $(\mathcal{A}, \varphi)$ is a self-adjoint operator $X \in \mathcal{A}$.

c) A stochastic process on $(\mathcal{A}, \varphi)$ is a family $(X_t)_{t\in T}$ of random variables $X_t \in \mathcal{A}$ $(t \in T)$.

d) The distribution of a random variable X on $(\mathcal{A}, \varphi)$ is the probability measure ν on the spectrum of X determined by

$$\varphi(X^n) = \int x^n dv(x) \quad \text{for all } n \in \mathbb{N}_0.$$

We should point out that there are also a lot of quantum probabilistic investigations of more general, non-tracial situations, see e.g. [61], [87].

We will only consider centered Gaussian processes, thus a q-Gaussian process will be totally determined by its covariance. Since we would like to have realizations of our processes on separable Hilbert spaces, our admissible covariances are not just positive definite functions, but they should admit a separable representation.

Definition (2.1.23)[60]: A function $c: T \times T \to \mathbb{R}$ is called covariance function, if there exists a separable real Hilbert space $\mathcal{H}$ and vectors $f_t \in \mathcal{H}$ for all $t \in T$ such that

$$c(s,t) = \langle f_s, f_t \rangle \ (s,t \in \mathcal{H}).$$

Definition (2.1.24)[60]: Let $c: T \times T \to \mathbb{R}$ be a covariance function corresponding to a real Hilbert space $\mathcal{H}$ and vectors $f_t \in \mathcal{H} (t \in T)$. Then we put for all $t \in T$

$$X_t := \omega(f_t) \in \Gamma_q(\mathcal{H})$$

and call the process $(X_t)_{t\in T}$ on $(\Gamma_q(\mathcal{H}), \mathrm{E})$ the q-Gaussian process with covariance c. see Frisch and Bourret [77].

We can now define q-analogues of all classical Gaussian processes, just by choosing the appropriate covariance. In the following we consider three prominent examples.

Definitions (2.1.25)[60]: a) The q-Gaussian process $\left(X_t^{qBM}\right)_{t\in[0,\infty)}$ with covariance

$$c(s,t) = \min\ (s,t) \qquad (0 \le s,t < \infty)$$

is called q-Brownian motion.

b) The q-Gaussian process $\left(X_t^{qBB}\right)_{t\in[0,1]}$ with covariance

$$c(s,t) = s(1-t)\ (0 \le s \le t \le 1)$$

is called q-Brownian bridge.

c) The q-Gaussian process $\left(X_t^{qOU}\right)_{t\in\mathbb{R}}$ with covariance

$$c(s,t) = e^{-|t-s|}\ (s,t \in \mathbb{R})$$

is called q-Ornstein-Uhlenbeck process.See [102], [107].

Definition (2.1.26)[60]: Let $(\mathcal{A}, \varphi)$ be a probability space and $(X_t)_{t\in T}$ a stochastic process on $(\mathcal{A}, \varphi)$. Denote by

$$\mathcal{A}_{t]} := \mathrm{vN}(X_u \mid u \leq t) \subset \mathcal{A}$$
$$\mathcal{A}_{[t]} := \mathrm{vN}(X_t) \subset \mathcal{A}.$$
We say that $(X_t)_{t\in T}$ is a Markov process if we have for all $s, t \in T$ with $s \leq t$ the property
$$\varphi[X \mid \mathcal{A}_s] \subset \mathcal{A}_{[s]} \text{ for all } X \in \mathcal{A}_{[t]}.$$
Now, the conditional expectations $\mathrm{E}\left[\cdot \mid \mathcal{A}_{s]}\right]$ in the case of q-Gaussian processes are quite easy to handle because they are nothing but the second quantization of projections in the underlying Hilbert space. Namely, consider a q-Gaussian process $(X_t)_{t\in T}$ corresponding to the real Hilbert space $\mathcal{H}$ and vectors $f_t(t \in T)$. Let us denote by
$$\mathcal{H}_{t]} := \mathrm{span}(f_u \mid u \leq t) \subset \mathcal{H}$$
$$\mathcal{H}_{[t]} := \mathbb{R}f_t \subset \mathcal{H}$$
the Hilbert space analogues of $\mathcal{A}_{t]}$ and $\mathcal{A}_{[t]}$, respectively. Then we have
$$\mathcal{A}_{t]} \cong \Gamma_q\left(\mathcal{H}_{t]}\right) \text{ and } \mathcal{A}_{[t]} \cong \Gamma_q\left(\mathcal{H}_{[t]}\right),$$
and $\mathrm{E}[\cdot \mid \mathcal{A}_{t]}] = \Gamma_q\left(P_{t]}\right)$ is the second quantization of the orthogonal projection
$$P_{t]}: \mathcal{H} \to \mathcal{H}_{t]}.$$
Thus we can translate the Markov property for q-Gaussian processes into the following Hilbert space level statement.

Proposition (2.1.27)[60]:. Let $(X_t)_{t\in T}$ be a q-Gaussian process as above. It has the Markov property if and only if
$$P_{s]}\mathcal{H}_{[t]} \subset \mathcal{H}_{[s]} \text{ for all s, } t \in T \text{ with } s \leq t.$$
Thus Markovianity is a property of the underlying Hilbert space and does not depend on q and we get as in the classical case the following characterization in terms of the covariance.

Proposition (2.1.28)[60]:. A q-Gaussian process with covariance c is Markovian if and only if we have for all triples $s, u, t \in T$ with $s \leq u \leq t$ that
$$c(t,s)c(u,u) = c(t,u)c(u,s).$$
Proof. See the proof of Theorem 3.9 in [102].

Corollary (2.1.29)[60]:. The q-Brownian motion $\left(X_t^{qBM}\right)_{t\in[0,\infty)}$, the q-Brownian bridge $\left(X_t^{qBB}\right)_{t\in[0,1]}$, and the q-Ornstein-Uhlenbeck process $\left(X_t^{qOU}\right)_{t\in\mathbb{R}}$ are all Markovian.

Analogously, we have all statements of the classical Gaussian processes which depend only on Hilbert space properties. Let us just state the characterization of the Ornstein-Uhlenbeck process as the only stationary Gaussian Markov process with continuous covariance and the characterization of martingales among the Gaussian processes.

Proposition (2.1.30)[60]:. Let $(X_t)_{t\in T}$ be a q-Gaussian process which is stationary, Markovian and whose covariance $c(s,t) = c'(t-s)$ is continuous. Then $X_t = \alpha X_{\beta t}^{qOU}$ for suitable $\alpha, \beta > 0$.

Proof. See the proof of the analogous statement for classical Gaussian processes, Corollary 4.10 in [102].

Definition (2.1.31)[60]:. Let $(X_t)_{t\in T}$ be a stochastic process on a probability space $(\mathcal{A}, \varphi)$ and let the notations be as in Definition (2.1.26). Then we say that $(X_t)_{t\in T}$ is a martingale if
$$\varphi\left[X_t \mid \mathcal{A}_{s]}\right] = X_s \text{ for all } s \leq t.$$
Proposition (2.1.32)[60]:. A q-Gaussian process is a martingale if and only if $P_{s]}f_t = f_s$ for all $s \leq t -$ which is the case if and only if $c(s,t) = c(s,s)$ for all $s \leq t$.

Proof. We have

$$\omega(f_s) = X_s = \mathrm{E}[X_t \mid \mathcal{A}_{s]}] = \Gamma_q(P_{s]})\omega(f_t) = \omega(P_{s]}f_t),$$

implying $P_{s]}f_t = f_s$.

We want to address the question whether our non-commutative stochastic processes can also be interpreted classically.

Definition (2.1.33)[60]: Let $(X_t)_{t\in T}$ be a stochastic process on some non-commutative probability space $(\mathcal{A},\varphi)$. We call a classical real-valued process $(\tilde{X}_t)_{t\in T}$ on some classical probability space $(\Omega,\mathfrak{A},P)$ a classical version of $(X_t)_{t\in T}$ if all time-ordered moments of $(X_t)_{t\in T}$ and $(\tilde{X}_t)_{t\in T}$ coincide, i.e. if we have for all $n\in\mathbb{N}$, all $t_1 \dots, t_n \in T$ with $t_1 \le \cdots \le t_n$, and all bounded Borel functions $h_1,\dots,h_n$ on $\mathbb{R}$ the equality

$$\varphi\left[h_1(X_{t_1})\dots h_n\left(X_{t_p}\right)\right] = \int h_1\left(\tilde{X}_{t_1}(\omega)\right)h_n\left(\tilde{X}_{t_p}(\omega)\right)dP(\omega).$$

It is clear that there is at most one classical version for a given non-commutative process $(X_t)_{t\in T}$. The problem consists in showing the existence. If we denote by $\mathbf{1}_B$ the characteristic function of a measurable subset B of $\mathbb{R}$, then we can construct the classical version $(\tilde{X}_t)_{t\in T}$ of $(X_t)_{t\in T}$ via Kolmogorov's existence theorem from the collection of all $\mu_{t_1,\dots,t_n}(n\in\mathbb{N}, t_1\le\cdots\le t_n) -$ which are for $B_1,\dots,B_n \subset \mathbb{R}$ defined by

$$\begin{aligned}\mu_{t_1,\dots,t_n}(B_1\times\cdots\times B_n) &= P(\tilde{X}_{t_1}\in B_1,\dots,\tilde{X}_{t_n}\in B_n)\\ &= \varphi[\mathbf{1}_{B_1}(X_{t_1})\dots\mathbf{1}_{B_n}(X_{t_n})]\end{aligned}$$

- if and only if all $\mu_{t_1,\dots,t_n}$ are probability measures. Whereas this is of course the case for μ_{t_1} and, in our tracial frame because of

$$\mu_{t_1,t_2}(B_1\times B_2) = \varphi[\mathbf{1}_{B_1}(X_{t_1})\mathbf{1}_{B_2}(X_{t_2})] = \varphi[\mathbf{1}_{B_1}(X_{t_1})\mathbf{1}_{B_2}(X_{t_2})\mathbf{1}_{B_1}(X_{t_1})],$$

also for μ_{t_1,t_2}, there is no apriori reason why it should be true for bigger n. And in general it is not. It is essentially the content of Bell's inequality that there are examples of non-commutative processes which possess no classical version $-$ for a discussion of these subjects see, e.g., [89].

But for special classes of non-commutative processes classical versions might exist. One prominent example of such a class are the Markov processes.

Definition (2.1.34)[60]: Let $(X_t)_{t\in T}$ be a Markov process on a probability space $(\mathcal{A},\varphi)$. Let, for $t\in T, \mathrm{spect}(X_t)$ and ν_t be the spectrum und the distribution, respectively, of the self-adjoint operator X_t. Denote by

$$L^\infty(X_t) := \mathrm{vN}(X_t) = L^\infty(\mathrm{spect}(X_t),\nu_t).$$

The operators

$$\mathcal{K}_{s,t}: L^\infty(X_t)\to L^\infty(X_s)\ (s\le t),$$

determined by

$$\varphi[h(X_t)\mid\mathcal{A}_{s]}] = \varphi[h(X_t)\mid\mathcal{A}_{[s]}] = (\mathcal{K}_{s,t}h)(X_s)$$

are called transition operators of the process $(X_t)_{t\in T}$, and, looked upon from the other side, the process $(X_t)_{t\in T}$ is called a dilation of the transistion operators $\mathcal{K} = (\mathcal{K}_{s,t})_{s\le t}$.

The following theorem is by now some kind of folklore in quantum probability, see, e.g. [61], [88],[62], [63]. We just indicate the proof for sake of completeness.

Theorem (2.1.35)[60]: If $(X_t)_{t\in T}$ is a Markov process on some probability space $(\mathcal{A},\varphi)$, then there exists a classical version $(\tilde{X}_t)_{t\in T}$ of $(X_t)_{t\in T}$, which is a classical Markov process.

Proof. One can express the time-ordered moments of a Markov process in terms of the transition operators via

$$\begin{aligned}\varphi[h_1(X_{t_1})\dots h_n(X_{t_n})] &= \varphi[h_1(X_{t_1})\dots h_n(X_{t_n}) \mid \mathcal{A}_{t_{n-1}}]\\ &= \varphi\left[h_1(X_{t_1})\dots h_{n-1}(X_{t_{n-1}})\varphi[h_n(X_{t_n}) \mid \mathcal{A}_{t_{n-1}}]\right]\\ &= \varphi[h_1(X_{t_1})\dots h_{n-1}(X_{t_{n-1}})(\mathcal{K}_{t_{n-1},t_n}h_n)(X_{t_{n-1}})]\\ &= \varphi[h_1(X_{t_1})\dots h_{n-2}(X_{t_{n-2}})(h_{n-1}\cdot\mathcal{K}_{t_{n-1},t_n}h_n)(X_{t_{n-1}})]\\ &= \cdots\\ &= \varphi\left[\left(h_1\cdot\mathcal{K}_{t_1,t_2}\left(h_2\cdot\mathcal{K}_{t_2,t_3}(h_3\cdot\dots)\right)\right)(X_{t_1})\right],\end{aligned}$$

from which it follows − because $\mathcal{K}_{s,t}$ preserves positivity − that the corresponding $\mu_{t_1,\dots,t_n}$ are probability measures. That the classical version is also a classical Markov process follows by the same formula.

Corollary (2.1.36)[60]: There exist classical versions of all q-Gaussian Markov processes. In particular, we have classical versions of the q-Brownian motion, of the q-Brownian bridge, and of the q-Ornstein-Uhlenbeck process.

We describe these classical versions more explicitly by calculating their transition probabilities in terms of the orthogonalizing measure ν_q and the kernel $p_r^{(q)}(x,y)$ of Theorem (2.1.8).

Theorem (2.1.37)[60]: Let $(X_t)_{t\in T}$ be a q-Gaussian Markov process with covariance c and put

$$\lambda_t := \sqrt{c(t,t)} \text{ and } \lambda_{s,t} := \frac{c(t,s)}{\sqrt{c(s,s)c(t,t)}}.$$

a) We have

$$L^\infty(X_t) = L^\infty\left(\left[-2\lambda_t/\sqrt{1-q}, 2\lambda_t/\sqrt{1-q}\right], \nu_q(dx/\lambda_t)\right).$$

b) If $\lambda_{s,t} = \pm 1$, then the transition operator $\mathcal{K}_{s,t}^{(q)}$ is given by

$$\left(\mathcal{K}_{s,t}^{(q)}h\right)(x) = h(\pm x\lambda_t/\lambda_s).$$

If $|\lambda_{s,t}| < 1$, then the transition operator $\mathcal{K}_{s,t}^{(q)}$ is given by

$$\left(\mathcal{K}_{s,t}^{(q)}h\right)(x) = \int h(y)k_{s,t}^{(q)}(x,dy),$$

where the transition probabilities $k_{s,t}^{(q)}$ are Feller kernels which have the explicit form

$$k_{s,t}^{(q)}(x,dy) = p_{\lambda_{s,t}}^{(q)}(x/\lambda_s, y/\lambda_t)\nu_q(dy/\lambda_t).$$

In particular, for $q = 0$ and $|\lambda_{s,t}| < 1$, we have the following transition probabilities for the free Gaussian Markov processes

$$k_{s,t}^{(0)}(x,dy) =$$

$$= \frac{1}{2-\lambda_t^2}\frac{(1-\lambda_{s,t}^2)\sqrt{4\lambda_t^2-y^2}\,dy}{(1-\lambda_s^2)^2-\lambda_s^2(1+\lambda_s^2)(x/\lambda_s)(y/\lambda_t)+\lambda_s^2\left((x^2/\lambda_s^2)+(y^2/\lambda_t^2)\right)}.$$

Recall that a kernel $k(x,dy)$ is called Feller, if the map $x \mapsto k(x,dy)$ is weakly continuous and $k(x,\cdot) \to 0$ weakly as $x \to \pm\infty$ − or equivalently that the corresponding operator $\mathcal{K}$ sends $C_0(\mathbb{R})$ to $C_0(\mathbb{R})$, see, e.g., [73].

Proof. a) This was shown in [67]; noticing the connection between q-relations and q-Hermite polynomials the assertion reduces essentially to part a) of Theorem (2.1.8).

a) By Proposition (2.1.7), we know

$$\Psi\left(f^{\otimes n}\right) = \| f \|^n H_n^{(q)}(\omega(f)/\| f \|).$$

Let our q-Gaussian process $(X_t)_{t\in T}$ now be of the form $X_t = \omega(f_t)$. Markovianity implies

$$P_{s]}f_t = \mu f_s \text{ where } \mu = \frac{\langle f_t, f_s\rangle}{\langle f_s, f_s\rangle} = \frac{c(t,s)}{c(s,s)}.$$

Because of

$$\mathrm{E}\left[\Psi\left(f_t^{\otimes n}\right) \mid \mathcal{A}_{s]}\right] = \Psi\left(\left(P_{s]}f_t\right)^{\otimes n}\right) = \mu^n \Psi\left(f_s^{\otimes n}\right)$$

we obtain with

$$\lambda_t := \|f_t\| = \sqrt{c(t,t)} \text{ and } \lambda_{s,t} := \mu\frac{\lambda_s}{\lambda_t} = \frac{c(t,s)}{\sqrt{c(s,s)c(t,t)}}$$

the formula

$$\begin{aligned}
\mathrm{E}\left[H_n^{(q)}(X_t/\lambda_t) \mid \mathcal{A}_{s]}\right] &= \frac{1}{\lambda_t^n}\mathrm{E}\left[\Psi\left(f_t^{\otimes n}\right) \mid \mathcal{A}_s\right] \\
&= \frac{\mu^n}{\lambda_t^n}\Psi\left(f_s^{\otimes n}\right) \\
&= \left(\mu\frac{\lambda_s}{\lambda_t}\right)^n H_n^{(q)}(X_s/\lambda_s) \\
&= \lambda_{s,t}^n H_n^{(q)}(X_s/\lambda_s),
\end{aligned}$$

implying

$$\mathcal{K}_{s,t}^{(q)}\left(H_n^{(q)}(\cdot/\lambda_t)\right) = \lambda_{s,t}^n H_n^{(q)}(\cdot/\lambda_s).$$

Let us now consider the canonical extension of our transition operators from the L^∞-spaces to the L^2-spaces, i.e.

$$\mathcal{K}_{s,t}^{(q)}: L^2(X_t) \to L^2(X_s).$$

If we use the fact that the rescaled q-Hermite polynomials $\left(H_n^{(q)}(\cdot/\lambda_t)/\sqrt{[n]!}\right)_{n\in\mathbb{N}_0}$ constitute an orthonormal basis of $L^2(X_t)$, we get directly the assertion in the case $\lambda_{s,t} = \pm 1$. (For $\lambda_{s,t} = -1$ one also has to note that $H_{2k}^{(q)}$ and $H_{2k+1}^{(q)}$ are even and odd polynomials, respectively.)

In the case $\left|\lambda_{s,t}\right| < 1$, our formula implies that $\mathcal{K}_{s,t}^{(q)}$ is a Hilbert-Schmidt operator, thus it has a concrete representation by a kernel $k_{s,t}^{(q)}$, which is given by

$$\begin{aligned}
k_{s,t}^{(q)}(x, dy) &= \sum_{n=0}^{\infty}\frac{\lambda_{s,t}^n}{[n]_q!}H_n^{(q)}(x/\lambda_s)H_n^{(q)}(y/\lambda_t)\nu_q(dy/\lambda_t) \\
&= \sum^{\infty}\lambda_s(x/\lambda_s, y/\lambda_t)\nu_q(dy/\lambda_t).
\end{aligned}$$

That our kernels are Feller follows from the fact that, by Theorem (2.1.20), our second quantization (i.e. our transition operators) restrict to the C^*-level (i.e. to continuous functions).

The formula for $k_{s,t}^{(0)}$ follows from the concrete form of $p_r^{(0)}$ of Theorem (2.1.8) and the fact that

$$\nu_0(dy) = \frac{1}{2\pi}\sqrt{4-y^2}dy \text{ for } y \in [-2,2].$$

The main formula of our proof, namely the action of the conditional expectation on the q-Hermite polynomials, says that we have some quite canonical martingales associated to q-Gaussian Markov processes - provided the factor $\lambda_{s,t}$ decomposes into a quotient $\lambda_{s,t} = \lambda(s)/\lambda(t)$. Since this can be assured by a corresponding factorization property of the covariance function − which is not very restrictive for Gaussian Markov processes, see Theorem 4.9 of [102] - we get the following corollary.

Corollary (2.1.38)[60]: Let $(X_t)_{t\in T}$ be a q-Gaussian process whose covariance factorizes for suitable functions g and f as

$$c(s,t) = g(s)f(t) \text{ for } s \le t.$$

Then, for all $n \in \mathbb{N}_0$, the processes $(M_n(t))_{t\in T}$ with

$$M_n(t) := (g(t)/f(t))^{n/2} H_n^{(q)}(X_t/\lambda_t)$$

are martingales.

Note that the assumption on the factorization of the covariance is in particular fulfilled for the q-Brownian motion, for the q-Ornstein-Uhlenbeck process, and for the q-Brownian bridge.

Proof. Our assumption on the covariance implies

$$\lambda_{s,t} = \sqrt{\frac{g(s)/f(s)}{g(t)/f(t)}},$$

hence our formula for the action of the conditional expectation on the q-Hermite polynomials can be written as

$$(g(t)/f(t))^{\frac{n}{2}} \mathrm{E}[H_n^{(q)}(X_t/\lambda_t) \mid \mathcal{A}_{s]}\Big] = (g(s)/f(s))^{n/2} H_n^{(q)}(X_s/\lambda_s),$$

which is exactly our assertion.

Example (2.1.39)[60]: Free Gaussian processes. We will now specialize the formula for $k_{s,t}^{(0)}$ to the case of the free Brownian motion, the free Ornstein-Uhlenbeck process and the free Brownian bridge. The transition probabilities for the two former cases were also derived by Biane [63] in the context of processes with free increments.

a) free Brownian motion: We have $c(s,t) = \min(s,t)$, thus

$$\lambda_t = \sqrt{t} \text{ and } \lambda_{s,t} = \sqrt{s/t}.$$

This yields

$$k_{s,t}(x,dy) = \frac{(t-s)}{(t-s)^2 - (t+s)xy + x^2t + y^2s} \frac{\sqrt{4t-y^2}dy}{2\pi}$$

for

$$x \in [-2\sqrt{s}, 2\sqrt{s}] \text{ and } y \in [-2\sqrt{t}, 2\sqrt{t}].$$

b) free Ornstein-Uhlenbeck process: We have $c(s,t) = e^{-|t-s|}$, thus

$$\lambda_t = 1 \text{ and } \lambda_{s,t} = e^{-|t-s|}.$$

Since this process is stationary, it suffices to consider the transition probabilities for $= 0$:

$$k_{0,t}(x,dy) = \frac{(e^{2t}-1)}{4\sinh^2 t - 2xy\cosh t + x^2 + y^2} \frac{\sqrt{4-y^2}dy}{2\pi} \text{ for } x,y \in [-2,2].$$

Let us also calculate the generator N of this process - which is characterized by

$$\mathcal{K}_{s,t} = e^{-(t-s)N}.$$

It has the property

$$NH_n^{(0)} = nH_n^{(0)} \ (n \in \mathbb{N}_0),$$

and differentiating the above kernel shows that it should be given formally by a kernel $-2/(y-x)^2$ with respect to ν_0. Making this more rigorous [108] yields that N has on functions which are differentiable the form

$$(Nh)(x) = xf'(x) - 2\int \frac{f(y) - f(x) - f'(x)(y-x)}{(y-x)^2} 2\nu_0(dy).$$

c) free Brownian bridge: We have $c(s,t) = s(1-t)$ for $s \leq t$, thus

$$\lambda_t = \sqrt{t(1-t)} \ \text{ and } \ \lambda_{s,t} = \sqrt{\frac{s(1-t)}{t(1-s)}}.$$

This yields

$$k_{s,t}(x, dy) =$$

$$= \frac{1-s}{1-t} \frac{(t-s)}{(t-s)^2 - (s+t-2st)xy + t(1-t)x^2 + s(1-s)y^2} \frac{\sqrt{4t(1-t) - y^2}dy}{2\pi},$$

for

$$x \in [-2\sqrt{s(1-s)}, 2\sqrt{s(1-s)}]$$

$$\text{and } \ y \in [-2\sqrt{t(1-t)}, 2\sqrt{t(1-t)}].$$

Example (2.1.40)[60]: Fermionic Gaussian processes. For illustration, we also want to consider the fermionic ($q = -1$) analogue of Gaussian processes. Although this case has not been included in our frame everything works similar, the only difference is that in the Fock space we get a kernel of our scalar product consisting of anti-symmetric tensors. This is responsible for the fact that the corresponding (-1)-Hermite polynomials collapse just to

$$H_0^{(-1)}(x) = 1 \ \text{ and } \ H_1^{(-1)}(x) = x.$$

The corresponding measure ν_{-1} is not absolutely continuous with respect to the Lebesgue measure anymore, but collapses to

$$\nu_{-1}(dx) = \frac{1}{2}\big(\delta_{-1}(dx) + \delta_{+1}(dx)\big).$$

This yields

$$p_r^{(-1)}(x,y) = H_0^{(-1)}(x)H_0^{(-1)}(y) + rH_1^{(-1)}(x)H_1^{(-1)}(y) = 1 + rxy,$$

giving as transition probabilities

$$k_{s,t}^{(-1)}(x,dy) = \frac{1}{2}\left(1 + \frac{c(s,t)}{c(s,s)c(t,t)}xy\right)\left(\delta_{-\sqrt{c(t,t)}}(dy) + \delta_{+\sqrt{c(t,t)}}(dy)\right).$$

a) fermionic Brownian motion: X_t can only assume the values $+\sqrt{t}$ and $-\sqrt{t}$ and the transition probabilities are given by the table

$k_{s,t}$	$\sqrt{t}$	$-\sqrt{t}$
$\sqrt{s}$	$\frac{1}{2}(1+\sqrt{s/t})$	$\frac{1}{2}(1-\sqrt{s/t})$.
$-\sqrt{s}$	$\frac{1}{2}(1-\sqrt{s/t})$	$\frac{1}{2}(1+\sqrt{s/t})$

This case coincides with the corresponding $c = -1$ case of the Azéma martingale, see [97].

b) fermionic Ornstein-Uhlenbeck process: This stationary process lives on the two values $+1$ and -1 with the following transition probabilities

$$\begin{array}{ccc} k_{s,t} & 1 & -1 \\ 1 & \frac{1}{2}\left(1+e^{-(t-s)}\right) & \frac{1}{2}\left(1-e^{-(t-s)}\right). \\ -1 & \frac{1}{2}\left(1-e^{-(t-s)}\right) & \frac{1}{2}\left(1+e^{-(t-s)}\right) \end{array}$$

This classical two state Markov realization of the corresponding fermionic relations has been known for a long time, see [77].

c) fermionic Brownian bridge: X_t can only assume the values $+\sqrt{t(1-t)}$ and $-\sqrt{t(1-t)}$ and the transition probabilities are given by the table

$$\begin{array}{ccc} k_{s,t} & \sqrt{t(1-t)} & -\sqrt{t(1-t)} \\ \sqrt{s(1-s)} & \frac{1}{2}\left(1+\sqrt{\frac{s(1-t)}{t(1-s)}}\right) & \frac{1}{2}\left(1-\sqrt{\frac{s(1-t)}{t(1-s)}}\right). \\ -\sqrt{s(1-s)} & \frac{1}{2}\left(1-\sqrt{\frac{s(1-t)}{t(1-s)}}\right) & \frac{1}{2}\left(1+\sqrt{\frac{s(1-t)}{t(1-s)}}\right) \end{array}$$

Example (2.1.41)[60]: Hypercontractivity. Consider the q-Ornstein-Uhlenbeck process with stationary transition operators $\mathcal{K}_t^{(q)} := \mathcal{K}_{s,s+t}^{qOU}$. Note that this q-OrnsteinUhlenbeck semigroup is nothing but the second quantization of the simplest contraction, namely with the one-dimensional real Hilbert space $\mathcal{H} = \mathbb{R}$ and the corresponding identity operator $\mathbf{1}: \mathbb{R} \to \mathbb{R}$ we have

$$\Gamma_q(\mathbb{R}) \cong L^\infty\left(-2/\sqrt{1-q}, 2/\sqrt{1-q}, \nu_q(dx)\right) \text{ and } \Gamma_q(e^{-t}\mathbf{1}) \cong \mathcal{K}_t^{(q)}.$$

We have seen that the $\mathcal{K}_t^{(q)}$ are, for all $t > 0$, contractions on L^2 and on L^∞ (and thus, by duality and interpolation, on all L^p). In the classical case $q = 1$ (and also for $q = -1$) it is known [101],[94],[95],[80],[71] that much more is true, namely the Ornstein-Uhlenbeck semigroup is also hypercontractive, i.e. it is bounded as a map from L^2 to L^4 for sufficiently large t. Having the concrete form of the kernel

$$k_t^{(q)}(x, dy) = p_{e^{-t}}^{(q)}(x, y)\nu_q(dy)$$

of $\mathcal{K}_t^{(q)}$ it is easy to check that we also have hypercontractivity for all $-1 < q < 1$ Even more, we can show that $\mathcal{K}_t^{(q)}$ is bounded from L^2 to L^∞ for $t > 0$, i.e. we have what one might call 'ultraconctractivity' - which is, of course, not given for $q = \pm 1$. This ultracontractivity follows from the estimate

$$\left\|\mathcal{K}_t^{(q)} h\right\|_\infty \le \alpha(t, q)^{1/2} \parallel h \parallel_2 \quad \text{where } \alpha(t, q) := \sup_{x\in[-2,2]} \sup_{y\in[-2,2]} p_{e^{-t}}^{(q)}(x, y)$$

and from the explicit form of $p_r^{(q)}$ from Theorem (2.1.8), which ensures that $\alpha(t, q)$ is finite for $t > 0$ and $-1 < q < 1$. One may also note that for small t the leading term of $\alpha(t, q)^{1/2}$ is of order $t^{-3/2}$.

Section (2.2): The q-Deformed von Neumann Algebra

For $H_{\mathbb{R}}$ be a real Hilbert space and $H_{\mathbb{C}}$ its complexification. Let T be a Yang-Baxter operator on $H_{\mathbb{C}} \otimes H_{\mathrm{C}}$ with $\| T \| < 1$. Let $\mathcal{F}_T(H_{\mathbb{C}})$ be the associated deformed Fock space and $\Gamma_T(H_{\mathbb{R}})$ the von Neumann algebra generated by the corresponding deformed gaussian random variables, introduced by Bozejko and Speicher [68] (also see [60]). In addition, we will assume that T is tracial, i.e that the vacuum expectation is a trace on $\Gamma_T(H_{\mathbb{R}})$ (cf [68]). Under these assumptions, it was proved in [68] that $\Gamma_T(H_{\mathrm{R}})$ is not injective as soon as $\dim H_{\mathbb{R}} > \frac{16}{(1-q)^2}$, where $\| T \| = q$. Since then the problem whether $\Gamma_T(H_{\mathbb{R}})$ is not injective as soon as $\dim H_{\mathbb{R}} \geq 2$ had been left open. We emphasize that this problem remained open even in the particular case of the qdeformation, that is when $T = q\sigma$, where σ is the reflexion : $\sigma(\xi \otimes \eta) = \eta \otimes \xi$. Recall that the free von Neumann algebra $\Gamma_0(H_{\mathrm{R}})$ (corresponding to $T = 0$) is not injective as soon as $n = \dim H_{\mathbb{R}} \geq 2$, for $\Gamma_0(H_{\mathrm{R}})$ is isomorphic to the free group von Neumann algebra $VN(\mathbb{F}_n)$(cf. [19]). The main result solves the above problem.

To explain the idea of our proof we first recall the main ingredient of the proof of the non injectivity theorem in [68]. It is the following vector-valued non-commutative Khintchine inequality. Let $(e_i)_{i\in I}$ be an orthonormal basis of $H_{\mathbb{R}}$. Let K be a complex Hilbert space and $B(K)$ the space of all bounded operators on K. Then for any finitely supported family $(a_i)_{i\in I} \subset B(K)$

$$\max\left\{\left\|\sum_{i\in I} a_i^* a_i\right\|_{B(K)}^{\frac{1}{2}}, \left\|\sum_{i\in I} a_i a_i^*\right\|_{B(K)}^{\frac{1}{2}}\right\} \leq \left\|\sum_{i\in I} a_i \otimes G(e_i)\right\|$$
$$\leq \frac{2}{\sqrt{1-q}} \max\left\{\left\|\sum_{i\in I} a_i^* a_i\right\|_{B(K)}^{\frac{1}{2}}, \left\|\sum_{i\in I} a_i a_i^*\right\|_{B(K)}^{\frac{1}{2}}\right\}$$

where $G(e) = a^*(e) + a(e)$ is the deformed gaussian variable associated with a vector $e \in H_{\mathbb{R}}$. Using this Khintchine inequality and the equivalence between the injectivity and the semidiscreteness, one easily deduces the non-injectivity of $\Gamma_T(H_{\mathbb{R}})$ as soon as $\dim H_{\mathrm{R}} > \frac{16}{(1-q)^2}$.

The proof of our non-injectivity theorem follows the same pattern. We will first need to extend the preceding vector-valued non-commutative Khintchine inequality to Wick products. It is well known that for any ξ, a finite linear combination of elementary tensors, there is a unique operator $W(\xi) \in \Gamma_T(H_{\mathbb{R}})$ such that $W(\xi)\Omega = \xi$. Instead of the previous inequality, the main ingredient of our proof is the following. Let $n \geq 1$. Let $(\xi_i)_{|i]=n}$ be an orthonormal basis of $H_{\mathbb{C}}^{\otimes n}$ and $(\alpha_i) \subset B(K)$ a finitely supported family. Then

$$\max_{0\leq k\leq n}\left\{\left\|\sum_{|\underline{i}|=n} \alpha_{\underline{i}} \otimes R_{n,k}^* \xi_{\underline{i}}\right\|\right\} \leq \left\|\sum_{|\underline{i}|=n} \alpha_{\underline{i}} \otimes \left(\xi_{\underline{i}}\right)\right\|$$
$$\leq (n+1)C_q \max_{0\leq k\leq n}\left\{\left\|\sum_{|\underline{i}|=n} \alpha_{\underline{i}} \otimes R_{n,k}^* \xi_{\underline{i}}\right\|\right\} \qquad (1)$$

where the norms in the left and right handside have to be taken in $B(K) \otimes_{\min} H_c^{\otimes n-k} \otimes_h H_c^{\otimes k}$ (see Theorem (2.2.6) below for the precise statement). Inequality (1) is the vector-valued version of Bozejko's ultracontractivity inequality proved in [126] and thus it solves a problem posed in [126]. Using (1) and a careful analysis on the norms of Wick products on a same level, we deduce our non-injectivity result.

We devoted to necessary definitions and preliminaries on the deformation by a Yang-Baxter operator and the associated von Neumann algebra. We also include a brief discussion on the simplest case, the free case, i.e. when $T = 0$. All our results and arguments become very simple in this case, for instance, inequality (1) above is then easy to state and prove. The proof of the non-injectivity of $\Gamma_0(H_{\mathbb{R}})$ can be done in just a few lines. The reason why we have decided to include such a discussion on the free case is the fact that it already contains the main idea for the general case. We will establish (1) and prove the non-injectivity of $\Gamma_T(H_{\mathbb{R}})$. The last aims at proving the non-injectivity of the Araki-Woods factors $\Gamma_q(H, U_t)$ introduced by Hiai in [123]. Note that Hiai proved a non-injectivity result with a condition on the dimension of the spectral sets of the positive generator of U_t, which is similar to that of [68]. The problem is left open whether the dimension can go down to 2. Although we cannot completely solve this, our method permits to improve in some sense the criterion for non-injectivity given in [123].

Recall that the free Fock space associated with $H_{\mathbb{R}}$ is given by

$$\mathcal{F}_0(H_{\mathbb{C}}) = \bigoplus_{n\geq 0} H_{\mathbb{C}}^{\otimes n}$$

where $H_{\mathbb{C}}^{\otimes 0}$ is by definition $\mathbb{C}\Omega$ with Ω a unit vector called the vacuum.

A Yang-Baxter operator on $H_{\mathbb{C}} \otimes H_{\mathbb{C}}$ is a self-adjoint contraction satisfying the following braid relation:

$$(I \otimes T)(T \otimes I)(I \otimes T) = (T \otimes I)(I \otimes T)(T \otimes I)$$

For $n \geq 2$ and $1 \leq k \leq n-1$ we define T_k on $H_{\mathbb{C}}^{\otimes n}$ by

$$T_k = I_{H_{\mathbb{C}}^{k-1}} \otimes T \otimes I_{H_{\mathbb{C}}^{n-k-1}}$$

Let S_n be the group of permutations on a set of n elements. A function φ is defined on S_n by quasi-multiplicative extension of:

$$\varphi(\pi_k) = T_k$$

where $\pi_k = (k, k+1)$ is the transposition exchanging k and $k+1, 1 \leq k \leq n-1$. The symmetrizator $P_T^{(n)}$ is the following operator defined on $H_{\mathbb{C}}^{\otimes n}$ by:

$$P_T^{(n)} = \sum_{\sigma\in S_n} \varphi(\sigma)$$

$P_T^{(n)}$ is a positive operator on $H_{\mathbb{C}}^{\otimes n}$ for any Yang-Baxter operator T and is strictly positive if T is strictly contractive (cf. [68]). In the latter case we are allowed to define a new scalar product on $H_{\mathbb{C}}^{\otimes n}$(for ≥ 2) by:

$$\langle \xi, \eta \rangle_T = \left\langle \xi, P_T^{(n)} \eta \right\rangle$$

The associated norm is denoted by $\|\cdot\|_T$. The deformed Fock space associated with T is then defined by

$$\mathcal{F}_T(H_{\mathrm{C}}) = \bigoplus_{n\geq 0} H_{\mathbb{C}}^{\otimes n}$$

where $H_{\mathbb{C}}^{\otimes n}$ is now equipped with our deformed scalar product for $n \geq 2$. From now on we will only consider a strictly contractive Yang-Baxter T and $\| T \| \leq q < 1$.

For $f \in H_{\mathbb{R}}, a^*(f)$ will denote the creation operator associated with f, and $a(f)$ its adjoint with respect to the T-scalar product:

$$a^*(f)(f_1 \otimes \cdots \otimes f_n) = f \otimes f_1 \otimes \cdots \otimes f_n$$

For $f \in H_{\mathbb{R}}$ the deformed gaussian is the following hermitian operator:

$$G(f) = a^*(f) + a(f)$$

We are interested in $\Gamma_T(H_{\mathrm{R}})$ which is the von Neumann algebra generated by all gaussians $G(f)$ for $f \in H_{\mathbb{R}}$:

$$\Gamma_T(H_{\mathbb{R}}) = \{G(f): f \in H_{\mathbb{R}}\}'' \subset B\big(\mathcal{F}_T(H_{\mathbb{C}})\big)$$

Let $(e_i)_{i \in I}$ be an orthonormal basis of $H_{\mathbb{R}}$ and set

$$t_{ij}^{sr} = \langle e_s \otimes e_r, T(e_i \otimes e_j) \rangle$$

Then the following deformed commutation relations hold:

$$a(e_i)a^*(e_j) - \sum_{r,s \in I} t_{js}^{ir} a^*(e_r)(e_s) = \delta_{ij}$$

Moreover if the following condition holds

$$\langle e_s \otimes e_r, Te_i \otimes e_j \rangle = \langle e_r \otimes e_j, Te_s \otimes e_i \rangle$$

which is equivalent to the cyclic condition :

$$t_{ij}^{sr} = t_{si}^{rj}$$

then the vacuum is cyclic and separating for $\Gamma_T(H_{\mathbb{R}})$ and the vacuum expectation is a faithful trace on $\Gamma_T(H_{\mathbb{R}})$ that will be denoted by τ. If this cyclic condition holds we say that T is tracial, and from now on we will always assume that T has this property.

We will denote by $\Gamma_T^\infty(H_{\mathbb{R}})$ the subspace $\Gamma_T(H_{\mathbb{R}})\Omega$ of $\mathcal{F}_T(H_{\mathbb{C}})$. Since Ω is separating for $\Gamma_T(H_{\mathbb{R}})$, for every $\xi \in \Gamma_T^\infty(H_{\mathbb{R}})$ there exists a unique operator $W(\xi) \in \Gamma_T(H_{\mathbb{R}})$ such that

$$W(\xi)\Omega = \xi$$

W is called Wick product.

The right creation operator, $a_r^*(f)$, is defined by the following formula:

$$a_r^*(f)(f_1 \otimes \cdots \otimes f_n) = f_1 \otimes \cdots \otimes f_n \otimes f$$

We will also denote by $a_r(f)$ the right annihilation operator, which is its adjoint with respect to the T-scalar product, by $G_r(f)$ the right gaussian operator, and by $\Gamma_{T,r}(H_{\mathbb{R}})$ the von Neumann algebra generated by all right gaussians. It is easy to see that $\Gamma_{T,r}(H_{\mathbb{R}}) \subset \Gamma_T(H_{\mathbb{R}})'$. Actually, by Tomita's theory, we have

$$\Gamma_{T,r}(H_{\mathbb{R}}) = S\Gamma_T(H_{\mathbb{R}})S = \Gamma_T(H_{\mathbb{R}})'$$

where S is the anti linear operator on $\mathcal{F}_T(H_{\mathbb{C}})$ (which is actually an anti unitary) defined by

$$S(f_1 \otimes \cdots \otimes f_n) = f_n \otimes \cdots \otimes f_1$$

for any $f_1, \cdots, f_n \in H_{\mathbb{R}}$. Since Ω is also separating for $\Gamma_{T,r}(H_{\mathbb{R}})$ we can define the right Wick product, that will be denoted by $W_r(\xi)$. For any $\xi \in \Gamma_T^\infty(H_{\mathbb{R}})$ we have

$$(W(\xi))^* = W(S\xi) \text{ and } SW(\xi)S = W_r(S\xi)$$

Some particular cases of deformation have been studied in the literature. Let $(q_{ij})_{i,j \in I}$ be a hermitian matrix such that $\sup_{i,j}|q_{ij}| < 1$. Define

$$Te_i \otimes e_j = q_{ij} e_j \otimes e_i$$

Then T is a strictly contractive Yang-Baxter operator, and it is tracial if and only if the q_{ij} are real. Our deformed Fock space is then a realisation of the following q_{ij}-relations :

$$a(e_i)a^*(e_j) - q_{ij}a^*(e_j)a(e_i) = \delta_{ij}$$

In the special case where all q_{ij} are equal, we obtain the well known q-relations.

Let us define the following selfadjoint unitary on the free Fock space :

$$\forall f_1, \ldots, f_n \in H_{\mathrm{C}},\ U(f_1 \otimes \cdots \otimes f_n) = f_n \otimes \cdots \otimes f_1$$

Since $UP_T^{(n)} = P_T^{(n)}U$(cf. [124]), U is also a selfadjoint unitary on each T-Fock space. Given vectors $f_1, \dots, f_n$ in $H_{\mathbb{R}}$ we define :

$$a^*(f_1 \otimes \cdots \otimes f_n) = a^*(f_1) \dots a^*(f_n) \text{ and } a(f_1 \otimes \cdots \otimes f_n) = a(f_1) \dots a(f_n)$$

For $0 \le k \le n$, let $R_{n,k}$ be the operator on $H_{\mathbb{C}}^{\otimes n}$ given by

$$R_{n,k} = \sum_{\sigma \in S_n / S_{n-k} \times S_k} \varphi(\sigma^{-1})$$

where the sum runs over the representatives of the right cosets of $S_{n-k} \times S_k$ in S_n with minimal number of inversions. Then

$$P_T^{(n)} = R_{n,k}\left(P_T^{(n-k)} \otimes P_T^{(k)}\right) \text{ and } \|R_{n,k}\| \le C_q \tag{2}$$

where $C_q = \prod_{n=1}^{\infty}(1-q^n)^{-1}$ (cf. [126] and [124]). It follows that

$$P_T^{(n)} \le C_q P_T^{(n-k)} \otimes P_T^{(k)} \tag{3}$$

It also follows that a^*, respectively a, extend linearly, respectively antilinearly, and continuously to $H_{\mathbb{C}}^{\otimes n}$ for every $n \ge 1$. Then for each vector $\xi \in H_{\mathbb{C}}^{\otimes n}$ we have

$$\|a^*(\xi)\| \le C_q^{\frac{1}{2}} \| \xi \|_T \text{ and } \left(a^*(\xi)\right)^* = a(U\xi). \tag{4}$$

Let $n \ge 1$ and $1 \le k \le n$, $H_{\mathbb{C}}^{\otimes n-k} \otimes H_{\mathbb{C}}^{\otimes k}$ will be the Hilbert tensor product of the Hilbert spaces $H_{\mathbb{C}}^{\otimes k}$ and $H_{\mathbb{C}}^{\otimes n-k}$ where both $H_{\mathbb{C}}^{\otimes k}$ and $H_{\mathbb{C}}^{\otimes n-k}$ are equipped with the T-scalar product.

Lemma (2.2.1)[113]: There is a positive constant $D_{q,n,k}$ such that

$$P_T^{(n-k)} \otimes P_T^{(k)} \le D_{q,n,k} P_T^{(n)}$$

Consequently for every $n \ge 1$ and $1 \le k \le n$, $H_{\mathbb{C}}^{\otimes n}$ and $H_{\mathbb{C}}^{\otimes k} \otimes H_{\mathbb{C}}^{\otimes n-k}$ are algebraically the same and their norms are equivalent.

Proof. It was shown in [125] that there is a positive constant $\omega(q)$ such that

$$P_T^{(n-1)} \otimes I \le \omega(q)^{-1} P_T^{(n)}$$

Since $U\left(P_T^{(n-1)} \otimes I\right)U = I \otimes P_T^{(n-1)}$ we also have

$$I \otimes P_T^{(n-1)} \le \omega(q)^{-1} P_T^{(n)} \tag{5}$$

Fix some $k, 2 \le k \le n-1$, using (3) and (4) we get:

$$\begin{aligned} P_T^{(n-k+1)} \otimes P_T^{(k-1)} &\le C_q P_T^{(n-k)} \otimes I \otimes P_T^{(k-1)} \\ &\le C_q \omega(q)^{-1} P_T^{(n-k)} \otimes P_T^{(k)} \end{aligned}$$

Thus by iteration it follows that for $0 \le k \le n$:

$$P_T^{(n-k)} \otimes P_T^{(k)} \le \omega(q)^{-1}\left(C_q \omega(q)^{-1}\right)^{n-k} P_T^{(n)} \tag{6}$$

Since $U\left(P_T^{(n-k)} \otimes P_T^{(k)}\right)U = P_T^{(k)} \otimes P_T^{(n-k)}$ it follows from (6) that

$$P_T^{(k)} \otimes P_T^{(n-k)} \le \omega(q)^{-1}\left(C_q \omega(q)^{-1}\right)^{n-k} P_T^{(n)}$$

Combining this last inequality and (6) we finally obtain :

$$P_T^{(n-k)} \otimes P_T^{(k)} \le \omega(q)^{-1}\left(C_q \omega(q)^{-1}\right)^{\min\,(k,n-k)} P_T^{(n)} \tag{7}$$

Then the desired result follows from (3) and (7).

For $k \ge 0$ let us now define on the family of finite linear combinations of elementary tensors of length not less than k the following operator U_k :

$$U_k(f_1 \otimes \cdots \otimes f_n) = a^*(f_1 \otimes \cdots \otimes f_{n-k}) a\left(\bar{f}_{n-k+1} \otimes \cdots \otimes \bar{f}_n\right)$$

where $\overline{\xi + \iota\eta} = \xi - i\eta$ for all $\xi, \eta \in H_{\mathbb{R}}$.

Fix n and k with $n \geq k$. Let $\mathcal{J}: H_{\mathbb{C}}^{\otimes k} \to \overline{H_{\mathbb{C}}^{\otimes k}}$ be the conjugation (which is an anti isometry). For any $f_1, \cdots, f_n, \mathcal{J}$ is defined by $\mathcal{J}(f_1 \otimes \cdots \otimes f_n) = \overrightarrow{\overline{f_1} \otimes \cdots \otimes \overline{f_n}}$. It is clear that U_k extends boundedly to $H_{\mathbb{C}}^{\otimes n-k} \otimes H_{\mathbb{C}}^{\otimes k}$ by the formula :

$$U_k = M(a^* \otimes a\mathcal{J})$$

where M is the multiplication operator from $B(\mathcal{F}_T(H_{\mathbb{C}})) \otimes_{\min} B(\mathcal{F}_T(H_{\mathbb{C}}))$ to $B(\mathcal{F}_T(H_{\mathbb{C}}))$ defined by $M(A \otimes B) = AB$. Moreover, by (4) we have

$$\|U_k\| \leq \| M \| \cdot \|a^* \otimes a\mathcal{J}\| \leq C_q$$

where U_k is viewed as an operator from $H_{\mathbb{C}}^{\otimes n-k} \otimes H_{\mathbb{C}}^{\otimes k}$ to $B(\mathcal{F}_T(H_{\mathbb{C}}))$.

In the following lemma we state an extension of the Wick formula (Theorem 3 in [124]). We deduce it as an easy consequence of the original Wick formula and of our previous discussion.

Lemma (2.2.2)[113]: Let $n \geq 1$ and $\xi \in H_{\mathbb{C}}^{\otimes n}$, then $H_{\mathbb{C}}^{\otimes n} \subset \Gamma_T^{\infty}(H_{\mathbb{R}})$ and we have the following Wick formula:

$$W(\xi) = \sum_{k=0}^{n} U_k R_{n,k}^*(\xi) \tag{8}$$

Moreover

$$\| \xi \|_q \leq \| W(\xi) \| \leq C_q^{\frac{3}{2}}(n+1) \| \xi \|_q \tag{9}$$

Proof. The usual Wick formula is the following (cf [126] and [124]): $\forall f_1, \dots, f_n \in H_{\mathbb{C}}$ we have

$$W(f_1 \otimes \cdots \otimes f_n) = \sum_{k=0}^{n} \sum_{\sigma \in S_n/S_{n-k} \times S_k} U_k \varphi(\sigma)(f_1 \otimes \cdots \otimes f_n)$$

Hence (8) holds for every $\xi \in \mathcal{A}_n = \{$ linear combinations of elementary tensors of length $n\}$. By Lemma (2.2.1) and our previous discussion, the right handside of (8) is continuous from $H_{\mathbb{C}}^{\otimes n}$ to $B(\mathcal{F}_T(H_{\mathbb{C}}))$. Since Ω is separating, it follows that $H_{\mathbb{C}}^{\otimes n} \subset \Gamma_T^{\infty}(H_{\mathbb{R}})$ and that (8) extends by density from $\mathcal{A}_n$ to $H_C^{\otimes n}$. Actually, our argument shows that for any $\xi \in H^{\otimes n}, W(\xi)$ belongs to $C_T^*(H_{\mathbb{R}})$ which is the C^*-algebra generated by the T-gaussians.

Since for any $\xi \in H_{\mathbb{C}}^{\otimes n}, W(\xi)\Omega = \xi$, the left inequality in (9) holds. We have just showed that W is bounded from $H_{\mathbb{C}}^{\otimes n}$ to $B(\mathcal{F}_T(H_{\mathbb{C}}))$. Hence, there is a constant $B_{q,n}$ such that for any $\xi \in H_C^{\otimes n}$ we have $\| W(\xi) \| \leq B_{q,n} \| \xi \|_q$. To end the proof of (9) we now give a precise estimate of $B_{q,n}$. Let $\xi \in H_{\mathbb{C}}^{\otimes n}$, by (8) and (3) we have

$$\| W(\xi) \| \leq \sum_{k=0}^{n} \|U_k R_{n,k}^*(\xi)\| \leq C_q \sum_{k=0}^{n} \|R_{n,k}^*(\xi)\|_{H_{\mathbb{C}}^{\otimes n-k} \otimes H_{\mathbb{C}}^{\otimes k}} \tag{10}$$

It remains to compute the norm of $R_{n,k}^*$ as an operator from $H_{\mathbb{C}}^{\otimes n}$ to $H_{\mathbb{C}}^{\otimes n-k} \otimes H_{\mathbb{C}}^{\otimes k}$. Let $\eta \in H_{\mathbb{C}}^{\otimes n}$ we have, by (2) and (3)

$$\begin{aligned} \|R_{n,k}^*\eta\|_{H_{\mathbb{C}}^{\otimes n-k} \otimes H_{\mathbb{C}}^{\otimes k}}^2 &= \left\langle P_T^{(n-k)} \otimes P_T^{(k)} R_{n,k}^*\eta, R_{n,k}^*\eta \right\rangle_0 \\ &= \left\langle P_T^{(n)}\eta, R_{n,k}^*\eta \right\rangle_0 \leq \| \eta \|_T \|R_{n,k}^*\eta\|_T \end{aligned}$$

On the other hand,

$$\begin{aligned}\|R_{n,k}^*\eta\|_T^2 &= \left\langle P_T^{(n)} R_{n,k}^*\eta, R_{n,k}^*\eta\right\rangle_0 \leq C_q \left\langle P_T^{(n-k)} \otimes P_T^{(k)} R_{n,k}^*\eta, R_{n,k}^*\eta\right\rangle_0 \\ &\leq C_q \left\langle P_T^{(n)}\eta, R_{n,k}^*\eta\right\rangle_0 \\ &\leq C_q \parallel \eta \parallel_T \|R_{n,k}^*\eta\|_T\end{aligned}$$

Hence it follows that $\|R_{n,k}^*\eta\|_T \leq C_q \parallel \eta \parallel_T$ and $\|R_{n,k}^*\eta\|^2_{H_{\mathbb{C}}^{\otimes n-k}\otimes H_{\mathbb{C}}^{\otimes k}} \leq C_q\|\eta\|_T^2$. Thus $\|R_{n,k}^*\| \leq C_q^{\frac{1}{2}}$ as an operator from $H_{\mathbb{C}}^{\otimes n}$ to $H_{\mathbb{C}}^{\otimes n-k} \otimes H_{\mathbb{C}}^{\otimes k}$. From (10) and this last estimate, follows the second inequality in (9).

The remainder is devoted to a simple proof of the non-injectivity of the free von Neumann algebra $\Gamma_0(H_{\mathbb{R}})(\dim H_{\mathbb{R}} \geq 2)$. The main ingredient is the vector valued Bozejko inequality (Lemma (2.2.3) below), which is the free Fock space analogue of the corresponding inequality for the free groups proved by Haagerup and Pisier in [122] and extended by Buchholz in [118] (see also [117]). Note also that the inequality (11) below was first proved in [122] in the case $n = 1$ (i.e. for free gaussians) and that a similar inequality holds for products of free gaussians (see [118]).

We will need the following notations: $(e_i)_{i\in I}$ will denote an orthonormal basis of $H_{\mathbb{R}}$, and for a multi-index $\underline{i}$ of length $n, \underline{i} = (i_1, \dots, i_n) \in I^n, e_{\underline{i}} = e_{i_1} \otimes \cdots \otimes e_{i_n}$. $(e_{\underline{i}})_{|\underline{i}|=n}$ is a real orthonormal basis of $H_{\mathbb{C}}^{\otimes n}$ equipped with the free scalar product and $(e_{\underline{i}})_{|\underline{i}|\geq 0}$ is a real orthonormal basis of the free Fock space.

Lemma (2.2.3)[113]: Let $n \geq 1, K$ a complex Hilbert space and $(\alpha_{\underline{i}})_{|\underline{i}|=n}$ a finitely supported family of $B(K)$. Then:

$$\max_{0\leq k\leq n}\left\{\left\|\left(\alpha_{\underline{j},\underline{l}}\right)_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}}\right\|\right\} \leq \left\|\sum_{|\underline{i}|=n} \alpha_{\underline{i}} \otimes (e_{\underline{i}})\right\|$$

$$\leq (n+1) \max_{0\leq k\leq n}\left\{\left\|\left(\alpha_{\underline{j},\underline{l}}\right)_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}}\right\|\right\} \tag{11}$$

Proof. We write

$$\sum_{|\underline{i}|=n} \alpha_{\underline{i}} \otimes W(e_{\underline{i}}) = \sum_{k=0}^{n} F_k$$

where

$$F_k = \sum_{|\underline{l}|\leq n-k} \alpha_{\underline{j},\underline{l}} \otimes a^*\left(e_{\underline{j}}\right) a\left(e_{\underline{l}}\right)$$

we have

$$F_k = \left(\dots I_K \otimes a^*\left(e_{\underline{j}}\right) \dots\right)_{|\underline{j}|=n-k} \left(\alpha_{\underline{j},\underline{l}} \otimes I_{\mathcal{F}_0(H_{\mathbb{C}})}\right)_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}} \begin{pmatrix} \vdots \\ I_K \otimes a(e_{\underline{l}}) \\ \vdots \end{pmatrix}_{|\underline{l}|=k}$$

that is, F_k is a product of three matrices, the first is a row indexed by $\underline{j}$, the third a column indexed by $\underline{l}$. Note that

$$\left\|\left(\dots a^*\left(e_{\underline{j}}\right)\dots\right)_{|\underline{j}|=n-k}\right\|^2 = \left\|\sum_{|\underline{j}|=n-k} a^*\left(e_{\underline{j}}\right)\left(a^*\left(e_{\underline{j}}\right)\right)^*\right\|$$

$$= \left\|\sum_{|\underline{j}|=n-k} a^*\left(e_{\underline{j}}\right)a\left(Ue_{\underline{j}}\right)\right\|$$

It is easy to see that $\sum_{|\underline{j}|=n-k} a^*\left(e_{\underline{j}}\right)a\left(Ue_{\underline{j}}\right)$ is the orthogonal projection on $\oplus_{p\geq n-k}H^{\otimes p}$.

$$\left\|\left(\dots a^*\left(e_{\underline{j}}\right)\dots\right)_{|\underline{j}|=n-k}\right\| \leq 1$$

Therefore

$$\|F_k\| = \left\|\left(\dots I_K\otimes a^*\left(e_{\underline{j}}\right)\dots\right)_{|\underline{j}|=n-k}\right\| \cdot \left\|\left(\alpha_{\underline{j},\underline{l}}\otimes I_{\mathcal{F}_0(H_{\mathbb{C}})}\right)_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}}\right\|$$

$$\cdot \left\|\begin{pmatrix}\vdots\\ I_K\otimes a\left(e_{\underline{l}}\right)\\ \vdots\end{pmatrix}_{|\underline{l}|=k}\right\|$$

$$\left\|\left(\dots a^*\left(e_{\underline{j}}\right)\dots\right)_{|\underline{j}|=n-k}\right\| \cdot \left\|\left(\alpha_{\underline{j},\underline{l}}\right)_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}}\right\| \cdot \left\|\left(\dots a^*\left(Ue_{\underline{l}}\right)\dots\right)_{|\underline{l}|=k}\right\|$$

$$\cdot \left\|\left(\alpha_{\underline{j},\underline{l}}\right)_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}}\right\|$$

It follows that

$$\sum_{|\underline{j}|=n} \alpha_{\underline{i}}\otimes W\left(e_{\underline{i}}\right)\Big\| \leq \sum_{k=0}^{n}\Big\|F_k\Big\| \leq (n+1)\max_{0\leq k\leq n}\left\|\left(\alpha_{\underline{j},\underline{l}}\right)_{|\underline{j}|=n-k}\right\|$$

To prove the first inequality, fix $0\leq k_0\leq n$ and consider $\left(v_{\underline{p}}\right)_{|\underline{p}|=k_0}$ such that $\sum_{|\underline{p}|=k_0}\left\|v_{\underline{p}}\right\|^2 < +\infty$. Let $\eta = \sum_{|\underline{p}|=k_0} v_{\underline{p}}\otimes Ue_{\underline{p}}$. We have:

$$\left\|\sum_{|\underline{i}|=n}\alpha_{\underline{i}}\otimes W\left(e_{\underline{i}}\right)\eta\right\|^2 = \sum_{k=0}^{n}\|F_k\eta\|^2 \geq \|F_{k_0}\eta\|^2$$

$$= \left\|\sum_{\substack{|\underline{j}|n-k_0\\|\underline{i}|=k_0}} \alpha_{\underline{j},\underline{l}}v_{\underline{l}}\otimes e_{\underline{j}}\right\|^2$$

$$= \sum_{|\underline{j}|=n-k_0} \left\| \sum_{|\underline{l}|=k_0} \alpha_{\underline{j},\underline{l}} v_{\underline{l}} \right\|^2$$

$$= \left\| \left(\alpha_{\underline{j},\underline{l}}\right)_{\substack{|\underline{j}|=n-k_0 \\ |\underline{l}|=k_0}} \begin{pmatrix} \vdots \\ v_{\underline{l}} \\ \vdots \end{pmatrix}_{|\underline{l}|=k_0} \right\|^2$$

Then the result follows.

Using Lemma (2.2.3), it is now easy to prove that $\Gamma_0(H_{\mathbb{R}})$ is not injective as soon as $\dim H_{\mathbb{R}} \geq 2$. Suppose that $\Gamma_0(H_{\mathbb{R}})$ is injective and $\dim H_{\mathbb{R}} \geq 2$. Choose two orthonormal vectors e_1 and e_2 in $H_{\mathbb{R}}$. For $n \geq 1$ we have by semi-discreteness (which is equivalent to the injectivity):

$$\tau\left(\sum_{|\underline{i}|=n} W(e_{\underline{i}})^* W(e_{\underline{i}})\right) \leq \left\| \sum_{|\underline{i}|=n} \overline{W(e_{\underline{\iota}})} \otimes W(e_{\underline{i}}) \right\|$$

where in the above sums, the index $\underline{i} \in \{1,2\}^n$. However,

$$\tau\left(\sum_{|\underline{i}|=n} W(e_{\underline{i}})^* W(e_{\underline{i}})\right) = \sum_{|\underline{i}|=n} \langle W(e_{\underline{i}})\Omega, W(e_{\underline{i}})\Omega \rangle_0$$

$$= \sum_{|\underline{i}|=n} \|e_{\underline{i}}\|^2 = 2^n$$

On the other hand, by Lemma (2.2.3),

$$\left\| \sum_{|\underline{i}|=n} \overline{W(e_{\underline{\iota}})} \otimes W(e_{\underline{i}}) \right\| \leq (n+1) \max_{0 \leq k \leq n} \left\{ \left\| \left(\overline{W\left(e_{\underline{j},\underline{l}}\right)} \right)_{\substack{|\underline{j}|=n-k \\ |\underline{l}|=k}} \right\| \right\}$$

$$\leq (n+1) \left(\sum_{|\underline{i}|=n} \|W(e_{\underline{i}})\|^2 \right)^{\frac{1}{2}}$$

$$\leq (n+1)(2^n (n+1)^2)^{\frac{1}{2}}$$

$$\leq (n+1)^2 2^{\frac{n}{2}}$$

Combining the preceding inequalities, we get $2^n \leq (n+1)^2 2^{\frac{n}{2}}$ which yields a contradiction for sufficiently large n. Therefore, $\Gamma_0(H_{\mathrm{R}})$ is not injective if $\dim H_{\mathbb{R}} \geq 2$.

In the following we state and prove the generalized inequality (1). It actually solves a question of Marek Bozejko (in [126] page 210) whether it is possible to find an operator coefficient version of the following inequality (this is inequality (9) in Lemma (2.2.2)):

$$\left\| \sum_{|\underline{i}|=n} \alpha_{\underline{i}} e_{\underline{i}} \right\| \leq \left\| \sum_{|\underline{i}|=n} \alpha_{\underline{i}} W(e_{\underline{i}}) \right\| \leq C_q^{\frac{3}{2}} (n+1) \left\| \sum_{|\underline{i}|=n} \alpha_{\underline{i}} e_{\underline{i}} \right\| \qquad (12)$$

where $(\alpha_i)_i$ is a finitely supported family of complex numbers. Inequality (12) was proved in [126] for the q-deformation, and generalized in [124] for the Yang-Baxter deformation.

First, we need to recall some basic notions from operator space theory. See [121] and [125] for more information. Given K a complex Hilbert space, we can equip K with the column, respectively the row, operator space structure denoted by K_c, respectively K_r, and defined by

$$K_c = B(\mathbb{C}, K) \text{ and } K_r = B(K^*, \mathbb{C}).$$

Moreover, we have $K_c^* = \bar{K}_r$ as operator spaces.

Given two operator spaces E and F, let us briefly recall the definition of the Haagerup tensor product of E and F. $E \otimes F$ will denote the algebraic tensor product of E and F. For $n \geq 1$ and $x = (x_{i,j})$ belonging to $M_n(E \otimes F)$ we define

$$\| x \|_{(h,n)} = \inf \left\{ \| y \|_{M_{n,r}(E)} \| z \|_{M_{r,n}(F)} \right\}$$

where the infimum runs over all $r \geq 1$ and all decompositions of x of the form

$$x_{i,j} = \sum_{k=1}^{r} y_{i,k} \otimes z_{k,j}.$$

By Ruan's theorem, this sequence of norms define an operator space structure on the completion of $E \otimes F$ equipped with $\|\cdot\|_h = \|\cdot\|_{(h,1)}$. The resulting operator space, which is called the Haagerup tensor product of E and F is denoted by $E \otimes_h F$.

In this setting, a bilinear map $u: E \times F \to B(K)$ is said to be completely bounded, in short c.b, if and only if the associated linear map $\hat{u}: E \otimes F \to B(K)$ extends completely boundedly to $E \otimes_h F$. We define $\| u \|_{cb} = \| \hat{u} \|_{cb}$. This notion goes back to Christensen and Sinclair [120]. We will often use the following classical identities for hilbertian operator spaces:

$$K_c \otimes_{\min} H_r = K_c \otimes_h H_r = \mathcal{K}(\bar{H}, K),$$

where $\mathcal{K}$ stands for the compact operators and

$$K_c \otimes_{\min} H_c = K_c \otimes_h H_c = (K \otimes_2 H)_c$$

and similarly for rows using duality.

There is another notion of complete boundedness for bilinear maps, called jointly complete boundedness. Let E, F be operator spaces, K a complex Hilbert space, and $u: E \times F \to B(K)$ a bilinear map. u is said to be jointly completely bounded (in short j.c.b) if and only if for any C*-algebras B_1 and B_2, u can be boundedly extended to a bilinear map $(u)_{B_1,B_2} : E \otimes_{\min} B_1 \times F \otimes_{\min} B_2 \to B(K) \otimes_{\min} B_1 \otimes_{\min} B_2$ taking $(e \otimes b_1, f \otimes b_2)$ to $u(e,f) \otimes b_1 \otimes b_2$.

We put $\| u \|_{jcb} = \sup_{B_1,B_2} \|(u)_{B_1,B_2}\|$. Observe that in this definition B_1 and B_2 can be replaced by operator spaces.

We will need the fact that every bilinear c.b map is a j.c.b map with $\|u\|_{jcb} \leq \| u \|_{cb}$. Let K be a complex Hilbert space and $u: B(K) \times K_c \to K_c$ the bilinear map taking (φ, k) to $\varphi(k)$. Then it is easy to see that u is a norm one bilinear cb map. To simplify our notations, $H_{\mathbb{C}}$ will be, most of the time, replaced by H in the rest. For the same reason we will denote by $H_c^{\otimes n}$ (respectively $H_r^{\otimes n}$) the column Hilbert space $\left(H_{\mathbb{C}}^{\otimes n}\right)_c$(respectively the row Hilbert space $\left(H_{\mathbb{C}}^{\otimes n}\right)_r$).

Lemma (2.2.4)[113]: Let $n \geq 1$. The mappings $a^*: H_c^{\otimes n} \to B(\mathcal{F}_T(H_{\mathbb{C}}))$ and $a: \bar{H}_r^{\otimes n} \to B(\mathcal{F}_T(H_{\mathbb{C}}))$ are completely bounded with cb-norms less than $\sqrt{C}_q$.

Proof. Let us start with the proof of the statement concerning a^*. Let $n \geq 1, K$ a complex Hilbert space and $(\alpha_i)_{|i|=n}$ a finitely supported family of $B(K)$ such that

$$\sum_{|\underline{i}|=n} \alpha_{\underline{i}} \otimes e_{\underline{i}} \|_{B(K)\otimes_{\min} H_c} < 1.$$

Then, since the maps $a^*(e_i)$ acts diagonally with respect to degrees of tensors in $\mathcal{F}_T(H_{\mathbb{C}})$,

$$\sum_{|\underline{i}|=n} \alpha_{\underline{i}} \otimes a^*(e_{\underline{i}}) \Big\|_{B(K)\otimes_{\min} B(\mathcal{F}_T(H_{\mathbb{C}}))} = \sup_{k\geq 0} \Big\| \sum_{|\underline{i}|=n} \alpha_{\underline{i}} \otimes a^*(e_{\underline{i}}) \|_{B(K)\otimes_{\min} B(H^{\otimes k}, H^{\otimes n+k})}$$

To compute the right term, fix $k \geq 0$ and let $\left(\xi_{\underline{j}}\right)_{|\underline{j}|=k}$ be a finitely supported family of vectors in K such that

$$\sum_{|\underline{j}|=k} \xi_{\underline{j}} \otimes e_{\underline{j}} \|_{K\otimes_2 H^{\otimes k}} < 1.$$

By (3) we have

$$\left\|\sum_{\underline{i},\underline{j}} \alpha_{\underline{i}}\left(\xi_{\underline{j}}\right) \otimes e_{\underline{i}} \otimes e_{\underline{j}}\right\|_{K\otimes_2 H^{\otimes n+k}} \leq C_q^{\frac{1}{2}} \left\|\sum_{\underline{i},\underline{j}} \alpha_{\underline{i}}\left(\xi_{\underline{j}}\right) \otimes e_{\underline{i}} \otimes e_{\underline{j}}\right\|_{K\otimes_2 H^{\otimes n}\otimes_2 H^{\otimes k}}.$$

Let $u: B(K) \times K_c \to K_c$ given by $(\varphi, \xi) \mapsto \varphi(\xi)$. Recall that $\| u \|_{cb} = 1$. Consequently, $\| u \|_{jcb} \leq 1$. Therefore, we deduce

$$\begin{aligned}
&\left\|\sum_{\underline{i},\underline{j}} \alpha_{\underline{i}}\left(\xi_{\underline{j}}\right) \otimes e_{\underline{i}} \otimes e_{\underline{j}}\right\|_{K\otimes_2 H^{\otimes n}\otimes_2 H^{\otimes k}} \\
&= \left\|\sum_{\underline{i},\underline{j}} \alpha_{\underline{i}}\left(\xi_{\underline{j}}\right) \otimes e_{\underline{i}} \otimes e_{\underline{j}}\right\|_{K_c\otimes_{\min} H_c^{\otimes n}\otimes_{\min} H_c^{\otimes k}} \\
&= \left\|(u)_{H_c^{\otimes n}, H_c^{\otimes k}k}\left(\sum_{\underline{i}} \alpha_{\underline{i}} \otimes e_{\underline{i}}, \sum_{\underline{j}} \xi_{\underline{j}} \otimes e_{\underline{j}}\right)\right\| \\
&\leq \| u \|_{jcb} \left\|\sum_{\underline{i}} \alpha_{\underline{i}} \otimes e_{\underline{i}}\right\|_{B(K)\otimes_{\min} H_c^{\otimes n}} \left\|\sum_{\underline{j}} \xi_{\underline{j}} \otimes e_j\right\|_{K_c\otimes_{\min} H_c^{\otimes k}} \\
&\leq 1
\end{aligned}$$

By the result just proved, for any complex Hilbert space K and for any finitely supported family $\left(\alpha_{\underline{i}}\right)_{|\underline{i}|=n}$ of $B(K)$ we have

$$\left\|\sum_{|\underline{i}|=n} \alpha_i \otimes a^*\left(e_{\underline{i}}\right)\right\|_{B(K)\otimes_{\min} B(\mathcal{F}_T(H_{\mathbb{C}}))} \leq \sqrt{C_q} \left\|\sum_{|\underline{i}|=n} \alpha_{\underline{i}} \otimes e_{\underline{i}}\right\|_{B(K)\otimes_{\min} H_c^{\otimes n}}$$

Taking adjoints on both sides we get

$$\left\|\sum_{|\underline{i}|=n} \alpha_i^* \otimes a\left(\overline{e}_{\underline{i}}\right)\right\|_{B(K)\otimes_{\min} B(\mathcal{F}_T(H_C))} \leq \sqrt{C_q} \left\|\sum_{|\underline{i}|=n} \alpha_{\underline{i}} \otimes \overline{e}_{\underline{i}}\right\|_{B(K)\otimes_{\min} \bar{H}_r^{\otimes n}}$$

Changing α_i^* to α_i and using the fact that U (reversing the order of tensor) is a complete isometry on $H_r^{\overline{\otimes n}^{1}}$, we get that for any finitely supported family $(\alpha_i)_{|i|=n}$ of $B(K)$ we have

$$\left\|\sum_{|\underline{i}|=n} \alpha_{i} \otimes a(\overline{e_{\underline{l}}})\right\|_{B(K)\otimes_{\min}B(\mathcal{F}_T(H_C))} \leq \sqrt{C_q}\left\|\sum_{|\underline{i}|=n} \alpha_{\underline{i}} \otimes \overline{e_{\underline{l}}}\right\|_{B(K)\otimes_{\min}\bar{H}_r^{\otimes n}}.$$

In other words,

$$a: \bar{H}_r^{\otimes n} \to B(\mathcal{F}_T(H_{\mathbb{C}}))$$

is also completely bounded with norm less than $\sqrt{C}_q$.

Corollary (2.2.5)[113]: For any $n \geq 0$, and any $k \in \{0 \dots n\}$,

$$U_k: H_c^{\otimes n-k} \otimes_h H_r^{\otimes k} \to B(\mathcal{F}_T(H_{\mathbb{C}}))$$

is completely bounded with cb-norm less than C_q.

Proof. Let us denote by M the multiplication map $B(\mathcal{F}_T(H_{\mathbb{C}})) \otimes_h B(\mathcal{F}_T(H_{\mathbb{C}})) \to B(\mathcal{F}_T(H_{\mathbb{C}}))$ given by $A \otimes B \mapsto AB, M$ is obviously completely contractive. We have the formula

$$U_k = M(a^* \otimes a\mathcal{J})$$

if $\mathcal{J}: H^{\otimes k} \to \bar{H}^{\otimes k}$ is the conjugation (which is a complete isometry). By injectivity of the Haagerup tensor product and by Lemma (2.2.4) we deduce that

$$\|a^* \otimes a\mathcal{J}\|_{cb} \leq C_q$$

Then

$$\|U_k\|_{cb} \leq \| M \|_{cb} \|a^* \otimes a\mathcal{J}\|_{cb} \leq C_q$$

Recall that, by definition, $\Gamma_T^\infty(H_{\mathrm{R}})$ is identified with $\Gamma_T(H_{\mathbb{R}})$ by the mapping sending ξ to $W(\xi)$. Thus $\Gamma_T^\infty(H_{\mathbb{R}})$ inherits the operator space structure of $\Gamma_T(H_{\mathbb{R}})$. In particular for all $n \geq 0, H^{\otimes n}$ will be equipped with the operator space structure of $E_n = \{W(\xi), \xi \in H^{\otimes n}\}$.

Theorem (2.2.6) below was first obtained via elementary, but long, computations. In the version presented here, we have chosen to follow an approach indicated to us by Eric Ricard. This approach is much more transparent but involves some notions of operator space theory.

Theorem (2.2.6)[113]: Let K be a complex Hilbert space. Then for all $n \geq 0$ and for all $\xi \in B(K) \otimes_{\min} H^{\otimes n}$ we have

$$\max_{0\leq k\leq n} \|(Id \otimes R_{n,k}^*)(\xi)\| \leq \| (Id \otimes W)(\xi) \|_{\min} \leq C_q(n+1) \max_{0\leq k\leq n} \|(Id \otimes R_{n,k}^*)(\xi)\| \quad (13)$$

where Id denotes the identity mapping of $B(K)$, and where the norm $\|(Id \otimes R_{n,k}^*)(\xi)\|$ is that of $B(K) \otimes \min H_c^{\otimes n-k} \otimes \min H_r^{\otimes k}$.

Proof. For the second inequality, we use the Wick formula :

$$W|_{H^{\otimes n}} = \sum_{k=0}^{n} U_k R_{n,k}^*.$$

Let $\xi \in B(K) \otimes_{\min} H^{\otimes n}$, then by Corollary (2.2.5)

$$\| (Id \otimes W)(\xi) \|_{\min} \leq C_q \sum_{k=0}^{n} \|(Id \otimes R_{n,k}^*)(\xi)\|$$

which yields the majoration.

For the minoration, for $x \in H_c^{\otimes n-k} \otimes H_r^{\otimes k} \subset B(\bar{H}^{\otimes k}, H^{\otimes n-k})$, we claim that

$$P_{n-k} U_k(x)|_{H^{\otimes}k} = x(U\mathcal{J}) \quad (14)$$

where P_{n-k} is the projection on tensors of rank $n-k$ in $\mathcal{F}_T(H_{\mathbb{C}})$. Assuming this claim and recalling that U and $\mathcal{J}$ are (anti)-isometry, we get that for any $x \in B(K) \otimes_{\min} H_c^{\otimes n-k} \otimes_{\min} H_r^{\otimes k}$

$$\| x \|_{B(K)\otimes_{\min} H_c^{\otimes n-k} \otimes_{\min} H_r^{\otimes k}} \leq \|P_{n-k}\|_{B(\mathcal{F}_T(H_{\mathbb{C}}))} \|(Id \otimes U_k)(x)\|_{B(K)\otimes_{\min} B(\mathcal{F}_T(H_{\mathbb{C}}))}$$

The conclusion follows applying this inequality to $x = \left(Id \otimes R_{n,k}^*\right)(\xi)$ To prove (14), it suffices to consider an elementary tensor product with entries in any basis of H, say $x = e_{\underline{i}} \otimes e_{\underline{j}}$. Consider $e_{\underline{l}} \in H^{\otimes k}$, a length argument gives that $a\left(\mathcal{J} e_{\underline{j}}\right) \cdot e_{\underline{l}}$ is of the form $\lambda\Omega$, with

$$\lambda = \left\langle a\left(\mathcal{J} e_{\underline{j}}\right) \cdot e_{\underline{l}}, \Omega\right\rangle = \left\langle e_{\underline{l}}, \mathcal{J} U e_{\underline{j}}\right\rangle$$

We deduce that

$$P_{n-k} U_k \left(e_i \otimes e_{\underline{j}}\right) \cdot e_{\underline{l}} = \left\langle e_{\underline{l}}, U\mathcal{J} e_{\underline{j}}\right\rangle e_{\underline{i}}$$

On the other hand, viewing x as an operator, we compute

$$x(\mathcal{J}U) \cdot e_{\underline{l}} = x \cdot \left(\mathcal{J} U e_{\underline{l}}\right) = \left\langle e_{\underline{j}}, \mathcal{J} U e_{\underline{l}}\right\rangle e_{\underline{i}}$$

But since U is unitary and $\mathcal{J}$ antiunitary,

$$\left\langle e_{\underline{j}}, \mathcal{J} U e_{\underline{l}}\right\rangle = \left\langle e_{\underline{l}}, U \mathcal{J} e_{\underline{j}}\right\rangle$$

This ends the proof.

The following theorem is the main result.

Theorem (2.2.7)[113]: $\Gamma_T(H_{\mathbb{R}})$ is not injective as soon as $\dim(H_{\mathbb{R}}) \geq 2$.

Proof. Let $d \leq \dim H_{\mathbb{R}}$. For all $n \geq 0, \left(\xi_{\underline{i}}\right)_{|\underline{i}|=n}$ will denote a real orthonormal family of $H^{\otimes n}$ equipped with the T-scalar product of cardinal d^n. For example one can take $\xi_i = \left(P_T^{(n)}\right)^{-\frac{1}{2}} e_i$. Suppose that $\Gamma_T(H_{\mathbb{R}})$ is injective. Fix $n \geq 1$. By injectivity we have,

$$\tau\left(\sum_{|\underline{i}|=n} W\left(\xi_{\underline{i}}\right)^* W\left(\xi_{\underline{i}}\right)\right) \leq \left\|\sum_{|\underline{i}|=n} \overline{W\left(\xi_{\underline{\imath}}\right)} \otimes W\left(\xi_{\underline{i}}\right)\right\|$$

It is clear that

$$\tau\left(\sum_{|\underline{i}|=n} W\left(\xi_{\underline{i}}\right)^* W\left(\xi_{\underline{i}}\right)\right) = d^n$$

On the other hand, applying twice (13) consecutively

$$\sum_{|\underline{i}|=n} \overline{W\left(\xi_{\underline{\imath}}\right)} \otimes W\left(\xi_{\underline{i}}\right) \| \leq (n+1)^2 C_q^2 \max_{0\leq k,k'\leq n} \left\{\left\|\sum_{|\underline{i}|=n} \overline{R_{n,k'}^*\left(\xi_{\underline{\imath}}\right)} \otimes R_{n,k}^*\left(\xi_{\underline{i}}\right)\right\|\right\}$$

The norms are computed in $\bar{H}_c^{\otimes n-k'} \otimes_{\min} \bar{H}_r^{\otimes k'} \otimes_{\min} H_c^{\otimes n-k} \otimes_{\min} H_r^{\otimes k}$ for fixed k and k'. We can rearrange this tensor product and use the comparison with the Hilbert Schmidt norm: Let $t = \sum_{|\underline{i}|=n} \overline{R_{n,k'}^*\left(\xi_{\underline{\imath}}\right)} \otimes R_{n,k}^*\left(\xi_{\underline{i}}\right)$

$$\| t \|_{\bar{H}_c^{\otimes n-k'} \otimes_{\min}} \bar{H}_r^{\otimes k'} \otimes \min H_c^{\otimes n-k} \otimes \min H_r^{\otimes k}$$

$$=\| t \|_{\left(\bar{H}^{\otimes n-k'} \otimes_2 H^{\otimes n-k}\right)_c \otimes \min} \left(\bar{H}^{\otimes k'} \otimes_2 H^{\otimes k}\right)_r$$
$$\leq \| t \|_{\left(\bar{H}^{\otimes n-k'} \otimes_2 H^{\otimes n-k}\right) \otimes_2 \left(\bar{H}^{\otimes k'} \otimes_2 H^{\otimes k}\right)}$$
$$\leq \| t \|_{H^{\otimes n-k'} \otimes_2 H^{8k'} \otimes_2 H^{\otimes n-k} \otimes_2 H^{\otimes k}}$$

Finally, we use the estimates on $R_{n,k}^*$:

$$\| t \|_{\bar{H}_c^{\otimes n-k'} \otimes_{\min} H_r^{\otimes k'} \otimes_{\min} H_c^{\otimes n-k} \otimes_{\min} H_r^{\otimes k}} \leq \| t \|_{\bar{H}^{\otimes n-k'} \otimes_2 \bar{H}^{\otimes k'} \otimes_2 H^{\otimes n-k} \otimes_2 H^{\otimes k}}$$
$$\leq C_q \left\| \sum_{|\underline{i}|=n} \overline{\xi_{\underline{\imath}}} \otimes \xi_{\underline{i}} \right\|_{H^n \otimes_2 H^n}$$

But by the choice of ξ_i: $\left\| \sum_{|\underline{i}|=n} \overline{\xi_{\underline{\imath}}} \otimes \xi_{\underline{i}} \right\|_{\overline{H^n} \otimes_2 H^n} = d^{n/2}$.

Combining all inequalities above, we deduce

$$d^n \leq C_q^3 (n+1)^2 d^{n/2}$$

which yields a contradiction when n tends to infinity as soon as $d \geq 2$.

Let $C_T^*(H_{\mathbb{R}})$ be the C^*-algebra generated by all gaussians $G(f)$ for $f \in H_{\mathbb{R}}$. The preceding theorem implies directly that $C_T^*(H_{\mathbb{R}})$ is not nuclear as soon as $\dim(H_{\mathbb{R}}) \geq 2$ (cf. [119] Corollary 6.5). Actually the preceding argument can be modified to prove that $C_T^*(H_{\mathbb{R}})$ does not have the weak expectation property as soon as $\dim H_{\mathbb{R}} \geq 2$. Recall that a C^*-algebra A has the weak expectation property (WEP in short) if and only if the canonical inclusion $A \to A^{**}$ factorizes completely contractively through $B(K)$ for some complex Hilbert space K. By the results of Haagerup (cf. [125] Chapter 15) a C^*-algebra A has the WEP if and only if for all finite family $x_1, \dots, x_n$ in A

$$\left\| \sum_{i=1}^n x_i \otimes \overline{x_\imath} \right\|_{A \otimes_{\max} \bar{A}} = \left\| \sum_{i=1}^n x_i \otimes \overline{x_\imath} \right\|_{A \otimes_{\min} \bar{A}} \tag{15}$$

Corollary (2.2.8)[113]: $C_T^*(H_{\mathbb{R}})$ does not have the WEP as soon as $\dim H_{\mathbb{R}} \geq 2$.

Proof. Let us use the same notations as in the preceding proof and suppose that $C_T^*(H_{\mathbb{R}})$ has the WEP. Fix $n \geq 1$, by (15) we have

$$\left\| \sum_{|\underline{i}|=n} W(\xi_{\underline{i}}) \otimes \overline{W(\xi_{\underline{\imath}})} \right\|_{C_T^*(H_{\mathbb{R}}) \otimes_{\max} C_T^*(H_{\mathbb{R}})}$$
$$\leq \left\| \sum_{|\underline{i}|=n} W(\xi_{\underline{i}}) \otimes \overline{W(\xi_{\underline{\imath}})} \right\|_{C_T^*(H_{\mathbb{R}}) \otimes_{\min} \overline{C_T^*(H_{\mathbb{R}})}} \tag{16}$$

To estimate from below the left handside of (16) observe that $\Phi: \overline{C_T^*(H_{\mathbb{R}})} \to C_T^*(H_{\mathrm{R}})'$ taking $\overline{W(\xi)}$ to $\mathcal{J}UW(\xi)\mathcal{J}U = W_r(\mathcal{J}U\xi)$ is a *- representation. Thus

$$\left\| \sum_{|\underline{i}|=n} W(\xi_{\underline{i}}) \otimes \overline{W(\xi_{\underline{\imath}})} \right\|_{C_T^*(H_{\mathbb{R}}) \otimes \max C_T^*(H_{\mathbb{R}})}$$
$$= \left\| \sum_{|\underline{i}|=n} W(\xi_{\underline{i}}) \otimes W_r(\mathcal{J}U\xi_{\underline{i}}) \right\|_{C_T^*(H_{\mathbb{R}}) \otimes \max C_T^*(H_{\mathrm{R}}))'}$$

$$\geq \left\| \sum_{|\underline{i}|=n} W(\xi_{\underline{i}}) W_r(\mathcal{J}U\xi_{\underline{i}}) \right\|_{B(\mathcal{F}_T(H_{\mathbb{C}}))}$$
$$\geq \sum_{|\underline{i}|=n} \langle \mathcal{J}U\xi_{\underline{i}}, W(\xi_{\underline{i}})^* \Omega \rangle_T$$
$$\geq \sum_{|\underline{i}|=n} \langle \mathcal{J}U\xi_{\underline{i}}, W(\mathcal{J}U\xi_{\underline{i}}) \Omega \rangle_T$$
$$\geq \sum_{|\underline{i}|=n} \|\mathcal{J}U\xi_{\underline{i}}\|_T^2 = d^n$$

Then we can finish the proof as for Theorem (2.2.7).

For this we mainly see [123] where the q-Araki-Woods algebras are defined as a generalization of the q - deformed case of Bożejko and Speicher on the one hand, and the quasi-free case of Shlyakhtenko (cf. [15]) on the other. Let H_{R} be a real Hilbert space given with U_t, a strongly continuous group of orthogonal transformations on $H_{\mathbb{R}}.U_t$ can be extended to a unitary group on the complexification $H_{\mathbb{C}}$. Let A be its positive non-singular generator on $H_{\mathbb{C}}: U_t = A^{it}$. A new scalar product $\langle .,, \rangle_U$ is defined on $H_{\mathbb{C}}$ by the following relation:

$$\langle \xi, \eta \rangle_U = \langle 2A(1+A)^{-1}\xi, \eta \rangle$$

We will denote by H the completion of $H_{\mathbb{C}}$ with respect to this new scalar product.
For a fixed $q \in]-1,1[$, we now consider the q-deformed Fock space associated with H and we denote it by $\mathcal{F}_q(H)$. Recall that it is the Fock space with the following Yang-Baxter deformation T defined by:

$$T: H \otimes H \longrightarrow H \otimes H$$
$$\xi \otimes \eta \mapsto q\eta \otimes \xi$$

Or equivalently, for every $n \geq 2$ and $\sigma \in S_n$ we have

$$\varphi(\sigma) = q^{i(\sigma)} U_\sigma$$

where $i(\sigma)$ denotes the number of inversions of the permutation σ and U_σ is the unitary on $H^{\otimes n}$ defined by

$$U_\sigma(f_1 \otimes \cdots \otimes f_n) = f_{\sigma^{-1}(1)} \otimes \cdots \otimes f_{\sigma^{-1}(n)}$$

In this setting, the q-Araki-Woods algebra is the following von Neumann algebra

$$\Gamma_q(H_{\mathbb{R}}, U_t) = \{G(h), h \in H_{\mathbb{R}}\}'' \subset B\left(\mathcal{F}_q(H_{\mathbb{C}})\right)$$

Let $H'_{\mathbb{R}} = \{g \in H, \langle g, h \rangle_U \in \mathbb{R} \text{ for all } h \in H_{\mathbb{R}}\}$ and

$$\Gamma_{q,r}(H'_{\mathbb{R}}, U_t) = \{G_r(h), h \in H'_{\mathbb{R}}\}''$$

where $G_r(h)$ is the right gaussian corresponding to the right creation operator.

Since $\Gamma_{q,r}(H'_{\mathbb{R}}, U_t) \subset \Gamma_q(H_{\mathbb{R}}, U_t)', \overline{H_{\mathbb{R}} + \imath H_{\mathbb{R}}} = H$ and $\overline{H'_{\mathbb{R}} + \imath H'_{\mathbb{R}}} = H$(cf. [15]), it is easy to deduce that Ω is cyclic and separating for both $\Gamma_q(H_{\mathbb{R}}, U_t)$ and $\Gamma_{q,r}(H'_{\mathbb{R}}, U_t)$. So Tomita's theory can apply : recall that the anti-linear operator S is the closure of the operator defined by :

$$S(x\Omega) = x^*\Omega \text{ for all } x \in \Gamma_q(H_{\mathbb{R}}, U_t)$$

Let $S = J\Delta^{\frac{1}{2}}$ be its polar decomposition. J and Δ are called respectively the modular conjugation and the modular operator. The following explicit formulas hold (cf. [123] and [15])

$$S(h_1 \otimes \cdots \otimes h_n) = h_n \otimes \cdots \otimes h_1 \text{ for all } h_1, \ldots, h_n \in H_{\mathbb{R}}$$
Δ is the closure of the operator $\oplus_{n=0}^{\infty} (A^{-1})^{\otimes n}$ and
$$J(h_1 \otimes \cdots \otimes h_n) = A^{-\frac{1}{2}} h_n \otimes \cdots \otimes A^{-\frac{1}{2}} h_1 \text{ for all } h_1, \ldots, h_n \in H_{\mathbb{R}} \cap \operatorname{dom} A^{-\frac{1}{2}}$$
By Tomita's theory, we have
$$\Gamma_q(H_{\mathbb{R}}, U_t)' = J\Gamma_q(H_{\mathbb{R}}, U_t)J$$
Let $h \in H_{\mathbb{R}}$, as in [15] we have $Jh \in H_{\mathbb{R}'}$, then, since Ω is separating for $\Gamma_{q,r}(H'_{\mathbb{R}}, U_t)$, we obtain that $JG(h)J = G_r(Jh) \in \Gamma_{q,r}(H'_{\mathbb{R}}, U_t)$, so that
$$\Gamma_q(H_{\mathbb{R}}, U_t)' = \Gamma_{q,r}(H'_{\mathbb{R}}, U_t)$$
Moreover, if $\xi \in \Gamma_q(H_{\mathbb{R}}, U_t)\Omega$, then $J\xi \in \Gamma_{q,r}(H'_{\mathbb{R}}, U_t)\Omega$ and since Ω is separating, we get $JW(\xi)J = W_r(J\xi)$.

Recall that if U_t is non trivial, the vacuum expectation φ is no longer tracial and is called the q-quasi-free state. In fact in most cases (cf. [123] Theorem 3.3), Araki-Woods factors are type III von Neumann algebras.

When A is bounded, it is clear that our preliminaries are still valid with minor changes. For example we should get an extra $\|A^{-1}\|^{k/2} = \| A \|^{k/2}$ in the estimation of $\|U_k\|$. Note, in particular, that the Wick formula, as stated in Lemma (2.2.2), is still true, and that the following analogue of Bożejko's scalar inequality holds: (proved in [123])

If A is bounded, $(\eta_u)_{u \in U}$ is a family of vectors in $H^{\otimes n}$ and $(\alpha_u)_{u \in U}$ a finitely supported family of complex numbers then :
$$\left\| \sum_{u \in U} \alpha_u \eta_u \right\|_q \leq \left\| \sum_{u \in U} \alpha_u W(\eta_u) \right\| \leq C_{|q|}^{\frac{3}{2}} \mid \frac{| A \|^{\frac{n+1}{2}} - 1}{\| A \|^{\frac{1}{2}} - 1} \left\| \sum_{u \in U} \alpha_u \eta_u \right\|_q \quad (17)$$
It is also a straightforward verification that Lemma (2.2.4), still hold in this setting. Observe also that U is a unitary on $\mathcal{F}_q(H)$: this follows from the fact that for every $n \geq 1, P_q^{(n)}, A^{\otimes n}$ and U commute on $H^{\otimes n}$. Note that $\mathcal{J}$ is no more an anti unitary from $H^{\otimes k}$ to $H^{\otimes k}$, but since $U_k(I \otimes S) = M(a^* \otimes aU)$, we can deduce, as in the proof of Corollary (2.2.5), that $U_k(I \otimes S)$: $H_c^{\otimes n-k} \otimes_h \overline{H_r^{\otimes k}} \to B(\mathcal{F}_T(H_{\mathbb{C}}))$ is completely bounded with norm less than C_q, where I stands for the identity of $H_c^{\otimes n-k}$. Following the same lines as in the proof of Theorem (2.2.6) we get:

Theorem (2.2.9)[113]: Assume A is bounded. Let K be a complex Hilbert space. Then for all $n \geq 0$ and for all $\xi \in B(K) \otimes_{\min} H^{\otimes n}$ we have
$$\max_{0<k<n} \| \left(Id \otimes \left((I \otimes S)R_{n,k}^*\right)(\xi) \right\| \leq \| (Id \otimes W)(\xi) \|_{\min} \quad (18)$$
$$\leq C_q(n+1) \max_{0 \leq k \leq n} \| \left(Id \otimes \left((I \otimes S)R_{n,k}^*\right)(\xi)\right) \|$$
where Id denotes the identity mapping of $B(K), I$ the identity of $H_c^{\otimes n-k}$, and where the norms of the left and right handsides are taken in $B(K) \otimes \min H_c^{\otimes n-k} \otimes_{\min} \overline{H_r^{\otimes k}}$.

It is known (cf. [123]) that if U_t has a non trivial continuous part then $\Gamma_q(H_{\mathbb{R}}, U_t)$ is not injective. Using our techniques we are able to state a non-injectivity criterion similar to that of [123] but independent of q.

Corollary (2.2.10)[113]: If either
$$\dim E_A(\{1\})H_{\mathbb{C}} \geq 2$$
or for some $T > 1$

$$\frac{\dim E_A(]1,T])H_{\mathbb{C}}}{T^2} > \frac{1}{2}$$

where E_A is the spectral projection of A, then $\Gamma_q(H_{\mathbb{R}}, U_t)$ is non injective.

Proof. We can assume that U_t is almost periodic, then we can write

$$(H_{\mathbb{R}}, U_t) = \left(\hat{H}_{\mathbb{R}}, \mathrm{Id}_{\hat{H}_{\mathbb{R}}}\right) \bigoplus_{\alpha\in\Lambda} \left(H_{\mathbb{R}}^{(\alpha)}, U_t^{(\alpha)}\right)$$

where

$$H_{\mathbb{R}}^{(\alpha)} = \mathbb{R}^2,\ U_t^{(\alpha)} = \begin{pmatrix} \cos(t\ln\lambda_\alpha) & -\sin(t\ln\lambda_\alpha) \\ \sin(t\ln\lambda_\alpha) & \cos(t\ln\lambda_\alpha) \end{pmatrix},\ \lambda_\alpha > 1$$

Thus the eigenvalues of the generator $A^{(\alpha)}$ of $U_t^{(\alpha)}$ are λ_α and λ_α^{-1}.

If $\dim E_A(\{1\})H_{\mathrm{C}} \geq 2$ then $\dim \hat{H}_{\mathbb{R}} \geq 2$ and since U_t is trivial on $\hat{H}_{\mathbb{R}}$, the non-injectivity follows from Theorem (2.2.7) .

For the remaining case we first suppose that $\dim H_{\mathbb{R}} = 2, U_t$ is not trivial and that $\Gamma_q(H_{\mathbb{R}}, U_t)$ is injective. For all $n \geq 1, A^{\otimes n}$ is a positive operator on $H^{\otimes n}$ equipped with the deformed scalar product, we will denote by λ and λ^{-1} the eigenvalues of A with $\lambda > 1$ and by $\left(\xi_{\underline{i}}\right)_{|\underline{i}|=n}$ an orthonormal basis of eigenvectors of $A^{\otimes n}$ associated to the eigenvalues $\left(\lambda_{\underline{i}}\right)_{|\underline{i}|=n}$. Since $\Gamma_q(H_{\mathbb{R}}, U_t)$ is semidiscrete we must have for every $n \geq 1$

$$\begin{aligned}\left\|\sum_{|\underline{i}|=n} W_r(J\xi_{\underline{i}})W(\xi_{\underline{i}})\right\| &\leq \left\|\sum_{|\underline{i}|=n} W_r(J\xi_{\underline{i}}) \otimes W(\xi_{\underline{i}})\right\| \\ &= \left\|\sum_{|\underline{i}|=n} JW(\xi_{\underline{i}})J \otimes W(\xi_{\underline{i}})\right\|\end{aligned}$$

It is easily seen that

$$\begin{aligned}\left\|\sum_{|\underline{i}|=n} W_r(J\xi_{\underline{i}})W(\xi_{\underline{i}})\right\| &\geq \sum_{|\underline{i}|=n} \left\langle \Omega, W_r(J\xi_{\underline{i}})W(\xi_{\underline{i}})\Omega\right\rangle_q \\ &= \sum_{|\underline{i}|=n} \left\langle JW(\xi_{\underline{i}})^* J\Omega, W(\xi_{\underline{i}})\Omega\right\rangle_q \\ &= \sum_{|\underline{i}|=n} \left\langle \Delta^{\frac{1}{2}}\xi_{\underline{i}}, \xi_{\underline{i}}\right\rangle_q = \mathrm{Trace}\left(\left(A^{-\frac{1}{2}}\right)^{\otimes n}\right) = \left(\lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}}\right)^n\end{aligned}$$

On the other hand, the map from $J\Gamma_q(H_{\mathbb{R}}, U_t)J$ to $\overline{\Gamma_q(H_{\mathrm{R}}, U_t)}$ taking $JW(\xi)J$ to $\overline{W(\xi)}$ is a *-isomorphism, hence

$$\left\|\sum_{|\underline{i}|=n} JW(\xi_{\underline{i}})J \otimes W(\xi_{\underline{i}})\right\|_{\min} = \left\|\sum_{|\underline{i}|=n} \overline{W(\xi_{\underline{i}})} \otimes W(\xi_{\underline{i}})\right\|$$

Applying (18) twice, and recalling that on $H^{\otimes k}, S = J\Delta^{\frac{1}{2}} = J\left(A^{\otimes k}\right)^{-\frac{1}{2}}$ and that $J\colon \overline{H_r^{\otimes k}} \to H_r^{\otimes k}$ is completely isometric, we get

$$\left\|\sum_{|\underline{i}|=n}\overline{W(\xi_{\underline{\iota}})}\otimes W(\xi_{\underline{i}})\right\|_{\min}\leq C_q^2(n+1)^2\max_{0\leq k,k'\leq n}\left\|\sum_{|\underline{i}|=n}\overline{(I\otimes S)R_{n,k'}^*(\xi_{\underline{\iota}})}\otimes(I\otimes S)R_{n,k}^*(\xi_{\underline{i}})\right\|$$

$$\leq C_q^2(n+1)^2\max_{0\leq k,k'\leq n}\left\|\sum_{|\underline{i}|=n}\overline{\left(I\otimes\left(A^{\otimes k'}\right)^{-\frac{1}{2}}\right)R_{n,k'}^*(\xi_{\underline{\iota}})}\otimes\left(I\otimes\left(A^{\otimes k}\right)^{-\frac{1}{2}}\right)R_{n,k}^*(\xi_{\underline{i}})\right\|$$

Where the norms are computed in $\overline{H_c^{\otimes n-k'}}\otimes_{\min}\overline{H_r^{\otimes k'}}\otimes_{\min}H_c^{\otimes n-k}\otimes_{\min}H_r^{\otimes k}$. For a fixed (k,k'), let us denote by

$$t=\sum_{|\underline{i}|=n}\overline{\left(I\otimes\left(A^{\otimes k'}\right)^{-\frac{1}{2}}\right)R_{n,k'}^*(\xi_{\underline{\iota}})}\otimes\left(I\otimes\left(A^{\otimes k}\right)^{-\frac{1}{2}}\right)R_{n,k}^*(\xi_{\underline{i}})$$

As in the proof of Theorem (2.2.7), we have the following Hilbert-Schmidt estimate:

$$\|t\|_{\overline{H^{\otimes n-k'}}\otimes_{\min}\overline{H_r^{\otimes k'}}\otimes_{\min}H_c^{\otimes n-k}\otimes_{\min}H_r^{\otimes k}}\leq\|t\|_{\overline{H^{\otimes n-k'}}\otimes_2\overline{H^{\otimes k'}}\otimes_2H^{\otimes n-k}\otimes_2H^{\otimes k}}$$

Recall that $R_{n,k}^*:H^{\otimes n}\to H^{\otimes n-k}\otimes_2 H^{\otimes k}$ is of norm less than $C_{|q|}^{\frac{1}{2}}$ and that $\left\|\left(A^{\otimes k}\right)^{-\frac{1}{2}}\right\|_{B(H^{\otimes k})}=\lambda^{\frac{k}{2}}$. Hence,

$$\|t\|_{\overline{H^{\otimes n-k'}}\otimes_2\overline{H^{\otimes k'}}\otimes_2H^{\otimes n-k}\otimes_2H^{\otimes k}}\leq C_{|q|}\lambda^n\left\|\sum_{|\underline{i}|=n}\overline{\xi_{\underline{\iota}}}\otimes\xi_{\underline{i}}\right\|_{\overline{H^{\otimes n}}\otimes H^{\otimes n}}$$

$$\leq C_{|q|}(\sqrt{2}\lambda)^n$$

Combining all inequalities we get

$$\left(\lambda^{\frac{1}{2}}+\lambda^{-\frac{1}{2}}\right)^n\leq C_{|q|}^3(n+1)^2(\sqrt{2}\lambda)^n.$$

We now return to the general case, we fix $T>1$ and we denote by $\lambda_1,\dots,\lambda_p$ the eigenvalues of A in $]1,T]$ counted with multiplicities. Thus we have $p=\dim E_A(]1,T])H_{\mathbb{C}}$. It is easy to deduce from our first step that for any $n\geq 1$ we have

$$\left(\sum_{i=1}^{p}\lambda_i^{\frac{1}{2}}+\lambda_i^{-\frac{1}{2}}\right)^n\leq C_{|q|}^3(n+1)^2(2p)^{\frac{n}{2}}T^n$$

Since for any i we have $\lambda_i^{\frac{1}{2}}+\lambda_i^{-\frac{1}{2}}\geq 2$ we deduce

$$(2p)^n\leq C_{|q|}^3(n+1)^2(2p)^{\frac{n}{2}}T^n$$

So we necessarily have

$$\frac{2p}{T^2}\leq 1$$

that is to say

$$\frac{\dim E_A]1,T]H_{\mathbb{C}}}{T^2}\leq\frac{1}{2}$$

Chapter 3
QWEP Property for q-Araki-Woods Algebras

We use this model and an elaborated ultraproduct procedure, to show that all q-Araki-Woods von Neumann algebras are QWEP. In addition we show the q-deformed Araki-Woods factors.

Section (3.1): q-Deformed Araki-Woods Factors

For $\mathcal{H}_{\mathbf{R}}$ be a separable real Hilbert space and U_t a strongly continuous one-parameter group of orthogonal transformations on $\mathcal{H}_{\mathbf{R}}$. By linearity U_t extends to a one-parameter unitary group on the complexified Hilbert space $\mathcal{H}_{\mathbf{C}} := \mathcal{H}_{\mathbf{R}} + i\mathcal{H}_{\mathbf{R}}$. Write $U_t = A^{it}$ with the generator A (a positive non-singular operator on $\mathcal{H}_{\mathbf{C}}$) and define an inner product $\langle \cdot,\cdot \rangle_U$ on $\mathcal{H}_{\mathbf{C}}$ by

$$\langle x, y\rangle_U = \langle 2A(1 + A)^{-1}x, y\rangle,\ x, y \in \mathcal{H}_{\mathbf{C}}.$$

Let $\mathcal{H}$ be the complex Hilbert space obtained by completing $\mathcal{H}_{\mathbf{C}}$ with respect to $\langle \cdot,\cdot \rangle_U$.

For $-1 < q < 1$ the q-Fock space $\mathcal{F}_q(\mathcal{H})$ was introduced in [129, 60] as follows.

Let $\mathcal{F}^{\text{finite}}(\mathcal{H})$ be the linear span of $f_1 \otimes \cdots \otimes f_n \in \mathcal{H}^{\otimes n} (n = 0,1, \dots)$ where $\mathcal{H}^{\otimes 0} = \mathbf{C}\Omega$ with vacuum Ω. The sesquilinear form $\langle \cdot,\cdot \rangle_q$ on $\mathcal{F}^{\text{finite}}(\mathcal{H})$ is given by

$$\langle f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_m\rangle_q = \delta_{nm} \sum_{\pi\in S_n} q^{i(\pi)}\langle f_1, g_{\pi(1)}\rangle_U \cdots \langle f_n, g_{\pi(n)}\rangle_U,$$

where $i(\pi)$ denotes the number of inversions of the permutation $\pi \in S_n$. For $-1 < q < 1$, $\langle \cdot,\cdot \rangle_q$ is strictly positive and the q-Fock space $\mathcal{F}_q(\mathcal{H})$ is the completion of $\mathcal{F}^{\text{finite}}(\mathcal{H})$ with respect to $\langle \cdot,\cdot \rangle_q$. Given $h \in \mathcal{H}$ the q-creation operator $a_q^*(h)$ and the q-annihilation operator $a_q(h)$ on $\mathcal{F}_q(\mathcal{H})$ are defined by

$$a_q^*(h)\Omega = h,$$

$$a_q^*(h)(f_1 \otimes \cdots \otimes f_n) = h \otimes f_1 \otimes \cdots \otimes f_n,$$

and

$$a_q(h)\Omega = 0,$$

$$a_q(h)(f_1 \otimes \cdots \otimes f_n) = \sum_{i=1}^{n} q^{i-1}\langle h, f_i\rangle_U f_1 \otimes \cdots \otimes f_{i-1} \otimes f_{i+1} \otimes \cdots \otimes f_n.$$

The operators $a_q^*(h)$ and $a_q(h)$ are bounded operators on $\mathcal{F}_q(\mathcal{H})$ and they are adjoins of each other (see [60, Remark 1.2]).

Following [15] we consider the von Neumann algebra $\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)''$, called a q deformed Araki-Woods algebra, generated on $\mathcal{F}_q(\mathcal{H})$ by

$$s_q(h) := a_q^*(h) + a_q(h),\ h \in \mathcal{H}_{\mathbf{R}}.$$

The vacuum state $\varphi\big(= \varphi_{q,U}\big) := \langle \Omega, \cdot\, \Omega\rangle_q$ on $\Gamma_q(\mathcal{H}_{\mathrm{R}}, U_t)''$ is called the q-quasi-free state.

Proposition (3.1.1)[127]: Ω is cyclic and separating for $\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)''$.

One can canonically extend U_t on $\mathcal{H}$ to a one-parameter unitary group (the so-called second quantization) $\mathcal{F}_q(U_t)$ on $\mathcal{F}_q(\mathcal{H})$ by

$$\mathcal{F}_q(U_t)\Omega = \Omega,$$

$$\mathcal{F}_q(U_t)(f_1 \otimes \cdots \otimes f_n) = (U_t f_1) \otimes \cdots \otimes (U_t f_n).$$

Notice $\mathcal{F}_q(U_t)a_q^*(h)\mathcal{F}_q(U_t)^* = a_q^*(U_t h)$ for $h \in \mathcal{H}$ so that

$$\mathcal{F}_q(U_t)s_q(h)\mathcal{F}_q(U_t)^* = s_q(U_t h),\ h \in \mathcal{H}_{\mathbf{R}}.$$

Thus, $\alpha_t := \operatorname{Ad}\mathcal{F}_q(U_t)$ defines a strongly continuous one-parameter automorphism group on $\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)''$.

Proposition (3.1.2)[127]: The q-quasi-free state φ on $\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)''$ satisfies the KMS condition with respect to α_t at $\beta = 1$.

Let $(\mathcal{K}_{\mathbf{R}}, V_t)$ be another pair of a separable real Hilbert space and a one-parameter group V_t of orthogonal transformations on $\mathcal{K}_{\mathbf{R}}$. Let $T: \mathcal{H}_{\mathbf{R}} \to \mathcal{K}_{\mathbf{R}}$ be a contraction such that $TU_t = V_t T$ for all $t \in \mathbf{R}$. By linearity T extends to a contraction $T: \mathcal{H}_{\mathrm{C}} \to \mathcal{K}_{\mathbf{C}}$ and it satisfies $TU_t = V_t T$ on $\mathcal{H}_{\mathrm{C}}$. Let B be the generator of V_t so that $V_t = B^{it}$. Since

$$TA(1+A)^{-1} = B(1+B)^{-1}T,$$

T can further extend to a contraction from $(\mathcal{H}, \langle\cdot,\cdot\rangle_U)$ to $(\mathcal{K}, \langle\cdot,\cdot\rangle_V)$. Then:

Proposition (3.1.3)[127]: There is a unique completely positive normal contraction $\Gamma_q(T)$: $\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)'' \to \Gamma_q(\mathcal{K}_{\mathbf{R}}, V_t)''$ such that

$$\big(\Gamma_q(T)x\big)\Omega = \mathcal{F}_q(T)(x\Omega),\ x \in \Gamma_q(\mathcal{H}_{\mathrm{R}}, U_t)'',$$

where $\mathcal{F}_q(T): \mathcal{F}_q(\mathcal{H}) \to \mathcal{F}_q(\mathcal{K})$ is given by

$$\mathcal{F}_q(T)(f_1 \otimes \cdots \otimes f_n) = (Tf_1) \otimes \cdots \otimes (Tf_n).$$

In this way, we have presented a q-analogue of Shlyakhtenko's free CAR functor; namely, a von Neumann algebra with a specified state, $\big(\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)'', \varphi\big)$, is associated to each real Hilbert space with a one-parameter group of orthogonal transformations, $(\mathcal{H}_{\mathbf{R}}, U_t)$, and a unital completely positive state-preserving map $\Gamma_q(T): \Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)'' \to \Gamma_q(\mathcal{K}_{\mathbf{R}}, V_t)''$ to every contraction $T: (\mathcal{H}_{\mathbf{R}}, U_t) \to (\mathcal{K}_{\mathbf{R}}, V_t)$.

When $q = 0, \Gamma(\mathcal{H}_{\mathbf{R}}, U_t)'' \equiv \Gamma_0(\mathcal{H}_{\mathbf{R}}, U_t)''$ is a free Araki-Woods factor (of type III) in [15]. On the other hand, when $U_t = \mathrm{id}$ a trivial action, $\Gamma_q(\mathcal{H}_{\mathbf{R}})'' \equiv \Gamma_q(\mathcal{H}_{\mathbf{R}}, \mathrm{id}\ )$ is a q-deformation of the free group factor in [60]; in particular, $\Gamma_0(\mathcal{H}_{\mathbf{R}})'' \cong L\big(\mathbb{F}_{\dim \mathcal{H}_{\mathbf{R}}}\big)$ a free group factor.

The following were proven in [68, 60], but it is still open whether $\Gamma_q(\mathcal{H}_{\mathbf{R}})''$ is a non-injective type II_1 factor whenever $\dim \mathcal{H}_{\mathbf{R}} \geq 2$.

1. If $-1 < q < 1$ and $\dim \mathcal{H}_{\mathbf{R}} > 16/(1-|q|)^2$, then $\Gamma_q(\mathcal{H}_{\mathbf{R}})''$ is not injective.
2. If $\dim \mathcal{H}_{\mathbf{R}} = \infty$, then $\Gamma_q(\mathcal{H}_{\mathbf{R}})$ is a factor (of type II_1) for all $-1 < q < 1$.

These results can be extended to $\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)''$ as follows.

Theorem (3.1.4)[127]: If there is $T \in [1, \infty)$ such that

$$\frac{\dim E_A([1,T])\mathcal{H}_{\mathbf{C}}}{T} > \frac{16}{(1-|q|)^2}$$

where E_A is the spectral measure of A, then $\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)''$ is not injective. In particular, $\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)''$ is not injective if A has a continuous spectrum.

Theorem (3.1.5)[127]: Assume that the almost periodic part of $(\mathcal{H}_{\mathbf{R}}, U_t)$ is infinite dimensional, that is, A has infinitely many mutually orthogonal eigenvectors. Then

$$\big(\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)''\big)'_{\varphi} \cap \Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)'' = \mathbf{C1},$$

where $\big(\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)''\big)_{\varphi}$ is the centralizer of $\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)''$ with respect to the vacuum state φ. In particular, $\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)''$ is a factor.

As usual let S_φ be the closure of the operator given by

$$S_\varphi(x\Omega) = x^*\Omega,\ x \in \Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)'',$$

and let $\Delta_\varphi, J_\varphi$ be the associated modular operator and the modular conjugation. Then the following are seen as in [15]: For $h_1, \ldots, h_n \in \mathcal{H}_{\mathbf{R}}$,

$$S_\varphi(h_1 \otimes h_2 \otimes \cdots \otimes h_n) = h_n \otimes h_{n-1} \otimes \cdots \otimes h_1,$$

and for $h_1, \ldots, h_n \in \mathcal{H}_{\mathbf{R}} \cap \operatorname{dom} A^{-1}$,

$$\Delta_\varphi(h_1 \otimes \cdots \otimes h_n) = (A^{-1}h_1) \otimes \cdots \otimes (A^{-1}h_n).$$

Noting that $\mathcal{D} := \{h + ig : h, g \in \mathcal{H}_{\mathrm{R}} \cap \operatorname{dom} A^{-1}\}$ is a core of A^{-1}(on $\mathcal{H}$) such that $U_t\mathcal{D} = \mathcal{D}$ for all $t \in \mathbf{R}$, we see that

$$\Delta_\varphi^{it} = \mathcal{F}_q(A^{-it}) = \mathcal{F}_q(U_{-t}),\ t \in \mathbf{R}.$$

By this and Theorem (3.1.5) we obtain the following type classification result:

Theorem (3.1.6)[127]: Assume that A has infinitely many mutually orthogonal eigenvectors. Let G be the closed multiplicative subgroup of $\mathbf{R}_+$generated by the spectrum of A ($U_t = A^{it}$). Then $\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)''$ is a non-injective factor of type II_1 or type III $(0 < \lambda \leq 1)$, and

$$\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)'' \text{ is } \begin{cases} \text{type } II_1 & \text{if } G = \{1\}, \\ \text{type III }_\lambda & \text{if } G = \{\lambda^n : n \in \mathbf{Z}\}(0 < \lambda < 1), \\ \text{type } III\ _1 & \text{if } G = \mathbf{R}_+. \end{cases}$$

This result for free Araki-Woods factors (in case of $q = 0$) was shown in [15, 14] generally when $\dim \mathcal{H}_{\mathbf{R}} \geq 2$. Moreover, it was shown as a consequence of Barnett's theorem that free Araki-Woods factors are full whenever U_t is almost periodic (i.e. the eigenvectors of A span $\mathcal{H}$). The assumption of Theorems (3.1.5) and (3.1.6) is a bit too restrictive while the following opposite extreme case is easy to see:

Proposition (3.1.7)[127]: If U_t has no eigenvectors, then $\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)''$ is a type $III\ _1$ factor.

It is worthwhile to note that the type $III\ _0$ case does not appear in the above type classifications.

For example, let $(\mathcal{H}_{\mathbf{R}}, U_t) = \oplus_{k=1}^\infty (\mathbf{R}^2, V_t)$ where $V_t := \begin{bmatrix} \cos(t \log \lambda) & -\sin(t \log \lambda) \\ \sin(t \log \lambda) & \cos(t \log \lambda) \end{bmatrix}$, $0 < \lambda \leq 1$, and write $(T_{q,\lambda}, \varphi_{q,\lambda}) := (\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)'', \varphi)$ with two parameters $q \in (-1,1)$ and $\lambda \in (0,1]$. For $0 < \lambda < 1, T_{q,\lambda}$ is a type III_λ q-deformed Araki-Woods factor. In particular when $q = 0, (T_{0,\lambda}, \varphi_{0,\lambda})$ coincides with the type III_λ free Araki-Woods factor $(T_\lambda, \varphi_\lambda)$ discussed in [11, 15]. For $\lambda = 1, T_{q,1}$ is the q-deformed type II_1 factor treated in [60].

The C^*-algebra $\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t), -1 < q < 1$, generated by $\{s_q(h) : h \in \mathcal{H}_{\mathbf{R}}\}$ on $\mathcal{F}_q(\mathcal{H})$ is considered as the q-analogue of the CAR algebra. From this point of view, the above $T_{q,\lambda}(0 < \lambda < 1)$ may be considered as the q-analogue of Powers' III_λ factor. In fact, we remark that, for $q = -1$, our construction of $T_{q,\lambda}$ provides Powers' III_λ factor. To be more precise, for given $(\mathcal{H}_{\mathbf{R}}, U_t)$, let $\Gamma_-(\mathcal{H}_{\mathbf{R}}, U_t)''$ denote the von Neumann algebra generated by $s_-(h) := a_-^*(h) + a_-(h)(h \in \mathcal{H}_{\mathbf{R}})$ on the Fermion Fock space $\mathcal{F}_-(\mathcal{H})$, where $a_-^*(h)$ and $a_-(h)$ are the Fermion (CAR) creation and annihilation operators.

If $(\mathcal{H}_{\mathbf{R}}, U_t) = \oplus_{k=1}^\infty \left(\mathcal{H}_{\mathbf{R}}^{(k)}, U_t^{(k)}\right)$ where $\mathcal{H}_{\mathbf{R}}^{(k)} = \mathbf{R}^2, U_t^{(k)} = \begin{bmatrix} \cos(t \log \lambda_k) & -\sin(t \log \lambda_k) \\ \sin(t \log \lambda_k) & \cos(t \log \lambda_k) \end{bmatrix}$ with $\lambda_k \leq 1$, then $(\Gamma_-(\mathcal{H}_{\mathbf{R}}, U_t)'', \varphi := \langle \Omega, \cdot\, \Omega \rangle_-)$becomes an Araki-Woods factor

$$\bigotimes_{k=1}^{\infty}\left(M_2(\mathbf{C}),\mathrm{Tr}\left(\cdot\begin{bmatrix}\frac{\lambda_k}{\lambda_k+1} & 0\\ 0 & \frac{1}{\lambda_k+1}\end{bmatrix}\right)\right).$$

Upon these considerations we called $\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)''$ a q-deformed Araki-Woods algebra.

When $T = e^{-t}1_{\mathcal{H}_{\mathbf{R}}}(t > 0)$, we obtain a semigroup $\Gamma_q(e^{-t})(t > 0)$ of completely positive normal contractions on $\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)''$. This is a non-tracial extension of q Ornstein-Uhlenbeck semigroup discussed in [128,115]. In the tracial case (i.e. the case of U_t being trivial), the ultracontractivity for $\Gamma_q(e^{-t})$ was proven in [115] as follows:

$$\|\Gamma_q(e^{-t})x\| \le C_{|q|}^{3/2}\sqrt{\frac{1+e^{-2t}}{(1-e^{-2t})^3}}\parallel x\Omega \parallel,\ x \in \Gamma_q(\mathcal{H}_{\mathbf{R}})''$$

with $C_{|q|}$ given below. In the non-tracial type III case, we have the following hypercontractivity property. This reduces to the above ultracontractivity when $A = 1$ or $\gamma = 0$.

Theorem (3.1.8)[127]: Assume that A is bounded (in particular, this is the case if $\dim \mathcal{H}_{\mathrm{R}} < +\infty$), and let $\gamma := \frac{1}{2}\log \parallel A \parallel$. If $-1 < q < 1$ and $t > \gamma$, then

$$\|\Gamma_q(e^{-t})x\| \le C_{|q|}^{3/2}\sqrt{\frac{1+e^{-(2t-\gamma)}}{(1-e^{-2t})(1-e^{-(2t-\gamma)})(1-e^{-2(t-\gamma)})}}\left\|\Delta_{\varphi}^{\theta/2}x\Omega\right\|$$

for all $x \in \Gamma_q(\mathcal{H}_{\mathrm{R}}, U_t)''$ and $0 \le \theta \le 1$, where

$$C_{|q|} := \frac{1}{\prod_{m=1}^{\infty}(1-|q|^m)}.$$

It might be expected that the hypercontractivity given in the above theorem is valid for the whole $t > 0$. However, the next proposition says that it is impossible to remove the assumption $t > \gamma$, so Theorem (3.1.8) seems more or less best possible. Also, it says that the hypercontractivity in the sense that $\|\Gamma_q(e^{-t})x\| \le C \parallel x\Omega \parallel_q$ holds for some $t > 0$ and for all $x \in \Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)''$ is impossible when A is unbounded; for example, this is the case when $U_t f = f(\cdot + t)$ on $\mathcal{H}_{\mathbf{R}} = L^2(\mathbf{R};\mathbf{R})$.

Proposition (3.1.9)[127]: Let $-1 < q < 1, 0 \le \theta \le 1$ and $t > 0$. If there exists a constant $c > 0$ such that

$$\|\Gamma_q(e^{-t})x\| \le c\left\|\Delta_{\varphi}^{\frac{\theta}{2}}x\Omega\right\|,\ x \in \Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)'',$$

then A is bounded and

$$\parallel A \parallel \le \exp\left(\frac{2t}{\max\{\theta, 1-\theta\}}\right).$$

It seems that it is convenient to consider the hypercontractivity of $\Gamma_q(T)$ in the setting of Kosaki's interpolated L^p-spaces. For a general von Neumann algebra $\mathcal{M}$ and $1 \le p \le \infty$ let $L^p(\mathcal{M})$ be Haagerup's L^p-space. Given a faithful normal state φ on $\mathcal{M}$ let h_φ denote the element of $L^1(\mathcal{M})(\cong \mathcal{M}_*)$ corresponding to φ. For each $1 < p < \infty$ and $0 \le \theta \le 1$, Kosaki's L^p-space $L^p(\mathcal{M};\varphi)_\theta$ with respect to φ is introduced as the complex interpolation space

$$C_{1/p}\left(h_\varphi^{\theta}\mathcal{M}h_\varphi^{1-\theta}, L^1(\mathcal{M})\right)$$

equipped with the complex interpolation norm $\|\cdot\|_{p,\theta}\left(=\|\cdot\|_{C_{1/p}}\right)$. Let $T:\mathcal{H}_{\mathbf{R}}\to\mathcal{K}_{\mathbf{R}}$ be a contraction with $TU_t=V_tT, t\in\mathbf{R}$. The adjoint operator $T^*:\mathcal{K}_{\mathbf{R}}\to\mathcal{H}_{\mathbf{R}}$ is also a contraction satisfying $T^*V_t=U_tT^*, t\in\mathbf{R}$. For each $-1<q<1$ let

$$\mathcal{M}\quad:=\Gamma_q(\mathcal{H}_{\mathbf{R}},U_t)'' \text{ with } \varphi=\langle\Omega,\cdot\,\Omega\rangle_{q,}$$

$$\mathcal{N}:=\Gamma_q(\mathcal{K}_{\mathbf{R}},V_t)'' \text{ with } \psi=\langle\Omega,\cdot\,\Omega\rangle_{q,}$$

where the vacuums in $\mathcal{F}_q(\mathcal{H})$ and in $\mathcal{F}_q(\mathcal{K})$ are denoted by the same Ω. Then, by Proposition (3.1.3) the completely positive normal contractions

$$\Gamma_q(T):\mathcal{M}\to\mathcal{N} \text{ and } \Gamma_q(T^*):\mathcal{N}\to\mathcal{M}$$

are determined by

$$\left(\Gamma_q(T)x\right)\Omega\quad=\mathcal{F}_q(T)(x\Omega),\qquad x\in\mathcal{M},$$

$$\left(\Gamma_q(T^*)y\right)\Omega\quad=\mathcal{F}_q(T^*)(y\Omega),\qquad y\in\mathcal{N}.$$

One can define the contraction $\omega\mapsto\omega\circ\Gamma_q(T^*)$ of $\mathcal{M}_*$ into $\mathcal{N}_*$. Via $\mathcal{M}_*\cong L^1(\mathcal{M})$ and $\mathcal{N}_*\cong L^1(\mathcal{N})$ this induces the contraction $\tilde{\Gamma}_q(T)$ of $L^1(\mathcal{M})$ into $L^1(\mathcal{N})$ as follows:

$$\tilde{\Gamma}_q(T)h_\omega=h_{\omega\circ\Gamma_9(T^*)},\ \omega\in\mathcal{M}_*.$$

We see that for every $0\le\theta\le1$ and $x\in\mathcal{M}$,

$$\tilde{\Gamma}_q(T)\left(h_\varphi^\theta x h_\varphi^{1-\theta}\right)=h_\psi^\theta\left(\Gamma_q(T)x\right)h_\psi^{1-\theta},$$

so that $\tilde{\Gamma}_q(T):L^1(\mathcal{M})\to L^1(\mathcal{N})$ is the (unique) continuous extension of the linear mapping from $h_\varphi^\theta\mathcal{M}h_\varphi^{1-\theta}(\subset L^1(\mathcal{M}))$ into $h_\psi^\theta\mathcal{N}h_\psi^{1-\theta}(\subset L^1(\mathcal{N}))$ given by

$$h_\varphi^\theta x h_\varphi^{1-\theta}\mapsto h_\psi^\theta\left(\Gamma_q(T)x\right)h_\psi^{1-\theta},\ x\in\mathcal{M}.$$

Moreover, the Riesz-Thorin theorem implies that for each $0\le\theta\le1$ and $1\le p\le\infty, \tilde{\Gamma}_q(T)$ maps $L^p(\mathcal{M};\varphi)_\theta$ into $L^p(\mathcal{N};\psi)_\theta$ and

$$\left\|\tilde{\Gamma}_q(T)a\right\|_{p,\theta}\le\|a\|_{p,\theta},\ a\in L^p(\mathcal{M};\varphi)_\theta.$$

The next theorem is shown by using Theorem (3.1.8).

Theorem (3.1.10)[127]: Assume that either $A\left(U_t=A^{it}\right)$ or $B\left(V_t=B^{it}\right)$ is bounded, and let $\rho:=\min\{\|A\|,\|B\|\}$. Let $T:\mathcal{H}_{\mathbf{R}}\to\mathcal{K}_{\mathbf{R}}$ be a bounded operator such that $TU_t=V_tT$ for all $t\in\mathbf{R}$ and $\|T\|<\rho^{-1}$. Then $\tilde{\Gamma}_q(T)$ maps $L^1(\mathcal{M})$ into $\bigcap_{0\le\theta\le1}h_\psi^\theta\mathcal{N}h_\psi^{1-\theta}$ and

$$\left\|\tilde{\Gamma}_q(T)a\right\|_{\infty,\theta}\le C_{|q|}^3\frac{1+\rho^{1/2}\|T\|}{(1-\|T\|)(1-\rho^{1/2}\|T\|)(1-\rho\|T\|)}\|a\|_1$$

for all $a\in L^1(\mathcal{M}), 0\le\theta\le1$.

Section (3.2): Asymptotic Matricial Models

Recall that a C^*-algebra has the weak expectation property (in short WEP) if the canonical inclusion from A into A^{**} factorizes completely contractively through some $B(H)$ (H Hilbert). A C^*-algebra is QWEP if it is a quotient by a closed ideal of an algebra with the WEP. The notion of QWEP was introduced by Kirchberg in [132]. Since then, it became an important notion in the theory of C^*-algebras. Very recently, Pisier and Shlyakhtenko [26] proved that Shlyakhtenko's free quasi-free factors are QWEP. This result plays an important role in their work on the operator space Grothendieck Theorem, as well as in the subsequent related works [134] and [140]. On the other hand, in [131] on the embedding of Pisier's operator Hilbertian space OH and the projection constant of OH_n, Junge used QWEP in a crucial way.

Hiai [23] introduced the so-called q-Araki-Woods algebras. Let $-1 < q < 1$, and let H_{R} be a real Hilbert space and $(U_t)_{t\in\mathbb{R}}$ an orthogonal group on H_{R}. Let $\Gamma_q(H_{\mathrm{R}}, (U_t)_{t\in\mathbb{R}})$ denote the associated q-Araki-Woods algebra. These algebras are generalizations of both Shlyakhtenko's free quasi-free factors (for $q = 0$), and Bożejko and Speicher's q-Gaussian algebras (for $(U_t)_{t\in\mathbb{R}}$ trivial). We prove that $\Gamma_q(H_{\mathrm{R}}, (U_t)_{t\in\mathbb{R}})$ is QWEP. This is an extension of Pisier Shlyakthenko's result for the free quasi-free factor (with $(U_t)_{t\in\mathbb{R}}$ almost periodic), already quoted above.

We recall some general background on q-Araki-Woods algebras and we give a proof of the main result in the particular case of Bożejko and Speicher's q-Gaussian algebras $\Gamma_q(H_{\mathbb{R}})$. The proof relies on an asymptotic random matrix model for standard q-Gaussians. The existence of such a model goes back to Speicher's central limit Theorem for mixed commuting/anti-commuting non-commutative random variables (see [139]). Alternatively, one can also use the Gaussian random matrix model given by Śniady in [138]. Notice that the matrices arising from Speicher's central limit Theorem may not be uniformly bounded in norm. Therefore, we have to cut them off in order to define a homomorphism from a dense subalgebra of $\Gamma_q(H_{\mathbb{R}})$ into an ultraproduct of matricial algebras. In this tracial framework it can be shown quite easily that this homomorphism extends to an isometric *-homomorphism of von Neumann algebras, simply because it is trace preserving. Thus $\Gamma_q(H_{\mathbb{R}})$ can be seen as a (necessarily completely complemented) subalgebra of an ultraproduct of matricial algebras. This solves the problem in the tracial case.

Moreover, in this (relatively) simple situation, we are able to extend the result to the C^*-algebra generated by all q-Gaussians, $C_q^*(H_{\mathbb{R}})$. Indeed, using the ultracontractivity of the q-Ornstein Uhlenbeck semi-group (see [115]) we establish that $C_q^*(H_{\mathbb{R}})$ is "weakly ucp complemented" in $\Gamma_q(H_{\mathbb{R}})$. This last fact, combined with the QWEP of $\Gamma_q(H_{\mathbb{R}})$, implies that $C_q^*(H_{\mathbb{R}})$ is also QWEP.

We adapt the proof of the more general type III q-Araki-Woods algebras. We start by recalling Raynaud's construction of the von Neumann algebra's ultraproduct when algebras are equipped with non-tracial states (see [136]). Then, we give some general conditions in order to define an embedding into such an ultraproduct, whose image is of a state preserving conditional expectation.

We define a twisted Baby Fock model, to which we apply Speicher's central limit Theorem. This provides us with an asymptotic random matrix model for (finite dimensional) q-Araki Woods algebras, generalizing the asymptotic model already introduced by Speicher and used by Biane in [128]. Using this asymptotic model, we then define an algebraic *-homomorphism from a dense subalgebra of $\Gamma_q(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$ into a von Neumann ultraproduct of finite dimensional C^*-algebras. Notice that the cut off argument requires some extra work (compare the proofs of Lemma (3.2.7) and Lemma (3.2.24)), for instance we need to use our knowledge of the modular theory at the Baby Fock level to conclude. We then apply the general results (Theorem (3.2.15)) to extend this algebraic *-homomorphism into a *-isomorphism from $\Gamma_q(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$ to the von Neumann algebra's ultraproduct, whose image is completely complemented. This allows us to show that $\Gamma_q(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$ is QWEP for $H_{\mathbb{R}}$ finite dimensional (see Theorem (3.2.26)). It implies, by inductive limit, that $\Gamma_q(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$ is QWEP when $(U_t)_{t\in\mathbb{R}}$ is almost periodic (see Corollary (3.2.27)).

We consider a general algebra $\Gamma_q(H_{\mathbb{R}},(U_t)_{t\in\mathbb{R}})$. We use a discretization procedure on the unitary group $(U_t)_{t\in\mathbb{R}}$ in order to approach $\Gamma_q(H_{\mathbb{R}},(U_t)_{t\in\mathbb{R}})$ by almost periodic q-Araki-Woods algebras. We then apply the general results and, we recover the general algebra as a complemented subalgebra of the ultraproduct of the discretized ones (see Theorem (3.2.30)). From this last fact follows the QWEP of $\Gamma_q(H_{\mathbb{R}},(U_t)_{t\in\mathbb{R}})$. However we were unable to establish the corresponding result for the C^*-algebra $C_q^*(H_{\mathbb{R}},(U_t)_{t\in\mathbb{R}})$. Indeed, if $(U_t)_{t\in\mathbb{R}}$ is not trivial then the ultracontractivity of the q-Ornstein-Uhlenbeck semi-group never holds in any right-neighborhood of zero (see [23]).

We highlight that the modular theory on the twisted Baby Fock algebras, on their ultraproduct, and on the q-Araki Woods algebras, are crucial tools in order to overcome the difficulties arising in the non-tracial case.

Marius Junge informed us that he had obtained our main result using his proof of the non-commutative L^1-Khintchine inequalities for q-Araki-Woods algebras. Junge's approach is slightly different but its main steps are the same as ours: the proof uses in a crucial way Speicher's central limit Theorem, an ultraproduct argument and modular theory.

We mainly follow the notations used in [15], [23] and [113]. Let $H_{\mathbb{R}}$ be a real Hilbert space and $(U_t)_{t\in\mathbb{R}}$ be a strongly continuous group of orthogonal transformations on $H_{\mathbb{R}}$. We denote by $H_{\mathbb{C}}$ the complexification of $H_{\mathbb{R}}$ and still by $(U_t)_{t\in\mathbb{R}}$ its extension to a group of unitaries on $H_{\mathbb{C}}$. Let A be the (unbounded) non degenerate positive infinitesimal generator of $(U_t)_{t\in\mathbb{R}}$.

$$U_t = A^{\mathrm{it}} \quad \text{for all } t \in \mathbb{R}$$

A new scalar product $\langle .,. \rangle_U$ is defined on H_{C} by the following relation:

$$\langle \xi,\eta\rangle_U = \langle 2A(1+A)^{-1}\xi,\eta\rangle$$

We denote by H the completion of H_{C} with respect to this new scalar product. For $q \in (-1,1)$ we consider the q-Fock space associated with H and given by:

$$\mathcal{F}_q(H) = \mathbb{C}\Omega \bigoplus_{n\geqslant 1} H^{\otimes n}$$

where $H^{\otimes n}$ is equipped with Bożejko and Speicher's q-scalar product (see [68]). The usual creation and annihilation operators on $\mathcal{F}_q(H)$ are denoted respectively by a^* and a (see [68]). For $f \in H_{\mathbb{R}}, G(f)$, the q-Gaussian operator associated to f, is by definition:

$$G(f) = a^*(f) + a(f) \in B\big(\mathcal{F}_q(H)\big)$$

The von Neumann algebra that they generate in $B\big(\mathcal{F}_q(H)\big)$ is the so-called q-Araki-Woods algebra: $\Gamma_q(H_{\mathrm{R}},(U_t)_{t\in\mathbb{R}})$. The q-Araki-Woods algebra is equipped with a faithful normal state φ which is the expectation on the vacuum vector Ω. We denote by W the Wick product ; it is the inverse of the mapping:

$$\begin{aligned} \Gamma_q(H_{\mathbb{R}},(U_t)_{t\in\mathbb{R}}) &\longrightarrow \Gamma_q(H_{\mathbb{R}},(U_t)_{t\in\mathbb{R}})\Omega \\ X &\mapsto X\Omega \end{aligned}$$

Recall that $\Gamma_q(H_{\mathbb{R}},(U_t)_{t\in\mathbb{R}}) \subset B\big(\mathcal{F}_q(H)\big)$ is the GNS representation of (Γ,φ). The modular theory relative to the state φ was computed in [23] and [15]. We now briefly recall their results. As usual we denote by S the closure of the operator:

$$S(x\Omega) = x^*\Omega \ \text{ for all } x \in \Gamma_q(H_{\mathrm{R}},(U_t)_{t\in\mathbb{R}})$$

Let $S = J\Delta^{\frac{1}{2}}$ be its polar decomposition. J and Δ are respectively the modular conjugation and the modular operator relative to φ. The following explicit formulas hold:

$$S(h_1 \otimes \cdots \otimes h_n) = h_n \otimes \cdots \otimes h_1 \ \text{ for all } h_1,\ldots,h_n \in H_{\mathbb{R}}$$

Δ is the closure of the operator $\bigoplus_{n=0}^{\infty}\ (A^{-1})^{\otimes n}$ and

$$J(h_1\otimes\cdots\otimes h_n)=A^{-\frac{1}{2}}h_n\otimes\cdots\otimes A^{-\frac{1}{2}}h_1\ \text{ for all } h_1,\dots,h_n\in H_{\mathbb{R}}\cap\operatorname{dom}A^{-\frac{1}{2}}$$

The modular group of automorphisms $(\sigma_t)_{t\in\mathbb{R}}$ on $\Gamma_q(H_{\mathbb{R}},(U_t)_{t\in\mathbb{R}})$ relative to φ is given by:

$$\sigma_t(G(f))=\Delta^{it}G(f)\Delta^{-it}=G(U_{-t}f)\ \text{ for all } t\in\mathbb{R}\ \text{ and all } f\in H_{\mathbb{R}}$$

In the following Lemma we state a well known formula giving, in particular, all moments of the q-Gaussians.

Lemma (3.2.1)[130]: Let $r\in\mathbb{N}_*$ and $(h_l)_{-r\leqslant l\leqslant r}$ be a family of vectors in $H_{\mathbb{R}}$. For all $l\in\{1,\dots,r\}$ consider the operator $d_l=a^*(h_l)+a(h_{-l})$. For all $(k(1),\dots,k(r))\in\{1,*\}^r$ we have:

$$\varphi(d_1^{k(1)}\dots d_r^{k(r)})=\begin{cases}0 & \text{if } r \text{ is odd}\\ \displaystyle\sum_{\substack{\mathcal{V}\ \text{2-partition}\\ \mathcal{V}=\{(s_t,t_l)_{l=1}^{l=p}\}\ \text{with } s_l<t_l}} q^{i(\mathcal{V})}\prod_{l=1}^{p}\varphi\left(d_{s_l}^{k(s_l)}d_{t_l}^{k(t_l)}\right) & \text{if } r=2p\end{cases}$$

where $i(\mathcal{V})=\#\{(k,l),s_k<s_l<t_k<t_l\}$ is the number of crossings of the 2-partition $\mathcal{V}$.

Therefore, we see that the distribution of a single gaussian does not depend on the group $(U_t)_{t\in\mathbb{R}}$. In the tracial case (thus in all cases), and when $\parallel f\parallel=1$, this distribution is the absolutely continuous probability measure ν_q supported on the interval $[-2/\sqrt{1-q},2/\sqrt{1-q}]$ whose orthogonal polynomials are the q-Hermite polynomials (see [60]). In particular, we have:

$$\text{For all } f\in H_{\mathbb{R}},\ \ \parallel G(f)\parallel=\frac{2}{\sqrt{1-q}}\parallel f\parallel_{H_{\mathbb{R}}}\qquad(1)$$

We now briefly recall a description of the von Neumann algebra $\Gamma_q(H_{\mathbb{R}},U_t)$ where $H_{\mathbb{R}}$ is an Euclidian space of dimension $2k(k\in\mathbb{N}_*)$. There exists $\left(H_j\right)_{1\leqslant j\leqslant k}$ a family of two dimensional spaces, invariant under $(U_t)_{t\in\mathbb{R}}$, and $\left(\lambda_j\right)_{1\leqslant j\leqslant k}$ some real numbers greater or equal to 1 such that for all $j\in\{1,\dots,k\}$,

$$H_{\mathbb{R}}=\bigoplus_{1\leqslant j\leqslant k}H_j\ \text{ and }\ U_{t|H_j}=\begin{pmatrix}\cos\left(t\ln\left(\lambda_j\right)\right) & -\sin\left(t\ln\left(\lambda_j\right)\right)\\ \sin\left(t\ln\left(\lambda_j\right)\right) & \cos\left(t\ln\left(\lambda_j\right)\right)\end{pmatrix}$$

We put $I=\{-k,\dots,-1\}\cup\{1,\dots,k\}$. It is then easily checked that the deformed scalar product $\langle .,.\rangle_U$ on the complexification $H_{\mathbb{C}}$ of $H_{\mathbb{R}}$ is characterized by the condition that there exists a basis $\left(f_j\right)_{j\in I}$ in $H_{\mathbb{R}}$ such that for all $(j,l)\in\{1,\dots,k\}^2$

$$\left\langle f_j,f_{-l}\right\rangle_U=\delta_{j,l}.\,i\frac{\lambda_j-1}{\lambda_j+1}\ \text{ and }\ \left\langle f_{\pm j},f_{\pm l}\right\rangle_U=\delta_{j,l}\qquad(2)$$

For all $j\in\{1,\dots,k\}$ we put $\mu_j=\lambda_j^{\frac{1}{4}}$. Let $\left(e_j\right)_{j\in I}$ be a real orthonormal basis of $\mathbb{C}^{2k}$ equipped with its canonical scalar product. For all $j\in\{1,\dots,k\}$ we put

$$\hat{f}_j=\frac{1}{\sqrt{\mu_j^2+\mu_j^{-2}}}\left(\mu_je_{-j}+\mu_j^{-1}e_j\right)\ \text{ and }\ \hat{f}_{-j}=\frac{i}{\sqrt{\mu_j^2+\mu_j^{-2}}}\left(\mu_je_{-j}-\mu_j^{-1}e_j\right)$$

It is easy to see that the conditions (2) are fulfilled for the family $(\hat{f}_j)_{j\in I}$. We will denote by H_R the Euclidian space generated by the family $(\hat{f}_j)_{j\in I}$ in $\mathbb{C}^{2k}$. This provides us with a realization of $\Gamma_q(H_\mathbb{R}, U_t)$ as a subalgebra of $B\left(\mathcal{F}_q(\mathbb{C}^{2k})\right)$. Indeed, $\Gamma_q(H_R, U_t) = \{G(\hat{f}_j), j \in I\}'' \subset B\left(\mathcal{F}_q(\mathbb{C}^{2k})\right)$. For all $j \in \{1, \dots, k\}$ put

$$f_j = \frac{\sqrt{\mu_j^2 + \mu_j^{-2}}}{2}\hat{f}_j \text{ and } f_{-j} = \frac{\sqrt{\mu_j^2 + \mu_j^{-2}}}{2}\hat{f}_{-j}$$

We define the following generalized semi-circular variable by:

$$c_j = G(f_j) + iG(f_{-j}) = W(f_j + if_{-j})$$

It is clear that $\Gamma_q(H_R, U_t) = \{c_j, j \in \{1, \dots, k\}\}'' \subset B\left(\mathcal{F}_q(\mathbb{C}^{2k})\right)$ and we can check that

$$c_j = \mu_j a(e_{-j}) + \mu_j^{-1} a^*(e_j) \tag{3}$$

Moreover, for all $j \in \{1, \dots, k\}$, c_j is an entire vector for $(\sigma_t)_{t\in\mathbb{R}}$ and we have, for all $\in \mathbb{C}$:

$$\sigma_z(c_j) = \lambda_j^{iz} c_j.$$

Recall that all odd *-moments of the family $(c_j)_{1\leqslant j\leqslant k}$ are zero. Applying Lemma (3.2.1) to the operators c_j we state, for further references, an explicit formula for the *-moments of $(c_j)_{1\leqslant j\leqslant k}$. In the following we use the convention $c^{-1} = c^*$ when there is no possible confusion.

Lemma (3.2.2)[130]: Let $r \in \mathbb{N}_*$, $(j(1), \dots, j(2r)) \in \{1, \dots, k\}^{2r}$ and $(k(1), \dots, k(2r)) \in \{\pm 1\}^{2r}$

$$\varphi\left(c_{j(1)}^{k(1)} \dots c_{j(2r)}^{k(2r)}\right) = \sum_{\substack{\mathcal{V}\ 2-\text{partition} \\ \mathcal{V}=\{(s_t,t_l)_{l=1}^{l=r}\} \text{ with } s_l<t_l}} q^{i(\mathcal{V})} \prod_{l=1}^{r} \varphi\left(c_{j(s_l)}^{k(s_l)} c_{j(t_l)}^{k(t_l)}\right)$$

$$= \sum_{\substack{\mathcal{V}\ 2-partition \\ \mathcal{V}=\{(s_t,t_l)_{l=1}^{l=r}\} with\ s_l<t_l}} q^{i(\mathcal{V})} \prod_{l=1}^{r} \mu_{j(s_l)}^{2k(s_l)} \delta_{k(s_l),-k(t_l)} \delta_{j(s_l),j(t_l)}$$

Proof. As said above this is a consequence of Lemma (3.2.1) and the explicit computation of covariances. Using (3) we have:

$$\begin{aligned}
\varphi\left(c_{j(1)}^{k(1)} c_{j(2)}^{k(2)}\right) &= \left\langle c_{j(1)}^{-k(1)}\Omega, c_{j(2)}^{k(2)}\Omega\right\rangle \\
&= \left\langle \mu_{j(1)}^{k(1)} e_{-k(1)j(1)}, \mu_{j(2)}^{-k(2)} e_{k(2)j(2)}\right\rangle \\
&= \mu_{j(1)}^{2k(1)} \delta_{k(1),-k(2)} \delta_{j(1),j(2)}
\end{aligned}$$

The symmetric Baby Fock (also known as symmetric toy Fock space) is at some point a discrete approximation of the bosonic Fock space (see [92]). In [128], Biane considered spin systems with mixed commutation and anti-commutation relations (which is a generalization of the symmetric toy Fock), and used it to approximate q-Fock space (via Speicher central limit Theorem). We recall the formal construction of [128]. Let I be a finite subset of $\mathbb{Z}$ and ϵ a function from $I \times I$ to $\{-1,1\}$ satisfying for all $(i,j) \in I^2, \epsilon(i,j) = \epsilon(j,i)$

and $\epsilon(i,i) = -1$. Let $\mathcal{A}(I,\epsilon)$ be the free complex unital algebra with generators $(x_i)_{i\in I}$ quotiented by the relations

$$x_i x_j - \epsilon(i,j) x_j x_i = 2\delta_{i,j} \text{ for } (i,j) \in I^2 \tag{4}$$

We define an involution on $\mathcal{A}(I,\epsilon)$ by $x_i^* = x_i$. For a subset $A = \{i_1, \dots, i_k\}$ of I with $i_1 < \cdots < i_k$ we put $x_A = x_{i_1} \dots x_{i_k}$, where, by convention, $x_\emptyset = 1$. Then $(x_A)_{A\subset I}$ is a basis of the vector space $\mathcal{A}(I,\epsilon)$. Let φ^ϵ be the tracial functional defined by $\varphi^\epsilon(x_A) = \delta_{A,\emptyset}$ for all $A \subset I$. $\langle x,y\rangle = \varphi^\epsilon(x^* y)$ defines a positive definite hermitian form on $\mathcal{A}(I,\epsilon)$. We will denote by $L^2(\mathcal{A}(I,\epsilon),\varphi^\epsilon)$ the Hilbert space $\mathcal{A}(I,\epsilon)$ equipped with $\langle .,. \rangle$. $(x_A)_{A\subset I}$ is an orthonormal basis of $L^2(\mathcal{A}(I,\epsilon),\varphi^\epsilon)$. For each $i \in I$, define the following partial isometries β_i^* and α_i^* of $L^2(\mathcal{A}(I,\epsilon),\varphi^\epsilon)$ by:

$$\beta_i^*(x_A) = \begin{cases} x_i x_A & \text{if} \quad i \notin A \\ 0 & \text{if} \quad i \in A \end{cases} \text{ and } \alpha_i^*(x_A) = \begin{cases} x_A x_i & \text{if} \quad i \notin A \\ 0 & \text{if} \quad i \in A \end{cases}$$

Note that their adjoints are given by:

$$\beta_i(x_A) = \begin{cases} x_i x_A & \text{if} \quad i \in A \\ 0 & \text{if} \quad i \notin A \end{cases} \text{ and } \alpha_i(x_A) = \begin{cases} x_A x_i & \text{if} \quad i \in A \\ 0 & \text{if} \quad i \notin A \end{cases}$$

β_i^* and β_i (respectively α_i^* and α_i) are called the left (respectively right) creation and annihilation operators at the Baby Fock level. In the next Lemma we recall from [128] the fundamental relations 1. and 2., and we leave the proof of 3., 4. and 5.

Lemma (3.2.3)[130]: The following relations hold:

1. For all $i \in I (\beta_i^*)^2 = \beta_i^2 = 0$ and $\beta_i \beta_i^* + \beta_i^* \beta_i = Id$.
2. For all $(i,j) \in I^2$ with $i \neq j \beta_i \beta_j - \epsilon(i,j)\beta_j \beta_i = 0$ and $\beta_i \beta_j^* - \epsilon(i,j)\beta_j^* \beta_i = 0$.
3. Same relations as in (a). and (b). with α in place of β.
4. For all $i \in I \beta_i^* \alpha_i^* = \alpha_i^* \beta_i^* = 0$ and for all $(i,j) \in I^2$ with $i \neq j \beta_i^* \alpha_j^* = \alpha_j^* \beta_i^*$.
5. For all $(i,j) \in I^2 \beta_i^* \alpha_j = \alpha_j \beta_i^*$.

It is easily seen, by (a) and (b) of Lemma (3.2.3), that the self adjoint operators defined by: $\gamma_i = \beta_i^* + \beta_i$ satisfy the following relation :

$$\text{for all } (i,j) \in I^2,\ \gamma_i \gamma_j - \epsilon(i,j)\gamma_j \gamma_i = 2\delta_{i,j}\mathrm{Id} \tag{5}$$

Let $\Gamma_I \subset B\big(L^2(\mathcal{A}(I,\epsilon),\varphi^\epsilon)\big)$ be the $*$-algebra generated by all $\gamma_i, i \in I$. Still denoting by φ^ϵ the vector state associated to the vector 1, it is known that φ^ϵ is a faithful normalized trace on the finite dimensional C^*-algebra Γ_I. Moreover, $\Gamma_I \subset B\big(L^2(\mathcal{A}(I,\epsilon),\varphi^\epsilon)\big)$ is the faithful GNS representation of $(\Gamma_I, \varphi^\epsilon)$ with cyclic and separating vector 1.

Then, it is clear from (4) and (5) of Lemma (3.2.3), that $\Gamma_{r,I} \subset \Gamma_I'$ (there is actually equality). Since 1 is clearly cyclic for $\Gamma_{r,I}$, then it is also cyclic for Γ_I', thus 1 is separating for Γ_I.

1. Let I and $J, I \subset J$, be some sets together with signs ϵ and ϵ' such that $\epsilon'_{|I\times I} = \epsilon$. It is clear that $L^2(\mathcal{A}(I,\epsilon),\varphi^\epsilon)$ embeds isometrically in $L^2\big(\mathcal{A}(J,\epsilon'),\varphi^{\epsilon'}\big)$. Set $K = J \setminus I$. Fix some total orders on I and K and consider the total order on J which coincides with the orders of I and K and such that any element of I is smaller than any element of K. The associated orthonormal basis of $L^2\big(\mathcal{A}(J,\epsilon'),\varphi^{\epsilon'}\big)$ is given by the family $(x_A x_B)_{A\in\mathcal{F}(I), B\in\mathcal{F}(K)}$ (where $\mathcal{F}(I)$, respectively $\mathcal{F}(K)$, denotes the set of finite subsets of I, respectively K). In particular with have the following Hilbertian decomposition:

$$L^2\big(\mathcal{A}(J,\epsilon'),\varphi^{\epsilon'}\big) = \bigoplus_{B\in\mathcal{F}(K)} L^2(\mathcal{A}(I,\epsilon),\varphi^\epsilon) x_B \tag{6}$$

For $j \in I$ we (temporarily) denote by $\tilde{\beta}_j$ the annihilation operator in $B\left(L^2(\mathcal{A}(I,\epsilon),\varphi^\epsilon)\right)$ and simply by β_j its analogue in $B\left(L^2\left(\mathcal{A}(J,\epsilon'),\varphi^{\epsilon'}\right)\right)$. Let $\tilde{C}_I$ (respectively C_J) be the C^*-algebra generated by $\{\tilde{\beta}_j, j \in I\}$ (respectively $\{\beta_j, j \in J\}$) in $B\left(L^2(\mathcal{A}(I,\epsilon),\varphi^\epsilon)\right)$ (respectively $B\left(L^2\left(\mathcal{A}(J,\epsilon'),\varphi^{\epsilon'}\right)\right)$). Consider also C_I the C^*-algebra generated by $\{\beta_j, j \in I\}$ in $B\left(L^2(\mathcal{A}(I,\epsilon),\varphi^\epsilon)\right)$. For $B = \{j_1,\dots,j_k\} \subset K$, with $j_1 < \dots < j_k$, let us denote by α_B the operator $\alpha_{j_1} \dots \alpha_{j_k}$. If $\tilde{T} \in \tilde{C}_I$ and if T denotes its counterpart in C_I, then it is easily seen that, with respect to the Hilbertian decomposition (6), we have

$$T = \bigoplus_{B \in \mathcal{F}(K)} \alpha_B^* \tilde{T} \alpha_B. \tag{7}$$

It follows that $\tilde{C}_I$ is *-isomorphic to $C_I \subset C_J$.

1. It is possible to find explicitly selfadjoint matrices satisfying the mixed commutation and anti-commutation relations (5) (see [139] and [128]). We choose to present this approach because it will be easier to handle the objects of modular theory in this abstract situation when we will deal with non-tracial von Neumann algebras.

We recall Speicher's central limit theorem which is specially designed to handle either commuting or anti-commuting (depending on a function) independent variables. Speicher's central limit theorem asserts that such a family of centered noncommutative variables which have a fixed covariance, and uniformly bounded *-moments, is convergent in *-moments, as soon as a combinatorial quantity associated with ϵ is converging. Moreover the limit *-distribution is only determined by the common covariance and the limit of the combinatorial quantity. We start by recalling some basic notions on independence and set partitions.

Definition (3.2.4)[130]: Let $(\mathcal{A},\varphi)$ be a *-algebra equipped with a state φ and $(\mathcal{A}_i)_{i\in I}$ a family of C^* subalgebras of $\mathcal{A}$. The family $(\mathcal{A}_i)_{i\in I}$ is said to be independent if for all $r \in \mathbb{N}_*$, $(i_1,\dots,i_r) \in I^r$ with $i_s \neq i_t$ for $s \neq t$, and all $a_{i_s} \in \mathcal{A}_{i_s}$ for $s \in \{1,\dots,r\}$ we have:

$$\varphi\left(a_{i_1} \dots a_{i_r}\right) = \varphi\left(a_{i_1}\right) \dots \varphi\left(a_{i_r}\right)$$

As usual, a family $(a_i)_{i\in I}$ of non-commutative random variables of $\mathcal{A}$ will be called independent if the family of C^*-subalgebras of $\mathcal{A}$ that they generate is independent.

On the set of p-uples of integers belonging to $\{1,\dots,N\}$ define the equivalence relation $\sim$ by:

$$\left(i(1),\dots,i(p)\right) \sim \left(j(1),\dots,j(p)\right)$$
$$\text{if } (i(l) = i(m) \Leftrightarrow j(l) = j(m)) \forall (l,m) \in \{1,\dots,p\}^2$$

Then the equivalence classes for the relation $\sim$ are given by the partitions of the set $\{1,\dots,p\}$. We denote by $V_1,\dots,V_r$ the blocks of the partition $\mathcal{V}$ and we call $\mathcal{V}$ a 2-partition if each of these blocks is of cardinal 2. The set of all 2-partitions of the set $\{1,\dots,p\}$ (even) will be denoted by $\mathcal{P}_2(1,\dots,p)$. For $\mathcal{V} \in \mathcal{P}_2(1,\dots,2r)$ let us denote by $V_l = (s_l,t_l), s_l < t_l$, for $l \in \{1,\dots,r\}$ the blocks of the partition $\mathcal{V}$. The set of crossings of $\mathcal{V}$ is defined by

$$I(\mathcal{V}) = \{(l,m) \in \{1,\dots,r\}^2, s_l < s_m < t_l < t_m\}$$

The 2-partition $\mathcal{V}$ is said to be crossing if $I(\mathcal{V}) \neq \emptyset$ and non-crossing if $I(\mathcal{V}) = \emptyset$.

Theorem (3.2.5)[130]: (Speicher) Consider k sequences $\left(b_{i,j}\right)_{(i,j)\in\mathbb{N}_+\times\{1,\dots,k\}}$ in a noncommutative probability space (B,φ) satisfying the following conditions:

1. The family $\left(b_{i,j}\right)_{(i,j)\in\mathbb{N}_*\times\{1,\dots,k\}}$ is independent.

2. For all $(i,j) \in \mathbb{N}_* \times \{1, \dots, k\}, \varphi(b_{i,j}) = 0$
3. For all $(k(1), k(2)) \in \{-1,1\}^2$ and $(j(1), j(2)) \in \{1, \dots, k\}^2$, the covariance $\varphi\left(b_{i,j(1)}^{k(1)} b_{i,j(2)}^{k(2)}\right)$ is independent of i and will be denoted by $\varphi\left(b_{j(1)}^{k(1)} b_{j(2)}^{k(2)}\right)$.
4. For all $w \in \mathbb{N}_*, (k(1), \dots, k(w)) \in \{-1,1\}^w$ and all $j \in \{1, \dots, k\}$ there exists a constant C such that for all $i \in \mathbb{N}_*, \left|\varphi\left(b_{i,j}^{k(1)} \dots b_{i,j}^{k(w)}\right)\right| \leqslant C$.
5. For all $(i(1), i(2)) \in \mathbb{N}_*^2$ there exists a sign $\epsilon(i(1), i(2)) \in \{-1,1\}$ such that for all $(j(1), j(2)) \in \{1, \dots, k\}^2$ with $(i(1), j(1)) \neq (i(2), j(2))$ and all $(k(1), k(2)) \in \{-1,1\}^2$ we have
$$b_{i(1),j(1)}^{k(1)} b_{i(2),j(2)}^{k(2)} - \epsilon(i(1), i(2)) b_{i(2),j(2)}^{k(2)} b_{i(1),j(1)}^{k(1)} = 0.$$
(notice that the function ϵ is necessarily symmetric in its two arguments).
6. For all $r \in \mathbb{N}_*$ and all $\mathcal{V} = \{(s_l, t_l)_{l=1}^{l=r}\} \in \mathcal{P}_2(1, \dots, 2r)$ the following limit exists
$$t(\mathcal{V}) = \lim_{N \to +\infty} \frac{1}{N^r} \sum_{\substack{i(s_1) \dots, i(s_r)=1 \\ i(s_l) \neq, i(s_m) \neq m}}^{N} \prod_{(l,m) \in I(\mathcal{V})} \epsilon(i(s_l), i(s_m)))$$

Let $S_{N,j} = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} b_{i,j}$. Then we have for all $p \in \mathbb{N}_*, (k(1), \dots, k(p)) \in \{-1,1\}^p$ and all $(j(1), \dots, j(p)) \in \{1, \dots, k\}^p$:
$$\lim_{N \to +\infty} \varphi\left(S_{N,j(1)}^{k(1)} \dots S_{N,j(p)}^{k(p)}\right) = \begin{cases} 0 & \text{if } p \text{ is odd} \\ \displaystyle\sum_{\substack{\mathcal{V} \in \mathcal{P}_2(1, \dots, 2r) \\ \mathcal{V} = \{(s_l, t_l)_{l=1}^{l=r}}} t(\mathcal{V}) \prod_{l=1}^{r} \varphi\left(b_{j(s_l)}^{k(s_l)} b_{j(t_l)}^{k(t_l)}\right) & \text{if } p = 2r \end{cases}$$

The following Lemma, proved in [139], guarantees the almost sure convergence of the quantity $t(\mathcal{V})$ provided that the function ϵ has independent entries following the same 2-points Dirac distribution:

Lemma (3.2.6)[130]: Let $q \in (-1,1)$ and consider a family of random variables $\epsilon(i,j)$ for $(i,j) \in \mathbb{N}_*$ with $i \neq j$, such that
1. For all $(i,j) \in \mathbb{N}_*$ with $i \neq j, \epsilon(i,j) = \epsilon(j,i)$
2. The family $(\epsilon(i,j))_{i>j}$ is independent
3. For all $(i,j) \in \mathbb{N}_*$ with $i \neq j$ the probability distribution of $\epsilon(i,j)$ is
$$\frac{1+q}{2} \delta_1 + \frac{1-q}{2} \delta_{-1}$$

Then, almost surely, we have for all $r \in \mathbb{N}_*$ and for all $\mathcal{V} \in \mathcal{P}_2(1, \dots, 2r)$
$$\lim_{N \to +\infty} \frac{1}{N^r} \sum_{\substack{i(s_1), \dots, i(s_r)=1 \\ i(s_l) \neq i(s_m) \text{ } for \text{ } l \neq m}} \prod_{(l,m) \in I(\mathcal{V})} \epsilon(i(s_l), i(s_m)) = q^{i(\mathcal{V})}$$

Alternatively, one can apply directly Speicher's theorem to families of mixed commuting /anticommuting creation operators as it is done in [139] and [128]. The limit $*$-moments are in this case the $*$-moments of classical q-creation operators.

We show that $\Gamma_q(H_{\mathbb{R}})$ is QWEP. In fact, by inductive limit, it is sufficient to prove it for $H_{\mathbb{R}}$ finite dimensional. Let $k \geqslant 1$. We will consider $\mathbb{R}^k$ as the real Hilbert space of dimension k, with the canonical orthonormal basis $(e_1, \dots, e_k)$, and $\mathbb{C}^k$, its complex counterpart. Let us fix $q \in (-1,1)$ and consider $\Gamma_q(\mathbb{R}^k)$ the von Neumann algebra generated by the q-Gaussians $G(e_1), \dots, G(e_k)$. We denote by τ the expectation on the vacuum vector, which is a trace in this particular case.

By the ending remark, there are Hermitian matrices, $g_{n,1}(\omega), \dots, g_{n,k}(\omega)$, depending on a random parameter denoted by ω and lying in a finite dimensional matrix algebra, such that their joint $*$-distribution converges almost surely to the joint $*$-distribution of the q-Gaussians in the following sense: for all polynomial P in k noncommuting variables, $\lim_{n\to\infty} \tau_n\left(P\big(g_{n,1}(\omega), \dots, g_{n,k}(\omega)\big)\right) = \tau\left(P\big(G(e_1), \dots, G(e_k)\big)\right)$ almost surely in ω.

We will denote by $\mathcal{A}_n$ the finite dimensional C^*-algebra generated by $g_{n,1}(\omega), \dots, g_{n,k}(\omega)$. We recall that these algebras are equipped with the trace τ_n defined by:

$$\tau_n(x) = \langle 1, x.1\rangle$$

Since the set of all monomials in k noncommuting variables is countable, we have for almost all ω,

$$\lim_{n\to\infty} \tau_n\left(P\big(g_{n,1}(\omega), \dots, g_{n,k}(\omega)\big)\right) = \tau\left(P\big(G(e_1), \dots, G(e_k)\big)\right) \text{ for all such monomials } P \tag{8}$$

A fortiori we can find an ω_0 such that (8) holds for ω_0. We will fix such an ω_0 and simply denote by $g_{n,i}$ the matrix $g_{n,i}(\omega_0)$ for all $i \in \{1, \dots, k\}$. With these notations, it is clear that, by linearity, we have for all polynomials P in k noncommuting variables,

$$\lim_{n\to\infty} \tau_n\left(P\big(g_{n,1}, \dots, g_{n,k}\big)\right) = \tau\left(P\big(G(e_1), \dots, G(e_k)\big)\right). \tag{9}$$

We need to have a uniform control on the norms of the matrices $g_{n,i}$. Let C be such that $\|G(e_1)\| < C$, we will replace the $g_{n,i}$'s by their truncations $\chi_{]-C,C[}\big(g_{n,i}\big)g_{n,i}$ (where $\chi_{]-C,C[}$ denotes the characteristic function of the interval $]-C,C[$). For simplicity $\chi_{]-C,C[}\big(g_{n,i}\big)g_{n,i}$ will be denoted by $\tilde{g}_{n,i}$. We now check that (9) is still valid for the $\tilde{g}_{n,i}$'s.

Lemma (3.2.7)[130]: With the notations above, for all polynomials P in k noncommuting variables we have

$$\lim_{n\to\infty} \tau_n\left(P\big(\tilde{g}_{n,1}, \dots, \tilde{g}_{n,k}\big)\right) = \tau\left(P\big(G(e_1), \dots, G(e_k)\big)\right). \tag{10}$$

Proof. We just have to prove that for all monomials P in k noncommuting variables we have

$$\lim_{n\to\infty} \tau_n\big[P\big(\tilde{g}_{n,1}, \dots, \tilde{g}_{n,k}\big) - P\big(g_{n,1}, \dots, g_{n,k}\big)\big] = 0.$$

Writing $g_{n,i} = \tilde{g}_{n,i} + \big(g_{n,i} - \tilde{g}_{n,i}\big)$ and developing using multilinearity, we are reduced to showing that the L^1-norms of any monomial in $\tilde{g}_{n,i}$ and $\big(g_{n,i} - \tilde{g}_{n,i}\big)$ (with at least one factor $\big(g_{n,i} - \tilde{g}_{n,i}\big)\big)$ tend to 0 . By the Hölder inequality and the uniform boundedness of the $\|\tilde{g}_{n,i}\|$'s, it suffices to show that for all $i \in \{1 \dots k\}$,

$$\lim_{n\to\infty} \tau_n\big(\big|\tilde{g}_{n,i} - g_{n,i}\big|^p\big) = 0 \text{ for all } p \geqslant 1. \tag{11}$$

Let us prove (11) for $i = 1$. We are now in a commutative setting. Indeed, let us introduce the spectral resolutions of identity, E_t^n (respectively E_t), of $g_{n,1}$ (respectively $G(e_1)\big)$. By (9) we have for all polynomials P

$$\lim_{n\to\infty} \tau_n\left(P\big(g_{n,1}\big)\right) = \tau\left(P\big(G(e_1)\big)\right).$$

We can rewrite this as follows: for all polynomials P

$$\lim_{n\to\infty}\int_{\sigma(g_{n,1})} P(t)\mathrm{d}\langle E_t^n\cdot 1,1\rangle = \int_{\sigma(G(e_1))} P(t)\mathrm{d}\langle E_t\cdot\Omega,\Omega\rangle.$$

Let μ_n (respectively) denote the compactly supported probability measure $\langle E_t^n\cdot 1,1\rangle$ (respectively $\langle E_t\cdot\Omega,\Omega\rangle$) on $\mathbb{R}$. With these notations our assumption becomes: for all polynomials P

$$\lim_{n\to\infty}\int P\,\mathrm{d}\mu_n = \int P\,\mathrm{d}\mu. \tag{12}$$

and (11) is equivalent to:

$$\lim_{n\to\infty}\int_{|t|\geqslant C} |t|^p\,\mathrm{d}\mu_n = 0 \text{ for all } p\geqslant 1. \tag{13}$$

Then the result follows from the following elementary Lemma. We give a proof for sake of completeness.

Lemma (3.2.8)[130]: Let $(\mu_n)_{n\geqslant 1}$ be a sequence of compactly supported probability measures on $\mathbb{R}$ converging in moments to a compactly supported probability measure μ on $\mathbb{R}$. Assume that the support of μ is included in the open interval $]-C,C[$. Then,

$$\lim_{n\to\infty}\int_{|t|\geqslant C}\mathrm{d}\mu_n = 0.$$

Moreover, let f be a borelian function on $\mathbb{R}$ such that there exist $M>0$ and $r\in\mathbb{N}$ satisfying $|f(t)|\leqslant M(t^{2r}+1)$ for all $t\geqslant C$. Then

$$\lim_{n\to\infty}\int_{|t|\geqslant C} f\,\mathrm{d}\mu_n = 0.$$

Proof. For the first assertion, let $C'<C$ such that the support of μ is included in $]-C',C']$. Let $\epsilon>0$ and an integer k such that $\left(\frac{C'}{C}\right)^{2k}\leqslant\epsilon$. Let $P(t)=\left(\frac{t}{C}\right)^{2k}$. It is clear that $\chi_{\{|t|\geqslant C\}}(t)\leqslant P(t)$ for all $t\in\mathbb{R}$ and that $\sup_{|t|<C'}P(t)\leqslant\epsilon$. Thus,

$$0\leqslant\limsup_{n\to\infty}\int_{|t|\geqslant C}\mathrm{d}\mu_n\leqslant\lim_{n\to\infty}\int P(t)\mathrm{d}\mu_n=\int P(t)\mathrm{d}\mu\leqslant\epsilon.$$

Since ϵ is arbitrary, we get $\lim_{n\to\infty}\int_{|t|\geqslant C}\mathrm{d}\mu_n = 0$.

The second assertion is a consequence of the first one. Let f be a borelian function on $\mathbb{R}$ such that there exist $M>0$ and $r\in\mathbb{N}$ satisfying $|f(t)|\leqslant M(t^{2r}+1)$ for all $t\in\mathbb{R}$. Using the Cauchy-Schwarz inequality we get:

$$\begin{aligned}0\leqslant\limsup_{n\to\infty}\int_{|t|\geqslant C}|f|\mathrm{d}\mu_n &\leqslant\limsup_{n\to\infty}\int_{|t|\geqslant C}M(t^{2r}+1)\mathrm{d}\mu_n\\ &\leqslant M\lim_{n\to\infty}\left(\int (t^{2r}+1)^2\,\mathrm{d}\mu_n\right)^{\frac{1}{2}}\lim_{n\to\infty}\left(\int_{|t|\geqslant C}\mathrm{d}\mu_n\right)^{\frac{1}{2}}\\ &\leqslant M\left(\int (t^{2r}+1)^2\,\mathrm{d}\mu\right)^{\frac{1}{2}}\lim_{n\to\infty}\left(\int_{|t|\geqslant C}\mathrm{d}\mu_n\right)^{\frac{1}{2}}=0\end{aligned}$$

Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}^*$ and consider the ultraproduct von Neumann algebra (see [135] section 9.10) N defined by

$$N=\left(\prod_{n\geqslant 1}\mathcal{A}_n\right)/I_{\mathcal{U}}$$

where $I_U = \{(x_n)_{n\geqslant 1} \in \prod_{n\geqslant 1} \mathcal{A}_n, \lim_U \tau_n(x_n^* x_n) = 0\}$. The von Neumann algebra N is equipped with the faithful normal and normalized trace $\tau((x_n)_{n\geqslant 1}) = \lim_U \tau_n(x_n)$ (which is well defined). Using the asymptotic matrix model for the q-Gaussians and by the preceding remark, we can define a $*$-homomorphism φ between the *-algebras $\mathcal{A}$ and N in the following way:

$$\varphi\left(P\big(G(e_1),\dots,G(e_k)\big)\right) = \left(P\big(\tilde{g}_{n,1},\dots,\tilde{g}_{n,k}\big)\right)_{n\geqslant 1}$$

for every polynomial P in k noncommuting variables. By Lemma (3.2.7), φ is trace preserving on $\mathcal{A}$. Since the $*$-algebra $\mathcal{A}$ is weak-* dense in $\Gamma_q(\mathbb{R}^k)$, φ extends naturally to a trace preserving homomorphism of von Neumann algebras, that is still denoted by φ (see Lemma (3.2.14) below for a more general result). It follows that $\Gamma_q(\mathbb{R}^k)$ is isomorphic to a sub-algebra of N which is the image of a conditional expectation (this is automatic in the tracial case). Since the $\mathcal{A}_n$'s are finite dimensional, they are injective, hence their product is injective and a fortiori has the WEP, and thus N is QWEP. Since $\Gamma_q(\mathbb{R}^k)$ is isomorphic to a sub-algebra of N which is the image of a conditional expectation, $\Gamma_q(\mathbb{R}^k)$ is also QWEP (see [133]). We have obtained the following:

Theorem (3.2.10)[130]: Let $H_{\mathbb{R}}$ be a real Hilbert space and $q \in (-1,1)$. The von Neumann algebra $\Gamma_q(H_{\mathbb{R}})$ is $QWEP$.

Proof. Our previous discussion implies the result for every finite dimensional $H_{\mathbb{R}}$. The general result is a consequence of the stability of QWEP by inductive limit (see [132] and ([133] Proposition 4.1 (iii)).

Let $C_q^*(H_{\mathbb{R}})$ be the C^*-algebra generated by all q-Gaussians:

$$C_q^*(H_{\mathbb{R}}) = C^*(\{G(f), f \in H_{\mathbb{R}}\}) \subset B\left(\mathcal{F}_q(H_{\mathbb{C}})\right).$$

We now deduce the following strengthening of Theorem (3.2.10).

Corollary (3.2.11)[130]: Let $H_{\mathbb{R}}$ be a real Hilbert space and $q \in (-1,1)$. The C^*-algebra $C_q^*(H_{\mathbb{R}})$ is QWEP.

Proof. This is a consequence of Theorem (3.2.10), Lemma (3.2.12) and Proposition 4.1 (ii) in [133].

Let A, B, with $A \subset B$ be C^*-algebras. Recall (from [133]) that A is said to be weakly cp complemented in B, if there exists a unital completely positive map $\Phi: B \longrightarrow A^{**}$ such that $\Phi_{|A} = \mathrm{id}_A$. Corollary (3.2.11) is then a consequence of the following Lemma.

Lemma (3.2.12)[130]: The C^*-algebra $C_q^*(H_{\mathrm{R}})$ is weakly cp complemented in the von Neumann algebra $\Gamma_q(H_{\mathbb{R}})$.

Proof. For any $t \in \mathbb{R}_+$ denote by Φ_t the unital completely positive maps which are the second quantization of $e^{-t}\mathrm{id}: H_{\mathrm{R}} \longrightarrow H_{\mathbb{R}}$ (see [60]):

$$\Phi_t = \Gamma_q(e^{-t}\mathrm{id}): \Gamma_q(H_{\mathbb{R}}) \longrightarrow \Gamma_q(H_{\mathbb{R}}), \text{ for all } t \geqslant 0.$$

$(\Phi_t)_{t\in\mathbb{R}_+}$ is a semi-group of unital completely positive maps which is also known as the q Ornstein-Uhlenbeck semi-group. By the well-known ultracontractivity of the semi-group $(\Phi_t)_{t\in\mathbb{R}_+}$ (see [115]), for all $t \in \mathbb{R}_+^*$ and all $W(\xi) \in \Gamma_q(H_{\mathbb{R}})$, we have

$$\|\Phi_t(W(\xi))\| \leqslant C_{|q|}^{\frac{3}{2}} \frac{1}{1-e^{-t}} \parallel \xi \parallel. \tag{14}$$

On the other hand, as a consequence of the Haagerup-Bożejko's inequality (see [115]), for every $n \in \mathbb{N}$ and for every $\xi_n \in H_{\mathbb{C}}^{\otimes n}$, we have $W(\xi_n) \in C_q^*(H_{\mathbb{R}})$. Fix $t \in \mathbb{R}_+^*, W(\xi) \in$

$\Gamma_q(H_{\mathbb{R}})$, and write $\xi = \sum_{n\in\mathbb{N}} \xi_n$ with $\xi_n \in H_{\mathbb{C}}^{\otimes n}$ for all n. From our last observation, for all $N \in \mathbb{N}$,

$$T_N = \Phi_t\left(W\left(\sum_{n=0}^{N} \xi_n\right)\right) = \sum_{n=0}^{N} e^{-tn} W(\xi_n) \in C_q^*(H_{\mathbb{R}}).$$

By (14), $\Phi_t(W(\xi))$ is the norm limit of the sequence $(T_N)_{N\in\mathbb{N}}$, so $\Phi_t(W(\xi))$ belongs to $C_q^*(H_{\mathbb{R}})$. It follows that Φ_t maps $\Gamma_q(H_{\mathbb{R}})$ into $C_q^*(H_{\mathbb{R}})$. Moreover, it is clear that

$$\lim_{t\to 0} \|\Phi_t(W(\xi)) - W(\xi)\| = 0, \text{ for all } W(\xi) \in C_q^*(H_{\mathbb{R}}). \tag{15}$$

Take $(t_n)_{n\in\mathbb{N}}$ a sequence of positive real numbers converging to 0 and fix $\mathcal{U}$ a free ultrafilter on $\mathbb{N}$. By w^*-compactness of the closed balls in $\left(C_q^*(H_{\mathbb{R}})\right)^{**}$, we can define the following mapping $\Phi\colon \Gamma_q(H_{\mathbb{R}}) \longrightarrow \left(C_q^*(H_{\mathbb{R}})\right)^{**}$ by

$$\Phi(W(\xi)) = w^* - \lim_{n,\mathcal{U}} \Phi_{t_n}(W(\xi)), \text{ for all } W(\xi) \in \Gamma_q(H_{\mathbb{R}}).$$

Φ is a unital completely positive map satisfying $\Phi_{|C_q^*(H_{\mathbb{R}})} = \mathrm{id}_{C_q^*(H_{\mathrm{R}})}$ by (15).

We start with a family $\left((\mathcal{A}_n, \varphi_n)\right)_{n\in\mathbb{N}}$ of von Neumann algebras equipped with normal faithful state φ_n. We assume that $\mathcal{A}_n \subset B(H_n)$, where the inclusion is given by the G.N.S. representation of $(\mathcal{A}_n, \varphi_n)$. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$, and let

$$\tilde{\mathcal{A}} = \prod_{n\in\mathbb{N}} \mathcal{A}_n/\mathcal{U}$$

be the C^*-ultraproduct over $\mathcal{U}$ of the algebras $\mathcal{A}_n$. We canonically identify $\tilde{\mathcal{A}} \subset B(H)$, where $H = \prod_G H_n/\mathcal{U}$ is the ultraproduct over $\mathcal{U}$ of the Hilbert spaces H_n. Following Raynaud (see 2 [136]), we define $\mathcal{A}$, the vN-ultraproduct over $\mathcal{U}$ of the von Neumann algebras $\mathcal{A}_n$, as the w^* closure of $\tilde{\mathcal{A}}$ in $B(H)$. Then the predual $\mathcal{A}_*$ of $\mathcal{A}$ is isometrically isomorphic to the Banach ultraproduct over $\mathcal{U}$ of the preduals $(\mathcal{A}_n)_*$

$$\mathcal{A}_* = \prod_{n\in\mathbb{N}} (\mathcal{A}_n)_*/\mathcal{U} \tag{16}$$

Let us denote by φ the normal state on $\mathcal{A}$ associated to $(\varphi_n)_{n\in\mathbb{N}}$. Note that φ is not faithful on $\mathcal{A}$, so we introduce $p \in \mathcal{A}$ the support of the state φ. Recall that for all $x \in \mathcal{A}$ we have $\varphi(x) = \varphi(xp) = \varphi(px)$, and that $\varphi(x) = 0$ for a positive x implies that $pxp = 0$. Denote by $(p\mathcal{A}p, \varphi)$ the induced von Neumann algebra $p\mathcal{A}p \subset B(pH)$ equipped with the restriction of the state φ. For each $n \in \mathbb{N}$, let $(\sigma_t^n)_{t\in\mathbb{R}}$ be the modular group of automorphisms of φ_n with the associated modular operator given by Δ_n. For all $t \in \mathbb{R}$, let $\left(\Delta_n^{it}\right)^{\cdot}$ be the associated unitary in $\prod_{n\in\mathbb{N}} B(H_n)/\mathcal{U} \subset B(H)$. Since $(\sigma_t^n)^{\cdot}_{n\in\mathbb{N}}$ is the conjugation by $\left(\Delta_n^{it}\right)^{\cdot}$, it follows that $(\sigma_t^n)^{\cdot}_{n\in\mathbb{N}}$ extends by w^*-continuity to a group of $*$-automorphisms of $\mathcal{A}$. Let $(\sigma_t)_{t\in\mathbb{R}}$ be the local modular group of automorphisms of $p\mathcal{A}p$. By Raynaud's result (see Theorem 2.1 in [136]), $p\mathcal{A}p$ is stable by $(\sigma_t^n)^{\cdot}_{n\in\mathbb{N}}$ and the restriction of $(\sigma_t^n)^{\cdot}_{n\in\mathbb{N}}$ to $p\mathcal{A}p$ coincides with σ_t.

We consider a von Neumann algebra $\mathcal{N} \subset B(K)$ equipped with a normal faithful state ψ. Let $\tilde{\mathcal{N}}$ be a w^*-dense *-subalgebra of $\mathcal{N}$ and Φ a *-homomorphism from $\tilde{\mathcal{N}}$ into $\mathcal{A}$ whose image will be denoted by $\mathcal{B}$ with w^*-closure denoted by $\mathcal{B}$:

$$\Phi\colon \tilde{\mathcal{N}} \subset \mathcal{N} \subset B(K) \longrightarrow \widetilde{\mathcal{B}} \subset \mathcal{A} \subset B(H) \text{ and } \overline{\tilde{\mathcal{N}}}^{w^*} = \mathcal{N},\ \overline{\widetilde{\mathcal{B}}}^{w^*} = \mathcal{B}$$

By a result of Takesaki (see [107]) there is a normal conditional expectation from $p\mathcal{A}p$ onto $p\mathcal{B}p$ if and only if $p\mathcal{B}p$ is stable by the modular group of φ (which is here given by Raynaud's results). Under this condition there will be a normal conditional expectation from $\mathcal{A}$ onto $p\mathcal{B}p$ and $p\mathcal{B}p$ will inherit some of the properties of $\mathcal{A}$. We would like to pull back these properties to $\mathcal{N}$ itself. It turns out that, with good assumptions on Φ (see Lemma (3.2.13) below), the compression from $\mathcal{B}$ onto $p\mathcal{B}p$ is a *-homomorphism. If in addition, we suppose that Φ is state preserving, then $p\Phi p$ can be extended into a w^*-continuous $*$-isomorphism between $\mathcal{N}$ and $p\mathcal{B}p$.

Lemma (3.2.13)[130]: In the following, (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e).:

1. For all $x \in \tilde{\mathcal{B}}$ there is a representative $(x_n)_{n\in\mathbb{N}}$ of x such that for all $n \in \mathbb{N}, x_n$ is entire for $(\sigma_t^n)_{t\in\mathbb{R}}$ and $\left(\sigma_{-i}^n(x_n)\right)_{n\in\mathbb{N}}$ is uniformly bounded.
2. For all $x \in \widetilde{\mathcal{B}}$ there exists $z \in \mathcal{A}$ such that for all $y \in \mathcal{A}$ we have $\varphi(xy) = \varphi(yz)$.
3. For all $(x, y) \in \mathcal{B}^2$: $\varphi(xpy) = \varphi(xy)$
4. For all $(x, y) \in \mathcal{B}^2$, pxyp = pxpyp, i.e the canonical application from $\mathcal{B}$ to $p\mathcal{B}p$ is a *-homomorphism.
5. $p \in \mathcal{B}'$.

Proof. (a)$\Longrightarrow$(b) Consider $x \in \tilde{\mathcal{B}}$ with a representative $(x_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}, x_n$ is entire for $(\sigma_t^n)_{t\in\mathbb{R}}$ and $\left(\sigma_{-i}^n(x_n)\right)_{n\in\mathbb{N}}$ is uniformly bounded. Denote by $z \in \mathcal{A}$ the class $\left(\sigma_{-i}^n(x_n)\right)^{\cdot}_{n\in\mathbb{N}}$. By w^*-density and continuity if suffices to consider an element y in $\tilde{\mathcal{A}}$ with representative $(y_n)_{n\in\mathbb{N}}$. Then,

$$\varphi(xy) = \lim_{n,\mathcal{U}} \varphi_n(x_n y_n) = \lim_{n,\mathcal{U}} \varphi_n\left(y_n \sigma_{-i}^n(x_n)\right) = \varphi(yz)$$

(b)$\Longrightarrow$(c) Here again it suffices to consider $(x, y) \in \widetilde{\mathcal{B}}^2$. By assumption there exists $z \in \mathcal{A}$ such that for all $t \in \mathcal{A}, \varphi(xt) = \varphi(tz)$. Applying our assumption for $t = py$ and $t = y$ successively, we obtain the desired result:

$$\varphi(xpy) = \varphi(pyz) = \varphi(yz) = \varphi(xy)$$

(c)$\Longrightarrow$(d) Let $x \in \mathcal{B}$. We have, by (c): $\varphi(x(1-p)x^*) = 0$. Since p is the support of φ and $x(1-p)x^* \geqslant 0$, this implies $px(1-p)x^*p = 0$. Thus for all $x \in \mathcal{B}$ we have

$$\mathrm{pxpx}^* p = pxx^*p$$

We conclude by polarization.

(d)$\Longrightarrow$(e) Let q be an orthogonal projection in $\mathcal{B}$. By (d), pqp is again an orthogonal projection and we claim that this is equivalent to $pq = qp$. Indeed, let us denote by x the contraction qp. Then $x^*x = pqp$ and since pqp is an orthogonal projection we have $|x| = pqp$. It follows that the polar decomposition of x is of the form $x = upqp$, with u a partial isometry. Computing x^2, we see that x is a projection:

$$x^2 = upqp(qp) = \mathrm{upq}\, p = x.$$

Since x is contractive, we deduce that x is an orthogonal projection and that $x^* = x$. Thus $pq = qp$. Since $\mathcal{B}$ is generated by its projections, we have $p \in \mathcal{B}'$.

(e) $\Longrightarrow$(c) This is clear.

We assume that one of the technical conditions of the previous Lemma is fulfilled. Let us denote by $\Theta = p\Phi p$. Θ is a $*$-homomorphism from $\tilde{\mathcal{N}}$, into $p\mathcal{A}p$.

$$\Theta = p\Phi p\colon \tilde{\mathcal{N}} \longrightarrow p\mathcal{A}p \subset B(pH).$$

We assume that Φ, and hence Θ, is state preserving. Then Θ can be extended into a (w^* continuous) *-isomorphism from $\mathcal{N}$ onto $p\mathcal{B}p$. This is indeed a consequence of the following well known fact:

Lemma (3.2.14)[130]: Let $(\mathcal{M},\varphi)$ and $(\mathcal{N},\psi)$ be von Neumann algebras equipped with normal faithful states. Let $\widetilde{\mathcal{M}}$,(respectively $\tilde{\mathcal{N}}$), be a w^* dense *-subalgebra of $\mathcal{M}$(respectively $\mathcal{N}$). Let Ψ be a $*$-homomorphism from $\widetilde{\mathcal{M}}$ onto $\tilde{\mathcal{N}}$ such that for all $m\in\widetilde{\mathcal{M}}$ we have $\psi(\Psi(m))=\varphi(m)$ (Ψ is state preserving). Then Ψ extends uniquely into a normal *-isomorphism between $\mathcal{M}$ and $\mathcal{N}$.

Proof. Since φ is faithful, we have for all $m\in\mathcal{M}, \| m \|=\lim_{n\to+\infty}\varphi((m^*m)^n)^{\frac{1}{2n}}$. Thus, since Ψ is state preserving, Ψ is isometric from $\widetilde{\mathcal{M}}$ onto $\tilde{\mathcal{N}}$. We put

$$\varphi\widetilde{\mathcal{M}}=\{\varphi.m,m\in\widetilde{\mathcal{M}}\}\subset\mathcal{M}_* \text{ and } \psi\tilde{\mathcal{N}}=\{\psi\cdot n,n\in\tilde{\mathcal{N}}\}\subset\mathcal{N}_*.$$

$\varphi\widetilde{\mathcal{M}}$ (respectively $\psi\tilde{\mathcal{N}}$) is dense in $\mathcal{M}_*$ (respectively $\mathcal{N}_*$). Let us define the following linear operator Ξ from $\psi\tilde{\mathcal{N}}$ onto $\widetilde{\mathcal{M}}$:

$$\Xi(\psi.\Psi(m))=\varphi.m \text{ for all } m\in\widetilde{\mathcal{M}}$$

Using Kaplansky's density Theorem and the fact that Ψ is isometric, we compute:

$$\Xi(\psi.\Psi(m))=\sup_{m_0\in\widetilde{\mathcal{M}},\|m_0\|\leqslant 1}\|\varphi(mm_0)\|=\sup_{m_0\in\mathcal{M},\|m_0\|\leqslant 1}\|\psi(\Psi(m)\Psi(m_0))\|$$
$$=\sup_{n_0\in\tilde{N},\|n_0\|\leqslant 1}\| \psi(\Psi(m)n_0) \|=\| \psi.\Psi(m) \|$$

So that Ξ extends into a surjective isometry from $\mathcal{N}_*$ onto $\mathcal{M}_*$. Moreover Ξ is the preadjoint of Ψ. Indeed we have for all $(m,m_0)\in\widetilde{\mathcal{M}}^2$:

$$\langle\psi\cdot\Psi(m),\Psi(m_0)\rangle=\psi(\Psi(m)\Psi(m_0))=\varphi(mm_0)=\langle\Xi(\psi\cdot\Psi(m)),m_0\rangle$$

Thus Ψ extends to a normal $*$-isomorphism between $\mathcal{N}$ and $\mathcal{M}$.

In the following Theorem, we sum up what we have proved in the previous discussion:

Theorem (3.2.15)[130]: Let $(\mathcal{N},\psi)$ and $(\mathcal{A}_n,\varphi_n)$, for $n\in\mathbb{N}$, be von Neumann algebras equipped with normal faithful states. Let $\mathcal{U}$ be a non trivial ultrafilter on $\mathbb{N}$, and $\mathcal{A}$ the von Neumann algebra ultraproduct over $\mathcal{U}$ of the $\mathcal{A}_n$'s. For all $n\in\mathbb{N}$ let us denote by $(\sigma_t^n)_{t\in\mathbb{R}}$ the modular group of φ_n and by φ the normal state on $\mathcal{A}$ which is the ultraproduct of the states $\varphi_n.p\in\mathcal{A}$ denote the support of φ. Consider $\tilde{\mathcal{N}}$ a w^*-dense $*$-subalgebra of $\mathcal{N}$ and a $*$-homomorphism Φ

$$\Phi:\tilde{\mathcal{N}}\subset\mathcal{N}\longrightarrow\mathcal{A}=\prod_{n,\mathcal{U}}\mathcal{A}_n$$

Assume Φ satisfies:

1. Φ is state preserving: for all $x\in\tilde{\mathcal{N}}$ we have
$$\varphi(\Phi(x))=\psi(x)$$
2. For all $(x,y)\in\Phi(\tilde{\mathcal{N}})^2$
$$\varphi(xy)=\varphi(xpy).$$
(Or one of the technical conditions of Lemma (3.2.13).)
3. For all $t\in\mathbb{R}$ and for all $y=(y_n)_{n\in\mathbb{N}}\in\Phi(\tilde{\mathcal{N}})$,
$$p\big(\sigma_t^n(y_n)\big)^{\cdot}_{n\in\mathbb{N}}p(=\sigma_t(pyp))\in p\mathcal{B}p$$

where $\mathcal{B}$ is the w^*-closure of $\Phi(\tilde{\mathcal{N}})$ in $\mathcal{A}$.

Then $\Theta=p\Phi p:\tilde{\mathcal{N}}\longrightarrow p\mathcal{A}p$ is a state preserving *-homomorphism which can be extended into a normal isomorphism (still denoted by Θ) between $\mathcal{N}$ and its image $\Theta(\mathcal{N})=p\mathcal{B}p$. Moreover there exists a (normal) state preserving conditional expectation from $\mathcal{A}$ onto $\Theta(\mathcal{N})$.

Corollary (3.2.16)[130]: Under the assumptions of the previous Theorem, $\mathcal{N}$ is QWEP provided that each of the $\mathcal{A}_n$ is QWEP.

Proof. This is a consequence of Kirchberg's results (see [132], [133]). First, $\prod_{n\in\mathbb{N}} \mathcal{A}_n$ is QWEP as a product of QWEP C^*-algebras ([133] Proposition 4.1(i)). Since $\tilde{\mathcal{A}}$ is a quotient of a QWEP C^*-algebra, it is also QWEP. It follows that $\mathcal{A}$ which is the w^*-closure of $\tilde{\mathcal{A}}$ in $B(H)$ is QWEP (by [133] Proposition 4.1 (iii)). Since there is a conditional expectation from $\mathcal{A}$ onto $p\mathcal{A}p, p\mathcal{A}p$ is QWEP (see [132]). Finally, by Theorem (3.2.15), $\mathcal{N}$ is isomorphic to a subalgebra of $p\mathcal{A}p$ which is the image of a (state preserving) conditional expectation, thus $\mathcal{N}$ inherits the QWEP property.

We show that $\Gamma_q(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$ is QWEP when $H_{\mathbb{R}}$ is finite dimensional. For notational purpose, it will be more convenient to deal with $\dim(H_{\mathbb{R}})$ even. This is not relevant in our context (see the remark after Theorem (3.2.26)). We put $\dim(H_{\mathbb{R}}) = 2k$. Notice that $\Gamma_q(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$ only depends on the spectrum of the operator A. The spectrum of A is given by the set $\{\lambda_1, \dots, \lambda_k\} \cup \{\lambda_1^{-1}, \dots, \lambda_k^{-1}\}$ where for all $j \in \{1, \dots, k\}, \lambda_j \geqslant 1$. We use the notation $\mu_j = \lambda_j^{\frac{1}{4}}$.

We start by adapting Biane's model to our situation. Let us denote by I the set $\{-k, \dots, -1\} \cup \{1, \dots, k\}$. We give us a function ϵ on $I \times I$ into $\{-1,1\}$ and we consider the associated complex *-algebra $\mathcal{A}(I, \epsilon)$. By analogy with (3), for all $j \in \{1, \dots, k\}$ we define the following generalized semi-circular variables acting on $L^2(\mathcal{A}(I, \epsilon), \varphi^\epsilon)$:

$$\gamma_i = \mu_i^{-1}\beta_i^* + \mu_i\beta_{-i} \text{ and } \delta_i = \mu_i\alpha_i^* + \mu_i^{-1}\alpha_{-i}$$

We denote by Γ (respectively Γ_r) the von Neumann algebra generated in $B\big(L^2(\mathcal{A}(I, \epsilon), \varphi^\epsilon)\big)$ by the γ_i (respectively δ_i). Γ_r is the natural candidate for the commutant of Γ in $B\big(L^2(\mathcal{A}(I, \epsilon), \varphi^\epsilon)\big)$. We need to show that the vector 1 is cyclic and separating for Γ. To do so we must assume that ϵ satisfies the following additional condition:

$$\text{For all } (i,j) \in I^2,\ \epsilon(i,j) = \epsilon(|i|, |j|) \tag{17}$$

This condition is in fact a necessary condition for $\Gamma_r \subset \Gamma'$ and for condition (i)(a) of Lemma (3.2.18) below.

Lemma (3.2.17)[130]: Under condition (17) the following relation holds:

$$\text{For all } i \in I,\ \alpha_i\beta_i^* + \alpha_{-i}^*\beta_{-i} = \beta_i^*\alpha_i + \beta_{-i}\alpha_{-i}^*$$

Proof. Let $i \in I$ and $A \subset I$. We have

$$(\alpha_i\beta_i^* + \alpha_{-i}^*\beta_{-i})(x_A) = \begin{cases} x_{-i}x_Ax_{-i} & \text{if } i \in A \text{ and } -i \in A \\ 0 & \text{if } i \in A \text{ and } -i \notin A \\ x_ix_Ax_i + x_{-i}x_Ax_{-i} & \text{if } i \notin A \text{ and } -i \in A \\ x_ix_Ax_i & \text{if } i \notin A \text{ and } -i \notin A \end{cases}$$

and

$$(\beta_i^*\alpha_i + \beta_{-i}\alpha_{-i}^*)(x_A) = \begin{cases} x_ix_Ax_i & \text{if } i \in A \text{ and } -i \in A \\ x_ix_Ax_i + x_{-i}x_Ax_{-i} & \text{if } i \in A \text{ and } -i \notin A \\ 0 & \text{if } i \notin A \text{ and } -i \in A \\ x_{-i}x_Ax_{-i} & \text{if } i \notin A \text{ and } -i \notin A \end{cases}$$

Thus, we need to study the following cases. Assume that $A = \{i_1, \dots, i_p\}$ where $i_1 < \dots < i_p$.

1. If i and $-i$ belong to A then there exists $(l,m) \in \{1,\dots,p\}, l < m$, such that $i_l = -i$ and $i_m = i$. Applying successively relations (4) and (17), we get:

$$\begin{aligned} x_{-i}x_Ax_{-i} &= \left(\prod_{q=1}^{l-1} \epsilon(i_q,-i)\right) x_{i_1} \dots x_{i_{l-1}} x_{i_{l+1}} \dots x_{i_p} x_{-i} \\ &= \left(\prod_{q=1}^{l-1} \epsilon(i_q,-i)\right)\left(\prod_{q=l+1}^{p} \epsilon(i_q,-i)\right) x_A = -\left(\prod_{q=1}^{p} \epsilon(i_q,-i)\right) x_A \\ &= -\left(\prod_{q=1}^{p} \epsilon(i_q,i)\right) x_A = x_i x_A x_i \end{aligned}$$

2. If i and $-i$ do not belong to A, we can check in a similar way that:

$$\begin{aligned} x_{-i}x_Ax_{-i} &= \left(\prod_{q=1}^{p} \epsilon(i_q,-i)\right) x_A = \left(\prod_{q=1}^{p} \epsilon(i_q,i)\right) x_A \\ &= x_i x_A x_i \end{aligned}$$

3. If $i \in A$ and $-i \notin A$, then there exists $l \in \{1,\dots,p\}$ such that $i_l = i$. We have:

$$\begin{aligned} x_i x_A x_i &= \left(\prod_{q=1}^{l-1} \epsilon(i_q,\hat{\imath})\right) x_{i_1} \dots x_{i_{l-1}} x_{i_{l+1}} \dots x_{i_p} x_i \\ &= \left(\prod_{q=1}^{l-1} \epsilon(i_q,i)\right)\left(\prod_{q=l+1}^{p} \epsilon(i_q,i)\right) x_A = -\left(\prod_{q=1}^{p} \epsilon(i_q,i)\right) x_A \\ &= -\left(\prod_{q=1}^{p} \epsilon(i_q,-i)\right) x_A = -x_{-i}x_Ax_{-i} \end{aligned}$$

This finishes the proof.

Lemma (3.2.18)[130]: By construction we have:

1. For all $(i,j) \in \{1,\dots,k\}^2, i \neq j$, the following mixed commutation and anti-commutation relations hold:
   (a) $\gamma_i\gamma_j - \epsilon(i,j)\gamma_j\gamma_i = 0$
   (b) $\gamma_i^*\gamma_j - \epsilon(i,j)\gamma_j\gamma_i^* = 0$
   (c) $(\gamma_i^*)^2 = \gamma_i^2 = 0$
   (d) $\gamma_i^*\gamma_i + \gamma_i\gamma_i^* = (\mu_i^2 + \mu_i^{-2})Id$.
2. Same relations as in (a) for the operators δ_i.
3. $\Gamma_r \subset \Gamma'$.
4. The vector 1 is cyclic and separating for both Γ and Γ_r.
5. $\Gamma \subset B\big(L^2(\mathcal{A}(I,\epsilon),\varphi^\epsilon)\big)$ is the (faithful) G.N.S representation of $(\Gamma,\varphi^\epsilon)$.

Proof. (i)(a) Thanks to (ii). of Lemma (3.2.3) and (17) we get:

$$\begin{aligned}\gamma_i\gamma_j = &\ \mu_i^{-1}\mu_j^{-1}\beta_i^*\beta_j^* + \mu_i\mu_j\beta_{-i}\beta_{-j} + \mu_i^{-1}\mu_j\beta_i^*\beta_{-j} + \mu_i\mu_j^{-1}\beta_{-i}\beta_j^* \\ = &\ \epsilon(i,j)\mu_i^{-1}\mu_j^{-1}\beta_j^*\beta_i^* + \epsilon(-i,-j)\mu_i\mu_j\beta_{-j}\beta_{-i} + \epsilon(i,-j)\mu_i^{-1}\mu_j\beta_{-j}\beta_i^* \\ &+ \epsilon(-i,j)\mu_i\mu_j^{-1}\beta_j^*\beta_{-i} \\ = &\ \epsilon(i,j)\left(\mu_i^{-1}\mu_j^{-1}\beta_j^*\beta_i^* + \mu_i\mu_j\beta_{-j}\beta_{-i} + \mu_i^{-1}\mu_j\beta_{-j}\beta_i^* + \mu_i\mu_j^{-1}\beta_j^*\beta_{-i}\right) \\ = &\ \epsilon(i,j)\gamma_j\gamma_i\end{aligned}$$

(i)(b) Is analogous to (a) and is left to the reader.

(i)(c) Using (a) and (b) of Lemma (3.2.3), and $\epsilon(i,-i) = \epsilon(i,i) = -1$ we get:

$$\begin{aligned}\gamma_i^2 &= \mu_i^{-2}(\beta_i^*)^2 + \mu_i^2\beta_{-i}^2 + \beta_i^*\beta_{-i} + \beta_{-i}\beta_i^* \\ &= \epsilon(i,-i)\beta_{-i}\beta_i^* + \beta_{-i}\beta_i^* = 0\end{aligned}$$

(i)(d) Using similar arguments, we compute:

$$\begin{aligned}\gamma_i^*\gamma_i + \gamma_i\gamma_i^* &= \mu_i^{-2}(\beta_i\beta_i^* + \beta_i^*\beta_i) + \mu_i^2(\beta_{-i}^*\beta_{-i} + \beta_{-i}\beta_{-i}^*) + \beta_i\beta_{-i} + \beta_{-i}\beta_i \\ &\quad +\beta_{-i}^*\beta_i^* + \beta_i^*\beta_{-i}^* \\ &= \left(\mu_i^{-2} + \mu_i^2\right)Id + (\epsilon(i,-i)+1)(\beta_i\beta_{-i} + \beta_{-i}^*\beta_i^*) = \left(\mu_i^{-2} + \mu_i^2\right)Id\end{aligned}$$

(ii) Is now clear from the proof of (i) since the relations for the α_i's are the same as the ones for the β_i's.

(iii) It suffices to show that for all $(i,j) \in \{1, \dots, k\}^2$ we have $\gamma_i\delta_j = \delta_j\gamma_i$ and $\gamma_i\delta_j^* = \delta_j^*\gamma_i$.

If $i \neq j$ then from (v) of Lemma (3.2.3) it is clear that $\gamma_i\delta_j = \delta_j\gamma_i$ and $\gamma_i\delta_j^* = \delta_j^*\gamma_i$.

If $i = j$ then using (iv) and (v) of Lemma (3.2.3) and Lemma (3.2.17) we obtain the desired result as follows:

$$\begin{aligned}\gamma_i\delta_i &= \beta_i^*\alpha_i^* + \beta_{-i}\alpha_{-i} + \mu_i^{-2}\beta_i^*\alpha_{-i} + \mu_i^2\beta_{-i}\alpha_i^* = \mu_i^{-2}\beta_i^*\alpha_{-i} + \mu_i^2\beta_{-i}\alpha_i^* \\ &= \mu_i^{-2}\alpha_{-i}\beta_i^* + \mu_i^2\alpha_i^*\beta_{-i} = \delta_i\gamma_i\end{aligned}$$

and

$$\begin{aligned}\gamma_i\delta_i^* &= \beta_i^*\alpha_i + \beta_{-i}\alpha_{-i}^* + \mu_i^{-2}\beta_i^*\alpha_{-i}^* + \mu_i^2\beta_{-i}\alpha_i \\ &= \alpha_i\beta_i^* + \alpha_{-i}^*\beta_{-i} + \mu_i^{-2}\beta_i^*\alpha_{-i} + \mu_i^2\beta_{-i}\alpha_i^* \\ &= \alpha_i\beta_i^* + \alpha_{-i}^*\beta_{-i} + \mu_i^{-2}\alpha_{-i}\beta_i^* + \mu_i^2\alpha_i^*\beta_{-i} = \delta_i^*\gamma_i\end{aligned}$$

(iv) It suffices to prove that for any $A \subset I$ we have $x_A \in \Gamma 1 \cap \Gamma_r 1$. Let $A \subset I$ and $(\chi_i)_{i\in I} \in \{0,1\}^I$ such that $\chi_i = 1$ if and only if $i \in A$. Then

$$\begin{aligned}x_A &= x_{-k}^{\chi_{-k}} \dots x_{-1}^{\chi_{-1}} x_1^{\chi_1} \dots x_k^{\chi_k} \\ &= (\mu_k^{-1}\gamma_k^*)^{\chi_{-k}} \dots (\mu_1^{-1}\gamma_1^*)^{\chi_{-1}}(\mu_1\gamma_1)^{\chi_1} \dots (\mu_k\gamma_k)^{\chi_k}1 \\ &= \mu_1^{\chi_1-\chi_{-1}} \dots \mu_k^{\chi_k-\chi_{-k}}\gamma_k^{-\chi_{-k}} \dots \gamma_1^{-\chi_{-1}}\gamma_1^{\chi_1} \dots \gamma_k^{\chi_k}1\end{aligned}$$

where by convention $\gamma_i^{-1} = \gamma_i^*$.

The same computation is valid for Γ_r and we obtain:

$$x_A = \mu_1^{\chi_{-1}-\chi_1} \dots \mu_k^{\chi_{-k}-\chi_k}\delta_k^{\chi_k} \dots \delta_1^{\chi_1}\delta_1^{-\chi_{-1}} \dots \delta_k^{-\chi_{-k}}1$$

It follows that the vector 1 is cyclic for both Γ and Γ_r. Since $\Gamma_r \subset \Gamma'$ then 1 is also cyclic for Γ' and thus separating for Γ. The same argument applies to Γ_r and thus 1 is also a cyclic and separating vector for Γ_r.

(v) This is clear from the just proved assertion and the fact that the state φ^ϵ is equal to the vector state associated to the vector 1.

By the Lemma just proved, we are in a situation where we can apply Tomita-Takesaki theory. As usual we denote by S the involution on $L^2(\mathcal{A}(I,\epsilon),\varphi^\epsilon)$ defined by: $S(\gamma 1) = \gamma^* 1$ for all $\gamma \in \Gamma$. Δ will denote the modular operator and J the modular conjugation. Recall that

$S = J\Delta^{\frac{1}{2}}$ is the polar decomposition of the antilinear operator S (which is here bounded since we are in a finite dimensional framework). We also denote by $(\sigma_t)_{t\in\mathbb{R}}$ the modular group of automorphisms of Γ associated to φ. Recall that for all $\gamma \in \Gamma$ and all $t \in \mathbb{R}$ we have $\sigma_t(\gamma) = \Delta^{it}\gamma\Delta^{-it}$.

Notation (3.2.19)[130]: In the following, for $A \subset I$ we denote by $(\chi_i)_{i\in I}$ the characteristic function of the set A: $\chi_i = 1$ if $i \in A$ and $\chi_i = 0$ if $i \notin A$. (We will not keep track of the dependance in A unless there could be some confusion.)

Proposition (3.2.20)[130]: The modular operators and the modular group of $(\Gamma, \varphi^\epsilon)$ are determined by:

1. J is the antilinear operator given by: for all $A \subset I$,
$$J(x_A) = J\left(x_{-k}^{\chi_{-k}} \dots x_{-1}^{\chi_{-1}} x_1^{\chi_1} \dots x_k^{\chi_k}\right) = x_{-k}^{\chi_k} \dots x_{-1}^{\chi_1} x_1^{\chi_{-1}} \dots x_k^{\chi_{-k}}$$
2. Δ is the diagonal and positive operator given by: for all $A \subset I$,
$$\Delta(x_A) = \Delta\left(x_{-k}^{\chi_{-k}} \dots x_{-1}^{\chi_{-1}} x_1^{\chi_1} \dots x_k^{\chi_k}\right) = \lambda_k^{(\chi_k-\chi_{-k})} \dots \lambda_1^{(\chi_1-\chi_{-1})} x_A$$
3. For all $j \in \{1 \dots, k\}, \gamma_j$ is entire for $(\sigma_t)_t$ and satisfies $\sigma_z(\gamma_j) = \lambda_j^{iz}\gamma_j$ for all $z \in \mathbb{C}$.

Proof. Let $A \subset I$. We have
$$x_A = x_{-k}^{\chi_{-k}} \dots x_{-1}^{\chi_{-1}} x_1^{\chi_1} \dots x_k^{\chi_k} = \mu_1^{\chi_1-\chi_{-1}} \dots \mu_k^{\chi_k-\chi_{-k}} \gamma_k^{-\chi_{-k}} \dots \gamma_1^{-\chi_{-1}} \gamma_1^{\chi_1} \dots \gamma_k^{\chi_k} 1$$
Thus,
$$\begin{aligned} S(x_A) &= \mu_1^{\chi_1-\chi_{-1}} \dots \mu_k^{\chi_k-\chi_{-k}} \left(\gamma_k^{-\chi_{-k}} \dots \gamma_1^{-\chi_{-1}} \gamma_1^{\chi_1} \dots \gamma_k^{\chi_k}\right)^* 1 \\ &= \mu_1^{\chi_1-\chi_{-1}} \dots \mu_k^{\chi_k-\chi_{-k}} \gamma_k^{-\chi_k} \dots \gamma_1^{-\chi_1} \gamma_1^{\chi_{-1}} \dots \gamma_k^{\chi_{-k}} 1 \\ &= \mu_1^{2(\chi_1-\chi_{-1})} \dots \mu_k^{2(\chi_k-\chi_{-k})} x_{-k}^{\chi_k} \dots x_{-1}^{\chi_1} x_1^{\chi_{-1}} \dots x_k^{\chi_{-k}} \end{aligned}$$
By uniqueness of the polar decomposition, we obtain the stated result. Let $j \in \{1 \dots k\}$ and $t \in \mathbb{R}$ we have:
$$\begin{aligned} \sigma_t(\gamma_j)1 &= \Delta^{it}\gamma_j\Delta^{-it}1 = \Delta^{it}\gamma_j 1 = \mu_j^{-1}\Delta^{it}x_j = \mu_j^{-1}\mu_j^{4it}x_j \\ &= \mu_j^{4it}\gamma_j 1 \end{aligned}$$
It follows, since 1 is separating for Γ, that $\sigma_t(\gamma_j) = \mu_j^{4it}\gamma_j$.

We use the twisted Baby Fock construction to obtain an asymptotic random matrix model for the q-Gaussian variables, via Speicher's central limit Theorem. Let us first check the independence condition:

Lemma (3.2.21)[130]: For all $j \in \{1, \dots, k\}$ let us denote by $\mathcal{A}_j$ the C^*-subalgebra of $B(L^2(\mathcal{A}(I,\epsilon), \varphi^\epsilon))$ generated by the operators β_j and β_{-j}. Then the family $(\mathcal{A}_j)_{1\leqslant j\leqslant k}$ is independent in $B(L^2(\mathcal{A}(I,\epsilon), \varphi^\epsilon))$. In particular, the family $(\gamma_j)_{1\leqslant j\leqslant k}$ is independent.

Proof. The proof proceeds by induction. Changing notation, it suffices to show that
$$\varphi^\epsilon(a_1 \dots a_{r+1}) = \varphi^\epsilon(a_1 \dots a_r)\varphi^\epsilon(a_{r+1})$$
where $a_l \in \mathcal{A}_l$ for all $l \in \{1, \dots, r+1\}$. Since a_{r+1} is a certain non-commutative polynomial in the variables $\beta_{r+1}, \beta_{r+1}^*, \beta_{-(r+1)}$, and $\beta_{-(r+1)}^*$, it is clear that there exists $\nu \in \operatorname{Span}\{x_{r+1}, x_{-(r+1)}, x_{-(r+1)}x_{r+1}\}$ such that
$$a_{r+1}1 = \langle 1, a_{r+1}1\rangle 1 + \nu$$
It is easy to see that $a_r^* \dots a_1^* 1 \in \operatorname{Span}\{x_B, B \subset \{-r, \dots, -1\} \cup \{1, \dots, r\}\}$, which is orthogonal to $\operatorname{Span}\{x_{r+1}, x_{-(r+1)}, x_{-(r+1)}x_{r+1}\}$. We compute:

$$\begin{aligned}\varphi^{\epsilon}(a_1 \dots a_{r+1}) &= \langle 1, a_1 \dots a_r a_{r+1} 1\rangle = \langle a_r^* \dots a_1^* 1, a_{r+1} 1\rangle \\ &= \langle a_r^* \dots a_1^* 1, 1\rangle\langle 1, a_{r+1} 1\rangle + \langle a_r^* \dots a_1^* 1, \nu\rangle = \langle 1, a_1 \dots a_r 1\rangle\langle 1, a_{r+1} 1\rangle \\ &= \varphi^{\epsilon}(a_1 \dots a_r)\varphi^{\epsilon}(a_{r+1})\end{aligned}$$

Let $q \in (-1,1)$ (this is Proposition 3 in [128]). Let us choose a family of random variables $(\epsilon(i,j))_{(i,j)\in\mathbb{N}^2, i\neq j}$ as in Lemma (3.2.6), and set $\epsilon(i,i) = -1$ for all $i \in \mathbb{N}_*$. For all $n \in \mathbb{N}_*$ we will consider the complex *-algebra $\mathcal{A}(I_n, \epsilon_n)$ where

$$I_n = \{1, \dots, n\} \times (\{-k, \dots, -1\} \cup \{1, \dots, k\})$$

and

$$\epsilon_n\big((i,j),(i',j')\big) = \epsilon(i,i') \text{ for all } \big((i,j),(i',j')\big) \in I_n^2.$$

Notice that the analogue of condition (17) is automatically satisfied. Indeed, we have:

$$\epsilon_n\big((i,j),(i',j')\big) = \epsilon_n\big((i,|j|),(i',|j'|)\big) \text{ for all } \big((i,j),(i',j')\big) \in I_n^2.$$

Let us remind that $\mathcal{A}(I_n, \epsilon_n)$ is the unital free complex algebra with generators $\big(x_{i,j}\big)_{(i,j)\in I_n}$ quotiented by the relations,

$$x_{i,j}x_{i',j'} - \epsilon(i,i')x_{i',j'}x_{i,j} = 2\delta_{(i,j),(i',j')}$$

and with involution given by $x_{i,j}^* = x_{i,j}$. For all $(i,j) \in \{1, \dots, n\} \times \{1, \dots, k\}$ let $\gamma_{i,j}$ be the "twisted semi-circular variable" associated to μ_j

$$\gamma_{i,j} = \mu_j^{-1}\beta_{i,j}^* + \mu_j\beta_{i,j}$$

We denote by $\Gamma_n \subset B\big(L^2(\mathcal{A}(I_n,\epsilon_n),\varphi^{\epsilon_n})\big)$ the von-Neumann algebra generated by the $\gamma_{i,j}$ for $(i,j) \in \{1, \dots, n\} \times \{1, \dots, k\}$. Observe that all our notations are consistent since $(\Gamma_n, \varphi^{\epsilon_n})$ is naturally embedded in $(\Gamma_{n+1}, \varphi^{\epsilon_{n+1}})$ (see following Lemma (3.2.3)). In fact all these algebras $(\Gamma_n, \varphi^{\epsilon_n})$ can be embedded in the bigger von Neumann algebra $(\Gamma, \varphi^{\bar{\epsilon}})$ which is the Baby Fock construction associated to the infinite set $\bar{I}$ and the sign function $\bar{\epsilon}$ given by

$$\bar{I} = \mathbb{N}_* \times (\{-k, \dots, -1\} \cup \{1, \dots, k\})$$

and

$$\bar{\epsilon}\big((i,j),(i',j')\big) = \epsilon(i,i') \text{ for all } \big((i,j),(i',j')\big) \in \bar{I}^2.$$

Let us denote by $s_{n,j}$ the following sum:

$$s_{n,j} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \gamma_{i,j}$$

We now check the hypothesis of Theorem (3.2.5) for the family $\big(\gamma_{i,j}\big)_{(i,j)\in\mathbb{N}_*\times\{1,\dots,k\}} \subset (\Gamma, \varphi^{\bar{\epsilon}})$.

1. The family is independent by Lemma (3.2.21).
2. It is clear that for all (i,j) we have $\varphi^{\bar{\epsilon}}\big(\gamma_{i,j}\big) = 0$.
3. Let $(j(1), j(2)) \in \{1, \dots, k\}$ and $i \in \mathbb{N}_*$. We compute and identify the covariance thanks to Lemma (3.2.2) :

$$\begin{aligned}\varphi^{\bar{\epsilon}}\left(\gamma_{i,j(1)}^{k(1)}\gamma_{i,j(2)}^{k(2)}\right) &= \left\langle \gamma_{i,j(1)}^{-k(1)}1, \gamma_{i,j(2)}^{k(2)}1\right\rangle = \left\langle \mu_{j(1)}^{k(1)}x_{-k(1)i,-k(1)j(1)}, \mu_{j(2)}^{-k(2)}x_{k(2)i,k(2)j(2)}\right\rangle \\ &= \mu_{j(1)}^{2k(1)}\delta_{k(2),-k(1)}\delta_{j(1),j(2)} = \varphi\left(c_{j(1)}^{k(1)}c_{j(2)}^{k(2)}\right)\end{aligned}$$

4. It is easily seen that $\varphi^{\bar{\epsilon}}\left(\gamma_{i,j}^{k(1)} \dots \gamma_{i,j}^{k(w)}\right)$ is independent of $i \in \mathbb{N}_*$.
5. This is a consequence of Lemma (3.2.18).
6. This follows from Lemma (3.2.6) almost surely.

Thus, by Theorem (3.2.5), we have, almost surely, for all $p \in \mathbb{N}_*, (k(1), \dots, k(p)) \in \{-1,1\}^p$ and all $(j(1), \dots, j(p)) \in \{1, \dots, k\}^p$:

$$\lim_{n\to+\infty} \varphi^{\bar{\epsilon}}\left(s_{n,j(1)}^{k(1)} \dots s_{n,j(p)}^{k(p)}\right) = \begin{cases} 0 & \text{if } p \text{ is odd} \\ \sum_{\substack{\mathcal{V}\in\mathcal{P}_2(1,\dots,2r) \\ \mathcal{V}=\{(s_l,t_l)\}_{l=1}^{l=r}}} q^{i(\mathcal{V})} \prod_{l=1}^{r} \varphi\left(c_{j(s_l)}^{k(s_l)} c_{j(t_l)}^{k(t_l)}\right) & \text{if } p = 2r \end{cases}$$

By Lemma (3.2.2) we see that all *-moments of the family $\left(s_{n,j}\right)_{j\in\{1,\dots,k\}}$ converge when n goes to infinity to the corresponding *-moments of the family $\left(c_j\right)_{j\in\{1,\dots,k\}}$:

Proposition (3.2.22)[130]: For all $p \in \mathbb{N}_*, (j(1), \dots, j(p)) \in \{1, \dots, k\}^p$ and for all $(k(1), \dots, k(p)) \in \{-1,1\}^p$ we have:

$$\lim_{n\to+\infty} \varphi^{\bar{\epsilon}}\left(s_{n,j(1)}^{k(1)} \dots s_{n,j(p)}^{k(p)}\right) = \varphi\left(c_{j(1)}^{k(1)} \dots c_{j(p)}^{k(p)}\right) \text{ almost surely} \quad (18)$$

For all $j \in \{1, \dots, k\}$ let us denote by $g_{n,j} = \operatorname{Re}\left(s_{n,j}\right)$ and $g_{n,-j} = \operatorname{Im}\left(s_{n,j}\right)$. By (18) we have that for all monomials P in $2k$ noncommuting variables:

$$\lim_{n\to+\infty} \varphi^{\bar{\epsilon}}\left(P\left(g_{n,-k}, \dots, g_{n,k}\right)\right) = \varphi\left(P\left(G(f_{-k}), \dots, G(f_k)\right)\right) \text{ almost surely} \quad (19)$$

Since the set of all non-commutative monomials is countable, we can find a choice of signs ϵ such that (19) is true for all P. In the sequel we fix such an ϵ and forget about the dependance on ϵ

Lemma (3.2.23)[130]: For all polynomials P in $2k$ noncommuting variables we have:

$$\lim_{n\to+\infty} \varphi\left(P\left(g_{n,-k}, \dots, g_{n,k}\right)\right) = \varphi\left(P\left(G(f_{-k}), \dots, G(f_k)\right)\right) \quad (20)$$

We are now ready to construct an embedding of $\Gamma_q(H_{\mathbb{R}}, U_t)$ into an ultraproduct of the finite dimensional von Neumann algebras Γ_n. To do so we need to have a uniform bound on the operators $g_{n,j}$. Let $C > 0$ such that for all $j \in I, \|G(f_j)\| < C$, as in the tracial case, we replace the $g_{n,j}$ by the their truncations $\tilde{g}_{n,j} = \chi_{]-C,C[}\left(g_{n,j}\right)g_{n,j}$. The following is the analogue of Lemma (3.2.7):

Lemma (3.2.24)[130]: For all polynomials P in $2k$ noncommuting variables we have:

$$\lim_{n\to+\infty} \varphi\left(P\left(\tilde{g}_{n,-k}, \dots, \tilde{g}_{n,k}\right)\right) = \varphi\left(P\left(G(f_{-k}), \dots, G(f_k)\right)\right) \quad (21)$$

Proof. It suffices to show that for all $(j(1), \dots, j(p)) \in I^p$ we have

$$\lim_{n\to+\infty} \varphi\left(\tilde{g}_{n,j(1)} \cdots \tilde{g}_{n,j(p)}\right) = \varphi\left(G\left(f_{j(1)}\right) \dots G\left(f_{j(p)}\right)\right)$$

By (20) it is sufficient to prove that

$$\lim_{n\to+\infty} \left|\varphi\left(g_{n,j(1)} \cdots g_{n,j(p)}\right) - \varphi\left(\tilde{g}_{n,j(1)} \cdots \tilde{g}_{n,j(p)}\right)\right| = 0$$

Using multi-linearity we can write

$$\begin{aligned} &\varphi\left(g_{n,j(1)} \cdots g_{n,j(p)}\right) - \varphi\left(\tilde{g}_{n,j(1)} \cdots \tilde{g}_{n,j(p)}\right) \\ &= \left|\sum_{l=1}^{p} \varphi\left[\tilde{g}_{n,j(1)} \cdots \tilde{g}_{n,j(l-1)}\left(g_{n,j(l)} - \tilde{g}_{n,j(l)}\right)g_{n,j(l+1)} \cdots g_{n,j(p)}\right]\right| \\ &\leqslant \sum_{l=1}^{p} \left|\varphi\left[\tilde{g}_{n,j(1)} \cdots \tilde{g}_{n,j(l-1)}\left(g_{n,j(l)} - \tilde{g}_{n,j(l)}\right)g_{n,j(l+1)} \cdots g_{n,j(p)}\right]\right| \end{aligned}$$

Fix $l \in \{1, \dots, p\}$, using the modular group we have:

$$\left|\varphi\left[\tilde{g}_{n,j(1)} \cdots \tilde{g}_{n,j(l-1)}\left(g_{n,j(l)} - \tilde{g}_{n,j(l)}\right)g_{n,j(l+1)} \cdots g_{n,j(p)}\right]\right|$$
$$= \left|\varphi\left[\sigma_i\left(g_{n,j(l+1)} \cdots g_{n,j(p)}\right)\tilde{g}_{n,j(1)} \cdots \tilde{g}_{n,j(l-1)}\left(g_{n,j(l)} - \tilde{g}_{n,j(l)}\right)\right]\right|$$

Estimating by Cauchy-Schwarz's inequality we obtain:

$$\left|\varphi\left[\sigma_i\left(g_{n,j(l+1)} \cdots g_{n,j(p)}\right)\tilde{g}_{n,j(1)} \cdots \tilde{g}_{n,j(l-1)}\left(g_{n,j(l)} - \tilde{g}_{n,j(l)}\right)\right]\right|$$
$$\leqslant \varphi\left[\sigma_i\left(g_{n,j(l+1)} \cdots g_{n,j(p)}\right)\tilde{g}_{n,j(1)} \cdots \tilde{g}^2_{n,j(l-1)} \cdots \tilde{g}_{n,j(1)}\sigma_{-i}\left(g_{n,j(p)} \cdots g_{n,j(l+1)}\right)\right]^{\frac{1}{2}}$$
$$\times \varphi\left[\left(g_{n,j(l)} - \tilde{g}_{n,j(l)}\right)^2\right]^{\frac{1}{2}}$$
$$\leqslant C^{l-1}\varphi\left[\sigma_i\left(g_{n,j(l+1)} \cdots g_{n,j(p)}\right)\sigma_{-i}\left(g_{n,j(p)} \cdots g_{n,j(l+1)}\right)\right]^{\frac{1}{2}}\varphi\left[\left(g_{n,j(l)} - \tilde{g}_{n,j(l)}\right)^2\right]^{\frac{1}{2}}$$

The conclusion follows from the convergence of this last term to 0. Indeed, by (22) there exists a polynomial in $2k$ non-commutative variables Q, independent on n, such that $Q\left(g_{n,-k} \cdots g_{n,k}\right) = \sigma_i\left(g_{n,j(l+1)} \cdots g_{n,j(p)}\right)\sigma_{-i}\left(g_{n,j(p)} \cdots g_{n,j(l+1)}\right)$. It follows by (20) that

$$\lim_{n\to+\infty} \varphi\left[\sigma_i\left(g_{n,j(l+1)} \cdots g_{n,j(p)}\right)\sigma_{-i}\left(g_{n,j(p)} \cdots g_{n,j(l+1)}\right)\right]$$
$$= \varphi\left(Q\left(G(f_{-k}) \dots G(f_k)\right)\right).$$

and by Lemma (3.2.8), $\varphi\left[\left(g_{n,j(l)} - \tilde{g}_{n,j(l)}\right)^2\right]$ converges to 0 when n goes to infinity.

Remark (3.2.25)[130]: For all $n \in \mathbb{N}_*$ and all $j \in I$ the element $g_{n,j}$ is entire for the modular group (this is always the case in a finite dimensional framework). By (iii) of Proposition (3.2.20), we have for all $j \in \{1, \dots, k\}$

$$\sigma_z\left(s_{n,j}\right) = \lambda_j^{iz} s_{n,j} \text{ for all } z \in \mathbb{C}$$

Thus for all $z \in \mathbb{C}$,

$$\sigma_z\left(g_{n,j}\right)$$
$$= \begin{cases} \cos\left(z\ln\left(\lambda_j\right)\right)g_{n,j} - \sin\left(z\ln\left(\lambda_j\right)\right)g_{n,-j} & \text{for all } j \in \{1, \dots, k\} \\ \sin\left(z\ln\left(\lambda_{-j}\right)\right)g_{n,-j} + \cos\left(z\ln\left(\lambda_{-j}\right)\right)g_{n,j} & \text{for all } j \in \{-1, \dots, -k\} \end{cases} \quad (22)$$

Let us denote by $\mathcal{P}$ the w^*-dense $*$-subalgebra of $\Gamma_q(H_{\mathbb{R}}, U_t)$ generated by the set $\{G(f_j), j \in I\}$. We know that $\mathcal{P}$ is isomorphic to the algebra of non-commutative polynomials in $2k$ variables (see the remark after Lemma (3.2.8)). Given $\mathcal{U}$ a non trivial ultrafilter on $\mathbb{N}$, it is thus possible to define the following $*$-homomorphism Φ from $\mathcal{P}$ into the von Neumann ultraproduct $\mathcal{A} = \prod_{n,\mathcal{U}} \Gamma_n$ by:

$$\Phi\left(P\left(G(f_{-k}), \dots, G(f_k)\right)\right) = \left(P\left(\tilde{g}_{n,-k}, \dots, \tilde{g}_{n,k}\right)\right)^{\cdot}_{n\in\mathbb{N}}$$

Indeed the right term is well defined since it is uniformly bounded in norm. Let us check the hypothesis of Theorem (3.2.15).

1. By Lemma (3.2.24),Φ is state preserving.
2. It is sufficient to check that condition (b) of Lemma (3.2.13) is satisfied for every generator $\Phi\left(G(f_j)\right), j \in I$. Let us fix $j \in I$ and recall that by (22) there are complex numbers ν_j and ω_j (independent of n) such that $\sigma^n_{-i}\left(g_{n,j}\right) = \nu_j g_{n,j} + \omega_j g_{n,-j}$. We show that condition of Lemma (3.2.13) is satisfied for $x = \Phi\left(G(f_j)\right)$ and $z = \nu_j\Phi\left(G(f_j)\right) + \omega_j\Phi\left(G(f_{-j})\right)$. By w^*-density it is sufficient to consider $y = (y_n)_{n\in\mathbb{N}} \in \tilde{\mathcal{A}}$. Using Lemma (3.2.24) we have:

$$\varphi\left(\Phi\left(G(f_j)\right)y\right) = \lim_{n,\mathcal{U}}\varphi_n(\tilde{g}_{n,j}y_n) = \lim_{n,\mathcal{U}}\varphi_n(g_{n,j}y_n) = \lim_{n,\mathcal{U}}\varphi_n\left(y_n\sigma_{-i}^n(g_{n,j})\right)$$
$$= \lim_{n,\mathcal{U}}\varphi_n\left(y_n(\nu_j g_{n,j} + \omega_j g_{n,-j})\right) = \lim_{n,\mathcal{U}}\varphi_n\left(y_n(\nu_j \tilde{g}_{n,j} + \omega_j \tilde{g}_{n,-j})\right)$$
$$= \varphi\left(y\left(\nu_j\Phi\left(G(f_j)\right) + \omega_j\Phi\left(G(f_{-j})\right)\right)\right)$$

3. It suffices to check that the intertwining condition given in the remark of Theorem (3.2.15) is satisfied for the generators $\Phi\left(G(f_j)\right) = (\tilde{g}_{n,j})^{\cdot}_{n\in N}$:

$$\text{for all } j \in I,\ \sigma_t\left(p\Phi\left(G(f_j)\right)p\right) = p\Phi\left(\sigma_t\left(G(f_j)\right)\right)p$$

To fix ideas we will suppose that $j \geqslant 0$. Recall that in this case for all $t \in \mathbb{R}$ and for all $n \in \mathbb{N}$, we have

$$\sigma_t^n(g_{n,j}) = \cos\left(t\ln(\lambda_j)\right)g_{n,j} - \sin\left(t\ln(\lambda_j)\right)g_{n,-j}.$$

Since the functional calculus commutes with automorphisms, for all $t \in \mathbb{R}$ and for all $n \in \mathbb{N}$, we have:

$$\sigma_t^n(\tilde{g}_{n,j}) = h\left(\sigma_t^n(g_{n,j})\right),$$

where $h(\lambda) = \chi_{]-C,C[}(\lambda)\lambda$, for all $\lambda \in \mathbb{R}$. But by Lemma 5.6,

$$\sigma_t^n(g_{n,j}) = \cos\left(t\ln(\lambda_j)\right)g_{n,j} - \sin\left(t\ln(\lambda_j)\right)g_{n,-j}$$

converges in distribution to

$$\cos\left(t\ln(\lambda_j)\right)G(f_j) - \sin\left(t\ln(\lambda_j)\right)G(f_{-j}) = \sigma_t\left(G(f_j)\right)$$

and $\left\|\sigma_t\left(G(f_j)\right)\right\| = \|G(f_j)\| < C$. Thus, by Lemma (3.2.8), we deduce that $\sigma_t^n(\tilde{g}_{n,j})$ converges in distribution to $\sigma_t\left(G(f_j)\right)$. On the other hand, by Lemma (3.2.24),

$$\cos\left(t\ln(\lambda_j)\right)\tilde{g}_{n,j} - \sin\left(t\ln(\lambda_j)\right)\tilde{g}_{n,-j}$$

also converges in distribution to

$$\cos\left(t\ln(\lambda_j)\right)G(f_j) - \sin\left(t\ln(\lambda_j)\right)G(f_{-j}) = \sigma_t\left(G(f_j)\right).$$

Let $y \in \mathcal{A}$, using Raynaud's results we compute:

$$\begin{aligned}\varphi\left(\sigma_t\left(p\Phi\left(G(f_j)\right)p\right)pyp\right) &= \varphi\left((\Delta_n^{it})^{\cdot}p\Phi\left(G(f_j)\right)p(\Delta_n^{-it})^{\cdot}pyp\right)\\ &= \varphi\left(p(\Delta_n^{it})^{\cdot}\Phi\left(G(f_j)\right)(\Delta_n^{-it})^{\cdot}pyp\right)\\ &= \varphi\left((\Delta_n^{it})^{\cdot}\Phi\left(G(f_j)\right)(\Delta_n^{-it})^{\cdot}py\right)\end{aligned}$$

Let $z = (z_n)_{n\in\mathbb{N}} \in \tilde{\mathcal{A}}$. By our previous observations, we have:

$$\begin{aligned}&\varphi\left((\Delta_n^{it})^{\cdot}\Phi\left(G(f_j)\right)(\Delta_n^{-it})^{\cdot}z\right)\\ &= \lim_{n,\mathcal{U}}\varphi_n(\Delta_n^{it}\tilde{g}_{n,j}\Delta_n^{-it}z_n)\\ &= \lim_{n,\mathcal{U}}\varphi_n(\sigma_t^n(\tilde{g}_{n,j})z_n)\\ &= \varphi\left(\sigma_t\left(G(f_j)\right)z\right)\\ &= \lim_{n,\mathcal{U}}\varphi_n\left(\left(\cos\left(t\ln(\lambda_j)\right)\tilde{g}_{n,j} - \sin\left(t\ln(\lambda_j)\right)\tilde{g}_{n,-j}\right)z_n\right)\\ &= \varphi\left(\left(\cos\left(t\ln(\lambda_j)\right)\Phi\left(G(f_j)\right) - \sin\left(t\ln(\lambda_j)\right)\Phi\left(G(f_{-j})\right)\right)z\right)\end{aligned}$$

$$= \varphi\left(\left(p\Phi\left(\sigma_t\left(G(f_j)\right)\right)p\right)zp\right)$$

By w^*-density and continuity, we can replace z by py in the previous equality, which gives:

$$\varphi\left(\sigma_t\left(p\Phi\left(G(f_j)\right)p\right)pyp\right) = \varphi\left(\left(p\Phi\left(\sigma_t\left(G(f_j)\right)\right)p\right)pyp\right).$$

Thus, taking $y = \sigma_t\left(p\Phi\left(G(f_j)\right)p\right) - p\Phi\left(\sigma_t\left(G(f_j)\right)\right)p$, and by the faithfulness of $\varphi(p \cdot p)$ we deduce that

$$\sigma_t\left(p\Phi\left(G(f_j)\right)p\right) = p\Phi\left(\sigma_t\left(G(f_j)\right)\right)p \in p\mathrm{Im}(\Phi)p$$

By Theorem (3.2.15), $\Theta = p\Phi p$ can be extended into a (necessarily injective because state preserving) w^*-continuous *-homomorphism from $\Gamma_q(H_{\mathbb{R}}, U_t)$ into $p\mathcal{A}p$ with a completely complemented image. By its Corollary (3.2.16), since the algebras Γ_n are finite dimensional and a fortiori are QWEP, it follows that $\Gamma_q(H_{\mathbb{R}}, U_t)$ is QWEP.

Theorem (3.2.26)[130]: If $H_{\mathbb{R}}$ is a finite dimensional real Hilbert space equipped with a group of orthogonal transformations $(U_t)_{t\in\mathbb{R}}$, then the von Neumann algebra $\Gamma_q(H_{\mathbb{R}}, U_t)$ is $QWEP$.

Corollary (3.2.27)[130]: If $(U_t)_{t\in\mathbb{R}}$ is almost periodic on $H_{\mathbb{R}}$, then $\Gamma_q(H_{\mathbb{R}}, U_t)$ is QWEP.

Proof. There exist an invariant real Hilbert space H_1, an orthogonal family of invariant 2 dimensional real Hilbert spaces $(H_\alpha)_{\alpha\in A}$ and real eigenvalues $(\lambda_\alpha)_{\alpha\in A}$ greater than 1 such that

$$H_{\mathbb{R}} = H_1 \bigoplus_{\alpha\in A} H_\alpha \text{ and } U_{t|H_1} = \mathrm{Id}_{H_1},\ U_{t|H_\alpha}$$
$$= \begin{pmatrix} \cos(t\ln(\lambda_\alpha)) & -\sin(t\ln(\lambda_\alpha)) \\ \sin(t\ln(\lambda_\alpha)) & \cos(t\ln(\lambda_\alpha)) \end{pmatrix}$$

In particular it is possible to find a net $\left(I_\beta\right)_{\beta\in B}$ of isometries from finite dimensional subspaces $H_\beta \subset H_{\mathbb{R}}$ into $H_{\mathbb{R}}$, such that for all $\beta \in B, H_\beta$ is stable by $(U_t)_{t\in\mathbb{R}}$ and $\bigcup_{B\in B} H_\beta$ is dense in $H_{\mathbb{R}}$. By second quantization, for all $\beta \in B$, there exists an isometric *-homomorphism $\Gamma_q(I_\beta)$ from $\Gamma_q\left(H_\beta, U_{t|H_\beta}\right)$ into $\Gamma_q(H_{\mathbb{R}}, U_t)$, and $\Gamma_q(H_{\mathbb{R}}, U_t)$ is the inductive limit (in the von Neumann algebra's sense) of the algebras $\Gamma_q\left(H_\beta, U_{t|H_\beta}\right)$. By the previous Theorem, for all $\beta \in B, \Gamma_q\left(H_\beta, U_{t|H_\beta}\right)$ is QWEP, thus $\Gamma_q(H_{\mathrm{R}}, U_t)$ is QWEP, as an inductive limit of QWEP von Neumann algebras.

We will derive the general case by discretization and an ultraproduct argument similar.

Let H_{R} be a real Hilbert space and $(U_t)_{t\in\mathbb{R}}$ a strongly continuous group of orthogonal transformations on $H_{\mathbb{R}}$. We denote by $H_{\mathbb{C}}$ the complexification of $H_{\mathbb{R}}$ and by $(U_t)_{t\in\mathbb{R}}$ its extension to a group of unitaries on $H_{\mathbb{C}}$. Let A be the (unbounded) non degenerate positive infinitesimal generator of $(U_t)_{t\in\mathbb{R}}$. For every $n \in \mathbb{N}_*$ let g_n be the bounded Borelian function defined by:

$$g_n = \chi_{]1,1+\frac{1}{2^n}[} + \left(\sum_{k=2^n+1}^{n2^n-1} \frac{k}{2^n}\chi_{[\frac{k}{2^n},\frac{k+1}{2^n}[}\right) + n\chi_{[n,+\infty[}$$

and

$$f_n(t) = g_n(t)\chi_{\{t>1\}}(t) + \frac{1}{g_n(1/t)}\chi_{\{t<1\}}(t) + \chi_{\{1\}}(t) \text{ for all } t \in \mathbb{R}_+$$

It is clear that

$$f_n(t) \nearrow t \text{ for all } t \geqslant 1 \text{ and } f_n(t) = \frac{1}{f_n(1/t)} \text{ for all } t \in \mathbb{R}_+^*. \qquad (23)$$

For all $n \in \mathbb{N}_*$, let A_n be the invertible positive and bounded operator on $H_\mathbb{C}$ defined by $A_n = f_n(A)$. Denoting by $\mathcal{J}$ the conjugation on $H_\mathbb{C}$, we know, by [15], that $\mathcal{J}A = A^{-1}\mathcal{J}$. By the second part of (23), it follows that for all $n \in \mathbb{N}_*$,

$$\mathcal{J}A_n = \mathcal{J}f_n(A) = f_n(A^{-1})\mathcal{J} = f_n(A)^{-1}\mathcal{J} = A_n^{-1}\mathcal{J} \qquad (24)$$

Consider the strongly continuous unitary group $(U_t^n)_{t\in\mathbb{R}}$ on $H_\mathbb{C}$ with positive non degenerate and bounded infinitesimal generator given by A_n. By definition, we have $U_t^n = A_n^{it}$. By (24), and since $\mathcal{J}$ is anti-linear, we have for all $n \in \mathbb{N}_*$ and all $\in \mathbb{R}$:

$$\mathcal{J}U_t^n = \mathcal{J}A_n^{it} = A_n^{it}\mathcal{J} = U_t^n\mathcal{J}$$

It follows that for all $n \in \mathbb{N}_*$ and for all $t \in \mathbb{R}$, $H_\mathbb{R}$ is globally invariant by U_t^n, thus we have

$$U_t^n(H_\mathbb{R}) = H_\mathbb{R}$$

Hence, $(U_t^n)_{t\in\mathbb{R}}$ induces a group of orthogonal transformations on $H_\mathbb{R}$ such that its extension on $H_\mathbb{C}$ has infinitesimal generator given by the discretized operator A_n. In the following we will index by $n \in \mathbb{N}_*$ the objects relative to the discretized von Neumann algebra $\Gamma_n = \Gamma_q(H_\mathbb{R}, (U_t^n)_{t\in\mathbb{R}})$. We simply set $\Gamma = \Gamma_q(H_\mathbb{R}, (U_t)_{t\in\mathbb{R}})$.

Moreover for all $n \in \mathbb{N}_*$ the scalar products $\langle .,. \rangle_{U^n}$ and $\langle .,, \rangle_{H_C}$ are equivalent on $H_\mathbb{C}$ since A_n is bounded.

Scholie (3.2.28)[130]: For all ξ and η in $H_\mathbb{C}$ we have:

$$\lim_{n\to+\infty} \langle \xi, \eta \rangle_{H_n} = \langle \xi, \eta \rangle_H$$

Proof. Let E_A be the spectral resolution of A. Take $\xi \in H_\mathbb{C}$ and denote by μ_ξ the finite positive measure on $\mathbb{R}_+$given by $\mu_\xi = \langle E_A(\cdot)\xi, \xi \rangle_{H_\mathbb{C}}$. Since for all $\lambda \in \mathbb{R}_+, \lim_{n\to+\infty} g \circ f_n(\lambda) = g(\lambda)$, and $g(\lambda) = 2\lambda/(1+\lambda)$ is bounded on $\mathbb{R}_+$, we have by the Lebesgue dominated convergence Theorem:

$$\| \xi \|_H^2 = \left\langle \frac{2A}{1+A}\xi, \xi \right\rangle_{H_\mathbb{C}} = \int_{\mathbb{R}_+} g(\lambda) \mathrm{d}\mu_\xi(\lambda)$$

$$= \lim_{n\to+\infty} \int_{\mathbb{R}_+} g \circ f_n(\lambda) \mathrm{d}\mu_\xi(\lambda) = \lim_{n\to+\infty} \left\langle \frac{2A_n}{1+A_n}\xi, \xi \right\rangle_{H_\mathbb{C}} = \lim_{n\to+\infty} \| \xi \|_{H_n}^2$$

and we finish the proof by polarization.

Let E be the vector space given by

$$E = \cup_{k\in\mathbb{N}_*} \chi_{\left[\frac{1}{k},k\right]}(A)(H_\mathbb{R})$$

We have

$$\mathcal{J}\chi_{\left[\frac{1}{k},k\right]}(A) = \chi_{\left[\frac{1}{k},k\right]}(A^{-1})\mathcal{J} = \chi_{\left[\frac{1}{k},k\right]}(A)\mathcal{J}$$

thus $E \subset H_\mathbb{R}$. Since A is non degenerate,

$$\overline{\cup_{k\in\mathbb{N}_*} \chi_{\left[\frac{1}{k},k\right]}(A)(H_\mathbb{C})} = \chi_{]0,+\infty[}(A)(H_\mathbb{C}) = H_\mathbb{C}$$

It follows that E is dense in $H_\mathbb{R}$. Let $(e_i)_{i\in I}$ be an algebraic basis of unit vectors of E and denote by $\mathcal{E}$ the algebra generated by the Gaussians $G(e_i)$ for $i \in I. \mathcal{E}$ is w^* dense in Γ and every element in $\mathcal{E}$ is entire for $(\sigma_t)_{t\in\mathbb{R}}$ (because for all $k \in \mathbb{N}_*, A$ is bounded and has a bounded inverse on $\chi_{\left[\frac{1}{k},k\right]}(A)(H_{\mathbb{C}})\Big)$. Denoting by W the Wick product in Γ, we have for all $i \in I$ and all $z \in \mathbb{C}$:

$$\sigma_z\big(G(e_i)\big) = W(U_{-z}e_i) = W\big(A^{-iz}e_i\big) \tag{25}$$

Since $H_{\mathbb{R}} \subset H$ and for all $n \in \mathbb{N}_*, H_{\mathbb{R}} \subset H_n$(isometrically), we have by (1)

$$\text{For all } (i,n) \in I \times \mathbb{N}_*, \quad \|G_n(e_i)\| = \frac{2}{\sqrt{1-q}} \tag{26}$$

Scholie (3.2.29)[130]: For all $r \in \mathbb{R}$ and for all $i \in I$ we have

$$\sup_{n\in\mathbb{N}_*} \|\sigma_{ir}^n\big(G_n(e_i)\big)\| < +\infty$$

Proof. Fix $i \in I$. By (25):

$$\begin{aligned}\|\sigma_{ir}^n\big(G_n(e_i)\big)\| &= \|W(A_n^r e_i)\| = \|a_n^*(A_n^r e_i) + a_n(\mathcal{J}A_n^r e_i)\| \\ &\leqslant C_{|q|}^{\frac{1}{2}}\left(\|A_n^r e_i\|_{H_n} + \|\mathcal{J}A_n^r e_i\|_{H_n}\right) \\ &\leqslant C_{|q|}^{\frac{1}{2}}\left(\|A_n^r e_i\|_{H_n} + \left\|\Delta_n^{\frac{1}{2}}A_n^r e_i\right\|_{H_n}\right) \\ &\leqslant C_{|q|}^{\frac{1}{2}}\left(\|A_n^r e_i\|_{H_n} + \left\|A_n^{r-\frac{1}{2}} e_i\right\|_{H_n}\right)\end{aligned}$$

Thus it suffices to prove that for all $r \in \mathbb{R}$ we have

$$\sup_{n\in\mathbb{N}_*} \|A_n^r e_i\|_{H_n} < +\infty$$

Let us denote by $\mu_i = \langle E_A(.)e_i, e_i\rangle_{H_C}$ and by $g_r(\lambda) = 2\lambda^{2r+1}/(1+\lambda)$. There exists $k \in \mathbb{N}_*$ such that $e_i \in \chi_{[1/k,k]}(A)(H_{\mathrm{R}})$, thus we have:

$$\|A_n^r e_i\|_{H_n}^2 = \langle g_r \circ f_n(A)e_i, e_i\rangle_{H_{\mathrm{C}}} = \int_{[1/k,k]} g_r \circ f_n(\lambda)\mathrm{d}\mu_i(\lambda)$$

It is easily seen that $(g_r \circ f_n)_{n\in\mathbb{N}_*}$ converges uniformly to g_r on $[1/k, k]$. The result follows by:

$$\begin{aligned}\lim_{n\to+\infty} \|A_n^r e_i\|_{H_n}^2 &= \lim_{n\to+\infty} \int_{[1/k,k]} g_r \circ f_n(\lambda)\mathrm{d}\mu_i(\lambda) \\ &= \int_{\left[\frac{1}{k},k\right]} g_r(\lambda)\mathrm{d}\mu_i(\lambda) = \|A^r e_i\|_H^2.\end{aligned}$$

Recall that $\mathcal{E}$ is isomorphic to the complex free $*$-algebra with $|I|$ generators. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}_*$, by (26) we can define a $*$-homomorphism Φ from $\mathcal{E}$ into the von Neumann algebra ultraproduct over $\mathcal{U}$ of the algebras Γ_n by:

$$\begin{aligned}\Phi: \mathcal{E} &\longrightarrow \mathcal{A} = \prod_{n,\mathcal{U}} \Gamma_n \\ G(e_i) &\longmapsto \big(G_n(e_i)\big)_{n\in\mathbb{N}_*}\end{aligned}.$$

We will now check the hypothesis of Theorem (3.2.15).

1. We first check that Φ is state preserving. It suffices to verify it for a product of an even number of Gaussians. Take $(i_1, \ldots, i_{2k}) \in I^{2k}$, we have by Scholie (3.2.28):

$$\varphi\left(G(e_{i_1})\dots G(e_{i_{2k}})\right) = \sum_{\substack{\mathcal{V}\in\mathcal{P}_2(1,\dots,k)\\ \mathcal{V}=\left((s(l),t(l))\right)_{l=1}^{l=k}}} q^{i(\mathcal{V})} \prod_{l=1}^{l=k} \left\langle e_{i_{s(l)}}, e_{i_{t(l)}}\right\rangle_H$$

$$= \lim_{n\to+\infty} \sum_{\substack{\mathcal{V}\in\mathcal{P}_2(1,\dots,k)\\ \mathcal{V}=\left((s(l),t(l))\right)_{l=1}^{l=k}}} q^{i(\mathcal{V})} \prod_{l=1}^{l=k} \left\langle e_{i_{s(l)}}, e_{i_{t(l)}}\right\rangle_{H_n}$$

$$= \lim_{n\to+\infty} \varphi_n\left(G_n(e_{i_1})\dots G_n(e_{i_{2k}})\right)$$

This implies, in particular that Φ is state preserving.

2. Condition (a) of Lemma (3.2.13) is satisfied by Scholie (3.2.29).
3. It suffices to check that for all $i \in I$ and all $t \in \mathbb{R}$, $\left(\sigma_t^n\left(G_n(e_i)\right)\right)^{\cdot}_{n\in\mathbb{N}_0} \in \overline{\operatorname{Im}\Phi}^{w^*}$. Fix $i \in I$ and $t \in \mathbb{R}$. For all $n \in \mathbb{N}_*$ we have

$$\|A_n^{-it}e_i - A^{-it}e_i\|_{H_\mathbb{R}}^2 = \int_{\mathbb{R}_+} \left|f_n^{-it}(\lambda) - \lambda^{-it}\right|^2 \mathrm{d}\mu_i(\lambda)$$

By the Lebesgue dominated convergence Theorem, it follows that

$$\lim_{n\to+\infty} \|A_n^{-it}e_i - A^{-it}e_i\|_{H_\mathbb{R}} = 0.$$

By (26) we deduce that

$$\lim_{n\to+\infty} \|G_n\left(A_n^{-it}e_i\right) - G_n\left(A^{-it}e_i\right)\| = 0$$

Thus we have

$$\left(\sigma_t^n\left(G_n(e_i)\right)\right)^{\cdot}_{n\in\mathbb{N}_*} = \left(G_n\left(A_n^{-it}e_i\right)\right)^{\cdot}_{n\in\mathbb{N}_*} = \left(G_n\left(A^{-it}e_i\right)\right)^{\cdot}_{n\in\mathbb{N}_*} \in \overline{\operatorname{Im}\Phi}^{\|\cdot\|} \subset \overline{\operatorname{Im}\Phi}^{w^*}.$$

By Theorem (3.2.15), we deduce our main Theorem:

Theorem (3.2.30)[130]: Let $H_\mathbb{R}$ be a real Hilbert space given with a group of orthogonal transformations $(U_t)_{t\in\mathbb{R}}$. Then for all $q \in (-1,1)$ the q-Araki- Woods algebra $\Gamma_q(H_\mathrm{R}, (U_t)_{t\in\mathbb{R}})$ is QWEP.

Chapter 4
Most One Cartan Subalgebra and Structural Results with Approximation Properties

We show that if a free ergodic measure-preserving action of a free group $F_r, 2 \leq r \leq 1$, on a probability space (X, μ) is profinite then the group measure space factor $L^\infty(X) \rtimes F_r$ has unique Cartan subalgebra, up to unitary conjugacy. As an application, we construct an example of a non-amenable solid II_1 factor N with full fundamental group, i.e. $F(N) = R_+^*$, which is not isomorphic to any interpolated free group factor $L(F_t)$, for $1 < t \leq +\infty$. We finally deduce that the type III_1 factors constructed by Connes in the '70s can never be isomorphic to any free Araki-Woods factor, which answers a question of Shlyakhtenko and Vaes.

Section (4.1): On a Class of II_1 Factors

A celebrated theorem of Connes ([28]) shows that all amenable II_1 factors are isomorphic to the approximately finite-dimensional (AFD) II_1 factor R of Murray and von Neumann ([161]). In particular, all II_1 group factors $L(\Gamma)$ associated with ICC (infinite conjugacy class) amenable groups Γ, and all group measure space II_1 factors $L^\infty(X) \rtimes \Gamma$ arising from free ergodic measure-preserving (m.p.) actions of countable amenable groups Γ on a probability space $\Gamma \curvearrowright X$, are isomorphic to R. Moreover, by [29], any decomposition of R as a group measure space algebra is unique, i.e. if $R = L^\infty(X_i) \rtimes \Gamma_i$, for some free ergodic measure-preserving actions $\Gamma_i \curvearrowright X_i, i = 1,2$, then there exists an automorphism of R taking $L^\infty(X_1)$ onto $L^\infty(X_2)$. In fact, any two Cartan subalgebras of R are conjugate by an automorphism of R.

Recall in this respect that a Cartan subalgebra A in a II_1 factor M is a maximal abelian $*$-subalgebra $A \subset M$ with normalizer $\mathcal{N}_M(A) = \{u \in (\mathcal{A}) \mid uAu^* = A\}$ generating M([35],[36],[36]). Its presence amounts to realizing M as a generalized, twisted-version of the group measure space construction, corresponding to the equivalence relation induced by the orbits of some ergodic m.p. action of a countable group, $\Gamma \curvearrowright X$, and a 2 -cocycle, with $A = L^\infty(X)$. Decomposing factors this way is important, especially if one can show uniqueness of their Cartan subalgebras, because then the classification of the factors reduces to the classification of the corresponding actions $\Gamma \curvearrowright X$ up to orbit equivalence ([36], [36]). But beyond the amenable case, very little is known about uniqueness, or possible nonexistence, of Cartan subalgebras in group factors, or other factors that are a priori constructed in different ways than as group measure space algebras.

We investigate Cartan decomposition properties for a class of nonamenable I_1 factors that are in some sense "closest to being amenable". Thus, we consider factors M which satisfy the complete metric approximation property (c.m.a.p.) of Haagerup ([153]), which requires existence of normal, finite rank, completely bounded (cb) maps $\phi_n: M \to M$, such that $\|\phi_n\|_{\mathrm{cb}} \leq 1$ and $\lim\|\phi_n(x) - x\|_2 = 0$, for all $x \in M$, where $\|\cdot\|_2$ denotes the Hilbert norm given by the trace of M (note that if ϕ_n could in addition be taken unital, M would follow amenable). This is the same as saying that the Cowling-Haagerup constant $\Lambda_{\mathrm{cb}}(M)$ equals 1 (see [147]). The prototype nonamenable $c.m.a.p.$ factors are the free group factors $L(\mathbb{F}_r), 2 \leq r \leq \infty$ ([153]). Like amenability, the c.m.a.p. passes to subfactors and is well-behaved to inductive limits and tensor products.

We in fact restrict our attention to c.m.a.p. factors of the form $M = Q \rtimes \mathbb{F}_r$ and to subfactors N of such M. The aim is to locate all (or prove possible absence of) diffuse AFD subalgebras $P \subset N$ whose normalizer $\mathcal{N}_N(P)$ generates N. Our general result along these lines shows:

Theorem (4.1.1)[141]: Let $\mathbb{F}_r \curvearrowright Q$ be an action of a free group on a finite von Neumann algebra. Assume $M = Q \rtimes \mathbb{F}_r$ has the complete metric approximation property. If $P \subset M$ is a diffuse amenable subalgebra and N denotes the von Neumann algebra generated by its normalizer $\mathcal{N}_M(P)$, then either N is amenable relative to Q inside M, or P can be embedded into Q inside M.

The amenability property of a von Neumann subalgebra $N \subset M$ relative to another von Neumann subalgebra $Q \subset M$ is rather self-explanatory: it requires existence of a norm-one projection from the basic construction algebra of the inclusion $Q \subset M$ onto N (see Definition (4.1.5)). The "embeddability of a subalgebra $P \subset M$ into another subalgebra $Q \subset M$ inside an ambient factor" is in the sense of [52], and roughly means that P can be conjugated into Q via a unitary element of M.

We mention three applications of the theorem, each corresponding to a particular choice of $\mathbb{F}_r \curvearrowright Q$ and solving well-known problems. Thus, taking $Q = \mathbb{C}$, we get:

Corollary (4.1.2)[141]: The normalizer of any diffuse amenable subalgebra P of a free group factor $L(\mathbb{F}_r)$ generates an amenable (thus AFD by [28]) von Neumann algebra.

If we take Q to be an arbitrary finite factor with $\Lambda_{\text{cb}}(Q) = 1$ and let $\mathbb{F}_r$ act trivially on it, then $M = Q \mathbin{\bar{\otimes}} L(\mathbb{F}_r), \Lambda_{\text{cb}}(M) = 1$ and the theorem implies:

Corollary (4.1.3)[141]: If Q is $a\text{II}_1$ factor with the complete metric approximation property then $Q \mathbin{\bar{\otimes}} L(\mathbb{F}_r)$ does not have Cartan subalgebras. Moreover, if $N \subset Q \mathbin{\bar{\otimes}} L(\mathbb{F}_r)$ is a subfactor of finite index [158], then N does not have Cartan subalgebras either.

This shows in particular that any factor of the form $(\mathbb{F}_r) \otimes R, L\left(\mathbb{F}_{r_1}\right) \mathbin{\bar{\otimes}} L\left(\mathbb{F}_{r_2}\right) \mathbin{\bar{\otimes}} \cdots$, and more generally any subfactor of finite index of such a factor, has no Cartan decomposition. Besides $Q = R, L(\mathbb{F}_r)$, other examples of factors with $\Lambda_{\text{cb}}(Q) = 1$ are the group factors $L(\Gamma)$ corresponding to ICC discrete subgroups Γ of $\text{SO}(1, n)$ and $\text{SU}(1, n)$ ([33], [32]), as well as any subfactor of a tensor product of such factors. None of the factors covered by Corollary (4.1.3) were known until now not to have Cartan decomposition.

Finally, if we take $\mathbb{F}_r \curvearrowright X$ to be a profinite $m.p.$ action on a probability measure space (X, μ), i.e. an action with the property that $L^\infty(X)$ is a limit of an increasing sequence of $\mathbb{F}_r$-invariant finite-dimensional subalgebras $Q_n \subset L^\infty(X)$,
then $M = L^\infty(X) \rtimes \mathbb{F}_r$ is an increasing limit of the algebras $Q_n \rtimes \mathbb{F}_r$, each one of which is an amplification of $L(\mathbb{F}_r)$. Since c.m.a.p. behaves well to amplifications and inductive limits, it follows that M has c.m.a.p., so by applying the theorem and (A.1 in [51]) we get:

Corollary (4.1.4)[141]: If $\mathbb{F}_r \curvearrowright X$ is a free ergodic measure-preserving profinite action, then $L^\infty(X)$ is the unique Cartan subalgebra of the $II_{\ 1}$-factor $L^\infty(X) \rtimes \mathbb{F}_r$, up to unitary conjugacy.

The above corollary produces the first examples of nonamenable II_1 factors with all Cartan subalgebras unitary conjugate. Indeed, the "unique Cartan decomposition" results in [51], [52], [156] only showed conjugacy of Cartan subalgebras satisfying certain properties. This was still enough for differentiating factors of the form $L^\infty(\mathbb{T}^2) \rtimes \mathbb{F}_r$ and calculating their fundamental group in [51], by using [150]. Similarly here, when combined with Gaboriau's results, Corollary (4.1.4) shows that any factor $L^\infty(X) \rtimes \mathbb{F}_r, 2 \leq r < \infty$, arising from a free ergodic profinite action $\mathbb{F}_r \curvearrowright X$, has trivial fundamental group. Also, if $\mathbb{F}_s \curvearrowright X$ is another such action, with $r < s \leq \infty$, then $L^\infty(X) \rtimes \mathbb{F}_r \not\simeq L^\infty(Y) \rtimes \mathbb{F}_s$. It can be shown that the factors considered in [51], [52], [156] cannot even be embedded into the factors arising from profinite actions of free groups. Note that the uniqueness of the Cartan subalgebras of the AFD factor R is up to conjugacy by automorphisms ([29]), but not up to unitary conjugacy, i.e. up to conjugacy by inner automorphisms. Indeed, by [36], [36] there exist uncountably many nonunitary conjugate Cartan subalgebras in R. Finally, note that Connes and Jones constructed examples of II_1 factors M with two Cartan subalgebras that are not conjugate by automorphisms of M ([30]).

Corollary (4.1.2) strengthens two well-known in-decomposability properties of free group factors: Voiculescu's result in [172], showing that $L(\mathbb{F}_r)$ has no Cartan subalgebras, which in fact exhibited the first examples of factors with no Cartan decomposition, and in [162], showing that the commutant in $L(\mathbb{F}_r)$ of any diffuse subalgebra must be amenable ($L(\mathbb{F}_r)$ are solid), which itself strengthened the in-decomposability of $L(\mathbb{F}_r)$ into tensor product of II_1 factors (primeness of free group factors) in [152].

One should point out that Connes already constructed in [146] a factor N that does not admit a "classic" group measure space decomposition $L^\infty(X) \rtimes \Gamma$.

His factor N is defined as the fixed point algebra of an appropriate finite group of automorphisms of $M = R \otimes L(\mathbb{F}_r)$. But it was left open whether N cannot be obtained as a generalized group measure space factor either, i.e. whether it does not have Cartan decomposition. Corollary (4.1.3) shows that indeed it does not.

The proof of the theorem follows a "deformation/rigidity" strategy, being inspired by arguments in [169] and [51]. A key role is played by a property of group actions $\Gamma \curvearrowright P$ called weak compactness, requiring $L^2(P)$ to be a limit of finite dimensional subspaces that are almost invariant to both the left multiplication by elements in P and to the Γ-action, in the Hilbert-Schmidt norm. In case $P = L^\infty(X)$, this property is weaker than profiniteness and compactness, and it is an orbit equivalence invariant. The first step towards proving the theorem is to show that if a II_1 factor M has c.m.a.p. then given any AFD subalgebra $P \subset M$ the action implemented on P by its normalizer, $\mathcal{N}_M(P) \curvearrowright P$, is weakly compact (see Theorem (4.1.22)). Note that this implies wreath product factors $M = B^\Gamma \rtimes \Gamma$, with Γ nonamenable and $B \neq \mathbb{C}$, can never have the c.m.a.p. In particular, $\Lambda_{\mathrm{cb}}(H \setminus \Gamma) > 1$, for all $H \neq 1$, a fact that was open until now.

To explain the rest of the argument, assume for simplicity $M = L(\mathbb{F}_r)$. Let $P \subset M$ be AFD diffuse, $N = \mathcal{N}_M(P)''$. Taking

$$\eta \in HS(L^2(M)) \simeq L^2(M) \mathbin{\bar{\otimes}} L^2(\bar{M})$$

to be Følner-type elements, as given by the weak compactness of $\mathcal{N}_M(P) \curvearrowright P$, and α_t the "malleable deformation" of $L(\mathbb{F}_r) * L(\mathbb{F}_r)$ in [168], [169], it follows that for t small the elements $(\alpha_t \otimes 1)(\eta) \in L^2(M * M) \mathbin{\bar{\otimes}} L^2(\bar{M})$ are still "almost invariant," in the above sense. We finally use this to prove that $L^2(N)$ is weakly contained in a multiple of the coarse bimodule $L^2(M) \mathbin{\bar{\otimes}} L^2(\bar{M})$, thus showing N is AFD by the characterizations of amenability in [28]. All this is the subject of Theorem (4.1.23).

We recall a number of known results needed in the proofs. This includes a discussion of relative amenability, intertwining lemmas and several facts on the complete metric approximation property. We prove that for each $2 \leq r \leq \infty$ there exist uncountably many non orbit equivalent profinite actions $\mathbb{F}_r \curvearrowright X$, which by Corollary (4.1.4) provide uncountably many nonisomorphic factors $L^\infty(X) \rtimes \mathbb{F}_r$ as well (see Corollary (4.1.39)).

We fix conventions for (semi-)finite von Neumann algebras, but before that we note that the symbol "Lim" will be used for a state on $\ell^\infty(\mathbb{N})$, or more generally on $\ell^\infty(I)$ with I

directed, which extends the ordinary limit, and that the abbreviation "u.c.p." stands for "unital completely positive." We say a map is normal if it is ultraweakly continuous. Whenever a finite von Neumann algebra M is being considered, it comes equipped with a distinguished faithful normal tracial state, denoted by τ. Any group action on a finite von Neumann algebra is assumed to preserve the tracial state τ. If $M = L(\Gamma)$ is a group von Neumann algebra, then the tracial state τ is given by $\tau(x) = \langle x\delta_1, \delta_1\rangle$ for $x \in L(\Gamma)$. Any von Neumann subalgebra $P \subset M$ is assumed to contain the unit of M and inherits the tracial state τ from M. The unique τ-preserving conditional expectation from M onto P is denoted by E_P. We denote by $\mathcal{J}(M)$ the center of M; by $\mathcal{U}(M)$ the group of unitary elements in M; and by

$$\mathcal{N}_M(P) = \{u \in \mathcal{u}(M): (\operatorname{Ad} u)(P) = P\}$$

the normalizing group of P in M, where $(\operatorname{Ad} u)(x) = uxu^*$. A maximal abelian von Neumann subalgebra $A \subset M$ satisfying $\mathcal{N}_M(A)'' = M$ is called a Cartan subalgebra. We note that if $\Gamma \curvearrowright X$ is an ergodic essentially-free probabilitymeasure-preserving action, then $A = L^\infty(X)$ is a Cartan subalgebra in the crossed product $L^\infty(X) \rtimes \Gamma$. (See [36],[36].)

See Section IX.2 of [58] for the details of the following facts on noncommutative L^p-spaces. Let $\mathcal{N}$ be a semi-finite von Neumann algebra with a faithful normal semi-finite trace Tr. For $1 \le p < \infty$, we define the L^p-norm on $\mathcal{N}$ by $\| x \|_p = \operatorname{Tr}(|x|^p)^{1/p}$. By completing $\{x \in \mathcal{N}: \| x \|_p < \infty\}$ with respect to the L^p-norm, we obtain a Banach space $L^p(\mathcal{N})$. We only need $L^1(\mathcal{N}), L^2(\mathcal{N})$ and $L^\infty(\mathcal{N}) = \mathcal{N}$. The trace Tr extends to a contractive linear functional on $L^1(\mathcal{N})$.

We occasionally write $\hat{x}$ for $x \in \mathcal{N}$ when viewed as an element in $L^2(\mathcal{N})$. For any $1 \le p, q, r \le \infty$ with $1/p + 1/q = 1/r$, there is a natural product map

$$L^p(\mathcal{N}) \times L^q(\mathcal{N}) \ni (x, y) \mapsto xy \in L^r(\mathcal{N})$$

which satisfies $\| xy \|_r \le \| x \|_p \| y \|_q$ for any x and y. The Banach space $L^1(\mathcal{N})$ is identified with the predual of $\mathcal{N}$ under the duality $L^1(\mathcal{N}) \times \mathcal{N} \ni (\zeta, x) \mapsto \operatorname{Tr}(\zeta x) \in \mathbb{C}$.

The Banach space $L^2(\mathcal{N})$ is identified with the GNS-Hilbert space of $(\mathcal{N}, \operatorname{Tr})$. Elements in $L^p(\mathcal{N})$ can be regarded as closed operators on $L^2(\mathcal{N})$ which are affiliated with $\mathcal{N}$ and hence in addition to the above-mentioned product, there are well-defined notion of positivity, square root, etc. We will use many times the generalized Powers-Størmer inequality (Theorem XI.1.2 in [58]):

$$\| \eta - \zeta \|_2^2 \le \|\eta^2 - \zeta^2\|_1 \le \| \eta + \zeta \|_2 \| \eta - \zeta \|_2 \qquad (1)$$

for every $\eta, \zeta \in L^2(\mathcal{N})_+$.The Hilbert space $L^2(\mathcal{N})$ is an $\mathcal{N}$-bimodule such that $\langle x\xi y, \eta\rangle = \operatorname{Tr}(x\xi y\eta^*)$ for $\xi, \eta \in L^2(\mathcal{N})$ and $x, y \in \mathcal{N}$. We recall that this gives the canonical identification between the commutant $\mathcal{N}'$ of $\mathcal{N}$ in $\mathbb{B}(L^2(\mathcal{N}))$ and the opposite von Neumann algebra $\mathcal{N}^{\text{op}} = \{x^{\text{op}}: x \in \mathcal{N}\}$ of $\mathcal{N}$. Moreover, the opposite von Neumann algebra $\mathcal{N}^{\circ p}$ is $*$-isomorphic to the complex conjugate von Neumann algebra $\bar{\mathcal{N}} = \{\bar{x}: x \in \mathcal{N}\}$ of $\mathcal{N}$ under the $*$-isomorphism $x^{\text{op}} \mapsto \bar{x}^*$.

Whenever $\mathcal{N}_0 \subset \mathcal{N}$ is a von Neumann subalgebra such that the restriction of Tr to $\mathcal{N}_0$ is still semi-finite, we identify $L^p(\mathcal{N}_0)$ with the corresponding subspace of $L^p(\mathcal{N})$. Anticipating a later use, we consider the tensor product von Neumann algebra $(\mathcal{N} \bar{\otimes} M, \operatorname{Tr} \otimes \tau)$ of a semi-finite von Neumann algebra $(\mathcal{N}, \operatorname{Tr})$ and a finite von Neumann algebra (M, τ). Then, $\mathcal{N} \cong \mathcal{N} \bar{\otimes} \mathbb{C}1 \subset \mathcal{N} \bar{\otimes} M$ and the restriction of $\operatorname{Tr} \otimes \tau$ to $\mathcal{N}$ is Tr.

Moreover, the conditional expectation $\mathrm{id} \otimes \tau: \mathcal{N} \bar{\otimes} M \to \mathcal{N}^-$ extends to a contraction from $L^1(\mathcal{N} \bar{\otimes} M) \to L^1(\mathcal{N})$.

Let $Q \subset M$ be finite von Neumann algebras. Then, the conditional expectation E_Q can be viewed as the orthogonal projection e_Q from $L^2(M)$ onto $L^2(Q) \subset L^2(M)$. It satisfies $e_Q x e_Q = E_Q(x) e_Q$ for every $x \in M$. The $ba-$ sic construction $\langle M, e_Q \rangle$ is the von Neumann subalgebra of $\mathbb{B}(L^2(M))$ generated by M and e_Q. We note that $\langle M, e_Q \rangle$ coincides with the commutant of the right Q-action in $\mathbb{B}(L^2(M))$. The linear span of $\{x e_Q y: x, y \in M\}$ is an ultraweakly dense *-subalgebra in $\langle M, e_Q \rangle$ and the basic construction $\langle M, e_Q \rangle$ comes together with the faithful normal semi-finite trace Tr such that $\mathrm{Tr}(xeQy) = \tau(xy)$. See Section 1.3 in [51] for more information on the basic construction.

We adapt here Connes' characterization of amenable von Neumann algebras to the relative situation. Recall that for von Neumann algebras $N \subset \mathcal{N}$, a state φ on $\mathcal{N}$ is said to be N-central if $\varphi \circ \mathrm{Ad}(u) = \varphi$ for any $u \in (\mathcal{N})$, or equivalently if $\varphi(ax) = \varphi(xa)$ for all $a \in N$ and $x \in \mathcal{N}$.

Definition (4.1.5)[141]: Let $Q, N \subset M$ be finite von Neumann algebras. We say N is amenable relative to Q inside M, denoted by $N \ll_M Q$, if any of the conditions in Theorem (4.1.6) holds. We say Q is co-amenable in M if $M \ll_M Q$ (cf. [167], [143])

Theorem (4.1.6)[141]: Let $Q, N \subset M$ be finite von Neumann algebras. Then, the following are equivalent:

1. There exists a N-central state φ on $\langle M, e_Q \rangle$ such that $\varphi|_M = \tau$.
2. There exists a N-central state φ on $\langle M, e_Q \rangle$ such that φ is normal on M and faithful on $\mathcal{Z}(N' \cap M)$.
3. There exists a conditional expectation Φ from $\langle M, e_Q \rangle$ onto N such that $\Phi|_M = E_N$.
4. There exists a net (ξ_n) in $L^2\langle M, e_Q \rangle$ such that $\lim_n \langle x\xi_n, \xi_n \rangle = \tau(x)$ for every $x \in M$ and that $\lim \|[u, \xi_n]\|_2 = 0$ for every $u \in N$.

Proof. The proof follows a standard recipe of the theory (cf. [28], [40], [167]). The implication (a)⇒(b) is obvious. To prove the converse, assume condition (b). Then, there exists $b \in L^1(M)_+$such that $\varphi(x) = \tau(bx)$ for $x \in M$. Since φ is N-central, one has $ubu^* = b$ for all $u \in \boldsymbol{U}(N)$, i.e. $b \in L^1(N' \cap M)$. We consider the directed set I of finite subsets of $\boldsymbol{u}(N' \cap M)$. For each element $i = \{u_1, \dots, u_n\} \in I$ and $m \in \mathbb{N}$, we define $b_i = n^{-1} \sum u_k b u_k^* \in L^1(N' \cap M)_+, c_{i,m} = \chi_{(1/m,\infty)}(b_i) b_i^{-1/2} \in N' \cap M$ and

$$\psi_{i,m}(x) = \frac{1}{n} \sum_{k=1}^{n} \varphi\left(u_k^* c_{i,m} x c_{i,m} u_k\right)$$

for $x \in \langle M, e_Q \rangle$. Since $c_{i,m} u_k \in N' \cap M$, the positive linear functionals $\psi_{i,m}$ are still N-central and $\psi_{i,m}(x) = \tau\left(\chi_{(1/m,\infty)}(b_i) x\right)$ for $x \in M$. We note that

$$\lim_i \lim_m \chi_{(1/m,\infty)}(b_i) = \lim_i s(b_i) = \lim_i \bigvee s(u_k b u_k^*) = z,$$

where $s(\cdot)$ means the support projection and z is the central support projection of b in $N' \cap M$. Since $\varphi(z^\perp) = \tau(bz^\perp) = 0$ and φ is faithful on $\mathcal{J}(N' \cap M)$, one has $z = 1$. Hence, the state $\psi = \mathrm{Lim}_i \, \mathrm{Lim}_m \, \psi_{i,m}$ on $\langle M, e_Q \rangle$ is N-central and satisfies $\psi|_M = \tau$. This proves (a).

We prove (a)⇒(d): Let a N-central state φ on $\langle M, e_Q\rangle$ be given such that $\varphi|_M = \tau$. Take a net (ζ_n) of positive norm-one elements in $L^1\langle M, e_Q\rangle$ such that $\mathrm{Tr}(\zeta_n \cdot)$ converges to φ pointwise. Then, for every $x \in \langle M, e_Q\rangle$ and $u \in (\mathcal{N})$, one has

$$\lim_n \mathrm{Tr}\big((\zeta_n - \mathrm{Ad}(u)\zeta_n)x\big) = \varphi(x) - \varphi(\mathrm{Ad}(u^*)(x)) = 0$$

by assumption. It follows that for every $u \in\sim (N)$, the net $\zeta_n - \mathrm{Ad}(u)(\zeta_n)$ in $L^1\langle M, e_Q\rangle$ converges to zero in the weak-topology. By the Hahn-Banach separation theorem, one may assume, by passing to convex combinations, that it converges to zero in norm. Thus, $\|[u, \zeta_n]\|_1 \to 0$ for every $u \in \mathcal{U}(N)$. By (1), if we define $\xi_n = \zeta_n^{1/2} \in L^2\langle M, e_Q\rangle$, then one has $\|[u, \xi_n]\|_2 \to 0$ for every $u \in \mathcal{U}(N)$.

Moreover, for any $x \in M$,

$$\lim_n \langle x\xi_n, \xi_n\rangle = \lim_n \mathrm{Tr}(\zeta_n x) = \varphi(x) = \tau(x).$$

We prove (d)⇒(c): For each $x \in \langle M, e_Q\rangle$, denote $\varphi(x) = \mathrm{Lim}_n\langle x\xi_n, \xi_n\rangle$ Note that φ is an N-central sate on $\langle M, e_Q\rangle$ with $\varphi_M = \tau$. Since

$$|\varphi(bcyz)| = |\varphi(cyzb)| \le \varphi(cyy^*c^*)^{1/2}\varphi(b^*z^*zb)^{1/2} \le \|b\|_2\|c\|_2\|y\|\|z\|$$

for every $b, c \in N$ and $y, z \in \langle M, e_Q\rangle$, one has $|\varphi(ax)| \le \| a \|_1 \| x \|$ for every $a \in N$ and $x \in \langle M, e_Q\rangle$. Hence, for every $x \in \langle M, e_Q\rangle$, we may define $\Phi(x) \in N = L^1(N)^*$ by the duality $\tau(a\Phi(x)) = \varphi(ax)$ for all $a \in N$. It is clear that Φ is a conditional expectation onto N such that $\Phi|_M = E_N$.

We prove (c)⇒(a): If there is a conditional expectation Φ from $\langle M, e_Q\rangle$ onto N such that $\Phi|_M = E_N$, then $\varphi = \tau \circ \Phi$ is an N-central state such that $\varphi|_M = \tau$.

Let $N_0 \subset M$ be a von Neumann subalgebra whose unit e does not coincide with the unit of M. We say N_0 is amenable relative to Q inside M, denoted by $N_0 \lessdot_M Q$, if $N_0 + \mathbb{C}(1-e) \lessdot_M Q$. We observe that $N_0 \lessdot_M Q$ if and only if there exists an N_0-central state φ on $e\langle M, e_Q\rangle e$ such that $\varphi(exe) = \tau(\text{ exe })/\tau(e)$ for $x \in M$.

Corollary (4.1.7)[141]: Let $Q_1, \dots, Q_k, N \subset M$ be finite von Neumann algebras and $\subset\subset (N)$ be a subgroup such that $\mathcal{Y}'' = N$. Assume that for every nonzero projection $p \in \mathcal{L}(N' \cap M)$, there exists a net (ξ_n) of vectors in a multiple of $\oplus_{j=1}^k L^2\left\langle M, e_{Q_j}\right\rangle$ such that:

1. $\limsup\|x\xi_n\|_2 \le \| x \|_2$ for all $x \in M$;
2. $\lim\inf\|p\xi_n\|_2 > 0$; and
3. $\lim\|[u, \xi_n]\|_2 = 0$ for every $u \in \mathcal{G}$.

Then, there exist projections $p_1, \dots, p_k \in \mathcal{E}(N' \cap M)$ such that $\sum_{j=1}^k p_j = 1$ and $Np_j \ll_M Q_j$ for every j.

Proof. We observe that if there exists an increasing net $(e_i)_i$ of projections in $\mathcal{Z}(N' \cap M)$ such that $Ne_i \lessdot_M Q$ for all i, then $Ne \lessdot_M Q$ for $e = \sup e_i$. Hence, by Zorn's lemma, there is a maximal k-tuple $(p_1, \dots, p_k)$ of projections in $\mathcal{Z}(N' \cap M)$ such that $\sum_j p_j \le 1$ and $Np_j \ll_M Q_j$ for every j. We prove that $\sum_j p_j = 1$. Suppose by contradiction that $p = 1 - \sum_j p_j \ne 0$, and take a net (ξ_n) as in the statement of the corollary. We may assume that all ξ_n 's are in a multiple of $L^2\langle M, eQ_j\rangle$ for some fixed $j \in \{1, \dots, k\}$. We define a state ψ on $\langle M, eQ_j\rangle$ by

$$\psi(x) = \mathrm{Lim}_n\|p\xi_n\|_2^{-2}\langle xp\xi_n, p\xi_n\rangle$$

for $x \in \left\langle M, e_{Q_j} \right\rangle$. It is not hard to see that $\psi(p) = 1, \psi \circ \mathrm{Ad}(u) = \psi$ for every $u \in \mathcal{G}$ and $\psi(x^*x) \leq (\liminf \|p\xi_n\|)^{-2} \parallel xp \parallel_2^2$ for every $x \in M$. It follows that $\psi|_M$ is normal and ψ is N-central. Let q be the minimal projection in $\mathcal{Z}(N' \cap M)$ such that $\psi(q) = 1$. We finish the proof by showing $Nr \ll_M Q_j$ for $r = p_j + q$ (which gives the desired contradiction to maximality). Since $Np_j \lessdot_M Q_j$, there is an Np_j-central state φ on $p_j\langle M, e_j\rangle p_j$ such that $\varphi(p_j x p_j) = \tau(p_j x p_j)/\tau(p_j)$ for $x \in M$. We fix a state extension $\tilde{\tau}$ of τ on $\left\langle M, e_{Q_j} \right\rangle$ and define a state $\tilde{\varphi}$ on $\left\langle M, e_{Q_j} \right\rangle$ by

$$\tilde{\varphi}(x) = \tau(p_j)\varphi(p_j x p_j) + \tau(q)\psi(qxq) + \tilde{\tau}((1-r)x(1-r))$$

for $x \in \left\langle M, e_{Q_j} \right\rangle$. The state $\tilde{\varphi}$ is $(Nr + \mathbb{C}(1-r))$-central, normal on M and faithful on $\mathcal{Z}((Nr + \mathbb{C}(1-r))' \cap M) = \ell(N' \cap M)r + \mathcal{Z}(M)(1-r)$. Hence Theorem (4.1.6) implies $Nr \ll_M Q_j$.

Compare the following result with [167] and [143].

Proposition (4.1.8)[141]: Let $P, Q, N \subset M$ be finite von Neumann algebras. Then, the following are true:

1. Suppose that $M = Q \rtimes \Gamma$ is the crossed product of Q by a group Γ. Then, $L(\Gamma) \ll_M Q$ if and only if Γ is amenable.
2. Suppose that Q is AFD. Then, $P \ll_M Q$ if and only if P is AFD.
3. If $N \lessdot_M P$ and $P \ll_M Q$, then $N \ll_M Q$.

Proof. Denote by λ_g the unitary element in M which implements the action of $g \in \Gamma$. Since $e_Q\lambda(g)e_Q = 0$ for $g \in \Gamma \setminus \{1\}$, the projections $\{\lambda_g e_Q \lambda_g^* : g \in \Gamma\}$ are mutually orthogonal and generate an isomorphic copy of $\ell^\infty(\Gamma)$ in $\langle M, e_Q\rangle$.

Hence, if there exists an $L(\Gamma)$-central state on $\langle M, e_Q\rangle$, then its restriction to $\ell^\infty(\Gamma)$ becomes a Γ-invariant mean. This proves the "only if" part of assertion (a).

The "if" part is trivial. The assertion (b) easily follows from the fact that $\langle M, e_Q\rangle$ is injective if (and only if) Q is AFD([28]).

Let us finally prove (c). Fix a conditional expectation Φ from $\langle M, e_Q\rangle$ onto P such that $\Phi|_M = E_P$. For $\xi = \sum_{i=1}^m a_i \otimes b_i \in M \otimes M$, we denote

$$\parallel \xi \parallel_2 = \left\| \sum_{i=1}^m a_i e_P b_i \right\|_{L^2\langle M, e_P\rangle} = \left(\sum_{i,j} \tau\left(b_i^* E_P(a_i^* a_j) b_j\right) \right)^{\frac{1}{2}}.$$

For $\xi = \sum_{i=1}^m a_i \otimes b_i$ and $\eta = \sum_{j=1}^n c_j \otimes d_j$ in $M \otimes M$, we define a linear functional $\varphi_{\eta,\xi}$ on $\langle M, e_Q\rangle$ by

$$\varphi_{\eta,\xi}(x) = \sum_{i,j} \tau\left(b_i^* \Phi(a_i^* x c_j) d_j\right).$$

We claim that $\|\varphi_{\eta,\xi}\| \leq \parallel \eta \parallel_2 \parallel \xi \parallel_2$. Indeed, if $\Phi(x) = V^*\pi(x)V$ is a Stinespring dilation, then one has

$$\varphi_{\eta,\xi}(x) = \left\langle \pi(x) \sum_j \pi(c_j) V d_j \hat{1}_P, \sum_i \pi(a_i) V b_i \hat{1}_P \right\rangle$$

and $\|\sum_i \pi(a_i)Vb_i\hat{1}_P\| = \| \xi \|_2$ and likewise for η. It follows that $\varphi_{\eta,\xi}$ is defined for $\xi,\eta \in L^2\langle M, e_P\rangle$ in such a way that $\|\varphi_{\eta,\xi}\| \leq \| \eta \|_2 \| \xi \|_2$. Now take a net of unit vectors (ξ_n) in $L^2\langle M, e_P\rangle$ satisfying condition (d) in Theorem (4.1.6), and let $\varphi = \operatorname{Lim} \varphi_{\xi_n,\xi_n}$ be the state on $\langle M, e_Q\rangle$. Then, one has

$$\varphi \circ \operatorname{Ad}(u) = \operatorname{Lim}_n \varphi_{\operatorname{Ad}(u)(\xi_n),\operatorname{Ad}(u)(\xi_n)} = \operatorname{Lim}_n \varphi_{\xi_n,\xi_n} = \varphi$$

for all $u \in (\mathcal{N})$ and

$$\varphi(x) = \operatorname{Lim}_n \langle x\xi_n, \xi_n\rangle_{L^2\langle N,e_P\rangle} = \tau(x)$$

for all $x \in M$. This proves that $N \ll_M Q$.

We extract from [51], [52] some results which are needed later. The following are Theorem A.1 in [51] and its corollary (also, a particular case of 2.1 in [52]).

Theorem (4.1.9)[141]: Let N be a finite von Neumann algebra and $P, Q \subset N$ be von Neumann subalgebras. Then, the following are equivalent:

1. There exists a nonzero projection $e \in \langle N, e_Q\rangle$ with $\operatorname{Tr}(e) < \infty$ such that the ultraweakly closed convex hull of $\{w^*ew : w \in (\mathcal{P})\}$ does not contain 0.
2. There exist nonzero projections $p \in P$ and $q \in Q$, a normal *-homomorphism $\theta : pPp \to qQq$ and a nonzero partial isometry $v \in N$ such that for all $x \in pPp$, $xv = v\theta(x)$ and $v^*v \in \theta(pPp)' \cap qNq, vv^* \in p(P' \cap N)p$.

Definition (4.1.10)[141]: Let $P, Q \subset N$ be finite von Neumann algebras. Following [52], we say that P embeds into Q inside N, and write $P \preceq_N Q$, if any of the conditions in Theorem (4.1.9) holds.

Let ϕ be a τ-preserving u.c.p. map on N. Then, ϕ extends to a contraction T_ϕ on $L^2(N)$ by $T_\phi(\hat{x}) = \widehat{\phi(x)}$. Suppose that $\phi|_Q = \operatorname{id}Q$. Then, ϕ automatically satisfies $\phi(axb) = a\phi(x)b$ for any $a, b \in Q$ and $x \in N$. It follows that $T_\phi \in \mathbb{B}(L^2(N))$ commutes with the right action of Q, i.e., $T_\phi \in \langle N, e_Q\rangle$. We say ϕ is compact over Q if T_ϕ belongs to the "compact ideal" of $\langle N, e_Q\rangle$ (see [51]). If ϕ is compact over Q, then for any $\varepsilon > 0$, the spectral projection $e = \chi_{[\varepsilon,1]}(T_\phi^* T_\phi) \in \langle N, e_Q\rangle$ has finite $\operatorname{Tr}(e)$ and

$$\langle w^*ew\hat{1}, \hat{1}\rangle_{L^2(N)} \geq \langle T_\phi^* T_\phi \hat{w}, \hat{w}\rangle_{L^2(N)} - \varepsilon = \| \phi(w) \|_2^2 - \varepsilon$$

for all $w \in \mathcal{U}(P)$. These observations imply the following corollary [51].

Corollary (4.1.11)[141]: Let $P, Q \subset N$ be finite von Neumann algebras. Suppose that ϕ is a τ-preserving $u.c.p.$ map on N such that $\phi|_Q = \operatorname{id}_Q$ and ϕ is compact over Q. If $\inf\{\| \phi(w) \|_2 : w \in \mathcal{U}(P)\} > 0$, then $P \preceq_N Q$.

Finally, recall that A.1 in [51] shows the following:

Lemma (4.1.12)[141]: Let A and B be maximal abelian *-subalgebras of a type II_1– factor N. If $A \preceq_N B$, then there exists a nonzero partial isometry $v \in N$ such that $v^*v \in A, vv^* \in B$ and $vAv^* = Bvv^*$. If, moreover, $\mathcal{N}_N(A)'', \mathcal{N}_N(B)''$ are factors (i.e. A, B are semiregular [35]), then v can be taken a unitary element.

Let Γ be a discrete group. For a function f on Γ, we write m_f for the multiplier on $\mathbb{C}\Gamma \subset L(\Gamma)$ defined by $m_f(g) = fg$ for $g \in \mathbb{C}\Gamma$. We simply write $\| f \|_{\text{cb}}$ for $\|m_f\|_{\text{cb}}$ and call it the Herz Schur norm. If $\| f \|_{\text{cb}}$ is finite and $f(1) = 1$, then m_f extends to a τ-preserving normal unital map on $L(\Gamma)$. See Sections 5 and 6 in [166] for an account of Herz-Schur multipliers.

Definition (4.1.13)[141]: A discrete group Γ is weakly amenable if there exist a constant $C \geq 1$ and a net (f_n) of finitely supported functions on Γ such that $\limsup \|f_n\|_{\text{cb}} \leq C$ and

$f_n \to 1$ pointwise. The Cowling-Haagerup constant $\Lambda_{\text{cb}}(\Gamma)$ of Γ is defined as the infimum of the constant C for which a net (f_n) as above exists.

We say a von Neumann algebra M has the (weak*) completely bounded approximation property if there exist a constant $C \geq 1$ and a net (ϕ_n) of normal finiterank maps on M such that $\limsup\|\phi_n\|_{\text{cb}} \leq C$ and $\|x - \phi_n(x)\|_2 \to 0$ for every $x \in M$. The Cowling-Haagerup constant $\Lambda_{\text{cb}}(M)$ of M is defined as the infimum of the constant C for which a net (ϕ_n) as above exists. Also, we say that M has the (weak*) complete metric approximation property (c.m.a.p.) if $\Lambda_{\text{cb}}(M) = 1$. Note that, by Connes' theorem [28], amenability trivially implies c.m.a.p.

By routine perturbation arguments, one may arrange ϕ_n 's in the above definition to be unital and trace-preserving when M is finite. We are interested here in the case $\Lambda_{\text{cb}}(M) = 1$, i.e. when M has the complete metric approximation property. We summarize below some known results in this direction. For part (g), recall that an action of a group Γ on a finite von Neumann algebra P is profinite if there exists an increasing sequence of Γ-invariant finite-dimensional von Neumann subalgebras $P_n \subset P$ that generate P. Note that this implies P is AFD. If $P = L^\infty(X)$ is abelian and $\Gamma \curvearrowright P$ comes from $a\, m.p.$ action $\Gamma \curvearrowright X$, then the profiniteness of $\Gamma \curvearrowright P$ amounts to the existence of a sequence of Γ-invariant finite partitions of X that generate the σ-algebra of measurable subsets of X.

Theorem (4.1.14)[141]:

1. $\Lambda_{\text{cb}}(L(\Gamma)) = \Lambda_{\text{cb}}(\Gamma)$ for any Γ.
2. If Γ is a discrete subgroup of $\text{SO}(1, n)$ or of $\text{SU}(1, n)$, then $\Lambda_{\text{cb}}(\Gamma) = 1$.
3. If Γ acts properly on a finite-dimensional CAT(0) cubical complex, then $\Lambda_{\text{cb}}(\Gamma) = 1$.
4. If $\Lambda_{\text{cb}}(\Gamma_i) = 1$ for $i = 1,2$, then $\Lambda_{\text{cb}}(\Gamma_1 \times \Gamma_2) = 1$ and $\Lambda_{\text{cb}}(\Gamma_1 * \Gamma_2) = 1$.
5. If $N \subset M$ are finite von Neumann algebras, then $\Lambda_{\text{cb}}(N) \leq \Lambda_{\text{cb}}(M)$. Moreover, if N, M are factors and $[M: N] < \infty$, then $\Lambda_{\text{cb}}(M) = \Lambda_{\text{cb}}(N)$ and $\Lambda_{\text{cb}}(M^t) = \Lambda_{\text{cb}}(M)$, for all $t > 0$.
6. Let M be a finite von Neumann algebra and (M_n) be an increasing net of von Neumann subalgebras of M such that $M = (\cup M_n)''$. Then,$\Lambda_{\text{cb}}(M) = \sup\Lambda_{\text{cb}}(M_n)$.
7. If P is a finite von Neumann algebra and $\Gamma \curvearrowright P$ is a profinite action, then $\Lambda_{\text{cb}}(P \rtimes \Gamma) = \Lambda_{\text{cb}}(\Gamma)$.

The assertions (a),(b),(c) and (d) are respectively due to [147], [33], [32], [38] and [170]. The rest are trivial. We will see in Corollary (4.1.19) that property (g) generalizes to compact actions of groups Γ, and even to actions of Γ that are "weakly compact", in the sense of Definition (4.1.17).

We prove a general property about normal amenable subgroups of groups with Λ_{cb}-constant equal to 1. While this property is a consequence of Theorem (4.1.22) (via (c)⇔(d) in Proposition (4.1.18)), we give here a direct proof in group-theoretic framework. To this end, note that if Λ ⊲ Γ is a normal subgroup then the semi-direct product group $\Lambda \rtimes \Gamma$ acts on Λ by $(a, g)b = agbg^{-1}$, for $(a, g) \in \Lambda \rtimes \Gamma$ and $b \in \Lambda$.

Proposition (4.1.15)[141]: Suppose that Γ has an infinite normal amenable subgroup Λ ⊲ Γ and that $\Lambda_{\text{cb}}(\Gamma) = 1$. Then there exists a $\Lambda \rtimes \Gamma$-invariant mean on $\ell^\infty(\Lambda)$ (i.e., Γ is co-amenable in $\Lambda \rtimes \Gamma$). In particular, Γ is inner-amenable. (See \$5 for the definition of inner-amenability.)

Proof. Let f_n be a net of finitely supported functions such that $\sup \|f_n\|_{\text{cb}} = 1$ and $f_n \to 1$ pointwise. By the Bożejko-Fendler theorem (Theorem 6.4 in [166]), there are Hilbert space vectors $\xi_n(a)$ and $\eta_n(b)$ of norm at most one such that $f_n(ab^{-1}) = \langle \eta_n(b), \xi_n(a) \rangle$ for all $a, b \in \Gamma$. Then, for every $g \in \Gamma$, one has

$$\lim_n \sup_{a\in\Gamma} \|\xi_n(ga) - \xi_n(a)\|^2$$
$$\leq \lim_n \sup_{a\in\Gamma} 2\left(\|\xi_n(ga) - \eta_n(a)\|^2 + \|\eta_n(a) - \xi_n(a)\|^2\right)$$
$$\leq \lim_n 2(2 - 2\Re) f_n(g) + 2 - 2\Re f_n(1)\Big) = 0,$$

and similarly $\lim_n \sup_{b\in\Gamma} \|\eta_n(gb) - \eta_n(b)\| = 0$ for every $g \in \Gamma$. It follows that

$$\lim_n \|f_n - f_n^g\|_{\mathrm{cb}} = 0$$

for every $g \in \Gamma$, where $f_n^g \in \mathbb{C}\Gamma$ is defined by $f_n^g(a) = f_n(gag^{-1})$. Now since Λ ⍰ Γ is amenable, the trivial representation $\tau_0 : C^*_{\mathrm{red}}(\Lambda) \to \mathbb{C}$ is continuous. We define a linear functional ω_n on $C^*_{\mathrm{red}}(\Lambda)$ by $\omega_n = \tau_0 \circ m_{f_n} \mid C^*_{\mathrm{red}}(\Lambda)$. Since f_n is finitely supported, ω_n is ultraweakly continuous on $L(\Lambda)$. We note that $\lim \omega_n(\lambda(a)) = 1$ for all $a \in \Lambda$ and

$$\lim_n \|\omega_n - \omega_n \circ \mathrm{Ad}(g)\| \leq \lim_n \|f_n - f_n^g\|_{\mathrm{cb}} = 0$$

for all $g \in \Gamma$. Since $\|\omega_n\| \leq 1$ and $\lim \omega_n(1) = 1$, we have $\lim \|\omega_n - |\omega_n|\| = 0$. We view $|\omega_n|$ as an element in $L^1(L(\Lambda))$ (which is $L^1(\widehat{\Lambda})$ if Λ is abelian) and consider $\zeta_n = |\omega_n|^{1/2} \in L^2(L(\Lambda)) = \ell^2(\Lambda)$. Then, the net (ζ_n) satisfies $\lim_n \langle \lambda(a)\zeta_n, \zeta_n \rangle = 1$ for all $a \in \Lambda$ and $\lim_n \|\zeta_n - \mathrm{Ad}(g)(\zeta_n)\|_2 = 0$ for all $g \in \Gamma$ by (1). Therefore, the state ω on $\ell^\infty(\Lambda) \subset \mathbb{B}(\ell^2(\Lambda))$ defined by

$$\omega(x) = \mathrm{Lim}_n \langle x\zeta_n, \zeta_n \rangle = \mathrm{Lim}_n \sum_{a\in\Lambda} x(a)\zeta_n(a)^2$$

is $\Lambda \rtimes \Gamma$-invariant. Since Λ is infinite, the Λ-invariant mean ω is singular, i.e, $\zeta_n \to 0$ weakly. This implies inner-amenability of Γ.

Recall that the wreath product $H \zeta \Gamma_0$ of a group H by a group Γ_0 is defined as the semi-direct product $\left(\oplus_{\Gamma_0} H\right) \rtimes \Gamma_0$ of $\oplus_{\Gamma_0} H$ by the shift action $\Gamma_0 \curvearrowright \oplus_{\Gamma_0} H$.

Corollary (4.1.16)[141]: If Γ_0 is nonamenable and $H \neq \{1\}$, then $\Lambda_{\mathrm{cb}}(H\zeta\Gamma_0) > 1$, i.e. $L(H > \Gamma_0)$ does not have c.m.a.p. Also, if Γ is a nonamenable group having a nontrivial normal amenable subgroup Λ such that the centralizer $\mathcal{Z}(a) = \{g \in \Gamma : ga = ag\}$ of any nonneutral element $a \in \Lambda$ is amenable, then $\Lambda_{\mathrm{cb}}(\Gamma) > 1$.

Proof. Suppose that Γ_0 is nonamenable and $\Lambda_{\mathrm{cb}}(H > \Gamma_0) = 1$. Passing to a subgroup if necessary, we may assume that H is cyclic. Thus $\Lambda = \oplus_{\Gamma_0} H$ is a nontrivial normal amenable subgroup of $\Gamma = H > \Gamma_0$ such that the centralizer of any nonneutral element of Λ is amenable (finite). It is thus sufficient to prove the second part of the statement. We consider Λ as a set on which Γ acts by conjugation.

Then, $\Lambda \setminus \{1\} = \sqcup_{a\in X} \Gamma/\ell(a)$ as a Γ-set, where X is a system of representatives of Γ-orbits of $\Lambda \setminus \{1\}$. We observe that there is a Γ-equivariant $u.c.p.$ map from $\ell^\infty(\Gamma)$ into $\ell^\infty(\Gamma/\ell(a))$, which is given by a fixed right $\mathcal{J}(a)$-invariant mean applied to each coset $g\mathcal{E}(a) \subset \Gamma$. Hence, there is a Γ-equivariant $u.c.p.$ map from $\ell^\infty(\Gamma)$ into $\ell^\infty(\Lambda \setminus \{1\})$. Since Γ is nonamenable, there is no Γ-invariant mean on $\Lambda \setminus \{1\}$. Hence, any Γ-invariant mean on Λ has to be concentrated on $\{1\}$. Such mean cannot be Λ-invariant. This is in contradiction with Proposition (4.1.15).

Definition (4.1.17)[141]: Let σ be an action of a group Γ on a finite von Neumann algebra P. Recall that σ is called compact if $\sigma(\Gamma) \subset \mathrm{Aut}(P)$ is pre-compact in the point-ultraweak topology. We call the action σ weakly compact if there exists a net (η_n) of unit vectors in $L^2(P \bar{\otimes} \bar{P})+$ such that:

1. $\|\eta_n - (v \otimes \bar{v})\eta_n\|_2 \to 0$ for every $v \in \boldsymbol{U}(P)$.
2. $\|\eta_n - (\sigma_g \otimes \bar{\sigma}_g)(\eta_n)\|_2 \to 0$ for every $g \in \Gamma$.
3. $\langle (x \otimes 1)\eta_n, \eta_n\rangle = \tau(x) = \langle \eta_n, (1 \otimes \bar{x})\eta_n\rangle$ for every $x \in P$ and every n.

Here, we consider the action σ on P as the corresponding unitary representation on $L^2(P)$. By the proof of Proposition (4.1.18), condition (c) can be replaced with a formally weaker condition

1. $\langle (x \otimes 1)\eta_n, \eta_n\rangle \to \tau(x)$ for every $x \in P$.

Weak compactness is manifestly weaker than profiniteness, which is why in an initial version, we called it weak profiniteness. We are very grateful to Adrian Ioana, who pointed out to us that the condition is even weaker than compactness (cf. (b) ⇒ (c) below) and suggested a change in terminology.

Proposition (4.1.18)[141]: Let σ be an action of a group Γ on a finite von Neumann algebra P and consider the following conditions:

1. The action σ is profinite.
2. The action σ is compact and the von Neumann algebra P is AFD.
3. The action σ is weakly compact.
4. There exists a state φ on $\mathbb{B}(L^2(P))$ such that $\varphi|_P = \tau$ and $\varphi \circ \mathrm{Ad}\, u = \varphi$ for all $u \in \mathcal{U}(P) \cup \sigma(\Gamma)$.
5. The von Neumann algebra $L(\Gamma)$ is co-amenable in $P \times \Gamma$.

Then, one has (a)⇒(b)⇒(c)⇔(d)⇔(e).

(Note that, by a result of Høegh-Krohn-Landstad-Størmer ([154]), if in the above statement we restrict our attention to ergodic actions $\Gamma \curvearrowright P$, then the condition that P is AFD in part (b) follows automatically from the assumption $\Gamma \curvearrowright P$ compact. We observe that weak compactness also implies that P is AFD by Connes' theorem ([28]).)

Proof. We have (a)⇒(b), by the definitions. We prove (b)⇒(d). Since P is AFD, there is a net Φ_n of normal $u.c.p.$ maps from $\mathbb{B}(L^2(P))$ into P such that $\tau \circ (\Phi_n \mid P) = \tau$ and $\|a - \Phi_n(a)\|_2 \to 0$ for all $a \in P$. Let G be the SOT-closure of $\sigma(\Gamma)$ in the unitary group on $L^2(P)$. By assumption, G is a compact group and has a normalized Haar measure m. We define a state φ_n on $\mathbb{B}(L^2(P))$ by

$$\varphi_n(x) = \int_G \tau \circ \Phi_n(gxg^{-1})d\, m(g).$$

It is clear that φ_n is Ad(\Gamma)-invariant and $\varphi_n \mid P = \tau$. We will prove that the net φ_n is approximately P-central. Let $\Phi_n(x) = V^*\pi(x)V$ be a Stinespring dilation. Then, for $x \in \mathbb{B}(L^2(P))$ and $a \in P$, one has

$$\begin{aligned}\|\Phi_n(xa) - \Phi_n(x)\Phi_n(a)\|_2 &= \|V^*\pi(x)(1 - VV^*)\pi(a)V\hat{1}\|_{L^2(P)} \\ &\leq \| x \| \, \|(1 - VV^*)^{1/2}\pi(a)V\hat{1}\|_{L^2(P)} \\ &= \| x \| \, \tau(\Phi_n(a^*a) - \Phi_n(a^*)\Phi_n(a))^{1/2} \\ &\leq 2 \| x \| \| a \|^{\frac{1}{2}} \|a - \Phi_n(a)\|_2^{\frac{1}{2}}.\end{aligned}$$

It follows that for every $x \in \mathbb{B}(L^2(P))$ and $a \in P$, one has

$$|\varphi_n(xa) - \varphi_n(ax)| \leq 4 \| x \| \| a \|^{\frac{1}{2}} \sup_{g \in G} \|gag^{-1} - \Phi_n(gag^{-1})\|_2^{\frac{1}{2}},$$

which converge to zero since $\{gag^{-1}: g \in G\}$ is compact in $L^2(P)$ and Φ_n 's are contractive on $L^2(P)$. Hence φ_n is approximately P-central and $\varphi = \mathrm{Lim}_n\, \varphi_n$ satisfies the requirement.

We prove (c)⇔(d). Take a net η_n satisfying conditions (a),(b) and (c') of Definition (4.1.17). We define a state φ on $\mathbb{B}(L^2(P))$ by $\varphi = \mathrm{Lim}_n\, \varphi_n$ with $\varphi_n(x) = \langle (x \otimes 1)\eta_n, \eta_n\rangle$. Then, for any $u \in \mathcal{U}(P) \cup \sigma(\Gamma)$, one has

$$\varphi(u^*xu) = \mathrm{Lim}_n \langle (x \otimes 1)(u \otimes \bar{u})\eta_n, (u \otimes \bar{u})\eta_n\rangle = \varphi(x)$$

by conditions (a) and (b) of Definition (4.1.17). That $\varphi|_P = \tau$ follows from (c'). Conversely, suppose now that φ is given. We recall that $\mathbb{B}(L^2(P))$ is canonically identified with the dual Banach space of the space $S_1(L^2(P))$ of trace class operators. Take a net of positive elements $T_n \in S_1(L^2(P))$ with $\mathrm{Tr}(T_n) = 1$ such that $\mathrm{Tr}(T_n x) \to \varphi(x)$ for every $x \in \mathbb{B}(L^2(P))$. Let $b_n \in L^1(P)_+$be such that $\mathrm{Tr}(T_n a) = \tau(b_n a)$ for $a \in P$. Since $\mathrm{Tr}(T_n a) \to \varphi(a) = \tau(a)$ for $a \in P$, the net (b_n) converges to 1 weakly in $L^1(P)$. Thus, by the Hahn-Banach separation theorem, one may assume, by passing to a convex combinations, that $\|b_n - 1\|_1 \to 0$.

By a routine perturbation argument, we may further assume that $b_n = 1$. We give an argument for this. Let $h(t) = \max\{1, t\}$ and $k(t) = \max\{1-t, 0\}$ be functions on $[0, \infty)$, and let $c_n = h(b_n)^{-1}$. We note that $0 \le c_n \le 1$ and $b_n c_n + k(b_n) = 1$. We define $T_n' = c_n^{1/2} T_n c_n^{1/2} + k(b_n)^{1/2} P_0 k(b_n)^{1/2}$, where P_0 is the orthogonal projection onto $\mathbb{C}\hat{1}$. Then, one has

$$\begin{aligned} \|T_n - T_n'\|_1 &\le 2\left\|T_n^{1/2} - c_n^{1/2} T_n^{1/2}\right\|_2 + \|k(b_n)\|_1 \\ &= 2\tau\left(b_n\left(1 - c_n^{1/2}\right)^2\right)^{1/2} + \|k(b_n)\|_1 \\ &\le 2\tau\big(b_n(1 - c_n)\big)^{1/2} + \|k(b_n)\|_1 \\ &\le 2\|b_n - 1\|_1^{\frac{1}{2}} + \|1 - b_n\|_1 \to 0. \end{aligned}$$

Hence, by replacing T_n with T_n', we may assume that $\mathrm{Tr}(T_n a) = \tau(a)$ for $a \in P$.
Since for every $x \in \mathbb{B}(L^2(P))$ and $u \in \mathcal{U}(P) \cup \sigma(\Gamma)$, one has

$$\mathrm{Tr}\big((T_n - \mathrm{Ad}(u)\, T_n)x\big) \to \varphi(x) - \varphi(\mathrm{Ad}(u^*)(x)) = 0,$$

by applying the Hahn-Banach separation theorem again, one may furthermore assume that $\|T_n - \mathrm{Ad}(u)(T_n)\|_{S_1} \to 0$ for every $u \in \mathcal{U}(P) \cup \sigma(\Gamma)$. Then by (1), the Hilbert-Schmidt operators $T_n^{1/2}$ satisfy $\left\|T_n^{1/2} - \mathrm{Ad}(u)\left(T_n^{1/2}\right)\right\|_{S_2} \to 0$ for every $u \in \mathcal{U}(P) \cup \sigma(\Gamma)$. Now, if we use the standard identification between $S_2(L^2(P))$ and $L^2(P \bar{\otimes} \bar{P})$ given by

$$S_2(L^2(P)) \ni \sum_k \langle \cdot, \eta_k\rangle \xi_k \mapsto \sum_k \xi_k \otimes \bar{\eta}_k \in L^2(P \bar{\otimes} \bar{P})$$

and view $T_n^{1/2}$ as an element $\zeta_n \in L^2(P \bar{\otimes} \bar{P})$, then we have $\langle (a \otimes 1)\zeta_n, \zeta_n\rangle = \tau(a) = \langle \zeta_n, (1 \otimes \bar{a})\zeta_n\rangle$ and $\|\zeta_n - (u \otimes \bar{u})\zeta_n\|_2 \to 0$ for every $u \in \mho(P) \cup \sigma(\Gamma)$.

Therefore, the net of $\eta_n = (\zeta_n \zeta_n^*)^{1/2} \in L^2(P \bar{\otimes} \bar{P})_+$verifies the conditions of weak compactness.

Finally, we prove (d)⇔(e). We consider $P \rtimes \Gamma$ as the von Neumann subalgebra of $\mathbb{B}\big(L^2(P) \bar{\otimes} \ell^2(\Gamma)\big)$ generated by $P \otimes \mathbb{C}1$ and $(\sigma \otimes \lambda)(\Gamma)$. This gives an identification between $L^2(P \rtimes \Gamma)$ and $L^2(P) \bar{\otimes} \ell^2(\Gamma)$. Moreover, the basic construction $\langle P \rtimes \Gamma, e_{L(\Gamma)}\rangle$ becomes $\mathbb{B}(L^2(P)) \bar{\otimes} L(\Gamma)$, since it is the commutant of the right $L(\Gamma)$-action (which is given by $(1 \otimes \rho)(\Gamma)$). Now suppose that φ is given as in condition (d). Then, $\tilde{\varphi} = \varphi \otimes \tau$

on $\mathbb{B}(L^2(P)) \bar{\otimes} L(\Gamma)$ is $\mathrm{Ad}(\mathcal{U}(P \bar{\otimes} \mathbb{C}1) \cup (\sigma \otimes \lambda)(\Gamma))$-invariant and $\tilde{\varphi}|_{P\rtimes\Gamma} = \tau$. This implies that $L(\Gamma)$ is co-amenable in $P \rtimes \Gamma$. Conversely, if $\tilde{\varphi}$ is a $(P \rtimes \Gamma)$-central state such that $\tilde{\varphi}|_{P\rtimes\Gamma} = \tau$, then the restriction φ of $\tilde{\varphi}$ to $\mathbb{B}(L^2(P))$ satisfies condition (d).

Note that by part (g) in Theorem (4.1.14), if $\Lambda_{\mathrm{cb}}(\Gamma) = 1$ and $\Gamma \curvearrowright P$ is a profinite action then $\Lambda_{\mathrm{cb}}(P \rtimes \Gamma) = 1$. More generally we have the following. (Compare this with [157].)

Corollary (4.1.19)[141]: Let Γ be weakly amenable and $\Gamma \curvearrowright P$ be a weakly compact action on an AFD von Neumann algebra. Then, $P \rtimes \Gamma$ has the completely bounded approximation property and $\Lambda_{\mathrm{cb}}(P \rtimes \Gamma) = \Lambda_{\mathrm{cb}}(\Gamma)$.

Proof. By Proposition (4.1.18), $L(\Gamma)$ is co-amenable in $P \rtimes \Gamma$. Hence, Theorem 4.9 of [143] implies that $\Lambda_{\mathrm{cb}}(P \rtimes \Gamma) = \Lambda_{\mathrm{cb}}(L(\Gamma)) = \Lambda_{\mathrm{cb}}(\Gamma)$.

Proposition (4.1.20)[141]: Let $P \subset M$ be an inclusion of finite von Neumann algebras such that $P' \cap M \subset P$. Assume the normalizer $\mathcal{N}_M(P)$ contains a subgroup $\mathcal{G}$ such that its action on P is weakly compact and $(P \cup \mathcal{G})'' = \mathcal{N}_M(P)''$. Then the action of $\mathcal{N}_M(P)$ on P is weakly compact. Moreover, if $\mathcal{N}_M(P) \curvearrowright P$ is weakly compact and $p \in \mathcal{P}(P)$ then $\mathcal{N}_{pMp}(pPp) \curvearrowright pPp$ is weakly compact.

Proof. We may clearly assume $\mathcal{N}_M(P)'' = M$. Denote by σ the action of $\mathcal{N}_M(P)$ on P. If $u \in \mathcal{N}_M(P)$, then by the conditions $P' \cap M = \mathcal{A}(P)$ and $(P \cup \mathcal{S})'' = M$ it follows that there exists a partition $\{p_i\}_i \subset \mathcal{Z}(P)$ and unitary elements $v_i \in P$ such that $v = \Sigma_i p_i v_i u_i$ for some $u_i \in \mathcal{G}$ (see e.g. [148]). Then $\sigma_v(x) = vxv^* = \Sigma_i p_i \sigma_{v_i u_i}(x)$. Let now $\eta_n \in L^2(P \bar{\otimes} \bar{P})_+$satisfy the conditions in Definition (4.1.17) for the action $\sigma_{|q_G}$. By Definition (4.1.17)(a) we have $\|\Sigma_i(p_i \otimes \bar{p}_i)\eta_n - \eta_n\|_2 \to 0$, and thus $\|(p_i \otimes \bar{p}_j)\eta_n\|_2 \to 0$, for all $i \neq j$. Since $q_i = \sigma_{u_i^*}(p_i)$ are mutually orthogonal as well, this also implies that for $i \neq j$ we have

$$\| \left(p_i \otimes \bar{p}_j\right) \quad \left(\sigma_{v_i u_i} \otimes \bar{\sigma}_{v_j u_j}\right)(\eta_n) \|_2$$

$$= \left\|\left(\sigma_{v_i u_i} \otimes \bar{\sigma}_{v_j u_j}\right)\left((q_i \otimes \bar{q}_j)\eta_n\right)\right\|_2 = \left\|(q_i \otimes \bar{q}_j)\eta_n\right\|_2 \to 0.$$

Also, since $w_i = u_i^* v_i u_i \in \Psi(P)$, we have $\| \sigma_{w_i} \otimes \bar{\sigma}_{w_i})(\eta_n) - \eta_n \|_2 \to 0$. Combining with condition Proposition (4.1.18)(b) on the action $\mathcal{G} \curvearrowright P$, one gets

$$\left\|(p_i \otimes \bar{p}_i)\left(\eta_n - (\sigma_{v_i u_i} \otimes \bar{\sigma}_{v_i u_i})(\eta_n)\right)\right\|_2 \to 0.$$

By Pythagoras' theorem, and using that $\sum_{i,j}\|p_i \otimes \bar{p}_j\|_2^2 = 1$, all this entails

$$\|\eta_n - (\sigma_v \otimes \bar{\sigma}_v)(\eta_n)\|_2^2 = \Sigma_{i,j}\left\|(p_i \otimes \bar{p}_j)\eta_n - (p_i \otimes \bar{p}_j)(\sigma_v \otimes \bar{\sigma}_v)(\eta_n)\right\|_2^2$$

$$= \Sigma_{i,j}\left\|(p_i \otimes \bar{p}_j)\eta_n - (p_i \otimes \bar{p}_j)\left(\sigma_{v_i u_i} \otimes \bar{\sigma}_{v_j u_j}\right)(\eta_n)\right\|_2^2 \to 0,$$

showing that $\mathcal{N}_M(P) \curvearrowright P$ satisfies Definition (4.1.17)(b), thus being weakly compact.

To see that weak compactness behaves well to reduction by projections, note that any $v \in \mathcal{N}_{pMp}(pPp)$ extends to a unitary in $\mathcal{N}_M(P)$. Thus, if φ satisfies Proposition (4.1.18)(d) for $\mathcal{N}_M(P) \curvearrowright P$ then $\varphi^p = \varphi(p \cdot p)$ clearly satisfies the same condition for $\mathcal{N}_{pMp}(pPp) \curvearrowright pPp$.

The above result shows in particular that if a measure-preserving action of a countable group Γ on a probability space (X, μ) is weakly compact (i.e., $\Gamma \curvearrowright L^\infty(X)$ weakly compact), then the action of its associated full group $[\Gamma]$, as defined in [148], is weakly compact. Thus, weak compactness is an orbit equivalence invariant for group actions, unlike profiniteness and compactness which are of course not. In fact, Proposition (4.1.20) shows that weak compactness is even invariant to stable orbit equivalence (also called measure equivalence).

An embedding of finite von Neumann algebras $P \subset M$ is called weakly compact if the action $\mathcal{N}_M(P) \curvearrowright P$ is weakly compact. The next result shows that the complete metric approximation property of a factor M imposes the weak compactness of all embeddings into M of AFD (in particular abelian) von Neumann algebras.

For the proof, we need the following consequence of Connes' theorem [28]. This is well-known, but we include a proof for the reader's convenience.

Lemma (4.1.21)[141]: Let M be a finite von Neumann algebra, $P \subset M$ be an AFD von Neumann subalgebra and $u \in \mathcal{N}_M(P)$. Then, the von Neumann algebra Q generated by P and u is AFD.

Proof. Since P is injective, the τ-preserving conditional expectation E_P from M onto P extends to a $u.c.p.$ map $\tilde{E}_P$ from $\mathbb{B}(L^2(M))$ onto P. We note that $\tilde{E}_P$ is a conditional expectation: $\tilde{E}_P(axb) = a\tilde{E}_P(x)b$ for every $a, b \in P$ and $x \in \mathbb{B}(L^2(M))$. We define a state σ on $\mathbb{B}(L^2(M))$ by

$$\sigma(x) = \mathrm{Lim}_n \frac{1}{n}\sum_{k=0}^{n-1} \tau\left(\tilde{E}_P(u^k x u^{-k})\right).$$

It is not hard to check that $\sigma|_M = \tau, \sigma \circ \mathrm{Ad}\, u = \sigma$ and $\sigma \circ \mathrm{Ad}\ v = \sigma$ for every $v \in \boldsymbol{U}(P)$. It follows that σ is a Q-central state with $\sigma|_Q = \tau$. By Connes' theorem, this implies that Q is AFD.

Theorem (4.1.22)[141]: Let M be a finite von Neumann algebra with the c.m.a.p., i.e. $\Lambda_{\mathrm{cb}}(M) = 1$. Then any embedding of an AFD von Neumann algebra $P \subset M$ is weakly compact, i.e., $\mathcal{N}_M(P) \curvearrowright P$ is weakly compact, for all $P \subset M$ AFD subalgebra.

Proof. First we note the following general fact: Let ω be a state on a C^*-algebra N and $u \in \boldsymbol{U}(N)$. We define $\omega_u(x) = \omega(xu^*)$ for $x \in N$. Then, one has

$$\max\{\|\omega - \omega_u\|, \|\ \omega - \omega \circ \mathrm{Ad}(u)\ \|\} \le 2\sqrt{2|1-\omega(u)|}. \qquad (2)$$

Indeed, one has $\|\xi_\omega - u^*\xi_\omega\|^2 = 2(1 - \Re\omega(u)) \le 2|1-\omega(u)|$, where ξ_ω is the GNS-vector for ω.

Let (ϕ_n) be a net of normal finite rank maps on M such that $\limsup\|\phi_n\|_{\mathrm{cb}} \le 1$ and $\|x - \phi_n(x)\|_2 \to 0$ for all $x \in M$. We observe that the net $(\tau \circ \phi_n)$ converges to τ weakly in M_*. Hence by the Hahn-Banach separation theorem, one may assume, by passing to convex combinations, that $\|\tau - \tau \circ \phi_n\| \to 0$. Let μ be the *-representation of the algebraic tensor product $M \otimes \bar{M}$ on $L^2(M)$ defined by

$$\mu\left(\sum_k a_k \otimes \bar{b}_k\right)\xi = \sum_k a_k \xi b_k^*.$$

We define a linear functional μ_n on $M \otimes \bar{M}$ by

$$\mu_n\left(\sum_k a_k \otimes \bar{b}_k\right) = \left\langle \mu\left(\sum_k \phi_n(a_k) \otimes \bar{b}_k\right)\hat{1}, \hat{1}\right\rangle_{L^2(M)} = \tau\left(\sum_k \phi_n(a_k) b_k^*\right).$$

Since ϕ_n is normal and of finite rank, μ_n extends to a normal linear functional on $M \bar{\otimes} \bar{M}$, which is still denoted by μ_n. For an AFD von Neumann subalgebra $Q \subset M$, we denote by μ_n^Q the restriction of μ_n to $Q \bar{\otimes} \bar{Q}$. Since Q is AFD, the *-representation μ is continuous with respect to the spatial tensor norm on $Q \otimes \bar{Q}$ and hence $\|\mu_n^Q\| \le \|\phi_n\|_{\mathrm{cb}}$. We denote $\omega_n^Q =$

$\|\mu_n^Q\|^{-1}|\mu_n^Q|$. Since $\limsup\|\mu_n^Q\| \leq 1$ and $\lim\mu_n^Q(1 \otimes 1) = 1$, the inequality (2), applied to ω_n^Q, implies that

$$\limsup_n \|\mu_n^Q - \omega_n^Q\| = 0. \quad (3)$$

Now, consider the case $Q = P$. Since $\mu_n^P(v \otimes \bar{v}) = \tau(\phi_n(v)v^*) \to 1$ for any $v \in \boldsymbol{U}(P)$, one has

$$\limsup_n \|\omega_n^P - (\omega_n^P)_{v\otimes\bar{v}}\| = 0 \quad (4)$$

by (2) and (3). Now, let $u \in \mathcal{N}_M(P)$ and consider the case $Q = \langle P, u\rangle$, which is AFD by Lemma (4.1.21). Since $\mu_n^{\langle P,u\rangle}(u \otimes \bar{u}) = \tau(\phi_n(u)u^*) \to 1$, one has

$$\limsup_n \left\|\mu_n^{\langle P,u\rangle} - \mu_n^{\langle P,u\rangle} \circ \mathrm{Ad}(u \otimes \bar{u})\right\| = 0 \quad (5)$$

by (2) and (3). But since $\left(\mu_n^{\langle P,u\rangle} \circ \mathrm{Ad}(u \otimes \bar{u})\right)\Big|_{P\bar{\otimes}\bar{P}} = \mu_n^P \circ \mathrm{Ad}(u \otimes \bar{u})$, one has

$$\limsup_n \|\omega_n^P - \omega_n^P \circ \mathrm{Ad}(u \otimes \bar{u})\| = 0 \quad (6)$$

by (3) and (5). Now, we view ω_n^P as an ζ_n element in $L^1(P \bar{\otimes} \bar{P}) +$ and let $\eta_n = \zeta_n^{1/2}$. By (1), the net η_n satisfies all the required conditions.

They will all follow from the following stronger version of the theorem stated:

Theorem (4.1.23)[141]: Let $\Gamma = \mathbb{F}_{r(1)} \times \cdots \times \mathbb{F}_{r(k)}$ be a direct product of finitely many free groups of rank $2 \leq r(j) \leq \infty$ and denote by Γ_j the kernel of the projection from Γ onto $\mathbb{F}_{r(j)}$. Let $M = Q \rtimes \Gamma$ be the crossed product of a finite von Neumann algebra Q by Γ (action need not be ergodic nor free). Let $P \subset M$ be such that P Let $\mathcal{G} \subset \mathcal{N}_M(P)$ be a subgroup which acts weakly compactly on P by conjugation, and denote $N = \mathcal{Y}'$. Then there exist projections $p_1, \dots, p_k \in \mathcal{Z}(N' \cap M)$ with $\sum_{j=1}^k p_j = 1$ such that $Np_j \prec_M Q \rtimes \Gamma_j$ for every j.

From the above result, we will easily deduce several (in)decomposability properties for certain factors constructed out of free groups and their profinite actions. Note that Corollaries (4.1.24) and (4.1.25) below are just Corollaries (4.1.2) and (4.1.3) in the introduction, while Corollary (4.1.34) is a generalization of Corollary (4.1.4) therein.

Corollary (4.1.24)[141]: If $P \subset L(\mathbb{F}_r)^t$ is a diffuse AFD von Neumann subalgebra of the amplification by some $t > 0$ of a free group factor $L(\mathbb{F}_r), 2 \leq r \leq \infty$, then $\mathcal{N}_{L(\mathbb{F}_r)^t}(P)''$ is AFD.

Proof. This is a trivial consequence of Theorem (4.1.22) and Theorem (4.1.23).

Note that the above corollary generalizes the (in)-decomposability results for free group factors in [162] and [172]. Indeed, Voiculescu's celebrated result in [172], showing that the normalizer of any amenable diffuse subalgebra $P \subset L(\mathbb{F}_r)$ cannot generate all $L(\mathbb{F}_r)$, follows from Corollary (4.1.24) because $L(\mathbb{F}_r)$ is nonAFD by [161]. Also, since any unitary element commuting with a subalgebra $P \subset L(\mathbb{F}_r)$ lies in the normalizer of P, Corollary (4.1.24) shows in particular that the commutant of any diffuse AFD subalgebra $P \subset L(\mathbb{F}_r)$ is amenable, i.e. $L(\mathbb{F}_r)$ is solid in the sense of [162], which amounts to the free group case of a result in [162]. Note however that the (in)-decomposability results in [172] and [162] cover much larger classes of factors, e.g. all free products of diffuse von Neumann algebras in [172] (for absence of Cartan subalgebras) and all II_1 factors arising from word-hyperbolic groups in [162] (for solidity).

Calling strongly solid (or s-solid) the factors satisfying the property that the normalizer of any diffuse amenable subalgebra generates an amenable von Neumann algebra, it would be interesting at this point to produce examples of II_1 factors that are s-solid, have both c.m.a.p. and Haagerup property, yet are not isomorphic to an amplification of a free group factor (i.e., to an interpolated free group factor [6], [10]).

Corollary (4.1.30) blow shows in particular that if Q is an arbitrary subfactor of a tensor product of free group factors, then $Q \bar{\otimes} L(\mathbb{F}_r)$ (or any of its finite index subfactors) has no Cartan subalgebras. When applied to $Q = R$, this shows that the subfactor $N \subset R \bar{\otimes} L(\mathbb{F}_r)$ with $N \nsim N^{\mathrm{op}}$ constructed in [172], as the fixed point algebra of an appropriate free action of a finite group on $R \bar{\otimes} L(\mathbb{F}_r)$ (which thus has finite index in $R \bar{\otimes} L(\mathbb{F}_r)\Big)$, does not have Cartan subalgebras.

Another class of factors without Cartan subalgebras is provided by part (2) of the next corollary.

Note that one can view part (a) of the Corollary (4.1.32) as a strong rigidity result, in the spirit of results in ([51], [52], [156]). Indeed, by taking $A = L^\infty(Y)$ to be Cartan in M^t, it follows that any isomorphism between group measure space II_1 factors $\theta: (L^\infty(X) \rtimes \Gamma)^t \simeq L^\infty(Y) \rtimes \Lambda$, with the "source" Γ a direct product of finitely many free groups and the "target" Λ arbitrary but the action $\Lambda \curvearrowright Y$ weakly compact (e.g. profinite, or compact), is implemented by a stable orbit equivalence of the free ergodic actions $\Gamma \curvearrowright X, \Lambda \curvearrowright Y$, up to perturbation by an inner automorphism and by an automorphism coming from a 1-cocycle of the target action.

Corollary (4.1.34) implies that any isomorphism between factors $M \in \wp\wp$ comes from an isomorphism of the orbit equivalence relations $\mathcal{R}_M$ associated with their unique Cartan decomposition. Hence, like in the case of the $\mathcal{HJ}$-factors in [51], invariants of equivalence relations, such as Gaboriau's cost and L^2-Betti numbers ([150]), are isomorphism invariants of II_1 factors in $\wp\wp$. The subfactor theory within the class $\wp\wp$ is particularly interesting: By Corollary (4.1.34) and its proof (see Proposition (4.1.33)), and Section 7 in [51], any irreducible inclusion of finite index $N \subset M$ in this class has a canonical decomposition $N \subset Q \subset P \subset M$, with $P \subset M$ coming from a subequivalence relation of $\mathcal{R}_M, N \subset Q$ from a quotient of $\mathcal{R}_Q$ and $Q \subset P$ from an irreducible $\mathrm{U}(n)$-valued 1 -cocycle for $\mathcal{R}_Q$.

Note that all factors in the class $\wp\wp$ have Λ_{cb}-constant equal to 1 by Theorem (4.1.14) and have Haagerup's compact approximation property by [153]. The sub-class of II_1 factors $L^\infty(X) \rtimes \mathbb{F}_r \in \wp\wp$, arising from free ergodic profinite probability-measure-preserving actions of free groups $\mathbb{F}_r \curvearrowright X$, is of particular interest, as they are inductive limits of (amplifications of) free group factors. We call such a factor $L^\infty(X) \rtimes \mathbb{F}_r$ an approximate free group factor of rank r. By Corollary (4.1.34), more than being in the class $\wp\wp$, such a factor has the property that any maximal abelian *-subalgebra with normalizer generating a von Neumann algebra with no amenable summand is unitary conjugate to $L^\infty(X)$. When combined with [150], we see that approximate free group factors of different rank are not isomorphic and that for $r < \infty$ they have trivial fundamental group. Also, they are prime by [164], in fact by Theorem (4.1.23) the normalizer (in particular the commutant) of any AFD II_1 subalgebra of such a factor must generate an AFD von Neumann algebra. We will construct uncountably many approximate free group factors and comment more on this class.

For the proof of Theorem (4.1.23), recall from [168], [169] the construction of 1-parameter automorphisms α_t ("malleable deformation") of $L\big(\mathbb{F}_r * \widetilde{\mathbb{F}}_r\big)$. Let $\widetilde{\mathbb{F}}_r$ be a copy of $\mathbb{F}_r$ and $a_1, a_2, \ldots$ (resp. $b_1, b_2, \ldots$) be the standard generators of $\mathbb{F}_r$ (resp. $\widetilde{\mathbb{F}}_r$) viewed as unitary elements in $L(\mathbb{F}_r * \widetilde{\mathbb{F}}_r)$. Let $h_s = (\pi\sqrt{-1})^{-1}\log b_s$, where log is the principal branch of the complex logarithm so that h_s is a selfadjoint element with spectrum contained in $[-1,1]$. For simplicity, we write $b_s^t (s = 1,2,\dots$ and $t \in \mathbb{R})$ for the unitary element $\exp(t\pi\sqrt{-1}h_s)$. The $*$-automorphism α_t is defined by $\alpha_t(a_s) = b_s^t a_s$ and $\alpha_t(b_s) = b_s$.

We adapt this construction to $\Gamma = \mathbb{F}_{r(1)} \times \cdots \times \mathbb{F}_{r(k)}$ acting on Q and $M = Q \rtimes \Gamma$. We extend the action $\Gamma \curvearrowright Q$ to that of

$$\tilde{\Gamma} = (\mathbb{F}_{r(1)} * \widetilde{\mathbb{F}}_{r(1)}) \times \cdots \times (\mathbb{F}_{r(k)} * \widetilde{\mathbb{F}}_{r(k)}),$$

where $\widetilde{\mathbb{F}}_{r(j)}$'s act trivially on Q. We denote by $a_{j,1}, a_{j,2}, \dots$ (resp. $b_{j,1}, b_{j,2}, \dots$) the standard generators of $\mathbb{F}_{r(j)}$ (resp. $\widetilde{\mathbb{F}}_{r(j)}$) We redefine the *-homomorphism

$$\alpha_t : M \to \widetilde{M} = Q \rtimes \tilde{\Gamma}$$

by $\alpha_t(x) = x$ for $x \in Q$ and $\alpha_t(a_{j,s}) = b_{j,s}^t a_{j,s}$ for each $1 \le j \le k$ and s. (We can define α_t on $\widetilde{M}$, but we do not need it.)

Let

$$\gamma(t) = \tau(b_{j,s}^t) = \frac{1}{2}\int_{-1}^{1} \exp(t\pi\sqrt{-1}h)dh = \frac{\sin(t\pi)}{t\pi} = \gamma(-t)$$

and $\phi_{j,\gamma(t)} : L(\mathbb{F}_{r(j)}) \to L(\mathbb{F}_{r(j)})$ be the Haagerup multiplier ([153]) associated with the positive type function $g \mapsto \gamma(t)^{|g|}$ on $\mathbb{F}_{r(j)}$. We may extend

$$\phi_{\gamma(t)} = \phi_{1,\gamma(t)} \otimes \cdots \otimes \phi_{k,\gamma(t)}$$

to M by defining $\phi_{\gamma(t)}(x\lambda(g)) = x\phi_{\gamma(t)}(\lambda(g))$ for $x \in Q$ and $\lambda(g) \in L(\Gamma)$. We relate α_t and $\phi_{\gamma(t)}$ as follows (cf. [49]).

Lemma (4.1.25)[141]: One has $E_M \circ \alpha_t = \phi_{\gamma(t)}$.

Proof. Since $E_M(x\lambda(g)) = xE_{L(\Gamma)}(\lambda(g))$ for $x \in Q$ and $\lambda(g) \in L(\tilde{\Gamma})$, one has $E_M \circ \alpha_t(x\lambda(g)) = xE_{L(\Gamma)}(\alpha_t(\lambda(g)))$ for $x \in Q$ and $\lambda(g) \in L(\Gamma)$. Hence it suffices to show $E_{L(\Gamma)} \circ \alpha_t = \phi_{\gamma(t)}$ on $L(\Gamma)$. Since all $E_{L(\Gamma)}, \alpha_t$ and $\phi_{\gamma(t)}$ split as tensor products, we may assume that $k = 1$. Since $a_1, \dots, b_1, \dots$ are mutually free, it is not hard to check

$$(E_{L(F_r)} \circ \alpha_t)(a_{i_1}^{\pm 1} \cdots a_{i_n}^{\pm 1}) = \gamma(t)^n a_{i_1}^{\pm 1} \cdots a_{i_n}^{\pm 1} = \phi_{\gamma(t)}(a_{i_1}^{\pm 1} \cdots a_{i_n}^{\pm 1})$$

for every reduced word $a_{i_1}^{\pm 1} \cdots a_{i_n}^{\pm 1}$ in $\mathbb{F}_r$.

In particular, the $u.c.p.$ map $E_M \circ \alpha_t$ on M is compact over Q provided that $r(j) < \infty$ for every j. In case of $r(j) = \infty$, we need a little modification: we replace the defining equation $\alpha_t(a_{j,s}) = b_{j,s}^t a_{j,s}$ with $\alpha_t(a_{j,s}) = b_{j,s}^{st} a_{j,s}$. Then, the $u.c.p.$ map $E_M \circ \alpha_t$ is compact over Q and $\alpha_t \to \mathrm{id}_M$ as $t \to 0$.

Let Γ_j be the kernel of the projection from Γ onto $\mathbb{F}_{r(j)}$ and $Q_j = Q \rtimes \Gamma_j \subset M$. We consider the basic construction $\langle M, e_{Q_j} \rangle$ of $(Q_j \subset M)$. Then, $L^2\langle M, e_{Q_j} \rangle$ is naturally an M-bimodule.

Lemma (4.1.26)[141]: Let $Q_j \subset M \subset \widetilde{M}$ be as above. Then, $L^2(\widetilde{M}) \ominus L^2(M)$ is isomorphic as an M-bimodule to a submodule of a multiple of $\oplus_{j=1}^{k} L^2\langle M, e_{Q_j} \rangle$.

Proof. Let $\tilde{\Gamma}_j$ be the kernel of the projection from $\tilde{\Gamma}$ onto $\mathbb{F}_{r(j)} * \widetilde{\mathbb{F}}_{r(j)}$. By permuting the position appropriately, we consider that $\tilde{\Gamma}_j \times \mathbb{F}_{r(j)} \subset \tilde{\Gamma}$ and $\bigcap \tilde{\Gamma}_j \times \mathbb{F}_{r(j)} = \Gamma$. Let $\tilde{Q}_j = Q \rtimes$

$\tilde{\Gamma}_j$ and $\widetilde{M}_j = Q \rtimes (\tilde{\Gamma}_j \times \mathbb{F}_{r(j)})$. Since $L^2(M) = \bigcap_{j=1}^{k} L^2(\widetilde{M}_j)$, it suffices to show $L^2(\widetilde{M}) \ominus L^2(\widetilde{M}_j)$ is isomorphic as an M-bimodule to a multiple of $L^2 \langle M, e_{Q_j} \rangle$.

We observe that

$$L^2(\widetilde{M}) \ominus L^2(\widetilde{M}_j) = \bigoplus_{d} \, [\tilde{Q}_j \lambda(\mathbb{F}_{r(j)} d \mathbb{F}_{r(j)})],$$

where the square bracket means the L^2-closure and the direct sum runs all over $d \in \mathbb{F}_{r(j)} * \widetilde{\mathbb{F}}_{r(j)}$ whose initial and final letters in the reduced form come from $\widetilde{\mathbb{F}}_{r(j)}$. Let $\pi_j: \mathbb{F}_{r(j)} * \widetilde{\mathbb{F}}_{r(j)} \to \mathbb{F}_{r(j)}$ be the projection sending $\widetilde{\mathbb{F}}_{r(j)}$ to $\{1\}$. It is not difficult to see that

$$x\lambda(gdh) \mapsto x\lambda(g) e_{Q_j} \lambda(\pi_j(d) h)$$

extends to an M-bimodule isometry from $[\tilde{Q}_j \lambda(\mathbb{F}_{r(j)} d \mathbb{F}_{r(j)})]$ onto $L^2 \langle M, e_{Q_j} \rangle$. We summarize the above two lemmas as follows.

Proposition (4.1.27)[141]: Let $Q \subset Q_j \subset M$ be as above. Then, there are a finite von Neumann algebra $\widetilde{M} \supset M$ and trace-preserving *-homomorphisms $\alpha_t: M \to \widetilde{M}$ such that:

1. $\lim_{t \to 0} \|\alpha_t(x) - x\|_2 \to 0$ for every $x \in M$;
2. $E_M \circ \alpha_t$ is compact over Q for every $t > 0$; and
3. $L^2(\widetilde{M}) \ominus L^2(M)$ is isomorphic as an M-bimodule to a submodule of a multiple of $\bigoplus_{j=1}^{k} L^2 \langle M, e_{Q_j} \rangle$.

We complete the proof of Theorem (4.1.23) in this abstract setting.

Theorem (4.1.28)[141]: Let $Q \subset Q_j \subset M$ be as in Proposition (4.1.27). Let $P \subset M$ be such that $P \nprec_M Q$. Let $\mathcal{G} \subset \mathcal{N}_M(P)$ be subgroup which acts weakly compactly on P by conjugation, and $N = \mathcal{G}''$. Then there exist projection $p_1, \dots, p_k \in L(N' \cap M)$ with $\sum_{j=1}^{k} p_j = 1$ such that $N p_j \ll_M Q_j$ for every j.

Proof. We may assume that $(\mathcal{P}) \subset \mathcal{G}$. We use Corollary (4.1.7) to conclude the relative amenability. Let a nonzero projection p in $L(N' \cap M)$, a finite subset $F \subset \mathcal{G}$ and $\varepsilon > 0$ be given arbitrary. It suffices to find $\xi \in \oplus \bigoplus_{j=1}^{k} L^2 \langle M, e_{Q_j} \rangle$ such that $\| x\xi \|_2 \le \| x \|_2$ for all $x \in M$, $\| p\xi \|_2 \ge \| p \|_2 / 8$ and $\| [\xi, u] \|_2^2 < \varepsilon$ for every $u \in F$.

Let $\delta = \| p \|_2 / 8$. We choose and fix $t > 0$ such that $\alpha = \alpha_t$ satisfies $\| p - \alpha(p) \|_2 < \delta$ and $\| u - \alpha(u) \|_2 < \varepsilon / 6$ for every $u \in F$. We still denote by α when it is viewed as an isometry from $L^2(M)$ into $L^2(\widetilde{M})$. Let (η_n) be the net of unit vectors in $L^2(P \mathbin{\bar{\otimes}} \bar{P})_+$ as in Definition (4.1.17) and denote

$$\tilde{\eta}_n = (\alpha \otimes 1)(\eta_n) \in L^2(\widetilde{M}) \mathbin{\bar{\otimes}} L^2(\bar{M}).$$

We note that

$$\|(x \otimes 1)\tilde{\eta}_n\|_2^2 = \tau\left(\alpha^{-1}\left(E_{\alpha(M)}(x^* x)\right)\right) = \| x \|_2^2 \tag{7}$$

for every $x \in \widetilde{M}$. In particular, one has

$$\|[u \otimes \bar{u}, \tilde{\eta}_n]\|_2 \le \|[u \otimes \bar{u}, \eta_n]\|_2 + 2 \| u - \alpha(u) \|_2 < \frac{\varepsilon}{2} \tag{8}$$

for every $u \in F$ and large enough $n \in \mathbb{N}$. We denote $\zeta_n = (e_M \otimes 1)(\tilde{\eta}_n)$ and $\zeta_n^{\perp} = \tilde{\eta}_n - \zeta_n$. Noticing that $L^2(M) \mathbin{\bar{\otimes}} L^2(\bar{M})$ is an $M \mathbin{\bar{\otimes}} \bar{M}$-bimodule, it follows from (8) that

$$\|[u \otimes \bar{u}, \zeta_n]\|_2^2 + \|[u \otimes \bar{u}, \zeta_n^{\perp}]\|_2^2 = \|[u \otimes \bar{u}, \tilde{\eta}_n]\|_2^2 < \frac{\varepsilon}{(2)^2} \tag{9}$$

for every $u \in F$ and large enough $n \in \mathbb{N}$. We claim that

$$\operatorname{Lim}_n \|(p \otimes 1)\zeta_n^{\perp}\|_2 > \delta. \tag{10}$$

Suppose this is not the case. Then, for any $v \in U(P)$, one has

$$\begin{aligned}
&\operatorname{Lim}_n \|(p \otimes 1)\tilde{\eta}_n - (e_M \alpha(v) p \otimes \bar{v})\zeta_n\|_2 \\
&\le \operatorname{Lim}_n \|(p \otimes 1)\tilde{\eta}_n - (e_M \alpha(v) p \otimes \bar{v})\tilde{\eta}_n\|_2 + \operatorname{Lim}_n \|(p \otimes 1)\zeta_n^{\perp}\|_2 \\
&\le \operatorname{Lim}_n \|(p \otimes 1)\tilde{\eta}_n - (e_M p \otimes 1)(\alpha(v) \otimes \bar{v})\tilde{\eta}_n\|_2 + \| [\alpha(v), p] \|_2 + \delta \\
&\le \operatorname{Lim}_n \|(p \otimes 1)\zeta_n^{\perp}\|_2 + \operatorname{Lim}_n \|\tilde{\eta}_n - (\alpha(v) \otimes \bar{v})\tilde{\eta}_n\|_2 + 2 \| p - \alpha(p) \|_2 + \delta \\
&\le 4\delta
\end{aligned}$$

since $pe_M = e_M p$. It follows that

$$\begin{aligned}
\|(E_M \circ \alpha)(v)p\|_2 &= \operatorname{Lim}_n \|\big((E_M \circ \alpha)(v)p \otimes \bar{v}\big)\tilde{\eta}_n\| \\
&\ge \operatorname{Lim}_n \|(e_M \otimes 1)\big((E_M \circ \alpha)(v)p \otimes \bar{v}\big)\tilde{\eta}_n\| \\
&= \operatorname{Lim}_n \|(e_M \alpha(v) p \otimes \bar{v})\zeta_n\| \\
&\ge \| p \|_2 - 4\delta > 0
\end{aligned} \tag{11}$$

for all $v \in (\mathcal{P})$. (One has $\|(E_M \circ \alpha)(vp)\|_2 \ge \| p \|_2 - 6\delta$ as well.) Since $E_M \circ \alpha$ is compact over Q, this implies $P \preceq_M Q$ by Corollary (4.1.11), contradicting the assumption. Thus by (9) and (10), there exists $n \in \mathbb{N}$ such that $\zeta = \zeta_n^{\perp} \in \big(L^2(\tilde{M}) \ominus L^2(M)\big) \bar{\otimes} L^2(\bar{M})$ satisfies $\| [u \otimes \bar{u}, \zeta] \|_2 < \varepsilon/2$ for every $u \in F$ and $\| (p \otimes 1)\zeta \|_2 \ge \delta$. We note that for all $x \in M$, equation (7) implies

$$\| (x \otimes 1)\zeta \|_2^2 = \|(e_M^{\perp} \otimes 1)(x \otimes 1)\tilde{\eta}_n\|_2^2 \le \|(x \otimes 1)\tilde{\eta}_n\|_2^2 = \| x \|_2^2. \tag{12}$$

By Proposition (4.1.27), we may view ζ as a vector (ζ_i) in $\oplus_i L^2\langle M, eQ_{j(i)}\rangle \bar{\otimes} L^2(\bar{M})$.

We consider $\zeta_i\zeta_i^* \in L^1\left(\left\langle M, e_{Q_{j(i)}}\right\rangle \bar{\otimes} \bar{M}\right)$ and define $\xi_i = \big((\mathrm{id} \otimes \tau)(\zeta_i\zeta_i^*)\big)^{1/2}$ and then $\xi = (\xi_i) \in \oplus_i L^2 \left\langle M, e_{Q_{j(i)}}\right\rangle$. Then, the inequality (12) implies

$$\| x\xi \|_2^2 = \sum_i \tau\big(x^* x (\mathrm{id} \otimes \tau)(\zeta_i\zeta_i^*)\big) = \| (x \otimes 1)\zeta \|_2^2 \le \| x \|_2^2,$$

and for all $x \in M$. In particular,

$$\| p\xi \|_2 = \| (p \otimes 1)\zeta \|_2 \ge \delta.$$

Finally, by (1), one has

$$\begin{aligned}
\| [\xi, u] \|_2^2 = \sum_i \|\xi_i - (\mathrm{Ad}\, u)(\xi_i)\|_2^2 &\le \sum_i \|\xi_i^2 - (\mathrm{Ad}\, u)(\xi_i^2)\|_1 \\
&\le \sum_i \|\zeta_i\zeta_i^* - \mathrm{Ad}(u \otimes \bar{u})\,(\zeta_i\zeta_i^*)\|_1 \\
&\le \sum_i 2\|\zeta_i\|_2 \|[u \otimes \bar{u}, \zeta_i]\|_2 \\
&\le 2 \| \zeta \|_2 \| [u \otimes \bar{u}, \zeta] \|_2 < \varepsilon
\end{aligned}$$

for every $u \in F$.

Before proving the corollaries to Theorem (4.1.23), we mention one more result in the spirit of Theorem (4.1.23). Its proof is similar to the above, but requires more involved technique from [156].

Theorem (4.1.29)[141]: Let $M = M_1 * M_2$ be the free product of finite von Neumann algebras and $P \subset M$ be a von Neumann subalgebra such that $P \npreceq_M M_i$ for $i = 1,2$. If the action of $\mathcal{G} \subset \mathcal{N}_M(P)$ on P is weakly compact, then $\mathcal{G}''$ is AFD.

Proof. We follow the proof of Theorem (4.1.23), but use instead the deformation α_t given in Lemma 2.2.2 in [156]. Let a nonzero projection $p \in L(\mathcal{G}' \cap M)$, a finite subset $F \subset \mathcal{G}$ and $\varepsilon > 0$ be given arbitrary. Since P for $i = 1,2$, one has

$$\lim_{t\to 0} \text{i} \ \{\|(E_M \circ \alpha_t)(vp)\|_2 : v \in \mathcal{U}(P)\} < (999/1000) \parallel p \parallel_2$$

by Proposition (4.1.20) and Theorem 4.3 in [156]. Hence, if we choose $\delta > 0$ small enough and $t > 0$ accordingly, then one obtains as in the proof of Theorem (4.1.23) that

$$\text{Lim}_n \|(p \otimes 1)\zeta_n^{\perp}\|_2 \geq \delta$$

for $\zeta_n^{\perp} = \big((1 - e_M) \otimes 1\big)\tilde{\eta}_n \in L^2(\widetilde{M} \ominus M) \bar{\otimes} L^2(\bar{M})$. Since $L^2(\widetilde{M} \ominus M)$ is a multiple of $L^2(M \bar{\otimes} M)$ as an M-bimodule, one obtains $\xi \in \oplus L^2(M \bar{\otimes} M)$ such that $\parallel x\xi \parallel_2 = \parallel \xi x \parallel_2 \leq \parallel x \parallel_2$ for all $x \in M$, $\parallel p\xi \parallel_2 \geq \delta$ and $\parallel [u, \xi] \parallel_2 < \varepsilon$ for every $u \in F$. This proves that $\mathcal{G}''$ is AFD.

Corollary (4.1.30)[141]: If Q is a type II_1-factor with c.m.a.p., then $Q \bar{\otimes} L(\mathbb{F}_r)$ does not have Cartan subalgebras. Moreover, if $N \subset Q \bar{\otimes} L(\mathbb{F}_r)$ is a subfactor of finite index, then N does not have Cartan subalgebras either.

Proof. Suppose there is a Cartan subalgebra $A \subset M$ where $M \subset N = Q \bar{\otimes} L(\mathbb{F}_r)$ is a subfactor of finite index. Since $\mathbb{F}_r$ is nonamenable, N is not amenable relative to Q, so by Proposition (4.1.8), M is not amenable relative to Q inside N. Hence, by Theorems (4.1.22) and (4.1.23) one has $A \preceq NQ$. By Theorem (4.1.9), this implies there exist projections $p \in A' \cap N, q \in Q$, an abelian von Neumann subalgebra $A_0 \subset qQq$ and a nonzero partial isometry $v \in N$ such that $p_0 = vv^* \in p(A' \cap N)p, q_0 = v^*v \in A_0' \cap qNq$ and $v^*(Ap_0)v = A_0q_0$.

Since $Q = L(\mathbb{F}_r)' \cap N$, by "shrinking" q if necessary we may clearly assume $q = \bigvee\{uq_0u^* : u \in \mathcal{U}\big(L(\mathbb{F}_r)\big)\}$. Since $L(\mathbb{F}_r)q$ is contained in $(A_0q)' \cap qNq$, this implies q_0 has central support 1 in the von Neumann algebra $(A_0q)' \cap qNq$. But $(A_0q_0)' \cap q_0Nq_0 = v^*(A' \cap N)v$ by spatiality and since $M \subset N$ has finite index, $A \subset A' \cap N$ has finite index as well (in the sense of [165]) so $A' \cap N$ is type I, implying $(A_0q_0)' \cap q_0Nq_0$ type I, and thus $(A_0q)' \cap qNq$ type I as well. But $L(\mathbb{F}_r) \simeq L(\mathbb{F}_r)q \subset (A_0q)' \cap qNq$, contradiction

For the proof of Corollary (4.1.32), we will need the following general observation.

Lemma (4.1.31)[141]: Let Γ be an ICC group and Γ o» X an ergodic measurepreserving action. Let $M = L^\infty(X) \rtimes \Gamma$. Then M is a factor. Moreover, the following conditions are equivalent:

1. $\Gamma \curvearrowright X$ is free.
2. $L^\infty(X)$ is maximal abelian (thus Cartan) in M.
3. There is a maximal abelian $*$-subalgebra $A \subset M$ such that $A \preceq_M L^\infty(X)$.

Proof. The first part is well-known, its proof being identical to the Murrayvon Neumann classical argument in [161], showing that if a group Γ is ICC then its group von Neumann algebra $L(\Gamma)$ is a factor.

The equivalence of (a) and (b) is a classical result of Murray and von Neumann, and (b)⇒(c) is trivial. To prove (c)⇒(b), denote $B = L^\infty(X)$ and let $A \subset M$ be maximal abelian satisfying $A \preceq MB$. Then there exists a nonzero partial isometry $v \in M$, projections $p \in A = A' \cap M, q \in B$ and a unital isomorphism θ of Ap onto a unital subalgebra B_0 of Bq such that $va = \theta(a)v$, for all $a \in Ap$. Denoting $q' = vv^* \in B_0' \cap qMq$, it follows that $q'(B_0' \cap qMq)q' = (B_0q')' \cap q'Mq'$. Since by spatiality $B_0q' = vAv^*$ is maximal abelian, this implies $q'(B_0' \cap qMq)q' = vAv^*$. Thus, $B_0' \cap qMq$ has a type I direct summand. Since $(Bq)' \cap qMq$ is a subalgebra of $B_0' \cap pMp$, it follows that $B' \cap M$ has a type I summand.

Since Γ acts ergodically on $\mathcal{A}(B' \cap M) \supset B$ (or else M would not be a factor), the algebra $B' \cap M$ is homogeneous of type I_n, for some $n < \infty$.

Note at this point that since all maximal abelian subalgebras of the type I summand of $B_0' \cap qMq$ containing q' are unitary conjugate (cf. [160]), we may assume that q' is in a maximal abelian algebra containing Bq. Thus, if £ the center of $B' \cap M$, then $£q' \subset q'(B_0' \cap qMq)q' = B_0 q' \subset Bq'$, showing that $£q' = Bq'$. Since $B, £$ are Γ-invariant with the corresponding Γ-actions ergodic, it follows that there exists a partition of 1 with projections of equal trace $p_1, \dots, p_m \in \mathcal{D}$ such that $£ = \Sigma_i B p_i$ and $E_B(p_i) = m^{-1}1$, for all i. Since $B' \cap M = \mathcal{L}' \cap M$ has an orthonormal basis over £ with n^2 unitary elements, this shows that $B' \cap M$ has a finite unitary orthonormal basis over B. But if $x \in (B' \cap M) \setminus B$, and $x = \Sigma_g a_g u_g$ is its Fourier series, with $a_g \neq 0$ for some $g \neq e$, then $p_g u_g \in B' \cap M$, where p_g denotes the support projection of a_g.

Now, since Γ is ICC there exist infinitely many $h_n \in \Gamma$ such that $g_n = h_n g h_n^{-1}$ are distinct. This shows that all $\sigma_{h_n}(p_g) u_{g_n} \subset B' \cap M$ are mutually orthogonal relative to B. By [165], this contradicts the finiteness of the index of $B \subset B' \cap M$.

Thus, we must have $B' \cap M = B$, showing that $\Gamma \curvearrowright X$ is free and $B = L^\infty(X)$ is maximal abelian, hence Cartan.

Corollary (4.1.32)[141]: Let $\Gamma = \mathbb{F}_{r(1)} \times \cdots \times \mathbb{F}_{r(k)}$, as in Theorem (4.1.23), and $\Gamma \curvearrowright X$ an ergodic probability-measure-preserving action. Then $M = L^\infty(X) \rtimes \Gamma$ is a II_1 factor and for each $t > 0$ we have:

1. Assume M^t has a maximal abelian $*$-subalgebra A such that $\mathcal{N}_{M^t}(A) \curvearrowright A$ is weakly compact and $N = \mathcal{N}_{M^t}(A)''$ is a subfactor of finite index in M^t. Then $\Gamma \curvearrowright X$ is necessarily a free action, $L^\infty(X)$ is Cartan in M and there exists a unitary element $u \in M^t$ such that $uAu^* = L^\infty(X)^t$.
2. Assume $\Gamma \curvearrowright X$ is profinite (or merely compact). Then M has a Cartan subalgebra if and only if $\Gamma \curvearrowright X$ is free.
3. Assume $\Gamma = \mathbb{F}_r$. If M^t has a weakly compact maximal abelian $*$-subalgebra A whose normalizer generates a von Neumann algebra without amenable direct summand, then $\Gamma \curvearrowright X$ follows free and A is unitary conjugate to $L^\infty(X)^t$.

Proof. The factoriality of M was shown in Lemma (4.1.31) above.

To prove part (a), note that $\mathcal{N}_{M^t}(A) \curvearrowright A$ weakly compact implies $\mathcal{N}_M(A^{1/t}) \curvearrowright A^{1/t}$ weakly compact, where $A^{1/t} \subset M$ is the semiregular maximal abelian *-subalgebra obtained by amplifying $A \subset M^t$ by $1/t$ (see Proposition (4.1.20) and the comments following its proof). This shows that it is sufficient to prove the case $t = 1$. Let Γ_j be as in Theorem (4.1.23). If $N = \mathcal{N}_M(A)'' \prec_M L^\infty(X) \rtimes \Gamma_j$ for some j, then by $[M:N] < \infty$ it follows that $M \prec_M L^\infty(X) \rtimes \Gamma_j$ as well. But this implies $\mathbb{F}_{r(j)}$ amenable, a contradiction. Thus, by Theorem (4.1.23) we have $A \preceq L^\infty(X)$ and the statement follows from Lemma (4.1.31).

Part (b) follows trivially from part (a), since $\Gamma \curvearrowright X$ compact implies M has c.m.a.p., by Proposition (4.1.18).

An obvious maximality argument shows that in order to prove (c) it is sufficient to show: (c') for all $p \in \mathcal{P}(A), p \neq 0, \exists v \in M^t$, nonzero partial isometry, such that $v^*v \in Ap, vAv^* \subset L^\infty(X)^t$. By amplifying $A \subset M^t$ by suitable integers, we see that in order to prove (c') for arbitrary $t > 0$, it is sufficient to prove it for $t = 1$. Since $N \prec_M L^\infty(X)$ would imply N amenable, by Theorem (4.1.23) we must have $A \preceq L^\infty(X)$. Then Lemma (4.1.31) implies $L^\infty(X)$ maximal abelian in M and Lemma (4.1.12) applies to get (c'), thus (c) as well.

The proof of Corollary (4.1.34) will follow readily from the next general "principle".

Proposition (4.1.33)[141]: Assume a II_1 factor M has the property:

1. $\exists\, A \subset M$ Cartan and any maximal abelian *-subalgebra $A_0 \subset M$ with $\mathcal{N}_M(A_0)''$ a subfactor of finite index in M is unitary conjugate to A.

Then any amplification and finite index extension/restriction of M satisfies (a) as well. Moreover, if M satisfies (a) and $N \subset M$ is an irreducible subfactor of finite index, then $[M: N]$ is an integer.

Proof. For the proof, we call an abelian von Neumann subalgebra B of a II_1, factor P virtually Cartan if it is maximal abelian and $Q = \mathcal{N}_P(B)''$ has finitedimensional center with $[qPq: Qq] < \infty$ for any atom $q \in \mathcal{L}(Q)$. We first prove that if $P \subset N$ is an inclusion of factors with finite index and $B \subset P$ is virtually Cartan in P then any maximal abelian *-subalgebra A of $B' \cap N$ is virtually Cartan in N.

To see this, note that, by commuting squares, the index of $B \subset B' \cap N$ (in the sense of [165]) is majorized by $[N: P] < \infty$, implying that $B' \cap N$ is a direct sum of finitely many homogeneous type I_{n_i} von Neumann algebras B_i, with $1 \le n_1 < n_2 < \cdots < n_k < \infty$. Since any two maximal abelian *-subalgebras of a finite type I von Neumann algebra are unitary conjugate and $\mathcal{N}_P(B)$ leaves $B' \cap N$ globally invariant, it follows that given any $u \in \mathcal{N}_P(B)$, there exists $v(u) \in \mathcal{U}(B' \cap N)$ such that $v(u)uAu^*v(u)^* = A$. Moreover, A is Cartan in $B' \cap N$, i.e. $\mathcal{N}_{B' \cap N}(A)'' = B' \cap N$. This shows in particular that the von Neumann algebra generated by $\mathcal{N}_N(A)$ contains $B' \cap N$ and $v(u)u$, and thus it contains u, i.e. $\mathcal{N}_P(B) \subset \mathcal{N}_N(A)''$.

Thus, the [165]-index of $\mathcal{N}_N(A)''$ in N is majorized by the index of P in N, and is thus finite. Since N is a factor, this implies $Q = \mathcal{N}_N(A)''$ has finite-dimensional center and $[qNq: Qq] < \infty$ for any atom in its center, i.e. A is virtually Cartan in N.

Now notice that since any unitary conjugacy of subalgebras $A, A_0 \subset M$ as in (a) can be "amplified" to a unitary conjugacy of A^t, A_0^t in M^t, property (a) is stable to amplifications. This also shows that (a) holds true for a factor M if and only if M satisfies:

2. $\exists\, A \subset M$ Cartan and any virtually Cartan subalgebra A_0 of M is unitary conjugate to A.

Since if a subfactor $N \subset M$ satisfies $[M: N] < \infty$ then $\langle M, e_N \rangle$ is an amplification of N (see e.g. [165]), it follows that in order to finish the proof of the statement it is sufficient to prove that if M satisfies (b) and $N \subset M$ is a subfactor with finite index, then N satisfies (b).

Let $A \subset M$ be a Cartan subalgebra of M. Let $P \subset N$ be such that $N \subset M$ is the basic construction of $P \subset N$ (cf. [158]). Thus P is isomorphic to an amplification of M and so it has a Cartan subalgebra $A_2 \subset P$. By the first part of the statement any maximal abelian subalgebra A_1 of $A_2' \cap N$ is virtually Cartan in N. Applying again the first part, any maximal abelian A_0 of $A_1' \cap M$ is virtually Cartan in M, so it is unitary conjugate to A. Thus, $A_0 \subset M$ follows Cartan.

Thus, $L^2(M) = \oplus\, u_n L^2(A_0)$, for some partial isometries $u_n \in M$ normalizing A_0. Since A_0 is a finitely generated A_1-module, it follows that each $u_n L^2(A_0)$ is finitely generated both as left and as right A_1 module, i.e. there exist finitely many $\xi_i, \xi_j' \in u_n L^2(A_0)$ such that $\Sigma_i \xi_i A_1$ and $\Sigma A_1 \xi_i'$ are dense in $u_n L^2(A_0)$. Thus, if we denote by $\mathcal{H}_n$ the closure of the range of the projection of $u_n L^2(A_0)$ onto $L^2(N)$ and by η_i, η_j' the projection of ξ_i, ξ_j' onto

$L^2(N)$, then $\mathcal{H}_n$ is a Hilbert A_1-bimodule generated as left Hilbert A_1-module by $\eta_i \in L^2(N)$ and as a right Hilbert A_1-module by $\eta'_j \in L^2(N)$. Moreover, since $\vee_n u_n L^2(A_0) = L^2(M)$, we have $\vee_n \mathcal{H}_n = L^2(N)$. Thus, by Section 1.4 in [51], A_1 is Cartan in N.

Note that the above argument shows that N has Cartan subalgebra, but also that any virtually Cartan subalgebra of N is in fact Cartan. If now $B_1 \subset N$ is another Cartan subalgebra of N, then let B_0 be a maximal abelian subalgebra of $B'_1 \cap M$. By the first part of the proof B_0 is virtually Cartan, so by (b) there exists $v \in \boldsymbol{U}(M)$ such that $vA_0v^* = B_0$. Thus, if we let $v_n = vu_n$ then $L^2(M) = \bigoplus_n v_n L^2(A_0) = \bigoplus_n L^2(B_0)v_n$. Since A_0 (resp. B_0) is a finitely generated A_1 (resp. B_1) module, there exist $\xi_i, \xi'_j \in v_n L^2(A_0) = L^2(B_0)v_n$ such that $\Sigma_i \xi_i A_1$ is dense in $v_n L^2(A_0)$ and $\Sigma_j B_1 \xi'_j$ is dense in $L^2(B_0)v_n$. But then exactly the same argument as above shows that $L^2(N)$ is spanned by Hilbert $B_1 - A_1$ bimodules $\mathcal{H}_n$ which are finitely generated both as right A_1 Hilbert modules and as left Hilbert B_1, modules. By Section 1.4 in [51], it follows that A_1, B_1 are unitary conjugate.

Finally, to see that for irreducible inclusions of factors $N \subset M$ satisfying (a), the index $[M:N]$ is an integer, when finite, let $N \subset Q \subset P \subset M$ be the canonical intermediate subfactors constructed in 7.1 of [51]. Then Q, P satisfy (a) as well and by 7.1 in [51] the Cartan subalgebra of P is maximal abelian and Cartan in M. Thus, as in the proof of $7.2.3°$ in [51], we have $[Q:N], [P:Q], [M:P] \in \mathbb{N}$, implying that $[M:N] \in \mathbb{N}$.

Corollary (4.1.34)[141]: Let $\Gamma = \mathbb{F}_{r(1)} \times \cdots \times \mathbb{F}_{r(k)}$ (as in Theorem (4.1.23), Corollary (4.1.32)) and $\Gamma \curvearrowright X$ a free ergodic profinite (or merely compact) action. Then, $L^\infty(X)$ is the unique Cartan subalgebra of the II_1-factor $L^\infty(X) \rtimes \Gamma$, up to unitary conjugacy. Moreover, if $\wp\wp$ denotes the class of all II_1 factors that can be embedded as subfactors of finite index in an amplification of some $L^\infty(X) \rtimes \Gamma$, with $\Gamma \curvearrowright X$ free ergodic compact action and Γ as above, then any $M \in \wp\wp$ has unique Cartan subalgebra, up to unitary conjugacy. The class $\wp\wp$ is closed under amplifications, tensor product and finite index extension/restriction. Also, if $M \in \wp\wp$ and $N \subset M$ is an irreducible subfactor of finite index, then $[M:N]$ is an integer.

Proof. Let $M = L^\infty(X) \rtimes \Gamma$ and assume $A \subset M$ is a Cartan subalgebra. By Proposition (4.1.18) and Corollary (4.1.19), M follows c.m.a.p. Thus, Theorem (4.1.22) applies to show that $\mathcal{N}_M(A) \curvearrowright A$ is weakly compact. Since $\mathbb{F}_{r(j)}$ are all nonamenable, $M = \mathcal{N}_M(A)''$ cannot be amenable relative to $L^\infty(X) \rtimes \Gamma_j$ (with Γ_j as defined in Theorem (4.1.23)), for all j. Hence, Theorem (4.1.23) implies $A \preceq ML^\infty(X)$.

Then Lemma (4.1.12) shows there is $u \in \boldsymbol{U}(M)$ such that $uAu^* = L^\infty(X)$, proving the first part of the statement. The rest is a consequence of Proposition (4.1.33).

We prove that there are uncountably many approximate free group factors of any rank $2 \leq n \leq \infty$. We do this by using a "separability argument," in the spirit of [167], [159], [163]. The proof is independent of the previous. The result shows in particular the existence of uncountably many orbit inequivalent profinite actions of $\mathbb{F}_n$. The fact that $\mathbb{F}_n$ has uncountably many orbit inequivalent actions was first shown in [151]. A concrete family of orbit inequivalent actions of $\mathbb{F}_n$ was recently obtained in [155]. Note that the actions $\mathbb{F}_n \curvearrowright X$ in [151] and [155] are not orbit equivalent to profinite actions (because they have quotients that are free and have relative property (T) in the sense of [51]).

Definition (4.1.35)[141]: We say a unitary representation $(\pi, \mathcal{H})$ of Γ has (resp. essential) spectral gap if there is a finite subset F of Γ and $\varepsilon > 0$ such that the self-adjoint operator

$$\frac{1}{2|F|}\sum_{g\in F}\left(\pi(g)+\pi(g^{-1})\right)$$

has (resp. essential) spectrum contained in $[-1,1-\varepsilon]$. We say such (F,ε) witnesses (resp. essential) spectral gap of $(\pi,\mathcal{H})$.

It is well-known that $(\pi,\mathcal{H})$ has spectral gap if and only if it does not contain approximate invariant vectors.

Definition (4.1.36)[141]: Let Γ be a group. We say Γ is inner-amenable ([149]) if the conjugation action of Γ on $\ell^2(\Gamma\setminus\{1\})$ does not have spectral gap.

Let $\{\Gamma_n\}$ be a family of finite index (normal) subgroups of Γ. We say Γ has the property (τ) with respect to $\{\Gamma_n\}$ if the unitary Γ-representation on

$$\bigoplus_n \ell^2(\Gamma/\Gamma_n)^o$$

has spectral gap, where $\ell^2(\Gamma/\Gamma_n)^o=\ell^2(\Gamma/\Gamma_n)\ominus\mathbb{C}1_{\frac{\Gamma}{\Gamma_n}}$.

Let I be a family of decreasing sequences

$$i=\left(\Gamma=\Gamma_0^{(i)}\geq\Gamma_1^{(i)}\geq\Gamma_2^{(i)}\geq\cdots\right)$$

of finite index normal subgroups of Γ such that $\bigcap\Gamma_n^{(i)}=\{1\}$. We allow the possibility that $\Gamma_n^{(i)}=\Gamma_{n+1}^{(i)}$. We say the family I is admissible if Γ has the property (τ) with respect to $\left\{\Gamma_m^{(i)}\cap\Gamma_n^{(j)}: i,j\in I, m,n\in\mathbb{N}\right\}$ and

$$\sup\left\{\left[\Gamma:\Gamma_m^{(i)}\Gamma_n^{(j)}\right]: m,n\in\mathbb{N}\right\}<\infty$$

for any $i,j\in I$ with $i\neq j$.

Lemma (4.1.37)[141]: Let $\Gamma\leq \mathrm{SL}(d,\mathbb{Z})$ with $d\geq 2$ be a finite index subgroup and

$$\Gamma_n=\Gamma\cap\ker\left(\mathrm{SL}(d,\mathbb{Z})\to\mathrm{SL}\left(d,\frac{\mathbb{Z}}{n\mathbb{Z}}\right)\right).$$

Let I be a family of infinite subsets of prime numbers such that $|i\cap j|<\infty$ for any $i,j\in I$ with $i\neq j$. (We note that there exists such an uncountable family I.) Associate each $i=\{p_1<p_2<\cdots\}\in I$ with the decreasing sequence of finite index normal subgroups $\Gamma_n^{(i)}=\Gamma_{i(n)}$ where $i(n)=p_1\cdots p_n$. Then, the family I is admissible.

Proof. First, we note that $\Gamma_m\cap\Gamma_n=\Gamma_{\gcd(m,n)}$. By the celebrated results of Kazhdan for $d\geq 3$ (see [144]) and Selberg for $d=2$ (see [42]) the group Γ has the property (τ) with respect to the family $\{\Gamma_n : n\in\mathbb{N}\}$. We observe that the index $\left[\Gamma:\Gamma_m^{(i)}\Gamma_n^{(j)}\right]$ is the cardinality of Γ-orbits of $\left(\Gamma/\Gamma_m^{(i)}\right)\times\left(\Gamma/\Gamma_n^{(j)}\right)$. Since

$$\mathrm{SL}(d,\mathbb{Z}/p_1\cdots p_l\mathbb{Z})=\prod_{k=1}^{l}\mathrm{SL}(d,\mathbb{Z}/p_k\mathbb{Z})$$

for any mutually distinct primes $p_1,\ldots,p_l$, one has a group isomorphism

$$\mathrm{SL}\left(d,\frac{\mathbb{Z}}{i(m)\mathbb{Z}}\right)\times\mathrm{SL}\left(d,\frac{\mathbb{Z}}{j(n)\mathbb{Z}}\right)\cong\mathrm{SL}\left(d,\frac{\mathbb{Z}}{k\mathbb{Z}}\right)\times\mathbf{SL}\left(d,\frac{\mathbb{Z}}{l\mathbb{Z}}\right),$$

where $k=\gcd(i(m),j(n))$ and $l=i(m)j(n)/\gcd(i(m),j(n))$. Since

$$\left(\Gamma/\Gamma_m^{(i)}\right)\times\left(\Gamma/\Gamma_n^{(j)}\right)\subset\mathrm{SL}(d,\mathbb{Z}/i(m)\mathbb{Z})\times\mathrm{SL}(d,\mathbb{Z}/j(n)\mathbb{Z})$$

as a Γ-set, one has

$$\left[\Gamma:\Gamma_m^{(i)}\Gamma_n^{(j)}\right] \le \left|\mathrm{SL}\left(d,\frac{\mathbb{Z}}{k\mathbb{Z}}\right)\right|\left[\mathrm{SL}\left(d,\frac{\mathbb{Z}}{l\mathbb{Z}}\right):\frac{\Gamma}{\Gamma_l}\right].$$

Therefore, the condition $\sup\left\{\left[\Gamma:\Gamma_m^{(i)}\Gamma_n^{(j)}\right]: m,n \in \mathbb{N}\right\} < \infty$ follows from the fact that $|i \cap j| < \infty$.

For example, we can take $\Gamma \le \mathrm{SL}(2,\mathbb{Z})$ to be $\left\langle\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix},\begin{pmatrix}1 & 0\\2 & 1\end{pmatrix}\right\rangle \cong \mathbb{F}_2$. By [171], one may relax the assumption that " $\Gamma \le \mathrm{SL}(d,\mathbb{Z})$ has finite index" to " $\Gamma \le \mathrm{SL}(d,\mathbb{Z})$ is co-amenable," so that one can take Γ to be isomorphic to $\mathbb{F}_\infty$.

Let $\mathcal{P} = (\Gamma_n)_{n=1}^{\infty}$ be a decreasing sequence of finite index subgroups of a group Γ. We write $X_{\mathcal{S}} = \lim_{\leftarrow}\Gamma/\Gamma_n$ for the projective limit of the finite probability space Γ/Γ_n with uniform measures. We note that $L^\infty(X_{\mathcal{S}}) = \left(\cup\ell^\infty(\Gamma/\Gamma_n)\right)''$, where the inclusion $\iota_n: \ell^\infty(\Gamma/\Gamma_n) \hookrightarrow \ell^\infty(\Gamma/\Gamma_{n+1})$ is given by $\iota_n(f)(g\Gamma_{n+1}) = f(g\Gamma_n)$. There is a natural action $\Gamma \curvearrowright L^\infty(X_{\mathcal{S}})$ which is ergodic, measurepreserving and profinite. (Any such action arises in this way.) The action is essentially-free if and only if

$$\text{for all } g \in \Gamma \setminus \{1\}\ |\{x \in X_{\mathcal{S}}: gx = x\}| = \lim_n \frac{\left|\left\{x\in\frac{\Gamma}{\Gamma_n}: gx = x\right\}\right|}{\left|\frac{\Gamma}{\Gamma_n}\right|} = 0. \quad (13)$$

This condition clearly holds if all Γ_n are normal and $\cap\Gamma_n = \{1\}$. We denote $A_{\mathcal{S}} = L^\infty(X_{\mathcal{S}})$ and $A_{\mathcal{S},n} = \ell^\infty(\Gamma/\Gamma_n) \subset A_{\mathcal{S}}$. Since

$$L^2(A_\varphi) \cong \mathbb{C}1 \oplus \bigoplus_{n=1}^{\infty}\left(L^2(A_{Y,n}) \ominus L^2(A_{\mathcal{S},n-1})\right) \subset \mathbb{C}1 \oplus \bigoplus_{n=1}^{\infty} \ell^2(\Gamma/\Gamma_n)^o$$

as a Γ-space, the action $\Gamma \curvearrowright A$ is strongly ergodic if Γ has the property (τ) with respect to $\mathcal{P}$.

Theorem (4.1.38)[141]: Let Γ be a countable group which is not inner-amenable, and I be an uncountable admissible family of decreasing sequences of finite index normal subgroups of Γ. Then, all $M_i = L(X_i) \rtimes \Gamma$ are full factors of type II_1 and the set $\{M_i: i \in I\}$ contains uncountably many isomorphism classes of von Neumann algebras.

Proof. That all M_i are full follows from [145]. Take a finite subset F of Γ and $\varepsilon > 0$ such that (F,ε) witnesses spectral gap for both non-inner-amenability and the property (τ) with respect to $\left\{\Gamma_m^{(i)} \cap \Gamma_n^{(j)}\right\}$. We write $\lambda_i(g)$ for the unitary element in M_i that implements the action of $g \in \Gamma$.

We claim that if $i \ne j$, then (F,ε) witnesses essential spectral gap of the unitary Γ-representation $\mathrm{Ad}(\lambda_i \otimes \lambda_j)$ on $L^2(M_i \bar{\otimes} M_j)$. First, we deal with the $\mathrm{Ad}(\lambda_i \otimes \lambda_i)(\Gamma)$-invariant subspace

$$L^2(A_i \bar{\otimes} A_j) \cong \mathbb{C}1 \oplus \bigoplus_{n=1}^{\infty}\left(L^2(A_{i,n} \bar{\otimes} A_{j,n}) \ominus L^2(A_{i,n-1} \bar{\otimes} A_{j,n-1})\right). \quad (14)$$

We note that the unitary Γ-representation on

$$L^2(A_{i,n} \bar{\otimes} A_{j,n}) \cong \ell^2\left(\left(\Gamma/\Gamma_n^{(i)}\right) \times \left(\Gamma/\Gamma_n^{(j)}\right)\right)$$

is contained in a multiple of $\ell^2\left(\Gamma/\left(\Gamma_n^{(i)} \cap \Gamma_n^{(j)}\right)\right)$. Hence if we show that the subspace of Γ-invariant vectors in $L^2(A_i \bar{\otimes} A_j)$ is finite-dimensional, then we can conclude by the property (τ) that (F,ε) witnesses essential spectral gap. Suppose $\xi \in L^2(A_{i,n} \bar{\otimes} A_{j,n})$ is Γ-

invariant. Since $\Gamma_n^{(i)}$ acts trivially on $L^2(A_{i,n})$, the vector ξ is $\mathrm{Ad}(1 \otimes \lambda_j)\left(\Gamma_n^{(i)}\right)$-invariant. The same thing is true for j. It follows that ξ is in the $\Gamma_n^{(i)}\Gamma_n^{(j)} \times \Gamma_n^{(i)}\Gamma_n^{(j)}$-invariant subspace, whose dimension is $\left[\Gamma : \Gamma_n^{(i)}\Gamma_n^{(j)}\right]^2$.

Since this number stays bounded as n tends to ∞, we are done. Second, we deal with the $\mathrm{Ad}(\lambda_i \otimes \lambda_j)(\Gamma)$-invariant subspace

$$\left(L^2(M_i) \ominus L^2(A_i)\right) \bar{\otimes} L^2(M_j) \cong \ell^2(\Gamma \setminus \{1\}) \bar{\otimes} L^2(A_i) \bar{\otimes} L^2(M_j), \tag{15}$$

where Γ acts on the right-hand side Hilbert space (which will be denoted by $\mathcal{H}$) as $\mathrm{Ad}(\lambda(g) \otimes \lambda_i(g) \otimes \lambda_j(g))$. For every vector $\xi \in \mathcal{H}$, we write it as $(\xi_g)_{g \in \Gamma \setminus \{1\}}$ with $\xi_g \in L^2(A_i) \bar{\otimes} L^2(M_j)$ and define $|\xi| \in \ell^2(\Gamma \setminus \{1\})$ by $|\xi|(g) = \|\xi_g\|$. It follows that

$$\begin{aligned}
&\Re\langle \mathrm{Ad}(\lambda(g) \otimes \lambda_i(g) \otimes \lambda_j(g))\xi, \xi\rangle \\
&\qquad = \Re \sum_{h \in \Gamma \setminus \{1\}} \langle \mathrm{Ad}(\lambda_i(g) \otimes \lambda_j(g))\xi_h, \xi_{ghg^{-1}}\rangle \\
&\qquad \leq \sum_{h \in \Gamma \setminus \{1\}} \|\xi_h\| \|\xi_{ghg-1}\| = \langle \mathrm{Ad}\,\lambda(g)|\xi|, |\xi|\rangle
\end{aligned}$$

for every $g \in \Gamma$ and $\xi \in \mathcal{H}$. Since (F, ε) witnesses spectral gap of the conjugation action on $\ell^2(\Gamma \setminus \{1\})$, it also witnesses spectral gap of the Γ-action on $\mathcal{H}$. Similarly, (F, ε) witnesses spectral gap of

$$L^2(M_i) \bar{\otimes} \left(L^2(M_j) \ominus L^2(A_j)\right). \tag{16}$$

Since the Hilbert spaces (14)-(16) cover $L^2(M_i \bar{\otimes} M_j)$, we conclude that (F, ε) witnesses essential spectral gap of the Γ-action $\mathrm{Ad}(\lambda_i \otimes \lambda_j)$. This argument is inspired by [145].

We claim that for any $i \in I$ and any unitary element $u(g) \in M_i$ with $\| \lambda_i(g) - u(g) \|_2 < \varepsilon/4$, the essential spectrum of the self-adjoint operator

$$h_F = \frac{1}{2|F|} \sum_{g \in F} \left(\mathrm{Ad}(\lambda_i(g) \otimes u(g)) + \mathrm{Ad}(\lambda_i(g^{-1}) \otimes u(g^{-1}))\right)$$

on $L^2(M_i \bar{\otimes} M_i)$ intersects with $[1 - \varepsilon/2, 1]$. We fix $i \in I$ and define for every $n \in \mathbb{N}$ the projection $\chi_n \in M_i \bar{\otimes} M_i$ by $\chi_n = \sum e_k \otimes e_k$, where $\{e_k\}$ is the set of nonzero minimal projections in $A_{i,n} \cong \ell^\infty\left(\Gamma/\Gamma_n^{(i)}\right)$. We normalize $\xi_n = \left[\Gamma : \Gamma_n^{(i)}\right]^{1/2} \chi_n$ so that $\|\xi_n\|_2 = 1$. Then, it is not hard to see

$$\mathrm{Ad}(\lambda_i(g) \otimes \lambda_i(g))\xi_n = \xi_n$$

for all $g \in \Gamma$, and

$$\|(1 \otimes a)\xi_n\|_2^2 = \| a \|_2^2 = \|\xi_n(1 \otimes a)\|_2^2$$

for all $a \in M_i$. It follows that

$$\begin{aligned}
\langle h_F \xi_n, \xi_n\rangle &= \frac{1}{|F|} \sum_{g \in F} \Re\langle \mathrm{Ad}(\lambda_i(g) \otimes u(g))\xi_n, \xi_n\rangle \\
&\geq \frac{1}{|F|} \sum_{g \in F} \left(1 - 2\|\lambda_i(g) - u(g)\|_2\right) > 1 - \varepsilon/2.
\end{aligned}$$

Since $\xi_n \to 0$ weakly as $n \to \infty$, the claim follows (cf. [42]).

From the above claims, we know that if $i \neq j$, then there is no $*$-isomorphism θ from M_i onto M_j such that $\|\theta(\lambda_i(g)) - \lambda_j(g)\|_2 < \varepsilon/4$ for all $g \in F$. Now, if the isomorphism classes of $\{M_i : i \in I\}$ were countable, then there would be M_0 and an uncountable subfamily $I_0 \subset I$ such that $M_i \cong M_0$ for all $i \in I_0$. Take an *isomorphism $\theta_i : M_i \to M_0$ for every $i \in I_0$. Since M_0^F is separable in $\|\cdot\|_2$-norm, there has to be $i, j \in I_0$ with $i \neq j$ such that

$$\max_{g \in F} \|\theta_i(\lambda_i(g)) - \theta_j(\lambda_j(g))\|_2 < \varepsilon/4,$$

in contradiction to the above.

When combined with Lemma (4.1.37), Theorem (4.1.38) shows in particular that any arithmetic property (T) group has uncountably many orbit inequivalent free ergodic profinite actions, thus recovering a result in [42]. However, [42] provides a "concrete" family (consequence of a cocycle superrigidity result for profinite actions of Kazhdan groups) rather than an "existence" result, as Theorem (4.1.38) does. But the consequence of Theorem (4.1.38) and Lemma (4.1.37) that is relevant here is the following:

Corollary (4.1.39)[141]: For each $2 \leq r \leq \infty$, there exist uncountably many nonisomorphic approximate free group factors of rank r. In particular, there exist uncountably many orbit inequivalent free ergodic profinite actions of $\mathbb{F}_r$.

As mentioned, all $L(\mathbb{F}_{1^+}^{r,t})$ have Haagerup's compact approximation property (by [153]), the complete metric approximation property (by Theorem (4.1.14)) and unique Cartan subalgebra, up to unitary conjugacy (by Corollary (4.1.34)). Also, by [164], the commutant of any hyperfinite subfactor of $L(\mathbb{F}_{1^+}^{r,9})$ must be an amenable von Neumann algebra, in particular $L(\mathbb{F}_{1^+}^{r,\mathcal{S}})$ is prime, i.e. it cannot be written as a tensor product of two II_1 factors. By $[Pop06a]$, since the factors $L(\mathbb{F}_{1^+}^{r,\mathcal{S}})$ have Haagerup property they cannot contain factors M which have a diffuse subalgebra with the relative property (T). In particular, the $\mathcal{H}\,\mathfrak{F} -$ factors considered in [51]) cannot be embedded into approximate free group factors. Same for the factors arising from Bernoulli actions of " w-rigid" groups in [168].

Corollary (4.1.34) combined with [150] shows that approximate free group factors of different rank are nonisomorphic, $L(\mathbb{F}_{1^+}^{r,\mathcal{S}}) \not\simeq L(\mathbb{F}_{1^+}^{s,\mathcal{S}})$, for all $2 \leq r \neq s \leq \infty$, and have trivial Murray-von Neumann fundamental group [161] when the rank is finite, $\mathcal{F}\left(L(\mathbb{F}_{1^+}^{s,\mathcal{S}})\right) = \{1\}$, for all $2 \leq r < \infty$. (Recall from [161] that if M is aII_1 factor then its fundamental group is defined by $\mathcal{F}(M) = \{t > 0 \mid M^t \simeq M\}$.) The first examples of factors with trivial fundamental group were constructed in [51], were it is shown that $\mathcal{F}(L^\infty(\mathbb{T}^2) \rtimes \mathbb{F}_r) = \{1\}$, for any finite $r \geq 2$, the action of $\mathbb{F}_r$ on $\mathbb{I}^2$ being inherited from the natural action $\mathrm{SL}(2,\mathbb{Z}) \curvearrowright \mathbb{I}^2 = \widehat{\mathbb{Z}^2}$, for some embedding $\mathbb{F}_r \subset \mathrm{SL}(2,\mathbb{Z})$.

One can show that amplifications of approximate free group factors are related by the formula $L(\mathbb{F}_{1^+}^{r,\mathcal{S}})^t = L\left(\mathbb{F}_{1^+}^{r',\mathcal{S}'}\right)$, with $r' = t^{-1}(r-1)+1$, whenever t^{-1} is an integer dividing the index of some $[\Gamma : \Gamma_n]$ in the decreasing sequence of groups $\mathcal{P} = (\Gamma_n)$, with $\mathcal{P}'$ appropriately derived from $\mathcal{S}$. It is not clear however if this is still the case for other values of t for which $t^{-1}(r-1)+1$ is still an integer.

Finally, note that $L(\mathbb{F}_{1^+}^{r,\mathcal{S}})$ is non Γ if and only if the action $\Gamma \curvearrowright X_{\mathcal{S}}$ has spectral gap. Indeed, since the acting group is $\mathbb{F}_r$, any asymptotically central sequence in $L(\mathbb{F}_{1^+}^{r,\mathcal{S}}) = L^\infty(X_{\mathcal{S}}) \rtimes \mathbb{F}_r$ must lie in $L^\infty(X_{\mathcal{Y}})$, so $L(\mathbb{F}_{1^+}^{r,\mathcal{A}})$ is non Γ if and only if $\mathbb{F}_r \curvearrowright X_{\mathcal{S}}$ is strongly ergodic, which by [142] is equivalent to $\mathbb{F}_r \curvearrowright X_{\mathcal{S}}$ having spectral gap. For each $2 \leq r \leq \infty$, one can easily produce sequences of subgroups $\mathcal{P} = (\Gamma_n)$ such that $\mathbb{F}_r \curvearrowright X_S$ does not have spectral gap, thus giving factors $L(\mathbb{F}_{1^+}^{r,S})$ with property Γ. On the other hand, as mentioned before, if $\mathbb{F}_r$ is embedded with finite index in $\mathrm{SL}(2,\mathbb{Z})$ (or merely embedded "co-amenably," see [171]) and $\mathcal{P} = (\Gamma_n)$ is given by congruence subgroups, then $\mathbb{F}_r \curvearrowright X_S$ has spectral gap by Selberg's theorem. Thus, the corresponding approximate free group factors $L(\mathbb{F}_{1^+}^{r,y})$ are non Γ. By Corollary (4.1.39) and its proof, there are uncountably many nonisomorphic such factors $L(\mathbb{F}_{1^+}^{r,S})$ for each $2 \le r \le \infty$. It is an open problem whether there exist solid factors within this class.

Section (4.2): Free Araki-Woods Factors and Their Continuous Cores

The free Araki-Woods factors were introduced by Shlyakhtenko in [15]. In the context of free probability theory, these factors can be regarded as the analogs of the hyperfinite factors coming from the CAR functor. To each real separable Hilbert space H_{R} together with an orthogonal representation (U_t) of $\mathbf{R}$ on $H_{\mathbf{R}}$, one can associate a von Neumann algebra denoted by $\Gamma(H_{\mathbf{R}}, U_t)''$, called the free A raki-Woods von Neumann algebra. The von Neumann algebra $\Gamma(H_{\mathbf{R}}, U_t)''$ comes equipped with a unique free quasi-free state denoted by φ_U, which is always normal and faithful on $\Gamma(H_{\mathbf{R}}, U_t)''$. If $\dim H_{\mathbf{R}} = 1$, then $\Gamma(\mathbf{R}, \mathrm{Id})'' \cong L^\infty[0,1]$. If $\dim H_{\mathbf{R}} \ge 2$, then $\mathcal{M} = \Gamma(H_{\mathbf{R}}, U_t)''$ is a full factor. In particular, $\mathcal{M}$ can never be of type III_0. The type classification of these factors is the following:

1. $\mathcal{M}$ is a type II_1 factor iff the representation (U_t) is trivial: in that case the functor Γ is Voiculescu's free Gaussian functor [19]. Then $\Gamma(H_{\mathbf{R}}, \mathrm{Id})'' \cong L(\mathbf{F}_{\dim H_{\mathrm{R}}})$.
2. $\mathcal{M}$ is a type III $_\lambda$ factor, for $0 < \lambda < 1$, iff the representation (U_t) is $\frac{2\pi}{|\log \lambda|}$-periodic.
3. $\mathcal{M}$ is a type III_1 factor iff (U_t) is non-periodic and non-trivial. Using free probability techniques, Shlyakhtenko obtained several remarkable classification results for $\Gamma(H_{\mathbf{R}}, U_t)''$. For instance, if the orthogonal representations (U_t) are almost periodic, then the free Araki-Woods factors $\mathcal{M} = \Gamma(H_{\mathbf{R}}, U_t)''$ are completely classified up to statepreserving *-isomorphism [15]: they only depend on Connes' invariant $\mathrm{Sd}(\mathcal{M})$ which is equal in that case to the (countable) subgroup $S_U \subset \mathbf{R}_+^*$ generated by the eigenvalues of (U_t). Moreover, the discrete core $\mathcal{M} \rtimes_\sigma \widehat{S_U}$ (where $\widehat{S_U}$ is the compact group dual of S_U) is *isomorphic to $L(\mathbf{F}_\infty) \bar{\otimes} \mathbf{B}(\ell^2)$. Shlyakhtenko showed in [14] that if (U_t) is the left regular representation, then the continuous core $M = \mathcal{M} \rtimes_\sigma \mathbf{R}$ is isomorphic to $L(\mathbf{F}_\infty) \bar{\otimes} \mathbf{B}(\ell^2)$ and the dual "trace-scaling" action (θ_s) is precisely the one constructed by Rădulescu [184]. For more on free Araki-Woods factors, we refer to [179],[1],[185],[12],[186],[13],[14],[15] and also to Vaes' Bourbaki seminar [18].

The free Araki-Woods factors as well as their continuous cores carry a malleable deformation in the sense of Popa. Then we will use the deformation/rigidity strategy together with the intertwining techniques in order to study the associated continuous cores. The high flexibility of this approach will allow us to work in a semifinite setting, so that we can obtain new structural/indecomposability results for the continuous cores of the free Araki-Woods factors. We first need to recall a few concepts. Following Ozawa [162], [164], a finite von Neumann algebra N is said to be:

1. solid if for any diffuse von Neumann subalgebra $A \subset N$, the relative commutant $A' \cap N$ is amenable;

2. semisolid if for any type II_1 von Neumann subalgebra $A \subset N$, the relative commutant $A' \cap N$ is amenable.

It is easy to check that solidity and semisolidity for II_1 factors are stable under taking amplification by any $t > 0$. Moreover, if N is a non-amenable II_1 factor, then solid $\Longrightarrow$ semisolid $\Longrightarrow$ prime. Recall in this respect that N is said to be prime if it cannot be written as the tensor product of two diffuse factors.

Ozawa discovered a class $\mathcal{S}$ of countable groups for which whenever $\Gamma \in \mathcal{S}$, the group von Neumann algebra $L(\Gamma)$ is solid [162]. He showed that the following countable groups belong to the class : the word-hyperbolic groups [162], the wreath products $\Lambda \wr \Gamma$ for Λ amenable and $\Gamma \in \mathcal{S}$[164], and $\mathbf{Z}^2 \rtimes \mathrm{SL}(2,\mathbf{Z})$[181]. He moreover proved that if $\Gamma \in \mathcal{S}$, then for any free, ergodic, p.m.p. action $\Gamma \curvearrowright (X,\mu)$, the corresponding II_1 factor $L^\infty(X,\mu) \rtimes \Gamma$ is semisolid [164]. Recall that a non-amenable solid II_1 factor does not have property Γ of Murray & von Neumann [162].

Definition (4.2.1)[173]: Let M be a II_∞ factor and let Tr be a fixed faithful normal semifinite trace on M. We shall say that M is solid (resp. semisolid) if for any non-zero projection $q \in M$ such that $\mathrm{Tr}(q) < \infty$, the II_1 factor qMq is solid (resp. semisolid).

Recall that an orthogonal/unitary representation (U_t) acting on H is said to be mixing if for any $\xi, \eta \in H, \langle U_t\xi, \eta\rangle \to 0$, as $|t| \to \infty$. The main result is the following:

Theorem (4.2.2)[173]: Let $\mathcal{M} = \Gamma(H_{\mathbf{R}}, U_t)''$ be a type III_1 free Araki-Woods factor. Then the continuous core $M = \mathcal{M} \rtimes_\sigma \mathbf{R}$ is a semisolid II_∞ factor. Since M is non-amenable, M is always a prime factor. If the representation (U_t) is moreover assumed to be mixing, then M is a solid II_∞ factor.

The proof of Theorem (4.2.2) follows Popa's deformation/rigidity strategy. This theory has been successfully used over the last eight years to give a plethora of new classification/rigidity results for crossed products/free products von Neumann algebras. We refer to [176],[180],[156],[182],[169],[52],[51],[168],[183],[189] for some applications of the deformation/rigidity technique. We point out that in the present, the rigidity part does not rely on the notion of (relative) property (T) but rather on a certain spectral gap property discovered by Popa in [182],[169]. Using this powerful technique, Popa was able to show for instance that the Bernoulli action of groups of the form $\Gamma_1 \times \Gamma_2$, with Γ_1 non-amenable and Γ_2 infinite is $\mathcal{U}_{\mathrm{fin}}$ -cocycle superrigid [182]. The spectral gap rigidity principle gave also a new approach to proving primeness and (semi)solidity for type $\mathrm{II}_1/\mathrm{III}$ factors [175],[176],[182],[169]. We briefly remind below the concepts that we will play against each other in order to prove Theorem (4.2.2):

1. The first ingredient we will use is the "malleable deformation" by automorphisms (α_t, β) defined on the free Araki-Woods factor $\mathcal{M} * \mathcal{M} = \Gamma(H_{\mathbf{R}} \oplus H_{\mathbf{R}}, U_t \oplus U_t)''$. This deformation naturally arises as the "second quantization" of the rotations/reflection defined on $H_{\mathbf{R}} \oplus H_{\mathbf{R}}$ that commute with $U_t \oplus U_t$. It was shown in [182] that such a deformation automatically features a certain "transversality property" (see Lemma 2.1 in [182]) which will be of essential use in our proof.
2. The second ingredient we will use is the spectral gap rigidity principle discovered by Popa in [182],[169]. Let $B \subset M_i$ be an inclusion of finite von Neumann algebras, for $i = 1,2$, with B amenable. Write $M = M_1 *_B M_2$. Then for any von Neumann subalgebra $Q \subset M_1$ with no amenable direct summand, the action by conjugation $\mathrm{Ad}(\mathcal{U}(Q)) \curvearrowright M$ has "spectral gap" relative to M_1: for any $\varepsilon > 0$, there exist $\delta > 0$

and a finite "critical" subset $F \subset \mathcal{U}(Q)$ such that for any $x \in (M)_1$ (the unit ball of M), if $\|uxu^* - x\|_2 \leq \delta, \forall u \in F$, then $\|x - E_{M_1}(x)\|_2 \leq \varepsilon$.

3. Let $M = \mathcal{M} \rtimes_\sigma \mathbf{R}$ be the continuous core the free Araki-Woods factor $\mathcal{M}$. Let $q \in M$ be a non-zero finite projection. A combination of (a) and (b) yields that for any $Q \subset qMq$ with no amenable direct summand, the malleable deformation (α_t) necessarily converges uniformly in $\|\cdot\|_2$ on $(Q' \cap qMq)_1$. Then, using Popa's intertwining techniques, one can locate the position of $Q' \cap qMq$ inside qMq.

The second result we provide a new example of a non-amenable solid II_1 factor. We first need the following:

Example (4.2.3)[173]: Using results of [174], we construct an example of an orthogonal representation (U_t) of $\mathbf{R}$ on a (separable) real Hilbert space $K_\mathbf{R}$ such that:

1. (U_t) is mixing.
2. The spectral measure of $\bigoplus_{n\geq 1} U_t^{\otimes n}$ is singular w.r.t. the Lebesgue measure on $\mathbf{R}$.

Shlyakhtenko showed in [14] that if the spectral measure of the representation $\bigoplus_{n\geq 1} U_t^{\otimes n}$ is singular w.r.t. the Lebesgue measure, then the continuous core of the free Araki-Woods factor $\Gamma(H_\mathbf{R}, U_t)''$ cannot be isomorphic to any $L(\mathbf{F}_t) \bar{\otimes} \mathbf{B}(\ell^2)$, for $1 < t \leq \infty$, where $L(\mathbf{F}_t)$ denote the interpolated free group factors [6],[10]. Therefore, we obtain:

Theorem (4.2.4)[173]: Let (U_t) be an orthogonal representation acting on K_R as in Example (4.2.3). Denote by $\mathcal{M} = \Gamma(K_\mathrm{R}, U_t)''$ the corresponding free Araki-Woods factor and by $M = \mathcal{M} \rtimes_\sigma \mathbf{R}$ its continuous core. Let $q \in L(\mathbf{R})$ be a non-zero projection such that $\mathrm{Tr}(q) < \infty$. Then the non-amenable II_1 factor qMq is solid, has full fundamental group, i.e. $\mathcal{F}(qMq) = \mathbf{R}^*_+$, and is not isomorphic to any interpolated free group factor $L(\mathbf{F}_t)$, for $1 < t \leq \infty$.

We recall the necessary background on free Araki-Woods factors as well as intertwining techniques for (semi)finite von Neumann algebras. We mainly devoted to the proof of Theorem (4.2.2), following the deformation/spectral gap rigidity strategy presented above. We construct Example (4.2.3) and deduce Theorem (4.2.4).

Let H_R be a real separable Hilbert space and let (U_t) be an orthogonal representation of $\mathbf{R}$ on $H_\mathbf{R}$ such that the map $t \mapsto U_t$ is strongly continuous. Let $H_\mathbf{C} = H_\mathbf{R} \otimes_\mathbf{R} \mathbf{C}$ be the complexified Hilbert space. We shall still denote by (U_t) the corresponding unitary representation of $\mathbf{R}$ on $H_\mathbf{C}$. Let A be the infinitesimal generator of (U_t) on H_C (Stone's theorem), so that A is the positive, self-adjoint, (possibly) unbounded operator on H_C which satisfies $U_t = A^{it}$, for every $t \in \mathbf{R}$. Define another inner product on $H_\mathbf{C}$ by

$$\langle \xi, \eta \rangle_U = \left\langle \frac{2}{1 + A^{-1}} \xi, \eta \right\rangle, \forall \xi, \eta \in H_\mathbf{C}.$$

Note that for any $\xi \in H_\mathbf{R}$, $\| \xi \|_U = \| \xi \|$; also, for any $\xi, \eta \in H_\mathbf{R}$, $\Re(\langle \xi, \eta \rangle_U) = \langle \xi, \eta \rangle$, where $\Re$ denotes the real part. Denote by H the completion of $H_\mathbf{C}$ w.r.t. the new inner product $\langle \cdot, \cdot \rangle_U$, and note that (U_t) is still a unitary representation on H. Introduce now the full Fock. space of H:

$$\mathcal{F}(H) = \mathbf{C}\Omega \oplus \bigoplus_{n=1}^{\infty} H^{\bar{\otimes} n}.$$

The unit vector Ω is called the vacuum vector. For any $\xi \in H$, we have the left creation operator

$$\ell(\xi): \mathcal{F}(H) \to \mathcal{F}(H): \begin{cases} \ell(\xi)\Omega = \xi. \\ \ell(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n. \end{cases}$$

For any $\xi \in H$, we denote by $s(\xi)$ the real part of $\ell(\xi)$ given by

$$s(\xi)=\frac{\ell(\xi)+\ell(\xi)^*}{2}.$$
The crucial result of Voiculescu [19] is that the distribution of the operator $s(\xi)$ w.r.t. the vacuum vector state $\varphi_U=\langle\cdot\,\Omega,\Omega\rangle_U$ is the semicircular law of Wigner supported on the interval $[-\|\xi\|,\|\xi\|]$.

Definition (4.2.5)[173]: (Shlyakhtenko, [15]). Let (U_t) be an orthogonal representation of $\mathbf{R}$ on the real Hilbert space $H_{\mathbf{R}}$. The free Araki-Woods von Neumann algebra associated with $H_{\mathbf{R}}$ and (U_t), denoted by $\Gamma(H_{\mathbf{R}},U_t)''$, is defined by
$$\Gamma(H_{\mathbf{R}},U_t)'':=\{s(\xi):\xi\in H_{\mathbf{R}}\}''.$$
The vector state $\varphi_U=\langle\cdot\,\Omega,\Omega\rangle_U$ is called the free quasi-free state. It is normal and faithful on $\Gamma(H_{\mathbf{R}},U_t)''$.

Recall that for any type III_1 factor $\mathcal{M}$, Connes-Takesaki's continuous decomposition [4],[187] yields
$$\mathcal{M}\,\bar{\otimes}\,\mathbf{B}(L^2(\mathbf{R}))\cong(\mathcal{M}\rtimes_\sigma\mathbf{R})\rtimes_\theta\mathbf{R},$$
where the continuous core $\mathcal{M}\rtimes_\sigma\mathbf{R}$ is a II_∞ factor and θ is the trace-scaling action [187]:
$$\mathrm{Tr}\big(\theta_s(x)\big)=e^{-s}\,\mathrm{Tr}(x),\forall x\in(\mathcal{M}\rtimes_\sigma\mathbf{R})_+,\forall s\in\mathbf{R}.$$
The fact that $\mathcal{M}\rtimes_\sigma\mathbf{R}$ does not depend on the choice of a f.n. state on $\mathcal{M}$ follows from Connes' Radon-Nikodym derivative theorem [4]. Moreover, for any non-zero finite projection $q\in M=\mathcal{M}\rtimes_\sigma\mathbf{R}$, the II_1 factor qMq has full fundamental group.

Following [3], a factor $\mathcal{M}$ (with separable predual) is said to be full if the subgroup of inner automorphisms $\mathrm{Inn}(\mathcal{M})\subset\mathrm{Aut}(\mathcal{M})$ is closed. Recall that $\mathrm{Aut}(\mathcal{M})$ is endowed with the u-topology: for any sequence (θ_n) in $\mathrm{Aut}(\mathcal{M})$,
$$\theta_n\to\mathrm{Id},\text{ as }n\to\infty\Leftrightarrow\|\varphi\circ\theta_n-\varphi\|\to0,\text{ as }n\to\infty,\forall\varphi\in\mathcal{M}_*.$$
Since $\mathcal{M}$ has a separable predual, $\mathrm{Aut}(\mathcal{M})$ is a polish group. For any II_1 factor N, N is full iff N does not have property Γ of Murray & von Neumann (see [3]).

Denote by $\pi:\mathrm{Aut}(\mathcal{M})\to\mathrm{Out}(\mathcal{M})$ the canonical projection. Assume $\mathcal{M}$ is a full factor so that $\mathrm{Out}(\mathcal{M})$ is a Hausdorff topological group. Fix a f.n. state φ on $\mathcal{M}$. Connes' invariant $\tau(\mathcal{M})$ is defined as the weakest topology on $\mathbf{R}$ that makes the map
$$\begin{aligned}\mathbf{R}&\to\mathrm{Out}(\mathcal{M})\\ t&\mapsto\pi\big(\sigma_t^\varphi\big)\end{aligned}$$
continuous. Note that this map does not depend on the choice of the f.n. state φ on $\mathcal{M}$[4]. Denote by $\mathcal{F}(U_t)=\oplus_{n\in\mathbf{N}}U_t^{\otimes n}$. The modular group σ^{φ_U} of the free quasi-free state is given by: $\sigma_t^{\varphi_U}=\mathrm{Ad}\big(\mathcal{F}(U_{-t})\big)$, for any $t\in\mathbf{R}$. The free Araki-Woods factors provided many new examples of full factors of type III [2],[4],[12]. We can summarize their general properties in the following theorem (see also Vaes' Bourbaki seminar [18]):

Theorem (4.2.6)[173]: (Shlyakhtenko, [12],[13],[14],[15]). Let (U_t) be an orthogonal representation of $\mathbf{R}$ on the real Hilbert space $H_{\mathbf{R}}$ with $\dim H_{\mathbf{R}}\geq2$. Denote by $\mathcal{M}=\Gamma(H_{\mathbf{R}},U_t)''$.

1. $\mathcal{M}$ is a full factor and Connes' invariant $\tau(\mathcal{M})$ is the weakest topology on $\mathbf{R}$ that makes the map $t\mapsto U_t$ strongly continuous.
2. $\mathcal{M}$ is of type II_1 iff $U_t=\mathrm{id}$ for every $t\in\mathbf{R}$. In this case, $\mathcal{M}\cong L\big(\mathbf{F}_{\dim(H_{\mathbf{R}})}\big)$.
3. $\mathcal{M}$ is of type $\mathrm{III}_\lambda(0<\lambda<1)$ iff (U_t) is periodic of period $\frac{2\pi}{|\log\lambda|}$.
4. $\mathcal{M}$ is of type III_1 in the other cases.
5. $\mathcal{M}$ has almost periodic states iff (U_t) is almost periodic.

Moreover, it follows from [185] that any free Araki-Woods factor $\mathcal{M}$ is generalized solid in the sense of [188]: for any diffuse von Neumann subalgebra $A \subset \mathcal{M}$ such that there exists a faithful normal conditional expectation $E: \mathcal{M} \to A$, the relative commutant $A' \cap \mathcal{M}$ is amenable.

Notice that the centralizer of the free quasi-free state $\mathcal{M}^{\varphi_U}$ may be trivial. This is the case for instance when the representation (U_t) has no eigenvectors. Nevertheless, the author recently proved in [179] that for any type III_1 free Araki-Woods factor $\mathcal{M}$, the bicentralizer is trivial, i.e. there always exists a faithful normal state ψ on $\mathcal{M}$ such that $\left(\mathcal{M}^{\psi}\right)' \cap \mathcal{M} = \mathbf{C}$. See [178] for more on Connes' bicentralizer problem.

Remark (4.2.7)[173]: ([15]). Explicitly the value of φ_U on a word in $s(\xi_\iota)$ is given by

$$\varphi_U\big(s(\xi_1)\cdots s(\xi_n)\big) = 2^{-n} \sum_{(\{\beta_i,\gamma_i\})\in \mathbb{NC}(n),\beta_i<\gamma_i} \prod_{k=1}^{n/2} \left\langle \xi_{\beta_k}, \xi_{\gamma_k}\right\rangle_U. \qquad (17)$$

for n even and is zero otherwise. Here $\mathrm{NC}(2p)$ stands for all the non-crossing pairings of the set $\{1, \dots, 2p\}$, i.e. pairings for which whenever $a < b < c < d$, and a, c are in the same class, then b, d are not in the same class. The total number of such pairings is given by the p-th Catalan number

$$C_p = \frac{1}{p+1}\binom{2p}{p}.$$

Recall that a continuous φ-preserving action (σ_t) of $\mathbf{R}$ on a von Neumann algebra $\mathcal{M}$ endowed with a f.n. state φ is said to be φ-mixing if for any $x, y \in \mathcal{M}$ with $\varphi(x) = \varphi(y) = 0$,

$$\varphi(\sigma_t(x)y) \to 0, \text{ as } |t| \to \infty. \qquad (2)$$

Proposition (4.2.8)[173]: Let $\mathcal{M} = \Gamma(H_{\mathbf{R}}, U_t)''$ be any free Araki-Woods factor and let φ_U be the free quasi-free state. Then

$$(U_t) \text{ is mixing } \Leftrightarrow \left(\sigma_t^{\varphi_U}\right) \text{ is } \varphi_U - \text{mixing}.$$

Proof. We prove both directions.

$\Leftarrow$ For any $\xi, \eta \in H_{\mathbf{R}}, \varphi_U(s(\xi)) = \varphi_U(s(\eta)) = 0$. Moreover,

$$\begin{aligned} \langle U_t\xi, \eta\rangle_U &= 4\varphi_U(s(U_t\xi)s(\eta)) \\ &= 4\varphi_U\big(\sigma_{-t}^{\varphi_U}(s(\xi))s(\eta)\big) \to 0, \text{ as } |t| \to \infty. \end{aligned}$$

It follows that (U_t) is mixing.

$\Rightarrow$ One needs to show that for any $x, y \in \mathcal{M}$,

$$\lim_{|t|\to\infty} \varphi_U\big(\sigma_t^{\varphi_U}(x)y\big) = \varphi_U(x)\varphi_U(y).$$

Note that

$$\mathrm{span}\{1, s(\xi_1)\cdots s(\xi_n): n \geq 1, \xi_1, \dots, \xi_n \in H_{\mathbf{R}}\}$$

is a unital *-strongly dense $*$-subalgebra of $\mathcal{M}$. Using Kaplansky density theorem, it suffices to check Equation (18) for $x, y \in \mathcal{M}$ of the following form:

$$\begin{aligned} x &= s(\xi_1)\cdots s(\xi_p) \\ y &= s(\eta_1)\cdots s(\eta_q). \end{aligned}$$

Assume that $p + q$ is odd. Then p or q is odd and we have $\varphi_U\big(\sigma_t^{\varphi_U}(x)y\big) = 0 = \varphi_U(x)\varphi_U(y)$ for any $t \in \mathbf{R}$.

Assume now that $p + q$ is even.

(a) Suppose that p, q are odd and write $p = 2k + 1, q = 2l + 1$. Then

$$\begin{aligned}\varphi_U\left(\sigma_t^{\varphi_U}(x)y\right) &= \varphi_U\left(s(U_{-t}\xi_1)\cdots s(U_{-t}\xi_{2k+1})s(\eta_1)\cdots s(\eta_{2l+1})\right)\\ &= 2^{-2(k+l+1)}\sum_{(\{\beta_i,\gamma_i\})\in NC(2(k+l+1)),\beta_i<\gamma_i}\ \prod_{j=1}^{k+l+1}\left\langle h_{\beta_j},h_{\gamma_j}\right\rangle_U,\end{aligned}$$

where the letter h stands for $U_{-t}\xi$ or η. Notice that since $2k+1$ and $2l+1$ are odd, for any non-crossing pairing $(\{\beta_i,\gamma_i\})\in \mathrm{NC}(2(k+l+1))$, there must exist some $j\in\{1,\dots,k+l+1\}$ such that $\left\langle h_{\beta_j},h_{\gamma_j}\right\rangle=\left\langle U_{-t}\xi_{\beta_j},\eta_{\gamma_j}\right\rangle$. Since we assumed that (U_t) is mixing, it follows that $\varphi_U\left(\sigma_t^{\varphi_U}(x)y\right)\to 0=\varphi_U(x)\varphi_U(y)$, as $|t|\to\infty$.

(b) Suppose that p,q are even and write $p=2k, q=2l$. Then

$$\begin{aligned}\varphi_U\left(\sigma_t^{\varphi_U}(x)y\right) &= \varphi_U\left(s(U_{-t}\xi_1)\cdots s(U_{-t}\xi_{2k})s(\eta_1)\cdots s(\eta_{2l})\right)\\ &= 2^{-2(k+l)}\sum_{(\{\beta_i,\gamma_i\})\in NC(2(k+l)),\beta_i<\gamma_i}\ \prod_{j=1}^{k+l}\left\langle h_{\beta_j},h_{\gamma_j}\right\rangle_U,\end{aligned}$$

where the letter h stands for $U_{-t}\xi$ or η. Note that for a non-crossing pairing $\nu=(\{\beta_i,\gamma_i\})\in \mathrm{NC}(2(k+l))$ such that an element of $\{1,\dots,2k\}$ and an element of $\{1,\dots,2l\}$ are in the same class, the proof of (a) yields that the corresponding product $\prod_{j=1}^{k+l}\left\langle h_{\beta_j},h_{\gamma_j}\right\rangle_U$ goes to 0, as $|t|\to\infty$. Thus, we just need to sum up over the noncrossing pairings ν of the form $\nu_1\times\nu_2$, where ν_1 is a non-crossing pairing on the set $\{1,\dots,2k\}$ and ν_2 is a non-crossing pairing on the set $\{1,\dots,2l\}$. Consequently, we get $\varphi_U\left(\sigma_t^{\varphi_U}(x)y\right)\to\varphi_U(x)\varphi_U(y)$, as $|t|\to\infty$. Therefore, $\left(\sigma_t^{\varphi U}\right)$ is mixing.

Proposition (4.2.9)[173]: Let $\mathcal{M}=\Gamma(H_{\mathrm{R}},U_t)''$. If (U_t) is mixing, then Connes' invariant $\tau(\mathcal{M})$ is the usual topology on R.

Proof. Let $\mathcal{M}=\Gamma(H_{\mathrm{R}},U_t)''$. Recall from Theorem (4.2.6) that $\tau(\mathcal{M})$ is the weakest topology on R that makes the map $t\mapsto U_t$ strongly continuous. Let (t_k) be a sequence in R such that $t_k\to 0$ w.r.t. the topology $\tau(\mathcal{M})$, as $k\to\infty$, i.e. $U_{t_k}\to Id$ strongly, as $k\to\infty$. Fix $\xi\in H_{\mathrm{R}}$, $\|\xi\|=1$. Since

$$\lim_{k\to\infty}\langle U_{t_k}\xi,\xi\rangle=1$$

and (U_t) is assumed to be mixing, it follows that (t_k) is necessarily bounded. Let $t\in$ R be any cluster point for the sequence (t_k). Then $U_t=\mathrm{Id}$. Since (U_t) is mixing, it follows that $t=0$. Therefore (t_k) converges to 0 w.r.t. the usual topology on R.

Let (B,τ) be a finite von Neumann algebra with a distinguished f.n. trace. Since τ is fixed, we simply denote $L^2(B,\tau)$ by $L^2(B)$. Let H be a right Hilbert B-module, i.e. H is a complex (separable) Hilbert space together with a normal *-representation $\pi\colon B^{\mathrm{op}}\to\mathrm{B}(H)$. For any $b\in B$, and $\xi\in H$, we shall simply write $\pi(b^{op})\xi=\xi b$. By the general theory, we know that there exists an isometry $v\colon H\to\ell^2\bar{\otimes}L^2(B)$ such that $v(\xi b)=v(\xi)b$, for any $\xi\in H, b\in B$. Since $p=vv^*$ commutes with the right B-action on $\ell^2\bar{\otimes}L^2(B)$, it follows that $p\in\mathbf{B}(\ell^2)\bar{\otimes}B$. Thus, as right B-modules, we have $H_B\simeq p\left(\ell^2\bar{\otimes}L^2(B)\right)_B$. On $\mathbf{B}(\ell^2)\bar{\otimes}B$, we define the following f.n. semifinite trace Tr (which depends on τ): for any $x=[x_{ij}]_{i,j}\in\left(\mathbf{B}(\ell^2)\bar{\otimes}B\right)_+$,

$$\mathrm{Tr}\left([x_{ij}]_{i,j}\right)=\sum_i\tau(x_{ii}).$$

We set $\dim(H_B) = \mathrm{Tr}(vv^*)$. Note that the dimension of H depends on τ but does not depend on the isometry v. Indeed take another isometry $: H \to \ell^2 \bar{\otimes} L^2(B)$, satisfying $w(\xi b) = w(\xi)b$, for any $\xi \in H, b \in B$. Note that $vw^* \in \mathbf{B}(\ell^2) \bar{\otimes} B$ and $w^*w = v^*v = 1$. Thus, we have

$$\mathrm{Tr}(vv^*) = \mathrm{Tr}(vw^*wv^*) = \mathrm{Tr}(wv^*vw^*) = \mathrm{Tr}(ww^*).$$

Assume that $\dim(H_B) < \infty$. Then for any $\varepsilon > 0$, there exists a central projection $z \in \mathcal{Z}(B)$, with $\tau(z) \geq 1 - \varepsilon$, such that the right B-module Hz is finitely generated, i.e. of the form $pL^2(B)^{\oplus n}$ for some projection $p \in \mathbf{M}_n(\mathbf{C}) \otimes B$. The non-normalized trace on $\mathbf{M}_n(\mathbf{C})$ will be denoted by Tr_n. For simplicity, we shall denote $B^n := \mathbf{M}_n(\mathbf{C}) \otimes B$.

In [52],[51], Popa introduced a powerful tool to prove the unitary conjugacy of two von Neumann subalgebras of a tracial von Neumann algebra (M, τ). If $A, B \subset (M, \tau)$ are two (possibly non-unital) von Neumann subalgebras, denote by $1_A, 1_B$ the units of A and B. Note that we endow the finite von Neumann algebra B with the trace $\tau(1_B \cdot 1_B)/\tau(1_B)$.

Theorem (4.2.10)[173]: (Popa, [52],[51]). Let $A, B \subset (M, \tau)$ be two (possibly non-unital) embeddings. The following are equivalent:

1. There exist $n \geq 1$, a (possibly non-unital) *-homomorphism $\psi: A \to B^n$ and a nonzero partial isometry $v \in \mathbf{M}_{1,n}(\mathbf{C}) \otimes 1_A M 1_B$ such that $xv = v\psi(x)$, for any $x \in A$.
2. The bimodule ${}_A L^2(1_A M 1_B)_B$ contains a non-zero sub-bimodule ${}_A H_B$ which satisfies $\dim(H_B) < \infty$.
3. There is no sequence of unitaries (u_k) in A such that $\|E_B(a^* u_k b)\|_2 \to 0$, as $k \to \infty$, for any $a, b \in 1_A M 1_B$.

If one of the previous equivalent conditions is satisfied, we shall say that A embeds into B inside M and denote $A \preceq_M B$.

Definition (4.2.11)[173]: (Popa & Vaes, [183]). Let $A \subset B \subset (N, \tau)$ be an inclusion of finite von Neumann algebras. We say that $B \subset N$ is weakly mixing through A if there exists a sequence of unitaries (u_k) in A such that

$$\|E_B(a^* u_k b)\|_2 \to 0, \text{ as } k \to \infty, \forall a, b \in N \ominus B.$$

The following result will be a crucial tool: it will allow us to control the relative commutant $A' \cap N$ of certain subalgebras A of a given von Neumann algebra N.

Theorem (4.2.12)[173]: (Popa, [52]). Let (N, τ) be a finite von Neumann algebra and $A \subset B \subset N$ be von Neumann subalgebras. Assume that $B \subset N$ is weakly mixing through A. Then for any sub-bimodule ${}_A H_B$ of ${}_A L^2(N)_B$ such that $\dim(H_B) < \infty$, one has $H \subset L^2(B)$. In particular, $A' \cap N \subset B$.

We will need to use Popa's intertwining techniques for semifinite von Neumann algebras. See Section 2 of [176] where such techniques were developed. Namely, let (M, Tr) be a von Neumann algebra endowed with a faithful normal semifinite trace Tr. We shall simply denote by $L^2(M)$ the $M - M$ bimodule $L^2(M, \mathrm{Tr})$, and by $\|\cdot\|_{2,\mathrm{Tr}}$ the L^2-norm associated with the trace Tr. We will use quite often the following inequality:

$$\| x\eta y \|_{2,\mathrm{Tr}} \leq \| x \|_\infty \| y \|_\infty \| \eta \|_{2,\mathrm{Tr}}, \forall \eta \in L^2(M), \forall x, y \in M,$$

where $\|\cdot\|_\infty$ denotes the operator norm. We shall say that a projection $p \in M$ is Tr-finite if $\mathrm{Tr}(p) < \infty$. Note that a non-zero Tr-finite projection p is necessarily finite and $\mathrm{Tr}(p \cdot p)/\mathrm{Tr}(p)$ is a f.n. (finite) trace on pMp. Remind that for any projections $p, q \in M$, we have $p \vee q - p \sim q - p \wedge q$. Then it follows that for any Tr-finite projections $p, q \in M, p \vee q$ is still Tr-finite and $\mathrm{Tr}(p \vee q) = \mathrm{Tr}(p) + \mathrm{Tr}(q) - \mathrm{Tr}(p \wedge q)$.

Note that if a sequence (x_k) in M converges to 0 strongly, as $k \to \infty$, then for any non-zero Tr-finite projection $q \in M$, $\|x_k q\|_{2,\,\mathrm{Tr}} \to 0$, as $k \to \infty$. Indeed,

$$\begin{aligned} x_k \to 0 \text{ strongly in } M \quad &\Leftrightarrow x_k^* x_k \to 0 \text{ weakly in } M \\ &\Rightarrow q x_k^* x_k q \to 0 \text{ weakly in } qMq \\ &\Rightarrow \mathrm{Tr}(q x_k^* x_k q) \to 0 \\ &\Rightarrow \|x_k q\|_{2,\mathrm{Tr}} \to 0. \end{aligned}$$

Moreover, there always exists an increasing sequence of Tr-finite projections (p_k) in M such that $p_k \to 1$ strongly, as $k \to \infty$.

Theorem (4.2.13)[173]:([176]). Let (M, Tr) be a semifinite von Neumann algebra. Let $B \subset M$ be a von Neumann subalgebra such that $\mathrm{Tr}_{|B}$ is still semifinite. Denote by $E_B: M \to B$ the unique Tr-preserving faithful normal conditional expectation. Let $q \in M$ be a non-zero Trfinite projection. Let $A \subset qMq$ be a von Neumann subalgebra. The following conditions are equivalent:

1. There exists a Tr-finite projection $p \in B, p \neq 0$, such that the bimodule ${}_A L^2(qMp)_{pBp}$ contains a non-zero sub-bimodule ${}_A H_{pBp}$ which satisfies $\dim\left(H_{pBp}\right) < \infty$, where pBp is endowed with the finite trace $\mathrm{Tr}(p \cdot p)/\mathrm{Tr}(p)$.
2. There is no sequence of unitaries (u_k) in A such that $E_B(x^* u_k y) \to 0$ strongly, as $k \to \infty$, for any $x, y \in qM$.

Definition (4.2.14)[173]: Under the assumptions of Theorem (4.2.13), if one of the equivalent conditions is satisfied, we shall still say that A embeds into B inside M and still denote $A \preceq_M B$.

Let H_{R} be a separable real Hilbert space ($\dim(H_{\mathrm{R}}) \geq 2$) and let (U_t) be an orthogonal representation of **R** on $H_{\mathbf{R}}$ that we assume to be neither trivial nor periodic. We set:

1. $\mathcal{M} = \Gamma(H_{\mathbf{R}}, U_t)''$ is the free Araki-Woods factor associated with $(H_{\mathbf{R}}, U_t), \varphi$ is the free quasi-free state and σ is the modular group of the state $\varphi. \mathcal{M}$ is necessarily a type III_1 factor since (U_t) is neither periodic nor trivial.
2. $M = \mathcal{M} \rtimes_\sigma \mathbf{R}$ is the continuous core of $\mathcal{M}$ and Tr is the semifinite trace associated with the state $\varphi. M$ is a II_∞ factor since $\mathcal{M}$ is a type III_1 factor.
3. Likewise $\widetilde{\mathcal{M}} = \Gamma(H_{\mathbf{R}} \oplus H_{\mathbf{R}}, U_t \oplus U_t)'', \tilde{\varphi}$ is the corresponding free quasi-free state and $\tilde{\sigma}$ is the modular group of $\tilde{\varphi}$.
4. $\widetilde{M} = \widetilde{\mathcal{M}} \rtimes_{\tilde{\sigma}} \mathbf{R}$ is the continuous core of $\widetilde{\mathcal{M}}$ and $\widetilde{Tr}$ is the f.n. semifinite trace associated with $\tilde{\varphi}$.

It follows from [15] that

$$\left(\widetilde{\mathcal{M}}, \tilde{\varphi}\right) \cong (\mathcal{M}, \varphi) * (\mathcal{M}, \varphi).$$

In the latter free product, we shall write $\mathcal{M}_1$ for the first copy of $\mathcal{M}$ and $\mathcal{M}_2$ for the second copy of $\mathcal{M}$. We regard $\mathcal{M} \subset \widetilde{\mathcal{M}}$ via the identification of $\mathcal{M}$ with $\mathcal{M}_1$.

Denote by (λ_t) the unitaries in $L(\mathbf{R})$ that implement the modular action σ on $\mathcal{M}$ (resp. $\tilde{\sigma}$ on $\widetilde{\mathcal{M}}$). Define the following faithful normal conditional expectations:

1. $E: M \to L(\mathbf{R})$ such that $E(x\lambda_t) = \varphi(x)\lambda_t$, for every $x \in \mathcal{M}$ and $t \in \mathbf{R}$;
2. $\tilde{E}: \widetilde{M} \to L(\mathbf{R})$ such that $\tilde{E}(x\lambda_t) = \tilde{\varphi}(x)\lambda_t$, for every $x \in \widetilde{\mathcal{M}}$ and $t \in \mathbf{R}$.

Then

$$\left(\widetilde{M}, \tilde{E}\right) \cong (M, E) *_{L(\mathbf{R})} (M, E).$$

Likewise, in the latter amalgamated free product, we shall write M_1 for the first copy of M and M_2 for the second copy of M. We regard $M \subset \widetilde{M}$ via the identification of M with M_1.

Notice that the conditional expectation E (resp. $\tilde{E}$) preserves the canonical semifinite trace Tr (resp. widetilder) associated with the state φ(resp. $\tilde{\varphi}$) (see [17]).

Consider the following orthogonal representation of $\mathbf{R}$ on $H_{\mathbf{R}} \oplus H_{\mathbf{R}}$:

$$V_s = \begin{pmatrix} \cos\left(\frac{\pi}{2}s\right) & -\sin\left(\frac{\pi}{2}s\right) \\ \sin\left(\frac{\pi}{2}s\right) & \cos\left(\frac{\pi}{2}s\right) \end{pmatrix}, \forall s \in \mathbf{R}.$$

Let (α_s) be the natural action on $(\widetilde{\mathcal{M}}, \tilde{\varphi})$ associated with (V_s): $\alpha_s = \mathrm{Ad}\big(\mathcal{F}(V_s)\big)$, for every $s \in \mathbf{R}$. In particular, we have

$$\alpha_s\left(s\begin{pmatrix}\xi\\ \eta\end{pmatrix}\right) = s\left(V_s\begin{pmatrix}\xi\\ \eta\end{pmatrix}\right), \forall s \in \mathbf{R}, \forall \xi, \eta \in H_{\mathbf{R}},$$

and the action (α_s) is $\tilde{\varphi}$-preserving. We can easily see that the representation (V_s) commutes with the representation $(U_t \oplus U_t)$. Consequently, (α_s) commutes with modular action $\tilde{\sigma}$. Moreover, $\alpha_1(x * 1) = 1 * x$, for every $a \in \mathcal{M}$. At last, consider the automorphism β defined on $(\widetilde{\mathcal{M}}, \tilde{\varphi})$ by:

$$\beta\left(s\begin{pmatrix}\xi\\ \eta\end{pmatrix}\right) = s\begin{pmatrix}\xi\\ -\eta\end{pmatrix}, \forall \xi, \eta \in H_{\mathbf{R}}.$$

It is straightforward to check that β commutes with the modular action $\tilde{\sigma}, \beta^2 = \mathrm{Id}, \beta_{|\mathcal{M}} = \mathrm{Id}_{\ |\mathcal{M}}$ and $\beta\alpha_s = \alpha_{-s}\beta, \forall s \in \mathbf{R}$. Since (α_s) and β commute with the modular action $\tilde{\sigma}$, one may extend (α_s) and β to $\widetilde{M}$ by $\alpha_{s|L(\mathbf{R})} = \mathrm{Id}_{L(\mathbf{R})}$, for every $s \in \mathbf{R}$ and $\beta_{|L(\mathbf{R})} = \mathrm{Id}_{L(\mathbf{R})}$.

Moreover (α_s, β) preserves the semifinite trace $\widehat{Tr}$. Let's summarize what we have done so far:

Proposition (4.2.15)[173]: The widetilder-preserving deformation (α_s, β) defined on $\widetilde{M}$ is s-malleable:

1. $\alpha_{s|L(\mathbf{R})} = \mathrm{Id}_{L(\mathbf{R})}$, for every $s \in \mathbf{R}$ and $\alpha_1\big(x *_{L(\mathbf{R})} 1\big) = 1 *_{L(\mathbf{R})} x$, for every $x \in M$.
2. $\beta^2 = \mathrm{Id}$ and $\beta_{|M} = \mathrm{Id}_{|M}$.
3. $\beta\alpha_s = \alpha_{-s}\beta$, for every $s \in \mathbf{R}$.

Denote by $E_M: \widetilde{M} \to M$ the canonical trace-preserving conditional expectation. Since $\widetilde{\mathrm{Tr}}_{|M} = \mathrm{Tr}$, we will simply denote by Tr the semifinite trace on $\widetilde{M}$. Remind that the smalleable deformation (α_s, β) automatically features a certain transversality property.

Proposition (4.2.16)[173]: (Popa, [182]). We have the following:

$$\|x - \alpha_{2s}(x)\|_{2,\mathrm{Tr}} \le 2\|\alpha_s(x) - E_M\big(\alpha_s(x)\big)\|_{2,\mathrm{Tr}}, \forall x \in L^2(M, \mathrm{Tr}), \forall s > 0. \quad (19)$$

The next proposition refered as the spectral gap property was first proved by Popa in [169] for free products of finite von Neumann algebras. We will need the following straightforward generalization:

Proposition (4.2.17)[173]: ([176]). We keep the same notation as before. Let $q \in M$ be a non-zero projection such that $\mathrm{Tr}(q) < \infty$. Let $Q \subset qMq$ be a von Neumann subalgebra with no amenable direct summand. Then for any free ultrafilter ω on $\mathbf{N}$, we have $Q' \cap (q\widetilde{M}q)^\omega \subset (qMq)^\omega$.

Let $q \in M$ be a non-zero projection such that $\mathrm{Tr}(q) < \infty$. Note that $\mathrm{Tr}(q \cdot q)/\mathrm{Tr}(q)$ is a finite trace on $q\widetilde{M}q$. If $Q \subset qMq$ has no amenable direct summand, then for any $\varepsilon > 0$, there exist $\delta > 0$ and a finite subset $F \subset \mathcal{U}(Q)$ such that for any $x \in (q\widetilde{M}q)_1$ (the unit ball w.r.t. the operator norm),

$$\| ux - xu \|_{2,\mathrm{Tr}} < \delta, \forall u \in F \Longrightarrow \|x - E_{qMq}(x)\|_{2,\mathrm{Tr}} < \varepsilon. \quad (20)$$

We will simply denote $ux - xu$ by $[u, x]$.

The following theorem is in some ways a reminiscence of a result of Ioana, Peterson & Popa, namely Theorem 4.3 of [156] and also Theorem 4.2 of [176]. The deformation/spectral gap rigidity strategy enables us to locate inside the core M of a free Araki-Woods factor the position of subalgebras $A \subset M$ with a large relative commutant $A' \cap M$.

Theorem (4.2.18)[173]: Let $\mathcal{M} = \Gamma(H_{\mathbf{R}}, U_t)''$ be a free Araki-Woods factor and $M = \mathcal{M} \rtimes_\sigma \mathbf{R}$ be its continuous core. Let $q \in L(\mathbf{R}) \subset M$ be a non-zero projection such that $\operatorname{Tr}(q) < \infty$. Let $Q \subset qMq$ be a von Neumann subalgebra with no amenable direct summand. Then $Q' \cap qMq \preceq_M L(\mathbf{R})$.

Proof. Let $q \in L(\mathbf{R})$ be a non-zero projection such that $\operatorname{Tr}(q) < \infty$. Let $Q \subset qMq$ be a von Neumann subalgebra with no amenable direct summand. Denote by $Q_0 = Q' \cap qMq$. We keep the notation introduced previously and regard $M \subset \widetilde{M} = M_1 *_{L(\mathbf{R})} M_2$ via the identification of M with M_1. Remind that $\alpha_{s|L(\mathbf{R})} = \operatorname{Id}_{L(\mathbf{R})}$, for every $s \in \mathbf{R}$. In particular $\alpha_s(q) = q$, for every $s \in \mathbf{R}$.

Step (1): Using the spectral gap condition and the transversality property of (α_t, β) to find $t > 0$ and a non-zero intertwiner v between Id and α_t.

Let $\varepsilon = \frac{1}{4} \| q \|_{2,\operatorname{Tr}}$. We know that there exist $\delta > 0$ and a finite subset $F \subset \mathcal{U}(Q)$, such that for every $x \in (q\widetilde{M}q)_1$,

$$\| [x, u] \|_{2,\operatorname{Tr}} \leq \delta, \forall u \in F \Longrightarrow \|x - E_{qMq}(x)\|_{2,\operatorname{Tr}} \leq \varepsilon.$$

Since $\alpha_t \to \operatorname{Id}$ pointwise $*$-strongly, as $t \to 0$, and since F is a finite subset of $Q \subset qMq$, we may choose $t = 1/2^k$ small enough ($k \geq 1$) such that

$$\max \left\{ \|u - \alpha_t(u)\|_{2,\operatorname{Tr}} : u \in F \right\} \leq \frac{\delta}{2}.$$

For every $x \in (Q_0)_1$ and every $u \in F \subset Q$, since $[u, x] = 0$, we have

$$\begin{aligned} \|[\alpha_t(x), u]\|_{2,\operatorname{Tr}} &= \|[\alpha_t(x), u - \alpha_t(u)]\|_{2,\operatorname{Tr}} \\ &\leq 2\|u - \alpha_t(u)\|_{2,\operatorname{Tr}} \\ &\leq \delta. \end{aligned}$$

Consequently, we get for every $x \in (Q_0)_1, \|\alpha_t(x) - E_{qMq}(\alpha_t(x))\|_{2,\operatorname{Tr}} \leq \varepsilon$. Using Proposition (4.2.16), we obtain for every $x \in (Q_0)_1$

$$\|x - \alpha_s(x)\|_{2,Tr} \leq \frac{1}{2} \| q \|_{2,Tr},$$

where $s = 2t$. Thus, for every $u \in \mathcal{U}(Q_0)$, we have

$$\begin{aligned} \|u^*\alpha_s(u) - q\|_{2,\operatorname{Tr}} &= \|u^*(\alpha_s(u) - u)\|_{2,Tr} \\ &\leq \|u - \alpha_s(u)\|_{2,\operatorname{Tr}} \\ &\leq \frac{1}{2} \| q \|_{2,\operatorname{Tr}}. \end{aligned}$$

Denote by $\mathcal{C} = \overline{\mathrm{c}}^w\{u^*\alpha_s(u) : u \in \mathcal{U}(Q_0)\} \subset qL^2(\widetilde{M})q$ the ultraweak closure of the convex hull of all $u^*\alpha_s(u)$, where $u \in \mathcal{U}(Q_0)$. Denote by a the unique element in $\mathcal{C}$ of minimal $\|\cdot\|_{2,\operatorname{Tr}}$– norm. Since $\| a - q \|_{2,\operatorname{Tr}} \leq 1/2 \| q \|_{2,\operatorname{Tr}}$, necessarily $a \neq 0$. Fix $u \in \mathcal{U}(Q_0)$. Since $u^*a\alpha_s(u) \in \mathcal{C}$ and $\|u^*a\alpha_s(u)\|_{2,\operatorname{Tr}} = \| a \|_{2,\operatorname{Tr}}$, necessarily $u^*a\alpha_s(u) = a$. Taking $v = \operatorname{pol}(a)$ the polar part of a, we have found a non-zero partial isometry $v \in q\widetilde{M}q$ such that

$$xv = v\alpha_s(x), \forall x \in Q_0. \tag{21}$$

Step (2): Proving $Q_0 \preceq_M L(\mathbf{R})$ using the malleability of (α_t, β). By contradiction, assume $Q_0 \npreceq_M L(\mathbf{R})$. The first task is to lift Equation (21) to $s = 1$. Note that it is enough to find a non-zero partial isometry $w \in q\tilde{M}q$ such that

$$xw = w\alpha_{2s}(x), \forall x \in Q_0.$$

Indeed, by induction we can go till $s = 1$ (because $= 1/2^{k-1}$). Remind that $\beta(z) = z$, for every $z \in M$. Note that $vv^* \in Q_0' \cap q\widetilde{M}q$. Since $Q_0 \npreceq_M L(\mathbf{R})$, we know from Theorem 2.4 in [176] that $Q_0' \cap q\widetilde{M}q \subset qMq$. In particular, $vv^* \in qMq$. Set $w = \alpha_s(\beta(v^*)v)$. Then,

$$\begin{aligned} ww^* &= \alpha_s(\beta(v^*)vv^*\beta(v)) \\ &= \alpha_s(\beta(v^*)\beta(vv^*)\beta(v)) \\ &= \alpha_s\beta(v^*v) \neq 0. \end{aligned}$$

Hence, w is a non-zero partial isometry in $q\widetilde{M}q$. Moreover, for every $x \in Q_0$,

$$\begin{aligned} w\alpha_{2s}(x) &= \alpha_s(\beta(v^*)v\alpha_s(x)) \\ &= \alpha_s(\beta(v^*)xv) \\ &= \alpha_s(\beta(v^*x)v) \\ &= \alpha_s(\beta(\alpha_s(x)v^*)v) \\ &= \alpha_s\beta\alpha_s(x)\alpha_s(\beta(v^*)v) \\ &= \beta(x)w \\ &= xw. \end{aligned}$$

Since by induction, we can go till $s = 1$, we have found a non-zero partial isometry $v \in q\widetilde{M}q$ such that

$$xv = v\alpha_1(x), \forall x \in Q_0. \tag{22}$$

Note that $v^*v \in \alpha_1(Q_0)' \cap qMq$. Moreover, since $\alpha_1 : q\widetilde{M}q \to q\widetilde{M}q$ is a *-automorphism, and $Q_0 \npreceq_M L(\mathbf{R})$, Theorem 2.4 in [176] gives

$$\begin{aligned} \alpha_1(Q_0)' \cap q\widetilde{M}q &= \alpha_1\big(Q_0' \cap q\widetilde{M}q\big) \\ &\subset \alpha_1(qMq). \end{aligned}$$

Hence $v^*v \in \alpha_1(qMq)$.

Since $Q_0 \npreceq_M L(\mathbf{R})$, we know that there exists a sequence of unitaries (u_k) in Q_0 such that $E_{L(\mathbf{R})}(x^*u_k y) \to 0$ strongly, as $k \to \infty$, for any $x, y \in qM$. We need to go further and prove the following:

Claim (4.2.19)[173]: $\forall a, b \in q\widetilde{M}q, \|E_{M_2}(a^*u_k b)\|_{2,\mathrm{Tr}} \to 0$, as $k \to \infty$.

Proof. Let $a, b \in (\widetilde{M})_1$ be either elements in $L(\mathbf{R})$ or reduced words with letters alternating from $M_1 \ominus L(\mathbf{R})$ and $M_2 \ominus L(\mathbf{R})$. Write $b = yb'$ with

1. $y = b$ if $b \in L(\mathbf{R})$;
2. $y = 1$ if b is a reduced word beginning with a letter from $M_2 \ominus L(\mathbf{R})$;
3. $y =$ the first letter of b otherwise.

Note that either $b' = 1$ or b' is a reduced word beginning with a letter from $M_2 \ominus L(\mathbf{R})$. Likewise write $a = a'x$ with

4. $x = a$ if $x \in L(\mathbf{R})$;
5. $x = 1$ if a is a reduced word ending with a letter from $M_2 \ominus L(\mathbf{R})$;
6. $x =$ the last letter of a otherwise.

Either $a' = 1$ or a' is a reduced word ending with a letter from $M_2 \ominus L(\mathbf{R})$. For any $z \in Q_0 \subset M_1, xzy - E_{L(\mathbf{R})}(xzy) \in M_1 \ominus L(\mathbf{R})$, so that

$$E_{M_2}(azb) = E_{M_2}\big(a'E_{L(\mathbf{R})}(xzy)b'\big).$$

Since $E_{L(\mathbf{R})}(xu_k y) \to 0$ strongly, as $k \to \infty$, it follows that $E_{M_2}(au_k b) \to 0$ strongly, as $k \to \infty$, as well. Thus, in the finite von Neumann algebra $q\widetilde{M}q$, we get $\|qE_{M_2}(au_k b)q\|_{2,\mathrm{Tr}} \to 0$ as $k \to \infty$.

Note that

$$\mathcal{A} := \mathrm{span}\{L(\mathbf{R}), (M_{i_1} \ominus L(\mathbf{R})) \cdots (M_{i_n} \ominus L(\mathbf{R})) : n \geq 1, i_1 \neq \cdots \neq i_n\}$$

is a unital *-strongly dense *-subalgebra of $\widetilde{M}$. What we have shown so far is that for any $a, b \in \mathcal{A}$, $\|qE_{M_2}(au_k b)q\|_{2,\mathrm{Tr}} \to 0$, as $k \to \infty$. Let now $a, b \in (\widetilde{M})_1$. By Kaplansky density theorem, let (a_i) and (b_j) be sequences in $(\mathcal{A})_1$ such that $a_i \to a$ and $b_j \to b$ strongly. Recall that (u_k) is a sequence in $Q_0 \subset q\widetilde{M}q$. We have

$$\begin{aligned}
&\|qE_{M_2}(au_k b)q\|_{2,\mathrm{Tr}} \\
&\leq \|qE_{M_2}(a_i u_k b_j)q\|_{2,\mathrm{Tr}} + \left\|qE_{M_2}\left(a_i u_k (b - b_j)\right)q\right\|_{2,\mathrm{Tr}} \\
&\quad + \left\|qE_{M_2}\left((a - a_i)u_k b_j\right)q\right\|_{2,\mathrm{Tr}} + \left\|qE_{M_2}\left((a - a_i)u_k (b - b_j)\right)q\right\|_{2,\mathrm{Tr}} \\
&\leq \|qE_{M_2}(a_i u_k b_j)q\|_{2,\mathrm{Tr}} + \|qa_i u_k (b - b_j)q\|_{2,\mathrm{Tr}} \\
&\quad + \|q(a - a_i)u_k b_j q\|_{2,\mathrm{Tr}} + \|q(a - a_i)u_k (b - b_j)q\|_{2,\mathrm{Tr}} \\
&\leq \|qE_{M_2}(a_i u_k b_j)q\|_{2,\mathrm{Tr}} + 3\|(b - b_j)q\|_{2,\mathrm{Tr}} + \|q(a - a_i)q\|_{2,\mathrm{Tr}}
\end{aligned}$$

Fix $\varepsilon > 0$. Since $a_i \to a$ and $b_j \to b$ strongly, let i_0, j_0 large enough such that

$$3\|(b - b_{j_0})q\|_{2,\mathrm{Tr}} + \|q(a - a_{i_0})q\|_{2,\mathrm{Tr}} \leq \varepsilon/2.$$

Now let $k_0 \in \mathbf{N}$ such that for any $k \geq k_0$

$$\|qE_{M_2}(a_{i_0} u_k b_{j_0})q\|_{2,T_r} \leq \varepsilon/2.$$

We finally get $\|qE_{M_2}(au_k b)q\|_{2,Tr} \leq \varepsilon$, for any $k \geq k_0$, which finishes the proof of the claim.

We remind that for any $x \in Q_0, v^*xv = \alpha_1(x)v^*v$. Moreover, $v^*v \in \alpha_1(qMq) \subset qM_2q$. So, for any $x \in Q_0, v^*xv \in qM_2q$. Since $\alpha_1(u_k) \in \mathcal{U}(qM_2q)$, we get

$$\begin{aligned}
\|v^*v\|_{2,\mathrm{Tr}} = \|\alpha_1(u_k)v^*v\|_{2,\mathrm{Tr}} &= \|E_{M_2}(\alpha_1(u_k)v^*v)\|_{2,\mathrm{Tr}} \\
&= \|E_{M_2}(v^*u_k v)\|_{2,\mathrm{Tr}} \to 0.
\end{aligned}$$

Thus $v = 0$, which is a contradiction.

Corollary (4.2.20)[173]: Let $\mathcal{M} = \Gamma(H_{\mathbf{R}}, U_t)''$ be a free Araki-Woods factor of type III $_1$. Then the continuous core $M = \mathcal{M} \rtimes_\sigma \mathbf{R}$ is a semisolid II_∞ factor. Since M is non-amenable, M is always a prime factor.

Proof. Let $q \in L(\mathbf{R})$ be a non-zero projection such that $\mathrm{Tr}(q) < \infty$. Denote by $N = qMq$ the corresponding II_1 factor and by $\tau = \mathrm{Tr}(q \cdot q)/\mathrm{Tr}(q)$ the canonical trace on N. By contradiction, assume that N is not semisolid. Then there exists $Q \subset N$ a nonamenable von Neumann subalgebra such that the relative commutant $Q' \cap N$ is of type II_1. Write $z \in \mathcal{Z}(Q)$ for the maximal projection such that Qz is amenable. Then $1 - z \neq 0$, the von Neumann algebra $Q(1 - z)$ has no amenable direct summand and $(Q' \cap N)(1 - z)$ is still of type II_1. We may choose a projection $q_0 \in Q(1 - z)$ such that $\tau(q_0) = 1/n$. Since N is a II_1 factor, we may replace Q by $\mathbf{M}_n(\mathbf{C}) \otimes q_0 Q q_0$, so that we may assume $Q \subset N$ has no amenable direct summand and $Q' \cap N$ is still of type II_1.

If we apply Theorem (4.2.18), it follows that $Q' \cap N \preceq_M L(\mathbf{R})$. We get a contradiction because $Q' \cap N$ is of type II_1 and $L(\mathbf{R})$ is of type I.

It follows from [12] that for any type III_1 factor $\mathcal{M}$, if the continuous core $M = \mathcal{M} \rtimes_\sigma \mathbf{R}$ is full, then Connes' invariant $\tau(\mathcal{M})$ is the usual topology on $\mathbf{R}$. Let now $\mathcal{M} = \Gamma(H_\mathbf{R}, U_t)''$ be a free Araki-Woods factor associated with (U_t) an almost periodic representation. Denote by $S_U \subset \mathbf{R}_+^*$ the (countable) subgroup generated by the point spectrum of (U_t). Then $\tau(\mathcal{M})$ is strictly weaker than the usual topology. More precisely, the completion of $\mathbf{R}$ w.r.t. the topology $\tau(\mathcal{M})$ is the compact group $\widehat{S_U}$ dual of S_U (see [3]). Therefore in this case, for any non-zero projection $q \in L(\mathbf{R})$ such that $\mathrm{Tr}(q) < \infty$, the II_1 factor qMq is semisolid, by Theorem (4.2.18), and has property Γ of Murray & von Neumann by the above remark.

The solidity of the continuous core M forces the centralizers on $\mathcal{M}$ to be amenable. Indeed, fix ψ any f.n. state on $\mathcal{M}$. Assume that the continuous core $M \simeq \mathcal{M} \rtimes_\sigma \psi\mathbf{R}$ is solid. Choose a non-zero projection $q \in L(\mathbf{R})$ such that $\mathrm{Tr}(q) < \infty$. Since $L(\mathbf{R})q$ is diffuse in $q(\mathcal{M} \rtimes_{\sigma^\psi} \mathbf{R})q$, its relative commutant must be amenable. In particular $\mathcal{M}^\psi \bar{\otimes} L(\mathbf{R})q$ is amenable. Thus, $\mathcal{M}^\psi$ is amenable.

Note that if the orthogonal representation (U_t) contains a $\frac{2\pi}{1\log\lambda}$-periodic subrepresentation $(V_t^\lambda), 0 < \lambda < 1$, of the form

$$V_t^\lambda = \begin{pmatrix} \cos(t\log\lambda) & -\sin(t\log\lambda) \\ \sin(t\log\lambda) & \cos(t\log\lambda) \end{pmatrix},$$

then the free Araki-Woods factor $\mathcal{M} = \Gamma(H_\mathbf{R}, U_t)''$ freely absorbs $L(\mathbf{F}_\infty)$ (see [15]) :

$$(\mathcal{M}, \varphi_U) * (L(\mathbf{F}_\infty), \tau) \cong (\mathcal{M}, \varphi_U).$$

In particular, the centralizer of the free quasi-free state $\mathcal{M}^{\varphi_U}$ is non-amenable since it contains $L(\mathbf{F}_\infty)$. Therefore, whenever (U_t) contains a periodic subrepresentation of the form (V_t^λ) for some $0 < \lambda < 1$, the continuous core of $\Gamma(H_\mathbf{R}, U_t)''$ is semisolid by Theorem (4.2.18) but can never be solid. However, when (U_t) is assumed to be mixing, we get solidity of the continuous core. Indeed in that case, we can control the relative commutant $A' \cap M$ of diffuse subalgebras $A \subset L(\mathbf{R}) \subset M$, where M is the continuous core of the free Araki-Woods factor associated with (U_t). Thus, the next theorem can be regarded as the analog of a result of Popa, namely Theorem 3.1 of [52] (see also Theorem D.4 in [189]).

Theorem (4.2.21)[173]: Let (U_t) be a mixing orthogonal representation of $\mathbf{R}$ on the real Hilbert space $H_\mathbf{R}$. Denote by $\mathcal{M} = \Gamma(H_\mathbf{R}, U_t)''$ the corresponding free Araki-Woods factor and by $M = \mathcal{M} \rtimes_\sigma \mathbf{R}$ its continuous core. Let $k \geq 1$ and let $q \in \mathbf{M}_k(\mathbf{C}) \otimes L(\mathbf{R})$ be a non-zero projection such that $T := (\mathrm{Tr}_k \otimes \mathrm{Tr})(q) < \infty$. Write $L(\mathbf{R})^T := q(\mathbf{M}_k(\mathbf{C}) \otimes L(\mathbf{R}))q$ and $M^T := q(\mathbf{M}_k(\mathbf{C}) \otimes M)q$. Let $A \subset L(\mathbf{R})^T$ be a diffuse von Neumann subalgebra.

Then for any sub-bimodule ${}_A H_{L(\mathbf{R})^T}$ of ${}_A L^2(M^T)_{L(\mathbf{R})^T}$ such that $\dim(H_{L(\mathbf{R})^T}) < \infty$, one has $H \subset L^2(L(\mathbf{R})^T)$. In particular $A' \cap M^T \subset L(\mathbf{R})^T$.

Proof. As usual, denote by (λ_t) the unitaries in $L(\mathbf{R})$ that implement the modular action σ on $\mathcal{M}$. Let $\Phi: L^\infty(\mathbf{R}) \to L(\mathbf{R})$ be the Fourier Transform so that $\Phi(e^{it}) = \lambda_t$, for every $t \in \mathbf{R}$. Let $T > 0$ and denote by $q = \Phi(\chi_{[0,T]})$. Notice that $L^\infty(\mathbf{R})\chi_{[0,T]} \cong L^\infty[0, T]$ and that

$$\mathrm{span}\left\{\sum_{k\in F} c_k e^{i\frac{2\pi}{T}k\cdot}\chi_{[0,T]} : F \subset \mathbf{Z} \text{ finite subset, } c_k \in \mathbf{C}, \forall k \in F\right\}$$

is a unital *-strongly dense *-subalgebra of $L^\infty(\mathbf{R})\chi_{[0,T]}$. Thus, using the isomorphism Φ, we get that

$$\mathcal{A} := \text{span}\left\{\sum_{k\in F} c_k\lambda_{\frac{2\pi}{T}k}q : F \subset \mathbf{Z} \text{ finite subset}, c_k \in \mathbf{C}, \forall k \in F\right\}$$

is a unital *-strongly dense *-subalgebra of $L(\mathbf{R})q$. Let (u_n) be bounded sequence in $L(\mathbf{R})q$ such that $u_n \to 0$ weakly, as $n \to \infty$, and $\|u_n\|_\infty \leq 1$, for every $n \in \mathbf{N}$. Using Kaplansky density theorem together with a standard diagonal process, choose a sequence $y_n \in \mathcal{A}$ such that $\|y_n\|_\infty \leq 1$, for every $n \in \mathbf{N}$, and $\|u_n - y_n\|_{2,\mathrm{Tr}} \to 0$, as $n \to \infty$. We will write $y_n = z_n q$ with

$$z_n = \sum_{k\in F_n} c_{k,n}\lambda_{\frac{2\pi}{T}k},$$

where $F_n \subset \mathbf{Z}$ is finite, $c_{k,n} \in \mathbf{C}$, for any $k \in F_n$ and any $n \in \mathbf{N}$. Using the T-periodicity, we have for any $n \in \mathbf{N}$,

$$\begin{aligned}\|z_n\|_\infty &= \|\Phi^{-1}(z_n)\|_\infty \\ &= \operatorname*{ess\,sup}_{x\in\mathbf{R}}\left|\sum_{k\in F_n} c_{k,n}e^{i\frac{2\pi}{T}kx}\right| \\ &= \operatorname*{ess\,sup}_{x\in[0,T]}\left|\sum_{k\in F_n} c_{k,n}e^{i\frac{2\pi}{T}kx}\right| \\ &= \left\|\Phi^{-1}(z_n)\chi_{[0,T]}\right\|_\infty \\ &= \|y_n\|_\infty \leq 1.\end{aligned}$$

Thus, the sequence (z_n) is uniformly bounded.
The first step of the proof consists in proving the following:

$$\left\|E_{L(\mathbf{R})q}(au_nb)\right\|_{2,\mathrm{Tr}} \to 0, \text{ as } n \to \infty, \forall a, b \in qMq \cap \ker\left(E_{L(\mathbf{R})q}\right).$$

Equivalently, we need to show that

$$\left\|qE_{L(\mathbf{R})}(au_nb)q\right\|_{2,\mathrm{Tr}} \to 0, \text{as } n \to \infty, \forall a, b \in \ker\left(E_{L(\mathbf{R})}\right). \qquad (23)$$

The first step of the proof is now divided in three different claims that will lead to proving (23). First note that

$$\mathcal{E} := \text{span}\left\{\sum_{t\in F} x_t\lambda_t : F \subset \mathbf{R} \text{ finite subset}, x_t \in \mathcal{M} \text{ with } \varphi(x_t) = 0, \forall t \in F\right\}$$

is *-strongly dense in ker $\left(E_{L(\mathbf{R})}\right)$ by Kaplansky density theorem. We first prove the following:

Claim (4.2.22)[173]: If $\left\|qE_{L(\mathbf{R})}(xu_ny)q\right\|_{2,\mathrm{Tr}} \to 0$, as $n \to \infty, \forall x, y \in \mathcal{M}$ with $\varphi(x) = \varphi(y) = 0$, then (23) is satisfied.

Proof. Assume $\left\|qE_{L(\mathbf{R})}(xu_ny)q\right\|_{2,\mathrm{Tr}} \to 0$, as $n \to \infty, \forall x, y \in \mathcal{M}$ with $\varphi(x) = \varphi(y) = 0$. First take $a \in \mathcal{E}$ that we write $a = \sum_{s\in F} x_s\lambda_s$, with $F \subset \mathbf{R}$ finite subset, such that $x_s \in \mathcal{M}, \varphi(x_s) = 0$, for every $s \in F$. Then take $b \in \ker\left(E_{L(\mathbf{R})}\right)$ and let $\left(b_j\right)_{j\in J}$ bea sequence in $\mathcal{E}$ such that $b - b_j \to 0$ *-strongly, as $j \to \infty$. Since $\|u_n\|_\infty \leq 1$, we get for any $n \in \mathbf{N}$ and any $j \in J$,

$$\left\|qE_{L(\mathbf{R})}(au_nb)q\right\|_{2,\,\mathrm{Tr}}$$

$$\leq \|qE_{L(\mathbf{R})}(au_nb_j)q\|_{2,\mathrm{Tr}} + \left\|qE_{L(\mathbf{R})}\left(au_n(b-b_j)\right)q\right\|_{2,\mathrm{Tr}}$$
$$\leq \|qE_{L(\mathbf{R})}(au_nb_j)q\|_{2,\mathrm{Tr}} + \| a \|_\infty \|(b-b_j)q\|_{2,\mathrm{Tr}}$$

Fix $\varepsilon > 0$. Since $b - b_j \to 0$ $*$-strongly, as $j \to \infty$, fix $j_0 \in J$ such that $\| a \|_\infty \|(b - b_{j_0})q\|_{2,\mathrm{Tr}} \leq \varepsilon/2$. Write $b_{j_0} = \sum_{t\in F'} y_t\lambda_t$, with $F' \subset \mathbf{R}$ finite subset, such that $y_t \in \mathcal{M}, \varphi(y_t) = 0$, for every $t \in F'$. Therefore, for any $n \in \mathbf{N}$,

$$\begin{aligned}\|qE_{L(\mathbf{R})}(au_nb_{j_0})q\|_{2,Tr} &\leq \sum_{(s,t)\in F\times F'} \|qE_{L(\mathbf{R})}(x_s\lambda_s u_n y_t\lambda_t)q\|_{2,\mathrm{Tr}} \\ &= \sum_{(s,t)\in F\times F'} \|\lambda_s qE_{L(\mathbf{R})}(\sigma_{-s}(x_s)u_n y_t)q\lambda_t\|_{2,\mathrm{Tr}} \\ &= \sum_{(s,t)\in F\times F'} \|qE_{L(\mathbf{R})}(\sigma_{-s}(x_s)u_n y_t)q\|_{2,\mathrm{Tr}}.\end{aligned}$$

Since $\varphi(\sigma_{-s}(x_s)) = \varphi(y_t) = 0$, for any $(s,t) \in F \times F'$, using the assumption of the claim, there exists $n_0 \in \mathbf{N}$ large enough such that for any $n \geq n_0$, $\|qE_{L(\mathbf{R})}(au_nb_{j_0})q\|_{2,Tr} \leq \varepsilon/2$.

Thus, for any $n \geq n_0$, $\|qE_{L(\mathbf{R})}(au_nb)q\|_{2,\mathrm{Tr}} \leq \varepsilon$. This proves that for any $a \in \mathcal{E}$ and any $b \in \ker(E_{L(\mathbf{R})})$, $\|qE_{L(\mathbf{R})}(au_nb)q\|_{2,\mathrm{Tr}} \to 0$, as $n \to \infty$. If we do the same thing by approximating $a \in \ker(E_{L(\mathbf{R})})$ with elements in $\mathcal{E}$, using the fact that $u_n \in (L(\mathbf{R})q)_1$, we finally get the claim.

We now replace the sequence (u_n) by (z_n), use the mixing property of the modular action σ and prove the following:

Claim (4.2.23)[173]L: $\forall a, b \in (\mathcal{M})_1$ with $\varphi(a) = \varphi(b) = 0$, $\|qE_{L(\mathbf{R})}(az_nb)q\|_{2,\mathrm{Tr}} \to 0$, as $n \to \infty$.

Proof. Fix $a, b \in (\mathcal{M})_1$ such that $\varphi(a) = \varphi(b) = 0$. Fix $\varepsilon > 0$. For any $n \in \mathbf{N}$, we have

$$\begin{aligned}\|qE_{L(\mathbf{R})}(az_nb)q\|_{2,\mathrm{Tr}}^2 &= \left\|\sum_{k\in F_n} c_{k,n}\varphi\left(a\sigma_{\frac{2\pi}{T}k}(b)\right)\lambda_{\frac{2\pi}{T}k}q\right\|_{2,\mathrm{Tr}}^2 \\ &= \mathrm{Tr}(q)\sum_{k\in F_n} |c_{k,n}|^2 \left|\varphi\left(a\sigma_{\frac{2\pi}{T}k}(b)\right)\right|^2.\end{aligned}$$

Moreover for any $n \in \mathbf{N}$,

$$\mathrm{Tr}(q)\sum_{k\in F_n} |c_{k,n}|^2 = \|z_nq\|_{2,\mathrm{Tr}}^2 \leq \mathrm{Tr}(q)\,\|z_nq\|_\infty^2 \leq T.$$

Since the modular group σ is φ-mixing (because (U_t) is assumed to be mixing), there exists a finite subset $K \subset \mathbf{Z}$ such that for any $k \in \mathbf{Z} \setminus K$, $\left|\varphi\left(a\sigma_{\frac{2\pi}{T}k}(b)\right)\right| \leq \varepsilon/\sqrt{2T}$. Thus,

$$\|qE_{L(\mathbf{R})}(az_nb)q\|_{2,\mathrm{Tr}} \leq \left\|\sum_{k\in K\cap F_n} c_{k,n}\lambda_{\frac{2\pi}{T}k}q\right\|_{2,\mathrm{Tr}} + \varepsilon/2.$$

Since $u_n - z_nq \to 0$ strongly and $u_n \to 0$ weakly, as $n \to \infty$, it follows that $z_nq \to 0$ weakly, as $n \to \infty$. In particular there exists n_0 large enough such that for any $n \geq n_0$, for any $k \in K \cap F_n$, $|c_{k,n}| \leq \varepsilon/(2|K| \, \| q \|_{2,Tr})$. Thus, for any $n \geq n_0$

$$\|qE_{L(\mathbf{R})}(az_nb)q\|_{2,\mathrm{Tr}} \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This proves that $\|qE_{L(\mathbf{R})}(az_nb)q\|_{2,T\mathrm{Tr}} \to 0$, as $n \to \infty$.

The last claim consists in going back to the sequence (u_n) and proving the following:

Claim (4.2.24)[173]: $\forall a, b \in (\mathcal{M})_1$ with $\varphi(a) = \varphi(b) = 0$, $\|qE_{L(\mathbf{R})}(au_nb)q\|_{2,\mathrm{Tr}} \to 0$, as $n \to \infty$.

Proof. Applying once more Kaplansky density theorem, we can find a sequence $(q_i)_{i\in I}$ in $L(\mathbf{R})$ such that

1. $q_i = \sum_{t\in F_i} d_t\lambda_t$, with $F_i \subset \mathbf{R}$ finite subset, $d_t \in \mathbf{C}$, for any $t \in F_i$ and for any $i \in I$;
2. $\|q_i\|_\infty \leq 1$, for any $i \in I$;
3. $q - q_i \to 0$ *-strongly, as $i \to \infty$.

Fix now $a, b \in (\mathcal{M})_1$ such that $\varphi(a) = \varphi(b) = 0$. Using the fact that

$$\| a \|_\infty, \| b \|_\infty, \| q \|_\infty, \|z_n\|_\infty \leq 1, \forall n \in \mathbf{N},$$

we get for any $n \in \mathbf{N}$ and any $i \in I$,

$$\begin{aligned}
&\|qE_{L(\mathbf{R})}(au_nb)q\|_{2,\mathrm{Tr}} \\
&\leq \|qE_{L(\mathbf{R})}(a(u_n - z_nq)b)q\|_{2,\mathrm{Tr}} + \|qE_{L(\mathbf{R})}(az_nqb)q\|_{2,\mathrm{Tr}} \\
&\leq \|u_n - z_nq\|_{2,\mathrm{Tr}} + \|qE_{L(\mathbf{R})}(az_n(q - q_i)b)q\|_{2,\mathrm{Tr}} \\
&\quad + \|qE_{L(\mathbf{R})}(az_nq_ib)q\|_{2,\mathrm{Tr}} \\
&\leq \|u_n - z_nq\|_{2,\mathrm{Tr}} + \|(q - q_i)bq\|_{2,\mathrm{Tr}} \\
&\quad + \sum_{t\in F_i} |d_t| \|qE_{L(\mathbf{R})}(az_n\sigma_t(b))\lambda_tq\|_{2,\mathrm{Tr}} \\
&\leq \|u_n - z_nq\|_{2,\mathrm{Tr}} + \|(q - q_i)bq\|_{2,\mathrm{Tr}} \\
&\quad + \sum_{t\in F_i} |d_t| \|qE_{L(\mathbf{R})}(az_n\sigma_t(b))q\|_{2,\mathrm{Tr}}.
\end{aligned}$$

Since $q - q_i \to 0$ $*$-strongly, as $i \to \infty$, it follows that $\|(q - q_i)bq\|_{2,\mathrm{Tr}} \to 0$, as $i \to \infty$. Fix $\varepsilon > 0$. Then, take $i_0 \in I$ such that $\|(q - q_{i_0})bq\|_{2,\mathrm{Tr}} \leq \varepsilon/3$. Since $\|u_n - z_nq\|_{2,\mathrm{Tr}} \to 0$, as $n \to \infty$ and using Claim (4.2.23), we may choose n_0 large enough such that for any $n \geq n_0$,

$$\begin{aligned}
\|u_n - z_nq\|_{2,\mathrm{Tr}} &\leq \varepsilon/3 \\
\sum_{t\in F_{i_0}} |d_t| \|qE_{L(\mathbf{R})}(az_n\sigma_t(b))q\|_{2,\mathrm{Tr}} &\leq \varepsilon/3.
\end{aligned}$$

Consequently, for any $n \geq n_0$, we get $\|qE_{L(\mathbf{R})}(au_nb)q\|_{2,\mathrm{Tr}} \leq \varepsilon$. Therefore, we have proven $\|qE_{L(\mathbf{R})}(au_nb)q\|_{2,\mathrm{Tr}} \to 0$, as $n \to \infty$.

Thanks to Claims (4.2.22) and (4.2.24), it is then clear that (23) is satisfied. This finishes the first step of the proof.

The last step of the proof consists in using Theorem (4.2.12). Let $k \geq 1$ and $q \in \mathbf{M}_k(\mathbf{C}) \otimes L(\mathbf{R})$ be a non-zero projection such that $T := (\mathrm{Tr}_k \otimes \mathrm{Tr})(q) < \infty$. Since $\mathbf{M}_k(\mathbf{C}) \otimes M$ is a II_∞ factor, there exists a unitary $u \in \mathcal{U}(\mathbf{M}_k(\mathbf{C}) \otimes M)$ such that

$$q = u \begin{pmatrix} q_0 & & 0 \\ & \ddots & \\ 0 & & q_0 \end{pmatrix} u^*$$

where $q_0 = \Phi(\chi_{[0,T/k]}) \in L(\mathbf{R})$. Using the spatiality of $\mathrm{Ad}(u)$ on $\mathbf{M}_k(\mathbf{C}) \otimes M$, we may assume without loss of generality that

$$q = \begin{pmatrix} q_0 & & 0 \\ & \ddots & \\ 0 & & q_0 \end{pmatrix}$$

In particular, $q \in \mathbf{M}_k(\mathbf{C}) \otimes L(\mathbf{R})q_0$. Define $M^T := q(\mathbf{M}_k(\mathbf{C}) \otimes M)q$ and $L(\mathbf{R})^T := q(\mathbf{M}_k(\mathbf{C}) \otimes L(\mathbf{R}))q$. Let $A \subset L(\mathbf{R})^T$ be a diffuse von Neumann subalgebra. Choose a sequence of unitaries (u_n) in A such that $u_n \to 0$ weakly, as $n \to \infty$. Thus, we can write $u_n = \left[u_n^{i,j}\right]_{i,j}$ where $u_n^{i,j} \in L(\mathbf{R})q_0$ and $\left\|u_n^{i,j}\right\|_\infty \leq 1$, for any $n \in \mathbf{N}$ and any $i,j \in \{1,\dots,k\}$. Moreover, $u_n^{i,j} \to 0$ weakly, as $n \to \infty$, in $L(\mathbf{R})q_0$, for any $i,j \in \{1,\dots,k\}$. Thus, using the first step of the proof, it becomes clear that the inclusion $L(\mathbf{R})^T \subset M^T$ is weakly mixing through A in the sense of Definition (4.2.11). Thus, using Theorem (4.2.12), it follows that for any ${}_A H_{L(\mathbf{R})^T}$ sub-bimodule of ${}_A L^2(M^T)_{L(\mathbf{R})^T}$ such that $\dim\left(H_{L(\mathbf{R})^T}\right) < \infty$, one has $H \subset L^2(L(\mathbf{R})^T)$. In particular $A' \cap M^T \subset L(\mathbf{R})^T$.

Claim (4.2.25)[173]: Let $\mathcal{M} = \Gamma(H_{\mathbf{R}}, U_t)''$ be a free Araki-Woods factor such that the orthogonal representation (U_t) is mixing. Then the continuous core $M = \mathcal{M} \rtimes_\sigma \mathbf{R}$ is a solid II_∞ factor.

Proof. Let $q \in L(\mathbf{R})$ be a non-zero projection such that $\mathrm{Tr}(q) < \infty$. Denote by $N = qMq$ the corresponding II_1 factor. By contradiction assume that N is not solid. Then there exists a non-amenable von Neumann subalgebra $Q \subset N$ such that the relative commutant $Q' \cap N$ is diffuse. Since N is a II_1 factor, using the same argument as in the proof of Corollary (4.2.20), we may assume that Q has no amenable direct summand and $Q_0 = Q' \cap N$ is still diffuse.

Since Q has no amenable direct summand, Theorem (4.2.18) yields $Q_0 \preceq_M L(\mathbf{R})$. Thus using Theorem (4.2.13), we know that there exists a non-zero projection $p \in L(\mathbf{R})$ such that $\mathrm{Tr}(p) < \infty$, and $Q_0 \preceq_{eMe} L(\mathbf{R})p$ where $e = p \vee q$. Consequently, there exist $n \geq 1$, a (possibly non-unital) $*$-homomorphism $\psi: Q_0 \to \mathbf{M}_n(\mathbf{C}) \otimes L(\mathbf{R})p$ and a non-zero partial isometry $v \in \mathbf{M}_{1,n}(\mathbf{C}) \otimes qMp$ such that

$$xv = v\psi(x), \forall x \in Q_0.$$

We moreover have

$$vv^* \in Q_0' \cap qMq \text{ and } v^*v \in \psi(Q_0)' \cap \psi(q)(\mathbf{M}_n(\mathbf{C}) \otimes pMp)\psi(q).$$

Write $Q_1 = Q_0' \cap qMq$ and notice that $Q \subset Q_1$. Since $\psi(Q_0)$ is diffuse and $v^*v \in \psi(Q_0)' \cap \psi(q)(\mathbf{M}_n(\mathbf{C}) \otimes pMp)\psi(q)$, Theorem (4.2.21) yields $v^*v \in \psi(q)(\mathbf{M}_n(\mathbf{C}) \otimes L(\mathbf{R})p)\psi(q)$, so that we may assume $v^*v = \psi(q)$. For any $y \in Q_1$, and any $x \in Q_0$,

$$\begin{aligned} v^*yv\psi(x) &= v^*yxv \\ &= v^*xyv \\ &= \psi(x)v^*yv. \end{aligned}$$

Thus, $v^*Q_1v \subset \psi(Q_0)' \cap v^*v(\mathbf{M}_n(\mathbf{C}) \otimes pMp)v^*v$. Since $\psi(Q_0)$ is diffuse, Theorem (4.2.21) yields $v^*Q_1v \subset v^*v(\mathbf{M}_n(\mathbf{C}) \otimes L(\mathbf{R})p)v^*v$. Since Q has no amenable direct summand and $Q \subset Q_1$ is a unital von Neumann subalgebra, it follows that Q_1 has no amenable direct summand either. Thus the von Neumann algebra $vv^*Q_1vv^*$ is non-amenable. But $\mathrm{Ad}(v^*): vv^*Mvv^* \to v^*v(\mathbf{M}_n(\mathbf{C}) \otimes pMp)v^*v$ is a *-isomorphism and

$$\mathrm{Ad}(v^*)(vv^*Q_1vv^*) \subset v^*v(\mathbf{M}_n(\mathbf{C}) \otimes L(\mathbf{R})p)v^*v.$$

Since $v^*v(\mathbf{M}_n(\mathbf{C}) \otimes L(\mathbf{R})p)v^*v$ is of type I, hence amenable, we get a contradiction.

Since the left regular representation (λ_t) of $\mathbf{R}$ acting on $L^2_{\mathbf{R}}(\mathbf{R}$, Lebesgue) is mixing, the continuous core M of $\Gamma\left(L^2_{\mathbf{R}}(\mathbf{R}, \text{Lebesgue }), \lambda_t\right)''$ is solid. We partially retrieve a previous result of Shlyakhtenko [14] where he proved in this case that $M \cong L(\mathbf{F}_\infty) \bar{\otimes} \mathbf{B}(\ell^2)$, which is solid by [162]. We will give an example of a non-amenable solid II_1 factor with full fundamental group which is not isomorphic to any interpolated free group factor $L(\mathbf{F}_t)$, for $1 < t \leq \infty$.

Note that the mixing property of the representation (U_t) is not a necessary condition for the solidity of the continuous core M. Indeed, take $U_t = \mathrm{Id} \oplus \lambda_t$ on $H_{\mathbf{R}} = \mathbf{R} \oplus L^2_{\mathbf{R}}(\mathbf{R}$, Lebesgue). Then (U_t) is not mixing, but the continuous core M of $\Gamma(H_{\mathbf{R}}, U_t)''$ is still isomorphic to $L(\mathbf{F}_\infty) \bar{\otimes} \mathbf{B}(\ell^2)$[186].

Write λ for the Lebesgue measure on the real line $\mathbf{R}$. Let μ be a symmetric (positive) probability measure on $\mathbf{R}$, i.e. $\mu(X) = \mu(-X)$, for any Borel subset $X \subset \mathbf{R}$. Consider the following unitary representation $\left(U_t^\mu\right)$ of $\mathbf{R}$ on $L^2(\mathbf{R}, \mu)$ given by:

$$\left(U_t^\mu f\right)(x) = e^{itx} f(x), \forall f \in L^2(\mathbf{R}, \mu), \forall t, x \in \mathbf{R}. \qquad (24)$$

Define the Hilbert subspace of $L^2(\mathbf{R}, \mu)$

$$K_{\mathbf{R}}^\mu := \left\{f \in L^2(\mathbf{R}, \mu) : f(x) = \overline{f(-x)}, \forall x \in \mathbf{R}\right\}. \qquad (25)$$

Since μ is assumed to be symmetric, the restriction of the inner product to K_{R} is real-valued. Indeed, for any $f, g \in K_{\mathbf{R}}^\mu$,

$$\begin{aligned} \langle f, g\rangle &= \int_{\mathbf{R}} f(x)\overline{g(x)} d\mu(x) \\ &= \int_{\mathbf{R}} f(-x)\overline{g(-x)} d\mu(-x) \\ &= \int_{\mathbf{R}} \overline{f(x)} g(x) d\mu(x) \\ &= \overline{\langle f, g\rangle}. \end{aligned}$$

Moreover the representation $\left(U_t^\mu\right)$ leaves $K_{\mathbf{R}}^\mu$ globally invariant. Thus, $\left(U_t^\mu\right)$ restricted to $K_{\mathbf{R}}^\mu$ becomes an orthogonal representation. Define the Fourier Transform of the probability measure μ by:

$$\tilde{\mu}(t) = \int_{\mathbf{R}} e^{itx} d\mu(x), \forall t \in \mathbf{R}.$$

We shall identify $\widehat{\mathbf{R}}$ with $\mathbf{R}$ in the usual way, such that

$$\hat{f}(t) = \int_{\mathbf{R}} e^{itx} f(x) d\lambda(x), \forall t \in \mathbf{R}, \forall f \in L^1(\mathbf{R}, \lambda).$$

Proposition (4.2.26)[173]: Let μ be a symmetric probability measure on $\mathbf{R}$. Then

$$\left(U_t^\mu\right) \text{ is mixing } \Leftrightarrow \tilde{\mu}(t) \rightarrow 0, \text{as } |t| \rightarrow \infty.$$

Proof. We prove both directions.

$\Longrightarrow$ Assume $\left(U_t^\mu\right)$ is mixing. Let $f = \mathbf{1}_{\mathbf{R}} \in L^2(\mathbf{R}, \mu)$ be the constant function equal to 1 . Then

$$\begin{aligned} \tilde{\mu}(t) &= \int_{\mathbf{R}} e^{itx} d\mu(x) \\ &= \left\langle U_t^\mu f, f\right\rangle \rightarrow 0, \text{ as } |t| \rightarrow \infty. \end{aligned}$$

$\Leftarrow$ Assume $\tilde{\mu}(t) \to 0$, as $|t| \to \infty$. Let $f, g \in L^2(\mathbf{R}, \mu)$. Then $h := f\bar{g} \in L^1(\mathbf{R}, \mu)$. Since the set $\{f \in C_0(\mathbf{R}) : \hat{f} \in L^1(\mathbf{R}, \lambda)\}$ is dense in $L^1(\mathbf{R}, \mu)$, we may choose a sequence (h_n) in $C_0(\mathbf{R})$ such that $\|h - h_n\|_{L^1(\mathbf{R},\mu)} \to 0$, as $n \to \infty$, and $\hat{h}_n \in L^1(\mathbf{R}, \lambda)$, for any $n \in \mathbf{N}$. Define

$$\tilde{h}(t) = \int_{\mathbf{R}} e^{itx} h(x) d\mu(x), \forall t \in \mathbf{R}$$

$$\tilde{h}_n(t) = \int_{\mathbf{R}} e^{itx} h_n(x) d\mu(x), \forall t \in \mathbf{R}, \forall n \in \mathbf{N}.$$

Since $\|h - h_n\|_{L^1(\mathbf{R},\mu)} \to 0$, as $n \to \infty$, it follows that $\|\tilde{h} - \tilde{h}_n\|_\infty \to 0$, as $n \to \infty$. Since $\hat{h}_n \in L^1(\mathbf{R}, \lambda)$, we know that

$$h_n(x) = C \int_{\mathbf{R}} e^{-ixu} \hat{h}_n(u) d\lambda(u), \forall x \in \mathbf{R}.,$$

where C is a universal constant that only depends on the normalization of the Lebesgue measure λ on $\mathbf{R}$. Therefore, for any $t \in \mathbf{R}$ and any $n \in \mathbf{N}$,

$$\begin{aligned}
\tilde{h}_n(t) &= \int_{x\in\mathbf{R}} e^{itx} h_n(x) d\mu(x) \\
&= C \int_{x\in\mathbf{R}} \left(\int_{u\in\mathbf{R}} e^{i(t-u)x} \hat{h}_n(u) d\lambda(u) \right) d\mu(x) \\
&= C \int_{u\in\mathbf{R}} \hat{h}_n(u) \left(\int_{x\in\mathbf{R}} e^{i(t-u)x} d\mu(x) \right) d\lambda(u) \\
&= C \int_{u\in\mathbf{R}} \hat{h}_n(u) \tilde{\mu}(t-u) d\lambda(u) \\
&= C\left(\hat{h}_n * \tilde{\mu}\right)(t),
\end{aligned}$$

where $*$ is the convolution product. Since $\tilde{\mu} \in C_0(\mathbf{R})$ and $\hat{h}_n \in L^1(\mathbf{R}, \lambda)$, it is easy to check that $\hat{h}_n * \tilde{\mu} \in C_0(\mathbf{R})$. Consequently, $\tilde{h}_n \in C_0(\mathbf{R})$ and since $\|\tilde{h} - \tilde{h}_n\|_\infty \to 0$, as $n \to \infty$, it follows that $\tilde{h} \in C_0(\mathbf{R})$. But for any $t \in \mathbf{R}$,

$$\begin{aligned}
\langle U_t^\mu f, g \rangle &= \int_{\mathbf{R}} e^{itx} f(x) \overline{g(x)} d\mu(x) \\
&= \tilde{h}(t).
\end{aligned}$$

Thus, the unitary representation (U_t^μ) is mixing.

For a measure ν on $\mathbf{R}$, define the measure class of ν by:

$$\mathcal{C}_\nu := \{\nu' : \nu' \text{ is absolutely continuous w.r.t. } \nu\}.$$

Definition (4.2.27)[173]: Let (V_t) be a unitary representation of $\mathbf{R}$ on a separable Hilbert space H.

Denote by B the infinitesimal generator of (V_t), i.e. B is the positive, self-adjoint (possibly) unbounded operator on H such that $V_t = B^{it}$, for every $t \in \mathbf{R}$. We define the spectral measure of the representation (V_t) as the spectral measure of the operator B and denote it by $\mathcal{C}_V$.

The measure class $\mathcal{C}_V$ can also be defined as the smallest collection of all the measures ν on $\mathbf{R}$ such that:

1. If $\nu \in \mathcal{C}_V$ and ν' is absolutely continuous w.r.t. ν, then $\nu' \in \mathcal{C}_V$;
2. For any unit vector $\eta \in H$, the probability measure associated with the positive definite function $t \mapsto \langle V_t \eta, \eta \rangle$ belongs to $\mathcal{C}_V$.

Since H is separable, there exists a measure ν that generates $\mathcal{C}_V$, i.e. $\mathcal{C}_V$ is the smallest collection of measures on **R** satisfying (a) and containing ν. We will refer to this particular measure ν as the "spectral measure" of the representation (V_t) and simply denote it by ν.

Let μ be a symmetric probability measure on **R** and consider the unitary representation $\left(U_t^\mu\right)$ on $L^2(\mathbf{R},\mu)$ as defined in (24). Then for any unit vector $f \in L^2(\mathbf{R},\mu)$,

$$\left\langle U_t^\mu f, f\right\rangle = \int_{\mathbf{R}} e^{itx}|f(x)|^2 d\mu(x), \forall t \in \mathbf{R}.$$

Since the probability measure $|f(x)|^2 d\mu(x)$ is absolutely continuous w.r.t. $d\mu(x)$, it is clear that the spectral measure of $\left(U_t^\mu\right)$ is μ. More generally, we have the following:

Proposition (4.2.28)[173]: Let μ be a symmetric probability measure on **R**. Consider the unitary representation $\left(U_t^\mu\right)$ defined on $L^2(\mathbf{R},\mu)$ by (24). Then for any $n \geq 1$, the spectral measure of the n-fold tensor product $\left(U_t^\mu\right)^{\otimes n}$ is the n-fold convolution product

$$\mu^{*n} = \underbrace{\mu * \cdots * \mu}_{n \text{ times}}.$$

Erdös showed in [177] that the symmetric probability measure μ_θ, with $\theta = 5/2$, obtained as the weak limit of

$$\left(\frac{1}{2}\delta_{-\theta^{-1}} + \frac{1}{2}\delta_{\theta^{-1}}\right) * \cdots * \left(\frac{1}{2}\delta_{-\theta^{-n}} + \frac{1}{2}\delta_{\theta^{-n}}\right)$$

has a Fourier Transform

$$\tilde{\mu}_\theta(t) = \prod_{n\geq 1} \cos\left(\frac{t}{\theta^n}\right)$$

which vanishes at infinity, i.e. $\tilde{\mu}(t) \to 0$, as $|t| \to \infty$, and μ_θ is singular w.r.t. the Lebesgue measure λ.

Example (4.2.29)[173]: Modifying the measure μ_θ, Antoniou & Shkarin (see Theorem 2.5,v in [174]) constructed an example of a symmetric probability μ on **R** such that:

1. The Fourier Transform of μ vanishes at infinity, i.e. $\tilde{\mu}(t) \to 0$, as $|t| \to \infty$.
2. For any $n \geq 1$, the n-fold convolution product μ^{*n} is singular w.r.t. the Lebesgue measure λ.

Let μ be a symmetric probability measure on **R** as in Example (4.2.29). Proposition (4.2.26) and Proposition (4.2.28) yields that the unitary representation $\left(U_t^\mu\right)$ defined on $L^2(\mathbf{R},\mu)$ by (24) satisfies:

1. $\left(U_t^\mu\right)$ is mixing.
2. The spectral measure of $\bigoplus_{n\geq 1}\left(U_t^\mu\right)^{\otimes n}$ is singular w.r.t. the Lebesgue measure λ.

Let now $\mathcal{M} = \Gamma(H_{\mathbf{R}}, U_t)''$ and let $M = \mathcal{M} \rtimes_\sigma \mathbf{R}$ be the continuous core. Let $q \in L(\mathbf{R})$ be a non-zero projection such that $\mathrm{Tr}(q) < \infty$. Denote by $N = qMq$ the corresponding II_1 factor. Using free probability techniques such as the free entropy, Shlyakhtenko (see Theorem 9.12 in [13]) showed that if the spectral measure of the unitary representation $\bigoplus_{n\geq 1} U_t^{\otimes n}$ is singular w.r.t. the Lebesgue measure λ, then for any finite set of generators $X_1, \dots, X_n$ of N, the free entropy dimension satisfies

$$\delta_0(X_1, \dots, X_n) \leq 1.$$

In particular, N is not isomorphic to any interpolated free group factor $L(\mathbf{F}_t)$, for $1 < t \leq \infty$. Combining these two results together with Theorem (4.2.21), we obtain the following:

Theorem (4.2.30)[173]: Let μ be a symmetric probability measure on **R** as in Example (4.2.29). Let $\mathcal{M} = \Gamma\left(K_{\mathbf{R}}^\mu, U_t^\mu\right)''$ be the free Araki-Woods factor associated with the orthogonal representation (U_t^{μ}) acting on the real Hilbert space $K_{\mathbf{R}}^{\mu}$, as defined in $(8-9)$. Let $M = \mathcal{M} \rtimes_\sigma \mathbf{R}$ be the continuous core. Fix a non-zero projection $q \in L(\mathbf{R})$ such that $\mathrm{Tr}(q) < \infty$, and denote by $N = qMq$ the corresponding II_1 factor. Then

1. N is non-amenable and solid.
2. N has full fundamental group, i.e. $\mathcal{F}(N) = \mathbf{R}_+^*$.
3. N is not isomorphic to any interpolated free group factor $L(\mathbf{F}_t)$, for $1 < t \leq \infty$.

We believe that all the free Araki-Woods factors $\mathcal{M} = \Gamma(H_{\mathbf{R}}, U_t)''$ have the complete metric approximation property (c.m.a.p.), i.e. there exists a sequence $\Phi_n: \mathcal{M} \to \mathcal{M}$ of finite rank, completely bounded maps such that $\Phi_n \to \mathrm{Id}$ ultraweakly pointwise, as $n \to \infty$, and $\mathrm{limsup}_{n\to\infty} \|\Phi_n\|_{\mathrm{cb}} \leq 1$. If $\mathcal{M} = \Gamma(H_{\mathbf{R}}, U_t)''$ had the c.m.a.p. then by [143], the continuous core $M = \mathcal{M} \rtimes_\sigma \mathbf{R}$ would have the c.m.a.p., as well as the II_1 factor qMq, for $q \in M$ non-zero finite projection. On the other hand, the wreath product II_1 factors $L(\mathbf{Z}\{\mathbf{F}_n)$ do not have the c.m.a.p., for any $2 \leq n \leq \infty$, by [141]. Thus, we conjecture that the solid II_1 factors constructed in Theorem (4.2.30) are not isomorphic to $L(\mathbf{Z})\mathbf{F}_n)$, for any $2 \leq n \leq \infty$.

Section (4.3): Absence of Cartan Subalgebra for Free Araki-Woods Factors

The free Araki-Woods factors were introduced by Shlyakhtenko [15]. In the context of free probability theory, these factors can be regarded as analogs of the hyperfinite factors coming from the CAR functor. To each real separable Hilbert space $H_{\mathbf{R}}$ together with an orthogonal representation (U_t) of $\mathbf{R}$ on $H_{\mathbf{R}}$, one associates [15] a von Neumann algebra denoted by $\Gamma(H_{\mathbf{R}}, U_t)''$, called the free Araki-Woods von Neumann algebra. The von Neumann algebra $\Gamma(H_{\mathbf{R}}, U_t)''$ comes equipped with a unique free quasi-free state, which is always normal and faithful. If $\dim H_{\mathbf{R}} = 1$, then $\Gamma(\mathbf{R}, \mathrm{Id})'' \cong L^\infty([0,1])$. If $\dim H_{\mathbf{R}} \geq 2$, then $\mathcal{M} = \Gamma(H_{\mathbf{R}}, U_t)''$ is a full factor. In particular, $\mathcal{M}$ can never be of type III_0. The type classification of these factors is the following:

1. $\mathcal{M}$ is a type II_1 factor if and only if the representation (U_t) is trivial: in that case the functor Γ is Voiculescu's free Gaussian functor [197]. Then $\Gamma(H_{\mathbf{R}}, 1)'' \cong L(\mathbf{F}_{\dim H_{\mathbf{R}}})$ is a free group factor.
2. $\mathcal{M}$ is a type III_λ factor, for $0 < \lambda < 1$, if and only if the representation (U_t) is $\frac{2\pi}{|\log \lambda|}$-periodic.
3. $\mathcal{M}$ is a type III_1 factor if and only if (U_t) is nonperiodic and nontrivial.

Let us start by recalling some fundamental structural results for free group factors. In their breakthrough [141], Ozawa and Popa showed that the free group factors $L(\mathbf{F}_n)$ are strongly solid, i.e. the normalizer $\mathcal{N}_{L(\mathbf{F}_n)}(P) = \{u \in \mathcal{U}(L(\mathbf{F}_n)): uPu^* = P\}$ of any diffuse amenable subalgebra $P \subset L(\mathbf{F}_n)$ generates an amenable von Neumann algebra, thus hyperfinite by Connes' result [198]. This strengthened two well-known indecomposability results for free group factors: Voiculescu's celebrated result in [172], showing that $L(\mathbf{F}_n)$ has no Cartan subalgebra, which in fact exhibited the first examples of factors with no Cartan decomposition; and Ozawa's result in [162], showing that the commutant in $L(\mathbf{F}_n)$ of any diffuse subalgebra must be amenable ($L(\mathbf{F}_n)$ are solid).

For the type III free Araki-Woods factors $\mathcal{M} = \Gamma(H_{\mathbf{R}}, U_t)''$, Shlyakhtenko obtained several remarkable classification results using free probability techniques:

1. When (U_t) are almost periodic, the free Araki-Woods factors are completely classified up to state-preserving *-isomorphism [15]: they only depend on Connes' invariant $\mathrm{Sd}(\mathcal{M})$ which is equal in that case to the (countable) subgroup $S_U \subset$

$\mathbf{R}_+$generated by the eigenvalues of (U_t). Moreover, the discrete core $\mathcal{M} \rtimes_\sigma \widehat{S_U}$ (where $\widehat{S_U}$ is the Pontryagin dual of S_U) is *isomorphic to $L(\mathbf{F}_\infty) \bar{\otimes} \mathbf{B}(\ell^2)$.

2. If (U_t) is the left regular representation, then the continuous core $M = \mathcal{M} \rtimes_\sigma \mathbf{R}$ is *-isomorphic to $L(\mathbf{F}_\infty) \bar{\otimes} \mathbf{B}(\ell^2)$[14] and the dual "trace-scaling" action (θ_s) is precisely the one constructed by Rădulescu [184].

For more on free Araki-Woods factors, see [173],[179],[180],[1],[12], [186],[202],[13],45],[46] and also to Vaes' Bourbaki seminar [18].

We deal with approximation properties for $\Gamma(H_{\mathbf{R}}, U_t)''$. Recall that a von Neumann algebra $\mathcal{N}$ is said to have the complete metric approximation property (c.m.a.p.) [196] if there exists a net of normal finite rank completely bounded maps $\Phi_n: \mathcal{N} \to \mathcal{N}$ such that

1. $\Phi_n(x) \to x$ $*$-strongly, for every $x \in \mathcal{N}$;
2. $\|\Phi_n\|_{\mathrm{cb}} \leq 1$, for every n.

Haagerup first established in [178] that the free group factors $L(\mathbf{F}_n)$ have the metric approximation property. His idea was to use radial multipliers on $\mathbf{F}_n$. In a subsequent unpublished work with Szwarc (see [195]), a complete description of completely bounded radial multipliers was obtained, showing that $L(\mathbf{F}_n)$ has the complete metric approximation property. Along the same pattern, we start by characterizing appropriate radial multipliers on $\Gamma(H_{\mathbf{R}}, U_t)''$. At the L^2-level, that is on the Fock space, they just act diagonally on tensor powers of H. They allow us to reduce the question of the approximation property to a finite length situation, which is enough to conclude for almost periodic representations (U_t). To proceed to the general case, we use completely positive maps arising from the second quantization functor. The novelty here is that it holds true under a milder assumption than the usual one [46],[1], and we obtain:

Theorem (4.3.1)[190]: All the free Araki-Woods factors have the complete metric approximation property.

The free Araki-Woods factors $\Gamma(H_{\mathbf{R}}, U_t)''$ as well as their continuous cores carry a free malleable deformation (α_t) in the sense of Popa: it naturally arises from the second quantization of the rotations defined on $H_{\mathbf{R}} \oplus H_{\mathbf{R}}$ that commute with $U_t \oplus U_t$. Using Ozawa-Popa's techniques [141],[198], we will then apply the deformation/rigidity strategy together with the intertwining techniques in order to study $\Gamma(H_{\mathbf{R}}, U_t)''$. The high flexibility of this approach will allow us to work in a semifinite setting, so that we can obtain new structural/indecomposability results for the free ArakiWoods factors as well as their continuous cores. Recall in that respect that a von Neumann subalgebra $A \subset \mathcal{M}$ is said to be a Cartan subalgebra if the following conditions hold:

1. A is maximal abelian, i.e. $A = A' \cap \mathcal{M}$.
2. There exists a faithful normal conditional expectation $E: \mathcal{M} \to A$.
3. The normalizer $\mathcal{N}_{\mathcal{M}}(A) = \{u \in \mathcal{U}(\mathcal{M}): uAu^* = A\}$ generates $\mathcal{M}$.

It follows from [201] that in that case, $L^\infty(X, \mu) = A \subset \mathcal{M} = L(\mathcal{R}, \omega)$ is the von Neumann algebra of a nonsingular equivalence relation $\mathcal{R}$ on the standard probability space (X, μ) up to a scalar 2 -cocycle ω for $\mathcal{R}$.

Shlyakhtenko showed [202] that the unique type III $_\lambda$ free Araki-Woods factor $(0 < \lambda < 1)$ has no Cartan subalgebra. We generalize this result and prove the analog of the strong solidity [141] for all the free Araki-Woods factors. Our second result is the following global dichotomy result for conditioned diffuse subalgebras of free Araki-Woods factors.

We can deduce from Theorem (4.3.1) and Theorem (4.3.37) new classification results for the free Araki-Woods factors. First recall that a factor $\mathcal{N}$ is said to be full if the subgroup of inner automorphisms $\mathrm{Inn}(\mathcal{N})$ is closed in $\mathrm{Aut}(\mathcal{N})$. Write $\pi: \mathrm{Aut}(\mathcal{N}) \to \mathrm{Out}(\mathcal{N})$ for the quotient map. For a full type III_1 factor $\mathcal{N}$, Connes' invariant $\tau(\mathcal{N})$ is defined as the weakest topology on $\mathbf{R}$ that makes the map $t \mapsto \pi(\sigma_t^{\varphi}) \in \mathrm{Out}(\mathcal{N})$ continuous. In [3], Connes constructed type III_1 factors $\mathcal{N}$ with prescribed τ invariant. Recall his construction. Let μ be a finite Borel measure on $\mathbf{R}_+$such that $\int \lambda \mathrm{d}\mu(\lambda) < \infty$. We will normalize μ so that $\int (1 + \lambda)\mathrm{d}\mu(\lambda) = 1$. Define the unitary representation (U_t) of $\mathbf{R}$ on the real Hilbert space $L^2(\mathbf{R}_+, \mu)$ by $(U_t\xi)(\lambda) = \lambda^{it}\xi(\lambda)$. We will assume that (U_t) is not periodic. Define on $P = \mathbf{M}_2(\mathbf{C}) \otimes L^\infty(\mathbf{R}_+, \mu)$ the faithful normal state φ by

$$\varphi\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \int f_{11}(\lambda)\mathrm{d}\mu(\lambda) + \int \lambda f_{22}(\lambda)\mathrm{d}\mu(\lambda).$$

Let $\mathbf{F}_n$ be acting by Bernoulli shift on

$$\mathcal{P}_\infty = \overline{\bigotimes_{g \in \mathbf{F}_n}} (P, \varphi).$$

Denote by $\mathcal{N} = \mathcal{P}_\infty \rtimes \mathbf{F}_n$ the corresponding crossed product. By the general theory, $\mathcal{N}$ is a type III $_1$ factor. Connes showed that $\mathcal{N}$ is a full factor and $\tau(\mathcal{N})$ is the weakest topology that makes the map $t \mapsto U_t$ *-strongly continuous. In particular, if (U_t) is the left regular representation, then $\tau(\mathcal{N})$ is the usual topology and $\mathcal{N}$ has no almost periodic state. Observe that $\mathcal{N}$ has a Cartan subalgebra A given by

$$A = \overline{\bigotimes_{g \in \mathbf{F}_n}} \mathrm{Diag}_2\big(L^\infty(\mathbf{R}_+, \mu)\big).$$

The following Corollary answers a question of Shlyakhtenko (see [13], Problem 8.7]) and Vaes (see [18], Remarque 2.8]).

Corollary (4.3.2)[190]: The type III_1 factors constructed by Connes are never isomorphic to any free Araki-Woods factor. More generally, they cannot be conditionally embedded into a free Araki-Woods factor.

The continuous cores $M = \Gamma(H_{\mathbf{R}}, U_t)'' \rtimes_\sigma \mathbf{R}$ of the free Araki-Woods factors were shown to be semisolid for every orthogonal representation (U_t) and solid when (U_t) is strongly mixing (see [173], Theorem 1.1]). They moreover have the c.m.a.p. by Theorem (4.3.1). Using a similar strategy as in [198], we obtain new structural results for the continuous cores of the free Araki-Woods factors.

The proof of Theorem (4.3.37) and Theorem (4.3.39) is a combination of ideas and techniques of [176],[198],[173],[141],[198] and rely on Theorem (4.3.1). Note that Theorem (4.3.39) allows us to obtain other new classification results. Indeed let $\mathrm{SL}_n(\mathbf{Z}) \curvearrowright \mathbf{R}^n$ be the linear action. Observe that it is an infinite measure-preserving free ergodic action. Thus the corresponding crossed product von Neumann algebra $Q_n = L^\infty(\mathbf{R}^n) \rtimes \mathrm{SL}_n(\mathbf{Z})$ is a II_∞ factor, which is nonamenable for $n \geq 3$. Since the dilation $d_t: \mathbf{R}^n \ni x \mapsto tx \in \mathbf{R}^n$ (for $t > 0$) commutes with $\mathrm{SL}_n(\mathbf{Z})$, it gives a trace-scaling action $(\theta_t): \mathbf{R}_+ \curvearrowright Q_n$. Theorem (4.3.39) implies in particular that the type III_1 factors $Q_n \rtimes_{(\theta_t)} \mathbf{R}_+$obtained this way cannot be isomorphic to any free Araki-Woods factor.

Using ([174], Theorem 2.5,v) (see also the discussion in [198],[4.2]), we can construct an example of an orthogonal representation (U_t) of $\mathbf{R}$ on a (separable) real Hilbert space $H_{\mathbf{R}}$ such that:

1. (U_t) is strongly mixing.
2. The spectral measure of $\bigoplus_{n \geq 1} U_t^{\otimes n}$ is singular with respect to the Lebesgue measure on $\mathbf{R}$.

Shlyakhtenko showed ([13], Theorem 9.12) that if the spectral measure of the representation $\bigoplus_{n\geq 1} U_t^{\otimes n}$ is singular with respect to the Lebesgue measure, then the continuous core of the free Araki-Woods factor $\Gamma(H_{\mathbf{R}}, U_t)''$ cannot be isomorphic to any $L(\mathbf{F}_t) \bar{\otimes} \mathbf{B}(\ell^2)$, for $1 < t \leq \infty$, where $L(\mathbf{F}_t)$ denote the interpolated free group factors [6],[10]. Therefore, we obtain:

Corollary (4.3.3)[190]: Let (U_t) be an orthogonal representation acting on $H_{\mathbf{R}}$ as above. Denote by $\mathbb{M} = \Gamma(H_{\mathbf{R}}, U_t)''$ the corresponding free Araki-Woods factor and by $M = \mathcal{M} \rtimes_\sigma \mathbf{R}$ its continuous core. Let $p \in M$ be a nonzero finite projection and write $N = pMp$. We have

1. N is a nonamenable strongly solid II_1 factor with the c.m.a.p. and the Haagerup property.
1. N is not isomorphic to any interpolated free group factor $L(\mathbf{F}_t)$, for $1 < t \leq \infty$;
2. $N \bar{\otimes} \mathbf{B}(\ell^2)$ is endowed with a continuous trace-scaling action, in particular $\mathcal{F}(N) = \mathbf{R}_+$.

We recall a number of known results needed in the proofs. This includes a discussion of intertwining techniques for semifinite von Neumann algebras as well as several facts on the non-commutative flow of weights, Cartan subalgebras and the complete metric approximation property. Theorem (4.3.1) is proven.

Let $P \subset \mathcal{M}$ be an inclusion of von Neumann algebras. The normalizer of P inside $\mathcal{M}$ is defined as

$$\mathcal{N}_{\mathcal{M}}(P) := \{u \in \mathcal{U}(\mathcal{M}) : \mathrm{Ad}(u)P = P\},$$

where $\mathrm{Ad}(u) = u \cdot u^*$. The inclusion $P \subset \mathcal{M}$ is said to be regular if $\mathcal{N}_{\mathcal{M}}(P)'' = \mathcal{M}$. The groupoid normalizer of P inside $\mathcal{M}$ is defined as

$$\mathcal{GN}_{\mathcal{M}}(P) := \left\{v \in \mathcal{M} \text{ partial isometry} : vPv^* \subset P, v^*Pv \subset P\right\}.$$

The quasi-normalizer of P inside $\mathcal{M}$ is defined as

$$\mathcal{QN}_{\mathcal{M}}(P) := \left\{a \in \mathcal{M} : \exists b_1, \dots, b_n \in \mathcal{M}, aP \subset \sum_i Pb_i, Pa \subset \sum_i b_i P\right\}.$$

The inclusion $P \subset \mathcal{M}$ is said to be quasi-regular if $\mathcal{QN}_{\mathcal{M}}(P)'' = \mathcal{M}$. Moreover,

$$P' \cap \mathcal{M} \subset \mathcal{N}_{\mathcal{M}}(P)'' \subset \mathcal{GN}_{\mathcal{M}}(P)'' \subset \mathcal{QN}_{\mathcal{M}}(P)''.$$

In ([188] Theorem 2.1, [204] Theorem A.1), Popa introduced a powerful tool to prove the unitary conjugacy of two von Neumann subalgebras of a tracial von Neumann algebra (M, τ). We will make intensively use of this technique. If $A, B \subset (M, \tau)$ are (possibly non-unital) von Neumann subalgebras, denote by 1_A (resp. 1_B) the unit of A (resp. B).

Theorem (4.3.4)[190]: (Popa, [188],[204]). Let (M, τ) be a finite von Neumann algebra. Let $A, B \subset M$ be possibly nonunital von Neumann subalgebras. The following are equivalent:

1. There exist $n \geq 1$, a possibly nonunital $*$-homomorphism $\psi: A \to \mathbf{M}_n(\mathbf{C}) \otimes B$ and a nonzero partial isometry $v \in \mathbf{M}_{1,n}(\mathbf{C}) \otimes 1_A M 1_B$ such that $xv = v\psi(x)$, for any $x \in A$.
2. There is no sequence of unitaries (u_k) in A such that
$$\lim_{k\to\infty} \|E_B(a^* u_k b)\|_2 = 0, \forall a, b \in 1_A M 1_B.$$

If one of the previous equivalent conditions is satisfied, we shall say that A embeds into B inside M and denote $A \preceq_M B$. For simplicity, we shall write $M^n := \mathbf{M}_n(\mathbf{C}) \otimes M$.

We will need to extend Popa's intertwining techniques to semifinite von Neumann algebras. Let (M, Tr) be a von Neumann algebra endowed with a semifinite faithful normal trace. We shall simply denote by $L^2(M)$ the $M, M-$ bimodule $L^2(M, \mathrm{Tr})$, and by $\|\cdot\|_{2,\mathrm{Tr}}$ the L^2-norm associated with Tr. We will use the following well-known inequality ($\|\cdot\|_\infty$ is the operator norm):

$$\| x\xi y \|_{2,\mathrm{Tr}} \leq \| \xi \|_{2,\mathrm{Tr}} \| x \|_\infty \| y \|_\infty, \forall \xi \in L^2(M), \forall x, y \in M.$$

We shall say that a projection $p \in M$ is Tr-finite if $\mathrm{Tr}(p) < \infty$. Then p is necessarily finite. Moreover, pMp is a finite von Neumann algebra and $\tau := \mathrm{Tr}(p \cdot p)/\mathrm{Tr}(p)$ is a faithful normal tracial state on pMp. Recall that for any projections $p, q \in M$, we have $p \vee q - p \sim q - p \wedge q$. Then it follows that for any Tr-finite projections $p, q \in M, p \vee q$ is still Tr-finite and $\mathrm{Tr}(p \vee q) = \mathrm{Tr}(p) + \mathrm{Tr}(q) - \mathrm{Tr}(p \wedge q)$.

Note that if a sequence (x_k) in M converges to 0 *-strongly, then for any nonzero Tr-finite projection $q \in M, \|x_k q\|_{2,\mathrm{Tr}} + \|q x_k\|_{2,\mathrm{Tr}} \to 0$. Indeed,

$$\begin{aligned} x_k \to 0 * - \text{strongly in } M &\Leftrightarrow x_k^* x_k + x_k x_k^* \to 0 \text{ weakly in } M \\ &\Rightarrow q x_k^* x_k q + q x_k x_k^* q \to 0 \text{ weakly in } qMq \\ &\Rightarrow \mathrm{Tr}(q x_k^* x_k q) + \mathrm{Tr}(q x_k x_k^* q) \to 0 \\ &\Leftrightarrow \mathrm{Tr}\big((x_k q)^*(x_k q)\big) + \mathrm{Tr}((q x_k)^* q x_k) \to 0 \\ &\Leftrightarrow \|x_k q\|_{2,Tr} + \|q x_k\|_{2,\mathrm{Tr}} \to 0. \end{aligned}$$

Moreover, there always exists an increasing sequence of Tr-finite projections (p_k) in M such that $p_k \to 1$ strongly.

Intertwining techniques for semifinite von Neumann algebras were developed in [176]. The following result due to S. Vaes is a slight improvement of ([176], Theorem 2.2) that will be useful in the sequel.

Lemma (4.3.5)[190]: (Vaes, [204]). Let (M, Tr) be a semifinite von Neumann algebra. Let $B \subset M$ be a von Neumann subalgebra such that $\mathrm{Tr}_{|B}$ is still semifinite. Let $p \in M$ be a nonzero projection such that $\mathrm{Tr}(p) < \infty$ and $A \subset pMp$ a von Neumann subalgebra. Then the following are equivalent:

1. For every nonzero projection $q \in B$ with $\mathrm{Tr}(q) < \infty$, we have

$$A \npreceq_{eMe} qBq, \text{ where } e = p \vee q,$$

in the usual sense for finite von Neumann algebras.

2. There exists a sequence of unitaries (u_n) in A such that

$$\lim_n \|E_B(x^* u_n y)\|_{2,\mathrm{Tr}} = 0, \forall x, y \in M.$$

If these conditions hold, we write $A \npreceq_M B$ and otherwise we write $A \preceq_M B$.

Proof. We prove both directions.

(a) $\Leftarrow$ (b). Take a nonzero projection $q \in B$ such that $\mathrm{Tr}(q) < \infty$ and set $e = p \vee q$. Write $\lambda = \mathrm{Tr}(e)$. For all $x, y \in pMq$, using the $\|\cdot\|_2$-norm with respect to the normalized trace on eMe, we have

$$\left\|E_{qBq}(x^* u_n y)\right\|_2 = \lambda^{-1/2} \|E_B(x^* u_n y)\|_{2,\mathrm{Tr}} \to 0.$$

This means exactly that $A \npreceq_{eMe} qBq$.

(a) $\Rightarrow$ (b). Let (q_n) be an increasing sequence of projections in B such that $q_n \to 1$ strongly and $\mathrm{Tr}(q_n) < \infty$. Set $e_n = p \vee q_n$. Let $\{x_k : k \in \mathbf{N}\}$ be a $*$-strongly dense subset of $(M)_1$ (the unit ball of M). Since $A \npreceq e_n M e_n\ q_n B q_n$, we can take a unitary $u_n \in \mathcal{U}(A)$ such that

$$\|E_B(q_n x_i u_n x_j q_n)\|_{2,\mathrm{Tr}} < \frac{1}{n}, \forall 1 \leq i, j \leq n.$$

Note that $u_n = p u_n p$.

Let $\varepsilon > 0$ and fix $x, y \in (M)_1$. Since $q_m \to 1$ strongly and since $\mathrm{Tr}(p) < \infty$, take $m \in \mathbf{N}$ large enough such that

$$\|q_m x p - x p\|_{2,\mathrm{Tr}} + \|p y q_m - p y\|_{2,\mathrm{Tr}} < \varepsilon.$$

Since $\mathrm{Tr}(q_m) < \infty$, next choose $i, j \in \mathbf{N}$ such that

$$\|q_m x p - q_m x_i\|_{2,\mathrm{Tr}} + \|p y q_m - x_j q_m\|_{2,\mathrm{Tr}} < \varepsilon.$$

Now, for every $n \in \mathbf{N}$, we have

$$\begin{aligned} \|E_B(x u_n y)\|_{2,\mathrm{Tr}} &= \|E_B(x p u_n p y)\|_{2,Tr} \\ &\leq \|E_B(q_m x p u_n p y q_m)\|_{2,\mathrm{Tr}} + \varepsilon \\ &\leq \|E_B(q_m x_i u_n x_j q_m)\|_{2,\mathrm{Tr}} + 2\varepsilon. \end{aligned}$$

Therefore, if $n \geq \max\{m, i, j\}$, we get

$$\|E_B(x u_n y)\|_{2,\mathrm{Tr}} \leq \frac{1}{n} + 2\varepsilon.$$

Write Tr_n for the non-normalized faithful trace on $\mathbf{M}_n(\mathbf{C})$. The faithful normal semifinite trace $\mathrm{Tr}_n \otimes \mathrm{Tr}$ on $\mathbf{M}_n(\mathbf{C}) \otimes M$ will be simply denoted by Tr. Observe that if $A \preceq_M B$ in the sense of Lemma (4.3.5), then there exist $n \geq 1$, a nonzero projection $q \in B^n$ such that $\mathrm{Tr}(q) < \infty$, a nonzero partial isometry $v \in \mathbf{M}_{1,n}(\mathbf{C}) \otimes M$ and a unital *-homomorphism $\psi: A \to qB^n q$ such that $xv = v\psi(x), \forall x \in A$. In the case when A and B are maximal abelian, one can get a more precise result. This is an analog of a result by Popa ([204], Theorem A.1) for semifinite von Neumann algebras.

Proposition (4.3.6)[190]: Let (M, Tr) be a semifinite von Neumann algebra. Let $B \subset M$ be a maximal abelian von Neumann subalgebra such that $\mathrm{Tr}_{|B}$ is still semifinite. Let $p \in M$ be a non-zero projection such that $\mathrm{Tr}(p) < \infty$ and $A \subset pMp$ a maximal abelian von Neumann subalgebra. The following are equivalent:

1. $A \preceq_M B$ in the sense of Lemma (4.3.5).
2. There exists a nonzero partial isometry $v \in M$ such that $vv^* \in A, v^*v \in B$ and $v^*Av = Bv^*v$.

Proof. We only need to prove (a) $\Longrightarrow$ (b). The proof is very similar to the one of ([204] Theorem A.1). We will use exactly the same reasoning as in the proof of ([188] Theorem C.3).

Since $A \preceq_M B$ in the sense of Lemma (4.3.5), we can find $n \geq 1$, a nonzero Trfinite projection $q \in \mathbf{M}_n(\mathbf{C}) \otimes B$, a nonzero partial isometry $w \in \mathbf{M}_{1,n}(\mathbf{C}) \otimes pM$ and a unital $*$-homomorphism $\psi: A \to q(\mathbf{M}_n(\mathbf{C}) \otimes B)q$ such that $xw = w\psi(x)$, $\forall x \in A$. Since we can replace q by an equivalent projection in $\mathbf{M}_n(\mathbf{C}) \otimes B$, we may assume $q = \mathrm{Diag}_n(q_1, \dots, q_n)$ (see for instance second item in ([188] Lemma C.2). Observe now that $\mathrm{Diag}_n(q_1 B, \dots, q_n B)$ is maximal abelian in $q(\mathbf{M}_n(\mathbf{C}) \otimes B)q$.

Since B is abelian, $q(\mathbf{M}_n(\mathbf{C}) \otimes B)q$ is of finite type I. Since A is abelian, up to unitary conjugacy by a unitary in $q(\mathbf{M}_n(\mathbf{C}) \otimes B)q$, we may assume that $\psi(A) \subset \mathrm{Diag}_n(q_1 B, \dots, q_n B)$ (see [188] Lemma C.2). We can now cut down ψ and w by one of projections $(0, \dots, q_i, \dots, 0)$ and assume $n = 1$ from the beginning.

Write $e = ww^* \in A$ (since $A' \cap pMp = A$) and $f = w^*w \in \psi(A)' \cap qMq$. By spatiality, we have

$$f(\psi(A)' \cap qMq)f = (\psi(A)f)' \cap fMf = (w^*Aw)' \cap fMf = w^*Aw,$$

which is abelian. Let $Q := \psi(A)' \cap qMq$, which is a finite von Neumann algebra. Since $Bq \subset Q$ is maximal abelian and $f \in Q$ is an abelian projection, ([188], Lemma C.2) yields a partial isometry $u \in Q$ such that $uu^* = f$ and $u^*Qu \subset Bq$. Define now $v = wu$. We get

$$v^*Av = u^*w^*Awu = u^*f(\psi(A)' \cap qMq)fu \subset Bq.$$

Moreover $vv^* = wuu^*w^* = wfw^* = e \in A$. Since v^*Av and Bv^*v are both maximal abelian, we get $v^*Av = Bv^*v$.

Let $\mathcal{M}$ be a von Neumann algebra. Let φ be a faithful normal state on $\mathcal{M}$. Denote by $\mathcal{M}^\varphi$ the centralizer and by $M = \mathcal{M} \rtimes_{\sigma^\varphi} \mathbf{R}$ the core of $\mathcal{M}$, where σ^φ is the modular group associated with the state φ. Denote by $\pi_{\sigma^\varphi}: \mathcal{M} \to M$ the representation of $\mathcal{M}$ in its core M, i.e. $\pi_{\sigma^\varphi}(x) = \left(\sigma_{-t}^\varphi(x)\right)_{t\in\mathbf{R}}$ for every $x \in \mathcal{M}$, and denote by $\lambda^\varphi(s)$ the unitaries in $L(\mathbf{R})$ implementing the action σ^φ. Consider the dual weight $\hat{\varphi}$ on M (see [186]) which satisfies the following:

$$\begin{aligned} \sigma_t^{\hat{\varphi}}(\pi_{\sigma^\varphi}(x)) &= \pi_{\sigma^\varphi}\left(\sigma_t^\varphi(x)\right), \forall x \in \mathcal{M} \\ \sigma_t^{\hat{\varphi}}(\lambda^\varphi(s)) &= \lambda^\varphi(s), \forall s \in \mathbf{R}. \end{aligned}$$

Note that $\hat{\varphi}$ is a semifinite faithful normal weight on M. Write θ^φ for the dual action of σ^φ on M, where we identify $\mathbf{R}$ with its Pontryagin dual. Take now h_φ a nonsingular positive self-adjoint operator affiliated with $L(\mathbf{R})$ such that $h_\varphi^{is} = \lambda^\varphi(s)$, for any $s \in \mathbf{R}$. Define $\mathrm{Tr}_\varphi := \hat{\varphi}\left(h_\varphi^{-1}\cdot\right)$. We get that Tr_φ is a semifinite faithful normal trace on M and the dual action θ^φ scales the trace Tr_φ:

$$\mathrm{Tr}_\varphi \circ \theta_s^\varphi(x) = e^{-s}\,\mathrm{Tr}_\varphi(x), \forall x \in M_+, \forall s \in \mathbf{R}.$$

Moreover, the canonical faithful normal conditional expectation $E_{L(\mathbf{R})}: M \to L(\mathbf{R})$ defined by $E_{L(\mathbf{R})}(x\lambda^\varphi(s)) = \varphi(x)\lambda^\varphi(s)$ preserves the trace Tr_φ, i.e.

$$\mathrm{Tr}_\varphi \circ E_{L(\mathbf{R})}(x) = \mathrm{Tr}_\varphi(x), \forall x \in M_+.$$

There is also a functorial construction of the core of the von Neumann algebra $\mathcal{M}$ which does not rely on the choice of a particular state φ on $\mathcal{M}$(see [4], [192],[193]). This is called the noncommutative flow of weights. We will simply denote it by $(\mathcal{M} \subset M, \theta, \mathrm{Tr})$, where M is the core of $\mathcal{M}, \theta$ is the dual action of $\mathbf{R}$ on the core M and Tr is the semifinite faithful normal trace on M such that $\mathrm{Tr} \circ \theta_s = e^{-s}\mathrm{Tr}$, for any $s \in \mathbf{R}$. Let φ be a faithful normal state on $\mathcal{M}$. It follows from ([193], Theorem 3.5) and ([16], Theorem XII.6.10) that there exists a natural *-isomorphism

$$\Pi_\varphi: \mathcal{M} \rtimes_{\sigma^\varphi} \mathbf{R} \to M$$

such that

$$\begin{aligned} \Pi_\varphi \circ \theta^\varphi &= \theta \circ \Pi_\varphi \\ \mathrm{Tr}_\varphi &= \mathrm{Tr} \circ \Pi_\varphi \\ \Pi_\varphi(\pi_{\sigma^\varphi}(\mathcal{M})) &= \mathcal{M}. \end{aligned}$$

Let now φ, ψ be two faithful normal states on $\mathcal{M}$. Through the *-isomorphism $\Pi_{\varphi,\psi} := \Pi_\psi^{-1} \circ \Pi_\varphi: \mathcal{M} \rtimes_{\sigma^\varphi} \mathbf{R} \to \mathcal{M} \rtimes_{\sigma^\psi} \mathbf{R}$, we will identify

$$\left(\pi_{\sigma^\varphi}(\mathcal{M}) \subset \mathcal{M} \rtimes_{\sigma^\varphi} \mathbf{R}, \theta^\varphi, \mathrm{Tr}_\varphi\right) \text{ with } \left(\pi_{\sigma^\psi}(\mathcal{M}) \subset \mathcal{M} \rtimes_{\sigma^\psi} \mathbf{R}, \theta^\psi, \mathrm{Tr}_\psi\right).$$

In the sequel, we will refer to the triple $(\mathcal{M} \subset M, \theta, \mathrm{Tr})$ as the noncommutative flow of weights. By Takesaki's Duality Theorem [186], we have

$$(\mathcal{M} \rtimes_\sigma \mathbf{R}) \rtimes_{(\theta_s)} \mathbf{R} \cong \mathcal{M} \mathbin{\bar{\otimes}} \mathbf{B}\left(L^2(\mathbf{R})\right).$$

In particular, $\mathcal{M}$ is amenable if and only if $M = \mathcal{M} \rtimes_{\sigma} \mathbf{R}$ is amenable. The following well-known proposition will be useful.

Proposition (4.3.7)[190]: Let φ be a faithful normal state on $\mathcal{M}$. Let $M = \mathcal{M} \rtimes_{\sigma^\varphi} \mathbf{R}$ be as above. Then $L(\mathbf{R})' \cap M = \mathcal{M}^\varphi \bar{\otimes} L(\mathbf{R})$. In particular, if $\mathcal{M}^\varphi = \mathbf{C}$ then $L(\mathbf{R})$ is maximal abelian in M.

Proof. We regard $M = \mathcal{M} \rtimes_{\sigma^\varphi} \mathbf{R}$ generated by $\pi(x) = \left(\sigma_{-t}^{\varphi}(x)\right)_{t\in\mathbf{R}}$, for $x \in \mathcal{M}$, and $1 \otimes \lambda^\varphi(t)$, for $t \in \mathbf{R}$. Therefore $M \subset \mathcal{M}\overline{\mathbf{B}}(L^2(\mathbf{R}))$. Since $L(\mathbf{R}) \subset \mathbf{B}(L^2(\mathbf{R}))$ is maximal abelian, we get $L(\mathbf{R})' \cap M \subset \mathcal{M} \bar{\otimes} L(\mathbf{R})$.

Denote by $\hat{\varphi}$ the dual weight of φ on M (see e.g. [186]). The following relations are true: for every $s, t \in \mathbf{R}$, for every $x \in \mathcal{M}$,

$$\begin{aligned} \sigma_t^{\hat{\varphi}}(\pi(x)) &= \pi\left(\sigma_t^{\varphi}(x)\right) \\ \sigma_t^{\hat{\varphi}}(1 \otimes \lambda^\varphi(s)) &= 1 \otimes \lambda^\varphi(s) \\ \Delta_{\hat{\varphi}}^{it} &= \Delta_{\varphi}^{it} \otimes 1. \end{aligned}$$

Since $(1 \otimes \lambda^\varphi(s))_{s\in\mathbf{R}}$ is a 1-cocycle for $\left(\sigma_t^{\hat{\varphi}}\right)$, ([4], Théorème 1.2.4) implies that the faithful normal semifinite weight Tr given by $\sigma_t^{\mathrm{Tr}} = (1 \otimes \lambda^\varphi(t))^* \sigma_t^{\hat{\varphi}} (1 \otimes \lambda^\varphi(t))$ is a trace on M. This implies that $L(\mathbf{R})' \cap M$ is exactly the centralizer of the weight $\hat{\varphi}$. Since $\Delta_{\hat{\varphi}}^{it} = \Delta_{\varphi}^{it} \otimes 1$, for every $t \in \mathbf{R}$, we get $L(\mathbf{R})' \cap M \subset \mathcal{M}^\varphi \bar{\otimes} \mathbf{B}(L^2(\mathbf{R}))$. Thus $L(\mathbf{R})' \cap M = \mathcal{M}^\varphi \bar{\otimes} L(\mathbf{R})$.

Definition (4.3.8)[190]: Let $\mathcal{M}$ be any von Neumann algebra. A von Neumann subalgebra $A \subset \mathcal{M}$ is said to be a Cartan subalgebra if the following conditions hold:

1. A is maximal abelian, i.e. $A = A' \cap \mathcal{M}$.
2. There exists a faithful normal conditional expectation $E: \mathcal{M} \to A$.
3. The normalizer $\mathcal{N}_{\mathcal{M}}(A) = \{u \in \mathcal{U}(\mathcal{M}): uAu^* = A\}$ generates $\mathcal{M}$.

Let $A \subset \mathcal{M}$ be a Cartan subalgebra. Let τ be a faithful normal tracial state on A. Then $\varphi = \tau \circ E$ is a faithful normal state on $\mathcal{M}$. Moreover $A \subset \mathcal{M}^\varphi$, where $\mathcal{M}^\varphi$ denotes the centralizer of φ. Write $\left(\sigma_t^\varphi\right)$ for the modular automorphism group. Denote by $M = \mathcal{M} \rtimes_{\sigma^\varphi} \mathbf{R}$ the continuous core and write $\lambda^\varphi(t)$ for the unitaries in M which implement the modular action. The following proposition is well-known and will be a crucial tool in order to prove Theorem (4.3.37).

Proposition (4.3.9)[190]: The von Neumann subalgebra $A \bar{\otimes} L(\mathbf{R}) \subset \mathcal{M} \rtimes_\sigma \mathbf{R}$ is a Cartan subalgebra.

Proof. Since $A \subset M$ and $L(\mathbf{R}) \subset \mathbf{B}(L^2(\mathbf{R}))$ are both maximal abelian, it follows that $A \bar{\otimes} L(\mathbf{R})$ is maximal abelian in $\mathcal{M} \mathbin{\overline{\otimes}} \mathbf{B}(L^2(\mathbf{R}))$. Therefore $A \bar{\otimes} L(\mathbf{R})$ is maximal abelian in $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbf{R}$.

The faithful normal conditional expectation $F: \mathcal{M} \rtimes_{\sigma^\varphi} \mathbf{R} \to A \bar{\otimes} L(\mathbf{R})$ is given by: $F(x\lambda^\varphi(t)) = E(x)\lambda^\varphi(t), \forall x \in \mathcal{M}, \forall t \in \mathbf{R}$. Observe that F preserves the canonical trace Tr_φ.

It remains to show that $A \bar{\otimes} L(\mathbf{R})$ is regular in $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbf{R}$. Recall that $A \subset \mathcal{M}^\varphi$, so that $a\lambda^\varphi(t) = \lambda^\varphi(t)a$, for every $t \in \mathbf{R}$ and every $a \in A$. For every $t \in \mathbf{R}$, every $u \in \mathcal{N}_{\mathcal{M}}(A)$ and every $a \in A$, we have

$$\begin{aligned}\sigma_t^{\varphi}(u)u^*a &= \sigma_t^{\varphi}(u)(u^*au)u^* \\ &= \sigma_t^{\varphi}(uu^*au)u^* \\ &= a\sigma_t^{\varphi}(u)u^*,\end{aligned}$$

so that $\sigma_t^{\varphi}(u)u^* \in A' \cap \mathcal{M} = A$. We moreover have

$$u(a\lambda^{\varphi}(t))u^* = (uau^*)u\lambda^{\varphi}(t) = (uau^*)\left(u\sigma_t^{\varphi}(u^*)\right)\lambda^{\varphi}(t),$$

so that $u(A \bar{\otimes} L(\mathbf{R}))u^* = A \bar{\otimes} L(\mathbf{R})$. Consequently, $A \bar{\otimes} L(\mathbf{R}) \subset \mathcal{M} \rtimes_{\sigma^{\varphi}} \mathbf{R}$ is regular.

Assume that $\mathcal{M}$ is a type II von Neumann algebra. Then $M = \mathcal{M} \rtimes_{\sigma} \mathbf{R}$ is still of type II. Assume now that $\mathcal{M}$ is a type III von Neumann algebra. Then M is of type II_{∞}. Let $p \in A \bar{\otimes} L(\mathbf{R})$ be a nonzero projection such that $\mathrm{Tr}(p) < \infty$, so that pMp is of type II_1. The next proposition shows that $(A \bar{\otimes} L(\mathbf{R}))p \subset pMp$ is a Cartan subalgebra.

Proposition (4.3.10)[190]: Let N be a type II_{∞} von Neumann algebra with a faithful normal semifinite trace Tr. Let $B \subset N$ be a maximal abelian *-subalgebra for which $\mathrm{Tr}_{|B}$ is still semifinite. Let $p \in B$ be a nonzero projection such that $\mathrm{Tr}(p) < \infty$. Then $\mathcal{N}_{pMp}(Bp)'' = p\mathcal{N}_M(B)''p$.

Proof. The equality $(pBp)' \cap pMp = p(B' \cap M)p$ is well-known (see for instance ([200], Lemma 2.1). Thus, Bp is maximal abelian in pMp. Let $u \in \mathcal{N}_M(B)$. We have

$$pup(Bp) = puBp = pBup = (Bp)pup.$$

It follows that $p\mathcal{N}_M(B)''p \subset \mathcal{QN}_{pMp}(Bp)''$. The normalizer and the quasi-normalizer of a maximal abelian subalgebra generate the same von Neumann algebra (see [201], Theorem 2.7). Thus $\mathcal{QN}_{pMp}(Bp)'' = \mathcal{N}_{pMp}(Bp)''$ and $p\mathcal{N}_M(B)''p \subset \mathcal{N}_{pMp}(Bp)''$. Let now $v \in \mathcal{N}_{pMp}(Bp)$. Define $u = v + (1-p) \in \mathcal{U}(M)$. It is clear that $u \in \mathcal{N}_M(B)$ and $pup = v$. Therefore $\mathcal{N}_{pMp}(Bp)'' \subset p\mathcal{N}_M(B)''p$, which finishes the proof.

Definition (4.3.11)[190]: (Haagerup, [196]). A von Neumann algebra $\mathcal{N}$ is said to have the (weak") complete bounded approximation property if there exist a constant $C \geq 1$ and a net of normal finite rank completely bounded maps $\Phi_n: \mathcal{N} \to \mathcal{N}$ such that

1. $\Phi_n(x) \to x$ $*$-strongly, for every $x \in \mathcal{N}$;
2. $\limsup_n \|\Phi_n\|_{\mathrm{cb}} \leq C$.

The Cowling-Haagerup constant $\Lambda_{\mathrm{cb}}(\mathcal{N})$ is defined as the infimum of the constants C for which a net (Φ_n) as above exists. Also we say that $\mathcal{N}$ has the (weak*) complete metric approximation property (c.m.a.p.) if $\Lambda_{\mathrm{cb}}(\mathcal{N}) = 1$.

Theorem (4.3.12)[190]: The following are true.

1. $\Lambda_{\mathrm{cb}}(p\mathcal{M}p) \leq \Lambda_{\mathrm{cb}}(\mathcal{M})$, for every projection $p \in \mathcal{M}$.
2. If $\mathcal{N} \subset \mathcal{M}$ such that there exists a conditional expectation $E: \mathcal{M} \to \mathcal{N}$, then $\Lambda_{\mathrm{cb}}(\mathcal{N}) \leq \Lambda_{\mathrm{cb}}(\mathcal{M})$.
3. If $\mathcal{M}$ is amenable then $\Lambda_{\mathrm{cb}}(\mathcal{M}) = 1$.
4. Denote by σ the modular automorphism group on $\mathbb{M}$. Then $\Lambda_{\mathrm{cb}}(\mathcal{M}) = \Lambda_{\mathrm{cb}}(\mathcal{M} \rtimes_{\sigma} \mathbf{R})$
5. If $\mathcal{M}_i$ is amenable for every $i \in I$, then $\Lambda_{\mathrm{cb}}(*_{i\in I}\, \mathcal{M}_i) = 1$.

Proof. (i), (ii), (iv) follow from [143]. The equivalence between semidiscreteness and amenability [198] gives (iii). Finally (v) is due to [170].

Recall now the construction of the free ArakiWoods factors due to Shlyakhtenko [15]. Let $H_{\mathbf{R}}$ be a real separable Hilbert space and let (U_t) be an orthogonal representation of $\mathbf{R}$ on $H_{\mathbf{R}}$. Let $H = H_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$ be the complexified Hilbert space. Let J be the canonical anti-unitary involution on H defined by:

$$J(\xi + i\eta) = \xi - i\eta, \forall \xi, \eta \in H_{\mathbf{R}}.$$

If A is the infinitesimal generator of (U_t) on H, we recall that $j: H_{\mathbf{R}} \to H$ defined by $j(\zeta) = \left(\frac{2}{A^{-1}+1}\right)^{1/2} \zeta$ is an isometric embedding of $H_{\mathbf{R}}$ into H. Moreover, we have $JAJ = A^{-1}$. Let $K_{\mathbf{R}} = j(H_{\mathbf{R}})$. It is easy to see that $K_{\mathbf{R}} \cap iK_{\mathbf{R}} = \{0\}$ and $K_{\mathbf{R}} + iK_{\mathbf{R}}$ is dense in H. Write $I = JA^{-1/2}$. Then I is a conjugate-linear closed invertible operator on H satisfying $I = I^{-1}$ and $I^*I = A^{-1}$. Such an operator is called an involution on H. Moreover, $K_{\mathbf{R}} = \{\xi \in \mathrm{dom}(I): I\xi = \xi\}$. We introduce the full Fock space of :

$$\mathcal{F}(H) = \mathbf{C}\Omega \oplus \bigoplus_{n=1}^{\infty} H^{\otimes n}.$$

The unit vector Ω is called the vacuum vector. For any $\xi \in H$, define the left creation operator $\ell(\xi): \mathcal{F}(H) \to \mathcal{F}(H)$

$$\begin{cases} \ell(\xi)\Omega = \xi, \\ \ell(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n. \end{cases}$$

We have $\| \ell(\xi) \|_\infty = \| \xi \|$ and $\ell(\xi)$ is an isometry if $\| \xi \| = 1$. For any $\xi \in H$, we denote by $s(\xi)$ the real part of $\ell(\xi)$ given by

$$s(\xi) = \frac{\ell(\xi) + \ell(\xi)^*}{2}.$$

The crucial result of Voiculescu [197] is that the distribution of the operator $s(\xi)$ with respect to the vacuum vector state $\chi(x) = \langle x\Omega, \Omega\rangle$ is the semicircular law of Wigner supported on the interval $[-\| \xi \|, \| \xi \|]$.

Definition (4.3.12)[190]: (Shlyakhtenko, [15]). Let (U_t) be an orthogonal representation of $\mathbf{R}$ on the real Hilbert space $H_{\mathbf{R}}$. The free Araki-Woods von Neumann algebra associated with $(H_{\mathbf{R}}, U_t)$, denoted by $\Gamma(H_{\mathbf{R}}, U_t)''$, is defined by

$$\Gamma(H_{\mathbf{R}}, U_t)'' := \{s(\xi): \xi \in K_{\mathbf{R}}\}''.$$

We will denote by $\Gamma(H_{\mathbf{R}}, U_t)$ the C^*-algebra generated by the (ξ) 's for all $\xi \in K_{\mathbf{R}}$.

The vector state $\chi(x) = \langle x\Omega, \Omega\rangle$ is called the free quasi-free state and is faithful on $\Gamma(H_{\mathbf{R}}, U_t)''$. Let $\xi, \eta \in K_{\mathbf{R}}$ and write $\zeta = \xi + i\eta$. We have

$$2s(\xi) + 2is(\eta) = \ell(\zeta) + \ell(I\zeta)^*.$$

Thus, $\Gamma(H_{\mathbf{R}}, U_t)''$ is generated as a von Neumann algebra by the operators of the form $\ell(\zeta) + \ell(I\zeta)^*$ where $\zeta \in \mathrm{dom}(I)$. Note that the modular group $\left(\sigma_t^{\chi}\right)$ of the free quasi-free state χ is given by $\sigma_{-t}^{\chi} = \mathrm{Ad}\big(\mathcal{F}(U_t)\big)$, where $\mathcal{F}(U_t) = 1 \oplus \bigoplus_{n\geq 1} U_t^{\otimes n}$.

In particular, it satisfies

$$\sigma_{-t}^{\chi}(\ell(\zeta) + \ell(I\zeta)^*) = \ell(U_t\zeta) + \ell(IU_t\zeta)^*, \forall \zeta \in \mathrm{dom}(I), \forall t \in \mathbf{R}.$$

The free Araki-Woods factors provided many new examples of full factors of type III [2],[4],[12]. We can summarize the general properties of the free Araki-Woods factors in the following theorem (see also [18]):

Theorem (4.3.13)[190]: (Shlyakhtenko, [12],[13],[45],[46]). Let (U_t) be an orthogonal representation of $\mathbf{R}$ on the real Hilbert space $H_{\mathbf{R}}$ with $\dim H_{\mathbf{R}} \geq 2$. Denote by $\mathcal{M} := \Gamma(H_{\mathbf{R}}, U_t)''$.

1. $\mathcal{M}$ is a full factor and Connes' invariant $\tau(\mathcal{M})$ is the weakest topology on $\mathbf{R}$ that makes the map $t \mapsto U_t$ $*$-strongly continuous.
2. $\mathcal{M}$ is of type II_1 if and only if $U_t = 1$, for every $t \in \mathbf{R}$.
3. $\mathcal{M}$ is of type III $_\lambda (0 < \lambda < 1)$ if and only if (U_t) is periodic of period $\frac{2\pi}{|\log \lambda|}$.
4. $\mathcal{M}$ is of type III_1 in the other cases.

5. The factor $\mathcal{M}$ has almost periodic states if and only if (U_t) is almost periodic.

Shlyakhtenko moreover showed [12] that every free Araki-Woods factor $\mathcal{M} = \Gamma(H_{\mathbf{R}}, U_t)''$ is generalized solid in the sense of [162],[187]: for every diffuse subalgebra $A \subset \mathcal{M}$ for which there exists a faithful normal conditional expectation $E: \mathcal{M} \to A$, the relative commutant $A' \cap \mathcal{M}$ is amenable. [179] showed that every type III_1 free Araki-Woods factor has trivial bicentralizer [178].

There are not so many ways to produce concrete examples of completely bounded maps on free Araki-Woods von Neumann algebras. When (U_t) is trivial, one recovers the free group algebras, and harmonic analysis joins the game with Fourier multipliers. On $\mathbf{F}_\infty$, multipliers that only depend on the length are said to be radial. Haagerup and Szwarc obtained a very nice characterization of them. Their approach was based on a one-to-one correspondence between Fourier multipliers on a group G and Schur multipliers on $\mathbf{B}(\ell^2(G))$ established by Gilbert. Their idea was to look for a description of Schur multipliers obtained this way and they managed to do so for more general multipliers related to homogeneous trees. The key point is to find a shift algebra that is preserved by those Schur multipliers. This technique or some variations have operated with success on other groups [195],[205].

The free semicircular random variables and the canonical generators of $\mathbf{F}_\infty$ have different shape but there is a natural length for both of them which is related to freeness. This notion still makes sense after the quasi-free deformation and one can hope to have nice multipliers. We follow the scheme of Haagerup and Szwarc, but Gilbert's theorem is missing here (there is no easy way to extend multipliers). We obtain exactly the same characterization and the parallel with Schur multipliers is very striking. This is the first step towards the approximation property that originates from [196], where it was shown that the projection onto tensors of a fixed given length is bounded. Haagerup's ideas turned out to be efficient to prove various approximation properties in relationship with Khintchine type inequalities (see [191],[113]). The second step consists in using functorial completely positive maps called second quantizations (see [46],[60]). The new point is that we show that the second quantization is valid under a milder assumption than the one in [15].

The C^*-algebra $\Gamma(H_{\mathbf{R}}, U_t)$ is generated by real parts of some left creation operators. Since we look for completely bounded maps on free ArakiWoods algebras, it seems natural to try to find them as restrictions on some larger algebra. This is why we are interested in basic properties of the algebra generated by creation operators.

Let H be a complex Hilbert space and $\mathcal{F}(H)$ the corresponding full Fock space. We write $\mathcal{T}(H)$ for the C^*-algebra generated by all the left creation operators $\mathcal{F}(H) = \langle \ell(e): e \in H \rangle$. It is easy to verify that for any $, f \in H$:

$$\ell(f)^* \ell(e) = \langle f, e \rangle.$$

In fact, this property completely characterizes the algebra $\mathcal{T}(H)$. Indeed, in the sense of [199], $\mathcal{T}(H)$ is a Toeplitz algebra and satisfies the following universal property (see [199] Theorem 3.4): if $u: H \to \mathbf{B}(K)$ is a linear map (for some Hilbert space K) so that $u^*(f)u(e) = \langle f, e \rangle$, then there is a unique *-homomorphism $\pi: \mathcal{T}(H) \to \mathbf{B}(K)$ so that $\pi(\ell(e)) = u(e)$.

When $H = \mathbf{C}$, we will simply denote $\mathcal{T}(\mathbf{C})$ by $\mathcal{T}$: this is the universal C^* algebra generated by a shift operator S (a nonunitary isometry). We will need the following very elementary estimates about creation operators:

Lemma (4.3.14)[190]: For orthonormal families $(e_i),(f_i)$ in H and $\alpha_i \in \mathbf{C}$ with $|\alpha_i| \le 1$, we have

$$\left\|\frac{1}{n}\sum_{i=1}^{n} \alpha_i \ell(e_i)\ell(f_i)^*\right\|_\infty \le \frac{1}{n} \text{ and } \left\|\frac{1}{n}\sum_{i=1}^{n} \alpha_i \ell(e_i)\ell(f_i)\right\|_\infty \le \frac{1}{\sqrt{n}}.$$

Proof. Let $(e_i),(f_i)$ be orthonormal families in H and $\alpha_i \in \mathbf{C}$ with $|\alpha_i| \le 1$. The first inequality follows from

$$\begin{aligned}\left(\frac{1}{n}\sum_{i=1}^{n} \alpha_i \ell(e_i)\ell(f_i)^*\right)\left(\frac{1}{n}\sum_{i=1}^{n} \alpha_i \ell(e_i)\ell(f_i)^*\right)^* &= \frac{1}{n^2}\sum_{i=1}^{n} |\alpha_i|^2 \ell(e_i)\ell(e_i)^* \\ &\le \frac{1}{n^2}\sum_{i=1}^{n} \ell(e_i)\ell(e_i)^* \\ &\le \frac{1}{n^2}.\end{aligned}$$

The second inequality follows from

$$\begin{aligned}\left(\frac{1}{n}\sum_{i=1}^{n} \alpha_i \ell(e_i)\ell(f_i)\right)^*\left(\frac{1}{n}\sum_{i=1}^{n} \alpha_i \ell(e_i)\ell(f_i)\right) &= \frac{1}{n^2}\sum_{i=1}^{n} |\alpha_i|^2 \ell(f_i)^*\ell(f_i) \\ &\le \frac{1}{n^2}\sum_{i=1}^{n} \ell(f_i)^*\ell(f_i) \\ &= \frac{1}{n}.\end{aligned}$$

We come back to free Araki-Woods algebras: $\Gamma(H_{\mathbf{R}},U_t) = \langle s(\xi): \xi \in K_{\mathbf{R}}\rangle$ is the C^*-algebra generated by the (ξ) 's for all $\xi \in K_{\mathbf{R}}$, and $\Gamma(H_{\mathbf{R}},U_t)''$ is the corresponding von Neumann algebra. Given any vector e in $K_{\mathbf{R}} + iK_{\mathbf{R}}$, we will simply write $\bar{e}$ for $I(e)$ as $I(h+ik) = h - ik$, for $h,k \in K_{\mathbf{R}}$.

The vacuum vector Ω is separating and cyclic for $\Gamma(H_{\mathbf{R}},U_t)''$. Consequently any $x \in \Gamma(H_{\mathbf{R}},U_t)''$ is uniquely determined by $\xi = x\Omega \in \mathcal{F}(H)$, so we will write $x = W(\xi)$. Note that for $\xi \in K_{\mathbf{R}}$, we recover the semicircular random variables $W(\xi) = 2s(\xi)$ generating $\Gamma(H_{\mathbf{R}},U_t)''$. It readily yields $W(e) = \ell(e) + \ell(\bar{e})^*$, for every $e \in K_{\mathbf{R}} + iK_{\mathbf{R}}$.

Given any vectors e_k belonging to $K_{\mathbf{R}} + iK_{\mathbf{R}}$, it is easy to check that $e_1 \otimes \cdots \otimes e_n$ lies in $\Gamma(H_{\mathbf{R}},U_t)\Omega$. Moreover we have a nice description of $W(e_1 \otimes \cdots \otimes e_n)$ in terms of the $\ell(e_k)$'s called the Wick formula. Since it plays a crucial role in our arguments, we state it as a lemma.

Lemma (4.3.15)[190]: (Wick formula). For any $(e_i)_{i\in\mathbf{N}}$ in $K_{\mathbf{R}} + iK_{\mathbf{R}}$ and any $n \ge 0$:

$$W(e_1 \otimes \cdots \otimes e_n) = \sum_{k=0}^{n} \ell(e_1)\cdots\ell(e_k)\ell(\bar{e}_{k+1})^*\cdots\ell(\bar{e}_n)^*.$$

Proof. We prove it by induction on n. For $n = 0,1$, we have $W(\Omega) = 1$ and we observed that $W(e_i) = \ell(e_i) + \ell(\bar{e}_i)^*$.

Next, for $e_0 \in K_{\mathbf{R}} + iK_{\mathbf{R}}$, we have

$$\begin{aligned}W(e_0)W(e_1 \otimes \cdots \otimes e_n)\Omega &= W(e_0)(e_1 \otimes \cdots \otimes e_n) \\ &= (\ell(e_0) + \ell(\bar{e}_0)^*)e_1 \otimes \cdots \otimes e_n \\ &= e_0 \otimes e_1 \otimes \cdots \otimes e_n + \langle \bar{e}_0, e_1\rangle e_2 \otimes \cdots \otimes e_n.\end{aligned}$$

Hence

$$W(e_0 \otimes \cdots \otimes e_n) = W(e_0)W(e_1 \otimes \cdots \otimes e_n) - \langle \bar{e}_0, e_1 \rangle W(e_2 \otimes \cdots \otimes e_n),$$

but using the assumption for n and $n-1$ and the commutation relations

$$\ell(\bar{e}_0)^* W(e_1 \otimes \cdots \otimes e_n) = \langle \bar{e}_0, e_1 \rangle W(e_2 \otimes \cdots \otimes e_n) + \ell(\bar{e}_0)^* \ell(\bar{e}_1)^* \cdots \ell(\bar{e}_n)^*.$$

Finally $\ell(e_0)W(e_1 \otimes \cdots \otimes e_n)$ gives the first n terms in the Wick formula for order $n+1$.

This formula expresses $W(e_1 \otimes \cdots \otimes e_n)$ as an element of $\mathcal{F}(H)$ and has many consequences such as Khintchine type inequalities in [191],[113],[60] for instance. We let

$$\mathcal{W} = \mathrm{span}\{W(e_1 \otimes \cdots \otimes e_n) : n \geq 0, e_k \in K_{\mathbf{R}} + iK_{\mathbf{R}}\}.$$

It is a dense *-subalgebra of $\Gamma(H_{\mathbf{R}}, U_t)$.

We will use the notion of completely bounded maps (see [135]). We will not need very much beyond definitions and the fact that bounded functionals are automatically completely bounded (with the same norm).

The construction of our radial multipliers relies on some functionals on $\mathcal{T}$. Let $\varphi: \mathbf{N} \to \mathbf{C}$ be a function. The radial functional γ associated to φ is defined on $\mathrm{span}\{S^i S^{*j}\} \subset \mathcal{F}$ by $\gamma(S^i S^{*j}) = \varphi(i+j)$.

The C^*-algebra $\mathcal{T}$ admits very few irreducible representations (the identity and its characters). It is thus possible to compute exactly the norm of such radial linear forms, see ([195] Proposition 1.8 and Theorem 1.3) and [205]:

Proposition (4.3.16)[190]: The functional γ extends to a bounded map on $\mathcal{T}$ if and only if $B = [\varphi(i+j) - \varphi(i+j+2)]_{i,j\geq 0}$ is a trace-class operator. If this is the case, then there are constants $c_1, c_2 \in \mathbf{C}$ and a unique $\psi: \mathbf{N} \longrightarrow \mathbf{C}$ such that

$$\forall n \in \mathbf{N}, \varphi(n) = c_1 + c_2(-1)^n + \psi(n), \text{ and } \lim_n \psi(n) = 0.$$

Moreover

$$\| \gamma \|_{\mathcal{T}^*} = |c_1| + |c_2| + \| B \|_1,$$

where $\| B \|_1$ is the trace norm of B.

We say that γ is the radial functional associated to φ. The definition of multipliers on $\Gamma(H_{\mathbf{R}}, U_t)''$ follows the same scheme. Define m_φ on $\mathcal{W}$ by

$$\mathrm{m}_\varphi\big(W(e_1 \otimes \cdots \otimes e_n)\big) = \varphi(n) W(e_1 \otimes \cdots \otimes e_n).$$

Lemma (4.3.17)[190]: Let $\varphi: \mathbf{N} \to \mathbf{C}$ be any function. If m_φ can be extended to a completely contractive map on $\Gamma(H_{\mathbf{R}}, U_t)$, then there is a unique normal completely contractive extension of m_φ from $\Gamma(H_{\mathbf{R}}, U_t)''$ to $\Gamma(H_{\mathbf{R}}, U_t)''$.

Proof. This is a standard fact. The space $\mathcal{W}$ is norm dense in $\Gamma(H_{\mathbf{R}}, U_t)$ which is weak-* dense in $\Gamma(H_{\mathbf{R}}, U_t)''$. So $\mathcal{W}$ is also norm dense in $\Gamma(H_{\mathbf{R}}, U_t)''_*$ using the basic embedding $\Gamma(H_{\mathbf{R}}, U_t)'' \to \Gamma(H_{\mathbf{R}}, U_t)''_*$ given by $j(x)(y) = \chi(xy)$ (where χ denotes the free quasi-free state). By a duality argument, $\mathrm{m}_\varphi: \mathcal{W} \to \mathcal{W}$ extends uniquely to a completely contractive map on $\Gamma(H_{\mathbf{R}}, U_t)''_*$, say T. Thus T^* is the only operator that satisfies the conclusion.

If m_φ is completely bounded on $\Gamma(H_{\mathbf{R}}, U_t)$, we say that m_φ is a radial multiplier on $\Gamma(H_{\mathbf{R}}, U_t)''$.

Theorem (4.3.18)[190]: Let $\varphi: \mathbf{N} \to \mathbf{C}$ be any function and $H_{\mathbf{R}}$ an infinite dimensional real Hilbert space with a one-parameter group (U_t) of orthogonal transformations. Then φ defines a completely bounded radial multiplier on $\Gamma(H_{\mathbf{R}}, U_t)''$ if and only if the radial functional γ on $\mathcal{T}$ associated to φ is bounded. Moreover

$$\|\mathrm{m}_\varphi\|_{\mathrm{cb}} = \| \gamma \|_{\mathcal{T}^*}.$$

Thanks to Proposition (4.3.16), we have an explicit formula for $\| \gamma \|_{\mathcal{F}^*}$.

Proof of the upper bound. We assume that φ gives a bounded functional γ on $\mathcal{T}$. By the universal property of $\mathcal{T}(H)$, there is a $*$-homomorphism

$$\pi: \begin{array}{ll} \mathcal{F}(H) & \to \mathcal{F}(H) \otimes_{\min} \mathcal{F} \\ \ell(\xi) & \mapsto \ell(\xi) \otimes S \end{array}$$

So the map $\mathrm{m}_\varphi = (\mathrm{Id} \otimes \gamma)\pi: \mathcal{T}(H) \to \mathcal{T}(H)$ is completely bounded on $\mathcal{T}(H)$ with norm $\| \gamma \|_{\mathcal{T}^*}$. We have, for all $n \in \mathbf{N}$ and all $e_k \in K_{\mathbf{R}} + iK_{\mathbf{R}}$:

$$\mathrm{m}_\varphi(\ell(e_1) \cdots \ell(e_k)\ell(\bar{e}_{k+1})^* \cdots \ell(\bar{e}_n)^*) = \varphi(n)\ell(e_1) \cdots \ell(e_k)\ell(\bar{e}_{k+1})^* \cdots \ell(\bar{e}_n)^*.$$

Recall that the Wick formula (Lemma (4.3.15)) says

$$W(e_1 \otimes \cdots \otimes e_n) = \sum_{k=0}^{n} \ell(e_1) \cdots \ell(e_k)\ell(\bar{e}_{k+1})^* \cdots \ell(\bar{e}_n)^*.$$

Thus we derive that $m_\varphi\big(W(e_1 \otimes \cdots \otimes e_n)\big) = \varphi(n)W(e_1 \otimes \cdots \otimes e_n)$. So m_φ is bounded on $\Gamma(H_{\mathbf{R}}, U_t)$ and is a radial multiplier.

To check the necessity of the condition, the idea is similar to [195] or [205]. We find a shift algebra on which m_φ acts. We start by taking an orthonormal system $(e_i)_{i\geq 1}$ in $K_{\mathbf{R}} + iK_{\mathbf{R}}$ such that $\langle \bar{e}_i, \bar{e}_j \rangle = \langle \bar{e}_i, e_j \rangle = 0$ for all $i \neq j$ and $\|\bar{e}_i\| \leq 1$ (this is possible by the Gram-Schmidt algorithm). Consider the following element for ≥ 1:

$$S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \ell(e_i) \otimes W(e_i) \in \mathcal{T}(H) \otimes \mathbf{B}\big(\mathcal{F}(H)\big).$$

Lemma (4.3.19)[190]: For all $n \geq 1$,

$$\|S_n^* S_n - 1\|_\infty \leq \frac{3}{\sqrt{n}}.$$

Proof. We have

$$\begin{aligned} W(e_i)^* W(e_i) &= \big(\ell(e_i)^* + \ell(\bar{e}_i)\big)(\ell(e_i) + \ell(\bar{e}_i)^*) \\ &= 1 + \ell(\bar{e}_i)\ell(e_i) + \ell(\bar{e}_i)\ell(\bar{e}_i)^* + \ell(e_i)^*\ell(\bar{e}_i)^* \\ &= 1 + W(\bar{e}_i \otimes e_i). \end{aligned}$$

It follows that

$$\begin{aligned} S_n^* S_n &= \frac{1}{n} \sum_{i,j=1}^{n} \ell(e_i)^* \ell\big(e_j\big) \otimes W(e_i)^* W\big(e_j\big) \\ &= 1 \otimes 1 + \frac{1}{n} \sum_{i=1}^{n} 1 \otimes (\ell(\bar{e}_i)\ell(e_i) + \ell(\bar{e}_i)\ell(\bar{e}_i)^* + \ell(e_i)^*\ell(\bar{e}_i)^*). \end{aligned}$$

Lemma (4.3.14) yields

$$\left\| \frac{1}{n} \sum_{i=1}^{n} 1 \otimes W(\bar{e}_i \otimes e_i) \right\|_\infty \leq \frac{3}{\sqrt{n}},$$

so that we get the estimate.

For convenience, we will use a standard multi-index notation, we write i for $(i_1, \ldots, i_n) \in \mathbf{N}^n$ and $|\underline{i}| = n$. For $\alpha, \beta \geq 0$, set

$$\begin{aligned} e_{\underline{i}}^{\alpha,\beta} &= e_{i_1} \otimes \cdots \otimes e_{i_\alpha} \otimes \bar{e}_{i_{\alpha+1}} \cdots \otimes \bar{e}_{i_{\alpha+\beta}} \\ V_{\alpha,\beta}^n &= n^{-\frac{\alpha+\beta}{2}} \sum_{i_1,\ldots,i_{\alpha+\beta}=1}^{n} \ell\big(e_{i_1}\big) \cdots \ell\big(e_{i_\alpha}\big) \ell\big(e_{i_{\alpha+1}}\big)^* \cdots \ell\left(e_{i_{\alpha+\beta}}\right)^* \otimes W\left(e_{\underline{i}}^{\alpha,\beta}\right), \end{aligned}$$

if $\alpha+\beta>0$ and $V_{0,0}^n=1\otimes 1$.

Lemma (4.3.20)[190]: For all $\alpha,\beta\geq 0$,

$$S_n^\alpha S_n^{*\beta}-V_{\alpha,\beta}^n=O\left(\frac{1}{\sqrt{n}}\right).$$

Proof. We do it by induction on $\alpha+\beta$. When $\alpha+\beta\leq 1$, there is equality. Assume this holds for (α,β), we prove it for $(\alpha+1,\beta)$. First, $S_nV_{\alpha,\beta}^n$ is equal to

$$n^{-\frac{\alpha+\beta+1}{2}}\sum_{i_0,\ldots,i_{\alpha+\beta}=1}^{n}\ell(e_{i_0})\cdots\ell(e_{i_\alpha})\ell(e_{i_{\alpha+1}})^*\cdots\ell\left(e_{i_{\alpha+\beta}}\right)^*\otimes W(e_{i_0})W\left(e_{\underline{i}}^{\alpha}\right).$$

Recall the identity

$$W(h)W(h_1\otimes\cdots)=W(h\otimes h_1\otimes\cdots)+\langle\bar{h},h_1\rangle W(h_2\otimes\cdots)$$

used in the proof of the Wick formula. Therefore

$$S_nV_{\alpha,\beta}^n=V_{\alpha+1,\beta}^n+\left(\frac{1}{n}\sum_{i=1}^{n}\left\langle\bar{e}_i,e_i^{(*)}\right\rangle\ell(e_i)\ell(e_i)^{(*)}\otimes 1\right)V_{\tilde{\alpha},\tilde{\beta}}^n$$

where $(*)=1,\tilde{\alpha}=\alpha-1,\tilde{\beta}=\beta$ and $e_i^{(*)}=e_i$ if $\alpha>0$, and $\ell(e_i)^{(*)}=\ell(\bar{e}_i)^*,\tilde{\alpha}=0,\tilde{\beta}=\beta-1$ and $e_i^{(*)}=\bar{e}_i$ if $\alpha=0$. We have by Lemma (4.3.14) $\frac{1}{n}\sum_{i=1}^n\left\langle\bar{e}_i,e_i^{(*)}\right\rangle\ell(e_i)\ell(\bar{e}_i)^*=O\left(\frac{1}{n}\right)$ and $\frac{1}{n}\sum_{i=1}^n\left\langle\bar{e}_i,e_i^{(*)}\right\rangle\ell(e_i)\ell(e_i)=O\left(\frac{1}{\sqrt{n}}\right)$. This yields $S_nV_{\alpha,\beta}^n-V_{\alpha+1,\beta}^n=O\left(\frac{1}{\sqrt{n}}\right)$. According to Lemma (4.3.19) and the induction hypothesis, S_n and then $V_{a,b}^n$ for $a+b\leq\alpha+\beta$ are uniformly bounded in n. Consequently,

$$S_n^{\alpha+1}S_n^{*\beta}-V_{\alpha+1,\beta}^n=S_n\left(S_n^\alpha S_n^{*\beta}-V_{\alpha,\beta}^n\right)+O\left(\frac{1}{\sqrt{n}}\right)=O\left(\frac{1}{\sqrt{n}}\right).$$

The other case $(\alpha,\beta+1)$ is obtained by taking adjoints. '

Proof of the lower bound. Assume m_φ is a completely bounded multiplier on the free Araki-Woods factor $\Gamma(H_\mathrm{R},U_t)''$. Let $\mathfrak{U}$ be a nontrivial ultrafilter on N. Set $\mathcal{B}=\mathcal{T}(H)\otimes\mathrm{B}(\mathcal{F}(H))$ so that $S_n\in\mathcal{B}$. Consider the C^*-algebra $\mathcal{A}=\prod_{\mathfrak{U}}\mathcal{B}$ and T the ultrapower of $\mathrm{Id}\otimes\mathrm{m}_\varphi$. The element $S=(S_n)\in\mathcal{A}$ satisfies $S^*S=1$ by Lemma (4.3.19). As $(\mathrm{Id}\otimes m_\varphi)\left(V_{\alpha,\beta}^n\right)=\varphi(\alpha+\beta)V_{\alpha,\beta}^n$, we get by Lemma (4.3.20), $T\left(S^\alpha S^{*\beta}\right)=\varphi(\alpha+\beta)S^\alpha S^{*\beta}$. Taking a particular non constant φ (that does exist), this shows that S is non unitary and S is a shift. Thus, T leaves $\mathcal{T}=\langle S\rangle$ invariant. By composing it with the trivial character ω of $\mathcal{T}\left(\omega\left(S^\alpha S^{*\beta}\right)=1\right)$, we obtain that $\gamma=\omega T$ is a bounded functional on $\mathcal{T}$ with $\|\gamma\|_{\mathcal{T}^*}\leq\left\|\mathrm{m}_\varphi\right\|_{\mathrm{cb}}$.

A linear map between C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ is decomposable if it is a linear combination of completely positive maps from $\mathcal{A}$ to $\mathcal{B}$. Any functional can be decomposed into sums of states, so we have:

Corollary (4.3.21)[190]: Any radial multiplier on $\Gamma(H_\mathrm{R},U_t)$ is decomposable from $\Gamma(H_\mathrm{R},U_t)$ into $\mathcal{T}(H)$.

More generally, a function $\varphi\colon\mathrm{N}\to\mathrm{C}$ defines a radial multiplier on $\mathcal{T}(H)$ if the map T_φ given by

$$T_\varphi(\ell(e_1)\cdots\ell(e_k)\ell(e_{k+1})^*\cdots\ell(e_n)^*)=\varphi(n)\ell(e_1)\cdots\ell(e_k)\ell(e_{k+1})^*\cdots\ell(e_n)^*$$

extends to a completely bounded map on $\mathcal{T}(H)$. The above proof actually gives the following

Corollary (4.3.22)[190]: For $\varphi: \mathbf{N} \to \mathbf{C}$, we have $\|\mathrm{m}_\varphi\|_{\mathrm{cb}} = \|T_\varphi\|_{\mathrm{cb}} = \|T_\varphi\|$.

Taking $\delta_{\le d}(n) = \delta_{n\le d}$, the corresponding multiplier P_d on $\Gamma(H_{\mathbf{R}}, U_t)$ is called the projection onto words of length less than d. Thanks to Proposition (4.3.16), we get:

Corollary (4.3.23)[190]: For any orthogonal group (U_t) on an infinite dimensional real Hilbert space $H_{\mathbf{R}}$,

$$\|P_d\|_{\mathrm{cb}(\Gamma(H_{\mathbf{R}},U_t))} \underset{d\to\infty}{\sim} \frac{4}{\pi} d.$$

Proof. We apply Theorem (4.3.18) and Proposition (4.3.16) to this particular radial function. It is clear that $c_1 = c_2 = 0$. It remains to estimate the trace norm of $B = \sum_{i=0}^{d} e_{i,d-i} + \sum_{i=0}^{d-1} e_{i,d-1-i}$. To do so, $B + e_{d,d}$ is unitarily equivalent to a circulant matrix of size $d+1$, $\mathrm{Id}_{\ d+1} + J_{d+1}$ where $J_{d+1} = \sum_{i=0}^{d} e_{i,i+1}$. The singular values of B are exactly $1 + e^{\frac{2i\pi k}{d+1}}$, for $k = 0, \dots, d$. We get that $\| B \|_1/d$ tends to $\int_0^1 \left|1 + e^{2i\pi t}\right| \mathrm{d}t = \frac{4}{\pi}$.

Corollary (4.3.24)[190]: For any orthogonal group (U_t) on an infinite dimensional real Hilbert space $H_{\mathbf{R}}$, there are finitely supported functions $\varphi_n: \mathbf{N} \to \mathbf{R}$ such that $\lim_n \| \mathrm{m}_{\varphi_n}\|_{\mathrm{cb}} = 1$ and $\lim_n \varphi_n(k) = 1$ for all $k \ge 0$.

Proof. This is an argument due to Haagerup [196] (see also [170]). Using Corollary (4.3.26) below or Theorem (4.3.18), the contraction $H \ni \xi \mapsto e^{-t}\xi \in H$ gives rise to a unital completely positive multiplier m_{ψ_t} on $\Gamma(H_{\mathbf{R}}, U_t)''$ (for $t \ge 0$) where $\psi_t(k) = e^{-kt}$. Since

$$\psi_t = \sum_d e^{-dt}\delta_d = \sum_d e^{-dt}(\delta_{\le d} - \delta_{\le d-1}),$$

the polynomial estimate gives that

$$\limsup_{d\to\infty} \left\|\mathrm{m}_{\psi_t}(1 - P_d)\right\|_{\mathrm{cb}} \le \limsup_{d\to\infty} \sum_{k\ge d} e^{-kt}\|P_{k+1} - P_k\|_{\mathrm{cb}} = 0.$$

For every $n \ge 1$, choose d_n large enough so that $\left\|\mathrm{m}_{\psi_{1/n}}\left(1 - P_{d_n}\right)\right\|_{\mathrm{cb}} \le 1/n$. The net of the form $\varphi_n = \psi_{1/n}\delta_{\le d_n}$ satisfies the conclusion of the corollary.

We follow a very typical approach. We first establish a second quantization procedure on free Araki-Woods von Neumann algebras, which generalizes [46],[1]. Then, to get the approximation property, we just need to cut them with some radial multipliers to get finite rank maps. Let H and K be Hilbert spaces and let $T: H \to K$ be a contraction. We will denote the corresponding first quantization $\mathcal{F}(H) \to \mathcal{F}(K)$ by

$$\tilde{\Gamma}(T) = 1 \oplus \bigoplus_{n\ge1} T^{\otimes n}.$$

Theorem (4.3.25)[190]: Let H and K be Hilbert spaces and $T: H \to K$ be a contraction. Then there is a unique unital completely positive map $\Gamma(T): \mathcal{T}(H) \to \mathcal{T}(K)$ such that

$$\Gamma(T)(\ell(h_1)\cdots\ell(h_k)\ell(h_{k+1})^*\cdots\ell(h_n)^*) = \\ \ell\big(T(h_1)\big)\cdots\ell\big(T(h_k)\big)\ell\big(T(h_{k+1})\big)^*\cdots\ell\big(T(h_n)\big)^*$$

for all $h_i \in H$.

Proof. This is again a consequence of the universal property of $\mathcal{T}(H)$. It is clear that if $\Gamma(T)$ and $\Gamma(S)$ exist then $\Gamma(ST) = \Gamma(S)\Gamma(T)$. So by the general form of a contraction, one just needs to prove the result when T is either an inclusion from H to K, or a unitary on H, or an orthogonal projection from H to K.

If T is an inclusion, this is just the universal property of $\mathcal{T}(H)$ (note that $\Gamma(T)$ is an injective *-representation). We emphasize that if $H \subset K$ and $h \in H$, then $\ell(h)$ has a priori two different meanings as a creation operator on $\mathcal{F}(H)$ or $\mathcal{F}(K)$. The universal property tells us that there is no difference at the C^*-level.

If T is a unitary, this is also the universal property, but in this case $\Gamma(T)$ is nothing but the restriction of the conjugation by the unitary $\tilde{\Gamma}(T)$ on the full Fock space $\mathcal{F}(H)$.

If T is an orthogonal projection from H to K, we write $j: K \to H$ for the inclusion. The first quantization $\tilde{\Gamma}(j) = \iota$ is also an inclusion of $\mathcal{F}(K)$ into $\mathcal{F}(H)$, the orthogonal projection ι^* is exactly $\tilde{\Gamma}(T)$. To avoid any confusion, for $k \in K$ write $\ell_K(k): \mathcal{F}(K) \to \mathcal{F}(K)$ for the creation operator on $\mathcal{F}(K)$ and $\ell_H(k)$: $\mathcal{F}(H) \to \mathcal{F}(H)$ for the creation operator on $\mathcal{F}(H)$. For $h \in H$ and $k \in K$, we have $\ell_H(h)^*k = \langle h, k\rangle\Omega = \langle T(h), k\rangle\Omega = \ell_H(T(h))^*k = \ell_K(T(h))^*k$. This yields

$$\iota^*\ell_H(h_1)\cdots\ell_H(h_k)\ell_H(h_{k+1})^*\cdots\ell_H(h_n)^*\iota =$$
$$\ell_K\big(T(h_1)\big)\cdots\ell_K\big(T(h_k)\big)\ell_K\big(T(h_{k+1})\big)^*\cdots\ell_K\big(T(h_n)\big)^*.$$

Hence $\Gamma(T)(x) = \iota^* x\iota$, for all $x \in \mathcal{T}(H)$. It is then clear that $\Gamma(T): \mathcal{T}(H) \to \mathcal{T}(K)$ is completely positive.

The second quantization is usually stated for maps such that $AT = TA$ which is a somewhat strong assumption [15]. This was the main obstacle to prove approximation properties for general free Araki-Woods algebras as there can be no finite rank T satisfying that condition.

Corollary (4.3.26)[190]: Let $T: H \to H$ be a contraction so that ITI $= T$. Then $\Gamma(T)$ leaves $\Gamma(H_{\mathbf{R}}, U_t)$ invariant and $\Gamma(T)$ extends to a normal completely positive map on $\Gamma(H_{\mathbf{R}}, U_t)''$ so that

$$\Gamma(T)W(\xi) = W\big(\tilde{\Gamma}(T)\xi\big), \forall \xi \in \Gamma(H_{\mathbf{R}}, U_t)''\Omega.$$

Proof. If $ITI = T$, this implies that for all $\xi \in K_{\mathbf{R}} + iK_{\mathbf{R}}$, we have $T(\bar{\xi}) = \overline{T(\xi)}$. So by the Wick formula for e_i in $K_{\mathbf{R}} + iK_{\mathbf{R}}$, we have

$$\begin{aligned}\Gamma(T)W(e_1 \otimes \cdots \otimes e_n) &= \sum_{k=0}^{n} \ell\big(T(e_1)\big)\cdots\ell\big(T(e_k)\big)\ell\big(T(\bar{e}_{k+1})\big)^*\cdots\ell\big(T(\bar{e}_n)\big)^* \\ &= \sum_{k=0}^{n} \ell\big(T(e_1)\big)\cdots\ell\big(T(e_k)\big)\ell\big(\overline{T(e_{k+1})}\big)^*\cdots\ell\big(\overline{T(e_n)}\big)^* \\ &= W\big(T(e_1) \otimes \cdots \otimes T(e_n)\big).\end{aligned}$$

As the set of such elements is linearly dense in $\Gamma(H_{\mathbf{R}}, U_t)$, we get that $\Gamma(H_{\mathbf{R}}, U_t)$ is stable by $\Gamma(T)$. The normal extension is done as in Lemma (4.3.17).

Proposition (4.3.27)[190]: There is a net of finite rank contractions $(T_k)_k$ converging to the identity on H pointwise, such that $T_k = IT_kI$, for every k.

Proof. Let $\big(\mathbf{1}_{[\lambda,\infty]}(A)\big)_{\lambda\geq 0}$ be the spectral projections of A. Since $IAI = A^{-1}$, we get

$$I\mathbf{1}_{[\lambda,\infty[}(A)(H) = \mathbf{1}_{[0,1/\lambda]}(A)(H).$$

Recall that $I = JA^{-1/2}$ is the polar decomposition of I. We also have $JAJ = A^{-1}$ and J is an anti-unitary that sends $\mathbf{1}_{[\lambda,\beta]}(A)(H)$ to $\mathbf{1}_{[1/\beta,1/\lambda]}(A)(H)$.

Fix $\lambda > 1$ and $0 < \delta < 1$. Take a subspace E in $\mathbf{1}_{[\lambda,\lambda+\delta]}(A)(H)$ and denote by P the orthogonal projection onto E. We show that IPI is almost the orthogonal projection JPJ. Indeed, we have

$$IPI = JA^{-1/2}\mathbf{1}_{[\lambda,\lambda+\delta]}(A)P\mathbf{1}_{[\lambda,\lambda+\delta]}(A)JA^{-1/2}\mathbf{1}_{\left[\frac{1}{\lambda+\delta},\frac{1}{\lambda}\right]}(A).$$

Moreover

$$\left\|A^{-1/2}\mathbf{1}_{[\lambda,\lambda+\delta]}(A)-\frac{1}{\sqrt{\lambda}}\mathbf{1}_{[\lambda,\lambda+\delta]}(A)\right\|_\infty \le \frac{\delta}{2\sqrt{\lambda}^3}$$

$$\left\|A^{-1/2}\mathbf{1}_{\left[\frac{1}{\lambda+\phi},\frac{1}{\lambda}\right]}(A)-\sqrt{\lambda}\mathbf{1}_{\left[\frac{1}{\lambda+\beta},\frac{1}{\lambda}\right]}(A)\right\|_\infty \le \frac{\delta}{2\sqrt{\lambda}}.$$

The triangle inequality gives

$$\| IPI - JPJ \|_\infty \le \frac{\delta}{2\lambda}+\frac{\delta}{2\lambda}+\frac{\delta^2}{4\lambda^2}\le\frac{2\delta}{\lambda}.$$

Summarizing, for any finite dimensional subspace $E\subset \mathbf{1}_{[\lambda,\lambda+\delta]}(A)(H)$ and corresponding projections $P_E, T_E=\frac{1}{1+\frac{2\delta}{\lambda}}(P_E\oplus IP_EI)$ is a finite rank contraction that satisfies $IT_EI=T_E$ and

$$\|T_E-(P_E\oplus JP_EJ)\|_\infty<\frac{4\delta}{\lambda}.$$

Observe that for operators S and T which have orthogonal left and right supports, we denote the sum $S+T$ by $S\oplus T$.

Take F a finite dimensional subspace of H and fix $\varepsilon>0$. Then there exists $n\in\mathbf{N}$ such that for all $f\in F$, we have $\|\mathbf{1}_{[e^{-n},e^{n}]}(A)f-f\|\le(\varepsilon/3)\|f\|$. Set $\lambda_k=e^{nk/N}$, for $1\le k\le N$ for some large N chosen later. Let P_k be the orthogonal projection onto $\mathbf{1}_{\left[\frac{1}{\lambda_{k+1}},\frac{1}{\lambda_k}\right]}(A)(H)\oplus\mathbf{1}_{[\lambda_k,\lambda_{k+1}]}(A)(H)$ for $k\ge1$, and P_0 be the projection onto the eigenspace of A for 1. Observe that $\frac{\lambda_{k+1}-\lambda_k}{\lambda_k}=e^{n/N}-1$.

By the above construction, for each $1\le k\le N$, we can find a finite rank contraction T_k on $P_k(H)$ such that $IT_kI=T_k$ and for every $f\in F$,

$$\|T_k(P_kf)-P_kf\|\le 4\left(e^{n/N}-1\right)\|P_kf\|.$$

For $k=0$, as I is an anti-unitary on $P_0(H)$, we take T_0 the orthogonal projection onto $P_0(F)+IP_0(F)$, it satisfies the above properties with $k=0$.

Set $T=\oplus_{k=0}^N T_k$, which is a finite rank contraction as the T_k 's act on orthogonal subspaces. Moreover $ITI=T$ and for all $f\in F$, gathering the estimates

$$\| T(f)-f \|\le 4\left(e^{n/N}-1\right)\| f \| + \left\|\mathbf{1}_{[e^{-n/N},e^{n/N}]\setminus\{1\}}(A)f\right\|+(2\varepsilon/3)\| f \|.$$

Letting $N\to\infty$, this upper bound can be made smaller than $\varepsilon\| f \|$. So we get the conclusion with a net index by finite dimensional subspace of H and $\varepsilon>0$.

Theorem (4.3.28)[190]: (Theorem (4.3.1)). The von Neumann algebra $\Gamma(H_{\mathbf{R}},U_t)''$ has the complete metric approximation property.

Proof. Using the contractions of the previous Proposition, the net $\left(\Gamma(T_k)\right)_k$ is made of unital completely positive maps which tend pointwise to the identity. Let $\left(\mathrm{m}_{\varphi_n}\right)$ be the multipliers from Corollary (4.3.24). Since

$$\left(\mathrm{m}_{\varphi_n}\circ\Gamma(T_k)\right)\left(W(e_{\underline{i}})\right)=\varphi_n\left(|\underline{i}|\right)W\left(\tilde{\Gamma}(T_k)e_{\underline{i}}\right),$$

the net $\left(\mathrm{m}_{\varphi_n}\circ\Gamma(T_k)\right)_{n,k}$ are normal finite rank completely bounded maps which satisfy:

1. $\lim_n\lim_k\left(\mathrm{m}_{\varphi_n}\circ\Gamma(T_k)\right)=\mathrm{Id}$ pointwise $*$-strongly and

2. $\lim_n \lim_k \| \mathrm{m}_{\varphi_n} \circ \Gamma(T_k)\|_{\mathrm{cb}} = 1$

The proof is complete.

There is another approximation property that turns out to be useful. A von Neumann algebra M satisfies the Haagerup property if there exists a net $(u_i)_{i\in I}$ of normal completely positive maps from M to M such that

1. for all $x \in M, u_i(x) \to x\sigma$-weakly.
2. for all $\xi \in L^2(M)$ and $i \in I$ the map $x \mapsto u_i(x)\xi$ is compact from M to $L^2(M)$.

Theorem (4.3.29)[190]: The von Neumann algebra $\Gamma(H_{\mathbf{R}}, U_t)''$ has the Haagerup property.

Proof. This is just a variation. As above, with the finite rank maps of the previous Proposition, it is easy to check that $\left(\Gamma(e^{-t}T_k)\right)_{t>0,k\in\mathbf{N}}$ is a net of unital completely positive maps that tends to the identity pointwise with respect to the σ-weak topology. It remains only to check the second point.

We use the notation of the proof of Corollary (4.3.24). We have $\Gamma(e^{-t}) = \mathrm{m}_{\psi_t}$ and

$$\lim_{d\to\infty} \|\mathrm{m}_{\psi_t}(1 - P_d)\|_{\mathrm{cb}} = 0.$$

So $\Gamma(e^{-t}T_k) = \mathrm{m}_{\psi_t}(1 - P_d)\Gamma(T_k) + P_d\Gamma(e^{-t}T_k)$, as $P_d\Gamma(e^{-t}T_k)$ is finite rank, $\Gamma(e^{-t}T_k)$ is a limit in norm of finite rank operators so is compact from $\Gamma(H_{\mathbf{R}}, U_t)''$ to $\Gamma(H_{\mathbf{R}}, U_t)''$. In particular, its composition with the evaluation on a vector $\xi \in L^2(\Gamma(H_{\mathbf{R}}, U_t)'')$ is also compact.

Let $H_{\mathbf{R}}$ be a separable real Hilbert space ($\dim H_{\mathbf{R}} \geq 2$) together with (U_t) an orthogonal representation of $\mathbf{R}$ on $H_{\mathbf{R}}$. We set:

1. $\mathcal{M} = \Gamma(H_{\mathbf{R}}, U_t)''$ the free Araki-Woods factor associated with $(H_{\mathbf{R}}, U_t)$. Denote by χ the free quasi-free state and by σ the modular group of the state χ.
2. $M = \mathcal{M} \rtimes_\sigma \mathbf{R}$ is the continuous core of $\mathcal{M}$ and Tr is the semifinite trace associated with the state χ.
3. Likewise $\widetilde{\mathcal{M}} = \Gamma(H_{\mathbf{R}} \oplus H_{\mathbf{R}}, U_t \oplus U_t)'', \tilde{\chi}$ is the corresponding free quasi-free state and $\tilde{\sigma}$ is the modular group of $\tilde{\chi}$.
4. $\widetilde{M} = \widetilde{\mathcal{M}} \rtimes_{\tilde{\sigma}} \mathbf{R}$ is the continuous core of $\widetilde{\mathcal{M}}$ and $\widetilde{\mathrm{Tr}}$ is the semifinite trace associated with $\tilde{\chi}$.

It follows from [15] that

$$\widetilde{\mathcal{M}} \cong \mathcal{M} * \mathcal{M}.$$

In the latter free product, we shall write $\mathcal{M}_1$ for the first copy of $\mathcal{M}$ and $\mathcal{M}_2$ for the second copy of $\mathcal{M}$. We regard $\mathcal{M} \subset \widetilde{\mathcal{M}}$ via the identification of $\mathcal{M}$ with $\mathcal{M}_1$. Denote by (λ_t) the unitaries in $L(\mathbf{R})$ that implement the modular action σ on $\mathcal{M}$ (resp. $\tilde{\sigma}$ on $\widetilde{M}$). Define the following faithful normal conditional expectations:

1. $E: M \to L(\mathbf{R})$ such that $E(x\lambda_t) = \chi(x)\lambda_t$, for every $x \in \mathcal{M}$ and $t \in \mathbf{R}$;
2. $\tilde{E}: \widetilde{M} \to L(\mathbf{R})$ such that $\tilde{E}(x\lambda_t) = \tilde{\chi}(x)\lambda_t$, for every $x \in \widetilde{\mathbb{M}}$ and $t \in \mathbf{R}.$

Then

$$\left(\widetilde{M}, \tilde{E}\right) \cong (M, E) *_{L(\mathbf{R})} (M, E).$$

Likewise, in the latter amalgamated free product, we shall write M_1 for the first copy of M and M_2 for the second copy of M. We regard $M \subset \widetilde{M}$ via the identification of M with M_1. Notice that the conditional expectation E (resp. $\tilde{E}$) preserves the canonical semifinite trace Tr (resp. Tr) associated with the state χ(resp. $\tilde{\chi}$) (see [196]).

Consider the following orthogonal representation of $\mathbf{R}$ on $H_{\mathbf{R}} \oplus H_{\mathbf{R}}$:

$$V_s = \begin{pmatrix} \cos\left(\frac{\pi}{2}s\right) & -\sin\left(\frac{\pi}{2}s\right) \\ \sin\left(\frac{\pi}{2}s\right) & \cos\left(\frac{\pi}{2}s\right) \end{pmatrix}, \forall s \in \mathbf{R}.$$

Let (α_s) be the natural action on $(\widetilde{\mathcal{M}}, \tilde{\chi})$ associated with (V_s):

$$\alpha_s = \Gamma(V_s), \forall s \in \mathbf{R}.$$

In particular, we have

$$\alpha_s\left(W\begin{pmatrix}\xi\\ \eta\end{pmatrix}\right) = W\left(V_s\begin{pmatrix}\xi\\ \eta\end{pmatrix}\right), \forall s \in \mathbf{R}, \forall \xi, \eta \in H_{\mathbf{R}},$$

and the action (α_s) is $\tilde{\chi}$-preserving. We can easily see that the representation (V_s) commutes with the representation $(U_t \oplus U_t)$. Consequently, (α_s) commutes with modular action $\tilde{\sigma}$. Moreover, $\alpha_1(x * 1) = 1 * x$, for every $x \in \mathcal{M}$. At last, consider the automorphism β defined on $(\widetilde{\mathcal{M}}, \tilde{\chi})$ by:

$$\beta\left(W\begin{pmatrix}\xi\\ \eta\end{pmatrix}\right) = W\begin{pmatrix}\xi\\ -\eta\end{pmatrix}, \forall \xi, \eta \in H_{\mathbf{R}}.$$

It is straightforward to check that β commutes with the modular action $\tilde{\sigma}, \beta^2 = \mathrm{Id}$, $\beta_{|\mathcal{M}} = \mathrm{Id}_{\mathcal{M}}$ and $\beta\alpha_s = \alpha_{-s}\beta, \forall s \in \mathbf{R}$. Since (α_s) and β commute with the modular action $\tilde{\sigma}$, one may extend (α_s) and β to $\widetilde{M}$ by $\alpha_{s|L(\mathbf{R})} = \mathrm{Id}_{L(\mathbf{R})}$, for every $s \in \mathbf{R}$ and $\beta_{|L(\mathbf{R})} = \mathrm{Id}_{L(\mathbf{R})}$. Moreover (α_s, β) preserves the semifinite trace $\widetilde{\mathrm{Tr}}$. We summarize what we have done so far:

Proposition (4.3.30)[190]: The $\{\widetilde{Tr}$-preserving deformation$\}$ (α_s, β) defined on $\widetilde{M} = M *_{L(R)} M$ is s-malleable:

1. $\alpha_{s|L(\mathbf{R})} = \mathrm{Id}_{L(\mathbf{R})}$, for every $s \in \mathbf{R}$ and $\alpha_1\left(x *_{L(\mathbf{R})} 1\right) = 1 *_{L(\mathbf{R})} x$, for every $x \in M$.
2. $\beta^2 = \mathrm{Id}$ and $\beta_{|M} = \mathrm{Id}_M$.
3. $\beta\alpha_s = \alpha_{-s}\beta$, for every $s \in \mathbf{R}$. Denote by $E_M: \widetilde{M} \to M$ the canonical trace preserving conditional expectation. Since $\widetilde{\mathrm{Tr}}_{|M} = \mathrm{Tr}$, we will simply denote by Tr the semifinite trace on $\widetilde{M}$. Recall that the s-malleable deformation (α_s, β) automatically features a certain transversality property.

Proposition (4.3.31)[190]: (Popa, [182]). We have the following:

$$\|x - \alpha_{2s}(x)\|_{2,\mathrm{Tr}} \leq 2\|\alpha_s(x) - (E_M \circ \alpha_s)(x)\|_{2,\mathrm{Tr}}, \forall x \in L^2(M, \mathrm{Tr}), \forall s > 0. \quad (26)$$

The following theorem is in some ways reminiscent of a result by Ioana, Peterson and Popa, namely ([156] Theorem 4.3) (see also [176] Theorem 4.2] and [173] Theorem 3.4).

Proposition (4.3.32)[190]: Let $\mathcal{M} = \Gamma(H_{\mathbf{R}}, U_t)''$ and $M = \mathcal{M} \rtimes_\sigma \mathbf{R}$ be as above. Let $p \in L(\boldsymbol{R}) \subset M$ be a nonzero projection such that $\mathrm{Tr}(p) < \infty$. Let $P \subset pMp$ be a von Neumann subalgebra such that the deformation (α_t) converges uniformly in $\|\cdot\|_{2,\mathrm{Tr}}$ on $\mathcal{U}(P)$. Then $P \preceq_M L(\mathbf{R})$.

Proof. Let $p \in L(\mathbf{R})$ be a nonzero projection such that $\mathrm{Tr}(p) < \infty$. Let $P \subset pMp$ be a von Neumann subalgebra such that (α_t) converges uniformly in $\|\cdot\|_{2,\mathrm{Tr}}$ on $\mathcal{U}(P)$. We keep the notation introduced previously and regard $M \subset \widetilde{M} = M_1 *_{L(\mathbf{R})} M_2$ via the identification of M with M_1. Recall that $\alpha_{s|L(\mathbf{R})} = \mathrm{Id}_{L(\mathbf{R})}$, for every $s \in \mathbf{R}$. In particular, $\alpha_s(p) = p$, for every $s \in \mathbf{R}$.

Step (1): Using the uniform convergence on $\mathcal{U}(P)$ to find $t > 0$ and a nonzero intertwiner v between Id and α_t. The first step uses a standard functional analysis trick. Let $\varepsilon = \frac{1}{2} \parallel p \parallel_{2,Tr}$. We know that there exists $s = 1/2^k$ such that $\forall u \in \mathcal{U}(P)$,

$$\|u - \alpha_s(u)\|_{2,\mathrm{Tr}} \leq \frac{1}{2} \parallel p \parallel_{2,\mathrm{Tr}},$$

Thus, $\forall u \in \mathcal{U}(P)$, we have

$$\begin{aligned} \|u^*\alpha_s(u) - p\|_{2,Tr} &= \|u^*(\alpha_s(u) - u)\|_{2,\mathrm{Tr}} \\ &\leq \|u - \alpha_s(u)\|_{2,Tr} \\ &\leq \frac{1}{2} \parallel p \parallel_{2,Tr.} \end{aligned}$$

Denote by $\mathcal{C} = \overline{co}^w\{u^*\alpha_s(u) : u \in \mathcal{U}(P)\} \subset pL^2(\widetilde{M})p$ the ultraweak closure of the convex hull of all $u^*\alpha_s(u)$, where $u \in \mathcal{U}(P)$. Denote by a the unique element in $\mathcal{C}$ of minimal $\|\cdot\|_{2,\mathrm{Tr}}-$ norm. Since $\parallel a - p \parallel_{2,\mathrm{Tr}} \leq 1/2 \parallel p \parallel_{2,\mathrm{Tr}}$, necessarily $a \neq 0$. Fix $u \in \mathcal{U}(P)$. Since $u^*a\alpha_s(u) \in \mathcal{C}'$ and $\|u^*a\alpha_s(u)\|_{2,\mathrm{Tr}} = \parallel a \parallel_{2,\mathrm{Tr}}$, necessarily $u^*a\alpha_s(u) = a$. Taking $v = \mathrm{pol}(a)$ the polar part of a, we have found a nonzero partial isometry $v \in p\widetilde{M}p$ such that

$$xv = v\alpha_s(x), \forall x \in P. \tag{27}$$

Note that $vv^* \in P' \cap p\widetilde{M}p$ and $v^*v \in \alpha_s(P)' \cap p\widetilde{M}p$.

Step (2): Proving $P \preceq_M L(\mathbf{R})$ using the malleability of (α_t, β). The rest of the proof, is very similar to the reasoning in ([168], Lemma 4.8, Theorem 6.1), ([188], Theorem 4.1) and ([156], Theorem 4.3) (see also [180], Theorem 5.6) and ([173], Theorem 3.4). For the sake of completeness, we will give a detailed proof.

By contradiction, assume $P \npreceq_M L(\mathbf{R})$. The first task is to lift Equation (27) to $s = 1$. Note that it is enough to find a nonzero partial isometry $w \in p\widetilde{M}p$ such that

$$xw = w\alpha_{2s}(x), \forall x \in P.$$

Indeed, by induction we can go till $s = 1$ (because $s = 1/2^k$). Recall that $\beta(z) = z$, for every $z \in M$. Recall that $vv^* \in P' \cap p\widetilde{M}p$. Since $P \npreceq_M L(\mathbf{R})$, we know from [6, Theorem 2.4] that $P' \cap p\widetilde{M}p \subset pMp$. In particular, $vv^* \in pMp$. Set $w = \alpha_s(\beta(v^*)v)$. Then

$$\begin{aligned} ww^* &= \alpha_s(\beta(v^*)vv^*\beta(v)) \\ &= \alpha_s(\beta(v^*)\beta(vv^*)\beta(v)) \\ &= \alpha_s\beta(v^*v) \neq 0. \end{aligned}$$

Hence, w is a nonzero partial isometry in $p\widetilde{M}p$. Moreover, for every $x \in P$,

$$\begin{aligned} w\alpha_{2s}(x) &= \alpha_s(\beta(v^*)v\alpha_s(x)) \\ &= \alpha_s(\beta(v^*)xv) \\ &= \alpha_s(\beta(v^*x)v) \\ &= \alpha_s(\beta(\alpha_s(x)v^*)v) \\ &= \alpha_s\beta\alpha_s(x)\alpha_s(\beta(v^*)v) \\ &= \beta(x)w \\ &= xw. \end{aligned}$$

Since by induction, we can go till $s = 1$, we have found a nonzero partial isometry $v \in p\widetilde{M}p$ such that

$$xv = v\alpha_1(x), \forall x \in P. \tag{28}$$

Note that $v^*v \in \alpha_1(P)' \cap pMp$. Moreover, since $\alpha_1 : p\widetilde{M}p \to p\widetilde{M}p$ is a *-automorphism, and $P \npreceq_M L(\mathbf{R})$, ([176] Theorem 2.4) gives

$$\begin{aligned}\alpha_1(P)' \cap p\widetilde{M}p &= \alpha_1\left(P' \cap p\widetilde{M}p\right)\\ &\subset \alpha_1(pMp).\end{aligned}$$

Hence $v^*v \in \alpha_1(pMp)$.

Since $P \nprec_M L(\mathbf{R})$, we know that there exists a sequence of unitaries (u_k) in P such that $\lim_k \|E_{L(\mathbf{R})}(x^*u_k y)\|_{2,\mathrm{Tr}} \to 0$, for any $x, y \in M$. We need to go further and prove the following:

Claim (4.3.33)[190]: $\forall a, b \in \widetilde{M}, \lim_k \|E_{M_2}(a^*u_k b)\|_{2,\mathrm{Tr}} = 0$.

Proof. Let $a, b \in (\widetilde{M})_1$ be either elements in $L(\mathbf{R})$ or reduced words with letters alternating from $M_1 \ominus L(\mathbf{R})$ and $M_2 \ominus L(\mathbf{R})$. Write $b = yb'$ with

1. $y = b$ if $b \in L(\mathbf{R})$;
2. $y = 1$ if b is a reduced word beginning with a letter from $M_2 \ominus L(\mathbf{R})$;
3. $y =$ the first letter of b coming from $M_1 \ominus L(\mathbf{R})$ otherwise.

Note that either $b' = 1$ or b' is a reduced word beginning with a letter from $M_2 \ominus L(\mathbf{R})$. Likewise write $a = a'x$ with

1. $x = a$ if $x \in L(\mathbf{R})$;
2. $x = 1$ if a is a reduced word ending with a letter from $M_2 \ominus L(\mathbf{R})$;
3. $x =$ the last letter of a coming from $M_1 \ominus L(\mathbf{R})$ otherwise.

Either $a' = 1$ or a' is a reduced word ending with a letter from $M_2 \ominus L(\mathbf{R})$. For any $z \in M_1, xzy - E_{L(\mathbf{R})}(xzy) \in M_1 \ominus L(\mathbf{R})$, so that

$$E_{M_2}(azb) = E_{M_2}\left(a'E_{L(\mathbf{R})}(xzy)b'\right).$$

Since $\lim_k \|E_{L(\mathbf{R})}(xu_k y)\|_{2,\mathrm{Tr}} = 0$, it follows that $\lim_k \|E_{M_2}(au_k b)\|_{2,\mathrm{Tr}} = 0$ as well.

Note that

$$\mathcal{A} := \mathrm{span}\left\{L(\mathbf{R}), \left(M_{i_1} \ominus L(\mathbf{R})\right) \cdots \left(M_{i_n} \ominus L(\mathbf{R})\right) : n \geq 1, i_1 \neq \cdots \neq i_n\right\}$$

is a unital *-strongly dense *-subalgebra of $\widetilde{M}$. What we have shown so far is that for any $a, b \in \mathcal{A}, \|E_{M_2}(au_k b)\|_{2,\mathrm{Tr}} \to 0$, as $k \to \infty$. Let now $a, b \in (\widetilde{M})_1$ By Kaplansky density theorem, let (a_i) and (b_j) be sequences in $(\mathcal{A})_1$ such that $a_i \to a$ and $b_j \to b$ *-strongly. Recall that (u_k) is a sequence in $P \subset p\widetilde{M}p$ with $\mathrm{Tr}(p) < \infty$. We have

$$\begin{aligned}\|E_{M_2}(au_k b)\|_{2,\mathrm{Tr}} &\leq \|E_{M_2}(a_i u_k b_j)\|_{2,\mathrm{Tr}} + \left\|E_{M_2}\left(a_i u_k (b - b_j)\right)\right\|_{2,\mathrm{Tr}}\\ &\quad + \left\|E_{M_2}\left((a - a_i)u_k b_j\right)\right\|_{2,\mathrm{Tr}} + \left\|E_{M_2}\left((a - a_i)u_k(b - b_j)\right)\right\|_{2,\mathrm{Tr}}\\ &\leq \|E_{M_2}(a_i u_k b_j)\|_{2,\mathrm{Tr}} + \|a_i u_k p(b - b_j)\|_{2,\mathrm{Tr}}\\ &\quad + \|(a - a_i)pu_k b_j\|_{2,\mathrm{Tr}} + \|(a - a_i)u_k p(b - b_j)\|_{2,\mathrm{Tr}}\\ &\leq \|E_{M_2}(a_i u_k b_j)\|_{2,\mathrm{Tr}} + 2\|p(b - b_j)\|_{2,\mathrm{Tr}} + \|(a - a_i)p\|_{2,\mathrm{Tr}}\end{aligned}$$

Fix $\varepsilon > 0$. Since $a_i \to a$ and $b_j \to b$ *-strongly, let i_0, j_0 large enough such that

$$2\|p(b - b_{j_0})\|_{2,\mathrm{Tr}} + \|(a - a_{i_0})p\|_{2,\mathrm{Tr}} \leq \varepsilon/2.$$

Now let $k_0 \in \mathbf{N}$ such that for any $k \geq k_0$,

$$\|E_{M_2}(a_{i_0} u_k b_{j_0})\|_{2,\mathrm{Tr}} \leq \varepsilon/2.$$

We finally get $\|E_{M_2}(au_k b)\|_{2,\mathrm{Tr}} \leq \varepsilon$, for any $k \geq k_0$, which finishes the proof of the claim.

Recall that for any $x \in P, v^*xv = \alpha_1(x)v^*v$, by Equation (28). Moreover, $v^*v \in \alpha_1(pMp) \subset pM_2p$. So, for any $x \in P, v^*xv \in pM_2p$. Since $\alpha_1(u_k) \in \mathcal{U}(pM_2p)$, we get

$$\begin{aligned}\|v^*v\|_{2,\mathrm{Tr}} &= \|\alpha_1(u_k)v^*v\|_{2,\mathrm{Tr}} \\ &= \|E_{M_2}(\alpha_1(u_k)v^*v)\|_{2,\mathrm{Tr}} \\ &= \|E_{M_2}(v^*u_kv)\|_{2,\mathrm{Tr}} \to 0.\end{aligned}$$

Thus $v = 0$, which is a contradiction.

Corollary (4.3.34)[190]: Let $\mathcal{M} = \Gamma(H_{\mathbf{R}}, U_t)''$ and $M = \mathcal{M} \rtimes_\sigma \mathbf{R}$ be as above. Let $p \in L(\mathbf{R}) \subset M$ be a nonzero projection such that $\mathrm{Tr}(p) < \infty$. Let $P \subset pMp$ be a von Neumann subalgebra such that $P \npreceq_M L(\mathbf{R})$. Then there exist $0 < \kappa < 1$, a sequence (t_k) of positive reals and a sequence (u_k) of unitaries in $\mathcal{U}(P)$ such that $\lim_k t_k = 0$ and $\|(E_M \circ \alpha_{t_k})(u_k)\|_{2,\mathrm{Tr}} \leq \kappa \| p \|_{2,\mathrm{Tr}}$, for every $k \in \mathbf{N}$.

Proof. Assume $P \npreceq_M L(\mathbf{R})$. Using Proposition (4.3.32), we obtain that the deformation (α_t) does not converge uniformly on $\mathcal{U}(P)$. Combining this with Inequality (26) in Proposition (4.3.31), we get that there exist $0 < c < 1$, a sequence of positive reals (t_k) and a sequence of unitaries (u_k) in $\mathcal{U}(P)$ such that $\lim_k t_k = 0$ and $\|\alpha_{t_k}(u_k) - (E_M \circ \alpha_{t_k})(u_k)\|_{2,\mathrm{Tr}} \geq c \| p \|_{2,\mathrm{Tr}}, \forall k \in \mathbf{N}$. Since $\|\alpha_{t_k}(u_k)\|_{2,\mathrm{Tr}} = \| p \|_{2,\mathrm{Tr}}$ by Pythagora's theorem we obtain

$$\|(E_M \circ \alpha_{t_k})(u_k)\|_{2,\mathrm{Tr}} \leq \kappa \| p \|_{2,\mathrm{Tr}}, \forall k \in \mathbf{N}.$$

where $\kappa = \sqrt{1 - c^2}$.

Let M, N, P be any von Neumann algebras. For any M, N-bimodules H, K, denote by π_H (resp. π_K) the associated *-representation of the algebraic tensor product $M \odot N^{\mathrm{op}}$ on H (resp. on K). We say that H is weakly contained in K and denote it by $H \subset_{\mathrm{weak}} K$ if $\|\pi_H(T)\|_\infty \leq \|\pi_K(T)\|_\infty$, for every $T \in M \odot N^{\circ p}$. Recall that $H \subset_{\mathrm{weak}} K$ if and only if H lies in the closure (for the Fell topology) of all finite direct sums of copies of K. Let H, K be M, N-bimodules. The following are true:

1. Assume that $H \subset_{\mathrm{weak}} K$. Then, for any N, P-bimodule L, we have $H \otimes_N L \subset_{\mathrm{weak}} K \otimes_N L$, as M, P-bimodules. Likewise, for any P, M-bimodule L we have $L \otimes_M H \subset_{\mathrm{weak}} L \otimes_M K$, as P, N-bimodules (see [143], Lemma 1.7).
2. A von Neumann algebra B is amenable if and only if $L^2(B) \subset_{\mathrm{weak}} L^2(B) \otimes L^2(B)$, as B, B-bimodules.

Let B, M, N be von Neumann algebras such that B is amenable. Let H be any M, B-bimodule and let K be any B, N-bimodule. Then, as M, N-bimodules, we have $H \otimes_B K \subset_{\mathrm{weak}} H \otimes K$ (straightforward consequence of (a) and (b)).

Let $\mathcal{M} = \Gamma(H_{\mathbf{R}}, U_t)''$ be a free Araki-Woods factor. Denote by $M = \mathcal{M} \rtimes_\sigma \mathbf{R}$ its continuous core.

Lemma (4.3.35)[190]: Let $p \in L(\mathbf{R})$ be a nonzero projection such that $\mathrm{Tr}(p) < \infty$. The pM_1p, pM_1p-bimodule $\mathcal{H} = L^2(p\widetilde{M}p) \ominus L^2(pM_1p)$ is weakly contained in the coarse bimodule $L^2(pM_1p) \otimes L^2(pM_1p)$.

Proof. Set $B = L(\mathbf{R})$. Let $p \in L(\mathbf{R})$ be a nonzero projection such that $\mathrm{Tr}(p) < \infty$. By definition of the amalgamated free product $\widetilde{M} = M_1 * L(\mathbf{R}) M_2$ (see [197] and [196]), we have as pM_1p, pM_1p-bimodules

$$L^2(p\widetilde{M}p) \ominus L^2(pM_1p) \cong \bigoplus_{n \geq 1} \mathcal{H}_n,$$

where

$$\mathcal{H}_n = L^2(pM_1) \otimes_B \overbrace{(L^2(M_2) \ominus L^2(B)) \otimes_B \cdots \otimes_B (L^2(M_2) \ominus L^2(B))}^{2n-1} \otimes_B L^2(M_1 p).$$

Since $B = L(\mathbf{R})$ is amenable, the identity bimodule $L^2(B)$ is weakly contained in the coarse bimodule $L^2(B) \otimes L^2(B)$. From the standard properties of composition and weak containment of bimodules, it follows that as pM_1p, pM_1p-bimodules

$$\mathcal{H}_n \subset_{\text{weak}} L^2(pM_1) \otimes \overbrace{\left(L^2(M_2) \ominus L^2(B)\right) \otimes \cdots \otimes \left(L^2(M_2) \ominus L^2(B)\right)}^{2n-1} \otimes L^2(M_1 p).$$

Consequently, we obtain as pM_1p, pM_1p-bimodules

$$\mathcal{H} = L^2\left(p\widetilde{M}p\right) \ominus L^2(pM_1p) \subset_{\text{weak}} \bigoplus L^2(pM_1) \otimes L^2(M_1p).$$

Moreover, as a left pM_1p-module, $L^2(pM_1)$ is contained in $\oplus L^2(pM_1p)$. Likewise, the right pM_1p-module $L^2(M_1p)$ is contained in $\oplus L^2(pM_1p)$. Therefore, we get as pM_1p, pM_1p-bimodules

$$\mathcal{H} = L^2\left(p\widetilde{M}p\right) \ominus L^2(pM_1p) \subset_{\text{weak}} \bigoplus L^2(pM_1p) \otimes L^2(pM_1p).$$

Let $\mathcal{M} = \Gamma(H_{\mathbf{R}}, U_t)''$ be a free Araki-Woods factor. Since $\mathcal{M}$ has the complete metric approximation property by Theorem (4.3.1), so do its core $M = \mathcal{M} \rtimes_\sigma \mathbf{R}$ and pMp, for any Tr-finite nonzero projection $p \in M$ by Theorem (4.3.12).

Theorem (4.3.36)[190]: Let $\mathcal{M} = \Gamma(H_{\mathbf{R}}, U_t)''$ be a free Araki-Woods factor. Denote by χ the corresponding free quasi-free state and by $M = \mathcal{M} \rtimes_{\sigma^x} \mathbf{R}$ the continuous core. Let $p \in L(\mathbf{R})$ be a nonzero projection such that $\mathrm{Tr}(p) < \infty$. Let $P \subset pMp$ be an amenable von Neumann subalgebra. If $P \npreceq_M L(\mathbf{R})$, then $\mathcal{N}_{pMp}(P)''$ is amenable.

Proof. The proof is a generalization of the one of [198], building on the work of Ozawa and Popa (see [141], Theorem 4.9) and [198]. What is shown in [198], is the following. Assume that $P \subset N$ are finite von Neumann algebras such that P is amenable and N has the c.m.a.p. Assume moreover that there are a finite von Neumann algebra $N \subset \widetilde{N}$ and trace-preserving *-homomorphisms $\alpha_t: N \to \widetilde{N}$ such that:

1. $\lim_{t\to 0} \|\alpha_t(x) - x\|_2 = 0$, for every $x \in N$.
2. There exists $0 < \kappa < 1$, a sequence of positive reals (t_k) and a sequence of unitaries (u_k) in $\mathcal{U}(P)$ such that $\lim_k t_k = 0$ and $\|(E_N \circ \alpha_{t_k})(u_k)\|_{2,\mathrm{Tr}} \leq \kappa \parallel p \parallel_{2,\,\mathrm{Tr}}$, for every $k \in \mathbf{N}$.
3. The N, N-bimodule $L^2(\widetilde{N}) \ominus L^2(N)$ is weakly contained in the coarse bimodule $L^2(N) \otimes L^2(N)$.

Then $\mathcal{N}_N(P)''$ is amenable.

Now let $\mathcal{M} = \Gamma(H_{\mathbf{R}}, U_t)''$ be a free Araki-Woods factor. Denote by χ the corresponding free quasi-free state and by $M = \mathcal{M} \rtimes_\sigma \chi\mathbf{R}$ the continuous core. Let $p \in L(\mathbf{R})$ be a nonzero projection such that $\mathrm{Tr}(p) < \infty$. We know that $N = pMp$ has the c.m.a.p. since both $\mathcal{M}$ and M have the c.m.a.p. (by Theorem (4.3.1)). Let $P \subset pMp$ be an amenable von Neumann subalgebra. The malleable deformation (α_t) clearly satisfies (a). Since $P \npreceq_M L(\mathbf{R})$, Corollary (4.3.34) yields (b). Lemma (4.3.35) finally yields (c). Therefore $\mathcal{N}_{pMp}(P)''$ is amenable.

Theorem (4.3.37)[190]: Let $\mathcal{M} = \Gamma(H_{\mathbf{R}}, U_t)''$ be any free Araki-Woods factor. Let $\mathcal{N} \subset \mathcal{M}$ be a diffuse von Neumann subalgebra for which there exists a faithful normal conditional expectation $E: \mathcal{M} \to \mathcal{N}$. Then either $\mathcal{N}$ is hyperfinite or $\mathcal{N}$ has no Cartan subalgebra.

Proof. Let $\mathcal{M}$ be a von Neumann algebra and let φ, ψ be two faithful normal states on $\mathcal{M}$. Recall that through the natural *-isomorphism

$$\Pi_{\varphi,\psi}: \mathcal{M} \rtimes_{\sigma^\varphi} \mathbf{R} \to \mathcal{M} \rtimes_{\sigma^\psi} \mathbf{R},$$

we will identify

$$\left(\pi_{\sigma^\varphi}(\mathcal{M}) \subset \mathcal{M} \rtimes_{\sigma^\varphi} \mathbf{R}, \theta^\varphi, \mathrm{Tr}_\varphi\right) \text{ with } \left(\pi_{\sigma^\psi}(\mathcal{M}) \subset \mathcal{M} \rtimes_{\sigma^\psi} \mathbf{R}, \theta^\psi, \mathrm{Tr}_\psi\right),$$

and simply denote it by $(\mathcal{M} \subset M, \theta, \mathrm{Tr})$, where θ is the dual action of $\mathbf{R}$ on the core M and Tr is the semifinite faithful normal trace on M such that $\mathrm{Tr} \circ \theta_s = e^{-s}\mathrm{Tr}$, for any $s \in \mathbf{R}$.

However, we need to pay attention to the following: whereas the inclusion $\mathcal{M} \subset M$ does not depend on the state, there are a priori two different copies of the abelian von Neumann algebra $L(\mathbf{R})$ inside M. To avoid any confusion, we will denote by $\lambda^\varphi(s)$ (resp. $\lambda^\psi(s)$) the unitaries implementing the modular action σ^φ (resp. σ^ψ) on $\mathcal{M}$. The following technical Proposition will be useful, as it explains why we do not have to worry very much about the state.

Proposition (4.3.38)[190]: Let $\mathcal{M}$ be a von Neumann algebra. Let $A \subset \mathcal{M}$ be a separable diffuse von Neumann subalgebra. Then, for any nonzero projection $p \in A' \cap M$ with $\mathrm{Tr}(p) < \infty$, and any faithful normal state φ on $\mathcal{M}$, we have

$$Ap \nprec_M \lambda^\varphi(\mathbf{R})''.$$

Proof. Fix φ a faithful normal state on $\mathcal{M}$ and p a nonzero Tr-finite projection in M. Since $A \subset \mathcal{M}$ is diffuse and separable, any maximal abelian *-subalgebra in A is separable and diffuse, and thus isomorphic to $L^\infty([0,1])$. Therefore there exists a sequence of unitaries (u_n) in A such that $u_n \to 0$ weakly. Observe that $Ap \subset pMp$ is a von Neumann subalgebra and that $(u_n p)$ are unitaries in Ap.

Let (q_m) be an increasing sequence of projections in $\lambda^\varphi(\mathbf{R})''$ such that $q_m \to 1$ strongly and $\mathrm{Tr}(q_m) < \infty$. Let $x, y \in (M)_1$ and $\varepsilon > 0$. Since $\mathrm{Tr}(p) < \infty$, choose $m \in \mathbf{N}$ large enough such that

$$\|q_m x^* p - x^* p\|_{2,\mathrm{Tr}} + \|p y q_m - p y\|_{2,\mathrm{Tr}} < \varepsilon.$$

Observe now that the unital *-algebra

$$\mathcal{E} := \left\{\sum_{s \in S} x_s \lambda^\varphi(s) : S \subset \mathbf{R} \text{ finite}, x_s \in \mathcal{M}\right\}$$

is *-strongly dense in M, so that one can find nets $(x_i)_{i\in I}$ and $\left(y_j\right)_{j\in J}$ in $(\mathcal{E})_1$ such that $x_i \to px$ and $y_j \to py$ $*$-strongly. Since now $\mathrm{Tr}(q_m) < \infty$, one can find $(i,j) \in I \times J$, such that

$$\|q_m x^* p - q_m x_i^*\|_{2,\mathrm{Tr}} + \left\|p y q_m - y_j q_m\right\|_{2,\mathrm{Tr}} < \varepsilon.$$

For simplicity of notation write $L(\mathbf{R}) := \lambda^\varphi(\mathbf{R})''$. For every $n \in \mathbf{N}$, we get

$$\begin{aligned} \left\|E_{L(\mathbf{R})}(x^* p u_n p y)\right\|_{2,\mathrm{Tr}} &\leq \left\|E_{L(\mathbf{R})}(q_m x^* p u_n p y q_m)\right\|_{2,\mathrm{Tr}} + \varepsilon \\ &\leq \left\|E_{L(\mathbf{R})}\left(q_m x_i^* u_n y_j q_m\right)\right\|_{2,Tr} + 2\varepsilon. \end{aligned}$$

Since $x_i, y_j \in (\mathcal{E})_1$, write

$$\begin{aligned} x_i &= \sum_{s \in S} x_s \lambda^\varphi(s) \\ y_j &= \sum_{t \in T} y_t \lambda^\varphi(t), \end{aligned}$$

where $S, T \subset \mathbf{R}$ are finite and $x_s, y_t \in \mathcal{M}$. Therefore

$$E_{L(\mathbf{R})}(q_m x_i^* u_n y_j q_m) = \sum_{(s,t)\in S\times T} \varphi(x_s^* u_n y_t)\lambda^\varphi(t-s)q_m.$$

Since φ is a faithful normal state on $\mathcal{M}$, one may regard $A \subset \mathcal{M} \subset \mathbf{B}(L^2(\mathcal{M},\varphi))$. Since $u_n \to 0$ weakly in A, there exists $n_0 \in \mathbf{N}$ large enough such that $\forall n \geq n_0$, $\forall (s,t) \in S \times T$

$$|\varphi(x_s^* u_n y_t)| \leq \frac{\varepsilon}{\|q_m\|_{2,\mathrm{Tr}}(|S|\cdot|T|+1)}.$$

We get, for every $n \geq n_0$,

$$\|E_{L(\mathbf{R})}(q_m x_i^* u_n y_j q_m)\|_{2,\mathrm{Tr}} \leq \varepsilon.$$

Therefore, we have for every $n \geq n_0$

$$\|E_{L(\mathbf{R})}(x^* p u_n p y)\|_{2,\mathrm{Tr}} \leq 3\varepsilon.$$

By (27) of Lemma (4.3.5), we get $Ap \npreceq_M \lambda^\varphi(\mathbf{R})''$.

We are now ready to prove Theorem (4.3.37). We will denote by χ the corresponding free quasi-free state on $\mathcal{M}$. We prove the result by contradiction. Assume that there exists a diffuse nonamenable von Neumann subalgebra $\mathcal{N} \subset \mathcal{M}$ together with $E: \mathcal{M} \to \mathcal{N}$ a faithful normal conditional expectation such that $\mathcal{N}$ has a Cartan subalgebra $A \subset \mathcal{N}$. Observe that A is necessarily diffuse. Denote by $F: \mathcal{N} \to A$ the faithful normal conditional expectation. Choose a faithful normal trace τ on A. Write $\psi = \tau \circ F \circ E$. Observe that ψ is a faithful normal state on $\mathcal{M}$ such that $\psi \circ E = \psi$ and $A \subset \mathcal{N}^\psi$. Set $M = \mathcal{M} \rtimes_{\sigma^\psi} \mathbf{R}$ and $N = \mathcal{N} \rtimes_{\sigma^\psi} \mathbf{R}$ and notice that $\lambda^\psi(\mathbf{R})'' \subset A' \cap M$. Observe that since $\mathcal{N}$ is a nonamenable von Neumann algebra, its core N is nonamenable as well. Take a nonzero Trfinite projection $p \in \lambda^\psi(\mathbf{R})''$ large enough such that pNp is nonamenable. Since $\left(A \bar{\otimes} \lambda^\psi(\mathbf{R})''\right)(1 \otimes p) \subset pNp$ is regular and pNp is nonamenable, Theorem (4.3.36) implies that $\left(A \bar{\otimes} \lambda^\psi(\mathbf{R})''\right)(1 \otimes p) \preceq_M \lambda^\chi(\mathbf{R})''$ and thus $A(1 \otimes p) \preceq_M \lambda^\chi(\mathbf{R})''$. Since A is diffuse, this contradicts Proposition (4.3.38).

Theorem (4.3.39)[190]: Let (U_t) be a nontrivial nonperiodic orthogonal representation of $\mathbf{R}$. Denote by $\mathbb{M} = \Gamma(H_\mathbf{R}, U_t)''$ the corresponding type III_1 free Araki-Woods factor. Denote by $M = \mathcal{M} \rtimes_\sigma \mathbf{R}$ its continuous core, which is a type II_∞ factor. Let $p \in M$ be a nonzero finite projection and write $N = pMp$.

1. For any maximal abelian *-subalgebra $A \subset N, \mathcal{N}_N(A)''$ is amenable. In particular, N has no Cartan subalgebra.
2. Assume that
   1. either (U_t) is strongly mixing;
   2. or $U_t = \mathbf{R} \oplus V_t$, where (V_t) is strongly mixing. Then for any diffuse amenable von Neumann subalgebra $P \subset N, \mathcal{N}_N(P)''$ is amenable, i.e. N is strongly solid.

Proof. Let $\mathcal{M} = \Gamma(H_\mathbf{R}, U_t)''$ be a free Araki-Woods factor. As usual, denote by $M = \mathcal{M} \rtimes_\sigma \mathbf{R}$ its continuous core, where σ is the modular group associated with the free quasi-free state χ. Let $p \in L(\mathbf{R}) := \lambda^\chi(\mathbf{R})''$ be a nonzero projection such that $\mathrm{Tr}(p) < \infty$.

(a) By contradiction, assume that there exists a maximal abelian *-subalgebra $A \subset pMp$ for which $\mathcal{N}_{pMp}(A)''$ is not amenable. Write $p - z \in \mathcal{Z}\left(\mathcal{N}_{pMp}(A)''\right)$ for the maximal projection such that $\mathcal{N}_{pMp}(A)''(p-z)$ is amenable. Then $z \neq 0$ and $\mathcal{N}_{pMp}(A)''z$ has no amenable direct summand. Notice that

$$\mathcal{N}_{pMp}(A)''z \subset \mathcal{N}_{zMz}(Az)''.$$

Since this is a unital inclusion (with unit z), $\mathcal{N}_{zMz}(Az)''$ has no amenable direct summand either. Moreover, $Az \subset zMz$ is still maximal abelian. Since $L(\mathbf{R})$ is diffuse, $\mathrm{Tr}_{|L(\mathbf{R})}$ is semifinite and M is a type II_∞ factor, we can find a projection $p_0 \in L(\mathbf{R})$ such that $p_0 \leq p$ and a unitary $u \in \mathcal{U}(M)$ such that $uzu^* = p_0$ Observe that $A_0 = uAzu^* \subset p_0 M p_0$ is maximal abelian and $\mathcal{N}_{p_0 M p_0}(A_0)''$ has no amenable direct summand. Therefore, we may assume without loss of generality that $p = p_0$, i.e. $A \subset pMp$ is a maximal abelian *-subalgebra for which $\mathcal{N}_{pMp}(A)''$ has no amenable direct summand.

Theorem (4.3.36) yields $A \preceq_M L(\mathbf{R})$. Thus there exists $n \geq 1$, a nonzero Tr-finite projection $q \in L(\mathbf{R})^n$, a nonzero partial isometry $v \in \mathbf{M}_{1,n}(\mathbf{C}) \otimes pM$ and a unital *-homomorphism $\psi: A \to L(\mathbf{R})^n$ such that $xv = v\psi(x), \forall x \in A$. Write $q = \psi(p), q' = v^*v$. Note that $vv^* \in A' \cap pMp = A$ and $q' \in \psi(A)' \cap qM^nq$. It follows that $q'(\psi(A)' \cap qM^nq)q' = (\psi(A)q')' \cap q'M^nq'$. Since by spatiality $\psi(A)q' = v^*Av$ is maximal abelian, we get $q'(\psi(A)' \cap qM^nq)q' = \psi(A)q' = v^*Av$. Thus $\psi(A)' \cap qM^nq$ has a type I abelian direct summand. Moreover,

$$q\left(\mathcal{M}^\chi \mathbin{\bar{\otimes}} L(\mathbf{R})\right)^n q \subset q(L(\mathbf{R})' \cap M)^n q \subset \psi(A)' \cap qM^nq.$$

Recall that one of the following situations holds:

1. (U_t) contains a trivial or periodic subrepresentation of dimension 2. In that case, $L(\mathbf{F}_2) \subset \mathcal{M}^\chi$.
2. $(U_t) = \mathbf{R} \oplus (V_t)$, where (V_t) is weakly mixing. In that case, $\mathcal{M}^\chi = L(\mathbf{Z})$.
3. (U_t) is weakly mixing and then $\mathcal{M}^\chi = \mathbf{C}$.

The subcase (i) cannot occur because otherwise $\psi(A)' \cap qM^nq$ would be of type II. Assume now that (ii) occurs. We have $(U_t) = \mathbf{R} \oplus (V_t)$ where (V_t) is weakly mixing. Then we have

$$\mathcal{M} = \Gamma(H_{\mathbf{R}}, U_t)'' \simeq \Gamma(K_{\mathbf{R}}, V_t)'' * L(\mathbf{Z}),$$

and [203] implies that $L(\mathbf{Z})$ is maximal abelian in $\mathcal{M}$. Therefore $B = L(\mathbf{Z}) \mathbin{\bar{\otimes}} L(\mathbf{R})$ is maximal abelian in M. Since $A \preceq_M L(\mathbf{R})$, we get $A \preceq_M B$. Since $A \subset pMp$ and $B \subset M$ are both maximal abelian, Proposition (4.3.6) yields $n \geq 1$, a nonzero partial isometry $v \in pM$ such that $vv^* \in A, v^*v \in B$ and $v^*Av = Bv^*v$. By spatiality, we get

$$\mathrm{Ad}(v^*)(\mathcal{N}_{vv^*Mvv^*}(Avv^*)'') = \mathcal{N}_{v^*vMv^*v}(Bv^*v)''.$$

On the one hand, $\mathcal{N}_{vv^*Mvv^*}(Avv^*)'' = vv^*\mathcal{N}_{pMp}(A)''vv^*$ is not amenable, since $\mathcal{N}_{pMp}(A)''$ has no amenable direct summand. On the other hand, since $L(\mathbf{Z}) = \mathcal{M}^\chi$ is diffuse, Proposition (4.3.38) implies $Bv^*v = (L(\mathbf{Z}) \mathbin{\bar{\otimes}} L(\mathbf{R}))v^*v \npreceq_M L(\mathbf{R})$. Theorem (4.3.36) implies that $\mathcal{N}_{v^*vMv^*v}(Bv^*v)''$ is amenable. We have reached a contradiction.

Assume at last that (iii) occurs. Since (U_t) is weakly mixing, it follows that $\mathcal{M}^\chi = \mathbf{C}$ and $L(\mathbf{R})$ is maximal abelian in M by Proposition (4.3.10). Proposition (4.3.6) yields $n \geq 1$, a nonzero partial isometry $v \in pM$ such that $vv^* \in A, v^*v \in L(\mathbf{R})$ and $v^*Av = L(\mathbf{R})v^*v$. By spatiality, we get

$$\mathrm{Ad}(v^*)(\mathcal{N}_{vv^*Mvv^*}(Avv^*)'') = \mathcal{N}_{v^*vMv^*v}(L(\mathbf{R})v^*v)''.$$

On the one hand, $\mathcal{N}_{vv^*Mvv^*}(Avv^*)'' = vv^*\mathcal{N}_{pMp}(A)''vv^*$ is not amenable, since $\mathcal{N}_{pMp}(A)''$ has no amenable direct summand. On the other hand, since (U_t) is weakly mixing, $L(\mathbf{R})$ is singular in M, i.e. $\mathcal{N}_M(L(\mathbf{R}))'' = L(\mathbf{R})$. Therefore $\mathcal{N}_{v^*vMv^*v}(L(\mathbf{R})v^*v)'' = L(\mathbf{R})v^*v$. We have reached again a contradiction.

(2-i) Assume that (U_t) is strongly mixing. Let $P \subset pMp$ be a unital diffuse amenable von Neumann subalgebra. By contradiction, assume that $\mathcal{N}_{pMp}(P)''$ is not amenable. With the same reasoning as before, we may assume that $\mathcal{N}_{pMp}(P)''$ has no amenable direct summand.

Theorem (4.3.36) yields $P \preceq_M L(\mathbf{R})$. Thus there exist $n \geq 1$, a nonzero Tr-finite projection $q \in L(\mathbf{R})^n$, a nonzero partial isometry $v \in \mathbf{M}_{1,n}(\mathbf{C}) \otimes pM$ and a unital *-homomorphism $\psi: P \to qL(\mathbf{R})^n q$ such that $xv = v\psi(x), \forall x \in P$. Note that $vv^* \in P' \cap pMp \subset \mathcal{N}_{pMp}(P)''$ and $v^*v \in \psi(P)' \cap qM^n q$. Since $\psi(P) \subset qL(\mathbf{R})^n q$ is a unital diffuse von Neumann subalgebra and the action $\mathbf{R} \curvearrowright \mathcal{M}$ is strongly mixing (see [173], Proposition 2.4 and Theorem 3.7) yields $\mathcal{QN}_{qM^nq}(\psi(P))'' \subset qL(\mathbf{R})^n q$. Thus we may assume that $v^*v = q$. Let $u \in \mathcal{N}_{pMp}(P)$. We have

$$\begin{aligned} v^*uv\psi(P) &= v^*uPv \\ &= v^*Puv \\ &= \psi(P)v^*uv. \end{aligned}$$

Hence $v^*\mathcal{N}_{pMp}(P)''v \subset \mathcal{QN}_{qM^nq}(\psi(P))'' \subset qL(\mathbf{R})^n q$. But

$$\mathrm{Ad}(v^*): vv^*\mathcal{N}_{pMp}(P)''vv^* \to qL(\mathbf{R})^n q$$

is a unital $*$-isomorphism. Since $\mathcal{N}_{pMp}(P)''$ has no amenable direct summand, $vv^* \cdot \mathcal{N}_{pMp}(P)''vv^*$ is not amenable. This contradicts the fact that $qL(\mathbf{R})^n q$ is amenable.

(2-ii) Assume that $U_t = \mathrm{R} \oplus V_t$ where (V_t) is strongly mixing. Observe that we have $\Gamma(H_{\mathbf{R}}, U_t)'' = \Gamma(K_{\mathbf{R}}, V_t)'' * L(\mathbf{Z})$. If we denote by u a generating Haar unitary for $L(\mathbf{Z})$ and by $\mathcal{Q}_\infty = *_{n\in\mathbf{Z}}\, u^n\Gamma(K_{\mathbf{R}}, V_t)''u^{-n}$ the infinite free product, we may regard $\Gamma(H_{\mathbf{R}}, U_t)''$ as the crossed product

$$\Gamma(H_{\mathbf{R}}, U_t)'' = \mathcal{Q}_\infty \rtimes \mathbf{Z}$$

where the action $\mathbf{Z} \curvearrowright \mathcal{Q}_\infty$ is the free Bernoulli shift. Observe that the modular group (σ_t^χ) acts trivially on $L(\mathbf{Z})$. Moreover, (σ_t^χ) acts diagonally on $\mathcal{Q}_\infty$ in the following sense. Denote by ψ the free quasi-free state on $\Gamma(K_{\mathbf{R}}, V_t)''$. Let $y_1, \dots, y_k \in \Gamma(K_{\mathbf{R}}, V_t)'' \ominus \mathbf{C}, n_1 \neq \cdots \neq n_k, x_i = u^{n_i}y_iu^{-n_i}$ and write $x = x_1 \cdots x_k$ for the corresponding reduced word in $\mathcal{Q}_\infty$. Then we have

$$\sigma_t^\chi(x) = u^{n_1}\sigma_t^\psi(y_1)u^{-n_1} \cdots u^{n_k}\sigma_t^\psi(y_k)u^{-n_k}.$$

The core M is therefore given by

$$M = \mathcal{Q}_\infty \rtimes (\mathbf{Z} \times \mathbf{R}).$$

Since (V_t) is assumed to be strongly mixing, it is straightforward to check that the action $\mathbf{Z} \times \mathbf{R} \curvearrowright \mathcal{Q}_\infty$ is strongly mixing (see [173]).

We are now ready to prove that pMp is strongly solid. Assume by contradiction that it is not. As we did before, let $P \subset pMp$ be a unital diffuse amenable von Neumann subalgebra such that $\mathcal{N}_{pMp}(P)''$ has no amenable direct summand. Theorem (4.3.36) yields $P \preceq_M L(\mathbf{R})$ and hence $P \preceq_M L(\mathbf{Z}) \bar{\otimes} L(\mathbf{R})$. Thus there exists $n \geq 1$, a nonzero Tr-finite projection $q \in (L(\mathbf{Z}) \overline{\otimes} L(\mathbf{R}))^n$, a nonzero partial isometry $v \in \mathbf{M}_{1,n}(\mathbf{C}) \otimes pM$ and a unital $*$-homomorphism $\psi: P \to q(L(\mathbf{Z}) \bar{\otimes} L(\mathbf{R}))^n q$ such that $xv = v\psi(x), \forall x \in P$. Note that $vv^* \in P' \cap pMp \subset \mathcal{N}_{pMp}(P)''$ and $v^*v \in \psi(P)' \cap qM^n q$. Since $\psi(P) \subset q(L(\mathbf{Z}) \bar{\otimes} L(\mathbf{R}))^n q$ is a unital diffuse von Neumann subalgebra and the action $\mathbf{Z} \times \mathbf{R} \curvearrowright \mathcal{Q}_\infty$ is strongly mixing, [173] yields $v^*\mathcal{N}_{pMp}(P)''v \subset q(L(\mathbf{Z}) \bar{\otimes} L(\mathbf{R}))^n q$. But

$$\mathrm{Ad}(v^*): vv^*\mathcal{N}_{pMp}(P)''vv^* \to q(L(\mathbf{Z}) \bar{\otimes} L(\mathbf{R}))^n q$$

is a unital *-isomorphism. Since $\mathcal{N}_{pMp}(P)''$ has no amenable direct summand, $vv^*\mathcal{N}_{pMp}(P)''vv^*$ is not amenable. This contradicts the fact that $q(L(\mathbf{Z}) \bar{\otimes} L(\mathbf{R}))^n q$ is amenable.

A free malleable deformation for (amalgamated) free products of von Neumann algebras was discovered in [156]. Using ideas and techniques of [176],[197],[156],[141], we obtain the following indecomposability results for free products of von Neumann algebras:

Theorem (4.1.40)[190]: Let $(\mathcal{M}_i, \varphi_i)$ be a family of von Neumann algebras endowed with faithful normal states. Denote by $(\mathcal{M}, \varphi) = *_{i \in I} (\mathcal{M}_i, \varphi_i)$ their free product.

1. Assume that $\mathcal{M}$ has the complete metric approximation property. Then either $\mathcal{M}$ is amenable or $\mathcal{M}$ has no Cartan subalgebra.
2. Assume that each $\mathcal{M}_i$ is hyperfinite. Let $\mathcal{N} \subset \mathcal{M}$ be a diffuse von Neumann subalgebra for which there exists a faithful normal conditional expectation $E: \mathcal{M} \to \mathcal{N}$. Then either $\mathcal{N}$ is hyperfinite or $\mathcal{N}$ has no Cartan subalgebra.

Observe that in (b), a free product of hyperfinite von Neumann algebras automatically has the complete metric approximation property by [170].

Chapter 5
Mixing Subalgebras and Generator Masas

We introduce for a finite von Neumann algebra M and von Neumann subalgebras A, B of M, a notion of weak mixing of $B \subset M$ relative to A. We show that weak mixing of B ⊂ M relative to a subalgebra $A \subset B$ is equivalent to the following property: if x ∈ M and there exist a finite number of elements $x_1, \ldots, x_n \in M$ such that $Ax \subset \sum_{i=1}^{n} x_i B$, then $x \in B$. We conclude with an assortment of further examples of mixing subalgebras arising from the amalgamated free product and crossed product constructions. We show that if the orthogonal representation is not ergodic then these von Neumann algebras are factors whenever $\dim(H_R) \geq 2$ and $q \in (-1, 1)$. In such case, the centralizer of the q-quasi free state has trivial relative commutant. In the process, we study 'generator MASAs' in these factors and establish that they are strongly mixing.

Section (5.1): Finite von Neumann Algebras

In [211], Jolissaint and Stalder defined weak mixing and mixing for abelian von Neumann subalgebras of finite von Neumann algebras. These properties arose as natural extensions of corresponding notions in ergodic theory in the following sense: If σ is a measure preserving action of a countable discrete abelian group Γ_0 on a finite measure space (X, μ), then the action is (weakly) mixing in the sense of [207] if and only if the abelian von Neumann subalgebra $L(\Gamma_0)$ is (weakly) mixing in the crossed product finite von Neumann algebra $L^\infty(X, \mu) \rtimes \Gamma_0$.

We extend the definitions of weak mixing and mixing to general von Neumann subalgebras of finite von Neumann algebras, and study various algebraic and analytical properties of these subalgebras. In a forthcoming note, the authors will specialize to the study of mixing properties of maximal abelian von Neumann subalgebras. If B is a von Neumann subalgebra of a finite von Neumann algebra M, and $\mathbb{E}_B$ denotes the usual trace-preserving conditional expectation onto B, we call B a weakly mixing subalgebra of M if there exists a sequence of unitary operators $\{u_n\}$ in B such that

$$\lim_{n\to\infty} \|\mathbb{E}_B(xu_n y) - \mathbb{E}_B(x)u_n\mathbb{E}_B(y)\|_2 = 0, \ \forall x, y \in M.$$

We call B a mixing subalgebra of M if the above limit is satisfied for all elements x, y in M and all sequences of unitary operators $\{u_n\}$ in B such that $\lim_{n\to\infty} u_n = 0$ in the weak operator topology. When B is an abelian algebra, our definition of weak mixing is precisely the weak asymptotic homomorphism property introduced by Robertson, Sinclair and Smith [216]. Although our definitions of weak mixing and mixing are slightly different from those of Jolissaint and Stalder, our definitions coincide with theirs in the setting of the action of a countable discrete group on a probability space. Using arguments similar to those in the proofs of Proposition 2.2 and Proposition 3.6 of [211], one can show:

Proposition (5.1.1)[206]: If σ is a measure preserving action of a countable discrete group Γ_0 on a finite measure space (X, μ), then the action is (weakly) mixing in the sense of [207] if and only if the von Neumann subalgebra $L(\Gamma_0)$ is (weakly) mixing in the crossed product finite von Neumann algebra

$$L^\infty(X, \mu) \rtimes \Gamma_0.$$

For an inclusion of finite von Neumann algebras $B \subset M$, we call a unitary operator $u \in M$ a normalizer of B in M if $uBu^* = B$[35]. Clearly, every unitary in B satisfies this condition; the subalgebra B is said to be singular in M if the only normalizers of B in M are elements of B. There is a close relationship between the concepts of weak mixing and singularity.

Sinclair and Smith [218] noted one connection in proving that weakly mixing von Neumann subalgebras are singular in their containing algebras. The converse was proved by Sinclair, Smith, White and Wiggins [221] under the assumption that the subalgebra is also masa (maximal abelian self-adjoint subalgebra) in the ambient von Neumann algebra. In other words, the measure preserving action of a countable discrete abelian group Γ_0 on a finite measure space (X, μ) is weakly mixing if and only if the associated von Neumann algebra $L(\Gamma_0)$ is singular in $L^\infty(X, \mu) \rtimes \Gamma_0$. This provides an operator algebraic characterization of weak mixing in the abelian setting, which is the main motivation for the study undertaken here. In contrast to the abelian case, Grossman and Wiggins [209] showed that for general finite von Neumann algebras, weakly mixing is not equivalent to singularity, so techniques beyond those known for singular subalgebras are required. In what follows, we develop basic theory for mixing properties of general subalgebras of finite von Neumann algebras. This leads to a number of new observations about mixing properties of subalgebras and group actions, a characterization of weakly mixing subalgebras in terms of their finite bimodules, and a variety of new examples of inclusions of von Neumann algebras satisfying mixing conditions.

We show that if B is a diffuse finite von Neumann algebra, then

$$B^\omega \ominus B = \{x \in B^\omega : \tau_\omega(x^* b) = 0, \forall b \in B\}$$

is the weak operator closure of the linear span of unitary operators in $B^\omega \ominus B$, where B^ω is the ultra-power algebra of B.

We prove that if B is a mixing von Neumann subalgebra of a finite von Neumann algebra M, one has

$$\lim_{n\to\infty} \|\mathbb{E}_B(x b_n y) - \mathbb{E}_B(x) b_n \mathbb{E}_B(y)\|_2 = 0, \ \forall x, y \in M,$$

if $\{b_n\}$ is a bounded sequence of operators in B such that $\lim_{n\to\infty} b_n = 0$ in the weak operator topology. As applications, we show that if B is mixing in M, k is a positive integer, and $e \in B$ is a projection, then $M_k(\mathbb{C}) \otimes B$ is mixing in $M_k(\mathbb{C}) \otimes M$ and eBe is mixing in eMe. We also show that, in contrast to weakly mixing masas, one cannot distinguish mixing masas by the presence or absence of centralizing sequences in the masa for the containing II_1 factor.

We concerns the special case of inclusions of group von Neumann algebras. We extend some results of [211] for abelian subgroups to the case of a general inclusion of countable, discrete groups $\Gamma_0 \subset \Gamma$ in showing that $L(\Gamma_0)$ is mixing in $L(\Gamma)$ if and only if $g\Gamma_0 g^{-1} \cap \Gamma_0$ is a finite group for every $g \in \Gamma \setminus \Gamma_0$. These two conditions are seen to be equivalent the property that for every diffuse von Neumann subalgebra A of B and every $y \in M$, $yAy^* \subset B$ implies $y \in B$. Some applications to ergodic theory are given. In particular, Theorem (5.1.16) generalizes results of Kitchens and Schmidt [213] and Halmos [210].

We introduce and study the concept of relative weak mixing for a triple of finite von Neumann algebras, and obtain several characterizations of weakly mixing triples. It turns out that relative weak mixing of an inclusion $B \subset M$ with respect to a von Neumann subalgebra $A \subset B$ is closely related to the bimodule structure between the two subalgebras A and B. In particular, we show that $B \subset M$ is weakly mixing relative to A if and only the following property holds: if $x \in M$ satisfies $Ax \subset \sum_{i=1}^{n} x_i B$ for a finite number of elements $x_1, \dots, x_n$ in M, then $x \in B$.

The results show that mixing von Neumann subalgebras have hereditary properties which are notably different from those of general singular subalgebras. We also consider the relationship between mixing and normalizers; in particular, we show that subalgebras of mixing algebras inherit a strong singularity property from the containing algebra. Finally, we provide an assortment of new examples of mixing von Neumann subalgebras which arise from the amalgamated free product and crossed product constructions.

We collect here some basic facts about finite von Neumann algebras. Throughout, M is a finite von Neumann algebra with a given faithful normal trace τ. Denote by $L^2(M) = L^2(M,\tau)$ the Hilbert space obtained by the GNS-construction of M with respect to τ. The image of $x \in M$ via the GNS-construction is denoted by $\hat{x}$, and the image of a subset L of M is denoted by $\hat{L}$. The trace norm of $x \in M$ is defined by $\| x \|_2 = \| x \|_{2,\tau} = \tau(x^*x)^{1/2}$. Suppose that B is a von Neumann subalgebra of M. Then there exists a unique faithful normal conditional expectation $\mathbb{E}_B$ from M onto B preserving τ. Let e_B be the projection of $L^2(N)$ onto $L^2(B)$. Then the von Neumann algebra $\langle M, e_B \rangle$ generated by M and e_B is called the basic construction of M, which plays a crucial role in the study of von Neumann subalgebras of finite von Neumann algebras. There is a unique faithful tracial weight Tr on $\langle M, e_B \rangle$ such that

$$\mathrm{Tr}(x e_B y) = \tau(xy),\ \forall x, y \in M.$$

For $\xi \in L^2(\langle M, e_B \rangle, \mathrm{Tr})$, define $\| \xi \|_{2,\mathrm{Tr}} = \mathrm{Tr}(\xi^* \xi)^{1/2}$. For more details of the basic construction, see [208],[158],[165],[219]. For a detailed account of finite von Neumann algebras and the theory of masas, see [219].

Let M be a finite von Neumann algebra with a faithful normal trace τ, and let B be a von Neumann subalgebra of M. We denote by $M \ominus B$ the orthngonal complement of B in M with respect to the standard inner product on M, that is,

$$M \ominus B = \{x \in M : \tau(x^* b) = 0 \text{ for all } b \in B\}.$$

Then $x \in M \ominus B$ if and only if $\mathbb{E}_B(x) = 0$, where $\mathbb{E}_B$ is the trace-preserving conditional expectation of M onto B. Note that if $x \in M \ominus B$, then $\tau(x) = \tau(\mathbb{E}_B(x)) = 0$, so the unique positive element in $M \ominus B$ is 0. On the other hand, it is easy to see that $M \ominus B$ is the linear span of self-adjoint elements in $M \ominus B$.

We will use the fact that a bounded sequence (b_n) in a finite von Neumann algebra B converges to 0 in the weak operator topology if and only if it defines an element of the ultrapower B^ω which is orthogonal to B in the above sense. A key step in the proof of Theorem (5.1.10) will then be to approximate an arbitrary $z \in B^\omega \ominus B$ by linear combinations of unitary operators in $B^\omega \ominus B$. That such an approximation is possible is the main technical result.

When $B \subset M$ comes from an inclusion of countable discrete groups, there is an obvious dense linear subspace of $\ominus B$: if G is a subgroup of a discrete group Γ, then $\mathcal{L}(\Gamma) \ominus \mathcal{L}(G)$ is the weak closure of the linear span of unitary operators corresponding to elements in $\Gamma \setminus G$. Although in the case of a general inclusion $B \subseteq M$, such a canonical set of unitaries is not available, we nevertheless obtain a partial answer to the following question: If B is a subalgebra of a diffuse finite von Neumann algebra M such that $eMe \neq eBe$ for every nonzero projection $e \in B$, is $M \ominus B$ the weak closure of the linear span of unitaries in $M \ominus B$?

The assumption that $eMe \neq eBe$ for every nonzero projection $e \in B$ is necessary, as is the assumption that M is diffuse. For instance, if $M = \mathbb{C} \oplus \mathbb{C}$ and $B = \mathbb{C}$ and $\tau(1 \oplus 0) \neq \tau(0 \oplus 1)$, then there are no unitary operators in $M \ominus B$.

Let $(M)_1$ be the operator norm-closed unit ball of M, and let

$$\Lambda = \{x \in (M)_1 : x = x^*, \mathbb{E}_B(x) = 0\}.$$

Then Λ is a convex set which is closed, hence also compact, in the weak operator topology. By the Krein-Milman Theorem, Λ is the weak operator closure of the convex hull of its extreme points. Thus, we need only characterize the extreme points of Λ.

Lemma (5.1.2)[206]: Suppose that for every nonzero projection $p \in M$, there exists a nonzero element $x_p \in pMp$ satisfying $\mathbb{E}_B(x_p) = 0$. Then the extreme points of Λ are

$$\left\{2e - 1: e \in M \text{ a projection, with } \mathbb{E}_B(e) = \frac{1}{2}\right\}.$$

Proof. If $e \in M$ is a projection with $\mathbb{E}_B(e) = \frac{1}{2}$, then it is easy to see that the operator $u = 2e - 1 \in \Lambda$ is an extreme point of the unit ball $(M)_1$, hence also an extreme point of Λ. On the other hand, suppose that $a \in \Lambda$ is an extreme point of Λ, but is not of the form $2e - 1$, for some projection $e \in M$, as above. By the spectral decomposition theorem, there exists an $\epsilon > 0$ and a nonzero spectral projection e of a such that

$$(-1 + \epsilon)e \le ae \le (1 - \epsilon)e.$$

By assumption, there is a nonzero self-adjoint element $x \in eMe$ such that $\mathbb{E}_B(x) = 0$. By multiplying by a scalar, we may insist that $-\epsilon e \le x \le \epsilon e$ Then $a + x, a - x \in \Lambda$ and $a = \frac{1}{2}(a + x) + \frac{1}{2}(a - x)$, so a is not an extreme point of Λ, contradicting our assumption. Therefore, $a = 2e - 1$ for some projection $e \in M$. Since $\mathbb{E}_B(a) = 0, \mathbb{E}_B(e) = \frac{1}{2}$. This completes the proof.

The following example shows that the assumptions of the above lemma are essential.

Example (5.1.3)[206]: In the inclusion $\mathbb{C} \subset M_3(\mathbb{C})$, there is no projection $e \in M_3(\mathbb{C})$ satisfing $\tau(e) = \frac{1}{2}$. In this case, the partial isometry

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

is an extreme point of Λ.

Corollary (5.1.4)[206]: Let M be a diffuse finite von Neumann algebra with a faithful normal trace τ. Then $M \ominus \mathbb{C}1$ is the weak operator closure of the linear span of self-adjoint unitary operators in $M \ominus \mathbb{C}1$.

Proof. For every nonzero projection $p \in M, pMp$ is diffuse and hence $pMp \neq \mathbb{C}p$. So there is a nonzero operator $x_p \in pMp$ with $\tau(x_p) = 0$ By Lemma (5.1.2), $M \ominus \mathbb{C}1$ is the weak operator closure of the linear span of self-adjoint unitary operators in $M \ominus \mathbb{C}1$.

For the next result, recall that every diffuse finite von Neumann algebra N with faithful trace τ contains a Haar unitary, that is, a unitary element $u \in N$ such that $\tau(u^n) = 0$ for all $n \in \mathbb{N}$.

Lemma (5.1.5)[206]: Suppose B is a diffuse finite von Neumann algebra with a faithful normal trace τ. For $\epsilon > 0$ and $x_1, \dots, x_n \in B$, there exists a Haar unitary operator $u \in B$ such that

$$|\tau(x_i u^*)| < \epsilon,\ 1 \le i \le n.$$

Proof. Since B is diffuse, B contains a Haar unitary operator v. Note that $v^n \to 0$ in the weak operator topology. So there exists an N such that

$$|\tau(x_i (v^N)^*)| < \epsilon,\ 1 \le i \le n.$$

Let $u = v^N$. Then u is a Haar unitary operator and the lemma follows.

Given a separable diffuse von Neumann algebra B with faithful normal trace τ and an ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$, denote by B^ω the corresponding ultrapower algebra, and the induced faithful normal trace by τ_ω (see [217]). We again use the standard notation of $(B^\omega \ominus B)_1$ for the norm-closed unit ball of $B^\omega \ominus B$. The following proposition is the main result.

Proposition (5.1.6)[206]: Suppose B is a separable diffuse finite von Neumann algebra with a faithful normal trace τ. Then $(B^\omega \ominus B)_1$ is the trace norm closure of the convex hull of self-adjoint unitary operators in $B^\omega \ominus B$.

Proof. We claim that for every nonzero projection $p \in B^\omega$, there exists a nonzero element x_p in $pB^\omega p$ such that $\mathbb{E}_B(x_p) = 0$, where $\mathbb{E}_B$ is the conditional expectation of B^ω onto B preserving τ_ω. Let $p = (p_n) \in B^\omega$, where $p_n \in B$ is a projection with $\tau(p_n) = \tau_\omega(p) > 0$. Since B is separable, there is a sequence $\{y_k\}$ in B which is dense in the trace norm. We may assume that $y_1 = 1$. By Lemma (5.1.5), for any initial segment $\{y_1, \dots, y_n\}$ of the dense sequence, there is a Haar unitary operator $u_n \in p_n B p_n$ such that

$$|\tau(p_n y_i p_n u_n^*)| < \frac{1}{n}, \quad \forall 1 \le i \le n.$$

Now define an element x_p of B^ω by $x_p = (u_n)$. Then

$$\|x_p\|_2^2 = \lim_{n\to\omega} \|u_n\|_2^2 = \tau(p) > 0.$$

Hence, $x_p \neq 0$ and $x_p \in pB^\omega p$. Note that for each $k \in \mathbb{N}$, we have

$$\tau_\omega\left(y_k (x_p)^*\right) = \tau_\omega\left((p y_k p)(x_p)^*\right) = \lim_{n\to\omega} \tau(p_n y_k p_n u_n^*) = 0.$$

Since $\{y_k\}$ is dense in B in the trace norm topology, $\tau_\omega\left(y(x_p)^*\right) = 0$ for all $y \in B$. This implies $\mathbb{E}_B(x_p) = 0$. By Lemma (5.1.2), $(B^\omega \ominus B)_1$ is the weak operator closure of the convex hull of self-adjoint unitary operators in $B^\omega \ominus B$. Note that $(B^\omega \ominus B)_1$ is a convex set, so its weak operator closure coincides with its closure in the strong operator and trace norm topologies. This proves the result.

Corollary (5.1.7)[206]: Suppose B is a separable diffuse finite von Neumann algebra with a faithful normal trace τ. Then $B^\omega \ominus B$ is the weak operator closure of the linear span of self-adjoint unitary operators in $B^\omega \ominus B$.

Using a similar approach, we can also prove the following result.

Proposition (5.1.8)[206]: If M is a separable type II_1 factor and B is an abelian von Neumann subalgebra of M, then $M \ominus B$ is the weak operator closure of the linear span of unitary operators in $M \ominus B$.

It is not clear whether Proposition (5.1.8) holds for nonabelian subalgebras. We are unable, for instance, to establish the conclusion of the result when B is a hyperfinite subfactor of a nonhyperfinite type II_1 factor M, e.g. $L\mathbb{F}_2$.

Let M be a finite von Neumann algebra with a faithful normal trace τ, and let B be a von Neumann subalgebra of M.

Definition (5.1.9)[206]: An algebra B is a mixing von Neumann subalgebra of M if

$$\lim_{n\to\infty} \|\mathbb{E}_B(x u_n y) - \mathbb{E}_B(x) u_n \mathbb{E}_B(y)\|_2 = 0$$

holds for all $x, y \in M$ and every sequence of unitary operators $\{u_n\}$ in B such that $\lim_{n\to\infty} u_n = 0$ in the weak operator topology. If B is a mixing von Neumann subalgebra of M, then we say $B \subseteq M$ a mixing inclusion of finite von Neumann algebras.

It is easy to see that B is a mixing von Neumann subalgebra of M if and only if for all elements x, y in M with $\mathbb{E}_B(x) = \mathbb{E}_B(y) = 0$, one has

$$\lim_{n\to\infty} \|\mathbb{E}_B(x u_n y)\|_2 = 0$$

whenever $\{u_n\}$ is a sequence of unitary operators in B such that $\lim_{n\to\infty} u_n = 0$ in the weak operator topology.

The following theorem, which is the main result, provides a useful equivalent condition for mixing inclusions of finite von Neumann algebras.

Theorem (5.1.10)[206]: If B is a mixing von Neumann subalgebra of M and $x, y \in M$ with $\mathbb{E}_B(x) = \mathbb{E}_B(y) = 0$, then

$$\lim_{n\to\infty} \|\mathbb{E}_B(xb_n y)\|_2 = 0$$

whenever $\{b_n\}$ is a bounded sequence of operators in Buch that $\lim_{n\to\infty} b_n = 0$ in the weak operator topology.

Proof. Let ω be a free ultrafilter of the set of natural numbers and let M^ω be the ultrapower algebra of M. Then M^ω is a finite von Neumann algebra with a faithful normal trace τ_ω. We can identify B^ω with a von Neumann subalgebra of M^ω in the natural way. Every bounded sequence (b_n) in B defines an element z of B^ω. We may assume that $\| z \| \leq 1$. It is easy to see that $\lim_{n\to\omega} b_n = 0$ in the weak operator topology if and only if

$$\tau_\omega(zb) = 0,\ \forall b \in B.$$

Recall that $M \ominus B = \{x \in M: \tau(x^* b) = 0 \text{ for all } b \in B\}$. It is easy to see that Definition (5.1.9) is equivalent to the following: For any x, y in $M \ominus B$, and any unitary operator $u \in B^\omega \ominus B$, one has $\mathbb{E}_{B^\omega}(xuy) = 0$.

Note that B is a diffuse subalgebra of M. Indeed, suppose $p \in B$ is a minimal projection. Since B is mixing, then in particular we have that $B' \cap M \subset B$, so Theorem 12.2.4 of [219] implies that there exists a masa A of M such that $p \in A \subset B$. But then p is a minimal projection of A, a contradiction. Thus, Proposition (5.1.6) applies, and $(B^\omega \ominus B)_1$ is the trace norm closure of the convex hull of unitary operators in $B^\omega \ominus B$. Let $\epsilon > 0$. Then there exist unitary operators $u_1, \ldots, u_n$ in $B^\omega \ominus B$ and positive numbers $\alpha_1, \ldots, \alpha_n$ with $\alpha_1 + \cdots + \alpha_n = 1$ such that

$$\left\| z - \sum_{k=1}^{n} \alpha_k u_k \right\|_{2,\tau_\omega} < \epsilon.$$

For any elements x and y of $M \ominus B$,

$$\begin{aligned} \|\mathbb{E}_{B^\omega}(xzy)\|_{2,\tau_\omega} &= \left\| \mathbb{E}_{B^\omega}\left(x \left(z - \sum_{k=1}^{n} \alpha_k u_k \right) y \right) \right\|_{2,\tau_\omega} \\ &\leq \left\| x \left(z - \sum_{k=1}^{n} \alpha_k u_k \right) y \right\|_{2,\tau_\omega} \\ &\leq \| x \| \cdot \left\| z - \sum_{k=1}^{n} \alpha_k u_k \right\|_{2,\tau_\omega} \cdot \| y \| \\ &\leq \epsilon \| x \| \| y \|. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $\mathbb{E}_{B^\omega}(xzy) = 0$, which is equivalent to

$$\lim_{n\to\infty} \|\mathbb{E}_B(xb_n y)\|_2 = 0.$$

Two applications of the above theorem are the following.

Corollary (5.1.11)[206]: If B is a mixing von Neumann subalgebra of M and k is a positive integer, then $M_k(\mathbb{C}) \otimes B$ is mixing in $M_k(\mathbb{C}) \otimes M$.

Proof. Note that $x = (x_{ij}) \in (M_k(\mathbb{C}) \otimes M) \ominus (M_k(\mathbb{C}) \otimes B)$ if and only if $x_{ij} \in M \ominus B$ for all $1 \leq i,j \leq k$. Moreover, $b_n = (b_{ij}^n) \in M_k(\mathbb{C}) \otimes B$ converges to 0 in the weak operator topology if and only if b_{ij}^n converges to 0 in the weak operator topology for all $1 \leq i,j \leq k$. Now the corollary follows from Theorem (5.1.10).

Corollary (5.1.12)[206]: If B is a mixing von Neumann subalgebra of M and e is a projection of B, then eBe is mixing in eMe.

Proof. Let (b_n) be a bounded sequence in eBe which converges to 0 in the weak operator topology. For $x,y \in eMe \ominus eBe$, we have $x,y \in M \ominus B$. By Theorem (5.1.10),

$$\lim_{n\to\infty} \|\mathbb{E}_{eBe}(xb_ny)\|_2 = \lim_{n\to\infty} \|\mathbb{E}_B(xb_ny)\|_2 = 0.$$

It is well-known that the presence of centralizing sequences in a masa for its containing II_1 factor is a conjugacy invariant for the masa. More generally, it is possible to build nonconjugate masas of a II_1 factor by controlling the existence of centralizing sequences in various cutdowns of each masa. Sinclair and White [220] developed this technique to produce uncountably many nonconjugate weakly mixing masas in the hyperfinite II_1 factor with the same Pukánszky invariant. The final result implies that, in contrast to the larger class of weakly mixing masas, there is no hope of distinguishing mixing masas along these lines. Following the notation of [220], for a von Neumann subalgebra B of a II_1 factor M, we denote by $\Gamma(B)$ the maximal trace of a projection $e \in B$ for which e Be contains a nontrivial centralizing sequences for eMe.

Proposition (5.1.13)[206]: If B is a mixing subalgebra of a type II_1 factor M and $eBe \neq e$ Me for each nonzero projection $e \in B$, then $\Gamma(B) = 0$.

Proof. By Corollary (5.1.12), we need only show that there is no nontrivial sequence $\{b_n\}$ in B which is centralizing for M. Suppose $\{b_n\} \subset B$ is such a centralizing sequence for M. We may assume that $\tau(b_n) = 0$ for each n. Suppose that $\lim_{n\to\omega} b_n = z \in B$ in the weak operator topology. Then for all $x \in M$,

$$zx = \lim_{n\to\omega} b_nx = \lim_{n\to\omega} xb_n = xz.$$

Since M is a type II_1 factor, $z = \tau(z)1 = 0$. Hence $\lim_{n\to\omega} b_n = 0$ in the weak operator topology. Choose a nonzero element $x \in M$ such that $\tau(xb) = 0$ for all $b \in B$. Note that

$$\begin{aligned}\|xb_n - b_nx\|_2^2 &= \|xb_n\|_2^2 + \|b_nx\|_2^2 - 2\mathrm{Re}\,\tau(b_n^*x^*b_nx)\\ &\geq \tau(b_n^*x^*xb_n) - 2\mathrm{Re}\,\tau\big(b_n^*\mathbb{E}_B(x^*b_nx)\big)\\ &= \tau(x^*xb_nb_n^*) - 2\,\mathrm{Re}\,\tau\big(b_n^*\mathbb{E}_B(x^*b_nx)\big).\end{aligned}$$

Since $\{b_n\}$ is a central sequence of $M, \{b_nb_n^*\}$ is also a central sequence of M. The uniqueness of the trace on M implies that

$$\lim_{n\to\omega} \tau(x^*xb_nb_n^*) = \lim_{n\to\omega} \tau(x^*x)\tau(b_nb_n^*) = \lim_{n\to\omega} \parallel x \parallel_2^2 \cdot \|b_n\|_2^2.$$

By Theorem (5.1.10),

$$0 = \lim_{n\to\infty} \|xb_n - b_nx\|_2 \geq \parallel x \parallel_2 \lim_{n\to\infty} \|b_n\|_2,$$

which implies that $\lim_{n\to\omega} \|b_n\|_2 = 0$. This completes the proof.

Corollary (5.1.14)[206]: If B is a mixing masa of a type II_1 factor M, then $\Gamma(B) = 0$.

We apply our operator-algebraic machinery to the special case of mixing inclusions of von Neumann algebras that arise from actions of countable, discrete groups. This direction was taken up in [211], where it was shown that, for an infinite abelian subgroup Γ_0 of a countable group Γ, the inclusion $L(\Gamma_0) \subset L(\Gamma)$ is mixing if and only if the following condition (called (ST)) is satisfied:

For every finite subset C of $\Gamma \setminus \Gamma_0$, there exists a finite exceptional set $E \subset \Gamma_0$ such that $g\gamma h \notin \Gamma_0$ for all $\gamma \in \Gamma_0 \setminus E$ and $g, h \in C$.

Theorem (5.1.16) supplies a similar characterization for the case in which Γ_0 is not abelian, and also establishes a connection between the group normalizer of the subgroup Γ_0 and the "analytic" normalizer of its associated group von Neumann algebra. The key observation required is the following, which shows that mixing subalgebras satisfy a much stronger form of singularity.

Theorem (5.1.15)[206]: Let B be a mixing von Neumann subalgebra of M, and suppose that A is a diffuse von Neumann subalgebra of B. If $y \in M$ satisfies $yAy^* \subseteq B$, then $y \in B$.

Proof. We may assume that A is a diffuse abelian von Neumann algebra. Then A is generated by a Haar unitary operator w. In particular, $\lim_{n\to\infty} w^n = 0$ in the weak operator topology. Let $x \in M$ and $\mathbb{E}_B(x) = 0$.

Then

$$|\tau(xy)|^2 \leq \|\mathbb{E}_{A'\cap M}(xy)\|_2^2.$$

Note that

$$\mathbb{E}_{A'\cap M}(xy) = \lim_{n\to\omega} \frac{\sum_{k=1}^{n} w^k(xy)(w^*)^k}{n}$$

in the weak operator topology. Hence,

$$\begin{aligned}
|\tau(xy)|^2 &\leq \|\mathbb{E}_{A'\cap M}(xy)\|_2^2 \\
&\leq \lim_{n\to\omega} \left\| \frac{\sum_{k=1}^{n} w^k(xy)(w^*)^k}{n} \right\|_2^2 \\
&= \lim_{n\to\omega} \frac{1}{n^2} \sum_{i,j=1}^{n} \tau\big(w^i(xy)(w^*)^i w^j(y^*x^*)(w^*)^j\big) \\
&\leq \lim_{n\to\omega} \frac{1}{n^2} \sum_{i,j=1}^{n} \left|\tau\big(x\big(yw^{j-i}y^*\big)x^*(w^*)^{j-i}\big)\right| \\
&\leq \lim_{n\to\omega} \frac{1}{n^2} \sum_{i,j=1}^{n} \left\|\mathbb{E}_B\big(x\big(yw^{j-i}y^*\big)x^*(w^*)^{j-i}\big)\right\|_2 \\
&= \lim_{n\to\omega} \frac{1}{n^2} \sum_{i,j=1}^{n} \left\|\mathbb{E}_B\big(x\big(yw^{j-i}y^*\big)x^*\big)\right\|_2.
\end{aligned}$$

By hypothesis, $yw^ny^* \in B$. Note that $\lim_{n\to\infty} yw^ny^* = 0$ in the weak operator topology. By Theorem (5.1.10),

$$\lim_{n\to\infty} \|\mathbb{E}_B(x(yw^ny^*)x^*)\|_2 = 0.$$

So

$$|\tau(xy)|^2 \leq \lim_{n\to\omega} \frac{1}{n^2} \sum_{i,j=1}^{n} \left\|\mathbb{E}_B\big(x\big(yw^{j-i}y^*\big)x^*\big)\right\|_2 = 0.$$

Therefore, $\tau(xy) = 0$ for all $y \in M \ominus B$. This implies that $y \in B$.

Theorem (5.1.16)[206]: Let $M = L(\Gamma)$ and $B = L(\Gamma_0)$. Then the following conditions are equivalent:

1. $B = L(\Gamma_0)$ is mixing in $M = L(\Gamma)$.

2. $g\Gamma_0 g^{-1} \cap \Gamma_0$ is a finite group for every $g \in \Gamma \setminus \Gamma_0$.
3. For every diffuse von Neumann subalgebra A of B and every unitary operator $v \in M$, if $vAv^* \subseteq B$, then $v \in B$.
4. For every diffuse von Neumann subalgebra A of B and every operator $y \in M$, if $yAy^* \subseteq B$, then $y \in B$.

Proof. (a) ⇒ (d) follows from Theorem (5.1.15) and (d)⇒(c) is trivial.

(c)⇒(b) Suppose $M = L(\Gamma)$ and $B = L(\Gamma_0)$. Suppose for some $g \in \Gamma \setminus \Gamma_0$,, $g\Gamma_0 g^{-1} \cap \Gamma_0$ is an infinite group. Let $\Gamma_1 = \Gamma_0 \cap g^{-1}\Gamma_0 g = g^{-1}(g\Gamma_0 g^{-1} \cap \Gamma_0)g$ Then Γ_1 is an infinite group, and $g\Gamma_1 g^{-1} \subseteq \Gamma_0$. So $\lambda(g)L(\Gamma_1)\lambda(g^{-1}) \subseteq L(\Gamma_0)$. By the third statement, $\lambda(g) \in L(\Gamma_0)$ and $g \in \Gamma_0$. This is a contradiction.

(b)⇒(a) First, we show that if $g_1, g_2 \in \Gamma \setminus \Gamma_0$, then $g_1\Gamma_0 g_2 \cap \Gamma_0$ is a finite set. Suppose $h_1, h_2 \in \Gamma_0$ and $g_1h_1g_2, g_1h_2g_2 \in \Gamma_0$. Then
$$g_1h_1h_2^{-1}g_1^{-1} = g_1h_1g_2(g_1h_2g_2)^{-1} \in \Gamma_0 \cap g_1\Gamma_0 g_1^{-1}.$$
Since $\Gamma_0 \cap g_1\Gamma_0 g_1^{-1}$ is a finite group,
$$\{h_1h_2^{-1}: h_1, h_2 \in \Gamma_0 \text{ and } g_1h_1g_2, g_1h_2g_2 \in \Gamma_0\}$$
is a finite set. Hence, $g_1\Gamma_0 g_2 \cap \Gamma_0$ is a finite set.

Let $\{v_n\}$ be a sequence of unitary operators in B such that $\lim_{n\to\infty} v_n = 0$ in the weak operator topology. Write $v_n = \sum_{k=1}^{\infty} \alpha_{n,k}\lambda(h_k)$. Then for each k, $\lim_{n\to\infty} \alpha_{n,k} = 0$. Suppose $g_1, g_2 \in \Gamma \setminus \Gamma_0$. There exists an N such that for all $m \geq N$, $g_1h_mg_2 \notin \Gamma_0$. Hence,
$$\|\mathbb{E}_B(g_1v_ng_2)\|_2 = \sum_{i=1}^{N} \|\alpha_{n,i}\mathbb{E}_B(g_1\lambda(h_i)g_2)\|_2 \leq \sum_{i=1}^{N} |\alpha_{n,i}| \to 0$$
when $n \to \infty$. M is mixing relative to B.

We now apply Theorem (5.1.16) to the group-theoretic situation arising from a semidirect product $\Gamma = G \rtimes \Gamma_0$, where Γ_0 is an infinite group. Let $\sigma_h(g) = \mathrm{hgh}^{-1}$ for $h \in \Gamma_0$ and $g \in G$. Then σ_h is an automorphism of G. Note that $hg = hgh^{-1}h = \sigma_h(g)h$ for $h \in \Gamma_0$ and $g \in G$.

Proposition (5.1.17)[206]: Let $M = L(G \rtimes \Gamma_0)$ and $B = L(\Gamma_0)$. Then B is mixing in M if and only if for each $g \in G, g \neq e$, the group
$$\{h \in \Gamma_0: \sigma_h(g) = g\}$$
is finite.

Proof. Let $g \in G$ and $h \in \Gamma_0$. Suppose $h \in g\Gamma_0 g^{-1} \cap \Gamma_0$. Then $ghg^{-1} \in \Gamma_0$. Note that $\mathrm{ghg}^{-1} = hh^{-1}\mathrm{ghg}^{-1} = h(\sigma_{h^{-1}}(\mathrm{g})\mathrm{g}^{-1})$. So $\mathrm{ghg}^{-1} \in \Gamma_0$ implies that $\sigma_{h^{-1}}(g)g^{-1} \in \Gamma_0 \cap G = \{e\}$, i.e., $\sigma_{h^{-1}}(g) = g$ and hence $\sigma_h(g) = g$ Conversely, suppose $\sigma_h(g) = g$. Then $\sigma_{h^{-1}}(g) = g$ and hence $ghg^{-1} = h\sigma_{h-1}(g)g^{-1} = h \in \Gamma_0 \cap g\Gamma_0 g^{-1}$. This proves
$$\{h \in \Gamma_0: \sigma_h(g) = g\} = \{h \in \Gamma_0: h \in g\Gamma_0 g^{-1} \cap \Gamma_0\}.$$
Suppose B is mixing in M. By (b) of Theorem (5.1.16), $g\Gamma_0 g^{-1} \cap \Gamma_0$ is a finite group for every $g \in G$ with $g \neq e$. So the group $\{h \in H: \sigma_h(g) = g\}$ is finite. Conversely, suppose that for each $g \in G, g \neq e$, the group $\{h \in \Gamma_0: \sigma_h(g) = g\}$ is finite. Our previous observations then imply that the group $g\Gamma_0 g^{-1} \cap \Gamma_0$ is finite. A group element of $\Gamma \setminus \Gamma_0$ can be written as $h, g \in G$ $g \neq e, h \in \Gamma_0$. Note that
$$gh\Gamma_0 h^{-1}g^{-1} \cap \Gamma_0 = g\Gamma_0 g^{-1} \cap \Gamma_0$$
is finite. So B is mixing in M by (b) of Theorem (5.1.16).

Recall that the action σ of a group H on a finite von Neumann algebra N is called ergodic if $\sigma_h(x) = x$ for all $h \in H$ implies that $x = \lambda 1$. The following result extends Theorem 2.4 of [213] to the noncommutative setting.

Corollary (5.1.18)[206]: Let $M = L(G \rtimes \Gamma_0)$ and $B = L(\Gamma_0)$. Suppose Γ_0 is a finitely generated, infinite, abelian group or Γ_0 is a torsion free group. Then B is mixing in M if and only if every element $h \in \Gamma_0$ of infinite order is ergodic on $L(G)$.

Proof. If B is mixing in M, then clearly every element $h \in \Gamma_0$ of infinite order is ergodic on $L(G)$. Now suppose every element $h \in \Gamma_0$ of infinite order is ergodic on $L(G)$. If B is not mixing in M, then there is a $g \in G$, $g \neq e$, such that $\{h \in \Gamma_0 : \sigma_h(g) = g\}$ is an infinite group. Under the above hypotheses on Γ_0, there exists an element h_0 of infinite order such that $\sigma_{h_0}(g) = g$. This implies that the action of h_0 on $L(G)$ is not ergodic, which is a contradiction.

Corollary (5.1.19)[206]: Let $M = L(G \rtimes \mathbb{Z})$ and $B = L(\mathbb{Z})$. Then the following conditions are equivalent:

1. The action of $\mathbb{Z}$ on $L(G)$ is mixing, i.e., B is mixing in M.
2. The action of $\mathbb{Z}$ on $L(G)$ is weakly mixing, i.e., B is weakly mixing in M.
3. The action of $\mathbb{Z}$ on $L(G)$ is ergodic.
4. For every $g \in G, g \neq e$, the orbit $\{\sigma_h(g)\}$ is infinite.
5. For every $g \in G, g \neq e, \{h \in \mathbb{Z} : \sigma_h(g) = g\} = \{e\}$.

Proof. Let γ be a generator of $\mathbb{Z}$. Clearly (a) ⇒ (b) ⇒ (c).

(c) ⇒ (d) Suppose $\sigma_{\gamma^n}(g) = g$ and n is the minimal positive integer satisfies this condition. Let $x = L_g + L_{\sigma_\gamma(g)} + \cdots + L_{\sigma_{\gamma^{n-1}}(g)}$. Then $x \in L(G)$, $x \neq \lambda 1$, and $\sigma_h(x) = x$ for all $h \in \mathbb{Z}$. This implies that the action of $\mathbb{Z}$ on $L(G)$ is not ergodic.

(d) ⇒ (e) Suppose $\sigma_{\gamma^n}(g) = g$ for some positive integer n. Then the orbit $\{\sigma_h(g)\}$ has at most n elements.

(e) ⇒ (a) follows from Proposition (5.1.17).

A special case of Corollary (5.1.19) implies the following classical result of Halmos [210].

Corollary (5.1.20)[206]: (Halmos's Theorem). Let X be a compact abelian group, and $T: X \to X$ a continuous automorphism. Then T is mixing if and only if T is ergodic.

Proof. By the Pontryagin duality theorem, the dual group G of X is a discrete abelian group. Furthermore, there is an induced action of $\mathbb{Z}$ on G, and the action is unitarily conjugate to the action of T on X. Now the corollary follows from Corollary (5.1.19).

Suppose M is a finite von Neumann algebra with a faithful normal trace τ, and A, B are von Neumann subalgebras of M. We say $B \subset M$ is weakly mixing relative to A if there exists a sequence of unitary operators $u_n \in A$ such that

$$\lim_{n\to\infty} \|\mathbb{E}_B(xu_ny) - \mathbb{E}_B(x)u_n\mathbb{E}_B(y)\|_2 = 0, \ \forall x, y \in M.$$

So B is weakly mixing in M if and only if $B \subset M$ is weakly mixing relative to B. Since every diffuse von Neumann algebra contains a sequence of unitary operators converging to 0 in the weak operator topology, B is mixing in M implies that $B \subset M$ is weakly mixing relative to A for all diffuse von Neumann subalgebras A of B.

It is easy to see that $B \subset M$ is weakly mixing relative to A if and only if there exists a sequence of unitary operators $u_n \in A$ such that for all elements x, y in M with $\mathbb{E}_B(x) = \mathbb{E}_B(y) = 0$, one has

$$\lim_{n\to\infty} \|\mathbb{E}_B(xu_ny)\|_2 = 0.$$

The main result is the following, which is inspired by [52].

Corollary (5.1.21)[206]: Let M be a finite von Neumann algebra with a faithful normal trace τ, and let B be a von Neumann subalgebra of M. Then the following conditions are equivalent:

1. B is a weakly mixing von Neumann subalgebra of M.
2. If $x \in M$ satisfies $Bx \subset \sum_{i=1}^{n} x_i B$ for a finite number of elements $x_1, \dots, x_n \in M$, then $x \in B$.

The following corollary gives an operator algebraic characterization of weakly mixing actions of countable discrete groups.

Corollary (5.1.22)[206]: If σ is a measure preserving action of a countable discrete group Γ_0 on a finite measure space (X, μ), then weak mixing of σ is equivalent to the following property: if $x \in L^\infty(X, \mu) \rtimes \Gamma_0$ and $L(\Gamma_0)x \subset \sum_{i=1}^{n} x_i L(\Gamma_0)$ for a finite number of elements $x_1, \dots, x_n$ in $L^\infty(X, \mu) \rtimes \Gamma_0$, then $x \in L(\Gamma_0)$.

Corollary (5.1.23)[206]: Let M be a finite von Neumann algebra with a faithful normal trace τ, and let B be a mixing von Neumann subalgebra of M. If $A \subset B$ is a diffuse von Neumann subalgebra and $x \in M$ satisfies $Ax \subset \sum_{i=1}^{n} x_i B$ for a finite number of elements $x_1, \dots, x_n \in M$, then $x \in B$.

Lemma (5.1.24)[206]: Let $p \in \langle M, e_B \rangle$ be a finite projection, $p \le 1 - e_B$, and $\epsilon > 0$. Then there exist $x_1, \dots, x_n \in M \ominus B$, and projections $f_1, \dots, f_n \in B$ such that $\mathbb{E}_B\left(x_j^* x_i\right) = \delta_{ij} f_i$, and

$$\left\| p - \sum_{i=1}^{n} x_i e_B x_i^* \right\|_{2,\mathrm{Tr}} < \epsilon.$$

Proof. Let $q = e_B + p$. Then q is a finite projection in $\langle M, e_B \rangle$. By Lemma 1.8 of [215], there are $x_0, x_1, \dots, x_n \in M, x_0 = 1$, such that $\mathbb{E}_B\left(x_j^* x_i\right) = \delta_{ij} f_i$ for $0 \le i, j \le n$ and

$$\left\| q - \sum_{i=0}^{n} x_i e_B x_i^* \right\|_{2,\mathrm{Tr}} < \epsilon.$$

Clearly,

$$\left\| p - \sum_{i=1}^{n} x_i e_B x_i^* \right\|_{2,\mathrm{Tr}} < \epsilon.$$

Suppose that $\mathcal{H} \subset L^2(M)$ is a right B-module. Let $\mathcal{L}_B(L^2(B), \mathcal{H})$ be the set of bounded right B-modular operators from $L^2(B)$ into $\mathcal{H}$. The dimension of $\mathcal{H}$ over B is defined as

$$\dim_B(\mathcal{H}) = \mathrm{Tr}(1),$$

where Tr is the unique tracial weight on B' satisfying the following condition

$$\mathrm{Tr}(x^* x) = \tau(x x^*),\ \forall x \in \mathcal{L}_B(L^2(B), \mathcal{H}).$$

We say $\mathcal{H}$ is a finite right B-module if $\mathrm{Tr}(1) < \infty$. For details on finite modules, we refer the reader to appendix A of [188].

Suppose that $\mathcal{H} \subset L^2(M)$ is a right B-module. We say that $\mathcal{H}$ is finitely generated if there exist finitely many elements $\xi_1, \dots, \xi_n \in \mathcal{H}$ such that $\mathcal{H}$ is the closure of $\sum_{i=1}^{n} \xi_i B$. A set $\{\xi_i\}_{i=1}^{n}$ is called an orthonormal basis of $\mathcal{H}$ if $\mathbb{E}_B\left(\xi_i^* \xi_j\right) = \delta_{ij} p_i \in B, p_i^2 = p_i$, and for every $\xi \in \mathcal{H}$ we have

$$\xi = \sum_{i} \xi_i E_B(\xi_i^* \xi).$$

Let p be the orthogonal projection of $L^2(M)$ onto $\mathcal{H}$. Then $p = \sum_{i=1}^{n} \xi_i e_B \xi_i$, where $\xi_i \in L^2(M)$ is viewed as an unbounded operator affilated with M. Every finitely generated right B module has an orthonormal basis. For finitely generated right B-modules, see 1.4.1 of [51].

The following lemma is proved by Vaes in [188] (see Lemma A.1).

Lemma (5.1.25)[206]: Suppose $\mathcal{H}$ is a finite right B-module. Then there exists a sequence of projections z_n of $Z(B) = B' \cap B$ such that $\lim_{n\to\infty} z_n = 1$ in the strong operator topology and, for each n, there exists a projection $p_n \in M_{k_n}(B)$ such that $\mathcal{H}z_n$ is unitarily equivalent to the $p_n M_{k_n}(B) p_n$ B-bimodule $p_n\left(L^2(B)^{(n)}\right)$. In particular, $\mathcal{H}z_n$ is a finitely-generated right B-module.

The following lemma is motivated by Lemma 1.4.1 of [51].

Lemma (5.1.26)[206]: Suppose $\mathcal{H} \subset L^2(M)$ is an A-B-bimodule, which is finitely generated as a right B-module. Let p denote the orthogonal projection of $L^2(M)$ onto $\mathcal{H}$. Then there exists a sequence of projections z_n in $A' \cap M$ such that $\lim_{n\to\infty} z_n = 1$ in the strong operator topology and for each n, there exist a finite number of elements $x_{n,1}, \dots, x_{n,k} \in M$ such that

$$z_n p z_n(\hat{x}) = \sum_{i=1}^{k} x_{n,i} \widehat{\mathbb{E}_B\left(x_{n,\iota}^* x\right)},\ \forall x \in M.$$

Proof. Let $\{\xi_i\}_{i=1}^{k} \subset \mathcal{H} \subset L^2(M,\tau)$ be an orthonormal basis for $\mathcal{H}$, i.e., $\mathcal{H} = \bigoplus_{i=1}^{k} [\xi_i B]$. As in 1.4.1 of [51], the projection p from $L^2(M)$ onto $\mathcal{H}$ has the form $p = \sum_{i=1}^{k} \xi_i e_B \xi_i^*$, where $\xi_i \in L^2(M)$ is viewed as an unbounded operator affilated with M. Since $\mathcal{H}$ is a left A-submodule of $L^2(M)$, in particular it is an invariant subspace for the von Neumann algebra A, so the projection $p: L^2(M) \to \mathcal{H}$ commutes with A. Thus, $p \in A' \cap \langle M, e_B\rangle$. For $a \in A$, we have

$$a\left(\sum_{i=1}^{n} \xi_i e_B \xi_i^*\right) = \left(\sum_{i=1}^{n} \xi_i e_B \xi_i^*\right) a$$

and, applying the pull down map to both sides, we obtain

$$a\left(\sum_{i=1}^{n} \xi_i \xi_i^*\right) = \left(\sum_{i=1}^{n} \xi_i \xi_i^*\right) a.$$

Hence $aq = qa$ for all spectral projections q of $\xi_i \xi_i^*$. Since $\sum_{i=1}^{n} \xi_i \xi_i^*$ is a densely defined operator affilated with $M, q \in A' \cap M$. We thus obtain a sequence of projections $z_n \in A' \cap M$ such that $\lim_{n\to\infty} z_n = 1$ in the strong operator topology and $\sum_{i=1}^{k} z_n \xi_i \xi_i^* z_n$ is a bounded operator for each n. Let $x_{n,i} = z_n \xi_i, 1 \le i \le k$. Then $x_{n,i} \in M$ and

$$z_n p z_n(\hat{x}) = \sum_{i=1}^{k} z_n \xi_i e_B \xi_i^* z_n(\hat{x}) = \sum_{i=1}^{k} x_{n,i} e_B x_{n,i}^*(\hat{x}) = \sum_{i=1}^{k} x_{n,i} \widehat{\mathbb{E}_B\left(x_{n,\iota}^* x\right)}$$

for all $x \in M$.

Theorem (5.1.27)[206]: Let M be a finite von Neumann algebra with a faithful normal trace τ, and let A, B be von Neumann subalgebras of M with $A \subset B$. Then the following conditions are equivalent:

1. $B \subset M$ is weakly mixing relative to A, i.e., there exists a sequence of unitary operators $\{u_k\}$ in A such that

$$\lim_{k\to\infty} \|\mathbb{E}_B(x u_k y)\|_2 = 0,\ \forall x, y \in M \ominus B.$$

2. If $z \in A' \cap \langle M, e_B \rangle$ satisfies $\mathrm{Tr}(z^*z) < \infty$, then $e_B z e_B = z$.
3. If $p \in A' \cap \langle M, e_B \rangle$ satisfies $\mathrm{Tr}(p) < \infty$, then $e_B p e_B = p$.
4. If $x \in M$ satisfies $Ax \subset \sum_{i=1}^{n} x_i B$ for a finite number of elements $x_1, \dots, x_n \in M$, then $x \in \bar{B}$

Proof. (a)⇒(b) Suppose $e_B z e_B = z$ is not true. We may assume that $(1-e_B)z \neq 0$ (otherwise, consider $z(1-e_B)\big)$. Replacing z by a nonzero spectral projection of $(1-e_B)zz^*(1-e_B)$ corresponding to an interval $[c,1]$ with $c > 0$, we may assume that $z = p \neq 0$ is a subprojection of $1 - e_B$.

Let $\epsilon > 0$. By Lemma (5.1.24), there is a natural number n and $x_1, \dots, x_n \in M \ominus B$ such that $\mathbb{E}_B\big(x_j^* x_i\big) = \delta_{ij} f_i$, where f_i is a projection in B, and

$$\left\| p - \sum_{i=1}^{n} x_i e_B x_i^* \right\|_{2,\mathrm{Tr}} < \epsilon/2.$$

Let $p_0 = \sum_{i=1}^{n} x_i e_B x_i^*$. Then p_0 is a projection. Note that $u_k p u_k^* = p$. So

$$\|u_k p_0 u_k^* - p_0\|_{2,Tr} \leq \|u_k (p_0 - p) u_k^*\|_{2,Tr} + \|p_0 - p\|_{2,Tr} < \epsilon.$$

Therefore,

$$\begin{aligned} 2\|p_0\|_{2,\mathrm{Tr}}^2 &= \|u_k p_0 u_k^* - p_0\|_{2,\mathrm{Tr}}^2 + 2\mathrm{Tr}(u_k p_0 u_k^* p_0) \\ &= \|u_k p_0 u_k^* - p_0\|_{2,\mathrm{Tr}}^2 + 2 \sum_{1 \le i,j \le n} \mathrm{Tr}\big(u_k x_i e_B x_i^* u_k^* x_j e_B x_j^*\big) \\ &\leq \epsilon^2 + 2 \sum_{1 \le i,j \le n} \tau\big(\mathbb{E}_B\big(x_i^* u_k^* x_j\big) x_j^* u_k x_i\big) \\ &\leq \epsilon^2 + 2 \sum_{1 \le i,j \le n} \big\|\mathbb{E}_B\big(x_j^* u_k x_i\big)\big\|_{2,\tau}^2. \end{aligned}$$

By the assumption of the lemma, $2\sum_{1 \le i,j \le n} \big\|\mathbb{E}_B\big(x_j^* u_k x_i\big)\big\|_{2,\tau}^2 \to 0$ when $k \to \infty$. Hence, $\|p_0\|_{2,\mathrm{Tr}} \leq \epsilon$. Since $\epsilon > 0$ was arbitrary, this says $p = 0$ This is a contradiction.

(b) ⇒ (a) Suppose (a) is false. Then there exist $\epsilon_0 > 0$ and $x_1, \dots, x_n \in N \ominus B$ such that $\sum_{1 \le i,j \le n} \big\|\mathbb{E}_B\big(x_i u x_j^*\big)\big\|_{2,\tau}^2 \geq \epsilon_0$ for all $u \in \mathcal{U}(A)$. Let $z = \sum_{i=1}^{n} x_i^* e_B x_i$. Then $z \perp e_B, \mathrm{Tr}(z) < \infty$, and

$$\begin{aligned} \mathrm{Tr}(zuzu^*) &= \sum_{i,j=1}^{n} \mathrm{Tr}\big(x_i^* e_B x_i u x_j^* e_B x_j u^*\big) = \sum_{i,j=1}^{n} \mathrm{Tr}\big(E_B\big(x_i u x_j^*\big) e_B x_j u^* x_i^*\big) \\ &= \sum_{i,j=1}^{n} \tau\big(E_B\big(x_i u x_j^*\big) x_j u^* x_i^*\big) = \sum_{i,j=1}^{n} \big\|E_B\big(x_i u x_j^*\big)\big\|_2^2 \geq \epsilon, \end{aligned}$$

for all $u \in \mathcal{U}(A)$. Let Γ_z be the weak operator closure of the convex hull of $\{uzu^* : u \in \mathcal{U}(A)\}$. Then there exists a unique element $y \in \Gamma_z$ such that $\| y \|_{2,\mathrm{Tr}} = \min\big\{\| x \|_{2,Tr} : x \in \Gamma_z\big\}$. The uniqueness implies that $uyu^* = y$ for all $u \in \mathcal{U}(A)$ and hence $y \in A' \cap \langle N, e_B \rangle$. Since $\mathrm{Tr}(zuzu^*) \geq \epsilon_0, \mathrm{Tr}(zy) \geq \epsilon_0 > 0$. So $y > 0$ and $y \perp e_B$. Note that

$$\mathrm{Tr}(y^2) \leq \| y \| \,\mathrm{Tr}(y) \leq \| y \| \,\mathrm{Tr}(z) < \infty.$$

This contradicts the assumption of (b).

(b)⇔ (c) is easy to see.

(c)⇒ (d) Suppose $Ax \subset \sum_{i=1}^{n} x_i B$. Let $\mathcal{H}$ be the closure of $\widehat{AxB}$ in $L^2(N,\tau)$. Then $\mathcal{H}$ is a left A finitely generated right B bimodule. Let p be the projection of $L^2(N,\tau)$ onto $\mathcal{H}$. Then $p \in A' \cap \langle N, e_B \rangle$ is a finite projection of $\langle N, e_B \rangle$. By the assumption of (c),$p \le e_B$. So $\hat{x} = p(\hat{x}) = e_B(\hat{x}) \in \hat{B}$ and $x \in B$.

(d)⇒(c) Suppose $p \in A' \cap \langle M, e_B \rangle$ satisfies $\mathrm{Tr}(z^* z) < \infty$. Then $\mathcal{H} = pL^2(M)$ is a left A finite right B bimodule. By Lemma (5.1.25), we may assume that $\mathcal{H}$ is a left A finitely generated right B bimodule. By Lemma (5.1.26), there exists a sequence of projections z_n in $A' \cap M$ such that $\lim_{n\to\infty} z_n = 1$ in the strong operator topology and for each n, there exist $x_{n,1}, \dots, x_{n,k} \in M$ such that

$$z_n p z_n(\hat{x}) = \sum_{i=1}^{k} x_{n,i} \mathbb{E}_B\widehat{(x_{n,\iota}^* x)}, \text{ for all } x \in M.$$

Note that $z_n p z_n \in A' \cap \langle M, e_N \rangle$, and for every $x \in M$,

$$A(z_n p z_n(\hat{x})) = (z_n p z_n)(\widehat{Ax}) \subset \sum_{i=1}^{n} \widehat{x_{n,\iota} B}.$$

By the assumption of (4), $z_n p z_n(\hat{x}) \in \hat{B} \subset L^2(B)$ for every $x \in M$. Hence, for each $\xi \in L^2(M), z_n p z_n(\xi) \in L^2(B)$. Since $\lim_{n\to\infty} z_n = 1$ in the strong operator topology, $p(\xi) = \lim_{n\to\infty} z_n p z_n(\xi) \in L^2(B)$, i.e., $p \le e_B$.

We explore the hereditary properties of mixing subalgebras of finite von Neumann algebras; that is, we show that if $B \subset M$ is a mixing inclusion, then the properties of an inclusion $B_1 \subset B$ can force certain mixing properties on the inclusion $B_1 \subset M$. In particular, Proposition (5.1.28) below allows us to construct examples of weakly mixing subalgebras which are not mixing. We also use the crossed product and amalgamated free product constructions to produce further examples of mixing inclusions.

Proposition (5.1.28)[206]: Let B be a mixing von Neumann subalgebra of M, and let B_1 be a diffuse von Neumann subalgebra of B. We have the following:

1. $B_1' \cap M = B_1' \cap B$.
2. If B_1 is singular in B, then B_1 is singular in M.
3. $\mathcal{N}_M(B_1)'' \subseteq B$, where $\mathcal{N}_M(B_1) = \{u \in \mathcal{U}(M) : uB_1u^* = B_1\}$.
4. If B_1 is weakly mixing in B, then B_1 is weakly mixing in M.
5. If B_1 is mixing in B, then B_1 is mixing in M.

Proof. (a)-(c) follow from Theorem (5.1.15).

(d) By Corollary (5.1.21), we need to show that if $x \in M$ satisfies $B_1 x \subset \sum_{i=1}^{n} x_i B_1$ for a finite number of elements $x_1, \dots, x_n \in M$, then $x \in B_1$. Note that B is mixing in M. By Corollary (5.1.23), $x \in B$. Let $b_i = \mathbb{E}_B(x_i)$ for $1 \le i \le n$. Applying $\mathbb{E}_B$ to both sides of the inclusion $B_1 x \subset \sum_{i=1}^{n} x_i B_1$ we have $B_1 x \subset \sum_{i=1}^{n} b_i B_1$. Since B_1 is weakly mixing in $B, x \in B_1$ by Corollary (5.1.21).

(e) Suppose B_1 is mixing in B and u_n is a sequence of unitary operators in B_1 with $\lim_{n\to\infty} u_n = 0$ in the weak operator topology. For $x, y \in M$, we have

$$\lim_{n\to\infty} \|\mathbb{E}_B(x u_n y) - \mathbb{E}_B(x) u_n \mathbb{E}_B(y)\|_2 = 0$$

since B is mixing in M. Applying $\mathbb{E}_{B_1}$ to $\mathbb{E}_B(x u_n y) - \mathbb{E}_B(x) u_n \mathbb{E}_B(y)$, we have

$$\lim_{n\to\infty} \left\|\mathbb{E}_{B_1}(x u_n y) - \mathbb{E}_{B_1}(\mathbb{E}_B(x) u_n \mathbb{E}_B(y))\right\|_2 = 0. \tag{1}$$

Since B_1 is mixing in B,

$$\lim_{n\to\infty} \left\|\mathbb{E}_{B_1}(\mathbb{E}_B(x) u_n \mathbb{E}_B(y)) - \mathbb{E}_{B_1}(x) u_n \mathbb{E}_{B_1}(u)\right\|_2 = 0. \tag{2}$$

Combining (1) and (2), we have

$$\lim_{n\to\infty} \|\mathbb{E}_{B_1}(xu_ny) - \mathbb{E}_{B_1}(x)u_n\mathbb{E}_{B_1}(u)\|_2 = 0,$$

which implies that B_1 is mixing in M.

Proposition (5.1.29)[206]: Let M be a type II_1 factor with the faithful normal trace τ, and let B be a proper subfactor of M. If $\{u_n\}$ is a sequence of unitary operators in B such that for all elements x, y in M with $\mathbb{E}_B(x) = \mathbb{E}_B(y) = 0$, one has

$$\lim_{n\to\infty} \|\mathbb{E}_B(xu_ny)\|_2 = 0,$$

then $\lim_{n\to\infty} u_n = 0$ in the weak operator topology.

Proof. Note that B is weakly mixing in M and hence singular in M. In particular $B' \cap M = \mathbb{C}1$. Let ω be a non principal ultrafilter of $\mathbb{N}$ and suppose $\lim_{n\to\omega} u_n = b$ in the weak operator topology. For x, y in M with $\mathbb{E}_B(x) = \mathbb{E}_B(y) = 0$

$$\mathbb{E}_B(xby) = \lim_{n\to\omega} \mathbb{E}_B(xu_ny) = 0.$$

Let $b = u|b|$ be the polar decomposition of b. Note that

$$\mathbb{E}_B(xu^*) = \mathbb{E}_B(x)u^* = 0.$$

Hence,

$$\mathbb{E}_B(x|b|y) = \mathbb{E}_B(xu^*u|b|y) = \mathbb{E}_B(xu^*by) = 0.$$

Let $x = y^*$. Then $\mathbb{E}_B(y^*|b|y) = 0$ and hence $y^*|b|y = 0$. This implies that $|b|y = 0$ for all $y \in M$ with $\mathbb{E}_B(y) = 0$. For $b' \in B, \mathbb{E}_B(b'y) = b'\mathbb{E}_B(y) = 0$. Hence, $|b|b'y = 0$. This implies that $|b|R(b'y) = 0$, where $R(b'y)$ is the range projection of $b'y$. Let $p = \vee_{b'\in B} R(b'y)$. Then $|b|p = 0$. On the other hand, $0 \neq p \in B' \cap M$, so $p = 1$. We then have $|b| = 0$, and $b = 0$. Therefore, $\lim_{n\to\omega} u_n = 0$ in the weak operator topology. Since ω is an arbitrary non principal ultrafilter of $\mathbb{N}$, $\lim_{n\to\infty} u_n = 0$ in the weak operator topology.

Lemma (5.1.30)[206]: Let B be a von Neumann subalgebra of M. Then the following conditions are equivalent:

1. B is atomic type I.
2. For every bounded sequence $\{x_n\}$ in M with $\lim_{n\to\infty} x_n = 0$ in the weak operator topology, $\lim_{n\to\infty} \|\mathbb{E}_B(x_n)\|_2 = 0$.

Proof. (a)⇒(b) Since B is a finite atomic type I von Neumann algebra, $B = \bigoplus_{k=1}^{N} M_{n_k}(\mathbb{C})$, where $1 \le N \le \infty$. So there exists a sequence of finite rank central projections $p_n \in B$ such that $p_n \to 1$ in the strong operator topology. Therefore, $\tau(p_n) \to 1$. Let $\{x_n\}$ be a bounded sequence in M with $x_n \to 0$ in the weak operator topology, and let $\epsilon > 0$. We may assume that $\|x_n\| \le 1$. Choose p_k such that $\tau(1 - p_k) < \epsilon^2/4$. Note that the map $x \in M \to p_k\mathbb{E}_B(x)$ is a finite rank operator. There is an $m > 0$ such that for all $n \ge m, \|p_k\mathbb{E}_B(x_n)\|_2 < \epsilon/2$. Then

$$\|\mathbb{E}_B(x_n)\|_2 \le \|p_k\mathbb{E}_B(x_n)\|_2 + \|(1 - p_k)\mathbb{E}_B(x_n)\|_2 \le \epsilon/2 + \epsilon/2 = \epsilon.$$

This proves that $\|\mathbb{E}_B(x_n)\|_2 \to 0$.

(b)⇒(a) If M is not atomic type I, then there is a nonzero central projection $p \in M$ such that pM is diffuse. Thus, there is a Haar unitary operator $v \in pM$. Note that $v^n \to 0$ in the weak operator topology. But $\|\mathbb{E}_B(v^n)\|_2 = \|v^n\|_2 = \tau(p)^{1/2}$ does not converge to 0. This contradicts (b).

Proposition (5.1.31)[206]: Let $M = M_1 *_A M_2$ be the amalgamated free product of diffuse finite von Neumann algebras (M_1, τ_1) and (M_2, τ_2) over an atomic f nite von Neumann algebra A. Then M_1 is a mixing von Neumann subalgebra of M.

Proof. The following spaces are mutually orthogonal with respect to the unique trace τ on M: $M_2 \ominus A, (M_1 \ominus A) \otimes (M_2 \ominus A), (M_2 \ominus A) \otimes (M_1 \ominus A),$
$(M_1 \ominus A) \otimes (M_2 \ominus A) \otimes (M_1 \ominus A), \cdots$ Furthermore, the trace-norm closure of the linear span of the above spaces is $L^2(M,\tau) \ominus L^2(M_1,\tau)$. Suppose $\{u_n\}$ is a sequence of unitary operators in M_1 satisfying $\lim_{n\to\infty} u_n = 0$ in the weak operator topology. To prove M_1 is a mixing von Neumann subalgebra of M, we need only to show for x in each of the above spaces, we have

$$\lim_{n\to\infty} \left\|\mathbb{E}_{M_1}(xu_nx^*)\right\|_2 = 0.$$

We will give the proof for x in one of the following spaces:

$$(M_1 \ominus A) \otimes (M_2 \ominus A) \text{ and } (M_2 \ominus A) \otimes (M_1 \ominus A).$$

The other cases can be proved similarly.

Suppose $x = x_1y_1$, where $x_1 \in M_1 \ominus A$ and $y_1 \in M_2 \ominus A$. Then

$$xu_nx^* = x_1y_1\big(u_n - \mathbb{E}_A(u_n)\big)y_1^*x_1 + x_1y_1\mathbb{E}_A(u_n)y_1^*x_1^*.$$

Note that $\mathbb{E}_{M_1}\big(x_1y_1\big(u_n - \mathbb{E}_A(u_n)\big)y_1^*x_1\big) = 0$ and $\lim_{n\to\infty}\|E_A(u_n)\|_2 = 0$ by Lemma (5.1.30). So

$$\lim_{n\to\infty} \left\|\mathbb{E}_{M_1}(xu_nx^*)\right\|_2 = 0.$$

Suppose $x = y_1x_1$, where $x_1 \in M_1 \ominus A$ and $y_1 \in M_2 \ominus A$. Then
$xu_nx^* = y_1x_1u_nx_1^*y_1 = y_1\big(x_1u_nx_1^* - \mathbb{E}_A(x_1u_nx_1^*)\big)y_1^* - y_1\mathbb{E}_A(x_1u_nx_1^*)y_1^*.$
Note that

$$\mathbb{E}_{M_1}\big(y_1\big(x_1u_nx_1^* - \mathbb{E}_A(x_1u_nx_1^*)\big)y_1^*\big) = 0 \text{ and } \lim_{n\to\infty}\|\mathbb{E}_A(x_1u_nx_1^*)\|_2 = 0$$

by Lemma (5.1.30). So

$$\lim_{n\to\infty} \left\|\mathbb{E}_{M_1}(xu_nx^*)\right\|_2 = 0.$$

Note, in particular, that Proposition (5.1.31) implies that if A is a diffuse mixing masa in a finite von Neumann algebra M_1, and M_2 is also diffuse, then A is mixing in the free product $M_1 * M_2$.

Now let B be a diffuse finite von Neumann algebra with a faithful normal trace τ, and let G be a countable discrete group. Let $*_{g\in G} B_g$ be the free product von Neumann algebra, where B_g is a copy of B for each g. The shift transformation $\sigma(g)\big((x_h)\big) = \big(x_{g^{-1}h}\big)$ defines an action of G on $*_{g\in G} B_g$. Let $M = *_{g\in G} B_g \rtimes G$. Then M is a type II_1 factor and we can identify B with B_e.

Proposition (5.1.32)[206]: The above algebra B is a mixing von Neumann subalgebra of M.

Proof. Suppose v_g is the classical unitary operator corresponding to the action g in M. Then for every (x_h) in $*_{g\in G} B_g$,

$$v_g(x_h)v_g^{-1} = \Big(\sigma_g(x_h)\Big) = \big(x_{g^{-1}h}\big).$$

Suppose $b_n \in B = B_e, b_n \to 0$ in the weak operator topology, $g \neq e$, and $x_h \in B_h$. We may assume $\tau(b_n) = 0$ for each n. Note that

$$x_hv_gv_nv_g^*x_h^* = x_h\sigma_g(b_n)x_h^*.$$

If $h \neq e$, it is clear that $x_h\sigma_g(b_n)x_h^*$ is free with $B = B_e$ and hence orthogonal to B. If $h = e$, direct computations show that $x_e\sigma_g(b_n)x_e^*$ is orthogonal to $B = B_e$. So we have

$$\mathbb{E}_B\big(x_hv_gb_nv_g^*x_h^*\big) = \mathbb{E}_B\big(x_h\sigma_g(b_n)x_h^*\big) = \tau\big(x_h\sigma_g(b_n)x_h^*\big) = \tau\big(\sigma_g(b_n)x_h^*x_h\big) = \tau\Big(b_n\sigma_{g^{-1}}(x_h^*x_h)\Big),$$

and this last expression above converges to zero. Note that the linear span of the above elements $x_h v_g$ is dense in $M \ominus B$ in the weak operator topology. This proves that B is mixing in M.

Section (5.2): q-Deformed Araki–Woods Von Neumann Algebras and Factoriality

In free probability, Voiculescu's C^*-free Gaussian functor associates a canonical C^*-algebra denoted by $\Gamma(\mathcal{H}_{\mathbb{R}})$ to a real Hilbert space $\mathcal{H}_{\mathbb{R}}$, the former being generated by $s(\xi), \xi \in \mathcal{H}_{\mathbb{R}}$, where each $s(\xi)$ is the sum of creation and annihilation operators on the full Fock space of the complexification of $\mathcal{H}_{\mathbb{R}}$. The associated von Neumann algebra $\Gamma(\mathcal{H}_{\mathbb{R}})''$ is isomorphic to $L(\mathbb{F}_{\dim(\mathcal{H}_{\mathbb{R}})})$ and is the central object in the study of free probability (see [233] for more on the subject). There are three interesting types of deformations of Voiculescu's free Gaussian functor each of which has a real Hilbert space $\mathcal{H}_{\mathbb{R}}$ as the initial input data: (i) the q-Gaussian functor due to Bożejko and Speicher for $-1 < q < 1$ (see [66]), (ii) a functor due to Shlyakhtenko (see [15]) which is a free probability analog of the construction of quasi free states on the CAR and CCR algebras and (iii) the third one is a combination of the first two and is due to Hiai (see [123]); the associated von Neumann algebras are respectively called BożejkoSpeicher factors (or q-Gaussian von Neumann algebras), free Araki-Woods factors and q-deformed Araki-Woods von Neumann algebras.

Frisch and Bourret in [77] had considered operators satisfying the q-canonical commutation relations:

$$l(e)l(f)^* - ql(f)^*l(e) = \langle e, f\rangle I, -1 < q < 1.$$

The existence of such operators on an 'appropriate Fock space' was proved by Bożejko and Speicher in [66] and these operators have importance in particle statistics [230], [231]. Since then many experts have studied the q-Gaussian von Neumann algebras. Structural properties of the q-Gaussian algebras have been studied in [223], [66], [60], [68], [227], [137], [113], [239], [185], [238]. A short summary of the results obtained in these studies are as follows. For $\dim(\mathcal{H}_{\mathbb{R}}) \geq 2$, the q-Gaussian von Neumann algebras $\Gamma_q(\mathcal{H}_{\mathbb{R}})$ are non-injective, solid, strongly solid, non Γ factors with w^*-completely contractive approximation property. Further, $\Gamma_q(\mathcal{H}_{\mathbb{R}}) \cong L(\mathbb{F}_{\dim(\mathcal{H}_{\mathbb{R}})})$ for values of q sufficiently close to zero [232]. The Shlyakhtenko functor in [15] associates a C^*-algebra $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)$ to a pair $(\mathcal{H}_{\mathbb{R}}, U_t)$, where $\mathcal{H}_{\mathbb{R}}$ is a real Hilbert space and (U_t) is a strongly continuous real orthogonal representation of $\mathbb{R}$ on $\mathcal{H}_{\mathbb{R}}$. The von Neumann algebras $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)''$ obtained this way i.e., the free Araki-Woods von Neumann algebras are full factors of type $\mathrm{III}_\lambda, 0 < \lambda \leq 1$, when (U_t) is non-trivial and $\dim(\mathcal{H}_{\mathbb{R}}) \geq 2$ [15]. These von Neumann algebras are type III counterparts of the free group factors. In short, they satisfy the complete metric approximation property, lack Cartan subalgebras, are strongly solid, and, they satisfy Connes' bicentralizer problem when they are type III_1 (see [179],[190],[224]). They have many more interesting properties.

The third functor mentioned above is the q-deformed functor due to Hiai for $-1 < q < 1$ (see [123]). Hiai's functor is the main topic. It is a combination of Bożejko Speicher's functor and Shlyakhtenko's functor. This functor, like the Shlyakhtenko's functor, associates a C^*-algebra $\Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)$ to a pair $(\mathcal{H}_{\mathbb{R}}, U_t)$, where $\mathcal{H}_{\mathbb{R}}$ is a real Hilbert space and (U_t) is a strongly continuous orthogonal representation of $\mathbb{R}$ on $\mathcal{H}_{\mathbb{R}}$ as before. The associated von Neumann algebras in this construction a priori depend on $q \in (-1,1)$ and are represented in standard form on 'twisted full Fock spaces' that carry the spectral data of (U_t) and connects it to the modular theory of this particular standard representation in a manner such that the canonical creation and annihilation operators satisfy the q-canonical commutation relations of Frisch and Bourret. Hiai's functor coincides with Bożejko-Speicher's functor when (U_t) is trivial and also coincides with Shlyakhtenko's functor when $q = 0$. Note that $\Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)''$ is abelian when $\dim(\mathcal{H}_{\mathbb{R}}) = 1$, so the situation becomes interesting when $\dim(\mathcal{H}_{\mathrm{R}}) \geq 2$.

Assume $\dim(\mathcal{H}_{\mathbb{R}}) \geq 2$. Unlike the free Araki-Woods factors, not much is known about the q-deformed Araki-Woods von Neumann algebras. Hiai proved amongst other things that when the almost periodic part of (U_t) is infinite dimensional, the centralizer of the q-quasi free state (vacuum state) has trivial relative commutant and thus decided factoriality of the ambient von Neumann algebra $\Gamma_q(\mathcal{H}_{\mathrm{R}}, U_t)''$ (Thm. 3.2, [123]). Thus, he was also able to decide the type of these factors under the same hypothesis imposed (Thm. 3.3, [123]). He also exhibited non-injectivity of $\Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)''$ depending on the 'thickness of the spectrum of the analytic generator of (U_t)' (Thm. 2.3, [123]). Recently, Nelson generalized the techniques of free monotone transport originally developed in [232] beyond the tracial case. Using this powerful tool he proved that $\Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)'' \cong \Gamma_0(\mathcal{H}_{\mathbb{R}}, U_t)''$ (the latter being the free Araki-Woods factors) around a small interval centred at 0, and hence decided factoriality (Thm. 4.5,4.6, [236]). Thus, even factoriality of $\Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)''$ is not known to hold in general. We investigate the factoriality of $\Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)''$. The main result is the following:

Theorem (5.2.1)[222]: For any strongly continuous orthogonal representation $t \mapsto U_t$, of $\mathbb{R}$ on a separable real Hilbert space $\mathcal{H}_{\mathbb{R}}$ with $\dim(\mathcal{H}_{\mathbb{R}}) \geq 2$ and for all $q \in (-1,1)$, the q-deformed Araki-Woods von Neumann algebras $\Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)''$ are factors, if there exists a unit vector $\xi_0 \in \mathcal{H}_{\mathbb{R}}$ such that $U_t\xi_0 = \xi_0$ for all $t \in \mathbb{R}$.

The main result in [137] which proves the factoriality of $\Gamma_q(\mathcal{H}_{\mathbb{R}})$ uses MASAs. The proof of Theorem (5.2.1) also uses MASAs but from a different point of view. Note that, if a finite von Neumann algebra contains a diffuse MASA so that the orthocomplement of the associated Jones' projection (with respect to a faithful normal tracial state) as a bimodule over the MASA is a direct sum of coarse bimodules, then the ambient von Neumann algebra must be a factor. Thus, our proof depends on singular MASAs (and this is natural as we are dealing with algebras which are similar to free group factors [172]). So, our techniques are more close to understanding the measure-multiplicity invariant of a MASA that was introduced in [228]. The MASAs that we work with lie in the centralizer of the q-quasi free state. We call these generator MASAs, as these MASAs are indeed the analogue of generator MASAs in the free group factors. The generator MASAs in the free group factors have vigorous mixing properties. So, to compare, we investigate mixing properties of generator MASAs in $\Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)''$ and show that the left-right measure of these MASAs (see [234] for Defn.) are Lebesgue absolutely continuous. The second reason of working with MASAs is forced, for this enables us to decide the factoriality of the centralizer of the q-quasi free state and thereby facilitate the computation of the S-invariant of the ambient factor.

We collect all the necessary material that is needed to address the problem. We contains an account of Hiai's construction, associated modular theory, description of the commutant and other technical details. A convenient description of the centralizer of the q-quasi free state is required. The centralizer depends entirely on the almost periodic component of (U_t) and its GNS space is described in Theorem (5.2.11). We investigate the properties of the generator abelian algebras which are indispensable ingredients in our arguments. In Theorem (5.2.13), we establish that a canonical self-adjoint generator of

$\Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)''$ generates a diffuse abelian algebra (generator MASA) having conditional expectation that preserves the vacuum state if and only if the generator lies in the centralizer of $\Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)''$ with respect to the same state.

By making a short account on how to regard a GNS space of an arbitrary von Neumann algebra equipped with a faithful normal state as a standard bimodule over a MASA, when the MASA comes from the centralizer of the associated state. We also discuss strong mixing of MASAs (lying inside the centralizer) with respect to a particular faithful normal state and also highlight on calculating left-right measures of MASAs. In Theorem (5.2.17) and Theorem (5.2.18), we show that for a generator algebra (MASA) in $\Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)''$ that possess conditional expectation preserving the vacuum state, the left-right measure is indeed Lebesgue absolutely continuous for all $q \in (-1,1)$. This justifies the term 'generator MASA: This statement is an indication that $\Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)''$ will share many properties of the free group factors even when q is away from 0 (the case when q is close to ± 1 is probably more interesting from the point of view of physics) and is a reflection of a deep theorem of Voiculescu on the subject [172]. It readily follows that if the fixed point subspace of (U_t) is at least two dimensional, then the centralizer of the vacuum state has trivial relative commutant and hence $\Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)''$ is a factor (Corollary (5.2.20)).

We establish factoriality of $\Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)''$ in Theorem (5.2.22) and Theorem (5.2.23), when $\dim(\mathcal{H}_{\mathbb{R}}) \geq 2$ and $q \in (-1,1)$, in the case when (U_t) is not ergodic or has a non-trivial weakly mixing component. We extend the statement of Corollary (5.2.20) in Theorem (5.2.24) to show that the centralizer of the vacuum state has trivial relative commutant when $\dim(\mathcal{H}_{\mathbb{R}}) \geq 2$, the fixed point subspace of (U_t) is at least one dimensional and the dimension of the almost periodic part of (U_t) is at least two dimensional. Finally, we characterize the type of the factors obtained via Hiai's construction in Theorem (5.2.25) and Theorem (5.2.26) under the assumption that (U_t) is almost periodic with a non-trivial fixed point or has a weakly mixing component. The results are analogous to the ones found in Thm. 3.3 [123].

We collect some well known facts about the q-deformed Araki-Woods von Neumann algebras constructed by Hiai in [123] that will be indispensable for our purpose. For detailed exposition, see [123]. As a convention, all Hilbert spaces are separable, all von Neumann algebras have separable preduals and inner products are linear in the second variable.

Let $\mathcal{H}_{\mathbb{R}}$ be a real Hilbert space and let $t \mapsto U_t, t \in \mathbb{R}$, be a strongly continuous orthogonal representation of $\mathbb{R}$ on $\mathcal{H}_{\mathbb{R}}$. Let $\mathcal{H}_{\mathbb{C}} = \mathcal{H}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ denote the complexification of $\mathcal{H}_{\mathbb{R}}$. Denote the inner product and norm on $\mathcal{H}_{\mathbb{C}}$ by $\langle \cdot,\cdot \rangle_{\mathcal{H}_{\mathbb{C}}}$ and $\|\cdot\|_{\mathcal{H}_{\mathbb{C}}}$ respectively.

Identify $\mathcal{H}_{\mathbb{R}}$ in $\mathcal{H}_{\mathbb{C}}$ by $\mathcal{H}_{\mathbb{R}} \otimes 1$. Thus, $\mathcal{H}_{\mathbb{C}} = \mathcal{H}_{\mathbb{R}} + i\mathcal{H}_{\mathbb{R}}$, and as a real Hilbert space the inner product of $\mathcal{H}_{\mathbb{R}}$ in $\mathcal{H}_{\mathbb{C}}$ is given by $\Re\langle \cdot,\cdot \rangle_{\mathcal{H}_{\mathbb{C}}}$. Consider the bounded anti-linear operator $\mathcal{J}: \mathcal{H}_{\mathbb{C}} \to \mathcal{H}_{\mathbb{C}}$ given by $\mathcal{J}(\xi + i\eta) = \xi - i\eta, \xi, \eta \in \mathcal{H}_{\mathbb{R}}$, and note that $\mathcal{J}\xi = \xi$ for $\xi \in \mathcal{H}_{\mathbb{R}}$. Moreover,

$$\langle \xi, \eta \rangle_{\mathcal{H}_{\mathbb{C}}} = \overline{\langle \eta, \xi \rangle}_{\mathcal{H}_{\mathbb{C}}} = \langle \eta, \mathcal{J}\xi \rangle_{\mathcal{H}_{\mathbb{C}}}, \text{ for all } \xi \in \mathcal{H}_{\mathbb{C}}, \eta \in \mathcal{H}_{\mathbb{R}}.$$

Linearly extend the flow $t \mapsto U_t$ from $\mathcal{H}_{\mathbb{R}}$ to a strongly continuous one parameter group of unitaries in $\mathcal{H}_{\mathbb{C}}$ and denote the extensions by U_t for each t with abuse of notation. Let A denote the analytic generator and H the associated Hamiltonian of the extension. Then A is positive, nonsingular and self-adjoint, while H is self-adjoint. Since $\mathcal{H}_{\mathbb{R}}$ reduces U_t for all $t \in \mathbb{R}$, so $\mathcal{H}_{\mathbb{R}}$ reduces iH as well. Denoting $\mathfrak{D}(\cdot)$ to be the domain of an (unbounded) operator, one notes that $\mathfrak{D}(H)=\mathfrak{D}(iH)$ and H maps $\mathfrak{D}(H)\cap\mathcal{H}_{\mathbb{R}}$ into $i\mathcal{H}_{\mathbb{R}}$. It follows that $\mathcal{J}H=-H\mathcal{J}$ and $\mathcal{J}A=A^{-1}\mathcal{J}$.

Introduce a new inner product on $\mathcal{H}_{\mathbb{C}}$ by $\langle\xi,\eta\rangle_U=\left\langle\frac{2}{1+A^{-1}}\xi,\eta\right\rangle_{\mathcal{H}_{\mathbb{C}}}$, $\xi,\eta\in\mathcal{H}_{\mathbb{C}}$, and let $\|\cdot\|_U$ denote the associated norm on $\mathcal{H}_{\mathbb{C}}$. Let $\mathcal{H}$ denote the complex Hilbert space obtained by completing $(\mathcal{H}_{\mathbb{C}},\|\cdot\|_U)$. The inner product and norm of $\mathcal{H}$ will respectively be denoted by $\langle\cdot,\cdot\rangle_U$ and $\|\cdot\|_U$ as well. Then, $(\mathcal{H}_{\mathbb{R}},\|\cdot\|_{\mathcal{H}c})\ni\xi\mapsto\xi\in(\mathcal{H}_{\mathbb{C}},\|\cdot\|_U)\subseteq(\mathcal{H},\|\cdot\|_U)$, is an isometric embedding of the real Hilbert space $\mathcal{H}_{\mathbb{R}}$ in $\mathcal{H}$ (in the sense of [15]). With abuse of notation, we will identify $\mathcal{H}_{\mathbb{R}}$ with its image $\iota(\mathcal{H}_{\mathbb{R}})$. Then, $\mathcal{H}_{\mathbb{R}}\cap i\mathcal{H}_{\mathbb{R}}=\{0\}$ and $\mathcal{H}_{\mathbb{R}}+i\mathcal{H}_{\mathbb{R}}$ is dense in $\mathcal{H}$ (see pp. 332 [15]).

It is now appropriate to record a subtle point which will be crucial in our attempt to describe the centralizers of the q-deformed Araki-Woods von Neumann algebras. As A is affiliated to $vN(U_t: t\in\mathbb{R})$, so note that

$$\langle U_t\xi,U_t\eta\rangle_U=\langle\xi,\eta\rangle_U,\ \text{for } \xi,\eta\in\mathcal{H}_{\mathbb{C}}. \tag{3}$$

Consequently, (U_t) extends to a strongly continuous unitary representation $(\widetilde{U}_t)$ of $\mathbb{R}$ on $\mathcal{H}$. Let $\tilde{A}$ be the analytic generator associated to $(\widetilde{U}_t)$, which is obviously an extension of A. From the definition of $\langle\cdot,\cdot\rangle_U$ on $\mathcal{H}_{\mathbb{C}}$, it follows that if μ is the spectral measure of A, then $\nu=f\mu$ is the spectral measure of $\tilde{A}$, where $f(x)=\frac{2x}{1+x}$ for $x\in\mathbb{R}_{\geq 0}$, and by the spectral theorem (direct integral form), the multiplicity functions in the associated direct integrals remain the same. Note that $L^2(F,\mu_{|F})\subseteq L^2(F,\nu_{|F})$ for all Borel subsets F of $(0,\infty)$. But, as f is increasing, it follows that $L^2(F,\mu_{|F})=L^2(F,\nu_{|F})$ (as a vector space) when $F\subseteq[\lambda,\infty)$ is measurable for all $\lambda>0$. Moreover, $0<\lambda$ is an atom of μ if and only if it is an atom of ν. Thus, if E_A and $E_{\tilde{A}}$ denote the associated projection-valued spectral measures, then $E_A([\lambda,\infty))(\mathcal{H}_{\mathbb{C}})=E_{\tilde{A}}([\lambda,\infty))(\mathcal{H})$ and $E_A(\lambda)(\mathcal{H}_{\mathbb{C}})=E_{\tilde{A}}(\lambda)(\mathcal{H})$ for all $\lambda>0$. We record the following in the form of a proposition.

Proposition (5.2.2)[222]: Any eigenvector of $\tilde{A}$ is an eigenvector of A corresponding to the same eigenvalue.

Since the spectral data of A and $\tilde{A}$ (and hence of (U_t) and $(\widetilde{U}_t)$) are essentially the same, and $\widetilde{U}_t,\tilde{A}$ are respectively extensions of U_t, A for all $t\in\mathbb{R}$, so we would now write $\tilde{A}=A$ and $\widetilde{U}_t=U_t$ for all $t\in\mathbb{R}$.

Given a complex Hilbert space and $-1<q<1$, the notion of q-Fock space $\mathcal{F}_q(\cdot)$ was introduced in [66]. The q-Fock space $\mathcal{F}_q(\mathcal{H})$ of $\mathcal{H}$ is constructed as follows. Let Ω be a distinguished unit vector in $\mathbb{C}$ usually referred to as the vacuum vector. Denote $\mathcal{H}^{\otimes 0}=\mathbb{C}\Omega$, and, for $n\geq 1$, let $\mathcal{H}^{\otimes n}=\text{span}_{\mathbb{C}}\{\xi_1\otimes\cdots\otimes\xi_n:\xi_i\in\mathcal{H}\text{ for }1\leq i\leq n\}$ denote the algebraic tensor products. Let $\mathcal{F}_{fin}(\mathcal{H})=\text{span}_{\mathbb{C}}\{\mathcal{H}^{\otimes n}: n\geq 0\}$. For $n,m\geq 0$ and $f=\xi_1\otimes\cdots\otimes\xi_n\in\mathcal{H}^{\otimes n}$, $g=\zeta_1\otimes\cdots\otimes\zeta_m\in\mathcal{H}^{\otimes m}$, the association

$$\langle f,g\rangle_q=\delta_{m,n}\sum_{\pi\in S_n}q^{i(\pi)}\langle\xi_1,\zeta_{\pi(1)}\rangle_U\cdots\langle\xi_n,\zeta_{\pi(n)}\rangle_U, \tag{4}$$

where $i(\pi)$ denotes the number of inversions of the permutation $\pi\in S_n$, defines a positive definite sesquilinear form on $\mathcal{F}_{\text{fin}}(\mathcal{H})$ and the q-Fock space $\mathcal{F}_q(\mathcal{H})$ is the completion of $\mathcal{F}_{\text{fin}}(\mathcal{H})$ with respect to the norm $\|\cdot\|_q$ induced by $\langle\cdot,\cdot\rangle_q$.

For $n \in \mathbb{N}$, let $\mathcal{H}^{\otimes_q n} = \overline{\mathcal{H}^{\otimes n}\ \|\cdot\|_q}$. For our purposes, it is important to note that $\langle\cdot,\cdot\rangle_q$ and $\langle\cdot,\cdot\rangle_0$ are equivalent on $\mathcal{H}^{\otimes n}$ and $\langle\cdot,\cdot\rangle_0$ is the inner product of the standard tensor product. Thus, rephrasing and combining two lemmas of [66] one has the following.

Lemma (5.2.3)[222]: The map id: $(\mathcal{H}^{\otimes n}, \|\cdot\|_q) \to (\mathcal{H}^{\otimes n}, \|\cdot\|_0)$, given by $id(\xi_1 \otimes \cdots \otimes \xi_n) = (\xi_1 \otimes \cdots \otimes \xi_n)$, where $\xi_i \in \mathcal{H}, 1 \le i \le n$, extends uniquely to a bounded and invertible linear map $T: (\mathcal{H}^{\otimes_q n}, \|\cdot\|_q) \to (\mathcal{H}^{\otimes n}, \|\cdot\|_0)$ for $-1 < q < 1$.

Proof. Following [66], every $\pi \in S_n$ induces an unitary operator on $\mathcal{H}^{\otimes_0 n}$ given by $U_\pi(\xi_1 \otimes \cdots \otimes \xi_n) = \xi_{\pi(1)} \otimes \cdots \otimes \xi_{\pi(n)}, \xi_i \in \mathcal{H}, 1 \le i \le n$. Let $P_q = \sum_{\pi \in S_n} q^{i(\pi)} U_\pi$. Then $P_q \in \mathbf{B}(\mathcal{H}^{\otimes n})$ and by Lemma 3 and Lemma 4 of [66], P_q is strictly positive for $-1 < q < 1$ and $\langle f, g\rangle_q = \langle f, P_q g\rangle_0$ for all $f, g \in \mathcal{H}^{\otimes n}$. Consequently, P_q is injective and hence invertible. It follows that

$$\frac{1}{\left\|P_q^{-\frac{1}{2}}\right\|} \| f \|_0 \le \| f \|_q \le \|P_q\|^{\frac{1}{2}} \| f \|_0 \text{ , for } f \in \mathcal{H}^{\otimes n}. \quad (5)$$

The rest is obvious.

The following norm inequalities will be crucial (cf. [60],[66], and [137]):

1. If $\xi \in \mathcal{H}$ and $\| \xi \|_U = 1$, then

$$\|\xi^{\otimes n}\|_q^2 = [n]_q!, \quad (6)$$

where $[n]_q := 1 + q + \cdots + q^{(n-1)}, [n]_q! := \prod_{j=1}^n [j]_q$, for $n \ge 1$, and $[0]_q := 0, [0]_q! := 1$ by convention.

1. If $\xi_1, \cdots, \xi_n, \xi \in \mathcal{H}$ with $\|\xi_j\|_U = \| \xi \|_U = 1$ for all $1 \le j \le n$, then the following estimate holds:

$$\|\xi_1 \otimes \cdots \otimes \xi_n \otimes \xi^{\otimes m}\|_q \le C_q^{\frac{n}{2}} \sqrt{[m]_q!}, m \ge 0, \quad (7)$$

where $C_q = \prod_{i=1}^\infty \frac{1}{(1-|q|^i)}$.

For $\xi \in \mathcal{H}$, the left q-creation and q-annihilation operators on $\mathcal{F}_q(\mathcal{H})$ are respectively defined by:

$c_q(\xi)\Omega = \xi$

$c_q(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n,$

and,

$c_q(\xi)^*\Omega = 0,$

$$c_q(\xi)^*(\xi_1 \otimes \cdots \otimes \xi_n) = \sum_{i=1}^n q^{i-1} \langle \xi, \xi_i\rangle_U \xi_1 \otimes \cdots \otimes \xi_{i-1} \otimes \xi_{i+1} \otimes \cdots \otimes \xi_n, \quad (8)$$

where $\xi_1 \otimes \cdots \otimes \xi_n \in \mathcal{H}^{\otimes_q n}$ for $n \ge 1$. The operators $c_q(\xi)$ and $c_q(\xi)^*$ are bounded on $\mathcal{F}_q(\mathcal{H})$ and they are adjoints of each other. Moreover, they satisfy the following q-commutation relations:

$$c_q(\xi)^* c_q(\zeta) - q c_q(\zeta) c_q(\xi)^* = \langle \xi, \zeta\rangle_U 1, \text{ for all } \xi, \zeta \in \mathcal{H}.$$

The following observation will be crucial for our purpose.

Lemma (5.2.4)[222]: Let $\xi, \xi_i, \eta_j \in \mathcal{H}$, for $1 \le i \le n, 1 \le j \le m$. Then

$$c_q(\xi)^*\big((\xi_1 \otimes \cdots \otimes \xi_n) \otimes (\eta_1 \otimes \cdots \otimes \eta_m)\big)$$

$$= \left(c_q(\xi)^*(\xi_1 \otimes \cdots \otimes \xi_n)\right) \otimes (\eta_1 \otimes \cdots \otimes \eta_m)$$
$$+ q^n(\xi_1 \otimes \cdots \otimes \xi_n) \otimes \left(c_q(\xi)^*(\eta_1 \otimes \cdots \otimes \eta_m)\right).$$

Proof. The proof follows easily from Eq. (5).

Following [15] and [123], consider the C^*-algebra $\Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t) := C^*\{s_q(\xi): \xi \in \mathcal{H}_{\mathbb{R}}\}$ and the von Neumann algebra $\Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)''$, where

$$s_q(\xi) = c_q(\xi) + c_q(\xi)^*, \xi \in \mathcal{H}_{\mathbb{R}}.$$

$\Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)''$ is known as the q-deformed Araki-Woods von Neumann algebra (see [123]).

The vacuum state $\varphi_{q,U} := \langle \Omega, \cdot\, \Omega \rangle_q$ (also called the q-quasi free state), is a faithful normal state of $\Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)''$ and $\mathcal{F}_q(\mathcal{H})$ is the GNS Hilbert space of $\Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)''$ associated to $\varphi_{q,U}$. Thus, $\Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)''$ acting on $\mathcal{F}_q(\mathcal{H})$ is in standard form [233].

We will use the symbols $\langle \cdot,\cdot \rangle_q$ and $\|\cdot\|_q$ respectively to denote the inner product and two-norm of elements of the GNS Hilbert space.

Most of what is taken from [15],[123]. We need to have a convenient description of the commutant and centralizer of $\Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)''$ (which has been recorded in the case $q = 0$ in [15] and a similar collection of operators in the commutant has been identified in [123]). Thus, we need to record some facts related to the modular theory of the q-quasi free state $\varphi_{q,U}$. Let $J_{\varphi_{q,U}}$ and $\Delta_{\varphi_{q,U}}$ respectively denote the modular conjugation and modular operator associated to $\varphi_{q,U}$ and let $S_{\varphi_{q,U}} = J_{\varphi_{q,U}} \Delta_{\varphi_{q,U}}^{\frac{1}{2}}$ Then, for $n \in \mathbb{N}$,

$$\begin{aligned} &J_{\varphi_{q,U}}(\xi_1 \otimes \cdots \otimes \xi_n) = A^{-\frac{1}{2}}\xi_n \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_1, \forall \xi_i \in \mathcal{H}_{\mathbb{R}} \cap \mathfrak{D}\left(A^{-\frac{1}{2}}\right); \\ &\Delta_{\varphi_{q,U}}(\xi_1 \otimes \cdots \otimes \xi_n) = A^{-1}\xi_1 \otimes \cdots \otimes A^{-1}\xi_n, \forall \xi_i \in \mathcal{H}_{\mathbb{R}} \cap \mathfrak{D}(A^{-1}); \\ &S_{\varphi_{q,U}}(\xi_1 \otimes \cdots \otimes \xi_n) = \xi_n \otimes \cdots \otimes \xi_1, \forall \xi_i \in \mathcal{H}_{\mathbb{R}}. \end{aligned} \tag{9}$$

The modular automorphism group $\left(\sigma_t^{\varphi_{q,U}}\right)$ of $\varphi_{q,U}$ is given by $\sigma_{-t}^{\varphi_{q,U}} = \mathrm{Ad}\left(\mathcal{F}(U_t)\right)$, where $\mathcal{F}(U_t) = id \oplus \bigoplus_{n\geq 1} U_t^{\otimes_q n}$, for all $t \in \mathbb{R}$. In particular,

$$\sigma_{-t}^{\varphi_{q,U}}\left(s_q(\xi)\right) = s_q(U_t\xi), \text{ for all } \xi \in \mathcal{H}_{\mathbb{R}}. \tag{10}$$

Now we proceed to describe the commutant of $\Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)''$. Consider the set

$$\mathcal{H}_{\mathbb{R}}' = \left\{\xi \in \mathcal{H}: \langle \xi, \eta \rangle_U \in \mathbb{R} \text{ for all } \eta \in \mathcal{H}_{\mathbb{R}}\right\}.$$

Note that $\overline{\mathcal{H}_{\mathbb{R}}' + \iota\mathcal{H}_{\mathbb{R}}'} = \mathcal{H}$ and $\mathcal{H}_{\mathbb{R}}' \cap i\mathcal{H}_{\mathbb{R}}' = \{0\}$. Let $\zeta \in \mathfrak{D}\left(A^{-1/2}\right) \cap \mathcal{H}_{\mathbb{R}}$. Note that for all $\eta \in \mathcal{H}_{\mathbb{R}}$, one has

$$\begin{aligned} \left\langle A^{-\frac{1}{2}}\zeta, \eta \right\rangle_U &= \left\langle \frac{2A^{-\frac{1}{2}}}{1 + A^{-1}}\zeta, \eta \right\rangle_{\mathcal{H}_{\mathrm{C}}} = \left\langle \eta, \mathcal{J}\frac{2A^{-\frac{1}{2}}}{1 + A^{-1}}\zeta \right\rangle_{\mathcal{H}_{\mathbb{C}}} \\ &= \left\langle \eta, \frac{2A^{\frac{1}{2}}}{1 + A}\zeta_{\mathcal{H}_{\mathrm{C}}} = \left\langle \frac{2}{1 + A^{-1}}\eta, A^{-\frac{1}{2}}\zeta \right\rangle_{\mathcal{H}_{\mathbb{C}}} \right. \\ &= \left\langle \eta, A^{-\frac{1}{2}}\zeta \right\rangle_U. \end{aligned} \tag{11}$$

From Eq. (11), it follows that

$$A^{-1/2}\zeta \in \mathcal{H}'_{\mathbb{R}} \text{ for all } \zeta \in \mathfrak{D}\left(A^{-\frac{1}{2}}\right) \cap \mathcal{H}_{\mathbb{R}}. \tag{12}$$

Also note that for $\eta, \xi \in \mathfrak{D}(A^{-1}) \cap \mathcal{H}_{\mathbb{R}}$, one has

$$\begin{aligned} \langle \eta, \xi \rangle_U &= \left\langle \frac{2}{1+A^{-1}}\eta, \xi \right\rangle_{\mathcal{H}_{\mathbb{C}}} = \left\langle \xi, \mathcal{J}\frac{2}{1+A^{-1}}\eta \right\rangle_{\mathcal{H}_{\mathbb{C}}} \\ &= \left\langle \xi, \frac{2}{1+A}\eta \right\rangle_{\mathcal{H}_{\mathbb{C}}} = \left\langle \frac{2}{1+A^{-1}}\xi, A^{-1}\eta \right\rangle_{\mathcal{H}_{\mathbb{C}}} \\ &= \langle \xi, A^{-1}\eta \rangle_U = \left\langle A^{-\frac{1}{2}}\xi, A^{-\frac{1}{2}}\eta \right\rangle_U \left(\text{as } \mathfrak{D}(A^{-1}) \subseteq \mathfrak{D}\left(A^{-\frac{1}{2}}\right)\right). \end{aligned} \tag{13}$$

Now for $\xi \in \mathcal{H}$, define the right creation operator $r_q(\xi)$ on $\mathcal{F}_q(\mathcal{H})$ by

$$\begin{aligned} &r_q(\xi)\Omega = \xi, \\ &r_q(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = \xi_1 \otimes \cdots \otimes \xi_n \otimes \xi, \xi_i \in \mathcal{H}, n \geq 1. \end{aligned} \tag{14}$$

Clearly, $r_q(\xi) = \jmath c_q(\xi)\jmath^*$, where $\jmath: \mathcal{F}_q(\mathcal{H}) \to \mathcal{F}_q(\mathcal{H})$ is the unitary defined by

$$\jmath(\xi_1 \otimes \cdots \otimes \xi_n) = \xi_n \otimes \cdots \otimes \xi_1, \text{ where } \xi_i \in \mathcal{H} \text{ for all } 1 \leq i \leq n, n \geq 1, \tag{12}$$
$$\jmath(\Omega) = \Omega.$$

Therefore, $r_q(\xi)$ is a bounded operator on $\mathcal{F}_q(\mathcal{H})$ and its adjoint $r_q(\xi)^*$ is given by

$$r_q(\xi)^*\Omega = 0, \tag{15}$$

$$r_q(\xi)^*(\xi_1 \otimes \cdots \otimes \xi_n) = \sum_{i=1}^{n} q^{n-i}\langle \xi, \xi_i \rangle_U \xi_1 \otimes \cdots \otimes \xi_{i-1} \otimes \xi_{i+1} \otimes \cdots \otimes \xi_n, \xi_i \in \mathcal{H}, n \geq 1.$$

Write $d_q(\xi) = r_q(\xi) + r_q(\xi)^*, \xi \in \mathcal{H}$. It is easy to observe that $\{d_q(\xi): \xi \in \mathcal{H}'_{\mathbb{R}}\}'' \subseteq \Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)'$. The following result establishes that the reverse inclusion is also true and its proof is similar to the one obtained in ([15]).

Theorem (5.2.5)[222]: Suppose $\xi \in \mathfrak{D}(A^{-1}) \cap \mathcal{H}_{\mathbb{R}}$. Then $J_{\varphi_{q,U}} s_q(\xi) J_{\varphi_{q,U}} = d_q\left(A^{-\frac{1}{2}}\xi\right)$. Moreover, $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)' = \{d_q(\xi): \xi \in \mathcal{H}'_{\mathbb{R}}\}''$.

Proof. Fix $n \geq 1$ and let $\eta_1, \eta_2, \cdots, \eta_n \in \mathfrak{D}(A^{-1}) \cap \mathcal{H}_{\mathbb{R}}$. Then from Eq. (8), we have

$$\begin{aligned} &J_{\varphi_{q,U}} s_q(\xi)(\eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_n) \\ &= J_{\varphi_{q,U}}\left(\sum_{i=1}^{n} q^{(i-1)}\langle \xi, \eta_i \rangle_U \eta_1 \otimes \cdots \otimes \eta_{i-1} \otimes \eta_{i+1} \cdots \otimes \eta_n\right) \\ &\quad + J_{\varphi_{q,U}}(\xi \otimes \eta_1 \otimes \cdots \otimes \eta_n) \\ &= \sum_{i=1}^{n} q^{(i-1)}\langle \eta_i, \xi \rangle_U A^{-\frac{1}{2}}\eta_n \otimes \cdots \otimes A^{-\frac{1}{2}}\eta_{i+1} \otimes A^{-\frac{1}{2}}\eta_{i-1} \otimes \cdots \otimes A^{-\frac{1}{2}}\eta_1 \\ &\quad + A^{-\frac{1}{2}}\eta_n \otimes \cdots \otimes A^{-\frac{1}{2}}\eta_1 \otimes A^{-\frac{1}{2}}\xi \left(\text{since } \mathfrak{D}(A^{-1}) \subseteq \mathfrak{D}\left(A^{-\frac{1}{2}}\right)\right) \\ &= \sum_{i=1}^{n} q^{(i-1)}\langle \xi, A^{-1}\eta_i \rangle_U A^{-\frac{1}{2}}\eta_n \otimes \cdots \otimes A^{-\frac{1}{2}}\eta_{i+1} \otimes A^{-\frac{1}{2}}\eta_{i-1} \otimes \cdots \otimes A^{-\frac{1}{2}}\eta_1 \end{aligned}$$

$$+ A^{-\frac{1}{2}}\eta_n \otimes \cdots \otimes A^{-\frac{1}{2}}\eta_1 \otimes A^{-\frac{1}{2}}\xi \text{ (by Eq. (12))}$$
$$= \sum_{i=1}^{n} q^{(i-1)} \left\langle A^{-\frac{1}{2}}\xi, A^{-\frac{1}{2}}\eta_i \right\rangle_U A^{-\frac{1}{2}}\eta_n \otimes \cdots \otimes A^{-\frac{1}{2}}\eta_{i+1} \otimes A^{-\frac{1}{2}}\eta_{i-1} \otimes \cdots \otimes A^{-\frac{1}{2}}\eta_1$$
$$+ A^{-\frac{1}{2}}\eta_n \otimes \cdots \otimes A^{-\frac{1}{2}}\eta_1 \otimes A^{-\frac{1}{2}}\xi \left(\text{since } \mathfrak{D}(A^{-1}) \subseteq \mathfrak{D}\left(A^{-\frac{1}{2}}\right)\right)$$
$$= d_q\left(A^{-\frac{1}{2}}\xi\right) J_{\varphi_{q,U}}(\eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_n) \qquad \text{(from Eq. (13) and Eq. (15)).}$$
It follows that $J_{\varphi_{q,U}} s_q(\xi) J_{\varphi_{q,U}} = d_q\left(A^{-\frac{1}{2}}\xi\right)$.

Since $\Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)''$ is in standard form in $\mathcal{F}_q(\mathcal{H})$, so from the fundamental theorem of Tomita-Takesaki theory $\Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)' = J_{\varphi_{q,U}} \Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)'' J_{\varphi_{q,U}}$. Again from Eq. (9), one has $A^{-\frac{1}{2}}\xi \in \mathcal{H}'_{\mathbb{R}}$ for all $\xi \in \mathfrak{D}\left(A^{-\frac{1}{2}}\right) \cap \mathcal{H}_{\mathbb{R}}$.

By what we have proved so far, it follows that $\left\{J_{\varphi_{q,U}} s_q(\xi) J_{\varphi_{q,U}} : \xi \in \mathfrak{D}(A^{-1}) \cap \mathcal{H}_{\mathbb{R}}\right\} \subseteq \left\{d_q(\xi) : \xi \in \mathcal{H}'_{\mathrm{R}}\right\}''$. Note that from Eq. (7) it follows that, if $\mathcal{H}_{\mathbb{R}} \ni \xi_n \to \xi \in \mathcal{H}_{\mathbb{R}}$ in $\|\cdot\|_{\mathcal{H}_C}$ (equivalently in $\|\cdot\|_U$), then $s_q(\xi_n) \to s_q(\xi)$ in $\|\cdot\|$ $\left(\text{as} \|s_q(\zeta)\| = \frac{2}{\sqrt{1-q}} \| \zeta \|_U \text{ for all } \zeta \in \mathcal{H}_{\mathrm{R}}\right)$. Consequently, $\mathfrak{D}(A^{-1}) \cap \mathcal{H}_{\mathbb{R}}$ being dense in $\mathcal{H}_{\mathbb{R}}$, it follows that $\Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)' \subseteq \left\{d_q(\xi) : \xi \in \mathcal{H}'_{\mathbb{R}}\right\}''$. Since the reverse inclusion is straightforward to check, the proof is complete.

We are interested in the factoriality of $\Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)''$ and the orthogonal representation remains arbitrary but fixed. Thus, to reduce notation, we will write $M_q = \Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)''$ and $\varphi = \varphi_{q,U}$. We will also denote $J_{\varphi_{q,U}}$ by J and $\Delta_{\varphi_{q,U}}$ by Δ. As Ω is separating for both M_q and M'_q, for $\zeta \in M_q\Omega$ and $\eta \in M'_q\Omega$ there exist unique $x_\zeta \in M_q$ and $x'_\eta \in M'_q$ such that $\zeta = x_\zeta\Omega$ and $\eta = x'_\eta\Omega$. In this case, we will write
$$s_q(\zeta) = x_\zeta \text{ and } d_q(\eta) = x'_\eta. \tag{16}$$
Thus, for example, as $\xi \in M_q\Omega$ for every $\xi \in \mathcal{H}_{\mathbb{R}}$, so $s_q(\xi + i\eta) = s_q(\xi) + i s_q(\eta)$ for all $\xi, \eta \in \mathcal{H}_{\mathbb{R}}$.

Note that $c_q(\xi)$ and $r_q(\xi)$ are bounded operators for all $\xi \in \mathcal{H}$. Write
$$\tilde{s}_q(\xi) = c_q(\xi) + c_q(\xi)^* \text{ and } \tilde{d}_q(\xi) = r_q(\xi) + r_q(\xi)^*, \xi \in \mathcal{H}.$$
Note that if $\xi \in \mathcal{H}_{\mathbb{R}}$, then $\tilde{s}_q(\xi) = s_q(\xi)$, and if $\xi \in \mathcal{H}'_{\mathbb{R}}$ then $\tilde{d}_q(\xi) = d_q(\xi)$. If $\xi = \xi_1 + i\xi_2$ for $\xi_1, \xi_2 \in \mathcal{H}_{\mathbb{R}}$ and $\xi_2 \neq 0$, then note that $\tilde{s}_q(\xi) \neq s_q(\xi)$.

Write $\mathcal{Z}(M_q) = M_q \cap M'_q$. Let $M_q^{\varphi} = \{x \in M_q : \sigma_t^{\varphi}(x) = x \text{ for all } t \in \mathbb{R}\}$ denote the centralizer of M_q associated to the state φ. For $\xi \in \mathcal{H}_{\mathbb{R}}$, denote $M_\xi = vN(s_q(\xi))$. Note that M_ξ is abelian as $s_q(\xi)$ is self-adjoint. To understand the Hilbert space $\mathcal{F}_q(\mathcal{H})$ as a bimodule over M_ξ, it will be convenient for us to work with appropriate choice of orthonormal basis of $\mathcal{H}_{\mathbb{R}}$ with respect to $\langle\cdot,\cdot\rangle_{\mathcal{H}_C}$. Recall that $x \in M_q$ is analytic with respect to (σ_t^{φ}) if and only if the function $\mathbb{R} \ni t \mapsto \sigma_t^{\varphi}(x) \in M_q$ extends to a weakly entire function. We say that a vector $\xi \in \mathcal{H}_{\mathbb{R}}$ is analytic, if $s_q(\xi)$ is analytic for (σ_t^{φ}).

Proposition (5.2.6)[222]: $\mathcal{H}_{\mathbb{R}}$ has an orthonormal basis with respect to $\langle\cdot,\cdot\rangle_{\mathcal{H}_C}$ comprising of analytic vectors. Further, if $\xi_0 \in \mathcal{H}_{\mathbb{R}}$ be a unit vector such that $U_t\xi_0 = \xi_0$ for all $t \in \mathbb{R}$, then such an orthonormal basis of $\mathcal{H}_{\mathbb{R}}$ can be chosen so that it includes ξ_0.

Proof. Note that $U_t = A^{it}$ for all $t \in \mathbb{R}$. For $\zeta \in \mathcal{H}_\mathbb{R}$ and $r > 0$, let $\zeta_r = \sqrt{\frac{r}{\pi}} \int_\mathbb{R} e^{-rt^2} U_t \zeta dt$. It is well known that $\zeta_r \to \zeta$ in $\|\cdot\|_{\mathcal{H}_C}$ (equivalently in $\|\cdot\|_U$ as the vectors involved are real) as $r \to 0$. As (U_t) reduces $\mathcal{H}_\mathrm{R}$ and $\zeta \in M_q\Omega$, so $\zeta_r \in M_q\Omega$ for all $r > 0$. Fix $r > 0$. Consider $s_q(\zeta_r) \in M_q$ (as defined in Eq. (16)). Then, by Eq. (9) it follows that

$$\sigma_s^\varphi\left(s_q(\zeta_r)\right) = s_q\left(\sqrt{\frac{r}{\pi}} \int_\mathbb{R} e^{-r(t+s)^2} U_t \zeta dt\right), s \in \mathbb{R}. \tag{17}$$

Note that $f_{\zeta_r}(z) = \sqrt{\frac{T}{\pi}} \int_\mathbb{R} e^{-r(t+z)^2} U_t \zeta dt \in \mathcal{H}_\mathbb{C}$ for all $z \in \mathbb{C}$. Thus, $s_q\left(f_{\zeta_r}(z)\right)$ is defined by Eq. (16) and belongs to M_q. It is easy to see that $s_q\left(f_{\zeta_r}(\cdot)\right): \mathbb{C} \to M_q$ is an analytic extension of $\mathbb{R} \ni s \mapsto \sigma_s^\varphi\left(s_q(\zeta_r)\right)$. Thus, ζ_r is analytic.

Let $\mathfrak{D}_0 = \mathrm{span}_\mathbb{R}\{\zeta_r : r > 0, \zeta \in \mathcal{H}_\mathbb{R}\}$. Note that $\mathfrak{D}_0$ (consisting of analytic vectors) is dense in $\left(\mathcal{H}_\mathbb{R}, \langle\cdot,\cdot\rangle_{\mathcal{H}_\mathbb{C}}\right)$. Finally, use the fact that any dense subspace of a separable (real) Hilbert space has an orthonormal basis consisting of elements from the dense subspace. The rest is clear.

Lemma (5.2.7)[222]: Let $\xi_0 \in \mathcal{H}_\mathbb{R}$ be a unit vector such that $U_t\xi_0 = \xi_0$ for all t. Then the following hold.

(a) For $\eta \in \mathcal{H}_\mathbb{R} + i\mathcal{H}_\mathbb{R}$ one has

$$\langle \xi_0, \eta \rangle_U = \langle \xi_0, \eta \rangle_{\mathcal{H}_\mathbb{C}}.$$

(b) Let $\xi_1, \cdots, \xi_n \in \mathcal{H}_\mathbb{R}$ be non-zero vectors. If $k \geq 1$, then

$$\left\langle \xi_0^{\otimes k}, \xi_1 \otimes \cdots \otimes \xi_n \right\rangle_q = 0,$$

if and only if $n \neq k$ or $\langle \xi_0, \xi_i \rangle_{\mathcal{H}_C} = 0$ for at least one i.

(c) Let $\xi_1, \cdots, \xi_n \in \mathcal{H}_\mathbb{R} \cap \mathfrak{D}\left(A^{-\frac{1}{2}}\right)$ be non-zero vectors. If $k \geq 1$, then

$$\left\langle \xi_0^{\otimes k}, A^{-\frac{1}{2}}\xi_1 \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_n \right\rangle_q = 0,$$

if and only if $n \neq k$ or $\langle \xi_0, \xi_i \rangle_{\mathcal{H}_C} = 0$ for at least one i.

Proof. (a). Note that $\frac{2}{1+A^{-1}}\xi_0 = \xi_0$. Thus, the result follows from the definition of $\langle\cdot,\cdot\rangle_U$.

(b). Note that

$$\begin{aligned}
\left\langle \xi_0^{\otimes k}, \xi_1 \otimes \cdots \otimes \xi_n \right\rangle_q &= \delta_{n,k}\left\langle \xi_0^{\otimes n}, \xi_1 \otimes \cdots \otimes \xi_n \right\rangle_q \\
&= \delta_{n,k} \sum_{\pi \in S_n} q^{i(\pi)} \prod_{j=1}^{n} \left\langle \xi_0, \xi_{\pi(j)} \right\rangle_U \text{ (by Eq.(4))} \\
&= \delta_{n,k} \sum_{\pi \in S_n} q^{i(\pi)} \prod_{j=1}^{n} \left\langle \frac{2}{1+A^{-1}}\xi_0, \xi_{\pi(j)} \right\rangle_{\mathcal{H}_\mathbb{C}} \\
&= \delta_{n,k} \sum_{\pi \in S_n} q^{i(\pi)} \prod_{j=1}^{n} \left\langle \xi_0, \xi_{\pi(j)} \right\rangle_{\mathcal{H}_\mathbb{C}} \\
&= \delta_{n,k} \prod_{j=1}^{n} \left\langle \xi_0, \xi_j \right\rangle_{\mathcal{H}_\mathbb{C}} \sum_{\pi \in S_n} q^{i(\pi)}
\end{aligned}$$

$$= \delta_{n,k} \prod_{j=1}^{n} \langle \xi_0, \xi_j \rangle_{\mathcal{H}_{\mathbb{C}}} [n]_q!.$$

The rest is immediate.

(c). First note that as $\xi_i \in \mathcal{H}_{\mathbb{R}}$, so $A^{-\frac{1}{2}}\xi_i \in \mathcal{H}_{\mathbb{C}}$. Observe that

$$\begin{aligned}
\left\langle \xi_0^{\otimes k}, A^{-\frac{1}{2}}\xi_1 \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_n \right\rangle_q &= \delta_{n,k} \left\langle \xi_0^{\otimes n}, A^{-\frac{1}{2}}\xi_1 \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_n \right\rangle_q \\
&= \delta_{n,k} \sum_{\pi \in S_n} q^{i(\pi)} \prod_{j=1}^{n} \left\langle \xi_0, A^{-\frac{1}{2}}\xi_{\pi(j)} \right\rangle_U \text{ (by Eq.(4))} \\
&= \delta_{n,k} \sum_{\pi \in S_n} q^{i(\pi)} \prod_{j=1}^{n} \langle \xi_0, \xi_{\pi(j)} \rangle_U \\
&= \langle \xi_0^{\otimes k}, \xi_1 \otimes \cdots \otimes \xi_n \rangle_q.
\end{aligned}$$

Thus, the result follows from (b) above.

A convenient description of the centralizer M_q^{φ} is a component we need to decide the factoriality and type of M_q. We borrow ideas from Thm. 2.2 of [123] and show that the centralizer of M_q depends on the almost periodic part of the orthogonal representation (U_t). We need some intermediate results.

Lemma (5.2.8)[222]: The following hold.

1. The vector $\xi_1 \otimes \cdots \otimes \xi_n \in M_q\Omega$ for any $\xi_i \in \mathcal{H}_{\mathbb{R}}, 1 \le i \le n$ and $n \in \mathbb{N}$.
2. The vector $\xi_1 \otimes \cdots \otimes \xi_n \in M_q'\Omega$ for any $\xi_i \in \mathfrak{D}\left(A^{-\frac{1}{2}}\right) \cap \mathcal{H}_{\mathbb{R}}, 1 \le i \le n$ and $n \in \mathbb{N}$.

Proof. In both cases, the proof proceeds by induction.

(a) Let $n = 1$. Then by definition of M_q it follows that $\xi = s_q(\xi)\Omega \in M_q\Omega$ for all $\xi \in \mathcal{H}_{\mathrm{R}}$. Now suppose that $\xi_1 \otimes \cdots \otimes \xi_t \in M_q\Omega$ for all $\xi_j \in \mathcal{H}_{\mathbb{R}}, 1 \le j \le t$ and for all $1 \le t \le n$. Let $\xi_{n+1} \in \mathcal{H}_{\mathrm{R}}$. Then from Eq. (7) we have,

$$\xi_1 \otimes \cdots \otimes \xi_n \otimes \xi_{n+1} = s_q(\xi_1)s_q(\xi_2 \otimes \cdots \otimes \xi_{n+1})\Omega$$
$$-\sum_{l=2}^{n+1} q^{l-2}\langle \xi_1, \xi_l \rangle_U \xi_2 \otimes \cdots \otimes \xi_{l-1} \otimes \xi_{l+1} \otimes \cdots \otimes \xi_{n+1}.$$

But the right hand side of the above expression lies in $M_q\Omega$ by the induction hypothesis. Thus, $\xi_1 \otimes \cdots \otimes \xi_n \in M_q\Omega$ for $\xi_i \in \mathcal{H}_{\mathbb{R}}, 1 \le i \le n$ and for all $n \in \mathbb{N}$.

(b) Let $\xi \in \mathfrak{D}\left(A^{-\frac{1}{2}}\right) \cap \mathcal{H}_{\mathbb{R}}$. By Eq. (8), it follows that $J(\mathcal{H}_{\mathbb{R}}) \subseteq \mathcal{H}_{\mathbb{C}}$. Thus, write $J\xi = \eta_1 + i\eta_2$ with $\eta_1, \eta_2 \in \mathcal{H}_{\mathbb{R}}$. Then $s_q(\eta_j) \in M_q$ for $j = 1,2$, thus $J\xi = \left(s_q(\eta_1) + is_q(\eta_2)\right)\Omega \in M_q\Omega$. Note that $Js_q(J\xi)J\Omega = \xi$. Consequently, $\xi \in M_q'\Omega$ by the fundamental theorem of Tomita-Takesaki theory. Like before, assume that $\xi_1 \otimes \cdots \otimes \xi_t \in M_q'\Omega$ for all $\xi_j \in \mathcal{H}_{\mathbb{R}} \cap \mathfrak{D}\left(A^{-\frac{1}{2}}\right), 1 \le j \le t$ and for all $1 \le t \le n$.

Fix $\xi_{n+1} \in \mathfrak{D}\left(A^{-\frac{1}{2}}\right) \cap \mathcal{H}_{\mathbb{R}}$ and let $J\xi_{n+1} = A^{-\frac{1}{2}}\xi_{n+1} = \eta_{n+1}^1 + i\eta_{n+1}^2$ with $\eta_{n+1}^1, \eta_{n+1}^2 \in \mathcal{H}_{\mathbb{R}}$ (see Eq. (6)). Then for $\xi_i \in \mathfrak{D}\left(A^{-\frac{1}{2}}\right) \cap \mathcal{H}_{\mathbb{R}}$ for all $1 \le i \le n$, from Eq. (7), Eq. (8), and the fact that $J^2 = 1$, it follows that

$$
\begin{aligned}
&J s_q(J\xi_{n+1}) J d_q(\xi_1 \otimes \cdots \otimes \xi_n)\Omega \\
&\qquad = J s_q(J\xi_{n+1}) J(\xi_1 \otimes \cdots \otimes \xi_n) \\
&\qquad = J\left(\left(c_q(\eta_{n+1}^1) + i c_q(\eta_{n+1}^2)\right)\left(A^{-\frac{1}{2}}\xi_n \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_1\right)\right) \\
&\qquad\quad + J\left(\left(c_q(\eta_{n+1}^1)^* + i c_q(\eta_{n+1}^2)^*\right)\left(A^{-\frac{1}{2}}\xi_n \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_1\right)\right) \\
&\qquad = J\left((\eta_{n+1}^1 + i\eta_{n+1}^2) \otimes \left(A^{-\frac{1}{2}}\xi_n \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_1\right)\right) \\
&\qquad\quad + J\left(\left(c_q(\eta_{n+1}^1)^* + i c_q(\eta_{n+1}^2)^*\right)\left(A^{-\frac{1}{2}}\xi_n \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_1\right)\right) \\
&\qquad = J\left(A^{-\frac{1}{2}}\xi_{n+1} \otimes A^{-\frac{1}{2}}\xi_n \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_1\right) \\
&\qquad\quad + J\left(\left(c_q(\eta_{n+1}^1)^* + i c_q(\eta_{n+1}^2)^*\right)\left(A^{-\frac{1}{2}}\xi_n \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_1\right)\right) \\
&\qquad = \xi_1 \otimes \cdots \otimes \xi_n \otimes \xi_{n+1} \\
&\qquad\quad + J\left(\left(c_q(\eta_{n+1}^1)^* + i c_q(\eta_{n+1}^2)^*\right)\left(A^{-\frac{1}{2}}\xi_n \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_1\right)\right).
\end{aligned}
$$

Using the induction hypothesis, Eq. (7) and decomposing vectors in $\mathcal{H}_{\mathbb{C}}$ into real and imaginary parts, it is straightforward to check that

$$J\left(\left(c_q(\eta_{n+1}^1)^* + i c_q(\eta_{n+1}^2)^*\right)\left(A^{-\frac{1}{2}}\xi_n \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_1\right)\right) \in M_q'\Omega.$$

Hence, $\xi_1 \otimes \cdots \otimes \xi_n \otimes \xi_{n+1} \in M_q'\Omega$. Now use induction to complete the proof.

In the next Lemma, we make use of Lemma (5.2.8) to show how certain operators in M_q act on simple tensors.

Lemma (5.2.9)[222]: Let $\xi, \xi_i \in \mathcal{H}_{\mathbb{R}}$ for $1 \le i \le n$ be such that $\langle \xi_i, \xi \rangle_U = 0$ for $1 \le i \le n$. Then,

$$s_q(\xi_1 \otimes \cdots \otimes \xi_n)\left(\xi^{\otimes k}\right) = \xi_1 \otimes \cdots \otimes \xi_n \otimes \xi^{\otimes k}, \text{ for all } k \ge 0.$$

Proof. Note that by Lemma (5.2.8), it follows that $s_q(\xi_1 \otimes \cdots \otimes \xi_n) \in M_q$. The result is clearly true for $k = 0$ by definition (see Eq. (16)). We will only prove this result for $k = 1$. For $k \ge 2$, the argument is similar.

We use induction. Let $n = 1$, then note that,

$$s_q(\xi_1)\xi = \xi_1 \otimes \xi + \langle \xi_1, \xi \rangle_U \Omega = \xi_1 \otimes \xi, \text{ by Eq. (7)}.$$

Now suppose that the result is true for all $1 \le m \le n$. Let $\xi_{n+1} \in \mathcal{H}_{\mathbb{R}}$ be such that $\langle \xi_{n+1}, \xi \rangle_U = 0$. Then, from Eq. (7) and the proof of Lemma (5.2.8), we have

$$
\begin{aligned}
s_q(\xi_1 \otimes \cdots \otimes \xi_n \otimes \xi_{n+1}) &= s_q(\xi_1) s_q(\xi_2 \otimes \cdots \otimes \xi_{n+1}) \\
&\quad - \sum_{l=2}^{n+1} q^{l-2} \langle \xi_1, \xi_l \rangle_U s_q(\xi_2 \otimes \cdots \otimes \xi_{l-1} \otimes \xi_{l+1} \otimes \cdots \otimes \xi_{n+1})
\end{aligned}
$$

Consequently, by using the induction hypothesis, one has

$$
\begin{aligned}
& s_q(\xi_1 \otimes \cdots \otimes \xi_n \otimes \xi_{n+1})\xi \\
= \;& s_q(\xi_1) s_q(\xi_2 \otimes \cdots \otimes \xi_{n+1})\xi \\
& - \sum_{l=2}^{n+1} q^{l-2} \langle \xi_1, \xi_l \rangle_U s_q(\xi_2 \otimes \cdots \otimes \xi_{l-1} \otimes \xi_{l+1} \otimes \cdots \otimes \xi_{n+1})\xi \\
= \;& s_q(\xi_1)(\xi_2 \otimes \cdots \otimes \xi_{n+1} \otimes \xi) \\
& - \sum_{l=2}^{n+1} q^{l-2} \langle \xi_1, \xi_l \rangle_U (\xi_2 \otimes \cdots \otimes \xi_{l-1} \otimes \xi_{l+1} \otimes \cdots \otimes \xi_{n+1} \otimes \xi) \\
= \;& \xi_1 \otimes \cdots \otimes \xi_n \otimes \xi_{n+1} \otimes \xi, \text{ by Eq. (7).}
\end{aligned}
$$

This completes the proof.

Since $t \mapsto U_t, t \in \mathbb{R}$, is a strongly continuous orthogonal representation of $\mathbb{R}$ on the real Hilbert space $\mathcal{H}_{\mathbb{R}}$, there is a unique decomposition (cf. [15]),

$$
(\mathcal{H}_{\mathbb{R}}, U_t) = \left(\bigoplus_{j=1}^{N_1} (\mathbb{R}, \mathrm{id}) \right) \oplus \left(\bigoplus_{k=1}^{N_2} \big(\mathcal{H}_{\mathbb{R}}(k), U_t(k)\big) \right) \oplus \big(\widetilde{\mathcal{H}}_{\mathbb{R}}, \widetilde{U}_t\big), \tag{18}
$$

where $0 \leq N_1, N_2 \leq \aleph_0$,

$$
\mathcal{H}_{\mathbb{R}}(k) = \mathbb{R}^2,\; U_t(k) = \begin{pmatrix} \cos(t\log\lambda_k) & -\sin(t\log\lambda_k) \\ \sin(t\log\lambda_k) & \cos(t\log\lambda_k) \end{pmatrix}, \lambda_k > 1,
$$

and $\big(\widetilde{\mathcal{H}}_{\mathbb{R}}, \widetilde{U}_t\big)$ corresponds to the weakly mixing component of the orthogonal representation; thus $\widetilde{\mathcal{H}}_{\mathbb{R}}$ is either 0 or infinite dimensional.

If $N_1 \neq 0$, let $e_j = 0 \oplus \cdots \oplus 0 \oplus 1 \oplus 0 \oplus \cdots \oplus 0 \in \oplus_{j=1}^{N_1} \mathbb{R}$, where 1 appears at the j-th place for $1 \leq j \leq N_1$. Similarly, if $N_2 \neq 0$, let $f_k^1 = 0 \oplus \cdots \oplus 0 \oplus \begin{pmatrix} 1 \\ 0 \end{pmatrix} \oplus 0 \oplus \cdots \oplus 0 \in \oplus_{k=1}^{N_2} \mathcal{H}_{\mathbb{R}}(k)$ and $f_k^2 = 0 \oplus \cdots \oplus 0 \oplus \begin{pmatrix} 0 \\ 1 \end{pmatrix} \oplus 0 \oplus \cdots \oplus 0 \in \oplus_{k=1}^{N_2} \mathcal{H}_{\mathrm{R}}(k)$ be vectors with non-zero entries in the k-th position for $1 \leq k \leq N_2$. Denote

$$
e_k^1 = \frac{\sqrt{\lambda_k + 1}}{2}(f_k^1 + i f_k^2) \text{ and } e_k^2 = \frac{\sqrt{\lambda_k^{-1} + 1}}{2}(f_k^1 - i f_k^2),
$$

thus $e_k^1, e_k^2 \in \mathcal{H}_{\mathbb{R}}(k) + i\mathcal{H}_{\mathbb{R}}(k)$ are orthonormal basis of $(\mathcal{H}_{\mathbb{R}}(k) + i\mathcal{H}_{\mathbb{R}}(k), \langle \cdot,\cdot \rangle_U)$ for $1 \leq k \leq N_2$. Fix $1 \leq k \leq N_2$. The analytic generator $A(k)$ of $(U_t(k))$ is given by

$$
A(k) = \frac{1}{2} \begin{pmatrix} \lambda_k + \frac{1}{\lambda_k} & i\left(\lambda_k - \frac{1}{\lambda_k}\right) \\ -i\left(\lambda_k - \frac{1}{\lambda_k}\right) & \lambda_k + \frac{1}{\lambda_k} \end{pmatrix}.
$$

Moreover,

$$
A(k) e_k^1 = \frac{1}{\lambda_k} e_k^1 \text{ and } A(k) e_k^2 = \lambda_k e_k^2.
$$

Write $\mathcal{S} = \{e_j : 1 \leq j \leq N_1\} \cup \{e_k^1, e_k^2 : 1 \leq k \leq N_2\}$ if $N_1 \neq 0$ or $N_2 \neq 0$, else set $\mathcal{S} = \{0\}$. If $\mathcal{S} \neq \{0\}$, then $\mathcal{S}$ is an orthogonal set in $(\mathcal{H}_{\mathbb{C}}, \langle \cdot,\cdot \rangle_U)$ and the space of eigenvectors of the analytic generator A of (U_t) is contained in span $\mathcal{S}$. In the event $\mathcal{S} \neq \{0\}$, rename the elements of the set $\mathcal{S}$ as $\zeta_1, \zeta_2, \cdots$, i.e., $\mathcal{S} = \{\zeta_i : 1 \leq i \leq N_1 + 2N_2\}$, whence $A\zeta_l = \beta_l \zeta_l$ with $\beta_l \in \mathcal{E}_A$ for all l, where $\mathcal{E}_A = \{1\} \cup \{\lambda_k : 1 \leq k \leq N_2\} \cup \left\{\frac{1}{\lambda_k} : 1 \leq k \leq N_2\right\}$.

It is to be understood that when $N_1 = \infty$ (resp. $N_2 = \infty$), the constraints $j \leq N_1$ and $i \leq N_1 + 2N_2$ (resp. $k \leq N_2$ and $i \leq N_1 + 2N_2$) (in defining $\mathcal{S}$ and $\mathcal{E}_A$) is replaced by $j < N_1$ and $i < N_1 + 2N_2$ (resp. $k < N_2$ and $i < N_1 + 2N_2$).

The following result must be known.

Proposition (5.2.10)[222]: Let $(\rho, \mathcal{H})$ be a strongly continuous unitary representation of a separable locally compact abelian group G on a Hilbert space $\mathcal{H}$. For $n \geq 1$ and $q \in (-1,1)$, let $\rho^{\otimes_q n}$ be the n-fold amplification of ρ on $\mathcal{H}^{\otimes_q n}$ defined by

$$\rho^{\otimes_q n}(g)(\xi_1 \otimes \cdots \otimes \xi_n) = \rho(g)\xi_1 \otimes \cdots \otimes \rho(g)\xi_n, g \in G, \xi_i \in \mathcal{H} \text{ for } 1 \leq i \leq n.$$

Then $\left(\rho^{\otimes_q n}, \mathcal{H}^{\otimes_q n}\right)$ is a strongly continuous unitary representation of G. Let $\eta \in \mathcal{H}^{\otimes_q n}$ be an eigenvector of $\rho^{\otimes_q n}$ with associated character $\chi \in \hat{G}$. Let

$$\mathfrak{e}_\chi = \left\{\xi_1 \otimes \cdots \otimes \xi_n : \xi_i \in \mathcal{H}, \exists \chi_i \in \hat{G} \text{ such that } \rho(\cdot)\xi_i = \chi_i(\cdot)\xi_i, 1 \leq i \leq n, \prod_{i=1}^{n} \chi_i = \chi\right\}.$$

Then, $\eta \in \overline{\operatorname{span} \mathfrak{e}_\chi}$.

Proof. First of all, note that Eq. (4) forces that $\rho^{\otimes_q n}$ is a strongly continuous unitary representation of G. Note that by Lemma (5.2.3), the operator $T: \left(\mathcal{H}^{\otimes_q n}, \|\cdot\|_q\right) \to \left(\mathcal{H}^{\otimes_0 n}, \|\cdot\|_0\right)$ defined by $T(\xi_1 \otimes \cdots \otimes \xi_n) = \xi_1 \otimes \cdots \otimes \xi_n$, for all $\xi_i \in \mathcal{H}, 1 \leq i \leq n$, is bounded and invertible. Moreover, $T^{-1}\rho^{\otimes n}(\cdot)T = \rho^{\otimes q^n}(\cdot)$. Consequently, the spectral properties of $\rho^{\otimes qn}$ and $\rho^{\otimes_0 n}$ are identical. Also note that $\rho^{\otimes 0n}$ is the usual tensor product representation on the usual tensor product of Hilbert spaces.

The result now follows clearly from considering the direct integral version of the spectral theorem of tensor product of unitary operators.

Theorem (5.2.11)[222]: Let

$$\mathcal{W}_0 = \begin{cases} \left\{\zeta_{i_1} \otimes \cdots \otimes \zeta_{i_n} : \zeta_{i_j} \in \mathcal{S}, 1 \leq i_j \leq N_1 + 2N_2, \prod_{j=1}^{n} \beta_{i_j} = 1, n \in \mathbb{N}\right\}, \\ \qquad \text{if } \max\ (N_1, N_2) < \infty; \\ \left\{\zeta_{i_1} \otimes \cdots \otimes \zeta_{i_n} : \zeta_{i_j} \in \mathcal{S}, 1 \leq i_j < N_1 + 2N_2, \prod_{j=1}^{n} \beta_{i_j} = 1, n \in \mathbb{N}\right\}, \\ \qquad \text{if } \max\ (N_1, N_2) = \infty. \end{cases}$$

Let $\mathcal{W} = \mathbb{C}\Omega \oplus \overline{\operatorname{span} \mathcal{W}_0}^{\|\cdot\|_q}$. Then, $M_q^\varphi \Omega = \mathcal{W} \cap M_q\Omega$.

Proof. Decomposing vectors in $\mathcal{S}$ into real and imaginary parts and using Lemma (5.2.8), it follows that $\mathcal{W}_0 \subseteq M_q\Omega$. Fix $n \in \mathbb{N}$ and let $1 \leq i_1, \cdots, i_n \leq N_1 + 2N_2$ or $1 \leq i_1, \cdots, i_n < N_1 + 2N_2$ (as the case may be), be such that $\beta_{i_1} \cdots \beta_{i_n} = 1$. Pick $\zeta_{i_j} \in \mathcal{S}$ for $1 \leq j \leq n$ Consider $x = s_q\left(\zeta_{i_1} \otimes \cdots \otimes \zeta_{i_n}\right) \in M_q$. As $\sigma_{-t}^\varphi = \operatorname{Ad}\left(\mathcal{F}(U_t)\right)$ (see Eq. (6), (7)), so

$$\begin{aligned}
\sigma_{-t}^{\varphi}(x)\Omega &= \mathcal{F}(U_t)x\mathcal{F}(U_t)^*\Omega = \mathcal{F}(U_t)x\Omega = \mathcal{F}(U_t)\big(\zeta_{i_1}\otimes\cdots\otimes\zeta_{i_n}\big)\\
&= U_t\zeta_{i_1}\otimes\cdots\otimes U_t\zeta_{i_n}\\
&= \big(\beta_{i_1}\cdots\beta_{i_n}\big)^{it}\big(\zeta_{i_1}\otimes\cdots\otimes\zeta_{i_n}\big), (\text{ since } U_t = A^{it})\\
&= s_q\big(\zeta_{i_1}\otimes\cdots\otimes\zeta_{i_n}\big)\Omega\\
&= x\Omega, \text{ for all } t\in\mathbb{R}.
\end{aligned}$$

Consequently, $x = s_q\big(\zeta_{i_1}\otimes\cdots\otimes\zeta_{i_n}\big)\in M_q^{\varphi}$. Therefore, conclude that $\mathcal{W}\cap M_q\Omega\subseteq M_q^{\varphi}\Omega$.

For the reverse inclusion, let $y\in M_q^{\varphi}$ and write $y\Omega=\sum_{n=0}^{\infty}\eta_n$, where $\eta_n\in\mathcal{H}^{\otimes_q n}$ for all $n\geq 0$ and the series converges in $\|\cdot\|_q$. It is enough to show that $\eta_n\in\mathcal{W}$ for all $n\geq 0$. Again, note that

$$\begin{aligned}
\sum_{n=0}^{\infty}\eta_n &= \sigma_{-t}^{\varphi}(y)\Omega = \mathcal{F}(U_t)y\mathcal{F}(U_t)^*\Omega\\
&= \mathcal{F}(U_t)y\Omega\\
&= \mathcal{F}(U_t)\sum_{n=0}^{\infty}\eta_n = \sum_{n=0}^{\infty}\mathcal{F}(U_t)\eta_n, \text{ for all } t\in\mathbb{R}.
\end{aligned}$$

Since $\mathcal{F}(U_t)\mathcal{H}^{\otimes_q n} = \mathcal{H}^{\otimes_q}n$ for all $n\geq 0$ and for all $t\in\mathbb{R}$, so we have $\mathcal{F}(U_t)\eta_n=\eta_n$ for all n and for all $t\in\mathbb{R}$. Fix $n\geq 1$ such that $\eta_n\neq 0$. Therefore, by Proposition (5.2.2) and Proposition (5.2.10), it follows that there exist $\zeta_{k,l}^{(n)}\in\mathcal{S}$ and $\beta_{k,l}^{(n)}\in\mathcal{E}_A$ with $A\zeta_{k,l}^{(n)} = \beta_{k,l}^{(n)}\zeta_{k,l}^{(n)}$ for $1\leq k\leq n$ and scalars $c_{n,l}, l\in\mathbb{N}$, such that $\eta_n=\sum_l c_{n,l}\left(\zeta_{1,l}^{(n)}\otimes\cdots\otimes\zeta_{n,l}^{(n)}\right)$ and $\prod_{k=1}^{n}\beta_{k,l}^{(n)}=1$ for all l; the series above converges in $\|\cdot\|_q$. Consequently, $\eta_n\in\mathcal{W}$ for all $n\geq 0$ and the proof is complete.

We investigate the von Neumann subalgebras M_ξ for $\xi\in\mathcal{H}_{\mathrm{R}}$, and record some of their properties. This is a preparatory and the aforesaid subalgebras play a major role in deciding the factoriality of M_q.

In the case when $q=0, t\mapsto U_t$ is the identity representation of $\mathbb{R}$ and $\dim(\mathcal{H}_{\mathbb{R}})\geq 2$, it is well known that $M_0=\Gamma_0(\mathcal{H}_{\mathbb{R}}, id_t)\cong L\mathbb{F}_{\dim(\mathcal{H}_{\mathbb{R}})}$ (see [19]). In that case, for all $0\neq\xi\in\mathcal{H}_{\mathbb{R}}$, the algebra M_ξ is a maximal injective (see [237]), strongly mixing MASA, for which the orthocomplement of the associated Jones' projection regarded as an M_ξ-bimodule is an infinite direct sum of coarse bimodules (see [206],[228]). Moreover, if $\xi_1,\xi_2\in\mathcal{H}_{\mathbb{R}}$ are nonzero elements such that $\langle\xi_1,\xi_2\rangle_{\mathcal{H}_{\mathbb{C}}}=0$, then M_{ξ_1} and M_{ξ_2} are free and outer conjugate [19].

Note that if $0\neq\xi\in\mathcal{H}_{\mathbb{R}}$ and $U_t\xi=\xi$ for all $t\in\mathbb{R}$, then $s_q(\xi)\in M_q^{\varphi}$ (from Eq. (7)). So $J\xi = Js_q(\xi)\Omega = s_q(\xi)^*\Omega = s_q(\xi)\Omega=\xi$.

By Eq. (1.2) of [123], for $\xi\in\mathcal{H}_{\mathbb{R}}$ with $\|\xi\|_U=1$, the moments of the operator $s_q(\xi)$ with respect to the q-quasi free state $\varphi(\cdot)=\langle\Omega,\cdot\,\Omega\rangle_q$ are given by

$$\varphi\big(s_q(\xi)^n\big)=\begin{cases}0, & \text{if } n \text{ is odd,}\\ \displaystyle\sum_{\mathcal{V}=\{\pi(r),\kappa(r)\}_{1\leq r\leq\frac{n}{2}}} q^{c(\mathcal{V})}, & \text{if } n \text{ is even,}\end{cases}$$

where the summation is taken over all pair partitions $\mathcal{V} = \{\pi(r), \kappa(r)\}_{1\leq r\leq\frac{n}{2}}$ of $\{1,2,\cdots,n\}$ with $\pi(r) < \kappa(r)$ and $c(\mathcal{V})$ is the number of crossings of $\mathcal{V}$, i.e.,

$$c(\mathcal{V}) = \#\{(r,s): \pi(r) < \pi(s) < \kappa(r) < \kappa(s)\}.$$

So, it follows that for $\xi \in \mathcal{H}_{\mathbb{R}}$ with $\| \xi \|_U = 1$, the distribution of the single q-Gaussian $s_q(\xi)$ does not depend on the group (U_t). In the tracial case, and thus in all cases, this distribution obeys the q-semicircular law ν_q which is absolutely continuous with respect to the uniform measure supported on the interval $\left[-\frac{2}{\sqrt{1-q}}, \frac{2}{\sqrt{1-q}}\right]$. The associated orthogonal polynomials are q-Hermite polynomials $H_n^q, n \geq 0$. For the density function of ν_q and the recurrence relations defining the q-Hermite polynomials, see Defn. 1.9 and Thm. 1.10 of [60] (also see [130],[19]). Hence, $M_\xi \cong L^\infty\left(\left[-\frac{2}{\sqrt{1-q}}, \frac{2}{\sqrt{1-q}}\right], \nu_q\right)$, thus M_ξ is diffuse and $\left\{H_n^q\left(s_q(\xi)\right)\Omega: n \geq 0\right\}$, is a total orthogonal set of vectors in $\overline{M_\xi\Omega}\ \|\cdot\|_q^\beta$. Write $\mathcal{E}_\xi = \left\{\xi^{\otimes n}: n \geq 0\right\}$.

Lemma (5.2.12)[222]: The following hold.

1. Let $\xi \in \mathcal{H}_{\mathbb{R}}$ be a unit vector such that $U_t\xi = \xi$ for all $t \in \mathbb{R}$. Then, $\mathcal{E}_\xi \subseteq M_q\Omega \cap M_q'\Omega$.
2. Let $\xi \in \mathcal{H}_{\mathbb{R}}$ be a unit vector. Then, $\overline{M_\xi\Omega}^{\|\cdot\|_q} = \overline{\operatorname{span}\mathcal{E}_\xi}^{\|\cdot\|_q}$.

Proof. (a) This follows directly from Lemma (5.2.8) as $\xi \in \mathfrak{D}\left(A^{-\frac{1}{2}}\right)$.

(b) From the Wick product formula in Prop. 2.9 of [60], it follows that $\xi^{\otimes n} = H_n^q\left(s_q(\xi)\right)\Omega$ for all $n \geq 0$ (by convention $\xi^{\otimes 0} = \Omega$). Thus, $\xi^{\otimes n} \in M_\xi\Omega$ for all $n \geq 0$. It is now clear that $\overline{\operatorname{span}\mathcal{E}_\xi}^{\|\cdot\|_q} \subseteq \overline{M_\xi\Omega}\ \|\cdot\|_q$. Now use Stone-Weierstrass and Kaplansky density theorems or the fact that $\overline{M_\xi\Omega}^{\|\cdot\|_q} \cong L^2\left(\left[-\frac{2}{\sqrt{1-q}}, \frac{2}{\sqrt{1-q}}\right], \nu_q\right)$ to establish the reverse inclusion.

The next theorem is known in the case $q = 0$. When $q = 0$, one uses freeness and results from [15] to obtain a proof of it.

Theorem (5.2.13)[222]: Let $\xi \in \mathcal{H}_{\mathbb{R}}$ be a unit vector. There exists unique φ-preserving faithful normal conditional expectation $\mathbb{E}_\xi: M_q \to M_\xi$ if and only if $s_q(\xi) \in M_q^\varphi$, equivalently $U_t\xi = \xi$ for all $t \in \mathbb{R}$.

Proof. Suppose there exists a conditional expectation $\mathbb{E}_\xi: M_q \to M_\xi$ such that $\varphi\left(\mathbb{E}_\xi(x)\right) = \varphi(x)$, for all $x \in M_q$. Clearly, $\mathbb{E}_\xi$ is faithful and normal. By Takesaki's theorem [107], we have $\sigma_t^\varphi\left(M_\xi\right) = M_\xi$ for all $t \in \mathbb{R}$. Moreover, from [107] we have $\mathbb{E}_\xi \circ \sigma_t^\varphi = \sigma_t^\varphi \circ \mathbb{E}_\xi$ for all $t \in \mathbb{R}$. Thus

$$\mathbb{E}_\xi\left(\sigma_t^\varphi\left(s_q(\xi)\right)\right) = \sigma_t^\varphi\left(\mathbb{E}_\xi\left(s_q(\xi)\right)\right) \text{ for all } t.$$

Let $P_\xi: L^2\left(M_q, \varphi\right) \to \overline{M_\xi\Omega}^{\|\cdot\|_q}$ denote the orthogonal projection $\left(L^2\left(M_q, \varphi\right) = \mathcal{F}_q(\mathcal{H})\right)$.

Since $\varphi\left(s_q(\xi)\right) = 0$, so $\varphi\left(\sigma_t\left(s_q(\xi)\right)\right) = 0$ for all $t \in \mathbb{R}$ as well. Thus, using Lemma (5.2.12) and expanding in terms of orthonormal basis, we have $\sigma_t^\varphi\left(s_q(\xi)\right)\Omega = \sum_{n=1}^\infty a_n(t)\xi^{\otimes n}$, $a_n(t) \in \mathbb{C}$, for all $t \in \mathbb{R}$. Hence, from Eq. (9), we have

$$
\begin{aligned}
U_{-t}\xi &= s_q(U_{-t}\xi)\Omega = \sigma_t^{\varphi}\big(s_q(\xi)\big)\Omega \\
&= \sigma_t^{\varphi}\left(\mathbb{E}_{\xi}\big(s_q(\xi)\big)\right)\Omega = \mathbb{E}_{\xi}\left(\sigma_t^{\varphi}\big(s_q(\xi)\big)\right)\Omega \\
&= P_{\xi}\sigma_t^{\varphi}\big(s_q(\xi)\big)P_{\xi}\Omega \\
&= P_{\xi}\sum_{n=1}^{\infty} a_n(t)\xi^{\otimes n} \\
&= \sum_{n=1}^{\infty} a_n(t)\xi^{\otimes n}.
\end{aligned}
$$

Consequently, $a_n(t) = 0$ for all $n \geq 2$ from Eq. (6), and

$$U_{-t}\xi = a_1(t)\xi = \lambda_t\xi, \text{ for all } t \in \mathbb{R}.$$

As Ω is separating for M_q, it follows that $\sigma_t^{\varphi}\big(s_q(\xi)\big) = \lambda_t s_q(\xi)$. Thus, $\lambda_t\lambda_s = \lambda_{t+s}$ for all $t, s \in \mathbb{R}, \lambda_0 = 1, \lambda_t \in \{\pm 1\}$ (as $s_q(\xi)$ is self-adjoint) and $t \mapsto \lambda_t$ is continuous. Since the image of a connected set under a continuous map is connected, so either $\lambda_t = 1$ for all t or $\lambda_t = -1$ for all t. But $\lambda_0 = 1$, so $\lambda_t = 1$ for all t. Hence, $s_q(\xi) \in M_q^{\varphi}$.

Conversely, suppose $s_q(\xi) \in M_q^{\varphi}$. Then $M_{\xi} \subseteq M_q^{\varphi}$ and the modular group fixes M_q^{φ} pointwise. Now use Takesaki's theorem [107] to finish the proof.

We end with the following observation.

Lemma (5.2.14)[222]: For $\eta \in M_q'\Omega$ and $\zeta \in M_q\Omega$ one has $s_q(\zeta)\eta = d_q(\eta)\zeta$. In particular, for $\eta \in \mathcal{Z}\big(M_q\big)\Omega$ the same holds.

Proof. First note that the operators in the statement are defined by Eq. (14). Now $s_q(\zeta)\eta = s_q(\zeta)d_q(\eta)\Omega = d_q(\eta)s_q(\zeta)\Omega = d_q(\eta)\zeta$.

We intend to show that for any unit vector $\xi_0 \in \mathcal{H}_{\mathbb{R}}$ with $U_t\xi_0 = \xi_0$ for all $t \in \mathbb{R}$, the abelian algebra M_{ξ_0} of M_q is a MASA and possesses vigorous mixing properties. Needless to say, such a MASA is then singular from [36],[235],[206]. In order to do so, we need some general facts on MASAs. Most of these facts appear in the framework of finite von Neumann algebras. But, the MASAs of interest in M_q lie in the centralizer M_q^{φ} by Theorem (5.2.13); so we can freely invoke most of these techniques (used for finite von Neumann algebras) in our setup as well. We recall without proofs some facts that will be required, as a detailed exposition would be a digression. The proofs of these facts are analogous to the ones for the tracial case.

Let M be a von Neumann algebra equipped with a faithful normal state φ. Let M act on the GNS Hilbert space $L^2(M,\varphi)$ via left multiplication and let $\|\cdot\|_{2,\varphi}$ denote the norm of $L^2(M,\varphi)$. Let $J_{\varphi}, \Omega_{\varphi}$ respectively denote the associated modular conjugation operator and the vacuum vector, and let $\left(\sigma_t^{\varphi}\right)_{t\in\mathbb{R}}$ denote the modular automorphisms associated to φ. Let $A \subseteq M$ be a diffuse abelian von Neumann subalgebra contained in $M^{\varphi} = \{x \in M: \sigma_t^{\varphi}(x) = x \forall t \in \mathbb{R}\}$. Then there exists a unique faithful, normal and φ-preserving conditional expectation $\mathbb{E}_A$ from M on to A [107]. Let $L^2(A,\varphi) = \overline{A\Omega_{\varphi}}^{\|\cdot\|_{2,\varphi}}$. Denote $\mathcal{A} = \left(A \cup J_{\varphi}AJ_{\varphi}\right)''$. Then $\mathcal{A}$ is abelian, so its commutant is a type I algebra. Note that $A' \cap M$ is globally invariant under $\left(\sigma_t^{\varphi}\right)$, thus there exists a unique faithful, normal and φ-preserving conditional expectation from M on to $A' \cap M$ (see [107]), and the associated Jones' projection $e_{A'\cap M} \in \mathcal{A}$ [219, Lemma 7.1.1] and is a central projection of $\mathcal{A}'$. (This fact will not be directly used, it is worth mentioning as it is this fact for which the theory of bimodules of MASAs works and is indispensable). This algebra $\mathcal{A}$ has been studied extensively by many experts of MASAs to understand the size of normalizers, orbit equivalence, mixing properties and to provide invariants of MASAs. In short, $\mathcal{A}$ captures the structure of $L^2(M,\varphi)$ as a $A-A$ bimodule (see Ch. 6,7 [219]). With the setup as above we define the following:

Definition (5.2.15)[222]: (cf. [206]) A diffuse abelian subalgebra $B \subseteq M$ with a φ-preserving normal conditional expectation $\mathbb{E}_B$ is said to be φ-strongly mixing in M if $\|\mathbb{E}_B(xa_ny)\|_{2,\varphi} \to 0$ for all $x,y \in M$ with $\mathbb{E}_B(x) = 0 = \mathbb{E}_B(y)$, whenever $\{a_n\}$ is a bounded sequence in B that goes to 0 in the w.o.t.

The above definition appears in [206] of finite von Neumann algebras, but the definition is valid in general and thus we do not assume traciality to define the property of φ-strongly mixing here. Moreover, by a polarization identity it is enough to check the convergence of $\mathbb{E}_B(xa_nx^*)$ in Defn. (5.2.15) for all $x \in M$ such that $\mathbb{E}_B(x) = 0$.

Let M_a denote the *-subalgebra of all entire (analytic) elements of M with respect to (σ_t^φ). For $x \in M$ and $y \in M_a$, define

$$T_{x,y}: L^2(A,\varphi) \to L^2(A,\varphi) \text{ by } T_{x,y}(a\Omega_\varphi) = \mathbb{E}_A(xay)\Omega_\varphi, a \in A. \quad (19)$$

Note that $T_{x,y}$ is bounded. Indeed, as $y \in M_a$ so $y^* \in \mathfrak{D}(\sigma_z^\varphi)$ for all $z \in \mathbb{C}$. Hence, $J_\varphi \sigma_{-\frac{i}{2}}\varphi^\varphi(y^*)J_\varphi a\Omega_\varphi = ay\Omega_\varphi$ for all $a \in A$, where $(\sigma_z^\varphi)_{z\in C}$ denotes the analytic continuation of (σ_t^φ) (see [229]). Thus,

$$\begin{aligned}\|\mathbb{E}_A(xay)\Omega_\varphi\|_{2,\varphi} &\le \|xay\Omega_\varphi\|_{2,\varphi} \\ &\le \| x \| \, \|ay\Omega_\varphi\|_{2,\varphi} \\ &\le \| x \| \left\|J_\varphi \sigma^\varphi_{-\frac{i}{2}}(y^*)J_\varphi\right\| \|a\Omega_\varphi\|_{2,\varphi} \\ &= \| x \| \left\|\sigma^\varphi_{-\frac{i}{2}}(y^*)\right\| \|a\Omega_\varphi\|_{2,\varphi}, \text{ for all } a \in A.\end{aligned}$$

One can identify $A \cong L^\infty(X,\lambda)$, where X is a standard Borel space and λ is a nonatomic probability measure on X. The left-right measure of A is the measure (strictly speaking the measure class) on $X \times X$ obtained from the direct integral decomposition of $L^2(M,\varphi) \ominus L^2(A,\varphi)$ over $X \times X$ so that $\mathcal{A}(1-e_A)$ is the algebra of diagonalizable operators with respect to the decomposition [235], [206] (e_A denoting the Jones' projection associated to). The process to calculate the left left-right measure is similar to the discussion laid out in [234]. Many more details of the same are discussed in [234].

If A is identified with $L^\infty([a,b],\lambda)$ where λ is the normalized Lebesgue measure (or Lebesgue equivalent), then from the results of [235] (specifically Thm. 2.1), it follows that the left-right measure of A is Lebesgue absolutely continuous when T_{x,y^*} is Hilbert Schmidt for x,y varying over a set S such that $\mathbb{E}_A(x) = 0 = \mathbb{E}_A(y)$ for all $x,y \in S$ and the span of $S\Omega$ is dense in $L^2(A,\varphi)^\perp$. (Note that the arguments of [235] use the unit interval. It was so chosen to make a standard frame of reference. However, the arguments of relating to absolute continuity of measures do not depend on the choice of the interval. Neither do the same arguments to prove Thm. 2.1 in [235] require that A is a MASA; it only involved measure theory.) From Thm. 4.4 and Rem. 4.5 of [225] (similarly the proof of Thm. 4.4 of [225] uses measure theory and not that the diffuse abelian algebra there is a MASA), it follows that A is φ-strongly mixing in M if the left-right measure of A is Lebesgue absolutely continuous. Thus, one has:

Theorem (5.2.16)[222]: Let $A \subseteq M$ be a diffuse abelian algebra such that $A \subseteq M^{\varphi}$ and the left-right measure of A is Lebesgue absolutely continuous. Then, A is φ-strongly mixing in M. In particular, A is a singular MASA in M.

Proof. We only need to show that A is a singular MASA in M. Let $x \in A' \cap M$. Let $y = x - \mathbb{E}_A(x)$. For $a \in A$, one has $ay = a(x - \mathbb{E}_A(x)) = ax - \mathbb{E}_A(ax) = xa - \mathbb{E}_A(xa) = ya$. Since A is diffuse choose a sequence of unitaries $u_n \in A$ such that $u_n \to 0$ in w.o.t. Since A is φ-strongly mixing in M (by the previous discussion) it follows that $\lim_n \|\mathbb{E}_A(yy^*)\|_{2,\varphi} = \lim_n \|\mathbb{E}_A(yu_ny^*)\|_{2,\varphi} = 0$. Since $\mathbb{E}_A$ is faithful, it follows that $y = 0$ Thus, A is a MASA.

That A is singular follows from results of [229],[36] and [234].

We are now ready to prove that if $\xi_0 \in \mathcal{H}_{\mathrm{R}}$ is a unit vector such that $U_t\xi_0 = \xi_0$ for all $t \in \mathbb{R}$, then M_{ξ_0} is φ-strongly mixing in M_q. Let $\mathbb{E}_{\xi_0}$ denote the unique φ-preserving, faithful, normal conditional expectation from M_q onto M_{ξ_0} (see Theorem (5.2.13)). Extend ξ_0 to an orthonormal basis

$$\mathcal{O} = \left\{\xi_k : \xi_k \text{ analytic}, 0 \leq k \leq \dim(\mathcal{H}_{\mathbb{R}}) - 1\right\}$$

of $\mathcal{H}_{\mathbb{R}}$ with respect to $\langle\cdot,\cdot\rangle_{\mathcal{H}_{\mathrm{C}}}$ consisting of analytic vectors as described in Proposition (5.2.6). Fix $\xi_{i_j} \in \mathcal{O}$ for $1 \leq j \leq n$. Note that as the analytic elements form a (w^*-dense) *-subalgebra, so $s_q(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n})$ is analytic with respect to (σ_t^{φ}) from (the proof of) Lemma (5.2.8). It follows that $s_q\left(A^{-\frac{1}{2}}\xi_k\right)$ is also analytic with respect to (σ_t^{φ}) for all $\xi_k \in \mathcal{O}$. Thus, by (the proof of) Lemma (5.2.8), it follows that $s_q\left(A^{-\frac{1}{2}}\xi_{i_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{i_n}\right)$ is also analytic with respect to (σ_t^{φ}). Moreover, from Lemma (5.2.12) and Lemma (5.2.7) it follows that $\mathbb{E}_{\xi_0}\left(s_q(\xi_{i_1} \otimes \cdots \otimes \xi_{i_n})\right) = 0$ forces that at least one letter ξ_{i_j} must be different from ξ_0. Furthermore, from Lemma (5.2.7) it follows that $A^{-\frac{1}{2}}\xi_{i_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{i_n} \in \mathcal{F}_q(\mathcal{H}) \ominus L^2(M_{\xi_0}, \varphi)$ if and only if $\xi_{i_1} \otimes \cdots \otimes \xi_{i_n} \in \mathcal{F}_q(\mathcal{H}) \ominus L^2(M_{\xi_0}, \varphi)$.

We have the following theorem.

Theorem (5.2.17)[222]: Let $t \mapsto U_t$ be a strongly continuous orthogonal representation of $\mathbb{R}$ on a real Hilbert space $\mathcal{H}_{\mathbb{R}}$ with $\dim(\mathcal{H}_{\mathbb{R}}) \geq 2$. Suppose there exists a unit vector $\xi_0 \in \mathcal{H}_{\mathbb{R}}$ such that $U_t\xi_0 = \xi_0$ for all $t \in \mathbb{R}$. Let $x = s_q(\xi_{i_1} \otimes \cdots \otimes \xi_{i_m})$ and $y = s_q\left(A^{-\frac{1}{2}}\xi_{j_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{j_k}\right)$ be such that $\mathbb{E}_{\xi_0}(x) = 0 = \mathbb{E}_{\xi_0}(y)$, where $\xi_{i_u}, \xi_{j_v} \in \mathcal{O}$ for $1 \leq u \leq m$ and $1 \leq v \leq k$.

Then, $T_{x,y}$ is a Hilbert-Schmidt operator.

Proof. First of all, as $U_t\xi_0 = \xi_0$ for all $t \in \mathbb{R}$, so $M_{\xi_0} \subseteq M_q$ is a diffuse abelian algebra in M_q lying in M_q^{φ}. By the previous discussion, it follows that x, y are analytic with respect to (σ_t^{φ}). Thus, $T_{x,y} \in \mathbf{B}\left(L^2(M_{\xi_0}, \varphi)\right)$.

Also note that $\xi_{j_1} \otimes \cdots \otimes \xi_{j_k} \in M_q\Omega \cap M_q'\Omega$ from Lemma (5.2.8). From Lemma (5.2.12), it follows that $H_n^q\left(s_q(\xi_0)\right)\Omega = \xi_0^{\otimes n}$ for all $n \geq 0$. Note that $d_q\left(A^{-\frac{1}{2}}\xi_{j_1} \otimes \cdots \otimes\right.$

$A^{-\frac{1}{2}}\xi_{j_k}\Big) \in M_q'$ by Theorem (5.2.5). Let $e_{\xi_0}: L^2(M_q,\varphi) \to L^2(M_{\xi_0},\varphi)$ denote the Jones' projection associated to M_{ξ_0}. Then from Eq. (19), we have

$$T_{x,y}\left(H_n^q\left(s_q(\xi_0)\right)\Omega\right) = e_{\xi_0}\left(xH_n^q\left(s_q(\xi_0)\right)s_q\left(A^{-\frac{1}{2}}\xi_{j_1}\otimes\cdots\otimes A^{-\frac{1}{2}}\xi_{j_k}\right)\Omega\right) \quad (20)$$

$$= e_{\xi_0}\left(xH_n^q\left(s_q(\xi_0)\right)\left(A^{-\frac{1}{2}}\xi_{j_1}\otimes\cdots\otimes A^{-\frac{1}{2}}\xi_{j_k}\right)\right)$$

$$= e_{\xi_0}\left(xH_n^q\left(s_q(\xi_0)\right)d_q\left(A^{-\frac{1}{2}}\xi_{j_1}\otimes\cdots\otimes A^{-\frac{1}{2}}\xi_{j_k}\right)\Omega\right)$$

(from Eq. (16) and Lemma (5.2.14))

$$= e_{\xi_0}\left(xd_q\left(A^{-\frac{1}{2}}\xi_{j_1}\otimes\cdots\otimes A^{-\frac{1}{2}}\xi_{j_k}\right)H_n^q\left(s_q(\xi_0)\right)\Omega\right)$$

$$= e_{\xi_0}\left(xd_q\left(A^{-\frac{1}{2}}\xi_{j_1}\otimes\cdots\otimes A^{-\frac{1}{2}}\xi_{j_k}\right)\xi_0^{\otimes n}\right)$$

$$= e_{\xi_0}\left(s_q(\xi_{i_1}\otimes\cdots\otimes\xi_{i_m})d_q\left(A^{-\frac{1}{2}}\xi_{j_1}\otimes\cdots\otimes A^{-\frac{1}{2}}\xi_{j_k}\right)\xi_0^{\otimes n}\right), n \geq 0$$

Now from Lemma 3.1 of [123], we have

$$s_q(\xi_{i_1}\otimes\cdots\otimes\xi_{i_m}) = \sum\sum q^{\aleph(K,I)}c_q\left(\xi_{i_{\kappa(1)}}\right)\cdots c_q\left(\xi_{i_{\kappa(n_1)}}\right)c_q\left(\xi_{i_{\pi(1)}}\right)^*\cdots c_q\left(\xi_{i_{\pi(n_2)}}\right)^* \text{ and}$$

$$d_q\left(A^{-\frac{1}{2}}\xi_{j_1}\otimes\cdots\otimes A^{-\frac{1}{2}}\xi_{j_k}\right) = \sum\sum q^{N(K',I')}r_q\left(A^{-\frac{1}{2}}\xi_{j_{k(1)}}\right)\cdots r_q\left(A^{-\frac{1}{2}}\xi_{j_{k(m_1)}}\right)r_q\left(A^{-\frac{1}{2}}\xi_{j_{*(1)}}\right)^*\cdots r_q\left(A^{-\frac{1}{2}}\xi_{j_{*(m_2)}}\right)^*,$$

where the first sum varies over the pairs (n_1,n_2) and (K,I) restricted to the following conditions:

$$n_1,n_2 \geq 0,\ n_1+n_2 = \mathrm{m}; \quad \text{and,} \quad \begin{array}{c} K = \{\kappa(1),\cdots,\kappa(n_1): \kappa(1)\leq\cdots\leq\kappa(n_1)\}, \\ I = \{\pi(1),\cdots,\pi(n_2): \pi(1)\leq\cdots\leq\pi(n_2)\}, \\ K\cup I = \{1,\cdots,m\}, K\cap I = \emptyset, \end{array} \quad (21)$$

and $\aleph(K,I) = \#\{(r,s): 1\leq r\leq n_1, 1\leq s\leq n_2, \kappa(r) > \pi(s)\}$. Similarly, the expansion of $d_q\left(A^{-\frac{1}{2}}\xi_{j_1}\otimes\cdots\otimes A^{-\frac{1}{2}}\xi_{j_k}\right)$ above is in terms of $m_1,m_2\geq 0, m_1+m_2 = k, K', I', \aleph(K',I'), \tilde{\kappa}, \tilde{\pi}$ and $r_q\left(A^{-\frac{1}{2}}\xi_{j_{k(C)}}\right)$ and $r_q\left(A^{-\frac{1}{2}}\xi_{j_{\pi(\cdot)}}\right)^*$ defined analogous to Eq. (21).

Note that $\left\|\xi_0^{\otimes n}\right\|_q^2 = [n]_q!$ for all $n\geq 0$ (see Eq. (7)). Again from Lemma (5.2.12), it follows that $\left\{\frac{1}{\sqrt{[n]_q!}}\xi_0^{\otimes n}: n\geq 0\right\}$ is an orthonormal basis of $L^2(M_{\xi_0},\varphi)$. Thus, to show $T_{x,y}$ is a Hilbert-Schmidt operator we need to show that $\sum_{n=0}^{\infty}\frac{1}{[n]_q!}\left\|T_{x,y}(\xi_0^{\otimes n})\right\|_q^2 < \infty$. But since $s_q(\xi_{i_1}\otimes\cdots\otimes\xi_{i_m})$ and $d_q\left(A^{-\frac{1}{2}}\xi_{j_1}\otimes\cdots\otimes A^{-\frac{1}{2}}\xi_{j_k}\right)$ split as finite sums, so from Eq. (20) it is enough to show that for each fixed $n_1,n_2,m_1,m_2,\kappa,\pi,\tilde{\kappa},\tilde{\pi}$ (in Eq. (21)), if

$$\zeta_n = e_{\xi_0}\Bigg(\left(c_q(\xi_{i_{\kappa(1)}})\cdots c_q(\xi_{i_{\kappa(n_1)}})c_q(\xi_{i_{\pi(1)}})^*\cdots c_q(\xi_{i_{\pi(n_2)}})\right)^*$$

$$\cdot\, r_q\left(A^{-\frac{1}{2}}\xi_{j_{k(1)}}\right)\cdots r_q\left(A^{-\frac{1}{2}}\xi_{j_{\kappa(m_1)}}\right)r_q\left(A^{-\frac{1}{2}}\xi_{j_{*(1)}}\right)^*\cdots r_q\left(A^{-\frac{1}{2}}\xi_{j_{\pi(m_2)}}\right)^*\Bigg)\xi_0^{\otimes n}\Bigg), n\geq 0,$$

then $\sum_{n=0}^{\infty} \frac{1}{[n]_q!} \|\zeta_n\|_q^2 < \infty$. Renaming indices, we may write

$$\zeta_n = e_{\xi_0} \quad \Big(\big(c_q(\xi_{i_1}) \cdots c_q(\xi_{i_l}) c_q(\xi_{i_{l+1}})^* \cdots c_q(\xi_{i_m}) \big)^* \cdot r_q\left(A^{-\frac{1}{2}}\xi_{j_1}\right) \cdots r_q\left(A^{-\frac{1}{2}}\xi_{j_p}\right) r_q\left(A^{-\frac{1}{2}}\xi_{j_{p+1}}\right)^* \cdots r_q\left(A^{-\frac{1}{2}}\xi_{j_k}\right)^* \Big) \xi_0^{\otimes n} \Big), n \geq 0.$$

For $\xi_{j'} \in \mathcal{O}$, since $\langle \xi_{j'}, \xi_0 \rangle_q = 0$ for $j' \neq 0$ (by Lemma (5.2.7)), (and hence $\left\langle A^{-\frac{1}{2}}\xi_0, A^{-\frac{1}{2}}\xi_{j'} \right\rangle_q = 0$ for $j' \neq 0$ by Eq. (12)), so

$$r_q\left(A^{-\frac{1}{2}}\xi_{j'}\right)^* \xi_0^{\otimes n} = r_q\left(A^{-\frac{1}{2}}\xi_{j'}\right)^* \left(A^{-\frac{1}{2}}\xi_0\right)^{\otimes n} = 0$$

for all $n \geq 0$ and $j' \neq 0$. Since at least one letter in $A^{-\frac{1}{2}}\xi_{j_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{j_k}$ is different from ξ_0 and $A^{-\frac{1}{2}}\xi_0 = \xi_0$, so ζ_n can be non-zero only when $\xi_{j_{p+1}} = \cdots = \xi_{j_k} = \xi_0$. Write $\delta = \prod_{w=p+1}^{k} \delta_{\xi_{jw}, \xi_0}$. Hence, from Eq. (13) and Eq. (15) we have

$$\begin{aligned} \zeta_n = \quad & \delta \prod_{t=n-(k-p)}^{n} (1 + q + \cdots + q^{t-1}) \\ & \cdot e_{\xi_0} \Big(\big(c_q(\xi_{i_1}) \cdots c_q(\xi_{i_l}) c_q(\xi_{i_{l+1}})^* \cdots c_q(\xi_{i_m})^* \big) \\ & \cdot \left(\xi_0^{\otimes(n-(k-p))} \otimes A^{-\frac{1}{2}}\xi_{j_p} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{j_1} \right) \Big) \qquad (22) \\ = \quad & \delta \frac{[n]_q!}{[n-(k-p)]_q!} \\ & e_{\xi_0} \Big(\big(c_q(\xi_{i_1}) \cdots c_q(\xi_{i_l}) c_q(\xi_{i_{l+1}})^* \cdots c_q(\xi_{i_m})^* \big) \\ & \left(\xi_0^{\otimes(n-(k-p))} \otimes A^{-\frac{1}{2}}\xi_{j_p} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{j_1} \right) \Big). \end{aligned}$$

By hypothesis at least one letter in $A^{-\frac{1}{2}}\xi_{j_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{j_p}$ is different from $\xi_0 \left(= A^{-\frac{1}{2}}\xi_0\right)$.

Therefore, the constraints for ζ_n to be non-zero are $i_r = 0$ for all $1 \leq r \leq l, \#\{i_r : l+1 \leq r \leq m, i_r \neq 0\} \geq 1$ (counted with multiplicities) and the expression

$$c_q(\xi_{i_1}) \cdots c_q(\xi_{i_l}) c_q(\xi_{i_{l+1}})^* \cdots c_q(\xi_{i_m})^* \left(\xi_0^{\otimes(n-(k-p))} \otimes A^{-\frac{1}{2}}\xi_{j_p} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{j_1} \right)$$

has to lie in span $\mathcal{E}_{\xi_0}$ (see Lemma (5.2.12) and the discussion preceding it). By repeated application of Lemma (5.2.4), one obtains

$$c_q(\xi_{i_{l+1}})^* \cdots c_q(\xi_{i_m})^* (\overbrace{\xi_0^{\otimes(n-(k-p))}} \otimes \overbrace{\left(A^{-\frac{1}{2}}\xi_{J_p} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{J_1}\right)})$$

$$c_q(\xi_{i_{l+1}})^* \cdots c_q(\xi_{i_{m-1}})^* (\overbrace{\left(c_q(\xi_{\iota_m})^* \xi_0^{\otimes(n-(k-p))}\right)}) \otimes \overbrace{\left(A^{-\frac{1}{2}}\xi_{J_p} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{J_1}\right)}$$

$$+ q^{(n-(k-p))} \overbrace{\xi_0^{\otimes(n-(k-p))}} \otimes \overbrace{c_q(\xi_{\iota_m})^* \left(A^{-\frac{1}{2}}\xi_{J_p} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{J_1}\right)})$$

$$= \sum_{r_1=0}^{1} \cdots \sum_{r_{m-l}=0}^{1} c_{r_1,\cdots,r_{m-l}} \cdot \left(\prod_{w=1}^{m-l} \left(c_q\left(\xi_{i_{l+w}}\right)^*\right)^{(1-r_w)}\right) \xi_0^{\otimes(n-(k-p))} \otimes \left(\prod_{w=1}^{m-l} \left(c_q\left(\xi_{i_{l+w}}\right)^*\right)^{r_w}\right)\left(A^{-\frac{1}{2}}\xi_{j_p} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{j_1}\right),$$

where $c_{r_1,\cdots,r_{m-l}} \in \mathbb{R}$ for $(r_1,\cdots,r_{m-l}) \in \{0,1\}^{m-l}$ are calculated as follows.
Given a $(m-l)$-bit string $(r_1,\cdots,r_{m-l})$, let $s_w = \#$ of zeros in $\{r_w, r_{w+1},\cdots,r_{m-l}\}$ for $1 \le w \le m-l$. Then, clearly $s_{m-l} = 1 - r_{m-l}$ and by induction it follows that $s_{m-l-1} = (1-r_{m-l}) + (1-r_{m-l-1}),\cdots, s_1 = (1-r_{m-l}) + (1-r_{m-l-1}) + \cdots + (1-r_1)$.

Thus, repeated application of Lemma (5.2.4) in Eq. (23) entail that

$$\begin{aligned} c_{r_1,\cdots,r_{m-1}} &= q^{(n-(k-p))(\sum_{w=1}^{m-l} r_w) - \sum_{w=1}^{m-l} r_w s_w} \\ &= q^{(n-(k-p))(\sum_{w=1}^{m-l} r_w) - \sum_{w=1}^{m-l} r_w((m-l)-w+1-\sum_{w'=w}^{m-1} r_{w'})} \\ &= q^{((n-(k-p))-(m-l)-1)(\sum_{w=1}^{m-l} r_w) + \sum_{w=1}^{m-l} w r_w + \sum_{w=1}^{m-l} (\sum_{w'=w}^{m-l} r_{w'}) r_w}. \end{aligned}$$

The above formula for $c_{r_1,\cdots,r_{m-1}}$ can be obtained by drawing a binary tree of height $(m-l)$ with weights attached along edges in such a way that it encodes the tensoring on the left or on the right following Lemma (5.2.4). It is to be noted that the largest power of q that appears in Eq. (23) is $(n-(k-p))(m-l)$ which appears when $r_w = 1$ for all w and the smallest power of q is 0 and it occurs when $r_w = 0$ for all w.

Further, notice that since $\#\{i_r : l+1 \le r \le m, i_r \ne 0\} \ge 1$, i.e., there is at least one r_0 with $l+1 \le r_0 \le m$ such that $\xi_{i_{r_0}} \perp \xi_0$ (in $\langle\cdot,\cdot\rangle_U$), so

$$\left(c_q\left(\xi_{i_{l+1}}\right)^* \cdots c_q\left(\xi_{i_{m-1}}\right)^* c_q\left(\xi_{i_m}\right)^*\right)\xi_0^{\otimes(n-(k-p))} \otimes \left(A^{-\frac{1}{2}}\xi_{j_p} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{j_1}\right) = 0.$$

Therefore, the expression in Eq. (23) has at most 2^{m-l-1} many non-zero terms each with scalar coefficients of the form q^d, where $d \ge ((n-(k-p)) - (m-l-1))$. Consequently, by Eq. (6), Eq. (7), Eq. (14) and Eq. (22), we conclude that there is a positive constant $K(l,m,p,q)$ independent of n and $N_0 \in \mathbb{N}$ such that

$$\|\zeta_n\|_q^2 \le K(l,m,p,q) q^{2n} \left(\frac{[n]_{|q|}!}{[n-(k-p)]_{|q|}!}\sqrt{[n-N_0]_q!}\right)^2, \text{ for all } n > N_0.$$

Define a sequence $\{a_n\}$ of real numbers as follows:

$$a_n = \begin{cases} 1, & \text{if } 0 \le n \le N_0, \\ \dfrac{1}{[n]_q!}|q|^{2n}\left(\dfrac{[n]_{|q|}!}{[n-(k-p)]_{|q|}!}\sqrt{[n-N_0]_q!}\right)^2, & \text{otherwise.} \end{cases}$$

Note that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = |q|^2 < 1$. Consequently, by ratio test $\sum_{n\ge1} a_n < \infty$. Since the sequence $\{a_n\}$ eventually dominates the tail of the sequence $\left\{\frac{1}{[n]_{q!}}\|\zeta_n\|_q^2\right\}$ modulo a scalar multiple, the proof is complete.

Thus, we have the following results.

Theorem (5.2.18)[222]: Let $t \mapsto U_t$ be a strongly continuous orthogonal representation of $\mathbb{R}$ on a real Hilbert space $\mathcal{H}_\text{R}$ with $\dim(\mathcal{H}_\mathbb{R}) \ge 2$. Let $\xi_0 \in \mathcal{H}_\mathbb{R}$ be a unit vector such that

$U_t\xi_0 = \xi_0$ for all $t \in \mathbb{R}$. Then, M_{ξ_0} is a \varphi-strongly mixing MASA in M_q whose left-right measure is Lebesgue absolutely continuous.

Proof. In this proof, we repeatedly use Eq. (8), the right multiplication of elements of M_{ξ_0} from [229] and the fact that the analytic extension of (σ_t^φ) is algebraic on the analytic elements of M_q. Fix $m, p \in \mathbb{N}$. Note that if $\xi_{i_1}, \cdots, \xi_{i_m} \in \mathcal{O}$ and $\xi_{j_1}, \cdots, \xi_{j_p} \in \mathcal{O}$, and $x = s_q(\xi_{i_1} \otimes \cdots \otimes \xi_{i_m})$ and $y = s_q\left(A^{-\frac{1}{2}}\xi_{j_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{j_p}\right)$ be such that $\mathbb{E}_{\xi_0}(x) = 0 = \mathbb{E}_{\xi_0}(y)$, then by Theorem (5.2.17) it follows that $T_{x,y}, T_{x^*,y}$ are Hilbert-Schmidt operators. Consequently, letting $a = \frac{2}{\sqrt{1-q}}$, there exists $f \in L^2(\nu_q \otimes \nu_q)$ such that for all $n, k \geq 0$ one has

$$
\begin{aligned}
&\int_{-a}^{a}\int_{-a}^{a} H_n^q(t)H_k^q(s)f(t,s)d\nu_q(t)d\nu_q(s) \\
&= \left\langle s_q(\xi_{i_1} \otimes \cdots \otimes \xi_{i_m})\Omega, H_n^q\left(s_q(\xi_0)\right) s_q\left(A^{-\frac{1}{2}}\xi_{j_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{j_p}\right) H_k^q\left(s_q(\xi_0)\right)\Omega\right\rangle_q \\
&= \left\langle s_q(\xi_{i_1} \otimes \cdots \otimes \xi_{i_m})\Omega, H_n^q\left(s_q(\xi_0)\right) s_q\left(A^{-\frac{1}{2}}\xi_{j_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{j_p}\right) JH_k^q\left(s_q(\xi_0)\right)J\Omega\right\rangle_q \\
&= \left\langle s_q(\xi_{i_1} \otimes \cdots \otimes \xi_{i_m})\Omega, H_n^q\left(s_q(\xi_0)\right) JH_k^q\left(s_q(\xi_0)\right)Js_q\left(A^{-\frac{1}{2}}\xi_{j_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{j_p}\right)\Omega\right\rangle_q \\
&= \left\langle H_n^q\left(s_q(\xi_0)\right) s_q(\xi_{i_1} \otimes \cdots \otimes \xi_{i_m}) H_k^q\left(s_q(\xi_0)\right)\Omega, s_q\left(A^{-\frac{1}{2}}\xi_{j_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{j_p}\right)\Omega\right\rangle_q \\
&= \left\langle H_n^q\left(s_q(\xi_0)\right) s_q(\xi_{i_1} \otimes \cdots \otimes \xi_{i_m}) H_k^q\left(s_q(\xi_0)\right)\Omega, \Delta^{\frac{1}{2}}\left(\xi_{j_1} \otimes \cdots \otimes \xi_{j_p}\right)\right\rangle_q (by\ Eq.(6)) \\
&= \left\langle \Delta^{\frac{1}{4}}\left(H_n^q\left(s_q(\xi_0)\right) s_q(\xi_{i_1} \otimes \cdots \otimes \xi_{i_m}) H_k^q\left(s_q(\xi_0)\right)\right)\Omega, \Delta^{\frac{1}{4}}\left(\xi_{j_1} \otimes \cdots \otimes \xi_{j_p}\right)\right\rangle_q \\
&= \left\langle \sigma_{-\frac{i}{4}}^\varphi\left(H_n^q\left(s_q(\xi_0)\right) s_q(\xi_{i_1} \otimes \cdots \otimes \xi_{i_m}) H_k^q\left(s_q(\xi_0)\right)\right)\Omega, \Delta^{\frac{1}{4}}\left(\xi_{j_1} \otimes \cdots \otimes \xi_{j_p}\right)\right\rangle_q \\
&= \left\langle H_n^q\left(s_q(\xi_0)\right) \sigma_{-\frac{i}{4}}^\varphi\left(s_q(\xi_{i_1} \otimes \cdots \otimes \xi_{i_m})\right) H_k^q\left(s_q(\xi_0)\right)\Omega, \Delta^{\frac{1}{4}}\left(\xi_{j_1} \otimes \cdots \otimes \xi_{j_p}\right)\right\rangle_q \\
&\quad (\text{ as } s_q(\xi_0) \in M_q^\varphi) \\
&= \left\langle \sigma_{-\frac{i}{4}}^\varphi\left(s_q(\xi_{i_1} \otimes \cdots \otimes \xi_{i_m})\right)\Omega, H_n^q\left(s_q(\xi_0)\right)\left(\sigma_{-\frac{i}{4}}^\varphi\left(s_q\left(\xi_{j_1} \otimes \cdots \otimes \xi_{j_p}\right)\right) H_k^q\left(s_q(\xi_0)\right)\Omega\right\rangle_q
\end{aligned}
$$

From the above argument, it follows that $T_{z^*,w}$ is also a Hilbert-Schmidt operator, where $z = \sigma_{-\frac{i}{4}}^\varphi\left(s_q(\xi_{i_1} \otimes \cdots \otimes \xi_{i_m})\right)$ and $w = \sigma_{-\frac{i}{4}}^\varphi\left(s_q\left(\xi_{j_1} \otimes \cdots \otimes \xi_{j_p}\right)\right)$, as it is an integral operator given by a square integrable kernel.

Now use the discussion preceding Theorem (5.2.17), Eq. (8) and the fact that the complex span of

$$\left\{\sigma_{-\frac{i}{4}}^\varphi\left(s_q(\xi_{i_1} \otimes \cdots \otimes \xi_{i_m})\right) : \xi_{i_j} \in \mathcal{O}, 1 \leq j \leq m, \xi_{i_j} \neq \xi_0 \text{ for at least one } \xi_{i_j}, m \in \mathbb{N}\right\}$$

is dense in $\mathcal{F}_q(\mathcal{H}) \ominus L^2(M_{\xi_0}, \varphi)$ to conclude that the left-right measure of M_{ξ_0} is Lebesgue absolutely continuous. The rest is immediate from Theorem (5.2.16).

The results obtained so far can thus be summarized as follows.

Corollary (5.2.19)[222]: Let $t \mapsto U_t$ be a strongly continuous orthogonal representation of $\mathbb{R}$ on a real Hilbert space $\mathcal{H}_\mathbb{R}$ with $\dim(\mathcal{H}_\mathbb{R}) \geq 2$. Let $\xi_0 \in \mathcal{H}_\mathbb{R}$ be a unit vector. Then the following are equivalent:

1. $s_q(\xi_0) \in M_q^\varphi$;
2. $U_t\xi_0 = \xi_0$, for all $t \in \mathbb{R}$;
3. there exists a faithful normal conditional expectation $\mathbb{E}_{\xi_0}: M_q \to M_{\xi_0}$ such that $\varphi\left(\mathbb{E}_{\xi_0}(x)\right) = \varphi(x)$ for all $x \in M_q$;
4. M_{ξ_0} is a φ-strongly mixing MASA in M_q.

Proof. The conditions in the statement are equivalent from Theorem (5.2.13) and Theorem (5.2.18).

Hiai proved that if the almost periodic part of the orthogonal representation is infinite dimensional, then the centralizer M_q^φ has trivial relative commutant, i.e., $\left(M_q^\varphi\right)' \cap M_q = \mathrm{C}1$ (Thm. 3.2 [123]). Now we show that the same result is true under a weaker hypothesis as well.

Corollary (5.2.20)[222]: Let $t \mapsto U_t$ be a strongly continuous orthogonal representation of $\mathbb{R}$ on a real Hilbert space $\mathcal{H}_\mathbb{R}$ with $\dim(\mathcal{H}_\mathbb{R}) \geq 2$. Suppose there exist unit vectors $\xi_i \in \mathcal{H}_\mathbb{R}$ such that $U_t\xi_i = \xi_i, i = 1,2$, for all $t \in \mathbb{R}$, and $\langle \xi_1, \xi_2 \rangle_U = 0$. Then, $\left(M_q^\varphi\right)' \cap M_q = \mathbb{C}1$.

Proof. By Theorem (5.2.13) and Theorem (5.2.18), it follows that $M_{\xi_i} \subseteq M_q^\varphi$ is a masa in M_q for $i = 1,2$. Let $x \in \left(M_q^\varphi\right)' \cap M_q$. Then $x \in M_{\xi_1} \cap M_{\xi_2}$ and hence $x\Omega \in \overline{\operatorname{span} \mathcal{E}_{\xi_1}}^{\|\cdot\|_q} \cap \overline{\operatorname{span} \mathcal{E}_{\xi_2}}^{\|\cdot\|_q}$ from Lemma (5.2.12). But from Eq. (4), it follows that $\overline{\operatorname{span} \mathcal{E}_{\xi_1}}^{\|\cdot\|_q} \cap \overline{\operatorname{span} \mathcal{E}_{\xi_2}}^{\|\cdot\|_q} = \mathbb{C}\Omega$. As Ω is a separating vector for M_q the result follows.

We extend the previous efforts to decide the factoriality of M_q. We establish that M_q is a factor when $\dim(\mathcal{H}_\mathbb{R}) \geq 2$ and (U_t) is not ergodic or has a nontrivial weakly mixing component.

Our approach to prove factoriality is fundamentally along the lines of Éric Ricard [137]. As discussed the approach is to use ideas coming from Ergodic theory, namely, strong mixing. Our idea stems from the following observation. If a finite von Neumann algebra contains a diffuse MASA for which the orthocomplement of the associated Jones' projection is a coarse bimodule, then the von Neumann algebra must be a factor [225]. But for the MASA M_{ξ_0}, instead of showing that the orthocomplement of the Jones' projection is a coarse bimodule over M_{ξ_0}, we only settled with absolute continuity in Theorem (5.2.17) and Theorem (5.2.18) to avoid cumbersome calculations. We use the fact that M_{ξ_0} is a masa in M_q as obtained, to decide factoriality of M_q in the case when (U_t) has a non-trivial fixed vector.

The arguments needed to prove factoriality of M_q are divided into two cases, one dealing with the discrete part of the spectrum of A corresponding to the eigenvalue 1 and the other dealing with the continuous part of the spectrum.

Definition (5.2.21)[222]: A strongly continuous orthogonal representation (V_t) of $\mathbb{R}$ on a real Hilbert space $\mathcal{K}_\mathbb{R}$ is said to be weakly mixing if for any two nonzero vectors $\xi, \eta \in \mathcal{K}_\mathbb{R}$ one has

$$\lim_{T} \frac{1}{2T} \int_{-T}^{T} |\langle V_t \xi, \eta \rangle|^2 dt \to 0 \text{ as } T \to \infty.$$

Theorem (5.2.22)[222]: Let $t \mapsto U_t$ be a strongly continuous orthogonal representation of $\mathbb{R}$ on a real Hilbert space $\mathcal{H}_{\mathbb{R}}$. Suppose that the invariant subspace of weakly mixing vectors in $\mathcal{H}_{\mathbb{R}}$ is non-trivial. Then M_q is a factor.

Proof. Decompose $\mathcal{H}_{\mathbb{R}} = \mathcal{H}_c \oplus \mathcal{H}_{wm}$ (direct sum taken in $\langle \cdot,\cdot \rangle_{\mathcal{H}_c}$), where $\mathcal{H}_c$ and $\mathcal{H}_{wm}$ are closed invariant subspaces of the orthogonal representation consisting of compact and weakly mixing vectors respectively. First of all note that $\mathcal{H}_{\mathbb{R}}$ is infinite dimensional as $\mathcal{H}_{wm} \neq 0$. If $\mathcal{H}_c = 0$, then by Eq. (9) and Theorem (5.2.11), (σ_t^{φ}) acts ergodically on M_q. Consequently, M_q is a III_1 factor [187]. Note that this was also proved in [123].)

Let $\mathcal{H}_c \neq 0$. Then M_q^{φ} is non-trivial from Theorem (5.2.11). Let $\xi \in \mathcal{H}_{wm}$ be a unit analytic vector (see Proposition (5.2.6)). Note that $\mathcal{Z}(M_q) \subseteq M_q^{\varphi}$. Borrowing notations from Theorem (5.2.11) and the discussion preceding it, we have the following. For $\zeta_{i_j} \in \mathcal{S}, 1 \leq i_j \leq N_1 + 2N_2$ (or $1 \leq i_j < N_1 + 2N_2$ as the case may be) for $1 \leq j \leq n$ and $\prod_{j=1}^{n} \beta_{i_j} = 1$, note that $\zeta_{i_1} \otimes \cdots \otimes \zeta_{i_n} \in M_q^{\varphi}\Omega$. Note that the real and imaginary parts of ζ_{i_j} are analytic and individually orthogonal to ξ with respect to $\langle \cdot,\cdot, \rangle_U$ and $\langle \cdot,\cdot \rangle_{\mathcal{H}_{\mathbb{C}}}$ for all $1 \leq j \leq n$. Then, decomposing vectors into real and imaginary parts and using Eq. (7) and Lemma (5.2.9), it follows that $s_q(\xi) s_q(\zeta_{i_1} \otimes \cdots \otimes \zeta_{i_n})\Omega = \xi \otimes \zeta_{i_1} \otimes \cdots \otimes \zeta_{i_n}$, while $s_q(\zeta_{i_1} \otimes \cdots \otimes \zeta_{i_n}) s_q(\xi)\Omega = \zeta_{i_1} \otimes \cdots \otimes \zeta_{i_n} \otimes \xi$. This observation forces that if $a \in \mathcal{Z}(M_q)$ then $s_q(\xi) a\Omega = \xi \otimes a\Omega$, while $a s_q(\xi)\Omega = a\Omega \otimes \xi$.

Indeed, as $a \in M_q^{\varphi}$, so by Theorem (5.2.11) there is a sequence $\{s_l\}$ of linear combinations of elements of the form $s_q(\zeta_{i_1} \otimes \cdots \otimes \zeta_{i_n})$ (as before) such that $s_l \to a$ in s.o.t as $l \to \infty$. So $s_l\Omega \to a\Omega$ in $\|\cdot\|_q$ and thus $s_q(\xi) s_l \Omega \to s_q(\xi) a\Omega$ in $\|\cdot\|_q$. But $s_q(\xi) s_l\Omega = \xi \otimes s_l\Omega$ for l and $c_q(\xi)$ being continuous, it follows that $\xi \otimes s_l\Omega \to \xi \otimes a\Omega$. This proves $s_q(\xi) a\Omega = \xi \otimes a\Omega$.

A symmetric argument using the continuity of $r_q(\xi)$ proves that $a s_q(\xi)\Omega = a\Omega \otimes \xi$. Thus, $s_q(\xi)$ cannot commute with a unless a is a scalar multiple of 1, as Ω is a separating vector for M_q. This completes the argument.

Theorem (5.2.23)[222]: Let $\mathcal{H}_{\mathbb{R}}$ be a real Hilbert space with $\dim(\mathcal{H}_{\mathbb{R}}) \geq 2$. Let $t \mapsto U_t$ be a strongly continuous orthogonal representation of $\mathbb{R}$ on $\mathcal{H}_{\mathbb{R}}$. Suppose there exists a unit vector $\xi_0 \in \mathcal{H}_{\mathbb{R}}$ such that $U_t \xi_0 = \xi_0$ for all $t \in \mathcal{H}_{\mathbb{R}}$. Then M_q is a factor.

Proof. Let $x \in \mathcal{Z}(M_q)$. We will show that x is a scalar multiple of 1. By Theorem (5.2.18), $M_{\xi_0} \subseteq M_q$ is a diffuse masa with a unique φ-preserving faithful normal conditional expectation. Thus, $\mathcal{Z}(M_q) \subseteq M_{\xi_0}$ and hence $x \in M_{\xi_0}$. As seen in the proof of Lemma (5.2.12), $H_n^q\left(s_q(\xi_0)\right)\Omega = \xi_0^{\otimes n}$ for all $n \geq 0$. Consequently, $x\Omega \in \overline{\operatorname{span} \mathcal{E}_{\xi_0}}^{\|\cdot\|_q}$ from Lemma (5.2.12) and hence

$$x\Omega = \sum_{n=0}^{\infty} a_n \xi_0^{\otimes n} = \sum_{n=0}^{\infty} a_n H_n^q\left(s_q(\xi_0)\right)\Omega, a_n \in \mathbb{C},$$

where the series converges in $\|\cdot\|_q$.

Since $\dim(\mathcal{H}_{\mathrm{R}}) \geq 2$, so there exists an analytic vector $\xi_1 \in \mathcal{H}_{\mathbb{R}}$ (see Proposition (5.2.6)) such that $\langle \xi_0, \xi_1 \rangle_{\mathcal{H}_C} = 0$. Hence, from Eq. (7) and Lemma (5.2.7) it follows that

$$\begin{aligned} s_q(\xi_1)x\Omega &= \sum_{n=0}^{\infty} a_n s_q(\xi_1) H_n^q\left(s_q(\xi_0)\right)\Omega \\ &= \sum_{n=0}^{\infty} a_n s_q(\xi_1)\xi_0^{\otimes n} = \sum_{n=0}^{\infty} a_n(\xi_1 \otimes \xi_0^{\otimes n}). \end{aligned}$$

Again, from Eq. (7), $H_n^q\left(s_q(\xi_0)\right)s_q(\xi_1)\Omega = \xi_0^{\otimes n} \otimes \xi_1$ for all $n \geq 0$. To see this, we use induction. For $n = 0$, the conclusion is obvious, and for $n = 1$ the same follows from Lemma (5.2.7). Assume that the result is true for $k = 0,1,\cdots,n$. Note that the q-Hermite polynomials obey the following recurrence relations:

$H_0^q(x) = 1, H_1^q(x) = x$ and

$xH_n^q(x) = H_{n+1}^q(x) + [n]_q H_{n-1}^q(x), n \geq 1, x \in \left[-\frac{2}{\sqrt{1-q}}, \frac{2}{\sqrt{1-q}}\right]$ [60] Defn. 1.9.

Thus, by functional calculus one has

$$\begin{aligned} H_{n+1}^q\left(s_q(\xi_0)\right)\xi_1 &= s_q(\xi_0)H_n^q\left(s_q(\xi_0)\right)\xi_1 - [n]_q H_{n-1}^q\left(s_q(\xi_0)\right)\xi_1 \\ &= s_q(\xi_0)\left(\xi_0^{\otimes n} \otimes \xi_1\right) - [n]_q\left(\xi_0^{\otimes(n-1)} \otimes \xi_1\right) \\ &= \xi_0^{\otimes(n+1)} \otimes \xi_1, \text{by Eq. (7) and Lemma (5.2.8).} \end{aligned}$$

Thus, by induction the above conclusion follows. (This can also be proved by Lemma (5.2.7) and Lemma (5.2.9)).

Note that x is a limit in s.o.t. of a sequence of operators from the linear span of $\left\{H_n^q\left(s_q(\xi_0)\right): n \geq 0\right\}$. Consequently, $xs_q(\xi_1)\Omega \in \overline{\operatorname{span}\left\{\xi_0^{\otimes n} \otimes \xi_1 : n \geq 0\right\}}^{\|\cdot\|_q}$. Therefore, $xs_q(\xi_1) = s_q(\xi_1)x$ forces that $a_n = 0$ for all $n \geq 1$. Thus, $x\Omega = a_0\Omega$ and hence $x = a_0 1$ as Ω is separating for M_q. So the proof is complete.

We discuss the factoriality of the centralizer M_q^{φ} of the q-deformed Araki-Woods von Neumann algebra M_q. By Theorem (5.2.18), it follows that if the point spectrum of the analytic generator A of (U_t) is $\{1\}$ and is of simple multiplicity, then M_q^{φ} is a masa in M_q. Thus, for the centralizer to be large, the almost periodic part of (U_t) need to be reasonably large.

For a short account on bicentralizers that follows, see [178]. Let M be a separable type III_1 factor and let ψ be a faithful normal state on M. Denote $[x,y] = xy - yx$ and $[x,\psi] = x\psi - \psi x$ for $x, y \in M$. The asymptotic centralizer of ψ is defined to be

$$\mathrm{AC}_{\psi} = \left\{(x_n) \in \ell^{\infty}(\mathbb{N}, M) : \|[x_n, \psi]\| \to 0 \text{ as } n \to \infty\right\}.$$

Observe that AC_{ψ} is a unital C^*-subalgebra of $\ell^{\infty}(\mathbb{N}, M)$. The bicentralizer of ψ is defined by

$$B_{\psi} = \left\{y \in M : [y, x_n] \to 0 \text{ ultrastrongly as } n \to \infty \text{ for all } (x_n) \in \mathrm{AC}_{\psi}\right\}.$$

Note that B_{ψ} is a von Neumann subalgebra of M which is globally invariant with respect to the modular automorphism group $\left(\sigma_t^{\psi}\right)$. Further, $B_{\psi} \subseteq \left(M^{\psi}\right)' \cap M$. The type III_1 factor M is said to have trivial bicentralizer if $B_{\psi} = \mathbb{C}1$ for any faithful normal state ψ of M. The bicentralizer problem of Connes is open and asks if every separable type III_1 factor has trivial bicentralizer.

Theorem (5.2.24)[222]: Let $\mathcal{H}_{\mathbb{R}}$ be a real Hilbert space such that $\dim(\mathcal{H}_{\mathbb{R}}) \geq 2$. Let (U_t) be a strongly continuous real orthogonal representation of $\mathbb{R}$ on $\mathcal{H}_{\mathbb{R}}$ such that:

1. there exists a unit vector $\xi_0 \in \mathcal{H}_{\mathbb{R}}$ satisfying $U_t\xi_0 = \xi_0$ for all $t \in \mathbb{R}$,
2. the almost periodic part of (U_t) is at least two dimensional.

Then,

$$\left(M_q^{\varphi}\right)' \cap M_q = \mathbb{C}1.$$

In particular, the centralizer M_q^{φ} of M_q is a factor. Moreover, if M_q is a III_1 factor then it has trivial bicentralizer.

Proof. Under the stated hypothesis, if the almost periodic part of (U_t) is two dimensional then (U_t) admits two orthogonal invariant vectors. Then the result follows directly from Corollary (5.2.20). In the remaining case the argument is as follows.

First of all, note that from Corollary (5.2.19), the von Neumann algebra $M_{\xi_0} = vN\left(s_q(\xi_0)\right) \subseteq M_q^{\varphi}$ is a MASA in M_q with a unique φ-preserving faithful normal conditional expectation $\mathbb{E}_{\xi_0}: M_q \to M_{\xi_0}$. Therefore, $\left(M_q^{\varphi}\right)' \cap M_q \subseteq M_{\xi_0}$. Let $x \in \left(M_q^{\varphi}\right)' \cap M_q$.

Since the dimension of the almost periodic part of (U_t) is at least two, so from Theorem (5.2.11), it follows that there exist vectors $\zeta_i \in \mathcal{H}_C$ (with real and imaginary parts individually analytic), $1 \leq i \leq k$, such that $\zeta_1 \otimes \cdots \otimes \zeta_k \in M_q^{\varphi}\Omega$ and ζ_i and as well as its real and imaginary parts are orthogonal to ξ_0 for all $1 \leq i \leq k$, with respect to $\langle\cdot,\cdot\rangle_{\mathcal{H}_{\mathbb{C}}}$ (as well as orthogonal in $\langle\cdot,\cdot\rangle_U$, as $\dim(\mathcal{H}_{\mathbb{R}}) \geq 2$). Let $y = s_q(\zeta_1 \otimes \cdots \otimes \zeta_k) \in M_q^{\varphi}$.

As seen in the proof of Lemma (5.2.12), $H_n^q\left(s_q(\xi_0)\right)\Omega = \xi_0^{\otimes n}$ for all $n \geq 0$. Consequently, $x\Omega \in \overline{\operatorname{span} \mathcal{E}_{\xi_0}}^{\|\cdot\|_q}$ from Lemma (5.2.12) and hence,

$$x\Omega = \sum_{n=0}^{\infty} a_n \xi_0^{\otimes n} = \sum_{n=0}^{\infty} a_n H_n^q\left(s_q(\xi_0)\right)\Omega, a_n \in \mathbb{C},$$

where the series converges in $\|\cdot\|_q$. Moreover, decomposing vectors into real and imaginary parts and using Lemma (5.2.9), it follows that

$$yx\Omega \in \overline{\operatorname{span}\left\{\zeta_1 \otimes \cdots \otimes \zeta_k \otimes \xi_0^{\otimes n} : n \geq 0\right\}}^{\|\cdot\|_q}$$

Further, decomposing vectors into real and imaginary parts and using Eq. (5) and Lemma (5.2.7) it follows that $s_q(\xi_0)(\zeta_1 \otimes \cdots \otimes \zeta_k) = \xi_0 \otimes \zeta_1 \otimes \cdots \otimes \zeta_k$. Assume that $s_q\left(H_m^q(\xi_0)\right)(\zeta_1 \otimes \cdots \otimes \zeta_k) = \xi_0^{\otimes m} \otimes \zeta_1 \otimes \cdots \otimes \zeta_k$, for $m = 0,1,\cdots,n$. Using the recurrence relations of q-Hermite polynomials (as in the proof of Theorem (5.2.23)), Eq. (7), Lemma (5.2.7) and the induction hypothesis, it follows that $H_n^q\left(s_q(\xi_0)\right)(\zeta_1 \otimes \cdots \otimes \zeta_k) = \xi_0^{\otimes n} \otimes \zeta_1 \otimes \cdots \otimes \zeta_k$, for all $n \geq 0$. Now note that

$$\begin{aligned} xy\Omega &= Jy^*Jx\Omega = \sum_{n=0}^{\infty} a_n Jy^*JH_n^q\left(s_q(\xi_0)\right)\Omega = \sum_{n=0}^{\infty} a_n H_n^q\left(s_q(\xi_0)\right)y\Omega \\ &= \sum_{n=0}^{\infty} a_n \left(\xi_0^{\otimes n} \otimes (\zeta_1 \otimes \cdots \otimes \zeta_k)\right). \end{aligned}$$

Since, $xy = yx$, so $a_n = 0$ for all $n \neq 0$. Thus, the first statement follows.

The final statement is a direct consequence of Connes-Størmer transitivity theorem [9]. (This can also be deduced from the fact that $B_\varphi \subseteq \left(M_q^\varphi\right)' \cap M_q.$)

We describe the type of M_q under the same constraints by showing that the type depends on the spectral information of A as expected. To begin with, we recall some well known facts about the S invariant of Connes.

The S invariant of a factor M was defined in [66] to be the intersection over all faithful normal semifinite (f.n.s.) weights ϕ of the spectra of the associated modular operators Δ_ϕ. Further, M is a type III factor if and only if $0 \in S(M)$ and in this case Connes classified type III factors using their S invariant as follows:

$$S(M) = \begin{cases} [0,\infty), & \text{if } M \text{ is type III}_1, \\ \{0,1\}, & \text{if } M \text{ is type III}_0, \\ \{\lambda^n : n \in \mathbb{Z}\} \cup \{0\}, & \text{if } M \text{ is type III}_\lambda, 0 < \lambda < 1. \end{cases}$$

Also, recall from [66] that for a fixed faithful normal state (resp. f.n.s. weight) ϕ on M, the S invariant can be written as

$$S(M) = \cap \left\{ \operatorname{Sp}\left(\Delta_{\phi_p}\right) : 0 \neq p \in \mathcal{P}\left(\mathcal{Z}\left(M^\phi\right)\right) \right\},$$

$\mathcal{P}\left(\mathcal{Z}\left(M^\phi\right)\right)$ denoting the lattice of projections in the center of the centralizer M^ϕ and $\phi_p = \phi_{|pMp}$. So, let ϕ be a faithful normal state on M and let $0 \neq p \in M^\phi$ be a projection. Let Δ_{ϕ_p} and $\left(\sigma_t^{\phi_p}\right)$ respectively denote the modular operator and the modular automorphism group of the corner pMp associated to the positive functional ϕ_p. When $p = 1$, write Δ_{ϕ_1} and $\left(\sigma_t^{\phi_1}\right)$ respectively as Δ_ϕ and $\left(\sigma_t^\phi\right)$. It is clear that $\sigma_t^{\phi_p}(pxp) = p\sigma_t^\phi(x)p$ for all $x \in M$ and $t \in \mathbb{R}$. It is also easy to check that $\sigma_t^{\phi_p}$ is implemented by $\Delta_{\phi_p}^{it} = p\Delta_\phi^{it}p$ for all $t \in \mathbb{R}$.

Theorem (5.2.25)[222]: Let (U_t) be a strongly continuous real orthogonal representation of $\mathbb{R}$ on a real Hilbert space $\mathcal{H}_\mathbb{R}$ such that the weakly mixing component of (U_t) is non-trivial. Then M_q is a type III_1 factor.

Proof. Recall the definition of weak mixing from Definition (5.2.21). By the hypothesis it follows that $\mathcal{H}_\mathbb{R}$ is infinite dimensional. We need to show that $S\left(M_q\right) = [0,\infty)$. So, let $0 \neq p \in \mathcal{P}\left(\mathcal{Z}\left(M_q^\varphi\right)\right)$. By the hypothesis and Proposition (5.2.2), there exists $0 \neq \xi \in \mathcal{H}_\mathbb{R} \subseteq \mathcal{H}_\mathbb{C} \subseteq \mathcal{F}_q(\mathcal{H})$ such that

$$\frac{1}{2T}\int_{-T}^{T} |\langle U_t\xi, \xi\rangle_U|^2 dt \to 0, \text{ as } T \to \infty \left(\text{ see Eq. (3)}\right).$$

Thus, by Eq. (4), Eq. (8), Eq. (9) and the discussion following it, one has

$$\frac{1}{2T}\int_{-T}^{T} \left|\langle \mathcal{F}(U_t)\xi, \xi\rangle_q\right|^2 dt \to 0, \text{ as } T \to \infty.$$

Consequently, if μ_ξ denotes the elementary spectral measure (on $\mathbb{R}$) associated to ξ of the representation $\{t \mapsto \mathcal{F}(U_t) : t \in \mathbb{R}\}$, then μ_ξ is non-atomic (from Eq. (9)).

If $p \neq 1$, note that $p\xi, (1-p)\xi$ are non-zero vectors. Indeed, if $\zeta \in M_q^\varphi\Omega$ is such that $s_q(\zeta) = p$ (see Eq. (14)), then by Theorem (5.2.11) (as in the proof of Theorem (5.2.22)), it follows that $p\xi = \zeta \otimes \xi \neq 0$. Similar is the argument for $(1-p)\xi$. Let $\mu_p\xi, \mu_{(1-p)\xi}$

respectively denote the elementary spectral measures of $\{t \mapsto \mathcal{F}(U_t): t \in \mathbb{R}\}$ associated to the vectors $p\xi$ and $(1-p)\xi$. Note that $\mu_{p\xi}$ is the elementary spectral measure of $t \mapsto p\Delta^{it}p \left(= \Delta_{\varphi_p}^{it}\right), t \in \mathbb{R}$, corresponding to the vector $p\xi$, and the former implements $\left(\sigma_t^{\varphi_p}\right)$. Also, as $p \in M_q^{\varphi}$, so the range of p is an invariant subspace of $\{\mathcal{F}(U_t): t \in \mathbb{R}\}$. Hence,

$$\langle \mathcal{F}(U_t)p\xi, (1-p)\xi \rangle_q = 0, \text{ for all } t \in \mathbb{R}.$$

Consequently, $\mu_{\xi} = \mu_{p\xi} + \mu_{(1-p)\xi}$, thus $\mu_{p\xi}$ and $\mu_{(1-p)\xi}$ are both non-atomic.

Note that the weakly mixing component of $\{t \mapsto \mathcal{F}(U_t): t \in \mathbb{R}\}$ is invariant under the anti-unitary J. This follows by using the fact that $J\Delta^{it}J = \Delta^{it}$ for all $t \in \mathbb{R}$ and by the definition of weak mixing. Thus, $\mu_{pJ\xi}$ is non-zero and non-atomic. Note that both ξ and $J\xi$ are vectors in the 1-particle space $\mathcal{H}$ of $\mathcal{F}_q(\mathcal{H})$. This forces that the spectral measure of the action $\{t \mapsto \mathcal{F}(U_t): t \in \mathbb{R}\}$ when restricted to the 1-particle space $\mathcal{H}$ contains a non-trivial non-atomic component μ on both sides of 0 by an application of the Stone-Weierstrass theorem. Since, $\mathcal{F}(U_t) = id \oplus \bigoplus_{n\geq 1} U_t^{\otimes_q n}, t \in \mathbb{R}$, it follows that $\operatorname{Sp}\left(\Delta_{\varphi_p}\right) = [0, \infty)$. Thus, the result follows.

Now we turn to the case when the orthogonal representation is almost periodic.

Theorem (5.2.26)[222]: Let (U_t) be a strongly continuous almost periodic orthogonal representation of $\mathbb{R}$ on a real Hilbert space $\mathcal{H}_{\mathbb{R}}$ such that $\dim(\mathcal{H}_{\mathbb{R}}) \geq 2$ and such that there exists a unit vector $\xi_0 \in \mathcal{H}_{\mathbb{R}}$ with $U_t\xi_0 = \xi_0$ for all $t \in \mathbb{R}$. Let G be the closed subgroup of $\mathbb{R}_+^{\times}$generated by the spectrum of A. Then,

$$M_q \text{ is } \begin{cases} \text{type III}_1 & \text{if } G = \mathbb{R}_+^{\times}, \\ \text{type III}_\lambda & \text{if } G = \lambda^{\mathbb{Z}}, 0 < \lambda < 1, \\ \text{type II}_1 & \text{if } G = \{1\}. \end{cases}$$

The type II_1 case corresponds to $(U_t) = (id)$ and thus M_q is the Bożejko-Speicher's II_1 factor.

Proof. The hypothesis forces that if $\dim(\mathcal{H}_{\mathbb{R}}) = 2$, then M_q is a II_1 factor from Corollary (5.2.20) and there is nothing to prove. If $\dim(\mathcal{H}_{\mathbb{R}}) \geq 3$, then by Theorem (5.2.24) it follows that $\left(M_q^{\varphi}\right)' \cap M_q = \mathbb{C}1$. Thus, M_q^{φ} is a factor, and hence $S(M_q)$ is completely determined by $\operatorname{Sp}(\Delta)$. Now use the fact that $\mathcal{F}(U_t) = id \oplus \bigoplus_{n\geq 1} U_t^{\otimes_q n}, t \in \mathbb{R}$, and Proposition (5.2.10) to complete the proof.

Chapter 6
Free and q-Araki-Woods Algebras and Factors

We show that all q-Araki-Woods algebras possess the Haagerup approximation property. We show that any amenable von Neumann subalgebra of any free Araki–Woods factor that is globally invariant under the modular automorphism group of the free quasi-free state is necessarily contained in the almost periodic free summand. We show that the canonical ultraweakly dense C^*-subalgebras of q-Araki-Woods algebras are always QWEP.

Section (6.1): Extension of Second Quantisation and Haagerup Approximation Property

The Haagerup approximation property, along with amenability and weak amenability, started its life as an approximation property of (discrete) groups, although it was always intimately connected with operator algebras, beginning from its first appearance in [153]. This connection was further developed by Choda (cf. [241]), who defined the respective property for tracial von Neumann algebras and proved that a group von Neumann algebra (of a discrete group) possesses the Haagerup property if and only if so does the underlying group. The situation in the general locally compact case is, however, not that pleasant. It resembles the situation with amenability - injectivty of group von Neumann algebra captures amenability of the group in the discrete case, but not in general.

Ever since the advent of locally compact quantum groups and their approximation properties (cf. [243]), it has become crucial to extend many notions beyond the case of finite von Neumann algebras. As in the classical case, there is no hope to define the Haagerup property of a general locally compact quantum group only via its von Neumann algebra. In the discrete case this should be feasible. [243] prove the proposed equivalence for unimodular discrete quantum groups. The theory of quantum groups, however, has the unusual feature allowing discrete groups to be non-unimodular. This is a clear motivation to investigate the possibility of extending the definition to the case of non-tracial von Neumann algebras. Recently, two equivalent axiomatisations of the Haagerup property of general von Neumann algebras have been established (cf. [242] and [244]).

Whenever a new property is defined, it is useful to have a host of examples to confirm that the definition is a reasonable one. We prove that a wide class of type III von Neumann algebras, the so-called q-Araki-Woods algebras introduced by Hiai in [123] (based on earlier work of Shlyakhtenko, cf. [15]), possess the Haagerup approximation property. It is a natural extension of the fact that the q-Gaussian algebras of Bożejko and Speicher (cf. [60]) possess the Haagerup property, which seems to be a folklore result. One can also view as a contribution to the study of the structure of q-Araki-Woods algebras. A lot is known about their predecessors, the q-Gaussian algebras. They are known to be factors (cf. [137]), they are non-injective (cf. [113]), they possess the completely contractive approximation property (cf. [223]). In the case of q-ArakiWoods algebras we have only partial results, e.g. a recent development in the study of factoriality (cf. [222] and [245]). The best known result about non-injectivity was obtained by Nou in ([113], Corollary 3). So far, the CCAP has been obtained only for free Araki-Woods algebras ([190]); the general case, however, is likely to require new methods. We hope that this article will prompt further study of q-Araki-Woods algebras. Let us give a brief overview. We introduce the necessary definitions and tools. We provide an extension of the second quantisation procedure, necessary for the proof of the Haagerup approximation property. The basic idea is that second quantisation allows us to build approximants on the level of the Hilbert space, which is easier than working directly on the level of the von Neumann algebra.

We recall the construction of the q-Araki-Woods algebras and the definition of the Haagerup approximation property.

The material about, q-Araki-Woods algebras that follows, with a much more detailed exposition, can be easily found in [123].

We start from a real separable Hilbert space $\mathcal{H}_{\mathbb{R}}$ equipped with a one-parameter group of orthogonal transformations $(\mathrm{U}_t)_{t\in\mathbb{R}}$. This extends to a unitary group on the complexification, denoted $\mathcal{H}\mathbb{C}$, that has the form $\mathrm{U}_t = A^{it}$ for some positive, injective operator A. We define a new inner product on $\mathcal{H}_{\mathbb{C}}$ by $\langle x, y\rangle_{\mathrm{U}} := \left\langle \frac{2A}{1+A} x, y\right\rangle$. The completion with respect to this inner product is denoted by $\mathcal{H}$. Let us denote by I the conjugation on $\mathcal{H}_{\mathbb{C}}$ −it is a closed operator on $\mathcal{H}$ because the new inner product coincides with the old one on $\mathcal{H}_{\mathbb{R}}$. Consider now the q-Fock space $\mathcal{F}_q(\mathcal{H})$ (cf. [60]).

Definition (6.1.1)[240]: For any $h \in \mathcal{H}_{\mathbb{R}}$ define $s_q(h) = a_q^*(h) + a_q(h)$, where $a_q^*(h)$ and $a_q(h)$ are the creation and annihilation operators on $\mathcal{F}_q(\mathcal{H})$. q-Araki-Woods algebra is the von Neumann algebra generated by the set of operators $\{s_q(h): h \in \mathcal{H}_{\mathbb{R}}\}$. We will denote it by $\Gamma_q(\mathcal{H})$.

There are two special cases considered previously:

1. If the the group $(\mathrm{U}_t)_{t\in\mathbb{R}}$ is trivial, i.e. $\mathrm{U}_t = \mathrm{Id}$, then we denote the algebra $\Gamma_q(\mathcal{H})$ by $\Gamma_q(\mathcal{H}_{\mathbb{R}})$ and call it a q-Gaussian algebra (cf. [60]);
2. If $q = 0$, then $\Gamma_0(\mathcal{H})$ is called a free Araki-Woods factor; they were introduced earlier by Shlyakhtenko (cf. [15])

Definition (6.1.2)[240]: Let $T: \mathcal{K} \to \mathcal{H}$ be a contraction between two Hilbert spaces. Then there exists a contraction $\mathcal{F}_q(T): \mathcal{F}_q(\mathcal{K}) \to \mathcal{F}_q(\mathcal{H})$, called first quantisation of T, which is defined on finite tensors by $\mathcal{F}_q(T)(v_1 \otimes \cdots \otimes v_n) = Tv_1 \otimes \cdots \otimes Tv_n$.

We will follow the approach of Caspers and Skalski (cf. [242]); for a different approach, based on standard forms, see [244].

Definition (6.1.3)[240]: Let (M, φ) be a von Neumann algebra (with separabla predual) equipped with a normal, faithful, semifinite weight φ. It has Haagerup approximation property if there exists a sequence of unital, normal, completely positive (unital, completely positive will be abbreviated to ucp from now on) maps $(T_k: M \to M)_{k\in\mathbb{N}}$ such that:

1. $\varphi \circ T_k \leqslant \varphi$ for all $k \in \mathbb{N}$;
2. GNS-implementations $T_k: L^2(\mathrm{M}, \varphi) \to L^2(\mathrm{M}, \varphi)$ are compact and converge to $\mathbb{1}_{L^2(M,\varphi)}$ strongly.

We will prove that second quantisation can be defined for arbitrary contractions on $\mathcal{H}_{\mathbb{R}}$ that extend to contractions on $\mathcal{H}$; this condition will be written succinctly as $ITI = T$, where the left-hand side is understood as the closure of the product. Motivation comes from [190], where the analogous generalisation of second quantisation is an indispensable tool for obtaining approximation properties in the free case. Before we give the details of the proof, let us first recall how to show that the second quantisation is always available in the case of q-Gaussian algebras so that the similarities and the differences are clearly visible (cf. [60], Theorem 2.11). Before that, we need to recall the Wick formula (cf. [60], Proposition 2.7).

Lemma (6.1.4)[240]: Suppose that $e_1, \dots, e_n \in \mathcal{H}_{\mathbb{C}}$. Then

$$W(e_1 \otimes \cdots \otimes e_n) = \sum_{k=0}^{n} \sum_{i_1,\ldots,i_k,j_{k+1},\ldots,j_n} a_q^*(e_{i_1}) \ldots a_q^*(e_{i_k}) a_q(Ie_{j_{k+1}}) \ldots a_q(Ie_{j_n}) q^{i(I_1,I_2)}, \quad (1)$$

where $I_1 = \{i_1 < \cdots < i_k\}$ and $I_2 = \{j_{k+1} < \cdots < j_n\}$ form a partition of the set $\{1, \ldots, n\}$ and $i(I_1, I_2)$ is the number of crossings between I_1 and I_2, equal to $\sum_{l=1}^{k}(i_l - l)$.

Theorem (6.1.5)[240]: ([60], Theorem 2.11). Let $\mathcal{K}_\mathbb{R}$ and $\mathcal{H}_\mathbb{R}$ be real Hilbert spaces and let $T: \mathcal{K}_\mathbb{R} \to \mathcal{H}_\mathbb{R}$ be a contraction. Then there exists a ucp map $\Gamma_q(T): \Gamma_q(\mathcal{K}_\mathbb{R}) \to \Gamma_q(\mathcal{H}_\mathbb{R})$ such that $\Gamma_q(T)W(e_1 \otimes \cdots \otimes e_n) = W(Te_1 \otimes \cdots \otimes Te_n)$ for any $e_1, \ldots, e_n \in \mathcal{K}_\mathbb{R}$. Moreover, this map preserves the vacuum state.

Proof. To prove the existence, we will first dilate T to an orthogonal transformation U_T, i.e. define $U_T = \begin{bmatrix} (\mathbb{1}_{\mathcal{K}_\mathrm{R}} - T^*T)^{\frac{1}{2}} & T^* \\ T & -(\mathbb{1}_{\mathcal{H}_\mathbb{R}} - TT^*)^{\frac{1}{2}} \end{bmatrix}$, an orthogonal operator on $\mathcal{K}_\mathbb{R} \oplus \mathcal{H}_\mathbb{R}$ such that $T = PU_{T}\iota$, where $\iota: \mathcal{K}_\mathbb{R} \to \mathcal{K}_\mathbb{R} \oplus \mathcal{H}_\mathbb{R}$ is the inclusion onto the first summand and $P: \mathcal{K}_\mathbb{R} \oplus \mathcal{H}_\mathbb{R} \to \mathcal{H}_\mathbb{R}$ is the orthogonal projection onto the second summand. We will define separately $\Gamma_q(\iota), \Gamma(U_T)$, and $\Gamma_q(P)$ and then define $\Gamma_q(T) := \Gamma_q(P)\Gamma_q(U_T)\Gamma_q(\iota)$. The maps $\Gamma_q(P)$ and $\Gamma_q(U_T)$ are easy to define, so we will start with them. We define $\Gamma_q(P)x := \mathcal{F}_q(P)x\mathcal{F}_q(P)^*$. This is a normal ucp map from $B\left(\mathcal{F}_q(\mathcal{K}_\mathbb{C} \oplus \mathcal{H}_\mathbb{C})\right)$ to $B\left(\mathcal{F}_q(\mathcal{H}_\mathbb{C})\right)$, we just have to check that it maps $W(e_1 \otimes \cdots \otimes e_n)$ to $W(Pe_1 \otimes \cdots \otimes Pe_n)$. To this end, we will use the Wick formula (1). It suffices to show that

$$\Gamma_q(P)\left(a_q^*(v_1) \ldots a_q^*(v_k) a_q(v_{k+1}) \ldots a_q(v_n)\right) = a_q^*(Pv_1) \ldots a_q^*(Pv_k) a_q(Pv_{k+1}) \ldots a_q(Pv_n).$$

We will use the fact that $a_q(v)\mathcal{F}_q(T) = \mathcal{F}_q(T)a_q(T^*v)$ and $\mathcal{F}_q(T)a_q^*(v) = a_q^*(Tv)\mathcal{F}_q(T)$. An easy application of this shows that $\mathcal{F}_q(P)a_q^*(v_1) \ldots a_q^*(v_k)a_q(v_{k+1}) \ldots a_q(v_n)\mathcal{F}_q(P)^*$ is equal to $a_q^*(Pv_1) \ldots a_q^*(Pv_k)\mathcal{F}_q(PP^*)a_q(Pv_{k+1}) \ldots a_q(Pv_n)$ and we are done, because $PP^* = \mathbb{1}_{\mathcal{H}_\mathbb{R}}$. We define $\Gamma_q(U_T)$ analogously: $\Gamma_q(U_T)x = \mathcal{F}_q(U_T)x\mathcal{F}_q(U_T)^*$. The same computation as in the case of P shows that $\Gamma_q(U_T)W(e_1 \otimes \cdots \otimes e_n) = W(U_Te_1 \otimes \cdots \otimes U_Te_n)$.

Now we have to deal with $\Gamma_q(\iota)$. Since $u^* \neq \mathbb{1}_{\mathcal{K}_\mathbb{R} \oplus \mathcal{H}_\mathbb{R}}$, the previous approach does not work. We know, however, that $\Gamma_q(\iota)$ ought to be the inclusion of $\Gamma_q(\mathcal{K}_\mathbb{R})$ onto a von Neumann subalgebra of $\Gamma_q(\mathcal{K}_\mathbb{R} \oplus \mathcal{H}_\mathbb{R})$ generated by the operators $\{s_q(v): v \in \mathcal{K}_\mathbb{R} \oplus \{0\} \subset \mathcal{K}_\mathbb{R} \oplus \mathcal{H}_\mathbb{R}\}$; denote the latter by $\Gamma_q(\mathcal{K}_\mathbb{R}, \mathcal{K}_\mathbb{R} \oplus \mathcal{H}_\mathbb{R})$. To construct $\Gamma_q(\iota)$, we will define a map from $\Gamma_q(\mathcal{K}_\mathbb{R}, \mathcal{K}_\mathbb{R} \oplus \mathcal{H}_\mathbb{R})$ onto $\Gamma_q(\mathcal{K}_\mathbb{R})$ and show that it is an injective, hence isometric, *-homomorphism, therefore it has an inverse, which will be the sought $\Gamma_q(\iota)$. So far, we have a map $\Gamma_q(\iota^*): \Gamma_q(\mathcal{K}_\mathrm{R} \oplus \mathcal{H}_\mathbb{R}) \to \Gamma_q(\mathcal{K}_\mathbb{R})$. Let us show that this map, when restricted to $\Gamma_q(\mathcal{K}_\mathbb{R}, \mathcal{K}_R \oplus \mathcal{H}_\mathbb{R})$, is a $*$-homomorphism. To show that, note that every member of the generating set of $\Gamma_q(\mathcal{K}_\mathbb{R}, \mathcal{K}_\mathbb{R} \oplus \mathcal{H}_\mathbb{R})$ preserves the subspace $\mathcal{F}_q(\mathcal{K}_\mathbb{C}) \subset \mathcal{F}_q(\mathcal{K}_\mathbb{C} \oplus \mathcal{H}_\mathbb{C})$; it follows that every element of $\Gamma_q(\mathcal{K}_\mathbb{R}, \mathcal{K}_\mathbb{R} \oplus \mathcal{H}_\mathbb{R})$ enjoys this property. Let us take two elements $x, y \in \Gamma_q(\mathcal{K}_\mathbb{R}, \mathcal{K}_\mathbb{R} \oplus \mathcal{H}_\mathbb{R})$ and compute

$$\Gamma_q(\iota^*)(xy) = \mathcal{F}_q(\iota^*)xy\mathcal{F}_q(\iota) = \mathcal{F}_q(\iota^*)x\mathcal{F}_q(\iota)\mathcal{F}_q(\iota^*)y\mathcal{F}_q(\iota),$$

where the second equality follows from the fact that $\mathcal{F}_q(\iota^*)$ is the orthogonal projection from $\mathcal{F}_q(\mathcal{K}_{\mathbb{C}} \oplus \mathcal{H}_{\mathbb{C}})$ onto $\mathcal{F}_q(\mathcal{K}_{\mathbb{C}})$ and the image of $y\mathcal{F}_q(\iota)$ is contained in $\mathcal{F}_q(\mathcal{K}_{\mathbb{C}})$. Therefore $\Gamma_q(\iota^*): \Gamma_q(\mathcal{K}_{\mathbb{R}}, \mathcal{K}_{\mathbb{R}} \oplus \mathcal{H}_{\mathbb{R}}) \to \Gamma_q(\mathcal{K}_{\mathbb{R}})$ is a *-homomo-rphism. We will check now that it is injective. Suppose then that $\Gamma_q(\iota^*)x = 0$ for some $x \in \Gamma_q(\mathcal{K}_{\mathbb{R}}, \mathcal{K}_{\mathbb{R}} \oplus \mathcal{H}_{\mathbb{R}})$. It follows that $\Gamma_q(\iota^*)x\Omega = 0$. We have $\Gamma_q(\iota^*)x\Omega = \mathcal{F}_q(\iota^*)x\mathcal{F}_q(\iota)\Omega$ and $\mathcal{F}_q(\iota)\Omega = \Omega$, seen now as the vacuum vector in $\mathcal{F}_q(\mathcal{K}_{\mathbb{C}} \oplus \mathcal{H}_{\mathbb{C}})$. But we already know that $x\Omega \in \mathcal{F}_q(\mathcal{K}_{\mathbb{C}}) \subset \mathcal{F}_q(\mathcal{K}_{\mathbb{C}} \oplus \mathcal{H}_{\mathbb{C}})$, so from $\mathcal{F}_q(\iota^*)x\Omega = 0$ we can deduce that $x\Omega = 0$, therefore $x = 0$ as Ω is a separating vector for $\Gamma_q(\mathcal{K}_{\mathbb{R}} \oplus \mathcal{H}_{\mathbb{R}})$. We proved that $\Gamma_q(\iota^*): \Gamma_q(\mathcal{K}_{\mathbb{R}}, \mathcal{H}_{\mathbb{R}} \oplus \mathcal{H}_{\mathbb{R}}) \to \Gamma_q(\mathcal{K}_{\mathbb{R}})$ is an isometric *-isomorphism, hence it has an inverse and we call this inverse $\Gamma_q(\iota)$; it is clear that $\Gamma_q(\iota)W(e_1 \otimes \cdots \otimes e_n) = W(\iota e_1 \otimes \cdots \otimes \iota e_n)$. It is easy to see that the vacuum state is preserved, so this finishes the proof.

The following extension, with almost the same proof, is due to Hiai (cf. [123]):

Proposition (6.1.6)[240]: Let $(\mathcal{K}_{\mathbb{R}}, (\mathrm{U}_t)_{t\in\mathbb{R}})$ and $(\mathcal{H}_{\mathbb{R}}, (\mathrm{V}_t)_{t\in\mathbb{R}})$ be two real Hilbert spaces equipped with one parameter groups of orthogonal transformations. Suppose that $T: \mathcal{K}_{\mathbb{R}} \to \mathcal{H}_{\mathbb{R}}$ is a contraction such that $T\mathrm{U}_t = \mathrm{V}_t T$ for all $t \in \mathbb{R}$. Then there is a normal ucp map $\Gamma_q(T): \Gamma_q(\mathcal{K}) \to \Gamma_q(\mathcal{H})$ extending $W(e_1 \otimes \cdots \otimes e_n) \mapsto W(Te_1 \otimes \cdots \otimes Te_n)$.

Proof. We decompose $T = PU_T\iota$ as previously; the exact form of this decomposition is important. We equip the space $\mathcal{K}_{\mathbb{R}} \oplus \mathcal{H}_{\mathbb{R}}$ with the orthogonal group $(\mathrm{U}_t \oplus \mathrm{V}_t)_{t\in\mathbb{R}}$. Note that the completion of $\mathcal{K}_{\mathbb{C}} \oplus \mathcal{H}_{\mathbb{C}}$ with respect to the inner product defined by $(\mathrm{U}_t \oplus \mathrm{V}_t)_{t\in\mathbb{R}}$ is naturally identified with $\mathcal{K} \oplus \mathcal{H}$. Then the three maps P, U_T, and ι intertwine the orthogonal groups and, therefore, extend to contractions between appropriate Hilbert spaces. The rest of the proof is exactly the same as previously.

We would like to state now our extension of the second quantisation (with the same minimal requirements as in [190]).

Theorem (6.1.7)[240]: Suppose that $T: \mathcal{K} \to \mathcal{H}$ is a contraction such that $T = JTI$, where I is the conjugation on $\mathcal{K}_{\mathbb{C}}$ and J is the conjugation on $\mathcal{H}_{\mathrm{C}}$. Then the assignment $W(e_1 \otimes \cdots \otimes e_n) \mapsto W(Te_1 \otimes \cdots \otimes Te_n)$ extends to a normal ucp map $\Gamma_q(T): \Gamma_q(\mathcal{K}) \to \Gamma_q(\mathcal{H})$ that preserves the vacuum state.

Proof. We start similarly as in the proof of Theorem (6.1.5); dilate T to a unitary U_T on $\mathcal{K} \oplus \mathcal{H}$ given by $\begin{bmatrix} (\mathbb{1}_{\mathcal{K}} - T^*T)^{\frac{1}{2}} & T^* \\ T & -(\mathbb{1}_{\mathcal{H}} - TT^*)^{\frac{1}{2}} \end{bmatrix}$ so that $T = PU_{T'}$, where $\iota: \mathcal{K} \to \mathcal{K} \oplus \mathcal{H}$ is the natural inclusion and $P: \mathcal{K} \oplus \mathcal{H} \to \mathcal{H}$ is the orthogonal projection. Note that only U_T depends on T, so it is easy to see that ι and P come from maps of real Hilbert spaces $\mathcal{K}_{\mathbb{R}}, \mathcal{K}_{\mathbb{R}} \oplus \mathcal{H}_{\mathbb{R}}$, and $\mathcal{H}_{\mathbb{R}}$ and they intertwine the orthogonal groups $(\mathrm{U}_t)_{t\in\mathbb{R}}, (\mathrm{U}_t \oplus \mathrm{V}_t)_{t\in\mathbb{R}}$ and $(\mathrm{V}_t)_{t\in\mathbb{R}}$. Therefore there is no problem with defining the second quantisation for these maps (Proposition (6.1.6)). We get a ucp map $\Gamma_q(\iota): \Gamma_q(\mathcal{K}) \to \Gamma_q(\mathcal{K} \oplus \mathcal{H})$. The condition $JTI = T$ is not self-adjoint, hence in general U_T does not commute with $I \oplus J$, so there is no hope of defining a map $\Gamma_q(U_T): \Gamma_q(\mathcal{K} \oplus \mathcal{H}) \to \Gamma_q(\mathcal{K} \oplus \mathcal{H})$. However, there is a map $\mathcal{T}_q(U_T): \mathrm{B}(\mathcal{F}_q(\mathcal{K} \oplus \mathcal{H})) \to \mathrm{B}(\mathcal{F}_q(\mathcal{K} \oplus \mathcal{H}))$ given by conjugation $x \mapsto \mathcal{F}_q(U)x\mathcal{F}_q(U_T)^*$. One can easily check that

$$\mathcal{T}_q(U_T)\left(a_q^*(e_1) \dots a_q^*(e_k)a_q(e_{k+1}) \dots a_q(e_n)\right) = a_q^*(U_Te_1) \dots a_q^*(U_Te_k)a_q(U_Te_{k+1}) \dots a_q(U_Te_n).$$

So far, we have a normal ucp map $\mathcal{T}_q(U_T)\circ\Gamma_q^2(\iota):\Gamma_q(\mathcal{K})\to B(\mathcal{F}_q(\mathcal{K}\oplus\mathcal{H}))$. We now have to deal with the projection P. As in the case of a unitary operator, we get a map $\mathcal{T}_q(P): B(\mathcal{F}_q(\mathcal{K}\oplus\mathcal{H}))\to B(\mathcal{F}_q(\mathcal{H}))$ given by $x\mapsto\mathcal{F}_q(P)x\mathcal{F}_q(P)^*$. It is a simple matter to check that in this case we still have

$$\mathcal{T}_q(P)\left(a_q^*(e_1)\dots a_q^*(e_k)a_q(e_{k+1})\dots a_q(e_n)\right)$$
$$=a_q^*(Pe_1)\dots a_q^*(Pe_k)a_q(Pe_{k+1})\dots a_q(Pe_n).$$

Finally, we obtain a (normal) ucp map

$$v:=\mathcal{T}_q(P)\circ\mathcal{T}_q(U_T)\circ\Gamma_q(\iota):\Gamma_q(\mathcal{K})\to B(\mathcal{F}_q(\mathcal{H}))$$

that has the property that $v(W(e_1\otimes\cdots\otimes e_n))$ is equal to

$$\sum_{k=0}^{n}\sum_{i_1,\dots,i_k,j_{k+1},\dots,j_n} a_q^*(PU_T\iota e_{i_1})\dots a_q^*(PU_T\iota e_{i_k})a_q(PU_T\iota Ie_{j_{k+1}})\dots a_q(PU_T\iota Ie_{j_n})q^{i(I_1,I_2)}.$$

Since $T=PU_T\iota$ satisfies $JTI=T$ this is equal to $W(Te_1\otimes\cdots\otimes Te_n)$, therefore the image of v is contained in $\Gamma_q(\mathcal{H})$ and we define $\Gamma_q(T):=v$.

We would like to present a second approach to the extended second quantisation. Let us start with a definition.

Definition (6.1.8)[240]: Let $\mathcal{H}$ be a (complex) Hilbert space. Let $\mathcal{F}_q(\mathcal{H})$ be the q-Fock space over $\mathcal{H}$. We define the q-Toeplitz algebra $\mathcal{T}_q(\mathcal{H})$ to be the C^*-algebra generated by the creation operators $a_q^*(v)$ inside $B(\mathcal{F}_q(\mathcal{H}))$. If $\mathcal{K}\subset\mathcal{H}$ is a closed subspace, we define $\mathcal{T}_q(\mathcal{K},\mathcal{H})$ to be the C^*-subalgebra of $\mathcal{T}_q(\mathcal{H})$ generated by the set $\{a_q^*(v): v\in\mathcal{K}\}$.

We would like to note that both $\Gamma_q(P)$ and $\Gamma_q(U_T)$ (denoted then by $\mathcal{T}_q(P)$ and $\mathcal{T}_q(U_T)$) can be defined on the level of the algebra $\mathcal{T}_q(\mathcal{H})$ by the same formula. If we could do that also for $\Gamma_q(\iota)$, we would be able to obtain a second quantisation procedure on the level of the q-Toeplitz algebra. The reasons for seeking such a generalisation are twofold. First, it is interesting in its own right because better understanding of the structure of the q-Toeplitz algebra has potential applications to the study of radial multipliers (cf. [190] for the free case). Second, it allows us to use the approach of Houdayer and Ricard (cf. [190], Theorem 3.15 and Corollary 3.16) to extend the second quantisation for the q-Araki-Woods algebras. Let us point out what obstacle has to be overcome. To show that we can define $\mathcal{T}_q(\iota)$, we would like to show that the *-homomorphism $\mathcal{T}_q(\iota^*)$: $\mathcal{T}_q(\mathcal{K},\mathcal{K}\oplus\mathcal{H})\to\mathcal{T}_q(\mathcal{K})$ is injective. This is the hard part, because now the vacuum vector Ω is not separating anymore. The kernel $\ker\left(\mathcal{T}_q(\iota^*)\right)$ is formed by elements vanishing on the subspace $\mathcal{F}_q(\mathcal{K})\subset\mathcal{F}_q(\mathcal{K}\oplus\mathcal{H})$. We will now state the triviality of the kernel explicitly.

To make the theorem look plausible, we would like to state a lemma saying that the linear span of the products of generators of the q-Toeplitz algebra, a dense $*$-subalgebra of it, does not contain any nontrivial element of the kernel - this shows that there are no obvious candidates for the elements of the kernel. Before that, let us introduce some useful notation.

Definition (6.1.9)[240]: Let $\mathcal{H}$ be a complex Hilbert space and let $\bar{\mathcal{H}}$ be its complex conjugate space. We define maps $a_q^*:\mathcal{H}^{\otimes k}\to B(\mathcal{F}_q(\mathcal{H}))$ and $a_q:\bar{\mathcal{H}}^{\otimes k}\to B(\mathcal{F}_q(\mathcal{H}))$ (the tensor products are simply algebraic tensor products) to be the linear extensions of the maps given on simple tensors by $a_q^*(e_1\otimes\cdots\otimes e_k)=a_q^*(e_1)\dots a_q^*(e_k)$ and $a_q(\bar{e}_1\otimes\cdots\otimes\bar{e}_k)=a_q(e_1)\dots a_q(e_k)$. For any $\mathbf{v}_k\otimes\bar{\mathbf{w}}_{n-k}\in\mathcal{H}^{\otimes k}\otimes\bar{\mathcal{H}}^{\otimes(n-k)}$ we also define $A_{k,n}(\mathbf{v}_k\otimes\bar{\mathbf{w}}_{n-k}):=a_q^*(\mathbf{v}_k)a_q(\bar{\mathbf{w}}_{n-k})$. Let us also define the space

$$\mathrm{T}(\mathcal{H}) := \bigoplus_{n=0}^{\infty}\left(\bigoplus_{k=0}^{n} \mathcal{H}^{\otimes k} \otimes \bar{\mathcal{H}}^{\otimes n-k}\right),$$

where the direct sums and tensor products are algebraic. Direct sum of all the operators $A_{k,n}$ will be denoted by $A: \mathrm{T}(\mathcal{H}) \to \mathcal{T}_q(\mathcal{H})$. Note that if $\iota: \mathcal{K} \to \mathcal{H}$ is an inclusion of Hilbert spaces, then A can be equally well viewed as a map from $\mathrm{T}(\mathcal{K})$ to $\mathcal{T}_q(\mathcal{K}, \mathcal{H})$, for which we will use the same notation.

Using basic algebraic manipulations, we can obtain the following lemma.

Lemma (6.1.10)[240]: Let $\iota: \mathcal{K} \to \mathcal{H}$ be an inclusion of Hilbert spaces. Then the mapping $A : \mathrm{T}(\mathcal{K}) \to \mathcal{T}_q(\mathcal{K}, \mathcal{H})$ is injective. As a consequence, any x in the range of A is not in the kernel of the map $\mathcal{T}_q(\iota^*): \mathcal{T}_q(\mathcal{K}, \mathcal{H}) \to \mathcal{T}_q(\mathcal{K})$.

Before proving Theorem (6.1.11), we need just one more lemma, which we precede with introduction of convenient notation.

Elements of the form $a_q^*(\mathbf{v}_n) a_q(\bar{\mathbf{w}}_n)$, where $\mathbf{v}_n, \mathbf{w}_n \in \mathcal{H}^{\otimes n}$ will be called elements of length n, and their non-closed linear span will be denoted by $\left(\mathcal{T}_q(\mathcal{H})\right)_n$. Note that, by Lemma (6.1.10), the subspaces $\left(\mathcal{T}_q(\mathcal{H})\right)_n$ are linearly independent for different n, therefore the notion of length is well defined. We will also find it useful to specify the notation for the orthogonal projections $P_n: \mathcal{F}_q(\mathcal{H}) \to \mathcal{H}_q^{\otimes n}$, where $\mathcal{H}_q^{\otimes n}$ denotes the n-fold tensor power of $\mathcal{H}$ endowed with a q-deformed inner product. Let us also introduce the maps $R_{n+k,k}^*: \mathcal{H}_q^{\otimes(n+k)} \to \mathcal{H}_q^{\otimes n} \otimes \mathcal{H}_q^{\otimes k}$(cf. [113], Lemma 2) by their action on simple tensors:

$$R_{n+k,k}^*(v_1 \otimes \cdots \otimes v_{n+k}) = \sum_{|I_1|=n, |I_2|=k} q^{i(I_1, I_2)} v_{i_1} \otimes \cdots \otimes v_{i_n} \otimes v_{j_{n+1}} \otimes \cdots \otimes v_{j_{n+k}},$$

with the same notation as in Lemma (6.1.4).

Theorem (6.1.11)[240]: Let $\mathcal{K}$ and $\mathcal{H}$ complex Hilbert spaces, with inclusion $\iota: \mathcal{K} \to \mathcal{H}$. Then the $*$-homomorphism $\mathcal{T}_q(\iota^*): \mathcal{T}_q(\mathcal{K}, \mathcal{H}) \to \mathcal{T}_q(\mathcal{K})$ is injective.

Proof. First, we would like to show that the task of proving triviality of the kernel can be reduced to a slightly easier one. To show that the kernel is trivial, it suffices to look at positive elements, since the kernel is an ideal, in particular a C^* algebra, therefore it is spanned by positive elements. Suppose that x is in the kernel and is positive. There is an action of the circle group (in our case it is the interval $[0, 2\pi]$ with endpoints identified) on $\mathcal{T}_q(\mathcal{K}, \mathcal{H})$ given by $t \overset{\alpha}{\to} \mathcal{F}_q(e^{it}) x \mathcal{F}_q(e^{-it})$. This action leaves the kernel invariant, therefore the element $\mathbb{E}x := \frac{1}{2\pi}\int_0^{2\pi} \mathcal{F}_q(e^{it}) x \mathcal{F}_q(e^{-it}) \mathrm{d}t$ is also in the kernel and is invariant by the action of the circle group defined above (this action is used by Pimsner in [199] to show the universality of the usual Toeplitz algebra). It is a simple matter to check that the fixed point subalgebra is equal to the closed linear span of the elements of the form $a_q^*(\mathbf{v}_n) a_q(\bar{\mathbf{w}}_n)$, where $\mathbf{v}_n \in \mathcal{H}^{\otimes n}, \bar{\mathbf{w}}_n \in \bar{\mathcal{H}}^{\otimes n}$ and n ranges over non-negative integers, and $\mathbb{E}$ is a faithful conditional expectation onto this fixed point subalgebra. So it suffices to show that there are no non-zero positive elements in this fixed point subalgebra that are in the kernel.

Suppose that x is in the kernel and belongs to the subalgebra fixed by the circle action. We need to show that $x_{|\mathcal{H}^{\otimes n}} = 0$ for any $n \geqslant 0$; note that, as vector spaces, $\mathcal{H}_q^{\otimes n} = \mathcal{H}^{\otimes n}$, which follows from [113]. We will prove the statement inductively. Fix a sequence $(x_k)_{k \in \mathbb{N}}$ that approximates x in norm and is contained in the non-closed sum of the subspaces

$\left(\mathcal{T}_q(\mathcal{H})\right)_n$. Therefore every x_k admits a decomposition $x_k = \sum_{l=0}^{n_k} x_k^{(l)}$, where $x_k^{(l)} \in \left(\mathcal{T}_q(\mathcal{H})\right)_l$ and n_k is the smallest number such that $x_k^{(l)} = 0$ for $l > n_k$.

We would like to now state explicitly the statement we intend to prove by induction: For every $n \in \mathbb{N} \cup \{0\} x_{|\mathcal{H}^{\otimes n}} = 0$ and $\lim_{k\to\infty} \left\|\sum_{l=0}^{n} x_k^{(l)}\right\| = 0$. Let us start with $n = 0$. Our inductive statement for $n = 0$ means just that $P_0 x P_0 = 0$ and $\lim_{k\to\infty} \left\|x_k^{(0)}\right\| = 0$.

The first part translates to $x\Omega = 0$ and it follows from the fact that $\Omega \in \mathcal{K}$ and x belongs to the kernel. For the other part, it follows from Lemma (6.1.13) that an element y_l of length l satisfies an inequality $\|y_l\| \leqslant C(q)\|P_l y_l P_l\|$, where $C(q)$ is a positive constant depending only on q. In our case we get $\left\|x_k^{(0)}\right\| \leqslant C(q)\left\|P_0 x_k^{(0)} P_0\right\|$. We know that $P_0 x P_0 = 0$ and $P_0 x_k P_0$ converges to $P_0 x P_0$ in norm. However, $P_0 x_k P_0 = P_0 x_k^{(0)} P_0$, since elements of length greater than 0 annihilate the range of P_0.

Assume now that our statement has been proved for $m < n -$ we would like to show that it is also true for n. Use the decomposition $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^\perp$ to write $\mathcal{H}^{\otimes n} = \mathcal{K}^{\otimes n} \oplus \mathcal{H}'$, where $\mathcal{H}'$ is a direct sum of tensor products of the spaces $\mathcal{K}$ and $\mathcal{K}^\perp$, where at least one factor is equal to $\mathcal{K}^\perp$. We would like to show that x restricted to each of the tensor products vanishes. Since x is in the kernel, we get it for $x_{|\mathcal{K}^{\otimes n}}$. Let $\mathcal{K}'$ be any other summand. We will show that $x_k(\mathbf{e})$ converges to 0 for any simple tensor $\mathbf{e} \in \mathcal{K}'$. Since $\mathbf{e}$ is of length n, we get $x_k(\mathbf{e}) = \sum_{l=0}^{n} x_k^{(l)}(\mathbf{e})$. By the inductive assumption, we know that $\sum_{l=0}^{n-1} x_k^{(l)}$ converges in norm to 0, so we are left with $x_k^{(n)}(\mathbf{e})$. But every summand in $x_k^{(n)}$ is of the form $a_q^*(\mathbf{v}_n) a_q(\overline{\mathbf{w}}_n)$ and $a_q^*(\mathbf{v}_n) a_q(\overline{\mathbf{w}}_n)\mathbf{e} = \langle \Sigma \mathbf{w}_n, \mathbf{e}\rangle \mathbf{v}_n$, where $\Sigma: \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}$ is the flip map, taking $h_1 \otimes \cdots \otimes h_n$ to $h_n \otimes \cdots \otimes h_1$. Since e possesses a vector from $\mathcal{K}^\perp$ in its tensor decomposition, $\langle \Sigma \mathbf{w}_n, \mathbf{e}\rangle = 0$. It follows that $x_{|\mathcal{H}^{\otimes n}} = 0$. This implies that $\lim_{k\to\infty} \|P_n x_k P_n\| = 0$. Since $P_n x_k P_n = \sum_{l=0}^{n} P_n x_k^{(l)} P_n$ and $\lim_{k\to\infty} \left\|\sum_{l=0}^{n-1} x_k^{(l)}\right\| = 0$, we get that $\lim_{k\to\infty} \left\|P_n x_k^{(n)} P_n\right\| = 0$. Using Lemma (6.1.13), we conclude that $\lim_{k\to\infty} \left\|x_k^{(n)}\right\| = 0$ therefore $\lim_{k\to\infty} \left\|\sum_{l=0}^{n} x_k^{(l)}\right\| = 0$.

Lemma (6.1.12)[240]: Let $\mathcal{H}$ be a Hilbert space. Suppose that A, B are positive operators on $\mathcal{H}$ and T is a bounded operator such that $A = BT$. Then $A \leqslant \| T \| B$.

Proof. By taking the adjoint, we get $A = T^* B$, hence $A^2 = BTT^*B$. It follows that $A^2 \leqslant \| T \|^2 B^2$. The majorisation $A \leqslant \| T \| B$ is implied by the operator monotonicity of the square root.

Lemma (6.1.13)[240]: Suppose that $x \in \left(\mathcal{T}_q(\mathcal{H})\right)_n$. Then:

1. $P_{n+k} x P_{n+k} = \mathrm{Id}_{n,k}(P_n x P_n \otimes \mathrm{Id}_k) R_{n+k,k}^*$, where $\mathrm{Id}_{n,k}: \mathcal{H}_q^{\otimes n} \otimes \mathcal{H}_q^{\otimes k} \to \mathcal{H}_q^{\otimes n+k}$ is the extension of the identity map $\mathcal{H}^{\otimes n} \otimes \mathcal{H}^{\otimes k} \to \mathcal{H}^{\otimes n+k}$, defined on algebraic tensor products.
2. $\| x \| \leqslant C(q)\|P_n x P_n\|$, where $C(q)$ is a positive constant depending only on q. Consequently, $\| x \| \simeq \|P_n x P_n\|$.

Proof. (i) Fix x of length n - it is a linear combination of elements of the form $a_q^*(\mathbf{v}_n) a(\overline{\mathbf{w}}_n)$. Since the formula

$$P_{n+k} x P_{n+k} = (P_n x P_n \otimes \mathrm{Id}_k) R_{n+k,k}^*, \tag{2}$$

is linear in x, it suffices to prove it for x of the form $a_q^*(\mathbf{v}_n) a(\overline{\mathbf{w}}_n)$, where $\mathbf{v}_n = v_1 \otimes \cdots \otimes v_n$ and $\mathbf{w}_n = w_n \otimes \cdots \otimes w_1$ are simple tensors. Fix $\mathbf{e} \in \mathcal{H}^{\otimes n+k}$; we have to check that

$xe = (x \otimes \mathrm{Id}_k)R^*_{n+k,k}e$. Note that the action of creation operators does not depend on the tensor power on which they act - it always boils down to tensoring by a vector on the left. Therefore we need only to concern ourselves with annihilation operators. We would like to express q-annihilation operators $a_q(v)$ in terms of free annihilation operators $a(v) := a_0(v)$. Note that $a^*_q(v) = a^*(v)$, so for any finite tensors $\mathbf{y}$ and $\mathbf{z}$ we get $\langle \mathbf{z}, a_q(v)\mathbf{y}\rangle_q = \langle a^*(v)\mathbf{z}, \mathbf{y}\rangle_q$.

Let P^n_q be the positive operator on $\mathcal{H}^{\otimes n}$ defining the q-deformed inner product; let us denote by P_q the direct sum of all the operators P^n_q. Using the definition of the q-deformed inner product, we arrive at

$$\langle \mathbf{z}, P_q a_q(v)\mathbf{y}\rangle_0 = \langle P_q a^*(v)\mathbf{z}, \mathbf{y}\rangle_0.$$

It follows that $P_q a_q(v) = \left(P_q a^*(v)\right)^* = a(v)P_q$, so $a_q(v) = P_q^{-1}a(v)P_q$. If we restrict this equality to $\mathcal{H}^{\otimes n}$, we get $a_q(v)_{|\mathcal{H}^{\otimes n}} = \left(P^{n-1}_q\right)^{-1}a(v)P^n_q$. Let us compute the left-hand side of (2) :

$$a_q(w_n) \ldots a_q(w_1)\mathbf{e} = \left(P^k_q\right)^{-1}a(w_n) \ldots a(w_1)P^{n+k}_q\mathbf{e}.$$

This formula follows from the fact that first we change $a_q(w_1)$ to $\left(P^{n+k-1}_q\right)^{-1}a(w_1)P^{n+k}_q$, but then $a_q(w_2)$ has to be changed to $\left(P^{n+k-2}_q\right)^{-1}a(w_2)P^{n+k-1}_q$ and there is a cancellation between $a(w_2)$ and $a(w_1)$; using this fact repeatedly, we obtain the above formula. To calculate the right-hand side, recall (cf. [113], Formula 2 on page 21) that we have an equality $P^{n+k}_q = \left(P^n_q \otimes P^k_q\right)R^*_{n+k,k}$, so $R^*_{n+k,k} = \left(\left(P^n_q\right)^{-1} \otimes \left(P^k_q\right)^{-1}\right)P^{n+k}_q$. It leads us to:

$$\begin{aligned}
&\left(a_q(w_n) \ldots a_q(w_1) \otimes \mathrm{Id}_k\right)\left(\left(P^n_q\right)^{-1} \otimes \left(P^k_q\right)^{-1}\right)P^{n+k}_q\mathbf{e}\\
&= \left(a_q(w_n) \ldots a_q(w_1)\left(P^n_q\right)^{-1} \otimes \left(P^k_q\right)^{-1}\right)P^{n+k}_q\mathbf{e}\\
&= \left(a(w_n) \ldots a(w_1) \otimes \left(P^k_q\right)^{-1}\right)P^{n+k}_q\mathbf{e}.
\end{aligned}$$

We now only need to understand that this is exactly the same formula. It follows from the fact that the free annihilation operators act only on the n leftmost vectors, so the operator $\left(P^k_q\right)^{-1}$ in both situations acts only on the k rightmost ones.

(ii) First of all, since the spaces $\mathcal{H}^{\otimes k}_q$ are left invariant by x, we have $\| x \| = \sup_{k\geqslant 0}\|P_k x P_k\|$. Because $P_k x P_k = 0$ for $k < n$, we actually get $\| x \| = \sup_{k\geqslant 0}\|P_{n+k} x P_{n+k}\|$. We just have to show that $\|P_{n+k} x P_{n+k}\| \leqslant C(q)\|P_n x P_n\|$. From the first part of the proof we get that $\|P_{n+k} x P_{n+k}\| \leqslant \|\mathrm{Id}_{n,k}\| \cdot \|R^*_{n+k,n}\| \cdot \|P_n x P_n\|$. It is known (cf. [113], Formula 2 on page 21) that $\|R^*_{n+k,n}\| \leqslant C(q)$, where $C(q) = \prod_{k=1}^{\infty}(1-|q|^k)^{-1}$ and $R^*_{n+k,n}$ is seen as an operator on $\mathcal{H}^{\otimes(n+k)}$, where $\mathcal{H}^{\otimes(n+k)}$ is equipped with the standard inner product, not the q-deformed one. It follows from Lemma (6.1.12) that $P^{(n+k)}_q \leqslant C(q)P^{(n)}_q \otimes P^{(k)}_q$ as operators on $\mathcal{H}^{\otimes(n+k)}$, since $P^{(n+k)}_q = \left(P^{(n)}_q \otimes P^{(k)}_q\right)R^*_{n+k,k}$. Because $P^{(n+k)}_q$ defines the inner product on $\mathcal{H}^{\otimes(n+k)}_q$, and $P^{(n)}_q \otimes P^{(k)}_q$ defines the inner product on $\mathcal{H}^{\otimes n}_q \otimes \mathcal{H}^{\otimes k}_q$, it follows that the identity map $\mathrm{Id}_{n,k}: \mathcal{H}^{\otimes n}_q \otimes \mathcal{H}^{\otimes k}_q \to \mathcal{H}^{\otimes(n+k)}_q$ has norm not greater than $\sqrt{C(q)}$. Finally, $\left(\mathrm{Id}_{n,k}\right)^* = R^*_{n+k,n}$, so $\|R^*_{n+k,n}\| \leqslant \sqrt{C(q)}$ as an operator mapping $\mathcal{H}^{\otimes(n+k)}_q$ to $\mathcal{H}^{\otimes n}_q \otimes \mathcal{H}^{\otimes k}_q$.

This shows that $\|P_{n+k} x P_{n+k}\| \leqslant C(q)\|P_n x P_n\|$.

To prove the Haagerup property, we need to use one more lemma.

Lemma (6.1.14)[240]: (Houdayer-Ricard, [190]). There exists a sequence $(T_k)_{k\in\mathbb{N}}$ of finite rank contractions on $\mathcal{H}$ such that $IT_kI = T_k$ and $\lim_{k\to\infty} T_k = 1$ strongly.

Theorem (6.1.15)[240]: Let $(\mathcal{H}_{\mathrm{R}}, (\mathrm{U}_t)_{t\in\mathbb{R}})$ be a separable, real Hilbert space equipped with a one parameter group of orthogonal transformations $(\mathrm{U}_t)_{t\in\mathbb{R}}$. Then the q-Araki-Woods algebra $\Gamma_q(\mathcal{H}_{\mathbb{R}}, (\mathrm{U}_t)_{t\in\mathbb{R}})''$ has the Haagerup approximation property.

Proof. Consider the ucp maps $v_{k,t} := \Gamma_q(e^{-t}T_k) -$ they preserve the vacuum state. We would like to prove that the GNS-implementations of these maps converge strongly to identity and are compact. First of all, by definition, the GNS implementations of these maps are equal to $\mathcal{F}_q(e^{-t}T_k)$. Let us then check compactness. Recall that we denote by $P_n : \mathcal{F}_q(\mathcal{H}) \to \mathcal{F}_q(\mathcal{H})$ the orthogonal projection onto first n summands in the direct sum decomposition of the Fock space. Since T_k is a finite-rank operator, so is $P_n\mathcal{F}_q(e^{-t}T_k)$. We have to show that the norm of $P_n^{\perp}\mathcal{F}_q(e^{-t}T_k)$ converges to 0, when $n \to \infty$. First of all, let us reduce to the case $q = 0$. Operator P_q preserves all the tensor powers appearing in the direct sum decomposition of the Fock space, therefore it commutes with $P_n^{\perp}$. It also commutes with the first quantisation operators $\mathcal{F}_q(e^{-t}T_k)$. It follows from Lemma 1.4 in [60] that the norm of $P_n^{\perp}\mathcal{F}_q(e^{-t}T_k)$ does not change if we compute it on the free Fock space $\mathcal{F}_0(\mathcal{H})$; this is the norm that we will estimate. This is easy when T_k is self-adjoint and we will now show that one can assume that. Indeed, the first quantisation on the level of the Fock space interacts nicely with taking the adjoint, so we get (by the C^*-identity)

$$\|P_n^{\perp}\mathcal{F}_0(e^{-t}T_k)\|^2 = \|P_n^{\perp}\mathcal{F}_0(e^{-2t}T_k^*T_k)P_n^{\perp}\|.$$

Now $T_k^*T_k$ is a finite rank positive contraction, so there is an orthonormal basis $(e_i)_{i\in\mathbb{N}}$ of $\mathcal{H}$ such that $T_k^*T_ke_i = \lambda_i e_i$ and $\lambda_i \in [0,1]$. From the orthonormal basis of eigenvectors of $T_k^*T_k$ we can build an orthonormal basis of $\mathcal{F}_0(\mathcal{H})$, using tensor powers; for a multi-index $I^n = \{i_1, \dots, i_k\}$ we will denote $e_I = e_{i_1} \otimes \cdots \otimes e_{i_k}$ and $\lambda_I = \lambda_{i_1} \cdots \lambda_{i_k}$. We can now estimate the norm of $P_n^{\perp}\mathcal{F}_q(e^{-2t}T_k^*T_k)P_n^{\perp}$. Let $v \in \mathcal{F}_0(\mathcal{H})$ be written as $v = \sum a_I e_I$, then

$$\begin{aligned}\|P_n^{\perp}\mathcal{F}_0(e^{-2t}T_k^*T_k)P_n^{\perp}v\|^2 &= \left\|\sum_{|I|>n} e^{-2|I|}a_I\lambda_I e_I\right\|^2 \\ &= \sum_{|I|>n} e^{-4|I|}|a_I|^2|\lambda_I|^2 \\ &\leqslant e^{-4(n+1)} \| v \|^2,\end{aligned}$$

because $|\lambda_I| \leqslant 1$. The fact that the operators $\mathcal{F}_q(e^{-t}T_k)$ converge strongly to the identity when $t \to 0$ and $k \to \infty$ is clear; it can be easily checked on finite simple tensors and this suffices, since they are all contractive. This ends the proof.

Section (6.2): Structure of Modular Invariant Subalgebras

Free Araki-Woods factors were introduced by Shlyakhtenko in [15]. In the framework of Voiculescu's free probability theory, these factors can be regarded as the type III counterparts of free group factors using Voiculescu's free Gaussian functor [22], [19]. Following [15], to any orthogonal representation $U: \mathbf{R} \curvearrowright H_{\mathbf{R}}$ on a real Hilbert space, one associates the free Araki- Woods von- Neumann algebra $\Gamma(H_{\mathbf{R}}, U)''$. The von Neumann algebra $\Gamma(H_{\mathbf{R}}, U)''$ comes equipped with a unique free quasi-free state φ_U which is always normal and faithful. We have $\Gamma(H_{\mathbf{R}}, U)'' \cong \mathrm{L}(\mathbf{F}_{\dim(H_{\mathbf{R}})})$ when $U = 1_{H_{\mathbf{R}}}$ and $\Gamma(H_{\mathbf{R}}, U)''$ is a full type III factor when $U \neq 1_{H_{\mathbf{R}}}$.

Let $U: \mathbf{R} \curvearrowright H_{\mathbf{R}}$ be any orthogonal representation. Using Zorn's lemma, we may decompose $H_{\mathbf{R}} = H_{\mathbf{R}}^{\mathrm{ap}} \oplus H_{\mathbf{R}}^{\mathrm{wm}}$ and $U = U^{\mathrm{wm}} \oplus U^{\mathrm{ap}}$ where $U^{\mathrm{ap}}: \mathbf{R} \curvearrowright H_{\mathrm{R}}^{\mathrm{ap}}$ (resp. $U^{\mathrm{wm}}: \mathbf{R} \curvearrowright H_{\mathrm{R}}^{\mathrm{wm}}$) is the almost periodic (resp. weakly mixing) subrepresentation of $U: \mathbf{R} \curvearrowright H_{\mathbf{R}}$. Write $M = \Gamma(H_{\mathbf{R}}, U)'', N = \Gamma\left(H_{\mathbf{R}}^{\mathrm{ap}}, U^{\mathrm{ap}}\right)''$ and $P = \Gamma(H_{\mathbf{R}}^{\mathrm{wm}}, U^{\mathrm{wm}})''$ so that we have the following free product splitting

$$(M, \varphi_U) = (N, \varphi_{U^{\mathrm{ap}}}) * (P, \varphi_{U^{\mathrm{wm}}}).$$

We provide a general structural decomposition for any von Neumann subalgebra $Q \subset M$ that is globally invariant under the modular automorphism group σ^{φ_U} and shows that when Q is moreover assumed to be amenable then Q sits inside N. Our main theorem generalizes ([252], Theorem C) to arbitrary free Araki-Woods factors.

The main theorem should be compared to ([249], Theorem D) which provides a similar result for crossed product II_1 factors arising from free Bogoljubov actions of amenable groups.

The core of our argument is Theorem (6.2.2) which generalizes ([252], Theorem 4.3) to arbitrary free Araki-Woods factors. Let us point out that Theorem (6.2.2) is reminiscent of Popa's asymptotic orthogonality property in free group factors [237] which is based on the study of central sequences in the ultraproduct framework. Unlike other results on this theme [249], [250], [253], we do not assume here that the subalgebra $Q \subset M$ has a diffuse intersection with the free summand N of the free product splitting $(M, \varphi_U) = \left(N, \varphi_{U^{\mathrm{ap}}}\right) * (P, \varphi_{U^{\mathrm{wm}}})$ and so we cannot exploit commutation relations of Q-central sequences with elements in N. Instead, we use the facts that Q admits central sequences that are invariant under the modular automorphism group σ^{φ_U} of the ultraproduct state φ_U^{ω} and that the modular automorphism group σ^{φ_U} is weakly mixing on P.

For any von Neumann algebra M, we denote by $\mathcal{Z}(M)$ the centre of M, by $\mathcal{U}(M)$ the group of unitaries in M, by $\mathrm{Ball}(M)$ the unit ball of M with respect to the uniform norm and by $(M, \mathrm{L}^2(M), J, \mathrm{L}^2(M)_+)$the standard form of M. We say that an inclusion of von Neumann algebras $P \subset M$ is with expectation if there exists a faithful normal conditional expectation $\mathrm{E}_P: M \to P$. All the von Neumann algebras we consider are always assumed to σ-finite.

For M be any σ-finite von Neumann algebra with predual M_* and $\varphi \in M_*$ any faithful state. We write $\| x \|_{\varphi} = \varphi(x^* x)^{1/2}$ for all $x \in M$. Recall that on $\mathrm{Ball}(M)$, the topology given by $\|\cdot\|_{\varphi}$ coincides with the σ-strong topology. Denote by $\xi_{\varphi} \in \mathrm{L}^2(M)_+$the unique representing vector of φ. The mapping $M \to \mathrm{L}^2(M): x \mapsto x\xi_{\varphi}$ defines an embedding with dense image such that $\| x \|_{\varphi} = \|x\xi_{\varphi}\|_{L^2(M)}$ for all $x \in M$. We denote by σ^{φ} the modular automorphism group of the state φ. The centralizer M^{φ} of the state φ is by definition the fixed point algebra of (M, σ^{φ}). Recall from ([248], Section 2.1) that two subspaces $E, F \subset H$ of a Hilbert space are said to be ε-orthogonal for some $0 \leq \varepsilon \leq 1$ if $|\langle \xi, \eta \rangle| \leq \varepsilon \| \xi \| \| \eta \|$ for all $\xi \in E$ and all $\eta \in F$. We will then simply write $E \perp_{\varepsilon} F$.

Let M be any σ-finite von Neumann algebra and $\omega \in \beta(\mathbf{N}) \setminus \mathbf{N}$ any nonprincipal ultrafilter. Define

$$\mathcal{I}_{\omega}(M) = \left\{(x_n)_n \in \ell^{\infty}(M): x_n \to 0 \ * \text{-strongly as } n \to \omega\right\}$$

$$\mathcal{M}^{\omega}(M) = \left\{(x_n)_n \in \ell^{\infty}(M): (x_n)_n \mathcal{I}_{\omega}(M) \subset \mathcal{I}_{\omega}(M) \text{ and } \mathcal{I}_{\omega}(M)(x_n)_n \subset \mathcal{I}_{\omega}(M)\right\}.$$

The multiplier algebra $\mathcal{M}^\omega(M)$ is a C*-algebra and $\mathcal{I}_\omega(M) \subset \mathcal{M}^\omega(M)$ is a norm closed twosided ideal. Following [254], we define the ultraproduct von Neumann algebra M^ω by $M^\omega := \mathcal{M}^\omega(M)/\mathcal{I}_\omega(M)$, which is indeed known to be a von Neumann algebra. We denote the image of $(x_n)_n \in \mathcal{M}^\omega(M)$ by $(x_n)^\omega \in M^\omega$.

For every $x \in M$, the constant sequence $(x)_n$ lies in the multiplier algebra $\mathcal{M}^\omega(M)$. We will then identify M with $(M + \mathcal{I}_\omega(M))/\mathcal{I}_\omega(M)$ and regard $M \subset M^\omega$ as a von Neumann subalgebra. The map $E_\omega: M^\omega \to M: (x_n)^\omega \mapsto \sigma$-weak $\lim_{n\to\omega} x_n$ is a faithful normal conditional expectation.

For every faithful state $\varphi \in M_*$, the formula $\varphi^\omega := \varphi \circ \mathrm{E}_\omega$ defines a faithful normal state on M^ω. Observe that $\varphi^\omega((x_n)^\omega) = \lim_{n\to\omega} \varphi(x_n)$ for all $(x_n)^\omega \in M^\omega$.

Let $Q \subset M$ be any von Neumann subalgebra with faithful normal conditional expectation $\mathrm{E}_Q: M \to Q$. Choose a faithful state $\varphi \in M_*$ in such a way that $\varphi = \varphi \circ \mathrm{E}_Q$. We have $\ell^\infty(Q) \subset \ell^\infty(M), \mathcal{I}_\omega(Q) \subset \mathcal{I}_\omega(M)$ and $\mathcal{M}^\omega(Q) \subset \mathcal{M}^\omega(M)$. We will then identify $Q^\omega = \mathcal{M}^\omega(Q)/\mathcal{I}_\omega(Q)$ with $(\mathcal{M}^\omega(Q) + \mathcal{I}_\omega(M))/\mathcal{I}_\omega(M)$ and be able to regard $Q^\omega \subset M^\omega$ as a von Neumann subalgebra. Observe that the norm $\|\cdot\|_{(\varphi|_Q)^\omega}$ on Q^ω is the restriction of the norm $\|\cdot\|_\varphi$ to Q^ω. Observe moreover that $\left(\mathrm{E}_Q(x_n)\right)_n \in \mathcal{I}_\omega(Q)$ for all $(x_n)_n \in \mathcal{I}_\omega(M)$ and $\left(\mathrm{E}_Q(x_n)\right)_n \in \mathcal{M}^\omega(Q)$ for all $(x_n)_n \in \mathcal{M}^\omega(M)$. Therefore, the mapping $\mathrm{E}_{Q\omega}: M^\omega \to Q^\omega: (x_n)^\omega \mapsto \left(\mathrm{E}_Q(x_n)\right)^\omega$ is a well-defined conditional expectation satisfying $\varphi^\omega \circ \mathrm{E}_{Q^\omega} = \varphi^\omega$. Hence, $\mathrm{E}_{Q^\omega}: M^\omega \to Q^\omega$ is a faithful normal conditional expectation. For more on ultraproduct von Neumann algebras, we refer the reader to [247], [254].

Let $H_{\mathbf{R}}$ be any real Hilbert space and $U: \mathbf{R} \curvearrowright H_{\mathbf{R}}$ any orthogonal representation. Denote by $H = H_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C} = H_{\mathbf{R}} \oplus \mathrm{i}H_{\mathbf{R}}$ the complexified Hilbert space, by $I: H \to H: \xi + \mathrm{i}\eta \mapsto \xi - \mathrm{i}\eta$ the canonical anti-unitary involution on H and by A the infinitesimal generator of $U: \mathbf{R} \curvearrowright H$, that is, $U_t = A^{\mathrm{it}}$ for all $t \in \mathbf{R}$. Moreover, we have $IAI = A^{-1}$. Observe that $j: H_{\mathbf{R}} \to H: \zeta \mapsto \left(\frac{2}{A^{-1}+1}\right)^{1/2} \zeta$ defines an isometric embedding of $H_{\mathbf{R}}$ into H. Put $K_{\mathbf{R}} := j(H_{\mathbf{R}})$. It is easy to see that $K_{\mathbf{R}} \cap \mathrm{i}K_{\mathbf{R}} = \{0\}$ and that $K_{\mathbf{R}} + \mathrm{i}K_{\mathbf{R}}$ is dense in H. Write $T = IA^{-1/2}$. Then T is a conjugate-linear closed invertible operator on H satisfying $T = T^{-1}$ and $T^*T = A^{-1}$. Such an operator is called an involution on H. Moreover, we have $\mathrm{dom}(T) = \mathrm{dom}\left(A^{-1/2}\right)$ and $K_{\mathbf{R}} = \{\xi \in \mathrm{dom}(T): T\xi = \xi\}$. In what follows, we will simply write

$$\overline{\xi + \imath\eta} := T(\xi + \mathrm{i}\eta) = \xi - \mathrm{i}\eta, \forall \xi, \eta \in K_{\mathbf{R}}.$$

We introduce the full Fock space of :

$$\mathcal{F}(H) = \mathbf{C}\Omega \oplus \bigoplus_{n=1}^{\infty} H^{\otimes n}.$$

The unit vector Ω is called the vacuum vector. For all $\xi \in H$, define the left creation operator $\ell(\xi): \mathcal{F}(H) \to \mathcal{F}(H)$ by

$$\begin{cases} \ell(\xi)\Omega = \xi, \\ \ell(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n. \end{cases}$$

We have $\| \ell(\xi) \|_\infty = \| \xi \|$ and $\ell(\xi)$ is an isometry if $\| \xi \| = 1$. For all $\xi \in K_{\mathbf{R}}$, put $W(\xi) := \ell(\xi) + \ell(\xi)^*$. The crucial result of Voiculescu ([19], Lemma 2.6.3) is that the distribution of the self-adjoint operator $W(\xi)$ with respect to the vector state $\varphi_U = \langle \cdot\, \Omega, \Omega \rangle$ is the semicircular law of Wigner supported on the interval $[-\| \xi \|, \| \xi \|]$.

Definition (6.2.1)[246]: (Shlyakhtenko, [15]). Let $H_{\mathbf{R}}$ be any real Hilbert space and $U: \mathbf{R} \curvearrowright H_{\mathbf{R}}$ any orthogonal representation. The free Araki-Woods von Neumann algebra associated with $U: \mathbf{R} \curvearrowright H_{\mathbf{R}}$ is defined by

$$\Gamma(H_{\mathbf{R}}, U)'' := \{W(\xi): \xi \in K_{\mathbf{R}}\}''.$$

We will denote by $\Gamma(H_{\mathbf{R}}, U)$ the unital C^*-algebra generated by 1 and by all the elements $W(\xi)$ for $\xi \in K_{\mathbf{R}}$.

The vector state $\varphi_U = \langle \cdot\, \Omega, \Omega\rangle$ is called the free quasi-free state and is faithful on $\Gamma(H_{\mathbf{R}}, U)''$. Let $\xi, \eta \in K_{\mathbf{R}}$ and write $\zeta = \xi + \mathrm{i}\eta$. Put

$$W(\zeta) := W(\xi) + \mathrm{i}W(\eta) = \ell(\zeta) + \ell(\bar{\zeta})^*.$$

Note that the modular automorphism group σ^{φ_U} of the free quasi-free state φ_U is given by $\sigma_t^{\varphi_U} = \mathrm{Ad}(\mathcal{F}(U_t))$, where $\mathcal{F}(U_t) = 1_{\mathbf{C}\Omega} \oplus \oplus_{n\geq 1} U_t^{\otimes n}$. In particular, it satisfies

$$\sigma_t^{\varphi_U}(W(\zeta)) = W(U_t\zeta), \forall \zeta \in K_{\mathbf{R}} + \mathrm{i}K_{\mathbf{R}}, \forall t \in \mathbf{R}.$$

It is easy to see that for all $n \geq 1$ and all $\zeta_1, \dots, \zeta_n \in K_{\mathbf{R}} + \mathrm{i}K_{\mathbf{R}}, \zeta_1 \otimes \cdots \otimes \zeta_n \in \Gamma(H_{\mathbf{R}}, U)''\Omega$.

When $\zeta_1, \dots, \zeta_n$ are all nonzero, we will denote by $W(\zeta_1 \otimes \cdots \otimes \zeta_n) \in \Gamma(H_{\mathbf{R}}, U)''$ the unique element such that

$$\zeta_1 \otimes \cdots \otimes \zeta_n = W(\zeta_1 \otimes \cdots \otimes \zeta_n)\Omega.$$

Such an element is called a reduced word. By ([252], Proposition 2.1(i)) (see also [248], Proposition 2.4), the reduced word $W(\zeta_1 \otimes \cdots \otimes \zeta_n)$ satisfies the Wick formula given by

$$W(\zeta_1 \otimes \cdots \otimes \zeta_n) = \sum_{k=0}^{n} \ell(\zeta_1) \cdots \ell(\zeta_k)\ell(\bar{\zeta}_{k+1})^* \cdots \ell(\bar{\zeta}_n)^*.$$

Note that since inner products are assumed to be linear in the first variable, we have $\ell(\xi)^*\ell(\eta) = \overline{\langle \xi, \eta\rangle} 1 = \langle \eta, \xi\rangle 1$ for all $\xi, \eta \in H$. In particular, the Wick formula from [252, Proposition 2.1 (ii)] is

$$\begin{aligned} &W(\xi_1 \otimes \cdots \otimes \xi_r)W(\eta_1 \otimes \cdots \otimes \eta_s) \\ &= W(\xi_1 \otimes \cdots \otimes \xi_r \otimes \eta_1 \otimes \cdots \otimes \eta_s) + \overline{\langle \bar{\xi}_r, \eta_1\rangle} W(\xi_1 \otimes \cdots \otimes \xi_{r-1})W(\eta_2 \otimes \cdots \otimes \eta_s)\end{aligned}$$

for all $\xi_1, \dots, \xi_r, \eta_1, \dots, \eta_s \in K_{\mathbf{R}} + \mathrm{i}K_{\mathbf{R}}$. We will repeatedly use this fact. We refer to [252] for further details.

Let $U: \mathbf{R} \curvearrowright H_{\mathbf{R}}$ be any orthogonal representation. Using Zorn's lemma, we may decompose $H_{\mathbf{R}} = H_{\mathbf{R}}^{\mathrm{ap}} \oplus H_{\mathbf{R}}^{\mathrm{wm}}$ and $U = U^{\mathrm{wm}} \oplus U^{\mathrm{ap}}$ where $U^{\mathrm{ap}}: \mathbf{R} \curvearrowright H_{\mathbf{R}}^{\mathrm{ap}}$ (resp. $U^{\mathrm{wm}}: \mathbf{R} \curvearrowright H_{\mathbf{R}}^{\mathrm{wm}}\big)$ is the almost periodic (resp. weakly mixing) subrepresentation of $U: \mathbf{R} \curvearrowright H_{\mathbf{R}}$. Write $M = \Gamma(H_{\mathbf{R}}, U)'', N = \Gamma\left(H_{\mathbf{R}}^{\mathrm{ap}}, U^{\mathrm{ap}}\right)''$ and $P = \Gamma(H_{\mathbf{R}}^{\mathrm{wm}}, U_t^{\mathrm{wm}})''$ so that

$$(M, \varphi_U) = (N, \varphi_{U^{\mathrm{ap}}}) * (P, \varphi_{U^{\mathrm{wm}}}).$$

For notational convenience, we simply write $\varphi := \varphi_U$.

The main result, Theorem (6.2.2) below, strengthens and generalizes [252].

Theorem (6.2.2)[246]: Keep the same notation as above. Let $\omega \in \beta(\mathbf{N}) \setminus \mathbf{N}$ be any nonprincipal ultrafilter. For all $a \in M \ominus N$, all $b \in M$ and all $x, y \in (M^\omega)^{\varphi\omega} \cap (M^\omega \ominus M)$, we have

$$\varphi^\omega(b^*y^*ax) = 0.$$

Proof. Denote as usual by $H := H_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$ the complexified Hilbert space and by $U: \mathbf{R} \curvearrowright H$ the corresponding unitary representation. Put $H^{\mathrm{ap}} := H_{\mathbf{R}}^{\mathrm{ap}} \otimes \mathrm{r}\mathbf{C}$ and $H^{\mathrm{wm}} := H_{\mathbf{R}}^{\mathrm{wm}} \otimes \mathbf{RC}$.

Put $K_{\mathbf{R}} := j(H_{\mathbf{R}}), K_{\mathbf{R}}^{\mathrm{ap}} = j\left(H_{\mathbf{R}}^{\mathrm{ap}}\right)$ and $K_{\mathbf{R}}^{\mathrm{wm}} := j\left(H_{\mathbf{R}}^{\mathrm{wm}}\right)$, where j is the isometric embedding $\xi \in H_{\mathbf{R}} \mapsto \left(\frac{2}{1+A^{-1}}\right)^{1/2} \xi \in H$. Denote by $\mathcal{H} = \mathcal{F}(H)$ the full Fock space of H.

For every $t \in \mathbf{R}$, put $\kappa_t = 1_{\mathbf{C}\Omega} \oplus \bigoplus_{n\geq 1} U_t^{\otimes n} \in \mathcal{U}(\mathcal{H})$. For every $t \in \mathbf{R}$ and every $x \in M$, we have $\sigma_t^{\varphi}(x)\Omega = \kappa_t(x\Omega)$. We will implicitly identify the full Fock space $\mathcal{F}(H)$ with the standard Hilbert space $\mathrm{L}^2(M)$ and the vacuum vector $\Omega \in \mathcal{H}$ with the canonical representing vector $\xi_{\varphi} \in \mathrm{L}^2(M)_+$.

Put $K_{\mathrm{an}} := \bigcup_{\lambda>1} \mathbf{1}_{[\lambda^{-1},\lambda]}(A)(K_{\mathrm{R}} + \mathrm{i}K_{\mathbf{R}})$. Observe that $K_{\mathrm{an}} \subset K_{\mathrm{R}} + \mathrm{i}K_{\mathrm{R}}$ is a dense subspace of elements $\eta \in K_{\mathbf{R}} + iK_{\mathrm{R}}$ for which the map $\mathbf{R} \to K_{\mathbf{R}} + \mathrm{i}K_{\mathbf{R}}: t \mapsto U_t\eta$ extends to an $(K_{\mathrm{R}} + \mathrm{i}K_{\mathbf{R}})$-valued entire analytic function and that $\overline{K_{\mathrm{an}}} = K_{\mathrm{an}}$. For all $\eta \in K_{\mathrm{an}}$, the element $W(\eta)$ is analytic with respect to the modular automorphism group σ^{φ} and we have $\sigma_z^{\varphi}(W(\eta)) = W\left(A^{\mathrm{i}z}\eta\right)$ for all $z \in \mathbf{C}$.

Denote by $\mathcal{W}$ the set of reduced words of the form $W(\xi_1 \otimes \cdots \otimes \xi_n)$ for which $\geq 1, \xi_1, \dots, \xi_n \in K_{\mathrm{an}}$. By linearity/density, in order to prove Theorem (6.2.2), we may assume without loss of generality that a and b are reduced words in $\mathcal{W}$. Since moreover $a \in M \ominus N$, we can assume that at least one of its letters ξ_i lies in $K_{\mathbf{R}}^{\mathrm{wm}} + \mathrm{i}K_{\mathbf{R}}^{\mathrm{wm}}$. More precisely, we can write

$$\begin{aligned} a &= a'W\left(\xi_1 \otimes \cdots \otimes \xi_p\right)a'' \\ b &= b'W\left(\eta_1 \otimes \cdots \otimes \eta_q\right)b'' \end{aligned}$$

with $p \geq 1, q \geq 0 a', a'', b', b''$ are reduced words in N with letters in $K_{\mathrm{an}} \cap \left(K_{\mathbf{R}}^{\mathrm{ap}} + \mathrm{i}K_{\mathrm{R}}^{\mathrm{ap}}\right)$, $\xi_2, \dots, \xi_{p-1}, \eta_2, \dots, \eta_{q-1} \in K_{\mathrm{an}}$ and $\xi_1, \xi_p, \eta_1, \eta_q \in K_{\mathrm{an}} \cap (K_{\mathbf{R}}^{\mathrm{wm}} + \mathrm{i}K_{\mathbf{R}}^{\mathrm{wm}})$. By convention, when $q = 0, W\left(\eta_1 \otimes \cdots \otimes \eta_q\right)$ is the trivial word 1 , so that $b = b'b''$.

Denote by $L \subset K_{\mathbf{R}}^{\mathrm{wm}} + \mathrm{i}K_{\mathrm{R}}^{\mathrm{wm}}$ the finite dimensional subspace generated by $\xi_1, \xi_p, \eta_1, \eta_q$ and such that $\bar{L} = L$. If $q = 0$, then L is simply the subspace generated by $\xi_1, \xi_p, \bar{\xi}_1, \bar{\xi}_p$ Denote by

1. $\mathcal{X}(1, r) \subset \mathcal{H}$ the closed linear subspace generated by all the reduced words of the form $e_1 \otimes \cdots \otimes e_n$ with $r \geq 0, n \geq r + 1, e_1, \dots, e_r \in K_{\mathbf{R}}^{\mathrm{ap}^{\circ}} + \mathrm{i}K_{\mathbf{R}}^{\mathrm{ap}}$ and $e_{r+1} \in L$. When $r = 0$ simply denote $\mathcal{X}_1 := \mathcal{X}(1,0)$.
2. $\mathcal{X}(2, r) \subset \mathcal{H}$ the closed linear subspace generated by all the reduced words of the form $e_1 \otimes \cdots \otimes e_n$ with $r \geq 0, n \geq r + 1, e_{n-r} \in L$ and $e_{n-r+1}, \dots, e_n \in K_{\mathbf{R}}^{\mathrm{ap}} + \mathrm{i}K_{\mathbf{R}}^{\mathrm{ap}}$. When $r = 0$, simply denote $\mathcal{X}_2 := \mathcal{X}(2,0)$.
3. $\mathcal{Y} \subset \mathcal{H}$ the closed linear subspace generated by all the reduced words of the form $e_1 \otimes \cdots \otimes e_n$ with $n \geq 1$ and $e_1, e_n \in L^{\perp}$.

Observe that we have the following orthogonal decomposition

$$\mathcal{H} = \mathbf{C}\Omega \oplus \overline{(\mathcal{X}_1 + \mathcal{X}_2)} \oplus \mathcal{Y}.$$

Claim (6.2.3)[246]: Let $\varepsilon \geq 0$ and $t \in \mathbf{R}$ such that $U_t(L) \perp_{\varepsilon/\dim L} L$. Then for all $i \in \{1,2\}$ and all $r \geq 0$, we have

$$\kappa_t\left(\mathcal{X}(i, r)\right) \perp_{\varepsilon} \mathcal{X}(i, r).$$

Proof. Choose an orthonormal basis $(\zeta_1, \dots, \zeta_{\dim L})$ of L. We first prove the claim for $i = 1$. We will identify $\mathcal{X}(1, r)$ with $L \otimes \left((H^{\mathrm{ap}})^{\otimes r} \otimes \mathcal{H}\right)$ using the following unitary defined by

$$\mathcal{V}(1, r): H \otimes \left(H^{\otimes r} \otimes \mathcal{H}\right) \to \mathcal{H}: \zeta \otimes \mu \otimes \nu \mapsto \mu \otimes \zeta \otimes \nu.$$

Observe that $\kappa_t \mathcal{V}(1, r) = \mathcal{V}(1, r)\left(U_t \otimes (U_t)^{\otimes r} \otimes \kappa_t\right)$ for every $t \in \mathbf{R}$. Let $\Xi_1, \Xi_2 \in \mathcal{X}(1, r)$ be such that $\Xi_1 = \sum_{i=1}^{\dim L} \zeta_i \otimes \Theta_i^1$ and $\Xi_2 = \sum_{j=1}^{\dim L} \zeta_j \otimes \Theta_j^2$ with $\Theta_i^1, \Theta_j^2 \in \left(H^{\mathrm{ap}}\right)^{\otimes r} \otimes \mathcal{H}$. We have $\kappa_t(\Xi_1) = \sum_{i=1}^{\dim L} U_t(\zeta_i) \otimes \kappa_t(\Theta_i^1)$ and hence

$$|\langle \kappa_t(\Xi_1), \Xi_2\rangle| \leq \sum_{i,j=1}^{\dim L} |\langle U_t(\zeta_i), \zeta_j\rangle \| \Theta_i^1 \| \| \Theta_j^2 \|.$$

Since $|\langle U_t(\zeta_i), \zeta_j\rangle| \leq \varepsilon / \dim L$, we obtain $|\langle \kappa_t(\Xi_1), \Xi_2\rangle| \leq \varepsilon \|\Xi_1\| \|\Xi_2\|$ by the Cauchy-Schwarz inequality. The proof of the claim for $i = 2$ is entirely analogous.

Given a closed subspace $\mathcal{K} \subset \mathcal{H}$, we denote by $P_{\mathcal{K}}: \mathcal{H} \to \mathcal{K}$ the orthogonal projection onto $\mathcal{K}$.

Claim (6.2.4)[246]: Take $z = (z_n)^\omega \in (M^\omega)^{\varphi^\omega}$ and let $w_1, w_2 \in N$ be any elements of the following form:

1. either $w_1 = 1$ or $w_1 = W(\zeta_1 \otimes \cdots \otimes \zeta_r)$ with $r \geq 1$ and $\zeta_1, \dots, \zeta_r \in K_{\text{an}} \cap \left(K_R^{\text{ap}} + iK_R^{\text{ap}}\right)$.
2. either $w_2 = 1$ or $w_2 = W(\mu_1 \otimes \cdots \otimes \mu_r)$ with $s \geq 1$ and $\mu_1, \dots, \mu_r \in K_{\text{an}} \cap \left(K_R^{\text{ap}} + iK_R^{\text{ap}}\right)$.

Then for all $i \in \{1,2\}$, we have $\lim_{n\to\omega} \|P_{\mathcal{X}_i}(w_1 z_n w_2 \Omega)\| = 0$.

Proof. Observe that $w_1 z_n w_2 \Omega = w_1 J \sigma^{\varphi}_{-\mathrm{i}/2}(w_2^*) J z_n \Omega$. Firstly, we have

$$\begin{aligned} P_{\mathcal{X}(1,r)}\left(J\sigma^{\varphi}_{-\mathrm{i}/2}(w_2^*) J z_n \Omega\right) &= J\sigma^{\varphi}_{-\mathrm{i}/2}(w_2^*) J P_{\mathcal{X}(1,r)}(z_n \Omega) \\ P_{\mathcal{X}(2,s)}(w_1 z_n \Omega) &= w_1 P_{\mathcal{X}(2,s)}(z_n \Omega). \end{aligned}$$

Secondly, for all $\Xi \in \mathcal{H}$, we have

$$P_{\mathcal{X}_1}(w_1 \Xi) = P_{\mathcal{X}_1}\left(w_1 P_{\mathcal{X}(1,r)}(\Xi)\right)$$
$$P_{\mathcal{X}_2}\left(J\sigma^{\varphi}_{-\mathrm{i}/2}(w_2^*) J \Xi\right) = P_{\mathcal{X}_2}\left(J\sigma^{\varphi}_{-\mathrm{i}/2}(w_2^*) J P_{\mathcal{X}(2,s)}(\Xi)\right).$$

This implies that

$$P_{\mathcal{X}_1}(w_1 z_n w_2 \Omega) = P_{\mathcal{X}_1}\left(w_1 J\sigma^{\varphi}_{-\mathrm{i}/2}(w_2^*) J P_{\mathcal{X}(1,r)}(z_n \Omega)\right)$$
$$P_{\mathcal{X}_2}(w_1 z_n w_2 \Omega) = P_{\mathcal{X}_2}\left(w_1 J\sigma^{\varphi}_{-\mathrm{i}/2}(w_2^*) J P_{\mathcal{X}(2,s)}(z_n \Omega)\right).$$

and we are left to show that $\lim_{n\to\omega} \|P_{\mathcal{X}(1,r)}(z_n \Omega)\| = \lim_{n\to\omega} \|P_{\mathcal{X}(2,s)}(z_n \Omega)\| = 0$.

Let $i \in \{1,2\}$ and $k \in \{r, s\}$. Fix $N \geq 0$. Since the orthogonal representation $U: \mathbf{R} \curvearrowright H_{\mathbf{R}}^{\text{wm}}$ is weakly mixing and $L \subset H^{\text{wm}}$ is a finite dimensional subspace, we may choose inductively $t_1, \dots, t_N \in \mathbf{R}$ such that $U_{t_{j_1}}(L) \perp_{(N \dim(L))^{-1}} U_{t_{j_2}}(L)$ for all $1 \leq j_1 < j_2 \leq N$. By Claim (6.2.3), this implies that

$$\kappa_{t_{j_1}}(\mathcal{X}(i,k)) \perp_{1/N} \kappa_{t_{j_2}}(\mathcal{X}(i,k)), \forall 1 \leq j_1 < j_2 \leq N.$$

For all $t \in \mathbf{R}$ and all $n \in \mathbf{N}$, we have

$$\begin{aligned} \|P_{\mathcal{X}(i,k)}(z_n \Omega)\|^2 &= \langle P_{\mathcal{X}(i,k)}(z_n \Omega), z_n \Omega\rangle \\ &= \left\langle \kappa_t\left(P_{\mathcal{X}(i,k)}(z_n \Omega)\right), \kappa_t(z_n \Omega)\right\rangle \ (\text{since } \kappa_t \in \mathcal{U}(\mathcal{H})) \\ &= \langle P_{\kappa_t(\mathcal{X}(i,k))}\left(\kappa_t(z_n \Omega)\right), \kappa_t(z_n \Omega)\rangle. \end{aligned}$$

By [247], for all $t \in \mathbf{R}$, we have $(z_n)^\omega = z = \sigma_t^{\varphi\omega}(z) = \left(\sigma_t^{\varphi}(z_n)\right)^\omega$. This implies that $\lim_{n\to\omega} \|\sigma_t^{\varphi}(z_n) - z_n\|_\varphi = 0$, and hence $\lim_{n\to\omega} \|\kappa_t(z_n \Omega) - z_n \Omega\| = 0$ for all $t \in \mathbf{R}$. In particular, since the sequence $(z_n \Omega)_n$ is bounded in $\mathcal{H}$, we deduce that for all $t \in \mathbf{R}$,

$$\lim_{n\to\omega} \|P_{\mathcal{X}(i,k)}(z_n \Omega)\|^2 = \lim_{n\to\omega} \langle P_{\kappa t(\mathcal{X}(i,k))}(z_n \Omega), z_n \Omega\rangle.$$

Applying this equality to our well chosen reals $(t_j)_{1\leq j\leq N}$, taking a convex combination and applying Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\lim_{n\to\omega}\|P_{\mathcal{X}(i,k)}(z_n\Omega)\|^2 &= \lim_{n\to\omega}\frac{1}{N}\sum_{j=1}^{N}\left\langle P_{\kappa t_j(\mathcal{X}(i,k))}(z_n\Omega), z_n\Omega\right\rangle \\ &= \lim_{n\to\omega}\frac{1}{N}\left\langle\sum_{j=1}^{N} P_{\kappa t_j}(\mathcal{X}(i,k))(z_n\Omega), z_n\Omega\right\rangle \\ &\leq \lim_{n\to\omega}\frac{1}{N}\left\|\sum_{j=1}^{N} P_{\kappa t_j(\mathcal{X}(i,k))}(z_n\Omega)\right\|\|z_n\|_{\varphi}.\end{aligned}$$

Then, for all $n\in\mathbf{N}$ we have,

$$\begin{aligned}\left\|\sum_{j=1}^{N} P_{\kappa t_j(\mathcal{X}(i,k))}(z_n\Omega)\right\| &= \sum_{j_1,j_2=1}^{N}\left\langle P_{\kappa_{t_1}}(\mathcal{X}(i,k))(z_n\Omega), P_{\kappa_{t_j}}(\mathcal{X}(i,k))(z_n\Omega)\right\rangle \\ &\leq \sum_{j=1}^{N}\left\|P_{\kappa_{t_j}(\mathcal{X}(i,k))}(z_n\Omega)\right\|^2 + \sum_{j_1\neq j_2}^{N}\frac{\|z_n\|_{\varphi}^2}{N} \\ &\leq N\|z_n\|_{\varphi}^2 + N^2\frac{\|z_n\|_{\varphi}^2}{N} \\ &= 2N\|z_n\|_{\varphi}^2.\end{aligned}$$

Altogether, we have obtained the inequality $\lim_{n\to\omega}\|P_{\mathcal{X}(i,k)}(z_n\Omega)\|^2 \leq \sqrt{2}\parallel z\parallel_{\varphi^\omega}^2/\sqrt{N}$. As N is arbitrarily large, this finishes the proof of Claim (6.2.4). The above argument is inspired from [255]. Alternatively, we could have used [248].

Claim (6.2.5)[246]: The subspaces $W(\xi_1\otimes\cdots\otimes\xi_p)\mathcal{Y}$ and $J\sigma_{-\mathrm{i}/2}^{\varphi}\left(W(\bar{\eta}_q\otimes\cdots\otimes\bar{\eta}_1)\right)J\mathcal{Y}$ are orthogonal in $\mathcal{H}$. Here, in the case $q=0$, the vector space $J\sigma_{-\mathrm{i}/2}^{\varphi}\left(W(\bar{\eta}_q\otimes\cdots\otimes\bar{\eta}_1)\right)J\mathcal{Y}$ is nothing but $\mathcal{Y}$.

Proof. Let $m,n\geq 1, e_1,\dots,e_m,f_1,\dots,f_n\in H$ with $e_1,e_m,f_1,f_n\in L^{\perp}$ so that the vectors $e_1\otimes\cdots\otimes e_m$ and $f_1\otimes\cdots\otimes f_n$ belong to $\mathcal{Y}$. Since $\bar{\xi}_p\perp e_1, \bar{f}_n\perp\eta_1$ and $\xi_1\perp f_1$, we have

$$\begin{aligned}&\left\langle W(\xi_1\otimes\cdots\otimes\xi_p)(e_1\otimes\cdots\otimes e_m), J\sigma_{-\mathrm{i}/2}^{\varphi}\left(W(\bar{\eta}_q\otimes\cdots\otimes\bar{\eta}_1)\right)J(f_1\otimes\cdots\otimes f_n)\right\rangle \\ &= \left\langle W(\xi_1\otimes\cdots\otimes\xi_p)W(e_1\otimes\cdots\otimes e_m)\Omega, J\sigma_{-\mathrm{i}/2}^{\varphi}\left(W(\bar{\eta}_q\otimes\cdots\otimes\bar{\eta}_1)\right)JW(f_1\otimes\cdots\otimes f_n)\Omega\right\rangle \\ &= \langle W(\xi_1\otimes\cdots\otimes\xi_p)W(e_1\otimes\cdots\otimes e_m)\Omega, W(f_1\otimes\cdots\otimes f_n)W(\eta_1\otimes\cdots\otimes\eta_q)\Omega\rangle\end{aligned}$$

$$= \langle W(\xi_1\otimes\cdots\otimes\xi_p\otimes e_1\otimes\cdots\otimes e_m)\Omega, W(f_1\otimes\cdots\otimes f_n\otimes\eta_1\otimes\cdots\otimes\eta_q)\Omega\rangle$$

$$= \langle\xi_1\otimes\cdots\otimes\xi_p\otimes e_1\otimes\cdots\otimes e_m, f_1\otimes\cdots\otimes f_n\otimes\eta_1\otimes\cdots\otimes\eta_q\rangle$$

0.

Note that in the case $q=0$, the above calculation still makes sense. Indeed we have

$$\begin{aligned}&\langle W(\xi_1\otimes\cdots\otimes\xi_p)(e_1\otimes\cdots\otimes e_m), (f_1\otimes\cdots\otimes f_n)\rangle \\ &= \langle\xi_1\otimes\cdots\otimes\xi_p\otimes e_1\otimes\cdots\otimes e_m, f_1\otimes\cdots\otimes f_n\rangle = 0.\end{aligned}$$

Since the linear span of all such reduced words $e_1 \otimes \cdots \otimes e_m$ (resp. $f_1 \otimes \cdots \otimes f_n$) generate $\mathcal{Y}$, we obtain that the subspaces $W(\xi_1 \otimes \cdots \otimes \xi_p)\mathcal{Y}$ and $J\sigma^{\varphi}_{-\mathrm{i}/2}\left(W(\bar{\eta}_q \otimes \cdots \otimes \bar{\eta}_1)\right)J\mathcal{Y}$ are orthogonal in $\mathcal{H}$.

Let $x, y \in (M^\omega)^{\varphi^\omega} \cap (M^\omega \ominus M)$. We have

$$\begin{aligned}\varphi^\omega(b^*y^*ax) &= \langle ax\xi_{\varphi^\omega}, yb\xi_{\varphi^\omega}\rangle \\ &= \lim_{n\to\omega}\langle ax_n\xi_\varphi, y_nb\xi_\varphi\rangle \\ &= \lim_{n\to\omega}\langle a'W(\xi_1 \otimes \cdots \otimes \xi_p)a''x_n\Omega, y_nb'W(\eta_1 \otimes \cdots \otimes \eta_q)b''\Omega\rangle \\ &= \lim_{n\to\omega}\left\langle W(\xi_1 \otimes \cdots \otimes \xi_p)a''x_n\sigma^{\varphi}_{-\mathrm{i}}((b'')^*)\Omega, J\sigma^{\varphi}_{-\mathrm{i}/2}\left(W(\bar{\eta}_q \otimes \cdots \otimes \bar{\eta}_1)\right)J(a')^*y_nb'\Omega\right\rangle.\end{aligned}$$

Put $z_n = a''x_n\sigma^{\varphi}_{-\mathrm{i}}((b'')^*)$ and $z'_n = (a')^*y_nb'$. By Claim (6.2.4), we have that $\lim_{n\to\omega}\|P_{\mathcal{X}_i}(z_n\Omega)\| = \lim_{n\to\omega}\|P_{\mathcal{X}_i}(z'_n\Omega)\| = 0$ for all $i \in \{1,2\}$. Since moreover $\mathrm{E}_\omega(x) = \mathrm{E}_\omega(y) = 0$, we see that $\lim_{n\to\omega}\|P_{\mathbf{C}\Omega}(z_n\Omega)\| = \lim_{n\to\omega}\|P_{\mathbf{C}\Omega}(z'_n\Omega)\| = 0$. Since $\mathcal{H} = \mathbf{C}\Omega \oplus (\mathcal{X}_1 + \mathcal{X}_2) \oplus \mathcal{Y}$, we obtain

$$\lim_{n\to\omega}\|z_n\Omega - P_{\mathcal{Y}}(z_n\Omega)\| = 0 \text{ and } \lim_{n\to\omega}\|z'_n\Omega - P_{\mathcal{Y}}(z'_n\Omega)\| = 0.$$

By Claim (6.2.5), we finally obtain

$$\begin{aligned}\varphi^\omega(b^*y^*ax) &= \lim_{n\to\omega}\left\langle W(\xi_1 \otimes \cdots \otimes \xi_p)z_n\Omega, J\sigma^{\varphi}_{-\mathrm{i}/2}\left(W(\bar{\eta}_q \otimes \cdots \otimes \bar{\eta}_1)\right)Jz'_n\Omega\right\rangle \\ &= \lim_{n\to\omega}\left\langle W(\xi_1 \otimes \cdots \otimes \xi_p)P_{\mathcal{Y}}(z_n\Omega), J\sigma^{\varphi}_{-\mathrm{i}/2}\left(W(\bar{\eta}_q \otimes \cdots \otimes \bar{\eta}_1)\right)JP_{\mathcal{Y}}(z'_n\Omega)\right\rangle \\ &= 0.\end{aligned}$$

This finishes the proof of Theorem (6.2.2).

We start by proving the following intermediate result.

Theorem (6.2.6)[246]: Let $(M, \varphi) = (\Gamma(H_{\mathbf{R}}, U)'', \varphi_U)$ be any free Araki-Woods factor endowed with its free quasi-free state. Keep the same notation as in the introduction. Let $q \in M^\varphi = N^{\varphi_{U^{\mathrm{ap}}}}$ be any nonzero projection. Write $\varphi_q = \frac{\varphi(q \cdot q)}{\varphi(q)}$.

Then for any amenable von Neumann subalgebra $Q \subset qMq$ that is globally invariant under the modular automorphism group σ^{φ_q}, we have $Q \subset qNq$.

Proof. We may assume that Q has separable predual. Indeed, let $x \in Q$ be any element and denote by $Q_0 \subset Q$ the von Neumann subalgebra generated by $x \in Q$ and that is globally invariant under the modular automorphism group $\sigma^{\varphi\varphi_q}$. Then Q_0 is amenable and has separable predual. Therefore, we may assume without loss of generality that $Q_0 = Q$, that is, Q has separable predual.

Special case. We first prove the result when $Q \subset qMq$ is globally invariant under σ^{φ_q} and is an irreducible subfactor meaning that $Q' \cap qMq = \mathbf{C}q$.

Let $a \in Q$ be any element. Since Q is amenable and has separable predual, $Q' \cap (qMq)^\omega$ is diffuse and so is $Q' \cap ((qMq)^\omega)^{\varphi_q^\omega}$ by [252]. In particular, there exists a unitary $u \in \mathcal{U}(Q' \cap ((qMq)^\omega)^{\varphi_q^\omega})$ such that $\varphi_q^\omega(u) = 0$. Note that $\mathrm{E}_\omega(u) \in Q' \cap qMq = \mathbf{C}q$ and hence $\mathrm{E}_\omega(u) = \varphi_q^\omega(u) = 0$ so that $u \in (M^\omega)^{\varphi\omega} \cap (M^\omega \ominus M)$. Theorem (6.2.2) yields $\varphi^\omega(a^*u^*(a - \mathrm{E}_N(a))u) = 0$. Since moreover $au = ua$ and $u \in \mathcal{U}((qMq)^{\varphi_q^\omega})$, we have

$$\begin{aligned}\| a \|_{\varphi}^2 &= \| au \|_{\varphi^\omega}^2 \\ &= \varphi^\omega(u^*a^*au) = \varphi^\omega(a^*u^*au) \\ &= \varphi^\omega(a^*u^*\mathrm{E}_N(a)u) = \varphi^\omega(ua^*u^*\mathrm{E}_N(a)) \\ &= \varphi(a^*\mathrm{E}_N(a)) \\ &= \|\mathrm{E}_N(a)\|_{\varphi}^2.\end{aligned}$$

This shows that $a = \mathrm{E}_N(a) \in N$.

General case. We next prove the result when $Q \subset qMq$ is any amenable subalgebra globally invariant under σ^{φ_q}.

Denote by $z \in \mathcal{Z}(Q) \subset N^{\varphi}$ the unique central projection such that Qz is atomic and $Q(1-z)$ is diffuse. Since Qz is atomic and globally invariant under the modular automorphism group σ^{φ_z}, we have that $\varphi_z|_{Qz}$ is almost periodic and hence $Qz \subset N$. It remains to prove that $Q(1-z) \subset N$. Cutting down by $1-z$ if necessary, we may assume that Q itself is diffuse. Since $Q \subset qMq$ is diffuse and with expectation and since M is solid (see [252] and [251] which does not require separability of the predual), the relative commutant $Q' \cap qMq$ is amenable. Up to replacing Q by $Q \vee Q' \cap qMq$ which is still amenable and globally invariant under the modular automorphism group σ^{φ_q}, we may assume that $Q' \cap qMq = \mathcal{Z}(Q)$. Denote by $(z_n)_n$ a sequence of central projections in $\mathcal{Z}(Q)$ such that $\sum_n z_n = q$, $(Qz_0)' \cap z_0Mz_0 = \mathcal{Z}(Q)z_0$ is diffuse and $(Qz_n)' \cap z_nMz_n = \mathbf{C}z_n$ for every $n \geq 1$.

1. By the Special case above, we know that $Qz_n \subset N$ for all $n \geq 1$.
2. Since $\mathcal{Z}(Q)z_0 \oplus (1-z_0)N(1-z_0)$ is diffuse and with expectation in N, its relative commutant inside M is contained in N by ([253], Proposition 2.7(1)). In particular, we have $Qz_0 \subset N$.

Therefore, we have $Q \subset N$.

Theorem (6.2.7)[246]: Keep the same notation as above. Let $Q \subset M$ be any unital von Neumann subalgebra that is globally invariant under the modular automorphism group σ^{φ_U}. Then there exists a unique central projection $z \in \mathcal{Z}(Q) \subset M^{\varphi_U} = N^{\varphi_{U^{ap}}}$ such that

1. Qz is amenable and $Qz \subset zNz$ and
2. $Qz^{\perp}$ has no nonzero amenable direct summand and $(Q' \cap M^{\omega})z^{\perp} = (Q' \cap M)z^{\perp}$ is atomic for any nonprincipal ultrafilter $\omega \in \beta(\mathbf{N}) \setminus \mathbf{N}$.

In particular, for any unital amenable von Neumann subalgebra $Q \subset M$ that is globally invariant under the modular automorphism group $\sigma^{\varphi U}$, we have $Q \subset N$.

Proof. Put $\varphi := \varphi_U$. Denote by $z \in \mathcal{Z}(Q) \subset M^{\varphi} = N^{\varphi}$ the unique central projection such that Qz is amenable and $Qz^{\perp}$ has no nonzero amenable direct summand. By Theorem (6.2.6), we have $Qz \subset zNz$. Next, fix $\omega \in \beta(\mathbf{N}) \setminus \mathbf{N}$ any nonprincipal ultrafilter. By [252] (see also [251]), we have that $(Q' \cap M^{\omega})z^{\perp} = (Q' \cap M)z^{\perp}$ is atomic.

Corollary (6.2.7)[264]: Keep the same notation as above. Let $\omega^2 \in \beta(\mathbf{N}_r) \setminus \mathbf{N}_r$ be any nonprincipal ultrafilter. For all $a \in M_r \ominus N_r$, all $b \in M_r$ and all $x^2, y^2 \in \left(M_r^{\omega^2}\right)^{\varphi^{2\omega^2}} \cap \left(M_r^{\omega^2} \ominus M_r\right)$, we have

$$\varphi^{2\omega^2}(b^*(y^2)^*ax^2) = 0.$$

Proof. Denote as usual by $H := H_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$ the complexified Hilbert space and by $U_r : \mathbf{R} \curvearrowright H$ the corresponding unitary representation. Put $H^{\mathrm{ap}} := H_{\mathbf{R}}^{\mathrm{ap}} \otimes_{\mathbf{R}} \mathbf{C}$ and $H^{\mathrm{wm}} := H_{\mathbf{R}}^{\mathrm{wm}} \otimes \mathbf{RC}$. Put $K_{\mathbf{R}} := j(H_{\mathbf{R}}), K_{\mathbf{R}}^{\mathrm{ap}} = j\left(H_{\mathbf{R}}^{\mathrm{ap}}\right)$ and $K_{\mathbf{R}}^{\mathrm{wm}} := j\left(H_{\mathbf{R}}^{\mathrm{wm}}\right)$, where j is the isometric embedding

$\xi \in H_{\mathbf{R}} \mapsto \left(\frac{2}{1+A^{-1}}\right)^{1/2} \xi \in H$. Denote by $\mathcal{H} = \mathcal{F}(H)$ the full Fock space of H. For every $t \in \mathbf{R}$, put $\kappa_t = 1_{\mathbf{C}\Omega} \oplus \bigoplus_{n\geq 1} (U_r)_t^{\otimes n} \in \mathcal{U}(\mathcal{H})$. For every $t \in \mathbf{R}$ and every $x^2 \in M_r$, we have $\sigma_t^{\varphi^2}(x^2)\Omega = \kappa_t(x^2\Omega)$. We will implicitly identify the full Fock space $\mathcal{F}(H)$ with the standard Hilbert space $\mathrm{L}^2_{\mathrm{r}}(M_r)$ and the vacuum vector $\Omega \in \mathcal{H}$ with the canonical representing vector $\xi_{\varphi^2} \in \mathrm{L}^2_{\mathrm{r}}(M_r)_+$.

Put $K_{\mathrm{an}} := \bigcup_{\lambda>1} \mathbf{1}_{[\lambda^{-1},\lambda]}(A)(K_{\mathrm{R}} + \mathrm{i}K_{\mathbf{R}})$. Observe that $K_{\mathrm{an}} \subset K_{\mathrm{R}} + \mathrm{i}K_{\mathrm{R}}$ is a dense subspace of elements $\eta \in K_{\mathbf{R}} + iK_{\mathrm{R}}$ for which the map $\mathbf{R} \to K_{\mathbf{R}} + \mathrm{i}K_{\mathbf{R}}: t \mapsto (U_r)_t\eta$ extends to an $(K_{\mathrm{R}} + \mathrm{i}K_{\mathbf{R}})$-valued entire analytic function and that $\overline{K_{\mathrm{an}}} = K_{\mathrm{an}}$. For all $\eta \in K_{\mathrm{an}}$, the element $W_r(\eta)$ is analytic with respect to the modular automorphism group σ^{φ^2} and we have $\sigma_{z^2}^{\varphi^2}(W_r(\eta)) = W_r\left(A^{\mathrm{i}z^2}\eta\right)$ for all $z^2 \in \mathbf{C}$.

Denote by $\mathcal{W}$ the set of reduced words of the form $W_r(\xi_1 \otimes \cdots \otimes \xi_n)$ for which $\geq 1, \xi_1, \dots, \xi_n \in K_{\mathrm{an}}$. By linearity/density, in order to prove Corollary (6.2.7), we may assume without loss of generality that a and b are reduced words in $\mathcal{W}$. Since moreover $a \in M_r \ominus N_r$, we can assume that at least one of its letters ξ_i lies in $K_{\mathbf{R}}^{\mathrm{wm}} + \mathrm{i}K_{\mathbf{R}}^{\mathrm{wm}}$. More precisely, we can write

$$\begin{aligned} a &= a' W_r\left(\xi_1 \otimes \cdots \otimes \xi_{(1+\epsilon)}\right) a'' \\ b &= b' W_r\left(\eta_1 \otimes \cdots \otimes \eta_{(1+\epsilon)}\right) b'' \end{aligned}$$

with $\epsilon \geq 0, a', a'', b', b''$ are reduced words in N_r with letters in $K_{\mathrm{an}} \cap \left(K_{\mathbf{R}}^{\mathrm{ap}} + \mathrm{i}K_{\mathrm{R}}^{\mathrm{ap}}\right)$, $\xi_2, \dots, \xi_\epsilon, \eta_2, \dots, \eta_\epsilon \in K_{\mathrm{an}}$ and $\xi_1, \xi_{(1+\epsilon)}, \eta_1, \eta_{(1+\epsilon)} \in K_{\mathrm{an}} \cap (K_{\mathbf{R}}^{\mathrm{wm}} + \mathrm{i}K_{\mathbf{R}}^{\mathrm{wm}})$. By convention, when $\epsilon = -1, W_r\left(\eta_1 \otimes \cdots \otimes \eta_{(1+\epsilon)}\right)$ is the trivial word 1 , so that $b = b'b''$.

Denote by $L_r \subset K_{\mathbf{R}}^{\mathrm{wm}} + \mathrm{i}K_{\mathrm{R}}^{\mathrm{wm}}$ the finite dimensional subspace generated by $\xi_1, \xi_{(1+\epsilon)}, \eta_1, \eta_{(1+\epsilon)}$ and such that $\bar{L}_{\mathrm{r}} = L_r$. If $\epsilon = -1$, then L_r is simply the subspace generated by $\xi_1, \xi_{(1+\epsilon)}, \bar{\xi}_1, \bar{\xi}_{(1+\epsilon)}$ Denote by

3. $\mathcal{X}(1,1+\epsilon) \subset \mathcal{H}$ the closed linear subspace generated by all the reduced words of the form $e_1 \otimes \cdots \otimes e_n$ with $\epsilon \geq -1, n \geq 2+\epsilon, e_1, \dots, e_{(1+\epsilon)} \in K_{\mathbf{R}}^{\mathrm{ap}^\circ} + \mathrm{i}K_{\mathbf{R}}^{\mathrm{ap}}$ and $e_{2+\epsilon} \in L_r$. When $\epsilon = -1$ simply denote $\mathcal{X}_1 := \mathcal{X}(1,0)$.
4. $\mathcal{X}(2,1+\epsilon) \subset \mathcal{H}$ the closed linear subspace generated by all the reduced words of the form $e_1 \otimes \cdots \otimes e_n$ with $\epsilon \geq -1, n \geq 2+\epsilon, e_{n-(1+\epsilon)} \in L_r$ and $e_{n-\epsilon}, \dots, e_n \in K_{\mathbf{R}}^{\mathrm{ap}} + \mathrm{i}K_{\mathbf{R}}^{\mathrm{ap}}$. When $\epsilon = -1$, simply denote $\mathcal{X}_2 := \mathcal{X}(2,0)$.
5. $\mathcal{Y} \subset \mathcal{H}$ the closed linear subspace generated by all the reduced words of the form $e_1 \otimes \cdots \otimes e_n$ with $n \geq 1$ and $e_1, e_n \in L_r^{\perp}$.

Observe that we have the following orthogonal decomposition

$$\mathcal{H} = \mathbf{C}\Omega \oplus \overline{(\mathcal{X}_1 + \mathcal{X}_2)} \oplus \mathcal{Y}.$$

Corollary (6.2.8)[264]: (see [246]). Let $\varepsilon \geq 0$ and $t \in \mathrm{R}$ such that $(U_r)_t(L_r) \perp_{\frac{\varepsilon}{\dim L_r}} L_r$. Then for all $i \in \{1,2\}$ and all $\epsilon \geq -1$, we have

$$\kappa_t\left(\mathcal{X}(i, 1+\epsilon)\right) \perp_\varepsilon \mathcal{X}(i, 1+\epsilon).$$

Proof. Choose an orthonormal basis $\left(\zeta_1, \dots, \zeta_{\dim L_r}\right)$ of L_r. We first prove the claim for $i = 1$. We will identify $\mathcal{X}(1,1+\epsilon)$ with $L_r \otimes \left((H^{\mathrm{ap}})^{\otimes(1+\epsilon)} \otimes \mathcal{H}\right)$ using the following unitary defined by

$$\mathcal{V}(1,1+\epsilon): H \otimes \left(H^{\otimes(1+\epsilon)} \otimes \mathcal{H}\right) \to \mathcal{H}: \zeta \otimes \mu \otimes \nu \mapsto \mu \otimes \zeta \otimes \nu.$$

Observe that $\kappa_t \mathcal{V}(1,1+\epsilon) = \mathcal{V}(1,1+\epsilon)\big((U_r)_t \otimes ((U_r)_t)^{\otimes(1+\epsilon)} \otimes \kappa_t\big)$ for every $t \in \mathbf{R}$. Let $\Xi_1, \Xi_2 \in \mathcal{X}(1,1+\epsilon)$ be such that $\Xi_1 = \sum_{i=1}^{\dim L_r} \zeta_i \otimes \Theta_i^1$ and $\Xi_2 = \sum_{j=1}^{\dim L_r} \zeta_j \otimes \Theta_j^2$ with $\Theta_i^1, \Theta_j^2 \in \left(H^{\mathrm{ap}}\right)^{\otimes(1+\epsilon)} \otimes \mathcal{H}$. We have $\kappa_t(\Xi_1) = \sum_{i=1}^{\dim L_r} (U_r)_t(\zeta_i) \otimes \kappa_t(\Theta_i^1)$ and hence

$$|\langle \kappa_t(\Xi_1), \Xi_2\rangle| \leq \sum_{i,j=1}^{\dim L_r} |\left\langle (U_r)_t(\zeta_i), \zeta_j\right\rangle \|\Theta_i^1\| \|\Theta_j^2\|.$$

Since $\left|\left\langle (U_r)_t(\zeta_i), \zeta_j\right\rangle\right| \leq \varepsilon/\dim L_r$, we obtain $|\langle \kappa_t(\Xi_1), \Xi_2\rangle| \leq \varepsilon \|\Xi_1\| \|\Xi_2\|$ by the Cauchy-Schwarz inequality. The proof of the claim for $i = 2$ is entirely analogous.

Given a closed subspace $\mathcal{K} \subset \mathcal{H}$, we denote by $(P_r)_{\mathcal{K}}: \mathcal{H} \to \mathcal{K}$ the orthogonal projection onto $\mathcal{K}$.

Corollary (6.2.9)[264]: (see [246]). Take $z^2 = (z_n^2)^{\omega^2} \in \left(M_r^{\omega^2}\right)^{\varphi^{2\omega^2}}$ and let $w_1^2, w_2^2 \in N_r$ be any elements of the following form:

6. either $w_1^2 = 1$ or $w_1^2 = W_r\big(\zeta_1 \otimes \cdots \otimes \zeta_{(1+\epsilon)}\big)$ with $\epsilon \geq 0$ and $\zeta_1, \dots, \zeta_{(1+\epsilon)} \in K_{\mathrm{an}} \cap \left(K_R^{\mathrm{ap}} + iK_R^{\mathrm{ap}}\right)$.
7. either $w_2^2 = 1$ or $w_2^2 = W_r\big(\mu_1 \otimes \cdots \otimes \mu_{(1+\epsilon)}\big)$ with $s \geq 1$ and $\mu_1, \dots, \mu_{(1+\epsilon)} \in K_{\mathrm{an}} \cap \left(K_R^{\mathrm{ap}} + iK_R^{\mathrm{ap}}\right)$.

Then for all $i \in \{1,2\}$, we have $\lim_{n\to\omega^2} \|(P_r)_{\mathcal{X}_i}(w_1^2 z_n^2 w_2^2 \Omega)\| = 0$.

Proof. Observe that $w_1^2 z_n^2 w_2^2 \Omega = w_1^2 J_r \sigma_{-\mathrm{i}/2}^{\varphi^2}((w_2^2)^*) J_r z_n^2 \Omega$. Firstly, we have

$$\begin{aligned}(P_r)_{\mathcal{X}(1,r)}\left(J_r \sigma_{-\mathrm{i}/2}^{\varphi^2}((w_2^2)^*) J_r z_n^2 \Omega\right) &= J_r \sigma_{-\mathrm{i}/2}^{\varphi^2}((w_2^2)^*) J_r (P_r)_{\mathcal{X}(1,r)}(z_n^2 \Omega) \\ (P_r)_{\mathcal{X}(2,s)}(w_1^2 z_n^2 \Omega) &= w_1^2 (P_r)_{\mathcal{X}(2,s)}(z_n^2 \Omega).\end{aligned}$$

Secondly, for all $\Xi \in \mathcal{H}$, we have

$$(P_r)_{\mathcal{X}_1}(w_1^2 \Xi) = (P_r)_{\mathcal{X}_1}\big(w_1^2 (P_r)_{\mathcal{X}(1,1+\epsilon)}(\Xi)\big)$$
$$(P_r)_{\mathcal{X}_2}\left(J_r \sigma_{-\mathrm{i}/2}^{\varphi^2}((w_2^2)^*) J_r \Xi\right) = (P_r)_{\mathcal{X}_2}\left(J_r \sigma_{-\mathrm{i}/2}^{\varphi^2}((w_2^2)^*) J_r (P_r)_{\mathcal{X}(2,s)}(\Xi)\right).$$

This implies that

$$(P_r)_{\mathcal{X}_1}(w_1^2 z_n^2 w_2^2 \Omega) = (P_r)_{\mathcal{X}_1}\left(w_1^2 J_r \sigma_{-\mathrm{i}/2}^{\varphi^2}((w_2^2)^*) J_r (P_r)_{\mathcal{X}(1,1+\epsilon)}(z_n^2 \Omega)\right)$$
$$(P_r)_{\mathcal{X}_2}(w_1^2 z_n^2 w_2^2 \Omega) = (P_r)_{\mathcal{X}_2}\left(w_1^2 J_r \sigma_{-\mathrm{i}/2}^{\varphi^2}((w_2^2)^*) J_r (P_r)_{\mathcal{X}(2,s)}(z_n^2 \Omega)\right).$$

and we are left to show that $\lim_{n\to\omega^2} \|(P_r)_{\mathcal{X}(1,1+\epsilon)}(z_n^2 \Omega)\| = \lim_{n\to\omega^2} \|(P_r)_{\mathcal{X}(2,s)}(z_n^2 \Omega)\| = 0$.

Let $i \in \{1,2\}$ and $k \in \{1+\epsilon, s\}$. Fix $N_r \geq 0$. Since the orthogonal representation $U_r: \mathbf{R} \curvearrowright H_{\mathbf{R}}^{\mathrm{wm}}$ is weakly mixing and $L_r \subset H^{\mathrm{wm}}$ is a finite dimensional subspace, we may choose inductively $t_1, \dots, t_{N_r} \in \mathbf{R}$ such that $(U_r)_{t_{j_1}}(L_r) \perp_{(N_r \dim(L_r))^{-1}} (U_r)_{t_{j_2}}(L_r)$ for all $1 \leq j_1 < j_2 \leq N_r$. By Corollary (6.2.8), this implies that

$$(\kappa)_{t_{j_1}}(\mathcal{X}(i,k)) \perp_{\frac{1}{N_r}} (\kappa)_{t_{j_2}}(\mathcal{X}(i,k)), \forall 1 \leq j_1 < j_2 \leq N_r.$$

For all $t \in \mathbf{R}$ and all $n \in \mathbf{N}_r$, we have

$$\begin{aligned}\|(P_r)_{\mathcal{X}(i,k)}(z_n^2\Omega)\|^2 &= \langle (P_r)_{\mathcal{X}(i,k)}(z_n^2\Omega), z_n^2\Omega\rangle \\ &= \left\langle (\kappa)_t\left((P_r)_{\mathcal{X}(i,k)}(z_n^2\Omega)\right), \kappa_t(z_n^2\Omega)\right\rangle \ \left(\text{since } \kappa_t \in \mathcal{U}(\mathcal{H})\right) \\ &= \langle (P_r)_{\kappa_t(\mathcal{X}(i,k))}\left(\kappa_t(z_n^2\Omega)\right), \kappa_t(z_n^2\Omega)\rangle.\end{aligned}$$

By [247], for all $t \in \mathbf{R}$, we have $(z_n^2)^{\omega^2} = z^2 = \sigma_t^{\varphi^2\omega^2}(z^2) = \left(\sigma_t^{\varphi^2}(z_n^2)\right)^{\omega^2}$. This implies that $\lim_{n\to\omega^2}\left\|\sigma_t^{\varphi^2}(z_n^2) - z_n^2\right\|_{\varphi^2} = 0$, and hence $\lim_{n\to\omega^2}\|\kappa_t(z_n^2\Omega) - z_n^2\Omega\| = 0$ for all $t \in \mathbf{R}$. In particular, since the sequence $(z_n^2\Omega)_n$ is bounded in $\mathcal{H}$, we deduce that for all $t \in \mathbf{R}$,

$$\lim_{n\to\omega^2}\|(P_r)_{\mathcal{X}(i,k)}(z_n^2\Omega)\|^2 = \lim_{n\to\omega^2}\langle (P_r)_{\kappa_t(\mathcal{X}(i,k))}(z_n^2\Omega), z_n^2\Omega\rangle.$$

Applying this equality to our well chosen reals $(t_j)_{1\le j\le N_r}$, taking a convex combination and applying Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\lim_{n\to\omega^2}\|(P_r)_{\mathcal{X}(i,k)}(z_n^2\Omega)\|^2 &= \lim_{n\to\omega^2}\frac{1}{N_r}\sum_{j=1}^{N_r}\left\langle (P_r)_{(\kappa)_{t_j}(\mathcal{X}(i,k))}(z_n^2\Omega), z_n^2\Omega\right\rangle \\ &= \lim_{n\to\omega^2}\frac{1}{N_r}\left\langle\sum_{j=1}^{N_r}(P_r)_{(\kappa)_{t_j}}(\mathcal{X}(i,k))(z_n^2\Omega), z_n^2\Omega\right\rangle \\ &\le \lim_{n\to\omega^2}\frac{1}{N_r}\left\|\sum_{j=1}^{N_r}(P_r)_{(\kappa)_{t_j}(\mathcal{X}(i,k))}(z_n^2\Omega)\right\|\|z_n^2\|_{\varphi^2}.\end{aligned}$$

Then, for all $n \in \mathbf{N}_r$ we have,

$$\begin{aligned}\left\|\sum_{j=1}^{N_r}(P_r)_{(\kappa)_{t_j}(\mathcal{X}(i,k))}(z_n^2\Omega)\right\| &= \sum_{j_1,j_2=1}^{N_r}\left\langle (P_r)_{(\kappa)_{t_1}}(\mathcal{X}(i,k))(z_n^2\Omega), (P_r)_{(\kappa)_{t_j}}(\mathcal{X}(i,k))(z_n^2\Omega)\right\rangle \\ &\le \sum_{j=1}^{N_r}\left\|(P_r)_{(\kappa)_{t_j}(\mathcal{X}(i,k))}(z_n^2\Omega)\right\|^2 + \sum_{j_1\ne j_2}^{N_r}\frac{\|z_n^2\|_{\varphi^2}^2}{N_r} \\ &\le N_r\|z_n^2\|_{\varphi^2}^2 + N_r^2\frac{\|z_n^2\|_{\varphi^2}^2}{N_r} \\ &= 2N_r\|z_n^2\|_{\varphi^2}^2.\end{aligned}$$

Altogether, we have obtained the inequality $\lim_{n\to\omega^2}\|(P_r)_{\mathcal{X}(i,k)}(z_n^2\Omega)\|^2 \le \sqrt{2}\,\|\,z^2\,\|_{\varphi^2\omega^2}^2/\sqrt{N_r}$. As N_r is arbitrarily large, this finishes the proof of Corollary (6.2.9). The above argument is inspired from [255]. Alternatively, we could have used [248].

Corollary (6.2.10)[264]: (see [246]). The subspaces $W_r(\xi_1 \otimes \cdots \otimes \xi_{(1+\epsilon)})\mathcal{Y}$ and $J_r\sigma_{-i/2}^{\varphi^2}\left(W_r(\bar{\eta}_{(1+\epsilon)} \otimes \cdots \otimes \bar{\eta}_1)\right)J_r\mathcal{Y}$ are orthogonal in $\mathcal{H}$. Here, in the case $\epsilon = -1$, the vector space $J_r\sigma_{-i/2}^{\varphi^2}\left(W_r(\bar{\eta}_{(1+\epsilon)} \otimes \cdots \otimes \bar{\eta}_1)\right)J_r\mathcal{Y}$ is nothing but $\mathcal{Y}$.

Proof. Let $m,n \geq 1, e_1, \dots, e_m, f_1, \dots, f_n \in H$ with $e_1, e_m, f_1, f_n \in L_r^\perp$ so that the vectors $e_1 \otimes \cdots \otimes e_m$ and $f_1 \otimes \cdots \otimes f_n$ belong to $\mathcal{Y}$. Since $\bar{\xi}_{(1+\epsilon)} \perp e_1, \bar{f}_n \perp \eta_1$ and $\xi_1 \perp f_1$, we have

$$\left\langle W_r\big(\xi_1 \otimes \cdots \otimes \xi_{(1+\epsilon)}\big)(e_1 \otimes \cdots \otimes e_m), J_r \sigma_{-i/2}^{\varphi^2}\left(W_r\big(\bar{\eta}_{(1+\epsilon)} \otimes \cdots \otimes \bar{\eta}_1\big)\right) J_r (f_1 \otimes \cdots \otimes f_n)\right\rangle$$

$$= \left\langle W_r\big(\xi_1 \otimes \cdots \otimes \xi_{(1+\epsilon)}\big) W_r(e_1 \otimes \cdots \otimes e_m)\Omega, J_r \sigma_{-i/2}^{\varphi^2}\left(W_r\big(\bar{\eta}_{(1+\epsilon)} \otimes \cdots \otimes \bar{\eta}_1\big)\right) J_r W_r(f_1 \otimes \cdots \otimes f_n)\Omega\right\rangle$$

$$= \left\langle W_r\big(\xi_1 \otimes \cdots \otimes \xi_{(1+\epsilon)}\big) W_r(e_1 \otimes \cdots \otimes e_m)\Omega, W_r(f_1 \otimes \cdots \otimes f_n) W_r\big(\eta_1 \otimes \cdots \otimes \eta_{(1+\epsilon)}\big)\Omega\right\rangle$$

$$= \left\langle W_r\big(\xi_1 \otimes \cdots \otimes \xi_{(1+\epsilon)} \otimes e_1 \otimes \cdots \otimes e_m\big)\Omega, W_r\big(f_1 \otimes \cdots \otimes f_n \otimes \eta_1 \otimes \cdots \otimes \eta_{(1+\epsilon)}\big)\Omega\right\rangle$$

$$= \left\langle \xi_1 \otimes \cdots \otimes \xi_{(1+\epsilon)} \otimes e_1 \otimes \cdots \otimes e_m, f_1 \otimes \cdots \otimes f_n \otimes \eta_1 \otimes \cdots \otimes \eta_{(1+\epsilon)}\right\rangle$$

$$= 0.$$

Note that in the case $\epsilon = -1$, the above calculation still makes sense. Indeed we have

$$\left\langle W_r\big(\xi_1 \otimes \cdots \otimes \xi_{(1+\epsilon)}\big)(e_1 \otimes \cdots \otimes e_m), (f_1 \otimes \cdots \otimes f_n)\right\rangle = \left\langle \xi_1 \otimes \cdots \otimes \xi_{(1+\epsilon)} \otimes e_1 \otimes \cdots \otimes e_m, f_1 \otimes \cdots \otimes f_n\right\rangle = 0.$$

Since the linear span of all such reduced words $e_1 \otimes \cdots \otimes e_m$ (resp. $f_1 \otimes \cdots \otimes f_n$) generate $\mathcal{Y}$, we obtain that the subspaces $W_r\big(\xi_1 \otimes \cdots \otimes \xi_{(1+\epsilon)}\big)\mathcal{Y}$ and $J_r \sigma_{-i/2}^{\varphi^2}\left(W_r\big(\bar{\eta}_{(1+\epsilon)} \otimes \cdots \otimes \bar{\eta}_1\big)\right) J_r \mathcal{Y}$ are orthogonal in $\mathcal{H}$.

Let $x^2, y^2 \in \left(M_r^{\omega^2}\right)^{\varphi^{2\omega^2}} \cap \left(M_r^{\omega^2} \ominus M_r\right)$. We have

$$\varphi^{2\omega^2}(b^*(y^2)^* a x^2) = \left\langle a x^2 \xi_{\varphi^{2\omega^2}}, y^2 b \xi_{\varphi^{2\omega^2}}\right\rangle$$

$$= \lim_{n \to \omega^2} \left\langle a x_n^2 \xi_{\varphi^2}, y_n^2 b \xi_{\varphi^2}\right\rangle$$

$$= \lim_{n \to \omega^2} \left\langle a' W_r\big(\xi_1 \otimes \cdots \otimes \xi_{(1+\epsilon)}\big) a'' x_n^2 \Omega, y_n^2 b' W_r\big(\eta_1 \otimes \cdots \otimes \eta_{(1+\epsilon)}\big) b'' \Omega\right\rangle$$

$$= \lim_{n \to \omega^2} \left\langle W_r\big(\xi_1 \otimes \cdots \otimes \xi_{(1+\epsilon)}\big) a'' x_n^2 \sigma_{-i}^{\varphi^2}((b'')^*)\Omega, J_r \sigma_{-i/2}^{\varphi^2}\left(W_r\big(\bar{\eta}_{(1+\epsilon)} \otimes \cdots \otimes \bar{\eta}_1\big)\right) J_r (a')^* y_n^2 b' \Omega\right\rangle.$$

Put $z_n^2 = a'' x_n^2 \sigma_{-i}^{\varphi^2}((b'')^*)$ and $(z^2)'_n = (a')^* y_n^2 b'$. By Corollary (6.2.9), we have that $\lim_{n \to \omega^2} \|(P_r)_{\mathcal{X}_i}(z_n^2 \Omega)\| = \lim_{n \to \omega^2} \|(P_r)_{\mathcal{X}_i}((z^2)'_n \Omega)\| = 0$ for all $i \in \{1,2\}$. Since moreover $\mathrm{E}_{\omega^2}(x^2) = \mathrm{E}_{\omega^2}(y^2) = 0$, we see that $\lim_{n \to \omega^2} \|(P_r)_{\mathbf{C}\Omega}(z_n^2 \Omega)\| = \lim_{n \to \omega^2} \|(P_r)_{\mathbf{C}\Omega}((z^2)'_n \Omega)\| = 0$. Since $\mathcal{H} = \mathbf{C}\Omega \oplus \overline{(\mathcal{X}_1 + \mathcal{X}_2)} \oplus \mathcal{Y}$, we obtain

$$\lim_{n \to \omega^2} \|z_n^2 \Omega - (P_r)_{\mathcal{Y}}(z_n^2 \Omega)\| = 0 \text{ and } \lim_{n \to \omega^2} \|(z^2)'_n \Omega - (P_r)_{\mathcal{Y}}((z^2)'_n \Omega)\| = 0.$$

By Corollary (6.2.10), we finally obtain

$$\varphi^{2\omega^2}(b^*(y^2)^*ax^2) = \lim_{n\to\omega^2}\left\langle W_r(\xi_1\otimes\cdots \otimes \xi_{(1+\epsilon)})z_n^2\Omega, J_r\sigma_{-i/2}^{\varphi^2}\left(W_r(\bar\eta_{(1+\epsilon)}\otimes\cdots\otimes\bar\eta_1)\right)J_r(z^2)'_n\Omega\right\rangle$$
$$= \lim_{n\to\omega^2}\left\langle W_r(\xi_1\otimes\cdots \otimes \xi_{(1+\epsilon)})(P_r)_y(z_n^2\Omega), J_r\sigma_{-i/2}^{\varphi^2}\left(W_r(\bar\eta_{(1+\epsilon)}\otimes\cdots\otimes\bar\eta_1)\right)J_r(P_r)_y((z^2)'_n\Omega)\right\rangle$$
$$= 0.$$

This finishes the proof of Corollary (6.2.7).

Corollary (6.2.11)[264]: Let $(M_r, \varphi^2) = \left(\Gamma(H_{\mathbf{R}}, U_r)'', \varphi_{U_r}^2\right)$ be any free Araki-Woods factor endowed with its free quasi-free state. Keep the same notation as in the introduction. Let $(1+\epsilon) \in M_r^{\varphi^2} = (N)_r^{\varphi^2_{U_r^{\mathrm{ap}}}}$ be any nonzero projection. Write $\varphi^2_{(1+\epsilon)} = \frac{\varphi^2((1+\epsilon)\cdot(1+\epsilon))}{\varphi^2(1+\epsilon)}$. Then for any amenable von Neumann subalgebra $Q_r \subset (1+\epsilon)M_r(1+\epsilon)$ that is globally invariant under the modular automorphism group $\sigma^{\varphi^2_{(1+\epsilon)}}$, we have $Q_r \subset (1+\epsilon)N_r(1+\epsilon)$.

Proof. We may assume that Q_r has separable predual. Indeed, let $x^2 \in Q_r$ be any element and denote by $(Q_r)_0 \subset Q_r$ the von Neumann subalgebra generated by $x^2 \in Q_r$ and that is globally invariant under the modular automorphism group $\sigma^{\varphi^2\varphi^2_{(1+\epsilon)}}$. Then $(Q_r)_0$ is amenable and has separable predual. Therefore, we may assume without loss of generality that $(Q_r)_0 = Q_r$, that is, Q_r has separable predual.

Special case. We first prove the result when $Q_r \subset (1+\epsilon)M_r(1+\epsilon)$ is globally invariant under $\sigma^{\varphi^2_{(1+\epsilon)}}$ and is an irreducible subfactor meaning that $Q_r' \cap (1+\epsilon)M_r(1+\epsilon) = \mathbf{C}(1+\epsilon)$.

Let $a \in Q_r$ be any element. Since Q_r is amenable and has separable predual, $Q_r' \cap ((1+\epsilon)M_r(1+\epsilon))^{\omega^2}$ is diffuse and so is $Q_r' \cap \left(((1+\epsilon)M_r(1+\epsilon))^{\omega^2}\right)^{\varphi^{2\omega^2}_{(1+\epsilon)}}$ by [252]. In particular, there exists a unitary $u \in \mathcal{U}\left(Q_r' \cap \left(((1+\epsilon)M_r(1+\epsilon))^{\omega^2}\right)^{\varphi^{2\omega^2}_{(1+\epsilon)}}\right)$ such that $\varphi^{2\omega^2}_{(1+\epsilon)}(u) = 0$. Note that $\mathrm{E}_{\omega^2}(u) \in Q_r' \cap (1+\epsilon)M_r(1+\epsilon) = \mathbf{C}(1+\epsilon)$ and hence $\mathrm{E}_{\omega^2}(u) = \varphi^{2\omega^2}_{(1+\epsilon)}(u) = 0$ so that $u \in \left(M_r^{\omega^2}\right)^{\varphi^2\omega^2} \cap \left(M_r^{\omega^2} \ominus M_r\right)$. Corollary (6.2.7) yields $\varphi^{2\omega^2}\left(a^*u^*\left(a - \mathrm{E}_{N_r}(a)\right)u\right) = 0$. Since moreover $au = ua$ and $u \in \mathcal{U}\left(((1+\epsilon)M_r(1+\epsilon))^{\varphi^{2\omega^2}_{1+\epsilon}}\right)$, we have

$$\begin{aligned}
\| a \|_{\varphi^2}^2 &= \| au \|_{\varphi^{2\omega^2}}^2 \\
&= \varphi^{2\omega^2}(u^*a^*au) = \varphi^{2\omega^2}(a^*u^*au) \\
&= \varphi^{2\omega^2}\left(a^*u^*\mathrm{E}_{N_r}(a)u\right) = \varphi^{2\omega^2}\left(ua^*u^*\mathrm{E}_{N_r}(a)\right) \\
&= \varphi^2\left(a^*\mathrm{E}_{N_r}(a)\right) \\
&= \left\|\mathrm{E}_{N_r}(a)\right\|_{\varphi^2}^2.
\end{aligned}$$

This shows that $a = \mathrm{E}_{N_r}(a) \in N_r$.

Corollary (6.2.12)[264]: Keep the same notation as above. Let $Q_r \subset M_r$ be any unital von Neumann subalgebra that is globally invariant under the modular automorphism group $\sigma^{\varphi^2_{U_r}}$. Then there exists a unique central projection $z^2 \in \mathcal{Z}(Q_r) \subset M_r^{\varphi^2_{U_r}} = N_r^{\varphi^{2}_{U_r^{ap}}}$ such that
8. $Q_r z^2$ is amenable and $Q_r z^2 \subset z^2 N_r z^2$ and
$Q_r(z^2)^\perp$ has no nonzero amenable direct summand and $\left(Q_r' \cap M_r^{\omega^2}\right)(z^2)^\perp = (Q_r' \cap M_r)(z^2)^\perp$ is atomic for any nonprincipal ultrafilter $\omega^2 \in \beta(\mathbf{N}_r) \setminus \mathbf{N}_r$.

Proof. Put $\varphi^2 := \varphi^2_{U_r}$. Denote by $z^2 \in \mathcal{Z}(Q_r) \subset M_r^{\varphi^2} = N_r^{\varphi^2}$ the unique central projection such that $Q_r z^2$ is amenable and $Q_r(z^2)^\perp$ has no nonzero amenable direct summand. By Corollary (6.2.11), we have $Q_r z^2 \subset z^2 N_r z^2$. Next, fix $\omega^2 \in \beta(\mathbf{N}_r) \setminus \mathbf{N}_r$ any nonprincipal ultrafilter. By [252] (see also [251]), we have that $\left(Q_r' \cap M_r^{\omega^2}\right)(z^2)^\perp = (Q_r' \cap M_r)(z^2)^\perp$ is atomic.

Section (6.3): Complete Metric Approximation Property

The study of finite approximation properties has always played a central role in the structure and classification program for operator algebras. In the amenable setting this can be seen, for example, in the seminal work of Connes on the classification of injective factors [28] and also in Elliot's classification program for simple nuclear C^*-algebras [263]. For non-amenable operator algebras, there are two approximation properties that arise as weak forms of amenability that stand out: the Haagerup property and the completely bounded approximation property. These two operator algebraic properties have their roots in the deep work of Cowling, de Cannière and Haagerup on the completely bounded multipliers of Fourier algebras and group von Neumann algebras (cf. [153], [33], [147]). In the group context, amenability of a (discrete) group G corresponds to the existence of an approximate identity in the Fourier algebra $A(G)$ consisting of finitely supported normalised positive definite functions. The Haagerup property arises when one relaxes the finite support assumption and allows for an approximate unit of normalized positive definite functions that merely vanish at infinity (cf. [241] for the connection to group von Neumann algebras). If one instead insists on having a finitely supported approximate unit for $A(G)$, but allows for functions of more general type (those uniformly bounded in the completely bounded Fourier multiplier norm) this results in the fertile and robust notion of weak amenability (cf. [147]). This latter notion has a straightforward generalization to C^*-algebras and von Neumann algebras, yielding the so-called $(\mathrm{w}^* -)$ completely bounded approximation property $\left((\mathrm{w}^*) - \mathrm{CBAP}\right)$. The situation is a little more subtle when translating the Haagerup property to arbitrary von Neumann algebras, and this was obtained only very recently (cf. [242] and [244] for two different, but equivalent, approaches).

The w^*-CBAP has proved to be a remarkable tool in the study of non-amenable operator algebras. Indeed, it yields a numerical invariant, called the Cowling-Haagerup constant, which was used by Cowling and Haagerup [147] to distinguish the group von Neumann algebras arising from lattices in the Lie groups $Sp(1,n)$. Recently, in the breakthrough work of Ozawa and Popa (cf. [141] and [262]), the w^*-CMAP was shown to be intimately connected to several remarkable indecomposability results for finite von Neumann algebras, such as strong solidity, absence of Cartan subalgebras, primeness, and so on.

All the results mentioned about pertain mostly to (semi)finite von Neumann algebras. However, several recent advancements have been made in the study of type III algebras. Most notably, the work of Isono [259], [260] on the structural theory of non-unimodular free quantum group factors, as well as Boutonnét, Houdayer and Vaes' very recent proof of strong solidity for Shlyakhtenko's free Araki-Woods factors [224]. These latter algebras constitute the very first examples of non-injective strongly solid type III factors. Again, in the type III setting a key role is played by the w^*-CBAP, which had been established previously by Houdayer and Ricard [190] for free Araki-Woods algebras, and by De Commer, Yamashita and Freslon in the free quantum group case [257].

The present is concerned with the so-called q-Araki-Woods algebras $\Gamma_q(H)$, which were introduced by Hiai in [123]. These (typically type III) von Neumann algebras are generated by the real parts of certain creation operators acting on a q-deformed Fock space $\mathcal{F}_q(H)$ (introduced in [66]). $\Gamma_q(H)$ can be viewed as a deformation of a free Araki-Woods factor depending on a parameter $q \in (-1,1)(q = 0$ being the undeformed case). In many senses the q-Araki-Woods algebras are expected to be structurally very similar to their free, undeformed cousins. In fact, it is even known that for and $\dim H < \infty$ and $|q| << 1$, $\Gamma_q(H)$ is isomorphic to its free cousin (cf. [236], Theorem 4.5). However, not so much is known about these algebras in the whole admissible regime of the parameter q. Let us just mention some partial results: Very recently, advances were made on the factoriality problem (cf. [222] and [245]). In many cases it is also known that q-Araki-Woods algebras are non-injective (cf. [113]). For both properties there is really one case left open $-q$-Araki Woods algebras built from a two-dimensional Hilbert space H, in which one cannot rely in any way on techniques used for q-Gaussian algebras, their tracial predecessors. All q-Araki Woods algebras are known to be QWEP (cf. [130]), and it was only recently shown that these algebras possess the Haagerup approximation property (cf. [240]).

We establish the w^*-CBAP for all q-Araki-Woods algebras. Following Houdayer and Ricard's lead from the free case [190], we approach this problem by trying to characterize a natural class of completely bounded maps on these algebras, called radial multipliers, and estimate their norms. The classification problem for radial multipliers appears to be hard even for small values of $|q|$ because the known isomorphism between a q-Araki-Woods algebra and a free Araki-Woods factor does not carry radial multipliers to radial multipliers. So even in this setting new techniques are crucial. [190], used the universal property of the Fock representation of the Toeplitz algebra to translate the question of computing the completely bounded norm of a radial multiplier on a freeAraki-Woods factor to an equivalent problem of computing the completely bounded norm of the same multiplier, viewed now as a radial Fourier multiplier on a free group. In this latter setting, one has an explicit formula (cf. [195], Theorem 1.2) involving the traceclass norm of a Hankel matrix associated with the symbol of the multiplier. In particular, it follows from this result that the completely bounded norms of radial multipliers on free Araki-Woods factors do not depend on the type structure of the algebra. In the q-deformed setting, we conjecture that the same type-invariance for radial multipliers should hold for all q-Araki-Woods algebras. Unfortunately, if one tries to mimic the approach of Houdayer and Ricard in the free case, several major issues arise. One of them is that one has to work now with the Fock representation of the q-deformed Toeplitz algebras, and it is an interesting open problem to settle the universality question for the Fock representation here. We follow a different route, inspired by transference principles for multipliers. More precisely, we develop a non-tracial version of an ultraproduct embedding theorem of Junge and Zeng for mixed q-Gaussian algebras [261]. Our construction (Theorem (6.3.19)) yields a q-quasi-free state-preserving embedding of an arbitrary $\Gamma_q(H)$ into an ultraproduct of tensor products of tracial q-

Gaussian algebras and other q-Araki-Woods algebras. Using Theorem (6.3.19), we show that it is possible to transfer radial multipliers on (tracial) q-Gaussian algebras to arbitrary q-Araki-Woods algebras in such a way that the completely bounded norm does not increase (Theorem (6.3.24)). Our transference result provides strong evidence towards the conjecture that radial multipliers on q-Araki-Woods algebras do not depend on the type structure, and we fully expect (but are unable to prove at this time) that our transference principle should be isometric and bijective.

In any case, Theorem (6.3.24) does provide us with some new examples of completely bounded radial multipliers on q-Araki-Woods algebras. These are the projections onto Wick words of a given finite length. Upper bounds for the norms of such multipliers were obtained previously for q-Gaussian algebras by [223]. These norm estimates together with the extended second quantisation functor [240] turn out to be exactly what we need to establish the main result: the w^*-CBAP for all q-Araki-Woods algebras. In fact, just as in the free case, we obtain the completely contractive version of this property:

As an application of the above result, we are able to answer affirmatively a question left open by Nou ([130,] Remark after Theorem 6.3), concerning whether or not the canonical w^*-dense C^*-subalgebras $\mathcal{A}_q(\mathrm{H}) \subseteq \Gamma_q(\mathrm{H})$ are always QWEP; see Corollary 5.3. It is our hope that Theorem (6.3.27) will lead to a deeper understanding of the structure of q-Araki-Woods algebras. In particular, we expect this result to be a fundamental tool in the applications of deformation/rigidity tools to these algebras.

Let us conclude with a description of the layout of the main body. We introduce the relevant notation and background on operator spaces, von Neumann ultraproducts, and q-Araki-Woods algebras. We construct our ultraproduct embedding and apply it to obtain the transference principle for radial multipliers. Finally, we present the proof of Theorem (6.3.27).

Throughout, inner products on complex Hilbert spaces are always taken to be conjugate-linear in the left variable. The algebraic tensor product of two complex vector spaces V, W will always be denoted by $V \odot W$, and elementary tensors in $V \odot W$ will also be denoted using the symbol $\odot$. Given a natural number $n \in \mathbb{N}$, we denote by $[n]([n]_0)$ the ordered set $\{1,2,\ldots,n\}(\{0,1,2,\ldots,n\})$. Given $n, d \in \mathbb{N}$ we will interchangeably view multi-indices $k = (k(1), k(2), \ldots, k(d)) \in [n]^d$ as functions $k: [d] \to [n]$. Given $d \in \mathbb{N}$, we denote by $\mathcal{P}(d)$ the lattice of partitions of the ordered set $[d]$, and by $\mathcal{P}_2(d) \subset \mathcal{P}(d)$ the subset of pair partitions (i.e., partitions of $[d]$ into disjoint subsets ("blocks") of size 2). The partial order $\leqslant$ on $\mathcal{P}(d)$ is given by the usual refinement order on partitions, and given $\pi, \sigma \in \mathcal{P}(d)$, we denote by $\pi \vee \sigma \in \mathcal{P}(d)$ the lattice theoretic join of π and σ with respect to the partial order $\leqslant$. The number of blocks of a partition σ will be denoted by $|\sigma|$. Finally, given a multi-index $k: [d] \to [n]$, we denote by $\ker k \in \mathcal{P}(d)$ the partition defined by level sets of k : that is, $1 \leqslant r, s \leqslant d$ belong to the same block of $\ker k$ iff $k(r) = k(s)$.

Some amount of the theory of operator spaces is necessary for our work; even the statement of the main result uses notions from this field. Recall that an operator space is a Banach space X endowed with a specific choice of norms on the matricial spaces $\mathrm{M}_n(X) := \mathrm{M}_n \odot X$ satisfying the so-called Ruan axioms, ensuring that it comes from an isometric embedding of XintoB(H), the C^*-algebra of bounded linear operators on some Hilbert space H. Given a pair of operator spaces X, Y and a linear map $T: X \to Y$, the cb norm of T is given by

$$\| T \|_{cb} := \sup_{n \in \mathbb{N}} \| \mathrm{Id}_n \odot T : \mathrm{M}_n \odot X \to \mathrm{M}_n \odot Y \|.$$

If $\| T \|_{cb} < \infty$, we say that T is completely bounded (cb). We can now define the approximation properties that we are interested in. Let X be an operator space. We say that X possesses the completely bounded approximation property if there exists a net $(\Phi_i)_{i\in I}$ of finite rank completely bounded maps on X such that $\sup_{i\in I}\|\Phi_i\|_{cb} < \infty$, and $\lim_{i\in I}\|\Phi_i(x) - x\| = 0$ for every $x \in X$. If we can find a net $(\Phi_i)_{i\in I}$ such that $\|\Phi_i\|_{cb} \leqslant 1$ then we say that X has the complete metric approximation property. For a dual operator space X (i.e. $X \simeq (X_*)^*$ for some operator space X_*), there is a suitable analogue of this approximation property which takes into account this additional structure. Namely, we say that X has the w^*-complete metric approximation property if there exists a net $(\Phi_i)_{i\in I}$ of finite rank w^*-continuous completely bounded maps on X such that $\|\Phi_i\|_{cb} \leqslant 1$ for each $i \in I$, and $\lim_{i\in I}\Phi_i(x) = x$ (weak-*) for every $x \in X$.

We need to discuss two operator space structures associated with a given Hilbert space.

Definition (6.3.1)[256]: Let H be a complex Hilbert space. We define the following operator space structures on H :

1. the column Hilbert space structure H_c is given by the identification $\mathrm{H} \simeq \mathrm{B}(\mathbb{C}, \mathrm{H})$;
2. the row Hilbert space structure H_r is given by the identification $\mathrm{H} \simeq \mathrm{B}(\overline{\mathrm{H}}, \mathbb{C})$.

(cf. [258], Theorem 3.4.1 and Proposition 3.4.2).

These Hilbertian operator spaces will turn out to be critical for obtaining a right formulation of the non-commutative Khintchine inequalities (cf. Proposition 2.17).

In the theory of operator spaces there is a variety of different tensor products, analogous to tensor products of Banach spaces. There is, however, one tensor product that stands out and does not have a Banach space theoretic counterpart - the Haagerup tensor product.

Definition (6.3.2)[256]: Let X and Y be operator spaces. We define a bilinear map $\mathrm{M}_{n,r}(X) \times \mathrm{M}_{r,n}(Y) \ni (x, y) \mapsto x \cdot y \in \mathrm{M}_n(X \odot Y)$ to be the bilinear extension of the assignment $(A \odot x, B \odot y) \mapsto (AB, x \odot y)$. For any $z \in \mathrm{M}_n(X \odot Y)$ we define the norm

$$\| z \|_{h,n} := \inf \left\{\| x \| \| y \| : z = x \cdot y, x \in \mathrm{M}_{n,r}(X), y \in \mathrm{M}_{r,n}(Y), r \in \mathbb{N}\right\}.$$

This sequence of norms on the matricial spaces $\mathrm{M}_n(X \odot Y)$ satisfies Ruan's axioms and therefore defines an operator space structure on $X \odot Y$, called the Haagerup tensor product. The completions with respect to the norms $\|\cdot\|_{h,n}$ will be denoted $\mathrm{M}_n(X \otimes_h Y)$. For more information on the Haagerup tensor product, consult [258] and [135].

Later on we will need the following proposition.

Proposition (6.3.3)[256]: (Proposition 9.3.4 from [258]). Let K and H be complex Hilbert spaces. Then the assignment $\mathrm{H} \odot \overline{\mathrm{K}} \ni \xi \odot \eta \mapsto |\xi\rangle\langle\eta| \in \mathrm{K}(\mathrm{K}, \mathrm{H})$ (the compact operators) extends to a complete isometry $\mathrm{H}_c \otimes_h \overline{\mathrm{K}}_r \simeq \mathrm{K}(\mathrm{K}, \mathrm{H})$.

We present here a construction due to Hiai (cf. [123]), which builds upon previous developments: q-Gaussian algebras of Bożejko and Speicher (cf. [60]) and free Araki-Woods factors defined by Shlyakhtenko (cf. [15]).

The starting point is a real Hilbert space $\mathrm{H}_{\mathbb{R}}$ equipped with a continuous one parameter group of orthogonal transformations $(U_t)_{t\in\mathbb{R}}$. The extension of $(U_t)_{t\in\mathbb{R}}$ to a unitary group on $\mathrm{H}_{\mathbb{C}}$, the complexification of $\mathrm{H}_{\mathbb{R}}$, will be still denoted by $(U_t)_{t\in\mathbb{R}}$. By Stone's theorem, there exists an injective, positive operator A on H_{C} such that $U_t = A^{it}$. On $\mathrm{H}_{\mathbb{C}}$ we define a new inner product $\langle \xi \mid \eta\rangle_U := \left\langle \xi \mid \frac{2A}{1+A}\eta \right\rangle$ and denote by H the completion of $\mathrm{H}_{\mathbb{C}}$ with respect to this inner product. Note that the norms defined by $\langle\cdot|\cdot\rangle_U$ and $\langle\cdot|\cdot\rangle$ coincide on $\mathrm{H}_{\mathbb{R}}$. This implies that I, the complex conjugation on $\mathrm{H}_{\mathbb{C}}$, is a closed operator on H with dense domain $\mathrm{H}_{\mathbb{C}}$.

Next we form the q-Fock space $\mathcal{F}_q(\mathrm{H})$. Since we will have to delve deeper into its structure later on, we will present the construction here. First, let us fix $q \in (-1,1)$. For any n we define $P_q^n: \mathrm{H}^{\odot n} \to \mathrm{H}^{\odot n}$ by

$$P_q^n(e_1 \odot \cdots \odot e_n) = \sum_{\sigma \in S_n} q^{i(\sigma)} e_{\sigma(1)} \odot \cdots \odot e_{\sigma(n)}, \quad (3)$$

where $i(\sigma) := |\{(i,j) \in [n]^2 : i < j \text{ and } \sigma(i) > \sigma(j)\}|$ is the number of inversions. This operator is (strictly) positive definite (cf. [66, Proposition 1]), so it defines an inner product on $\mathrm{H}^{\odot n}$ by $\langle \xi \mid \eta \rangle_q := \langle \xi \mid P_q^n \eta \rangle$; the completion with respect to this inner product will be denoted by $\mathrm{H}_q^{\otimes n}$. The q-Fock space is defined by the orthogonal direct sum $\mathcal{F}_q(\mathrm{H}) := \bigoplus_{n \geqslant 0} \mathrm{H}_q^{\otimes n}$. For our purposes, there are two important sets of operators defined on the q-Fock space. For any $\xi \in \mathrm{H}$ we define the q-creation operator $a_q^*(\xi) \in \mathrm{B}(\mathcal{F}_q(\mathrm{H}))$ by

$$a_q^*(\xi)(e_1 \odot \cdots \odot e_n) = \xi \odot e_1 \odot \cdots \odot e_n$$

and the q-annihilation operator $a_q(\xi) = (a_4^*(\xi))^* \in \mathrm{R}(\mathcal{F}_q(\mathrm{H}))$. It is known (cf. [60], Remark 1.2) that

$$\|a_q(\xi)\| = \|a_q^*(\xi)\| = \begin{cases} \|\xi\|, & 0 \geqslant q > -1 \\ (1-q)^{-1/2} \|\xi\| & 0 < q < 1. \end{cases} \quad (\xi \in \mathrm{H}).$$

We are now ready to define q-Araki-Woods algebras.

Definition (6.3.4)[256]: Let $(\mathrm{H}_{\mathbb{R}}, (U_t)_{t \in \mathbb{R}})$ be a real Hilbert space endowed with a one-parameter group of orthogonal transformations. Let H be the complex Hilbert space obtained as the completion of $\mathrm{H}_{\mathbb{C}}$ with respect to $\langle \cdot | \cdot \rangle_U$. For any $\xi \in \mathrm{H}_{\mathbb{R}}$ we define $s_q(\xi) \in \mathrm{B}(\mathcal{F}_q(\mathrm{H}))$ by $s_q(\xi) = a_q^*(\xi) + a_q(\xi)$. We define the q-Araki-Woods algebra $\Gamma_q(\mathrm{H})$ to be the von Neumann algebra generated by the set $\{s_q(\xi) : \xi \in \mathrm{H}_{\mathbb{R}}\}$ inside $\mathrm{B}(\mathcal{F}_q(\mathrm{H}))$.

In the special case $U_t = 1$ we obtain the q-Gaussian algebras of Bożejko and Speicher and we will denote them, following the tradition, by $\Gamma_q(\mathrm{H}_{\mathbb{R}})$ (cf. [60], Definition 2.1).

There is a distinguished vector Ω in $\mathcal{F}_q(\mathrm{H})$, called the vacuum vector, which is equal to $1 \in \mathbb{C} \simeq \mathrm{H}_q^{\otimes 0} \subset \mathcal{F}_q(\mathrm{H})$. It is not hard to see that Ω is cyclic and separating for $\Gamma_q(\mathrm{H})$. In fact, one can verify that the algebraic direct sum $\bigoplus_{n \geqslant 0} \mathrm{H}_{\mathbb{C}}^{\odot n}$ is contained in $\Gamma_q(\mathrm{H})\Omega$. Using the generator A, one can explicitly identify a big enough subset of the commutant $\Gamma_q(\mathrm{H})'$ for which Ω is cyclic (cf. [15], Lemma 3.1), so Ω is also separating for $\Gamma_q(\mathrm{H})$. It follows that the normal state $\chi(\cdot) = \langle \Omega \mid \cdot\, \Omega \rangle$ is faithful on $\Gamma_q(\mathrm{H})$ (called the q-quasi-free state) and $\mathcal{F}_q(\mathrm{H})$ can be identified with the GNS Hilbert space associated with χ. What is more, the commutant can be identified with the version of our algebra acting on the right, but in this case not only one has to use right versions of $s_q(\xi)$ but also the real Hilbert space that one draws the vectors from needs to be changed. We record here for later use the so-called Wick formula, which describes the joint moments of the generators $\{s_q(\xi)\}_{\xi \in \mathrm{H}_{\mathbb{R}}}$ with respect to χ. Theorem 2.7 ([123], [130]). For any $d \in \mathbb{N}$ and any $e_1, \ldots, e_d \in \mathrm{H}_{\mathbb{R}}$, we have

$$\chi\left(s_q(e_1) s_q(e_2) \ldots s_q(e_d)\right) = \sum_{\sigma \in \mathcal{P}_2(d)} q^{\iota(\sigma)} \prod_{(r,t) \in \sigma} \langle e_r \mid e_t \rangle_U,$$

where $\iota(\sigma)$ denotes the number of crossings in the pairing $\sigma \in \mathcal{P}_2(d)$, and $(r,t) \in \sigma$ indicates that $1 \leqslant r < t \leqslant d$ are paired together by σ. If d is odd, we interpret the above (empty) sum as 0 .

Since $\bigoplus_{n \geqslant 0} \mathrm{H}_{\mathbb{C}}^{\odot n} \subset \Gamma_q(\mathrm{H})\Omega \subset \mathcal{F}_q(\mathrm{H})$, we are allowed to make the following definition.

Definition (6.3.5)[256]: Let $\xi \in \bigoplus_{n \geqslant 0} \mathrm{H}_{\mathbb{C}}^{\odot n}$. Then there is exactly one operator $W(\xi) \in \Gamma_q(\mathrm{H})$, called the Wick word associated with ξ, such that $W(\xi)\Omega = \xi$.

This definition will help us in constructing maps on $\Gamma_q(\mathrm{H})$ from operators on H. Let us first recall a version of this construction on the level of the q-Fock space (cf. [60], Lemma 1.4).

Definition (6.3.6)[256]: Let $T: \mathrm{K} \to \mathrm{H}$ be a contraction between complex Hilbert spaces. Then the assignment

$$\mathcal{F}_q(T)(e_1 \odot \cdots \odot e_n) = Te_1 \odot \cdots \odot Te_n$$

extends to a contraction $\mathcal{F}_q(T): \mathcal{F}_q(\mathrm{K}) \to \mathcal{F}_q(\mathrm{H})$, called the first quantisation of T.

On the level of the von Neumann algebra $\Gamma_q(\mathrm{H})$ it is tempting to extend the assignment

$$W(e_1 \otimes \cdots \otimes e_n) \mapsto W(Te_1 \otimes \cdots \otimes Te_n)$$

to a nice map on $\Gamma_q(\mathrm{H})$. It turns out that under a mild additional assumption on T the extension exists and is a normal, unital, completely positive map. The next proposition is an extension of Theorem 2.11 from [60], which is an analogous result for q-Gaussian algebras.

Proposition (6.3.7)[256]: ([240], Theorem 3.4). Let $(\mathrm{K}_{\mathbb{R}}, (V_t)_{t \in \mathbb{R}})$ and $(\mathrm{H}_{\mathbb{R}}, (U_t)_{t \in \mathbb{R}})$ be real Hilbert spaces equipped with respective one-parameter orthogonal groups. Construct out of them complex Hilbert spaces K and H. Suppose that $T: \mathrm{K} \to \mathrm{H}$ is a contraction such that $T(\mathrm{K}_{\mathbb{R}}) \subset \mathrm{H}_{\mathbb{R}}$ (a condition written more succinctly in the form $ITJ = T$, where J and I are complex conjugations on $\mathrm{K}_{\mathbb{C}}$ and $\mathrm{H}_{\mathbb{C}}$, respectively). Then the assignment $W(e_1 \otimes \cdots \otimes e_n) \mapsto W(\mathrm{T}e_1 \otimes \cdots \otimes \mathrm{T}e_n)$ extends to a normal ucp map $\Gamma_q(T): \Gamma_q(\mathrm{K}) \to \Gamma_q(\mathrm{H})$ that preserves the vacuum state. The maps $\Gamma_q(T)$ is called the second quantisation of T.

To fulfill the purpose, that is to prove the w*-complete metric approximation property for the q-Araki-Woods algebras, we need to expand our knowledge of the Wick words. Let us start with the celebrated Wick formula. The proof of the following result can be found in [60] in the tracial case. The general case follows along the same lines. See also [190].

Proposition (6.3.8)[256]: (Wick formula). Suppose that $e_1, \ldots, e_n \in \mathrm{H}_{\mathbb{C}}$. Then

$$W(e_1 \odot \cdots \odot e_n) = \sum_{k=0}^{n} \sum_{i_1,\ldots,i_{n-k},j_{n-k+1},\ldots,j_n} a_q^*(e_{i_1}) \ldots a_q^*(e_{i_{n-k}}) a_q(Ie_{j_{n-k+1}}) \ldots a_q(Ie_{j_n}) q^{i(I_1,I_2)}, \quad (4)$$

where $I_1 = \{i_1 < \cdots < i_{n-k}\}$ and $I_2 = \{j_{n-k+1} < \cdots < j_n\}$ form a partition of the set $[n]$ and $i(I_1, I_2) = \sum_{l=1}^{n-k}(i_l - l)$ is the number of inversion of the permutation defined by I_1 and I_2. In particular, we have $W(e) = s_q(e)$ for any $e \in \mathrm{H}_{\mathbb{R}}$.

We will be concerned with the subspaces $\Gamma_q^n(\mathrm{H})$ of $\Gamma_q(\mathrm{H})$ spanned by the sets $\{W(\xi): \xi \in \mathrm{H}_{\mathbb{C}}^{\odot n}\}$; elements of these subspaces will be called Wick words of length n. We will also denote by $\tilde{\Gamma}_q(\mathrm{H}) \subseteq \Gamma_q(\mathrm{H})$ the (non-closed) linear span of $\left(\Gamma_q^n(\mathrm{H})\right)_{n \in \mathbb{N}_0}$. Note that $\tilde{\Gamma}_q(\mathrm{H})$ is a w*-dense *-subalgebra of $\Gamma_q(\mathrm{H})$, called the algebra of Wick words. Note that if $\xi = e_1 \odot \cdots \odot e_n$, where $e_1, \ldots, e_n \in \mathrm{H}_{\mathbb{R}}$ then $W(\xi) - s_q(e_1) \ldots s_q(e_n)$ is a sum of Wick words of length strictly smaller than n, so inductively one can show that $\tilde{\Gamma}_q(\mathrm{H})$ is the same as the *-

algebra generated by $\{s_q(\xi): \xi \in \mathrm{H}_{\mathbb{R}}\}$. Let now $(e_i)_{i\in I}$ be a fixed orthonormal basis for $\mathrm{H}_{\mathbb{R}}$. Then the algebra of Wick words $\tilde{\Gamma}_q(\mathrm{H})$ is *-isomorphic to the *-algebra of noncommutative polynomials $\mathbb{C}\langle (X_i)_{i\in I} \mid X_i = X_i^*\rangle$. The isomorphism in this case is given by $X_i \mapsto s_q(e_i) = W(e_i)$. See [130] for details. At times we will also need to consider the C^*-completion $\mathcal{A}_q(\mathrm{H})$ of $\tilde{\Gamma}_q(\mathrm{H})$. The most important part of the proof of the main theorem is providing an estimate (which must grow at most polynomially in n) for the cb norm of the projection from $\tilde{\Gamma}_q(\mathrm{H})$ onto $\Gamma_q^n(\mathrm{H})$. Therefore we need to understand the operator space structure of these spaces. This will be acccomplished by reformulating the Wick formula so that it is more amenable to operator space theoretic techniques, following Nou's lead (cf. [113]). We first define some relevant maps.

Definition (6.3.9)[256]: Let H be a complex Hilbert space coming from a pair $(\mathrm{H}_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$. We define maps $\mathcal{J}, U$ and S on the algebraic direct sum $\bigoplus_{n\geqslant 0} \mathrm{H}_{\mathbb{C}}^{\odot n}$ by

1. $\mathcal{J}(e_1 \odot \cdots \odot e_n) := Ie_1 \odot \cdots \odot Ie_n$;
2. $U(e_1 \odot \cdots \odot e_n) = e_n \odot \cdots \odot e_1$;
3. $S = \mathcal{J}U$.

The antilinear map $\mathcal{J}$ is a natural extension of the complex conjugation on $\mathrm{H}_{\mathbb{C}}$, thereby it should be really viewed as a closed linear operator from $\mathcal{F}_q(\mathrm{H})$ to $\mathcal{F}_q(\overline{\mathrm{H}})$ mapping $e_1 \otimes \cdots \otimes e_n$ to $\overline{Ie_1} \otimes \cdots \otimes \overline{Ie_n}$. The flip map U actually extends to a unitary on $\mathcal{F}_q(\mathrm{H})$. The last map, S, is a conjugation relevant to the Tomita-Takesaki theory. For future reference, let us point out that the modular automorphism group $(\sigma_t)_{t\in\mathbb{R}}$ associated to the q-quasi-free state χ was computed in [15], [123], and is given by

$$\sigma_t\left(s_q(\xi)\right) = s_q(U_{-t}\xi) = s_q\left(A^{-it}\xi\right) \ (\xi \in \mathrm{H}_{\mathbb{C}}).$$

We still need two more maps for our reformulation of the Wick formula.

Definition (6.3.10)[256]: Fix $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$ such that $0 \leqslant k \leqslant n$. We define the map $R_{n,k}^*: \mathrm{H}_q^{\otimes n} \to \mathrm{H}_q^{\otimes(n-k)} \otimes_h \mathrm{H}_q^{\otimes k}$ by specifying its values on a dense subspace:

$$R_{n,k}^*(e_1 \odot \cdots \odot e_n) := \sum_{i_1,\dots,i_{n-k},j_{n-k+1},\dots,j_n} q^{i(I_1,I_2)}\left(e_{i_1} \odot \cdots \odot e_{i_{n-k}}\right) \otimes_h \left(e_{j_{n-k+1}} \odot \cdots \odot e_{j_n}\right).$$

We also define $U_{n,k}: \left(\mathrm{H}_q^{\otimes(n-k)}\right)_c \otimes_h \left(\overline{\mathrm{H}}_q^{\otimes k}\right)_r \to \mathrm{B}\left(\mathcal{F}_q(\mathrm{H})\right)$ by

$$U_{n,k}\left((e_1 \odot \cdots \odot e_{n-k}) \otimes_h (\overline{e_{n-k+1}} \odot \cdots \odot \overline{e_n})\right): = a_q^*(e_1) \dots a_q^*(e_{n-k}) a_q(e_{n-k+1}) \dots a_q(e_n).$$

We are now ready to state the reformulated Wick formula and the corresponding Khintchine inequality.

Proposition (6.3.11)[256]: For any $\xi \in \mathrm{H}_{\mathbb{C}}^{\odot n}$ we have $W(\xi) = \sum_{k=0}^n U_{n,k}(\mathbb{1}_{n-k} \odot \mathcal{J})R_{n,k}^*(\xi)$, where $\mathbb{1}_{n-k}$ is the identity map on $\mathrm{H}_{\mathbb{C}}^{\odot(n-k)}$.

Corollary (6.3.12)[256]: ([113], Theorem 3). Let K be a Hilbert space. If $\xi \in \mathrm{B}(\mathrm{K}) \odot \mathrm{H}_{\mathbb{C}}^{\odot n}$ then

$$\max_{0\leqslant k\leqslant n}\left\|\left(\mathrm{Id} \odot (\mathbb{1}_{n-k} \odot \mathcal{J})R_{n,k}^*\right)(\xi)\right\| \leqslant \| (\mathrm{Id} \odot W)(\xi) \| \tag{5}$$

$$\| (\mathrm{Id} \odot W)(\xi) \| \leqslant C(q)(n+1)\max_{0\leqslant k\leqslant n}\left\|\left(\mathrm{Id} \odot (\mathbb{1}_{n-k} \odot \mathcal{J})R_{n,k}^*\right)(\xi)\right\|. \tag{6}$$

The norm $\| (\mathrm{Id} \odot W)(\xi) \|$ is computed in $\mathrm{B}(\mathrm{K}) \otimes_{\min} \Gamma_q(\mathrm{H})$, and the other norms are computed in $\mathrm{B}(\mathrm{K}) \otimes_{\min} \left(\mathrm{H}_q^{\otimes(n-k)}\right)_c \otimes_h \left(\overline{\mathrm{H}}_q^{\otimes k}\right)_r$.

Proof. Inequality (4) follows from the Wick formula, complete boundedness of $U_{n,k}$ and the triangle inequality, as in the proof of Theorem 1 in [113]. The proof of (3) is also a repetition of the argument in Nou's.

We will be primarily interested in a special class of completely bounded linear maps on q-Araki-Woods algebras, called radial multipliers. In the following, we fix an arbitrary q-Araki-Woods algebra $\Gamma_q(\mathrm{H})$.

Definition (6.3.13)[256]: Let $\varphi: \mathbb{N}_0 \to \mathbb{C}$ be a bounded function. The (w^*-densely defined) linear map $\mathrm{m}_\varphi: \tilde{\Gamma}_q(\mathrm{H}) \to \tilde{\Gamma}_q(\mathrm{H})$ given by

$$\mathrm{m}_\varphi(W(\xi)) - \varphi(m)W(\xi) \left(\xi \in (\mathrm{H}_\mathbb{C})^{\odot m}\right)$$

is called the radial multiplier with symbol φ. If m_φ extends to a completely bounded map $\mathrm{m}_\varphi: \mathcal{A}_q(\mathrm{H}) \to \mathcal{A}_q(\mathrm{H})$, we call m_φ a completely bounded radial multiplier on $\Gamma_q(\mathrm{H})$.

In the course of the proof of the complete metric approximation property for q-ArakiWoods algebras we will need the following result obtained by the first-named author.

Theorem (6.3.14)[256]: ([223], Proposition 3.3 and the remark following it). Let $\mathrm{H}_\mathbb{R}$ be a real Hilbert space and let $\Gamma_q(\mathrm{H}_\mathbb{R})$ be the q-Gaussian algebra associated with it. Fix $n \in \mathbb{N}$ and and consider the radial multiplier m_φ associated to the Kronecker delta symbol $\varphi_n(k) = \delta_n(k) = \delta_{n,k}$. Then m_φ is a cb radial multiplier and corresponds to the projection P_n of $\Gamma_q(\mathrm{H}_\mathbb{R})$ onto the ultraweakly closed span of $\left\{W(\xi): \xi \in \mathrm{H}^{\odot n}\right\}$. Moreover, we have $\|\mathrm{m}_{\varphi_n}\|_{cb} \leqslant C(q)^2(n+1)^2$.

We will mostly follow [247]. Ultraproducts of von Neumann algebras are very useful, e.g. in the study of central sequences in connection with property Γ. The original construction was applicable only in the case of tracial algebras. The main difference in the type III case is that there are two different notions of ultraproducts, each havings its own virtues.

We start with a definition due to Ocneanu [254], which is closer to the ultraproduct of tracial von Neumann algebras. We fix a sequence $(\mathrm{M}_n, \varphi_n)_{n\in\mathbb{N}}$ of von Neumann algebras equipped with normal faithful states, and a non-principal ultrafilter ω on $\mathbb{N}$. Recall that if all the states were tracial, the ultraproduct would be defined as the direct product

$$\ell^\infty(\mathbb{N}, \mathbf{M}_n) := \left\{(x_n) \in \prod_{n\in\mathbb{N}} \mathbf{M}_n : \sup_{n\in\mathbb{N}} \|x_n\| < \infty\right\}$$

quotiented by the ideal of L^2-null sequences, i.e. sequences $(x_n) \in \ell^\infty(\mathbb{N}, \mathbf{M}_n)$ such that $\lim_{n\to\omega} \varphi_n(x_n^* x_n) = 0$. The problem in the non-tracial case is that this subspace is just a left ideal and there is no reason why we should prefer $\lim_{n\to\omega} \varphi_n(x_n^* x_n)$ to $\lim_{n\to\omega} \varphi(x_n x_n^*)$. This little nuisance can be taken care of by defining $\| x \|_\varphi^\# := \left(\varphi(x^*x + xx^*)\right)^{\frac{1}{2}}$ and working with the condition $\lim_{n\to\omega} \|x_n\|_{\varphi_n}^\# = 0$ instead. This, unfortunately, gives rise to another problem −the subspace

$$\mathrm{I}_\omega(\mathrm{M}_n, \varphi_n) := \left\{(x_n) \in \ell^\infty(\mathbb{N}, \mathrm{M}_n) : \lim_{n\to\omega} \|x_n\|_{\varphi_n}^\# = 0\right\}$$

is still not an ideal. We need to find the largest subalgebra inside $\ell^\infty(\mathbb{N}, \mathrm{M}_n)$ in which $\mathrm{I}_\omega(\mathrm{M}_n, \varphi_n)$ is an ideal. This leads us to the next definition.

Definition (6.3.15)[256]: Let $(\mathrm{M}_n, \varphi_n)$ be a sequence of von Neumann algebras equipped with normal faithful states. Define

$$\mathcal{M}^\omega(\mathrm{M}_n, \varphi_n) := \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, \mathrm{M}_n) : (x_n)\mathrm{l}_\omega \subset \mathrm{I}_\omega, \mathrm{I}_\omega(x_n) \subset \mathrm{I}_\omega\}.$$

Then $\mathcal{M}^\omega(\mathrm{M}_n, \varphi_n)$ is a C^*-algebra in which $\mathrm{I}_\omega(\mathrm{M}_n, \varphi_n)$ is a closed ideal. Therefore we can form the quotient

$$(\mathrm{M}_n, \varphi_n)^\omega := \mathcal{M}^\omega(\mathrm{M}_n, \varphi_n)/\mathrm{I}_\omega(\mathrm{M}_n, \varphi_n)$$

which is, a priori, a C^*-algebra but actually turns out to be a von Neumann algebra (cf. [254], Proposition on page 32), called the Ocneanu ultraproduct of the sequence $(\mathrm{M}_n, \varphi_n)_{n\in\mathbb{N}}$. The image of a sequence $(x_n)_{n\in\mathbb{N}} \in \mathcal{M}^\omega(\mathrm{M}_n, \varphi_n)$ in the quotient algebra $(\mathrm{M}_n, \varphi_n)^\omega$ will be denoted by $(x_n)^\omega$.

Despite being a natural generalisation of the tracial ultraproduct, the Ocneanu ultraproduct suffers from being inadequate for the purpose of non-commutative integration. One particular problem is that the Banach space ultraproduct of preduals is usually bigger than the predual of the Ocneanu ultraproduct.

There is a different construction that, as shown in [136], interacts nicely with ultraproducts of non-commutative L^p-spaces. Once again, we start from a sequence $(\mathrm{M}_n, \varphi_n)_{n\in\mathbb{N}}$ of von Neumann algebras endowed with normal faithful states. Using the GNS construction, we view $\mathrm{M}_n \subset \mathrm{B}(\mathrm{H}_n)$. Let $(\mathrm{M}_n)_\omega$ denote the Banach space ultraproduct of the sequence $(\mathrm{M}_n)_{n\in\mathbb{N}}$, which is a C^*-algebra. Let $(\mathrm{H}_n)_\omega$ be the ultraproduct of the corresponding GNS Hilbert spaces. Then we can view $(\mathrm{M}_n)_\omega$ as acting on $(\mathrm{H}_n)_\omega$ via

$$(x_n)_\omega(\xi_n)_\omega := (x_n\xi_n)_\omega. \quad (7)$$

It is not hard to see that this is well defined (by the joint continuity of the map $\mathrm{B}(\mathrm{H}) \times \mathrm{H} \ni (x, \xi) \mapsto x\xi \in \mathrm{H}$).

Definition (6.3.16)[256]: Let $(\mathrm{M}_n, \varphi_n)_{n\in\mathbb{N}}$ be a sequence of von Neumann algebras equipped with normal faithful states, represented faithfully on the GNS Hilbert spaces, i.e. $\mathrm{M}_n \subset \mathrm{B}(\mathrm{H}_n)$. The Raynaud ultraproduct is defined as the weak closure inside $\mathrm{B}((H_n)_\omega)$ of the image of the natural diagonal representation (5) of the C^*-ultraproduct $(\mathrm{M}_n)_\omega$ on $(\mathrm{H}_n)_\omega$; it is denoted by $\prod^\omega(\mathrm{M}_n, \varphi_n)$.

There is a nice relationship between the two constructions which is summarised in the following theorem.

Theorem (6.3.17)[256]: ([247], Theorem 3.7). Let $(\mathrm{M}_n, \varphi_n)_{n\in\mathbb{N}}$ be a sequence of von Neumann algebras equipped with normal faithful states. Let $\mathrm{H}_n := L^2(\mathrm{M}_n, \varphi_n)$ be the GNS-Hilbert space associated with the state φ_n on M_n, so we have $\prod^\omega(\mathrm{M}_n, \varphi_n) \subset \mathrm{B}((\mathrm{H}_n)_\omega)$. Let $\mathrm{M}^\omega := (\mathrm{M}_n, \varphi_n)^\omega$ and $\varphi^\omega := (\varphi_n)^\omega$. Define a map $w: L^2(\mathrm{M}^\omega, \varphi^\omega) \hookrightarrow (\mathrm{H}_n)_\omega$ from the GNS-Hilbert space of $(\mathrm{M}^\omega, \varphi^\omega)$ given by

$$w\left((x_n)^\omega(\xi_{\varphi^\omega})\right) := \left(x_n\xi_{\varphi_n}\right)_\omega,$$

where ξ (with an appropriate subscript) is the cyclic vector coming from the GNS construction. Then w is an isometry and $w^*\left(\prod^\omega(\mathrm{M}_n, \varphi_n)\right)w = \mathrm{M}^\omega$.

We would now like to describe a useful theorem from [130] concerning embeddings into ultraproducts.

Theorem (6.3.18)[256]: ([130], Theorem 4.3). Let (N, ψ) and $(\mathrm{M}_n, \varphi_n)_{n\in\mathbb{N}}$ be von Neumann algebras equipped with normal faithful states. Let ω be a non-principal ultrafilter on $\mathbb{N}$ and let $\prod^\omega(\mathrm{M}_n, \varphi_n)$ be the Raynaud ultraproduct. Let $(\sigma_t^n)_{t\in\mathbb{R}}$ denote the modular group of φ_n. Let $p \in \prod^\omega(\mathrm{M}_n, \varphi_n)$ denote the support of the ultraproduct state $(\varphi_n)_\omega$. Suppose that $\widetilde{\mathrm{N}} \subset \mathrm{N}$ is a weak*-dense *-subalgebra of N and we are given a *-homomorphism

$$\Phi: \widetilde{N} \to \prod^\omega (\mathrm{M}_n, \varphi_n).$$

Assume that Φ satisfies the following conditions:

1. It is state preserving, i.e. $(\varphi_n)_\omega(\Phi(x)) = \psi(x)$ for any $x \in \widetilde{\mathrm{N}}$
2. For any $x \in \Phi(\widetilde{\mathrm{N}})$ there is a representative $(x_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, \mathrm{M}_n)$ such that x_n is analytic for $(\sigma_t^n)_{t\in\mathbb{R}}$ and the sequence $\left(\sigma_{-i}^n(x_n)\right)_{n\in\mathbb{N}}$ is bounded (cf. [130], Lemma 4.1).
3. For all $t \in \mathbb{R}$ and for all $y = (y_n)_\omega \in \Phi(\widetilde{N})$, irrespective of the choice of the representative $(y_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, \mathbf{M}_n)$, we have
$$p\left(\sigma_t^n(y_n)\right)_\omega p \in p\,\mathrm{B}p,$$

where B is the w^*-closure of $\Phi(\widetilde{\mathrm{N}})$.

Then the map $\Theta := p\Phi p: \widetilde{\mathbf{N}} \to p\left(\prod^\omega(\mathrm{M}_n, \varphi_n)\right)p$ is a state-preserving *-homomorphism that can be extended to a normal *-isomorphism from N onto $\mathrm{B}p$. Moreover, there exists a normal, state-preserving conditional expectation from $\prod^\omega(\mathrm{M}_n, \varphi_n)$ onto $\Theta(\mathrm{N})$.

We end with a simple remark strengthening the connection between Theorem (6.3.18) and the Ocneanu ultraproduct. Suppose that $(x_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, \mathrm{M}_n)$ is a representative of an element $x \in (\mathrm{M}_n)_\omega$ such that the sequence $\left(\sigma_{-i}^n(x_n)\right)_{n\in\mathbb{N}}$ is bounded. Then the sequence $(x_n)_{n\in\mathbb{N}}$ belongs to $\mathcal{M}^\omega(\mathrm{M}_n, \varphi_n)$, so it defines an element of the Ocneanu ultraproduct. Indeed, suppose that $(y_n) \in \mathrm{I}_\omega(\mathrm{M}_n, \varphi_n)$. We would like to check that $\lim_{n\to\omega} \|x_n y_n\|_{\varphi_n}^\# = 0$. It boils down to checking that $\lim_{n\to\omega} \varphi_n(y_n^* x_n^* x_n y_n) = 0$ and $\lim_{n\to\omega} \varphi_n(x_n y_n y_n^* x_n^*) = 0$. The first equality is easy to verify because $y_n^* x_n^* x_n y_n \leqslant \|x_n\|^2 y_n^* y_n$ and the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded. For the second one we will use the KMS condition:
$$\varphi_n(x_n y_n y_n^* x_n^*) = \varphi_n\left(y_n y_n^* x_n^* \sigma_{-i}^n(x_n)\right).$$
Note that $z_n := x_n^* \sigma_{-i}^n(x_n)$ is a bounded sequence. If we denote $u_n = \sqrt{y_n y_n^*}$ then we have to bound $\varphi(u_n^2 y_n)$. By the Cauchy-Schwarz inequality we get
$$|\ \varphi_n\left(u_n(u_n z_n)\right) \leqslant \varphi_n(u_n^2)\varphi_n(z_n^* u_n^2 z_n).$$
By assumption we have $\lim_{n\to\omega} \varphi_n(u_n^2) = \lim_{n\to\omega} \varphi_n(y_n y_n^*) = 0$. The second term can be bounded above by the norm $\|z_n^* u_n^2 z_n\|$ that is bounded, so the product converges to zero.

We prove a result which shows that an arbitary q-Araki-Woods algebra embeds in a state preserving way into an ultraproduct of tensor products of q-Gaussian algebras and q-Araki-Woods algebras. This result will be key to our establishment of a transference principle for completely bounded radial multipliers in the following.

Let $\Gamma_q(\mathrm{H})$ be a fixed q-Araki-Woods algebra for some $q \in (-1,1)$, and write $q = q_0 q_1$ for some $|q| < q_0 < 1$. For any $m \in \mathbb{N}$, we let $\Gamma_{q0}(\mathbb{R}^m)$ be a q-Gaussian algebra and $\Gamma_{q_1}(\mathrm{H} \otimes \mathbb{C}^m)$ be a q-Araki-Woods algebra, where the inner product on $\mathrm{H} \otimes \mathbb{C}^m$ is the tensor product of the given deformed inner product on H and the nondeformed one on $\mathbb{C}^m$. In other words, if $(U_t)_t \curvearrowright \mathrm{H}_\mathbb{R}$ is the orthogonal group associated to $\Gamma_q(\mathrm{H})$, then $(U_t \otimes 1)_t \curvearrowright \mathrm{H}_\mathbb{R} \otimes \mathbb{R}^m$ is the orthogonal group associated to $\Gamma_{q_1}(\mathrm{H} \otimes \mathbb{C}^m)$. Denote by $\chi, \chi_{0,m}$ and $\chi_{1,m}$ the q-quasi-free states on $\Gamma_q(\mathrm{H}), \Gamma_{q_0}(\mathbb{R}^m)$ and $\Gamma_{q_1}(\mathrm{H} \otimes \mathbb{C}^m)$, respectively. For each m, fix an orthonormal basis $(e_1, \ldots, e_m)$ of $\mathbb{R}^m$ and define
$$u_m(\xi) := \frac{1}{\sqrt{m}} \sum_{k=1}^{m} W(e_k) \otimes W(\xi \odot e_k) \in \Gamma_{q_0}(\mathbb{R}^m) \bar{\otimes} \Gamma_{q_1}(\mathrm{H} \otimes \mathbb{C}^m)\ (\xi \in \mathrm{H}_\mathbb{C}).$$

Finally, we fix a non-principal ultrafilter ω on $\mathbb{N}$, form the corresponding (Raynaud) ultraproduct

$$\mathrm{A}=\prod^{\omega}\left(\Gamma_{q_0}(\mathbb{R}^m)\mathbin{\bar{\otimes}}\Gamma_{q_1}(\mathrm{H}\otimes\mathbb{C}^m),\chi_{0,m}\otimes\chi_{1,m}\right),$$

and let $p\in\mathrm{A}$ be the support of the ultraproduct state $\left(\chi_{0,m}\otimes\chi_{1,m}\right)_{\omega}$.

With the above notation fixed, we can now state our embedding result.

Theorem (6.3.19)[256]: (a) The mapping $W(\xi)\mapsto(u_m(\xi))_\omega\in\mathrm{A}\ (\xi\in\mathrm{H}_{\mathbb{C}})$ extends uniquely to a state-preserving *-homomorphism $\pi_\omega:\left(\tilde{\Gamma}_q(\mathrm{H}),\chi\right)\to\left(\mathrm{A},\left(\chi_{0,m}\otimes\chi_{1,m}\right)_\omega\right.$.

(b) The map $\Theta:=p\pi_\omega(\cdot)p:\tilde{\Gamma}_q(\mathrm{H})\to p\,\mathrm{A}p$ extends to a normal state-preserving *-isomorphism

$$\Theta:\Gamma_q(\mathrm{H})\to\Theta\left(\Gamma_q(\mathrm{H})\right)\subseteq p\,\mathrm{A}p.$$

Moreover, $\Theta(\Gamma_q(\mathrm{H}))$ is the range of a normal state-preserving conditional expectation $E:\mathrm{A}\to\Theta(\Gamma_q(\mathrm{H}))$.

Proof. (a). Recall that the algebra of Wick words is *-isomorphic to the *-algebra of noncommutative polynomials, so any *-homomorphism $\pi_\omega:\tilde{\Gamma}_q(\mathrm{H})\to\mathrm{A}$ is uniquely determined by specifying the images $\left(\pi_\omega(W(e_i))\right)_{i\in I}\subset\mathrm{A}$. Thus to conclude that the claimed π_ω exists and is well-defined, we just need to check that each sequence $(u_m(\xi))_{m\in\mathbb{N}}(\xi\in\mathrm{H}_{\mathbb{C}})$ is normbounded and hence defines an element $(u_m(\xi))_\omega\in\mathrm{A}$. To this end, we apply (the $n=1$ version of) Corollary (6.3.12) with coefficients $W(\xi\odot e_k)\in\mathrm{B}(\mathrm{K})=\mathrm{B}\left(\mathcal{F}_{q_1}(\mathrm{H}\otimes\mathbb{C}^m)\right)$ (see also [113], Page 17) to conclude that

$$\begin{aligned}\|u_m(\xi)\|&\leqslant 2(1-q_0)^{\frac{-1}{2}}m^{\frac{-1}{2}}\\&\qquad\max\left\{\left\|\sum_{k=1}^{m}W(\xi\odot e_k)^*W(\xi\odot e_k)\right\|^{\frac{1}{2}},\left\|\sum_{k=1}^{m}W(\xi\odot e_k)W(\xi\odot e_k)^*\right\|^{\frac{1}{2}}\right\}\\&\leqslant 2(1-q_0)^{\frac{-1}{2}}\|W(\xi\odot e_1)\|.\end{aligned}$$

Finally we check that π_ω is state-preserving. By linearity, it suffices to show that for any $d\in\mathbb{N}$ and $\xi_1,\dots,\xi_d\in\mathrm{H}_{\mathbb{R}}$, we have

$$\lim_{m\to\infty}\left(\chi_{0,m}\otimes\chi_{1,m}\right)\left(u_m(\xi_1)\cdot\ldots\cdot u_m(\xi_d)\right)=\chi\left(W(\xi_1)\cdot\ldots\cdot W(\xi_d)\right).$$

Fixing m and considering the terms on the left-hand side above, we have

$$\begin{aligned}&\left(\chi_{0,m}\otimes\chi_{1,m}\right)\left(u_m(\xi_1)\cdot\ldots\cdot u_m(\xi_d)\right)\\&\quad=m^{-d/2}\sum_{k:[d]\to[m]}\chi_{0,m}\left(W\left(e_{k(1)}\right)\cdot\ldots\cdot W\left(e_{k(d)}\right)\right)\\&\qquad\qquad\chi_{1,m}\left(W\left(\xi_1\odot e_{k(1)}\right)\cdot\ldots\cdot W\left(\xi_d\odot e_{k(d)}\right)\right)\\&\quad=m^{-d/2}\sum_{k:[d]\to[m]}\left(\sum_{\substack{\sigma\in\mathcal{P}_2(d)\\ker k\geqslant\sigma}}q_0^{\iota(\sigma)}\right)\end{aligned}$$

$$\left(\sum_{\sigma'\in\mathcal{P}_2(d)} q_1^{\iota(\sigma')} \prod_{(r,t)\in\sigma'} \left\langle \xi_r \odot e_{k(r)} \mid \xi_t \odot e_{k(t)}\right\rangle_U\right)$$

$$= m^{-d/2} \sum_{k:[d]\to[m]} \left(\sum_{\substack{\sigma\in\mathcal{P}_2(d)\\ kerk\geqslant\sigma}} q_0^{\iota(\sigma)}\right)\left(\sum_{\substack{\sigma'\in\mathcal{P}_2(d)\\ kerk\geqslant\sigma'}} q_1^{\ell(\sigma')} \prod_{(r,t)\in\sigma'} \langle \xi_r \mid \xi_t\rangle_U\right)$$

$$= \sum_{\sigma,\sigma'\in\mathcal{P}_2(d)} q_0^{\iota(\sigma)} q_1^{\ell(\sigma')} \prod_{(r,t)\in\sigma'} \langle \xi_r \mid \xi_t\rangle_U \sum_{\substack{k:[d]\to[m]\\ k\in rk\geqslant\sigma, \ker k\geqslant\sigma'}} m^{-d/2}$$

$$= \sum_{\sigma,\sigma'\in\mathcal{P}_2(d)} q_0^{\iota(\sigma)} q_1^{\iota(\sigma')} \prod_{(r,t)\in\sigma'} \langle \xi_r \mid \xi_t\rangle_U m^{-d/2+|\sigma\vee\sigma'|}.$$

Since

$$\lim_{m\to\infty} m^{-d/2+|\sigma\vee\sigma'|} = \delta_{\sigma,\sigma'}\ (\sigma,\sigma'\in\mathcal{P}_2(d)),$$

we conclude that

$$\lim_{m\to\infty}\left(\chi_{0,m}\otimes\chi_{1,m}\right)\left(u_m(\xi_1)\cdot\ldots\cdot u_m(\xi_d)\right) = \sum_{\sigma\in\mathcal{P}_2(d)} q^{\iota(\sigma)} \prod_{(r,t)\in\sigma} \langle \xi_r \mid \xi_t\rangle_U$$
$$= \chi\left(W(\xi_1)\cdot\ldots\cdot W(\xi_d)\right).$$

(b). To exhibit the desired properties of $\Theta := p\pi_\omega(\cdot)p$, we will verify conditions (i)-(iii) in Theorem (6.3.18) for the $*$-homomorphism π_ω. (i) follows immediately from part (a) of the present theorem. For (ii), we note that by linearity and multiplicativity of π_ω, it suffices to check condition (ii) on the generators $\pi_\omega(W(\xi)) = (u_m(\xi))_\omega, (\xi\in \mathrm{H}_\mathbb{C})$. However, there is a minor issue here coming from the fact that for arbitrary $\xi\in\mathrm{H}_\mathbb{C}$, there is no reason to expect elements $u_m(\xi)\in\Gamma_{q_0}(\mathbb{R}^m)\bar{\otimes}\Gamma_{q_1}(\mathrm{H}\otimes\mathbb{C}^m)$ to even be analytic, let alone the sequence $\left(\sigma_{-i}(u_m(\xi))\right)_{m\in\mathbb{N}}$ be uniformly bounded. To overcome this issue, put $\mathrm{H}_\mathbb{C}^{an} = \bigcup_{\lambda>1}\mathbf{1}_{[\lambda^{-1},\lambda]}(A)\mathrm{H}_\mathbb{C}$, where $\mathbf{1}_{[\lambda^{-1},\lambda]}(A)$ denotes the spectral projection of the analytic generator A corresponding to the interval $[\lambda^{-1},\lambda]$. Following [246], we see that $\mathrm{H}_\mathbb{C}^{an}\subset\mathrm{H}_\mathbb{C}$ is a dense linear subspace such that $I\mathrm{H}_\mathbb{C}^{an} = \mathrm{H}_\mathbb{C}^{an}$. Moreover, for each $\xi\in\mathrm{H}_\mathbb{C}^{an}$, we have that ξ (respectively $W(\xi)$) is analytic for the action of the unitary group $U_t = A^{it}$ (respectively the modular automorphism group σ_t), and

$$\sigma_z W(\xi) = W\left(A^{-iz}\xi\right)\ (z\in\mathbb{C}).$$

In our present setting, we shall restrict the domain of π_ω to the *-subalgebra $\tilde{\Gamma}_q(\mathrm{H})_{an}\subset\tilde{\Gamma}_q(\mathrm{H})$, consisting of linear combinations of Wick words of the form $W(\xi)$ with $\xi\in(\mathrm{H}_\mathbb{C}^{an})^{\odot n}$, $(n\in\mathbb{N}_0)$. Since $\tilde{\Gamma}_q(\mathrm{H})_{an}$ is still w*-dense in $\Gamma_q(\mathrm{H})$ and is generated by $(W(\xi))_{\xi\in\mathrm{H}_\mathbb{C}^{an}}$, we just have to show that the equivalence class representative $(u_m(\xi))_{m\in\mathbb{N}}$ for $\pi_\omega(W(\xi))$ satisfies condition (ii) of Theorem (6.3.19) for each $\xi\in\mathrm{H}_\mathbb{C}^{an}$. To this end, note that on $\Gamma_{q_0}(\mathbb{R}^m)\bar{\otimes}\Gamma_{q_1}(\mathrm{H}\otimes\mathbb{C}^m)$, we have

$$\sigma_t^m = \mathrm{id}_{\Gamma_{q_0}(\mathbb{R}^m)}\otimes\sigma_t^{\Gamma_{q_1}(\mathrm{H}\otimes\mathbb{C}^m)}\ \&\ \sigma_t^{\Gamma_{q_1}(\mathrm{H}\otimes\mathbb{C}^m)}(W(\xi\odot e))$$
$$= W\left(A^{-it}\xi\odot e\right)\ (\xi\in\mathrm{H}_\mathbb{C}, e\in\mathbb{C}^m).$$

It follows from these identities that if $\xi \in \mathrm{H}_{\mathbb{C}}^{an}$ and $e \in \mathbb{C}^m$, then elements $W(\xi \odot e)$ and $u_m(\xi)$ are analytic for their respective modular groups and

$$\begin{aligned}\sigma_z^m(u_m(\xi)) &= \left(\frac{1}{\sqrt{m}}\sum_{k=1}^{m} W(e_k) \otimes \sigma_z^{\Gamma_{q_1}(H\otimes\mathbb{C}^m)} W(\xi \odot e_k)\right) \\ &= \frac{1}{\sqrt{m}}\sum_{k=1}^{m} W(e_k) \otimes W\left(A^{-iz}\xi \odot e_k\right) = u_m\left(A^{-iz}\xi\right),\ (z \in \mathbb{C}).\end{aligned}$$

The uniform boundedness of the sequence $(\sigma_{-i}^m u_m(\xi))_{m\in\mathbb{N}}$ now follows along the same lines as that of $(u_m(\xi))_{m\in\mathbb{N}}$:

$$\sup_m \|\sigma_{-i}^m u_m(\xi)\| = \sup_m \|u_m(A^{-1}\xi)\| \leqslant 2(1-q_0)^{\frac{-1}{2}} \|W(A^{-1}\xi \odot e_1)\|.$$

For (iii), it again suffices by linearity and multiplicativity to verify that for all $\pi_\omega(W(\xi)) = (u_m(\xi))_\omega, (\xi \in \mathrm{H}_{\mathbb{C}})$,

$$p\left(\left(\sigma_t^m(u_m(\xi))\right)_\omega\right)p \in p\ \mathrm{B}p,$$

where B is the w^*-closure of $\pi_\omega\left(\tilde{\Gamma}_q(\mathrm{H})\right)$ in A. But this last point is obvious, because by the previous computation, $\sigma_t^m(u_m(\xi)) = u_m\left(A^{-it}\xi\right)$ for all m, giving

$$p\left(\left(\sigma_t^m(u_m(\xi))\right)_\omega\right)p = p\left(\left(u_m\left(A^{-it}\xi\right)\right)_\omega\right)p = p\pi_\omega\left(W\left(A^{-it}\xi\right)\right)p \in p\ \mathrm{B}p.$$

We use the ultraproduct embedding result (Theorem (6.3.19)) of the previous to establish the following transference result for radial multipliers on q-Araki-Woods algebras.

The main technical tool in establishing Theorem (6.3.24) is the following intertwining-type property for projections onto Wick words of a given length with respect to the ultraproduct embedding given by Theorem (6.3.19)

There are two things that have to be verified in Theorem (6.3.23). The first one, which is a routine check, is to prove that $(P_n \otimes \mathrm{Id})_\omega$ (and therefore also the composition $p(P_n \otimes \mathrm{Id})_\omega p$) is a well-defined map on the (Raynaud) ultraproduct A. Using Theorem (6.3.14), we can show that $(P_n \otimes \mathrm{Id})_\omega$ is well defined on the C^*-ultraproduct $\widetilde{\mathrm{A}} \subset \mathrm{A}$. To conclude, we have to verify that it extends to a normal map on A. Since we are dealing with the Raynaud ultraproduct, the predual of our ultraproduct is equal to the Banach space ultraproduct of preduals. On each level we can take the predual map of $(P_n \otimes \mathrm{Id})_{m\in\mathbb{N}}$ and use this sequence to obtain a map Ψ on the ultraproduct of L^1-spaces, the predual of the ultraproduct. The dual of Ψ coincides with $(P_n \otimes \mathrm{Id})_\omega$ on the C^*-ultraproduct, hence it is its unique normal extension. A similar argument is presented, for instance, in [190].

The second step in proving Theorem (6.3.23) is to understand the images of Wick words under the $*$-homomorphism $\pi_\omega: \tilde{\Gamma}_q(\mathrm{H}) \to \mathrm{A}$. To accomplish this, for any $d \in \mathbb{N}$ and $\xi_1, \dots, \xi_d \in \mathrm{H}_{\mathbb{C}}$, we define elements $W^s(\xi_1 \odot \cdots \odot \xi_d) \in \mathrm{A}$ by setting
$W^s(\xi_1 \odot \cdots \odot \xi_d)$.

$$:= \left(m^{-\frac{d}{2}} \sum_{\substack{k:[d]\to[m] \\ \text{injective}}} W\left(e_{k(1)}\right) \dots W\left(e_{k(d)}\right) \otimes W\left(\xi_1 \odot e_{k(1)}\right) \dots W\left(\xi_d \odot e_{k(d)}\right)\right)_\omega$$

Because we are summing over distinct indices, the vectors $e_{k(1)}, \dots, e_{k(d)}$ are pairwise orthogonal, so $W\left(e_{k(1)}\right) \dots W\left(e_{k(d)}\right) = W\left(e_{k(1)} \odot \cdots \odot e_{k(d)}\right)$. One can then use the

Khintchine inequality (Corollary (6.3.12)) to prove that the sequence defining $W^s(\xi_1 \odot \cdots \odot \xi_d)$ is uniformly bounded, hence defines a legitimate element of the ultraproduct. We will not give more details here because in the next proposition we show that $\pi_\omega\big(W(\xi_1 \odot \cdots \odot \xi_d)\big) = W^s(\xi_1 \odot \cdots \odot \xi_d)$, so it definitely is an element of the ultraproduct.

Theorem (6.3.20)[256]: Let $\xi_1, \dots, \xi_d \in \mathrm{H}_{\mathbb{C}}$. Let π_ω be as in Theorem (6.3.19). Then $\pi_\omega\big(W(\xi_1 \odot \cdots \odot \xi_d)\big) = W^s(\xi_1 \odot \cdots \odot \xi_d)$.

Proof. We proceed by induction on $d \in \mathbb{N}_0$. The base cases $d = 0,1$ are obvious from the definitions. Now assume that the claimed formula is true for all lengths $0 \leqslant d' \leqslant d$, and consider the $d+1$ case. Fix $\xi_0, \xi_1, \dots, \xi_d \in \mathrm{H}_{\mathbb{C}}$. It then follows from Proposition (6.3.8) that the following relation holds.

$$\begin{aligned} W(\xi_0 \odot \cdots \odot \xi_d) &= W(\xi_0)W(\xi_1 \odot \cdots \odot \xi_d) \\ &\quad - \sum_{l=1}^{d} q^{l-1} \langle I\xi_0 \mid \xi_l \rangle_U W\big(\xi_1 \odot \dots \odot \hat{\xi}_l \odot \dots \odot \xi_d\big), \end{aligned}$$

where, as usual, $\hat{\xi}_l$ means that the tensor factor ξ_l is deleted from the simple tensor under consideration. Applying π_ω to this relation and using our induction hypothesis, we have

$$\begin{aligned} \pi_\omega\big(W(\xi_0 \odot \dots \odot \xi_d)\big) &= W^s(\xi_0)W^s(\xi_1 \odot \dots \odot \xi_d) \\ &\quad - \sum_{l=1}^{d} q^{l-1} \langle I\xi_0 \mid \xi_l \rangle_U W^s\big(\xi_1 \odot \dots \odot \hat{\xi}_l \odot \dots \odot \xi_d\big). \quad (8) \end{aligned}$$

Next, we expand the first term on the right-hand side in the above equation:

$$\begin{aligned} &W^s(\xi_0)W^s(\xi_1 \odot \cdots \odot \xi_d) \\ &= \left(m^{-\frac{1}{2}} \sum_{k(0)=1}^{m} W\big(e_{k(0)}\big) \otimes W\big(\xi_0 \odot e_{k(0)}\big) \right)_\omega \\ &\times \left(m^{-\frac{d}{2}} \sum_{\substack{k:[d]\to[m] \\ k \text{ injective}}} W\big(e_{k(1)}\big) \dots W\big(e_{k(d)}\big) \otimes W\big(\xi_1 \odot e_{k(1)}\big) \dots W\big(\xi_d \odot e_{k(d)}\big) \right)_\omega \\ &= \left(m^{-\frac{d+1}{2}} \sum_{\substack{k:[d]_0\to[m] \\ k \text{ injective}}} W\big(e_{k(0)}\big) \dots W\big(e_{k(d)}\big) \otimes W\big(\xi_0 \odot e_{k(0)}\big) \dots W\big(\xi_d \odot e_{k(d)}\big) \right)_\omega \\ &+ \left(m^{-\frac{d+1}{2}} \sum_{k(0)=1}^{m} \sum_{l=1}^{d} \sum_{\substack{k:[d]\to[m] \\ k \text{ injective} \\ k(l)=k(0)}} W\big(e_{k(0)}\big) \dots W\big(e_{k(d)}\big) \otimes W\big(\xi_0 \odot e_{k(0)}\big) \dots W\big(\xi_d \odot e_{k(d)}\big) \right)_\omega \end{aligned}$$

$= W^s(\xi_0 \odot \xi_1 \odot \cdots \odot \xi_d)$ (this is the first term in the preceding sum)

$$+\sum_{l=1}^{d}\left(m^{-\frac{d+1}{2}}\sum_{k(0)=1}^{m}\sum_{\substack{k:[d]\to[m]\\ k\text{ injective}\\ k(l)=k(0)}} W(e_{k(0)})\dots W(e_{k(d)})\otimes W(\xi_0\odot e_{k(0)})\dots W(\xi_d\odot e_{k(d)})\right)_{\omega}$$

The first term is already a part of what we wanted, but we also have to deal with the second term. Note that for $k(0)=k(l)$ and $k(1)\neq\cdots\neq k(d)$ we have

$$W(e_{k(0)})\dots W(e_{k(d)}) = W(e_{k(0)}\odot\cdots\odot e_{k(d)}) + q_0^{l-1}W(e_{k(0)})\dots\widehat{W(e_{k(l)})}\dots W(e_{k(d)}) \quad (9)$$

And

$$W(\xi_0\odot e_{k(0)})\dots W(\xi_d\odot e_{k(d)})) \quad (10)$$

$$= W\left((\xi_0\odot e_{k(0)})\odot\cdots\odot(\xi_d\odot e_{k(d)})\right) + \langle I\xi_0\odot e_{k(0)} \mid \xi_l\odot e_{k(l)}\rangle_U q_1^{l-1}W(\xi_0\odot e_{k(0)})\dots W(\widehat{\xi_l\odot e_{k(l)}})\dots W(\xi_d\odot e_{k(d)})$$

$$= W\left((\xi_0\odot e_{k(0)})\odot\cdots\odot(\xi_d\odot e_{k(d)})\right) + \langle I\xi_0 \mid \xi_l\rangle_U q_1^{l-1}W(\xi_0\odot e_{k(0)})\dots W(\widehat{\xi_l\odot e_{k(l)}})\dots W(\xi_d\odot e_{k(d)}).$$

Indeed, if $(v_1,\dots,v_n)\subseteq \mathrm{H}_{\mathbb{C}}$ is a family of orthogonal vectors then $W(v_1)\dots W(v_n)=W(v_1\odot\cdots\odot v_n)$, as we remarked earlier. In our case we have a sequence $(w,v_1,\dots,v_d)$, where Iw is orthogonal to all vectors v_j for $j\neq l$, so we get

$$\begin{aligned} W(w)W(v_1)\dots W(v_d)\Omega &= (a^*(w)+a(Iw))v_1\odot\cdots\odot v_d \\ &= w\odot v_1\odot\cdots\odot v_d + a(Iw)(v_1\odot\cdots\odot v_d) \\ &= w\odot v_1\odot\cdots\odot v_d + q_1^{l-1}\langle Iw \mid v_l\rangle_U v_1\odot\cdots\odot\hat{v}_l\odot\cdots\odot v_d, \end{aligned}$$

hence the formula above. Tensoring $W(e_{k(0)})\dots W(e_{k(d)})$ with $W(\xi_0\odot e_{k(0)})\dots W(\xi_d\odot e_{k(d)})$ (keeping in mind that $q_0q_1=q$) gives us four terms, one of which is

$$q^{l-1}\langle I\xi_0 \mid \xi_l\rangle_U W(e_{k(1)})\dots\widehat{W(e_{k(l)})}\dots W(e_{k(d)}) \otimes W(\xi_0\odot e_{k(0)})\dots W(\widehat{\xi_l\odot e_{k(l)}})\dots W(\xi_d\odot e_{k(d)})$$

and we will deal with the three other terms later. To these expressions we need to apply the sum $\sum_{l=1}^{d} m^{-\frac{d+1}{2}}\sum_{k(0)=1}^{m}\sum_{\substack{k:[d]\to[m]\\ k\text{ injective}\\ k(l)=k(0)}}$ the condition $k(l)=k(0)$ and perform the sum over $k(0)$ immediately, resulting in a sum $\sum_{l=1}^{d} m^{-\frac{d-1}{2}}\sum_{k:[d]\setminus l\}\to[m]}$. Without the sum over l, this is the sum over $d-1$ distinct indices appearing in the definition of W^s, so we get the sum

$$\sum_{l=1}^{d} q^{l-1}\langle I\xi_0 \mid \xi_l\rangle_U W^s(\xi_0\odot\cdots\odot\hat{\xi}_l\odot\cdots\odot\xi_d).$$

To sum up, we have checked so far that

$$W^s(\xi_0)W^s(\xi_1\odot\cdots\odot\xi_d) = W^s(\xi_0\odot\cdots\odot\xi_d) + \sum_{l=1}^{d} q^{l-1}\langle I\xi_0 \mid \xi_l\rangle_U W^s(\xi_0\odot\cdots\odot\hat{\xi}_l\odot\cdots\odot\xi_d) + R,$$

where R is the "remainder" term that will turn out to be a zero element of the ultraproduct. Inserting this into (8) we get that

$$\pi_\omega\big(W(\xi_1 \odot \cdots \odot \xi_d)\big) = W^s(\xi_1 \odot \cdots \odot \xi_d) + R,$$

so if we can check that R is really a zero element then this ends the proof.
Let us just recall that R comes from the three neglected so far terms arising from tensoring $W\big(e_{k(0)}\big) \dots W\big(e_{k(d)}\big)$ with $W\big(\xi_0 \odot e_{k(0)}\big) \dots W\big(\xi_d \odot e_{k(d)}\big)$. It can be written as

$$R = \left(m^{-\frac{d+1}{2}} \sum_{l=1}^{d} \big(R_{1,l}(m) + \langle I\xi_0 \mid \xi_l \rangle q_1^{l-1} R_{2,l}(m) + q_0^{l-1} R_{3,l}(m)\big) \right)_\omega ,$$

where

$$R_{1,l}(m) = \sum_{k(0)=1}^{m} \sum_{\substack{k:[d]\to[m] \\ k \text{ injective} \\ k(l)=k(0)}} W\big(e_{k(0)} \odot \dots \odot e_{k(d)}\big) \otimes W\Big(\big(\xi_0 \odot e_{k(0)}\big) \odot \dots \odot \big(\xi_d \odot e_{k(d)}\big)\Big),$$

$$R_{2,l}(m) = \sum_{k(0)=1}^{m} \sum_{\substack{k:[d]\to[m] \\ k \text{ injective} \\ k(l)=k(0)}} W\big(e_{k(0)} \odot \dots \odot e_{k(d)}\big) \otimes W\big(\xi_1 \odot e_{k(1)}\big) \odot W\big(\widehat{\xi_l \odot} e_{k(l)}\big) \odot W\big(\xi_d \odot e_{k(d)}\big),$$

and

$$R_{3,l}(m) = \sum_{k(0)=1}^{m} \sum_{\substack{k:[d]\to[m] \\ k \text{ injective} \\ k(l)=k(0)}} W\big(e_{k(1)}\big) \dots \widehat{W\big(e_{k(l)}\big)} \dots W\big(e_{k(d)}\big) \otimes W\Big(\big(\xi_0 \odot e_{k(0)}\big) \odot \dots \odot \big(\xi_d \odot e_{k(d)}\big)\Big).$$

Recall the formulas (9) and (10). After tensoring the right-hand sides we get four terms, one of which was already incorporated in the proof of Theorem (6.3.20). The other three are:

$$W\big(e_{k(0)} \odot \dots \odot e_{k(d)}\big) \otimes W\Big(\big(\xi_0 \odot e_{k(0)}\big) \odot \dots \odot \big(\xi_d \odot e_{k(d)}\big)\Big)$$

$$q_1^{l-1} \langle I\xi_0 \mid \xi_l \rangle W\big(e_{k(0)} \odot \dots \odot e_{k(d)}\big) \otimes W\big(\xi_1 \odot e_{k(1)}\big) \dots W\big(\widehat{\xi_l \odot} e_{k(l)}\big) \dots W\big(\xi_d \odot e_{k(d)}\big),$$

and

$$q_0^{l-1} W\big(e_{k(1)}\big) \dots \widehat{W\big(e_{k(l)}\big)} \dots W\big(e_{k(d)}\big) \otimes W\Big(\big(\xi_0 \odot e_{k(0)}\big) \odot \dots \odot \big(\xi_d \odot e_{k(d)}\big)\Big).$$

To obtain R, we just need to take sums over appropriate sets of indices.
We will now examine properties of R. Since q_0, q_1, and the range of summation over l is fixed, to show that R is a zero element in the ultraproduct, it suffices to show that $\lim_{m\to\infty} m^{-\frac{d+1}{2}} \|R_{i,l}\| = 0$ for any $1 \leqslant i \leqslant 3$ and $l \in [d]$. We will use Nou's noncommutative Khintchine inequality for this (Corollary (6.3.12)), but before that we need to obtain a bound for the coefficients.

Lemma (6.3.21)[256]: There exists a constant $D(d) > 0$ (depending only on the initial choice of $\xi_1, \dots, \xi_d \in H_{\mathbb{C}}$) such that for all $m \in \mathbb{N}$ and all $k: [d] \to [m]$, the following inequalities hold:

$$\left\| W\left((\xi_0 \odot e_{k(0)}) \odot \cdots \odot (\xi_d \odot e_{k(d)}) \right) \right\| \leqslant D(d)$$

$$\| W(\xi_1 \odot e_{k(1)}) \dots W(\widehat{\xi_l \odot e_{k(l)}}) \dots W(\xi_d \odot e_{k(d)}) \| \leqslant D(d)$$

$$\| W(e_{k(1)}) \dots \widehat{W(e_{k(l)})} \dots W(e_{k(d)}) \| \leqslant D(d).$$

Proof. The second and third inequality will follow if we can show that there is a constant $D > 0$ such that $\|W(\xi_r \odot e_{k(r)})\|, \|W(e_{k(r)})\| \leqslant D$ (independently of $r \in [d]$). But the existence of D follows from the simple fact for any q-Araki-Woods algebra $\Gamma_q(H)$ and $\xi \in H_C$, we have $\| W(\xi) \|_{\Gamma_q(\mathbf{H})} \leqslant \|a_q^*(\xi)\| + \|a_q(I\xi)\| \leqslant 2(1-|q|)^{-1/2} \max\{\| \xi \|, \| I\xi \|\}$. Now consider the first inequality. By the Khintchine inequality with $K = \mathbb{C}$ (Corollary (6.3.12)), the left-hand side is bounded by

$$C(q_1)(d+1) \max_{0 \leqslant l \leqslant d} \left\| (\mathbb{1}_{d-l} \odot \mathcal{J}) \left(R_{d,l}^* \left((\xi_0 \odot e_{k(0)}) \odot \cdots \odot (\xi_d \odot e_{k(d)}) \right) \right) \right\|.$$

Writing the above $(\mathbb{1}_{d-l} \odot \mathcal{J}) R_{d,k}^*$ terms as sums of simple tensors, one easily sees that the corresponding norms are bounded by a constant depending only on d. (Note that the unboundedness of $\mathcal{J}$ plays no role here, as $\xi_0, \dots, \xi_d \in H_{\mathbb{C}}$ remain fixed.)

We need one more proposition. In the following, $m \in \mathbb{N}$ and $\xi_0, \dots, \xi_d$ are fixed as usual. Let I_l denote the set of indices $(k(0), \dots, k(d)) \in [m]^{d+1}$ that are pairwise distinct except for the pair $(k(0), k(l))$; a generic element of I_l will be called **i** and the corresponding tensor $e_{k(0)} \otimes \cdots \otimes e_{k(d)}$ will also be denoted by i. We will denote $W(e_{k(0)} \otimes \cdots \otimes e_{k(d)})$ by $W_{\mathbf{i}}$ and $W\left((\xi_0 \odot e_{k(0)}) \otimes \cdots \otimes (\xi_d \odot e_{k(d)}) \right)$ by $W_{\mathbf{i}}^{\xi}$.

Proposition (6.3.22)[256]: Given any Hilbert space K and any family of operators $(A_i)_{i \in I_l} \subset B(K)$, the following inequalities hold.

$$\left\| \sum_{I_l} A_i \otimes W_i \right\| \leqslant C(d) \sup_{i \in I_l} \|A_i\| m^{\frac{d}{2}}$$

$$\left\| \sum_{I_l} A_i \otimes W_i^{\xi} \right\| \leqslant C(d) \sup_{i \in I_l} \|A_i\| m^{\frac{d}{2}},$$

where $C(d) > 0$ depends only on d and the choice of vectors $\xi_0, \xi_1, \dots, \xi_d \in H_{\mathbb{C}}$.

Proof. The proofs of both inequalities are essentially the same. We will deal with the first one; to obtain a proof of the second one has to apply conjugation in some places but since we are dealing with a fixed number of vectors $\xi_0, \dots, \xi_d$, the unboundedness of conjugation does not play any role. By the Khintchine inequality (Corollary (6.3.12)) we need to deal with

$$\max_{0 \leqslant k \leqslant d+1} \left\| \sum_{I_l} A_{\mathbf{i}} \otimes R_{d+1,k}^*(\mathbf{i}) \right\|,$$

up to a d-dependent constant.

Since $R_{d+1,k}^*$ is a sum of operators that only permute vectors, and the coefficients of this sum are summable, we just need to take care of a single term of the form

$$\max_{0\leqslant k\leqslant d+1}\left\|\sum_{I_l} A_{\mathbf{i}}\otimes\sigma(\mathbf{i})_{(d+1,k)}\right\|,$$

where σ denotes the action of the permutation and the decoration $(d+1,k)$ reminds us of the fact that $\sigma(\mathbf{i})$ is viewed now as an element of $\mathbf{H}_c^{\otimes(d+1-k)}\otimes_h \mathrm{H}_r^{\otimes k}$. Whatever the σ, the tensor $\sigma(\mathbf{i})_{d+1,k}$ is always of the form $e_{i_0}\odot\cdots\odot e_{i_{d-k}}\otimes e_{i_{d-k+1}}\odot\cdots\odot e_{i_d}$, where for different indices $\mathbf{i}$ and $\mathbf{i}'$ these tensors are different. The key property that we will need is that we have two orthonormal systems $(v_s)_{s\in S}\subset \mathrm{H}^{\otimes(d+1-k)}$ and $(w_j)_{j\in J}\subset \mathrm{H}^{\otimes k}$ such that for any $\mathbf{i}\in I_l$ we have $\sigma(\mathbf{i})_{d+1,k}=v_s\otimes w_j$ for some $s\in S$ and $j\in J$. Therefore we can get rid of the sign σ and just consider

$$\max_{0\leqslant k\leqslant d+1}\left\|\sum_{I_l} A_{\mathbf{i}}\otimes\mathbf{i}_{(d+1,k)}\right\|,$$

Since we are dealing with tensor powers of H equipped with q-deformed inner products, we would rather have families $(v_s')_{s\in S}$ and $(w_j')_{j\in J}$ that are orthonormal in $\mathrm{H}_q^{\otimes(d+1-k)}$ and $\mathrm{H}_q^{\otimes k}$, respectively. To achieve this, we will use the operators defining the q-deformed inner products, P_q^{d+1-k} and P_q^k. Let $\xi(\mathbf{i})_{d+1,k}$ be tensors defined by $\left(\left(P_q^{d+1-k}\right)^{\frac{1}{2}}\otimes\left(P_q^{k}\right)^{\frac{1}{2}}\right)\left(\xi(\mathbf{i})_{d+1,k}\right)=\mathbf{i}_{d+1,k}$.

Then we can write $\xi(\mathbf{i})_{d+1,k}=v_s'\otimes w_j'$ for some tensors v_s' and w_j' coming from orthonormal families in $\mathrm{H}_q^{\otimes(d+1-k)}$ and $\mathrm{H}_q^{\otimes k}$. Since the row/column Hilbert spaces are homogeneous operator spaces (and Haagerup tensor product allows tensoring cb maps) we can bound $\max_{0\leqslant k\leqslant d+1}\|\sum_{I_l}A_{\mathrm{i}}\otimes\mathbf{i}_{d+1,k}\|$ by $\max_{0\leqslant k\leqslant d+1}\|\sum_{I_l}A_{\mathrm{i}}\otimes\xi(\mathbf{i})_{d+1,k}\|$, up to a d-dependent constant coming from the norms of $\left(P_q^{d+1-k}\right)^{\frac{1}{2}}$ and $\left(P_q^{k}\right)^{\frac{1}{2}}$. Because we are using the Haagerup tensor product, we have the following completely isometric isomorphism $\mathrm{H}_c\otimes_h\overline{\mathrm{K}}_r\simeq\mathcal{K}(\mathrm{K},\mathrm{H})$. Under this identification the tensors $\xi(\mathbf{i})_{d+1,k}$ correspond to matrix units in $\mathcal{K}\left(\mathrm{H}_q^{\otimes k},\mathrm{H}_q^{\otimes(d+1-k)}\right)$. This means that the operators $A_{\mathbf{i}}$ fill different entries in a large operator matrix. By comparing the operator norm with the Hilbert-Schmidt norm we get the estimate

$$\left\|\sum_{\mathbf{i}\in I_l} A_{\mathbf{i}}\otimes W_{\mathbf{i}}\right\|\leqslant C(d)\left(\sum_{\mathbf{i}\in I_l}\|A_{\mathbf{i}}\|^2\right)^{\frac{1}{2}}\leqslant C(d)\left(|I_l|\sup_{\mathbf{i}\in I_l}\|A_{\mathbf{i}}\|^2\right)^{\frac{1}{2}},$$

which can be further bounded by

$$C(d)\left(m^d\sup_{\mathbf{i}\in I_l}\|A_{\mathbf{i}}\|^2\right)^{\frac{1}{2}}=C(d)m^{\frac{d}{2}}\sup_{\mathbf{i}\in I_l}\|A_{\mathbf{i}}\|.$$

Finally, to conclude that $R=0$ in the ultraproduct, we just observe that each component $R_{i,l}=\left(R_{i,l}(m)\right)_{m\in\mathbb{N}}$ is a a sequence of terms of the form appearing in Proposition (6.3.22) with coefficients $(A_{\mathbf{i}}(m))_{m\in\mathbb{N},\mathbf{i}\in I_l}$ uniformly bounded in $\mathbf{i}$ and m by the constant $D(d)$ from Lemma (6.3.21), so the norm $m^{-\frac{d+1}{2}}R_{i,l}(m)$ is bounded from above by $C(d)D(d)m^{-\frac{1}{2}}$, and hence tends to zero. This finishes the proof of Theorem (6.3.20). With this tool at hand, we prove Theorem (6.3.23).

Theorem (6.3.23)[256]: Let $\Gamma_q(\mathrm{H})$ be a q-Araki-Woods algebra. Let $P_n: \Gamma_q(\mathrm{H}) \to \Gamma_q(\mathrm{H})$ be the projection onto the ultraweakly closed span of $\{W(\xi): \xi \in \mathrm{H}_{\mathbb{C}}^{\odot n}\}$. Then, using the notation from Theorem (6.3.19), we have

$$\Theta \circ P_n = p(P_n \otimes \mathrm{Id})_\omega p \circ \Theta. \tag{11}$$

Proof. Let $W(\xi)$ be a Wick word associated with $\xi \in \mathrm{H}_{\mathbb{C}}^{\odot d}$. Then we easily obtain $\pi_\omega(P_n W(\xi)) = \delta_{n,d} W^s(\xi)$. On the other hand, let us first apply π_ω to obtain $W^s(\xi)$. Since, as we already remarked earlier, $W(e_{k(1)}) \dots W(e_{k(d)}) = W(e_{k(1)} \otimes \cdots \otimes e_{k(d)})$, the operators acted on by the P_n part of the operator $(P_n \otimes \mathrm{Id})_\omega$ are exactly of length n. Therefore $(P_n \otimes \mathrm{Id})_\omega W^s(\xi) = \delta_{n,d} W^s(\xi)$. By linearity, this implies that $\pi_\omega \circ P_n = (P_n \otimes \mathrm{Id})_\omega \circ \pi_\omega$ on the algebra of Wick words $\tilde{\Gamma}_q(\mathrm{H})$. Compressing by the support projection p, we then obtain

$$\Theta \circ P_n = p(P_n \otimes \mathrm{Id})_\omega \circ \pi_\omega(\cdot) p = p(P_n \otimes \mathrm{Id})_\omega p \circ \Theta \text{ on } \tilde{\Gamma}_q(\mathrm{H}),$$

where in the second equality we used the fact that $p \in \pi_\omega(\tilde{\Gamma}_q(\mathrm{H}))'$ (see [130, Lemma 4.1]). Since the desired equality holds on the ultraweakly dense subset $\tilde{\Gamma}_q(\mathrm{H})$, and all maps under consideration are normal, equality holds everywhere.

Let us now furnish a proof of the transference result for radial multipliers.

Theorem (6.3.24)[256]: Let $\varphi: \mathbb{N} \to \mathbb{C}$ be a function such that the associated radial multipliers $\mathrm{m}_\varphi: \Gamma_q(\mathbb{R}^m) \to \Gamma_q(\mathbb{R}^m)$ have completely bounded norms uniformly bounded in m. Then the radial multiplier defined by φ on any q-Araki-Woods algebra $\Gamma_q(\mathrm{H})$ is completely bounded and

$$\begin{aligned}\|\boldsymbol{m}_\varphi: \Gamma_q(H) \to \Gamma_q(H)\|_{cb} &\leqslant \sup_{m \in \mathbb{N}} \|m_\varphi: \Gamma_q(\mathbb{R}^m) \to \Gamma_q(\mathbb{R}^m)\|_{cb} \\ &= \|m_\varphi: \Gamma_q(\ell_{2,\mathbb{R}}) \to \Gamma_q(\ell_{2,\mathbb{R}})\|_{cb}.\end{aligned}$$

Proof. From Theorem (6.3.23) we get that $\Phi \circ \mathrm{m}_\varphi(x) = p(\mathbf{m}_\varphi \otimes \mathrm{Id})_\omega p \circ \Phi(x)$ for any $x = W(\xi)$ with $\xi \in (\mathrm{H}_{\mathbb{C}})^{\odot d}$. By linearity we can extend this equality to all $x \in \tilde{\Gamma}_q(\mathrm{H})$. It follows that we have control on the cb norm of m_φ acting on the norm-closure of finite Wick words, i.e. on the C^*-algebra $\mathcal{A}_q(\mathrm{H})$. Since m_φ is automatically normal (cf. [190, Lemma 3.4]), it extends to a normal map on $\Gamma_q(\mathrm{H})$ with the same cb norm, so we get

$$\|\mathbf{m}_\varphi: \Gamma_q(\mathrm{H}) \to \Gamma_q(\mathrm{H})\|_{\mathrm{cb}} \leqslant \sup_{m \in \mathbb{N}} \|\mathrm{m}_\varphi: \Gamma_q(\mathbb{R}^m) \to \Gamma_q(\mathbb{R}^m)\|_{\mathrm{cb}}.$$

Since $\Gamma_q(\mathbb{R}^m)$ is a subalgebra of $\Gamma_q(\mathbb{R}^{m+1})$ which is the range of a normal faithful tracepreserving conditional expectation that intertwins the action of m_φ, the sequence of norms on the right-hand side is non-decreasing, so

$$\|\mathrm{m}_\varphi: \Gamma_q(\mathrm{H}) \to \Gamma_q(\mathrm{H})\|_{\mathrm{cb}} \leqslant \lim_{m \to \infty} \|\mathrm{m}_\varphi: \Gamma_q(\mathbb{R}^m) \to \Gamma_q(\mathbb{R}^m)\|_{\mathrm{cb}}.$$

By the same token, this limit is not greater than $\|\mathrm{m}_\varphi: \Gamma_q(\ell_{2,\mathbb{R}}) \to \Gamma_q(\ell_{2,\mathbb{R}})\|_{\mathrm{cb}}$. Since the union of the algebras $\Gamma_q(\mathbb{R}^m)$ is strongly dense in $\Gamma_q(\ell_{2,\mathbb{R}})$, the union of the preduals is normdense in the predual of $\Gamma_q(\ell_{2,\mathbb{R}})$. Therefore the limit of norms is equal to the norm of the multplier defined on $L^1\left(\Gamma_q(\ell_{2,\mathbb{R}})\right)$. By dualising, we get that

$$\lim_{m \to \infty} \|m_\varphi: \Gamma_q(\mathbb{R}^m) \to \Gamma_q(\mathbb{R}^m)\|_{\mathrm{cb}} = \|\mathrm{m}_\varphi: \Gamma_q(\ell_{2,\mathbb{R}}) \to \Gamma_q(\ell_{2,\mathbb{R}})\|_{\mathrm{cb}}.$$

Let us conclude with an application to the extension of Theorem (6.3.14) to general q-Araki-Woods algebras.

Corollary (6.3.25)[256]: Let $\Gamma_q(\mathrm{H})$ be a q-Araki-Woods algebra. Let P_n be the projection onto Wick words of length n, defined by $P_n W(\xi) = \delta_{n,d} W(\xi)$, where $\xi \in \mathrm{H}_{\mathbb{C}}^{\odot d}$. Then P_n extends to a completely bounded, normal map on $\Gamma_q(\mathrm{H})$ and $\|P_n\|_{cb} \leqslant C(q)^2(n+1)^2$.

Proof. We just observe that $P_n = \mathrm{m}_{\varphi_n}$, where φ_n is the Kroenecker delta-function $\varphi_n(k) = \delta_{k,n}$. We obtain $\|P_n\colon \Gamma_q(\mathrm{H}) \to \Gamma_q(\mathrm{H})\|_{\mathrm{cb}} \leqslant \| P_n\colon \Gamma_q(\ell_{2,\mathbb{R}}) \to \Gamma_q(\ell_{2,\mathbb{R}}) \|_{\mathrm{cb}} \leqslant C(q)^2(n+1)^2$. The last will be devoted to the proof of the complete metric approximation property for $\Gamma_q(\mathrm{H})$.

Before proving our main result, we need to recall one more lemma.

Lemma (6.3.26)[256]: ([190, Proposition 3.17]). Let H be the Hilbert space constructed from the pair $(\mathrm{H}_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$. Let I be the complex conjugation on $\mathrm{H}_{\mathbb{C}}$. Then there exists a net $(T_i)_{i\in I}$ of finite-rank contractions on H that satisfy $IT_iI = T_i$, i.e. preserve $\mathrm{H}_{\mathbb{R}}$, and converge strongly to identity.

Theorem (6.3.27)[256]: Let $\Gamma_q(\mathrm{H})$ be a q-Araki-Woods algebra. Then $\Gamma_q(\mathrm{H})$ has the w^*-complete metric approximation property.

Proof. We define a net $\Gamma_{n,t,i} := \Gamma_q(e^{-t}T_i)Q_n$, where $n \in \mathbb{N}, t > 0, i \in I$, the finite-rank maps T_i come from the previous lemma, and $Q_n = P_0 + \cdots + P_n = m_{\chi_{\{0,1,\ldots,n\}}}$ is the radial multiplier which projects onto Wick words of length at most n. Each $\Gamma_{n,t,i}$ is a finite rank map on $\Gamma_q(\mathrm{H})$; indeed, Q_n tells us that we have only Wick words of bounded length and T_i tells us that we can only draw vectors from a finite dimensional Hilbert space, so we are left with a space of the form $\bigoplus_{d=0}^{n} (\mathbb{C}^m)^{\otimes d}$, which is finite-dimensional. We will pass to a limit with $i \to \infty, n \to \infty$ and $t \to 0$. The rate of convergences of t and n will not be independent and will be chosen in a way that assures the convergence $\|\Gamma_{n,t,i}\|_{cb} \to 1$.

Let us check now that it is possible, using a standard argument of Haagerup (note that $\Gamma_q(e^{-t})P_k = e^{-kt}P_k$):

$$\begin{aligned}
\|\Gamma_{n,t,i}\|_{cb} &= \|\Gamma_q(e^{-t}T_i)Q_n\|_{cb} \\
&\leqslant \|\Gamma_q(e^{-t})Q_n\|_{cb} \\
&\leqslant \|\Gamma_q(e^{-t})\|_{cb} + \|\Gamma_q(e^{-t})(\mathbb{1} - Q_n)\|_{cb} \\
&\leqslant 1 + \sum_{k>n} e^{-kt}\|P_k\|_{cb} \\
&\leqslant 1 + C(q)^2 \sum_{k>n} e^{-kt}(k+1)^2.
\end{aligned}$$

Since the series $\sum_{k\geqslant 0} e^{-kt}(k+1)^2$ is convergent, for any $t > 0$ the sum will tend to zero when $n \to \infty$. Therefore we can choose the parameters $i, n \to \infty$ and $t \to 0$ such that the completely bounded norms of the operators $\Gamma_{n,t,i}$ tend to 1. Then the operators $\frac{\Gamma_{n,t,i}}{\|\Gamma_{n,t,i}\|_{cb}}$ are completely contractive. We have to check that they converge ultraweakly to $\mathbb{1}$. Since the denominators converge to 1 and the net is uniformly bounded, it suffices to prove strong convergence on a linearly dense set. It is very easy to verify that the convergence holds for finite simple tensors, so this ends the proof.

Let us state two corollaries of (the proof) of this theorem.

Corollary (6.3.28)[256]: Let H be the Hilbert space constructed from the pair $(\mathrm{H}_R, (U_t)_{t\in\mathbb{R}})$. Consider the σ-weakly dense C^*-algebra $\mathcal{A}_q(\mathrm{H}) \subseteq \Gamma_q(\mathrm{H})$ generated by the set $\{W(\xi): \xi \in \mathrm{H}_{\mathbb{R}}\} \subset \mathrm{B}(\mathcal{F}_q(\mathrm{H}))$. This C^*-algebra has the complete metric approximation property.

Proof. Consider once again the maps $\Gamma_{n,t,i} := \Gamma_q(e^{-t}T_i)Q_n$. The ranges of these maps are contained in $\tilde{\Gamma}_q(\mathrm{H})$, the bounds for the norms remain the same, so it suffices to check the pointwise convergence in norm. Since the maps are uniformly bounded, it suffices to check the convergence on a linearly dense set, hence we may assume that $x = W(\xi_1 \odot \cdots \odot \xi_k)$. If n is large enough the Q_n that appears in the definition of $\Gamma_{n,t,i}$ has no effect on x, so we get

$$\Gamma_{n,t,i}x - x = e^{-kt}W(T_i\xi_1 \odot \cdots \odot T_i\xi_k) - W(\xi_1 \odot \cdots \odot \xi_k).$$

This last expression is easily seen to converge to zero in norm as $t \to 0$ and $i \to \infty$. This can be seen either using the Khintchine inequality (Corollary (6.3.12)), or just by expressing $W(\xi_1 \odot \cdots \odot \xi_k)$ as a non-commutative polynomial in $a_q(\xi_r)$'s and $a_q^*(\xi_r)$'s and invoking the fact that

$$\lim_i \|a_q^*(T_i\xi_k) - a_q^*(\xi_k)\| = \lim_i \|a_q(T_i\xi_k) - a_q(\xi_k)\| \leqslant (1-|q|)^{-1/2} \lim_i \|T_i\xi_k - \xi_k\| \to 0.$$

Corollary (6.3.29)[256]: The C^*-algebra $\mathcal{A}_q(\mathrm{H})$ is $QWEP$.

Proof. We will show that $\mathcal{A}_q(\mathrm{H})$ is weakly cp complemented in the von Neumann algebra $\Gamma_q(\mathrm{H})$, meaning that there exists a ucp map $\Phi: \Gamma_q(\mathrm{H}) \to (\mathcal{A}_q(\mathrm{H}))^{**}$ such that $\Phi|_{\mathcal{A}_q(\mathrm{H})} = \mathrm{Id}$.

Let $(\Phi_i)_{i\in I}$ be the net of maps implementing at the same time the w^*-complete metric approximation property of $\Gamma_q(\mathrm{H})$ and the complete metric approximation property of $\mathcal{A}_q(\mathrm{H})$. Using this net, we get maps $\Phi_i: \Gamma_q(\mathrm{H}) \to (\mathcal{A}_q(\mathrm{H}))^{**}$, as Φ_i maps $\Gamma_q(\mathrm{H})$ into $\mathcal{A}_q(\mathrm{H})$. There exists a cluster point of this net in the point-weak*-topology and this cluster point is obviously a ucp map that is equal to identity, when restricted to $\mathcal{A}_q(\mathrm{H})$, because the net $(\Phi_i)_{i\in I}$ converges pointwise to identity on $\mathcal{A}_q(\mathrm{H})$. Since all q-Araki-Woods algebras are QWEP (cf. [130]) and this property descends to subalgebras that are weakly cp complemented (cf. [133, Proposition 4.1 (ii)]), we get the claimed result.

Corollary (6.3.30)[264]: (i) The mapping

$$W(\xi) \mapsto (u_m(\xi))_{\omega^2-1} \in \mathrm{A}\ (\xi \in \mathrm{H}_{\mathbb{C}})$$

extends uniquely to a state-preserving *-homomorphism $\pi_{\omega^2-1}: (\tilde{\Gamma}_{\epsilon-1}(\mathrm{H}), \chi) \to (\mathrm{A}, (\chi_{0,m} \otimes \chi_{1,m})_{\omega^2-1}$.

(ii) The map $\Theta := p\pi_{\omega^2-1}(\cdot)p: \tilde{\Gamma}_{\epsilon-1}(\mathrm{H}) \to p\,\mathrm{A}p$ extends to a normal state-preserving *-isomorphism

$$\Theta: \Gamma_{\epsilon-1}(\mathrm{H}) \to \Theta(\Gamma_{\epsilon-1}(\mathrm{H})) \subseteq p\,\mathrm{A}p.$$

Moreover, $\Theta(\Gamma_{\epsilon-1}(\mathrm{H}))$ is the range of a normal state-preserving conditional expectation $E: \mathrm{A} \to \Theta(\Gamma_{\epsilon-1}(\mathrm{H}))$.

Proof. (i). Recall that the algebra of Wick words is *-isomorphic to the *-algebra of noncommutative polynomials, so any *-homomorphism $\pi_{\omega^2-1}: \tilde{\Gamma}_{\epsilon-1}(\mathrm{H}) \to \mathrm{A}$ is uniquely determined by specifying the images $\left(\pi_{\omega^2-1}(W(e_i))\right)_{i\in I} \subset \mathrm{A}$. Thus to conclude that the claimed π_{ω^2-1} exists and is well-defined, we just need to check that each sequence $(u_m(\xi))_{m\in\mathbb{N}} (\xi \in \mathrm{H}_{\mathbb{C}})$ is normbounded and hence defines an element $(u_m(\xi))_{\omega^2-1} \in \mathrm{A}$. To this end, we apply (the $n=1$ version of) Corollary (6.3.29) with coefficients $W(\xi\odot e_{1+\epsilon})\in \mathrm{B}(\mathrm{K})=\mathrm{B}\left(\mathcal{F}_{q_1}(\mathrm{H}\otimes\mathbb{C}^m)\right)$ (see also [113]) to conclude that

$$\|u_m(\xi)\|\leqslant 2(1-q_0)^{\frac{-1}{2}}m^{\frac{-1}{2}}\max\left\{\left\|\sum_{\epsilon=0}^{m}W(\xi\odot e_{1+\epsilon})^*W(\xi\odot e_{1+\epsilon})\right\|^{\frac{1}{2}},\left\|\sum_{\epsilon=0}^{m}W(\xi\odot e_{1+\epsilon})W(\xi\odot e_{1+\epsilon})^*\right\|^{\frac{1}{2}}\right\}$$

$$\leqslant 2(1-q_0)^{\frac{-1}{2}}\|W(\xi\odot e_1)\|.$$

Finally we check that π_{ω^2-1} is state-preserving. By linearity, it suffices to show that for any $1+3\epsilon\in\mathbb{N}$ and $\xi_1,\dots,\xi_{1+3\epsilon}\in\mathrm{H}_{\mathbb{R}}$, we have

$$\lim_{m\to\infty}\left(\chi_{0,m}\otimes\chi_{1,m}\right)\left(u_m(\xi_1)\cdot\ldots\cdot u_m(\xi_{1+3\epsilon})\right)=\chi\left(W(\xi_1)\cdot\ldots\cdot W(\xi_{1+3\epsilon})\right).$$

Fixing m and considering the terms on the left-hand side above, we have

$$\left(\chi_{0,m}\otimes\chi_{1,m}\right)\left(u_m(\xi_1)\cdot\ldots\cdot u_m(\xi_{1+3\epsilon})\right)$$

$$=m^{-1+3\epsilon/2}\sum_{k:[1+3\epsilon]\to[m]}\chi_{0,m}\left(W\left(e_{k(1)}\right)\cdot\ldots\cdot W\left(e_{k(1+3\epsilon)}\right)\right)\chi_{1,m}\left(W\left(\xi_1\odot e_{k(1)}\right)\cdot\ldots\cdot W\left(\xi_{1+3\epsilon}\odot e_{k(1+3\epsilon)}\right)\right)$$

$$=m^{-\frac{1+3\epsilon}{2}}\sum_{k:[1+3\epsilon]\to[m]}\left(\sum_{\substack{\sigma\in\mathcal{P}_2(1+3\epsilon)\\ ker k\geqslant\sigma}}q_0^{\iota(\sigma)}\right)\left(\sum_{\sigma'\in\mathcal{P}_2(1+3\epsilon)}q_1^{\iota(\sigma')}\prod_{(1+\epsilon,t)\in\sigma'}\left\langle\xi_{1+\epsilon}\odot e_{k(1+\epsilon)}\mid\xi_t\odot e_{k(t)}\right\rangle_U\right)$$

$$=m^{-\frac{1+3\epsilon}{2}}\sum_{k:[1+3\epsilon]\to[m]}\left(\sum_{\substack{\sigma\in\mathcal{P}_2(1+3\epsilon)\\ ker k\geqslant\sigma}}q_0^{\iota(\sigma)}\right)\left(\sum_{\substack{\sigma'\in\mathcal{P}_2(1+3\epsilon)\\ ker k\geqslant\sigma'}}q_1^{\ell(\sigma')}\prod_{(1+\epsilon,t)\in\sigma'}\langle\xi_{1+\epsilon}\mid\xi_t\rangle_U\right)$$

$$=\sum_{\sigma,\sigma'\in\mathcal{P}_2(1+3\epsilon)}q_0^{\iota(\sigma)}q_1^{\ell(\sigma')}\prod_{(1+\epsilon,t)\in\sigma'}\langle\xi_{1+\epsilon}\mid\xi_t\rangle_U\sum_{\substack{k:[1+3\epsilon]\to[m]\\ k\in(1+\epsilon)k\geqslant\sigma,\ker k\geqslant\sigma'}}m^{-\frac{1+3\epsilon}{2}}$$

$$=\sum_{\sigma,\sigma'\in\mathcal{P}_2(1+3\epsilon)}q_0^{\iota(\sigma)}q_1^{\iota(\sigma')}\prod_{(1+\epsilon,t)\in\sigma'}\langle\xi_{1+\epsilon}\mid\xi_t\rangle_U m^{-\frac{1+3\epsilon}{2}+|\sigma\vee\sigma'|}.$$

Since

$$\lim_{m\to\infty}m^{-\frac{1+3\epsilon}{2}+|\sigma\vee\sigma'|}=\delta_{\sigma,\sigma'}\ (\sigma,\sigma'\in\mathcal{P}_2(1+3\epsilon)),$$

we conclude that

$$\lim_{m\to\infty}\left(\chi_{0,m}\otimes\chi_{1,m}\right)\left(u_m(\xi_1)\cdot\ldots\cdot u_m(\xi_{1+3\epsilon})\right)=\sum_{\sigma\in\mathcal{P}_2(1+3\epsilon)}(\epsilon-1)^{\iota(\sigma)}\prod_{(1+\epsilon,t)\in\sigma}\langle\xi_{1+\epsilon}\mid\xi_t\rangle_U=\chi\left(W(\xi_1)\cdot\ldots\cdot W(\xi_{1+3\epsilon})\right).$$

(ii). To exhibit the desired properties of $\Theta := p\pi_{\omega^2-1}(\cdot)p$, we will verify conditions (i)-(iii) in Theorem (6.3.27) for the $*$-homomorphism π_{ω^2-1}. (i) follows immediately from part (1) of the present theorem. For (ii), we note that by linearity and multiplicativity of π_{ω^2-1}, it suffices to check condition (ii) on the generators $\pi_{\omega^2-1}(W(\xi)) = (u_m(\xi))_{\omega^2-1}, (\xi \in \mathrm{H}_{\mathbb{C}})$. However, there is a minor issue here coming from the fact that for arbitrary $\xi \in \mathrm{H}_{\mathbb{C}}$, there is no reason to expect elements $u_m(\xi) \in \Gamma_{q_0}(\mathbb{R}^m) \bar{\otimes} \Gamma_{q_1}(\mathrm{H} \otimes \mathbb{C}^m)$ to even be analytic, let alone the sequence $\left(\sigma_{-i}(u_m(\xi))\right)_{m\in\mathbb{N}}$ be uniformly bounded. To overcome this issue, put $\mathrm{H}_{\mathbb{C}}^{an} = \bigcup_{\lambda>1} \mathbf{1}_{[\lambda^{-1},\lambda]}(A)\mathrm{H}_{\mathbb{C}}$, where $\mathbf{1}_{[\lambda^{-1},\lambda]}(A)$ denotes the spectral projection of the analytic generator A corresponding to the interval $[\lambda^{-1},\lambda]$. Following [246, Theorem 3.1], we see that $\mathrm{H}_{\mathbb{C}}^{an} \subset \mathrm{H}_{\mathbb{C}}$ is a dense linear subspace such that $I\mathrm{H}_{\mathbb{C}}^{an} = \mathrm{H}_{\mathbb{C}}^{an}$. Moreover, for each $\xi \in \mathrm{H}_{\mathbb{C}}^{an}$, we have that ξ (respectively $W(\xi)$) is analytic for the action of the unitary group $U_t = A^{it}$ (respectively the modular automorphism group σ_t), and

$$\sigma_z W(\xi) = W\left(A^{-iz}\xi\right) \ (z \in \mathbb{C}).$$

Now we shall restrict the domain of π_{ω^2-1} to the *-subalgebra $\tilde{\Gamma}_{\epsilon-1}(\mathrm{H})_{an} \subset \tilde{\Gamma}_{\epsilon-1}(\mathrm{H})$, consisting of linear combinations of Wick words of the form $W(\xi)$ with $\xi \in (\mathrm{H}_{\mathbb{C}}^{an})^{\odot n}$, $(n \in \mathbb{N}_0)$. Since $\tilde{\Gamma}_{\epsilon-1}(\mathrm{H})_{an}$ is still w*-dense in $\Gamma_{\epsilon-1}(\mathrm{H})$ and is generated by $(W(\xi))_{\xi\in\mathrm{H}_{\mathbb{C}}^{an}}$, we just have to show that the equivalence class representative $(u_m(\xi))_{m\in\mathbb{N}}$ for $\pi_{\omega^2-1}(W(\xi))$ satisfies condition (ii) of Corollary (6.3.30) for each $\xi \in \mathrm{H}_{\mathbb{C}}^{an}$. To this end, note that on $\Gamma_{q_0}(\mathbb{R}^m) \bar{\otimes} \Gamma_{q_1}(\mathrm{H} \otimes \mathbb{C}^m)$, we have

$$\sigma_t^m = \mathrm{id}_{\Gamma_{q_0}(\mathbb{R}^m)} \otimes \sigma_t^{\Gamma_{q_1}(\mathrm{H}\otimes\mathbb{C}^m)} \ \& \ \sigma_t^{\Gamma_{q_1}(\mathrm{H}\otimes\mathbb{C}^m)}(W(\xi \odot e)) = W\left(A^{-it}\xi \odot e\right) \ (\xi \in \mathrm{H}_{\mathbb{C}}, e \in \mathbb{C}^m).$$

It follows from these identities that if $\xi \in \mathrm{H}_{\mathbb{C}}^{an}$ and $e \in \mathbb{C}^m$, then elements $W(\xi \odot e)$ and $u_m(\xi)$ are analytic for their respective modular groups and

$$\sigma_z^m(u_m(\xi)) = \left(\frac{1}{\sqrt{m}}\sum_{\epsilon=0}^{m} W(e_{1+\epsilon}) \otimes \sigma_z^{\Gamma_{q_1}(H\otimes\mathbb{C}^m)} W(\xi \odot e_{1+\epsilon})\right)$$
$$= \frac{1}{\sqrt{m}}\sum_{\epsilon=0}^{m} W(e_{1+\epsilon}) \otimes W\left(A^{-iz}\xi \odot e_{1+\epsilon}\right) = u_m\left(A^{-iz}\xi\right), \ (z \in \mathbb{C}).$$

The uniform boundedness of the sequence $(\sigma_{-i}^m u_m(\xi))_{m\in\mathbb{N}}$ now follows along the same lines as that of $(u_m(\xi))_{m\in\mathbb{N}}$:

$$\sup_m \|\sigma_{-i}^m u_m(\xi)\| = \sup_m \|u_m(A^{-1}\xi)\| \leqslant 2(1-q_0)^{\frac{-1}{2}} \|W(A^{-1}\xi \odot e_1)\|.$$

For (iii), it again suffices by linearity and multiplicativity to verify that for all $\pi_{\omega^2-1}(W(\xi)) = (u_m(\xi))_{\omega^2-1}, (\xi \in \mathrm{H}_{\mathbb{C}})$,

$$p\left(\left(\sigma_t^m(u_m(\xi))\right)_{\omega^2-1}\right)p \in p\,\mathrm{B}p,$$

where B is the w^*-closure of $\pi_{\omega^2-1}\left(\tilde{\Gamma}_{\epsilon-1}(\mathrm{H})\right)$ in A. But this last point is obvious, because by the previous computation, $\sigma_t^m(u_m(\xi)) = u_m\left(A^{-it}\xi\right)$ for all m, giving

$$p\left(\left(\sigma_t^m(u_m(\xi))\right)_{\omega^2-1}\right)p = p\left(\left(u_m\left(A^{-it}\xi\right)\right)_{\omega^2-1}\right)p = p\pi_{\omega^2-1}\left(W\left(A^{-it}\xi\right)\right)p \in p\,\mathrm{B}p.$$

Corollary (6.3.31)[264]: (see [256]). Let $\xi_1, \ldots, \xi_{1+3\epsilon} \in \mathrm{H}_{\mathbb{C}}$. Let π_{ω^2-1} be as in Corollary (6.3.30). Then $\pi_{\omega^2-1}\left(W(\xi_1 \odot \cdots \odot \xi_{1+3\epsilon})\right) = W^{1+2\epsilon}(\xi_1 \odot \cdots \odot \xi_{1+3\epsilon})$.

Proof. We proceed by induction on $1+3\epsilon \in \mathbb{N}_0$. The base cases $1+3\epsilon = 0,1$ are obvious from the definitions. Now assume that the claimed formula is true for all lengths $\epsilon \geq 0$, and consider the $2(1+\epsilon)$ case. Fix $\xi_0, \xi_1, \dots, \xi_{1+3\epsilon} \in \mathrm{H}_{\mathbb{C}}$. It then follows from Proposition (6.3.22) that the following relation holds.

$$\begin{aligned} &W(\xi_0 \odot \cdots \odot \xi_{1+3\epsilon}) \\ &\qquad = W(\xi_0)W(\xi_1 \odot \cdots \odot \xi_{1+3\epsilon}) \\ &\qquad - \sum_{l=1}^{1+3\epsilon} (\epsilon-1)^{l-1} \langle I\xi_0 \mid \xi_l \rangle_U W\big(\xi_1 \odot \dots \odot \hat{\xi}_l \odot \dots \odot \xi_{1+3\epsilon}\big), \end{aligned}$$

where, as usual, $\hat{\xi}_l$ means that the tensor factor ξ_l is deleted from the simple tensor under consideration. Applying π_{ω^2-1} to this relation and using our induction hypothesis, we have

$$\begin{aligned} &\pi_{\omega^2-1}\big(W(\xi_0 \odot \dots \odot \xi_{1+3\epsilon})\big) \\ &\quad = W^{1+2\epsilon}(\xi_0)W^{1+2\epsilon}(\xi_1 \odot \dots \odot \xi_{1+3\epsilon}) \\ &\quad - \sum_{l=1}^{1+3\epsilon} (\epsilon-1)^{l-1} \langle I\xi_0 \mid \xi_l \rangle_U W^{1+2\epsilon}\big(\xi_1 \odot \dots \odot \hat{\xi}_l \odot \dots \odot \xi_{1+3\epsilon}\big). \qquad (12) \end{aligned}$$

Next, we expand the first term on the right-hand side in the above equation:

$$\begin{aligned} &W^{1+2\epsilon}(\xi_0)W^{1+2\epsilon}(\xi_1 \odot \cdots \odot \xi_{1+3\epsilon}) \\ &= \left(m^{-\frac{1}{2}} \sum_{k(0)=1}^{\mathrm{m}} W\big(e_{k(0)}\big) \otimes W\big(\xi_0 \odot e_{k(0)}\big) \right)_{\omega^2-1} \\ &\times \left(m^{-\frac{1+3\epsilon}{2}} \sum_{\substack{k:[1+3\epsilon]\to[m] \\ k \text{ injective}}} W\big(e_{k(1)}\big) \dots W\big(e_{k(1+3\epsilon)}\big) \right. \\ &\left. \otimes W\big(\xi_1 \odot e_{k(1)}\big) \dots W\big(\xi_{1+3\epsilon} \odot e_{k(1+3\epsilon)}\big) \right)_{\omega^2-1} \\ &= \left(m^{-\frac{1+3\epsilon+1}{2}} \sum_{\substack{k:[1+3\epsilon]_0\to[m] \\ k \text{ injective}}} W\big(e_{k(0)}\big) \dots W\big(e_{k(1+3\epsilon)}\big) \right. \\ &\left. \otimes W\big(\xi_0 \odot e_{k(0)}\big) \dots W\big(\xi_{1+3\epsilon} \odot e_{k(1+3\epsilon)}\big) \right)_{\omega^2-1} \end{aligned}$$

$$+\left(m^{-\frac{2+3\epsilon}{2}} \sum_{k(0)=1}^{m} \sum_{l=1}^{1+3\epsilon} \sum_{\substack{k:[1+3\epsilon]\to[m] \\ k \text{ injective} \\ k(l)=k(0)}} W(e_{k(0)}) \dots W(e_{k(1+3\epsilon)}) \right.$$

$$\left. \otimes W(\xi_0 \odot e_{k(0)}) \dots W(\xi_{1+3\epsilon} \odot e_{k(1+3\epsilon)}) \right)_{\omega^2-1}$$

$$= W^{1+2\epsilon}(\xi_0 \odot \xi_1 \odot \cdots \odot \xi_{1+3\epsilon}) \text{ (this is the first term in the preceding sum)}$$

$$+\sum_{l=1}^{1+3\epsilon} \left(m^{-\frac{2+3\epsilon}{2}} \sum_{k(0)=1}^{m} \sum_{\substack{k:[1+3\epsilon]\to[m] \\ k \text{ injective} \\ k(l)=k(0)}} W(e_{k(0)}) \dots W(e_{k(1+3\epsilon)}) \right.$$

$$\left. \otimes W(\xi_0 \odot e_{k(0)}) \dots W(\xi_{1+3\epsilon} \odot e_{k(1+3\epsilon)}) \right)_{\omega^2-1}$$

The first term is already a part of what we wanted, but we also have to deal with the second term. Note that for $k(0) = k(l)$ and $k(1) \neq \cdots \neq k(1+3\epsilon)$ we have

$$\begin{aligned} W(e_{k(0)}) \dots W(e_{k(1+3\epsilon)}) & \\ &= W(e_{k(0)} \odot \cdots \odot e_{k(1+3\epsilon)}) \\ &+ q_0^{l-1} W(e_{k(0)}) \dots \widehat{W(e_{k(l)})} \dots W(e_{k(1+3\epsilon)}) \quad (13) \end{aligned}$$

and

$$\begin{aligned} &W(\xi_0 \odot e_{k(0)}) \dots W(\xi_{1+3\epsilon} \odot e_{k(1+3\epsilon)})) \\ &= W\left((\xi_0 \odot e_{k(0)}) \odot \cdots \odot (\xi_{1+3\epsilon} \odot e_{k(1+3\epsilon)})\right) \\ &+\langle I\xi_0 \odot e_{k(0)} \mid \xi_l \odot e_{k(l)}\rangle_U q_1^{l-1} W(\xi_0 \odot e_{k(0)}) \dots W(\widehat{\xi_l \odot} e_{k(l)}) \dots W(\xi_{1+3\epsilon} \odot e_{k(1+3\epsilon)}) \\ &= W\left((\xi_0 \odot e_{k(0)}) \odot \cdots \odot (\xi_{1+3\epsilon} \odot e_{k(1+3\epsilon)})\right) \\ &+\langle I\xi_0 \mid \xi_l\rangle_U q_1^{l-1} W(\xi_0 \odot e_{k(0)}) \dots W(\widehat{\xi_l \odot} e_{k(l)}) \dots W(\xi_{1+3\epsilon} \odot e_{k(1+3\epsilon)}). \end{aligned}$$

Indeed, if $(v_1, \dots, v_n) \subseteq \mathrm{H}_{\mathbb{C}}$ is a family of orthogonal vectors then $W(v_1) \dots W(v_n) = W(v_1 \odot \cdots \odot v_n)$, as we remarked earlier. In our case we have a sequence $(w, v_1, \dots, v_{1+3\epsilon})$, where Iw is orthogonal to all vectors v_j for $j \neq l$, so we get

$$
\begin{aligned}
W(w)W(v_1)\ldots W(v_{1+3\epsilon})\Omega \quad &= (a^*(w)+a(Iw))v_1 \odot \cdots \odot v_{1+3\epsilon} \\
&= w \odot v_1 \odot \cdots \odot v_{1+3\epsilon} + a(Iw)(v_1 \odot \cdots \odot v_{1+3\epsilon}) \\
&= w \odot v_1 \odot \cdots \odot v_{1+3\epsilon} + q_1^{l-1}\langle Iw \mid v_l\rangle_U v_1 \odot \cdots \odot \hat{v}_l \odot \cdots \odot
\end{aligned}
$$

hence the formula above. Tensoring $W(e_{k(0)})\ldots W(e_{k(1+3\epsilon)})$ with $W(\xi_0 \odot e_{k(0)})\ldots W(\xi_{1+3\epsilon} \odot e_{k(1+3\epsilon)})$ (keeping in mind that $q_0q_1 = \epsilon - 1$) gives us four terms, one of which is

$$
(\epsilon-1)^{l-1}\langle I\xi_0 \mid \xi_l\rangle_U W(e_{k(1)})\ldots \widehat{W(e_{k(l)})}\ldots W(e_{k(1+3\epsilon)})
$$
$$
\otimes W(\xi_0 \odot e_{k(0)})\ldots W(\widehat{\xi_l \odot} e_{k(l)})\ldots W(\xi_{1+3\epsilon} \odot e_{k(1+3\epsilon)})
$$

and we will deal with the three other terms later. To these expressions we need to apply the sum $\sum_{l=1}^{1+3\epsilon} m^{-\frac{2+3\epsilon}{2}} \sum_{k(0)=1}^{m} \sum_{\substack{k:[1+3\epsilon]\to[m] \\ k \text{ injective} \\ k(l)=k(0)}}$ the condition $k(l)=k(0)$ and perform the sum over $k(0)$ immediately, resulting in a sum $\sum_{l=1}^{1+3\epsilon} m^{-\frac{3\epsilon}{2}} \sum_{k:[1+3\epsilon]\setminus l\}\to[m]}$. Without the sum over l, this is the sum over 3ϵ distinct indices appearing in the definition of $W^{1+2\epsilon}$, so we get the sum

$$
\sum_{l=1}^{1+3\epsilon} (\epsilon-1)^{l-1}\langle I\xi_0 \mid \xi_l\rangle_U W^{1+2\epsilon}(\xi_0 \odot \cdots \odot \hat{\xi}_l \odot \cdots \odot \xi_{1+3\epsilon}).
$$

To sum up, we have checked so far that

$$
\begin{aligned}
&W^{1+2\epsilon}(\xi_0)W^{1+2\epsilon}(\xi_1 \odot \cdots \odot \xi_{1+3\epsilon}) \\
&\qquad = W^{1+2\epsilon}(\xi_0 \odot \cdots \odot \xi_{1+3\epsilon}) \\
&\qquad + \sum_{l=1}^{1+3\epsilon} (\epsilon-1)^{l-1}\langle I\xi_0 \mid \xi_l\rangle_U W^{1+2\epsilon}(\xi_0 \odot \cdots \odot \hat{\xi}_l \odot \cdots \odot \xi_{1+3\epsilon}) + R,
\end{aligned}
$$

where R is the "remainder" term that will turn out to be a zero element of the ultraproduct. Inserting this into (12) we get that

$$
\pi_{\omega^2-1}(W(\xi_1 \odot \cdots \odot \xi_{1+3\epsilon})) = W^{1+2\epsilon}(\xi_1 \odot \cdots \odot \xi_{1+3\epsilon}) + R,
$$

so if we can check that R is really a zero element then this ends the proof.

Let us just recall that R comes from the three neglected so far terms arising from tensoring $W(e_{k(0)})\ldots W(e_{k(1+3\epsilon)})$ with $W(\xi_0 \odot e_{k(0)})\ldots W(\xi_{1+3\epsilon} \odot e_{k(1+3\epsilon)})$. It can be written as

$$
R = \left(m^{-\frac{2+3\epsilon}{2}} \sum_{l=1}^{1+3\epsilon} \left(R_{1,l}(m) + \langle I\xi_0 \mid \xi_l\rangle q_1^{l-1} R_{2,l}(m) + q_0^{l-1} R_{3,l}(m)\right) \right)_{\omega^2-1},
$$

where

$$
R_{1,l}(m) = \sum_{k(0)=1}^{m} \sum_{\substack{k:[1+3\epsilon]\to[m] \\ k \text{ injective} \\ k(l)=k(0)}} W(e_{k(0)} \odot \ldots \odot e_{k(1+3\epsilon)})
$$
$$
\otimes W\left((\xi_0 \odot e_{k(0)}) \odot \ldots \odot (\xi_{1+3\epsilon} \odot e_{k(1+3\epsilon)})\right),
$$

$$R_{2,l}(m)=\sum_{k(0)=1}^{m}\sum_{\substack{k:[1+3\epsilon]\to[m]\\ k\text{ injective}\\ k(l)=k(0)}} W\left(e_{k(0)}\odot\ldots\odot e_{k(1+3\epsilon)}\right)\otimes W\left(\xi_1\odot e_{k(1)}\right)$$
$$\odot\, W\left(\widehat{\xi_l\odot e_{k(l)}}\right)\odot W\left(\xi_{1+3\epsilon}\odot e_{k(1+3\epsilon)}\right),$$
and
$$R_{3,l}(m)=\sum_{k(0)=1}^{m}\sum_{\substack{k:[1+3\epsilon]\to[m]\\ k\text{ injective}\\ k(l)=k(0)}} W\left(e_{k(1)}\right)\ldots\widehat{W\left(e_{k(l)}\right)}\ldots W\left(e_{k(1+3\epsilon)}\right)$$
$$\otimes\, W\left(\left(\xi_0\odot e_{k(0)}\right)\odot\ldots\odot\left(\xi_{1+3\epsilon}\odot e_{k(1+3\epsilon)}\right)\right).$$
Recall the formulas (13) and (14). After tensoring the right-hand sides we get four terms, one of which was already incorporated in the proof of Corollary (6.3.31). The other three are:
$$W\left(e_{k(0)}\odot\ldots\odot e_{k(1+3\epsilon)}\right)\otimes W\left(\left(\xi_0\odot e_{k(0)}\right)\odot\ldots\odot\left(\xi_{1+3\epsilon}\odot e_{k(1+3\epsilon)}\right)\right)$$
$$q_1^{l-1}\langle I\xi_0\mid\xi_l\rangle W\left(e_{k(0)}\odot\ldots\odot e_{k(1+3\epsilon)}\right)\otimes W\left(\xi_1\odot e_{k(1)}\right)\ldots W\left(\widehat{\xi_l\odot e_{k(l)}}\right)\ldots W\left(\xi_{1+3\epsilon}\odot e_{k(1+3\epsilon)}\right)$$
and
$$q_0^{l-1}W\left(e_{k(1)}\right)\ldots\widehat{W\left(e_{k(l)}\right)}\ldots W\left(e_{k(1+3\epsilon)}\right)$$
$$\otimes\, W\left(\left(\xi_0\odot e_{k(0)}\right)\odot\ldots\odot\left(\xi_{1+3\epsilon}\odot e_{k(1+3\epsilon)}\right)\right).$$
To obtain R, we just need to take sums over appropriate sets of indices.
We will now examine properties of R. Since q_0, q_1, and the range of summation over l is fixed, to show that R is a zero element in the ultraproduct, it suffices to show that $\lim_{m\to\infty} m^{-\frac{2+3\epsilon}{2}}\|R_{i,l}\|=0$ for any $1\leqslant i\leqslant 3$ and $l\in[1+3\epsilon]$. We will use Nou's noncommutative Khintchine inequality for this (Corollary (6.3.29)), but before that we need to obtain a bound for the coefficients.

Corollary (6.3.32)[264]: There exists a constant $D(1+3\epsilon)>0$ (depending only on the initial choice of $\xi_1,\ldots,\xi_{1+3\epsilon}\in H_{\mathbb{C}}$) such that for all $m\in\mathbb{N}$ and all $k:[1+3\epsilon]\to[m]$, the following inequalities hold:
$$\left\|W\left(\left(\xi_0\odot e_{k(0)}\right)\odot\cdots\odot\left(\xi_{1+3\epsilon}\odot e_{k(1+3\epsilon)}\right)\right)\right\|\leqslant D(1+3\epsilon)$$
$$\left\|W\left(\xi_1\odot e_{k(1)}\right)\ldots W\left(\widehat{\xi_l\odot e_{k(l)}}\right)\ldots W\left(\xi_{1+3\epsilon}\odot e_{k(1+3\epsilon)}\right)\right\|\leqslant D(1+3\epsilon)$$
$$\left\|W\left(e_{k(1)}\right)\ldots\widehat{W\left(e_{k(l)}\right)}\ldots W\left(e_{k(1+3\epsilon)}\right)\right\|\leqslant D(1+3\epsilon).$$

Proof. The second and third inequality will follow if we can show that there is a constant $D>0$ such that $\|W(\xi_{1+\epsilon}\odot e_{k(1+\epsilon)})\|,\|W(e_{k(1+\epsilon)})\|\leqslant D$ (independently of $1+\epsilon\in[1+3\epsilon]$). But the existence of D follows from the simple fact for any $(\epsilon-1)$ −Araki-Woods algebra $\Gamma_{\epsilon-1}(H)$ and $\xi\in H_{C}$, we have $\|W(\xi)\|_{\Gamma_{\epsilon-1}(\mathbf{H})}\leqslant\|a^*_{\epsilon-1}(\xi)\|+\|a_{\epsilon-1}(I\xi)\|\leqslant 2(1-|\epsilon-1|)^{-1/2}\max\{\|\xi\|,\|I\xi\|\}$. Now consider the first inequality. By the Khintchine inequality with $K=\mathbb{C}$ (Corollary (6.3.29)), the left-hand side is bounded by

$$C(q_1)(2+3\epsilon)\max_{0\leqslant l\leqslant 1+3\epsilon}\left\|(\mathbb{1}_{1+3\epsilon-l}\odot \mathcal{J})\left(R^*_{1+3\epsilon,l}\left(\left(\xi_0\odot e_{k(0)}\right)\odot\cdots\odot\left(\xi_{1+3\epsilon}\odot e_{k(1+3\epsilon)}\right)\right)\right)\right\|.$$

Writing the above $(\mathbb{1}_{1+3\epsilon-l}\odot \mathcal{J})R^*_{1+3\epsilon,k}$ terms as sums of simple tensors, one easily sees that the corresponding norms are bounded by a constant depending only on $1+3\epsilon$. (Note that the unboundedness of $\mathcal{J}$ plays no role here, as $\xi_0,\dots,\xi_{1+3\epsilon}\in \mathrm{H}_{\mathbb{C}}$ remain fixed.)

We need one more proposition. In the following, $m\in\mathbb{N}$ and $\xi_0,\dots,\xi_{1+3\epsilon}$ are fixed as usual. Let I_l denote the set of indices $(k(0),\dots,k(1+3\epsilon))\in[m]^{2+3\epsilon}$ that are pairwise distinct except for the pair $(k(0),k(l))$; a generic element of I_l will be called **i** and the corresponding tensor $e_{k(0)}\otimes\cdots\otimes e_{k(1+3\epsilon)}$ will also be denoted by i. We will denote $W\left(e_{k(0)}\otimes\cdots\otimes e_{k(1+3\epsilon)}\right)$ by $W_{\mathbf{i}}$ and $W\left(\left(\xi_0\odot e_{k(0)}\right)\otimes\cdots\otimes\left(\xi_{1+3\epsilon}\odot e_{k(1+3\epsilon)}\right)\right)$ by $W_{\mathbf{i}}^{\xi}$.

Corollary (6.3.33)[264]: (see [256]). Given any Hilbert space K and any family of operators $(A_i)_{i\in I_l}\subset \mathrm{B(K)}$, the following inequalities hold.

$$\left\|\sum_{I_l} A_i\otimes W_i\right\|\leqslant C(1+3\epsilon)\sup_{i\in I_l}\|A_i\|m^{\frac{1+3\epsilon}{2}}$$

$$\left\|\sum_{I_l} A_i\otimes W_i^{\xi}\right\|\leqslant C(1+3\epsilon)\sup_{i\in I_l}\|A_i\|m^{\frac{1+3\epsilon}{2}},$$

where $C(1+3\epsilon)>0$ depends only on $1+3\epsilon$ and the choice of vectors $\xi_0,\xi_1,\dots,\xi_{1+3\epsilon}\in \mathrm{H}_{\mathbb{C}}$.

Proof. The proofs of both inequalities are essentially the same. We will deal with the first one; to obtain a proof of the second one has to apply conjugation in some places but since we are dealing with a fixed number of vectors $\xi_0,\dots,\xi_{1+3\epsilon}$, the unboundedness of conjugation does not play any role. By the Khintchine inequality (Corollary (6.3.29)) we need to deal with

$$\max_{0\leqslant k\leqslant 2+3\epsilon}\left\|\sum_{I_l} A_{\mathbf{i}}\otimes R^*_{2+3\epsilon,k}(\mathbf{i})\right\|,$$

up to a $(1+3\epsilon)$-dependent constant.

Since $R^*_{2+3\epsilon,k}$ is a sum of operators that only permute vectors, and the coefficients of this sum are summable, we just need to take care of a single term of the form

$$\max_{0\leqslant k\leqslant 2+3\epsilon}\left\|\sum_{I_l} A_{\mathbf{i}}\otimes \sigma(\mathbf{i})_{(2+3\epsilon,k)}\right\|,$$

where σ denotes the action of the permutation and the decoration $(2+3\epsilon,k)$ reminds us of the fact that $\sigma(\mathbf{i})$ is viewed now as an element of $\mathbf{H}_c^{\otimes(2+3\epsilon-k)}\otimes_h \mathrm{H}_{1+\epsilon}^{\otimes k}$. Whatever the σ, the tensor $\sigma(\mathbf{i})_{2+3\epsilon,k}$ is always of the form $e_{i_0}\odot\cdots\odot e_{i_{1+3\epsilon-k}}\otimes e_{i_{1+3\epsilon-k+1}}\odot\cdots\odot e_{i_{1+3\epsilon}}$, where for different indices $\mathbf{i}$ and $\mathbf{i}'$ these tensors are different. The key property that we will need is that we have two orthonormal systems $(v_{1+2\epsilon})_{1+2\epsilon\in S}\subset \mathrm{H}^{\otimes(2+3\epsilon-k)}$ and $\left(w_j\right)_{j\in J}\subset \mathrm{H}^{\otimes k}$ such that for any $\mathbf{i}\in I_l$ we have $\sigma(\mathbf{i})_{2+3\epsilon,k}=v_{1+2\epsilon}\otimes w_j$ for some $1+2\epsilon\in S$ and $j\in J$. Therefore we can get rid of the sign σ and just consider

$$\max_{0\leqslant k\leqslant 2+3\epsilon}\left\|\sum_{I_l} A_{\mathbf{i}}\otimes \mathbf{i}_{(2+3\epsilon,k)}\right\|,$$

Since we are dealing with tensor powers of H equipped with $(\epsilon-1)$ −deformed inner products, we would rather have families $(v'_{1+2\epsilon})_{1+2\epsilon\in S}$ and $(w'_j)_{j\in J}$ that are orthonormal in $\mathrm{H}_{\epsilon-1}^{\otimes(2+3\epsilon-k)}$ and $\mathrm{H}_{\epsilon-1}^{\otimes k}$, respectively. To achieve this, we will use the operators defining the $(\epsilon-1)$ −deformed inner products, $P_{\epsilon-1}^{2+3\epsilon-k}$ and $P_{\epsilon-1}^{k}$. Let $\xi(\mathbf{i})_{2+3\epsilon,k}$ be tensors defined by $\left(\left(P_{\epsilon-1}^{2+3\epsilon-k}\right)^{\frac{1}{2}}\otimes\left(P_{\epsilon-1}^{k}\right)^{\frac{1}{2}}\right)\left(\xi(\mathbf{i})_{2+3\epsilon,k}\right)=\mathbf{i}_{2+3\epsilon,k}$.

Then we can write $\xi(\mathbf{i})_{2+3\epsilon,k}=v'_{1+2\epsilon}\otimes w'_j$ for some tensors $v'_{1+2\epsilon}$ and w'_j coming from orthonormal families in $\mathrm{H}_{\epsilon-1}^{\otimes(2+3\epsilon-k)}$ and $\mathrm{H}_{\epsilon-1}^{\otimes k}$. Since the row/column Hilbert spaces are homogeneous operator spaces (and Haagerup tensor product allows tensoring cb maps) we can bound $\max_{0\leqslant k\leqslant 2+3\epsilon}\left\|\sum_{I_l}A_{\mathrm{i}}\otimes \mathbf{i}_{2+3\epsilon,k}\right\|$ by $\max_{0\leqslant k\leqslant 2+3\epsilon}\left\|\sum_{I_l}A_{\mathrm{i}}\otimes \xi(\mathbf{i})_{2+3\epsilon,k}\right\|$, up to a $1+3\epsilon$-dependent constant coming from the norms of $\left(P_{\epsilon-1}^{2+3\epsilon-k}\right)^{\frac{1}{2}}$ and $\left(P_{\epsilon-1}^{k}\right)^{\frac{1}{2}}$. Because we are using the Haagerup tensor product, we have the following completely isometric isomorphism $\mathrm{H}_c\otimes_h\overline{\mathrm{K}}_{1+\epsilon}\simeq\mathcal{K}(\mathrm{K},\mathrm{H})$. Under this identification the tensors $\xi(\mathbf{i})_{2+3\epsilon,k}$ correspond to matrix units in $\mathcal{K}\left(\mathrm{H}_{\epsilon-1}^{\otimes k},\mathrm{H}_{\epsilon-1}^{\otimes(2+3\epsilon-k)}\right)$. This means that the operators $A_{\mathbf{i}}$ fill different entries in a large operator matrix. By comparing the operator norm with the Hilbert-Schmidt norm we get the estimate

$$\left\|\sum_{\mathbf{i}\in I_l} A_{\mathbf{i}}\otimes W_{\mathbf{i}}\right\|\leqslant C(1+3\epsilon)\left(\sum_{\mathbf{i}\in I_l}\|A_{\mathbf{i}}\|^2\right)^{\frac{1}{2}}\leqslant C(1+3\epsilon)\left(|I_l|\sup_{\mathbf{i}\in I_l}\|A_{\mathbf{i}}\|^2\right)^{\frac{1}{2}},$$

which can be further bounded by

$$C(1+3\epsilon)\left(m^{1+3\epsilon}\sup_{\mathbf{i}\in I_l}\|A_{\mathbf{i}}\|^2\right)^{\frac{1}{2}}=C(1+3\epsilon)m^{\frac{1+3\epsilon}{2}}\sup_{\mathbf{i}\in I_l}\|A_{\mathbf{i}}\|.$$

Finally, to conclude that $R=0$ in the ultraproduct, we just observe that each component $R_{i,l}=\left(R_{i,l}(m)\right)_{m\in\mathbb{N}}$ is a a sequence of terms of the form appearing in Corollary (6.3.33) with coefficients $(A_{\mathbf{i}}(m))_{m\in\mathbb{N},\mathbf{i}\in I_l}$ uniformly bounded in $\mathbf{i}$ and m by the constant $D(1+3\epsilon)$ from Corollary (6.3.32), so the norm $m^{-\frac{2+3\epsilon}{2}}R_{i,l}(m)$ is bounded from above by $C(1+3\epsilon)D(1+3\epsilon)m^{-\frac{1}{2}}$, and hence tends to zero. This finishes the proof of Corollary (6.3.31). With this tool at hand, we prove Corollary (6.3.34) (see [256]).

Corollary (6.3.34)[264]: Let $\Gamma_{\epsilon-1}(\mathrm{H})$ be a $(\epsilon-1)$ −Araki-Woods algebra. Let $P_n\colon\Gamma_{\epsilon-1}(\mathrm{H})\to\Gamma_{\epsilon-1}(\mathrm{H})$ be the projection onto the ultraweakly closed span of $\{W(\xi)\colon\xi\in\mathrm{H}_{\mathbb{C}}^{\odot n}\}$. Then, using the notation from Corollary (6.3.30), we have

$$\Theta\circ P_n=p(P_n\otimes\mathrm{Id})_{\omega^2-1}p\circ\Theta. \qquad (14)$$

Proof. Let $W(\xi)$ be a Wick word associated with $\xi\in\mathrm{H}_{\mathbb{C}}^{\odot(1+3\epsilon)}$. Then we easily obtain $\pi_{\omega^2-1}(P_nW(\xi))=\delta_{n,1+3\epsilon}W^{1+2\epsilon}(\xi)$. On the other hand, let us first apply π_{ω^2-1} to obtain $W^{1+2\epsilon}(\xi)$. Since, as we already remarked earlier, $W\left(e_{k(1)}\right)\dots W\left(e_{k(1+3\epsilon)}\right)=W\left(e_{k(1)}\otimes\cdots\otimes e_{k(1+3\epsilon)}\right)$, the operators acted on by the P_n part of the operator $(P_n\otimes\mathrm{Id})_{\omega^2-1}$ are exactly of length n. Therefore $(P_n\otimes\mathrm{Id})_{\omega^2-1}W^{1+2\epsilon}(\xi)=\delta_{n,1+3\epsilon}W^{1+2\epsilon}(\xi)$. By linearity, this implies that $\pi_{\omega^2-1} \circ P_n = (P_n \otimes \mathrm{Id})_{\omega^2-1} \circ \pi_{\omega^2-1}$ on the algebra of Wick words $\tilde{\Gamma}_{\epsilon-1}(\mathrm{H})$. Compressing by the support projection p, we then obtain

$$\Theta \circ P_n = p(P_n \otimes \mathrm{Id})_{\omega^2-1} \circ \pi_{\omega^2-1}(\cdot)p = p(P_n \otimes \mathrm{Id})_{\omega^2-1} p \circ \Theta \text{ on } \tilde{\Gamma}_{\epsilon-1}(\mathrm{H}),$$

where in the second equality we used the fact that $p \in \pi_{\omega^2-1}\left(\tilde{\Gamma}_{\epsilon-1}(\mathrm{H})\right)'$ (see [130, Lemma 4.1]). Since the desired equality holds on the ultraweakly dense subset $\tilde{\Gamma}_{\epsilon-1}(\mathrm{H})$, and all maps under consideration are normal, equality holds everywhere.

Corollary (6.3.35)[264]: Let $\varphi: \mathbb{N} \to \mathbb{C}$ be a function such that the associated radial multipliers $\mathrm{m}_\varphi: \Gamma_{\epsilon-1}(\mathbb{R}^m) \to \Gamma_{\epsilon-1}(\mathbb{R}^m)$ have completely bounded norms uniformly bounded in m. Then the radial multiplier defined by φ on any $(\epsilon-1)$ −Araki-Woods algebra $\Gamma_{\epsilon-1}(\mathrm{H})$ is completely bounded and

$$\left\|\mathbf{m}_\varphi: \Gamma_{\epsilon-1}(\mathrm{H}) \to \Gamma_{\epsilon-1}(\mathrm{H})\right\|_{\mathrm{cb}} \leqslant \sup_{m\in\mathbb{N}} \left\|\mathrm{m}_\varphi: \Gamma_{\epsilon-1}(\mathbb{R}^m) \to \Gamma_{\epsilon-1}(\mathbb{R}^m)\right\|_{\mathrm{cb}}$$
$$= \left\|\mathrm{m}_\varphi: \Gamma_{\epsilon-1}(\ell_{2,\mathbb{R}}) \to \Gamma_{\epsilon-1}(\ell_{2,\mathbb{R}})\right\|_{\mathrm{cb}}.$$

The main technical tool in establishing Corollary (6.3.35) is the following intertwining-type property for projections onto Wick words of a given length with respect to the ultraproduct embedding given by Corollary (6.3.30)

Proof. From Corollary (6.3.34) we get that $\Phi \circ \mathrm{m}_\varphi(x_m) = p\left(\mathbf{m}_\varphi \otimes \mathrm{Id}\right)_{\omega^2-1} p \circ \Phi(x_m)$ for any $x_m = W(\xi)$ with $\xi \in (\mathrm{H}_\mathbb{C})^{\odot(1+3\epsilon)}$. By linearity we can extend this equality to all $x_m \in \tilde{\Gamma}_{\epsilon-1}(\mathrm{H})$. It follows that we have control on the cb norm of m_φ acting on the norm-closure of finite Wick words, i.e. on the C^*-algebra $\mathcal{A}_{\epsilon-1}(\mathrm{H})$. Since m_φ is automatically normal (cf. [190, Lemma 3.4]), it extends to a normal map on $\Gamma_{\epsilon-1}(\mathrm{H})$ with the same cb norm, so we get

$$\left\|\mathbf{m}_\varphi: \Gamma_{\epsilon-1}(\mathrm{H}) \to \Gamma_{\epsilon-1}(\mathrm{H})\right\|_{\mathrm{cb}} \leqslant \sup_{m\in\mathbb{N}} \left\|\mathrm{m}_\varphi: \Gamma_{\epsilon-1}(\mathbb{R}^m) \to \Gamma_{\epsilon-1}(\mathbb{R}^m)\right\|_{\mathrm{cb}}.$$

Since $\Gamma_{\epsilon-1}(\mathbb{R}^m)$ is a subalgebra of $\Gamma_{\epsilon-1}(\mathbb{R}^{m+1})$ which is the range of a normal faithful tracepreserving conditional expectation that intertwins the action of m_φ, the sequence of norms on the right-hand side is non-decreasing, so

$$\left\|\mathrm{m}_\varphi: \Gamma_{\epsilon-1}(\mathrm{H}) \to \Gamma_{\epsilon-1}(\mathrm{H})\right\|_{\mathrm{cb}} \leqslant \lim_{m\to\infty} \left\|\mathrm{m}_\varphi: \Gamma_{\epsilon-1}(\mathbb{R}^m) \to \Gamma_{\epsilon-1}(\mathbb{R}^m)\right\|_{\mathrm{cb}}.$$

By the same token, this limit is not greater than $\left\|\mathrm{m}_\varphi: \Gamma_{\epsilon-1}(\ell_{2,\mathbb{R}}) \to \Gamma_{\epsilon-1}(\ell_{2,\mathbb{R}})\right\|_{\mathrm{cb}}$. Since the union of the algebras $\Gamma_{\epsilon-1}(\mathbb{R}^m)$ is strongly dense in $\Gamma_{\epsilon-1}(\ell_{2,\mathbb{R}})$, the union of the preduals is normdense in the predual of $\Gamma_{\epsilon-1}(\ell_{2,\mathbb{R}})$. Therefore the limit of norms is equal to the norm of the multplier defined on $L^1\left(\Gamma_{\epsilon-1}(\ell_{2,\mathbb{R}})\right)$. By dualising, we get that

$$\lim_{m\to\infty} \left\|m_\varphi: \Gamma_{\epsilon-1}(\mathbb{R}^m) \to \Gamma_{\epsilon-1}(\mathbb{R}^m)\right\|_{\mathrm{cb}} = \left\|\mathrm{m}_\varphi: \Gamma_{\epsilon-1}(\ell_{2,\mathbb{R}}) \to \Gamma_{\epsilon-1}(\ell_{2,\mathbb{R}})\right\|_{\mathrm{cb}}.$$

Corollary (6.3.36)[264]: Let $\Gamma_{\epsilon-1}(\mathrm{H})$ be a $(\epsilon-1)$-Araki-Woods algebra. Let P_n be the projection onto Wick words of length n, defined by $P_n W(\xi) = \delta_{n,1+3\epsilon} W(\xi)$, where $\xi \in \mathrm{H}_\mathbb{C}^{\odot(1+3\epsilon)}$. Then P_n extends to a completely bounded, normal map on $\Gamma_{\epsilon-1}(\mathrm{H})$ and $\|P_n\|_{cb} \leqslant C(\epsilon-1)^2(n+1)^2$.

Proof. We just observe that $P_n = \mathrm{m}_{\varphi_n}$, where φ_n is the Kroenecker delta-function $\varphi_n(k) = \delta_{k,n}$. By Theorems 2.20 and (6.3.35), we obtain $\|P_n: \Gamma_{\epsilon-1}(\mathrm{H}) \to \Gamma_{\epsilon-1}(\mathrm{H})\|_{\mathrm{cb}} \leqslant \| P_n: \Gamma_{\epsilon-1}(\ell_{2,\mathbb{R}}) \to \Gamma_{\epsilon-1}(\ell_{2,\mathbb{R}}) \|_{\mathrm{cb}} \leqslant C(\epsilon-1)^2(n+1)^2$.

Now we devoted to the proof of the complete metric approximation property for $\Gamma_{\epsilon-1}(\mathrm{H})$.

Corollary (6.3.37)[264]: Let $\Gamma_{\epsilon-1}(\mathrm{H})$ be a $(\epsilon-1)$ −Araki-Woods algebra. Then $\Gamma_{\epsilon-1}(\mathrm{H})$ has the w^*-complete metric approximation property.

Proof. We define a net $\Gamma_{n,t,i} := \Gamma_{\epsilon-1}(e^{-t}T_i)Q_n$, where $n \in \mathbb{N}, t > 0, i \in I$, the finite-rank maps T_i come from the previous lemma, and $Q_n = P_0 + \cdots + P_n = m_{\chi_{\{0,1,\ldots,n\}}}$ is the radial multiplier which projects onto Wick words of length at most n. Each $\Gamma_{n,t,i}$ is a finite rank map on $\Gamma_{\epsilon-1}(\mathrm{H})$; indeed, Q_n tells us that we have only Wick words of bounded length and T_i tells us that we can only draw vectors from a finite dimensional Hilbert space, so we are left with a space of the form $\bigoplus_{\epsilon=-\frac{1}{3}}^{n}(\mathbb{C}^m)^{\otimes(1+3\epsilon)}$, which is finite-dimensional. We will pass to a limit with $i \to \infty, n \to \infty$ and $t \to 0$. The rate of convergences of t and n will not be independent and will be chosen in a way that assures the convergence $\|\Gamma_{n,t,i}\|_{cb} \to 1$.
Let us check now that it is possible, using a standard argument of Haagerup (note that $\Gamma_{\epsilon-1}(e^{-t})P_k = e^{-kt}P_k$):

$$
\begin{aligned}
\|\Gamma_{n,t,i}\|_{cb} &= \|\Gamma_{\epsilon-1}(e^{-t}T_i)Q_n\|_{cb} \\
&\leqslant \|\Gamma_{\epsilon-1}(e^{-t})Q_n\|_{cb} \\
&\leqslant \|\Gamma_{\epsilon-1}(e^{-t})\|_{cb} + \|\Gamma_{\epsilon-1}(e^{-t})(\mathbb{1} - Q_n)\|_{cb} \\
&\leqslant 1 + \sum_{k>n} e^{-kt}\|P_k\|_{cb} \\
&\leqslant 1 + C(\epsilon-1)^2 \sum_{k>n} e^{-kt}(k+1)^2.
\end{aligned}
$$

Since the series $\sum_{k\geqslant 0} e^{-kt}(k+1)^2$ is convergent, for any $t > 0$ the sum will tend to zero when $n \to \infty$. Therefore we can choose the parameters $i, n \to \infty$ and $t \to 0$ such that the completely bounded norms of the operators $\Gamma_{n,t,i}$ tend to 1. Then the operators $\frac{\Gamma_{n,t,i}}{\|\Gamma_{n,t,i}\|_{cb}}$ are completely contractive. We have to check that they converge ultraweakly to $\mathbb{1}$. Since the denominators converge to 1 and the net is uniformly bounded, it suffices to prove strong convergence on a linearly dense set. It is very easy to verify that the convergence holds for finite simple tensors, so this ends the proof.

Corollary (6.3.38)[264]: (see [256]). Let H be the Hilbert space constructed from the pair $(\mathrm{H}_R, (U_t)_{t\in\mathbb{R}})$. Consider the σ-weakly dense C^*-algebra $\mathcal{A}_{\epsilon-1}(\mathrm{H}) \subseteq \Gamma_{\epsilon-1}(\mathrm{H})$ generated by the set $\{W(\xi): \xi \in \mathrm{H}_{\mathbb{R}}\} \subset \mathrm{B}(\mathcal{F}_{\epsilon-1}(\mathrm{H}))$. This C^*-algebra has the complete metric approximation property.

Proof. Consider once again the maps $\Gamma_{n,t,i} := \Gamma_{\epsilon-1}(e^{-t}T_i)Q_n$. The ranges of these maps are contained in $\tilde{\Gamma}_{\epsilon-1}(\mathrm{H})$, the bounds for the norms remain the same, so it suffices to check the pointwise convergence in norm. Since the maps are uniformly bounded, it suffices to check the convergence on a linearly dense set, hence we may assume that $x_m = W(\xi_1 \odot \cdots \odot \xi_k)$. If n is large enough the Q_n that appears in the definition of $\Gamma_{n,t,i}$ has no effect on x_m, so we get

$$\Gamma_{n,t,i}x_m - x_m = e^{-kt}W(T_i\xi_1 \odot \cdots \odot T_i\xi_k) - W(\xi_1 \odot \cdots \odot \xi_k).$$

This last expression is easily seen to converge to zero in norm as $t \to 0$ and $i \to \infty$. This can be seen either using the Khintchine inequality, or just by expressing $W(\xi_1 \odot \cdots \odot \xi_k)$ as a non-commutative polynomial in $a_{\epsilon-1}(\xi_{1+\epsilon})$'s and $a^*_{\epsilon-1}(\xi_{1+\epsilon})$'s and invoking the fact that

$$
\begin{aligned}
\lim_i \|a^*_{\epsilon-1}(T_i\xi_k) - a^*_{\epsilon-1}(\xi_k)\| &= \lim_i \|a_{\epsilon-1}(T_i\xi_k) - a_{\epsilon-1}(\xi_k)\| \\
&\leqslant (1 - |\epsilon - 1|)^{-1/2} \lim_i \|T_i\xi_k - \xi_k\| \to 0.
\end{aligned}
$$

Corollary (6.3.39)[264]: (see [256]). The C^*-algebra $\mathcal{A}_{\epsilon-1}(\mathrm{H})$ is $QWEP$.

Proof. We will show that $\mathcal{A}_{\epsilon-1}(\mathrm{H})$ is weakly cp complemented in the von Neumann algebra $\Gamma_{\epsilon-1}(\mathrm{H})$, meaning that there exists a ucp map $\Phi: \Gamma_{\epsilon-1}(\mathrm{H}) \to (\mathcal{A}_{\epsilon-1}(\mathrm{H}))^{**}$ such that $\Phi|_{\mathcal{A}_{\epsilon-1}(\mathrm{H})} = \mathrm{Id}$.
Let $(\Phi_i)_{i\in I}$ be the net of maps implementing at the same time the w^*-complete metric approximation property of $\Gamma_{\epsilon-1}(\mathrm{H})$ and the complete metric approximation property of $\mathcal{A}_{\epsilon-1}(\mathrm{H})$. Using this net, we get maps $\Phi_i: \Gamma_{\epsilon-1}(\mathrm{H}) \to (\mathcal{A}_{\epsilon-1}(\mathrm{H}))^{**}$, as Φ_i maps $\Gamma_{\epsilon-1}(\mathrm{H})$ into $\mathcal{A}_{\epsilon-1}(\mathrm{H})$. There exists a cluster point of this net in the point-weak*-topology and this cluster point is obviously a ucp map that is equal to identity, when restricted to $\mathcal{A}_{\epsilon-1}(\mathrm{H})$, because the net $(\Phi_i)_{i\in I}$ converges pointwise to identity on $\mathcal{A}_{\epsilon-1}(\mathrm{H})$. Since all $(\epsilon - 1) -$Araki-Woods algebras are QWEP (cf. [130]) and this property descends to subalgebras that are weakly cp complemented (cf. [113, Proposition 4.1 (ii)]), we get the claimed result.

List of Symbols

Symbol		Page
CAR	: Cartan	1
dim	: dimension	2
ℓ^2	: Hilbert space of sequences	2
min	: minimum	3
$\oplus$	: Direct sum	3
$\otimes$	: Tensor product	3
$\mathcal{F}(H)$	: Full Fock space	3
L^∞	: Essential Lebesgue space	4
s.o	: strong operator	7
Ker	: Kernel	10
ICC	: infinite conjugecy class	13
p.m.p	: probability-measure preserving	13
Ind	: induced	14
ℓ^∞	: Essential Banach space	16
u.c.p	: unital completely positive	16
tr	: trace	16
diag	: diagonal	17
L^1	: Lebesgue on the real line	17
dom	: domain	19
sup	: Supremum	19
max	: maximum	24
m-a. e	: measurable almost everywhere	25
Aut	: Automorphism	25
q^{BM}	: q-Brownian	42
q^{OV}	: q-Ornstein-Uhlenberk	42
L^p	: Lebesgue space	48
VN	: Von Neumann	49
inf	: infimum	57
c. b	: completely bounded	57
j. c. b	: jointly completely bounded	57
WEP	: weak expectation property	61
QWEP	: quotient weak expectation property	67
KMS	: Kubo-Martin-Schwinger	68
OH	: Operator Hilbertian	71
GNS	: Gelfand-Naimark-Segal	73
AFD	: approximately finite-dimensional	96
mp	: measure -preserving	97
c. m. a. p	: complete metric approximation property	97
$\ominus$	: Direct difference	113
$g$cd	: greatest common divisor	120
Inn	: Inner	127
out	: outer	127

f. n	: faithful normal	127
CN	: Non-Crossing	128
co	: closure	163
pol	: polar	164
$\odot$	: Algebraic Tensor product	164
s. o. t	: algebraic strong operator topology	166
f. n. s	: faithful normal semi finite	213
CCAP	: completely contractive approximation property	217
CBAP	: completely bounded approximation property	239

References

1. Cyril Houdayer, On Some free Products of Von Neumann Algebras Which are Free Araki-woods Factors, Int. Math. Res. Notices. Vol. 2007, article ID rnm098, 21 pages.
2. L. Barnett, Free product von Neumann algebras of type III. Proc. Amer. Math. Soc. 123 (1995), 543–553.
3. A. Connes, Almost periodic states and factors of type III_1. J. Funct. Anal. 16 (1974), 415–445.
4. A. Connes, Une classification des facteurs de type III. Ann. Sci. École Norm. Sup. 6 (1973), 133–252.
5. K. Dykema, Free products of finite-dimensional and other von Neumann algebras with respect to non-tracial states. Free probability theory (Waterloo, ON, 1995) Fields Inst. Commun. 12 Amer. Math. Soc., Providence, RI, 1997, pp. 41–88.
6. K. Dykema, Interpolated free group factors. Pacific J. Math. 163 (1994), 123–135.
7. K. Dykema, Factoriality and Connes' invariant $T(M)$ for free products of von Neumann algebras. J. reine angew. Math. 450 (1994), 159–180.
8. K. Dykema, Free products of hyperfinite von Neumann algebras and free dimension. Duke Math. J. 69 (1993), 97–119.
9. S. Popa, Markov traces on universal Jones algebras and subfactors of finite index. Invent. Math. 111, (1993), 375–405.
10. F. Rădulescu, Random matrices, amalgamated free products and subfactors of the von Neumann algebra of a free group, of noninteger index. Invent. Math. 115, (1994), 347–389.
11. F. Rădulescu, A type III_λ factor with core isomorphic to the von Neumann algebra of a free group, tensor $B(H)$. Recent Advances in Operator Algebras (Orléans, 1992). Astérisque 232 (1995), 203–209.
12. D. Shlyakhtenko, On the Classification of Full Factors of Type III. Trans. Amer. Math. Soc. 356 (2004), 4143–4159.
13. D. Shlyakhtenko, A-valued semicircular systems. J. Funct. Anal. 166 (1999), 1–47.
14. D. Shlyakhtenko, Some applications of freeness with amalgamation. J. Reine Angew. Math. 500 (1998), 191–212.
15. D. Shlyakhtenko, Free quasi-free states. Pacific J. Math. 177 (1997), 329–368.
16. M. Takesaki, Theory of Operator Algebras II. EMS 125. Springer-Verlag, Berlin, Heidelberg, New- York, 2000.
17. Y. Ueda, Amalgamated free products over Cartan subalgebra. Pacific J. Math. 191 (1999), 359–392.
18. S. Vaes, États quasi-libres libres et facteurs de type III (d'aprés D. Shlyakhtenko). Séminaire Bourbaki, exposé 937, Astérisque 299 (2005), 329–350.
19. D.-V. Voiculescu, K.J. Dykema & A. Nica, Free random variables. CRM Monograph Series 1. American Mathematical Society, Providence, RI, 1992.
20. D.-V. Voiculescu, Circular and semicircular systems and free product factors. Operator algebras, Unitary representations, Enveloping Algebras, and Invariant Theory, Progress in Mathematics 92. Birkhäuser, Boston, (1990), 45–60.
21. D.-V. Voiculescu, Multiplication of certain noncommuting random variables. J. Operator Theory 18 (1987), 223–235.

22. D.-V. Voiculescu, Symmetries of some reduced free product C^*-algebras. Operator algebras and Their Connections with Topology and Ergodic Theory, Lecture Notes in Mathematics 1132. Spriger-Verlag, (1985), 556–588.
23. Narutaka Ozawa and Sorin Popa, On a class of II_1 Factors with at Most one Cartan Subalgebra II, Amer. J. Math. 132 (2010), 841–866.
24. M. E. B. Bekka, Amenable unitary representations of locally compact groups. Invent. Math. 100 (1990), 383–401.
25. M.E.B. Bekka and A. Valette, Kazhdan's property (T) and amenable representations. Math. Z. 212 (1993), 293–299.
26. N.P. Brown and N. Ozawa; C^*-algebras and finite-dimensional approximations. Grad. Stud. Math., 88. Amer. Math. Soc., Providence, RI, 2008.
27. P.-A. Cherix, M. Cowling, P. Jolissaint, P. Julg, and A. Valette, Groups with the Haagerup property. Gromov's a-T-menability. Progress in Mathematics, 197. Birkhäuser Verlag, Basel, 2001.
28. A. Connes, Classification of injective factors: cases $II_1, II_\infty, III_\lambda, \lambda \neq 1$. Ann. Math. 104 (1976), 73–115.
29. A. Connes, J. Feldman and B. Weiss, An amenable equivalence relation is generated by a single transformation. Ergodic Theory Dynam. Systems 1 (1981), 431–450.
30. A. Connes and V. Jones, A II1 factor with two nonconjugate Cartan subalgebras. Bull. Amer. Math. Soc. 6 (1982), 211–212.
31. Y. de Cornulier, Y. Stalder and A. Valette, Proper actions of lamplighter groups associated with free groups. C. R. Math. Acad. Sci. Paris 346 (2008), 173–176.
32. M. Cowling, Harmonic analysis on some nilpotent Lie groups (with application to the representation theory of some semisimple Lie groups). Topics in modern harmonic analysis, Vol. I, II (Turin/Milan, 1982), 81–123, Ist. Naz. Alta Mat. Francesco Severi, Rome, 1983.
33. J. de Cannière and U. Haagerup, Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups. Amer. J. Math. 107 (1985), 455–500.
34. E.B. Davies and J.M. Lindsay, Noncommutative symmetric Markov semigroups. Math. Z. 210 (1992), 379–411.
35. J. Dixmier, Sous-anneaux abéliens maximaux dans les facteurs de type fini. Ann. Of Math. (2) 59 (1954), 279–286.
36. J. Feldman and C. C. Moore, Ergodic equivalence relations, cohomology, and von Neumann algebras. I and II. Trans. Amer. Math. Soc. 234 (1977), 289–359.
37. A. Furman, Orbit equivalence rigidity. Ann. of Math. (2) 150 (1999), 1083–1108.
38. E. Guentner and N. Higson, Weak amenability of $CAT(0)$ cubical groups. Preprint 2007, Geom. Dedicata, to appear. arXiv math/0702568.
39. A. Guichardet, Sur la cohomologie des groupes topologiques. II. Bull. Sci. Math. (2) 96 (1972), 305–332.
40. U. Haagerup, Injectivity and decomposition of completely bounded maps. Operator algebras and their connections with topology and ergodic theory (Buşteni, 1983), 170–222, Lecture Notes in Math., 1132, Springer, Berlin, 1985.
41. P. de la Harpe, A.G. Robertson, A. Valette, On the spectrum of the sum of generators for a finitely generated group. Israel J. Math. 81 (1993), 65–96.
42. A. Ioana, Cocycle superrigidity for profinite actions of property (T) groups. Preprint. arXiv:0805.2998

43. A. Lubotzky, Discrete groups, expanding graphs and invariant measures. With an appendix by Jonathan D. Rogawski. Progress in Mathematics, 125. Birkhäuser Verlag, Basel, 1994.
44. A. Lubotzky and A. Żuk, On property (τ). In preperation. Currently downloadable from http://www.ma.huji.ac.il/~alexlub/
45. N. Mizuta, A Bożejko-Picardello type inequality for finite dimensional $\mathrm{CAT}(0)$ cube complexes. J. Funct. Anal., 254 (2008), 760–772.
46. N. Monod and Y. Shalom, Orbit equivalence rigidity and bounded cohomology. Ann. of Math. (2) 164 (2006), 825–878.
47. G. A. Niblo and L. D. Reeves, The geometry of cube complexes and the complexity of their fundamental groups. Topology 37 (1998), 621–633.
48. N. Ozawa and S. Popa, On a class of II_1 factors with at most one Cartan subalgebra. Ann. of Math. (2), to appear.
49. J. Peterson, L^2-rigidity in von Neumann algebras. Invent. Math. 175 (2009), 417–433.
50. G. Pisier, Factorization of linear operators and geometry of Banach spaces. CBMS Regional Conference Series in Mathematics, 60. American Mathematical Society, Providence, RI, 1986.
51. S. Popa, On a class of type II_1 factors with Betti numbers invariants. Ann. Math. (2) 163 (2006), 809–899.
52. S. Popa, Strong rigidity of II_1 factors arising from malleable actions of w-Rigid groups I. Invent. Math. 165 (2006), 369–408.
53. S. Popa, Cocycle and orbit equivalence superrigidity for malleable actions of w-rigid groups, Invent. Math. 170 (2007), 243–295.
54. J.-L. Sauvageot, Quantum Dirichlet forms, differential calculus and semigroups. Quantum probability and applications, V (Heidelberg, 1988), 334–346, Lecture Notes in Math., 1442, Springer, Berlin, 1990.
55. J.-L. Sauvageot, Strong Feller semigroups on C^*-algebras. J. Operator Theory 42 (1999), 83–102.
56. Y. Shalom, Rigidity of commensurators and irreducible lattices. Invent. Math. 141 (2000), 1–54.
57. Y. Shalom, Rigidity, unitary representations of semisimple groups, and fundamental groups of manifolds with rank one transformation group. Ann. of Math. (2) 152 (2000), 113–182.
58. M. Takesaki, Theory of operator algebras. II. Encyclopaedia of Mathematical Sciences, 125. Operator Algebras and Non-commutative Geometry, 6. Springer-Verlag, Berlin, 2003.
59. R.J. Zimmer, Ergodic theory and semisimple groups. Monographs in Mathematics, 81. Birkhäuser Verlag, Basel, 1984.
60. Marek Bożejko, Burkhard Kümmerer, Roland Speicher, q-gaussian processes: non-commutative and classical aspects, *Commun. Math. Phys.* 185 (1997), 129-154.
61. L. Accardi, A. Frigerio, and J.T. Lewis, Quantum stochastic processes, Publ. RIMS 18 (1982), 97–133.
62. B.V.R. Bhat and K.R. Parthasarathy, Markov dilations of nonconservative dynamical semigroups and a quantum boundary theory, Ann. Inst. Henri Poincaré 31 (1995), 601– 651.
63. Ph. Biane, On processes with free increments (1995), preprint.

64. Ph. Biane, Free Brownian motion, free stochastic calculus and random matrices (1995), preprint.
65. Ph. Biane, Quantum Markov processes and group representations (1995), preprint.
66. M. Bożejko and R. Speicher, An example of a generalized Brownian motion, Commun. Math. Phys. 137 (1991), 519–531.
67. M. Bożejko and R. Speicher, An example of a generalized Brownian motion II, Quantum Probability and Related Topics VII (L. Accardi, ed.), World Scientific, Singapore, 1992, pp. 219–236.
68. M. Bożejko and R. Speicher, Completely positive maps on Coxeter groups, deformed commutation relations, and operator spaces, Math. Ann. 300 (1994), 97–120.
69. M. Bożejko and R. Speicher, Interpolations between bosonic and fermionic relations given by generalized Brownian motions, Math. Z. (to appear).
70. D.M. Bressoud, A simple proof of Mehler's formula for q-Hermite polynomials, Indiana Univ. Math. J. 29 (1980), 577–580.
71. E.A. Carlen and E.H. Lieb, Optimal hypercontractivity for Fermi fields and related non-commutative integration inequalities, Commun. Math. Phys. 155 (1993), 27–46.
72. J. Cuntz, Simple C^*-algebras generated by isometries, Commun. Math. Phys 57 (1977), 173-185.
73. C. Dellacherie and P.A. Meyer, Probabilités et potentiel, Hermann, Paris.
74. K. Dykema and A. Nica, On the Fock representation of the q-commutation relations, J. reine angew. Math. 440 (1993), 201–212.
75. D.E. Evans, On On, Publ. RIMS 16 (1980), 915–927.
76. D.I. Fivel, Interpolation between Fermi and Bose statistics using generalized commutators, Phys. Rev. Lett. 65 (1990), 3361–3364, Erratum 69 (1992), 2020.
77. U. Frisch and R. Bourret, Parastochastics, J. Math. Phys. 11 (1970), 364–390.
78. G. Gasper and M. Rahman, Basic hypergeometric functions, Cambridge U.P., Cambridge, 1990.
79. O.W. Greenberg, Particles with small violations of Fermi or Bose statistics, Phys. Rev. D 43 (1991), 4111–4120.
80. L. Gross, Existence and uniqueness of physical ground states, J. Funct. Anal. 10 (1972), 52–109.
81. P.R. Halmos, Normal dilations and extensions of operators, Summa Brasiliensis Math. 2 (1950), 125–134.
82. R.L. Hudson and K.R. Parthasarathy, Quantum Ito's formula and stochastic evolution, Commun. Math. Phys. 93 (1984), 301–323.
83. E.M. Ismail, D. Stanton, and G. Viennot, The combinatorics of q-Hermite polynomials and Askey-Wilson integral, Europ. J. Comb. 8 (1987), 379–392.
84. P.E.T. Jørgensen, L.M. Schmitt, and R.F. Werner, q-canonical commutation relations and stability of the Cuntz algebra, Pac. J. Math. 165 (1994), 131–151.
85. P.E.T. Jørgensen, L.M. Schmitt, and R.F. Werner, Positive representations of general commutation relations allowing Wick ordering, J. Funct. Anal. 134 (1995), 3–99.
86. P.E.T. Jørgensen and R.F. Werner, Coherent states on the q-canonical commutation relations, Commun. Math. Phys. 164 (1994), 455-471.
87. B. Kümmerer, Markov dilations on W^*-algebras, J. Funct. Anal. 63 (1985), 139–177.
88. B. Kümmerer, Survey on a theory of non-commutative stationary Markov processes, Quantum Probability and Applications III (L. Accardi, W.v. Waldenfels, ed.), Springer, Berlin, 1988, pp. 228–244.

89. B. Kümmerer and H. Maassen, Elements of Quantum Probability (1995), preprint.
90. B. Kümmerer and R. Speicher, Stochastic integration on the Cuntz algebra O_∞, J. Funct. Anal. 103 (1992), 372–408.
91. H. van Leeuwen and H. Maassen, A q-deformation of the Gauss distribution, J. Math. Phys. 36 (1995), 4743–4756.
92. P.A. Meyer, Quantum probability for probabilists, Lecture Notes in Mathematics, vol. 1538, Springer, Heidelberg, 1993.
93. J.S. Møller, Second quantization in a quon-algebra, J. Phys. A 26 (1993), 4643–4652.
94. E. Nelson, Construction of quantum fields from Markoff fields, J. Funct. Anal. 12 (1973), 97–112.
95. E. Nelson, The free Markoff field, J. Funct. Anal. 12 (1973), 211-227.
96. P. Neu and R. Speicher, Spectra of Hamiltonians with generalized single-site dynamical disorder, Z. Phys. B 95 (1994), 101–111.
97. K.R. Parthasarathy, Azéma martingales and quantum stochastic calculus, Proc. R. C. Bose Symposium, Wiley Eastern, 1990, pp. 551–569.
98. K.R. Parthasarathy, An introduction to quantum stochastic calculus, Monographs in Mathematics, vol. 85, Birkhäuser, Basel, 1992.
99. L.J. Rogers, On a three-fold symmetry in the elements of Heine series, Proc. London Math. Soc. 24 (1893), 171–179.
100. M. Schürmann, Quantum q-white noise and a q-central limit theorem, Commun. Math. Phys. 140 (1991), 589–615.
101. B. Simon, The $P(\phi)_2$ Euclidean (Quantum) Field Theory, Princeton University Press, Princeton, 1974.
102. B. Simon, Functional Integration and Quantum Physics, Academic Press, New York, 1979.
103. R. Speicher, Generalized statistics of macroscopic fields, Lett. Math. Phys. 27 (1993), 97–104.
104. R. Speicher, On universal products, The Fields Institute Communications (to appear).
105. S. Stanciu, The energy operator for infinite statistics, Commun. Math. Phys. 147 (1992), 211–216.
106. G. Szego, Ein Beitrag zur Theorie der Thetafunktionen, Sitz. Preuss. Akad. Wiss. Phys. Math. Ll 19 (1926), 242–252.
107. M. Takesaki, Conditional expectations in von Neumann algebras, J. Funct. Anal. 9 (1972), 306–321.
108. W. von Waldenfels, Fast positive Operatoren, Z. Wahrscheinlic-hkeitstheorie verw. Geb. 4 (1965), 159–174.
109. R.F. Werner, The free quon gas suffers Gibb’s paradox, Phys. Rev. D 48 (1993), 2929–2934.
110. I.F. Wilde, The free Fermi field as a Markov field, J. Funct. Anal. 15 (1974), 12–21.
111. T. Yu and Z.-Y. Wu, Construction of the Fock-like space for quons, Science in China (Series A) 37 (1994), 1472–1483.
112. D. Zagier, Realizability of a model in infinite statistics, Commun.Math. Phys. 147 (1992),199–210.
113. Alexandre Nou, Non-injectivity of the q-deformed von Neumann algebra, *Math. Ann.*, 330(1):17–38, 2004.

114. M. Bożejko, Completely positive maps on Coxeter groups and the ultracontractivity of the q-Ornstein-Uhlenbeck semigroup, *Banach Center Publications*, vol. 43 (1999), 87-93; Institute of Mathematics, Polish Academy of Sciences, Warsawa.
115. M. Bożejko, Ultracontractivity and strong Sobolev inequality for q-Ornstein-Uhlenbeck semigroup $(-1 < q < 1)$, *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, 2(1999), 203-220.
116. M. Bożejko, R. Speicher, Interpolation between bosonic and fermionic relation given by generalized Brownian motions, *Math. Z.* 222 (1996), 135-160.
117. A. Buchholz, Free Khintchine-Haagerup inequality, *Proc. AMS*
118. A. Buchholz, L_∞-Khintchine-Bonami inequality in the free probability, *Quantum Probability, Banach Center Publications* Volume 43 (1998), 105-109.
119. M-D. Choi, E. Effros, Injectivity and Operator spaces, *Journal of functional analysis* 24 (1977), 156-209
120. E.Christensen, A.Sinclair, Representations of completely bounded multilinear operators, *Journal of Functional analysis* 72 (1987), 151-181
121. E. Effros, Z.-J. Ruan, Operators spaces, *Oxford University Press*, Oxford, 2000.
122. U. Haagerup, G. Pisier, Bounded linear operators between C^*-algebras, *Duke Math. J.* 71 (1993), 889-925
123. F. Hiai, $q-$Deformed Araki-Woods algebras, In *Operator algebras and mathematical physics (Constant¸a, 2001)*, pages 169–202. Theta, Bucharest, 2003.
124. I. Krolak, Wick product for commutation relations connected with Yang-Baxter operators and new constructions of factors, *Commun. Math. Phys.* 210, 685-701 (2000)
125. G. Pisier, An introduction to operator spaces, *Cambridge Press. 2003*
126. G. Pisier, D. Shlyakhtenko, Grothendieck's Theorem for Operator Spaces, *Inventiones Mathematicae*, 150(1):185–217, 2002.
127. Fumio Hiai, q-deformed Araki-Woods factors, Theory of Operator Algebras and its Applications, (2002), 1250: 82-88.
128. P. Biane, Free hypercontractivity, Comm. Math. Phys. 184 (1997), 457-474.
129. M. M. Bożejko, and R. Speicher, An example of a generalized Brownian motion, Comm. Math. Phys. 137 (1991), 519-531.
130. Alexandre Nou, Asymptotic matricial models and QWEP property for q-Araki-Woods algebras, J. Funct. Anal. 232(2) (2006) 295–327.
131. M. Junge. Embedding of the operator space OH and the logarithmic 'little Grothendieck inequality'. *To appear*.
132. E. Kirchberg. On nonsemisplit extensions, tensor products and exactness of group C^*-algebras. *Invent. Math.*, 112(3):449–489, 1993.
133. N. Ozawa. About the QWEP conjecture. *Inter. J. of Math.*, 15(5):501–530, 2004.
134. G. Pisier. Completely bounded maps into certain Hilbertian operator spaces. *To appear*.
135. G. Pisier. *Introduction to operator space theory*, volume 294 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2003.
136. Y. Raynaud. On ultrapowers of non-commutative L_p spaces. *J. Operator Theory*, 48(1):41–68, 2002.
137. E. Ricard. Factoriality of q-Gaussian von Neumann algebras. Comm. Math. Phys. 257(3) (2005) 659–665.

138. P. Śniady. Gaussian random matrix models for q-deformed Gaussian variables. *Comm. Math. Phys.*, 216(3):515–537, 2001.
139. R. Speicher. A non-commutative central limit theorem. *Math. Z.*, 209(1):55–66, 1992.
140. Q. Xu. Operator space Grothendieck inequalities for non-commutative L_p-spaces. *To appear*.
141. Narutaka Ozawa and Sorin Popa, On a class of II_1 factors with at most one Cartan subalgebra, Annals of Mathematics, 172 (2010), 713–749
142. M. Abert and G. Elek, Dynamical properties of groups acting on profinite spaces, preprint. arXiv 1005.3188
143. C. Anantharaman-Delaroche, Amenable correspondences and approximation properties for von Neumann algebras, Pacific J. Math. 171 (1995), 309–341.
144. B. Bekka, P. De La Harpe, and A. VALETTE, Kazhdan's Property (T), New Math. Mongr. 11, Cambridge Univ. Press, Cambridge, 2008.
145. M. Choda, Inner amenability and fullness, Proc. Amer. Math. Soc. 86 (1982), 663–666.
146. A. Connes, Sur la classification des facteurs de type II, C. R. Acad. Sci. Paris Sér. A-B 281 (1975), Aii, A13–A15.
147. M. Cowling and U. Haagerup, Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one, Invent. Math. 96 (1989), 507–549.
148. H. A. Dye, On groups of measure preserving transformation. I, Amer. J. Math. 81 (1959), 119–159.
149. E. G. Effros, Property Γ and inner amenability, Proc. Amer. Math. Soc. 47 (1975), 483–486.
150. D. Gaboriau, Invariants l^2 de relations d'équivalence et de groupes, Publ. Math. Inst. Hautes Études Sci. (2002), 93–150.
151. D. Gaboriau and S. Popa, An uncountable family of nonorbit equivalent actions of $\mathbb{F}_n$, J. Amer. Math. Soc. 18 (2005), 547–559.
152. L. Ge, Applications of free entropy to finite von Neumann algebras. II, Ann. of Math. 147 (1998), 143–157.
153. U. Haagerup, An example of a nonnuclear C^*-algebra, which has the metric approximation property, Invent. Math. 50 (1978/79), 279–293.
154. R. Høegh-Krohn, M. B. Landstad, and E. Størmer, Compact ergodic groups of automorphisms, Ann. of Math. 114 (1981), 75–86.
155. A. Ioana, Non-orbit equivalent actions of $\mathbb{F}_n$, Ann. Sci. Éc. Norm. Supér. 42 (2009), 675– 696.
156. A. Ioana, J. Peterson, and S. POPA, Amalgamated free products of weakly rigid factors and calculation of their symmetry groups, Acta Math. 200 (2008), 85–153.
157. P. Jolissaint, Actions of dense subgroups of compact groups and II1-factors with the Haagerup property, Ergodic Theory Dynam. Systems 27 (2007), 813–826.
158. V. F. R. Jones, Index for subfactors, Invent. Math. 72 (1983), 1–25.
159. M. Junge and G. Pisier, Bilinear forms on exact operator spaces and $B(H) \otimes B(H)$, Geom. Funct. Anal. 5 (1995), 329–363.
160. R. V. Kadison, Diagonalizing matrices, Amer. J. Math. 106 (1984), 1451–1468.
161. F. J. Murray and J. Von Neumann, On rings of operators. IV, Ann. of Math. 44 (1943), 716–808.
162. N. Ozawa, Solid von Neumann algebras, Acta Math. 192 (2004), 111–117.

163. N. Ozawa, There is no separable universal II_1-factor, Proc. Amer. Math. Soc. 132 (2004), 487–490.
164. N. Ozawa, A Kurosh-type theorem for type II_1 factors, Internat. Math. Res. Not. (2006), Art. ID 97560, 21.
165. M. Pimsner and S. Popa, Entropy and index for subfactors, Ann. Sci. École Norm. Sup. 19 (1986), 57–106.
166. G. Pisier, Similarity Problems and Completely Bounded Maps, expanded ed., Lecture Notes in Math. 1618, Springer-Verlag, New York, 2001, includes the solution to "The Halmos problem".
167. S. Popa, Correspondences, INCREST Preprint, 1986.
168. S. Popa, Some rigidity results for non-commutative Bernoulli shifts, J. Funct. Anal. 230 (2006), 273–328.
169. S. Popa, On Ozawa's property for free group factors, Int. Math. Res. Not. 11 (2007), 10pp.
170. É. Ricard and Q. Xu, Khintchine type inequalities for reduced free products and applications, J. Reine Angew. Math. 599 (2006), 27–59.
171. Y. Shalom, Expander graphs and amenable quotients, in Emerging Applications of Number Theory (Minneapolis, MN, 1996), IMA Vol. Math. Appl. 109, Springer-Verlag, New York, 1999, pp. 571–581.
172. D. Voiculescu, The analogues of entropy and of Fisher's information measure in free probability theory. III. The absence of Cartan subalgebras, Geom. Funct. Anal. 6 (1996), 172–199.

1. Cyril Houdayer, Structural Results for Free Araki-Woods Factors and Their Continuous Cores, J. Inst. Math. Jussieu 9 (2010), 741–767.
2. I. Antoniou & S. A. Shkarin, Decay measures on locally compact abelian topological groups. Proc. Roy. Soc. Edinburgh 131 (2001), 1257–1273.
3. I. Chifan & A. Ioana, Ergodic subequivalence relations induced by a Bernoulli action. arXiv:0802.2353
4. I. Chifan & C. Houdayer, Bass Serre rigidity results in von Neumann algebras. Duke Math. J.,153 (2010), 23–54.
5. P. Erdös, On a family of symmetric Bernoulli convolutions. Amer. J. Math. 61 (1939), 974–976.
6. U. Haagerup, Connes' bicentralizer problem and uniqueness of the injective factor of type III_1. Acta Math. 69 (1986), 95–148.
7. C. Houdayer, Free Araki-Woods factors and Connes' bicentralizer problem. Proc. Amer. Math. Soc. 137 (2009), 3749-3755.
8. C. Houdayer, Construction of type II_1 factors with prescribed countable fundamental group. J. reine angew Math. 634 (2009), 169-207.
9. N. Ozawa, An example of a solid von Neumann algebra. Hokkaido Math. J., 38 (2009), 557–561.
10. S. Popa, On the superrigidity of malleable actions with spectral gap. J. Amer. Math. Soc. 21 (2008), 981–1000.
11. S. Popa & S. Vaes, Strong rigidity of generalized Bernoulli actions and computations of their symmetry groups. Adv. Math. 217 (2008), 833–872.
12. F. Rădulescu, A one-parameter group of automorphisms of $L(F_\infty) \otimes B(H)$ scaling the trace. C. R. Acad. Sci. Paris Sr. I Math. 314 (1992), 1027–1032.

13. D. Shlyakhtenko, Some estimates for non-microstates free entropy dimension, with applications to q-semicircular families. Int. Math. Res. Notices 51 (2004), 2757–2772.
14. D. Shlyakhtenko, On multiplicity and free absorption for free Araki-Woods factors. math.OA/0302217
15. M. Takesaki, Duality for crossed products and structure of von Neumann algebras of type III. Acta Math. 131 (1973), 249–310.
16. S. Vaes & R. Vergnioux, The boundary of universal discrete quantum groups, exactness and factoriality. Duke Math. J. 140 (2007), 35-84.
17. S. Vaes, Rigidity results for Bernoulli actions and their von Neumann algebras (after S. Popa). Séminaire Bourbaki, exposé 961. Astérisque 311 (2007), 237-294.
18. Cyril Houdayer and Éric Ricard, Approximation Properties and Absence of Cartan Subalgebra for Free Araki-Woods Factors, Adv. Math., 228(2):764–802, 2011.
19. A. Buchholz, *Operator Khintchine inequality in non-commutative probability*. Math. Ann. 319 (2001), 1–16.
20. A. Connes, M. Takesaki, *The flow of weights on factors of type* III. Tôhoku Math. J. 29 (1977), 473–575.
21. A.J. Falcone, M. Takesaki, *Non-commutative flow of weights on a von Neumann algebra*. J. Funct. Anal. 182 (2001), 170–206.
22. J. Fang, *On completely singular von Neumann subalgebras*. Proc. Edinb. Math. Soc. 52 (2009), 607–618.
23. U. Haagerup, T. Steenstrup, R. Szwarc, *Schur multipliers and spherical functions on homogeneous trees*. Internat. J. Math. 21 (2010). 1337–1382.
24. U. Haagerup, *An example of non-nuclear C^*-algebra which has the metric approximation property*. Invent. Math. 50 (1979), 279–293.
25. C. Houdayer, *Strongly solid group factors which are not interpolated free group factors*. Math. Ann. 346 (2010), 969-989.
26. C. Houdayer, D. Shlyakhtenko, *Strongly solid II_1 factors with an exotic MASA*. Int. Math. Res. Not. IMRN 2011, no. 6, 1352-1380.
27. M.V. Pimsner, *A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed products by* Z. Free probability theory (Waterloo, ON, 1995), 189–212, Fields Inst. Commun., 12, Amer. Math. Soc., Providence, RI, 1997.
28. S. Popa, *On a problem of R.V. Kadison on maximal abelian $*$-subalgebras in factors*. Invent. Math. 65 (1981), 269-281.
29. S. Popa, D. Shlyakhtenko, *Cartan subalgebras and bimodule decompositions of II_1 factors*, Math. Scand. 92 (2003) 93–102.
30. D. Shlyakhtenko, *Prime type* III *factors*. Proc. Nat. Acad. Sci., 97 (2000), 12439–12441.
31. Y. Ueda, *Remarks on free products with respect to non-tracial states*. Math. Scand. 88(2001), 111–125.
32. S Vaes, *Personal communication.*
33. J. Wysoczański, *A characterization of radial Herz-Schur multipliers on free products of discrete groups*. J. Funct. Anal. 129, (1995), 268-292.
34. Jan Cameron, Junsheng Fang and Kunal Mukherjee, Mixing subalgebras of finite von Neumann algebras, New York J. Math. 19 (2013) 343-366.
35. Bergelson, Vitaly; Rosenblatt, Joseph. Mixing actions of groups. Illinois J. Math 32 (1988), no. 1, 65-80.

36. Christensen, Erik. Subalgebras of a finite algebra. Math. Ann. 243 (1979), no. 1,17-29.
37. Grossman, Pinhas; Wiggins, Alan. Strong singularity for subfactors. Bull. London Math. Soc. 42 (2010), no. 4, 607-620.
38. Halmos, Paul R. On automorphisms of compact groups. Bull. Amer. Math. Soc. 49 (1943), 619-624.
39. Jolissaint, Paul; Stalder, Yves. Strongly singular MASAs and mixing actions in finite von Neumann algebras. Ergodic Theory Dynam. Systems 28 (2008), no. 6, 1861-1878.
40. Kadison, Richard V.; Ringrose, John R. Fundamentals of the theory of operator algebras. Vol. II. Advanced Theory. Graduate Studies in Mathematics, 16. American Mathematical Society, Providence, RI, 1997. pp. i-xxii and 399-1074.
41. Kitchens, Bruce; Schmidt, Klaus. Automorphisms of compact groups. Ergodic Theory Dynam. Systems 9 (1989), no. 4, 691-735.
42. Nadkarni, M. G. Spectral theory of dynamical systems. Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser Verlag, Basel, 1998. x+182 pp.
43. Popa, Sorin. A short proof of "injectivity implies hyperfiniteness" for finite von Neumann algebras. J. Operator Theory 16 (1986), no. 2, 261-272.
44. Robertson, Guyan; Sinclair, Allan M.; Smith, Roger R. Strong singularity for subalgebras of finite factors. Internat. J. Math. 14 (2003), 235-258.
45. Sakai, S. The Theory of W^* Algebras. Lecture notes. Yale University, New Haven, CT, 1962.
46. Sinclair, A. M.; Smith, R. R. Strongly singular masas in type II_1 factors. Geom. Funct. Anal. 12 (2002), no. 1, 199-216.
47. Sinclair, Allan M.; Smith, Roger R. Finite von Neumann algebras and masas. London Mathematical Society Lecture Note Series, 351. Cambridge University Press, Cambridge, 2008. x+400 pp.
48. Sinclair, Allan; White, Stuart. A continuous path of singular masas in the hyperfinite II_1 factor. J. Lond. Math. Soc. (2) 75 (2007), no. 1, 243-254.
49. Sinclair, Allan M.; Smith, Roger R.; White, Stuart A.; Wiggins, Alan. Strong singularity of singular masas in II_1 factors. Illinois J. Math. 51 (2007), no. 4,1077-1084.
50. Panchugopal Bikram, Kunal Mukherjee, Generator masas in q-deformed Araki–Woods von Neumann algebras and factoriality, 2016. arXiv:1606.04752.
51. S. Avesec, Strong solidity of the q-Gaussian algebras for all $-1 < q < 1$, preprint, arXiv:1110.4918, 2011.
52. R. Boutonnet, C. Houdayer, S. Vaes, Strong solidity of free Araki–Woods factors, preprint, arXiv:1512.04820, 2015.
53. J. Cameron, J. Fang, K. Mukherjee, Mixing and weak mixing abelian subalgebras of type II_1 factors, preprint.
54. A. Connes, E. Størmer, Homogeneity of the state space of factors of type III_1, J. Funct. Anal. 28 (1978) 187–196.
55. Y. Dabrowski, A free stochastic partial differential equation, Ann. Inst. Henri Poincaré Probab. Stat. 50(4) (2014) 1404–1455.
56. K. Dykema, A. Sinclair, R. Smith, Values of the Pukánszky invariant in free group factors and the hyperfinite factor, J. Funct. Anal. 240(2) (2006) 373–398.

57. T. Falcone, L^2-von Neumann modules, their relative tensor products and the spatial derivative, Illinois J. Math. 44(2) (2000) 407–437.
58. O.W. Greenberg, Q-mutators and violations of statistics, University of Maryland, Preprint 91-034, 1990 (UM-PP-91-034, C90-04-16.1).
59. O. Greenberg, D. Greenberger, T. Greenbergest, (Para) bosons, (Para) fermions, quons and other beasts in the menagerie of particle statistics, arXiv:hep-ph/9306225, 1993.
60. A. Guionnet, D. Shlyakhtenko, Free monotone transport, Invent. Math. 197(3) (2014) 613–661.
61. U. Haagerup, The standard form of von Neumann Algebras, Math. Scand. 37 (1975) 271–283.
62. K. Mukherjee, Masas and bimodule decompositions of II_1 factors, Quart. J. Math. 62(2) (2009) 451–486.
63. K. Mukherjee, Singular masas and measure-multiplicity invariant, Houston J. Math. 39(2) (2013) 561–598.
64. B. Nelson, Free monotone transport without a trace, Comm. Math. Phys. 334 (2015) 1245–1298.
65. S. Popa, Maximal injective subalgebras in factors associated with free groups, Adv. Math. 50(1) (1983) 27–48.
66. D. Shlyakhtenko, Lower estimates on microstates free entropy dimension, Anal. PDE 2(2) (2009) 119–146.
67. P. Śniady, Factoriality of Bożejko–Speicher von Neumann algebras, Comm. Math. Phys. 246(3) (2004) 561–567.
68. Mateusz Wasilewski, q-Araki-Woods Algebras: Extension Of Second Quantisation And Haagerup Approximation Property, Arxiv:1605.06034v3 [Math.OA] 6 Oct 2016.
69. Marie Choda. Group factors of the Haagerup type. *Proc. Japan Acad. Ser. A Math. Sci.*, 59(5):174–177, 1983.
70. Martijn Caspers and Adam Skalski. The Haagerup Property for Arbitrary von Neumann Algebras. *Int. Math. Res. Not. IMRN*, (19):9857–9887, 2015.
71. Matthew Daws, Pierre Fima, Adam Skalski, and Stuart White. The Haagerup property for locally compact quantum groups. *J. Reine Angew. Math.*, 711:189–229, 2016.
72. Rui Okayasu and Reiji Tomatsu. Haagerup approximation property for arbitrary von Neumann algebras. *Publ. Res. Inst. Math. Sci.*, 51(3):567–603, 2015.
73. Adam Skalski and Simeng Wang. Remarks on factoriality and q-deformations, 2016. arXiv:1607.04027.
74. Rémi Boutonnet and Cyril Houdayer, Structure of Modular Invariant Subalgebras In Free Araki–Woods Factors, Anal. PDE, 9(8):1989–1998, 2016.
75. H. Ando, U. Haagerup, Ultraproducts of von Neumann algebras. J. Funct. Anal. 266 (2014), 6842– 6913.
76. C. Houdayer, A class of II_1 factors with an exotic abelian maximal amenable subalgebra. Trans. Amer. Math. Soc. 366 (2014), 3693–3707.
77. C. Houdayer, Structure of II_1 factors arising from free Bogoljubov actions of arbitrary groups. Adv. Math. 260 (2014), 414–457.
78. C. Houdayer, Gamma stability in free product von Neumann algebras. Comm. Math. Phys. 336 (2015), 831–851.

79. C. Houdayer, Y. Isono, Bi-exact groups, strongly ergodic actions and group measure space type III factors with no central sequence. To appear in Comm. Math. Phys. arXiv:1510.07987
80. C. Houdayer, S. Raum, Asymptotic structure of free Araki–Woods factors. Math. Ann. 363 (2015), 237–267.
81. C. Houdayer, Y. Ueda, Asymptotic structure of free product von Neumann algebras. Math. Proc. Cambridge Philos. Soc. 161 (2016), 489–516.
82. A. Ocneanu, Actions of discrete amenable groups on von Neumann algebras. Lecture Notes in Mathematics, 1138. Springer-Verlag, Berlin, 1985. iv+115 pp.
83. Chenxu Wen, Maximal amenability and disjointness for the radial masa. J. Funct. Anal. 270 (2016),787–801.
84. Stephen Avsec, Michael Brannan, and Mateusz Wasilewski, Complete Metric Approximation Property For q-Araki-Woods Algebras, arXiv:1703.01317v2 [math.OA] 15 May 2020.
85. Kenny De Commer, Amaury Freslon, and Makoto Yamashita. CCAP for universal discrete quantum groups. Comm. Math. Phys., 331(2):677–701, 2014. With an appendix by Stefaan Vaes.
86. Edward G. Effros and Zhong-Jin Ruan. Operator spaces, volume 23 of London Mathematical Society Monographs. New Series. The Clarendon Press, Oxford University Press, New York, 2000.
87. Yusuke Isono. Examples of factors which have no Cartan subalgebras. Trans. Amer. Math. Soc., 367(11):7917–7937, 2015.
88. Yusuke Isono. Some prime factorization results for free quantum group factors. J. Reine Angew. Math., 722:215–250, 2017.
89. Marius Junge and Qiang Zeng. Ultraproduct methods for mixed q-Gaussian algebras, 2015. arXiv:1505.07852.
90. Narutaka Ozawa. Weak amenability of hyperbolic groups. Groups Geom. Dyn., 2(2):271–280, 2008.
91. Andrew S. Toms. On the classification problem for nuclear C^*-algebras. Ann. of Math. (2), 167(3):1029–1044, 2008.
92. Shawgy Hussein and Elsadig Ahmed, Absence of Caran Subalgebra and Generator Masa is q-Deformed with Approximation Property for q-Araki-Woods Algebras, Ph.D. Thesis Sudan University of Science and Technology, Sudan (2021).