Sudan University of Science and Technology
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L^p-Harmonic Forms and Estimates on Minimal Hypersurfaces and Square Root of Elliptic Systems with Dvoretzky Theorem

الصيغ التوافقية - L^p والتقديرات على الفضاءات المفرطة الأصغرية والجذر التربيعي للأنظمة الناقصية مع مبرهنة دفورتزكي

A Thesis Submitted in Fulfillment of the Requirements for the Degree of Ph.D in Mathematics

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Dedication

To my Family.

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I would like to thank with all sincerity Allah, and my family for their supports throughout my study. Many thanks are due to my thesis guide, Prof. Dr. Shawgy Hussein AbdAlla.

Abstract

We study the minimal hypersurfaces and L^2-, L^p- harmonic 1-forms on submanifolds with finite total curvature on minimal hypersurfaces with finite total curvature, finite index and first eigenvalue of a stable minimal hypersurface. We also study the L^p p-harmonic 1-forms on submanifolds in a Hadamard manifold. We start by the Hardy inequality for functions vanishing on a part of the boundary to show the square roots of elliptic second order divergence operators on strongly Lipschitz domains on L^2 and L^p theory hence extended to L^p-estimates for the square root problem for second-order divergence form operators and of elliptic systems with mixed boundary conditions. The small ball probability and pointwise estimates for marginals of convex bodies with the Hastings additivity counterexample by Dvoretzky theorem are characterized. We deal with Dvoretzky theorem on almost spherical sections of convex bodies and for subspaces of the L^p-space.

الخلاصة

قمنا بدراسة الفضاءات المفرطة الأصغرية والصيغ – 1 التوافقية – L^2، – L^p على متعدد الطيات الجزئي على الفضاءات المفرطة الأصغرية مع الأنحناء الكلي المنتهي والدليل المنتهي والقيمة الذاتية الأولي للفضاء المفرط الأصغري المستقر. أيضاً قمنا بدراسة الصيغ -1 التوافقية - L^p p على متعدد الطيات الجزئي في متعدد طيات هادامارد. بدأنا بواسطة متباينة هاردي للدوال المتلاشية على جزء الحدية لتوضيح الجزر التربيعي لمؤثرات تباعد الرتبة الثانية الناقصية على مجالات لبيشيتز القوية على نظرية L^2 و L^p ومن ثم التمديد لتقديرات - L^p لمسألة الجزر التربيعي لمؤثرات صيغ تباعد الرتبة – الثانية و الأنظمة الناقصية مع شرط الحدية المختلط. تم تشخيص احتمالية الكرة الصغرى والتقديرات النقطية وحافات الاجسام المحدبة مع المثال العكسي الجمعي لهاستينجس بواسطة مبرهنة دفورتزكي. تعاملنا مع مبرهنة دفورتزكي على المقاطع الكروية تقريباً للأجسام المحدبة و للفضاءات الجزئية لفضاءات – L^p.

Introduction

We show the minimal hypersurfaces with finite index was obtained. For $x : M^m \to \overline{M}, m \geq 3$, be an isometric immersion of a complete noncompact manifold M in a complete simply connected manifold $\overline{M}$ with sectional curvature satisfying $-k^2 \leq K_{\overline{M}} \leq 0$, for some constant k. Assume that the immersion has finite total curvature in the sense that the traceless second fundamental form has finite L^m-norm. If $K_{\overline{M}} \equiv 0$, assume further that the first eigenvalue of the Laplacian of M is bounded from below by a suitable constant.

We study L^p estimates for square roots of second order elliptic non necessarily selfadjoint operators in divergence form $L = -div\,(A\nabla)$ on Lipschitz domains subject to Dirichlet or to Neumann boundary conditions,pursuing [94] where we considered operators on $\mathbb{R}^n$. We obtain among other things $\left\|L^{\frac{1}{2}}f\right\|_p \leq c\|\nabla f\|_p$ for all $1 < p < \infty$ if L is real symmetric and the domain bounded, which is new for $1 < p < 2$.

Large deviation estimates are by now a standard tool in Asymptotic Convex Geometry, contrary to small deviation results. We present a novel application for a small deviations inequality to a problem that is related to the diameters of random sections of high dimensional convex bodies.

We estimate the bottom of the spectrum of the Laplace operator on a stable minimal hypersurface in a negatively curved manifold. We also derive various vanishing theorems for L^p harmonic 1-forms on minimal hypersurfaces in terms of the bottom of the spectrum of the Laplace operator. For M^m $(m \geq 3)$ be an m-dimensioanal complete noncompact oriented submanifold with finite total curvature, in a Hadamard manifold N^{m+n} with the sectional curvature satisfying $-k^2 < K_N \leq 0$, where k is a positive constant. For N be a complete simply connected Riemannian manifold with sectional curvature K_N satisfying $-k^2 \leq K_N \leq 0$ for a nonzero constant k. We show that if M is an $n(\geq 3)$-dimensional complete minimal hypersurface with finite index in N, then the space of L^p harmonic 1-forms on M must be finite dimensional for certain $p > 0$ provided the bottom of the spectrum of the Laplace operator is sufficiently large.

We show the Kato conjecture for square roots of elliptic second order non-self-adjoint operators in divergence form $L = -div(\nabla)$ on strongly

Lipschitz domains in $\mathbb{R}^n$, $n > 2$, subject to Dirichlet or to Neumann boundary conditions. We show that, under general conditions, the operator $(-\nabla \cdot \mu\nabla + 1)^{\frac{1}{2}}$ with mixed boundary conditions provides a topological isomorphism between $W_D^{1,p}(\Omega)$ and $L^p(\Omega)$, for $p \in]1,2[$ if one presupposes that this isomorphism holds true for $p = 2$. We focus on L^p-estimates for the square root of elliptic systems of second order in divergence form on a bounded domain. We treat complex bounded measurable coefficients and allow for mixed Dirichlet/Neumann boundary conditions on domains beyond the Lipschitz class. If there is an associated bounded semigroup on L^{p_0}, then we show that the square root extends for all $p \in (p_0, 2)$ to an isomorphism between a closed subspace of $W^{1,p}$ carrying the boundary conditions and L^p. This result is sharp and extrapolates to exponents slightly above 2.

We show a pointwise version of the multi-dimensional central limit theorem for convex bodies. Namely, let μ be an isotropic, log-concave probability measure on $\mathbb{R}^n$. For a typical subspace $E \subset \mathbb{R}^n$ of dimension nc, consider the probability density of the projection of μ onto E. We show that for any $2 < p < \infty$ and for every n-dimensional subspace X of L^p, represented on $\mathbb{R}^n$, whose unit ball B_X is in Lewis' position one has the following two-level Gaussian concentration inequality: $\mathbb{P}(|\|Z\| - \mathbb{E}\|Z\|| > \varepsilon\mathbb{E}\|Z\|) \leq C \exp\left(-c\, min\left\{\alpha_p \varepsilon^2 n, (\varepsilon n)^{\frac{2}{p}}\right\}\right)$, $0 < \varepsilon < 1$, where Z is the standard n-dimensional Gaussian vector, $\alpha_p > 0$ is a constant depending only on p and $C, c > 0$ are absolute constants. We show optimal lower bound on the dimension of random almost spherical sections for these spaces.

The Contents

Chapter 1
Minimal Hypersurfaces and L^2-Harmonic 1-Forms

We show that the space of the L^2 harmonic 1-forms on M has finite dimension. Moreover, there exists a constant $\Lambda > 0$, explicitly computed, such that if the total curvature is bounded from above by Λ then there are no nontrivial L^2-harmonic 1-forms on M.

Section (1.1): Finite Index

In Cao-Shen-Zhu [2], they proved that a complete, immersed, stable minimal hypersurface M^n of $\mathbb{R}^{n+1}$ with $n \geq 3$ must have only one end.

When $n = 2$, it was proved independently by do Carmo-Peng [3] and Fischer-Colbrie-Schoen [5] that a complete, immersed, oriented stable minimal surface in $\mathbb{R}^3$ must be a plane. Later Gulliver [7] and Fischer-Colbrie [4] proved that if a complete, immersed, minimal surface in $\mathbb{R}^3$ has finite index, then it must be conformally equivalent to a compact Riemann surface with finitely many punctures.

Fischer-Colbrie actually proved this for minimal surfaces in a complete manifold with non-negative scalar curvature. In any event, a corollary is that if a complete, immersed, oriented minimal surface in $\mathbb{R}^3$ has finite index then it must have finitely many ends. We generalize this result for finitely many ends to higher dimensional minimal hypersurfaces in Euclidean space. We show that the first L^2-Betti number of such a manifold must be finite.

The strategy of Cao-Shen-Zhu was to utilize a result a Schoen-Yau [13] asserting that a complete, stable minimal hypersurface of $\mathbb{R}^{n+1}$ cannot admit a non-constant harmonic function with finite Dirichlet integral. Assuming that M has more than one end, Cao-Shen-Zhu constructed a non-constant harmonic function with finite Dirichlet integral. This approach very much fits into the scheme studied in [11], [12]. In fact, they showed that the number of non-parabolic ends of any complete Riemannian manifold is bounded above by the dimension of the space of bounded harmonic functions with finite Dirichlet integral.

The proof of Cao-Shen-Zhu can be modified to show that each end of a complete, immersed, minimal submanifold must be non-parabolic. Due to this connection with harmonic functions, we refine the argument of Schoen-Yau to obtain an estimate of the space of harmonic functions with finite Dirichlet integral.

Unfortunately, our estimate depends on the geometry of M on a compact subset, whose existence is guaranteed by the finite index assumption. While we succeeded in proving finite index implies finitely many ends, it is unclear if one can actually estimate the number of ends by the index directly.

Let us first recall (see [12] and [10]) that an end E of a complete manifold M is non-parabolic means that E admits a positive Green's function with Neumann boundary condition. First, we will recall a theorem in [12].

Theorem (1.1.1)[1]: Let M be a complete Riemannian manifold. Let $\mathcal{H}_D^0(M)$denote the space of bounded harmonic functions with finite Dirichlet integral. Then the number of non-parabolic ends of M is at most the dimension of $\mathcal{H}_D^0(M)$.Observing that if u is a harmonic function with finite Dirichlet integral then its exterior differential du is an L^2 harmonic 1-form. Moreover, $du = 0$ if and only if u is identically constant. Hence

$$dim\mathcal{H}_D^0(M) \leq dimH^1(L^2(M)) + 1.$$

Using this inequality, we can state Theorem (1.1.1) in terms of the first L^2 Betti number.
Corollary (1.1.2)[1]: Let M be a complete Riemannian manifold. Let $H^1(L^2(M))$ be the first L^2-cohomology of M. Then the number of non-parabolic ends of M is bounded from above by $dimH^1(L^2(M))+1$.

This corollary enables us to estimate the number of ends of a minimal hyper-surface if we can show that all its ends are non-parabolic. In fact, it was proved in [2] that this is the case for minimal submanifolds of $\mathbb{R}^{n+1}$. We provide a presentation which extract the main points of the proof and stated it for more general situations in terms of non-parabolicity.

Theorem (1.1.3)[1]: Let M^n be a complete, immerse, minimal sub-manifold of $\mathbb{R}^N$. If $n \geq 3$, then each end of M must be non-parabolic.

Proof. Let E be an end of M. For R sufficiently large, let us consider the set $E_R = E \cap B_p(R)$, where $B_p(R)$ is the geodesic ball of radius R in M centered at some point $p \in M$. Let us denote by r the distance function of M to the point p.
Suppose the function f_R is the solution of the equation

$$\Delta f_R = 0 \; on \; E_R,$$
$$f_R = 1 \;\; f_R \;\; on \;\; \partial E,$$

and

$$f_R = 0 \; on \; E \cap \partial B_p(R).$$

By the maximum principle, f_R is uniformly bounded between 0 and 1. This bound and the gradient estimate implies that the sequence f_R converges uniformly on compact subsets of E to a harmonic function f with boundary condition

$$f = 1 \; on \; \partial E.$$

Moreover, f with satisfy the bounds

$$0 \leq f \leq 1.$$

If we can show that f is non-constant, then E will be non-parabolic (see [11] and [10]).
For a fixed $0 < R_0 < R$ such that $E_{R_0} \neq \emptyset$, let Á be a non-negative cut-off function satisfying the properties that

$$\phi = 1 \; on \; E_R \backslash E_{R_0},$$
$$\phi = 0 \; on \; \partial E,$$

and

$$|\nabla \phi| \leq C_1.$$

The Sobolev inequality of Michael-Simon [13], integration by parts, and the fact that f_R is harmonic, imply that

$$\left(\int_{E_R} (\phi f_R)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq C \int_{E_R} |\nabla(\phi f_R)|^2$$
$$= C\left(\int_{E_R} |\nabla \phi|^2 f_R^2 + 2\int_{E_R} \phi f_R \langle \nabla \phi, \nabla f_R\rangle + \int_{E_R} \phi^2 |\nabla f_R|^2\right)$$
$$= C\left(\int_{E_R} |\nabla \phi|^2 f_R^2 + \frac{1}{2}\int_{E_R} \langle \nabla(\phi)^2, \nabla(f_R^2)\rangle + \int_{E_R} \phi^2 |\nabla f_R|^2\right) = C\int_{E_R} |\nabla \phi|^2 f_R^2.$$

In particular, for a fixed R_1 satisfying $R_0 < R_1 < R$, we have

$$\int_{E_{R_1}/E_{R_0}} f_R^{\frac{2n}{n-2}} \le C_2 \int_{E_{R_0}} f_R^2 .$$

If the limiting function f is identically constant, then f must be identically 1because of its boundary condition. Letting $R \to \infty$, we obtain

$$(V_E(R_1) - V_E(R_0))^{\frac{n-2}{n}} \le CV_E(R_0).$$

where $V_E(r)$ denotes the volume of the set E_r. Since $R_1 > R_0$ is arbitrary, this implies that E must have finite volume. However, since an end of a minimal sub-manifold must have infinite volume, this contradicts the assumption that $f = 1$, and the theorem is proved.

It is clear in the above argument that this theorem can be generalized to an arbitrary Riemannian manifold.

Corollary (1.1.4)[1]: Let E be an end of a complete Riemannian manifold. Suppose for some $v \ge 1$, E satisfies a Sobolev type inequality of the form

$$\left(\int_E |u|^{2v}\right)^{\frac{1}{v}} \le C \int_E |\nabla u|^2$$

for all compactly supported function $u \in W_{1,2}(E)$ defined on E, then E must either have finite volume or be non-parabolic.

We would like to remark that it was proved independently by Grigor'yan [6] and Varopoulos [14] that if a manifold is non-parabolic then its volume growth must satisfy

$$\int_r^\infty \frac{tdt}{V_p(t)} < \infty. \qquad (1)$$

In particular, when combine with Corollary (1.1.4), this implies that if and end satisfies a Sobolev type inequality as hypothesized in Corollary (1.1.4), then it must either have finite volume or its volume growth must be at least quadratic satisfying (1).

We prove the main result.

Theorem (1.1.5)[1]: Let M^n be a complete, immersed, oriented minimal hypersurface in $\mathbb{R}^{n+1}$ with $n \ge 3$. Suppose M has finite index. Then M must have finite first L^2-Betti number, i.e. $dimH^1(L^2(M)) < \infty$. In particular, M must have finitely many ends.

Proof. The assumption that M has finite index implies that there exists a compact set $\Omega \subset M$ such that $M\backslash\Omega$ is stable. In particular, we may assume that - $\Omega \subset B_p(R_0)$ for some geodesic ball centered at $p \in M$ of radius R_0. The monotonicity of eigenvalues implies that $M\backslash B_p(R_0)$ is stable. In particular, if $|A|^2$ denotes the square of the length of the second fundamental form of M, then the stability inequality [13] asserts that

$$\int_{M\backslash B_p(R_0)} \psi^2 |A|^2 \le \int_{M\backslash B_p(R_0)} |\nabla\psi|^2 \qquad (2)$$

for all compactly supported function ψ on $M\backslash B_p(R_0)$.

For any L^2 harmonic 1-form ω defined on M, let us denote

$$h = |\omega|$$

to be the length of the ω. The Bochner formula (see [9]) asserts that

$$\Delta h^2 \ge 2Ric(\omega,\omega) + 2|\nabla\omega|^2 \qquad (3)$$

where Ric denotes the Ricci curvature of M and $\nabla\omega$ is the covariant derivative of ω. Using the Gauss curvature equation, we conclude that

$$\mathrm{Ric}(\omega,\omega) \geq -|A|^2\, h^2. \tag{4}$$

Since ω is an L^2 harmonic 1-form, it must be both closed and co-closed. In par-ticular, in terms of an orthonormal co-frame $\{\omega_1, \dots, \omega_n\}$, we can write $\omega = a_i\omega_i$.
Then the closed condition is given by

$$a_{i,j} = a_{j,i}$$

and the co-closed condition is given by

$$\sum_{i=1}^{n} a_{i,i} = 0.$$

On the other hand,

$$|\nabla\omega|^2 = \sum_{i,j} a_{i,j}^2$$

$$\geq \sum_{j=1}^{n} a_{1,j}^2 + \sum_{\alpha=2}^{n} a_{\alpha,1}^2 + \sum_{\alpha=2}^{n} a_{\alpha,\alpha}^2$$

$$\geq \sum_{j=1}^{n} a_{1,j}^2 + \sum_{\alpha=2}^{n} a_{\alpha,1}^2 + \frac{1}{n-1} left\left(\sum_{\alpha=2}^{n} a_{\alpha,\alpha} \text{ right}\right)^2 .$$

Using both the closed and co-closed conditions, we conclude that

$$|\nabla\omega|^2 \geq \frac{n}{n-1}\sum_{j=1}^{n} a_{1,j}^2 \ . \tag{5}$$

However, at any fixed point $x \in M$, if we choose an orthonormal co-frame such that $|\omega|\omega_1 = \omega$, then

$$|\nabla(h^2)|^2 = 4\sum_{j=1}^{n}\left(a_1 a_{1,j}\right)^2$$

$$\leq 4h^2 \sum_{j=1}^{n} a_{1,j}^2 \ .$$

Combining with (3), (4), and (5), we obtain

$$\Delta h \geq -|A|^2 h + \frac{|\nabla h|^2}{(n-1)h}. \tag{6}$$

By choosing $\psi = \phi h$ with ϕ being a non-negative compactly supported function on $M\backslash B_p(R_0)$ (2) becomes

$$\int_{M\backslash B_p(R_0)} \phi^2\, |A|^2 h^2 \ \leq \int_{M\backslash B_p(R_0)} |\nabla\phi|^2\, h^2 + 2\int_{M\backslash B_p(R_0)} \phi h \langle\nabla\phi, \nabla h\rangle$$

$$+ \int_{M\backslash B_p(R_0)} \phi^2\, |\nabla h|^2$$

$$= \int_{M\backslash B_p(R_0)} |\nabla\phi|^2\, h^2 - \int_{M\backslash B_p(R_0)} \phi^2\, h\Delta h.$$

Combining with (6), we have

$$\int_{M\setminus B_p(R_0)} \phi^2 |\nabla h|^2 \le (n-1) \int_{M\setminus B_p(R_0)} |\nabla\phi|^2 h^2. \qquad (7)$$

On the other hand, the Sobolev inequality for minimal submanifold [13] implies that

$$\left(\int_{M\setminus B_p(R_0)} (\phi h)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \le C \int_{M\setminus B_p(R_0)} |\nabla(\phi h)|^2 = 2C \int_{M\setminus B_p(R_0)} \phi^2 |\nabla h|^2$$
$$+2C \int_{M\setminus B_p(R_0)} |\nabla\phi|^2 h^2$$

Combining with (7), we obtain

$$\left(\int_{M\setminus B_p(R_0)} (\phi h)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \le 2nC \int_{M\setminus B_p(R_0)} |\nabla\phi|^2 h^2. \qquad (8)$$

For $R > R_0 + 1$, let us choose Á satisfying the properties that

$$\phi = \begin{cases} 0 \text{ on} B_p(R_0) \\ 1 \text{ on } B_p(R)\setminus B_p(R_0+1) \\ 0 \text{ on } M \setminus B_p(2R), \end{cases}$$
$$|\nabla\phi| \le C_3 \text{ on } B_p(R_0+1)\setminus B_p(R_0)$$

and

$$|\nabla\phi| \le C_3 R^{-1} \text{ on } B_p(2R)\setminus B_p(R)$$

for some constant $C_3 > 0$. Applying this to (8), we have

$$\left(\int_{B_p(R)\setminus B_p(R_0+1)} h^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \le C_4 \int_{B_p(R_0+1)\setminus B_p(R_0)} h^2 + C_4 R^{-2} \int_{B_p(2R)\setminus B_p(R)} h^2 .$$

Since by the assumption h is in L^2, letting $R \to \infty$, the second term tends to 0 and we conclude that

$$\left(\int_{M\setminus B_p(R_0+1)} h^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \le C_4 \int_{B_p(R_0+2)\setminus B_p(R_0+1)} h^2 . \qquad (9)$$

On the other hand, the Schwarz inequality asserts that

$$\int_{B_p(R_0+2)\setminus B_p(R_0+1)} h^2 \le V_p^{\frac{2}{n}}(R_0+2)\left(\int_{B_p(R)\setminus B_p(R_0+1)} h^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}$$

Together with (9), we conclude that there exists a constant $C_5 > 0$ depending on $V_p(R_0 + 2)$ such that

$$\int_{B_p(R_0+2)} h^2 \le C_5 \int_{B_p(R_0+1)} h^2. \qquad (10)$$

The fact that h satisfies the differential inequality (6) implies that we can apply the Moser iteration argument (see [9]) and conclude that

$$h^2(x) \le C_6 \int_{B_x(1)} h^2$$

where $C_6 > 0$ depends only on n and the upper bound of $|A|^2$ on $Bx(1)$. In partic-ular, if $x \in B_p(R_0 + 1))$ has the property that

$$h^2(x) = \sup_{B_p(R_0+1)} h^2 ,$$

$$then \sup_{B_p(R_0+1)} h^2 \le C_6 \int_{B_p(R_0+2)} h^2 .$$

Combining with (10), this implies that there exists constant $C_7 >$ depending only on $n, V_p(R_0 + 2)$, and $\sup_{B_p(R_0+1)} |A|^2 \sup_{B_p(R_0+2)} |A|^2$, such that

$$\sup_{B_p(R_0+1)} h^2 \le C_7 \int_{B_p(R_0+1)} h^2. \tag{11}$$

We are now ready to show that $H^1(L^2(M))$ is finite dimensional. It suffices to show that any finite dimensional subspace K of $H^1(L^2(M))$ must have its dimension bounded by a fixed constant. Let k be the dimension of K. Let us consider the bi-linear form defined on K given by

$$\int_{B_p(R_0+1)} \langle \omega, \theta \rangle .$$

Note that if Z

$$\int_{B_p(R_0+1)} |\omega|^2 = 0$$

for some $\omega \in K$, then by unique continuationω must be identically 0. This implies that the quadratic form is an inner product defined on K.

According to Lemma 11 of [8], there exits an $\omega \in K$ such that

$$k \int_{B_p(R_0+1)} |\omega|^2 \le V_p(R_0 + 1)(\min\{n, k\}) \sup_{B_p(R_0+1)} |\omega|^2 .$$

However, combining with (11) we conclude that

$k \le C_8$ with $C_8 > 0$ depending only on n, $V_p(R_0 + 2)$, and $\sup_{B_p(R_0+2)} |A|^2$. The theorem follows by applying Corollary (1.1.2) and Theorem (1.1.3).

Section (1.2): Submanifolds with Finite Total Curvature

For $x: M^m \to \bar{M}$ be an isometric immersion of an m-dimensional manifold M in a Riemannian manifold.$\bar{M}$. Let I denote the second fundamental form and $H = \frac{1}{m} tr(\mathbb{I})$ the mean curvature vector field of the immersion x. The traceless second fundamental form Φ is defined by

$$\Phi(X, Y) = \mathbb{I}(X, Y) - \langle X, Y \rangle H,$$

for all vector fields X, Y on M, where $\langle , \rangle$is the metric of M. A simple computation shows that

$$|\Phi|^2 = |\mathbb{I}|^2 - m|H|^2.$$

In particular, $|\Phi| \equiv 0$ if and only if the immersion x is totally umbilical. We say that the immersion x has finite total curvature if the L^m-norm of the traceless second fundamental form is finite (see [30]), that is,

$$\|\Phi\|_{L^m(M)} = \left(\int_M |\Phi|^m dM \right)^{\frac{1}{m}} < +\infty,$$

Where dM stands for the volume element of M.

Topological and metric properties of complete submanifolds with finite total curvature have been a field of active research since the work of Gauss. For instance, let M^2 be a complete surface isometrically immersed in a Euclidean space R^n with finite total curvature. By celebrated results of Huber [28], Osserman [33], and Chern–Osserman [36], it is known that if the immersion is minimal then it is proper, M is homeomorphic to a compact surface M punctured at finitely many points, and the Gauss map extends continuously to all points of $\mathcal{M}$. See also White [27] and Müller–˘Sverák [30] for the non-minimal case. In higher dimension, the description of the topology is more involved, and there exist many interesting related to this subject (see, [9], [1], [32], [34], [32]).

We are interested in the study of cohomological aspects of noncompact submanifolds with finite total curvature. More specifically, assume that M is a complete noncompact manifold and consider the space of the L^2-harmonic 1-forms on M

$$\mathcal{H}^1(M)\left\{= \omega | \int_M \omega \wedge * \, \omega = \int_M |\omega|^2 \, dM < \infty \text{and } d\omega = d * \omega = 0 \right\}.$$

It is well known that the space $\mathcal{H}^1(M)$ is isomorphic to the first-reduced L^2-cohomology group of M (see [35]). Moreover, the dimension of $\mathcal{H}^1(M)$ gives an upper bound to the number of non-parabolic ends of$M$. In fact, $if \mathcal{H}_D^0(M)$ denotes the space of the harmonic functions on M with finite Dirichlet integral then $u \; \mathcal{H}_D^0(M)$
if and only if its differential exterior $du \in \mathcal{H}^1(M)$. Thus

$$dim\mathcal{H}_D^0(M) \leq dim\mathcal{H}^1(M) + 1.$$

On the other hand, an important result of Li–Tam (Theorem 2.1 of [12]) states that the number $\boldsymbol{e}(M)$ of nonparabolic ends of a complete manifold M satisfies $\boldsymbol{e}(M) \leq \dim\mathcal{H}_D^0(M)$.

In [35], Theorem 3.5, Carron proved that if $M^m, m \geq 3$, is a complete noncompact submanifold of $\mathbb{R}^n$ with finite total curvature and finite total mean curvature (i.e., the L^m-norm of the mean curvature vector is finite) then each space of reduced L^2-cohomology on M has finite dimension. Under the same conditions, and using techniques of harmonic functions as in [12], Fu and Xu also proved that $\mathcal{H}^1(M)$is finite dimensional (see [27]). Since the ends of complete noncompact submanifolds in $\mathbb{R}^n$ with finite total mean curvature are nonparabolic, they actually conclude that M must have finitely many ends.

Our first result is an improvement and a generalization of the Carron and Fu–Xu theorems. We recall that a Riemannian manifold is called a Hadamard manifold if it is complete, simply connected, and has nonpositive sectional curvature.

Corollary (1.2.1) Let $x: M^m \to \mathbb{R}^n, m \geq 3$, be an isometric immersion of a complete noncompact manifold M^m in the Euclidean space $\mathbb{R}^n$. If the total curvature of x satisfies

$$\|\Phi\|_{L^m(M)} < \frac{\frac{m}{(m-1)\sqrt{S}}}{\sqrt{1+\frac{(m-1)(m-2)^2}{4(m2-3m+1),}}}$$

then there are no nontrivial L^2-harmonic 1-forms on M.

Corollary (1.2.2) Let $x: M^m \to \bar{M}, m \geq 3$, be an isometric immersion of a complete noncompact manifold M in a Hadamard manifold $\bar{M}$ with sectional curvature satisfying $-\kappa^2 \leq K_{\bar{M}} \leq 0$, for some constant $k \neq 0$. Assume that $\lambda_1(M) > \frac{(m-1)^2k^2}{m}$ If the total curvature of x satisfies

$$\|\Phi\|_{L^m(M)} < \frac{m}{(m-1)\sqrt{S}} 1 -$$

$$\Lambda = \frac{\frac{m}{(m-1)\sqrt{S}}\sqrt{1-\frac{(m-1)^2k^2}{m\lambda_1(M)}}}{\sqrt{1+\frac{(m-1)(m-2)^2}{4\left(m^2-3m+1+\frac{(m-1)^2k^2}{\lambda_1(M)}\right)}}},$$

then there are no nontrivial L^2-harmonic 1-forms on M.

We obtain a general inequality for 1-forms on$M$ involving the geometry of immersion. We use such inequality to prove Theorem (1.2.3). We prove Theorem (1.2.4). We again use the inequality to give a proof of Theorem (1.2.4).

Let $x: M^m \to \bar{M}, m \geq 3$, be an isometric immersion of a complete noncompact manifold M in a Hadamard manifold $\bar{M}$ with sectional curvature satisfying $-\kappa^2 \leq K_{\bar{M}} \leq 0$, for some constant k. We obtain an integral inequality for L^2-harmonic 1-forms on M involving the geometry of x.

Given $\omega \in \mathcal{H}^1(M)$, we recall the refined Kato's inequality (see, for instance, Lemma 3.1 of [37]):

$$|\nabla|\omega||^2 \leq \frac{m-1}{m}|\nabla\omega|^2.$$

A direct computation yields

$$|\nabla\omega|^2 \leq 2\left(|\omega|\Delta|\omega| + \frac{m-1}{m}|\nabla\omega|^2\right).$$

Using Bochner's formula [31] (see also Lemma 3.2 of [9]), we obtain

$$|\omega|\Delta|\omega| \geq \frac{1}{m-1}\big|\nabla|\omega|\big|^2 + RicM(\omega^\#, \omega^\#), \tag{12}$$

where $\omega^\#$ is the dual vector field of ω. Under our hypothesis on the sectional curvature of $.\bar{M}$ we can estimate the Ricci curvature of M by using Proposition 2 of [36]:

$$Ric_M(\omega^\#, \omega^\#) \geq (m-1)(|H|^2 - k^2)|\omega|^2$$
$$-\frac{(m-1)}{m}|\Phi|^2|\omega|^2 - \frac{(m-2)\sqrt{m(m-1)}}{m}|H||\Phi||\omega|^2. \tag{13}$$

Using (12) and (13), we get

$$|\omega|\Delta|\omega| \geq \frac{1}{m-1}|\nabla\omega|^2 - \frac{m-1}{m}|\Phi|^2|\omega|^2 + (m-1)(|H|^2k^2)|\omega|^2$$

$$-\frac{(m-2)\sqrt{m(m-1)}}{m}|H||\Phi||\omega|^2 \qquad (14)$$

Let $\eta \in C_0^\infty(M)$ be a smooth function on M with compact support.We multiply both sides of (14) by η^2 and integrate by parts. For the sake of simplicity, henceforth we will omit the volume element in the integrals. So, we obtain

$$0 \leq -2\int_M \eta|\omega|\langle\nabla\eta, \nabla|\omega|\rangle - \frac{m-1}{m}\int_M \eta^2\big|\nabla|\omega|\big|^2$$
$$+\frac{(m-2)\sqrt{m(m-1)}}{m}\int_M \eta^2|H||\Phi||\omega|^2 + \frac{m-1}{m}\int_M \eta^2|\Phi|^2|\omega|^2$$
$$+(m-1)\int_M \eta^2(|H|^2-k^2)|\omega|^2. \qquad (15)$$

For each $a > 0$, we apply the Cauchy–Schwarz inequality in (15) to obtain

$$0 \leq -2\int_M \eta|\omega|\langle\nabla\eta, \nabla|\omega|\rangle - \frac{m-1}{m}\int_M \eta^2\big|\nabla|\omega|\big|^2$$
$$+\int_M \left((m-1)k^2 + \left(-(m-1)+\frac{a(m-2)\sqrt{m(m-1)}}{2m}\right)|H|^2\right)\eta^2|\omega|^2$$
$$+\left(\frac{(m-2)\sqrt{m(m-1)}}{2am}+\frac{m-1}{m}\right)\int_M \eta^2|\phi|^2|\omega|^2. \qquad (16)$$

On the other hand, since $m \geq 3$, we use the Hölder, Hoffman–Spruck [29], and Cauchy–Schwarz inequalities to get

$$\int_M \eta^2\,|\Phi|^2|\omega|^2 \leq \phi(\eta)\left(\int_M (\eta|\omega|)^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}}$$
$$\leq S\phi(\eta)\int_M (|\nabla(\eta|\omega|)|^2 + \eta^2|\omega|^2|H|^2)$$
$$\leq S\phi(\eta)\int_M \left(\left(1+\frac{1}{b}\right)|\omega|^2|\nabla\eta|^2 + (1+b)\eta^2\big|\nabla|\omega|\big|^2\right)$$
$$+S\phi(\eta)\int_M \eta^2|\omega|^2\,|H|^2, \qquad (17)$$

for all $b > 0$, where $\phi(\eta) = \left(\int_{supp(\eta)}|\Phi|^m\right)^{\frac{2}{m}}$ and $S = S(m) > 0$ is the constant in the Hoffman–Spruck inequality. Thus, using (16) and (17), we have

$$0 \leq -2\int_M \eta|\omega|\langle\nabla\eta, \nabla|\omega|\rangle - \frac{m}{m-1}\int_M \eta^2|\nabla\omega|^2 + A(m,a)\int_M |H|^2\eta^2|\omega|^2$$
$$+SB(m,a)\phi(\eta)\int_M \left(\left(1+\frac{1}{b}\right)|\omega|^2|\nabla\eta|^2 + (1+b)\eta^2\big|\nabla|\omega|\big|^2\right)$$

$$+SB(m,a)\phi(\eta)\int_M \eta^2|\omega|^2|H|^2+(m-1)k^2\int_M \eta^2|\omega|^2, \tag{18}$$

where $A(m,a)$ and $B(m,a)$ are given by

$$A(m,a)=-(m-1)+\frac{a(m-2)\sqrt{m(m-1)}}{2m}$$
$$B(m,a)=\frac{(m-2)\sqrt{m(m-1)}}{2am}+\frac{m-1}{m}. \tag{19}$$

We use the Cauchy–Schwarz inequality again to get

$$2\left|\int_M \eta|\omega|\,\langle\nabla\eta,\nabla|\omega|\rangle\right|\le c\int_M \eta^2\,|\nabla|\omega||^2+\frac{1}{c}\int_M |\omega|^2|\nabla\eta|^2, \tag{20}$$

for all $c>0$. Using (18) and (20), we obtain the following integral inequality:

$$C\int_M \eta^2\big|\nabla|\omega|\big|^2+D\int_M |H|^2\eta^2|\omega|^2\le E\int_M |\omega|^2|\nabla\eta|^2+(m-1)k^2\int_M \eta^2|\omega|^2, \tag{21}$$

where

$$-C=-C(m,a,b,c,\eta)=c+(1+b)SB(m,a)\phi(\eta)-\frac{m}{m-1},$$
$$-D=-D(m,a,\eta)=A(m,a)+SB(m,a)\phi(\eta),$$
$$E=E(m,a,b,c,\eta)=\frac{1}{c}+\left(1+\frac{1}{b}\right)SB(m,a)\phi(\eta). \tag{22}$$

Theorem (1.2.3)[15]: Let $x\colon M^m\to\bar{M}, m\ge 3$, be an isometric immersion of a complete noncompact manifold M in a Hadamard manifold .M with sectional curvature satisfying $-\kappa^2\le K_{\bar{M}}\le 0$, for some constant k. In the case $K_{\bar{M}}\not\equiv 0$, assume further that the first eigenvalue of the Laplace–Beltrami operator of M satisfies

$$\lambda_1(M)>\frac{(m-1)^2}{m}(\kappa^2-\inf|H|^2).$$

There exists a positive constant Λ such that if $\|\Phi\|_{L^m(M)}<\Lambda$ then there is no nontrivial L^2-harmonic 1-form on M. Furthermore, if k = 0 then Λ depends only on m; otherwise, Λ depends only on $m,k,\lambda_1(M)$, and $\inf|H|$.

It is a natural question to ask about the best constant Λ in Theorem (1.2.3). In the next result we present an explicit value for Λ depending on the case.

Proof. We will prove the existence of a positive constant Λ such that if

$\|\Phi\|_{L^m(M)}<\Lambda$ then $\mathcal{H}^1(M)=\{0\}$. Choose $0<d<\frac{1}{2}$

$,a\ =\ a(d)>0, and\ \Lambda=\Lambda(d)>0$ satisfying:

$$\begin{cases} d+(m\ -1)(1+d)d<\dfrac{m}{m-1}, \\ \dfrac{a(m-2)\sqrt{m(m-1)}}{2m}<(m-1)d, \\ SB(m,a)\Lambda^2<(m-1)d. \end{cases} \tag{23}$$

Now we set

$-\bar{C}=-\bar{C}(m,\Lambda,a,b,c)=c+(1\ +b)SB(m,a)\Lambda^2-\frac{m}{m-1}$, and

$$-\bar{D}=-\bar{D}(m,\Lambda,a)=\ A(m,a)+SB(m,a)\Lambda^2. \tag{24}$$

Using (19) and choosing $0 < c < d$ and $0 < b < d$ we get:

$$\bar{C} > \frac{m}{m-1} - d - (m-\ 1)(1\ + d)d > 0,$$
$$\bar{D} > (m-\ 1)(1\ - 2d) > 0.$$

Assume that the total curvature of x satisfies $\|\Phi\|_{L^m(M)} \leq \Lambda$. Plugging the above choices in (21) we obtain

$$\bar{C}\int_M \eta^2|\nabla|\omega||^2 + \bar{D}\int_M |H|^2\eta^2|\omega|^2$$
$$\leq \bar{E}\int_M |\omega|^2|\nabla\eta|^2 + (m-1)k^2\int_M \eta^2|\omega|^2, \tag{25}$$

where $\bar{E} = \frac{1}{c} + \left(1+\frac{1}{b}\right)SB(m,a)\Lambda^2$.

In particular, if $k = 0$, we obtain

$$\bar{C}\int_M \eta^2|\nabla\omega|^2 + \bar{D}\int_M |H|^2\eta^2|\omega|^2 \leq \bar{E}\int_M |\omega|^2|\nabla\eta|^2. \tag{26}$$

We will see later that this inequality is sufficient to prove our result in the case where $k = 0$. In this case, we also note that $\Lambda = \Lambda(\gamma)$ depends only on m. In order to deal with the case where $k \neq 0$, we need to introduce a new ingredient. We recall that the first eigenvalue $\lambda_1 = \lambda_1(M)$ of the Laplacian of M satisfies

$$\lambda_1\int_M \varphi^2 \leq \int_M |\nabla\phi|^2, \tag{27}$$

for all $\varphi \in C_0^\infty(M)$. Applying (27) with $\varphi = \eta|\omega|$ and once more using the Cauchy–Schwarz inequality we get for all $e > 0$

$$\lambda_1\int_M \eta^2|\omega|^2 \leq (1\ + e)\int_M \eta^2|\nabla\omega|^2 + \left(1+\frac{1}{e}\right)\int_M |\omega|^2|\nabla\eta|^2,$$

which implies that

$$\frac{\bar{C}\,\lambda_1}{1+e}\int_M \eta^2|\omega|^2 \leq \bar{C}\int_M \eta^2\big|\nabla|\omega|\big|^2 + \frac{\bar{C}}{e}\int_M |\omega|^2|\nabla\eta|^2. \tag{28}$$

Thus, using (25) and (28), we obtain

$$\frac{\bar{C}\,\lambda_1}{1+e}\int_M \eta^2|\omega|^2 \leq \bar{C}((m-1)k^2\bar{D}\inf|H|^2)\int_M \eta^2|\omega|^2\left(\bar{E}+\frac{\bar{C}}{e}\right)\int_M |\omega|^2|\nabla\eta|^2.$$

Note that

$$\frac{\bar{C}\,\lambda_1}{1+e} - ((m-1)k^2\bar{D}\inf|H|^2) \geq \left(\frac{m}{m-1} - d - (m-1)(1\ + d)d\right)\frac{\lambda_1}{1+e}$$
$$-(m-1)k^2\bar{D}\inf|H|^2). \tag{29}$$

Thus, if $\lambda_1 > \frac{(m-1)^2}{m}(k^2 -\ \inf|H|^2)$ then we can choose d and e, sufficiently small and depending on $m, k^2, \lambda_1,$ and $\inf|H|^2$, so that. $\frac{\bar{C}\,\lambda_1}{1+e} - (m-1)k^2 + \bar{D}\inf|H|^2 > 0$.

Hence we get

$$\int_M |\omega|^2\eta^2 \leq \bar{F}\int_M |\omega|^2|\nabla\eta|^2, \tag{30}$$

for some constant .$\bar{F} > 0$. In this case ($k \neq 0$), the constant Λ depends on m, k^2, λ_1, and $\inf |H|^2$.
For each $r > 0$, let B_r denote the geodesic ball of radius r on M centered at some fixed point and let $\eta_r \in C_0^\infty(M)$ be a smooth function such that

$$\begin{cases} 0 \leq \eta_r \leq 1 \text{ in} M, \\ \eta_r = 1 \text{ } in \text{ } B_r \text{ },\quad |\nabla \eta_r| \leq 2r^{-1} \text{ } in \text{ } M, \\ supp(\eta_r) = B_{2r}. \end{cases}$$

If $k = 0$ we use (26) with η_r to obtain

$$\bar{C} \int_{B_r} |\nabla|\omega||^2 + \bar{D} \int_{B_r} |H|^2|\omega|^2 \leq \bar{E} \frac{4}{r^2} \int_M |\omega|^2 .$$

Taking $r \to \infty$we get $|H||\omega| = |\nabla|\omega|| = 0$. Thus $|\omega|$ is constant. If ω is not identically zero then $H = 0$. In this case, since .M is a Hadamard manifold it is well known that M has infinite volume, which is a contradiction, since$\int_M |\omega|^2 < \infty$. If $k \neq 0$ then using (30) with η_r we get

$$\int_{B_r} |\omega|^2 \leq \bar{F} \frac{4}{r^2} \int_M |\omega|^2 .$$

Taking $r \to \infty$we have $\omega = 0$ and it finishes the proof.
Theorem (1.2.4)[15]: Let $S = S(m, 2)$ be the constant of Sobolev's inequality derived from [29]. Then the constant Λ as in Theorem (1.2.3)can be given explicitly as follows:
(i)If $k = 0$ and $H = 0$ then

$$\Lambda = \frac{m}{(m-1)\sqrt{S}}.$$

(ii) If $k = 0$ and H is arbitrary then

$$\Lambda = \frac{\frac{m}{(m-1)\sqrt{S}}}{\sqrt{1 + \frac{(m-1)(m-2)^2}{4(m^2 - 3m + 1)}}}.$$

(iii) If $k \neq 0$ and $H = 0$ then

$$\Lambda = \frac{m}{(m-1)\sqrt{S}} \sqrt{1 - \frac{(m-1)^2\kappa^2}{m\lambda_1(M)}}.$$

(iv) If $k \neq 0$ and $\inf |H| > \left(1 - \frac{m}{(m-1)^2}\right)^{-\frac{1}{2}} |k|$ then

$$\Lambda = \frac{\frac{m}{(m-1)\sqrt{S}}}{\sqrt{1 + \frac{(m-2)(m-1)^2}{4(m^2 - 3m + 1)} - \frac{(m-1)^2 k^2}{\inf |H|^2}}}.$$

(v) If $k \neq 0, H$ is arbitrary and $\lambda_1(M) > \frac{(m-1)^2 k^2}{m}$ then

$$\Lambda = \frac{\frac{m}{(m-1)\sqrt{S}}\sqrt{1-\frac{(m-1)^2k^2}{m\lambda_1(M)}}}{\sqrt{1+\frac{(m-2)(m-1)^2}{4\left(m^2-3m+1+\frac{(m-1)^2k^2}{\lambda_1(M)}\right)}}}.$$

(vi) If $k \neq 0, \inf|H| \leq \left(1-\frac{m}{(m-1)^2}\right)^{-\frac{1}{2}} |k| and\ \lambda_1 \leq \frac{(m-1)^2k^2}{m}$ then

$$\Lambda = \frac{m}{(m-1)\sqrt{S}} \frac{\sqrt{1-B\frac{(m-2)^2(m-1)}{4m}\inf|H|^2}}{\left(1+\frac{1}{B}\right)(\lambda_1(M)+\inf|H|^2},$$

where $B = -1+\sqrt{1+\frac{4mA}{(m-1)(m-2)^2\inf|H|^2}}$ and $A = \lambda_1 - \frac{(m-1)^2}{m}\ (k^2 - \inf|H|^2)$.

Proof. The key to the proof of Theorem (1.2.3)is the fact that there exists a constant $\Lambda > 0$ such that if $\|\Phi\|_{L^m}(M) < \Lambda$ then one of the inequalities (26) or (30) holds. We also proved that Λ depends only on m in the case where $k = 0$ and Λ depends on m, k^2, λ_1, and $\inf|H|$ in the case where $k \neq 0$. The goal is to give explicit estimates to Λ.

Using (19) and (24) we have

$$-\bar{C}(m,\Lambda,a,0,0) = S\Lambda^2B(m,a) - \frac{m}{m-1}$$
$$= S\Lambda^2\left(1+\frac{(m-2)}{2a}\sqrt{\frac{m}{m-1}}\right)\frac{m-1}{m} - \frac{m}{m-1}$$

and

$$-\bar{D}(m,\Lambda,a) = A(m,a) + S\Lambda^2B(m,a)$$
$$= \frac{m-1}{m}\left(-m+\frac{a(m-2)}{2}\sqrt{\frac{m}{m-1}}+S\Lambda^2\left(1+\frac{(m-2)}{2a}\sqrt{\frac{m}{m-1}}\right)\right).$$

Thus, using (24) and the continuity of $.\bar{C}$, it follows that there exist $\Lambda > 0$ and $a > 0$satisfying $.\bar{C} = \bar{C}(m,\Lambda,a,b,c) > 0$, for some $b > 0$ and $c > 0$, sufficiently small, if and only if

$$S\Lambda^2 < f_1(a) := \frac{\frac{m^2}{(m-1)^2}}{1+\frac{m-2}{2a}\sqrt{\frac{m}{m-1}}}. \tag{31}$$

Note that the function $f_1 : (0,\infty) \to \mathbb{R}$ is increasing and $\sup f_1 = \frac{m^2}{(m-1)^2}$. Assume that

$$\|\Phi\|_{L^m}(M) < \Lambda_1 := \frac{m}{(m-1)\sqrt{S}}$$

and take $\Lambda > 0$ so that $\|\Phi\|_{L^m}(M) < \Lambda < \Lambda_1$. Since $S\Lambda^2 < \sup f_1$ and $\phi(\eta) < \Lambda^2$, there exists $a_1 = a_1(m,\Lambda) > 0$ such that, for any $a > a_1$, there exist $b > 0$ and $c > 0$, sufficiently small, satisfying $C > \bar{C} > 0$ (see (22) and (24)). Thus, if $k = 0$ and $H = 0$, we obtain from (21) that inequality (26) holds. Thus, item (i) is proved.

Similarly, there exist$\Lambda > 0$ and$a > 0$ such that $.D = .D(m,\Lambda,a) > 0$ if and only if

$$S\Lambda^2 < f_2(a) := \frac{m - \frac{a(m-2)}{2}\sqrt{\frac{m}{m-1}}}{1 + \frac{(m-2)}{2a}\sqrt{\frac{m}{m-1}}}. \tag{32}$$

The function $f_2: (0, \infty) \to \mathbb{R}$ is concave and $\max f_2 = f_2\left(\sqrt{\frac{m}{m-1}}\right) = \frac{m}{m-1} > f_{21}\left(\sqrt{\frac{m}{m-1}}\right) = \frac{2m}{(m-1)^2}$. Thus, the maximum value $\Lambda^2 > 0$ that satisfies $.\bar{C} > 0$ and $\bar{D} > 0$, for any $0 < \Lambda < \Lambda^2$, and for some $a > 0, b > 0$, and $c > 0$, is obtained in the intersection point a_{12} of the graphs of f_1 and f_2. Namely,

$$a_{12} = \frac{2m(m^2 - 3m + 1)}{(m-2)(m-1)^2\sqrt{\frac{m}{m-1}}}.$$

Hence we set

$$\Lambda^2 := \sqrt{\frac{f_2(a_{12})}{S}} = \frac{m}{(m-\ 1)\sqrt{S}}\left(1 + \frac{(m-2)^2(m-1)}{4(m^2 - 3m + 1)}\right)^{-\frac{1}{2}}.$$

If $\|\Phi\|_{L^m}(M) < \Lambda < \Lambda^2$, then, taking $a = a_{12}$, we have $C > \bar{C} > 0$ and $D > \bar{D} > 0$, for suitable constants $b > 0$ and $d > 0$. Therefore, for $k = 0$ and H is arbitrary, inequality (26) holds. Thus, item (ii) is proved.

Now, we deal with the case where $k \neq 0$ and $H = 0$. Since we are assuming $\lambda_1 > \frac{(m-1)^2}{m}(k^2 - inf\ |H|^2)$ we immediately obtain that $\lambda_1 > 0$ and $0 < \frac{(m-1)^2}{m}\frac{k^2}{\lambda_1} < 1$.
Using (24) and the continuity of $.\bar{C}$, we see that there exist$\Lambda > 0$ and $a > 0$ satisfying $\bar{C} > 0$ and. $\frac{\bar{C}\lambda_1}{1+e} - (m-1)k^2 > 0$, for some $e > 0, b > 0$, and $c > 0$, sufficiently small, if and only if $\Lambda < \Lambda_1$ and

$$S\Lambda^2 < f_3(a) := \frac{m^2}{(m\ -1)^2}\frac{1 - \frac{(m-1)^2}{m}\frac{k^2}{\lambda_1}}{1 + \frac{(m-2)}{2a}\sqrt{\frac{m}{m-1}}}. \tag{33}$$

The function $f_3: (0, \infty) \to \mathbb{R}$ is increasing and $\sup f_3 = \frac{m^2}{(m-1)^2}\left(1 - \frac{(m-1)^2}{m}\frac{k^2}{\lambda_1}\right)$.

Set $\Lambda_3 := \frac{m}{(m-\ 1)\sqrt{S}}\left(1 + \frac{(m-1)^2}{m}\frac{k^2}{\lambda_1}\right)^{-\frac{1}{2}}$ and suppose that $\|\Phi\|_{L^m}(M) < \Lambda < \Lambda_3$. Since $S\Lambda^2 < \sup f_3 < S\Lambda_3^2 < S\Lambda_1^2$and $\phi(\eta) < \Lambda^2$, there exists $a_3 = a_3(m, k, \lambda_1, \Lambda) > 0$ 0such that, for any $a > a_3$, there exist $e > 0, b > 0$, and $c > 0$, sufficiently small, satisfying $C > \bar{C} > 0$ and.$\frac{\bar{C}\lambda_1}{1+e} - (m-1)k^2 > 0$. Thus, in the case $k \neq 0$ and $H = 0$inequality (30) holds and hence item (iii) follows.
Now, we assume that H is arbitrary. Using (24) we have that there exist $\Lambda > 0$ and $a > 0$ such that $\bar{C} > 0, \bar{D} > 0$, and. $\frac{\bar{C}\lambda_1}{1+e} - (m-1)k^2 + \bar{D}\inf|H|^2 > 0$, for some $e > 0, b > 0$, and $c > 0$, sufficiently small, if and only if $\Lambda < \Lambda^2$ and

$S\Lambda^2 < f_4(a)$:

$$= \frac{\frac{m^2}{(m-1)^2}\left(\lambda_1 - \frac{(m-1)^2}{m}(k^2 - \inf|H|^2)\right) - \frac{a(m-2)}{2}\inf|H|^2\sqrt{\frac{m}{m-1}}}{(\lambda_1 + \inf|H|^2)\left(1 + \frac{(m-2)}{2a}\sqrt{\frac{m}{m-1}}\right)}.$$

By a simple computation, we can rewrite f_4 in the following way:

$$f_4 = \frac{\inf|H|^2}{\lambda_1 + \inf|H|^2} f_2 + \frac{1}{\lambda_1 + \inf|H|^2}\left(\lambda_1 - \frac{(m-1)^2}{m}k^2\right) f_1$$

$$= f_1 + \frac{\inf|H|^2}{\lambda_1 + \inf|H|^2}(f_2 - f_1) - \frac{\frac{(m-1)^2}{m}\ k^2}{\lambda_1 + \inf|H|^2} f_1 \qquad (34)$$

$$= f_2 + \frac{\lambda_1}{\lambda_1 + \inf|H|^2}(f_1 - f_2) - \frac{\frac{(m-1)^2}{m}\ k^2}{\lambda_1 + \inf|H|^2} f_1. \qquad (35)$$

Using (34) we obtain that f_4 intersects f_1 if and only if $\inf|H|^2 > \left(1 - \frac{m}{(m-2)^2}\right)^{-1} k^2$. In this case, the intersection point of f_1 and f_4 is

$$a_{14} = \frac{2m\left(m^2 - 3m + 1 - \frac{(m-1)^2k^2}{\inf|H|^2}\right)}{(m-2)(m-1)^2\sqrt{\frac{m}{m-1}}} < a_{12}.$$

Using that $f_1 < f_2$ in $(0, a_{12})$, we obtain from (35) that $f_4 < f_2$ in $(0, a_{12})$. Thus, the maximum value $\Lambda_4 > 0$ such that .$\bar{C} > 0, \overline{D} > 0$, and. $\frac{\bar{C}\lambda_1}{1+e} - (m-1)k^2 + \overline{D}\inf|H|^2 > 0$, for all $0 < \Lambda < \Lambda_4$ and for some $a > 0, b > 0, c > 0$, and $e > 0$ is obtained considering

$$S\Lambda_4^2 = f_4(a_{14}) = \frac{m^2}{(m-1)^2}\left(1 + \frac{(m-2)^2(m-1)^2}{4m(m^2-3m+1) - \frac{(m-1)^2k^2}{\inf|H|^2}}\right)^{-1}. \qquad (36)$$

Thus, assume that $\inf|H|^2 > \left(1 - \frac{m}{(m-1)^2}\right)^{-1} k^2$ and $\|\Phi\|_{L^m}(M) < \Lambda_4$ Take $\Lambda > 0$ so that $\|\Phi\|_{L^m}(M) < \Lambda < \Lambda_4$ and let $a = a_{14}$. We obtain that $C > \bar{C} > 0$, $D > \overline{D} > 0$,And . $\frac{\bar{C}\lambda_1}{1+e} - (m-1)k^2 + \overline{D}\inf|H|^2 > 0$, for some $e > 0, b > 0$, and $c > 0$, sufficiently small. This implies that the inequality (30) holds. Item (iv) is proved.

Now, assume that $\inf|H|^2 \le \left(1 - \frac{m}{(m-1)^2}\right)^{-1} k^2$. This implies that $f_4 < f_1$, since $f_4$does not intersect f_1 and $\lim_{a\to\infty} f_4(a) < \lim_{a\to\infty} f_1(a)$. Using (35) we obtain that f_4 intersects f_2 if and only if $\lambda_1 > 0$. In this case, the intersection point of $f_4$and f_2 is

$$a_{24} = \frac{2m}{(m-2)\sqrt{\frac{m}{m-1}}}\left(1 - \frac{m}{(m-1)^2}\frac{k^2}{\lambda_1}\right).$$

Note also that $f_2(a) > 0$ if and only if $0 < a < \frac{2m}{(m-2)\sqrt{\frac{m}{m-1}}}$. This implies that $f_4(a_{24}) > 0$ if and only if$\lambda_1 > \frac{(m-1)^2}{m}k^2$. Thus, assume further that $\lambda_1 > \frac{(m-1)^2}{m}k^2$ and$\|\Phi\|_{L^m(M)} <$

$\Lambda_5 := \sqrt{\frac{f_4(a_{24})}{S}}$. Take $\Lambda > 0$ so that $\|\Phi\|_{L^m(M)} < \Lambda < \Lambda_4$ and let$a = a_{24}$. Using that $\phi(\eta) < \Lambda_2$ and $S\Lambda^2 < f_2(a_{24}) = f_4(a_{24}) < f_1(a_{24})$ we obtain that$C > \bar{C} > 0, D > \bar{D} > 0$, and. $\frac{\bar{C}\lambda_1}{1+e} - (m-1)c^2 + \bar{D}\inf|H|^2 > 0$, for some $e > 0, b > 0$, and $c > 0$, sufficiently small. This implies that inequality (30) holds. Item (v) is proved.

To finish the proof of Theorem (1.2.4) we assume that $k \neq 0$, inf $|H|^2 \leq \left(1 - \frac{m}{(m-1)^2}\right)^{-1} k^2$, and $\lambda_1 \leq \frac{(m-1)^2}{m} k^2$. This implies that $f_4 < f_1$ and $f_4 < f_2$. Furthermore, it holds that inf $|H| > 0$, since

$$\frac{(m-1)^2}{m} (k^2 - \lambda_1 + \inf|H|^2) < \lambda_1 \leq \frac{(m-1)^2}{m} k^2.$$

Thus, the maximum value Λ_6 such that $\bar{C} > 0, \bar{D} > 0$, and $\frac{\bar{C}\lambda_1}{1+e} - (m-1)k^2 + \bar{D}\inf|H|^2 > 0$, for some suitable constants $a > 0, b > 0, c > 0$, and $e > 0$, is obtained considering $S\Lambda_6^2 = \sup f_4$. Note also that sup $f_4 = f_4(a_4)$, where

$$a_4 = \frac{(m-2)}{2}\sqrt{\frac{m}{m-1}}\mathcal{B},$$

$$\mathcal{B} = -1 + 1 + \frac{4mA}{(m-1)(m-2)^2 \inf|H|^2}$$

And

$$A = \lambda_1 - \frac{(m-1)^2}{m}(k^2 - \inf|H|^2).$$

Thus, if

$$\|\Phi\|_{L^m} < \sqrt{\frac{f_4(a_4)}{S}} = \frac{m}{(m-1)\sqrt{S}}\sqrt{\frac{1 - \mathcal{B}\frac{(m-2)^2(m-1)}{4m}\inf|H|^2}{\left(1 + \frac{1}{B}\right)(\lambda_1 + \inf|H|^2)}},$$

Then, using similar arguments to the previous case and taking$a = a_4$, we obtain suitable positive constants b, c, and e such that $C > \bar{C} > 0, D > \bar{D} > 0$, and . $\frac{\bar{C}\lambda_1}{1+e} - (m-1)k^2 + \bar{D}\inf|H|^2 > 0$. This implies that inequality (30) holds. Item (vi) is proved.

Theorem (1.2.5)[15]: Let $x: M^m \to \bar{M}, m \geq 3$, be an isometric immersion of a complete noncompact manifold M in a Hadamard manifold $\bar{M}$ with sectional curvature satisfying $-k^2 \leq K_{\bar{M}} \leq 0$, for some constant k. In the case $K_{\bar{M}} \not\equiv 0$, assume further that the first eigenvalue of the Laplace–Beltrami operator of M satisfies

$$\lambda_1(M) > \frac{(m-1)^2}{m}\left(k^2 - \lim_{r(p)\to\infty} \inf|H(p)|^2\right),$$

where r stands for the distance in M from a fixed point. If x has finite total curvature, then the space $\mathcal{H}^1(M)$has finite dimension.

An interesting result of Anderson [12] shows that, for all $m \geq 3$ and$\kappa > m-2$, there exists a complete simply connected manifold M_κ^m with sectional curvature satisfying$-\kappa^2 \leq K \leq -1$ and such that $\dim\mathcal{H}^1(M_\kappa^m) = \infty$. Since .$\bar{M} = M_\kappa^m \times \mathbb{R}$ is a Hadamard manifold satisfying $-\kappa^2 \leq K_{\bar{M}} \leq 0$, and M_κ^m is a totally geodesic submanifold of M, we conclude that the hypothesis on the first eigenvalue in Theorem (1.2.5) is necessary.

In [35] Carron proved a gap theorem on the dimension of $\mathcal{H}^1(M)$.
More precisely, Carron proved that there exists a constant $\varepsilon(m)$ such that if $|\mathbb{I}| \leq \varepsilon(m)$, then all spaces of L^2-harmonic forms are trivial. In the next result we prove a gap theorem for immersion with small $\|\Phi\|_{L^m(M)}$. This result is also a generalizationof [37].

Proof. Consider $0 < d < \frac{1}{2}$, $a = a(d) > 0$, and $\Lambda = \Lambda(a) > 0$ as given in (23). Assume that the immersion *x* has finite total curvature. Fix $r_0 > 0$ so that

$$\|\Phi\|_{L^m}(M)\left(M - B_{r_0}\right)\sqrt{\Lambda}. \tag{37}$$

Let $\eta = \eta_{r_0} \in C_0^\infty(M)$ be any smooth function with compact support satisfying $\mathrm{supp}(\eta) \subset M - B_{r_0}$. Since $\phi(\eta) = \|\Phi\|_{L^m(\mathrm{supp}(\eta))}^2 < \Lambda$, we can proceed similarly as in (inequality (25)) to obtain

$$C\int_M \eta^2 \left|\nabla|\omega|\right|^2 + \bar{D}\int_M |H|^2\eta^2\,|\omega|^2 \leq E\int_M |\omega|^2|\nabla\eta|^2 + (m-1)k^2\int_M \eta^2\,|\omega|^2, \tag{38}$$

for all $\omega \in \mathcal{H}^1(M)$, where $\bar{D} = (m-1)(1-2d)$.
To deal with the case where $k \neq 0$, we assume further that

$$\lambda_1 > \frac{(m-1)^2}{m}k^2 - \lim_{r(p)\to\infty} \inf|H(p)|^2,$$

where r stands for the distance in M from a fixed point. It is easy to see that we can also consider $r_0 > 0$ sufficiently large satisfying

$$\lambda_1 > \frac{(m-1)^2}{m}\left(k^2 - \inf_{M-B_{r_0}} |H|^2\right).$$

We obtain

$$\int_M |\omega|^2\eta^2 \leq \bar{F}\int_M \omega^2\,|\nabla\eta|^2, \tag{39}$$

for some constant $\bar{F} > 0$.
It follows from the Cauchy–Schwarz and Hoffman–Spruck [29] inequalities that

$$\begin{aligned} S^{-1}\left(\int_M (\eta|\omega|)^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} &\leq \int_M |\nabla(\eta|\omega|)|^2 + \int_M \eta^2|\omega|^2|H|^2 \\ &\leq (1+s)\int_M \eta^2\left|\nabla|\omega|\right|^2 + \left(1+\frac{1}{s}\right)\int_M |\omega|^2|\nabla\eta|^2 \\ &\quad + \int_M \eta^2\,|\omega|^2|H|^2, \end{aligned} \tag{40}$$

for any $s > 0$. Using (38) and (40) we obtain

$$\begin{aligned} S^{-1}\left(\int_M (\eta|\omega|)^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} &\leq \left((1+s)C^{-1}\,\bar{D} + 1\right)\int_M |H|^2\eta^2\,|\omega|^2 \\ &\quad + \left(1+\frac{1}{s}\right)\int_M |\omega|^2|\nabla\eta|^2 \end{aligned}$$

$$\leq (1+\ s)C^{-1}(m-1)k^2\int_M \eta^2\,|\omega|^2, \tag{41}$$

for all $s > 0$. Since $m \geq 3$, we can choose d and s sufficiently small so that

$$(1+s)C^{-1}\overline{D}\ = (1+s)\left(\frac{m}{m-1} - d - (m-1)(1+d)d\right)^{-1}(m-1)(2d\ -1)$$
$$< -1. (42)$$

Thus, using (39) (in the case where$k \neq 0$), (41), and (42) we obtain the following inequality:

$$S^{-1}\left(\int_M (\eta|\omega|)^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} \leq \bar{A}\int_M |\omega|^2|\nabla\eta|^2, \tag{43}$$

for some constant $\bar{A} = \bar{A}(m) > 0$, for all $\omega \in \mathcal{H}^1(M)$.

From now on, the proof follows standard techniques (see, for instance, [1] after Eq. (2.7), [27] after Eq. (28), or [33] after Eq. (3.14)) and uses a Moser iteration argument and Lemma 11 of [8]. We include the proof here for the sake of completeness.

Take $r > r_0 + 1$ and let η be a smooth function satisfying the following conditions:

$$\begin{cases} \eta = 0\ in\ B_{r_0} \cup (M - B_{2r}), \\ \eta = 1\ in\ B_r - B_{r_0+1}, \\ |\nabla\eta_r| \leq c_1\ inB_{r_0+1} - B_{r_0}, \\ |\nabla\eta_r| \leq c_1 r^{-1}\ in\ B_{2r} - B_r, \end{cases}$$

for some positive constant c_1. Since $\operatorname{supp}(\eta) \subset M - B_{r_0}$, it follows from (43) that

$$\left(\int_{B_r - B_{r_0+1}} |\omega|^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} \leq \bar{A}\int_{B_{r_0+1}-B_{r_0}} |\omega|^2 + \frac{\bar{A}}{r^2}\int_{B_{2r}-B_r} |\omega|^2$$

Taking $r \to \infty$and using that $|\omega| \in L^2(M)$ we have

$$\left(\int_{M - B_{r_0+1}} |\omega|^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} \leq \bar{A}\int_{B_{r_0+1}-B_{r_0}} |\omega|^2. \tag{44}$$

Using Hölder's inequality we obtain

$$\int_{B_{r_0+2}} |\omega|^2 \leq \operatorname{vol}(B_{r_0+2})\frac{2}{m}\left(\int_{B_{r_0+2}-B_{r_0+1}} |\omega|^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} + \int_{B_{r_0+1}} |\omega|^2$$

Define $F = \left(1 + \bar{A}\operatorname{vol}(B_{r_0+2})^{\frac{2}{m}}\right)$. It follows from (44) that

$$\int_{B_{r_0+2}} |\omega|^2 \leq F\int_{B_{r_0+1}} |\omega|^2. \tag{45}$$

From inequality (14) we have the following:

$$|\omega|\Delta|\omega| \geq \frac{1}{m-1}\left|\nabla|\omega|\right|^2 - \Gamma|\omega|^2, \tag{46}$$

Where $\Gamma\colon M \to [0,\infty)$ is the function given by

$$\Gamma = \left|(m-1)(|H|^2-k^2)-\frac{m-1}{m}|\Phi|^2-\frac{(m-2)\sqrt{m(m-1)}}{m}\ |H||\Phi|.\right|$$

Fix $x \in M$ and take $\zeta \in C_0^1(B_1(x))$. Multiplying both sides of (46) by $\zeta^2|\omega|^{p-2}$, with $p \geq 2$, and integrating by parts we obtain

$$-2\int_{B_1(x)} \zeta|\omega|^{p-1}\langle\nabla\zeta,\nabla|\omega|\rangle$$

$$\geq \left(p-1+\frac{1}{m-1}\right)\int_{B_1(x)} |\omega|^{p-2}\zeta^2|\nabla|\omega|^2-\int_{B_1(x)} \Gamma\zeta^2|\omega|^p. \qquad (47)$$

Using the Cauchy–Schwarz inequality (with $\varepsilon = m-1$) we have

$$-2\zeta|\omega|^{p-1}\langle\nabla\zeta,\nabla|\omega|\rangle = 2\left\langle -|\omega|^{\frac{p}{2}}\nabla\zeta,|\omega|^{\frac{p}{2}-1}\zeta\nabla|\omega|\right\rangle$$

$$\leq (m-1)|\omega|^p|\nabla\zeta\ |^2+\frac{1}{m-1}|\omega|^{p-2}\zeta^2\ \nabla|\omega|^2.$$

Applying this inequality in (47) we obtain

$$(p-1)\int_{B_1(x)} |\omega|^{p-2}\zeta^2\,|\nabla|\omega|^2 \leq \int_{B_1(x)} \Gamma\zeta^2|\omega|^p+(m-1)\int_{B_1(x)} |\omega|^p\,|\nabla\zeta|^2. \qquad (48)$$

Using the Cauchy–Schwarz inequality (with $\varepsilon = 1/2$) we have

$$\int_{B_1(x)} \left|\nabla\left(\zeta|\omega|^{\frac{p}{2}}\right)\right|^2 \leq (p+1)\int_{B_1(x)} |\omega|^p\,|\nabla\zeta|^2$$

$$+\frac{p}{4}(p+1)\int_{B_1(x)} |\omega|^{p-2}\,\zeta^2\big|\nabla|\omega|\big|^2. \qquad (49)$$

Thus, using (48) and (49), we obtain

$$\int_{B_1(x)} \left|\nabla\left(\zeta|\omega|^{\frac{p}{2}}\right)\right|^2 \leq \int_{B_1(x)} A\Gamma\zeta^2|\omega|^p+\mathcal{B}|\omega|^p|\nabla\zeta|^2, \qquad (50)$$

where $\mathcal{A} = \frac{p}{4}(p+1)(p-1)^{-1}\ \leq p$and

$$\mathcal{B} = p+1+(m-1)\mathcal{A} \leq 1+mp.$$

In particular, $A, B \leq 2mp,$ since $p \geq 2$. Applying the Hoffmann-Spruck inequality [29] to the function $\zeta|\omega|^{p/2}$ and using (50) we get

$$S^{-1}\left(\int_{B_1(x)} \left(\zeta|\omega|^{\frac{p}{2}}\right)^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} \leq \int_{B_1(x)} \left|\nabla\left(\zeta|\omega|^{\frac{p}{2}}\right)\right|^2+|H|^2\left(\zeta|\omega|^{\frac{p}{2}}\right)^2$$

$$\leq \int_{B_1(x)} \left((A\Gamma+|H|^2)\zeta^2+\mathcal{B}|\nabla\zeta|^2\right)|\omega|^p.$$

For simplicity, we write*e*

$$\left(\int_{B_1(x)} \left(\zeta|\omega|^{\frac{p}{2}}\right)^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} \leq 2mpS\int_{B_1(x)} (G\zeta^2+|\nabla\zeta|^2)\,|\omega|^p. \qquad (51)$$

Where $G = \Gamma+|H|^2$.

Given an integer $k \geq 0$, we set $p_k = \frac{2m^k}{(m-2)^k}$ and $\rho_k = \frac{1}{2} + \frac{1}{2^{k+1}}$. Take a function $\zeta_k \in C_0^\infty(B_{\rho_k}(x))$ satisfying: $0 \leq \zeta_k \leq 1, \zeta_k = 1$ in $B\rho_{k+1}(x)$ and $|\nabla \zeta_k| \leq 2^{k+3}$. Using (51) with $p = p_k$ and $= \zeta k$, we obtain

$$\left(\int_{B_{\rho_{k+1}}(x)} |\omega|^{p_{k+1}}\right)^{\frac{1}{p_{k+1}}} \leq (2mp_k S)^{\frac{1}{p_k}} \int_{B_{\rho_k}(x)} \left((4^{k+3}+G)|\omega|^{p_k}\right)^{\frac{1}{p_k}}$$

$$\leq \left(2m\rho_k S 4^k \left(4^3 + \sup_{B_1(x)} G\right)\right)^{\frac{1}{\rho_k}} \left(\int_{B_{\rho_k}(x)} |\omega|^{p_k}\right)^{\frac{1}{p_k}}$$

$$\leq (\rho_k)^{\frac{1}{\rho_k}} (4^{k+k_0})^{\frac{1}{\rho_k}} \left(\int_{B_{\rho_k}(x)} |\omega|^{p_k}\right)^{\frac{1}{p_k}},$$

Where k_0 is an integer such that$2mS(4^3 + \sup_{B_1(x)} G) \leq 4^{k_0}$. By recurrence we obtain

$$\|\omega\|_{L^{p_{k+1}}}(B_{\frac{1}{2}}(x)) \leq \prod_{e=0}^{k} p_\ell^{\frac{1}{p_\ell}} 4^{\frac{\ell}{p_\ell}} 4^{\frac{k_0}{p_\ell}} \|\omega\|_{L^2(B_1(x))}.$$

Notice that $p_\ell^{\frac{1}{p_\ell}}, 4^{\frac{\ell}{p_\ell}} \leq B^{\frac{\ell}{2}a^\ell}$, and $4^{\frac{k_0}{p_\ell}} \leq B^{ba^\ell}$, where $a = (m-2)/m$ and B, b are suitable positive constants. Thus

$$\prod_{e=0}^{\infty} p_\ell^{\frac{1}{p_\ell}} 4^{\frac{\ell}{p_\ell}} 4^{\frac{k_0}{p_\ell}} \leq B^{\Sigma \ell^{a^\ell(\ell+b)}} < D.$$

Where $D > 0$ depends only on *m* and $\sup_{B_1}(x)G$. Taking $k \to \infty$, we obtain

$$\|\omega\|_{L^\infty\left(B_{\frac{1}{2}}(x)\right)} \leq D\|\omega\|_{L^2(B_1(x))}. \tag{52}$$

Now, take $y \in \bar{B}_{r_0+1}$ so that $\sup_{B_{r_0+1}} |\omega|^2 = |\omega(y)|^2$. since $B_1(y) \subset B_{r_0+2}$, using (52), we obtain

$$\sup_{B_{r_0+1}} |\omega|^2 \leq D\|\omega\|^2_{L^2(B_1(y))} \leq D\|\omega\|^2_{L^2(B_{r_0+2})}.$$

Thus, from (45), we have

$$\sup_{B_{r_0+1}} |\omega|^2 \leq \mathcal{E}\|\omega\|^2_{L^2(B_{r_0+1})}, \tag{53}$$

for all $\omega \in \mathcal{H}^1(M)$, where $\mathcal{E} > 0$ depends on $m, vol(B_{r_0+2})$, and $\sup_{B_{r_0+2}} G$.

Finally, let $\mathcal{V}$ be any finite-dimensional subspace of $\mathcal{H}^1(M)$. According to Lemma 11 of [8] there exists $\omega \in \mathcal{V}$ such that

$$\frac{\dim V}{vol(B_{r_0+1})} \|\omega\|^2_{L^2(B_{r_0+1})} \leq \sup_{B_{r_0+1}} |\omega|^2 (\min\{m, \dim V\}). \tag{54}$$

Using (53) and (54), we have that $\dim \mathcal{V} \leq c_0$, where$c_0$ depends only on $m, vol(B_{r_0+2})$, and $\sup_{B_{r_0+2}} G$. This implies that $\mathcal{H}^1(M)$ has finite dimension. This concludes the proof of Theorem (1.2.5).

Chapter 2
Hardy Inequality and Square Roots of Elliptic Second Order Divergence Operators

We develop a geometric framework for Hardy inequality on a bounded domain when the functions do vanish only on a closed portion of the boundary. We obtain similar results for perturbations of constant coefficients operators. The methods rely on a singular integral representation, Calderón-Zygmund theory and quadratic estimates. A feature of this study is the use of a commutator between the resolvent of the Laplacian (Dirichlet and Neumann) and partial derivatives which carries the geometry of the boundary.

Section (2.1): Functions Vanishing on a Part of the Boundary

Hardy's inequality is one of the classical items in analysis [65], [80]. Two milestones among many others in the development of the theory seem to be the result of Necas [79] that Hardy's inequality holds on strongly Lipschitz domains and the insight of Maz'ya [76], [77] that its validity depends on measure theoretic conditions on the domain. The geometric framework in which Hardy's inequality remains valid was enlarged up to the frontiers of what is possible – as long as the boundary condition is purely Dirichlet, see [63], [66], compare also [79], [69], [86]. Moreover, over the last years it became manifest that Hardy's inequality plays an eminent role in modern PDE theory, see e.g. [78], [83], [85], [89], [54], [61], [70], [72], [81], [84].

What has not been treated systematically is the case where only a part D of the boundary of the underlying domain Ω is involved, reflecting the Dirichlet condition of the equation on this part – while on $\partial\Omega\setminus D$ other boundary conditions may be imposed, compare [78], [84], [87], [62], [64]. We set up a geometric framework for the domain Ω and the Dirichlet boundary part D that allow to deduce the corresponding Hardy inequality

$$\int_{\Omega} \left|\frac{u}{dist_D}\right|^p dx \leq c \int_{\Omega} |\nabla u|^p \, dx.$$

As in the well established case $D = \partial\Omega$ we in essence only require that D is l-thick in the sense of [66]. This condition can be understood as an extremely weak compatibility condition between D and $\partial\Omega\setminus D$.

We reduce to the case $D = \partial\Omega$ by purely topological means, provided two major tools are applicable: An extension operator $\mathfrak{E}: W_D^{1,p}(\Omega) \to W_D^{1,p}(\mathbb{R}^d)$, the subscript D indicating the subspace of those Sobolev functions which vanish on D in an appropriate sense, and a Poincar$\acute{e}$ inequality on $W_D^{1,p}(\Omega)$. This abstract result is established. The partly implicit conditions are substantiated by more geometric assumptions that can be checked – more or less – by appearance. In particular, we prove that under the mere assumption that D is closed, every linear continuous extension operator $W_D^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^d)$ that is constructed by the usual procedure of gluing together local extension operators preserves the Dirichlet condition on D. This result even carries over to higher-order Sobolev spaces and sheds new light on some of the deep results on Sobolev extension operators obtained in [80].

It is ask, whether Hardy's inequality also characterizes the space $W_D^{1,p}(\Omega)$, i.e. whether the latter is precisely the space of those functions $u \in W^{1,p}(\Omega)$ for which $u/dist_D$belongs to $L^p(\Omega)$. Under very mild geometric assumptions we answer this question to the affirmative.

We attend to the naive intuition that the part of $\partial\Omega$ that is far away from D should only be circumstantial for the validity of Hardy's inequality and in fact we succeed to weaken the previously discussed geometric assumptions considerably.

We work in Euclidean space $\mathbb{R}^d, d \geq 1$. We use x, y, etc. for vectors in $\mathbb{R}^d$ and denote the open ball in $\mathbb{R}^d$ around x with radius r by $B(x,r)$. The letter c is reserved for generic constants that may change their value from occurrence to occurrence. Given $F \subset \mathbb{R}^d$ we write $dist_F$ for the function that measures the distance to F and diam(F) for the diameter of F.

We denote the underlying domain and its Dirichlet part by Ω and D. The various side results that are interesting in themselves and drop off on the way are identified by the use of Λ and E instead.

We introduce the common first-order Sobolev spaces of functions 'vanishing' on a part of the closure of the underlying domain that are most essential for the formulation of Hardy's inequality.

Definition (2.1.1)[39]: If Λ is an open subset of $\mathbb{R}^d$ and E is a closed subset of $\overline{\Lambda}$, then for $p \in [1,\infty[$ the space $W_E^{1,p}(\Lambda)$ is defined as the completion of

$$C_E^\infty(\Lambda) := \{v_{|\Lambda} : v \in C_0^\infty(\mathbb{R}^d), supp(v) \cap E = \emptyset\}$$

with respect to the norm $v \mapsto \left(\int_\Lambda |\nabla v|^p + |v|^p \, dx\right)^{\frac{1}{p}}$. More generally, for $k \in \mathbb{N}$ we define $W_E^{k,p}(\Lambda)$ as the closure of $C_E^\infty(\Lambda)$ with respect to the norm $\mapsto \left(\int_\Lambda \sum_{j=0}^k \left|D^j v\right|^p dx\right)^{\frac{1}{p}}$.

The situation we have in mind is of course when $\Lambda = \Omega$ and $E = D$ is the Dirichlet part D of the boundary $\partial\Omega$.

As usual, the Sobolev spaces $W^{k,p}(\Lambda)$ are defined as the space of those $L^p(\Lambda)$ functions whose distributional derivatives up to order k are in $L^p(\Lambda)$, equipped with the natural norm. Note that by definition $W_0^{k,p}(\Lambda) = W_{\partial(\Lambda)}^{k,p}$ but in general $W_{\emptyset(\Lambda)}^{k,p} \subsetneq W^{k,p}(\Lambda)$, cf. [77]

The following version of Hardy's inequality for functions vanishing on a part of the boundary is our main result.

Theorem (2.1.2)[39]: (A special Hardy inequality) Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $p \in]1,\infty[$. Let $D \subset \partial\Omega$ be l-thick for some $l \in]d-p,d]$ and assume that for every x $\in \overline{\partial\Omega\setminus D}$ there is an open neighborhood U_x of x such that $\Omega \cap U_x$ is a $W^{1,p}$-extension domain. Then there is a constant $c > 0$ such that

$$\int_\Omega \left|\frac{u}{dist_D}\right|^p dx \leq c \int_\Omega |\nabla u|^p \, dx, \qquad u \in W_D^{1,p}(\Omega).$$

Still, as we believe, the abstract framework traced out by the second and the third condition of Theorem (2.1.13) has the advantage that other sufficient geometric conditions for Hardy's inequality – tailor-suited for future applications – can be found much more easily. In fact the second condition is equivalent to the validity of Poincaré's inequality

$$\|u\|_{L^p} \leq c\|\nabla u\|_{L^p(\Omega)}, \qquad u \in W_D^{1,p}(\Omega),$$

that is clearly necessary for Hardy's inequality (4). We give a detailed discussion of Poincaré's inequality, see [88]. Concerning the third condition note carefully that we require the extension operator to preserve the Dirichlet boundary condition on D. Whereas extension of Sobolev functions is a well-established business, the preservation of traces is much more delicate.

We that under geometric assumptions very similar to those in Theorem (2.1.2) the space $W_D^{1,p}(\Omega)$ is the largest subspace of $W^{1,p}(\Omega)$in which Hardy's inequality can hold. This is made precise by the third main result.

We give the proof of the general Hardy inequality from Theorem

We recall the notions from geometric measure theory that are used to describe the regularity of the Dirichlet part D in Hardy's inequality. For $l \in]0,\infty[$ the l-dimensional Hausdorff measure of $F \subset \mathbb{R}^d$ is

$$\mathcal{H}_l(F) := \liminf_{\delta\to 0}\left\{\sum_{j=1}^{\infty} diam(F_j)^l \ : \ F_j \subset \mathbb{R}^d, diam(F_j) \leq \delta, F \subset \bigcup_{j=1}^{\infty} F_j\right\}$$

and its centered Hausdorff content is defined by

$$\mathcal{H}_l^\infty(F) := inf\left\{\sum_{j=1}^{\infty} r_j^l \ : \ x_j \in F, r_j > 0, F \subset \bigcup_{j=1}^{\infty} B(x_j, r_j)\right\}.$$

Definition (2.1.3)[39]: Let $l \in]0,\infty[$. A non-empty compact set $F \subset \mathbb{R}^d$ is called l-thick if there exist $R > 0$ and $\gamma > 0$ such that

$$\mathcal{H}_l^\infty\big(F \cap B(x,r)\big) \geq \gamma\, r^l \tag{1}$$

holds for all $x \in F$ and all $r \in]0,R]$. It is called l-set if there are two constants $c_0, c_1 > 0$ such that

$$c_0 r^l \leq \mathcal{H}_l(F \cap B(x,r)) \leq c_1 r^l$$

holds for all $x \in F$ and all $r \in]0,1]$. See ([71],[68])

Definition (2.1.4)[39]: A set $F \subset \mathbb{R}^d$ is porous if for some $\kappa \leq 1$ the following statement is true: For every ball $B(x,r)$ with $x \in \mathbb{R}^d$ and $0 < r \leq 1$ there is $y \in B(x,r)$ such that $B(y,\kappa r) \cap F = \emptyset$.

$$\int_{B(x,r)} dist(x,F)^{t-d}\, dx \leq c_t r^t, \quad x \in F, r > 0.$$

In particular, each l-set, $l \in]0,d[$, has Aikawa dimension equal to l and thus is porous [67].

For a later use we include a proof of the following two elementary facts. We remark that the first lemma is also implicit in [82].

Lemma (2.1.5)[39]: Let $l \in]0,\infty[$. If $F \subset \mathbb{R}^d$ is a compact l-set, then there are constants $c_0, c_1 > 0$ such that

$$c_0 r^l \leq \mathcal{H}_l^\infty(F \cap B(x,r)) \leq c_1 r^l$$

holds for all $r \in]0,1[$ and all $x \in F$. In particular, F is l-thick.

Proof. We prove $\mathcal{H}_l^\infty(A) \leq \mathcal{H}_l(A) \leq c\mathcal{H}_l^\infty(A)$ for all non-empty Borel subsets $A \subset F$.

First, fix $\varepsilon > 0$ and let $\{A_j\}_{j\in\mathbb{N}}$ be a covering of A by sets with diameter at most ε. If $A_j \cap A \neq \emptyset$, then A_j is contained in an open ball B_j centered in A and radius such that $r_j^l = diam(A_j)^l + \varepsilon 2^{-j}$. The so-obtained countable covering $\{B_j\}$ of A satisfies

$$\sum_{\substack{j\in\mathbb{N}\\ A_j\cap A\neq\emptyset}} diam(A_j)^l \geq \sum_{\substack{j\in\mathbb{N}\\ A_j\cap A\neq\emptyset}} (r_j^l - \varepsilon 2^{-j}) \geq \mathcal{H}_l^\infty(A) - \varepsilon.$$

Taking the infimum over all such coverings $\{A_j\}_{j\in\mathbb{N}}$ and passing to the limit $\varepsilon \to 0$ afterwards, $\mathcal{H}_l^\infty(A) \le \mathcal{H}_l(A)$ follows. Conversely, let $\{B_j\}_{j\in\mathbb{N}}$ be a covering of A by open balls with radii r_j centered in A. If $r_j \le 1$, then $\mathcal{H}_l(F \cap B_j) \le c r_j^l$ since by assumption F is an l-set, and if $r_j > 1$, then certainly $\mathcal{H}_l(F \cap B_j) \le \mathcal{H}_l(F) r_j^l$. Note carefully that $0 < \mathcal{H}_l(F) < \infty$ holds for F can be covered by finitely many balls with radius 1 centered in F. Altogether,

$$\sum_{j=1}^{\infty} r_j^l \ge c\sum_{j=1}^{\infty} \mathcal{H}_l(F \cap B_j) \ge c\mathcal{H}_l\left(F \cap \bigcup_{j=1}^{\infty} B_j\right) \ge c\mathcal{H}_l(A).$$

Passing to the infimum, $\mathcal{H}_l^\infty(A) \ge c\mathcal{H}_l(A)$ follows.

Lemma (2.1.6)[39]: If $F \subset \mathbb{R}^d$ is l-thick, then it is m-thick for every $m \in]0, l[$.

Proof. Inspecting the definition of thick sets, the claim turns out to be a direct consequence of the inequality

$$\sum_{j=1}^{N} r_j^m \ge \left(\sum_{j=1}^{N} r_j^l\right)^{\frac{m}{l}}$$

for positive real numbers $r_1, \dots, r_N$.

The results rely on deep insights from potential theory and we shall recall the necessary notions beforehand, see [77].

Definition (2.1.7)[39]: Let $\alpha > 0, p \in]1, \infty[$ and let $F \subset \mathbb{R}^d$. Denote by $G_\alpha := \mathcal{F}^{-1}\left((1 + |\xi|^2)^{-\frac{\alpha}{2}}\right)$ the Bessel kernel of order α. Then

$$C_{\alpha,p}(F) := inf \left\{\int_{\mathbb{R}^d} |f|^p \ : \ f \ge 0 \ on \ \mathbb{R}^d \ and \ G_\alpha * f \ge 1 \ on \ F\right\}$$

is called (α, p)-capacity of F. The corresponding Bessel potential space is
$H^{\alpha,p}(\mathbb{R}^d) := \{G_\alpha * f : f \in L^p(\mathbb{R}^d)\}$ $with\ norm \parallel G_\alpha * f \parallel_{H^{\alpha,p}(\mathbb{R}^d)} = \parallel f \parallel_p$.

It is well-known that for $k \in \mathbb{N}$ the spaces $H^{k,p}(\mathbb{R}^d)$ and $W^{k,p}(\mathbb{R}^d)$ coincide up to equivalent norms [83]. The capacities $C_{\alpha,p}$ are outer measures on $\mathbb{R}^d$ [77]. A property that holds true for all x in some set $E \subset \mathbb{R}^d$ but those belonging to an exceptional set $F \subset E$ with $C_{\alpha,p}(F) = 0$ is said to be true (α, p)-quasieverywhere on E, abbreviated (α, p)-q.e. A property that holds true (α, p)-q.e. also holds true (β, p)-q.e. if $\beta < 0$. This is an easy consequence of [77]. A more involved result in this direction is the following [77]

Lemma (2.1.8)[39]: Let $\alpha, \beta > 0$ and $1 < p, q < \infty$ be such that $\beta q < \alpha p < d$. Then each $C_{\alpha,p}$-nullset also is a $C_{\beta,q}$-nullset

There is also a close connection between capacities and Hausdorff measures, see [77] for an exhaustive discussion. Most important for us is the following comparison theorem. In the case $p \in]1, d]$ this is proved in [77] and if $p \in]d, \infty[$, then the result follows directly from [77].

Theorem (2.1.9)[39]: (Comparison Theorem) Let $1 < p < \infty$ and suppose $\alpha, l > 0$ are such that $d - l < \alpha p < \infty$. Then every $C_{\alpha,p}$-nullset is also a $\mathcal{H}_l$- and thus a $\mathcal{H}_l^\infty$-nullset.

Bessel capacities naturally occur when studying convergence of average integrals for Sobolev functions. In fact, if $\alpha > 0, p \in]1, \frac{d}{\alpha}]$ and $u \in H^{\alpha,p}(\mathbb{R}^d)$, then (α, p)-quasievery $y \in \mathbb{R}^d$ is a Lebesgue point for u in the L^p-sense, that is

$$\lim_{r\to 0} \frac{1}{|B(y,r)|} \int_{B(y,r)} u(x)\, dx = : u(y) \tag{2}$$

and

$$\lim_{r\to 0} \frac{1}{|B(y,r)|} \int_{B(y,r)} |u(x) - u(y)|^p\, dx = 0 \tag{3}$$

hold [77]. The (α, p)-quasieverywhere defined function u reproduces u within its $H^{\alpha,p}$-class. It gives rise to a meaningful (α, p)-quasieverywhere defined restriction $u|_E := u|_E$ of u to E whenever E has non-vanishing (α, p)-capacity. For convenience we agree upon that $u|_E = 0$ is true for all $u \in H^{\alpha,p}(\mathbb{R}^d)$ if E has zero (α, p)-capacity. Note also that these results remain true if $p \in]\frac{d}{\alpha}, \infty[$, since in this case u has a Hölder continuous representative u which then satisfies (2) and (3) for every $y \in \mathbb{R}^d$.

We obtain an alternate definition for Sobolev spaces with partially vanishing traces.

Definition (2.1.10)[39]: Let $k \in \mathbb{N}, p \in]1, \infty[$ and $E \subseteq \mathbb{R}^d$ be closed. Define

$$\mathcal{W}_E^{k,p}(\mathbb{R}^d) := \{u \in \mathcal{W}^{k,p}(\mathbb{R}^d) : D^\beta u|_E = 0 \; holds (k - |\beta|, p) - q.e. on\, E \; for\, all\, multiindices\, \beta, 0 \le |\beta| \le k - 1\}$$

and equip it with the $\mathcal{W}^{k,p}(\mathbb{R}^d)$-norm.

The following theorem of Hedberg and Wolff is also called (k, p)-synthesis.

Theorem (2.1.11)[39]: ([77]) The spaces $\mathcal{W}_E^{k,p}(\mathbb{R}^d)$ and $\mathcal{W}_E^{k,p}(\mathbb{R}^d)$ coincide whenever $k \in \mathbb{N}, p \in]1, \infty[$ and $E \subset \mathbb{R}^d$ is closed.

Hedberg and Wolff's theorem manifests the use of capacities in the study of traces of Sobolev functions. However, if one invests more on the geometry of E, e.g. if one assumes that it is an l-set, then by the subsequent recent result of Brewster, Mitrea, Mitrea and Mitrea capacities can be replaced by the l-dimensional Hausdorff measure at each occurrence.

Theorem (2.1.12)[39]: ([80]) Let $k \in \mathbb{N}, p \in]1, \infty[$ and let $E \subset \mathbb{R}^d$ be closed and additionally an l-set for some $l \in]d - p, d]$. Then

$$\mathcal{W}_E^{k,p}(\mathbb{R}^d) = \mathcal{W}_E^{k,p}(\mathbb{R}^d) = \{u \in \mathcal{W}^{k,p}(\mathbb{R}^d) : D^\beta u|_E = 0 \; holds\, \mathcal{H}_{d-1} - a.e. on\, E\, for\, all\, multiindices\, \beta, 0 \le |\beta| \le k - 1\},$$

where on the right-hand side $D^\beta u|_E = 0$ means, as before, that for $\mathcal{H}_{d-1}$-almost every $y \in E$ the average integrals $\frac{1}{|B(y,r)|} \int_{B(y,r)} D^\beta u(x)\, dx$ vanish in the limit $r \to 0$.

Theorem (2.1.13)[39]: Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $D \subset \partial\Omega$ be a closed part of the boundary and $p \in]1, \infty[$. Suppose that the following three conditions are satisfied.

(i) The set D is l-thick for some $l \in]d - p, d]$.

(ii) The space $W_D^{1,p}(\Omega)$ can be equivalently normed by $\|\nabla \cdot\|_{L^p(\Omega)}$.

(iii) There is a linear continuous extension operator $\mathfrak{E}: W_D^{1,p}(\Omega) \to W_D^{1,p}(\mathbb{R}^d)$.

Then there is a constant $c > 0$ such that Hardy's inequality

$$\int_\Omega \left|\frac{u}{dist_D}\right|^p dx \le c \int_\Omega |\nabla u|^p \, dx \qquad (4)$$

holds for all $u \in W_D^{1,p}(\Omega)$.

Of course the conditions (ii) and (iii) in Theorem (2.1.13) are rather abstract and should be supported by more geometrical ones. This will be the content , where we shall give an extensive kit of such conditions. In particular, we will obtain the following version of Hardy's inequality.

Proof. We will deduce Theorem (2.1.13) from the following proposition that states the assertion in the case $D = \partial\Omega$.

Proposition (2.1.14)[39]: ([66], see also [63]) Let $\Omega_\bullet \subseteq \mathbb{R}^d$ be a bounded domain and let $p \in]1,\infty[$. If $\partial\Omega_\bullet$ is l-thick for some $l \in]d-p,d]$, then Hardy's inequality is satisfied for all $u \in W_0^{1,p}(\Omega_\bullet)$, i.e. (4) holds with Ω replaced by $\Omega_\bullet$ and D by $\partial\Omega_\bullet$.

Below we will reduce to the case $D = \partial\Omega$ by purely topological means, so that we can apply Proposition (2.1.14) afterwards. We will repeatedly use the following topological fact.

(■) Let$\{M_\lambda\}_\lambda$be a family of connected subsets of a topological space. If $\bigcap_\lambda M_\lambda \ne \emptyset$, then $\bigcup_\lambda M_\lambda$ is again connected.

As required in Theorem (2.1.13) let now $\Omega \subseteq \mathbb{R}^d$ be a bounded domain and let D be a closed part of $\partial\Omega$. Then choose an open ball $B \supseteq \overline{\Omega}$ that, in what follows, will be considered as the relevant topological space. Consider

$$\mathcal{C} := \{M \subset B \setminus D : M open, connected\ and\ \Omega \subset M\}$$

and for the rest of the proof put

$$\Omega_\bullet := \bigcup_{M\in\mathcal{C}} M .$$

In the subsequent lemma we collect some properties of $\Omega_\bullet$. Our proof here is not the shortest possible, cf. [81] but it has, however, the advantage to give a description of $\Omega_\bullet$ as the union of Ω, the boundary part $\partial\Omega\setminus D$ and those connected components of $B\setminus\overline{\Omega}$ whose boundary does not consist only of points from D. This completely reflects the naive geometric intuition.

Lemma (2.1.15)[39]: It holds $\Omega \subseteq \Omega_\bullet \subseteq B$. Moreover, $\Omega_\bullet$ is open and connected and $\partial\Omega_\bullet = D$ in B.

Proof. The first assertion is obvious. By construction $\Omega_\bullet$ is open. Since all elements from $\mathcal{C}$ contain Ω the connectedness of $\Omega_\bullet$ follows by (■). It remains to show $\partial\Omega_\bullet = D$.

Let $x \in D$. Then x is an accumulation point of Ω and, since $\Omega \subseteq \Omega_\bullet$, also of $\Omega_\bullet$. On the other hand, $x \notin \Omega_\bullet$ by construction. This implies $x \in \partial\Omega_\bullet$ and so $D \subseteq \partial\Omega_\bullet$.

In order to show the inverse inclusion, we first show that points from $\partial\Omega \setminus D$ cannot belong to $\partial\Omega_\bullet$. Indeed, since D is closed, for $x \in \partial\Omega \setminus D$ there is a ball $B_x \subseteq B$ around x that does not intersect D. Since x is a boundary point of Ω , we have $B_x \cap \Omega \ne \emptyset$. Both Ω and B_x are connected, so (■) yields that $\Omega \cup B_x$ is connected. Moreover, this set is open, contains Ω and avoids D, so it belongs to $\mathcal{C}$ and we obtain $\Omega \cup B_x \subseteq \Omega_\bullet$. This in particular yields $x \in \Omega_\bullet$, so $x \notin \partial\Omega_\bullet$ since $\Omega_\bullet$ is open.

Summing up, we already know that $x \in \overline{\Omega}$ belongs to $\partial\Omega_\bullet$ if and only if $x \in D$. So, it remains to make sure that no point from $B\setminus\overline{\Omega}$ belongs to $\partial\Omega_\bullet$.

As $B\setminus\overline{\Omega}$ is open, it splits up into its open connected components $Z_0, Z_1, Z_2, \ldots$. There are possibly only finitely many such components but at least one. We will show in a first step that for all these components it holds $\partial Z_j \subseteq \partial\Omega$. This allows to distinguish the two cases $\partial Z_j \subseteq D$ and $\partial Z_j \cap (\partial\Omega \setminus D) \neq \emptyset$. In Steps 2 and 3 we will then complete the proof by showing that in both cases Z_j does not intersect $\partial\Omega_\bullet$.

Step 1: $\partial Z_j \subseteq \partial\Omega$ for all j.

First note that $\partial Z_j \cap \Omega = \emptyset$ for all j. Indeed, assuming this set to be non-empty and investing that Ω is open, we find that the set $Z_j \cap \Omega$ cannot be empty either and this contradicts the definition of Z_j.

Now, to prove the claim of Step 1, assume by contradiction that, for some , there is a point $x \in \partial Z_j$ that does not belong to $\partial\Omega$. By the observation above we then have $x \notin \overline{\Omega}$ and consequently there is a ball B_x around x that does not intersect $\overline{\Omega}$. Now, the set $B_x \cup Z_j$ is connected thanks to (5), avoids $\overline{\Omega}$ and includes Z_j properly. However, this contradicts the property of Z_j to be a connected component of $B\setminus\overline{\Omega}$.

Step 2: If $\partial Z_j \subseteq D$, then $\overline{\Omega}_\bullet \cap Z_j = \emptyset$.

We first note that it suffices to show $\Omega_\bullet \cap Z_j = \emptyset$. In fact, due to $\overline{\Omega}_\bullet = \partial\Omega_\bullet \cup \Omega_\bullet$ we then get $\overline{\Omega}_\bullet \cap Z_j = \emptyset$ since Z_j is open.

So, let us assume there is some $x \in \Omega_\bullet \cap Z_j$. Then $\Omega_\bullet \cup Z_j$ is connected due to (■). By assumption we have $\partial Z_j \subseteq D$ and by construction the sets Z_j and $\Omega_\bullet$ are both disjoint to D. So we can infer that $\partial Z_j \cap (\Omega_\bullet \cup Z_j) = \emptyset$ and this allows us to write

$$\Omega_\bullet \cup Z_j = \left(\Omega_\bullet \cup Z_j\right) \cap \left(Z_j \cup (B\setminus\bar{Z}_j)\right) = Z_j \cup \left(\Omega_\bullet \cap (B\setminus\bar{Z}_j)\right).$$

This is a decomposition of $\Omega_\bullet \cup Z_j$ into two open and mutually disjoint sets, so if we can show that both are nonempty then this yields a contradiction to the connectedness of $\Omega_\bullet \cup Z_j$ and the claim of Step 2 follows. Indeed, we even find

$$\Omega_\bullet \cap \left(B\setminus\bar{Z}_j\right) = \Omega_\bullet \setminus \bar{Z}_j = \Omega_\bullet \setminus \left(\partial Z_j \cup Z_j\right) \supset \setminus \left(D \cup Z_j\right) = \Omega \neq \emptyset,$$

since both D and Z_j do not intersect Ω.

Step 3: If $\partial Z_j \cap (\partial\Omega\setminus D) \neq \emptyset$, then $Z_j \subseteq \Omega_\bullet$.

Let $x \in \partial Z_j \cap (\partial\Omega \setminus D)$, and let B_x be a ball around x that does not intersect D. The point x is a boundary point of Z_j, so $B_x \cap Z_j \neq \emptyset$ and we obtain that $B_x \cup Z_j$ is connected by (■). By the same argument, also the set $B_x \cup \Omega$ is connected and putting these two together a third reiteration of the argument yields that $(B_x \cup \Omega) \cup (B_x \cup Z_j) = \Omega \cup B_x \cup Z_j$ is again connected. This last set is open and does not intersect D, so it belongs to $\mathcal{C}$ and we end up with $\Omega \cup B_x \cup Z_j \subseteq \Omega_\bullet$. In particular we have $Z_j \subseteq \Omega_\bullet$.

Remark (2.1.16)[39]: Conversely, it can be shown that the asserted properties characterize $\Omega_\bullet$ uniquely in the sense that if an open, connected subset $\Xi \supset \Omega$ of B additionally satisfies $\partial\Xi = D$, then necessarily $\Xi = \Omega_\bullet$. In fact, since $\Xi \cap D = \emptyset$ one has $\Xi \subset \Omega_\bullet$, due to the definition of $\Omega_\bullet$. In order to obtain the inverse inclusion we write

$$\Omega_\bullet = (\Omega_\bullet \cap \Xi) \cup (\Omega_\bullet \cap \partial\Xi) \cup \left(\Omega_\bullet \cap (B\setminus\bar{\Xi})\right) = \Xi \cup \left(\Omega_\bullet \cap (B\setminus\bar{\Xi})\right), \qquad (5)$$

since $\Omega_\bullet \cap \partial\Xi = \Omega_\bullet \cap D = \emptyset$. Both $\Xi = \Xi \cap \Omega_\bullet$ and $\Omega_\bullet \cap (B\setminus\bar{\Xi})$ are open in $\Omega_\bullet$, and $\Xi \supset \Omega$ is non-empty. Since $\Omega_\bullet$ is connected and $\Xi = \Xi \cap \Omega_\bullet$ is clearly disjoint to $\Omega_\bullet \cap (B\setminus\bar{\Xi})$, this latter set must be empty. Thus, (5) gives $\Xi = \Omega_\bullet$.

Corollary (2.1.17)[39]: Consider $\Omega_\bullet$ as a subset of $\mathbb{R}^d$. Then $\Omega_\bullet$ is open and connected. Moreover, either $\partial\Omega_\bullet = D$ or $\partial\Omega_\bullet = D \cup \partial B$.

Proof. It is clear that $\Omega_{\bullet}$ remains open. Assume that $\Omega_{\bullet}$ is not connected. Then there are disjoint open sets $U, V \subseteq \mathbb{R}^d$ such that $\Omega_{\bullet} = U \cup V$. However, the property $\Omega_{\bullet} \subseteq B$ then gives $\Omega_{\bullet} = \Omega_{\bullet} \cap B = (U \cap B) \cup (V \cap B)$, where $U \cap B$ and $V \cap B$ are open in B and disjoint to each other. This contradicts Lemma (2.1.15).

For the last assertion consider an annulus $A \subseteq B$ that is adjacent to ∂B and does not intersect $\overline{\Omega}$. Let Z_j be the connected component of $B \setminus \overline{\Omega}$ that contains A. We distinguish again the two cases of Step 2 and Step 3 in the proof of Lemma (2.1.15): If $\partial Z_j \subseteq D$, we have shown in Step 2 that Z_j is disjoint to $\Omega_{\bullet}$ and this implies $\partial\Omega_{\bullet} = \partial\Omega_{\bullet} \cap B = D$. In the second case, we infer from Step 3 in the above proof that $A \subseteq Z_j \subseteq \Omega_{\bullet}$ and this implies $\partial\Omega_{\bullet} = D \cup \partial B$.

We conclude the proof of Theorem (2.1.13). We first observe that in both cases appearing in Corollary (2.1.17) the set $\partial\Omega_{\bullet}$ is m-thick for some $m \in]d-p, d-1]$. In fact, D is l-thick for some $l \in]d-p, d]$ by assumption and using its local representation as the graph of a Lipschitz function, it can easily be checked that ∂B is a $(d-1)$-set, hence $(d-1)$- thick owing to Lemma (2.1.5). The claim follows from Lemma (2.1.6). Altogether, Proposition (2.1.14) applies to our special choice of $\Omega_{\bullet}$.

Now, let $\mathfrak{E}$ be the extension operator provided by Assumption (iii) of Theorem (2.1.13). In view of Corollary (2.1.17) we can define an extension operator $\mathfrak{E}_{\bullet} : W_D^{1,p}(\Omega) \to W_0^{1,p}(\Omega_{\bullet})$ as follows: If $\partial\Omega_{\bullet} = D$, then we put $\mathfrak{E}_{\bullet} v := \mathfrak{E} v|_{\Omega_{\bullet}}$ and if $\partial\Omega_{\bullet} = D \cup \partial B$, then we choose $\eta \in C_0^\infty(B)$ with the property $\eta \equiv 1$ on $\overline{\Omega}$ and put $\mathfrak{E}_{\bullet} v := (\eta E v)|_{\Omega_{\bullet}}$. This allows us to apply Proposition (2.1.14) to the functions $\mathfrak{E}_{\bullet} u \in W_0^{1,p}(\Omega_{\bullet})$, where u is taken from $W_D^{1,p}(\Omega)$. With a final help of Assumption (ii) in Theorem (2.1.13) this gives

$$\int_\Omega \left|\frac{u}{d_D}\right|^p dx \le \int_\Omega \left|\frac{u}{d_{\partial\Omega_{\bullet}}}\right|^p dx \le \int_{\Omega_{\bullet}} \left|\frac{\mathfrak{E}_{\bullet} u}{d_{\partial\Omega_{\bullet}}}\right|^p dx \le c \int_{\Omega_{\bullet}} |\nabla(\mathfrak{E}_{\bullet} u)|^p \, dx \le c\|\mathfrak{E}_{\bullet} u\|^p_{W_0^{1,p}(\Omega_{\bullet})}$$

$$\le c\|u\|^p_{W^{1,p}_{D(\Omega)}} \le c \int_\Omega |\nabla u|^p \, dx$$

for all $u \in W^{1,p}_{D(\Omega)}$ and the proof is complete.

Finally, instead of its l-thickness we can also require that D is an l-set – a condition that promises to be more common to applications. One access to such a result is to prove that the l-property of $\partial\Omega$ implies the p-fatness of $\mathbb{R}^d \setminus \Omega$ – a result which was first obtained by Maz'ya [78]. Knowing this, Hardy's inequality may then be deduced from the results in [69] or [86]. Our approach is quite different and simply rests on Proposition (2.1.14) and Lemma (2.1.5). So we can record the following.

Corollary (2.1.18)[39]: The assertion of Theorem (2.1.13) remains valid if instead of its l-thickness we require that D is an l-set.

We discuss the second condition in the main result Theorem (2.1.13), that is the extendability for $W_D^{1,p}(\Omega)$ within the same class of Sobolev functions. We develop three abstract principles concerning Sobolev extension.

(a) Dirichlet cracks can be removed: We open the possibility of passing from Ω to another domain $\Omega_\star$ with a reduced Dirichlet boundary part, while $\Gamma = \partial\Omega \setminus D$ remains part of $\partial\Omega_\star$. In most cases this improves the boundary geometry in the sense of Sobolev extendability, see the example in the following Figure.

(b) Sobolev extendability is a local property: We show that only the local geometry of the domain around the boundary part Γ plays a role for the existence of an extension operator.

(c) Preservation of traces: We prove under very general geometric assumptions that the extended functions do have the adequate trace behavior on D for every extension operator.

We believe that these results are of independent interest and therefore decided to directly present them for higher-order Sobolev spaces $W_E^{k,p}$. In the end we review some feasible commonly used geometric conditions which together with our abstract principles really imply the corresponding extendability.

As in Fig. (1) there may be boundary parts which carry a Dirichlet condition and belong to the inner of the closure of the domain under consideration. Then one can extend the functions on Λ by 0 to such a boundary part, thereby enlarging the domain and simplifying the boundary geometry. In the following we make this precise.

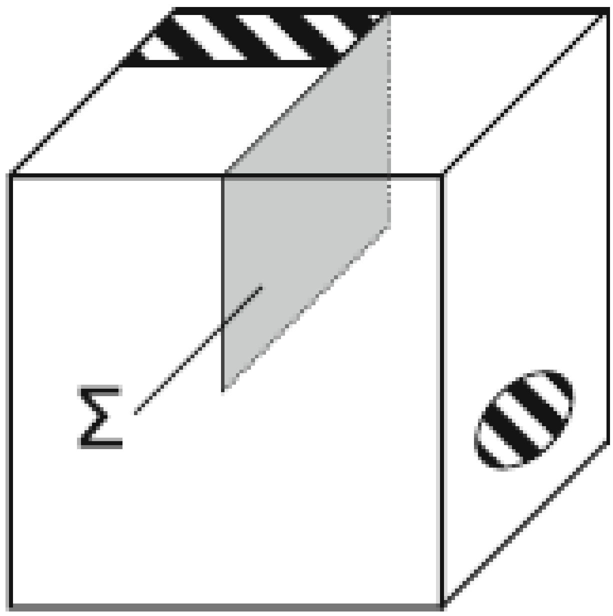


Fig. (1)[39]: the set ∑ does not belong to Ω, and carries-together with the striped parts – the Dirichlet condition

Lemma (2.1.19)[39]: Let $\Lambda \subset \mathbb{R}^d$ be a bounded domain and let $E \subset \partial\Lambda$ be closed. Define $\Lambda_\star$ as the interior of the set $\Lambda \cup E$. Then the following hold true.

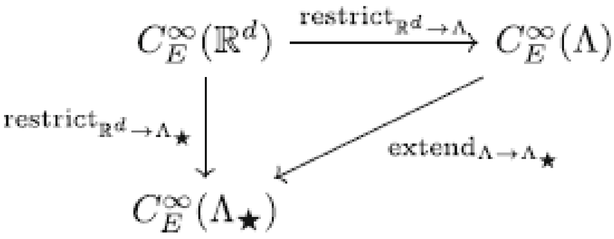
(i) The set $\Lambda_\star$ is again a domain, $\Xi := \partial\Lambda \setminus E$ is a (relatively) open subset of $\partial\Lambda_\star$ and $\partial\Lambda_\star = \Xi \cup (E \cap \partial\Lambda_\star)$.

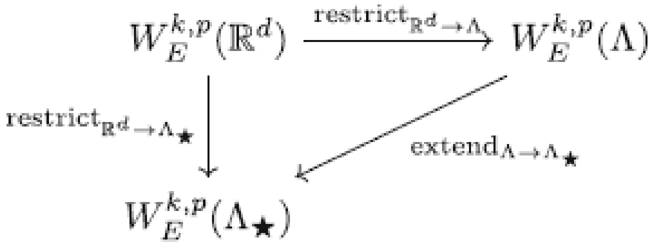
(ii) Let $k \in \mathbb{N}$ and $p \in [1, \infty[$. Extending functions from $W_E^{k,p}(\Lambda)$ by 0 to $\Lambda_\star$,one obtains an isometric extension operator $\mathrm{Ext}(\Lambda, \Lambda_\star)$ from $W_E^{k,p}(\Lambda)$ onto $W_E^{k,p}(\Lambda_\star)$.

Proof. (i) Due to the connectedness of Λ and the set inclusion $\Lambda \subset \Lambda_\star \subset \overline{\Lambda}$, the set $\Lambda_\star$ is also connected, and, hence a domain. Obviously, one has $\overline{\Lambda_\star} = \overline{\Lambda}$. This, together with the inclusion $\Lambda \subset \Lambda_\star$ leads to $\partial\Lambda_\star \subset \partial\Lambda$. Since $\Xi \cap \Lambda_\star = \emptyset$, one gets $\Xi \subset \partial\Lambda_\star$. Furthermore, Ξ was relatively open in $\partial\Lambda$, so it is relatively open also in $\partial\Lambda_\star$.

The last asserted equality follows from $\partial\Lambda_\star = (\Xi \cap \partial\Lambda_\star) \cup (E \cap \partial\Lambda_\star)$ and $\Xi \subset \partial\Lambda_\star$.

(ii) Consider any $\psi \in C_E^\infty(\mathbb{R}^d)$ and its restriction $\psi|_\Lambda$ to Λ. Since the support of ψ has a positive distance to E, one may extend $\psi|_\Lambda$by 0 to the whole of $\Lambda_\star$ without destroying the C^∞-property. Thus, this extension operator provides a linear isometry from $C_E^\infty(\Lambda)$ onto $C_E^\infty(\Lambda_\star)$ (if both are equipped with the $W^{k,p}$-norm). This extends to a linear extension operator $\operatorname{Ext}(\Lambda, \Lambda_\star)$ from $W_E^{k,p}(\Lambda)$ onto $W_E^{k,p}(\Lambda_\star)$, see the two following commutative diagrams:





The above considerations suggest the following procedure: extend the functions from $W_E^{k,p}(\Lambda)$ first to $\Lambda_\star$, and afterwards to the whole of $\mathbb{R}^d$. The next lemma shows that this approach is universal.

Lemma (2.1.20)[39]: Let $k \in \mathbb{N}$ and $p \in [1,\infty[$. Let $\Lambda \subset \mathbb{R}^d$ be a bounded domain, let $E \subset \partial\Lambda$ be closed and as before define $\Lambda_\star$ as the interior of the set $\Lambda \cup E$. Every linear, continuous extension operator $\mathfrak{F} : W_E^{k,p}(\Lambda) \to W_E^{k,p}(\mathbb{R}^d)$ factorizes as $\mathfrak{F} = \mathfrak{F}_\star\, Ext(\Lambda, \Lambda_\star)$ through a linear, continuous extension operator $\mathfrak{F}_\star : W_E^{k,p}(\Lambda_\star) \to W_E^{k,p}(\mathbb{R}^d)$.

Proof. Let $\mathfrak{S}$ be the restriction operator from $W_E^{k,p}(\Lambda_\star)$ to $W_E^{k,p}(\Lambda)$. Then we define, for every $\in W_E^{k,p}(\Lambda_\star), \mathfrak{F}_\star f := \mathfrak{F}\mathfrak{S}f$. We obtain $\mathfrak{F}_\star Ext(\Lambda, \Lambda_\star) = \mathfrak{F}\mathfrak{S}Ext(\Lambda, \Lambda_\star) = \mathfrak{F}$. This shows that the factorization holds algebraically. However, one also has

$$\|\mathfrak{F}_\star Ext(\Lambda, \Lambda_\star) f\|_{W_E^{k,p}(\mathbb{R}^d)} = \|\mathfrak{F}f\|_{W_E^{k,p}(\mathbb{R}^d)} \le \|\mathfrak{F}\|_{\mathcal{L}\left(W_E^{k,p}(\Lambda);\, W_E^{k,p}(\mathbb{R}^d)\right)} \|f\|_{W_E^{k,p}(\Lambda)}$$
$$= \|\mathfrak{F}\|_{\mathcal{L}\left(W_E^{k,p}(\Lambda);\, W_E^{k,p}(\mathbb{R}^d)\right)} \|Ext(\Lambda, \Lambda_\star) f\|_{W_E^{k,p}(\Lambda_\star)}.$$

Having extended the functions already to $\Lambda_\star$ one may proceed as follows: Since E is closed, so is $E_\star := E \cap \partial\Lambda_\star$. So, one can now consider the space $W_{E_\star}^{1,p}(\Lambda_\star)$ and has the task to establish an extension operator for this space – while afterwards one has to take into account that the original functions were 0 also on the set $E \cap \Lambda_\star$ and have not been altered by the extension operator thereon. However, note carefully that $E_\star := E \cap \partial \Lambda_\star$ may have a worse geometry than E. For example, take Fig. (2) and suppose that this time only Σ forms the whole Dirichlet part of the boundary. Then E is a $(d-1)$-set whereas even $\mathcal{H}_{d-1}(E_\star) = 0$ holds.

To sum up, if one aims at an extension operator $\mathfrak{E} : W_E^{k,p}(\Lambda) \to W_E^{k,p}(\mathbb{R}^d)$, one is free to modify the domain Λ to $\Lambda_\star$. In most cases this improves the local geometry concerning Sobolev extensions and we do not have examples where the situation gets worse. Though we do not claim that this is, in a whatever precise sense, the generic case.

Below, we make precise in which sense Sobolev extendability is a local property.

Definition (2.1.21)[39]: A domain $\Lambda \subset \mathbb{R}^d$ is a $W^{k,p}$-extension domain for given $k \in \mathbb{N}$ and $p \in [1,\infty[$ if there exists a continuous extension operator $\mathfrak{E}_{k,p} : W^{k,p}(\Lambda) \to$

$W^{k,p}(\mathbb{R}^d)$. If Λ is a $W^{k,p}$-extension domain for all k ∈ N and all p ∈ [1,∞[in virtue of the same extension operator, then is a universal Sobolev extension domain.

Proposition (2.1.22)[39]: Let $k \in \mathbb{N}$ and $p \in [1,\infty[$. Let Λ be a bounded domain and let E be a closed part of its boundary. Assume that for every $x \in \overline{\partial\Lambda \setminus E}$ there is an open neighborhood U_x of x such that $\Lambda \cap U_x$ is a $W^{k,p}$-extension domain. Then there is a continuous extension operator

$$\mathfrak{E}_{k,p} : W_E^{k,p}(\Lambda) \to W^{k,p}(\mathbb{R}^d).$$

Moreover, if each local extension operator $\mathfrak{E}_x$ maps the space $W_{E_x}^{k,p}(\Lambda \cap U_x)$ into $W_{E_x}^{k,p}(\mathbb{R}^d)$, where $E_x := \overline{E \cap U_x} \subset \partial(\Lambda \cap U_x)$, then also

$$\mathfrak{E}_{k,p} : W_E^{k,p}(\Lambda) \to W_E^{k,p}(\mathbb{R}^d).$$

Proof. For the construction of the extension operator let for every $x \in \overline{\partial\Lambda \setminus E}$ denote U_x the open neighborhood of x from the assumption. Let $U_{x_1}, \ldots, U_{x_n}$ be a finite subcovering of $\overline{\partial\Lambda \setminus E}$. Since the compact set $\overline{\partial\Lambda \setminus E}$ is contained in the open set $\bigcup_j U_{x_j}$, there is an $\varepsilon > 0$, such that the sets $U_{x_1}, \ldots, U_{x_n}$, together with the open set $U := \{y \in \mathbb{R}^d : dist(y, \overline{\partial\Lambda \setminus E}) > \varepsilon\}$, form an open covering of $\overline{\Lambda}$. Hence, on $\overline{\Lambda}$ there is a C_0^∞-partition of unity $\eta, \eta_1, \ldots, \eta_n$, with the properties supp$(\eta) \subset U$, supp$(\eta_j) \subset U_{x_j}$.

Assume $\psi \in C_E^\infty(\Lambda)$. Then $\eta\psi \in C_0^\infty(\Lambda)$. If one extends this function by 0 outside of Λ, then one obtains a function $\phi \in C_{\partial\Lambda}^\infty(\mathbb{R}^d) \subset C_E^\infty(\mathbb{R}^d) \subset W_E^{k,p}(\mathbb{R}^d)$ with the property $\|\phi\|_{W^{k,p}(\mathbb{R}^d)} = \|\eta\psi\|_{W^{k,p}(\Lambda)}$.

Now, for every fixed $j \in \{1, \ldots, n\}$, consider the function $\psi_j := \eta_j\psi \in W^{k,p}(\Lambda \cap U_{x_j})$. Since $\Lambda \cap U_{x_j}$ is a $W^{k,p}$-extension domain by assumption, there is an extension of ψ_j to a $W^{k,p}(\mathbb{R}^d)$-function ϕ_j together with an estimate $\|\phi_j\|_{W^{k,p}(\mathbb{R}^d)} \le c\|\psi_j\|_{W^{k,p}(\Lambda \cap U_{x_j})}$, where c is independent from ψ. Clearly, one has a priori no control on the behavior of ϕ_j on the set $\Lambda \setminus U_{x_j}$. In particular ϕ_j may there be nonzero and, hence, cannot be expected to coincide with $\eta_j\psi$ on the whole of Λ. In order to correct this, let ζ_j be a $C_0^\infty(\mathbb{R}^d)$-function which is identically 1 on supp(η_j) and has its support in U_{x_j}. Then $\eta_j\psi$ equals $\zeta_j\phi_j$ on all of Λ. Consequently, $\zeta_j\phi_j$ really is an extension of $\eta_j\psi$ to the whole of $\mathbb{R}^d$ which, additionally, satisfies the estimate

$$\|\zeta_j \phi_j\|_{W^{k,p}(\mathbb{R}^d)} \le c\|\phi_j\|_{W^{k,p}(\mathbb{R}^d)} \le c\|\eta_j\psi\|_{W^{k,p}(\Lambda \cap U_{x_j})} \le c\|\psi\|_{W^{k,p}(\Lambda)},$$

where c is independent from ψ. Thus, defining $\mathfrak{E}_{k,p}(\psi) = \phi + \sum_j \zeta_j \phi_j$ one gets a linear, continuous extension operator from $C_E^\infty(\Lambda)$ into $W^{k,p}(\mathbb{R}^d)$. By density, $\mathfrak{E}_{k,p}$ uniquely extends to a linear, continuous operator

$$\mathfrak{E}_{k,p} : W_E^{k,p}(\Lambda) \to W_E^{k,p}(\mathbb{R}^d).$$

Finally, assume that the local extension operators map $W_{E_{x_j}}^{k,p}(\Lambda \cap U_{x_j})$ into $W_{E_{x_j}}^{k,p}(\mathbb{R}^d)$. Using the notation above, this means that ϕ_j can be approximated in $W^{k,p}(\mathbb{R}^d)$ by a sequence from $C_{E_{x_j}}^\infty(\mathbb{R}^d)$. Since ζ_j is supported in U_{x_j}, multiplication by $\zeta_j \in C_0^\infty(\mathbb{R}^d)$ maps $C_{E_{x_j}}^\infty(\mathbb{R}^d)$ into $C_E^\infty(\mathbb{R}^d)$ boundedly with respect to the $W^{k,p}(\mathbb{R}^d)$-topology. Hence, $\zeta_j\phi_j \in W_E^{k,p}(\mathbb{R}^d)$. Since in any case $\phi \in W_E^{k,p}(\mathbb{R}^d)$, the conclusion follows.

Proposition (2.1.22) allows to construct Sobolev extension operators from $W_D^{k,p}(\Omega)$ into $W^{k,p}(\mathbb{R}^d)$ and gives a sufficient condition for preservation of the Dirichlet condition. We show that in fact every such extension operator has this feature. Recall that this is the crux of the matter in Assumption (iii) of Theorem (2.1.13). The key lemma is the following.

Lemma (2.1.23)[39]: Let $k \in \mathbb{N}$ and $p \in]1,\infty[$. Let $\Lambda \subset \mathbb{R}^d$ be a domain, let $E \subset \partial\Lambda$ be closed and let $\mathfrak{E}_{k,p}: W_E^{k,p}(\Lambda) \to W^{k,p}(\mathbb{R}^d)$ be a bounded extension operator. Any of the following conditions guarantees that $\mathfrak{E}_{k,p}$ in fact maps into $W_E^{k,p}(\mathbb{R}^d)$.

(i) For (k,p)-quasievery $y \in E$ balls around y in Λ have asymptotically nonvanishing relative volume, i.e.

$$lim \inf_{r\to 0} \frac{|B(y,r) \cap \Lambda)|}{r^d} > 0. \tag{6}$$

(ii) The set E is an l-set for some $l \in]d-p,d]$ and (6) holds for $\mathcal{H}_l$-almost every $y \in E$.

(iii) There exists $q > d$ such that $\mathfrak{E}_{k,p}$ maps $C_E^\infty(\Lambda)$ into $W^{k,q}(\mathbb{R}^d)$.

Proof. As $C_E^\infty(\Omega)$ is dense in $W_E^{k,p}(\Lambda)$ and since $\mathfrak{E}_{k,p}$ is bounded, it suffices to prove that given $v \in C_E^\infty$ the function $u := \mathfrak{E}_{k,p}v$ belongs to $W_E^{k,p}(\mathbb{R}^d)$. The proof of (i) is inspired by [88]. Easy modifications of the argument will yield (ii) and (iii).

(i) Fix an arbitrary multiindex β with $|\beta| \le k-1$. Let $\mathfrak{D}^\beta u$ be the representative of the distributional derivative $D^\beta u$ of u defined $(k-|\beta|,p)$-q.e. on $\mathbb{R}^d$ via

$$\mathfrak{D}^\beta u(y) := \lim_{r\to 0} \frac{1}{|B(y,r)|} \int_{B(y,r)} D^\beta\, u(x)\, dx\,.$$

Recall from (3) that then

$$\lim_{r\to 0} \frac{1}{|B(y,r)|} \int_{B(y,r)} \left| \mathfrak{D}^\beta\, u(x) - \mathfrak{D}^\beta\, u(y)\right| dx$$

$$\le \lim_{r\to 0} \left(\frac{1}{|B(y,r)|} \int_{B(y,r)} \left| \mathfrak{D}^\beta\, u(x) - \mathfrak{D}^\beta\, u(y)\right|^p dx \right)^{\frac{1}{p}} = 0. \tag{7}$$

holds for $(k-|\beta|,p)-q.e.\, y \in \mathbb{R}^d$. Since (6) holds for (k,p)-quasievery $y \in E$, it a fortiori holds for $(k-|\beta|,p)$-quasievery such y. Let now $N \subset \mathbb{R}^d$ be the exceptional set such that on $\mathbb{R}^d \setminus N$ the function $\mathfrak{D}^\beta u$ is defined and satisfies (7) and such that (6) holds for every $y \in E \setminus N$. Owing to Theorem (2.1.11) the claim follows once we have shown$\mathfrak{D}^\beta u(y) = 0$ for all $y \in E \setminus N$.

For the rest of the proof we fix $y \in E\setminus N$. For $r > 0$ we abbreviate $B(r) := B(y,r)$ and define

$$W_j := \{x \in \mathbb{R}^d \setminus N : |\mathfrak{D}^\beta u(x) - \mathfrak{D}^\beta\, u(y)| > 1/j\}. \tag{8}$$

Thanks to (7) for each $j \in \mathbb{N}$ we can choose some $r_j > 0$ such that $|B(r) \cap W_j| < 2^{-j}\, |B(r)|$ holds for all $r \in]0,r_j]$. Clearly, we can arrange that the sequence $\{r_j\}_j$ is decreasing. Now,

$$W := \bigcup_{j\in\mathbb{N}} \left\{ \left(B(r_j) \setminus B(r_{j+1})\right) \cap W_j \right\} \tag{9}$$

has vanishing Lebesgue density at y, i.e. $r^{-d}\,|B(r)\cap W|$ vanishes as r tends to 0: Indeed, if $r\in\,]r_{l+1},r_l]$, then

$$|B(r)\cap W|\le\left|(B(r)\cap W_l)\ \cup\bigcup_{j\ge l+1}\left(B(r_j)\cap W_j\right)\right|\le 2^{-l}|B(r)|+\sum_{j\ge l+1}2^{-j}\,|B(r_j)|$$
$$\le 2^{-l+1}\,|B(r)|.$$

Now, (6) allows to conclude

$$\liminf_{r\to 0}\frac{|B(r)\cap\Lambda\cap(\mathbb{R}^d\setminus W))|}{r^d}>0.$$

Since u is an extension of $v\in C_E^\infty(\Lambda)$ and y is an element of E it holds $\mathfrak{D}^\beta u=0$ a.e. on $B(r)\cap\Lambda$ with respect to the d-dimensional Lebesgue measure if $r>0$ is small enough. The previous inequality gives $|B(r)\cap\Lambda\cap(\mathbb{R}^d\setminus W))|>0$ if $r>0$ is small enough. In particular, there exists a sequence $\{x_j\}_j$ in $\mathbb{R}^d\setminus W$ approximating y such that$\mathfrak{D}^\beta u(x_j)=0$ for all $j\in\mathbb{N}$. Now, the upshot is that the restriction of $\mathfrak{D}^\beta u$ to $\mathbb{R}^d\setminus W$ is continuous at y since if $x\in\mathbb{R}^d\setminus W$ satisfies $|x-y|\le r_j$ then by construction $|\mathfrak{D}^\beta u(x)-\mathfrak{D}^\beta u(y)|\le 1/j$. Hence, $\mathfrak{D}^\beta u(y)=0$ and the proof is complete.

(ii) If E is an l-set for some $l\in\,]d-p,d]$, then we can appeal to Theorem (2.1.12) rather than Theorem (2.1.11) and the same argument as in (i) applies.

(iii) By assumption $u\in W_E^{k,q}(\mathbb{R}^d)$, where $q>d$. By Sobolev embeddings each distributional derivative $D^\beta u,|\beta|\le k-1$, has a continuous representative $\mathfrak{D}^\alpha u.$ As each $y\in E\subset\partial\Lambda$ is an accumulation point of $\Lambda\setminus E$ and since $\mathfrak{D}^\alpha u=\mathfrak{D}^\alpha v$ holds almost everywhere on Λ the representative $\mathfrak{D}^\alpha u$ must vanish everywhere on E and Theorem (2.1.11) yields $u\in W_E^{k,p}(\mathbb{R}^d)$ as required see [60].

We can now state and prove the remarkable result that every Sobolev extension operator that is constructed by localization techniques as in Proposition (2.1.22) preserves the Dirichlet condition.

Proposition (2.1.24)[39]:([56]) If a domain $\Lambda\subset\mathbb{R}^d$ is a $W^{k,p}$-extension domain for some $k\in\mathbb{N}$ and $p\in[1,\infty[$, then it is a d-set.

Theorem (2.1.25)[39]: Let $k\in\mathbb{N}$ and $p\in[1,\infty[$. Let Λ be a bounded domain and let E be a closed part of its boundary. Assume that for every $x\in\overline{\partial\Lambda\setminus E}$ there is an open neighborhood U_x of x such that $\Lambda\cap U_x$ is a $W^{k,p}$-extension domain. Then there exists a continuous extension operator

$$\mathfrak{E}_{k,p}:W_E^{k,p}(\Lambda)\to W_E^{k,p}(\mathbb{R}^d).$$

Proof. According to Proposition (2.1.22) it suffices to check that each local extension operator $\mathfrak{E}_x$ maps $W_{E_x}^{k,p}(\Lambda\cap U_x)$ into $W_{E_x}^{k,p}(\mathbb{R}^d)$, where $E_x:=\overline{E\cap U_x}\subset\partial(\Lambda\cap U_x)$. Owing to Proposition (2.1.24)the $W^{k,p}$-extension domain $\Lambda\cap U_x$ is a d-set and as such satisfies (6) around every of its boundary points. So, Lemma (2.1.23).(i) yields the claim.

We finally review common geometric conditions on the boundary part $\overline{\partial\Lambda\setminus E}$ such that the local sets $\Lambda\cap U_x$ really admit the Sobolev extension property required in Proposition (2.1.22).

A first condition, completely sufficient for the treatment of most real world problems, is the following Lipschitz condition.

Definition (2.1.26)[39]: A bounded domain $\Lambda \subset \mathbb{R}^d$ is called bounded Lipschitz domain if for each $x \in \partial\Lambda$ there is an open neighborhood U_x of x and a bi-Lipschitz mapping φ_x from U_x onto a cube, such that $\varphi_x(\Lambda \cap U_x)$ is the (lower) half cube and $\partial\Lambda \cap U_x$ is mapped onto the top surface of this half cube.

It can be proved by elementary means that bounded Lipschitz domains are $W^{1,p}$ extension domains for every $p \in [1, \infty[$, cf. e.g. [55] for the case $p = 2$. In fact, already the following (ε, δ)-condition of Jones [59] assures the existence of a universal Sobolev extension operator.

Definition (2.1.27)[39]: Let $\Lambda \subset \mathbb{R}^d$ be a domain and $\varepsilon, \delta > 0$. Assume that any two points $x, y \in \Lambda$, with distance not larger than δ, can be connected within Λ by a rectifiable arc γ with length $l(\gamma)$, such that the following two conditions are satisfied for all points z from the curve γ :

$$l(\gamma) \leq \frac{1}{\varepsilon} \| x - y\|, \quad and \quad \frac{\|x - z\|\|y - z\|}{\|x - y\|} \leq \frac{1}{\varepsilon}\, dist(z, \Lambda^c).$$

Then Λ is called (ε, δ)-domain.

Theorem (2.1.28)[39]: (Rogers) Each (ε, δ)-domain is a universal Sobolev extension domain.

Plugging in Rogers extension operator into Theorem (2.1.25) lets us re-discover [80] in case of bounded domains and p strictly between 1 and ∞. We even obtain a universal extension operator that simultaneously acts on all $W_E^{k,p}$ -spaces and at the same time our argument reveals that the preservation of the trace is irrespective of the specific structure of Jones' or Roger's extension operators.

We believe that this sheds some more light also on [80] though the argument cannot disclose the fundamental assertions on the support of the extended functions obtained in [80] by a careful analysis of Jones' extension operator. We summarize our observations in the following theorem.

Theorem (2.1.29)[39]: Let Λ be a bounded domain and let E be a closed part of its boundary. Assume that for every $x \in \overline{\partial\Lambda \setminus E}$ there is an open neighborhood U_x of x such that $\Lambda \cap U_x$ is a bounded Lipschitz or, more generally, an (ε, δ)-domain for some values $\varepsilon, \delta > 0$. Then there exists a universal operator $\mathfrak{E}$ that restricts to a bounded extension operator $W_E^{k,p}(\Lambda) \to W_E^{k,p}(\mathbb{R}^d)$ for each $k \in \mathbb{N}$ and each $p \in]1, \infty[$.

We will discuss sufficient conditions for Poincaré's inequality, thereby unwinding Assumption (ii) of Theorem (2.1.13). Our aim is not greatest generality as e.g. in [77] for functions defined on the whole of $\mathbb{R}^d$, but to include the aspect that our functions are only defined on a domain. Secondly, our interest is to give very general, but in some sense geometric conditions, which may be checked more or less 'by appearance' – at least for problems arising from applications.

The next proposition gives a condition that assures that a closed subspace of $W^{1,p}$ may be equivalently normed by the L^p-norm of the gradient of the corresponding functions only. We believe that this might also be of independent interest, compare also [88]. Throughout 1 denotes the function that is identically one.

Proposition (2.1.30)[39]: Let $\Lambda \subset \mathbb{R}^d$ be a bounded domain and suppose $p \in]1, \infty[$. Assume that X is a closed subspace of $W^{1,p}(\Lambda)$ that does not contain $\mathbb{1}$ and for which the restriction of the canonical embedding $W^{1,p}(\Lambda) \hookrightarrow L^p(\Lambda)$ to X is compact. Then X may be equivalently normed by $v \mapsto \left(\int_\Lambda |\nabla v|^p dx\right)^{\frac{1}{p}}$.

Proof. First recall that both X and $L^p(\Lambda)$ are reflexive. In order to prove the proposition, assume to the contrary that there exists a sequence $\{v_k\}_k$ from X such that

$$\frac{1}{k}\ \|v_k\|_{L^p(\Lambda)} \geq \ \|\nabla v_k\|_{L^p(\Lambda)}.$$

After normalization we may assume $\|v_k\|_{L^p(\Lambda)} = 1$ for every $k \in \mathbb{N}$. Hence, $\{\nabla v_k\}_k$ converges to 0 strongly in $L^p(\Lambda)$. On the other hand, $\{v_k\}_k$ is a bounded sequence in X and hence contains a subsequence $\{v_{k_l}\}_l$ that converges weakly in X to an element $v \in X$. Since the gradient operator $\nabla : X \to L^p(\Lambda)$ is continuous, $\{\nabla v_{k_l}\}_l$ converges to ∇v weakly in $L^p(\Lambda)$. As the same sequence converges to 0 strongly in $L^p(\Lambda)$, the function ∇v must be zero and hence v is constant. But by assumption X does not contain constant functions except for $v = 0$. So, $\{v_{k_l}\}_l$ tends to 0 weakly in X. Owing to the compactness of the embedding $X \hookrightarrow L^p(\Lambda)$, a subsequence of $\{v_{k_l}\}_l$ tends to 0 strongly in $L^p(\Lambda)$. This contradicts the normalization condition $\|v_{k_l}\|_{L^p(\Lambda)} = 1$.

In the case that E is l-thick, the following lemma presents two conditions that are particularly easy to check and entail $\mathbb{1} \notin W_E^{1,p}(\Lambda)$. Some knowledge on the common frontier of E and $\partial\Lambda\backslash E$ is required: Either not every point of E should lie thereon or $\partial\Lambda$ must not be too wild around this frontier.

Lemma (2.1.31)[39]: Let $p \in]1,\infty[$, let Λ be a bounded domain and let $E \subset \partial\Lambda$ be l-thick for some $l \in]d - p, d]$. Both of the following conditions assure $\mathbb{1} \notin W_E^{1,p}(\Lambda)$.

(i) The set E admits at least one relatively inner point x. Here, ‘relatively inner’ is with respect to $\partial\Lambda$ as ambient topological space.

(ii) For every $x \in \overline{\partial\Lambda\backslash E}$ there is an open neighborhood U_x of x such that $\Lambda \cap U_x$ is a $W^{1,p}$-extension domain.

Proof. We treat both cases separately.

(i) Assume the assertion was false and $\mathbb{1} \in W_E^{1,p}(\Lambda)$. Let x be the inner point of E from the hypotheses and let $B := B(x,r)$ be a ball that does not intersect $\partial\Lambda \setminus E$. Put $\frac{1}{2}B := B(x,\frac{r}{2})$ and let $\eta \in C_0^\infty(B)$ be such that $\eta \equiv 1$ on $\frac{1}{2}B$. We distinguish whether or not x is an interior point of $\overline{\Lambda}$.

First, assume it is not. For every $\psi \in C_E^\infty(\Lambda)$ the function $\eta\psi$ belongs to $W_0^{1,p}(\Lambda \cap B)$ and as such admits a $W^{1,p}$-extension $\widehat{\eta\psi}$ by zero to the whole of $\mathbb{R}^d$. In particular,

$$\widehat{\eta\psi}(y) = \begin{cases} \psi(y), & if \quad y \in \frac{1}{2}B \cap \Lambda \\ 0, & if \quad y \in \frac{1}{2}B \setminus \Lambda \end{cases}$$

and consequently,

$$\left\|\nabla\widehat{\eta\psi}\right\|_{L^p\left(\frac{1}{2}B\right)} = \|\nabla\psi\|_{L^p\left(\frac{1}{2}B\cap\Lambda\right)}.$$

Since by assumption $\mathbb{1}$ is in the $W^{1,p}(\Lambda)$-closure of $C_E^\infty(\Lambda)$ and since the mappings $W_E^{1,p}(\Lambda) \ni \psi \longmapsto \nabla\widehat{\eta\psi} \in L^p(\frac{1}{2}B)$ and $W_E^{1,p}(\Lambda) \ni \psi \longmapsto \nabla\psi \in L^p(\Lambda \cap \frac{1}{2}B)$ are continuous, the previous equation extends to $\psi = \mathbb{1}$:

$$\left\|\nabla\widehat{\eta\,\mathbb{1}}\right\|_{L^p\left(\frac{1}{2}B\right)} = \|\nabla\mathbb{1}\|_{L^p\left(\frac{1}{2}B\cap\Lambda\right)} = 0.$$

On the other hand x is not an inner point of $\overline{\Lambda}$ so that in particular $\frac{1}{2}B \setminus \overline{\Lambda}$ is nonempty. Since this set is open, $|\frac{1}{2}B \setminus \overline{\Lambda}| > 0$. Recall that by construction $\widehat{\eta \mathbb{1}} \in W^{1,p}(B)$ vanishes a.e. on $\frac{1}{2}B \setminus \overline{\Lambda}$. Hence, for some $c > 0$ the Poincaré inequality

$$\|\widehat{\eta \mathbb{1}}\|_{L^p(\frac{1}{2}B)} \leq c \, \|\nabla \widehat{\eta \mathbb{1}}\|_{L^p(\frac{1}{2}B)},$$

holds, cf. [88]. However, we already know that the right hand side is zero, whereas the left hand side equals $|\frac{1}{2}B \cap \Lambda|^{1/p}$, which is nonzero since $\frac{1}{2}B \cap \Lambda$ is nonempty and open – a contradiction.

Now, assume x is contained in the interior of $\overline{\Lambda}$. Upon diminishing B we may assume $B \subset \overline{\Lambda}$. For every $\psi \in C_E^\infty(\mathbb{R}^d)$ we have $\eta\psi \in C_E^\infty(\mathbb{R}^d)$ with an estimate

$$\|\eta\psi\|_{W^{1,p}(\mathbb{R}^d)} \leq c\|\psi\|_{W^{1,p}(B)} = c\left(\int_B |\psi|^p + |\nabla\psi|^p \, dx\right)^{1/p}$$

for some constant $c > 0$ depending only on η and p. By our choice of B split

$$B = B \cap \overline{\Lambda} = (B \cap \Lambda) \cup (B \cap \partial\Lambda) = (B \cap \Lambda) \cup (B \cap E).$$

Since ψ vanishes in a neighborhood of E,

$$\|\eta\psi\|_{W^{1,p}(\mathbb{R}^d)} \leq c\left(\int_{B\cap\Lambda} |\psi|^p + |\nabla\psi|^p \, dx\right)^{1/p} \leq c\|\psi\|_{W^{1,p}(\Lambda)}. \tag{10}$$

Taking into account $\eta \equiv 1$ on $\frac{1}{2}B$, the same reasoning gives

$$\int_{\frac{1}{2}B} |\nabla(\eta\psi)|^p \, dx = \int_{\frac{1}{2}B} |\nabla\psi|^p \, dx \leq \int_\Lambda |\nabla\psi|^p \, dx \, . \tag{11}$$

By assumption there is a sequence $\{\psi_j\}_j \subset C_E^\infty(\Lambda)$ tending to $\mathbb{1}$ in the $W^{1,p}(\Lambda)$- topology. Due to (10) and the choice of η, the sequence $\{\eta\psi_j\}_j \subset C_E^\infty(\mathbb{R}^d)$ then tends to some $u \in W_E^{1,p}(\mathbb{R}^d)$ satisfying $u = 1$ a.e. on $\frac{1}{2}B \cap \Lambda$. Due to (11), $\nabla u = 0$ a.e. on $\frac{1}{2}B$, meaning that u is constant on this set. Since $\frac{1}{2}B \cap \Lambda$ as a non-empty open set has positive Lebesgue measure, all this can only happen if $u = 1$ a.e. on $\frac{1}{2}B$. Hence,

$$\lim_{r\to 0} \frac{1}{|B(y,r)|} \int_{B(y,r)} u \, dx = 1$$

for every $y \in \frac{1}{3}B \cap E$, which by Theorem (2.1.11) is only possible if $C_{1,p}(\frac{1}{3}B \cap E) = 0$. By Theorem (2.1.11) this in turn implies $\mathcal{H}_l^\infty(\frac{1}{3}B \cap E) = 0$ in contradiction to the l-thickness of E.

(ii) Again assume the assertion was false. Then by (i) there exists some $x \in E$ that is not an inner point of E with respect to $\partial\Lambda$. Hence x is an accumulation point of $\partial\Lambda \setminus E$ and by assumption there is a neighborhood $U = U_x$ of x such that $\Lambda \cap U$ is a $W^{1,p}$ extension domain. We denote the corresponding extension operator by $\mathfrak{E}$. We shall localize the assumption $\mathbb{1} \in W_E^{1,p}(\Lambda)$ within U to arrive at a contradiction.

To this end, let $r_0 > 0$ be such that $\overline{B(x,r_0)} \subset U$ and let $\eta \in C_0^\infty(U)$ be such that $\eta \equiv 1$ on $B(x,r_0)$. Then also $\eta = \eta\mathbb{1} \in W_E^{1,p}(\Lambda)$ and in particular $\eta|_{\Lambda\cap U}$ belongs to $W_F^{1,p}(\Lambda\cap U)$, where $F := \overline{B(x,r_0/2) \cap E} \subset \partial(\Lambda\cap U)$. Recall from Proposition (2.1.24)that the $W^{1,p}$-extension domain $\Lambda\cap U$ satisfies in particular

$$\lim inf_{r\to 0} \frac{|B(y,r) \cap \Lambda \cap U)|}{r^d} > 0.$$

around every $y \in \partial(\Lambda\cap U)$. Thus, Lemma (2.1.23)(i) yields $u := \mathfrak{E}(\eta|_{\Lambda\cap U}) \in W_F^{1,p}(\mathbb{R}^d)$.

On the other hand, similar to the proof of Lemma (2.1.23) let $\mathfrak{u}$ be the representative of u that is defined by limits of integral means on the complement of some exceptional set N with $C_{1,p}(N) = 0$ and fix $y \in F \setminus N$. Take W as in (8) and (9). Repeating the arguments in the proof of Lemma (2.1.23) reveals that the restriction of $\mathfrak{u}$ to $\mathbb{R}^d\setminus W$ is continuous at y and that $|B(y,r) \cap \Lambda\cap U \cap (\mathbb{R}^d\setminus W)| > 0$ if $r > 0$ is small enough. By construction $\mathfrak{u} = 1$ a.e. on $B(y,r) \cap \Lambda \cap U \cap (\mathbb{R}^d\setminus W)$ if $r < r_0$. Hence, there is a sequence $\{x_j\}_j$ approximating y such that $\mathfrak{u}(x_j) = 1$ for every $j \in \mathbb{N}$. By continuity $\mathfrak{u}(y) = 1$ follows. This proves that $\mathfrak{u} = 1$ holds $(1,p)$-quasieverywhere on F.

By Theorem (2.1.11) this can only happen if $C_{1,p}(F) = 0$, which as in (i) contradicts the l-thickness of E.

Under the second assumption of Lemma (2.1.31) there exists a linear continuous Sobolev extension operator $\mathfrak{E}: W_E^{1,p}(\Lambda) \to W_E^{1,p}(\mathbb{R}^d)$, see Theorem (2.1.25). Then the compactness of the embedding $W_E^{1,p}(\Lambda) \hookrightarrow L^p(\Lambda)$ is classical and owing to Theorem (2.1.30) we can record the following special Poincar$\acute{e}$ inequality.

Proposition (2.1.32)[39]: Let $p \in]1,\infty[$ and let Λ be a bounded domain. Suppose that $E \subset \partial\Lambda$ is l-thick for some $l \in]d - p, d]$ and that for each $x \in \overline{\partial\Lambda\setminus E}$ there is an open neighborhood U_x of x such that $\Lambda\cap U_x$ is a $W^{1,p}$-extension domain. Then $W_E^{1,p}(\Lambda)$ may equivalently be normed by $v \longmapsto \left(\int_\Lambda |\nabla v|^p dx\right)^{\frac{1}{p}}$.

Now, also Theorem (2.1.2) follows. In fact, this result is just the synthesis of the above proposition with Theorems (2.1.13) and 3.9.

Theorem (2.1.33)[39]: Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $p \in]1,\infty[$. Let $D \subset \partial\Omega$ be porous and l-thick for some $l \in]d - p, d]$. Finally assume that for every x $\in \overline{\partial\Omega\setminus D}$ there is an open neighborhood U_x of x such that $\Omega\cap U_x$ is a $W^{1,p}$-extension domain. If $u \in W^{1,p}(\Omega)$ is such that $u/dist_D \in L^p(\Omega)$, then already $u \in W_D^{1,p}(\Omega)$.

Proof. The strategy of proof is to write u as the sum of $v \in W^{1,p}(\Omega)$ with $v/dist_{\partial\Omega} \in L^p(\Omega)$ and $w \in W^{1,p}$ with support within a neighborhood of $\overline{\partial\Omega \setminus D}$. Then v can be handled by the following classical result.

Proposition (2.1.34)[39]: ([49]) Let $\emptyset \subsetneq \Lambda \subsetneq \mathbb{R}^d$ be open and let $p \in]1,\infty[$. Then if $u \in W^{1,p}(\Lambda)$ and $u/\,dist_{\partial\Lambda} \in L^p(\Lambda)$, it follows $u \in W_0^{1,p}(\Lambda)$.

For w we can – since local extension operators are available – rely on the techniques developed. A key observation is an intrinsic relation between the property $\frac{u}{dist_D} \in L^p(\Omega)$ and Sobolev regularity of the function $\log(dist_D)$. In fact, a formal computation gives

$$\nabla\big(u\,log(dist_D)\big) = log(dist_D)\nabla u + \frac{u}{dist_D}\,\nabla\,dist_D\,.$$

Details are carried out in the following five consecutive steps.

Step 1: Splitting u and handling the easy term as in the proof of Proposition (2.1.22) for every $x \in \overline{\partial\Omega \setminus D}$, let U_x be the open neighborhood of x from the assumption, let $U_{x_1}, \dots, U_{x_n}$ be a finite subcovering of $\overline{\partial\Omega \setminus D}$ and let $\varepsilon > 0$ be such that the sets $U_{x_1}, \dots, U_{x_n}$, together with $U := \{y \in \mathbb{R}^d : dist(y, \overline{\partial\Omega \setminus D}) > \varepsilon\}$, form an open covering of $\overline{\Omega}$. Finally, let $\eta, \eta_1, \dots, \eta_n$ be a subordinated C_0^∞ -partition of unity. The described splitting is $u = v + w$, where $v := \eta u$ and $w := \sum_{j=1}^{n} \eta_j\, u = (1 - \eta)u$. Since

$$dist_{\partial\Omega}(x) \geq min\{\varepsilon, dist_D(x)\} \geq min\{\varepsilon\, diam(\Omega)^{-1}, 1\} \cdot dist_D(x)$$

holds for every $x \in supp(\eta) \cap \Omega$, the function $v \in W^{1,p}(\Omega)$ satisfies

$$\int_\Omega \left|\frac{v}{dist_{\partial\Omega}}\right|^p dx \leq c \int_\Omega \left|\frac{v}{dist_D}\right|^p dx \leq c \int_\Omega \left|\frac{u}{dist_D}\right|^p dx < \infty$$

by assumption on u. Now, Proposition (2.1.34) yields $v \in W_0^{1,p}(\Omega) \subset W_D^{1,p}(\Omega)$.

Step 2: Extending w by assumption the sets $\Omega \cap U_{x_j}$, $1 \leq j \leq n$, are $W^{1,p}$-extension domains. Since $w = (1 - \eta)u$, where $(1 - \eta)$ has compact support in the union of these domains, an extension $\widehat{w} \in W^{1,p}(\mathbb{R}^d)$ of $w \in W^{1,p}(\Omega)$ with compact support within $\bigcup_{j=1}^{n} U_{x_j}$ can be constructed just as in the proof of Proposition (2.1.22). Now, if we can show $w \in W_D^{1,p}(\Omega)$, then by Step 1 also $u = v + w$ belongs to this space.

Step 3: Estimating the trace of $\widehat{w}$

To prove $\widehat{w} \in W_D^{1,p}(\mathbb{R}^d)$ we rely once more on the techniques used in the proof of Lemma (2.1.23). So, let $\widehat{\mathfrak{w}}$ be the representative of $\widehat{w}$ defined on $\mathbb{R}^d \setminus N$ via

$$\widehat{\mathfrak{w}}(y) := \lim_{r \to 0} \frac{1}{|B(y,r)|} \int_{B(y,r)} \widehat{w}\, dx ,$$

where the exceptional set N is of vanishing $(1,p)$-capacity. Put

$$U_\star := \bigcup_{j=1}^{n} U_{x_j}, \quad \Omega_\star := \Omega \cap U_\star, \quad and \quad D_\star = \overline{D \cap U_\star} \subseteq \partial\Omega_\star.$$

Since $\widehat{w}$ has support in $U_\star$ it holds $\widehat{\mathfrak{w}}(y) = 0$ for every $y \in D \setminus D_\star$. For the rest of the step let $y \in D_\star \setminus N$.

By Proposition (2.1.24)each set $\Omega \cap U_{x_j}$ is a d-set and it can readily be checked that this property inherits to their union $\Omega_\star$. Hence, $\Omega_\star$ satisfies the asymptotically nonvanishing relative volume condition (6) around y with a lower bound c > 0 on the limes inferior that is independent of y and – just as in the proof of Lemma (2.1.23) – a set $W \subset \mathbb{R}^d$ can be constructed such that the restriction of $\widehat{\mathfrak{w}}$ to $\mathbb{R}^d \setminus W$ is continuous at y and such that $|B(y,r) \cap \Omega_\star \cap (\mathbb{R}^d \setminus W)| \geq cr^d/2$ if $r > 0$ is small enough. By these properties of W:

$$|\widehat{\mathfrak{w}}(y)| = \left|\lim_{r \to 0} \frac{1}{|B(y,r) \cap \Omega_\star \cap (\mathbb{R}^d \setminus W)|} \int_{B(y,r) \cap \Omega_\star \cap (\mathbb{R}^d \setminus W)} \widehat{\mathfrak{w}}\, dx\right|$$

$$\leq \lim sup_{r \to 0} \frac{2}{cr^d} \int_{B(y,r) \cap \Omega_\star} |\widehat{w}| dx$$

$$= \lim sup_{r\to 0} \frac{2}{cr^d} \int_{B(y,r)\cap\Omega_\star} |w|\, dx.$$

In order to force these mean-value integral to vanish in the limit $r \to 0$, introduce the function $\log(dist_D)^{-1}$, which is bounded above in absolute value by $|\, log\, r|^{-1}$ on $B(y,r)$ if $r < 1$. It follows

$$|\hat{\mathfrak{w}}(y)| \leq c \limsup_{r\to 0} |\, log\, r|^{-1} \left(\frac{1}{r^d} \int_{B(y,r)\cap\Omega_\star} |w\, log(dist_D)|\, dx \right). \qquad (12)$$

So, since $|\, log\, r|^{-1} \to 0$ as $r \to 0$ the function $\hat{\mathfrak{w}}$ vanishes at every $y \in D_\star \setminus N$ for which the mean value integrals on the right-hand side remain bounded as r tends to zero.

Step 4: Intermezzo on $w \log(dist_D)$

In this step we prove the following result.

Proposition (2.1.35)[39]: ([60]) Let $s \in]0,1[, p \in]1,\infty[$ and let $\Lambda \subset \mathbb{R}^d$ be a d-set. Then there exists a linear operator $\mathfrak{E}$ that extends every measurable function f on Λ that satisfies

$$\|f\|_{L^p(\Lambda)} + \left(\iint_{\substack{x,y\in\Lambda \\ |x-y|<1}} \frac{|f(x) - f(y)|^p}{|x - y|^{d+sp}}\, dx\, dy \right)^{\frac{1}{p}} < \infty$$

to a function $\mathfrak{E}f$ in the Besov space $B_s^{p,p}(\mathbb{R}^d)$ of all measurable functions g on $\mathbb{R}^d$ such that

$$\|g\|_{L^p(\mathbb{R}^d)} + \left(\iint_{x,y\in\mathbb{R}^d} \frac{|g(x) - g(y)|^p}{|x - y|^{d+sp}}\, dx\, dyx \right)^{\frac{1}{p}} < \infty.$$

Lemma (2.1.36)[39]: Let $p \in]1,\infty[$, let $\Lambda \subset \mathbb{R}^d$ be a bounded d-set, and let $E \subset \partial\Lambda$ be closed and porous. Suppose $u \in W^{1,p}(\Lambda)$ has an extension $v \in W^{1,p}(\mathbb{R}^d)$ and satisfies $\frac{u}{dist_E} \in L^p(\Lambda)$. If $r \in]1,p[$ and $s \in]0,1[$, then the function $|u\, log(dist_E)|$ defined on Λ has an extension in the Bessel potential space $H^{s,r}(\mathbb{R}^d)$ that is positive almost everywhere.

For the proof we need the following extension result of Jonsson and Wallin.

Proof. It suffices to construct an extension in $B_s^{p,p}(\mathbb{R}^d)$ with the respective properties. Moreover, by the reverse triangle inequality it is enough to construct any extension $f \in B_s^{p,p}(\mathbb{R}^d)$ of u log $dist_E$ – then $|f|$ can be used as the required extension of $|u\, log\, dist_E\, |$. These considerations and Proposition (2.1.35) show that the claim follows provided

$$\|u\, log(dist_D)\|_{L^r(\Lambda)} + \left(\iint_{\substack{x,y\in\Lambda \\ |x-y|<1}} \frac{|u(x) log(dist_E(x)) - u(y) log(dist_E(y))|^r}{|x - y|^{d+sr}}\, dx\, dy \right)^{\frac{1}{r}} \qquad (13)$$

is finite.

To bound the L^r norm on the left-hand side of (13) choose $q \in]1,\infty[$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and apply Hölder's inequality

$$\|u\, log(dist_E)\|_{L^r(\Lambda)} \leq \|u\|_{L^p(\Lambda)} \|\log(dist_D)\|_{L^q(\Lambda)}.$$

For the second term on the right-hand we use that the Aikawa dimension of the porous set E is strictly smaller than d. This entails for some $\alpha < d$ and some $x \in E$ the estimate

$$\int_{\Lambda} dist_E(x)^{\alpha-d}\,dx \le \int_{B(x,2\,diam\,\Lambda)} dist_E(x)^{\alpha-d}\,dx \le c_\alpha(2\,diam\,\Lambda)^\alpha < \infty.$$

Hence, some negative power of $dist_E$ is integrable on Λ and by subordination of logarithmic growth $log(dist_E) \in L^q(\Lambda)$ follows. Altogether, $u\,log(dist_E) \in L^r(\Lambda)$ taking care of the first term in (13).

By symmetry the domain of integration for the second term on the left-hand side of (13) can be restricted to $dist_E(x) \ge dist_E(y)$. By adding and subtracting the term $u(y)log(dist_E(x))$ it in fact suffices to prove that

$$\left(\int_{\Lambda}\int_{\Lambda} \frac{|u(x)-u(y)|^r}{|x-y|^{d+sr}} \left|log\big(dist_E(x)\big)\right|^r dx\,dy\right)^{1/r} \tag{14}$$

and

$$\left(\int_{\Lambda} |u(y)|^r \int_{\substack{x\in\Lambda \\ dist_E(x)\ge dist_E(y)}} \frac{\left|log\big(dist_E(x)\big) - log\big(dist_E(y)\big)\right|^r}{|x-y|^{d+sr}}\,dx\,dy\right)^{1/r} \tag{15}$$

are finite. Fix $s < t < 1$, write (14) in the form

$$\left(\int_{\Lambda}\int_{\Lambda} \frac{|u(x)-u(y)|^r\left|log\big(dist_E(x)\big)\right|^r}{|x-y|^{\frac{dr}{p}+tr}\,|x-y|^{\frac{dr}{q}+sr-tr}}\,dx\,dy\right)^{\frac{1}{r}}$$

and apply Hölder's inequality with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ to bound it by

$$\le \left(\int_{\Lambda}\int_{\Lambda} \frac{|u(x)-u(y)|^p}{|x-y|^{d+tp}}\,dx\,dy\right)^{\frac{1}{p}} \left(\int_{\Lambda}\int_{\Lambda} \frac{\left|log\big(dist_E(x)\big)\right|^q}{|x-y|^{d+(s-t)q}}\,dy\,dx\right)^{\frac{1}{q}}$$

$$\le \|log(dist_E)\|_{L^q(\Lambda)} \left(\int_{\Lambda}\int_{\Lambda} \frac{|u(x)-u(y)|^p}{|x-y|^{d+tp}}\,dx\,dy\right)^{\frac{1}{p}} \left(\int_{|y|\le diam(\Lambda)} \frac{1}{|y|^{d+(s-t)q}}\,dy\right)^{\frac{1}{q}}$$

Now, $log(dist_E) \in L^q(\Lambda)$ has been proved above and the third integral is absolutely convergent since $d + (s - t)q < d$. Finally note that by assumption u has an extension $v \in W^{1,p}(\mathbb{R}^d)$. Since $W^{1,p}(\mathbb{R}^d) \subset B_s^{p,p}(\mathbb{R}^d)$ the middle term above is finite as well.

We show that the most interesting term (15) is finite. Here, the additional assumptions on $u, s,$ and r enter the game. By the mean value theorem for the logarithm and since $dist_E$ is a contraction, the r-th power of this term is bounded above by

$$\int_{\Lambda} |u(y)|^r \int_{\substack{x\in\Lambda \\ dist_E(x)\ge dist_E(y)}} \frac{|\,dist_E(x) - dist_E(y)|^r}{dist_E(y)^r|x-y|^{d+sr}}\,dx\,dy$$

$$\le \int_{\Lambda} \left|\frac{u(y)}{dist_E(y)}\right|^r \int_{\Lambda} \frac{|x-y|^r}{|x-y|^{d+sr}}\,dx\,dy$$

$$\leq \int\limits_{\Lambda} \left|\frac{u(y)}{dist_E(y)}\right|^r dy \int\limits_{|x|\leq diam(\Lambda)} \frac{1}{|x|^{d+r(s-1)}}\, dx\,.$$

Now, the integral with respect to x is finite since $r(s-1) < 0$. The integral with respect to y is finite since by assumption $\frac{u}{dist_E}$ is p-integrable on the bounded domain Λ and thus r-integrable for every $r <$ p

On noting that by Definition (2.1.4) a subset of a porous set is again porous, Lemma (2.1.36) applies to the bounded d-set $\Omega_\star$ and the porous set $D_\star \subset D$. Moreover, $w = (1-\eta)u \in W^{1,p}(\Omega_\star)$ has the extension $\widehat{w} \in W^{1,p}(\mathbb{R}^d)$ and satisfies

$$\int\limits_{\Omega_\star} \left|\frac{w(x)}{dist_{D_\star}(x)}\right|^p dx \leq \|1-\eta\|_\infty \int\limits_{\Omega} \left|\frac{w(x)}{dist_D(x)}\right|^p dx < \infty.$$

Hence we can record:

Corollary (2.1.37)[39]: For every $r \in]1,p[$ and every $s \in]0,1[$ the function $|w\, log(dist_{D_\star})|$ defined on $\Omega_\star$ has an extension $f_{s,r} \in H^{s,r}(\mathbb{R}^d)$ that is positive almost everywhere.

Step 5: Re-inspecting the right-hand side of (12)

We return to (12). Given $r \in]1,p[$ and $s \in]0,1[$ let $f_{s,r} \in H^{s,r}(\mathbb{R}^d)$ be as in Corollary (2.1.37). By (3) we can infer

$$\limsup_{r\to 0} \frac{1}{r^d} \int\limits_{B(y,r)\cap\Omega_\star} |w\, log(dist_D)|\, dx \leq \limsup_{r\to 0} \frac{1}{r^d} \int\limits_{B(y,r)} f_{s,r}\, dx < \infty$$

for (s,r)-quasievery $y \in D_\star \setminus N$. By the conclusion of Step 3 this implies $\widehat{\mathfrak{w}}(y) = 0$ for (s,r)-quasievery $y \in D_\star \setminus N$. To proceed further, we distinguish two cases:

(i) It holds $p \leq d$. Since the product $sr < p \leq d$ can get arbitrarily close to p, Lemma (2.1.8) yields for every $r \in]1,p[$ that $\widehat{\mathfrak{w}} = 0$ holds $(1,r)$-quasieverywhere on $D_\star \setminus N$. Moreover, since $C_{1,p}(N) = 0$ by definition, $\widehat{\mathfrak{w}} = 0$ holds even $(1,r)$-quasieverywhere on $D_\star$.

(ii) It holds $p > d$. Then $\widehat{\mathfrak{w}}$ is the continuous representative of $\widehat{w} \in W^{1,p}(\mathbb{R}^d)$ and N is empty, see the beginning of Step 3. Moreover, we can choose s and r such that $d - l < sr$ and conclude from the comparison theorem, Theorem (2.1.9), that $\widehat{\mathfrak{w}}$ vanishes H_l^∞ -a.e. on $D_\star$. Since D is l-thick and $U_\star$ is open, for each $y \in D \cap U_\star$ the set $B(y,r) \cap D \cap U_\star$ coincides with $B(y,r) \cap D$ provided $r > 0$ is small enough and thus has strictly positive H_l^∞ -measure. So, the continuous function $\widehat{\mathfrak{w}}$ has to vanish everywhere on $D \cap U_\star$ as well as on the closure of the latter set – which by definition is $D_\star$.

Summing up, $\widehat{\mathfrak{w}} = 0$ has been shown to hold $(1,r)$-quasieverywhere on $D_\star$ for every $r \in]1,p[$. From the beginning of Step 3 we also know that $\widehat{\mathfrak{w}}$ vanishes everywhere on $D \setminus D_\star$ and as $\widehat{w} \in W^{1,p}(\mathbb{R}^d)$ has compact support, Hölder's inequality yields $\widehat{w} \in W^{1,r}(\mathbb{R}^d)$.

Combining these two observations with Theorem (2.1.11) we are eventually led to

$$\widehat{w} \in W^{1,p}(\mathbb{R}^d) \cap \bigcap_{1<r<p} W_D^{1,r}(\mathbb{R}^d). \qquad (16)$$

We continue by quoting the following result of Hedberg and Kilpeläinen.

Proposition (2.1.38)[39]: ([59]) Let $p \in]1,\infty[$ and let $\Lambda \subset \mathbb{R}^d$ be a bounded domain whose boundary is l-thick for some $l \in]d-p,d]$. Then

$$W^{1,p}(\Lambda) \cap \bigcap_{1<r<p} W_D^{1,r}(\Lambda) \subset W_0^{1,p}(\Lambda).$$

In order to apply this result to the case of mixed boundary conditions, we proceed similarly to the proof of Theorem (2.1.13): With $B \subset \mathbb{R}^d$ an open ball that contains the compact support of $\widehat{w}$ define again

$$\mathcal{C} := \{M \subset B \setminus D : M \; open, connected \; and \; \Omega \subset M\}$$

and

$$\Omega_\bullet := \bigcup_{M \in \mathcal{C}} M .$$

Then $\partial\Omega_\bullet \in \{D, D \cup \partial B\}$ by Corollary (2.1.17), subsequent to which it is also shown that $\partial\Omega_\bullet$ is m-thick for some $m \in]d-p,d]$. Finally, let $\eta \in C_0^\infty(B)$ be identically one on the support of $\widehat{w}$. As $\phi \mapsto (\eta\phi)|_{\Omega_\bullet}$ induces a bounded operator $W_D^{1,p}(\mathbb{R}^d) \to W_0^{1,p}(\Omega_\bullet)$, it follows from (16) that

$$\widehat{w}|_{\Omega_\bullet} = (\eta\widehat{w})|_{\Omega_\bullet} \in W^{1,p}(\Omega_\bullet) \cap \bigcap_{1<r<p} W_0^{1,r}(\Omega_\bullet)$$

and thus $\widehat{w}|_{\Omega_\bullet} \in W_0^{1,p}(\Omega_\bullet)$ thanks to Proposition (2.1.38). Since by construction $\Omega \subset \Omega_\bullet$ and $D \subset \partial\Omega_\bullet$, we eventually conclude

$$w = \widehat{w}|_\Omega \in W_D^{1,p}(\Omega)$$

and the proof is complete.

What is the most restricting condition in Theorem (2.1.13)?, the answer doubtlessly is the assumption that a global extension operator shall exist. Certainly, this excludes all geometries that include cracks not belonging completely to the Dirichlet boundary part as in Fig. (2).

Since the distance function $dist_D$ measures only the distance to the Dirichlet boundary part D, points in $\partial\Omega$ that are far from D should not be of great relevance in view of the Hardy inequality (4). In the following considerations we intend to make this precise. Let $U, V \subset \mathbb{R}^d$ be two open, bounded sets with the properties

$$D \subset U, \quad \bar{V} \cap D = \emptyset, \quad \bar{\Omega} \subset U \cup V. \tag{17}$$

The philosophy behind this is to take U as a small neighborhood of D which – desirably – excludes the 'nasty parts' of $\partial\Omega \setminus D$. More properties of U, V will be specified below.

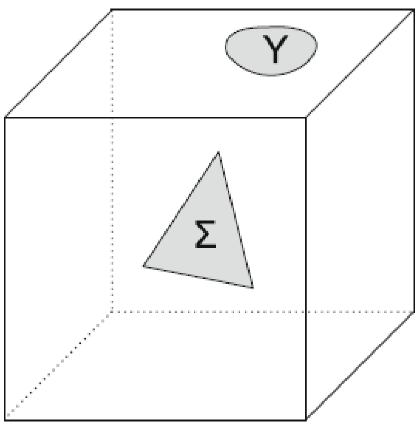


Fig. (2)[39]: The domain Ω is the cube minus the triangle Σ. The Dirichlet boundary part D consists exactly of the six outer sides of the cube minus the droplet Υ on the cover plate

Let $\eta_U \in C_0^\infty(U), \eta_V \in C_0^\infty(V)$ be two functions with $\eta_U + \eta_V = 1$ on $\overline{\Omega}$. Then one can estimate

$$\left(\int_\Omega |u|^p \, dist_D^{-p} \, dx\right)^{\frac{1}{p}} \le \left(\int_{U\cap\Omega} |\eta_U u|^p \, dist_D^{-p} \, dx\right)^{\frac{1}{p}} + \left(\int_{V\cap\Omega} |\eta_V u|^p \, dist_D^{-p} \, dx\right)^{\frac{1}{p}}.$$

Since $dist_D$ is larger than some $\varepsilon > 0$ on $\operatorname{supp}(\eta_V) \subset V$, the second term can be estimated by $\frac{1}{\varepsilon}\left(\int_\Omega |u|^p dx\right)^{\frac{1}{p}}$. If one assumes, as above, Poincaré's inequality, then this term may be estimated as required. In order to provide an adequate estimate also for the first term, we introduce the following assumption.

Assumption (2.1.39)[39]: The set U from above can be chosen in such a way that $\Lambda := \Omega \cap U$ is again a domain and if one puts $\Gamma := (\partial\Omega \setminus D) \cap U$ and $E := \partial\Lambda\setminus\Gamma$, then there is a linear, continuous extension operator $\mathfrak{F} : W_E^{1,p}(\Lambda) \to W_E^{1,p}(\mathbb{R}^d)$.

Clearly, this assumption is weaker than Condition (iii) in Theorem (2.1.13); in other words: Condition (iii) in Theorem (2.1.13) requires Assumption (2.1.39) to hold for an open set $U \supset \overline{\Omega}$.

We discuss the sense of Assumption (2.1.39) in extenso. It allows to focus on the extension not of the functions u but the functions $\eta_U u$, which live on a set whose boundary does (possibly) not include the 'nasty parts' of $\partial\Omega \setminus D$ that are an obstruction against a global extension operator. In detail: one first observes that, for $\eta = \eta_U \in C_0^\infty(U)$ and $v \in W_D^{1,p}(\Omega)$, the function $\eta_v|_\Lambda$ even belongs to $W_E^{1,p}(\Lambda)$, see [57]. Secondly, we have by the definition of E

$$\partial U \cap \Omega = (\partial U \cap \Omega) \setminus \Gamma \subset \partial\Lambda\setminus\Gamma = E.$$

This shows that the 'new' boundary part $\partial U \cap \Omega$ of Λ belongs to E and is, therefore, uncritical in view of extension. Thirdly, one has $D = D \cap U \subseteq \partial\Omega \cap U \subset \partial\Lambda$, and it is clear that the 'new Dirichlet boundary part' E includes the 'old' one D. Hence, the extension operator $\mathfrak{F}$ may be viewed also as a continuous one between $W_E^{1,p}(\Lambda)$ and $W_D^{1,p}(\mathbb{R}^d)$. Thus, concerning $v := \eta u = \eta_U u$ one is – mutatis mutandis – again in the situation: $\eta u \in W_E^{1,p}(\Lambda) \subset W_D^{1,p}(\Lambda)$ admits an extension $\mathfrak{F}(\eta u) \in W_E^{1,p}(\mathbb{R}^d) \subseteq W_D^{1,p}(\mathbb{R}^d)$, which satisfies the estimate $\|\mathfrak{F}(\eta u)\|_{W_D^{1,p}(\mathbb{R}^d)} \le c\|\eta u\|_{W_D^{1,p}(\Lambda)}$, the constant c being independent from u. This leads, as above, to a corresponding (continuous) extension operator $\mathfrak{F}_\bullet : W_E^{1,p}(\Lambda) \to W_0^{1,p}(\Lambda_\bullet)$. Here, of course, $\Lambda_\bullet$ has again to be defined as the connected component of $B\setminus D$ that contains Λ. Thus one may proceed again as in the proof of Theorem (2.1.13), and gets, for $u \in W_D^{1,p}(\Omega)$,

$$\int_\Omega \left(\frac{|\eta u|}{dist_D}\right)^p dx = \int_\Lambda \left(\frac{|\eta u|}{dist_D}\right)^p dx \le \int_{\Lambda_\bullet} \left(\frac{|\mathfrak{F}_\bullet(\eta u)|}{dist_{\partial\Lambda_\bullet}}\right)^p dx \le c\left\|\nabla\left(\mathfrak{F}_\bullet(\eta u)\right)\right\|_{L^p(\Lambda_\bullet)}^p$$

$$\le c\|\mathfrak{F}_\bullet(\eta u)\|_{W^{1,p}(\Lambda_\bullet)}^p \le c\|\eta u\|_{W^{1,p}(\Lambda)}^p \le c\left(\|u\|_{L^p(\Omega)}^p + \|\nabla u\|_{L^p(\Omega)}^p\right),$$

since ηu belongs to $W_E^{1,p}(\Lambda) \subset W_D^{1,p}(\Lambda)$. Exploiting a last time Poincaré's inequality, whose validity will be discussed in a moment, one gets the desired estimate.

When aiming at Poincaré's inequality, it seems convenient to follow again the argument in the proof of Proposition (2.1.30): as pointed out above, the property $\mathbb{1} \notin$

$W_D^{1,p}(\Omega)$ has to do only with the local behavior of Ω around the points of D, cf. Lemma (2.1.31).

Concerning the compactness of the embedding $W_D^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$, one does not need the existence of a global extension operator $\mathfrak{E} : W_D^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^d)$. In fact, writing for every $v \in W_D^{1,p}(\Omega)$ again $v = \eta_U v + \eta_V v$ and supposing Assumption (2.1.39), one gets the following:

If $\{v_k\}_k$ is a bounded sequence in $W_D^{1,p}(\Omega)$, then the sequence $\{\eta_U v_k|_\Lambda\}_k$ is bounded in $W_E^{1,p}(\Lambda)$. Due to the extendability property, this sequence contains a subsequence $\{\eta_U v_{k_l}|_\Lambda\}_l$ that converges in $L^p(\Lambda)$ to an element v_U . Thus, $\{\eta_U v_{k_l}\}_l$ converges to the function on Ω that equals v_U on Λ and 0 on $\Omega\backslash\Lambda$. The elements $\eta_V v_k$ in fact live on the set $\Pi := \Omega \cap V$ and are zero on $\Omega\backslash V$. In particular they are zero in a neighborhood of D. Moreover, they form a bounded subset of $W^{1,p}(\Pi)$. Therefore it makes sense to require that Π is again a domain, and, secondly that Π meets one of the well-known compactness criteria $W^{1,p}(\Pi) \hookrightarrow L^p(\Pi)$, cf. [77]. Keep in mind that such requirements are much weaker than the global $W^{1,p}$-extendability, and in particular include the example in Fig. (2), as long as the triangle Σ has a positive distance to the six outer sides of the cube. Resting on these criteria, one obtains again the convergence of a subsequence $\{\eta_V v_{k_l})|_\Pi\}_l$ that converges in $L^p(\Pi)$ towards a function v_V . The sequence $\{\eta_V v_{k_l}\}_l$ then converges in $L^p(\Omega)$ to a function that equals v_V on Π and zero on $\Omega\backslash V$.

Altogether, we have extracted a subsequence of $\{v_k\}_k$ that converges in $L^p(\Omega)$.

We summarize these considerations in the following theorem.

Theorem (2.1.40)[39]: Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $D \subset \partial\Omega$ be a closed part of the boundary. Suppose that the following three conditions are satisfied:

(i) The set D is l-thick for some $l \in]d-p, d]$.

(ii) The space $W_D^{1,p}(\Omega)$ can be equivalently normed by $\|\nabla \cdot\|_{L^p(\Omega)}$.

(iii) There are two open sets $U, V \subset \mathbb{R}^d$ that satisfy (17) and U satisfies Assumption (2.1.39).

Then there is a constant $c > 0$ such that Hardy's inequality

$$\int_\Omega \left|\frac{u}{dist_D}\right|^p dx \le c \int_\Omega |\nabla u|^p \, dx$$

holds for all $u \in W_D^{1,p}(\Omega)$.

Section (2.2): Strongly Lipschitz Domains

We consider the following class of elliptic operators. Let $A(x)$ be an $n \times n$ matrix function with bounded measurable complex-valued entries on $\mathbb{R}^n$ and assume the ellipticity condition $A + A^* \ge 2\delta\, Id$ almost everywhere for some $\delta > 0$, let Ω be a strongly Lipschitz domain of $\mathbb{R}^n$ and $V = H_0^1(\Omega)$ or $V = H^1(\Omega)$. Define $L = -\operatorname{div}(A\nabla)$ as the maximal accretive operator associated to the regularly accretive sesquilinear form $\int_\Omega A\nabla f\, \nabla g$ defined on $\times V$. Since the choice of the variational space defines the boundary condition,we shall speak of Dirichlet ($V = H_0^1(\Omega)$) and Neumann ($V = H^1(\Omega)$). We write $L = (A, \Omega, V)$ adopting the notation in [94],where we discussed the Kato problem for such operators,that is the comparisons between $\|L^{1/2} f\|_2$ and $\|\nabla f\|_2$.

We study the L^p theory of the square root of L for $p \neq 2$, that is,to compare $\|L^{1/2}f\|_p$ with $\|\nabla f\|_p$ or to compare the corresponding local norms obtained by adding $\|f\|_p$ to both terms. These questions are important for at least two reasons: (i) identifying the L^p-domain of $L^{1/2}$ with a "geometric" Sobolev space, and (ii) solving elliptic or parabolic PDE's associated with L on cylinder domains in $\mathbb{R}_+^{n+1}$ with bottom boundary a Lipschitz set in $\mathbb{R}^n$. Let us recall what is known about this problem.

First, in one dimension, this question is studied in [93] on the real line and, then, in [92] for any type of boundary condition on an interval: the above (local) L^p-norms are equivalent when $1 < p < \infty$.

When $n \geq 2$ (which is assumed from now on),we showed in [94] the following result on $\mathbb{R}^n$: Assume that $\nabla L^{-1/2}$,the "Riesz transform associated to L",is bounded on $L^2(\mathbb{R}^n)$ and that the kernel $K_t(x,y)$ of e^{-tL} satisfies Gaussian upper bounds and Hölder regularity in the second variable for all $t > 0$. Then $\nabla L^{-1/2}$ extends to a bounded operator on the Hardy space $H^1(\mathbb{R}^n)$,hence on $L^p(\mathbb{R}^n)$ for $1 < p < 2$. Furthermore,one cannot hope for positive results when $p > 2$ for the class of all elliptic operators by a counterexample of Kenig presented in [94]. Recently, using a generalization of Calderon-Zygmund theory, Duong and McIntosh were able to remove the Hölder regularity assumption and to prove $L^p(\mathbb{R}^n)$-boundedness when $1 < p < 2$ by interpolation with a weak type (1,1) estimate. Their technique also applies on arbitrary domains provided a suitable heat kernel upper bound holds. This is the case on Lipschitz domains for real symmetric operators subject to Dirichlet or Neumann boundary condition [101]. Jerison and Kenig [104] proved earlier L^p-boundedness of the Riesz transform of the Dirichlet Laplacian (that is $-\Delta = (I\,d, \Omega, H_0^1\,(\Omega))$ in our notation) on bounded Lipschitz domains for $1 < p < p_o$ for $p_o > 3$ (for $2 < p < p_o$ this requires specific features of potential theory). We learned that similar results to that of Jerison and Kenig for the Neumann Laplacian ($ie - \Delta = (I\,d, \Omega, H^1(\Omega))$) on bounded Lipschitz domains are in Mendez and Mitrea [108].

Converse inequalities of the type $\|L^{1/2}f\|_p \leq c\|\nabla f\|_p\,(*)$ are consequences of inequalities $\|\nabla f\|_{p'} \leq c\|(L^*)^{1/2}f\|_{p'}$ in the dual range,so that one obtains no better than $(*)$ for $p \geq 2$ by a duality argument. What about < 2 ? In [94],we showed $(*)$ for $1 < p < \infty$ in the $\mathbb{R}^n$ case provided that $\|L^{1/2}f\|_2 \leq c\|\nabla f\|_2$ holds and that the heat kernel $K_t(x,y)$ has Gaussian upper bounds and is Hölder continuous in the first variable for all $t > 0$. The idea of proof is the construction of a Calderon-Zygmund operator U for which $L^{1/2}f = U\nabla f$ when $f \in H^1(\mathbb{R}^n)$. On bounded Lipschitz domains,$(*)$ is proved with the full range of p's for the Dirichlet Laplacian [104] and for the Neumann Laplacian [108] (see [106],Problem 3.3.16,for earlier results).

We prove $(*)$ in the full range of p' s on any strongly Lipschitz domain for Dirichlet or Neumann elliptic operators subject to the same hypotheses as the ones described for $\mathbb{R}^n$. Our method relies on a singular integral approach and leads to a precise representation of the square root of L on special Lipschitz domains,namely $L^{1/2} = T_1\nabla + T_2(-\Delta)^{1/2} + H\nabla + BT\,r$. Let us describe the terms: T_1 and T_2 are Calderon-Zygmund operators and the boundedness of T_1 relies on a square function estimate; H is an operator of Hardy type: it is bounded on $L^p(\Omega)$ when $1 < p < \infty$; Tr is the trace operator,known to extend boundedly from $W^{1,p}(\Omega)$ onto the Besov space $B^{p,p}_{1-1/p}(\partial\Omega)$ when $1 < p < \infty$ (see [62]);

and B is an operator of Hardy type on the boundary: it is bounded from $B^{p,p}_{1-1/p}(\partial\Omega)$ to $L^p(\Omega)$. The last two terms arise from the study of a commutator (which appears in [95]) between partial derivatives and the resolvent of the Laplacian; their analysis bears on the geometry of the boundary. Here the Laplacian is taken with the same "boundary condition" as L (ie the same variational space). In the case of strongly Lipschitz domains,the analysis of the first two terms remains unchanged,while error terms appear in the last two due to localisation, and this brings a term $c\|f\|_p$ in the right hand side of (*).

Our approach has two features. First,it applies in the same flow to the Laplacian and general operators. Indeed,if $L = -\Delta$ there is such a representation with $T_2 = 0$. Secondly,except for technical arguments,the treatment of Dirichlet and Neumann boundary conditions is similar.

The difficult part of such a program is to establish the L^2-boundedness of T_1 because we are unable to check that "$T_1(20)$" is in BMO,hence to apply the David-Journé theorem. This boundedness is, in fact, equivalent to the $p = 2$ case of (*) via the use of quadratic functionals (when we assume heat kernel upper bounds).

Finally, our representation of square roots requires appropriate heat kernel estimates the validity of which being studied in [96]. They give localized L^2-estimates for the gradient of heat kernels, some of which are similar to the weighted L^2-estimates encountered,for example,when dealing with the Laplace-Beltrami on a manifold and obtained in via differential inequalities [102]. Here the approach relies on parabolic Caccioppoli inequalities which turn out to be valid for any elliptic operator.

We are given a complex elliptic operator $L = (A, \Omega, V)$. The notation $\|.\|_p$ stands for the norm on $L^p(\Omega)$. In order to state the main results,let us discuss the required hypotheses on L.

(H1)L^2 theory:

$$\|L^{1/2} f\|_2 \leq c\|\nabla f\|_2, \qquad \forall f \in V, \tag{18}$$

or

$$\|L^{1/2} f\|_2 \leq c(\|\nabla f\|_2 + \|f\|_2), \qquad \forall f \in V. \tag{19}$$

(H2) Gaussian upper bound: there exists $\tau \in (0,\infty]$ such that for all real $t \in (0,\tau]$,the kernel of e^{-tL},denoted by $K_t(x,y)$,is a measurable function on $\Omega \times \Omega$ and there exist constants $C_G > 0$ and $\alpha > 0$ such that

$$|K_t(x,y)| \leq \frac{C_G}{t^{n/2}} e^{-\frac{\alpha|x-y|^2}{t}}, \qquad t \in (0,\tau], a.e. \text{ on } \Omega \times \Omega. \tag{20}$$

(H3) Hölder regularity in the first variable: there exists $\tau \in (0,\infty]$ such that for all $y \in \Omega$ and all real $t \in (0,\tau], x \to K_t(x,y)$ is a Hölder continuous function in Ω and there exist constants $C_H > 0$ and $\mu > 0$ such that

$$|K_t(x,y) - K_t(x\,,y)| \leq \frac{C_H}{t^{n/2}} \frac{|x - x'|^{\mu}}{t^{\mu/2}}, t \in (0,\tau], x, x', y \in \Omega. \tag{21}$$

(H4) Uniform $L^p(\Omega)$ boundedness of $e^{-tL}, t > 0$,for all $1 < p < \infty$ if Ω unbounded.

The hypothesis (H1) not only implies that $V \subset \mathcal{D}(L^{1/2})$ which is a starting point,but is also equivalent to a key estimate proved.

Added in proof: the L^2-theory is now valid in full generality,hence (H1) is satisfied for all L as above (Square roots of elliptic second order divergence operators on strongly Lipschitz domains: L^2-theory,preprint. P. Auscher and Ph. Tchamitchian). This improves the conclusions of our Theorem (2.2.2).

The hypotheses (H2) and (H3) are used to obtain size and regularity estimates for various kernels. Such hypotheses depend on the coef- ficients,the domain and the boundary condition (see [96]). For example,they hold for real coefficients (again, Ω is a strongly Lipschitz domain and $V = H^1$ or H_0^1): the value of τ is always $+\infty$ except when Ω is bounded and V = $H^1(\Omega)$.

The last hypothesis is the least important; it is fulfilled if (H2) holds with $\tau = \infty$.

Finally,it is no loss of generality to assume $\tau = 1$ when it is finite.

Let $1 < p < \infty$. Define V^p (resp. $\dot{V}^p$) as the completion of $V \cap W^{1,p}(\Omega)$ under $\|\nabla f\|_p + \|f\|_p$ (resp. $\|\nabla f\|_p$). Since Ω is Lipschitz, $V^p = W^{1,p}(\Omega)$ when $V = H^1(\Omega)$ and $V^p = W_0^{1,p}(\Omega)$ when $V = H_0^1(\Omega)$. When Ω is furthermore bounded, $\dot{V}^p = \{f \in V^p;\ \int_\Omega f = 0\}$ when $V = H^1(\Omega)$ and $\dot{V}^p = V^p$ when $V = H_0^1(\Omega)$.

Corollary (2.2.1)[90]: Under the assumptions of Theorem (2.2.9), $L^{1/2}$ extends to a bounded operator from V^p to L^p (and from $\dot{V}^p$ to L^p if the homogeneous inequality holds).

We with our conclusions in the case of bounded domains. The case of other domains will be described.

Theorem (2.2.2)[90]: Let Ω be a bounded Lipschitz domain with Lipschitz constant M, A be an elliptic matrix with ellipticity constant δ and $L = (A, \Omega, V)$ be subject to Dirichlet or Neumann boundary condition. Then for all $1 < p < \infty$,

$$\left\|L^{1/2} f\right\|_p \leq c\|\nabla f\|_p, \qquad \forall f \in \dot{V}^p \tag{22}$$

holds under one of the following conditions:

(i) A is real-valued and symmetric,

(ii) $n = 2$ and $A \in ABMO_\varepsilon(\Omega)$ for some $\varepsilon = \varepsilon(\delta, M) > 0$,

(iii) $n \geq 3$, A real and $A \in ABMO_\varepsilon(\Omega)$ for some $\varepsilon = \varepsilon(n, \delta, M) > 0$,

(iv) $n \geq 3$, Ω is C^1 or has small enough Lipschitz constant, and the distance in BMO of A to vmo is small enough.

Proof. It amounts to checking the hypotheses of Theorem (2.2.9) for each item. First, (H1) is trivial for item 1 and already observed for the other items. The validity of (H2-H3) in the four items can be read of Theorem 7 in [96].

The definition of the Lipschitz constant will be recalled. The class $ABMO_\varepsilon(\Omega)$ is introduced in [97]: given $\varepsilon > 0$, a function $f \in BMO(\Omega)$ belongs to $ABMO_\varepsilon(\Omega)$ if there exists $\eta > 0$ such that

$$\sum_i \int_{Q_i \cap \Omega} \left|f - m_{Q_i \cap \Omega} f\right|^2 \leq \varepsilon |Q \cap \Omega|,$$

for all cubes $Q \in Q_\Omega$ with sidelength $\ell(Q) \leq 1$ and for all families $Q_i, i \in I$,of subcubes of Q having disjoint interiors,such that $\ell(Q_i) \leq \eta/(Q)$ and $Q_i \in Q_\Omega$. This class contains BMO-perturbations of the spaces BUC, vmo and MH^s whose elements are pointwise multipliers of the Sobolev space H^s . Theorem 28 in [97] states that (K_{loc}) holds when the coefficients of L belongs to some $ABMO_\varepsilon(\Omega)$ for small enough ε > 0,which implies that (H1) holds in item 2,3,and 4.

Eventually, we obtain

Theorem (2.2.3)[90]: Assume that one of the conditions in the above theorem holds. Then, there exists $\varepsilon = \varepsilon(L) > 0$ such that for $1 < p < 2 + \varepsilon$, the extension of $L^{1/2}$ to $\dot{V}^p$ is an isomorphism onto $L^p(\Omega)$, and $\|\nabla f\|_p$ and $\left\|L^{1/2} f\right\|_p$ are two equivalent norms on $\dot{V}^p$.

This result is optimal in the class of elliptic operators by adapting Kenig's counterexample (see [94]). Note that $\varepsilon(-\Delta) > 1$ if $n \geq 3$ for both the Dirichlet and Neumann Laplacians and $\varepsilon(-\Delta) > 2$ if $n = 2$ for the Dirichlet Laplacian from [104] and [108].

We denote by $-\Delta = (I\,d, \Omega, V)$ the negative Laplacian on Ω with same variational space (or,roughly,"same boundary condition") as $L = (A, \Omega, V)$. Let us begin with a formal analysis of $L^{1/2}$ whatever the choices of Ω and V may be. We compute $L^{1/2}f$ for f in an appropriate space, say V or a dense subspace of V:

$$L^{1/2} f = a \int_0^\infty t^3 L^2 e^{-2t^2 L} f \frac{dt}{t} \tag{23}$$

with $a^{-1} = \frac{1}{2} \int_0^\infty \zeta^{1/2} e^{-2\zeta} d\zeta$. The integral converges strongly in $L^2(\Omega)$ by the general theory recalled provided we assume that $V \subset D(L^{1/2})$.

Pick $T \in (0, \infty]$,so that

$$L^{1/2} f = a \int_0^T t^3 L^2 e^{-2t^2 L} f \frac{dt}{t} + a \int_T^\infty t^3 L^2 e^{-2t^2 L} f \frac{dt}{t}.$$

For $\leq T$, write

$$a t^3 L^2 e^{-2t^2 L} f = \tilde{\theta}_t \nabla f$$

where

$$\tilde{\theta}_t = -a t^3 L e^{-2t^2 L} \operatorname{div} A: L^2(\Omega, \mathbb{C}^n) \to L^2(\Omega). \tag{24}$$

Here, $-\operatorname{div}$ denotes the adjoint of $\nabla : V \to L^2(\Omega, \mathbb{C}^n)$; it is the distributional divergence in the Dirichlet case but not in the Neumann case. Next, we decompose

$$\tilde{\theta}_t \nabla f = \tilde{\theta}_t \nabla f + \tilde{\theta}_t R_t \nabla f + \tilde{\theta}_t \nabla(I - R_t) f + \tilde{\theta}_t [\nabla, R_t] f$$

where $R_t = (I - t^2 \Delta)^{-1}$. The commutator $\mathcal{C}_t = [\nabla, R_t]$ between the partial derivatives and the resolvent of $-\Delta$ is defined for $f \in V$ by

$$\mathcal{C}_t f = \left(D_j (I - t^2 \Delta)^{-1} f - (I - t^2 \Delta)^{-1} D_j f\right)_{1 \leq j \leq n}, \tag{25}$$

where $Dj = \frac{\partial}{\partial x_j}$. Thus,we have formally obtained

$$L^{1/2} f = \int_0^T \tilde{\theta}_t R_t \nabla f \frac{dt}{t} + \int_0^T \tilde{\theta}_t \nabla (I - R_t) f \frac{dt}{t} + \int_0^T \tilde{\theta}_t [\nabla, R_t] f \frac{dt}{t} + a \int_T^\infty t^3 L^2 e^{-2t^2 L} f \frac{dt}{t}. \tag{26}$$

Remark (2.2.4)[90]: If Ω is a special Lipschitz domain, (18) holds and $\tau = \infty$ in (H2) and (H3),then take $T = \infty$ in (26) so that the last integral disappears. In all other cases choose $T = 1$.

When $L = -\Delta$, one can do the same thing. Begin with

$$(-\Delta)^{1/2} f = b \int_0^\infty t^3 \Delta^2 e^{2t^2 \Delta} (1 - t^2 \Delta)^{-1} f \frac{dt}{t}$$

with $b^{-1} = \frac{1}{2} \int_0^\infty \zeta^{1/2} e^{-2\zeta} (1 + \zeta)^{-1} d\zeta$. Defining now $\tilde{\theta}_t = b t^3 \Delta e^{2t^2 \Delta^2} \operatorname{div}$,we have

$$b t^3 \Delta^2 e^{2t^2 \Delta} (I - t^2 \Delta)^{-1} f = \tilde{\theta}_t \nabla R_t f = \tilde{\theta}_t R_t \nabla f + \tilde{\theta}_t [\nabla, R_t] f$$

and we arrive at

$$(-\Delta)^{1/2} f = \int_0^T \tilde{\theta}_t R_t \nabla f \frac{dt}{t} + \int_0^T \tilde{\theta}_t [\nabla, R_t] f \frac{dt}{t}$$
$$+ b \int_T^\infty t^3 \Delta^2 e^{2t^2\Delta}(1 - t^2\Delta)^{-1} f \frac{dt}{t}. \quad (27)$$

As it is similar to (26) with the exception that one term is missing,the study of $(-\Delta)^{1/2}$ is a byproduct of our forthcoming analysis,which can be summed up as follows.

Lemma (2.2.5)[90]: If (H1), (H2) and (H3) hold then for all $1 < p < \infty$

$$\left\| \int_0^T \tilde{\theta}_t R_t \nabla f \frac{dt}{t} \right\|_p \leq c\|\nabla f\|_p \quad (28)$$

for all $f \in V \cap W^{1,p}(\Omega)$.

Lemma (2.2.6)[90]: If (H2) and (H3) hold then for all $1 < p < \infty$

$$\left\| \int_0^T \tilde{\theta}_t \nabla(I - R_t) f \frac{dt}{t} \right\|_p \leq c\left\|(-\Delta)^{1/2} f\right\|_p \quad (29)$$

for all $f \in V$ such that $(-\Delta)^{1/2} f \in L^p(\Omega)$.

Lemma (2.2.7)[90]: If (H2) holds then for all $1 < p < \infty$,

$$\left\| \int_0^T \tilde{\theta}_t [\nabla, R_t] f \frac{dt}{t} \right\|_p \leq c\|\nabla f\|_p \quad (30)$$

for $f \in V \cap W^{1,p}(\Omega)$ when Ω is a special Lipschitz domain and

$$\left\| \int_0^1 \tilde{\theta}_t [\nabla, R_t] f \frac{dt}{t} \right\|_p \leq c\|\nabla f\|_p + c\|f\|_p \quad (31)$$

for $f \in V \cap W^{1,p}(\Omega)$ when Ω is a general strongly Lipschitz domain.

Lemma (2.2.8)[90]: If (H4) holds when Ω is unbounded or if (H2) holds when Ω is bounded then

$$\left\| \int_1^\infty t^3 L^2 e^{-2t^2 L} f \frac{dt}{t} \right\|_p \leq c\|f\|_p \quad (32)$$

for all $1 < p < \infty$ and $f \in L^p(\Omega)$. Most is devoted to the proof of these results. Let us indicate how they imply the main theorem.

Theorem (2.2.9)[90]: Let $1 < p < \infty$. With the above notation, if (H1-4) hold then we have the a priori estimate

$$\left\|L^{1/2} f\right\|_p \leq c\|\nabla f\|_p + c\|f\|_p, \qquad \forall f \in V \cap W^{1,p}(\Omega). \quad (33)$$

Proof. It follows from these lemmata and (26) that

$$\left\|L^{1/2} f\right\|_p \leq c\left(\|\nabla f\|_p + \left\|(-\Delta)^{1/2} f\right\|_p + \|f\|_p\right)$$

for all $f \in V \cap W^{1,p}(\Omega)$ such that $(-\Delta)^{1/2} f \in L^p(\Omega)$. Applying Lemma (2.2.5), Lemma (2.2.7) and a trivial modification of Lemma (2.2.8) to the terms in (27) shows that

$$\left\|(-\Delta)^{1/2 f}\right\|_p \leq c\left(\|\nabla f\|_p + \|f\|_p\right)$$

for all $f \in V \cap W^{1,p}(\Omega)$. This proves (33).

Since we use them quite often,it is simpler to recall known results. Let $-\Delta = (I\,d, \Omega, V)$ where Ω is Lipschitz and $V = H^1(\Omega)$ or $V = H_0^1(\Omega)$.

As mentioned before,the kernel of $e^{t\Delta}$ always fulfills (H2) and (H3),and $\tau = \infty$ in all cases except when Ω is bounded and $V = H^1(\Omega)$.

As for the resolvent kernel $R_t(x, y)$ of $R_t = (I - t\text{^}2\Delta) - 1$,we have

$$|R_t(x,y)| \le ct^{-n} w_{n-2}\left(\frac{|x-y|}{t}\right) e^{-\gamma\frac{|x-y|}{t}}, \quad (34)$$

$$|R_t(x,y) - R_t(x,y')| \le ct^{-n} w_{n-2+\mu}\left(\frac{|x-y|}{t}\right)\left(\frac{|y-y'|}{t}\right)^{\mu} \quad (35)$$

when $|y - y| \le |x - y|/2$,and

$$|R_t(x,y) - R_t(x',y)| \le ct^{-n} w_{n-2+\mu}\left(\frac{|x-y|}{t}\right)\left(\frac{|x-x'|}{t}\right)^{\mu} \quad (36)$$

when $|x - x| \le |x - y|/2$,for some constants $c > 0, \gamma > 0$ and $\mu \in (0,1)$ depending on the domain **Ω**. Here and subsequently we set $w_s(u) = u - s$ if $s > 0$ and $w_0(u) = \ln\ (e + 1/u)$.

These estimates hold for $t > 0$ in all cases but when Δ is the Neumann Laplacian on Ω bounded for which the restriction $t \le t_o$ for any positive real to (e.g., $t_o = \tau^2$) is necessary (see [100],[103],[99]).

By routine arguments,these estimates are also valid for the kernel of $\left(I - t^2 e^{i^2\eta}\Delta\right)^{-1}$ uniformly for η in a compact subset of $]-\pi/2, \pi/2[$.

We are given an operator $L = (A, \Omega, V)$ with A elliptic with ellipticity constant δ, **Ω** a Lipschitz domain and $V = H_0^1(\Omega)$ or $H^1(\Omega)$. In what follows, I is an interval (0,τ) with τ finite or infinite.

Estimates in average on space derivatives of the heat kernel are of crucial importance. They follow from parabolic Caccioppoli inequalities.

Proposition (2.2.10)[90]: (Parabolic Caccioppoli inequality) Given $f \in L^2(\Omega)$, the solution $u_t = e^{-tL}f$ of the parabolic equation $\frac{\partial u_t}{\partial t} = -Lu_t$ in Ω satisfies

$$\int_\Omega |\varphi|^2\, |\nabla u_t|^2 \le 4\delta^{-4}\int_\Omega |u_t|^2\, |\nabla\varphi|^2 + 2\delta^{-1}\int_\Omega |u_t| \frac{\partial u_t}{\partial t}\, |\varphi|^2, \quad (37)$$

for any $\varphi \in C^1(\mathbb{R}^n)$, real-valued, bounded with bounded gradient.

Proof. See [96].

Let us also recall a well-known result that is a consequence of the analyticity of the semigroup generated by $-L$. A proof can be found in many places. See, e.g.,[91],[99],[94].

Lemma (2.2.11)[90]: There exists $\omega \in \left(0, \frac{\pi}{2}\right)$ such that $K_t(x,y)$ has a holomorphic extension $K_z(x,y)$ on the open sector defined by $z \in \mathbb{C}^*$ and $|\arg\ z| \le \omega$. Furthermore, if (H2-3) hold then for any $\nu \in \left(0, \frac{\pi}{2} - \omega\right)$, $K_z(x,y)$ satisfies (20) and (21) for $|\arg\ z| \le \nu$ with Re $z < \tau$ and one should replace t by Re z in the estimates. Consequently, for $k \in \mathbb{N}^*$, $t^k\left(\frac{d}{dt}\right)^k K_t(x,y)$, which is the kernel of $(-tL)^k e^{-tL}$, satisfies the same estimates as $K_t(x,y)$.

Proposition (2.2.12)[90]: Assume that (H2) holds. Then,

(i) For all $t \in I$ and almost all $x \in \Omega, y \to K_t(x,y) \in V \cap W^{1,p}(\Omega)$ when $1 < p \le 2$, and

$$\sup_{x\in\Omega} \left\|\nabla_y K_t(x,\cdot)\right\|_p \le c\, C_G\, t^{-\frac{1}{2}}\, t^{-\frac{n}{2}\left(1-\frac{1}{p}\right)}. \quad (38)$$

(ii) For all $x \in \Omega, t \in I$ and $r > 0$ one has

$$\left(\int_{\substack{r\le|x-y|\le 2r\\ y\in\Omega}} \left|\nabla_y K_t(x,y)\right|^2 dy\right)^{1/2} \le c\, C_G\, t^{-\frac{1}{2}-\frac{n}{4}} \left(\frac{r}{\sqrt{t}}\right)^{\frac{n-2}{2}} e^{-\frac{\beta r^2}{t}}, \quad (39)$$

where the constants c, β depend on the constant α in (20), n and δ. (iii) Assume furthermore that (H3) holds. For all $x, x' \in \Omega, z \in \mathbb{R}^n, t \in I$ and $r > 0$, one has

$$\left(\int_{\substack{|z-y|\le r\\ y\in\Omega}} \left|\nabla_y K_t(x,y) - \nabla_y K_t(x\,,y)\right|^2 dy\right)^{1/2}$$

$$\le c\,(C_H + C_G) t^{-\frac{1}{2}-\frac{n}{4}} \left(\frac{|x - x'|}{\sqrt{t}}\right)^{\eta} \inf\left(\left(\frac{r}{\sqrt{t}}\right)^{\frac{n-2}{2}}, 1\right). \quad (40)$$

The constants $c, \eta > 0$ depend only on the constant α in (20), μ in (21), n and δ.

Proof. For simplicity, we shall switch the roles of x and y, that is, we shall take gradient and do integration with respect to the first variable rather than the second (which amounts to changing L into its adjoint). Proof of (38). First ,the inequality (37) applies to $u_t(x) = K_t(x,y)$ with $\varphi = 1$. Indeed, by the semigroup property $u_t = e^{-t^2 L} u_{t/2}$ and $u_{t/2} \in L^2(\Omega)$ from the property (H2). Since $\|u_t\|_2 \le cC_G t^{-n/4}$ and,by the analyticity of the semigroup on $L^2(\Omega)$, $\left\|\frac{du_t}{dt}\right\|_2 \le cC_G t^{-1} t^{-n/4}$,we conclude that $\|\nabla u_t\|_2 \le cC_G t^{-\frac{1}{2}} t^{-n/4}$. This gives (38) for $p = 2$. Next,we show that for fixed $y \in \Omega$ and $t \in I$, $K_t(\cdot,y) \in W^{1,p}(\Omega)$ and estimate $\|\nabla_x K_t(\cdot,y)\|_p$ when $1 \le p < 2$. We do it for $p = 1$, the result follows by interpolation. It is clear that $K_t(\cdot,y) \in L^1(\Omega)$ from (H2). Next, Cauchy-Schwarz inequality from (39), whose proof is given below, gives us

$$\int_{r\le|x-y|\le 2r} t^{1/2} |\nabla_x K_t(x,y)|\, dx \le c \left(\frac{r}{\sqrt{t}}\right)^{n-1} e^{-\frac{\beta r^2}{t}}.$$

Summing these inequalities over $= 2j\,, j \in Z$, we find $\|\nabla_x K_t(\cdot,y)\|_1 \le ct^{-\frac{1}{2}}$.

Proof of (39). We have to bound $\int_{r\le|x-y|\le 2r} |\nabla_x K_t(x,y)|^2\, dx$ (all integrals occur on **Ω**) for fixed $y \in \Omega, t \in I$ and $r > 0$. Apply again the inequality (37) to $u_t(x) = K_t(x,y)$ and $\varphi(x)$ supported in $r/2 \le |x-y| \le 4r$ with $\varphi(x) = 1$ if $r \le |x-y| \le 2r$, $\|\varphi\|_\infty \le 1$ and $\|\nabla\varphi\|_\infty \le c/r$. On the support of φ we have

$$|u_t(x)| + \left|t\frac{\partial u_t(x)}{\partial t}\right| \le cC_G t^{-\frac{n}{2}} e^{-\frac{\alpha r^2}{4t}}.$$

Thus, we obtain

$$\int_{r\le|x-y|\le 2r} t\,|\nabla_x K_t(x,y)|^2\, dx \le \frac{cC_G^2 r^{n-2}}{t^{n-1}}\left(1 + \frac{r^2}{t}\right) e^{-\frac{\alpha r^2}{2t}}$$

$$\le cC_G^2\, t^{-\frac{n}{2}} \left(\frac{r}{\sqrt{t}}\right)^{n-2} e^{-\frac{\beta r^2}{t}}$$

for any $\beta < \alpha/2$. This is (39).

Proof of (40). We have to bound $\int_{|x-z|\le r} |\nabla_x K_t(x,y) - \nabla_x K_t(x,y\,)|^2\, dx$ where $y, y' \in \Omega, z \in \mathbb{R}^n, t \in I\,, r > 0$.

Set $u_t(x) = K_t(x,y) - K_t(x,y\,)$. The Hölder regularity in the second variable (since we switched variables) and interpolation with (20) gives us for any $s \in (0, 1)$,

$$u_t(x)| + \left| t\frac{\partial u_t(x)}{\partial t} \right|$$

$$\leq c C_H^s C_G^{1-s} t^{-\frac{n}{2}} \left(\frac{|y - y'|}{\sqrt{t}}\right)^{s\mu} \times \left(e^{-\frac{\alpha(1-s)|x-y|^2}{t}} + e^{-\frac{\alpha(1-s)|x-y'|^2}{t}} \right).$$

Now, apply (37) to $u_t(x)$ and $\varphi \equiv 1$ to obtain

$$\int_\Omega t\, |\nabla_x K_t(x,y) - \nabla_x K_t(x,y)|^2\, dx \leq c(C_H^s C_G^{1-s})^2\, t^{-\frac{n}{2}} \left(\frac{|y - y'|}{\sqrt{t}}\right)^{2s\mu}$$

hence (40) when $r^2 \geq t$.

Next, assume $r^2 \leq t$ and apply again (37) to $u_t(x)$ and $\varphi(x)$ supported in $|x - z| \leq 2r$ with $\varphi(x) = 1$ if $|x - z| \leq r, \|\varphi\|_\infty \leq 1$ and $\|\nabla\varphi\|_\infty \leq c/r$. Since

$$|u_t(x)| + \left| t\frac{\partial u_t(x)}{\partial t} \right| \leq c C_H\, t^{-\frac{n}{2}} \left(\frac{|y - y'|}{\sqrt{t}}\right)^{\mu}$$

we have

$$\int_{|x-z|\leq r} t\, |\nabla_x K_t(x,y) - \nabla_x K_t(x,y)|^2\, dx \leq c C_H^2 \frac{r^{n-2}}{t^{n-1}} \left(1 + \frac{r^2}{t}\right)\left(\frac{|y - y'|}{\sqrt{t}}\right)^{2\mu}$$

$$\leq 2c C_H^2\, t^{-\frac{n}{2}} \left(\frac{r}{\sqrt{t}}\right)^{n-2} \left(\frac{|y - y'|}{\sqrt{t}}\right)^{2\mu}.$$

This is (40) in this case.

We have

$$\tilde{\theta}_t : L^2(\Omega, \mathbb{C}^n) \to L^2(\Omega), \qquad \tilde{\theta}_t F = -at^3 L e^{-2t^2 L} \operatorname{div}(A F).$$

We also set

$$\theta_t : L^2(\Omega, \mathbb{C}^n) \to L^2(\Omega), \theta_t F = -t e^{-t^2 L} \operatorname{div}(A F).$$

By the functional calculus developed, they are well-defined and bounded operators with

$$\|\tilde{\theta}_t F\|_2 + \|\tilde{\theta}_t F\|_2 \leq c(\delta)\|F\|_2$$

for all $t > 0$, $c(\delta)$ depending only on ellipticity. Moreover,

$$\theta_t \nabla f = t e^{-t^{2L}L} f, \qquad f \in V, \tag{41}$$

$$\tilde{\theta}_t \nabla f = at^3 L e^{-2t^2 L} L f = -\frac{at}{4}\, t \left(\frac{d}{dt}\right) e^{-2t^2 L} L f, \qquad f \in V. \tag{42}$$

Let us study furthermore θ_t .

Proposition (2.2.13)[90]: Assume that (H2) holds. For $F \in L^2(\Omega, \mathbb{C}^n)$ and $\in I$, we have

$$\theta_t F(x) = t \int_\Omega A(y)F(y) \cdot \nabla_y K_{t^2}(x,y) dy$$

$$= t \sum_{1\leq j,k\leq n} \int_\Omega a_{j,k}(y) F_k(y) \frac{\partial K_{t^2}(x,y)}{\partial y_j}\, dy. \tag{43}$$

Letting $\theta_t(x,y)$ be the distributional kernel of θ_t we have

$$\theta_t(x,y) = t^T A(y) \nabla_y K_{t^2}(x,y), \tag{44}$$

where ${}^T A$ is the real transpose of A. Furthermore, for $t \in I$, θ_t extends to a bounded operator from $L^2(\Omega, \mathbb{C}^n)$ into $L^\infty(\Omega)$ with

$$\|\theta_t F\|_\infty \leq c\, C_G\, t^{-n/2} \|F\|_2 \tag{45}$$

and from $L^\infty(\Omega, \mathbb{C}^n)$ into $L^\infty(\Omega)$ with

$$\|\theta_t F\|_\infty \leq c\, C_G\, \|F\|_\infty. \tag{46}$$

Proof. By definition of θ_t ,for any $\varphi \in C_0^\infty(\Omega)$ and $F \in L^2(\Omega, \mathbb{C}^n)$, $\langle \theta_t F, \varphi \rangle = \langle AF , \nabla e^{-t^2 L^*} \varphi \rangle$!. Observe that

$$\left| A(y)F(y) \cdot \nabla_y K_{t^2}(x,y)\overline{\varphi(x)} \right| dxdy \le \|A\|_\infty \|F\|_2 \left\| \|\nabla + y\, K_{t^2}\|_{L_y^2} \right\|_{L_x^\infty} \|\varphi\|_1,$$

which is finite since $\left\| \|\nabla + y\, K_{t^2}\|_{L_y^2} \right\|_{L_x^\infty} \le ct^{-1-\frac{n}{2}}$. Thus,we have

$$\langle AF, \nabla e^{-t^2 L^*} \varphi \rangle = A(y)\nabla f(y) \cdot \nabla_y K_{t^2}(x,y)\varphi(x)\, dxdy.$$

where the integral exists in the Lebesgue sense. The formula (43) then follows from Fubini's theorem. The argument also yields (45). Lastly,(38) tells us that for almost all $x \in \Omega$ and $t \in I$, $\theta_t(x,y) \in L_y^1$ with $\|\theta_t(x,\cdot)\|_1 \le c$ uniformly. Hence,(46).

A similar result hold for $\tilde{\theta}_t$ whose kernel is given by

$$\tilde{\theta}_t(x,y) = -\frac{at}{4}\, {}^T A(y)\, \nabla_y \left[t \left(\frac{dt}{t} \right) K_{2t^2}(x,y) \right]$$

Notice that neither θ_t nor $\tilde{\theta}_t$ are bounded on L^p for p near 1. The smoothing procedure of the proposed algorithm to study $L^{1/2}$ circumvents this drawback.

Most of the kernel analysis will be based on the following technical lemma. Here, Ω is an arbitrary open set in $\mathbb{R}^n$.

Lemma (2.2.14)[90]: Assume we are given kernels $A_t(x,y)$ and $B_t(x,y)$ defined on $\Omega \times \Omega$, constants σ, s, η such that $\sigma > s + \eta > s \ge 0$ and $s + \eta \ge n$ with the following requirements:

(i) For all $x, x \in \Omega$, $z \in \mathbb{R}^n$, $r > 0$ and $t > 0$,

$$\int_{r \le |x-y| \le 2r} |A_t(x,y)| dy \le \left(\frac{r}{t} \right)^\sigma e^{-\frac{r}{t}}, \tag{47}$$

$$\int_{|z-y| \le r} |A_t(x\,,y) - A_t(x,y)|\, dy \le \inf \left(\left(\frac{r}{t} \right)^\sigma, 1 \right) \left(\frac{|x' - x|}{t} \right)^\eta. \tag{48}$$

(It is understood that the variable of integration is in Ω).

(ii) For all $x, y, y \in \Omega$ and $t > 0$,

$$|B_t(x,y)| \le t^{-n} w_s \left(\frac{|x-y|}{t} \right) e^{-\frac{|x-y|}{t}}, \tag{49}$$

and if $|y - y'| \le |x - y|/2$,

$$|Bt(x,y) - Bt(x,y)| \le t - nws + \eta\, |x - y|\, t\, |y - y|\, t\, \eta. \tag{50}$$

Then $\int_0^{+\infty} \int_\Omega A_t(x,z) B_t(z,y) dz \frac{dt}{t}$ is a Calderón-Zygmund kernel on $\Omega \times \Omega$.

A typical application will be $\sigma = n - 1, s = n - 2$ and $\eta \in (0, 1)$.

Proof. Let us begin with some useful consequences of the hypotheses. The assumption (47) on $A_t(x,y)$ imply

$$\int_\Omega |A_t(x,y)|\, dy \le c(n,\sigma) \tag{51}$$

and

$$\int_{|y-z| \le r} |A_t(x,y)| dy \le c(n,\sigma) \left(\frac{r}{t} \right)^\sigma. \tag{52}$$

Indeed,(51) follows by summing (47) over all dyadic rings $2^j \le |x - y| < 2^{j+1}$. Next observe that in (52), $x \in \Omega$ and $z \in \mathbb{R}^n$ are not correlated. If $|x - z| \ge 2r$ then $r \le |y - z| \le 3r$ when $|y - z| \le r$ so that (47) gives (52). If $|x - z| \le 2r$ then $|x -$

$y| \le 3r$ so that (52) follows by summing (47) over $r2^j \le |x - y| < r2^{j+1}$ for $= 1, 0, -1, \ldots$.

Now, let us show that $C_t(x,y) = \int_\Omega A_t(x,z)Bt(z,y)\,dz$ satisfies

$$|C_t(x,y)| \le ct^{-n} w_s \left(\frac{|x - y|}{t}\right) e^{-\frac{\gamma|x-y|}{t}}, \tag{53}$$

for some $c, \gamma > 0$. Once this is done, we get for $x \neq y$

$$\int_0^{+\infty} |C_t(x,y)| \frac{dt}{t} \le \frac{c}{|x - y|^n}$$

since $s < n$

Without loss of generality we may take $\Omega = \mathbb{R}^n$ by setting $A_t(x,z) = B_t(z,y) = 0$ if $z \notin \Omega$ and $t = 1$ since it does not play any role. We drop the subscript t in the notation and set $d = |x - y|$. We distinguish two regions of integration depending on :

(i) $z \in E$ defined by $|y - z| \ge d/2$. We have $|B(y,z)| \le w_s\left(\frac{d}{2}\right) e^{-d/2}$ by (49). Thus, by (51),

$$\int_E |A(x,z)B(z,y)|dz \le w_s(d/2)e^{-d/2} \int_\Omega |A(x,z)|\,dz \le c\, w_s(d/2)e^{-d/2}.$$

(ii) $z \in F$ defined by $|z - y| \le d/2$. Decompose further F as the union of the rings F_j defined by $d2^{-j-2} < |z - y| \le d2^{-j-1}$ with $= 0, 1, \ldots$. Then

$$\int_{F_j} |A(x,z)B(z,y)|\,dz \le w_s\left(d2^{-j-2}\right) |z - y| \le d2^{-j-1}\, |A(x,z)|\,dz.$$

We have two bounds for $A_j = \int_{|z-y| \le d2^{-j-1}} |A(x,z)|dz$. The first one comes from (52):

$$A_j \le c\left(d2^{-j-1}\right)^\sigma;$$

the second one from (47):

$$A_j \le c\left(\frac{d}{2}\right)^\sigma e^{-d/2},$$

since on the support of the integral we have $d/2 \le |x - z| \le 3d/2$. Hence, for $0 \le \theta \le 1$ we have

$$A_j \le c\left(d2^{-j-1}\right)^{\sigma\theta} \left((d/2)^\sigma e^{-d/2}\right)^{1-\theta}.$$

Pick $s/\sigma < \theta < 1$,then summing over all $j \ge 0$ yields

$$\int_F |A(x,z)B(z,y)|\,dz \le c\left((d/2)^\sigma e^{-d/2}\right)^{1-\theta} \sum_{j=0}^{\infty} \left(d2^{-j-1}\right)^{\sigma\theta} w_s\left(d2^{-j-2}\right)$$

$$\le c\, d^\sigma w_s(d) e^{-(1-\theta)d/2}.$$

Thus,(53) is proved.

Let us turn to the Hölder inequality in the second variable. It suffices to obtain

$$|C_t(x,y) - C_t(x,y')| \le cw_{s+\eta}\left(\frac{|x - y|}{t}\right)\left(\frac{|y - y'|}{t}\right)^\eta \tag{54}$$

for $x, y, y \in \Omega$ with $|y - y'| \le |x - y|/2$ since we can recover the exponential decay by interpolating with the upper bound (53),and then deduce (using $s + \eta < n$)

$$\int_0^{+\infty} |C_t(x,y) - C_t(x,y\,)| \frac{dt}{t} \le \frac{c|y - y'|^{\eta'}}{|x - y|^{n+\eta'}}$$

for $0 < \eta < \eta$. As before,set $t = 1$. Observe that the inequality $|B(z,y) - B(z,y)| \leq c\left(w_{s+\eta}(|z - y|) + w_{s+\eta}(|z - y|)\right)|y - y'|^{\eta}$, for all $z, y, y' \in \Omega$ holds. Thus

$$|C(x,y) - C(x,y')| \leq c|y - y'|^{\eta}\left(\int_{\Omega} |A(x,z)| w_{s+\eta}(|z - y|)\, dz + \int_{\Omega} |A(x,z)| w_{s+\eta}(|z - y'|)\, dz\right).$$

Split each integral as before (where $s + \eta$ replaces s) to obtain

$$|C(x,y) - C(x,y')| \leq c|y - y'|^{\eta}\left(w_{s+\eta}(|x - y|) + w_{s+\eta}(|x - y'|)\right),$$

for all $x, y, y \in \Omega$ and (54) is proved.

It remains to obtain the Hölder inequality in the first variable. Assume that $x, x, y \in \Omega$ and $|x - x| \leq |x - y|/2$. Set $d = |x - y|$ and assume $t = 1$. One has

$$|C(x,y) - C(x',y)| \leq \int_{\Omega} |A(x,z) - A(x,z)||B(z,y)|\, dz.$$

Now split the integral as above using the sets E, and F_j. If $z \in E$, then

$$\int_{E} |A(x,z) - A(x',z)||B(z,y)|\, dz \leq |x - x'|^{\eta} w_s(d/2).$$

For $j \leq 0$,

$$\int_{F_j} |A(x,z) - A(x',z)||B(z,y)| dz \leq |x - x'|^{\eta}\left(d2^{-j-1}\right)^{\sigma} w_s\left(d2^{-j-2}\right)$$

and since $\sigma < s$

$$\int_{F} |A(x,z) - A(x',z)||B(z,y)|\, dz \leq c|x - x'|^{\eta} d^{\sigma} w_s(d).$$

Hence,

$$|C(x,y) - C(x',y)| \leq c|x - x'|^{\eta} w_s(|x - y|).$$

Interpolating with the upper bound (53) yields

$$\int_0^{+\infty} |C_t(x,y) - C_t(x',y)| \frac{dt}{t} \leq \frac{c|x - x'|^{\eta}}{|x - y|^{n+\eta}} \tag{55}$$

for $0 < \eta' < \eta$ since $s < n$

Let us come to the analysis of $L^{1/2}$ and study the first two terms in (26) via Calderón-Zygmund theory. We shall need repeatedly the following lemma.

Lemma (2.2.15)[90]: Assume that (H2) and (H3) hold.

(i) Let $\Psi(\zeta) = a\zeta^2 e^{-2\zeta}$ where a is the constant defined in (23). Then, the kernel $L_t(x,y)$ of $\Psi(t^2 L)$ has upper bounds and Hölder estimates in the first variable as the ones for $K_{t^2}(x,y)$ in (H2) and (H3).

(ii) In the notation of Lemma (2.2.14), the kernels $\theta_t(x,y)$ and $\tilde{\theta}_t(x,y)$ of respectively $\theta_t = -te^{-t^2 L}\operatorname{div} A$ and $\tilde{\theta}_t = -at^3 Le^{-2t^2 L}\operatorname{div} A$ are of the form $cA_{\alpha t}(x,y)$ with parameter $\sigma = n - 1$ where $c, \alpha > 0$ are some normalizing constants.

Proof. The result on $\Psi(t^{2L})$ follows from Lemma (2.2.11). Using the explicit formulæ for the kernels of θ_t and $\tilde{\theta}_t$, (ii) is straightforward from Proposition (2.2.13),Proposition (2.2.12).

The proof of Lemma (2.2.5) relies on

Proposition (2.2.16)[90]: (H1), (H2) and (H3) imply that $T_1 = \int_0^T \tilde{\theta}_t R_t \frac{dt}{t}$ is a Calderon-Zygmund operator.

Proof. Let us first look at the kernel bounds: by the above lemma, the kernel of $\tilde{\theta}_t$ is of the form $cA_{\alpha t}(x,y)$ with $\sigma = n-1$ and $\alpha > 0$. Also the kernel of R_t has the form $cB_{\beta t}(x,y)$ with parameters $s = n - 2$ and $\eta = \mu$, and $c, \beta > 0$. The Calderon-Zygmund bounds on the kernel of T_1 follow by the lemma.

The main point is the L^2-boundedness of T_1. As mentioned, we use a reduction to quadratic estimates: this is the reason for our initial choice of writing $L^{1/2}$. First, factor $\tilde{\theta}_t$ as $\psi(t^2L)\theta_t$ where $\theta_t = -te^{-t^2L}\text{div}\, A$ and $\psi(\zeta) = a\zeta\, e^{-\zeta}$. Fix $F \in L^2(\Omega, \mathbb{C}^n)$ and $g \in L^2(\Omega)$. We have

$$\langle \tilde{\theta}_t R_t F, g\rangle = \langle \theta_t R_t F, \psi(t^2 L^*)g\rangle$$

so that

$$\int_0^T |\langle \tilde{\theta}_t R_t F, g\rangle| \frac{dt}{t} \le \left(\int_0^T \|\theta_t R_t F\|_2^2 \frac{dt}{t}\right)^{\frac{1}{2}} \left(\int_0^T \|\psi(t^2 L^*)g\|_2^2 \frac{dt}{t}\right)^{\frac{1}{2}}.$$

By functional calculus, $\left(\int_0^\infty \|\psi(t^2L^*)g\|_2^2 \frac{dt}{t}\right)^{\frac{1}{2}} \le c\|g\|_2$, so that everything reduces to the following result.

Next, let us prove Lemma (2.2.6). It is a consequence of

Proposition (2.2.17)[90]: For all $f \in V$,

$$\int_0^T \tilde{\theta}_t \nabla(I - R_t)\frac{dt}{t} = T_2(-\Delta)^{\frac{1}{2}}$$

where T_2 is bounded on $L^2(\Omega)$. Moreover, if (H2) and (H3) hold then T_2 is a Calderon-Zygmund operator.

Proof. We begin with the L^2-boundedness, following the same pattern as above. For $f \in V$,write

$$\tilde{\theta}_t \nabla\{I - (I - t^2\Delta)^{-1}\}f = \tilde{\theta}_t\, t\, \nabla \tilde{\psi}(-t^2\Delta)(-\Delta)^{\frac{1}{2}} f = \Psi(t^2L)\tilde{\psi}(-t^2\Delta)(-\Delta)^{\frac{1}{2}} f$$

where Ψ is defined in Lemma (2.2.15),i),and

$$\tilde{\psi}(\zeta) = (1 - \frac{(1+\zeta)^{-1}}{\zeta^{1/2}} = \frac{\zeta^{1/2}}{1+\zeta}.$$

Thus $T_2 = \int_0^T \Psi(t^2L)\tilde{\psi}(-t^2\Delta)\frac{dt}{t}$. Fix $f \in L^2(\Omega)$ and $g \in L^2(\Omega)$. Then,

$$\langle \Psi(t^2 L)\tilde{\psi}(-t^2\Delta)f, g\rangle = \langle \tilde{\psi}(-t^2\Delta)f, \Psi(t^2 L^*)g\rangle$$

so that

$$|\langle T_2 f, g\rangle| \le \left(\int_0^T \left\|\tilde{\psi}(-t^2\Delta)f\right\|_2^2 \frac{dt}{t}\right)^{\frac{1}{2}} \left(\int_0^T \|\Psi(t^2 L^*)g\|_2^2 \frac{dt}{t}\right)^{\frac{1}{2}}.$$

This is bounded by $c\|f\|_2\|g\|_2$ using square function estimates for L^* and $-\Delta$.

Let us turn to kernel estimates. Using the Cauchy formula,one has

$$\tilde{\psi}(-t^2\Delta) = \frac{1}{2\pi i}\int_\gamma \tilde{\psi}(\zeta)(\zeta + t^2\Delta)^{-1} d\zeta,$$

where γ is the path described counterclockwise made of two half rays $\zeta = re^{\pm i\nu}$, $r \ge 1$,and of the circular arc $\zeta = e^{i\eta}, |\eta| \le \nu$, for some fixed $\nu \in (0,\pi)$. Calling $M_t(x,y)$ the kernel $\tilde{\psi}(-t^2\Delta)$,the estimates on the kernel of $(\zeta + t^2\Delta)^{-1}$ and routine calculations imply the existence of two constants $c \geq 0$ and $\eta \in (0,1)$ such that for all $t > 0$ (or for $0 < t < 1$ when $V = H^1(\Omega)$ and Ω is bounded)

$$|M_t(x,y)| \leq \frac{c}{t|x - y|^{n-1}(1 + |x - y|/t)^2}, \qquad x,y \in \Omega,$$

and

$$|M_t(x,y) - Mt(x,y')| \leq \frac{c|y - y'|^\eta}{t|x - y|^{n-1+\eta}(1 + |x - y|/t)^2}, \quad x,y,y \in \Omega,$$

with $|y - y'| \leq |x - y|/2$. Since the kernel of $\Psi(t^2L)\,\tilde{\psi}(-t^2\Delta)$ is given by $\int_\Omega L_t(x,z)M_t(z,y)\,dz$, the estimates of Lemma (2.2.15),i),on $L_t(x,z)$ for t $\in (0,T)$ and the ones just obtained on $M_t(z,y)$ imply that T_2 has a Calderon-Zygmund kernel. The computations are in spirit of Lemma (2.2.14) but technically simpler since $L_t(x,y)$ has pointwise upper bounds. We skip details.

Theorem (2.2.18)[90]: Assume (H2) holds. Then (H1) implies that

$$\left(\int_0^T \|\theta_t R_t F\|_2^2 \frac{dt}{t}\right)^{\frac{1}{2}} \leq c\|F\|_2. \tag{56}$$

The proof of this result is the object. Note that it does not rely on the regularity of the heat kernel since (H3) is not assumed.

Proof. First,the results show that (H1) is equivalent to

$$\left(\int_0^T \left\|\tilde{\theta}_t F\right\|_2^2 \frac{dt}{t}\right)^{\frac{1}{2}} \leq C_1\|F\|_2. \tag{57}$$

Next,we need to recall the notion of Carleson measure. Say that a cube Q is in Q_Ω if Q has sides parallel to the axes and either (a) $Q \subset \Omega$ or (b) $Q \cap \partial\Omega \neq \emptyset, Q$ has centre in Ω and $\ell(Q) \leq \rho_0$ for some constant $\rho_0 > 0$ chosen so small to guarantee that $Q \cap \partial\Omega$ is (possibly up to a rotation in $\mathbb{R}^n$) contained in a Lipschitz graph. The choice of ρ_0 is as follows.

If Ω is a special Lipschitz domain, set $\rho_0 = \infty$. Otherwise, the part of $\partial\Omega$ outside a large enough ball B is (possibly up to a rotation) contained in a Lipschitz graph. Since one can cover $\partial\Omega \cap B$ by finitely many parts of Lipschitz graphs (possibly up to rotations), a compactness argument provides us with some suitable finite value for ρ_0.

As a consequence, there exists $\gamma > 0$ depending only on the Lipschitz constant of Ω such that if $Q \in Q_\Omega$ then $|Q| \leq \gamma\,|Q \cap \Omega|$.

For function $b(x,t) = b_t(x)$ defined for $(x,t) \in \Omega \times (0,T)$ and measurable, we set

$$|b_t|_c := \sup\left(\frac{1}{|Q \cap \Omega|}\int_0^{\ell(Q)\wedge T}\int_{Q\cap\Omega} |b_t(x)|^2 \frac{dxdt}{t}\right)^{\frac{1}{2}},$$

where the supremum is taken over those cubes in $Q \in Q_\Omega$ and $\ell(Q)$ stands for the sidelength of Q. We do not indicate T in the notation as the context will make it clear. The finiteness of $|b_t|_c$ means that $|b_t(x)|^{2\frac{dxdt}{t}}$ is a Carleson measure on $\Omega \times (0,T)$. With this definition, recall the Carleson inequality for later use:

$$\int_0^T \int_\Omega |P_t f(x)|^2 \,|b_t(x)|^2 \frac{dxdt}{t} \leq c(n,\gamma)|b_t|_c^2 \int_\Omega |P^* f(x)|^2\,dx, \tag{58}$$

where (P_t) is a family of operators and $P^* f\ (x) = \sup\ |P_t f\ (y)|$,the supremum being taken over all $(y,t)\ \in\ \Omega\ \times\ (0,T)$ with $|x\ -\ y|\ \leq\ t$.

With this in hand,we can continue our discussion by stating a simple lemma.

Lemma (2.2.19)[90]: Let $k_t(x,y)$ be kernels on $\Omega\ \times\ \Omega$ and $\varepsilon\ > 0$ such that

$$\int_{r\leq|x-y|\leq 2r} |k_t(x,y)|\, dy\ \leq \left(\frac{t}{r}\right)^{n+\varepsilon},$$

uniformly in $t\ \in\ (0,T), r > t$ and $x\ \in\ \Omega$. Assume there are operators k_t uniformly bounded on $L^2(\Omega)$ defined by $k_t f\ (x)\ =\ \int_\Omega\ k_t(x,y) f\ (y)\, dy$ when $\notin$ Supp f . Let $Q\ \in\ Q_\Omega$ and χ be the characteristic function of $Q\ \cap\ \Omega$. Then

$$\frac{1}{|Q\ \cap\ \Omega|}\int_0^{\ell(Q)\wedge T}\int_{Q\cap\Omega} \left|k_t\big(a(1\ -\ \chi\ )\big)(x)\right|^2 \frac{dxdt}{t}\ \leq\ c^2\ \|a\|_\infty^2,$$

where $c\ > 0$ depends on $\sup_{0<t<T}\ \|k_t\|_{2,2}, n$ and ε.

Proof. Extend $k_t(x,y)$ to be 0 if $(x,y)\ \notin\ \Omega\ \times\ \Omega$ or $t\ \geq\ T$. The argument follows in part the proof of Lemma 15 in Chapter 2 of [94]. Observe in the end that $|Q\ \cap\ \Omega|\ \geq\ \gamma\ |Q|$ since $Q\ \in\ Q_\Omega$.

In view of this lemma applied to $\theta_t(x,y)$ thanks to (39) and of (57), we easily obtain

$$\left|\theta_t\big(e_j\ \big)\right|_c\ \leq\ C_1\ +\ cC_G\ =\ C_2. \tag{59}$$

Here, $(e_1,\ldots,e_n)$ is the canonical basis of C_n, each vector being identified with a constant function.

The next step is to analyze $\theta_t(R_t)^k$ for some integer $k\ \in\ N^*$ chosen so as to insure good kernel estimates on $(R_t)^k$. More precisely, if $k\ >\ n/2$,it follows from the estimates (34) and (35) and standard computations that, there exists constants $c,\alpha\ >\ 0$ and $\mu\ \in\ (0,1)$ such that for all $x,y,y\ \in\ \Omega$ and $t\ \in\ (0,T)$,

$$|R_t^k\ (x,y)|\ \leq\ ct^{-n}e^{-\frac{\alpha|x-y|}{t}}, \tag{60}$$

$$\left|R_t^k\ (x,y) -\ R_t^k\ (x,y')\right|\ \leq\ ct^{-n}\left(\frac{|y\ -\ y'|}{t}\right)^{\mu}, \tag{61}$$

where $R_t^k\ (x,y)$ denotes the kernel of $(R_t)^k$. Thanks to the upper bound on its kernel,the maximal operator associated to $(R_t)^k$ is bounded on $L^2(\Omega)$,hence by the Carleson inequality (58) and (59)

$$\left(\int_0^T\ \left\|M_t(R_t)^k\ F\right\|_2^2 \frac{dt}{t}\right)^{\frac{1}{2}}\ \leq\ cC_2\|F\|_2\ =\ C_3\|F\|_2, \tag{62}$$

Where $M_t\ :\ L^2(\Omega,\mathbb{C}^n) \to\ L^2(\Omega)$ is defined for $F\ =\ (F_1,\ldots,F_n)$ by $M_t F\ (x)\ =\ \sum_{j=1}^n\ \Big(\theta_t\big(e_j\big)\Big)(x) F_j\ (x)$, for all $x\ \in\ \Omega$.

Now,we have the following lemma.

Proposition (2.2.20)[90]: The kernel $U_t(x,y)$ of $U_t\ = \{\ \theta_t\ -\ M_t\ \}(R_t)^k$ satisfies

$$|U_t(x,y)|\ \leq\ cC_G t^{-n}e^{-\frac{\alpha|x-y|}{t}}, \tag{63a}$$

$$|U_t(x,y')\ -\ U_t(x,y)|\ \leq\ cC_G t^{-n}\left(\frac{|y\ -\ y'|}{t}\right)^{\mu}, \tag{63b}$$

for some $\alpha > 0, \mu \in (0,1)$ and $c > 0$ whenever $t \in (0,T)$ and $x,y,y' \in \Omega$. Moreover, for $1 \le j \le n$, $\left|U_t(e_j)(x)\right|^2 \frac{dxdt}{t}$ are Carleson measures on $\Omega \times (0,T)$ and

$$\left|U_t(e_j)\right|_c \le cC_G \tag{64}$$

with c depending only on n, α and the Lipschitz constant of Ω.

Proof. $U_t(x,y)$ is $\mathbb{C}^n$-valued and its jth component is given by

$$U_{t,j}(x,y) = \int_\Omega (\theta_t(x,z)\cdot e_j)R_t^k(z,y)\,dz - \left(\theta_t(e_j)\right)(x)\,R_t^k(x,y).$$

By Proposition (2.2.13), the function $\left(\theta_t(e_j)\right)(x)$ is bounded on Ω uniformly in $t \in (0,T)$. Hence the estimates for $\left(\theta_t(e_j)\right)(x)R_t^k(x,y)$ are immediate.

The estimates for the other part can be obtained on applying part of Lemma (2.2.14) (since only (H2) is assumed) by writing $\theta_t(x,z) = cA_{\nu t}(x,y)$ with $\sigma = n-1$ and c depending on C_G,and $R_t^k(x,y) = cB_{\nu' t}(x,y)$ replacing $w_s(u)$ by 1 for all $u > 0$.

Under Neumann boundary condition,it is known that $(R_t)^k(20) = 1$ (conservation property),hence $U_t(e_j) = 0$ for all j ,which proves (64). Under Dirichlet boundary condition, the Carleson measure estimate requires a specific argument that is postponed.

By the T 1-theorem for Carleson measures in [98] adapted to the space of homogeneous type $(\Omega, |.|, dx)$ we deduce from the above result that

$$\int_0^T \left\| \{\theta_t - M_t\}(R_t)^k F\right\|_2^2 \frac{dt}{t} \le cC_G^2\|F\|_2^2.$$

Since $\theta_t(R_t)^k = M_t(R_t)^k + \{\theta_t - M_t\}(R_t)^k$ we have obtained

$$\int_0^T \left\|\theta_t(R_t)^k F\right\|_2^2 \frac{dt}{t} \le C_3\|F\|_2^2. \tag{65}$$

The last step is to come back to $\theta_t R_t$: we use functional calculus again. Let $\psi(\zeta) = (1+\zeta)^{-1} - (1+\zeta)^{-k}$, and notice that $|\psi(\zeta)| \le c \inf(|\zeta|, |\zeta|^{-1})$ for $\zeta > 0$, we have

$$\int_0^\infty \|\psi(-t\,2\,\Delta)g\|_2^2 \frac{dt}{t} = a(\psi)\|g\|_2^2, \tag{66}$$

with $a(\psi) = \frac{1}{2}\int_0^\infty |\psi(t)|^2 \frac{dt}{t} < \infty$. This and the uniform L^2-boundedness of θ_t (which depends only on ellipticity) give us

$$\int_0^T \left\|\theta_t\{R_t - (R_t)^k\}F\right\|_2^2 \frac{dt}{t} \le ca(\psi)C_G^2\|F\|_2^2,$$

where c depends only on n and δ.

We have obtained that

$$\int_0^T \|\theta_t R_t F\|_2^2 \frac{dt}{t} \le cC_G^2\|F\|_2^2.$$

as desired.

Let us mention that Theorem (2.2.18) has a converse. Namely,(56) implies (H1) (the condition (H2) is not used). Here is a proof.

We can write for all $f \in V$,

$$\theta_t\nabla f = \theta_t R_t \nabla f + \theta_t\nabla(I - R_t)f + \theta_t[\nabla, R_t]f.$$

Let us look at each term. By hypothesis,

$$\int_0^T \|\theta_t R_t \nabla f\|_2^2 \frac{dt}{t} \leq c\|\nabla f\|_2^2.$$

Next,write

$$\theta_t \nabla \{ I - (I - t^2 \Delta)^{-1}\} f = e\wedge - t^{2L} t^2 L \tilde{\psi}(-t^2 \Delta)(-\Delta)^{\frac{1}{2}} f$$

where $\tilde{\psi}$ is given in (38). Since $e^{-t^2L} t^2 L$ extends boundedly to $L^2(\Omega)$ uniformly in t,

$$\|\theta_t \nabla \{ I - (I - t^2\Delta)^{-1}\} f\|_2 \leq c(\delta) \left\|\tilde{\psi}(-t^2 \Delta)(-\Delta)^{\frac{1}{2}} f\right\|_2.$$

Hence,

$$\int_0^\infty \| \theta_t \nabla \{ I - (I - t^2 \Delta)^{-1}\} f\|_2^2 \frac{dt}{t} \leq c(\delta)^2 a(\tilde{\psi}) \|(-\Delta)^{1/2} f\|_2^2,$$

and we conclude for this term using that $\|(-\Delta)^{1/2} f \|_2 = \|\nabla f\|_2$ for all $f \in V$. For the last term,the uniform L^2-boundedness of θ_t gives us for all $t > 0$

$$\|\theta_t [\nabla, R_t] f\|_2^2 \leq c(\delta)\| [\nabla, Rt] f\|_2^2,$$

and the quadratic estimate follows from the commutator inequality

Proposition (2.2.21)[90]: ([95]) For all $f \in V$,

$$\int_0^{t_0} \|[\nabla, R_t] f\|_2^2 \frac{dxdt}{t} \leq c_0\|\nabla f\|_2^2 + c_1\|f\|_2^2,$$

where c_0 depends only on the Lipschitz constant of Ω and n. This is proved with $t_0 = \infty$ and $c_1 = 0$ when Ω is a special Lipschitz domain and also when Ω is a bounded domain with t_0 finite. The proof works for any other strongly Lipschitz domain with t_0 finite.

Hence, we have proved for all $f \in V$

$$\int_0^T \|\theta_t \nabla f\|_2^2 \frac{dxdt}{t} \leq c\|\nabla f\|_2^2 + c \|f\|_2^2,$$

where $c' = 0$ in the case of a special Lipschitz domain. This implies (H1) by Proposition (2.2.40).

We restrict our attention to the case of Dirichlet boundary condition. Set $g_t = (R_t)^k$ (20) and $\theta_{t,j} (x, y) = \theta_t(x, y) \cdot e_j$. Then for $t \in (0, T)$ and $x \in \Omega$

$$\left(U_t(e_j)\right)(x) = \int_\Omega \theta_{t,j} (x, y)\left(g_t(y) - g_t(x)\right) dy$$

$$= \int_\Omega \theta_{t,j} (x, y)(g_t(y) - 1) dy + \left(1 - g_t(x)\right) \int_\Omega \theta_{t,j} (x, y) dy.$$

Denote by $d(x) = \text{dist}\,(x, {}^c\Omega)$. That $|U_t(e_j)(x)^2 \frac{dxdt}{t}$ is a Carleson measure on $\Omega \times (0, T)$ is a consequence of the following lemmata.

Lemma (2.2.22)[90]: If $\alpha > 0$, then $e^{-\frac{\alpha d(x)}{t}} \frac{dxdt}{t}$ is a Carleson measure on $\Omega \times \mathbb{R}_+^*$ with norm that depends only on n, α and the Lipschitz constant of Ω.

Proof. It is based on the following geometrical observation. The Lipschitz condition on the boundary of Ω implies that for each cube $Q \in Q_\Omega$ and each $s > 0$,

$$m(s) = |\{x \in Q \cap \Omega\, ;\, d(x) < s\}| \leq c\ell(Q)^{n-1}s,$$

where c depends only on n and the Lipschitz constant of Ω. Also, if $f(x) = e^{-\frac{\alpha d(x)}{t}}$ then $f(x) > \lambda$ if, and only if, $d(x) < s$ with $= e^{-\frac{\alpha s}{t}}$. Therefore,

$$\int_{Q\cap\Omega} e^{-\frac{\alpha d(x)}{t}}\,dx = \int_0^1 |\{f > \lambda\}|\,d\lambda = \frac{\alpha}{t}\int_0^{+\infty} m(s)e^{-\frac{\alpha s}{t}}\,ds \le \frac{c\ell(Q)^{n-1}t}{\alpha},$$

and thus

$$\int_0^{\ell(Q)}\int_{Q\cap\Omega} e^{-\frac{\alpha d(x)}{t}}\frac{dxdt}{t} \le \frac{c|Q|}{\alpha} \le \frac{c\gamma|Q\cap\Omega|}{\alpha}.$$

The last inequality follows from the fact that $Q \in Q_\Omega$.

Lemma (2.2.23)[90]: If V_t is a linear operator whose kernel satisfies for all $r > 0$,

$$\sup_{x\in\Omega}\int_{\substack{r\le|x-y|\le 2r\\ y\in\Omega}} |V_t(x,y)|\,dy \le \inf\left(\left(\frac{r}{t}\right)\varepsilon, e^{-\frac{\alpha r}{t}}\right)$$

for some $\alpha, \varepsilon > 0$, then there exists $\alpha > 0$ such that for all $x \in \Omega$

$$\int_\Omega \left|V_t(x,y)e^{-\frac{\beta d(y)}{t}}\right| dy \le c(n,\alpha,\beta)e^{-\frac{\beta d(x)}{t}}$$

for all $\beta \in (0,\alpha)$.

Proof. The proof is straightforward once we observe that $|d(y) - d(x)| \le |y - x|$.

Lemma (2.2.24)[90]: There is $\alpha > 0$ such that for all $t \in (0,T)$ and $x \in \Omega$,

$$|1 - g_t(x)| \le ce^{-\frac{\alpha d(x)}{t}},$$

where c depends only on the Lipschitz constant of Ω.

Proof. Write

$$I - (R_t)^k = \sum_{j=0}^{k-1} (R_t)^j\,(I - R_t).$$

It follows from (34) and easy calculations that Lemma (2.2.23) applies to each $(R_t)^j$ so that we are reduced to proving

$$|v_t(x)| \le ce^{-\frac{\alpha d(x)}{t}}, \qquad x \in \Omega, 0 < t \le T, \tag{67}$$

where $v_t = 1 - R_t(20)$. This last estimate is obtained using elementary potential theory. Indeed, v_t is a bounded solution of the problem

$$\begin{cases} v_t - t^2\Delta v_t = 0, in\ \Omega,\\ v_t = 1, on\ \partial\Omega,\end{cases}$$

so that the minimum principle gives us $v_t \ge 0$ in **Ω**. Next, denote by $E(x)$ the fundamental solution of $I - \Delta$ on $\mathbb{R}^n$ that vanishes at ∞. Hence, for $t > 0$, $E_t(x) = t^{-n}E(x/t)$ is the fundamental solution of $I - t^2\Delta$ on $\mathbb{R}^n$ vanishing at ∞. Set

$$r_t(x) = \int_{c_\Omega} E_t(x - y)dy, \qquad x \in \Omega, t > 0.$$

This function is well-defined by the properties of E which we recall for convenience (see [109]): For all $x \ne 0$, $E(x) > 0$,and there exist constants $c_1, c_2, \alpha_1, \alpha_2 > 0$ such that

$$c_1 w_{n-2}(|x|)e^{-\alpha_1|x|} \le E(x) \le c_2 w_{n-2}(|x|)e^{-\alpha_2|x|}.$$

It is clear that $r_t - t^2\Delta r_t = 0$ in Ω and that, by a direct estimate,

$$r_t(x) \le ce^{-\frac{\alpha d(x)}{t}}, \qquad x \in \Omega, t > 0.$$

We claim that there exists a number $\kappa > 0$ such that $r_t \ge \kappa$ on $\partial\Omega$ for all $t \in (0,T]$. Admitting this claim, we deduce from the minimum principle that $\kappa^{-1}r_t - v_t \ge 0$ in Ω. Hence,for all $x \in \Omega$ and $t \in (0,T]$

$$v_t(x) \le \kappa^{-1} r_t(x) \le \kappa^{-1} ce^{-\frac{\alpha d(x)}{t}},$$

which is (67). Proof of the claim. We distinguish the special domains from the other cases. When Ω is a special Lipschitz domain,it has the exterior infinite cone condition. That is, there is a fixed cone Γ with vertex at 0 and aperture $a > 0$ such that for each $x \in \partial\Omega, \Gamma(x) = \{x\} + \Gamma \subset {}^c\Omega$. Since Γ is invariant under change of scales,it follows that for $x \in \partial\Omega$ and $t > 0$,

$$r_t(x) \geq \int_{\Gamma(x)} E_t(x-y)\,dy = \int_{\Gamma} E_t(z)\,dz = \int_{\Gamma} E(z)dz > 0.$$

When Ω is a strongly Lipschitz domain (bounded or not), it has the exterior truncated cone condition. That is,one can find an aperture $a > 0$ and a height $h > 0$ such that for each $x \in \partial\Omega$ there is a truncated cone cone $\Gamma_h(x) = \{x\} + \Gamma_h \subset {}^c\Omega$ obtained from a fixed cone Γ_h with vertex at 0, aperture $a > 0$ and height $h > 0$ by translation and rotation in $\mathbb{R}^n$. Thus, for $x \in \partial\Omega$,

$$r_t(x) \geq \int_{\Gamma_h} E_t(z)\,dz = \int_{\Gamma_{h/t}} E(z)\,dz > 0, t > 0.$$

This last quantity is a continuous function of t and it tends to $\int_\Gamma E(z)\,dz > 0$ as t tends to 0. Hence, inf $\{r_t(x);\ 0 < t \leq 1, x \in \partial\Omega\} > 0$.

Concerned with the analysis of the term in (26) involving the commutator $\mathcal{C}_t$ defined by (25),that is we prove Lemma (2.2.7). This is where the geometry of the boundary most enters. As indicated by this result,we begin with special Lipschitz domains,and then move to the general case.

Ω is a special Lipschitz domain. Let $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfying $\|\nabla\phi\|_\infty = M < \infty$ such that $\Omega = \{(x', x_n) \in \mathbb{R}^n ;\ x_n > \phi(x)\}$. The Lipschitz constant of Ω is M by definition. If $x = (x', x_n) \in \Omega$ then $\bar{x} = \big(x, \phi(x')\big)$ is its vertical projection on $\partial\Omega$ and $x^* = (x, 2\phi(x) - x_n)$ is its vertical symmetric across $\partial\Omega$. We shall consistently use the notation $\bar{x}$ to denote a point on $\partial\Omega$ (this confusion is convenient). Surface measure on $\partial\Omega$ is denoted by σ and $N(\bar{x})$ is the exterior unit normal at $\bar{x} \in \partial\Omega$. We shall often perform integration over Ω by writing $dy = d\sigma(\bar{y})du$ for $y = \bar{y} + ue_n, \bar{y} \in \partial\Omega$ and $u > 0$. Our analysis involves the following operators.

Definition (2.2.25)[90]: We say that H is a Hardy operator if H is an integral operator of the form $Hf(x) = \int_\Omega K(x,y)f(y)\,dy$ where $|K(x,y)| \leq C|x - y^*|^{-n}$ for almost every $(x,y) \in \Omega \times \Omega$.

Definition (2.2.26)[90]: We say that B is a boundary Hardy operator if it has the form $B_g(x) = \int_{\partial\Omega} K(x,\bar{y})\big(g(\bar{x}) - g(\bar{y})\big)\,d\sigma(\bar{y})$ where $|K(x,\bar{y})| \leq C|x - \bar{y}|^{-n}$ for almost every $(x,\bar{y}) \in \Omega \times \partial\Omega$.

Proposition (2.2.27)[90]: For $1 < p < \infty$,

$$\|Hf\|_p \leq c(p,M)\|f\|_p, \qquad \forall f \in L^p(\Omega). \tag{68}$$

$$\|B(Trf)\|_p \leq c(p,M)\|\nabla f\|_p, \qquad \forall f \in W^{1,p}(\Omega). \tag{69}$$

Here T r is the trace operator, which is bounded from $W^{1,p}(\Omega)$ onto the Besov space $B^{p,p}_{1-1/p}(\partial\Omega)$,so that the second inequality can be reformulated as the boundedness of B from that Besov space into $L^p(\Omega)$. The proof is postponed.

Let us come back to the commutators and distinguish each boundary condition.

Dirichlet boundary condition. We begin with

Lemma (2.2.28)[90]: ([95]) We have

$$[\nabla, R_t] = [\nabla, H_t] \tag{70}$$

where H_t is an integral operator with kernel $H_t(x,y)$ enjoying the upper bound

$$|H_t(x,y)| \leq \inf \left(E_t(x-y), cE_{at}(x-y^*)\right) \quad (71)$$

for all $t > 0$ and $x,y \in \Omega$, for some constants c and $a > 0$ depending only on the Lipschitz constant of Ω.

The kernel $E_t(x)$ was defined earlier. The idea of (70) is that the commutator annihilates the convolution part of R_t : in other words, Ht is nothing but $R_t - E_t$. Replacing R_t by H_t uses all the cancellation contained in the commutator, so it suffices to examinate the terms coming from ∇H_t and $H_t\nabla$ separately. Let us assume for the time being that $f \in C_0^1(\Omega)$.

First,write

$$\tilde{\theta}_t \nabla H_t(f) = \frac{1}{t} \Psi(t^2L)H_t(f).$$

Since f vanishes on $\partial\Omega$,we have by the fundamental theorem of calculus that $f = I_n(D_nf)$ where $D_n = \frac{\partial}{\partial x_n}$ and

$$I_n(g)(x) = I_n(g)(\bar{x} + ue_n) = \int_0^u g(\bar{x} + ve_n)\,dv. \quad (72)$$

Hence, we have established for all $f \in C_0^1(\Omega)$ that

$$\tilde{\theta}_t [\nabla, R_t]f = \frac{1}{t} \Psi(t^2L)H_t I_n(D_nf) - \tilde{\theta}_t H_t(\nabla f).$$

In view of the next result, the above representation will extend to all of H1 0 (Ω) by density, and with the help of Proposition (2.2.27) this proves Lemma (2.2.7) in the case of Dirichlet boundary condition.

Lemma (2.2.29)[90]: Under the condition (H2), $\int_0^T \frac{1}{t} \Psi(t^2L)H_t I_n \frac{dt}{t}$ and $\int_0^T \tilde{\theta}_t H_t \frac{dt}{t}$ are Hardy operators.

Proof. In the notation of Lemma (2.2.14), the upper bound $|H_t(x,y)| \leq cE_{at}(x-y^*)$ shows that it is of the form $cB_{\beta t}(x,y^*)$ with parameter $s = n-2$ for some $c,\beta > 0$, and by Lemma (2.2.15), the kernels of $\Psi(t^2L)$ and $\tilde{\theta}_t$ are of the form $cA_{\alpha t}(x,y)$ for some $c,\alpha > 0$ with parameters $\sigma = n$ and $\sigma = n-1$ respectively. Following the proof of (53), this gives an upper bound of the form $cB_{\beta t}(x,y^*)$ with parameter $s = n-2$ for the kernels of both $\Psi(t^2L)Ht$ and $\tilde{\theta}_t H_t$. Next, use the remark that if $S_t(x,y)$ is the kernel of some operator St then the kernel of S_t In is given by $\int_{y_n}^{\infty} S_t(x,\bar{y}+ue_n)\,du$. Direct calculations conclude the proof.

Neumann boundary condition.

Lemma (2.2.30)[90]: ([95]) For $f \in C_0^1(\Omega)$ and $x \in \Omega$, we have

$$[\nabla, R_t]f(x) = 2\int_\Omega E_t(x-y^*)\, D_nf(y)\tilde{N}(\bar{y})dy - \int_\Omega F_t(x,y)\nabla f(y)dy + \int_\Omega \nabla_x Ft(x,y)f(y)dy, \quad (73)$$

with $\tilde{N}(\bar{y}) = \sqrt{1+|\nabla\phi(y)|^2}N(\bar{y})$, when $\bar{y} = (y', \phi(y'))$. Here, $F_t(x,\cdot) \in H^1(\Omega)$, and satisfies the upper bound

$$|F_t(x,y)| \leq c\inf\left(E_{at}(x-y), E_{at}(x-y^*)\right), t > 0, x \in \Omega, y \in \Omega$$

for some $c > 0$ and $a > 0$ depending on the Lipschitz constant of Ω, and

$$\int_{\Omega} F_t(x,y)\, dy = 0, \qquad t > 0, x \in \Omega.$$

Let us remark that this lemma was proved under the assumption that **Ω** be smooth but easy limiting arguments show that this technical assumption can be removed. The idea of (73) is to compare $R_t(x,y)$ to the sum of $E_t(x - y)$ and $E_t(x - y^*)$ so as to obtain the mean value property.

In short, we write, for $f \in C_0^1(\overline{\Omega})$

$$[\nabla, R_t]f = 2E_{t,*}(\tilde{N}D_n f) - F_t(\nabla f) + \nabla F_t(f)$$

so that

$$\tilde{\theta}_t [\nabla, R_t]f = 2\tilde{\theta}_t E_{t,*}(\tilde{N}D_n f) - \tilde{\theta}_t F_t(\nabla f) + \tilde{\theta}_t \nabla F_t(f). \qquad (74)$$

Note that the multiplication by $\tilde{N}$ is harmless since it is bounded.

We focus on the last term in (74). Observe that

$$\tilde{\theta}_t \nabla F_t(f) = \frac{1}{t} \Psi(t^2 L) F_t(f).$$

The same argument as above shows that the kernel $S_t(x,y)$ of $\Psi(t^2 L)F_t$ has the same upper bound as $F_t(x,y)$. In particular, it is integrable with respect to y over **Ω**. Furthermore, by Fubini's theorem

$$\int_{\Omega} S_t(x,y)\, dy = \int_{\Omega} L_t(x,z) \int_{\Omega} F_t(z,y)\, dy\, dz = 0$$

using the mean value property of $F_t(x,y)$. Hence, we have for $x \in \Omega$,

$$\Psi(t^2 L)F_t(f)(x) = \int_{\Omega} S_t(x,y)\big(f(y) - f(\bar{x})\big)dy.$$

(Recall that $\bar{x}$ is the vertical projection of $\bar{x}$ onto Ω.) Using

$$f(y) - f(\bar{x}) = f(\bar{y}) - f(\bar{x}) + I_n(D_n f)(y)$$

and performing integration over **Ω** by writing $dy = d\sigma(\bar{y})du$ for $y = \bar{y} + u e_n$ we obtain

$$\frac{1}{t} \Psi(t^2 L)F_t(f)(x) = \int_{\partial\Omega} B_t(x,\bar{y})\big(f(\bar{y}) - f(\bar{x})\big)d\sigma(\bar{y}) + \frac{1}{t} \Psi(t^2 L)F_t\, I_n(D_n)(x)$$

so that

$$\bar{\theta}_t \nabla F_t(f)(x) = B_t(Trf)(x) + \frac{1}{t} \Psi(t^2 L)F_t\, I_n(D_n f)(x), \qquad (75)$$

where

$$B_t(x,\bar{y}) = \frac{1}{t} \int_0^{\infty} S_t(x, \bar{y} + u e_n)\, du.$$

Combining (74) and (75) gives us a representation for $\tilde{\theta}_t [\nabla, R_t](f)$ valid for all $f \in C_0^1(\Omega)$.

Lemma (2.2.31)[90]: Under the condition (H2), the integrals $\int_0^T \tilde{\theta}_t E_{t,*} \frac{dt}{t}$, $\int_0^T \tilde{\theta}_t F_t \frac{dt}{t}$ and $\int_0^T \frac{1}{t} \Psi(t^2 L)F_t\, I_n \frac{dt}{t}$ are Hardy operators, and $\int_0^T B_t \frac{dt}{t}$ is a boundary Hardy operator.

To prove the lemma, it suffices to carry out upper bounds for kernels by invoking Lemma (2.2.14) as in Lemma (2.2.29). We skip details. In view of this lemma, a density argument shows that this representation extends to all of $H^1(\Omega)$ and proves Lemma (2.2.7) in this case.

To analyze $\tilde{\theta}_t\, [\nabla, R_t]$ on general Lipschitz domains, we use a localization technique at the level of the commutator $C_t = [\nabla, R_t]$. This essentially involves comparing Laplacians defined on different sets and avoids the comparisons between general elliptic operators.

Recall that we assume (H2) with some τ finite, say $\tau = 1$ for simplicity.

We begin with the domain decomposition, following [109]. There is an integer s, a number $d > 0$ and for $0 \leq k \leq s, C_0^{\infty}(\mathbb{R}^n)$ real-valued functions χ_k and η_k, and open sets O_k, P_k, Ω_k with the following properties:

(i) $0 \leq k \leq s\ \chi_k(x) = 1$, for x in a neighborhood of Ω;

(ii) $\Omega_0 = \mathbb{R}^n$, $\operatorname{Supp} \chi_0 \subset O_0 \subset \overline{O_0} \subset P_0 \subset \overline{P_0} \subset \Omega$;

(iii) For $k \geq 1, \Omega_k$ is the image of a special Lipschitz domain under an orthogonal transformation in $\mathbb{R}^n$ such that $\operatorname{Supp} \chi_k \cap \Omega \subset \Omega_k \cap \Omega$;

(iv) For $k \geq 1, O_k$ and P_k are open neighborhoods of $\operatorname{Supp} \chi_k$ in $\mathbb{R}^n$ such that $\overline{O_k} \subset P_k, P_k \cap \Omega \subset \Omega_k \cap \Omega$ and $\partial\Omega \cap \overline{P_k} = \partial\Omega_k \cap \overline{P_k}$,at most one of the latter possibly infinite;

(v) For $k \geq 0$,$\operatorname{Supp} \eta_k \subset P_k, \eta_k = 1$ on a neighborhood of $\overline{O_k}, \eta_k \geq 0$ and $\|\eta_k\|_\infty = 1$;

(vi) For $k \geq 0$, $d(O_k, {}^cP_k) \geq d$ and $d(\operatorname{Supp} \chi_k, {}^cO_k) \geq d$.

Each patch (except for the 0th one) contributes to a Lipschitz constant M_k. The Lipschitz constant of Ω is by definition the least of max $(M_1, \ldots, M_s)$ taken over all such decompositions of Ω.

For $k = 0$,let $R_{t,0} = (I - t^2\Delta)^{-1}$ be the resolvent of the Laplacian on $\mathbb{R}^n$ and $V_0 = H^1(\mathbb{R}^n)$. Since $R_{t,0}$ is a convolution operator, $C_{t,0} = [\nabla, R_{t,0}] = 0$. For $k \geq 1$,let $C_{t,k} = [\nabla, R_{t,k}]$ denote the commutator defined on functions on Ω_k using the resolvent $R_{t,k} = (I - t^2\Delta)^{-1}$ of the Laplacian $-(Id, \Omega_k, V_k)$ where $V_k = H_0^1(\Omega_k)$ (resp. $H^1(\Omega_k)$) if $V = H_0^1(\Omega)$ (resp. $H^1(\Omega)$) . For $f \in V$, note that $\chi_k f \in V \cap V_k$ so that all operations will make sense. Now that these precautions are taken, fix $f \in V$ and since $f = \sum \chi_k f$,write for any $t > 0$,

$$\mathcal{C}_t f = \sum_{0\leq k\leq s} \eta_k \mathcal{C}_{t,k}(\chi_k f) + \sum_{0\leq k\leq s} (1 - \eta_k)\mathcal{C}_t(\chi_k f) + \sum_{0\leq k\leq s} \eta_k(\mathcal{C}_t - \mathcal{C}_{t,k}(\chi_k f)$$
$$= \mathrm{I} + \mathrm{II} + \mathrm{III}, \qquad (58)$$

and to each of those terms apply $\tilde{\theta}_t$ on the left. Fix a k once for all.

Term I. We assume that $k \neq 0$, otherwise there is nothing to do since $\mathcal{C}_{t,0} = 0$. Up to a rotation in $\mathbb{R}^n$ we may and do assume that Ω_k is a special Lipschitz domain.

Dirichlet boundary condition: With obvious notation and according to Lemma (2.2.28),we have $\mathcal{C}_{t,k} = \nabla H_{t,k} - H_{t,k}\nabla$ so that

$$\tilde{\theta}_t\, [\eta_k \mathcal{C}_{t,k}(\chi_k f) = \frac{1}{t}\, \Psi\, (t^2 L)\, [\eta_k H_{t,k}(\chi_k f\,)] - \tilde{\theta}_t\, [(\nabla_{\eta_k}) H_{t,k}(\chi_k f)]$$
$$+ \tilde{\theta}_t[\, \eta_k H_{t,k} \nabla(\chi_k f)]. \qquad (76)$$

Kernel analysis of each term yields Hardy type bounds in the set P_k and exponential bounds away from P_k. Let us be more explicit by treating the case of the kernel $P(x, y)$ of $P = \int_0^1 \tilde{\theta}_t\, \eta_k H_{t,k} \frac{dt}{t}$,which is the most singular one. It is given by

$$|P(x, y)| \leq \int_0^1 \int_\Omega \left|\tilde{\theta}_t\, (x, z)\eta_k(z) H_{t,k}(z, y)\right| dz \frac{dt}{t},$$

where we restrict ourselves to $x \in \Omega$ and $y \in \Omega \cap \operatorname{Supp} \chi_k$. First, replace η_k by $\|\eta_k\|_\infty$ as this function plays no role. If $x \in P_k$, then we use $|H_t, k(z,y)| \leq cE_{at}(z - y^*)$ so that

$$|P(x,y)| \leq c|x - y^*|^{-n}$$

from the results. If $x \notin P_k$ then $|H_{t,k}(z,y)| \leq E_t(z-y)$,Supp $\chi_k \subset O_k \subset P_k$ and $d(O_k, {}^cP_k) \geq d > 0$ lead to

$$|P\,(x,y)| \leq c(d)e^{-\alpha|x-y|}.$$

This easily yields $\|P\nabla(\chi_k f)\|_p \leq c\|\nabla(\chi_k f)\|_p$.

We do the same thing for the other terms in (76) using also the trick with the operator In defined in (72) on writing $\chi_k f = I_n D_n(\chi_k f)$.

Eventually,we have

$$\left\|\int_0^1 \tilde{\theta}_t\, \eta_k \mathcal{C}_{t,k}(\chi_k f)\frac{dt}{t}\right\|_p \leq c\|\nabla(\chi_k f)\|_p.$$

and c is proportional to C_G.

Neumann boundary condition: This is similar using the operators $F_{t,k}$, local Hardy and boundary Hardy estimates... We skip details.

Term II. This term receives the same analysis under both boundary conditions as we only use size estimates on the kernel of R_t . By definition of $\mathcal{C}_t$,

$$\tilde{\theta}_t\,(1 - \eta_k)\mathcal{C}_t(\chi_k f) = \tilde{\theta}_t\,(1 - \eta_k)\nabla R_t(\chi_k f) - \tilde{\theta}_t\,(1 - \eta_k)R_t\nabla(\chi_k f)\,.$$

The analysis in the proof of Lemma (2.2.14) applies to the kernel of $\tilde{\theta}_t\,(1 - \eta_k)R_t$. Moreover,since the supports of χ_k and $1 - \eta_k$ are at distance at least d,the integrand has support contained in $|y - z| \geq d$ and we get

$$\left|\int_\Omega \tilde{\theta}_t(x,z)(1 - \eta_k)(z)R_t(z,y)dz\right| \leq \int_{|y-z|\geq d} \left|\tilde{\theta}_t\,(x,z)\right||R_t(z,y)|\,dz$$
$$\leq ct^{-n}e^{-\frac{\beta(|x-y|+d)}{t}}$$

for some $\beta > 0$,whenever $x \in \Omega,\ y \in \Omega \cap \operatorname{Supp} \chi_k, t \leq 1$. Hence,

$$\left\|\tilde{\theta}_t\,(1 - \eta_k)R_t\nabla(\chi_k f)\right\|_p \leq ce^{-\frac{\beta d}{t}}\|\nabla(\chi_k f)\|_p.$$

The estimate for the kernel of $\tilde{\theta}_t\,(1 - \eta_k)\nabla R_t$ requires a specific argument and

$$\int_\Omega \tilde{\theta}_t\,(x,z)(1 - \eta_k)(z)\nabla_z R_t(z,y)dz \leq c(d)t^{-n-1}\,e^{-\frac{\beta(|x-y|+d)}{t}} \quad (77)$$

for some $\beta > 0$,whenever $x \in \Omega, y \in \Omega \cap \operatorname{Supp} \chi_k$ and $t \leq 1$. This implies

$$\left\|\tilde{\theta}_t\,(1 - \eta_k)\nabla R_t(\chi_k f)\right\|_p \leq ct^{-1}\,e^{-\frac{\beta d}{t}}\|\chi_k f\|_p.$$

Hence,

$$\int_0^1 \left\|\tilde{\theta}_t\,(1 - \eta_k)\mathcal{C}_t(\chi_k f)\right\|_p\frac{dt}{t} \leq c\|\chi_k f\|_p + c\|\nabla(\chi_k f)\|_p.$$

In order to prove (77),we use the following estimates:

$$\int_{|y-z|\geq r} |t\nabla_z R_t(z,y)|^2\,dz \leq \begin{cases} ct^{-n}\left(\frac{t}{r}\right)^{n-2} e^{-\frac{\alpha r}{t}}, \text{if } n > 2,\\ ct^{-2}\ln^2\left(e + \frac{t}{r}\right)e^{-\frac{\alpha r}{t}}, \text{if } n = 2.\end{cases} \quad (78)$$

$$\int_{|x-z|\sim r} \left|\tilde{\theta}_t\,(x,z)\right|^2 dz \leq ct^{-n}\left(\frac{r}{t}\right)^{n-2} e^{-\frac{\alpha r^2}{t^2}}, \quad (79)$$

$$\int_{|y-z|\le r} \left|\tilde{\theta}_t\,(x,z)\right|^2 dz \;\le\; ct^{-n}\left(\frac{r}{t}\right)^{n-2}. \qquad (80)$$

for some $c > 0$ and $\alpha > 0$ independent of $r > 0$, $t \in (0, 2]$, $x, y \in \Omega$. The proof of (78) is a consequence of (34) and elliptic Caccioppoli's inequality applied to $u(z) = R_t(z,y)$ that is a weak solution of $u - t^2\Delta u = 0$ in $\Omega \setminus \{y\}$. Indeed, Caccioppoli's inequality reads

$$\int |\nabla u|^2\,\varphi^2 \;\le\; C(\delta, \|A\|_\infty)\,|u|^2\,|\nabla\varphi|^2$$

for any real-valued smooth and bounded function with bounded gradient, and we choose φ to be supported in $|y - z| \ge r/2$ and $\varphi = 1$ on $|y - z| \ge r$. Equation (79) is nothing but a rewriting of (39). Eventually, (80) is a consequence of (79) when $n \ge 3$ while it follows from the $L^2 - L^\infty$ boundedness of $\tilde{\theta}_t$ when $n = 2$ (for $n = 2$, these estimates are not sharp but they suffice for our purpose). Since $y \in \operatorname{Supp}\chi_k$, we have

$$\int_\Omega \tilde{\theta}_t(x,z)(1 - \eta_k)(z)\nabla_z R_t(z,y)\,dz \;\le\; |y-z| \ge d\,|\tilde{\theta}_t(x,z)||\nabla_z R_t(z,y)|\,dz.$$

Call $Q_t(x,y)$ this last integral. Set $R = |x - y|$. Let us only consider the case $n > 2$ to simplify matters. First, assume $R \le d/2$. By Cauchy-Schwarz

$$Q_t(x,y) \;\le\; \sum_{j\ge 0}\left(\int_{F_j}\left|\tilde{\theta}_t(x,z)\right|^2 dz\right)^{\frac{1}{2}}\left(\int_{F_j}|\nabla_z R_t(z,y)|^2\,dz\right)^{\frac{1}{2}},$$

where F_j is the set of $z \in \Omega$ such that $d2^j \le |y - z| < d2^{j+1}$. On F_j we have $|x - z| \sim d2^j$. Using (78) and (79) yields a bound of the form $ct^{-n-1}\ln\left(e + \frac{t}{d}\right)e^{-\frac{\alpha d}{t}} \le c(d)t^{-n-1}e^{-\frac{\alpha d}{t}}$ as $t \le 1$.

Assume next that $R \ge d/2$. The integral over $|y - z| \ge 2R$ can be dealt with as above where F_j is now defined by $R2^{j+1} \le |y - z| < R2^{j+2}$. We obtain a bound $c(d)t^{-n-1}\ln\left(e + \frac{t}{R}\right)e^{-\frac{\alpha R}{t}}$.

Next, if $|x - z| \le R/2$, we have $R/2 \le |y - z| \le 3R/2$, hence by (78) and (80),

$$\int_{|y-z|\ge d, |x-z|\le R/2}\left|\tilde{\theta}_t(x,z)\right||\nabla_z R_t(z,y)|\,dz \;\le\; ct^{-n-1}\,e^{-\frac{\alpha R}{t}}.$$

Lastly, if $d \le |y - z| \le 2R$ and $|x - z| \ge R/2$, then $|x - z| \le 3R$ so that by (78) and (79)

$$\int_{2R\ge|y-z|\ge d, |x-z|\ge R/2}\ldots \le\; ct^{-n-1}\left(\frac{R}{t}\right)^{\frac{n-2}{2}} e^{-\frac{\alpha R^2}{t^2}}\left(\frac{t}{d}\right)^{\frac{n-2}{2}} e^{-\frac{\alpha d}{t}} \;\le\; c(d)t^{-n-1}\,e^{-\frac{\alpha R}{t}}.$$

Term III. The computations are similar for both boundary conditions.

Let $\tilde{O}_k$ be an open set such that $O_k \subset \tilde{O}_k \subset P_k$ and $d(c\,\tilde{O}_k, O_k) \ge d/2$, and let $\tilde{\eta}_k \in C_0^\infty(P_k)$ such that $\tilde{\eta}_k = 1$ on $\tilde{O}_k$. Let also $\tilde{\chi}_k \in C_0^\infty(O_k)$ such that $\tilde{\chi}_k = 1$ on $\operatorname{Supp}\chi_k$. We find

$$\tilde{\theta}_t\,[\eta_k(\mathcal{C}_t - \mathcal{C}_{t,k}(\chi_k f)$$
$$= \frac{1}{t}\,\Psi\,(t^2 L)\eta_k\big(R_t - R_{t,k}\big)(\chi_k f) - \tilde{\theta}_t\,\big(\nabla_{\eta_k}\big)\big(R_t - R_{t,k}\big)(\chi_k f\,)$$
$$- \tilde{\theta}_t\,\eta_k\big(R_t - R_{t,k}\big)\nabla(\chi_k f)$$
$$= \frac{1}{t}\,\Psi\,(t^2 L)\eta_k S_{t,k}(\chi_k f) - \tilde{\theta}_t\,\big(\nabla_{\eta_k}\big)S_{t,k}(\chi_k f\,)$$
$$- \tilde{\theta}_t\,\eta_k S_{t,k}\nabla(\chi_k f)\,, \tag{81}$$

where $S_{t,k}$ is the operator $\tilde{\eta}_k\big(R_t - R_{t,k}\big)\tilde{\chi}_k$ defined a priori on $L^2(\Omega)$. For $g \in L^2(\Omega)$ set $u_t = R_t(\tilde{\chi}_k g) \in V$, $u_{t,k} = R_{t,k}(\tilde{\chi}_k g) \in V_k$ and $w_t = \tilde{\eta}_k(u_t - u_{t,k}) = S_{t,k} g \in V \cap V_k$. We deduce from the equivalent definitions of u_t and $u_{t,k}$ by their variational formulation that

$$\int_\Omega (u_t - u_{t,k})\phi + t^2 \int_\Omega \nabla\big(u_t - u_{t,k}\big)\cdot\nabla\phi = 0$$

for all $\phi \in H_0^1(P_k) \cap V \cap V_k$. Now, let v be arbitrary in V. Then $\phi = \tilde{\eta}_k v \in H_0^1(P_k) \cap V \cap V_k$, so that

$$\int_\Omega w_t v + t^2 \int_\Omega \nabla w_t \cdot \nabla v$$
$$= t^2 \int_\Omega (u_t - u_{t,k})(\nabla\tilde{\eta}_k \cdot \nabla v) - t^2 \int_\Omega v\nabla\tilde{\eta}_k \cdot \nabla(u_t - u_{t,k}).$$

Since this last equality holds for all $v \in V$ and that $w_t \in V$, this characterizes w_t as

$$w_t = -t^2 R_t \operatorname{div}(\nabla\tilde{\eta}_k)\big(u_t - u_{t,k}\big) - t^2 R_t(\nabla\tilde{\eta}_k)\nabla(u_t - u_{t,k})$$
$$= -t^2 R_t \operatorname{div}(\nabla\tilde{\eta}_k)\big(R_t - R_{t,k}\big)(\tilde{\chi}_k g)$$
$$- t^2 R_t(\nabla\tilde{\eta}_k)\nabla\big(R_t - R_{t,k}\big)(\tilde{\chi}_k g)\,,$$

where it should be recalled that −div is the adjoint of ∇ on V .

At this point, the difference between R_t and $R_{t,k}$ in the last expression no longer plays any role, so that we obtain four terms to which we apply similar arguments. For example, the first one has kernel

$$t^2 \int_\Omega \nabla_z R_t(x,z)\nabla\tilde{\eta}_k(z)R_t(z,y)\tilde{\chi}_k(y)\,dz.$$

Because of the presence of η_k in (81), we take $x \in O_k \cap \Omega$. Also $y \in \operatorname{Supp}\tilde{\chi}_k \cap \Omega \subset O_k \cap \Omega$. Since $\operatorname{Supp}\nabla\_\tilde{\eta}_k \cap \tilde{O}_k = \emptyset$ we have $|x - z| \geq d/2$ and $|y - z| \geq d/2$ on the support of the integrand. Hence this last integral is not singular and routine calculations using the estimates (34) and (78) valid for $t \leq 1$ and $x, y \in O_k \cap \Omega$.

Hence, the kernel of $S_{t,k}$ satisfies

$$|S_{t,k}(x,y)| \leq c(d)t^{-n+1}e^{-\frac{\alpha(|x-y|+d)}{t}} \tag{82}$$

whenever $t \leq 1$ and $x, y \in O_k \cap \Omega$.

Now, Lemma (2.2.14) applies to the kernels of $\Psi\,(t^2L)\eta_k S_{t,k}, \tilde{\theta}_t\,(\nabla\eta_k)S_{t,k}$ and $\tilde{\theta}_t\,\eta_k S_{t,k}$. We obtain an upper bound of the form (82) valid for $t \leq 1, x \in \Omega$ and $y \in O_k \cap \Omega$. Gathering all terms we obtain

$$\big\|\tilde{\theta}_t\,\eta_k\big(C_t - C_{t,k}\big)(\chi_k f)\big\|_p \leq ce^{-\frac{\alpha d}{t}}\big(\|\nabla(\chi_k f)\|_p + \|\chi_k f\|_p\big).$$

whenever $t \leq 1$ for some $c > 0$ and $\alpha > 0$. This concludes our argument.

Let us take L, satisfying (H1-4). On the kth patch as defined above, one could have tried to construct an operator $L_k = (A_k, \Omega_k, V_k)$ satisfying (H1-4) which “coincides with

L on $\Omega \cap \Omega_k$" in order to apply the already proved result on special Lipschitz domains for each L_k. But we do not know how to extend A on Ω_k in the general case, the difficulty being to keep (H1) valid. However, this extension is possible for specific classes of L such as the ones in items 1, 2 and 3 of Theorem (2.2.2) and it leads to a simplification of the localisation argument. Let us explain the case of real symmetric operators.

Assume that A is real and symmetric on Ω with ellipticity constant δ. Define $\tilde{A}$ by $\tilde{A} = A$ on Ω and $\tilde{A} = \delta Id$ on $\mathbb{R}^n \setminus \overline{\Omega}$. So that $\tilde{A}$ is real and symmetric on $\mathbb{R}^n$ with ellipticity constant δ. Consider the domain decomposition as before and set $L_k = (A_k, \Omega_k, V_k)$ with A_k being the restriction of $\tilde{A}$ to Ω_k.

For $f \in V$, since $f = \tilde{\chi}_k f$, we may write

$$L^{1/2} f = \sum_{0 \le k \le s} \eta_k L_k^{1/2} (\chi_k f) + \sum_{0 \le k \le s} \eta_k \left(L^{1/2} - L_k^{1/2}\right)(\chi_k f) + \sum_{0 \le k \le s} (1 - \eta_k) L^{1/2} (\chi_k f).$$

We have already established the L^p estimates on $\mathbb{R}^n$ (see [94]) and on special Lipschitz domains and they are obviously invariant under rotations. Thus

$$\left\| \eta_k L_k^{1/2} (\chi_k f) \right\|_{L^p(\Omega_k)} \le c(p, \delta, \|A\|_\infty, M) \|\nabla(\chi_k f)\|_{L^p(\Omega_k)}.$$

Looking at supports, one can replace $L^p(\Omega_k)$ by $L^p(\Omega)$.

Next, fix $k \in \{0, \dots, n\}$. Set $u_t^k = (1 + t^2 L)^{-1} L(\chi_k f) \in V$, $v_k\, t = (1 + t^2 L_k)^{-1} L_k (\chi_k f) \in V_k$ and $w_k\, t = \eta_k (u_k\, t - v_k\, t)$. One has that

$$\eta_k \left(L^{1/2} - L_k^{1/2}\right)(\chi_k f) = \frac{2}{\pi} \int_0^\infty w_t^k \frac{dt}{t^2}. \tag{83}$$

Since $u_t^k = \chi_k f - (1 + t^2 L)^{-1} (\chi_k f)$ and the similar expression for $v_k\, t$,it follows from the L^q boundedness of the resolvents of L and L^k that for all $t > 0$ and $1 \le q \le \infty$,

$$\left\| w_t^k \right\|_{L^q(\Omega)} \le c \|\chi_k f\|_{L^q(\Omega)}. \tag{84}$$

Applying Lemma 4 in [97], one sees that

$$\left\| w_t^k \right\|_{L^2(\Omega)} \le \frac{c(m, n, \delta) t^m}{d^m} \|\chi_k f\|_{L^2(\Omega)} \tag{85}$$

for all positive integer m and all $t > 0$,where d appears in item 6 of the domain decomposition. Given $1 < p < \infty$ and $m > 0$, complex interpolation shows that

$$\left\| w_t^k \right\|_{L^p(\Omega)} \le \frac{c(m, n, \delta, p) t^m}{d^m} \|\chi_k f\|_{L^p(\Omega)} \tag{86}$$

for all $t > 0$ and one concludes from (83) that

$$\left\| \eta_k \left(L^{1/2} - L_k^{1/2}\right)(\chi_k f) \right\|_{L^p(\Omega)} \le \frac{c'}{d} \|\chi_k f\|_{L^p(\Omega)}.$$

Lastly, using $u_t^k = \chi_k f - (1 + t^2 L)^{-1} (\chi_k f)$ and $(1 - \eta_k)\chi_k = 0$, one finds

$$(1 - \eta_k) L^{1/2} (\chi_k f) = \frac{2}{\pi} \int_0^\infty (1 - \eta_k) u_t^k \frac{dt}{t^2} = -\frac{2}{\pi} \int_0^\infty g_t^k \frac{dt}{t^2}$$

where $g_t^k = (1 - \eta_k)(1 + t^2 L)^{-1} (\chi_k f)$.Again boundedness of the resolvent yields $\left\| g_t^k \right\|_{L^q(\Omega)} \le c \|\chi_k f\|_{L^q(\Omega)}$ for all $t > 0$ and $1 \le q \le \infty$, and an application of

Lemma 4 in [97] gives us $\left\|g_t^k\right\|_{L^2(\Omega)} \leq \frac{c(m,n,\delta)t^m}{d^m} \|\chi_k f\|_{L^2(\Omega)}$ for all $t > 0$ and positive integer m. Hence,by interpolation

$$\left\|(1 - \eta_k)L^{1/2}(\chi_k f)\right\|_{L^p(\Omega)} \leq \frac{c''}{d} \|\chi_k f\|_{L^p(\Omega)}.$$

Thus,

$$\left\|L^{1/2} f\right\|_{L^p(\Omega)} \leq \sum_{k=0}^{s} c\|\nabla(\chi_k f)\|_{L^p(\Omega)} + \frac{c}{d} \|\chi_k f\|_{L^p(\Omega)}. \qquad (87)$$

Here, we prove Proposition (2.2.27).

The following comments will simplify the proof. Since $\partial\Omega$ is a Lipschitz graph, by elementary geometry, any point $x \in \Omega$ has a unique decomposition as $x = \overline{x} + ue_n$ with $\bar{x} \in \partial\Omega$ and $u > 0$. In such a case, we have $x^* = \overline{x} - ue_n$. It will be useful to consider $\bar{x}$ and u as independent variable (by sweeping Ω with parallels to $\partial\Omega$). With this point of view any integral on Ω can be computed as $\int_\Omega f(x)\,dx = u > 0\ \partial\Omega\, f(\bar{x} + ue_n)\,du d\sigma(\bar{x})$. Observe also that for all $(x,y) \in \Omega \times \Omega, |x - y^*| \sim |x^* - y| \sim |\overline{x} - \overline{y}| + u + v$, where $x = \overline{x} + ue_n$, $\bar{x} \in \partial\Omega, u > 0$ and $y = \overline{y} + ve_n, \bar{y} \in \partial\Omega, v > 0$, and the constants of comparisons depend only on the Lipschitz constant of Ω.

Let us prove (68). If $x, y \in \Omega$,write $x = \overline{x} + ue_n$ and $y = \overline{y} + ve_n$. Then, there exists a constant $c > 0$ depending only on n and the Lipschitz constant of Ω such that for $u, v > 0$ fixed,

$$\int_{\partial\Omega} |x - y^*|^{-n}\, d\sigma(\bar{y}) + \int_{\partial\Omega} |x - y^*|^{-n}\, d\sigma(\bar{x}) \leq \frac{c}{u + v}.$$

Hence we obtain from Schur’s lemma

$$\int_{\partial\Omega}\int_{\partial\Omega} \frac{|f(\bar{y} + ue_n)||g(\bar{x} + ve_n)|}{|x - y^*|^n}\, d\sigma(\bar{y}) d\sigma(\overline{x}) \leq c\frac{|F(u)||G(v)|}{u + v},$$

where

$$F(u) = \left(\int_{\partial\Omega} |f(\bar{y} + ue_n)|^p d\sigma(\bar{y})\right)^{\frac{1}{p}},$$

$$G(v) = \left(\int_{\partial\Omega} |g(\bar{y} + ve_n)|^p\, d\sigma(\bar{y})\right)^{\frac{1}{p}},$$

and p' is the dual exponant to p. Integrating against $dudv$ yields

$$\left|\int_\Omega Hfg\right| \leq c\int_{u>0}\int_{v>0}\int_{\partial\Omega} \frac{\int_{\partial\Omega} |f(y)||g(x)|}{|x - y^*|^n}\, d\sigma(\bar{y})d\sigma(\overline{x})\, dudv$$

$$\leq c\int_{u>0}\int_{v>0} \frac{|F(u)||G(v)|}{u + v}\, dudv$$

$$\leq c\left(\int_{u>0} |F(u)|^p du\right)^{\frac{1}{p}}\left(\int_{v>0} |G(v)|^{p'}\, du\right)^{\frac{1}{p}} = c\|f\|_p\|g\|_{p'},$$

the last inequality being a consequence of Hardy inequality on $\mathbb{R}$.

We now prove (69). The trace theorem on Lipschitz domains (see [62]) asserts that any $f \in W^{1,p}(\Omega)$ has a trace on $\partial\Omega$ that belongs to a Besov space whose norm is given by the expression J and

$$J = \left(\int_{\partial\Omega} \int_{\partial\Omega} \frac{|Trf\,(\bar{x}) - Trf\,(\bar{y})|^p}{|\bar{x} \; - \bar{y}|^{\,n-2+p}} \, d\sigma(\bar{y}) d\sigma\,(\bar{x})\right)^{\frac{1}{p}} \le c(p,M)\|\nabla f\|_p.$$

It suffices to establish $|\int_{\Omega} \; Bf\,(x)g(x)\,dx| \le c(p,M)J\,\|g\|_{L^{p}\,(\Omega)}$ for $f \in C_0^1(\Omega)$ and to invoke the density of $C_0^1\,(\Omega)$ in $W^{1,p}(\Omega)$. Choose α,β,γ with the following requirements:

$$\frac{n-1}{p'} < \alpha < \frac{n-1}{p'} + \frac{1}{p}, \qquad \beta = n - \alpha, \qquad \gamma = \alpha - \frac{n-1}{p}.$$

Observe that $\alpha < \gamma p < 1$ and $\beta p + \gamma p - 1 = n - 2 + p$. Writing as above $x = \overline{x} + ue_n$ and using Hölder inequality, we have

$$\left|\int_{\Omega} Bf\,(x)g(x)dx\right| \le c \int_{\Omega} \int_{\partial\Omega} \frac{|f\,(\bar{x}) - f\,(\bar{y})|}{|x - \bar{y}|^n} \,|g(x)|\, dx d\sigma(\bar{y})$$

$$\le c\left(\int_{\Omega} \int_{\partial\Omega} \frac{|f\,(\bar{x}) - f\,(\bar{y})|^p}{|x - \overline{y}|^{\beta p}} \frac{dx d\sigma(\bar{y})}{u^{\gamma p}}\right)^{\frac{1}{p}}$$

$$\times\left(\int_{\Omega} \int_{\partial\Omega} \frac{u^{\gamma p'}}{|x - \overline{y}|^{\alpha p'}} \,|g(x)|^{p'}\, dx d\sigma(\bar{y})\right)^{\frac{1}{p'}}.$$

Now writing $dx = d\sigma(\bar{x})du$ in the first term and integrating first with respect to u we obtain cJ as an upper bound since

$$\int_0^{\infty} \frac{du}{|x - \overline{y}|^{\beta p} u^{\gamma p}} \le \frac{c}{|\,\overline{x} - \overline{y}|^{\beta p + \gamma p - 1}} = \frac{c}{|\,\overline{x} - \overline{y}|^{n-2+p}}.$$

The second term is dominated by $c\|g\|_{p'}$ since

$$\int_{\partial\Omega} \frac{u^{\gamma p'}}{|x - \overline{y}|^{\alpha p'}} d\sigma(\bar{y}) \le c.$$

To finish the proof of Theorem (2.2.9),it remains to prove Lemma (2.2.8) which we do now.

Assume first that Ω is unbounded. The analyticity of the semigroup e^{-tL} on L^2,(H4) and complex interpolation classically imply that the semigroup is bounded analytic on L^p for all $1 < p < \infty$. Hence there exists a constant c_p such that $\left\|t^4 L^2 e^{-2t^2 L} f\right\|_p \le \left\|c_p f\right\|_p$ for all $t > 0$ and $f \in L^p(\Omega)$. This yields the proposition in this case.

Next, assume that Ω is bounded. Set $h = \int_1^{\infty} \; t^3 L^2 e^{-2t^2 L} f \frac{dt}{t}$. Since Ω is bounded, we have $\|h\|_1 \le |\Omega|^{1/2}\|h\|_2$. Next, let $g = L^{1/2}e^{-L}f$ and write

$$h = \int_1^{\infty} \; t^3\, L^{3/2}\, e^{-(2t^2-1)L} g \frac{dt}{t} = \int_0^{\infty} \; m(s)s^{3/2}\, L^{3/2}\, e^{-sL} g \frac{ds}{s}$$

where we have set $s = 2t^2 - 1$ and the change of variable shows that $m(s)$ is a bounded function on $(0,\infty)$. By H^{∞} functional calculus (see (93)), this shows that $\|h\|_2 \le c\|g\|_2$. Now, $L^{1/2}e^{-(1/2)L}$ is bounded on $L^2(\Omega)$ by functional calculus,hence $\|g\|_2 \le c\left\|e^{-(1/2)L} f\,\right\|_2$. Lastly, (H2) implies that $e^{-(1/2)L}$ is bounded from $L^1(\Omega)$ to $L^2(\Omega)$. Thus, we have $\|h\|_1 \le c\|f\|_1$. Apply the same reasonning replacing L with L^* to obtain, by duality, $\|h\|_{\infty} \le c\|f\|_{\infty}$. Interpolation finishes the proof.

Next,we complete the proof of Theorem (2.2.3). We begin with

Theorem (2.2.32)[90]: Assume that Ω is bounded. In addition to the hypotheses in Theorem (2.2.9), assume that L^* satisfies (H1). Then, there exists $\varepsilon > 0$ (depending on L) such that $L^{1/2}$ extends to an isomorphism between $\dot{V}_p$ and $L^p(\Omega)$ for $1 < p < 2 + \varepsilon$. Furthermore, $\|\nabla f\|_p$ and $\left\|L^{1/2} f\right\|_p$ are equivalent norms on V_p.

Proof. First the assumptions (H1) for L and L^* show that $D\left(L^{1/2}\right) = V$ and that $\|\nabla f\|_2 \sim \left\|L^{1/2} f\right\|_2$. Next, we have to show that $\left\|\nabla L^{-1/2} f\right\|_p \leq c\|f\|_p$ for f in a dense subspace of $L^p(\Omega)$,say $L^2(\Omega) \cap L^p(\Omega)$. Write

$$\nabla L^{-1/2} f = c \int_0^\infty (t\nabla) e^{-t^2 L} f \frac{dt}{t}$$

where $c^{-1} = \int_0^\infty e^{-t^2} dt$. The integral for t small is handled by using Theorem 2 in [101] under the assumptions (H2) and (H1) for L^*,which are valid here (if L^* also satisfies (H3),then this can be obtained from the usual Hormander condition, see [94],pp. 162-163). Hence

$$\left\|\int_0^1 t\nabla e^{-t^2 L} f \frac{dt}{t}\right\|_p \leq c\|f\|_p. \tag{88}$$

The other part follows from

Lemma (2.2.33)[90]: Under (H1) for L^* and (H2), $\left\|\int_1^\infty t\nabla e^{-t^2 L} f \frac{dt}{t}\right\|_p \leq c\|f\|_p$ for all $1 \leq p \leq 2$ and $f \in L^2 \cap L^p(\Omega)$.

Hence, $\left\|\nabla L^{-1/2} f\right\|_p \leq c\|f\|_p$. Once this is established, it suffices then to adapt the arguments in [94], to obtain the ε (by analytic perturbation) and to prove invertibility. We skip details.

Proof. Let $h = \int_1^\infty t e^{-t^2 L} f \frac{dt}{t}$. First, $\|\nabla h\|_1 \leq |\Omega|^{1/2}\|\nabla h\|_2$ since Ω is bounded. Next, (H1) for L^* means by duality that $\|\nabla h\|_2 \leq c L^{1/2}\|h\|_2$ (on a bounded domain one can remove $\|h\|_2$). Now using the same idea as in the proof of Lemma (2.2.8),write

$$L^{1/2} h = c \int_1^\infty t L^{1/2} e^{-t^2 L} f \frac{dt}{t} = \int_0^\infty m(s) s^{1/2} L^{1/2} e^{-sL} e^{-(1/2)L} f \frac{ds}{s},$$

where $m(s)$ is a bounded function, so that $\left\|L^{1/2} h\right\|_2 \leq c\left\|e^{-(1/2)L} f\right\|_2$. It remains to invoke the $L^1 - L^2$ boundedness of $e^{-(1/2)L}$ by (H2). Hence $\|\nabla h\|_1 \leq c\|f\|_1$. The inequality for $p = 2$ is contained in the above argument and we conclude by y interpolation.

We summarize the study by the following result.

Theorem (2.2.34)[90]: Let Ω be a special Lipschitz domain and $L = (A, \Omega, V)$ for some $A \in \mathcal{A}$ with Dirichlet or Neumann boundary condition. Assume that the hypotheses (18), (H2) and $(H3)$ with $\tau = \infty$ hold. Then for all $f \in V$,

$$L^{1/2} f = T_1 \nabla f + T_2(-\Delta)^{1/2} f + H\nabla f + B(Trf), \tag{89}$$

where T_1, T_2 are Calderón-Zygmund operators and H is a Hardy operator and B is boundary Hardy operator. As a consequence, for all $1 < p < \infty$,

$$\left\|L^{1/2} f\right\|_p \leq c\|\nabla f\|_p, \qquad f \in V \cap W^{1,p}(\Omega). \tag{90}$$

The decomposition of $L^{1/2}$ comes from the proof of the four Lemmata to obtain Theorem (2.2.9). Let us recall that $T_2 = 0$ when $L = (Id, \Omega, V)$ is the negative Laplacian.

Theorem (2.2.35)[90]: In addition to the hypotheses in Theorem (2.2.34), assume that L^* satisfies $(H1)$. Then, there exists $\varepsilon > 0$ (depending on L) such that for $1 < p < 2 + \varepsilon, L^{1/2}$ extends to an isomorphism between $\dot{V}_p$ onto $L^p(\Omega)$, and $\|\nabla f\|_p$ and $\|L^{1/2}f\|_p$ are equivalent norms on $\dot{V}_p$.

Proof. From (H1) for L and L^* we have $\mathcal{D}(L^{1/2}) = V$ with $\|\nabla f\|_2 \sim \|L^{1/2}f\|_2 \sim \|(L^*)^{1/2}f\|_2$. Hence $L^{1/2}$ extends to a bounded and invertible operator from $\dot{V}_2$ to L^2. The rest of the proof is as in that of Theorem (2.2.3) and the range of integration in the inequality (88) is to be changed to $(0, +\infty)$.

Theorem (2.2.36)[90]: The conclusions of Theorem (2.2.35) are valid in any of the following situations:

(i) A is real-valued and symmetric,

(ii) A is an L^∞ or BMO (complex) perturbation of a real-valued and constant elliptic matrix,

(iii) A is an L^∞ or BMO perturbation of a complex-valued and constant elliptic matrix and the domain has small enough Lipschitz constant.

Proof. For real and symmetric operators (K) is trivial,and $(H2)$ and $(H3)$ are satisfied as mentioned previously. For the two other items, $(H2)$ and $(H3)$ follow from [96], Theorem 7. The problem is to obtain (18) and not just (19) as we proved in Theorem 2 of [97]. But a rescaling argument yields the better inequality. Indeed, denote by C the class of elliptic operators concerned with item 2: it is invariant under translations and dilations. For any f $\in V$ and any $L = (A, \Omega, V) \in C$,one has

$$\|L^{1/2}f\|_2 \leq c(\|\nabla f\|_2 + \|f\|_2)$$

$c = c(n, \delta, \|A\|_\infty, \|A\|_{BMO(\Omega)}, M)$ where δ is the ellipticity constant of A and M is the Lipschitz constant of Ω. Without loss of generality,we may restrict our attention to domains with boundary containing the origin. Then a dilation $x \to \lambda x$ in $\mathbb{R}^n$ (ie $A(x)$ changes to $A(\lambda x)$) preserves this class of domains and their Lipschitz constants,so that it leaves c unchanged,but leads to

$$\|L^{1/2}f\|_2 \leq c(\|\nabla f\|_2 + \lambda^{-1}\|f\|_2)$$

for any $f \in V$,any $L \in C$ and any $\lambda > 0$. Letting λ go to ∞ gives (18).

The same reasoning applies to the class of operators concerned with item 3.

We quote without proof the following result analogous to Theorem (2.2.2) on bounded domains.

Theorem (2.2.37)[90]: Let Ω be a special (or more generally unbounded) Lipschitz domain and $L = (A, \Omega, V)$ be subject to Dirichlet or Neumann boundary condition. Assume that A is real if $n \geq 3$ and complex if $n = 2$. There exists $\varepsilon = \varepsilon(\Omega, n, \delta) > 0$ such that if $A \in ABMO_\varepsilon(\Omega)$ then for $1 < p < \infty$

$$\|L^{1/2}f\|_p \leq c\|\nabla f\|_p + \|f\|_p, \qquad f \in V \cap W^{1,p}(\Omega). \tag{91}$$

For L be a one-one maximal-accretive operator of type ω for some $\omega \in [0, \pi/2)$ on a Hilbert space $\mathcal{H}$ [105]. It has an H^∞ functional calculus on $L^2(\Omega)$. This means that for any $\mu \in (\omega, \pi)$, for any $f \in H^\infty(\Gamma_\mu)$, that is f holomorphic and bounded in $\Gamma_\mu = \{z \in \mathbb{C} \setminus \{0\}; |\arg z| < \mu\}$ (where $|\arg z|$ is the argument in $(-\pi, \pi]$ of $z \in \mathbb{C}$),one can define a bounded operator $f(L)$ on $L^2(\Omega)$, with $f(L) \leq c_\mu \|f\|_{H^\infty(\Gamma_\mu)}$. In particular, L has a bounded (in fact,contracting) analytic semigroup e^{-zL} on $L^2(\Omega)$ defined on $\Gamma_{\pi/2} - \omega$.

Following [107],we say that $\psi \in \Psi(\Gamma_\mu)$ when $\psi \in H^\infty(\Gamma_\mu)$ and if there exists constants $c,\ s > 0$ such that for all $\zeta \in \Gamma_\mu, |\psi(\zeta)| \leq c|\zeta|^s\ (1 + |\zeta|)^{-2s}$. If $\psi \in \Psi(\Gamma_\mu), \psi(L)$ can be computed using the Cauchy formula

$$\psi(L) = \frac{1}{2\pi i}\int_\gamma (\zeta - L)^{-1}\psi(\zeta)\, d\zeta, \tag{92}$$

where the path γ is made of two rays $re^{\pm i\nu}$, $r \geq 0$ and $\omega < \nu < \mu$, and is described counterclockwise. Such functions can be used to generate functions in $H^\infty(\Gamma_\mu)$. For example,if $\psi \in \Psi(\Gamma_\mu)$ and $m: (0, +\infty) \to \mathbb{C}$ is bounded and measurable then the integral

$$f(\zeta) = \int_0^\infty \psi(t^2\zeta)m(t)\frac{dt}{t} \tag{93}$$

defines a function in $H^\infty(\Gamma_\mu)$. Moreover, the integrals $\int_\varepsilon^r \psi(t^2\zeta)m(t)\frac{dt}{t}$ converge uniformly to f on compact subsets of Γ_μ as $\varepsilon \to 0$ and $r \to +\infty$, thus one has

$$f(L) = \lim_{\substack{\varepsilon\to 0\\ r\to+\infty}} \int_\varepsilon^r \psi(t^2 L)m(t)\frac{dt}{t} = \int_0^\infty \psi(t^2 L)m(t)\frac{dt}{t}, \tag{94}$$

where the limit is in the strong topology of $\mathcal{H}$. In fact, for any $f \in H^\infty(\Gamma_\mu)$, there is such a representation for $f(L)$. We shall not need this fact, see [107].

The class $\Psi(\Gamma_\mu)$ can also be used to obtain double sided quadratic estimates of great. Indeed, for $\psi \in \Psi(\Gamma_\mu)$ not identically 0,there exists $c = c(\mu, \psi) > 0$ such that ([101],[107])

$$c\|u\|_{\mathcal{H}} \leq \left(\int_0^\infty \|\psi(t^2 L)u\|_{\mathcal{H}}^2 \frac{dt}{t}\right)^{\frac{1}{2}} \leq c^{-1}\|u\|_{\mathcal{H}}, u \in \mathcal{H}.$$

Common choices for ψ are often $e^{-\zeta}\zeta^{1/2}$ or $(1 + \zeta)^{-1}\zeta^{1/2}$ where $\zeta^{1/2}$ is the principal determination of the square root of ζ, but the freedom of choice is useful.

Note that if L is self-adjoint and non negative, by the Borel functional calculus,if ψ : $[0, \infty) \to \mathbb{C}$ is a Borel function satisfying

$$a(\psi) = \frac{1}{2}\int_0^\infty |\psi(t)|^2\frac{dt}{t} < \infty$$

then

$$\int_0^\infty \|\psi(t^2 L)u\|_{\mathcal{H}}^2\frac{dt}{t} = a(\psi)\|u\|_{\mathcal{H}}^2, \qquad u \in \mathcal{H}.$$

Back to the general case, denote by $L^{1/2}$ the unique maximal-accretive square root of L. Call $F(\Gamma_\mu)$ the set of holomorphic functions φ in Γ_μ for which there exists $c > 0$ such that for all $\zeta \in \Gamma_\mu, |\varphi(\zeta)| \leq c(1 + |\zeta|)^{-1}$. We shall constantly use the fact that for such φ, we have

$$\varphi(t^2 L)t^2 Lu = t^2 L\varphi(t^2 L)u, \qquad u \in \mathcal{D}(L),$$

the latter operator being bounded on $\mathcal{H}$ uniformly in t since $\zeta\varphi(\zeta) \in H^\infty(\Gamma_\mu)$. Hence, $\varphi(t^2L)tL$ extends to a bounded operator on $\mathcal{H}$ with

$$\|\varphi(t^2 L)tLu\|_{\mathcal{H}} \leq \frac{c(\delta, \varphi)}{t}\|u\|_{\mathcal{H}}. \tag{95}$$

Lemma (2.2.38)[90]: Let $\varphi \in F(\Gamma_\mu)$. If φ is not identically 0,

$$\|L^{1/2} u\|^2_{\mathcal{H}} \sim \int_0^\infty \|\varphi(t^2 L)tLu\|^2_{\mathcal{H}} \frac{dt}{t}, \qquad u \in \mathcal{D}(L). \tag{96}$$

If $c(\varphi) = \frac{1}{2}\int_0^\infty \varphi(\zeta)\zeta^{-1/2}\, d\zeta = 0$ then

$$\|L^{1/2}u\| = c(\varphi)^{-1}\int_0^\infty \varphi(t^2L)tLu\frac{dt}{t}, \qquad u \in \mathcal{D}(L). \tag{97}$$

Proof. Let $\psi(\zeta) = \varphi(\zeta)\zeta^{1/2}$. Then $\psi \in \Psi(\Gamma_\mu)$ and $\psi(t^2L)L^{1/2}u = \varphi(t^2L)tLu$ when $u \in \mathcal{D}(L)$. Then, use the quadratic estimates for the first relation and $\int_0^\infty \psi(t^2L)\frac{dt}{t} = c(\varphi)^{-1}I$ in the strong topology of $\mathcal{H}$.

Now let us assume that $L = D^*AD$. That is, given two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, a one-one operator $D: \mathcal{H} \to \mathcal{K}$ with dense domain V and a bounded and invertible accretive operator $A: \mathcal{K} \to \mathcal{K}$ with $A + A^* \geq 2\delta I$ in the sense of self-adjoint operators for some δ > 0, L is the unique maximal accretive operator on $\mathcal{H}$ associated with the sesquilinear regularly accretive form $\langle ADu, Dv\rangle$, $u, v \in V$. It can be shown that L is of type ω with $0 \leq w < \pi/2$, that $L = D^*AD$ in the sense of unbounded operators on $\mathcal{H}$ and that $\mathcal{D}(L)$ is dense in V equipped with the graph norm of D.

Let $\varphi \in F(\Gamma_\mu)$ with $\varphi(\zeta) \leq c_0(1 + |\zeta|)^{-1}$ for $\zeta \in \Gamma_\mu$. One can define $\varphi(t^2L)tD^*A$ as a bounded operator from $\mathcal{K}$ to $\mathcal{H}$ and obtain

$$\|\varphi(t^2L)tD^* Aw\|_{\mathcal{H}} \leq c\|w\|_{\mathcal{K}}$$

uniformly over $t > 0$,where c depends on c_0, μ, $\|A\|$ and δ.

Indeed, call V the dual of V ,extend D^* from $\mathcal{K}$ to V so that one has an extension of L from V into V' . Let $w \in \mathcal{K}$ and define $u_t \in V$ as the unique solution of

$$\langle u_t, v\rangle + t^2\langle ADu_t, Dv\rangle = t\langle Aw, Dv\rangle$$

for all $v \in V$. It is easy to show

$$\sqrt{\|u_t\|^2_{\mathcal{H}} + \delta t^2\|Du_t\|^2_{\mathcal{K}}} \leq \frac{\|A\|\|w\|_{\mathcal{H}}}{\delta^{1/2}}.$$

Setting $(1 + t^2L)^{-1}tD^*Aw = u_t$,we obtain a bounded operator from $\mathcal{K}$ to $\mathcal{H}$. Define then $\varphi(t^2L)tD^*A = \tilde{\varphi}(t^2L)(1 + t^2L)^{-1}tD^*A$ with $\tilde{\varphi}(\zeta) = (1 + \zeta)\varphi(\zeta)$. Since $\tilde{\varphi} \in H^\infty(\Gamma_\mu)$, we have proved our claim. Remark that

$$\varphi(t^2L)tLu = \varphi(t^2 L)tD^* A(Du), \qquad \forall\, u \in \mathcal{D}(L). \tag{98}$$

Indeed, it suffices to prove it when $\varphi(\zeta) = (1 + \zeta)^{-1}$. In this case,

$$\begin{aligned} v_t &= (1 + t^2 L)^{-1} t^2 Lu = t^2 L(1 + t^2 L)^{-1} u \\ &= u - (1 + t^2 L)^{-1}u. \end{aligned} \tag{99}$$

Next, set $u_t = (1 + t^2L) - 1t^2D^*A(Du)$. By definition,

$$\langle u_t, v\rangle + t^2\langle ADu_t, Dv\rangle = t^2\langle ADu, Dv\rangle$$

for all $v \in V$,so that

$$\langle (u_t - u), v\rangle + t^2\langle AD(u_t - u), Dv\rangle = \langle u, v\rangle$$

for all $v \in V$. This shows that $u_t - u \in \mathcal{D}(L)$ and that $(1 + t^2L)(u_t - u) = u$, hence $u_t = v_t$ as desired.

We know that $\varphi(t^2L)tL$ extends to all of $\mathcal{H}$,but the above remark shows that one can compute $\varphi(t^2L)tLu$ for $u \in V$ using (98).

Now, we choose $\varphi(\zeta) = e - \zeta$ for convenience. The following result is implicit in [94].

Proposition (2.2.39)[90]: The following are equivalent:

(i) $\|L^{1/2}u\|_{\mathcal{H}} \leq c\|Du\|_{\mathcal{K}}$ for all $u \in \mathcal{D}(L)$.

(ii) $\int_0^\infty \left\|e^{-t^2L}tD^*Aw\right\|_{\mathcal{H}}^2 \frac{dt}{t} \leq c\|w\|_{\mathcal{K}}^2$ for all $w \in \mathcal{K}$.

Proof. Using Lemma (2.2.38) and (98), it is quite clear that (ii) implies (i). Reciprocally, assume (i). This means that (ii) holds for any $w = Du$ when $u \in \mathcal{D}(L)$. Since $\mathcal{D}(L)$ is dense in V ,(ii) holds when $u \in V$.

Now,let $w \in \mathcal{K}$. Letting $\dot{V}$ be the completion of V under $\|Du\|$_$\mathcal{K}$, one can solve the equation $D^*Aw = D^*ADu$ for $u \in \dot{V}$ using the Lax-Milgram Lemma in $\dot{V}$. An approximation argument concludes the proof.

We deal with local analogues.

Proposition (2.2.40)[90]: The following are equivalent:

(i) $\left\|L^{1/2}u\right\|_{\mathcal{H}} \leq c(\|Du\|_{\mathcal{K}} + \|u\|_{\mathcal{H}})$ for all $u \in \mathcal{D}(L)$.

(ii) For all $\tau > 0$, $\int_0^\tau \left\| e^{-t^2L}tD^*Aw\right\|_{\mathcal{H}}^2 \frac{dt}{t} \leq c(\tau)\|w\|_{\mathcal{K}}^2$ for all $w \in \mathcal{K}$.

(iii) There exists $\tau > 0$ such that $\int_0^\tau \left\|e^{-t^2L}tD^*Aw\right\|_{\mathcal{H}}^2 \frac{dt}{t} \leq c(\tau)\|w\|_{\mathcal{K}}^2$ for all $w \in \mathcal{K}$.

Proof. Since $\left\|e^{-t^2L}tLu\right\|_{\mathcal{H}}^2 \leq ct^{-2}\|u\|_{\mathcal{H}}^2$,

$$\int_\tau^\infty \left\| e^{-t^2L}tLu\right\|_{\mathcal{H}}^2 \frac{dt}{t} \leq c\tau^{-1} \|u\|_{\mathcal{H}}^2.$$

This shows the equivalence between (ii) and (iii).

If (iii) holds then, by (98),

$$\int_0^\tau \left\|e^{-t^2L}tLu\right\|_{\mathcal{H}}^2 \frac{dt}{t} \leq c\|Du\|_{\mathcal{K}}^2,$$

for $u \in \mathcal{D}(L)$. Hence,(i) is proved.

Now, assume (i) and let us prove (iii) with $\tau = 1$. Since, by functional calculus, $\left\|L^{1/2}u\right\|_{\mathcal{H}} + \|u\|_{\mathcal{H}} \sim \left\| (L + 1)^{1/2}u\right\|$_$\mathcal{H}$, we have $\left\|(L + 1)^{1/2}u\right\|_{\mathcal{H}} \leq c\left\|\widetilde{D}u\right\|_{\mathcal{G}}$,where $\mathcal{G} = \mathcal{K} \oplus \mathcal{H}$ and $\widetilde{D} = (D, I)$ from $\mathcal{H}$ to $\mathcal{G}$. Let $\tilde{A} = A \oplus I$ on $\mathcal{G}$. One can then write $L + 1 = \widetilde{D}^*\tilde{A}\widetilde{D}$. The proposition above applies and we obtain that

$$\int_0^\infty \left\|e^{-t^2(L+1)}t\widetilde{D}^* \tilde{A}\widetilde{w} \right\|_{\mathcal{H}}^2 \frac{dt}{t} \leq c\|\widetilde{w}\|_{\mathcal{G}}^2, \qquad \forall \widetilde{w} \in \mathcal{G}. \tag{100}$$

Let $w \in K$ and $\widetilde{w} = w$ (ie, $u = 0 \in H$). Then $\widetilde{D}^*\tilde{A}\widetilde{w} = D^*Aw$. Next,

$$\begin{aligned}\left\|e^{-t^2L}tD^* Aw\right\|_{\mathcal{H}} &\leq \left(1 - e^{-t^2}\right)\left\|e^{-t^2L}tD^* Aw\right\|_{\mathcal{H}} + \left\|e^{-t^2(L+1)} tD^* Aw\right\|_{\mathcal{H}} \\ &\leq ct^2 \|w\|_{\mathcal{K}} + \left\| e^{-t^2(L+1)} t\widetilde{D}^* \tilde{A}\widetilde{w}\right\|_{\mathcal{H}}\end{aligned}$$

using the uniform boundedness of $e^{-t^2L}tD^*A$. This and (100) easily imply (iii).

Chapter 3
Dvoretzky Theorem

We show that the results imply an unexpected distinction between the lower and upper inclusions in Dvoretzky Theorem. A complete and rather simple proof of the famous Dvoretzky's theorem is presented.

Section (3.1): Small Ball Probability

In probability theory, the large deviation theory (or the tail probabilities) and the small deviation theory (or the small ball probabilities) are in a sense two complementary directions. The large deviation theory, which is a more classical direction, seeks to control the probability of deviation of a random variable X from its mean $M,$ i.e. one looks for upper bounds on Prob$(|X - M| > t)$. The small deviation theory seeks to control the probability of X being very small, i.e. it looks for upper bounds on $Prob(|X| < t)$. There are a number of excellent texts on large deviations, see [113] and [118]. A recent exposition of the state of the art in small deviation theory can be found in [122].

A modern powerful approach to large deviations is via the celebrated concentration of measure phenomenon. One of the early manifestations of this idea was $V.$ Milman's proof of Dvoretzky Theorem in the 1970s. Recall that Dvoretzky Theorem entails that any n-dimensional convex body of dimension c log n which is approximately a Euclidean ball. Since Milman's proof, the concentration of measure philosophy plays a major role in geometric functional analysis and in many other areas. A recent book by $M.$ Ledoux [121] gives an account of many ramifications of this method. A standard instance of the concentration of measure phenomenon is the case of a Lipschitz function

on the unit Euclidean sphere S^{n-1} . In view of the geometric applications, we shall state it for a norm k · k on R^n , or equivalently for its unit ball, which is a centrally-symmetric convex body $K \subset R^n$. We equip the sphere S^{n-1} with the unique rotation invariant probability measure σ. Two parameters are e response ble for many geometric properties of the convex body K; the maximal and the average values of the norm on the sphere S^{n-1} :

$$b = b(K) = \sup_{x\in S^{n-1}} \|x\|, \quad M = M(K) = \int_{S_{n-1}} \|x\| d\sigma(x). \tag{1}$$

The concentration of measure inequality, which appears $e.g.$ in [128] states that the norm is close to its mean M on most of the sphere. For any $t > 1$,

$$\sigma \{ x \in S^{n-1} : | \|x\| - M | > tM \} < \exp(-ct^2k) \tag{2}$$

Where

$$k = k(K) = n \left(\frac{M(K)}{b(K)}\right)^2$$

Here and thereafter the letters $c, C, c_0, \tilde{c}, c_1, c_2$ etc. denote some positive universal constants, whose values may be different in various appearances. The symbol $\approx$ denotes equivalence of two quantities up to an absolute constant factor, $i.e. a \approx b$ if $ca \le b \le Ca$ for some absolute constants $c, C > 0$.

The concentration of measure inequality can of course be interpreted as a large deviation inequality for the random variable $\|x\|$, and the connection to probability theory becomes even more sound when one recalls an analogous inequality for Gaussian measures, see [121]. The quantity $k(K)$ plays a crucial role in high dimensional convex geometry, as it is the critical dimension in Dvoretzky Theorem. We will call this dimension k(K) the

Dvoretzky dimension. Milman's proof of Dvoretzky Theorem [126] (see also [128]) provides accurate information regarding the dimension of the almost spherical of K. Milman's argument shows that if $l < ck(K)$, then with probability larger than $1 - e^{-c'l}$, a random l-dimensional subspace $E \in G_{n,l}$ satisfies

$$\frac{c}{M}(D^n \cap E) \subset K \cap E \subset \frac{c}{M}(D^n \cap E), \quad (3)$$

where $M = M(K)$, D^n denotes the unit Euclidean ball in R^n, and the randomness is induced by the unique rotation invariant probability measure on the grassmanian $G_{n,l}$ of l-dimensional subspaces in R^n.

The Dvoretzky dimension $k(K)$ was proved in [129] to be the exact critical dimension for a random to satisfy (3), in the following strong sense. If a random l −dimensional subspace $E \in G_{n,l}$ satisfies (3) with probability larger than, say, $1 - \frac{1}{n}$, , then necessarily $l < Ck(K)$. Thus a random of dimension $l < ck(K)$ is close to Euclidean with high probability, and a random of dimension $l > Ck(K)$ is typically far from Euclidean. These arguments completely clarify the question of the dimensions in which random of a given convex body are close to Euclidean. Once $b(K)$ and $M(K)$ are calculated, the behavior of a random is known. For instance, Dvoretzky dimension of the cube is $\approx \log n$, while the cross polytope $K = \{x \in R^n :$
$\sum |x_i| \leq 1\}$ has Dvoretzky dimension as large as k(K) ≈n.

We investigate Dvoretzky Theorem from a different direction, which does not involve the standard large deviations inequality (2). The second named conjectured that a phenomenon similar to the concentration of measure should also occur for the small ball probability, and he proved a weaker statement. The conjecture has been recently proved by R. Lata la and K. Oleszkiewitz [120], using the solution to the B-conjecture by Cordero, Fradelizi and Maurey [112]:

Theorem (3.1.1). (Small ball probability). For every $0 < \varepsilon < \frac{1}{2}$,

$$\sigma\{x \in S^{n-1} : \|x\| < \varepsilon M\} < \varepsilon^{ck}(K)$$

where $c > 0$ is a universal constant.

This theorem is related to the small ball probability (as a direction of the probability theory) in exactly the same way as the concentration of measure is related to large deviations. Here we apply Theorem (3.1.1) to study questions arising from Dvoretzky Theorem. We show that for some purposes, it is possible to relax the Dvoretzky dimension $k(K)$, replacing it by a quantity independent of the Lipschitzness of the norm (which is quantified by the Lipschitz constant $b(K)$). We wish to replace $k(K)$ by

$$d(K) = min\{-\log \sigma\left\{x \in S^{n-1} : \|x\| \leq \frac{1}{2}M\right\}, n\},$$

where log stands for the natural logarithm.Selecting $t = \frac{1}{2}$ in the concentration of measure inequality (2), we conclude that $d(K)$ must be at least of the same order of magnitude as Dvoretzky dimension $k(K)$:

$$d(K) \geq Ck(K).$$

The small ball Theorem (3.1.1) indeed holds with $d(K)$ (this is a part of the argument of Lata la and Oleszkiewicz, reproduced below). The resulting inequality can be viewed as Kahane-Khinchine type inequality for negative exponents:

For positive exponents, this inequality was proved in [124]: for $0 < k < ck(K)$,

$$cM < \left(\int_{Sn-1} \|x\|^{k}\, d\sigma(x)\right)^{\frac{1}{k}} < CM. \quad (4)$$

For negative exponents $-1 < \ell < 0$, inequality (4) follows from results of Guédon [117] that generalize Lovász-Simonovits inequality [123]. Proposition (3.1.3) extends (4) to the range $[-cd(K), ck(K)]$ (which of course includes the range $[-ck(K), ck(K)]$).
In Proposition (3.1.3), $\|x\|^{-1}$ can be regarded as the radius of the one-dimensional of the body K. Combining this with the recent inequality for diameters of due to [119], we are able to lift the dimension and thus compute the average diameter of l-dimensional of any centrally-symmetric convex body K.

The relation between Theorem (3.1.7) and Dvoretzky Theorem is clear. We show that for dimensions which may be much larger than $k(K)$, the upper inclusion in Dvoretzky Theorem (3) holds with high probability. This reveals an intriguing point in Dvoretzky Theorem. Milman's proof of Dvoretzky Theorem focuses on the left-most inclusion in (3). Once it is proved that the left-most inclusion in (3) holds with high probability, the right-most inclusion follows almost automatically.
Furthermore, Milman-Schechtman's argument [129] implies in fact that the left-most inclusion does not hold (with large probability) for dimensions larger than the Dvoretzky dimension. The reason that a random l-dimensional is far from Euclidean when $l > ck(K)$ is that a typical does not contain a sufficiently large Euclidean ball. In comparison, we observe that the upper inclusion in (3) holds for a much wider range of dimensions.
There are cases, such as the case of the cube, where the Dvoretzky dimension satisfies $k(K) \approx \log n$, while $d(K)$ is a polynomial in n. Hence, while the cube of dimension n^{c} are already contained in the appropriate Euclidean ball (for any fixed $c < 1$, independent of n), only when the dimension is $\approx \log n$, start to "fill from inside", and an isomorphic Euclidean ball is observed. The case of the cube is contained, using different terminology, in [125]. The fact that $d(K)$ is typically larger than $k(K)$ is a little unexpected. It implies that the correct upper bound for random of a convex body appears sometimes in much larger dimensions than those for which we have the lower bound.
In the past decade, diameters of random lower-dimensional of convex bodies attracted a considerable amount of attention, see in particular [114], [115], [116]. Theorem (3.1.7) is a significant addition to this line of results. It implies that diameters of random are equivalent for a wide range of dimensions – starting from dimension one, when the random diameter simply equals $\frac{1}{M(K)}$, and up to the critical dimension $d(K)$. Right after the proof of Theorem (3.1.7).

We discuss the negative moments of the norm, proving Proposition (3.1.3) and Theorem (3.1.1) by the Lata la-Oleszkiewicz argument. We perform the "dimension lift" and compute the average diameters of random, proving Theorem (3.1.7).

We begin by proving Proposition (3.1.3). This proposition is a reformulation of the "small ball probability conjecture". It was recently deduced by R. Lata la and K. Oleszkiewicz [120] from the B-conjecture proved by Cordero, Fradelizi and Maurey [112]. We will reproduce the Lata laoleszkiewicz argument here. We start with a standard and well-known lemma, on the close relation between the uniform measure σ on the sphere S^{n-1} and the standard gaussian measure γ on R^{n} . We include its proof.
Lemma (3.1.2)[111]: For every centrally-symmetric convex body K,

$$\frac{1}{2}\sigma(S^{n-1} \cap \frac{1}{2}K) \le \gamma(\sqrt{n}K) \le \sigma(S^{n-1} \cap 2K) + e^{-cn}$$

where $c > 0$ is a universal constant.

Proof. We will use the following two estimates on the Gaussian measure of the Euclidean ball,

$$\gamma(2\sqrt{n}D^n) > \frac{1}{2}, \gamma\left(\frac{1}{2}\sqrt{n}D^n\right) < e^{-cn}.$$

The first estimate is simply Chebychev's inequality, and the second follows from standard large deviation inequalities, $e.g.$ Cramer's Theorem [132]. Since K is star-shaped,

$$\gamma(\sqrt{n}K) \ge \gamma(2\sqrt{nD^n} \cap \sqrt{nK})$$
$$\ge \gamma(2\sqrt{nD^n})\sigma_1\left(2\sqrt{nS^{n-1}} \cap \sqrt{nK}\right)$$

where σ_1 denotes the probability rotation invariant measure on the sphere $2\sqrt{nS^{n-1}}$,

$$\ge \frac{1}{2}\sigma\left(S^{n-1} \cap \frac{1}{2}K\right).$$

This proves the lower estimate in the lemma.

For the upper estimate, note that no points of $\sqrt{n}K$ can lie outside both the ball $\frac{1}{2}\sqrt{nD^n}$ and the positive cone generated by $\frac{1}{2}S^{n-1} \cap \sqrt{nK}$. Adding the two measures together, we obtain

$$\gamma(\sqrt{nK}) \le \gamma\left(\frac{1}{2}\sqrt{nD^n}\right) + \sigma_2\left(\frac{1}{2}\sqrt{nS^{n-1}} \cap \sqrt{nK}\right)$$

where σ_2 denotes the probability rotation invariant measure on the sphere $\frac{1}{2}\sqrt{nS^{n-1}}$,

$$\le e^{-cn} + \sigma(S^{n-1} \cap 2K)$$

This completes the proof

Proposition (3.1.3)[111]: (Negative moments of a norm). Assume that $0 < \ell < cd(K)$. Then

$$cM < \left(\int_{S_{n-1}} \|x\|^{-\ell} \, d\sigma(x)\right)^{-\frac{1}{\ell}} < CM$$

where $c, C > 0$ are universal constants.

Proof. As usual, K will denote the unit ball of the norm k · k. The B-conjecture, proved in [112], asserts that the function $t \to \gamma(e^t K)$ is log-concave. This means that for any $a, b > 0$ and $0 < \lambda < 1$,

$$\gamma\left(a^{\lambda} b^{1-\lambda} K\right) \ge \gamma(aK)^{\lambda} \gamma(bK)^{1-\lambda}. \tag{5}$$

Let Med $= Med(K)$ be the median of the norm $\|\cdot\|$ on the unit sphere S^{n-1}. By Chebychev's inequality, $Med \le 2M(K)$. Set $L =$ Med $\cdot \sqrt{nK}$. According to Lemma (3.1.2),

$$\gamma(2L) \ge \frac{1}{2}\sigma(S^{n-1} \cap Med \cdot K) \ge \frac{1}{4} \tag{6}$$

by the definition of the median. On the other hand, again by Lemma (3.1.2),

$$\gamma\left(\frac{1}{8}L\right) \le \sigma\left(S^{n-1} \cap \frac{1}{4}Med \cdot K\right) + e^{-cn}$$
$$= \sigma\left(x \in S^{n-1} : \|x\| \le \frac{1}{4}Med\right) + e^{-cn} \tag{7}$$

$$\leq \sigma\left(x \in S^{n-1} \ : \|x\| \leq \frac{1}{2}M(K)\right) + e^{-cn} \leq e^{-d}(K) + e^{-c'n}$$
$$< 2e^{-Cd}(K)$$
because $d(K) \leq n$. We may assume that $< e\text{^} - 3$, and apply (5) for $a = \varepsilon, b = 2, \lambda = \frac{3}{log\frac{1}{\varepsilon}}$. This yields

$$\gamma(\varepsilon L) \frac{3}{log(1/\varepsilon)} \gamma(2L)\, 1 - \frac{3}{log(1/\varepsilon)} \leq \gamma\left(\varepsilon \frac{3}{log(1/\varepsilon)} 2^1 - \frac{3}{log(1/\varepsilon)} L\right)$$
$$\leq \gamma(\frac{1}{8} L)$$

Combining this with (6) and (7), we obtain that
$$\gamma\ (\varepsilon L) \leq 8e^{C'}d(K)\, log\varepsilon) \leq 8\varepsilon\text{^}cd\ (K) < (c'\varepsilon)\text{^}cd(K)$$
and according to Lemma (3.1.2) we can transfer this to the spherical measure, obtaining
$$\sigma(x \in S^{n-1} : \|x\| < \varepsilon M) < (C\varepsilon)^{cd(K)}$$
By integration by parts, this yields that for any $0 < \ell < \frac{cd(K)}{10}$,
$$\left(\int_{Sn-1} \left\|\frac{x}{M}\right\|^{-\ell} d\sigma(x)\right)^{\frac{1}{\ell}} \leq C,$$
which implies the left hand side of the inequality in Proposition (3.1.3). The right hand side follows easily by H¨older's inequality.
By Chebychev's inequality, Proposition (3.1.3) yields the desired tail inequality for the small ball probability:
Corollary (3.1.4)[111]: (The small ball probability). For every $0 < \varepsilon < \frac{1}{2}$,
$$\sigma\ \{x \in S^{n-1} : \|x\| < \varepsilon M\} < \varepsilon^{cd(K)} < \varepsilon^{c'k(K)}\ ,$$
where $c, c' > 0$ are universal constants.
Theorem (3.1.1) is contained in Corollary (3.1.4). Let us give some interpretation of the expression in Proposition (3.1.3). For a subspace $E \subset R^n$, let $S(E) = S^{n-1} \cap E$ and σE be the unique rotation invariant probability measure on the sphere $S(E)$. We will use the fact that $Vol(K) = ol(D^n)\ R \int_{S_{n-1}} \|x\|^{-n} d\sigma(x)$. The volume radius of a k-dimensional set T is defined as

$$v.rad.(T) = \left(\frac{Vol(T)}{Vol(D^k)}\right)^{\frac{1}{k}}$$

Thus

$$v.rad.(K) = \left(\int_{(Sn-1)} \|x\|^{-n} d\sigma(x)\right)^{1/n} .$$

By the rotation invariance of all the measures (as in [119]), we conclude that
$$\int_{Sn-1} \|x\|^{-k}\ d\sigma(x) = \int_{G_{n,k}} \int_{S(E)} \|x\|^{-k} d\sigma_E(x)\, d\mu(E)$$

$$= \int_{\mathrm{Gn,k}} v.rad.(K \cap E)^k \, d\mu(E), \quad (8)$$

where, as before, μ is the unique rotation invariant probability measure on $G_{n,k}$. Thus Proposition (3.1.3) asymptotically computes the average volume radius of random. This perfectly fits the estimates for diameters in [119], to be applied next.

We prove the main result, Theorem (3.1.7). We regard $\|x\|^{-1}$ as the radius of the one-dimensional spanned by x; thus Proposition (3.1.3) is an asymptotically sharp bound on the diameters of random one-dimensional. Theorem (3.1.7) extends this bound to k-dimensional, for all k up to the critical dimension $d(K)$. We start with a "dimension lift", which is based on the "low M estimate", Proposition 3.9 in [119] (the case $\lambda = \frac{1}{2}$ there). Here we estimate the L_k norm, rather than only the tail probability as in Proposition 3.9 in [119].

In order to prove Theorem (3.1.3) we need a standard lemma on the stability of the average norm M. We are unaware of a reference for the exact statement we need (a similar result appears e.g. in Lemma 6.6 of [127]), so a proof is provided. The average norm on a subspace $E \in G_{n,k}$ is denoted by $M_E = \int_{S(E)} \|x\| d\sigma_E(x)$.

Lemma (3.1.5)[111]: For every norm $\|\cdot\|$ on R^n and every integer $0 < k < n$

$$cM < \left(\int_{G_{n,k}} (ME)^{2k} \, d\mu(E) \right)^{\frac{1}{2k}} < CM \quad (9)$$

where $c, C > 0$ are universal constants.

Proof. The left hand size inequality in (9) follows easily from H¨older's inequality. In the proof of the right hand side inequality, we will use a variant of Raz's argument (see [131], [130]). We normalize so that $M = 1. Let\, X_1, .., X_k$ be k independent random vectors, distributed uniformly on S^{n-1} . It is well-known that a norm of a random vector on the sphere has a subgaussian tail (e.g. [124]. It actually follows from (2) above):

$$E \exp(s\|X_i\|) < exp\,(cs^2) \ for\ all\ i\ and\ all\ s > 1$$

which by independence implies

$$E \exp\left(s \cdot \frac{1}{k} \sum_{i=1}^{k} \|Xi\| \right) < \exp\left(\frac{Cs^2}{k} \right) for\ s > 1$$

Using Chebychev's inequality and optimizing over s (e.g. [128]), we obtain

$$Prob\ \left\{ \frac{1}{k} \sum_{i=1}^{k} \|X_i\| > Ct \right\} < exp\,(-t^2\, k)\ for\ t > 1 \quad (10)$$

Let E be the linear span of $X_1, .., X_k$. Then E is distributed uniformly in $G_{n,k}$ (up to an event of measure zero). Since for any two events one has $Prob(A) \leq \frac{Prob(B)}{Prob(B|A)}$, we conclude that

$$Prob\,\{M_E > 2ct\} \leq \frac{Probn\ \left\{ \frac{1}{k} \sum_{i=1}^{k} \|X_i\| > ct \right\}.}{Probn\ \left\{ \frac{1}{k}\ P \sum_{i=1}^{k} \|X_i\| > ct\,|\,|\,M_E > 2ct \right\}} \quad (11)$$

The numerator in (11) is bounded by (10). To bound the denominator from below, note that $\|Xi\| < C\sqrt{k}M_E$ pointwise for all i; This is a consequence of a simple comparison inequality for the Gaussian analogs of M and ME (see e.g. [128]). Let us fix a subspace $E \in G_{n,k}$. Note that, conditioning on $E = \text{span}\{X_1, \dots, X_k\}$, each of the vectors X_i is distributed uniformly in $S(E)$. Next, we estimate the robability probability $P_E = Prob\left\{\frac{1}{k} P \sum_{i=1}^{k} \|Xi\| > \frac{ME}{2} span\{X_1, .., X_k\} = E\right\}$ via Chebychev's inequality as

$$M_E = E\left(\frac{1}{k}\sum_{i=1}^{k} \|Xi\| \ span\{X_1, .., X_k\} = E\right) \leq C\sqrt{}kM_E P_E + \frac{ME}{2}(1 - P_E).$$

Hence $P_E \geq \frac{\bar{c}}{\sqrt{k}}$ for every $E \in G_{n,k}$. Thus,

denominator in (11) $\geq Prob\left\{(\frac{1}{k}\sum_{i=1}^{k} \|X_i\| > \frac{ME}{2} \mid M_E > 2ct)\right\}$

$$= \frac{1}{Prob\{E \in G_{n,k};\ M_E > 2ct\}}\int_{E\in G_{n,k}; M_E > 2ct} P_E \, d\mu(E)$$

$$\geq \min_{E\in G_{n,k}} P_E \geq \frac{\overline{c}}{\sqrt{k}}.$$

Combining this with (10) and (11) we get

$$Prob\{M_E > 2ct\} < c\sqrt{ke^{-t^2 k}} < e^{-Ct^2 k} \, for \ t > 1.$$

By integration by parts we obtain the desired estimate.

Proposition (3.1.6)[111]: (Dimension lift for diameters). Let $1 \leq k_0 < n$. Then for any integer $k < k_0/\ ,4$,

$$\left(\int_{Gn,k} diam(K \cap E)^k \, d\mu(E)\right)^{\frac{1}{k}} \leq CM(K)\left(\left(\int_{S^{n-1}} \|x\|^{-k_0} \, d\sigma(x)\right)\right)^{\frac{2}{k_0}}.$$

Proof. By H¨older inequality, the right hand side increases with k_0, hence we may assume that $k_0 = 4k$. We shall rely on the main result in [119], which claims that for any centrally-symmetric convex body $T \subset R^n$, and for all $0 < k \leq l < n$

$$v.rad.(T) > C\left(\int_{G_{n,k}} v.rad.(T \cap E)^l \, diam(T \cap E)^{n-l} \, d\mu(E)\right)^{\frac{1}{n}}. \tag{12}$$

We are going to apply (12) to $T = K \cap E$, for subspaces $\in G_{n,k_0}$. Denote by $G_{E,k}$ the grassmanian of all k-dimensional subspaces of E, equipped with the unique rotational invariant probability measure. Then by (8), (12) and the rotational invariance of all measures,

$$\int_{E\in G_{n,k}} v.rad.(K \cap E)^{2k} \, diam(K \cap E)^{2k} \, d\mu(E)$$

$$= \int_{E\in G_{n,4k}} \int_{FG_{E,k}} v.rad.(K \cap F)^{2k} \, diam(K \cap F)^{2k} \, d\mu E(F) d\mu(E)$$

$$\leq C^{k_0} \int_{E\in G_{n,k_0}} v.rad.(K \cap E)^{k_0} \, d\mu(E) = C^{k_0} \int_{S^{n-1}} \|x\|^{-k0} \, d\sigma(x).$$

Also, by the Cauchy-Schwartz inequality,

$$\left(\int_{E\in G_{n,k}} diam(K \cap E)^k \, d\mu(E)\right)^{\frac{1}{k}}$$

$$\leq \left(\left(\int_{E\in G_{n,k}} v.rad.(K \cap E)^{2k} \, diam(k \cap E)^{2k} \, d\mu(E)\right)^{\frac{1}{2k}} \left(\int_{E\in G_{n,k}} \frac{1}{v.rad.(K \cap E)^{2k}} \, d\mu(E)\right)^{\frac{1}{2k}}\right.$$

We will use the standard inequality $\frac{1}{v.rad.(K\cap E)} \leq ME$, which follows directly from $H\ddot{}$older inequality. Then,

$$\left(\int_{E\in G_{n,k}} diam(K \cap E)^k \, d\mu(E)\right)^{\frac{1}{k}}$$

$$\leq C \left(\int_{S^{n-1}} \|x\|^{-k_0} \, d\sigma(x)\right)^{\frac{2}{k_0}} \left(\int_{E\in G_{n,k}} (ME)^{2k} \, d\mu(E)\right)^{\frac{1}{2k}}$$

and the proposition follows by Lemma (3.1.5).

Theorem (3.1.7)[111]: (Diameters of random). Assume that $0 < \ell < \mathrm{cd}(K)$. Select a random ℓ-dimensional subspace $E \in G_{n,\ell}$ Then with probability larger than $1 - e^{-c'\ell}$,

$$K \cap E \subset \frac{C}{M} (D^n \cap E). \tag{13}$$

Furthermore,

$$\frac{\bar{c}}{M} < \left(\int_{G_{n,\ell}} diam(K \cap E)^{\ell} \, d\mu(E)\right)^{\frac{1}{\ell}} < \frac{\bar{C}}{M} \tag{14}$$

where μ is the unique rotation invariant probability measure on $G_{n,\ell}$, and $c, c', \bar{c}, C, \bar{C} > 0$ are universal constants.

Proof. It is sufficient to prove (14), since (13) follows by Chebychev's inequality. The left hand side inequality is clear. According to Proposition (3.1.6) and Proposition (3.1.3), the right hand side inequality of (14) follows, as

$$\left(\int_{G_{n,\ell}} diam(K \cap E)^{\ell} \, d\mu(E)\right)^{\frac{1}{\ell}} < CM(K)\left(\frac{C}{M(K)}\right)^2 \leq \frac{C'}{M(K)}.$$

Remark (3.1.8)[111]: (Optimality). The estimate $\ell < cd(K)$ in Theorem (3.1.7) is essentially optimal, for any centrally-symmetric convex $K \subset R^n$. Indeed, suppose that $\ell > d_u(K)$ for some $u \gg 1$. Then,

$$\left(\int_{S^{n-1}} \|x\|^{-\ell} \, d\sigma(x)\right)^{\frac{1}{\ell}} > \frac{u}{M} e^{-1} \gg \frac{1}{M}. \tag{15}$$

Since we always have diam$(K \cap E) \geq 2\, v.rad.(K \cap E)$, then by (15) and (8),

$$\left(\int_{G_{n,\ell}} diam(K \cap E)^{\ell}\, d\mu(E)\right)^{\frac{1}{\ell}} \gg \frac{1}{M}.$$

Thus, (14) cannot hold for $\ell > d_u(K)$ when $u \gg 1$.

Example (3.1.9)[111]: (of the cube). Suppose $K = B_\infty^n$ is a cube, the unit ball of l_∞^n, and let us estimate $d_u(K)$. It is well-known that $c\sqrt{\frac{\log n}{n}} < M(B_\infty^n) < C\sqrt{\frac{\log n}{n}}$. According to Lemma (3.1.2), we may equivalently carry out our computations in the Gaussian setting. Now, for $t > 0$,

$$\gamma\left(t\sqrt{\log n B_\infty^n}\right) = Y\prod_{i=1}^{n} Prob\{|X| \leq t\sqrt{\log n}\} = \left(1 - 2\Phi\left(t\sqrt{\log n}\right)\right)^n$$

where $X \sim N(0,1)$ is a standard normal random variable, and $\Phi(s) = R\int_s^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\, du$. For $s > 1$, a crude estimate gives $e^{-c}1^{s^2} < \Phi(s) < e^{-c}2^{s^2}$. Thus, when $t > \frac{10}{\sqrt{\log n}}$,

$$exp(-c_1' n^1 - c_2' t^2) < \gamma(t\sqrt{\log n B_\infty^n}) < exp(-c_1 n^1 - c_2 t^2).$$

Therefore, by Lemma (3.1.2),

$$c_1 n^1 - \frac{c1}{u^2} < d_u(B_\infty^n) < c_2 n^1 - \frac{c2}{u^2}$$

for any $u > 1$. Theorem (3.1.7) implies that random of this (polynomial!) dimension $d_u(B_\infty^n)$ have a diameter $\approx \frac{1}{M(B_\infty^n)}$.

Example (3.1.10)[111]: (of the l_p ball). Consider $= B_p^n$, the unit ball of l_p^n, for some fixed $1 \leq p < \infty$. In this case, the conclusion of Theorem (3.1.7) is well known. For $1 \leq p \leq 2$, we know that $k(B_p^n) > cn$ (e.g. [128]), hence also $d(B_p^n) > cn$ and the conclusion of the theorem follows from the classical Dvoretzky Theorem. In the case where $2 < p < \infty$, we have $B_p^n \subset \frac{cp}{M(B_p^n)} D^n$, and thus the conclusion of Theorem (3.1.7) is obvious in this case (with constants depending on p), and $d_{c_p}(B_p^n) > C_p n$, for some constants c_p, C_p depending only on p.

Section (3.2): Almost Spherical Sections of Convex Bodies

The following deep result is due to A. Dvoretzky [134].

Theorem (3.2.1)[133]: For every $\varepsilon > 0$ and for every integer k there exists an integer $n(k,\varepsilon)$ such that every n-dimensional Banach space H, with $n \geqq n(k,\varepsilon)$, contains a k-dimensional subspace E such that

$$d\left(E, I_2^k\right) \leqq 1 + \varepsilon$$

where d denotes the Banach-Mazur distance.

(In this and other standard notation we follow [138].)

We present here a complete proof of Dvoretzky's Theorem. The original proof is sophisticated and difficult. (Moreover, an approximation argument used in [134] without proof is not obvious. The details of it were supplied by T. Figiel [135].) V. D. Milman [127] gave a simplified exposition of Dvoretzky's proof. This proof relies on an

isoperimetric inequality of Levy whose proof (at least in Levy's original presentation) is not complete.
A functional analytic proof of Dvoretzky's theorem was recently given by L. Tzafriri [140] (his proof uses some results of Brunel and Sucheston). Tzafriri's proof shows only the existance of $n(k,\varepsilon_0)$ for all k and for a fixed positive ε_0.
Our proof appears much simpler than the original one. Although we exploit most of Dvoretzky's ideas, the most difficult part of his proof has been replaced by a simple geometric argument.
Let a norm $\|\ \|$ be defined on a subspace E of R^m. We say that $(E,\|\ \|)$ is e-euclidean if for some constant C,

$$c\|x\| \leqq \|x\|_2 \leqq c(1+\varepsilon)\|x\| \quad \text{for all } x \in E.$$

It is obvious that if E is e-euclidean, then

$$d\left(E, l_2^{dimE}\right) \leqq 1+\varepsilon.$$

We denote $S_E = \{x \in E: \|x\|_2 = 1\}$.
We can assume without loss of generality that $\|\ \|$ is uniformly smooth. Then there exists a unique mapping

$$T = T_E = T_{E\|\ \|}: S_E \to S_E$$

such that

$$(x, T_E x) = \|x\| \cdot \|T_E x\|_E^*$$

Where

$$\|x^*\|_E^* = \sup\{(x,x^*)\|x\|^{-1}: x \in E\}.$$

T has a clear geometric meaning: $T_E x\ \varepsilon$ he normalized vector that is normal to the hyperplane (in E) that supports a sphere of $\|\ \|$ If in x. It is obvious that $Tx = x$ for all $x \in S_E$, if and only if $\|\ \|$ is equal to $\lambda\|\ \|_2$ for some λ. It is also intuitively clear that if T is not far from the identity, then $\|\ \|$ should be ε-euclidean for a small ε.
We shall proceed from this idea. Let us denote, by $\phi_E(x)$, the angle between x and $T_E x$ and, by $\alpha_E(x)$, the cosine of this angle, that is,

$$\alpha_E(x) = (x, T_E x).$$

By Σ_E we shall denote the Stiefel manifold of all (ordered) pairs of orthonormal vectors in E.
In Σ_E we have σ_E, the unique rotation invariant normalized measure. For $\langle x,y\rangle \in \Sigma_E$ we put

$$\alpha_y(x) = \alpha_{[x,y]}(x).$$

It is obvious that

$$\alpha_E(x) = \inf\{\alpha_y(x): y \perp x, y \in S_E\}. \tag{16}$$

To fix attention, let

$$A_E = \left\{\langle x,y\rangle \in \Sigma_E: \alpha_y(x) > (1-\beta^2)^{\frac{1}{2}}\right\}. \tag{17}$$

It is intuitively clear that if β is small enough and $\sigma(A_E)$ is big enough, then E is ε-euclidean for a small ε. This will be made precise in our Proposition (3.2.6).
Now, let $m > k$, let $dim\, F = m$, and let Γ be the Grassman manifold of all k-dimensional subspaces of F. In Γ we have γ, the unique rotation invariant normalized measure. By uniqueness of σ_F, we have

$$\sigma_F(A_F) = \int_\Gamma \sigma_E(A_E)\, d\gamma(E). \tag{18}$$

But we have also, for the same reason,

$$\sigma_F(A_F) = \int_{S_F} \lambda_x(A_x) d\lambda(x) \tag{19}$$
where λ and λ_x are the normalized Lebesgue measures on S_F and
$$S_x = \{y \in S_F : (x, y) = 0\},$$
respectively, and
$$A_x = \{y \in S_x : \langle x, y \rangle \in A_F\}.$$
We use now the well-known Dvoretzky-Rogers Theorem the following form (see [134]).

Proposition (3.2.2)[133]: Let $n > 4m^2$. There exists an m-dimensional subspace F of n that is isometric to $F = (R^m, \| \|)$ with
$$\|x\|_2 \geqq \|x\| \geqq \|x\|_\infty \quad for \quad x \in F = R^m. \tag{20}$$
(Here $\|x\|_\infty = \sup|x_i|$ for $x = (x_1, x_2, \dots, x_m) \in R^m$.) From now on we shall deal with an F satisfying (20).

It appears now that $\lambda_x(A_x)$ can be suitably estimated in terms of $m, \|x\|_\infty$, and β only. Using a simple geometric argument we obtain namely (refer to Proposition (3.2.7))
$$\lambda_x(A_x) > 1 - 2\beta^{-1}\|x\|_\infty^{-1} \cdot m^{-\frac{1}{2}}. \tag{21}$$
Hence
$$\sigma_F(A_F) > 1 - 2\beta^{-1} \cdot m^{-\frac{1}{2}}\|x\|_\infty^{-1} d\lambda(x). \tag{22}$$
The Last integral appears also in Dvoretzky's proof. A delicate Dvoretzky's inequality gives exactly what we need:
$$\int_{S_F} \|x\|_\infty^{-1} d\lambda(x) = o\left(m^{-\frac{1}{2}}\right) \ (here\ m = \dim F)$$
or in its stronger version
$$\int_{S_F} \|x\|_\infty^{-1} d\lambda(x) = o\left(\frac{m}{2\log m}\right)^{\frac{1}{2}}. \tag{23}$$
(A short proof of (23) is presented in [135].)

We obtain now that for big m, the integral in (18) is arbitrarily close to 1 and therefore we can find $E \subset F$, $dim\ E = k$, with big enough $\sigma_E(A_E)$.

For $E \subset F = R^m$ we write:

λ_E = the normalized Lebesgue measure on $S_E = \{x \in E : \|x\|_2 = 1\}$;

Σ_E = the Stiefel manifold, that is, $\Sigma_E = \{\langle x, y \rangle \in S_E \times S_E : (x, y) = 0\}$;

σ_E = the normalized rotation invariant measure on Σ_E;

$E_x^\perp = \{y \in E : (x, y) = 0\}$;

$S_x^E = S_{E_x^\perp}$ and $\lambda_x^E = \lambda_{E_x^\perp}$.

We shall drop the index E whenever it does not lead to confusion.

On S we introduce the geodesic distance ρ, that is,
$$\rho\langle x, y \rangle = arc\cos(x, y), \|x - y\|_2 = 2\sin\frac{1}{2}\rho\langle x, y \rangle \leqq \rho\langle x, y \rangle.$$
Let $dim\ E = k$. It is obvious that λ_E of the fl-neighbourhood of a point in S_E depends only on k and β. The same can be said about λ_E of the β-neighbourhood of an equator in S_E (that is, of S_E by a hyperplane passing through zero). We shall denote the first number by $\mu(k, \beta)$ and the second one by $v(k, \beta)$.

We have the following estimates for μ and v (see [127]).
$$(k, \beta) \leqq (k-1)^{\frac{1}{2}} \sin^{k-1}\beta\,; \tag{24}$$

$$\mu(k,\beta) \geqq k^{\frac{1}{2}}\left(\frac{1}{4}\beta\right)^{k-1}; \tag{25}$$

$$v(k+2,\beta) \geqq 1-\sqrt{2}\,exp\left(-\frac{1}{2}\beta^2 k\right) \geqq 1-\beta^{-1}-k^{-1}. \tag{26}$$

Before we pass to Proposition (3.2.6), we shall need two lemmas. In their proofs we shall drop the subscript E wherever it should occur. We assume that β and η are sufficiently small.

Lemma (3.2.3)[133]: lf B is a β-net on S_E such that

$$(1+\eta)\inf\{\|x\|: x \in B\} \geqq \sup\{\|x\|: x \in B\} = r,$$

then E is $(2\eta + 5\beta)$-euclidean.

Proof. We can assume that $r = 1$. For every $y \in S$ we have

$$\|y\|^* \geqq sup\{(x,y)\cdot\|x\|^{-1}: x \in B\} \geqq \sup\{(x,y): x \in B\} \geqq \cos\beta > (1-\ \beta^2)^{-\frac{1}{2}}.$$

Therefore for every $x \in S$ wc have

$$\|x\| \leqq (1-\beta^2)^{-\frac{1}{2}}, \tag{27}$$

Now, let $z \in S$ and let $x \in B$ be such that $\rho\langle z,x\rangle \leqq \beta$. Let $y \in S$ be such that

$$\|x\| = (x,y)(\|y\|^*)^{-1}.$$

Thus we have for every $z \in S$:

$$\|z\| \geqq (z,y)(\|y\|^*)^{-1} = \|x\| + (z-x,y)(\|y\|^*)^{-1} \geqq (1+\eta)^{-1} - \|z-x\,\|_2(1-\beta^2)^{-\frac{1}{2}}$$

$$\geqq 1-\eta-2\beta(1-\beta^2)^{-\frac{1}{2}} > 1-\eta-3\beta. \tag{28}$$

Clearly, (27) and (28) imply

$$d\left(E, l_2^{dimE}\right) < (1-\beta^2)^{-\frac{1}{2}}(1-\eta-\beta)^{-1} \leqq (1+\beta)(1+\eta+3\beta)$$

$$= 1+\eta+4\beta+\beta(\eta+3\beta) \leqq 1+2\eta+5\beta.$$

Lemma (3.2.4)[133]: Let $\beta = 10^{-3}\varepsilon$. If the set

$$B = \left\{x \in S_E: \alpha_E(x) > (1-6^2\beta^2)^{\frac{1}{2}}\right\}$$

is a β-net in S_E, then E is ε-euclidean.

Proof. Let $z \in B$ be such that

$$\|z\| = \sup\{\|x\|: x \in B\}.$$

Take an arbitrary $x \in B$. Clearly there exists a sequence $x_0, \dots, x_p \in B$ such that

$$x_0 = \pm z, x_p = x, p \leqq \frac{1}{2}\pi\beta^{-1},$$

$$\rho\langle x_i, x_{i+1}\rangle \leqq 3\beta \quad for \quad i = 0, \dots p-1.$$

Since

$$\rho(x_i, Tx_i) \leqq 2\left(1-\alpha^2(x_i)\right)^{\frac{1}{2}} < 12\beta,$$

We obtain for $i = 0, \dots, p-1$

$$\rho(x_{i+1}, Tx_i) \leqq \rho(x_i + x_i) + \rho(x, Tx_i) < 15\beta.$$

Hence

$$\|x_{i+1}\| \geqq (x_{i+1}, Tx_i)(\|Tx_i\|^*)^{-1} \geqq (1-15^2\beta^2)^{\frac{1}{2}}\|x_i\|\alpha^{-1}(x_i) \geqq (1-15^2\beta^2)^{\frac{1}{2}}\|x_i\|.$$

Therefore (we use inequality $e^t \leqq 1+2t$ for $0 \leqq t \leqq 1$)

$$\|x\| \geqq (1-15^2\beta^2)^{\frac{1}{2}p}\|z\| \geqq \exp(-225\,\beta)\,\|z\| \geqq (1+450\,\beta)^{-1}\|z\|.$$

We take now $\eta =\ 450\beta$ and apply Lemma (3.2.4).

We have the following trivial lemma.

Lemma (3.2.5)[133]: Let $x \in E_1 \subset E$ and let π be the orthoaonal projection from onto E_1. Then

$$T_{E_1}x = \pi(T_E x)/\|\pi(T_E x)\|_2.$$

Consequently

$$\alpha_{E_1}(x) = \alpha_E(x)/\|\pi(T_E x)\|_2.$$

If now

$$E_1 = [x, y] \quad with \ \langle x, y\rangle \in \Sigma,$$

Then

$$\pi(T_E x) = (x, T_E x)x + (y, T_E x) \quad y = \alpha_E(x) \cdot x + (y, T_E x) \cdot y.$$

and finally

$$\alpha_y(x) = \alpha_E(x)(\alpha_E^2(x) + (y, T_E x)^2)^{-\frac{1}{2}}. \tag{29}$$

Proposition (3.2.6)[133]: Let $dim\ E = k$ and let $\beta = 10^{-3}\varepsilon$. If

$$\sigma_E(A_E) > 1 - \mu(k, \beta)\mu(k-1, \beta),$$

then E is e-euclidean.

Proof. Again we use uniqueness of σ and obtain

$$\sigma(A) = \int \lambda_x(A_x)d\lambda(x) \tag{30}$$

where $A_x = \{y \in S_x: \langle x, y\rangle \in A\}$. Put $B = \{x \in S: \lambda_x(A_x) > 1 - \mu(k-1,/\beta\}$. us notice that

A_x is a β-net in S_x for every $x \in B$.

From (30) it follows that

$$\lambda(B) > 1 - \mu(k, \beta)$$

and thus B is a β-net in S.

To apply Lemma (3.2.4) we should show that

$$\alpha(x) > (1 - 6^2\beta^2)^{\frac{1}{2}} \text{ for every } x \in B. \tag{31}$$

Let $x \in B$, let $z \in S_x$ be arbitrary, and let $y \in A_x$ be such that $\rho(y, z) < \beta$, hence $\|y - z\|_2 < \beta$. Assume $\beta < \frac{1}{4}$; it follows easily from (29) that

$(y, Tx) < 2\beta, \alpha(x) = (x, Tx) > \frac{1}{2}$ and

$(z, Tx) = (z - y, Tx) + (y, Tx) \leqq \|z - y\|_2 + 2\beta < 3\beta < 6\beta \cdot \alpha(x)$.

Hence, by (29),

$$\alpha_z(x) > (1 - 6^2\beta^2)^{\frac{1}{2}} \ for\ every\ \ z \in S_x.$$

This, by (32), gives us (31) and concludes the proof of Proposition (3.2.6).

Proposition (3.2.7)[133]: Inequality (22) holds.

Proof. It is clear that $\lambda_x(A_x)$ depends only on $\alpha(x)$; we have namely

$$\alpha_y(x) = \alpha(x)(\alpha^2(x) + \sin^2\theta\left(1 - \alpha^2(x)\right)^{-\frac{1}{2}}$$

where θ is the angle between y and the hyperplane $F_x^\perp \cap F_{Tx}^\perp$. Figures 1 and 2 should clarify the situation.

Thus $\alpha_y(x) > (1 - \beta^2)^{\frac{1}{2}}$ is satisfied provided

$$(1 + \sin^2\theta\, tg^2\phi)^{-1} > 1 - \beta^2. \tag{32}$$

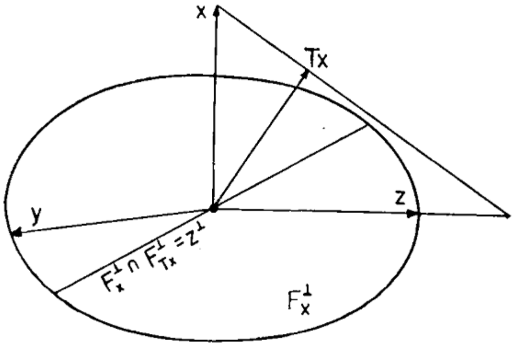


Fig. (1)[133]:

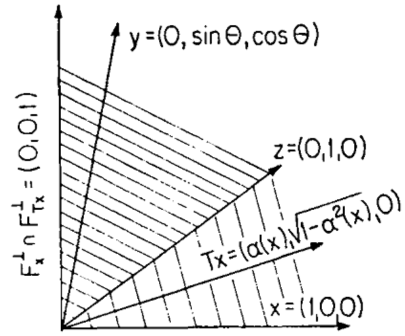


Fig. (2)[133]:

(Here $\phi = \phi(x) = \ arc \cos \alpha(x)$). Clearly, (32) is satisfied if

$$|\theta| < |\beta \ ctg \ \phi|.$$

From this and (26) it follows that (for $m \geqq 6$)

$$\lambda_x(A_x) \geqq v(m-1, |\beta \ ctg \ \phi|) \geqq 1 - 2\beta^{-1}m^{\frac{1}{2}}|tg \ \phi|. \tag{33}$$

By (20) we have $\|Tx\|_F^* \geqq \|Tx\|_2 = 1$ and hence

$$|ctg \ \phi(x)| > \cos\phi\,(x) = \alpha_F(x) = (x, Tx) \geq (x, Tx)(\|Tx\|_F^*)^{-1} = \|x\|$$
$$\geqq \frac{1}{2}\|x\|_\infty. \tag{34}$$

This gives (21) and, consequently, (22).

This, together with (23), completes the proof of Dvoretzky's Theorem.

A. An estimate for $n(k,\varepsilon)$. It is natural to ask what is the best possible value for $n(k,\varepsilon)$ in Theorem (3.2.1). This problem has also been discussed in [134] and [127]. The following upper estimates have been obtained there.

$$\ln[1], n(k,\varepsilon) \leqq \exp(C \cdot \varepsilon^{-2}k^2 \ln^2 k),$$
$$\ln[6], n(k,\varepsilon) \leqq \exp(C \cdot \varepsilon^{-2}k \ln \varepsilon^{-1}).$$

Here C is a constant.

Notice that the exponential order of magnitude of these estimates cannot be improved. For, taking $H = l_\infty^n$, we obtain the following lower estimate (see [127]):

$$n(k,\varepsilon) \geqq \exp(c \cdot k \ ln \ \varepsilon^{-1}).$$

Examining our proof, one obtains an estimate of essentially worse order of magnitude. We can, however, improve it by estimating more carefully the integral in (19).

By (26), (33), and (34) we obtain namely

$$\lambda_x(A_x) \geqq 1 - 2\exp\left(-\frac{1}{16}\beta^2\|x\|_\infty^2 m\right).$$

Therefore, by (19),

$$\sigma_F(A_F) \geqq 1 - 2\int_{S_F} \exp\left(-\frac{1}{16\beta^2}\|x\|_\infty^2 \, m\right) d\lambda(x). \tag{35}$$

Thus we want to estimate the integral

$$L_m = \int_{S^m} \exp\left(-\frac{1}{16\beta^2}\|x\|_\infty^2 m\right) d\lambda(x),$$

where S^m is the euclidean unit sphere in R^m and $\lambda = \lambda_m$ is the normalized Lebesgue measure on S^m.

In the remarkable Theorem 3(B) in [134], it is proved that for every $\varepsilon > 0$

$$\lim_{n\to\infty} \lambda_m \left\{x: \left(\frac{m}{2\log m - (1+\varepsilon)\log\log m}\right)^{\frac{1}{2}} \leqq \|x\|_\infty^{-1} \leqq \left(\frac{m}{2\log m - (1+\varepsilon)\log\log m}\right)^{\frac{1}{2}}\right\} = 1.$$

This means that values of the function $\|x\|_\infty^{-1}$ defined on S^m are assymptotically concentrated around the values $\left(\frac{m}{2\log m}\right)^{\frac{1}{2}}$ or, more precisely, around

$$\left(\frac{m}{2\log m - \log\log m}\right)^{\frac{1}{2}}.$$

In [135], Figiel proves a related result: For every function h of the form $h(t) = t^{-\alpha}, \alpha > 0$, we have

$$\int_{S^m} h(\|x\|) d\lambda(x) \leqq h\left(\left(\frac{m}{2\log m}\right)^{\frac{1}{2}}\right)\left(1 + O\left((\log m)^{-\frac{1}{2}}\right)\right) \qquad (36)$$

It seems thus that the same relation should hold for all h in a wide class of functions.

By a modification of Figiel's argument we shall strengthen his above-mentioned result. In the following lemma we actually prove more than is needed in applications to Dvoretzky's Theorem. We think, however, that the study of the integralgeometric properties of the function $\|x\|$ is of separate interest.

Lemma (3.2.8)[133]: Let $k = k_m = o(m)$ and let

$$J_m = \int_{S^m} \|x\|_\infty^{-2k} d\lambda(x).$$

(i) If $\varepsilon > 0$ and $k \leqq m^{\frac{\varepsilon}{4}}$, then

$$J_m \leqq \left(\frac{m}{2(1-a)\log m}\right)^k \left(1 + o\left(2^{-m^{\frac{\varepsilon}{4}}}\right)\right);$$

(ii) if $k = a_m (\log m)^{\frac{1}{2}}$, where $\{a_m\}$ is bounded, then

$$J_m \leqq \left(\frac{m}{2\log m}\right)^k \left(1 + a_m\sqrt{\pi} + o(\gamma^m)\right),$$

where γ is a universal constant less than 1.

Proof. We consider a function f defined on R^m by the formula

$$f(x) = \pi^{-\frac{m}{2}} \exp(-\|x\|_2^2)\, \|x\|_\infty^{-2k};$$

and we put

$$I_m = \int_{R^m} f(x) dx.$$

Let $S(t) = \{x \in R^m : \|x\|_2 = t\}$ and let us denote by dH_t, the $(m-1)$-dimensional Hausdorff measure induced by dx on $S(t)$. We have

$$I_m = \int_0^\infty dt \int_{S(t)} \pi^{-\frac{m}{2}} \exp(-t^2) \|x\|_\infty^{-2k} dh_t(x)$$
$$= \int_0^\infty exp(-t^2) t^{-2k} t^{m-1} \int_{S(t)} \|x\|_\infty^{-2k} dH_1(x).$$

We have $dH_t = 2\pi^{\frac{m}{2}} = \left(\Gamma\left(\frac{m}{2}\right)\right)^{-1} d\lambda$ and therefore

$$I_m = \left(\Gamma\left(\frac{m}{2}\right)\right)^{-1} \cdot J_m \cdot \int_0^\infty 2t^{m-2k-1} \exp(-t^2)\, dt \ = \Gamma(\frac{1}{2}m - k)/\Gamma\left(\frac{1}{2}m\right) \cdot J_m.$$

Hence

$$J_m = \frac{\Gamma\left(\frac{1}{2}m\right)}{\Gamma\left(\frac{1}{2}m - k\right)} \cdot I_m \leqq I_m \cdot \left(\frac{m}{2}\right)^k. \tag{37}$$

We shall thus estimate I_m. Let

$$\Psi(t) = \int_{\{x:\|x\|_\infty \leqq t\}} f(x) dx \quad and$$

$$\eta(t) = \pi^{-\frac{m}{2}} \int_{\{x:\|x\|_\infty \leqq t\}} \exp(-\|x\|_2^2)\, dx = \left(\left(\frac{2}{\sqrt{\pi}}\right) \int_0^t e^{-u^2} du\right)^m.$$

We have $\Psi'(t) = t^{-2k} \eta'(t)$. Integrating by parts, we obtain

$$I_m = \int_0^\infty \Psi'(t) dt = 2k \int_0^\infty t^{-2k-1} \eta(t) dt = 2k \int_0^\infty t^{-2k-1} \left(\frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2} du\right)^m dt.$$

Let $\varepsilon \geqq 0$; let $A = \left((1-\varepsilon)\log m\right)^{\frac{1}{2}}$. We split the last integral in the following way:

$$2k \int_0^\infty t^{-2k-1} \left(\left(\frac{2}{\sqrt{\pi}}\right) \int_0^t e^{-u^2} du\right)^m dt = 2k\left(\int_0^2 + \int_2^A + \int_A^\infty\right).$$

We will show that the two first integrals are of lower order of magnitude than the last one, which is estimated by

$$2k \int_0^\infty \ \leqq 2k \int_A^\infty t^{-2k-1} dt = A^{-2k} = \left((1-\varepsilon)\log m\right)^{-k}.$$

Concerning the first integral, put $\alpha = \left(\frac{2}{\sqrt{\pi}}\right) \int_0^2 e^{-u^2} du < 1$. We have clearly for every $t \in \langle 0,2 \rangle$,

$$2k \int_0^2 \ \leqq e^{-u^2} du \leqq \min\left\{\left(\frac{2}{\sqrt{\pi}}\right) t, \alpha\right\}.$$

Let r be a number such that $\alpha^r < \sqrt{\pi}/2$. then

$$2k \int_0^2 \ \leqq 2K \left(\frac{2}{\pi^{\frac{1}{2}}}\right)^{2k+1} \alpha^{m-2k-1} = O(\left(\frac{2}{\pi^{\frac{1}{2}}}\right)^{3k} \alpha^{m-2k-1} = O\left(\alpha^{m-3k(r+1)}\right).$$

It is clear that the last number is $o\left(\alpha^{\frac{m}{2}} (\log m)^{-k}\right)$ provided $k = o(m/\log log\, m)$.

To estimate the second integral, we consider two cases

(a) $\varepsilon > 0$. Then

$$2k\int_2^A \leqq \left(\left(\frac{2}{\sqrt{\pi}}\right)\int_0^A e^{-u^2}du\right)^m \cdot 2k\int_2^\infty t^{-2k-1}dt$$
$$= 2^{-2k}\left(1-\left(\frac{2}{\sqrt{\pi}}\right)\int_A^\infty e^{-u^2}du\right)^m$$
$$\leqq 2^{-2k}\left[1-\left(\frac{2}{\sqrt{\pi}}\right)\left(\left(1-\frac{\varepsilon}{2}\right)^{\frac{1}{2}}-(1-\varepsilon)^{\frac{1}{2}}\right)(\log m)^{\frac{1}{2}}e^{-\left(1-\frac{\varepsilon}{2}\right)\log}\right]^{\mathrm{m}}$$
$$\leqq 2^{2k}\left(1-\frac{\varepsilon}{3(\log m)^{\frac{1}{2}}}\cdot m^{\frac{\varepsilon}{2}}\cdot m^{-1}\right)^m \leqq 2^{-2k}\cdot 2^{-\left(\frac{\varepsilon}{3}\right)(\log m)^{\frac{1}{2}}m^{\frac{\varepsilon}{2}}}.$$

It is clear that if $k \leqq m^{\frac{\varepsilon}{4}}$, then the last number is $o\left(2^{-m^{\frac{\varepsilon}{4}}}\cdot(\log m)^{-k}\right)$.

(b) $\varepsilon = 0$. We use a trick due to Figiel:

$$2k\int_2^A = 2k\left(\frac{x}{\sqrt{\pi}2}\right)(m+1)^{-1}\int_2^A t^{-2k-1}e^{t^2}\cdot\frac{d}{dt}\left(\left(\frac{2}{\sqrt{\pi}}\right)\int_0^t e^{-u^2}du\right)^{m+1} dt$$
$$\leqq \sqrt{x}k(m+1)^{-1}\sup\left\{t^{-2k-1}e^{t^2}: 2\leqq t\leqq A\right\}$$

Since the function $\phi(t) = t^{-2k-1}e^{t^2}$ is convex, the last supremum is attained either for $t = 2$ or for $t = A$. We have (for $k > 10$),

$$\sqrt{\pi}k(m+1)^{-1}\phi(2) = \left(\sqrt{\pi}ke^4\right)2^{-2k-1}\cdot(m+1)^{-1} \leqq m^{-1}$$
$$\sqrt{\pi}k(m+1)^{-1}\phi(A) = \sqrt{\pi}k\cdot\log m^{-\frac{1}{2}}\cdot\left(\left(\frac{m}{m}+1\right)\log m\right)^{-k} \leqq a_m\sqrt{\pi}(\log m)^{-k}.$$

It is clear that the second number is bigger.

These estimates, together with (37), prove respectively (i) and (ii) of Lemma (3.2.8).

To estimate now our L_m, we use the following trivial inequality, valid for all $k > 0$ and for all $x > 0$:

$$e^{-x^2} \leqq k^k e^{-k}x^{-2k}.$$

(To prove it, we find that maximum of the function $x^{2k}e^\wedge - x^2$ is attained for $x = k^{\frac{1}{2}}$.)

Thus we obtain

$$L_m \leqq k^k(16e^{-1})^k\beta^{-2k}m^{-k}J_m$$

and, by Lemma (3.2.8)(i), with, $\varepsilon = \frac{1}{2}$,

$$L_m \leqq k^k(16e^{-1})\beta^{-2k}(\log m)^{-t}$$

for every $k \leqq m^{\frac{1}{8}}$.

It is obvious that the right-hand side attains its minimum for $k = 1/16\beta^2\log m$ and then we obtain

$$L_m \leqq \exp\left(-\frac{1}{16\beta^2\, log\, m}\right) = m^{-\frac{1}{16\beta^2}}.$$

By Proposition (3.2.6) and inequality (9'), we obtain

$$m \geqq \left(k\left(\frac{\beta}{4}\right)^{2k}\right)^{-\frac{16}{\beta^2}}.$$

If we now take into consideration Proposition (3.2.6), we obtain finally

$$n(k,\varepsilon) \leqq 4 \cdot\ k^{-16\cdot\frac{10^6}{t^2}} \cdot \left(\frac{4000}{\varepsilon}\right)^{32\cdot\frac{10^6 k}{\varepsilon^2}} \exp(C \cdot \varepsilon^{-2} k \ln \varepsilon^{-1}).$$

This is precisely Milman's estimate.

The complex case. With the following minor changes, our proof works also in the complex case.

It is easy to check that the proof of the Dvoretzky-Rogers Theorem is valid also in the complex case. We can thus work with a complex space F that, considered as a real space, is isometric to $(R^m, \|\ \|)$ with $\|\ \|$ satisfying (20). We can also assume $\|\ \|$ to be uniformly smooth.

For every real subspace E of $F = R^m$ we define $T_E, \alpha_E, \Sigma_E, \sigma_E, \alpha_y(x)$ and finally A_E exactly as in the beginning.

Let $m > 2k$ and let Γ_c be the Grassman manifold of all k-dimensional complex subspaces of F. Let γ_c be the unique normalized measure on Γ_c that is invariant under unitary transformations. By uniqueness of σ_F, we have

$$\sigma_F(A_F) = \int_{\Gamma_c} \sigma_E(A_E) d\gamma_c(E)\ and \tag{38}$$

$$\sigma_F(A_F) = \int_{S_F} \lambda_x(A_x) d\lambda(x). \tag{39}$$

The second formula is nothing but (19) and thus the argument of gives us again the inequality (22) (or (23), for better constant). If now

$$m \geqq \left(4k\left(\frac{\beta}{4}\right)^{4k}\right)^{-\frac{16}{\beta^2}},$$

the previous argument together with Proposition (3.2.6) and (38) ensure us the existence of a k-dimensional complex subspace E of F, that is ε-euclidean. But this implies immediately that

$$d_c\left(E, l_2^k(C)\right) < 1 + \varepsilon.$$

Here we denote

$d_c(E, E') = \inf\{\|T\|\|T^{-1}\|: T\ is\ a\ complex\ isomorphism\ between\ E\ and\ E'\}$ and

$$l_2^k(c) = (C^k, \|\ \|_2)$$

Where

$$\|(z_i)\|_2 = \left(\sum_{i=1}^{k} |z_i|^2\right)^{\frac{1}{2}}.$$

C. Dvoretzky's Theorem in special cases. We have actually proved the following Theorem.

Theorem (3.2.9)[133]: Let a norm $\|\ \|$ in R^m satisfy

$$\|x\|_2 \geqq \|x\| \geqq \omega(x) \quad for\ all \quad x \in R^m.$$

If

$$\int_{S^m} \exp(-10^{-6}\varepsilon^2 \omega^2(x) m)\, d\lambda(x) < k\left(\frac{\varepsilon}{4} \cdot 10^6\right)^{2k},$$

then there exists a k-dimensional subspace E of R^m such that $(E, \|\ \|)$ is ε- euclidean.

Thus, if we take in particular

$$\omega(x) = \|x\| = \|x\|_p,$$

with $p > 2$, we obtain

$$\int_{S^m} \exp(-\,10^{-6}\varepsilon^2\omega^2(x)m)\ d\lambda(x)\ \leqq \exp\left(-\,10^{-6}\varepsilon^2 m^{\frac{2}{p}}\right)$$

and therefore the numbers $m_p(k,\varepsilon) = \inf\{m: l_p^m \supset E, d(E, l_2^k) < 1+\varepsilon\}$ may be estimated by

$$m_p(k,\varepsilon) \leqq (10^7 k\varepsilon^{-2} \log \varepsilon^{-1})^{\frac{p}{2}}.$$

Another interesting case is that of a distorted norm in I_2^m. Let namely $\|\ \|$ be itself a $(K-1)$-euclidean norm, that is, $\|x\| \geqq \omega(x) = K^{-1}\|x\|_2$. We obtain then

$$\int_{S^m} \exp(10^{-6}\varepsilon^2\omega^2(x)m)\ d\lambda(x) \leqq \exp(-\,10^{-6}\varepsilon^2 K^{-2} m)$$

and $m^k(k,\varepsilon) = \inf\{m$: every m-dimensional $(K-1)$-euclidean space contains a k-dimensional ε-euclidean subspace$\}\leqq 10\varepsilon^{-2}\log\varepsilon^{-1} K^2 \cdot k$, that is, dependence is linear (this has been pointed out in [127]).

The infinite dimensional version. It is clear that Dvoretzky's Theorem is not valid in the infinite dimensional case. In fact there exist Banach spaces that do not contain subspaces isomorphic to l_2. It is natural therefore to consider only spaces that are isomorphic to l_2. Then we face the following problem which has been discussed in [136], [137], [139].

A norm $\|\ \|$ in l_p is called distorted if there exists an $\varepsilon > 0$ such that the norms $\|\ \|$ and $\|\ \|_p$ are not $(1+\varepsilon)$-equivalent on any ∞-dimensional subspace of I_p.

Chapter 4
L^p p-Harmonic 1-Forms

We show that as consequences, the corresponding Liouville type theorems for harmonic functions with finite L^p energy on minimal hypersurfaces in a Riemannian manifold are obtained. We show that if there exists point $q \in M$ such that $K_N(q) \neq 0$, assume further that the first eigenvalue of the Laplace-Beltrami operator of M is bounded by a suitable constant. We obtain that the dimension of $H^{1,p}(M)$ is finite, that is, dim $H^{1,p}(M) < \infty$. In particular, M has finitely many ends. These results can be regarded as an extension of Li–Wang (2002).

Section (4.1): First Eeigenvalue of a Stable Minimal Hypersurface

Hodge theory plays an important role in the topology of compact Riemannian manifolds. Unfortunately, the Hodge theory does not work anymore in noncompact manifolds. However, the L^2-Hodge theory works well in noncompact cases [142], [153]. In this direction, there are various results for L^2 harmonic1-forms on stable minimal hypersurfaces. Recall that a minimal hypersurface in a Riemannian manifold is called stable if the second variation of its volume is always nonnegative for any normal variation with compact support. More precisely, an n-dimensional minimal hypersurface M in a Riemannian manifold N is called stable if it holds that, for any compactly supported Lipschitz function f on M,

$$\int_M |\nabla f|^2 - \left(|A|^2 + \overline{Ric}(v,v)\right) f^2 dv \geq 0;$$

where v is the unit normal vector of $M, \overline{Ric}(v,v)$ denotes the Ricci curvature of N in the$v$ direction, $|A|^2$ is the square length of the second fundamental form A, and dv is the volume form for the induced metric on M.

Using the nonexistence of L^2 harmonic 1-forms, Palmer [170] proved that if there exists a codimension-one cycle on a complete minimal hypersurface M in Euclidean space, which does not separate M, M is unstable. Using Bochner's vanishing technique, Miyaoka [163] showed that a complete noncompact stable minimal hypersurface in a nonnegatively curved manifold has no nontrivial L^2harmonic 1-forms. Pigola, Rigoli, and Setti [172] gave general Liouville type results and the corresponding vanishing theorems on the L^2 co homology of stable minimal hypersurfaces. See [146], [173] for a survey in this area. While the L^2 theory is quite well understood, in the case $p \neq 2$, the L^p theory is less developed. See [176] for general L^p theory of differential forms on a manifold.

We estimate the smallest spectral value of the Laplace operator on a complete noncompact stable sminimal hypersurface in a Riemannian manifold under the assumption on L^p norm of the second fundamental form. Secondly, we obtain various vanishing theorems for L^p harmonic 1-forms on minimal hypersurfaces.

For M be a complete noncompact Riemannian manifold and let Ω be a compact domain in M. Let $\lambda_1(\Omega) > 0$ denote the first eigenvalue of the Dirichlet boundary value problem

$$\begin{cases} \Delta f + \lambda f = 0 & in \Omega, \\ f = 0 & on\ \partial\Omega, \end{cases}$$

where Δ denotes the Laplace operator on M. Then the first eigenvalue $\lambda_1(M)$ is defined by

$$\lambda_1(M) = \inf_{\Omega} \lambda_1(\Omega),$$

where the infimum is taken over all compact domains in M. Cheung and Leung in [143] gave the first eigenvalue estimate for an n-dimensional complete noncompact submanifold M with the norm of its mean curvature vector bounded in the hyperbolic space. In particular, they proved that if M is minimal, the first eigenvalue $\lambda_1(M)$satisfies

$$\frac{1}{4}(n-1)^2 \leq \lambda_1(M).$$

Note that this inequality is sharp because equality holds if M is totally geodesic [167]. This result was extended to an n-dimensional complete noncompact submanifold with the norm of its mean curvature vector bounded in a complete simply connected Riemannian manifold with sectional curvature bounded above by a negative constant. We have the following theorem.

Theorem [143], [179]. Let N be an n-dimensional complete simply connected Riemannian manifold with sectional curvature K_N satisfying $K_N \leq a^2 < 0$ for a positive constant $a > 0$. Let M be a an m-dimensional complete noncompact submanifold with bounded mean curvature vector H in N satisfying $|H| \leq b < (m-1)a$. Then

$$\frac{1}{4}[(m-1)-b]^2\, \lambda_1(M). \tag{1}$$

On the other hand, Candel [144] obtained an upper bound for the bottom of the spectrum of a complete simply connected stable minimal surface in 3-dimensional hyperbolic space. With finite L^2 norm of the second fundamental form, one may estimate an upper bound for the bottom of the spectrum of a stableminimal hypersurface in a Riemannian manifold with pinched negative sectional curvature [154], [178]. We estimate the bottom of the spectrum of the Laplace operator on stable minimal hypersurfaces under the assumption on the L^p norm of the second fundamental form. Indeed, we prove the following.

Let N be an $(n+1)$dimensional complete simply connected Riemannian manifold with sectional curvature satisfying $K_1 \leq K_N \leq K_2$, where K_1, K_2 are constants and $K_1 \leq K_2 < 0$. Let M be a complete stable non-totally geodesic minimal hypersurface in N. Assume that, for

$1-\sqrt{2/n} < p < 1+\sqrt{2/n},$

$$\lim_{R\to\infty} R^{-2} \int_{B(R)} |A|^{2p} = 0,$$

where $B(R)$is a geodesic ball of radius R on M. If $|\nabla K|^2 = \sum_{i;j;k;l;m} K_{i\,jklIm}^2 \leq K_3^2|A|^2$ for some constant $K_3 \geq 0$, we have

$$-K_2\frac{(n-1)^2}{4} \leq \lambda_1(M) \leq \frac{np^2\big(2K_3 - n(K_1+K_2)\big)}{2-n(p-1)^2}.$$

We proved that if M is an n-dimensional complete stable minimal hypersurface in hyperbolic space with$\lambda_1(M) > (2n-1)(n-1)$, there is no nontrivial L^2 harmonic 1-form on M. This result was generalized [154] to a complete stable minimal hypersurface in a Riemannian manifold with sectional curvature bounded below by a nonpositive constant. We prove an extended result for L^p harmonic 1-forms on a complete noncompact stable minimal hypersurface as follows.

Then there is no nontrivial L^{2p} harmonic 1-form on M. Yau proved that there are no nonconstant L^p harmonic functions on a complete Riemannian manifold for $1 < p < \infty$. Li and Schoen [163] proved that Yau's result is still true for L^p harmonic functions on a complete manifold of

Non negative Ricci curvature when $0 < p < \infty$. In the case of harmonic forms, see [156], [157] announced nonexistence of nontrivial L^p harmonic forms .$(1 \leq p < \infty)$ on complete Riemannian and Kählerian manifolds of nonnegative curvature. See also [149], [150], [151], [164], [165] for Liouville type theorems for harmonic functions on a complete Riemannian manifold. The Liouville property holds also for harmonic functions on minimal hypersurfaces in a Riemannian manifold. For instance, Schoen and Yau proved the Liouville type theorem on minimal hypersurfaces as follows.

Theorem (4.1.1)[141]: [13]. Let M be a complete noncompact stable minimal hypersurface in a Riemannian manifold with nonnegative sectional curvature. If f is a harmonic function on M with finite L^2 energy, f is constant.

Recall that a function f on a Riemannian manifold M has finite L^p energy if $|\nabla f| \in L^p(M)$. As an application of our theorem, we immediately obtain the following, which is a generalization of Schoen and Yau's result (see Corollary (4.1.17)).

Theorem (4.1.2)[141]: Let M be a complete noncompact stable minimal hypersurface in a Riemannian manifold with nonnegative sectional curvature with $\lambda_1(M) > 0$. Then there is no nontrivial harmonic function on M with finite L^p energy for $0 < p < n/(n-1) + \sqrt{2n}$. For $n \geq 3$, it is well known [2] that an n-dimensional complete stable minimal hypersurface M in Euclidean space cannot have more than one end.

This topological result was generalized to minimal hypersurfaces with finite index in Euclidean space and stable minimal hypersurfaces in a nonnegatively curved manifold by [1], [166]. If we assume that M has sufficiently small total scalar curvature instead of assuming that M is stable, we can also have the same conclusion [30], [34]. See also [171] for more general results related with L^p norm of the second fundamental form. In the same spirit, Yun [2002] proved that if $M \subset \mathbb{R}^{n+1}$ is a complete minimal hypersurface with sufficiently small total scalar curvature, there is no nontrivial L^2 harmonic 1-form on M. Yun's result was generalized [154] to a complete noncompact stable minimal hypersurface in a complete Riemannian manifold with sectional curvature bounded below by a nonpositive constant. The corresponding vanishing theorems for L^p harmonic 1-forms are obtained.

One crucial step in the proofs of our theorems is to obtain an inequality of Simons' type for $|\phi|^p$ rather than $|\phi|$, where ϕ is a geometric quantity which we want to analyze. This kind of inequalities has been used in [152], [155], [180]. Equipped with this Simons' type inequality, we extend the original Bochner technique to our cases.

Let M be an n-dimensional manifold immersed in an $(n+1)$-dimensional Riemannian manifold N. We choose a local vector field of orthonormal frames $e_1; \ldots, e_{n+1}$in N such that the vectors $e_1; \ldots, e_n$ are tangent to M and the vector e_{n+1} is normal to M. With respect to this frame field of N, let $K_{i\,jkl}$ be a curvature tensor of N. We denote by $K_{ijkl:m}$ the covariant derivative of K_{ijkl} . We follow the notation of [175].

Theorem (4.1.3)[141]: Let N be an $(n+1)$-dimensional complete simply connected Riemannian manifold with sectional curvature satisfying $K_1 \leq K_N \leq K_2$; where $K_1, K_2$are constants and $K_1 \leq K_2 < 0$. Let M be a complete stable non-totally geodesic minimal hypersurface in N. Assume that, for $1 - \sqrt{2/n} < p < 1 + \sqrt{2/n}$

$$\lim_{R\to\infty} R^{-2} \int_{B(R)} |A|^{2p} = 0.$$

where $B(R)$ is a geodesic ball of radius R on M. If $|\nabla K|^2 = \sum_{i;j;k;l;m} K_{i\,jkl;lm}^2 \leq K_3^2|A|^2$ for some constant $K_3 \geq 0$, we have

$$K_2\frac{(n-1)^2}{4} \leq \lambda_1(M) \leq \frac{np^2\big(2K_3 - n(K_1+K_2)\big)}{2-n(p-1)^2}.$$

Proof. As mentioned in the introduction, one sees that the lower bound of $\lambda_1(M)$ is given as $-K_2(n-1)^2/4$ from inequality (1) [143], [179]. Namely, the first eigenvalue of an n-dimensional minimal hypersurface in acomplete simply connected Riemannian manifold with sectional curvature bounded above by a negative constant K_2 is bounded below by $-K_2(n-1)^2/4$. Therefore, in the rest of the proof, we shall find the upper bound of the first eigenvalue $\lambda_1(M)$

By [175], we have

$$|A|\Delta|A|2K_3|A|^2 + (K_2-K_1)|A|^2 + |A|^4 \geq \sum h_{ijk}^2 - \big|\nabla|A|\big|^2$$

at all points where $|A| \neq 0$. Because $K_2 - K_1 \geq 0$, this inequality implies

$$|A|\Delta|A|2K_3|A|^2 - nK_2|A|^2 + |A|^4 \geq \sum h_{ijk}^2 - \big|\nabla|A|\big|^2 = |\nabla A|^2 - \big|\nabla|A|\big|^2.$$

Applying the Kato-type inequality

$$|\nabla A|^2 - \big|\nabla|A|\big|^2 \geq \frac{2}{n}\big|\nabla|A|\big|^2,$$

we get

$$|A|\Delta|A| + (2K_3 - nK_2)|A|^2 + |A|^4 \geq -\frac{2}{n}\big|\nabla|A|\big|^2.$$

For a positive number $p > 0$, we have

$$|A|^p\Delta|A|^p = |A|^p div(\nabla|A|^p)$$

$$= |A|^p div(\mathrm{p}|A|^{p-1}\nabla|A|)$$

$$= p(p-1)|A|^{2p-2}\big|\nabla|A|\big|^2 + p|A|^{2p-1}\Delta|A|$$

$$= \frac{p-1}{p}|\nabla|A|^p|^2 + \mathrm{p}|A|^{2p-2}|A|\Delta|A|.$$

It follows from inequality (2) that

$$|A|^p\Delta|A|^p \geq \frac{p-1}{p}|\nabla|A|^p|^2 + \frac{2p}{n}|A|^{2p-2}\big|\nabla|A|\big|^2 - p|A|^{2p+2} - p(2K_3 - nK_2)|A|^{2p}$$

$$= \frac{p-1}{p}|\nabla|A|^p|^2 + \frac{2}{np}|\nabla|A|^p|^2 - p|A|^{2p+2} - p(2K_3 - nK_2)|A|^{2p}. \quad (2)$$

Thus

$$|A|^p\Delta|A|^p + p(2K_3 - nK_2)|A|^{2p} + p|A|^{2p+2} \geq \left(1 - \frac{n-2}{np}\right)|\nabla|A|^p|^2.$$

Choose a Lipschitz function f with compact support in a geodesic ball $B(R)$ of radius R centered at a point $x \in M$. Multiplying both sides by f^2 and integrating over $B(R)$ we obtain

$$\int_{B(R)} f^2\,|A|^p\Delta|A|^p + p(2K_3 - nK_2)\int_{B(R)} f^2\,|A|^{2p} + \int_{B(R)} f^2\,|A|^{2p+2}$$

$$\geq \left(1 - \frac{n-2}{np}\right)\int_{B(R)} f^2\,|\nabla|A|^p|^2.$$

The divergence theorem yields

$$\int_{B(R)} f^2 |A|^p \Delta |A|^p = \int_{B(R)} div(f^2 |A|^p \nabla |A|^p) - \int_{B(R)} f^2 |\nabla |A|^p|^2 - 2 \int_{B(R)} f |A|^p \langle \nabla f, \nabla |A|^p \rangle$$

$$= - \int_{B(R)} f^2 |\nabla |A|^p|^2 - 2 \int_{B(R)} f |A|^p \langle \nabla f, \nabla |A|^p \rangle.$$

Therefore

$$\left(1 - \frac{n-2}{np}\right) \int_{B(R)} f^2 |\nabla |A|^p|^2 \leq p(2K_3 - nK_2) \int_{B(R)} f^2 |A|^{2p} + p \int_{B(R)} f^2 |A|^{2p+2}$$

$$- \int_{B(R)} f^2 |\nabla |A|^p|^2 - 2 \int_{B(R)} f |A|^p \langle \nabla f, \nabla |A|^p \rangle \qquad (3)$$

The stability of M implies that

$$\int_{B(R)} |\nabla f|^2 |A|^2 + \overline{Ric}(e_n + 1) f^2 \geq 0 \qquad (4)$$

for any compactly supported Lipschitz function f on M. From our assumption on the sectional curvature of N, we see that

$$nK_1 \leq \overline{Ric}(e_{n+1}) = R_{n+1,n+1,1+} + \cdots + R_{n+1,n+1,n} \leq nK_2.$$

Hence the stability inequality (4) gives

$$\int_M |\nabla f|^2 - (|A|^2 + nK_1) f^2 \geq 0 \qquad (5)$$

for any compactly supported Lipschitz function f on M. Choose a Lipschitz function f with compact support in a geodesic ball $B(R) \subset M$, as before. Replacing f by $|A|^p f$ in inequality (5), we have

$$\int_M |\nabla |A|^p f|^2 - (|A|^{2p+2} f^2 + nK_1 |A|^{2p}) f^2 \geq 0$$

Thus

$$\int_{B(R)} |\nabla |A|^p|^2 f^2 + \int_{B(R)} |\nabla f|^2 |A|^{2p} + 2 \int_{B(R)} f |A|^p \langle \nabla f, \nabla |A|^p \rangle$$

$$\geq \int_{B(R)} |A|^{2p+2} f^2 + nK_1 \int_{B(R)} |A|^{2p} f^2. \qquad (6)$$

Combining the inequalities (3) and (6), we get

$$\left(1 - \frac{n-2}{np}\right) \int_{B(R)} f^2 |\nabla |A|^p|^2$$

$$\leq p(2K_3 - nK_1 - nK_2) \int_{B(R)} f^2 |A|^{2p} (p-1) \int_{B(R)} f^2 |\nabla |A|^p|^2$$

$$+p \int_{B(R)} |\nabla f|^2 |A|^{2p} + 2(p-1) \int_{B(R)} f |A|^p \langle \nabla f, \nabla |A|^p \rangle. \qquad (7)$$

On the other hand, from the definition of $\lambda_1(M)$ and the domain monotonicity of eigenvalues, it follows that

$$\lambda_1(M) \leq \lambda_1(B(R)) \frac{\int_{B(R)} |\nabla f|^2}{\int_{B(R)} f^2} \tag{8}$$

for any compactly supported nonconstant Lipschitz function f on M. Substituting $|A|^p f$ for f in inequality (8), we see that

$$\lambda_1(M) \int_{B(R)} f^2 \, |A|^{2p} \leq \int_{B(R)} f^2 \, |\nabla(|A|^p f)|^2$$

$$= +p \int_{B(R)} f^2 |\nabla|A|^p|^2 + \int_{B(R)} |A|^{2p} |\nabla f|^2 + 2 \int_{B(R)} f \, |A|^p \langle \nabla f, \nabla |A|^p \rangle. \tag{9}$$

Plugging inequality (9) into (7), we have

$$\left(1 - \frac{n-2}{np}\right) \int_{B(R)} f^2 |\nabla|A|^p|^2 \leq \frac{p}{\lambda_1(M)} (2K_3 - nK_1 - nK_2)$$

$$\left(\int_{B(R)} f^2 |\nabla|A|^p|^2 + \int_{B(R)} f^2 |\nabla|A|^p|^2 + \int_{B(R)} f \, |A|^p \langle \nabla f, \nabla|A|^p \rangle \right)$$

$$+(p-1) \int_{B(R)} f^2 |\nabla|A|^p|^2 + p \int_{B(R)} |\nabla f|^2 |A|^{2p} + 2(p-1) \int_{B(R)} f \, |A|^p \langle \nabla f, \nabla|A|^p \rangle.$$

Thus

$$\left(1 - \frac{n-2}{np}\right) \int_{B(R)} f^2 |\nabla|A|^p|^2 \leq \left(\frac{p}{\lambda_1(M)} (2K_3 - nK_1 - nK_2) + p - 1 \right) \int_{B(R)} f^2 |\nabla|A|^p|^2$$

$$+ \left(\frac{p}{\lambda_1(M)} (2K_3 - nK_1 - nK_2) + p \right) \int_{B(R)} |\nabla f|^2 \, |A|^{2p}$$

$$+2 \left(\frac{p}{\lambda_1(M)} (2K_3 - nK_1 - nK_2) + p - 1 \right) \int_{B(R)} f \, |A|^p \langle \nabla f, \nabla|A|^p \rangle \tag{10}$$

Note that Young's inequality yields

$$2 \int_{B(R)} f \, |A|^p \langle \nabla f, \nabla|A|^p \rangle \leq \varepsilon \int_{B(R)} |\nabla f|^2 |A|^{2p} + \frac{1}{\varepsilon} \int_{B(R)} f^2 |\nabla|A|^p|^2 \tag{11}$$

for any $\varepsilon > 0$. From inequalities (10) and (11), it follows that

$$\left(1 - \frac{n-2}{np}\right) \int_{B(R)} f^2 |\nabla|A|^p|^2 \leq \left(\frac{p}{\lambda_1(M)} (2K_3 - nK_1 - nK_2) + p - 1 \right) \int_{B(R)} f^2 |\nabla|A|^p|^2$$

$$+ \left(\frac{p}{\lambda_1(M)} (2K_3 - nK_1 - nK_2) + p - 1 \right) \left(\varepsilon \int_{B(R)} |\nabla f|^2 |A|^{2p} + \frac{1}{\varepsilon} \int_{B(R)} f^2 |\nabla|A|^p|^2 \right).$$

Which yields that

$$\left[1-\frac{n-2}{np}\left(1+\frac{1}{\varepsilon}\right)\left(\frac{p}{\lambda_1(M)}(2K_3-nK_1-nK_2)+p-1\right)\right]\int\limits_{B(R)} f^2|\nabla|A|^p|^2$$
$$\leq\left[(1+\varepsilon)\left(\frac{p}{\lambda_1(M)}(2K_3-nK_1-nK_2)+p\right)-\varepsilon\right]\int\limits_{B(R)}|\nabla f|^2|A|^{2p}$$

For a contradiction, we suppose that

$$\lambda_1(M)>\frac{p(2K_3-nK_1-nK_2)}{1-\frac{n-2}{np}-(p-1)}=\frac{np^2\big(2K_3-n(K_1+K_2)\big)}{2-n(p-1)^2}$$

Note the assumption that $1-\sqrt{2/n}\ <\ p\ <1+\sqrt{2/n}$ is equivalent to

$$2-n(p-1)^2>0.$$

Choose a sufficiently large $\varepsilon>0$ satisfying

$$\left[1-\frac{n-2}{np}\left(1+\frac{1}{\varepsilon}\right)\left(\frac{p}{\lambda_1(M)}(2K_3-nK_1-nK_2)+p-1\right)\right]>0,$$

Since $|\nabla f|\leq 1/R$ by our choice of , one can conclude that, by letting $R\to\infty$,

$$\int\limits_M|\nabla|A|^p|^2=0,$$

where we used the growth condition on $\int_{B(R)}|A|^{2p}$. Thus we see that $|A|^{2p}$ is constant.

Since the volume of M is infinite [181], we get $|A|\equiv 0$. This implies that M is totally geodesic, which is impossible by our assumption. Therefore we obtain the upper bound$\lambda_1(M)$:

$$\lambda_1(M)\leq\frac{np^2\big(2K_3-n(K_1+K_2)\big)}{2-n(p-1)^2}.$$

Dung and [154] gave an estimate of the bottom of the spectrum for
the Laplace operator on a complete noncompact stable minimal hypersurface M in a complete simply connected Riemannian manifold with pinched negative sectional curvature under the assumption on L^2-norm of the second fundamental form A of M. In Theorem (4.1.3), if we take $p=1$, we get the following.

Corollary (4.1.4)[141]: [154]. Let N be an$(n+1)$-dimensional complete simply connected Riemannian manifold with sectional curvature satisfying $K_1\leq K_N\leq K_2$; where K_1,K_2 are constants and $K_1\leq K_2<0$. Let M be a complete stable non-totally geodesic minimal hypersurface in N. Assume that

$$\lim_{R\to\infty}R^2\int\limits_{B(R)}|A|^2=0,$$

where $B(R)$is a geodesic ball of radius R on M. If $|\nabla K|^2=\sum_{i,j,k,l,m}K^2_{i,j,k,l;m}K_3^2|A|^2$ for some constant $K_3>0$, we have

$$-K_2\frac{(n-1)^2}{4}\lambda_1(M)\leq\frac{\big(2K_3-n(K_1+K_2)\big)n}{2}.$$

In particular, if N is the $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$, one sees that $K_1=K_2=-1$, and hence $|\nabla K|^2=0$, that is, $K_3=0$. Moreover, it follows from McKean's result see [167] that the first eigenvalue $\lambda_1(M)$ of any complete totally geodesic hypersurface $M\subset\mathbb{H}^{n+1}$ satisfies $\lambda_1(M)=$

$(n-1)^2/4$. Therefore we have the following consequence which is an extension of the result in [178].

Corollary (4.1.5)[141]: Let M be a complete stable minimal hypersurface in $\mathbb{H}^{n+1}$ R with $\int_M |A|^{2p}\, dv < \infty$ for $\sqrt{2/n} < p < 1 + \sqrt{2/n}$.. Then we have

$$-K_2 \frac{(n-1)^2}{4} \lambda_1(M) \leq \frac{2n^2p^2}{2 - n(p-1)^2}.$$

As another application of Theorem (4.1.3), we have the following when $n < 8$.

Corollary (4.1.6)[141]: Let N be an .$(n+1)$-dimensional complete simply connected Riemannian manifold with sectional curvature satisfying $K_1 \leq K_N \leq K_2$, where K_1, K_2 are constants and $K_1 \leq K_2 < 0$ for $n < 8$. Let M be a complete stable non-totally geodesic minimal hypersurface in N. For $p = 1,2,3$, if $\int_M |A|^{2p} < \infty$, we have

$$-K_2 \frac{(n-1)^2}{4} \lambda_1(M) \leq \frac{np^2\big(2K_3 - n(K_1 + K_2)\big)}{2 - n(p-1)^2}.$$

Proof. Since $\sqrt{2/n} > 1/2$ when$n < 8$, the conclusion can be derived from Theorem (4.1.3).

Before we prove the vanishing theorems for L^p harmonic 1-forms on complete minimal hypersurface, we begin with some useful facts.

Lemma (4.1.7)[141]: [162]. Let M be an n-dimensional complete immersed minimal hypersurface in a Riemannian manifold N. If all the sectional curvatures of N are bounded below by a constant K,

$$Ric \geq (n-1)K - \frac{n-1}{n}|A|^2,$$

Lemma (4.1.8)[141]: [37]. Letω be a harmonic 1-form on an n-dimensional Riemannian manifold M. Then

$$|\nabla\omega|^2 - \big|\nabla|\omega|\big|^2 \geq \frac{n-1}{n}\big|\nabla|\omega|\big|^2. \tag{12}$$

We also need the following well-known Sobolev inequality on a Riemannian Manifold.

Lemma (4.1.9)[141]: [29]. Let M^n be a c omplete immersed minimal submanifold in a nonpositively curved manifold N^{n+p} , $n \geq 3$. Then, for any $\phi \in W_0^{1,2}(M)$, we have

$$\left(\int_M |\phi|^{2n/(n-2)}\, dv\right)^{(n-2)/n} \leq C_s \int_M |\nabla\phi|^2\, dv, \tag{13}$$

where C_s is the Sobolev constant which depends only on $n \geq 3$.

A complete Riemannian manifold M is called nonparabolic if it admits a nonconstant positive superharmonic function. Otherwise, M is said to be parabolic.

The following sufficient condition for parabolicity is well known.

Theorem (4.1.10)[141]: [158], [159], [161], [14]. Let M be a complete Riemannian manifold. If, for any point $p \in M$ and a geodesic ball$B_p(r)$,

$$\int_1^\infty \frac{r}{Vol\big(B_p(r)\big)} dr = \infty$$

M is parabolic.

It immediately follows from this result that if M is nonparabolic,

$$\int_1^{\infty} \frac{r}{Vol\left(B_p(r)\right)} dr < \infty,$$

and hence M has infinite volume. Moreover, if $\lambda_1(M) > 0, M$ is nonparabolic [160]. Therefore one can conclude the following.

Proposition (4.1.11)[141]: Let M be an n-dimensional complete noncompact Riemannian manifold with $\lambda_1(M) > 0$. Then $Vol(M) = \infty$.

Note that, in the case of submanifolds, see [147] proved that the volume $Vol\left(B_p(r)\right)$ of every complete noncompact submanifold M in the Euclidean or hyperbolic space grows at least as a linear function of r under the assumption that the mean curvature vector H of M is bounded in absolute value.

We state and prove vanishing theorems for L^p harmonic 1-forms on a complete noncompact stable minimal hypersurface.

Theorem (4.1.12)[141]: Let N be an$(n+1)$-dimensional complete Riemannian manifold with sectional curvature satisfying $K \le K_N$ where $K \le 0$ is a constant. Let M be a complete noncompact stable minimal hypersurface in N. Assume that, for

$$0 < p < n/(n-1) + \sqrt{2n},$$

$$\lambda_1(M) > -\frac{2n(n-1)^2\, p^2 K}{2n - [(n-1)p - n]^2}.$$

Then there is no nontrivial L^{2p} harmonic 1-form on$M$.

Proof. We consider two cases: $K < 0$ and $K = 0$.

Case 1: $K < 0$. Letω! be an L^{2p} harmonic 1-form on M, that is,

$$\Delta\omega = 0 \; and \int_M |\omega|^{2p}\, dv < \infty.$$

In an abuse of notation, we refer to both a harmonic 1-form and its dual harmonic vector field byω,. Bochner's formula yields

$$\Delta|\omega|^2 = 2\left(|\nabla\omega|^2 + Ric(\omega,\omega)\right).$$

Moreover,

$$\Delta|\omega|^2 = 2\left(|\omega|\Delta|\omega| + \left|\nabla|\omega|\right|^2\right).$$

Applying Lemma (4.1.7) and the Kato-type inequality (12), we see that

$$|\omega|\Delta|\omega| + \frac{n-1}{n}|A|^2|\omega|^2(n-1)K|\omega|^2 \ge \frac{1}{n-1}\left|\nabla|\omega|\right|^2. \tag{14}$$

For any positive number p, we have

$$\begin{aligned}|\omega|^p\Delta|\omega|^p &= |\omega|^p div(\nabla|\omega|^p) = |\omega|^p div(p|\omega|^{p-1}\nabla\omega)\\ &= p(p-1)|\omega|^{2p-2}\left|\nabla|\omega|\right|^2 + p|\omega|^{2p-1}\Delta|\omega|\\ &= \frac{p-1}{p}\,|\nabla|\omega|^p|^2 + p|\omega|^{2p-2}|\omega|\Delta|\omega|.\end{aligned}$$

Plugging inequality (14) into the above equality, we have

$$|\omega|^p\Delta|\omega|^p + p(n-1)\left(\frac{|A|^2}{n} - K\right)|\omega|^p \ge \left(1 - \frac{1}{p} + \frac{1}{p(n-1)}\right)\left|\nabla|\omega|\right|^2.$$

Choose a Lipschitz function f with compact support in a geodesic ball $B(R)$ of radius R centered at $p \in M$. Multiplying both side by f^2 and integrating over $B(R)$, we obtain

$$\left(1-\frac{1}{p}+\frac{1}{p(n-)}\right)\int_{B(R)} f^2\left|\nabla|\omega|\right|^2$$

$$\leq \int_{B(R)} f^2\,|\omega|^{\mathrm{p}}\Delta|\omega|^{\mathrm{p}}+\frac{p(n-1)}{n}\int_{B(R)} f^2\,|\mathrm{A}|^2|\omega|^{2\mathrm{p}}-\,p(n-1)K\int_{B(R)} f^2\,|\omega|^{2\mathrm{p}}.$$

The divergence theorem gives

$$\int_{B(R)} f^2\,|\omega|^{\mathrm{p}}\Delta|\omega|^{\mathrm{p}}=-\int_{B(R)} f^2\,|\nabla|\omega|^{\mathrm{p}}|^2-2\int_{B(R)} f\,|\omega|^{\mathrm{p}}\langle\nabla \mathrm{f},\nabla|\omega|^{\mathrm{p}}\rangle.$$

Thus

$$\left(1-\frac{1}{p}+\frac{1}{p(n-1)}\right)\int_{B(R)} f^2\,|\nabla|\omega|^{\mathrm{p}}|^2$$

$$\leq\frac{p(n-1)}{n}\int_{B(R)} f^2\,|\mathrm{A}|^2|\omega|^{2\mathrm{p}}-\,p(n-1)K\int_{B(R)} f^2\,|\omega|^{2\mathrm{p}}.$$

$$=-\int_{B(R)} f^2\,|\nabla|\omega|^{\mathrm{p}}|^2-2\int_{B(R)} f\,|\omega|^{\mathrm{p}}\langle\nabla \mathrm{f},\nabla|\omega|^{\mathrm{p}}\rangle. \tag{15}$$

Since M is stable,

$$\int_M |\nabla \mathrm{f}|^2-\left(|A|^2+\overline{Ric}(e_{n+1})\right)f^2\geq 0$$

for any compactly supported Lipschitz function f on M. From the assumption on the sectional curvature of N, it follows that

$$\int_M |\nabla \mathrm{f}|^2-(|A|^2+nK)f^2\geq 0$$

for any compactly supported Lipschitz function f on M. Replacing f by $|\omega|^{\mathrm{p}}f$ we have

$$\int_{B(R)} f^2\,|\nabla|\omega|^{\mathrm{p}}|^2+\int_{B(R)}|\nabla f|^2\,|\omega|^{2p}+2\int_{B(R)} f\,|\omega|^{\mathrm{p}}\langle\nabla \mathrm{f},\nabla|\omega|^{\mathrm{p}}\rangle\geq\int_{B(R)} f^2\,|\mathrm{A}|^2|\omega|^{2\mathrm{p}} \tag{16}$$

Combining the inequalities (15) and (16) gives

$$\left(1-\frac{1}{p}+\frac{1}{p(n-1)}\right)\int_{B(R)} f^2\,|\nabla|\omega|^{\mathrm{p}}|^2$$

$$\leq\frac{p(n-1)}{n}\left[\int_{B(R)} f^2\,|\nabla|\omega|^{\mathrm{p}}|^2+\int_{B(R)}|\nabla f|^2\,|\omega|^{2\mathrm{p}}\right.$$

$$\left.+\,2\int_{B(R)} f\,|\omega|^{\mathrm{p}}\langle\nabla \mathrm{f},\nabla|\omega|^{\mathrm{p}}\rangle\,p(n-1)-nK\int_{B(R)} f^2|\omega|^{2\mathrm{p}}\right]$$

$$-p(n-1)K\int_{B(R)} f^2\,|\omega|^{2\mathrm{p}}-\int_{B(R)} f^2\,|\nabla|\omega|^{\mathrm{p}}|^2-2\int_{B(R)} f\,|\omega|^{\mathrm{p}}\langle\nabla \mathrm{f},\nabla|\omega|^{\mathrm{p}}\rangle.$$

Hence

$$\left(1-\frac{1}{p}+\frac{1}{p(n-1)}\right)\int_{B(R)} f^2\,|\nabla|\omega|^{\mathrm{p}}|^2$$
$$\leq\left(\frac{p(n-1)}{n}-1\right)\int_{B(R)} f^2\,|\nabla|\omega|^{\mathrm{p}}|^2+\frac{p(n-1)}{n}\int_{B(R)}|\nabla f|^2\,|\omega|^{2\mathrm{p}}$$
$$-\,2p(n-1)K\int_{B(R)} f^2|\omega|^{2\mathrm{p}}$$
$$+2\left(\frac{p(n-1)}{n}-1\right)\int_{B(R)} f\,|\omega|^{\mathrm{p}}\langle\nabla \mathrm{f},\nabla|\omega|^{\mathrm{p}}\rangle. \tag{17}$$

Moreover, using the definition of the bottom of the spectrum, we see that

$$\lambda_1(M)\int_{B(R)}|\omega|^{2\mathrm{p}}f^2\int_{B(R)}|\nabla(|\omega|^{\mathrm{p}}f)|^2$$
$$=\int_{B(R)} f^2\,|\nabla|\omega|^{\mathrm{p}}|^2+\int_{B(R)}|\omega|^{2\mathrm{p}}|\nabla f|^2+2\int_{B(R)} f\,|\omega|^{\mathrm{p}}\langle\nabla \mathrm{f},\nabla|\omega|^{\mathrm{p}}\rangle, \tag{18}$$

From inequalities (17) and (18), it follows that

$$\left(1-\frac{1}{p}+\frac{1}{p(n-1)}\right)\int_{B(R)} f^2\,|\nabla|\omega|^{\mathrm{p}}|^2$$
$$\leq\left(\frac{p(n-1)}{n}-1-\frac{2p(n-1)K}{\lambda_1(M)}\right)\int_{B(R)} f^2\,|\nabla|\omega|^{\mathrm{p}}|^2$$
$$+\left(\frac{p(n-1)}{n}-1-\frac{2p(n-1)K}{\lambda_1(M)}\right)\int_{B(R)}|\nabla f|^2\,|\omega|^{2\mathrm{p}}$$
$$+\,2\left(\frac{p(n-1)}{n}-1-\frac{2p(n-1)K}{\lambda_1(M)}\right)\int_{B(R)} f\,|\omega|^{\mathrm{p}}\langle\nabla \mathrm{f},\nabla|\omega|^{\mathrm{p}}\rangle.$$

Applying Young's inequality, we have

$$2\int_{B(R)} f\,|\omega|^{\mathrm{p}}\langle\nabla \mathrm{f},\nabla|\omega|^{\mathrm{p}}\rangle\leq\varepsilon\int_{B(R)} f^2\,|\nabla|\omega|^{\mathrm{p}}|^2+\frac{1}{\varepsilon}\int_{B(R)}|\nabla f|^2\,|\omega|^{2\mathrm{p}}$$

for any $\varepsilon>0$. Thus

$$\left[2-\frac{1}{p}+\frac{1}{p(n-1)}+\frac{2p(n-1)K}{\lambda_1(M)}-\frac{p(n-1)}{n}-\varepsilon\left(\frac{p(n-1)}{n}-1-\frac{2p(n-1)K}{\lambda_1(M)}\right)\right]$$
$$\times\int_{B(R)} f^2\,|\nabla|\omega|^{\mathrm{p}}|^2$$
$$\leq\left[\frac{p(n-1)}{n}-\frac{2p(n-1)K}{\lambda_1(M)}+\frac{1}{\varepsilon}\left(\frac{p(n-1)}{n}-1-\frac{2p(n-1)K}{\lambda_1(M)}\right)\right]\int_{B(R)}|\nabla f|^2\,|\omega|^{2\mathrm{p}}.$$

Since

$$\lambda_1(M)>\frac{-2p(n-1)K}{2-1/p+1/\big(p(n-1)\big)-p(n-1)/n}=\frac{-2n(n-1)^2p^2K}{2n-[(n-1)p-n]^2}$$

by the hypothesis, one can choose a sufficiently small $\varepsilon>0$ satisfying that

$$\left[2-\frac{1}{p}+\frac{1}{p(n-1)}-\frac{2p(n-1)K}{\lambda_1(M)}+\frac{p(n-1)}{n}+\varepsilon\left(\frac{p(n-1)}{n}-1-\frac{2p(n-1)K}{\lambda_1(M)}\right)\right]>0.$$

Note that $\int_M|\omega|^{2p}<\infty$, since ! is an L^{2p} harmonic 1-form on M. Letting R tend to infinity, we obtain

$$\int_M |\nabla|\omega|^p|^2 = 0,$$

Which implies that $\big|\nabla|\omega|\big| \equiv 0$. Hence $|\omega| \equiv$ constant. From Proposition (4.1.11), itfollows that $|\omega| \equiv 0$..

Case 2: $K = 0$. Using the inequality (17) and Young's inequality, we obtain

$$\left[2 - \frac{1}{p} + \frac{1}{p(n-1)} - \frac{p(n-1)}{n} - \varepsilon\left(\frac{p(n-1)}{n} - 1\right)\right] \int_{B(R)} f^2 |\nabla|\omega|^p|^2$$
$$\leq \left[\frac{p(n-1)}{n} + \frac{1}{\varepsilon}\left(\frac{p(n-1)}{n} - 1\right)\right] \int_{B(R)} |\nabla f|^2 |\omega|^{2p}.$$

Since $0 < p < n/(n-1) + \sqrt{2n}$, one may choose a sufficiently small $\varepsilon > 0$ satisfying

$$2 - \frac{1}{p} + \frac{1}{p(n-1)} - \frac{p(n-1)}{n} + \varepsilon\left(\frac{p(n-1)}{n} - 1\right) > 0.$$

Letting R tend to infinity gives

$$\int_{B(R)} |\nabla|\omega|^p|^2 = o,$$

which implies that $|\omega| \equiv$ constant. From the assumption that$\lambda_1(M) > 0$ and Proposition (4.1.11), it follows that $|\omega| \equiv 0$.

As a consequence of Theorem (4.1.12), given a complete noncompact stable minimal hypersurface in a nonnegatively curved Riemannian manifold, one has the following result.

Corollary (4.1.13)[141]: Let N be an.$(n + 1)$-dimensionalcomplete nonnegatively curved Riemannian manifold. Let M be a complete noncompact stable minimal hypersurface in N with $\lambda_1(M) > 0$. If$n \leq 11$, there is no nontrivial L^p harmonic 1-form on M for any $0 < p \leq n$.

Proof. For $n \leq 11$, the inequality $2\big(n/(n-1) + \sqrt{2n}\big) \geq n$ holds. _

Corollary (4.1.14)[141]: Let N be an.$(n+1)$-dimensional complete nonnegatively curved Riemannian manifold. Let M be a complete noncompact stable minimal hypersurface in N with$\lambda_1(M) > 0$. If $n \leq 11$, there is no nontrivial L^2 harmonic 1-form on M.

In the case of L^2 harmonic 1-forms, Theorem (4.1.12) gives a generalization of [154] as follows.

Corollary (4.1.15)[141]: Let N be an $(n+1)$-dimensional complete Riemannian manifold with sectional curvature satisfying $K \leq K_N$ where $K < 0$ is a constant. Let M be a complete noncompact stable minimal hypersurface in N. Assume that

$$\lambda_1(M) > \frac{2n(n-1)^2 K}{2n-1}.$$

Then there are no nontrivial L^2 harmonic 1-forms on M.

In particular, if N is $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$, Corollary (4.1.15) improves the previous result of [177]. Related to this result, Cavalcante, Mirandola, and Vitório [15] obtained the vanishing theorem for L^2 harmonic 1-forms on complete noncompact submanifolds in a Cartan–Hadamard manifold.

Palmer [170] showed that if there exists a codimension-one cycle in a complete minimal hypersurface M in $\mathbb{R}^{n+1}$ which does not separate M, M is unstable. We obtain a generalization of Palmer's result as follows.

Corollary (4.1.16)[141]: Let N be an $(n+1)$-dimensional complete Riemannian manifold with sectional curvature satisfying $K \leq K_N$ where $K \leq 0$ is a constant. Let M be a complete noncompact minimal hypersurface in N. Assume that

$$\lambda_1(M) > \frac{2n(n-1)^2 K}{2n-1}.$$

Suppose that there exists a codimension-one cycle in M which does not separate M. Then M cannot be stable.

Proof. Suppose that M is stable in N. From [153], there exists a nontrivial L^2 harmonic 1-form on M, which is a contradiction to Corollary (4.1.15). Let M be a complete Riemannian manifold and let f be a harmonic function on M with finite L^p energy. Then the total differential df is obviously an L^p harmonic 1-form on M. As another application of Theorem (4.1.12), we prove the following Liouville type theorem for harmonic functions with finite L^p energy on a complete noncompact stable minimal hypersurface, which is a generalization of Schoen and Yau's result [183], as mentioned.

Corollary (4.1.17)[141]: Let N be an $(n+1)$-dimensional complete Riemannian manifold with sectional curvature satisfying $K \leq K_N$ where $K \leq 0$ is a constant. Let M be a complete noncompact stable minimal hypersurface in N. Assume that, for $0 < p < n\left(n/(n-1)+\sqrt{2n}\right)$,

$$\lambda_1(M) > \frac{2n(n-1)^2 p^2 K}{2n-[(n-1)p-n]^2}.$$

Then there is no nontrivial harmonic function on M with finite L^p energy.

So far, we have assumed that $\lambda_1(M) > 0$ for a complete noncompact stable minimal hypersurface M in a nonnegatively curved Riemannian manifold. However, we do not know whether the assumption that $\lambda_1(M) > 0$ is necessary or not. It would be interesting to remove the condition in these results.

We prove a vanishing theorem for L^p harmonic 1-forms on a complete stable minimal hypersurface M, assuming that M has sufficiently small total scalar curvature instead of assuming that M is stable.

Theorem (4.1.18)[141]: Let N be an $(n+1)$-dimensional complete simply connected Riemannian manifold with sectional curvature K_N satisfying that $K_1 \leq K_N \leq K_2 < 0$, where K_1, K_2 are constants and $n \geq 3$. Let M be a complete minimal hypersurface in N. Assume that $K := K_2 = K_1$ satisfies

$$K > \frac{4(n-2)}{(n-1)^2}.$$

For

$$\frac{(n-1)K}{4} - \frac{1}{2}\sqrt{\frac{(n-1)^2K^2}{4} - (n-2)K}$$

$$< p\frac{(n-1)K}{4} + \frac{1}{2}\sqrt{\frac{(n-1)^2K}{4} - (n-2)K},$$

Assume that

$$\left(\int_M |A|^n\right)^{n/2} < \frac{n(2p(n-1)-n+2-4p^2K)}{p^2(n-1)^2C_s},$$

where C_s is the Sobolev constant in [29]. Then there are no nontrivial L^{2p} harmonic 1-forms on M.

Proof. A similar argument as in the proof of Theorem (4.1.12) shows

$$|\omega|^{p}\Delta|\omega|^{p}+p(n-1)\left(\frac{|A|^2}{n}-K_1\right)|\omega|^{2p}\geq\left(1-\frac{1}{p}+\frac{1}{p(n-1)}\right)|\nabla|\omega|^{p}|^2$$

for any Lipschitz function f with compact support in a geodesic ball $B(R)$ of radius R centered at a point $p\in M$. Multiplying both sides byf^2, integrating over $B(R)$,and applying the divergence theorem, we see that

$$\left(1-\frac{1}{p}+\frac{1}{p(n-1)}\right)\int_{B(R)}f^2|\nabla|\omega|^{p}|^2$$
$$\leq\frac{p(n-1)}{n}\int_{B(R)}f^2|\mathrm{A}|^2|\omega|^{2p}-p(n-1)K_1\int_{B(R)}f^2|\omega|^{2p}$$
$$-\int_{B(R)}f^2|\nabla|\omega|^{p}|^2-2\int_{B(R)}f|\omega|^{p}\langle\nabla\mathrm{f},\nabla|\omega|^{p}\rangle. \tag{19}$$

On the other hand, the Sobolev inequality (13) implies that

$$\int_{B(R)}f^2|\mathrm{A}|^2|\omega|^{2p}\left(\int_M|A|^n\right)^{2/n}\left(\int_M(|\omega|^{p}\mathrm{f})^{(2n)/n-2}\right)^{(n-2)/n}$$
$$\leq C_s\left(\int_M|A|^n\right)^{2/n}\int_M|\nabla(|\omega|^{p}f)|^2$$
$$\leq C_s\left(\int_M|A|^n\right)^{2/n}\left(\int_{B(R)}f^2|\nabla|\omega|^{p}|^2+\int_{B(R)}|\nabla f|^2|\omega|^{2p}+2\int_{B(R)}f|\omega|^{p}\langle\nabla\mathrm{f},\nabla|\omega|^{p}\rangle\right).$$

Plugging this inequality into (19) gives

$$\left(1-\frac{1}{p}+\frac{1}{p(n-1)}\right)\int_{B(R)}f^2|\nabla|\omega|^{p}|^2\leq\frac{p(n-1)C_s}{n}\left(\int_M|A|^n\right)^{2/n}\int_{B(R)}f^2|\omega|^{2p}$$
$$+\left(\frac{p(n-1)C_s}{n}\left(\int_M|A|^n\right)^{2/n}-1\right)\int_{B(R)}f^2|\nabla|\omega|^{p}|^2$$
$$+2\left(\frac{p(n-1)C_s}{n}\left(\int_M|A|^n\right)^{2/n}-1\right)\int_{B(R)}f|\omega|^{p}\langle\nabla\mathrm{f},\nabla|\omega|^{p}\rangle.$$
$$-p(n-1)K_1\int_{B(R)}f^2|\omega|^{2p}. \tag{20}$$

An estimate (1) for the bottom of the spectrum yields

$$\frac{K_2(n-1)^2}{4}\leq\lambda_1(M)\leq\frac{\int_{B(R)}|\nabla(|\omega|^{p}\mathrm{f})|^2}{\int_{B(R)}(|\omega|^{p}f)^2},$$

which gives

$$\int_{B(R)} (|\omega|^{p} f)^2$$
$$\leq \frac{4}{K_2(n-1)^2}\left(\int_{B(R)} f^2|\nabla|\omega|^{p}|^2 + \int_{B(R)} u|\nabla f|^2|\omega|^{2p} + 2\int_{B(R)} f|\omega|^{p}\langle \nabla f, \nabla|\omega|^{p}\rangle\right). \tag{21}$$

Thus, from inequalities (20) and (21), it follows that

$$\left(1-\frac{1}{p}+\frac{1}{p(n-1)}\right)\int_{B(R)} f^2|\nabla|\omega|^{p}|^2$$
$$\leq B\int_{B(R)} f^2|A|^2|\omega|^{2p} + (B-1)\int_{B(R)} f^2|\nabla|\omega|^{p}|^2 + 2(B-1)\int_{B(R)} f|\omega|^{p}\langle \nabla f, \nabla|\omega|^{p}\rangle.$$

Where

$$B = \frac{p(n-1)C_s}{n}\left(\int_M |A|^n\right)^{2/n} + \frac{4p}{(n-1)}\frac{1}{K}.$$

Applying Young's inequality

$$2\int_{B(R)} f|\omega|^{p}\langle \nabla f, \nabla|\omega|^{p}\rangle \leq \varepsilon\int_{B(R)} f^2|\nabla|\omega|^{p}|^2 + \frac{1}{\varepsilon}\int_{B(R)} f^2|A|^2|\omega|^{2p}$$

for any $\varepsilon > 0$, we see that

$$\left(2-\frac{1}{p}+\frac{1}{p(n-1)}-B-\varepsilon(B-1)\right)\int_{B(R)} f^2|\nabla|\omega|^{p}|^2$$
$$\leq \left(B+\frac{1}{\varepsilon}(B-1)\right)\int_{B(R)} |\nabla f|^2|\omega|^{2p}.$$

From the assumption on the total curvature of M, one can make

$$\left(2-\frac{1}{p}+\frac{1}{p(n-1)}-B-\varepsilon(B-1)\right) > 0$$

by choosing a sufficiently small $\varepsilon > 0$. Letting $R \to \infty$ and using that ω is an L^{2p}harmonic 1-form, we conclude that

$$\int_M |\nabla|\omega|^{p}|^2 = 0.$$

The same argument as before shows that $|\omega| \equiv 0$.

Corollary (4.1.19)[141]: Let M be a complete minimal hypersurface in $\mathbb{H}^{n+1}$ satisfying

$$\left(\int_M |A|^n\right)^{2/n} < n\frac{(-4p^2+2p(n-1)-n+2)}{p^2(n-1)^2C_s}$$

for $\frac{1}{2} < p < \frac{n}{2} - 1$. Then there are no nontrivial L^{2p} harmonic 1-forms on M.

Corollary (4.1.20)[141]: Under the same conditions as in Theorem (4.1.18), there is no nontrivial harmonic function on M with finite L^p energy.

When the L^∞ norm of the second fundamental form of a complete minimal hypersurface is bounded, the following vanishing theorem holds.

Theorem (4.1.21)[141]: Let N be an.$(n+1)$-dimensional complete simply connected Riemannian manifold with sectional curvature K_N satisfying $K_1 \le K_N \le K_2 < 0$, where K_1, K_2 are constants and $n \ge 3$. Let M be a complete noncompact minimal hypersurface in N. Assume that $K \coloneqq K_2/K_1 > 4(n-2)/(n-1)^2$ and the second fundamental form A satisfies

$$|A|^2 \le C < \frac{4p^2K_1 - (2p(n-1) - n + 2K_2}{4p^2}$$

For

$$\frac{(n-1)K}{4} - \frac{1}{2}\sqrt{\frac{(n-1)^2K^2}{4} - (n-2)K}$$
$$< p < \frac{(n-1)K}{4} + \frac{1}{2}\sqrt{\frac{(n-1)^2K^2}{4} - (n-2)K}.$$

Then there are no nontrivial L^{2p} harmonic 1-forms on M.

Proof. A similar argument as before shows

$$\left(1 - \frac{1}{p} + \frac{1}{p(n-1)}\right)\int_{B(R)} f^2|\nabla|\omega|^p|^2$$
$$\le \frac{p(n-1)}{n} \le \int_{B(R)} f^2|A|^2|\omega|^{2p} - p(n-1)K_1\int_{B(R)} f^2|\omega|^{2p}$$
$$-\int_{B(R)} f^2|\nabla|\omega|^p|^2 - 2\int_{B(R)} f|\omega|^p\langle\nabla f, \nabla|\omega|^p\rangle.$$

Since $|A|^2 \le C$,

$$\left(2 - \frac{1}{p} + \frac{1}{p(n-1)}\right)\int_{B(R)} f^2|\nabla|\omega|^p|^2$$
$$\le \left(\frac{p(n-1)C}{n} - p(n-1)K_1\right)\int_{B(R)} f^2|\omega|^{2p} - 2\int_{B(R)} f|\omega|^p\langle\nabla f, \nabla|\omega|^p\rangle.$$

Using an estimate for the bottom of the spectrum and Young's inequality again, we have

$$\left(1-\frac{1}{p}+\frac{1}{p(n-1)}-D-\varepsilon(D-1)\right)\int_{B(R)} f^2|\nabla|\omega|^{\mathrm{p}}|^2$$
$$\leq\left(D+\frac{1}{\varepsilon}(D-1)\right)\int_{B(R)}|\nabla f|^2|\omega|^{2\mathrm{p}},$$

Where

$$D=\frac{-4}{(n-1)^2K_2}\left(\frac{p(n-1)C}{n}-p(n-1)K_1\right).$$

Since

$$C<\frac{4p^2K_1-(2p(n-1)-n+2)K_2}{4p^2},$$

by our assumption, we may choose a sufficiently small $\varepsilon>0$ satisfying

$$\left(2-\frac{1}{p}+\frac{1}{p(n-1)}-D-\varepsilon(D-1)\right)>0.$$

Thus we get

$$\int_{B(R)}|\nabla|\omega|^{\mathrm{p}}|^2=0.$$

by letting R tend to infinity. Hence $\omega\equiv 0$.

Corollary (4.1.22)[141]: Let M be a complete minimal hypersurface in $\mathbb{H}^{n+1}$ with the second fundamental form A satisfying

$$|A|^2\leq\ C<\frac{-4p^2+2p(n-1)-n+2}{4p^2},$$

for $\frac{1}{2}<p<\frac{n}{2}-1$. Then there are no nontrivial L^{2p} harmonic 1-forms on M.

Corollary (4.1.23)[141]: Under the same conditions as in Theorem (4.1.21), there is no nontrivial harmonic function on M with finite L^p energy.

We remark that there are lots of examples of minimal hypersurfaces with finite L^n or L^∞ norm of the second fundamental form in $\mathbb{H}^{n+1}$ [145], [169], [174], [178].

Section (4.2): Submanifolds in a Hadamard Manifold

The geometric structure and topological properties of submanifolds in various ambient space have been studied extensively during past few years. In [2], Cao, Shen and Zhu showed that a complete connected stable minimal hypersurface in Euclidean space must have exactly one end. Its strategy was to utilize a result of Schoen-Yau asserting that a complete stable minimal hypersurface in Euclidean space can not admit a non-constant harmonic function with finite integral [13]. Later Ni [31] proved that if n-dimensional complete minimal submanifold M in Euclidean space has sufficient small total scalar curvature (i.e.$\int_M\ |A|^n<C_1$), then M has only one end. In [34], Seo improved the upper bound C1. Due to this connection with harmonic functions, this allows one to estimate the number of ends of the above submanifold by estimating the dimension of the space of bounded harmonic function with finite Dirichlet integral [165]. In [27], Fu and Xu proved that a complete submanifold M^m with finite total curvature and some conditions on mean curvaute in an $(n+p)$-dimensional simply connected space form $M^{m+p}(c)$ must have finitely many ends. In [189], M.P. Cavalcante, H. Mirandola, F. Vitório proved that a complete submanifold M^m with finite total curvature and some conditions on the first

eigenvalue of the Laplace-Beltrami operator of M in an Hadamard manifold must have finitely many ends. For p-harmonic 1-forms, Zhang [189] obtained vanishing Manuscript results for p-harmonic 1-form. Chang [185] obtained the compactness for any bounded set of p-harmonic 1-forms.

For (M^m, g) be a Riemannian manifold, and let u be a real C^∞ function on Mm. Fix $p \in \mathbb{R}, p \geq 2$ and consider a compact domain $\Omega \subset M^m$. The p-energy of u on **Ω**, is defined to be

$$E_p(\Omega, u) = \frac{1}{p}\int_\Omega |\nabla u|^p .$$

The function u is said to be p-harmonic on M^m if u is a critical point of $E_p(\Omega,*)$ for every compact domain $\Omega \subset M^m$. Equivalently, u satisfies the Euler-Lagrange equation.

$$div(|\nabla|^{p-2}\nabla u) = 0.$$

Thus, the concept of p-harmonic function is a natural generalization of that of harmonic function, that is, of a critical point of the 2-energy functional.

Definition (4.2.1)[184]: A p-harmonic 1-form is a differentiable 1-form on M^m satisfying the following properties:

$$\begin{cases} d\omega = 0, \\ \delta(|\omega|^{p-2}\omega) = 0, \end{cases}$$

where δ is the codifferential operator. It is easy to see that the differential of a p-harmonic function is a p-harmonic 1-from.

We investigate the properties for p-harmonic 1-form on noncompact submanifolds with finite total curvature. We assume that M^m is a complete noncompact manifold and define the space of the L^p p-harmonic 1-froms on M by

$$H^{1,p}(M) = \left\{\omega \mid \int_M |\omega|^p \, dv < \infty, d\omega = 0 \text{ and } \delta(|\omega|^{p-2}\omega) = 0\right\}$$

where $p \geq 2$.

Theorem (4.2.2)[184]: Let $x : M^m \to N, m \geq 3$, be an isometric immersion of a complete noncompact manifold M in a Hadamard manifold N with the sectional curvature satisfying $-k^2 \leq K_N \leq 0$ for some constant k. In the case there exists point $q \in M$ such that $K_N(q) \neq 0$, assume further that the first eigenvalue of the Laplace-Beltrami operator of M satisfies

$$\lambda_1(M) > \frac{(m-1)^2 p^2 \left(k^2 - \lim_{\rho\to\infty} \inf |H|^2\right)}{4[(m-1)(p-1)+1]}.$$

where ρ stands for the distance in M from a fixed point. If M^m has finite total curvature, then the $dimH^{1,p}(M) < \infty$ for $p \geq 2$.

Corollary (4.2.3)[184]: Let $x : M^m \to R^{m+n}, m \geq 3$, be an isometric immersion of a complete noncompact manifold M in R^{m+n}. If M^m has finite total curvature, then the $dimH^{1,p}(M) < \infty$ for $p \geq 2$.

Corollary (4.2.4)[184]: Let $x : M^m \to H^{m+n}(-1), m \geq 3$, be an isometric immersion of a complete noncompact manifold M in $H^{m+n}(-1)$. Assume further that the first eigenvalue of the Laplace-Beltrami operator of M satisfies

$$\lambda_1(M) > \frac{(m-1)^2 p^2 \left(1 - \lim_{\rho\to\infty} \inf |H|^2\right)}{4[(m-1)(p-1)+1]}.$$

where ρ stands for the distance in M from a fixed point. If M^m has finite total curvature, then the $dimH^{1,p}(M) < \infty$ for $p \geq 2$.

From the proof of Theorem (4.2.2), we can obtain the following result.

Theorem (4.2.5)[184]: Let $x : M^m \to N, m \geq 3$, be an isometric immersion of a complete noncompact manifold M ia a Hadamard manifold N with the sectional curvature satisfying $-k^2 \leq K_N \leq 0$ for some constant k. In the case there exists point $q \in M$ such that $K_N(q) \neq 0$, assume further that the first eigenvalue of the Laplace-Beltrami operator of M satisfies

$$\lambda_1(M) > \frac{(m-1)^2 p^2 (k^2 - \inf|H|^2)}{4[(m-1)(p-1)+1]}.$$

If there exists a positive constant Λ such that $\|\Phi\|_{L^m(M)} < \Lambda$, then $H^{1,p}(M) = \{0\}$ for $p \geq 2$. Furthermore, if $k = 0$, then Λ depends only on m, p; otherwise, Λ depends only on m, k, p and $\inf|H|$.

Corollary (4.2.6)[184]: Let $x : M^m \to R^{m+n}, m \geq 3$, be an isometric immersion of a complete noncompact manifold M in R^{m+n}. If there exists a positive constant Λ depending on m, p, such that $\|\Phi\|_{L^m(M)} < \Lambda$, then $H^{1,p}(M) = \{0\}$ for $p \geq 2$.

Corollary (4.2.7)[184]: Let $x : M^m \to H^{m+n}(-1), m \geq 3$, be an isometric immersion of a complete noncompact manifold M in $H^{m+n}(-1)$. Assume further that the first eigenvalue of the Laplace-Beltrami operator of M satisfies

$$\lambda_1(M) > \frac{(m-1)^2 p^2 (1 - \inf|H|^2)}{4[(m-1)(p-1)+1]}.$$

If there exists a positive constant Λ depending on m, p, inf |H|, such that $\|\Phi\|_{L^m(M)} < \Lambda$, then $H^{1,p}(M) = \{0\}$ for $p \geq 2$.

Let M^m be a complete submanifold immersed in a Riemmannian manifold N^{m+n}. Fix a pint $x \in M$ and a local orthonormmal frame $\{e_1, \cdots, e_{m+n}\}$ of N^{m+n} such that $\{e_1, \cdots, e_m\}$ are tangent fields of M. For each $\alpha, m+1 \leq \alpha \leq m+n$, define a line map $A_\alpha : T_xM \to T_xM$ by $\langle A_\alpha X, Y\rangle = \langle \overline{\nabla}_X Y, e_\alpha\rangle$, where X, Y are tangent fields and ∇ is the Riemannian connection of N^{m+n}. Denote by $h_{ij}^\alpha = \langle A_\alpha e_i, e_j\rangle$. The squared norm $|A|^2$ of the second fundamental form and the mean curvature vector H are defined by

$$|A|^2 = \sum_{ij_\alpha} \left(h_{ij}^\alpha\right)^2 \quad H = \sum_\alpha H^\alpha e_\alpha = \frac{1}{m}\sum_{i\alpha} h_{ii}^\alpha e_\alpha.$$

The traceless second fundamental form Φ is defined by

$$\Phi(X, Y) = A(X, Y) - \langle X, Y\rangle H,$$

for all vector fields X, Y on M. A simple computation shows that

$$|\Phi|^2 = |A|^2 - m|H|^2,$$

which measures how much the immersion deviates from being totally umbilical. We say that M^m has finite total curvature if

$$\|\Phi\|_{L^m(M)} = \left(\int_M |\Phi|^m dv\right)^{\frac{1}{m}} < \infty.$$

We need the following results:

Lemma (4.2.8)[184]: [8] Let E be a finite dimensional subspace of the space L^2 q-forms on a compact Riemannian manifold Mfm. Then there exists $\omega \in E$ such that

$$\frac{dimE}{Vol(\widetilde{M})}\int_{\widetilde{M}} |\omega|^2 \, dv \leq \min\left\{\binom{m}{q}, dimE\right\} \sup_{\widetilde{M}} |\omega|^2.$$

From the above Lemma, we can prove the following result.

Lemma (4.2.9)[184]: Let E be a finite dimensional subspace of the space L^p q-forms on a compact Riemannian manifold $\widetilde{M}^m$. Then there exists $\omega \in E$ such that
$$\frac{dimE}{Vol(\widetilde{M})}\int_{\widetilde{M}} |\omega|^p\, dv \le \min\left\{\binom{m}{q}, dimE\right\} \sup_{\widetilde{M}} |\omega|^p ,$$
where C_p is a positive constant depending only p and $p \ge 2$.

Proof. Let $\{\omega_i\}_{i=1}^r, r = dimE,$ be a orthonormal basis of E. Define the function
$$F(x) = \left[\sum_{i=1}^r |\omega_i(x)|^2\right]^{\frac{p}{2}} .$$
Clearly, $F(x)$ is well defined under orthogonal change of basis. Since $E \neq 0$, $\|F\|_{L^\infty(\widetilde{M})} \neq 0$. Let $x_0 \in \widetilde{M}$ such that $F(x_0) = \|F\|_{L^\infty(\widetilde{M})}$. Define the subspace E_0 of E by
$$E_0 = \{\omega \in E | \omega(x_0) = 0\}.$$
By the choice of $x_0, E\_0 \neq E$. We claim that the orthogonal complement $E_0^\perp$ is of at most dimension $\binom{m}{q}$. In fact, if $\{\omega_\alpha\}_{\alpha=1}^s$ form an orthonormal basis for $E_0^\perp$ with $s > \binom{m}{q}$, then there exists $(a_\alpha)_{\alpha=1}^s \in R^s$ with $(a_\alpha) \neq 0$ such that $\sum_{\alpha=1}^s a_\alpha \omega_\alpha(x_0) = 0$. This is true because the dimension of the vector space of antisymmetric p-tensors on m-dimensional vector space is $\binom{m}{q}$. This implies $\sum_{\alpha=1}^s a_\alpha\omega_\alpha \in E_0$, which is a contradiction. Now we choose orthonormal basis for E such that $\{\omega_\alpha\}_{\alpha=1}^s$ form an orthonormal basis for $E_0^\perp$ and $\{\omega_i\}_{i=s+1}^r$ an orthonormal basis for E_0. Then
$$dimE = \int_{\widetilde{M}} \sum |\omega|^p\, dv \le \int_{\widetilde{M}} F(x)dv \le \|F\|_{L^\infty(\widetilde{M})} Vol(\widetilde{M}) = F(x_0)Vol(\widetilde{M})$$
$$= Vol(\widetilde{M})\left(\sum_{\alpha=1}^s \omega_\alpha(x_0)\right)^{\frac{p}{2}} \le C_p\binom{m}{q} Vol(\widetilde{M}) \max_\alpha \|\omega_\alpha\|^p_{L^\infty(\widetilde{M})} .$$
Since $\|\omega\|^p_{L^p(\widetilde{M})} \le Vol(\widetilde{M})\|\omega\|^p_{L^\infty(\widetilde{M})}$ for all $\omega \in E$, this proves the Lemma. In the following, we obtain a Kato type inequality for p-harmonic 1-form.

Lemma (4.2.10)[184]: Let ω be a p-harmonic 1-form on Riemannian manifold M^m. Then we have the following inequality
$$|\nabla(|\omega|^{p-2}\omega)|^2 \ge \left(1 + \frac{1}{(m-1)(p-1)^2}\right)|\nabla|\omega|^{p-1}|^2 , \qquad (22)$$
where $p \ge 2$.

Proof. When $p = 2, \omega$ is a 2-harmonic 1-form, i.e. harmonic 1-form, (22) is true. So we only need to prove the case for $p > 2$. We can choose a local orthonormal basis $e_1, \cdots, e_m$ with the dual basis $\theta_1, \cdots, \theta_m$ of M^m near a fixed point $q \in M$ such that $\nabla e_i\, e_j\, (q) = 0, \omega_1(q) = \omega(e_1)(q) = |\omega|(q)$ and $\omega(e_i) = \omega_i = 0$ for $i \ge 2$. Writing
$$\omega = \sum_{i=1}^m \omega_i\theta_i .$$
We have
$$d\omega = \sum_{i,j=1} \omega_{ij}\theta_j \wedge \theta_i$$

and

$$\delta(|\omega|^{p-2}\omega) = -|\omega|^{p-2}\sum_{i=1}^{m} [(p - 2)\nabla_i(\ln|\omega|)\omega_i + \omega_{ii}]$$

Since ω is a p-harmonic 1-form, that is, $d\omega = 0$ and $\delta(|\omega|^{p-2}\omega) = 0$, therefore

$$\omega_{ij} = \omega_{ji}$$

for $i, j = 1, \cdots, m$ and

$$\sum_{i=1}^{m} [(p - 2)\nabla_i(\ln|\omega|)\omega_i + \omega_{ii}] = 0$$

and

$$\nabla_{e_i}|\omega| = \nabla_i|\omega| = \nabla_i\left(\sqrt{\sum_{j=1}^{m} \omega_j^2}\right) = \frac{\sum \omega_j\omega_{ij}}{|\omega|} = \omega_{1i}\,.$$

At the point q, we compute,

$$\begin{aligned}
&|\nabla(|\omega|^{p-2}\omega)| - |\nabla|\omega|^{p-1}| \\
&= \sum_{i,j=1}^{m} |\omega|^{2(p-2)}\big[(p - 2)\nabla_i(\ln|\omega|)\omega_j + \omega_{ij}\big]^2 \\
&- \sum_{i,j=1}^{m} |\omega|^{2(p-2)}[(p - 2)\nabla_i(\ln|\omega|)|\omega| + \nabla_i|\omega|]^2 \\
&= \sum_{i,j=1}^{m} |\omega|^{2(p-2)}\big[(p - 2)\nabla_i(\ln|\omega|)\omega_j + \omega_{ij}\big]^2 \\
&- \sum_{i,j=1}^{m} |\omega|^{2(p-2)}[(p - 2)\nabla_i(\ln|\omega|)\omega_1 + \omega_{1i}]^2 \\
&\geq \sum_{i\neq 1} |\omega|^{2(p-2)}[(p - 2)\nabla_1(\ln|\omega|)\omega_i + \omega_{1i}]^2 \\
&+ \sum_{i\neq 1} |\omega|^{2(p-2)}[(p - 2)\nabla_i(\ln|\omega|)\omega_i + \omega_{ii}]^2 \\
&= \sum_{i\neq 1} |\omega|^{2(p-2)}[\omega_{1i}]^2 + \sum_{i\neq 1} |\omega|^{2(p-2)}[(p - 2)\nabla_i(\ln|\omega|)\omega_i + \omega_{ii}]^2 \\
&\geq \sum_{i\neq 1} |\omega|^{2(p-2)}[\omega_{1i}]^2 + \frac{1}{m - 1}|\omega|^{2(p-2)}\left[\sum_{i\neq 1} \big((p - 2)\nabla_i(\ln|\omega|)\omega_i + \omega_{ii}\big)\right]^2 \\
&= \sum_{i\neq 1} |\omega|^{2(p-2)}[\omega_{1i}]^2 + \frac{1}{m - 1}|\omega|^{2(p-2)}[-(p - 2)\nabla_1(\ln|\omega|)\omega_1 - \omega_{11}]^2 \\
&= \sum_{i\neq 1} |\omega|^{2(p-2)}[\omega_{1i}]^2 + (p - 1)^2\frac{1}{m - 1}|\omega|^{2(p-2)}\omega_{11}^2 \\
&\geq \sum_{i\neq 1} |\omega|^{2(p-2)}[\omega_{1i}]^2 + \frac{1}{m - 1}|\omega|^{2(p-2)}\omega_{11}^2
\end{aligned}$$

$$\geq \frac{1}{m-1}|\omega|^{2(p-2)}\sum_{i=1}^{m}\omega_{1i}^2$$
$$=\frac{1}{(m-1)(p-1)^2}|\nabla|\omega|^{p-1}|^2 .$$

This proves the Lemma.

Using Bochner's formula [20], we have the following results.

Lemma (4.2.11)[184]: Let ω be a p-harmonic 1-form on Riemannian manifold M^m. Then we have

$$\frac{1}{2}\triangle|\omega|^{2(p-2)}$$
$$= |\nabla(|\omega|^{p-2})\omega|^2 - \langle\delta d(|\omega|^{p-2}\omega),|\omega|^{p-2}\omega\rangle + |\omega|^{2(p-2)}Ric^M(\omega,\omega). \quad (23)$$

From (22) and (23), we have

$$|\omega|^{p-1}\triangle|\omega|^{p-1}$$
$$\geq \frac{1}{(m-1)(p-1)^2}|\nabla|\omega|^{p-1}|^2 - \langle\delta d(|\omega|^{p-2}\omega),|\omega|^{p-2}\omega\rangle, +|\omega|^{2(p-2)}Ric^M(\omega,\omega), \quad (24)$$

where ω is a p-harmonic 1-from on Riemannian manifold M^m.

Lemma (4.2.12)[184]: [188] Let M^m be an m-dimensional complete immersed minimal submanifold in a Hadamard manifold N with the sectional curvature satisfying $-k^2 \leq K_N \leq 0$ for some constant k. Then the Ricci curvature of M satisfies

$$Ric^M \geq (m-1)(|H|^2-k^2)-\frac{m-1}{m}|\Phi|^2-\frac{(m-2)\sqrt{m(m-1)}}{m}|H||\Phi| \quad (25)$$

Lemma (4.2.13)[184]: [29] Let M^m be an m-dimensional complete immersed minimal submanifold in N with nonpositive curvature, $m \geq 3$. Then for any $\phi \in C_0^\infty(M)$ we have

$$\left(\int_M|\phi|^{\frac{m}{m-1}}dv\right)^{\frac{m-1}{m}} \leq C'(m)\int_M(|\nabla\phi|+m|H||\varphi|)dv, \quad (26)$$

where $C'(m)$ depends only on m.

From Lemma (4.2.13), we have the following Sobolev inequality

$$\left(\int_M|\phi|^{\frac{2m}{m-2}}dv\right)^{\frac{m-2}{m}} \leq C(m)\int_M(|\nabla\phi|^2+|H|^2\varphi^2)dv, \quad (27)$$

where $C(m) > 0$ depends only on m.

Lemma (4.2.14)[184]: Let $f : M^m \to R$ be a smooth function on Riemannian manifold M, and ω be a closed 1-from on M. Then we

$|d(f\omega)| \leq |df||\omega|$.

Proof. We can choose a local orthonormal basis $e_1,\cdots,e_m$ with the dual basis $\theta_1,\cdots,\theta_m$. Writing

$$d(f\omega)=df\wedge\omega=\sum_{i,j=1}^{m}f_i\omega_j\theta_i\wedge\theta_j=\sum_{i<j}(f_i\omega_j-f_j\omega_i)\theta_i\wedge\theta_j .$$

Now we compute

$$|df|^2\,|\omega|^2 - |df \wedge \omega|^2 = \left(\sum_{i=1}^{m} f_i^2\right)\left(\sum_{j=1}^{m} \omega_j^2\right) - \sum_{i<j} \left(f_i\omega_j - f_j\omega_i\right)^2$$
$$= \sum_{i=1}^{m} f_i^2\omega_i^2 + \sum_{i\neq j}^{m} (f_i\omega_i)(f_j\omega_j) = (f_1\omega_1, \dots, f_m\omega_m) \geq 0$$

This proves the Lemma

Theorem (4.2.15)[184]: Let $x: M^m \to N, m \geq 3$, be an isometric immersion of a complete noncompact manifold M in a Hadamard manifold N with the sectional curvature satisfying $-k^2 \leq K_N \leq 0$ for some constan k. In the case there exist point $q \in M$ such that $K_N(q) \neq 0$, assume further that the first eigenvalue of the Laplace-Beltrami operator of M satisfies

$$\lambda_1(M) > \frac{(m-1)^2 p^2\left(k^2 - \lim_{\rho\to\infty} \inf |H|^2\right)}{4[(m-1)(p-1)+1]}.$$

where ρ stands for the distance in M from a fixed point. If M^m has finite total curvature, then the $dimH^{1,p}(M) < \infty$ for $p \geq 2$.

Proof. Assume that ω is a p-harmonic 1-from on M^m, i.e. ω ∈ H1,p(M). From (24) and (25), we have

$$|\omega|^{p-1} \triangle |\omega|^{p-1}$$
$$\geq -\langle \delta d(|\omega|^{p-2}\omega), |\omega|^{p-2}\omega\rangle + \frac{1}{(m-1)(p-1)^2} |\nabla|\omega|^{p-1}|^2$$
$$+(m-1)(|H|^2 - k^2)|\omega|^{2(p-1)} - \frac{m-1}{m} |\Phi|^2\,|\omega|^{2(p-1)}$$
$$-\frac{(m-2)\sqrt{m(m-1)}}{m} |H||\Phi||\omega|^{2(p-1)}$$
$$\geq -\langle \delta d(|\omega|^{p-2}\omega), |\omega|^{p-2}\omega\rangle + \frac{1}{m-1}\frac{4}{p^2} |\omega|^{p-2} \left|\nabla|\omega|^{\frac{p}{2}}\right|^2$$
$$+(m-1)(|H|^2 - k^2)|\omega|^{2(p-1)} - \frac{m-1}{m} |\Phi|^2|\omega|^{2(p-1)}$$
$$-\frac{(m-2)\sqrt{m(m-1)}}{m} |H||\Phi||\omega|^{2(p-1)}.$$

So we have

$$\begin{aligned} |\omega| \triangle |\omega|^{p-1} \geq& -\langle \delta d(|\omega|^{p-2}\omega), \omega\rangle + \frac{1}{m-1}\frac{4}{p^2} \left|\nabla|\omega|^{\frac{p}{2}}\right|^2 \\ &+(m-1)(|H|^2 - k^2)|\omega|^p - \frac{m-1}{m} |\Phi|^2\,|\omega|^p \\ &-\frac{(m-2)\sqrt{m(m-1)}}{m} |H||\Phi||\omega|^p . \end{aligned} \tag{28}$$

Fixed a point $x_0 \in M$ and denote by $\rho(x)$ the geodesic distance on M from x_0 to x. Let us choose $\eta \in C_0^\infty(M)$ satisfying

$$\eta = \begin{cases} 0 & \text{on } Bx_0(r_0) \cup \left(M \backslash Bx_0(2r)\right), \\ \rho(x_0,x) - r_0 & \text{on } Bx_0(r_0+1) \backslash Bx_0(r_0), \\ 1 & \text{on } Bx_0(r) \backslash Bx_0(r_0+1), \\ 2r - \rho(x_0,x)\, r & \text{on } Bx_0(2r) \backslash Bx_0(r), \end{cases}$$

where $r > r_0 + 1$ and r_0 will be determined later. Multiplying (28) by η 2 and integrating over on M, we have

$$-\int_M \eta^2 \nabla|\omega|\nabla|\omega|^{p-1} - 2\int_M \eta|\omega|\langle\nabla\eta, \nabla|\omega|^{p-1}\rangle + \int_M \eta^2 \langle\delta d(|\omega|^{p-2}\omega), \omega\rangle$$
$$\geq +\frac{1}{m-1}\frac{4}{p^2}\int_M \eta^2 \left|\nabla|\omega|^{\frac{p}{2}}\right|^2 + (m-1)\int_M (|H|^2 - k^2)|\omega|^p \eta^2 \quad (29)$$
$$-\frac{m-1}{m}\int_M |\Phi|^2 |\omega|^p \eta^2 - \frac{(m-2)\sqrt{m(m-1)}}{m}\int_M |H||\Phi||\omega|^p \eta^2 .$$

From Lemma (4.2.14), we have

$$\left|\int_M \eta^2 \langle\delta d(|\omega|^{p-2}\omega), \omega\rangle\right| = \left|\int_M \langle d(|\omega|^{p-2}\omega), d(\eta^2\omega)\rangle\right|$$
$$\leq \int_M |d(|\omega|^{p-2}\omega)||d(\eta^2\omega)| \leq 2\int_M \eta|d\eta||\omega|^2 |d|\omega|^{p-2}| \quad (30)$$
$$= \frac{4(p-2)}{p}\int_M \eta|\nabla\eta||\omega|^{\frac{p}{2}} \left|\nabla|\omega|^{\frac{p}{2}}\right|.$$

By direct computation, we get

$$-\int_M \eta^2 \nabla|\omega|\nabla|\omega|^{p-1} - 2\int_M \eta|\omega|\langle\nabla\eta, \nabla|\omega|^{p-1}\rangle$$
$$= -\frac{4(p-1)}{p^2}\int_M \eta^2 \left|\nabla|\omega|^{\frac{p}{2}}\right|^2 - \frac{4(p-1)}{p}\int_M \eta\langle\nabla\eta, \nabla|\omega|^{\frac{p}{2}}\rangle |\omega|^{\frac{p}{2}} \quad (31)$$
$$) \leq -\frac{4(p-1)}{p^2}\int_M \eta^2 \left|\nabla|\omega|^{\frac{p}{2}}\right|^2 + \frac{4(p-1)}{p}\int_M \eta|\nabla\eta||\omega|^{\frac{p}{2}} \left|\nabla|\omega|^{\frac{p}{2}}\right|.$$

From (29), (30) and (31), we have

$$0 \leq -\frac{4(p-1)}{p^2}\int_M \eta^2 \left|\nabla|\omega|^{\frac{p}{2}}\right|^2 + \frac{4(2p-3)}{p}\int_M \eta|\nabla\eta||\omega|^{\frac{p}{2}} \left|\nabla|\omega|^{\frac{p}{2}}\right|$$
$$-\frac{4}{p^2(m-1)}\int_M \eta^2 \left|\nabla|\omega|^{\frac{p}{2}}\right|^2 + (m-1)\int_M (k^2 - |H|^2)|\omega|^p \eta^2 \quad (32)$$
$$+\frac{m-1}{m}\int_M |\Phi|^2 |\omega|^p \eta^2 + \frac{(m-2)\sqrt{m(m-1)}}{m}\int_M |H||\Phi||\omega|^p \eta^2$$

For $\varepsilon_1 > 0, \varepsilon_2 > 0$, we apply the Cauchy-Schwarz inequality, we have

$$\left[\frac{4}{p^2}\frac{(m-1)(p-1)+1}{m-1} - \frac{4(2p-3)}{p}\varepsilon_1\right]\int_M \eta^2 \left|\nabla|\omega|^{\frac{p}{2}}\right|^2$$
$$\leq \frac{2p-3}{p}\frac{1}{\varepsilon_1}\int_M |\omega|^p |\nabla\eta|^2 \quad (33)$$
$$+\int_M \left[(m-1)k^2 + \left(-(m-1) + \frac{(m-2)\varepsilon_2\sqrt{m(m-1)}}{2m}\right)|H|^2\right]\eta^2 |\omega|^p$$
$$+\left(\frac{m-1}{m} + \frac{(m-2)\sqrt{m(m-1)}}{2m\varepsilon_2}\right)\int_M |\Phi|^2 |\omega|^p \eta^2 .$$

On the other hand, since $m \geq 3$, we use Hölder, Sobolev inequality (27), and Cauchy-Schwartz inequalities to obtain

$$
\begin{aligned}
\int_M |\Phi|^2 |\omega|^p \eta^2 &\leq \left(\int_{\mathrm{supp}(\eta)} |\Phi|^m \right)^{\frac{2}{m}} \left(\int_M \left(\eta |\omega|^{\frac{p}{2}}\right)^{\frac{2m}{m-2}} \right)^{\frac{m-2}{m}} \\
&\leq C(m) \left(\int_{\mathrm{supp}(\eta)} |\Phi|^m \right)^{\frac{2}{m}} \int_M \left(\left| \nabla \left(\eta |\omega|^{\frac{p}{2}} \right) \right|^2 + |H|^2 \eta^2 |\omega|^p \right) \\
&\leq \phi(\eta) \left[(1+\varepsilon_3) \int_M \eta^2 \left| \nabla |\omega|^{\frac{p}{2}} \right| + \left(1 + \frac{1}{\varepsilon_3}\right) \int_M |\omega|^p |\nabla \eta|^2 \right] \quad (34) \\
&\quad + \phi(\eta) \int_M \eta^2 |\omega|^p |H|^2
\end{aligned}
$$

for $\varepsilon_3 > 0$, where $\phi(\eta) = C(m) \left(\int_{\mathrm{supp}(\eta)} |\Phi|^m \right)^{\frac{2}{m}}$ and $C(m) > 0$ is the constant in the Hoffman-Spruck inequality. From (33) and (34), we have

$$
\begin{aligned}
&A \int_M \eta^2 \left| \nabla |\omega|^{\frac{p}{2}} \right|^2 + B \int_M \eta^2 |\omega|^p |H|^2 \\
&\quad \leq C \int_M |\omega|^p |\nabla \eta|^2 + (m-1)k^2 \int_M \eta^2 |\omega|^p \qquad (35)
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \frac{4}{p^2} \frac{(m-1)(p-1)+1}{m-1} - \frac{4(2p-3)}{p} \varepsilon_1 \\
&\quad - \left(\frac{m-1}{m} + \frac{(m-2)\sqrt{m(m-1)}}{2m\varepsilon_2} \right) \phi(\eta)(1+\varepsilon_3) \\
B &= (m-1) - \frac{(m-2)\varepsilon_2 \sqrt{m(m-1)}}{2m} \\
&\quad - \phi(\eta) \left(\frac{m-1}{m} + \frac{(m-2)\sqrt{m(m-1)}}{2m\varepsilon_2} \right) \\
C &= \frac{2p-3}{p} \frac{1}{\varepsilon_1} + \phi(\eta) \left(1 + \frac{1}{\varepsilon_3}\right) \left(\frac{m-1}{m} + \frac{(m-2)\sqrt{m(m-1)}}{2m\varepsilon_2} \right)
\end{aligned}
$$

We choose $0 < \varepsilon < \min\left\{ \frac{-(mp+5p-12) + \sqrt{(m-1)2(p^2+16p-16) + 8(m-1)(2p^2-3p+2) + 16(2p-3)^2}}{2(m-1)p}, \frac{1}{2} \right\}$, $\varepsilon_2 = \varepsilon_2(\varepsilon)$ and a positive constant $\Lambda(\varepsilon) > 0$ satisfying:

$$
\begin{aligned}
&\frac{4(2p-3)}{p} \varepsilon + (m-1)\varepsilon(1+\varepsilon) < \frac{4[(m-1)(p-1)+1]}{(m-1)p^2} \\
&\frac{(m-2)\varepsilon_2 \sqrt{m(m-1)}}{m} < (m-1)\varepsilon \qquad (36) \\
&\left(\frac{m-1}{m} + \frac{(m-2)\sqrt{m(m-1)}}{m\varepsilon_2} \right) \Lambda_2 < (m-1)\varepsilon.
\end{aligned}
$$

Since M has finite total curvature, we can fix r_1 large enough such that

$$\left(\int_{M\setminus Bx_0\,(r_1)} |\phi|^m\right)^{\frac{2}{m}} < \frac{\Lambda}{C(m)}\,. \tag{37}$$

Take $r_0 > r_1$, thus $\text{supp}(\eta) \subset M\setminus Bx_0\,(r_1)$ and $\phi(\eta) = C(m)\left(\int_{\text{supp}(\eta)} |\Phi|^m\right)^{\frac{2}{m}} < \Lambda$. Choosing $0 < \varepsilon_1 < \varepsilon$ and $0 < \varepsilon_3 < \varepsilon$, we have

$$A > \tilde{A} = \frac{4[(m-1)(p-1)+1]}{(m-1)p^2} - \frac{4(2p-3)}{p}\,\varepsilon - (m-1)\varepsilon(1+\varepsilon) > 0$$

$$B > \tilde{B} = (m-1)(1-2\varepsilon) > 0$$

$$C < \tilde{C} = \frac{(2p-3)}{p}\frac{1}{\varepsilon_1} + \Lambda^2\left(1+\frac{1}{\varepsilon_3}\right)\left(\frac{m-1}{m} + \frac{(m-2)\sqrt{m(m-1)}}{m\varepsilon_2}\right)$$

From (35) and the above inequalities, we have

$$\tilde{A}\int_M \eta^2\left|\nabla|\omega|^{\frac{p}{2}}\right|^2 + \tilde{B}\int_M \eta^2\,|\omega|^p\,|H|^2$$
$$\leq \tilde{C}\int_M |\omega|^p\,|\nabla\eta|^2 + (m-1)k^2\int_M \eta^2\,|\omega|^p\,. \tag{38}$$

If $k = 0$, from (38) and the definition of η, we have

$$\tilde{A}\int_M \eta^2\left|\nabla|\omega|^{\frac{p}{2}}\right|^2 + \tilde{B}\int_M \eta^2\,|\omega|^p\,|H|^2 \leq \tilde{C}\int_M |\omega|^p\,|\nabla\eta|^2\,. \tag{39}$$

From (27) and the Cauchy-Schwarz inequality, we have

$$C(m)^{-1}\left(\int_M \left(\eta|\omega|^{\frac{p}{2}}|\right)^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} \leq \int_M \left(\left|\nabla\left(\eta|\omega|^{\frac{p}{2}}\right)\right|^2 + |H|^2\,\eta^2\,|\omega|^p\right)$$
$$\leq (1+\varepsilon_5)\int_M \eta^2\left|\nabla|\omega|^{\frac{p}{2}}\right|^2 + \left(1+\frac{1}{\varepsilon_5}\right)\int_M |\omega|^p\,|\nabla\eta|^2 + \int_M \eta^2\,|H|^2\,|\omega|^p \tag{40}$$

for any ε5 > 0. From (39) and (40), we have

$$C(m)^{-1}\left(\int_M \left(\eta|\omega|^{\frac{p}{2}}\right)^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} \leq \left(1 - \frac{(1+\varepsilon_5)\tilde{B}}{\tilde{A}}\right)\int_M \eta^2\,|H|^2\,|\omega|^p$$
$$+\left(1+\frac{1}{\varepsilon_5} + (1+\varepsilon_5)\frac{\tilde{C}}{\tilde{A}}\right)\int_M |\omega|^p\,|\nabla\eta|^2\,. \tag{41}$$

If $k \neq 0$. We recall that the first eigenvalue $\lambda_1(M)$ of the Laplacian of M satisfies

$$\lambda_1(M)\int_M \varphi^2 \leq \int_M |\nabla\varphi|^2 \tag{42}$$

for any $\varphi \in C_0^\infty(M)$. Applying (69) with $\varphi = \eta|\omega|^{\frac{p}{2}}$, we have

$$\lambda_1(M)\int_M \eta^2\,|\omega|^p \leq \int_M \left|\nabla\left[\eta|\omega|^{\frac{p}{2}}\right]\right|^2$$
$$= \int_M \left[\eta^2\left|\nabla|\omega|^{\frac{p}{2}}\right| + 2\eta|\omega|^{\frac{p}{2}}\,\langle\nabla\eta, \nabla|\omega|^{\frac{p}{2}}\rangle + |\omega|^p\,|\nabla\eta|^2\right]$$

By using the Cauchy-Schwartz inequality, we have for $\varepsilon_4 > 0$

$$\lambda_1(M)\int_M \eta^2\,|\omega|^p \leq \int_M \left[(1+\varepsilon_4)\eta^2\left|\nabla|\omega|^{\frac{p}{2}}\right| + \left(1+\frac{1}{\varepsilon_4}\right)|\omega|^p\,|\nabla\eta|^2\right] \tag{43}$$

From (38) and (70), we have

$$\left[\frac{\tilde{A}\lambda_1(M)}{1 + \varepsilon_4} - \left[(m - 1)k^2 - \tilde{B} \inf_{\mathrm{supp}(\eta)} |H|^2\right]\right] \int_M \eta^2 |du|^p$$
$$\leq \left[\tilde{C} + \frac{\tilde{A}}{\varepsilon_4}\right] \int_M |du|^p |\nabla\eta|^2 \qquad (44)$$

Note that

$$\left[\frac{\tilde{A}\lambda_1(M)}{1 + \varepsilon_4} - \left[(m - 1)k^2 - \tilde{B} \inf_{\mathrm{supp}(\eta)} |H|^2\right]\right]$$
$$> \left[\frac{4[(m - 1)(p - 1) + 1]}{(m - 1)p^2} - \frac{4(2p - 3)}{p} \varepsilon - (m - 1)\varepsilon(1 + \varepsilon)\right]\frac{\lambda_1(M)}{1 + \varepsilon_4}$$
$$-(m - 1)\left(k^2 - (1 - 2\varepsilon) \inf_{\mathrm{supp}(\eta)} |H|^2\right)$$

Thus, if $\lambda_1(M) > \frac{(m-1)2p^2\left(k^2 - \inf_{\rho\to\infty} |H|^2\right)}{4[(m-1)(p-1)+1]}$, then we can choose $\varepsilon, \varepsilon_4$ small enough and depending on $m, p, k_2, \lambda_1(M)$ and $\inf_{\mathrm{supp}(\eta)} |H|^2$, so that $\left[\frac{\tilde{A}\lambda_1(M)}{1+\varepsilon_4} - \left[(m - 1)k^2 - \tilde{B} \inf_{\mathrm{supp}(\eta)} |H|^2\right]\right] > 0$. Then we have

$$\int_M \eta^2 |\omega|^p \leq \widetilde{D} \int_M |\omega|^p |\nabla\eta|^2 , \qquad (45)$$

where De is a positive constant. From (38) and (40), we have

$$C(m)^{-1} \left(\int_M \left(\eta|\omega|^{\frac{p}{2}}\right)^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} \leq \left(1 - (1 + \varepsilon_5)\frac{\tilde{B}}{\tilde{A}}\right) \int_M \eta^2 |H|^2 |\omega|^p$$
$$+\left(1 + \frac{1}{\varepsilon_5} + (1 + \varepsilon_5) \frac{\tilde{C}}{\tilde{A}}\right) \int_M |\omega|^p |\nabla\eta|^2$$
$$+ (1 + \varepsilon_5)(m - 1)k^2 \frac{1}{\tilde{A}} \int_M \eta^2 |\omega|^p . \qquad (46)$$

Since $m \geq 3$ and $p \geq 2$, we can choose ε and ε_5 small enough such that

$$(1 + \varepsilon_5)\frac{\tilde{B}}{\tilde{A}} > 1 \qquad (47)$$

From (41), (45), (46) and (47), we have

$$C(m)^{-1} \left(\int_M \left(\eta|\omega|^{\frac{p}{2}}|\right)^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} \leq \tilde{E}(m,p)\int_M |\omega|^p |\nabla\eta|^p \qquad (48)$$

for some constant $\tilde{E}(m,p) > 0$, for all $\omega \in H^{1,p}(M)$.
It follows from the definition of η and (48), we have

$$\left(\int_{Bx_0(r)\setminus Bx_0(r_0+1)} \left(\eta|\omega|^{\frac{p}{2}}|\right)^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} \leq C_1 \int_{Bx_0(r_0+1)\setminus Bx_0(r_0)} |\omega|^p \quad (49)$$
$$+\frac{C_1}{r^2} \int_{Bx_0(2r)\setminus Bx_0(r)} |\omega|^p .$$

Since |ω| ∈ L p (M), taking r → ∞, we have

$$\left(\int_{Bx_0\,(r) \setminus Bx_0\,(r_0+1)} \left(\eta|\omega|^{\frac{p}{2}}|\right)^{\frac{2m}{m-2}} \right)^{\frac{m-2}{m}} \leq C_1 \int_{Bx_0\,(r_0+1) \setminus Bx_0\,(r_0)} |\omega|^p \qquad (50)$$

It follows from the Hölder inequality that

$$\int_{Bx_0\,(r_0+2) \setminus Bx_0\,(r_0+1)} |\omega|^p$$

$$\leq Vol(Bx_0\,(r_0 + 2)) \left(\int_{Bx_0\,(r) \setminus Bx_0\,(r_0+1)} \left(\eta|\omega|^{\frac{p}{2}}|\right)^{\frac{2m}{m-2}} \right)^{\frac{m-2}{m}}. \qquad (51)$$

From (50) and (51), we have

$$\int_{Bx_0\,(r_0+2)} |\omega|^p \leq C_2(\int_{Bx_0\,(r_0+1)} |\omega|^p , \qquad (52)$$

where C_2 depends on Vol($Bx_0\,(r_0 + 2)$), m and p. From (28), we have

$$|\omega| \triangle |\omega|^{p-1} \geq -\langle \delta d(|\omega|^{p-2}\omega), \omega\rangle + \frac{4}{p^2(m - 1)} \left|\nabla|\omega|^{\frac{p}{2}}\right|^2 - \alpha|\omega|^p , (53)$$

where $\alpha : M \to [0, \infty)$ is the function given by

$$\alpha = \left|(m - 1)(|H|^2 - k^2) - \frac{m - 1}{m} |\Phi|^2 - \frac{(m - 2)\sqrt{m(m - 1)}}{m} |H||\Phi|\right|.$$

Fix $x \in M$ and take $\xi \in C_0^1\left(B_1(x)\right)$. Multiply both sides of (53) by $\xi^2 |\omega|^{\frac{pq}{2}-p}$, with $q \geq 2$, and integrating by parts we obtain

$$-\frac{4(p - 1)}{p} \int_{Bx(1)} \xi|\omega|^{\frac{pq}{2}-\frac{p}{2}} \langle \nabla\xi, \nabla|\omega|^{\frac{p}{2}}\rangle$$

$$\geq \left[\frac{2(p - 1)(q - 1)}{p} + \frac{4}{p^2(m - 1)}\right] \int_{B_x(1)} |\omega|^{\frac{pq}{2}-p} \left|\nabla|\omega|^{\frac{p}{2}}\right|^2 \xi^2$$

$$- \alpha \int_{B_x(1)} \xi^2 |\omega|^{\frac{pq}{2}} - \int_{B_x(1)} \langle d(|\omega|^{p-2}\omega), d\left(\xi^2 |\omega|^{\frac{pq}{2}-p}\omega\right)\rangle. (54)$$

From Lemma (4.2.14) and Cauchy-Schwatz inequality, we have

$$\int_{B_x(1)} \left|\langle d(|\omega|^{p-2}\omega), d\left(\xi^2 |\omega|^{\frac{pq}{2}-p}\omega\right)\rangle\right| \leq \int_{B_x(1)} \left|d(|\omega|^{p-2}\omega), d\left(\xi^2 |\omega|^{\frac{pq}{2}-p}\omega\right)\right|$$

$$\leq \int_{B_x(1)} |\nabla|\omega|^{p-2}||\omega|^2 \left|\left[d(\xi^2)|\omega|^{\frac{pq}{2}-p} + \xi^2\, d|\omega|^{\frac{pq}{2}-p}\right]\right| \qquad (55)$$

$$\leq \frac{4(p - 2)}{p} \int_{B_x(1)} \xi^2 |\omega|^{\frac{pq}{2}-\frac{p}{2}} |\nabla\xi| \left|\nabla|\omega|^{\frac{p}{2}}\right|$$

$$+ \frac{2(p - 2)(q - 2)}{p} \int_{B_x(1)} \xi^2 |\omega|^{\frac{pq}{2}-p} \left|\nabla|\omega|^{\frac{p}{2}}\right|^2$$

$$\leq \frac{2}{p^2(m - 1)} \int_{B_x(1)} |\omega|^{\frac{pq}{2}-p} \left|\nabla|\omega|^{\frac{p}{2}}\right|^2 \xi^2 + 2(p - 2)^2(m - 1) \int_{B_x(1)} |\nabla\xi|^2|\omega|^{\frac{pq}{2}}$$

$$+ \frac{2(p - 2)(q - 2)}{p} \int_{B_x(1)} \xi^2 |\omega|^{\frac{pq}{2}-p} \left|\nabla|\omega|^{\frac{p}{2}}\right|^2$$

and

$$-\frac{4(p-1)}{p}\int_{B_x(1)} \xi\ |\omega|^{\frac{pq}{2}-\frac{p}{2}} \langle \nabla\xi, \nabla|\omega|^{\frac{p}{2}}\rangle$$
$$\leq \frac{2}{p^2(m-1)}\int_{B_x(1)} |\omega|^{\frac{pq}{2}-p} \left|\nabla|\omega|^{\frac{p}{2}}\right|^2 \xi^2$$
$$+2(p-1)^2(m-1)\int_{B_x(1)} (1)|\nabla\xi|^2 |\omega|^{\frac{pq}{2}}. \tag{56}$$

From (54), (55) and (56), we have

$$\left[\frac{2(p-1)(q-1)}{p} - \frac{2(p-2)(q-2)}{p}\right]\int_{B_x(1)} |\omega|^{\frac{pq}{2}-p} \left|\nabla|\omega|^{\frac{p}{2}}\right|^2 \xi^2$$
$$\leq \alpha\int_{B_x(1)} \xi^2 |\omega|^{\frac{pq}{2}} + [2(p-1)^2 + 2(p-2)^2](m-1)\int_{B_x(1)} |\nabla\xi|^2 |\omega|^{\frac{pq}{2}}. \tag{57}$$

By using the Cauchy-Schwarz inequality, we have

$$\int_{B_x(1)} \left|\nabla\left(\xi|\omega|^{\frac{pq}{4}}\right)\right|^2 = \int_{B_x(1)} |\omega|^{\frac{pq}{2}} |\nabla\xi|2 + \frac{q^2}{4}\int_{B_x(1)} \xi^2 |\omega|^{\frac{pq}{2}-p} \left|\nabla|\omega|^{\frac{p}{2}}\right|^2$$
$$+q\int_{B_x(1)} \xi|\omega|^{\frac{pq}{2}-\frac{p}{2}} \langle\nabla\xi, \nabla|\omega|^{\frac{p}{2}}\rangle \tag{58}$$
$$\leq (1+q)\int_{B_x(1)} |\omega|^{\frac{pq}{2}} |\nabla\xi|^2 + \frac{q}{4}(q+1)\int_{B_x(1)} \xi^2 |\omega|^{\frac{pq}{2}-p} \left|\nabla|\omega|^{\frac{p}{2}}\right|^2.$$

From (57) and (58), we have

$$\int_{B_x(1)} \left|\nabla\left(\xi|\omega|^{\frac{pq}{4}}\right)\right|^2 \leq C_2\int_{B_x(1)} |\omega|^{\frac{pq}{2}} |\nabla\xi|^2 + C_3\int_{B_x(1)} \alpha\xi_2 |\omega|^{\frac{pq}{2}}, \tag{59}$$

where

$$C_2 = 1 + q + \frac{q}{4}(q+1)[2(p-1)^2 + 2(p-2)^2](m-1)$$
$$\left[\frac{2(p-1)(q-1)}{p} - \frac{2(p-2)(q-2)}{p}\right]^{-1} \leq C(p)mq,$$
$$C_3 = \frac{q}{4}(q+1)\left[\frac{2(p-1)(q-1)}{p} - \frac{2(p-2)(q-2)}{p}\right]^{-1} \leq C(p)q,$$

where $C(p)$ is a positive constant depending only on p. Applying (27) to $\xi|\omega|^{\frac{pq}{4}}$ and using (59), we have

$$\left(\int_{B_x(1)} \left(\xi|\omega|^{\frac{pq}{4}}\right)^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}}$$
$$\leq C(m)\int_{B_x(1)} \left|\nabla\left(\xi|\omega|^{\frac{pq}{4}}\right)\right|^2 + C(m)\int_{B_x(1)} |H|^2 \xi^2 |\omega|^{\frac{pq}{2}}$$
$$\leq \int_{B_x(1)} [C(m)C_3\alpha + C(m)|H|^2]\xi^2 |\omega|^{\frac{pq}{2}} + C(m)C_2\int_{B_x(1)} |\omega|^{\frac{pq}{2}} |\nabla\xi|^2$$

so we have

$$\left(\int_{B_x(1)} \left(\xi|\omega|^{\frac{pq}{4}}\right)^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} \leq qC_4 \int_{B_x(1)} [\xi^2 + |\nabla\xi|^2]|\omega|^{\frac{pq}{2}}, \qquad (60)$$

for a constant $C_4 > 0$ depending m, p, $\mathrm{Vol}(B_x(1))$, $\sup_{B_x(1)} \alpha$ and $\sup_{B_x(1)} |H|^2$. Given an integer $k \geq 0$, we set $q^k = \frac{2m^k}{(m-2)^k}$ and $\rho_k = \frac{1}{2} + \frac{1}{2^{k+1}}$. Take a function $\xi^k \in C_0^\infty\left(B_x(\rho_k)\right)$ satisfying $\xi_k \geq 0, \xi_k = 1$ on $B_x(\rho_{k+1})$ and $|\nabla\xi_k| \leq 2^{k+3}$. Replacing q and ξ in (60) by q_k and ξ_k respectively, we have

$$\left(\int_{B_x(\rho_{k+1})} |\omega|^{\frac{pq_{k+1}}{2}}\right)^{\frac{1}{q_{k+1}}} \leq (q_k C_4 4^{k+4})^{\frac{1}{q_k}} \left(\int_{B_x(\rho_k)} |\omega|^{\frac{pq_k}{2}}\right)^{\frac{1}{q_k}}. \quad (61)$$

Applying the Moser iteration to (61), we conclude that

$$|\omega|^p(x) \leq \|\omega\|^p_{L^\infty\left(B_x\left(\frac{1}{2}\right)\right)} \leq C_5 \int_{B_x(1)} |\omega|^p \qquad (62)$$

for a constant $C_5 > 0$ depending only on m, p, $\mathrm{Vol}(B_x(1))$, $\sup_{B_x(1)} \alpha$ and $\sup_{B_x(1)} |H|^2$. Take $x \in B_{x_0}(r_0 + 1)$ such that

$$|\omega|^p(x) = \sup_{B_{x_0}(r_0+1)} |\omega|^p. \qquad (63)$$

From (62) and (63), we have

$$\sup_{B_{x_0}(r_0+1)} |\omega|^p \leq C_5 \int_{B_{x_0}(r_0+2)} |\omega|^p. \qquad (64)$$

From (52) and (64), we have

$$\sup_{B_{x_0}(r_0+1)} |\omega|^p \leq C_6 \int_{B_{x_0}(r_0+1)} |\omega|^p, \qquad (65)$$

where $C_6 > 0$ is a constant depending on $m, p, \mathrm{Vol}(B_x(r_0 + 2))$, $\sup_{B_x(r_0+2)} \alpha$ and $\sup_{B_x(r_0+2)} |H|^2$.

Finally, let V be any finite-dimensional subspace of $H^{1,p}(M)$. From Lemma (4.2.9), there exists $\omega \in V$ such that

$$\frac{dimV}{Vol\left(B_{x_0}(r_0 + 1)\right)} \int_{B_{x_0}(r_0+1)} |\omega|^p \leq \min\left\{C_p\binom{m}{q}, dimV\right\} \sup_{B_{x_0}(r_0+1)} |\omega|^p. \qquad (66)$$

From (65) and (66), we have $dimV \leq C_7$, where $C_7 > 0$ depends only on on $m, p, Vol(B_x(r_0 + 2))$, $\sup_{B_x(r_0+2)} \alpha$ and $\sup_{B_x(r_0+2)} |H|^2$. This implies that $H^{1,p}(M)$ has finite dimension.

From the proof of the Theorem (4.2.15), we obtain the following result:

Theorem (4.2.16)[184]: Let $x : M^m \to N, m \geq 3$, be an isometric immersion of a complete noncompact manifold M ia a Hadamard manifold N with the sectional curvature satisfying $-k^2 \leq K_N \leq 0$ for some constant k. In the case there exists point $q \in M$ such that $K_N(q) \neq 0$, assume further that the first eigenvalue of the Laplace-Beltrami operator of M satisfies

$$\lambda_1(M) > \frac{(m-1)^2p^2(k^2 - \inf|H|^2)}{4[(m-1)(p-1)+1]}.$$

If there exists a positive constant Λ such that $\|\Phi\|_{L^m(M)} < \Lambda$, then $H^{1,p}(M) = \{0\}$ for p ≥ 2. Furthermore, if $k = 0$, then Λ depends only on m, p; otherwise, Λ depends only on m, k, p and $\inf|H|$.

Proof. Consider $0 < \varepsilon < \min\left\{-(mp+5p-12) + \frac{\sqrt{(m-1)^2(p^2+16p-16)+8(m-1)(2p^2-3p+2)+16(2p-3)^2}}{2(m-1)p}, \frac{1}{2}\right\}$, $\varepsilon_2 = \varepsilon_2(\varepsilon)$ and a positive constant $\Lambda(\varepsilon) > 0$ as given in (36). Assume that $\|\Phi\|_{L^m(M)} < \Lambda(\varepsilon)$. For a point x_0 and take a cut-off function η satisfying $0 \leq \eta \leq 1, \eta = 1$ on $B_{x_0}(r), \eta = 0$ on $M\backslash B_{x_0}(2r)$ and $|d\eta| \leq \frac{c}{r}$, where c is a positive constant. We can proceed similarly as in the proof of (38)

$Ae\ Z\ M\ \eta\ 2\ |\nabla|\omega|\ p\ 2\ |\ 2\ +\ Be\ Z\ M\ \eta\ 2\ |\omega|\ p\ |H|\ 2$

$\leq\ Ce\ Z\ M\ |\omega|\ p\ |\nabla\eta|\ 2\ +\ (m\ -\ 1)k\ 2\ Z\ M\ \eta\ 2\ |\omega|\ p\ .$ (67)

If $k = 0$, from (67) and the definition of η, we have

$$\tilde{A}\int_{B_{x_0}(r)} \left|\nabla|\omega|^{\frac{p}{2}}\right|^2 + \tilde{B}\int_{B_{x_0}(r)} |\omega|^p\,|H|^2 \leq \tilde{C}\frac{c^2}{r^2}\int_M |\omega|^p \quad (68)$$

Taking r → ∞, we have $\left|\nabla|\omega|^{\frac{p}{2}}\right| = |H||\omega|^{\frac{p}{2}} = 0$. Then |ω| is constant. If ω is not identically zero, then $H = 0$. Since N is a Hadamard manifold, we know that M has infinite volume, which is a contradiction, since $\int_M |\omega|^p < \infty$. So we obtain that $\omega = 0$.

If $k \neq 0$. We recall that the first eigenvalue $\lambda_1(M)$ of the Laplacian of M satisfies

$$\lambda_1(M)\int_M \varphi^2 \leq \int_M |\nabla\varphi|^2 \quad (69)$$

for any $\varphi \in C_0^\infty(M)$. Applying (69) with $\varphi = \eta|\omega|^{\frac{p}{2}}$, we have

$$\lambda_1(M)\int_M \eta^2\,|\omega|^p \leq \int_M \left|\nabla\left[\eta|\omega|^{\frac{p}{2}}\right]\right|^2$$

$$= \int_M \left[\eta^2\left|\nabla|\omega|^{\frac{p}{2}}\right| + 2\eta|\omega|^{\frac{p}{2}}\langle\nabla\eta, \nabla|\omega|^{\frac{p}{2}}\rangle + |\omega|^p\,|\nabla\eta|^2\right]$$

By using the Cauchy-Schwartz inequality, we have for $\varepsilon_4 > 0$

$\lambda1(M)\ Z\ M\ \eta\ 2\ |\omega|\ p$

$\leq\ Z\ M\ [(1\ +\ \varepsilon8)\eta\ 2\ |\nabla|\omega|\ p\ 2\ |\ +\ (1\ +\ 1\ \varepsilon8\ )|\omega|\ p\ |\nabla\eta|\ 2\ ]$ (70)

From (38) and (70), we have

$$\left[\frac{\tilde{A}\lambda_1(M)}{1+\varepsilon_8} - [(m-1)k^2 - \tilde{B}\inf|H|^2]\right]\int_M \eta^2\,|du|^p$$

$$\leq \left[\tilde{C} + \frac{\tilde{A}}{\varepsilon_8}\right]\int_M |du|^p\,|\nabla\eta|^2 \quad (71)$$

Note that

$$\left[\frac{\tilde{A}\lambda_1(M)}{1+\varepsilon_8} - [(m-1)k^2 - \tilde{B}\inf|H|^2]\right] >$$

$$\left[\frac{4[(m-1)(p-1)+1]}{(m-1)p^2} - \frac{4(2p-3)}{p}\varepsilon - (m-1)\varepsilon(1+\varepsilon)\right]\frac{\lambda_1(M)}{1+\varepsilon_8}$$

$$-(m-1)(k^2-(1-2\varepsilon)\inf|H|^2)$$

Thus, if $\lambda_1(M) > \frac{(m-1)^2p^2(k^2-\inf|H|^2)}{4[(m-1)(p-1)+1]}$., then we can choose $\varepsilon, \varepsilon_8$ small enough and depending on $m, p, k^2, \lambda_1(M)$ and $\inf|H|^2$, so that $\left[\frac{\tilde{A}\lambda_1(M)}{1+\varepsilon_8} - [(m-1)k^2 - \tilde{B}\inf|H|^2]\right] > 0$. Then we have

$$\int_M \eta^2 |\omega|^p \le \tilde{D}\int_M |\omega|^p |\nabla\eta|^2 \qquad (72)$$

From the definition of η and (72), we have

$$\int_{B_{x_0}(r)} |\omega|^p \le \tilde{D}\frac{c^2}{r^2}\int_M |\omega|^p \qquad (73)$$

Taking $r \to \infty$, we have $\omega = 0$.

Section (4.3): Minimal Hypersurfaces with Finite

It is well-known that a complete oriented stable minimal surface in $\mathbb{R}^3$ must be a plane, which was independently proved by do Carmo–Peng [3] and Fischer-Colbrie–Schoen [191]. Recall that a minimal submanifold is said to be stable if the second variation of its volume functional is always nonnegative for all normal variations with compact support. Later it was proved by Fischer-Colbrie [4] and Gulliver [7] that, for a complete oriented minimal surface in$\mathbb{R}^3$, the condition that it has finite index is equivalent to the condition that it has finite total curvature. This shows that a complete oriented minimal surface with finite index in $\mathbb{R}^3$ must have finitely many ends. Furthermore, do Carmo–da Silveira [191] proved the same result for Bryant surfaces in the 3-dimensional hyperbolic space. Recall that a surface with constant mean curvature 1 in the 3-dimensional hyperbolic space is called a Bryant surface.

In higher-dimensional cases, Cao–Shen–Zhu [2] proved that an $n(\geq 3)$-dimensional complete stable minimal hypersurface in $\mathbb{R}^{n+1}$ has only one end. Later Li–Wang [1] generalized this result to minimal hypersurfaces with finite index in$\mathbb{R}^{n+1}$. In fact, they proved that if M is a complete minimal hypersurface with finite index in $\mathbb{R}^{n+1}$for$n \geq 3$, then the dimension of the space ofL^2 harmonic 1-forms on M is finite. Since the number of ends of M is controlled by the dimension of the space of L^2 harmonic 1-forms [164], they were able to show that M has finitely many ends.

As mentioned in the above, L^2 harmonic 1-forms on a minimal hypersurface are useful to analyze the topology of the hypersurface. Palmer [170] proved that if there exists a codimension one cycle in a complete minimal hypersurface M in$\mathbb{R}^{n+1}$which does not separate M, then M must be unstable. In [168], Miyaoka proved that there is no nontrivial L^2 harmonic 1-forms on a complete stable minimal hypersurface in $\mathbb{R}^{n+1}$. In 2005, Pigola–Rigoli–Setti [172] obtained general Liouville type results and vanishing theorems on the L^2cohomology of stable minimal hypersurfaces. See also [164],[173] for a survey related to this subject. The second-named author in [177] obtained vanishing theorems for L^2 harmonic 1-forms on a complete stable minimal hypersurface M in hyperbolic space$\mathbb{H}^{n+1}$ with an assumption on the first eigenvalue of then Laplace. Beltrami operator on$M$. This result was extended to the case of stable minimal hypersurfaces in a Riemannian manifold with bounded sectional curvature [154]. WhileL^2 theory on manifold has been well understood, general L^p theory is relatively less developed (see [176]). We obtained vanishing results about L^p harmonic 1-forms on complete noncompact stable minimal hypersurfaces under the assumption on the bottom of the spectrum of the Laplace operator [141]. Recall that the bottom of the spectrum of the Laplace operator on a complete manifold M is defined by

$$\lambda_1(M) := \inf_{\Omega} \lambda_1(\Omega),$$

where the infimum is taken over all compact domains in M and $\lambda_1(\Omega)$ denotes the first eigenvalue of the Dirichlet boundary value problem on a compact domain $\Omega \subset M$

$$\begin{cases} \Delta f = -\lambda f & \text{in } \Omega \\ f = 0 & \text{on } \partial\Omega. \end{cases}$$

Furthermore, Cavalcante-Mirandola-Vitorio [15] proved that if M is a complete noncompact submanifold in a Cartan-Hadamard manifold with finite total curvature under additional assumption on $\lambda_1(M)$, then the space of L^2 harmonic 1-forms on M is finite dimensional. Later Han.Pan [184] generalized their result to L^p p-harmonic 1-forms on submanifolds in a Cartan-Hadamard manifold.

Motivated by Li-Wang's work [1], we investigate the space of L^p harmonic 1-forms on minimal hypersurfaces with finite index in a complete simply connected Riemannian manifold with nonpositive sectional curvature. We prove that if M is a complete minimal hypersurface with finite index in a complete simply connected Riemannian manifold with nonpositive sectional curvature, then the dimension of the space of L^p harmonic 1-forms on M is finite for certain $p > 0$ provided $\lambda_1(M)$ is sufficiently large (see Theorem (4.3.3)). As a consequence of our main theorem, we are able to show that if M is a complete minimal hypersurface with finite index in hyperbolic space $\mathbb{H}^{n+1}$ and if $\lambda_1(M)$ is bigger than $n(n-1)$, then the space of L^2 harmonic 1-forms on M is finite dimensional, and hence M must have finitely many ends (see Corollary (4.3.7)). This can be regarded as a generalization of Li-Wang's result [1].

We begin with the following useful facts in order to prove our main Theorem (4.3.4).

Lemma (4.3.1)[190]: ([192]). Let M be an n-dimensional complete noncompact Riemannian manifold. For $x \in M$ and a constant $\kappa \geq 0$, we assume that a Ricci curvature of M satisfies

$$\text{Ric} \geq -(n-1)\kappa$$

on the geodesic ball $B_x(4r)$ centered at p with radius $4r$. Let $0 < \delta < \frac{1}{2}$ and $\lambda > 0$ be fixed constants. Then there exists a positive constant $C = C(r, \delta, \lambda, \kappa)$ satisfying that if any nonnegative function $f \in C^\infty(B_x(2r))$ satisfying the differential inequality

$$\Delta f \geq -\lambda f,$$

then

$$\sup_{B_x((1-\delta)r)} f^2 \leq \frac{C}{\text{Vol}(B_x(r))} \int_{B_x} f^2,$$

where $\text{Vol}(B_x(r))$ denotes the volume of the geodesic ball $B_x(r)$.

Lemma (4.3.2)[190]: ([13,21]). Let K be a finite dimensional subspace of L^{2p} harmonic q-forms on an m-dimensional complete noncompact

Riemannian manifold M for any $p > 0$. Then there exists $\eta \in K$ such that

$$(\dim K)^{\min(1,p)} \int_{B_x(r)} |\eta|^{2p} \leq \text{Vol}(B_x(r)) \min\left\{\binom{m}{q}, \dim K\right\}^{\min\{1,p\}} \cdot \sup_{B_x(r)} |\eta|^{2p}$$

for any $x \in M$ and $r > 0$.

Adapting the argument of Li-Wang [1], we are now able to prove our main Theorem (4.3.4).

Theorem (4.3.3)[190]: Let N be an $(n+1)$-dimensional complete simply connected Riemannian manifold with sectional curvature K_N satisfying $-k^2 \leq K_N \leq 0$ for a nonzero constant k and $n \geq 3$. Let M be a complete noncompact minimal hypersurface with finite index in N. For $\frac{n-2}{n-1} < p < \frac{n}{n-1}$, assume that

$$\lambda_1(M) > \max\left\{\frac{k^2(n-1)^2p^2}{p(n-1)-n+2}.\frac{k^2n(n-1)p}{n-p(n-1)}\right\}.$$

Then

$$\dim \mathcal{H}^1\left(L^{2p}(M)\right) < \infty,$$

where $\mathcal{H}^1\left(L^{2p}(M)\right)$ denotes the space of L^{2p} harmonic 1-forms on M.

Proof. Let ω be a nontrivial L^{2p} harmonic 1-form on M, i.e.,

$$\Delta\omega = 0 \text{ and } \int_M |\omega|^{2p} < \infty.$$

Denote by $\omega^\#$ the dual harmonic vector field of ω. From Bochner formula, we see that

$$\Delta|\omega|^2 = 2(|\nabla\omega|^2 + \mathrm{Ric}(\omega^\#, \omega^\#)),$$

where Ric denotes the Ricci curvature on M. We recall the following Ricci curvature estimate of minimal hypersurfaces by Leung [162]

$$\mathrm{Ric}(\omega^\#, \omega^\#) \geq -k(n-1)|\omega|^2 - \frac{n-1}{n}|\mathrm{A}|^2|\omega|^2.$$

Thus

$$\Delta|\omega|^2 \geq 2\left(|\nabla\omega|^2 - k(n-1)|\omega|^2 - \frac{n-1}{n}|A|^2|\omega|^2\right). \tag{74}$$

Moreover, since

$$\Delta|\omega|^2 = 2\left(|\omega|\Delta|\omega| + \left|\nabla|\omega|\right|^2\right),$$

(74) becomes

$$|\omega|\Delta|\omega| + k^2(n-1)|\omega|^2 + \frac{n-1}{n}|A|^2|\omega|^2 \geq |\nabla\omega|^2 - \left|\nabla|\omega|\right|^2 \geq \frac{1}{n-1}\left|\nabla|\omega|\right|^2, \tag{75}$$

where we used the Kato-type inequality [173] for harmonic 1-forms in the last inequality. Furthermore

$$|\omega|^p\Delta|\omega|^p = \frac{p-1}{p}|\nabla|\omega|^p|^2 + p|\omega|^{2p-2}|\omega|\Delta|\omega|$$

for any $p > 0$ (see also [141]). Combining with (75), we get

$$|\omega|^p\Delta|\omega|^p + p(n-1)\left(\frac{|A|^2}{n} + k^2\right)|\omega|^{2p} \geq \left(1 - \frac{1}{p} + \frac{1}{p(n-1)}\right)|\nabla|\omega|^p|^2. \tag{76}$$

Since M has finite index, there exists a compact subset $\Omega \subset M$ such that $M\backslash\Omega$ is stable (see [4], [193] for example). In other words, for any compactly supported Lipschitz function f on $M\backslash\Omega$,

$$\int_{M\backslash\Omega} |\nabla f|^2 - (\overline{\mathrm{Ric}}(\nu) + |A|^2)f^2 \geq 0, \tag{77}$$

where $\overline{\mathrm{Ric}}(\nu)$ denotes the Ricci curvature of N in the direction of the unit vector ν normal to M and A denotes the second fundamental form on M. We note that, for any geodesic ball $B_x(R_0) \subset M$ centered at $p \in M$ of radius R_0 containing the compact set Ω , the region $M\backslash B_x(R_0)$ is stable by the monotonicity of eigenvalues of the stability operator. Thus we may assume that $\Omega = B_x(R_0)$ without loss of generality. The assumption on the sectional curvature of N implies that

$$\overline{\mathrm{Ric}}(\nu) \geq -nk^2.$$

Therefore the stability inequality (77) becomes

$$\int_{M\backslash B_x(R_0)} |\nabla f|^2 - (|A|^2 - nk^2) f^2 \geq 0 \tag{78}$$

for all compactly supported Lipschitz function f on $M\backslash B_x(R_0)$. Replacing f by $f|\omega|^p$ in (5), we get

$$\int_{M\backslash B_x(R_0)} f^2|\omega|^{2p}|A|^2 - nk^2 \int_{M\backslash B_x(R_0)} f^2|\omega|^{2p} \leq \int_{M\backslash B_x(R_0)} |\nabla(f|\omega|^p)|^p. \tag{79}$$

On the other hand, the domain monotonicity of eigenvalues implies that

$$\lambda_1(M) \leq \lambda_1\big(M\backslash B_x(R_0)\big) \leq \frac{\int_{M\backslash B_x(R_0)} |\nabla f|^2}{\int_{M\backslash B_x(R_0)} f^2}$$

for any compactly supported nonconstant Lipschitz function f on $M\backslash B_x(R_0)$. Substituting $f|\omega|^p$ for f in the inequality, we have

$$\int_{M\backslash B_x(R_0)} f^2|\omega|^{2p} \leq \frac{1}{\lambda_1(M)} \int_{M\backslash B_x(R_0)} |\nabla(f|\omega|^p)|^2. \tag{80}$$

From (79) and (80), it follows that

$$\begin{aligned}
&\int_{M\backslash B_x(R_0)} f^2|\omega|^{2p}|A|^2 \\
&\leq \left(1 + \frac{nk^2}{\lambda_1(M)}\right)\left(\int_{M\backslash B_x(R_0)} f^2|\nabla|\omega|^p|^2 + \int_{M\backslash B_x(R_0)} |\nabla f|^2|\omega|^{2p} \right. \\
&\qquad \left. + 2\int_{M\backslash B_x(R_0)} f|\omega|^p\langle \nabla f, \nabla|\omega|^p\rangle\right).
\end{aligned}$$

Applying the divergence theorem and using the relation (76), we get

$$\begin{aligned}
\int_{M\backslash B_x(R_0)} f^2|\omega|^{2p}|A|^2 &\leq \left(1 + \frac{nk^2}{\lambda_1(M)}\right)\left(\int_{M\backslash B_x(R_0)} |\nabla f|^2|\omega|^{2p} - \int_{M\backslash B_x(R_0)} f^2|\omega|^p\Delta|\omega|^p\right) \\
&\leq \left(1 + \frac{nk^2}{\lambda_1(M)}\right)\int_{M\backslash B_x(R_0)} |\nabla f|^2|\omega|^{2p} \\
&\quad -\left(1 + \frac{nk^2}{\lambda_1(M)}\right)\left(1 - \frac{1}{p} + \frac{1}{p(n-1)}\right)\int_{M\backslash B_x(R_0)} f^2|\nabla|\omega|^p|^2 \\
&\quad + p(n-1)\left(1 + \frac{nk^2}{\lambda_1(M)}\right)\int_{M\backslash B_x(R_0)} \left(\frac{|A|^2}{n} + k^2\right) f^2|\omega|^{2p}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\left(1 + \frac{nk^2}{\lambda_1(M)}\right)\left(1 - \frac{1}{p} + \frac{1}{p(n-1)}\right)\int_{M\backslash B_x(R_0)} f^2|\nabla|\omega|^p|^2 \\
&\leq \left(1 + \frac{nk^2}{\lambda_1(M)}\right)\int_{M\backslash B_x(R_0)} |\nabla f|^2|\omega|^{2p} \\
&\quad + p(n-1)k^2\left(1 + \frac{nk^2}{\lambda_1(M)}\right)\int_{M\backslash B_x(R_0)} f^2|\omega|^{2p}
\end{aligned}$$

$$+\left(\left(1+\frac{nk^2}{\lambda_1(M)}\right)\frac{p(n-1)}{n}-1\right)\int_{M\setminus B_x(R_0)} f^2|A|^2|\omega|^{2p}.$$

The assumption on $\lambda_1(M)$ gives

$$\left(1-\frac{1}{p}+\frac{1}{p(n-1)}\right)\int_{M\setminus B_x(R_0)} f^2|\nabla|\omega|^p|^2$$
$$\leq \int_{M\setminus B_x(R_0)} |\nabla f|^2|\omega|^{2p} + p(n-1)k^2\int_{M\setminus B_x(R_0)} f^2|\omega|^{2p}. \qquad (81)$$

Applying Young's inequality in (80), we have

$$\int_{M\setminus B_x(R_0)} f^2|\omega|^{2p} \leq \frac{1}{\lambda_1(M)}\left(1+\frac{1}{\varepsilon}\right)\int_{M\setminus B_x(R_0)} |\nabla f|^2|\omega|^{2p}$$
$$+\frac{1+\varepsilon}{\lambda_1(M)}\int_{M\setminus B_x(R_0)} f^2|\nabla|\omega|^p|^2$$

for any $\varepsilon > 0$. Combining this with (81) we get

$$\left(1-\frac{1}{p}+\frac{1}{p(n-1)}\frac{k^2p(n-1)(1+\varepsilon)}{\lambda_1(M)}\right)\int_{M\setminus B_x(R_0)} f^2|\nabla|\omega|^p|^2$$
$$\leq \left(1+\frac{k^2p(n-1)}{\lambda_1(M)}\left(1+\frac{1}{\varepsilon}\right)\right)\int_{M\setminus B_x(R_0)} |\nabla f|^2|\omega|^{2p}.$$

Using the assumption on $\lambda_1(M)$, we choose a sufficiently small $\varepsilon > 0$ such that

$$1-\frac{1}{p}+\frac{1}{p(n-1)}\frac{k^2p(n-1)(1+\varepsilon)}{\lambda_1(M)} > 0.$$

Then we obtain

$$\int_{M\setminus B_x(R_0)} f^2|\nabla|\omega|^p|^2 \leq C_0\int_{M\setminus B_x(R_0)} |\nabla f|^2|\omega|^{2p} \qquad (82)$$

for some positive constant C_0 which depends only on p, n, k and $\lambda_1(M)$. On the other hand, from the Sobolev inequality for minimal submanifolds [29], we have

$$\left(\int_{M\setminus B_x(R_0)} |f|\omega|^p|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq C_S\int_{M\setminus B_x(R_0)} |\nabla(f|\omega|^p)|^2$$
$$\leq 2C_S\int_{M\setminus B_x(R_0)} f^2|\nabla|\omega|^p|^2$$
$$+2C_S\int_{M\setminus B_x(R_0)} |\nabla f|^2|\omega|^{2p} \qquad (83)$$

where the Sobolev constant C_S depends only on n. Combining (82) and (10), we get

$$\left(\int_{M\setminus B_x(R_0)} (f|\omega|^p)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq C_1\int_{M\setminus B_x(R_0)} |\nabla f|^2|\omega|^{2p}, \qquad (84)$$

for some constant $C_1 \geq 2C_S(1+C_0)$. Now we choose our test function $0 \leq f \leq 1$ as follows: given $R > R_0 + 1$,

(a) $f = 1$ on $B_x(R)\setminus B_x(R_0+1)$
(b) $f = 0$ on $B_x(R_0)\cup\big(M\setminus B_x(2R)\big)$
(c) $|\nabla f| \leq C_2$ on $B_x(R_0+1)\setminus B_x(R_0)$
(d) $|\nabla f| \leq \frac{C_S}{R}$ |on $B_x(2R)\setminus B_x(R)$

for some constant $C_2 > 0$. Applying this test function f to(84), we get

$$\left(\int_{B_x(R)\setminus B_x(R_0+1)} |\omega|^{\frac{2pn}{n-2}}\right)^{\frac{n-2}{n}} \leq C_3 \int_{B_x(R_0+1)\setminus B_x(R_0)} |\omega|^{2p} + \frac{C_3}{R^2}\int_{B_x(2R)\setminus B_x(R)} |\omega|^{2p}.$$

Letting $R \to \infty$and using the assumption that$\int_m |\omega|^{2p} < \infty$, we obtain

$$\left(\int_{M\setminus B_x(R_0+1)} |\omega|^{2p}\right)^{\frac{n-2}{n}} \leq C_3 \int_{B_x(R_0+1)\setminus B_x(R_0)} |\omega|^{2p}.$$

Moreover, since

$$\int_{B_x(R_0+2)\setminus B_x(R_0+1)} |\omega|^{2p} \leq \mathrm{Vol}\big(B_x(R_0+2)\big)^{\frac{2}{n}} \left(\int_{B_x(R_0+2)\setminus B_x(R_0+1)} |\omega|^{\frac{2pn}{n-2}}\right)^{\frac{n-2}{n}}$$

by Hölder inequality, we conclude that

$$\int_{B_x(R_0+2)\setminus B_x(R_0+1)} |\omega|^{2p} \leq C_3.\mathrm{Vol}\big(B_x(R_0+2)\big)^{\frac{2}{n}} \int_{B_x(R_0+1)\setminus B_x(R_0)} |\omega|^{2p}, \quad (85)$$

where$\mathrm{Vol}\big(B_x(R_0+2)\big)$denotes the volume of the geodesic ball $B_x(R_0+2)$ on M. Adding

$$\int_{B_x(R_0+1)\setminus B_x(R_0)} |\omega|^{2p}$$

to both sides of (85), we get

$$\int_{B_x(R_0+2)\setminus B_x(R_0)} |\omega|^{2p} \leq \left(C_3.\mathrm{Vol}\big(B_x(R_0+2)\big)^{\frac{2}{n}} + 1\right) \int_{B_x(R_0+1)\setminus B_x(R_0)} |\omega|^{2p}.$$

Again adding$\int_{B_x(R_0)} |\omega|^{2p}$

$$\int_{B_x(R_0+2)} |\omega|^{2p} \leq C_4 \int_{B_x(R_0+1)} |\omega|^{2p}, \quad (86)$$

where $C_4 = C_3.\mathrm{Vol}\big(B_x(R_0+2)\big)^{\frac{2}{n}} + 1$.

On the other hand, since$|\omega|$ satisfies the differential inequality(76), Lemma (4.3.1) asserts that

$$\sup_{B_x((1-\delta)(R_0+2))} |\omega|^{2p} \leq C_5 \int_{B_x(R_0+2)} |\omega|^{2p}$$

for some positive constant$C_5 = C_5\big(n, \mathrm{Vol}\big(B_x(R_0+2)\big), \sup_{B_x(R_0+2)} |A|^2\big)$. For a sufficiently small$\delta > 0$ such that $(1-\delta)(R_0+2) > R_0+1$,

$$\sup_{B_x(R_0+2)} |\omega|^{2p} \leq C_5 \int_{B_x(R_0+2)} |\omega|^{2p}.$$

Together with(86), we obtain

$$\sup_{B_x(R_0+1)} |\omega|^{2p} \leq C_6 \int_{B_x(R_0+1)} |\omega|^{2p}. \quad (87)$$

where the constant$C_6 > 0$depends on $p, k, n,\ R_0, \lambda_1(M)$, the volume of the geodesic ball $B_x(R_0+2)$, and the supremum of$|A|^2$ on $B_x(R_0+2)$.

In order to prove that $\dim \mathcal{H}^1\big(L^{2P}(M)\big) < \infty$, let us consider any finite dimensional subspace$K \subset \mathcal{H}^1\big(L^{2P}(M)\big)$. It suffices to show that the dimension of K is bounded above by a constant, which is independent of K. According to Lemma (4.3.2), we see that there exists anL^{2p} harmonic 1-form $\omega \in K$such that

$$(\dim K)^{\min(1,p)} \int_{B_x(R_0+1)} |\omega|^{2p} \leq \mathrm{Vol}\big(B_x(R_0+1)\big) \min\{n, \dim K\}^{\min\{1,p\}} . \sup_{B_x(R_0+1)} |\omega|^{2p}.$$

From (87), it follows that

$$\dim K \leq \big(C_6 . \mathrm{Vol}\big(B_x(R_0+1)\big)\big)^{\frac{1}{\min\{1,p\}}} \min\{n, \dim K\},$$

which implies that $\dim K$ is bounded by a fixed constant. Since K is an arbitrary subspace of finite dimension, we get the conclusion.

Theorem (4.3.4)[190]: ([173], [194]). Let M be a complete Riemannian manifold and let E be a Riemannian vector bundle of rank 1 over$M$, whose sections are denoted by$\Gamma(E)$. Assume that $V \subset \Gamma(E)$is a subspace satisfying the following property:

(i) every section $\xi \in V$has the unique continuation property;

(ii) there exist a function $a \in C^0(M)$and constants $A, p, H \in \mathbb{R}$ satisfying $H \geq p \geq A+1$ and $p > 0$ such that for each $\xi \in V$, its norm$\psi = |\xi|$satisfies

$$\begin{cases} \psi\Delta\psi + a(x)\psi^2 + A|\nabla\psi|^2 \geq 0 \text{ weakly on} M \\ \displaystyle\int_{B_r} |\psi|^{2p} = o(r^{2p}) as\ r \to \infty. \end{cases}$$

If there exists a function$\varphi \in \mathrm{Lip}_{\mathrm{loc}}(M)$ satisfying

$$\Delta\varphi + Ha(x)\varphi \leq 0$$

weakly outside a compact set $K \subset M$, then$\dim V < \infty$.

Applying their theorem, one can obtain a finiteness result for our setting which is weaker than ours. To see this, letω be $a\ L^{2p}$ and the inequality (75). Choose

$$A = -\frac{1}{n-1}, H = p \geq \frac{n-2}{n-1}, a(x) = k^2(n-1) + \frac{n-1}{n}|A|^2.$$

Since the stability operator satisfies

$$\begin{aligned} 0 &\leq -\Delta - (\overline{\mathrm{Ric}}(\nu,\nu) + |A|^2) \\ &\leq -\Delta - (-nk^2 + |A|^2) \end{aligned}$$

outside some compact set $K \subset M$, the operator$L = -\Delta - Ha(x)$ satisfies

$$\begin{aligned} L &= -\Delta - k^2p(n-1) - \frac{p(n-1)}{n}|A|^2 \\ &= \frac{p(n-1)}{n}(-\Delta - |A|^2) - \frac{n-p(n-1)}{n}\Delta - k^2p(n-1) \\ &\geq \frac{p(n-1)}{n}(-nk^2) - \frac{n-p(n-1)}{n}\Delta - k^2p(n-1) \\ &= \frac{n-p(n-1)}{n}\left(-\Delta - \frac{2k^2n(n-1)p}{n-p(n-1)}\right) \geq 0 \end{aligned}$$

provided that, for$\frac{n-2}{n-1} \leq p \leq \frac{n}{n-1}$,

$$\lambda_1(M) \geq \frac{2k^2n(n-1)p}{n-p(n-1)}.$$

Here the condition on $\lambda_1(M)$ is stronger than ours. Therefore our theorem can be regarded as an improvement of Pigola–Rigoli–Setti's result for the space ofL^p harmonic 1-forms on complete minimal hypersurfaces with finite index in a Cartan–Hadamard manifold with pinched sectional curvature.

Let M be an $n(\geq 3)$-dimensional complete minimal hypersurface in a complete simply connected Riemannian manifold of nonpositive sectional curvature. Then, applying the argument of Cao–Shen–Zhu [2], we can see that each end of M is nonparabolic [1] (see also [195], [181]). Recall that a complete Riemannian manifold M is called nonparabolic if M admits a nonconstant positive superharmonic function. Denote by$\mathcal{H}_D^0(M)$the space of bounded harmonic functions with finite Dirichlet (i.e.,L^2) energy. According to Li–Tam [164], we see that the number of nonparabolic end of M is bounded above by the dimension of$\mathcal{H}_D^0(M)$, i.e.,

$$\#(\text{nonparabolic ends of } M) \leq \dim \mathcal{H}_D^0(M). \quad (88)$$

We also note that if$f \in \mathcal{H}_D^0(M)$, then $df \in \mathcal{H}^1\big(L^2(M)\big)$.Furthermore, $df = 0$ if and only if f is constant. Therefore

$$\dim \mathcal{H}_D^0(M) \leq \dim \mathcal{H}^1\big(L^2(M)\big) + 1. \quad (89)$$

By (88) and(89), we obtain

$$\#(\text{nonparabolic ends of } M) \leq \dim \mathcal{H}^1\big(L^2(M)\big) + 1,$$

which was proved by Li–Wang [1]. In case of L^2 harmonic 1-forms, we can obtain the following consequence, using the above observation and Theorem (4.3.3).

Corollary (4.3.5)[190]: Let N be an $(n+1)$-dimensional complete simply connected Riemannian manifold with sectional curvatureK_N satisfying $-k^2 \leq K_N \leq 0$for a nonzero constant k. Let M be a complete noncompact minimal hypersurface with finite index in N. Assume that$\lambda_1(M) > k^2 n(n-1)$. Then M must have finitely many ends.

Remark (4.3.6)[190]: In Euclidean space, Tysk [193] obtained that the index of a complete minimal hypersurface M in$\mathbb{R}^{n+1}$satisfies the following:

$$\text{lndex}(M) \leq C_n \int_M |A|^n, \quad (90)$$

whereC_n depends only on the dimension n. The proof of (90) uses the method of Li–Yau [196]. In fact, this inequality is still valid on a complete noncompact minimal hypersurface M in a Cartan- Hadamard manifold N with sectional curvature satisfying$-k^2 \leq K_N \leq 0$. If the second fundamental form A on M has finite L^n-norm, then M has finite index by(90), which implies that M has finitely many ends by Corollary (4.3.5). This result was already obtained by Cavalcante–Mirandola–Vitório [15] (see also[184]). However, unlike the 2-dimensional case, it is unknown whether the condition that a complete noncompact minimal hypersurface M in a Cartan–Hadamard manifold has finite index implies that the second fundamental form on M has finite L^n-norm.

When the ambient space is the hyperbolic space$\mathbb{H}^{n+1}$of constant sectional curvature-1, one can conclude the following:

Corollary (4.3.7)[190]: Let M be an n-dimensional complete noncompact minimal hypersurface with finite index in hyperbolic space$\mathbb{H}^{n+1}$. If$\lambda_1(M) > n(n-1)$, then

$$\dim \mathcal{H}^1\big(L^2(M)\big) < \infty.$$

Moreover, M has finitely many ends.

Corollary (4.3.8)[353]: Let N be an $(4+\epsilon)$-dimensional complete simply connected Riemannian manifold with sectional curvatureK_N satisfying$-k_r^2 \leq K_N \leq 0$for a nonzero constant k_r and $\epsilon \geq 0$. Let M_r be a complete noncompact minimal hypersurface with finite index in N. For $\epsilon > 0$, assume that

$$(\lambda_r)_1(M_r) > \max\left\{k_r^2(2+\epsilon)^2(1+\epsilon)^2. \frac{k_r^2(3+\epsilon)(2+\epsilon)(1+\epsilon)}{1-2\epsilon-\epsilon^2}\right\}.$$

Then

$$\dim \mathcal{H}^1\left(L^{2(1+\epsilon)}(M_r)\right) < \infty,$$

where $\mathcal{H}^1\left(L^{2(1+\epsilon)}(M_r)\right)$denotes the space of $L^{2(1+\epsilon)}$ harmonic 1-forms on M_r.

Proof. Let ω_r be a nontrivial $L^{2(1+\epsilon)}$ harmonic 1-form on M_r, i.e.,

$$\Delta\omega_r = 0 \text{ and} \int_{M_r} |\omega_r|^{2(1+\epsilon)} < \infty.$$

Denote by $\omega_r^{\#}$ the dual harmonic vector field of ω_r. From Bochner formula, we see that

$$\Delta|\omega_r|^2 = 2(|\nabla\omega_r|^2 + \mathrm{Ric}(\omega_r^{\#}, \omega_r^{\#})),$$

where Ric denotes the Ricci curvature on M_r. We recall the following Ricci curvature estimate of minimal hypersurfaces by Leung [162]

$$\mathrm{Ric}(\omega_r^{\#}, \omega_r^{\#}) \geq -k_r(2+\epsilon)|\omega_r|^2 - \frac{2+\epsilon}{3+\epsilon}|\mathrm{A}|^2|\omega_r|^2.$$

Thus

$$\Delta|\omega_r|^2 \geq 2\left(|\nabla\omega_r|^2 - k_r(2+\epsilon)|\omega_r|^2 - \frac{2+\epsilon}{3+\epsilon}|A|^2|\omega_r|^2\right). \quad (91)$$

Moreover, since

$$\Delta|\omega_r|^2 = 2\left(|\omega_r|\Delta|\omega_r| + \left|\nabla|\omega_r|\right|^2\right),$$

(91) becomes

$$|\omega_r|\Delta|\omega_r| + k_r^2(2+\epsilon)|\omega_r|^2 + \frac{2+\epsilon}{3+\epsilon}|A|^2|\omega_r|^2 \geq |\nabla\omega_r|^2 - \left|\nabla|\omega_r|\right|^2$$
$$\geq \frac{1}{2+\epsilon}\left|\nabla|\omega_r|\right|^2, \quad (92)$$

where we used the Kato-type inequality [173] for harmonic 1-forms in the last inequality. Furthermore

$$|\omega_r|^{1+\epsilon}\Delta|\omega_r|^{1+\epsilon} = \frac{\epsilon}{1+\epsilon}|\nabla|\omega_r|^{1+\epsilon}|^2 + (1+\epsilon)|\omega_r|^{2\epsilon}|\omega_r|\Delta|\omega_r|$$

for any $\epsilon \geq 0$ (see also [141]). Combining with (92), we get

$$|\omega_r|^{1+\epsilon}\Delta|\omega_r|^{1+\epsilon} + (1+\epsilon)(2+\epsilon)\left(\frac{|A|^2}{3+\epsilon} + k_r^2\right)|\omega_r|^{2(1+\epsilon)}$$
$$\geq \left(\frac{\epsilon(2+\epsilon)+1}{(1+\epsilon)(2+\epsilon)}\right)|\nabla|\omega_r|^{1+\epsilon}|^2. \quad (93)$$

Since M_r has finite index, there exists a compact subset $\Omega \subset M_r$ such that $M_r \backslash \Omega$ is stable (see [4], [193] for example). In other words, for any compactly supported Lipschitz function f^2 on $M_r \backslash \Omega$,

$$\int_{M_r \backslash \Omega} |\nabla f^2|^2 - (\overline{\mathrm{Ric}}(\nu_r) + |A|^2) f^4 \geq 0, \tag{94}$$

where $\overline{\mathrm{Ric}}(\nu_r)$ denotes the Ricci curvature of N in the direction of the unit vector ν_r normal to M_r and A denotes the second fundamental form on M_r. We note that, for any geodesic ball $B_{x_r}(R_0) \subset M_r$centered at $(1+\epsilon) \in M_r$ of radius R_0 containing the compact set Ω , the region $M_r \backslash B_{x_r}(R_0)$is stable by the monotonicity of eigenvalues of the stability operator. Thus we may assume that $\Omega = B_{x_r}(R_0)$ without loss of generality. The assumption on the sectional curvature of N implies that

$$\overline{\mathrm{Ric}}(\nu_r) \geq -(3+\epsilon) k_r^2.$$

Therefore the stability inequality (94) becomes

$$\int_{M_r \backslash B_{x_r}(R_0)} |\nabla f^2|^2 - (|A|^2 - (3+\epsilon) k_r^2) f^4 \geq 0 \tag{95}$$

for all compactly supported Lipschitz function f^2 on $M_r \backslash B_{x_r}(R_0)$. Replacing f^2 by $f^2 |\omega_r|^{1+\epsilon}$in(95), we get

$$\int_{M_r \backslash B_{x_r}(R_0)} f^4 |\omega_r|^{2(1+\epsilon)} |A|^2 - (3+\epsilon) k_r^2 \int_{M_r \backslash B_{x_r}(R_0)} f^4 |\omega_r|^{2(1+\epsilon)}$$
$$\leq \int_{M_r \backslash B_{x_r}(R_0)} |\nabla (f^2 |\omega_r|^{1+\epsilon})|^{1+\epsilon}. \tag{96}$$

On the other hand, the domain monotonicity of eigenvalues implies that

$$(\lambda_r)_1(M_r) \leq (\lambda_r)_1 \left(M_r \backslash B_{x_r}(R_0) \right) \leq \frac{\int_{M_r \backslash B_{x_r}(R_0)} |\nabla f^2|^2}{\int_{M_r \backslash B_{x_r}(R_0)} f^4}$$

for any compactly supported nonconstant Lipschitz function f^2 on $M_r \backslash B_{x_r}(R_0)$. Substituting $f^2 |\omega_r|^{1+\epsilon}$for f^2 in the inequality, we have

$$\int_{M_r \backslash B_{x_r}(R_0)} f^4 |\omega_r|^{2(1+\epsilon)} \leq \frac{1}{(\lambda_r)_1(M_r)} \int_{M_r \backslash B_{x_r}(R_0)} |\nabla (f^2 |\omega_r|^{1+\epsilon})|^2. \tag{97}$$

From (96) and (97), it follows that

$$\int_{M_r\setminus B_{x_r}(R_0)} f^4|\omega_r|^{2(1+\epsilon)}|A|^2$$
$$\leq\left(1+\frac{(3+\epsilon)k_r^2}{(\lambda_r)_1(M_r)}\right)\left(\int_{M_r\setminus B_{x_r}(R_0)} f^4|\nabla|\omega_r|^{1+\epsilon}|^2\right.$$
$$\left.+\int_{M_r\setminus B_{x_r}(R_0)}|\nabla f^2|^2|\omega_r|^{2(1+\epsilon)}+2\int_{M_r\setminus B_{x_r}(R_0)} f^2|\omega_r|^{1+\epsilon}\langle\nabla f^2,\nabla|\omega_r|^{1+\epsilon}\rangle\right).$$

Applying the divergence theorem and using the relation(3), we get

$$\int_{M_r\setminus B_{x_r}(R_0)} f^4|\omega_r|^{2(1+\epsilon)}|A|^2$$
$$\leq\left(1+\frac{(3+\epsilon)k_r^2}{(\lambda_r)_1(M_r)}\right)\left(\int_{M_r\setminus B_{x_r}(R_0)}|\nabla f^2|^2|\omega_r|^{2(1+\epsilon)}\right.$$
$$\left.-\int_{M_r\setminus B_{x_r}(R_0)} f^4|\omega_r|^{1+\epsilon}\Delta|\omega_r|^{1+\epsilon}\right)$$
$$\leq\left(1+\frac{(3+\epsilon)k_r^2}{(\lambda_r)_1(M_r)}\right)\int_{M_r\setminus B_{x_r}(R_0)}|\nabla f^2|^2|\omega_r|^{2(1+\epsilon)}$$
$$-\left(1+\frac{(3+\epsilon)k_r^2}{(\lambda_r)_1(M_r)}\right)\left(\frac{1+\epsilon}{2+\epsilon}\right)\int_{M_r\setminus B_{x_r}(R_0)} f^4|\nabla|\omega_r|^{1+\epsilon}|^2+(1$$
$$+\epsilon)(2+\epsilon)\left(1+\frac{(3+\epsilon)k_r^2}{(\lambda_r)_1(M_r)}\right)\int_{M_r\setminus B_{x_r}(R_0)}\left(\frac{|A|^2}{3+\epsilon}+k_r^2\right)f^4|\omega_r|^{2(1+\epsilon)}.$$

Therefore

$$\left(1+\frac{(3+\epsilon)k_r^2}{(\lambda_r)_1(M_r)}\right)\left(\frac{1+\epsilon}{2+\epsilon}\right)\int_{M_r\setminus B_{x_r}(R_0)} f^4|\nabla|\omega_r|^{1+\epsilon}|^2$$
$$\leq\left(1+\frac{(3+\epsilon)k_r^2}{(\lambda_r)_1(M_r)}\right)\int_{M_r\setminus B_{x_r}(R_0)}|\nabla f^2|^2|\omega_r|^{2(1+\epsilon)}+(1$$
$$+\epsilon)(2+\epsilon)k_r^2\left(1+\frac{(3+\epsilon)k_r^2}{(\lambda_r)_1(M_r)}\right)\int_{M_r\setminus B_{x_r}(R_0)} f^4|\omega_r|^{2(1+\epsilon)}$$
$$+\left(\left(1+\frac{(3+\epsilon)k_r^2}{(\lambda_r)_1(M_r)}\right)\frac{(1+\epsilon)(2+\epsilon)}{3+\epsilon}-1\right)\int_{M_r\setminus B_{x_r}(R_0)} f^4|A|^2|\omega_r|^{2(1+\epsilon)}.$$

The assumption on $(\lambda_r)_1(M_r)$ gives

$$\left(\frac{1+\epsilon}{2+\epsilon}\right)\int_{M_r\setminus B_{x_r}(R_0)} f^4|\nabla|\omega_r|^{1+\epsilon}|^2$$

$$\leq \int_{M_r\setminus B_{x_r}(R_0)} |\nabla f^2|^2|\omega_r|^{2(1+\epsilon)} + (1+\epsilon)(2+\epsilon)k_r^2\int_{M_r\setminus B_{x_r}(R_0)} f^4|\omega_r|^{2(1+\epsilon)}. \tag{98}$$

Applying Young's inequality in (97), we have

$$\begin{aligned}\int_{M_r\setminus B_{x_r}(R_0)} f^4|\omega_r|^{2(1+\epsilon)} \\ &\leq \frac{1}{(\lambda_r)_1(M_r)}\left(\frac{1+\epsilon}{\epsilon}\right)\int_{M_r\setminus B_{x_r}(R_0)} |\nabla f^2|^2|\omega_r|^{2(1+\epsilon)} \\ &\quad + \frac{1+\epsilon}{(\lambda_r)_1(M_r)}\int_{M_r\setminus B_{x_r}(R_0)} f^4|\nabla|\omega_r|^{1+\epsilon}|^2\end{aligned}$$

for any $\epsilon > 0$. Combining this with (98) we get

$$\begin{aligned}\left(\frac{\epsilon}{1+\epsilon} + \frac{k_r^2(1+\epsilon)}{(\lambda_r)_1(M_r)}\right)\int_{M_r\setminus B_{x_r}(R_0)} f^4|\nabla|\omega_r|^{1+\epsilon}|^2 \\ &\leq \left(1 + \frac{k_r^2(1+\epsilon)^2(2+\epsilon)}{(\lambda_r)_1(M_r)\epsilon}\right)\int_{M_r\setminus B_{x_r}(R_0)} |\nabla f^2|^2|\omega_r|^{2(1+\epsilon)}.\end{aligned}$$

Using the assumption on $(\lambda_r)_1(M_r)$, we choose a sufficiently small $\epsilon > 0$ such that

$$\frac{\epsilon}{1+\epsilon} + \frac{k_r^2(1+\epsilon)}{(\lambda_r)_1(M_r)} > 0.$$

Then we obtain

$$\int_{M_r\setminus B_{x_r}(R_0)} f^4|\nabla|\omega_r|^{1+\epsilon}|^2 \leq C_0\int_{M_r\setminus B_{x_r}(R_0)} |\nabla f^2|^2|\omega_r|^{2(1+\epsilon)} \tag{99}$$

for some positive constant C_0 which depends only on $1+\epsilon, 3+\epsilon, k_r$ and $(\lambda_r)_1(M_r)$. On the other hand, from the Sobolev inequality for minimal submanifolds [29], we have

$$\left(\int_{M_r\setminus B_{x_r}(R_0)} |f^2|\omega_r|^{1+\epsilon}|^{\frac{2(3+\epsilon)}{1+\epsilon}}\right)^{\frac{1+\epsilon}{3+\epsilon}} \leq C_S\int_{M_r\setminus B_{x_r}(R_0)} |\nabla(f^2|\omega_r|^{1+\epsilon})|^2$$

$$\leq 2C_S\int_{M_r\setminus B_{x_r}(R_0)} f^4|\nabla|\omega_r|^{1+\epsilon}|^2 + 2C_S\int_{M_r\setminus B_{x_r}(R_0)} |\nabla f^2|^2|\omega_r|^{2(1+\epsilon)}, \tag{100}$$

where the Sobolev constant C_S depends only on $3+\epsilon$. Combining (99) and(100), we get

$$\left(\int_{M_r\backslash B_{x_r}(R_0)}\left(f^2|\omega_r|^{1+\epsilon}\right)^{\frac{2(3+\epsilon)}{1+\epsilon}}\right)^{\frac{1+\epsilon}{3+\epsilon}} \leq C_1\int_{M_r\backslash B_{x_r}(R_0)}|\nabla f^2|^2|\omega_r|^{2(1+\epsilon)}, \quad (101)$$

for some constant $C_1 \geq 2C_S(1+C_0)$. Now we choose our test function $0 \leq f^2 \leq 1$ as follows: given $R > R_0 + 1$,

(a) $f^2 = 1$ on$B_{x_r}(R)\backslash B_{x_r}(R_0+1)$
(b) $f^2 = 0$ on $B_{x_r}(R_0) \cup \left(M_r\backslash B_{x_r}(2R)\right)$
(c) $|\nabla f^2| \leq C_2$ on $B_{x_r}(R_0+1)\backslash B_{x_r}(R_0)$
(d) $|\nabla f^2| \leq \frac{C_S}{R}$ |on $B_{x_r}(2R)\backslash B_{x_r}(R)$

for some constant$C_2 > 0$. Applying this test function f^2 to(101), we get

$$\left(\int_{B_{x_r}(R)\backslash B_{x_r}(R_0+1)}|\omega_r|^{2(3+\epsilon)}\right)^{\frac{1+\epsilon}{3+\epsilon}}$$
$$\leq C_3\int_{B_{x_r}(R_0+1)\backslash B_{x_r}(R_0)}|\omega_r|^{2(1+\epsilon)} + \frac{C_3}{R^2}\int_{B_{x_r}(2R)\backslash B_{x_r}(R)}|\omega_r|^{2(1+\epsilon)}.$$

Letting $R \to \infty$and using the assumption that $\int_{M_r}|\omega_r|^{2(1+\epsilon)} < \infty$, we obtain

$$\left(\int_{M_r\backslash B_{x_r}(R_0+1)}|\omega_r|^{2(1+\epsilon)}\right)^{\frac{1+\epsilon}{3+\epsilon}} \leq C_3\int_{B_{x_r}(R_0+1)\backslash B_{x_r}(R_0)}|\omega_r|^{2(1+\epsilon)}.$$

Moreover, since

$$\int_{B_{x_r}(R_0+2)\backslash B_{x_r}(R_0+1)}|\omega_r|^{2(1+\epsilon)}$$
$$\leq \mathrm{Vol}\left(B_{x_r}(R_0+2)\right)^{\frac{2}{3+\epsilon}}\left(\int_{B_{x_r}(R_0+2)\backslash B_{x_r}(R_0+1)}|\omega_r|^{2(3+\epsilon)}\right)^{\frac{1+\epsilon}{3+\epsilon}}$$

by Hölder inequality, we conclude that

$$\int_{B_{x_r}(R_0+2)\backslash B_{x_r}(R_0+1)}|\omega_r|^{2(1+\epsilon)}$$
$$\leq C_3.\mathrm{Vol}\left(B_{x_r}(R_0+2)\right)^{\frac{2}{3+\epsilon}}\int_{B_{x_r}(R_0+1)\backslash B_{x_r}(R_0)}|\omega_r|^{2(1+\epsilon)}, \quad (102)$$

where $\mathrm{Vol}\left(B_{x_r}(R_0+2)\right)$denotes the volume of the geodesic ball $B_{x_r}(R_0+2)$ on M_r. Adding

$$\int_{B_{x_r}(R_0+1)\setminus B_{x_r}(R_0)} |\omega_r|^{2(1+\epsilon)}$$

to both sides of (102), we get

$$\int_{B_{x_r}(R_0+2)\setminus B_{x_r}(R_0)} |\omega_r|^{2(1+\epsilon)} \le \left(C_3.\mathrm{Vol}\left(B_{x_r}(R_0+2)\right)^{\frac{2}{3+\epsilon}}+1\right)\int_{B_{x_r}(R_0+1)\setminus B_{x_r}(R_0)} |\omega_r|^{2(1+\epsilon)}.$$

Again adding $\int_{B_{x_r}(R_0)} |\omega_r|^{2(1+\epsilon)}$

$$\int_{B_{x_r}(R_0+2)} |\omega_r|^{2(1+\epsilon)} \le C_4 \int_{B_{x_r}(R_0+1)} |\omega_r|^{2(1+\epsilon)}, \tag{103}$$

where $C_4 = C_3.\mathrm{Vol}\left(B_{x_r}(R_0+2)\right)^{\frac{2}{3+\epsilon}}+1$.

On the other hand, since$|\omega_r|$ satisfies the differential inequality (93), Lemma 2.1 asserts that

$$\sup_{B_{x_r}((1-\delta)(R_0+2))} |\omega_r|^{2(1+\epsilon)} \le C_5 \int_{B_{x_r}(R_0+2)} |\omega_r|^{2(1+\epsilon)}$$

for some positive constant$C_5 = C_5\left(3+\epsilon, \mathrm{Vol}\left(B_{x_r}(R_0+2)\right), \sup_{B_{x_r}(R_0+2)}|A|^2\right)$. For a sufficiently small$\delta > 0$ such that $(1-\delta)(R_0+2) > R_0+1$,

$$\sup_{B_{x_r}(R_0+2)} |\omega_r|^{2(1+\epsilon)} \le C_5 \int_{B_{x_r}(R_0+2)} |\omega_r|^{2(1+\epsilon)}.$$

Together with(103), we obtain

$$\sup_{B_{x_r}(R_0+1)} |\omega_r|^{2(1+\epsilon)} \le C_6 \int_{B_{x_r}(R_0+1)} |\omega_r|^{2(1+\epsilon)}. \tag{104}$$

where the constant $C_6 > 0$depends on $1+\epsilon, k_r, 3+\epsilon,\ R_0, (\lambda_r)_1(M_r)$, the volume of the geodesic ball $B_{x_r}(R_0+2)$, and the supremum of$|A|^2$ on $B_{x_r}(R_0+2)$.

In order to prove that $\dim \mathcal{H}^1\left(L^{2(1+\epsilon)}(M_r)\right) < \infty$, let us consider any finite dimensional subspace $K \subset \mathcal{H}^1\left(L^{2(1+\epsilon)}(M_r)\right)$. It suffices to show that the dimension of K is bounded above by a constant, which is independent of K. According to Lemma (4.3.2), we see that there exists an $L^{2(1+\epsilon)}$ harmonic 1-form $\omega_r \in K$ such that

$$(\dim K)^{\min(1,1+\epsilon)} \int_{B_{x_r}(R_0+1)} |\omega_r|^{2(1+\epsilon)}$$
$$\leq \mathrm{Vol}\left(B_{x_r}(R_0+1)\right) \min\{3+\epsilon, \dim K\}^{\min\{1,1+\epsilon\}} . \sup_{B_{x_r}(R_0+1)} |\omega_r|^{2(1+\epsilon)}.$$

From (104), it follows that

$$\dim K \leq \left(C_6 . \mathrm{Vol}\left(B_{x_r}(R_0+1)\right)\right)^{\frac{1}{\min\{1,1+\epsilon\}}} \min\{3+\epsilon, \dim K\},$$

which implies that $\dim K$ is bounded by a fixed constant. Since K is an arbitrary subspace of finite dimension, we get the conclusion.

Chapter 5
On Strongly Lipschitz Domains with Mixed Boundary Conditions on L^p

We show a method reling on a transference procedure from the recent positive result on $\mathbb{R}^n$ in [199]. We show that the domain Ω is assumed to be bounded, and the Dirichlet part D of the boundary has to satisfy the well-known Ahlfors–David condition, whilst for the points from $\overline{\partial\Omega\backslash D}$ the existence of bi-Lipschitzian boundary charts is required. We obtain an optimal p-interval for the bounded H^∞-calculus on L^p. Estimates depend holomorphically on the coefficients, thereby making them applicable to questions of (non-autonomous) maximal regularity and optimal control. For completeness we also provide a short summary on the Kato square root problem in L^2 for systems with lower order terms in our setting.

Section (5.1): Square Roots of Elliptic Second Order Divergence Operators

For Ω be an open subset of $\mathbb{R}^n$, A a bounded uniformly elliptic complex matrix on Ω, and $L = -\text{div}(A\nabla)$ the elliptic second order divergence operator defined as the maximal-accretive operator associated with a regularly accretive sesquilinear form on a closed subspace V of $H^1(\Omega)$ containing $H_0^1(\Omega)$. The Kato conjecture amounts to showing that for any such L, the domain of the maximal-accretive square root $L^{1/2}$ of L agrees with V with equivalence of norms. One of Kato's questions was about perturbation theory for the square roots of real symmetric operators in order to study hyperbolic evolution equations with time-dependent coefficients. This conjecture is also related to other topics; see, e.g., [206].

In one dimension, the conjecture is now completely settled: for any Ω, V and L as above, the domain of $L^{1/2}$ agrees with V. The first solution when $\Omega = \mathbb{R}$ was given by Coifman, McIntosh and Meyer [201]. Their argument relied on translation invariance, so other methods needed to be devised when $\Omega \neq \mathbb{R}$. We used ad hoc wavelets in [200], while Auscher, McIntosh and Nahmod used a reduction from the case $\Omega = \mathbb{R}$ via [192]. In higher dimensions, when $\Omega = \mathbb{R}^n$, [94] and for a discussion on progress over the years about this problem until 1998. Very recently, the conjecture has been established in arbitrary dimensions by Hofmann, Lacey and McIntosh along with us [199] after it was proved for L^∞ of self-adjoint operators by Hofmann, Lewis and us [198].

When $\Omega \neq \mathbb{R}^n$, geometry at the boundary plays a role which prevents a straightforward generalisation of results and methods in $\mathbb{R}^n$. Not even the Kato conjecture for L^∞-perturbations of the Laplacian is known. Mclntosh proved it when the coefficients are in the space $\text{MHS}(\Omega)$ of pointwise multipliers of the Sobolev space $H^s(\Omega)$ [205] for some $s > 0$ and Ω strongly Lipschitz. This seems to be the best result currently available on strongly Lipschitz domains. We establish the following result.

Theorem (5.1.1)[197]: If $n > 2$, the Kato conjecture holds for any elliptic second order divergence operator -div($A\,\nabla$) subject to a Dirichlet or Neumann boundary condition on a strongly Lipschitz domain.

The meaning of a Dirichlet and Neumann boundary condition wilt be explained.

Although square roots are non-local operators, the proof of Theorem (5.1.1) follows procedures which are customary for boundary value problems: we transfer the result from $\mathbb{R}^n$ to $\mathbb{R}_+^n$ by a reflection principle; then to special Lipschitz domains by a bilipschitz change of variables; and eventually to general strongly Lipschitz domains by localisation.

This last step relies upon a kind of "weak" comparison principle for solutions of complex elliptic operators.

Our method does not seem to work for more general boundary conditions (e.g., for mixed Dirichlet-Neumann conditions).

By a strongly Lipschitz domain, we mean an open connected set in $\mathbb{R}^n$ whose boundary is a finite union of parts of rotated graphs of Lipschitz maps, at most one of which parts is possibly infinite. These include special Lipschitz domains (the open set above a Lipschitz graph), bounded Lipschitz domains and exterior Lipschitz domains.

For an open set Ω of $\mathbb{R}^n$, $\|f\|_p$ or $\|f\|_{L^{p},(\Omega)}$ denotes the usual norm in the Lebesgue space $L^p(\Omega)$ equipped with Lebesgue measure. We write $H^1(\Omega)$ for the usual Sobolev space with norm $(\|\nabla f\|_2^2 + \|f\|_2^2)\ 1/=$ and $H_0^1(\Omega)$ for the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$.

Denote by $\mathcal{A}$ the class of elliptic matrices in $L^\infty(\mathbb{R}^n, M_n(\mathcal{C}))$ with ellipticity constants $0 < \lambda, \Lambda < \infty$, that is, the best constants in the inequalities

$$\|A\|_\infty \leq \Lambda \text{ and } \forall \xi \in \mathcal{C}^n \ \mathrm{Re}\, A(x)\xi \cdot \overline{\xi} \geq \lambda |\xi|^2, \text{a.e. on } \mathbb{R}^n.$$

If A is merely given on Ω we tacitly require A to be the λ times identity matrix elsewhere.

Given $A \in \mathcal{A}$, an open set Ω of $\mathbb{R}^n$ and a closed subspace V of $H^1(\Omega)$ containing $H_0^1(\Omega)$, denote by L the maximal-accretive operator on $L^2(\Omega)$, with largest domain $\mathcal{D}(L) \subset V$, such that

$$\langle Lf, g\rangle = \int_\Omega A\nabla f \cdot \overline{\nabla g}, \qquad f \in \mathcal{D}(L), \qquad g \in V. \tag{1}$$

The domain of L is characterized by the following condition. Let $f \in V$; then $f \in \mathcal{D}(L)$ if and only if there exists a constant c such that for all $g \in V$,

$$\left|\int_\Omega A\nabla f \cdot \overline{\nabla g}\right| \leq c\|g\|_2. \tag{2}$$

It is known that $\mathcal{D}(L)$ is dense in V [203].

Set -div $= \nabla^*\colon L^2(\Omega, \mathcal{C}^n) \to V'$ the adjoint of $\nabla\colon V \to L^2(\Omega, \mathcal{C}^n)$. By density, we may extend L continuously from V to V'. We use the same letter to denote both L or its extension depending on the context. Instead of the customary notation -div$(A\nabla)$, we prefer to write L as the triplet (A, Ω, V) to indicate the matrix of coefficients A, the domain Ω and the boundary condition determined by the space V.

Any L as above possesses a unique maximal-accretive square root $L^{1/2}$, given by Kato's representation

$$L^{1/2}f = \frac{2}{\pi}\int_0^\infty (1 + t^2L)^{-1}tLf\,\frac{dt}{t}, \qquad f \in \mathcal{D}(L). \tag{3}$$

For $f \in \mathcal{D}(L)$, we have

$$(1 + t^2L)^{-1}Lf = L(1 + t^2L)^{-1}f = t^{-2}(f - (1 + t^2L)^{-1}f); \tag{4}$$

hence, $\|(1 + t^2L)^{-1}Lf\|_2 < \inf(\|Lf\|_2, 2\|f\|_2\, t^{-2})$, since the resolvent is L^2-contractive. The above integral converges in $L^2(\Omega)$-norm. Observe that for each $t > 0$, $(1 + t^2L)^{-1}tL$ extends to a bounded operator on $L^2(\Omega)$ with

$$\|(1 + t^2L)^{-1}tLf\|_2 \leq \frac{2}{t}\|f\|_2. \tag{5}$$

Note also that $f \in V$, then $(1 + t^2L)^{-1}tLf \in V$.

To tackle the Kato conjecture, it is enough to prove one of the inequalities

$$\left\|L^{1/2}f\right\|_2 \leq c\|\nabla f\|_2, f \in V, \tag{K}$$

$$\left\| L^{1/2} f \right\|_2 \sim c(\|\nabla f\|_2 + \|f\|_2), \qquad f \in V \qquad (\mathrm{K_{loc}})$$
(it suffices to do it a priori for $f \in \mathcal{D}(L)$). Indeed, it is well-known that (K) (resp. $(\mathrm{K_{loc}})$ for L and its adjoint imply that the domain of $L^{1/2}$ is V [204].

Here, Dirichlet boundary condition means $V = H_0^1(\Omega)$; Neumann: $V = H^1(\Omega)$. Assume Ω is strongly Lipschitz. In the first case, a function f is in the domain of L if $f \in H_0^1(\Omega)$ and the divergence of $A\nabla f$ in the distributional sense on Ω belongs to $L^2(\Omega)$. In the latter case, a function f is in the domain of L if $f \in H^1(\Omega)$, the divergence of $A\nabla f$ in the distributional sense on Ω belongs to $L^2(\Omega)$ and the conormal derivative of f at the boundary vanishes.

One can think of (K) as a homogeneous or global inequality and $(\mathrm{K_{loc}})$ as an inhomogeneous or local inequality.

When Ω is unbounded (e.g., special Lipschitz or an exterior domain), this does make a difference. In particular, we do not obtain (K) on an exterior domain while we expect it. This suggests finding a different argument. In the case of bounded domains, there is no distinction between (K) and $(\mathrm{K_{loc}})$. Indeed, when Ω is a bounded connected set with Lipschitz boundary, the Poincar- Wirtinger inequality yields that $\left(\int_\Omega |\nabla f|^2\right)^{1/2}$ is a norm on $H_0^1(\Omega)$ or on the subspace of functions in $H^1(\Omega)$ with vanishing mean. Thus, (K) and $(\mathrm{K_{loc}})$ are the same in the Dirichlet case. In the Neumann case, they are the same on functions with vanishing mean; this is harmless as V and L annihilate constants and, in fact, $\mathcal{N}(\nabla) = \mathcal{C} = \mathcal{N}(L)$. Another way of saying this is by factoring out $\mathcal{C}$: write $L^2(\Omega) = L_0^2(\Omega) \oplus \mathcal{C}$, where $L_0^2(\Omega)$ is the subspace of $L^2(\Omega)$ characterized by faf = 0; then the restriction of L to $\mathcal{D}(L) \cap L_0^2(\Omega)$ is one-one, and so is the restriction of V to $H^1 \cap L_0^2(\Omega)$.

We consider that L has the form D^*AD with D being a one-one operator and the abstract nonsense material contained in [94] applies.

To prove Theorem (5.1.1), we establish (K) or $(\mathrm{K_{loc}})$ (depending on Ω) for any elliptic operator $L = (A, \Omega, V)$ as above: $A \in \mathcal{A}, \Omega$ is a strongly Lipschitz domain and V is $H_0^1(Q)$ or $H^1(\Omega)$.

By [199], (K) holds for all elliptic operators of the form $\left(A, \mathbb{R}^n, H^1(\Omega)\right)$. The argument to obtain the conclusion on any strongly Lipschitz domain contains four steps: localization, change of variables, multiplicative perturbations and the study on the upper half-space. We take them in reverse order.

Step 1: Study on the upper half-space.
Pick a coordinate system $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$. Let
$$\Omega = \mathbb{R}_+^n = \{x \in \mathbb{R}^n;\ x_n > 0\}.$$
Define the orthogonai symmetry S of $\mathbb{R}^n$ across $\partial\mathbb{R}_+^n$ by
$$S(x_1, \ldots, x_{n-1}, x_n) : (x_1, \ldots, x_{n-1}, -x_n).$$
Denote by $I(f)(x) = f(x)$ the identity operator and by $J(f)(x) = f(Sx)$ the reflection operator for $f : \mathbb{R}^n \to \mathcal{C}$. The transformation , defined by
$$\mathcal{J}(f) = \frac{1}{\sqrt{2}}\left((I + J)(f)|_{\mathbb{R}_+^n}, (I - J)(f)|_{\mathbb{R}_+^n}\right)$$
is an isometry from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}_+^n) \oplus L^2(\mathbb{R}_+^n)$ with
$$\int_{\mathbb{R}^n} |f|^2 = \frac{1}{2}\int_{\mathbb{R}_+^n} - |(I + J)(f)|^2 + \int_{\mathbb{R}_+^n} |(I - J)(f)|^2$$
and from $H^1(\mathbb{R}^n)$ to $H^1(\mathbb{R}_+^n) \oplus H_0^1(\mathbb{R}_+^n)$ (homogeneous spaces) with

$$\int_{\mathbb{R}^n} |\nabla f|^2 = \frac{1}{2}\int_{\mathbb{R}^n_+} - |\nabla(I+J)(f)|^2 + \int_{\mathbb{R}^n_+} |\nabla(I-J)(f)|^2$$

This map is also onto in both cases, and its inverse is given by

$$\mathcal{J}^{-1}(\phi,\psi) = \frac{1}{\sqrt{2}}(\phi_\epsilon + \psi_o),$$

where for $f : \mathbb{R}^n_+ \to \mathcal{C}, f_e$ (resp., fo) is its even (resp., odd) extension to $\mathbb{R}^n$ defined by $f_e(X) = f(Sx)$ (resp., $f_o(X) = -f(Sx)$) if $x_n < 0$.

Given $A \in -\mathcal{A}$, define $A^\# \in \mathcal{A}$ by $A^\#(x) = A(z)$ if $x_n > 0$ and $A^\#(x) = SA(Sx)S$ if $x_n < 0$. Let $L_D = \left(A, \mathbb{R}^n, H_0^1(\mathbb{R}^n)\right)$, $L_N = \left(A, \mathbb{R}^n_+, H^1(\mathbb{R}^n_+)\right)$ and $L^\# = \left(A, \mathbb{R}^n, H^1(\mathbb{R}^n)\right)$; and let Q_D, Q_N and $Q^\#$ be the associated sesquilinear forms as in (1). The operator ,7 relates the forms by

$$Q^\#(f,g) = Q_N(f_N, g_N) + Q_D(f_D, g_D),$$

where $\mathcal{J}(f) = (f_N, f_D)$ and $\mathcal{J}(g) = (g_N, g_D)$. Using the characterization (2) of the domain of each operator, it is not difficult to show that

$$\mathcal{D}(L^\#) = \mathcal{J}^{-1}\big(\mathcal{D}(L_D) \oplus \mathcal{D}(L_N)\big)$$

and that It follows from the interpolation result of [204] that (K) holds for L ~ if and only if it holds for both L_N and L_D. Hence, we have proved that (K) holds for any $L == (A, \mathbb{R}^n_+, V)$.

Step 2: Perturbative multiplications. Assume that m is a positive real-valued function with $m, m^{-1} \in L^\infty(\mathbb{R}^n_+)$ and let $L = (A, \mathbb{R}^n_+, V)$, The operator mL is well-defined on $\mathcal{D}(L)$ and has a square root. We have that (K) for L is equivalent to (K) for mL. The proof of Lemma 14 in the Preliminaries of [94] given on $\mathbb{R}^n$ applies with the obvious changes.

Step 3: Bilipschitz change of variables. Assume that Ω is a special Lipschitz domain: if $\Phi: \mathbb{R}^{n-1} \to \mathbb{R}$ is a defining Lipschitz function of $\partial\Omega$ the Lipschitz constant is, by definition, the quantity $\|\nabla\Phi\|_\infty$.

Choose $\phi: \mathbb{R}^n \to \mathbb{R}$ to be a bilipschitz change of variables with $\phi(\mathbb{R}^n_+) = \Omega$ and $\partial\phi(\mathbb{R}^n_+) = \partial\Omega$. Define $Tf = f \circ \phi$. Let $L = (A, \Omega, V)$. Then one has $\mathcal{D}(L) = T^{-1}\left(\mathcal{D}\left(mL_\phi\right)\right)$ and

$$L = T^{-1}\big(mL_\phi\big)T,$$

where $L_\phi = \left(A_\phi, \mathbb{R}^n_+, T^{-1}(V)\right)$ with, for $x \in \mathbb{R}^n_+$,

$$A_\phi(x) = \left|\det J_\phi(x)\right|^T J_\phi^{-1}(x) A\big(\phi(x)\big), J_\phi^{-1}(x),$$

$J_\phi^{-1}(x)$ being the jacobian matrix of ϕ at x, ${}^T J_\phi(x)$ its transpose and $m(x) = \left|\det J_\phi(x)\right|^{-1}$. Note that $T^{-1}(V) = H_0^1(\mathbb{R}^n_+)$ if $V = H^1(\Omega)$ and $T^{-1}(V) = H_0^1(\mathbb{R}^n_+)$ if $V = H_0^1(\Omega)$.

From the first two steps, we deduce that (K) is valid for L.

Step 4: Localisation.

This relies on three lemmas, the first of which we only need for $k = 2$ being the key one. We stress that since the operators are complex, the usual comparison principles for weak solutions do not apply.

Lemma (5.1.2)[197]: Let Ω be an open set of $\mathbb{R}^n$ and $A \in \mathcal{A}$. Let V be a closed subspace of $H^1(\Omega)$ that contains $H_0^1(\Omega)$ such that $v \in V$ and $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ imply $v\eta_{|\Omega} \in V$. Let P be an open set of $\mathbb{R}^n$ and, for $t > 0$, let $u_t \in V$ be such that

$$\int_{\Omega} u_t\, \overline{v} + t^2 \int_{\Omega} A\nabla u_t \cdot \overline{\nabla v} = 0$$

for all $v \in V$ such that supp $v \subset P$. Let O be an open set with positive distance to cp (in particular, $O \subset P$). Then, for any $k \in \mathcal{N}^*$, we have

$$\int_{O\cap\Omega} |u_t|^2 \leq \frac{ct^{2k}}{d^{2k}} \int_{P\cap\Omega} |u_t|^2,$$

where $d = d({}^cP, O) > 0$ and c depends on n, k and the ellipticity constants of A.

Proof. The argument uses a Caccioppoli-type inequality. Let $\eta \in C_0^\infty(\mathbb{R}^n)$, where η is real-valued with sup $\eta \cap P$; then $v = u_t \eta^2_{|\Omega}$ is an appropriate test function in V. A calculation gives

$$\int_{\Omega} |u_t|^2\eta^2 + t^2 \int_{\Omega} A\nabla u_t \cdot \overline{\nabla u_t}\eta^2 = -2t^2 \int_{\Omega} A(\eta\nabla u_t) \cdot \overline{u_t \nabla \eta}.$$

Using ellipticity and $2|ab| \leq \epsilon|a|^2 + \epsilon^{-1}|b|^2$,we ontain for all $\epsilon > 0$

Choosing $\epsilon = \lambda/\Lambda$ leads to

$$\int_{\Omega} |u_t|^2\eta^2 \leq \delta t^2 \int_{\Omega} |u_t|^2 |\nabla\eta|^2.$$

We have set $\delta = \Lambda^2/\lambda$. Observe that this is valid for all η as above. Hence, applying this inequality to η^k, k integral, and iterating yields

$$\begin{aligned} \int_{\Omega} |u_t|^2\eta^{2k} &\leq \delta t^2 \int_{\Omega} |u_t|^2 k^2 \eta^{2(k-1)} |\nabla\eta|^2 \\ &\leq k^2\delta \|\nabla\eta\|_\infty^2 t^2 \int_{\Omega} |u_t|^2 \eta^{2(k-1)} \\ &\leq (k!)^2 (\delta\|\nabla\eta\|_\infty^2 t^2)^{k-1} \int_{\Omega} |u_t|^2\eta^2. \\ &\leq (k!)^2 (\delta\|\nabla\eta\|_\infty^2 t^2)^{k} \int_{P\cap\Omega} |u_t|^2. \end{aligned}$$

It remains to choose $\eta = 1$ on O with $\|\nabla\eta\|_\infty \sim 1/d$ to conclude.

This means that the well-known Gaffney Lemma [202] extends to any complex elliptic second order operator as above with the hypotheses on V in the lemma. Of course, if Ω is strongly Lipschitz or $\mathbb{R}^n$ and $V = H_0^1(\Omega)$ or $H^1(\Omega)$, then the lemma applies.

The first consequence is the treatment of operators with coefficients that agree on an open set. We use a formulation that takes into account interior and boundary estimates in the same flow.

Lemma (5.1.3)[197]: (Comparison principle)**.** Assume both domains Ω_α and Ω_β to be either $\mathbb{R}^n$ or strongly Lipschitz. Let $A_\alpha, A_\beta \in \mathcal{A}$ such that $L_\alpha = -\mathrm{div}(A_\alpha\nabla)$ and $L_\beta = -\mathrm{div}(A_\beta\nabla)$ are operators with $V_i : H^1(\Omega)$ (resp., $V_i : H^1(\Omega_i)$)for $i : \alpha, \Omega$. Let P be an open set of $\mathbb{R}^n$ such that $P \cap \partial\Omega_\alpha = P \cap \Omega_\beta$, $P \cap \Omega_\beta \subset \Omega_\alpha$ and that $A_\alpha = A_\beta$ on $P \cap \Omega_\beta$. Then for any $\chi \in C_0^\infty(P)$ and for any open set O of $\mathbb{R}^n$ such that $d = d({}^cP, O) > O$, we have

$$\int_0^\infty \left\| \left(1 + t^2 L_\beta\right)^{-1} tL_\beta(\chi f) - (1 + t^2 L_\alpha)^{-1} tL_\alpha(x f) \right\|_{L^2(O\cap\Omega_\beta)} \frac{dt}{t} \leq \frac{c\|\chi f\|_{L^2(\Omega_\beta)}}{d}$$

for all $f \in V_\beta$, where c depends only on n and the ellipticity constants of A_β.

Proof. First, note that the assumptions on f and χ insure that $\chi f \in V_\alpha \cap V_\beta$ (we are making a slight abuse of notation, as one should distinguish $f_\alpha = \chi_{|\Omega_\alpha} f$ from $f_\beta = \chi_{|\Omega_\beta} f$. Set $u_t^i = (1+t^2L_i)^{-1}tL_i(\chi f) \in V_i$, for/= a,/~, and $u_t = u_t^\beta - u_t^\alpha$. Since $\|u_t^i\|_{L^2(\Omega_i)} \le \frac{2}{t}\|\chi f\|_{L^2(\Omega_i)} = \frac{2}{t}\|\chi f\|_{L^2(\Omega_\beta)}$

$$\int_d^\infty \|u_t^i\|_{L^2(\Omega_i)} \frac{dt}{t} \le \frac{2}{d}\|\chi f\|_{L^2(\Omega_\beta)},$$

so it is enough to prove

$$\int_0^d \|u_t\|_{L^2(O\cap\Omega_\beta)} \frac{dt}{t} \le \frac{c}{d}\|\chi f\|_{L^2(\Omega_\beta)}.$$

The variational formulation tells us that for all $v \in V_i$,

$$\int_{\Omega_i} u_t^i \bar{v} + t^2 \int_{\Omega_i} A_i \nabla u_t^i \cdot \overline{\nabla v} = -t \int_{\Omega_i} A_i \nabla(\chi f) \cdot \overline{\nabla v};$$

and since $A_\alpha = A_\beta$ on $P \cap \Omega_\beta$, we obtain

$$\int_{\Omega_\beta} u_t \bar{v} + t^2 \int_{\Omega_\beta} A_\beta \nabla u_t \cdot \overline{\nabla v} = 0$$

for all $v \in V_\beta$ such that Supp $v \subset P$. We deduce from the previous lemma that

$$\|u_t\|_{L^2(O\cap\Omega_\beta)} < \frac{ct^2}{d^2}\|u_t\|_{L^2(O\cap\Omega_\beta)} \le \frac{4ct}{d^2}\|\chi f\|_{L^2(\Omega_\beta)},$$

and the conclusion follows readily.

Next, we can also obtain estimates taking care of non-local terms.

Lemma (5.1.4)[197]: (Off-diagonal estimates)**.** Let Ω be a strongly Lipschitz domain or $\mathbb{R}^n$ and $L = (A, \Omega, V)$ an elliptic operator on Ω with Dirichlet or Neumann boundary condition. Let E, F be two closed subsets of $\mathbb{R}^n$ such that $d = d(E, F) > 0$ and $\chi \in C_0^\infty(E)$. Then

$$\int_0^\infty \|(1 + t^2L)^{-1}tL(\chi f)\|_{L^2(F\cap\Omega)} \frac{dt}{t} \le \frac{c\|\chi f\|_{L^2(\Omega)}}{d}$$

for all $f \in V$. The constant c depends on n and the ellipticity constants of A.

Proof. Again, χf should be interpreted as $\chi_{|\Omega} f$. Using (5), we have

$$\int_d^\infty \|(1 + t^2L)^{-1}tL(\chi f)\|_{L^2(\Omega)} \frac{dt}{t} \le \|\chi f\|_{L^2(\Omega)}.$$

Next, using (4) and (Supp χ) $\cap F = 0$, we obtain

$$(1 + t^2L)^{-1}tL(\chi f) = \frac{(1 + t^2L)^{-1}(\chi f)}{t}, \qquad \text{on } F\cap\Omega$$

Hence, it suffices to prove ,

$$\int_0^d \frac{1}{t}\|(1 + t^2L)^{-1}(\chi f)\|_{L^2(F\cap\Omega)} \frac{dt}{t} \le \frac{c}{d}\|\chi f\|_{L^2(\Omega)}.$$

Setting $u_t = (1+t^2L)^{-1}(\chi f)$, we have for all $v \in V$ such $v = 0$ on F

$$\int_\Omega u_t \bar{v} + t^2 \int_\Omega A \nabla u_t \cdot \overline{\nabla v} = 0$$

By Lemma (5.1.2) applied with O a neighborhood of F and P a neighborhood of O such that $d(E, P) = d(E, F)/2 > 0$, and the $L^2(\Omega)$-contractivity of the resolvent, we obtain

$$\|u_t\|_{L^2(F\cap\Omega)} < \frac{ct^2}{d^2}\|u_t\|_{L^2(P\cap\Omega)} \le \frac{ct^2}{d^2}\|\chi f\|_{L^2(\Omega)},$$

The conclusion follows at once.

We are now ready to prove that (K_{loc}) holds on all strongly Lipschitz domains. Let $L = (A,\Omega,V)$ be defined on the strongly Lipschitz domain Ω with boundary condition space given by V. Following [12], there exist an integer s, a number $d > 0$ and for $0 < k < s, C_0^\infty(\mathbb{R}^n)$ real-valued functions χ_k and η_k, and open sets O_k, P_k, Ω_k with the following properties:

(i) $\sum_{0\le k\le s} \chi_k(x) = 1$, for x in a neighborhood of Ω ;

(ii) $\Omega_0 = \mathbb{R}^n$, Supp $\chi_0 \subset O_0 \subset \overline{O_0} \subset P_0 \subset \overline{P_0} \subset \Omega$;

(iii) For $k \ge 1, \Omega_k$ is the image of a special Lipschitz domain under an orthogonal transformation in $\mathbb{R}^n$ such that Supp $\chi_k \cap \Omega \subset \Omega_k \cap \Omega$;

(iv) for $k > 1, O_k$ and P_k are open neighborhoods of Supp χ_k in $\mathbb{R}^n$ such that $\overline{O_k} \subset P_k, P_k \cap \Omega \subset \Omega_k \cap \Omega$ and $\partial\Omega \cap P_k = \partial\Omega_k \cap \overline{P_k}$, at most one of the latter possibly infinite;

(v) for $k > 0$, Supp $\eta_k \subset P_k, \eta_k = 1$ on a neighborhood of $\Omega_k, \eta_k > 0$ and $\|\eta_k\|_\infty$;

(vi) for $k \ge 0, d(O_k, {}^cP_k) > d$ and $d(\text{Supp}\,\chi_k, {}^cO_k) \ge d$.

The Lipschitz constant of Ω is the infimum of $\max(M_1,\ldots,M_s)$, where M_k is the Lipschitz constant of Ω_k, taken over all possible decompositions of f~ in this way. Roughly, there is one interior piece and s boundary pieces to look at.

For $0 < k < s$, set $L_k = (A,\Omega_k,V_k)$, where $\Omega_0 = \mathbb{R}^n, V_0 = H^1(\mathbb{R}^n)$ and for $k > 1$, if $V = H^1(\Omega)$ (resp., $H^1(\Omega)$) then $V_k = H_0^1(\Omega_k)$ (resp., $H^\downarrow(\Omega_k)$). Note that if $f \in V$, then $\chi_k f \in V \cap V_k$, so that all operations make sense.

Now that these precautions are taken, fix $f \in \mathcal{D}(L) \subset V$; since $f = \sum \chi_k f$, we may write

$$L^{1/2} f = \sum \eta_k L_k^{1/2}(\chi_k f) + \sum_{0\le k\le s}\left(L^{1/2} - L_k^{1/2}\right)(\chi_k f) \sum_{0\le k\le s}(1-\eta_k)L^{1/2}(\chi_k f).$$

By the result on $\mathbb{R}^n$ and on special Lipschitz domains together with rotational invariance, the inequality (K) holds for L_k; hence

$$\left\|\eta_k L_k^{1/2}(\chi_k f)\right\|_{L^2(\Omega)} < c_k\|\nabla\chi_k f\|_{L^2(\Omega)}.$$

Note that c_k depends on n, λ, Λ and also on M_k if, in addition, $k > 1$.

Next, the comparison principle with $L_\alpha = L_k, L_\beta = L$, $P = P_k, O = O_k, \Omega_\alpha = \Omega_k$ and $\Omega_\beta = \Omega$ and the representation (3) for square roots yield

$$\left\|\eta_k(L^{1/2} - L_k^{1/2})(\chi_k f)\right\|_{L^2(\Omega)} < \frac{c'}{d}\|\chi_k f\|_{L^2(\Omega)}.$$

Finally, the off-diagonal estimates with $E = \text{Supp}\,\chi_k$ and $F = {}^cO_k$ and (3) imply

$$\left\|(1-\eta_k)L^{1/2}(\chi_k f)\right\|_{L^2(\Omega)} < \frac{c''}{d}\|\chi_k f\|_{L^2(\Omega)}.$$

Hence (K_{loc}) follows for L. This concludes the proof of Theorem (5.1.1).

Section (5.2): The Square Root Problem for Second-Order, Divergence form Operators

We identify the domain of the square root of a divergence form operator $-\nabla \cdot \mu\nabla + 1$ on $L^p(\Omega)$ as a Sobolev space $W^{1,p}\,D\,(\Omega)$ of differentiability order 1 for $p \in]1, 2]$. (The subscript D indicates the subspace of $W^{1,p}(\Omega\,)$whose elements vanish on the boundary part D). Our focus lies on nonsmooth geometric situations in $\mathbb{R}^d$ for $d \geq 2$. So, we allow for mixed boundary conditions and, additionally, deviate from the Lipschitz property of the domain in the following spirit: the boundary ∂ decomposes into a closed subset D (the Dirichlet part) and its complement, which may share a common frontier within $\partial\Omega$. Concerning D, we only demand that it satisfies the well-known Ahlfors–David condition (equivalently: is a $(d-1)$-set in the sense of Jonsson/Wallin [234]), and only for points from the complement, we demand bi-Lipschitzian charts around. As special cases, the pure Dirichlet ($D = \partial\Omega$) and pure Neumann case ($D = \emptyset$) are also included in our considerations. Finally, the coefficient function μ is just supposed to be real, measurable, bounded and elliptic in general, cf. Assumption (5.2.5). Together, this setting should cover nearly all geometries that occur in real-world problems—as long as the domain does not have irregularities like cracks meeting the Neumann boundary part $\partial\Omega\backslash D$. In particular, all boundary points of a polyhedral 3-manifold with boundary admit bi-Lipschitzian boundary charts—irrespective how 'wild' the local geometry is, cf. [233].

The identification of the domain for fractional powers of elliptic operators, in particular that of square roots, has a long history. Concerning Kato's square root problem—in the Hilbert space L^2—see, e.g. [199],[212],[242],[225] (here, only the non-selfadjoint case is of interest). Early efforts, devoted to the determination of domains for fractional powers in the non-Hilbert space case, seem to culminate in [245]. In recent years, the problem has been investigated in the case of L^p $(p \neq 2)$ for instance in [210],[94],[90],[232],[235],[104], but only the last three are dedicated to the case of a nonsmooth $\Omega = \mathbb{R}^d$. In [90], the domain is a strong Lipschitz domain and the boundary conditions are either pure Dirichlet or pure Neumann. Our result generalizes this to a large extent and, at the same time, gives a new proof for these special cases, using more 'global' arguments. Since, in the case of a non-symmetric coefficient function μ, for the nonsmooth constellations described above no general condition is known that assures $(-\nabla \cdot \mu\nabla + 1)^{1/2} : W_D^{1,2}\Omega \to L^2(\Omega)$ to be an isomorphism, this is supposed as one of our assumptions. This serves then as our starting point to show the corresponding isomorphism property of $(-\nabla \cdot \mu\nabla + 1)^{1/2} : W_D^{1,p}(\Omega) \to L^p(\Omega)$ for $p \in]1, 2[$. For the case $d = 1$, this is already known, even for all $p \in]1, \infty[$ and more general coefficient functions μ, cf. [211]. So we stick to the case $d \geq 2$.

Whilst the isomorphism property is already interesting in itself, our original motivation comes from applications: having the isomorphism $(-\nabla \cdot \mu\nabla + 1)^{1/2} : W_D^{1,p}(\Omega) \to L^p(\Omega)$ at hand, the adjoint isomorphism $\left((-\nabla \cdot \mu\nabla + 1)^{1/2}\right)^* = (-\nabla \cdot \mu T\,\nabla + 1)^{1/2} : L^q\,(\Omega) \to W_D^{-1,q}\,(\Omega)$ allows to carry over substantial properties of the operators $-\nabla \cdot \mu\nabla$ on the L^p-scale to the scale of $W_D^{-1,q}$ -spaces for $q \in [2, \infty[$. In particular, this concerns the H^∞-calculus and maximal parabolic regularity, which in turn is a powerful tool for the treatment of linear and nonlinear parabolic equations, see, e.g. [244] and [59].

After presenting some notation and general assumptions, we introduce the Sobolev scale $W_D^{1,p}$ (), $1 \le p \le \infty$, related to mixed boundary conditions and point out some of their properties. We define properly the elliptic operator under consideration and collect some known facts for it. The main result on the isomorphism property for the square root of the elliptic operator is precisely formulated. The following contain preparatory material for the proof of the main result, which is finished at the end. Some of these results have their own interest, such as Hardy's inequality for mixed boundary conditions that is proved and the results on real and complex interpolation for the spaces $W_D^{1,p}(\Omega), 1 \le p \le \infty$, so we shortly want to comment on these.

The proof of Hardy's inequality heavily rests on two things: first one uses an operator that extends functions from $W_D^{1,p}(\Omega)$ to $W_0^{1,p}(\Omega_\bullet)$, where $\Omega_\bullet$ is a domain containing . Then, one is in a situation where the deep results of Ancona [236], Lewis [70] and Wannebo [87], combined with Lehrbäck's [67] ingenious characterization of pfatness, may be applied.

The proof of the interpolation results, as well as other steps in the proof of the main result, is fundamentally based on an adapted Calderón–Zygmund decomposition for Sobolev functions. Such a decomposition was first introduced in [210] and has also successfully been used in [213], see also [214]. We have to modify it, since the main point here is, that the decomposition has to respect the boundary conditions. This is accomplished by incorporating Hardy's inequality into the controlling maximal operator. This result, which is at the heart of our considerations, is contained. All these preparations, together with off-diagonal estimates for the semigroup generated by our operator, lead to the proof of the main result. We draw some consequences, as already sketched above. After having finished, we got to know of [237]. There, among other deep things, Lemma (5.2.3) and the interpolation results are also proved— and this in an even much broader setting than ours.

We will use $x, y, \ldots$ for vectors in $\mathbb{R}^d$ and the symbol $B(x,r)$ stands for the ball in $\mathbb{R}^d$ around x with radiusr. For $, F \subseteq \mathbb{R}^d$, we denote by $d(E,F)$ the distance between E and F, and if $E = \{x\}$, then we write $d(x,F)$ or $d_F(x)$instead. Regarding our geometric setting, we suppose the following assumption.

Assumption (5.2.1)[206]: (i) Let $d \ge 2$, let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain and let D be a closed subset of the boundary $\partial\Omega$ (to be understood as the Dirichlet boundary part). For every $x \in \overline{\partial\Omega \setminus D}$, there exists an open neighbourhood U_x of x and a bi-Lipschitz map ϕ_x from U_x onto the cube $K :=]-1,1[^d$, such that the following three conditions are satisfied:

$$\phi_x(x) = 0,$$
$$\phi_x(U_x \cap \Omega) = \{x \in K : x_d < 0\} =: K_-,$$
$$\phi_x(U_x \cap \partial\Omega) = \{x \in K : x_d = 0\} =: \Sigma.$$

(ii) We suppose that D is either empty or satisfies the Ahlfors–David condition: there are constants $c_0, c_1 > 0$ and $r_{AD} > 0$, such that for all $x \in D$ and all $r \in]0, r_{AD}]$

$$c_0 r^{d-1} \le \mathcal{H}_{d-1}\big(D \cap B(x,r)\big) \le c_1 r^{d-1}, \qquad (6)$$

where $\mathcal{H}_{d-1}$ denotes (here and in the sequel) the $(d-1)$-dimensional Hausdorff measure, defined by

$$\mathcal{H}_{d-1}(A) := \liminf_{\varepsilon\to 0} \left\{ \sum_{j=1}^{\infty} \operatorname{diam}(A_j)^{d-1} : A_j \subseteq \mathbb{R}^d, \operatorname{diam}(A_j) \le \varepsilon, A \subseteq \bigcup_{j=1}^{\infty} A_j \right\}.$$

If B is a closed operator on a Banach space X, then we denote by $\operatorname{dom}_X(B)$ the domain of this operator. $\mathcal{L}(X,Y)$ denotes the space of linear, continuous operators from X into Y; if $X = Y$, then we abbreviate $\mathcal{L}(X)$. Furthermore, we will write $\langle \cdot,\cdot \rangle_{X'}$ for the pairing of elements of X and the dual space X' of X.

Finally, the letters c and C denote generic constants that may change value from occurrence to occurrence.

We will introduce the Sobolev spaces related to mixed boundary conditions and prove some results related to them that will be needed later.

If Υ is an open subset of $\mathbb{R}^d$ and F a closed subset of $\overline{\Upsilon}$, e.g. the Dirichlet part D of $\partial\Omega$, then for $1 \le q < \infty$, we define $W_F^{1,q}(\Upsilon)$ as the completion of

$$C_F^\infty(\Upsilon) := \{\psi|_\Upsilon : \psi \in C_0^\infty(\mathbb{R}^d), \operatorname{supp}(\psi) \cap F = \emptyset\} \tag{7}$$

with respect to the norm $\psi \mapsto \left(\int_\Upsilon |\nabla\psi|^q + |\psi|^q \, dx\right)^{1/q}$. For $1 < q < \infty$, the dual of this space will be denoted by $W_F^{-1,q'}(\Upsilon)$ with $1/q + 1/q' = 1$. Here, the dual is to be understood with respect to the extended L^2 scalar product, or, in other words: $W_F^{-1,q'}(\Upsilon)$ is the space of continuous antilinear forms on $W_F^{1,q}(\Upsilon)$.

Finally, we define the respective spaces for the case $q = \infty$. We set $W_F^{1,\infty}(\Upsilon) := \operatorname{Lip}_{\infty,F}(\Upsilon)$ with

$$\begin{aligned}\operatorname{Lip}_{\infty,F}(\Upsilon) &:= \{f|_\Upsilon : f \in (L^\infty \cap \operatorname{Lip})(\mathbb{R}^d), f|_F = 0\} \\ &= \{f \in (L^\infty \cap \operatorname{Lip})(\Upsilon), f|_F = 0\}.\end{aligned} \tag{8}$$

The norm on this space is

$$\|f\|_{L^\infty(\Upsilon)} + \sup_{x,y\in\Upsilon, x\neq y} \frac{|f(x) - f(y)|}{|x-y|}.$$

The last equality in (8) is a consequence of the Whitney extension theorem. We have $\operatorname{Lip}_{\infty,F}(\Upsilon) \subseteq \{f \in \mathcal{W}^{1,\infty}(\Upsilon) : f|_F = 0\}$ ($\mathcal{W}^{1,\infty}(\Upsilon)$ is defined using distributions) and the converse holds iff is uniformly locally convex by [32, Theorem 7].

In order to simplify notation, we drop the Ω in the notation of spaces, if misunderstandings are not to be expected. Thus, function spaces without an explicitly given domain are to be understood as function spaces on Ω.

Lemma (5.2.2)[206]: Let $\Upsilon \subseteq \mathbb{R}^d$ be a bounded domain and F a (relatively) closed subset of $\partial\Upsilon$, which is of Lebesgue measure 0. Then, $W_F^{1,\infty}(\Upsilon) \subseteq W_F^{1,q}(\Upsilon)$ for $1 \le q < \infty$.

Proof. Let $(\alpha_n)_n$ be the sequence of cut-off functions defined on $\mathbb{R}^+$ by

$$\alpha_n(t) = \begin{cases} 0, & \text{if } 0 \le t < 1/n, \\ nt - 1, & \text{if } 1/n \le t \le 2/n, \\ 1, & \text{if } t > 2/n. \end{cases}$$

Remark that for $t \neq 0$, the sequence $\alpha_n(t)$ tends to 1 as $n \to \infty$. Furthermore, for all $t \ge 0$, we have $0 \le t\alpha_n'(t) \le 2$ and the sequence $\left(t\alpha_n'(t)\right)_n$ tends to 0.

For $\in \mathbb{R}^d$, we set $w_n(x) := \alpha_n(d(x,F))$. Then, by the above considerations, $w_n \to 1$ almost everywhere as $n \to \infty$. The function $d(\cdot,F)$ is Lipschitzian with Lipschitz constant 1, and hence, it belongs to $W_{\text{loc}}^{1,\infty}(\mathbb{R}^d)$, cf. [245]. Since α is piecewise smooth, the usual chain rule for weak differentiation (cf. [29, Ch. 7.4 Thm. 7.8]) applies, which gives

$$|\nabla w_n(x)| = \left|\alpha_n'\big(d(x,F)\big)\right| |\nabla d(x,F)| \le \left|\alpha_n'\big(d(x,F)\big)\right|$$

almost everywhere on $\mathbb{R}^d$. Thus, $d(x,F)|\nabla w_n(x)|$ is bounded and converges to 0 almost everywhere as $n \to \infty$.

Let $g \in W_F^{1,\infty}(\Upsilon)$, which we consider as defined on $\mathbb{R}^d$. Since Υ is bounded, we may assume that g has compact support in some large ball B. Let $g_n := g w_n$. Then, g_n is compactly supported in B and in $\mathbb{R}^d \backslash F$. We claim that $g_n \to g$ in $W^{1,q}(\mathbb{R}^d)$. Indeed, $g - g_n = g(1 - w_n)$ and, by the dominated convergence theorem, $g(1 - w_n) \to 0$ in $L^q(\mathbb{R}^d)$, since $w_n \to 1$. Now, for the gradient, we have

$$\nabla g_n - \nabla g = (w_n - 1)\nabla g + g\nabla w_n.$$

Again by the dominated convergence theorem, the first term converges to 0 in $L^q(\mathbb{R}^d)$. It remains to prove that $\|g\nabla w_n\|_{L^q(\mathbb{R}^d)}$ converges to 0. We have

$$(g\nabla w_n)(x) = \begin{cases} 0, & \text{if } x \in F, \\ \dfrac{g(x)}{d(x,F)} d(x,F)\nabla w_n(x) & \text{a.e. on } \mathbb{R}^d \backslash F \end{cases}. \tag{9}$$

Since g is Lipschitz continuous on the whole of $\mathbb{R}^d$ and satisfies $g = 0$ on F, we find

$$\sup_{x\in\mathbb{R}^d} \left|\frac{g(x)}{d(x,F)}\right| = \sup_{x\in\mathbb{R}^d} \left|g(x) - \frac{g(x_*)}{x - x_*}\right| \le C,$$

where $x_* \in F$ denotes an element of F that realizes the distance of x to F. So both factors on the right-hand side in (9) are bounded and $d(x,F)\nabla w_n(x)$ goes to 0 almost everywhere as $n \to \infty$. Thus, since g has compact support, the dominated convergence theorem yields $g\nabla w_n \to 0$ in $L^q(\mathbb{R}^d)$.

Finally, it suffices to convolve this approximation with a smooth mollifying function that has small support to conclude $g \in W_F^{1,q}(\Upsilon)$.

Next, we establish the following extension property for function spaces on domains, satisfying just part (i) of Assumption (5.2.1). This has been proved in [244] for $q = 2$. We include a proof.

Lemma (5.2.3)[206]: Let Ω and D satisfy Assumption (5.2.1) (i). Then, there is a continuous extension operator $\mathfrak{E}$, which maps each space $W_D^{1,q}(\Omega)$ continuously into $W_D^{1,q}(\mathbb{R}^d)$, $q \in [1,\infty]$. Moreover, $\mathfrak{E}$ maps $L^q(\Omega)$ continuously into $L^q(\mathbb{R}^d)$ for $q \in [1,\infty]$.

Proof. For every $x \in \overline{\partial\Omega\backslash D}$, let the set U_x be an open neighbourhood that satisfies the condition from Assumption (5.2.1) (i). Let $U_{x_1}, \dots, U_{x_\ell}$ be a finite subcovering of $\overline{\partial\Omega\backslash D}$ and let $\eta \in C_0^\infty(\mathbb{R}^d)$ be a function that is identically one in a neighbourhood of $\overline{\partial\Omega\backslash D}$ and has its support in $U := \bigcup_{j=1}^{\ell} U_{x_j}$.

Assume $\psi \in C_D^\infty(\Omega)$; then, we can write $\psi = \eta\psi + (1 - \eta)\psi$. By the definition of $C_D^\infty(\Omega)$ and η, it is clear that the support of $(1 - \eta)\psi$ is contained in , and thus, this function may be extended by 0 to the whole space $\mathbb{R}^d$—whilst its $W^{1,q}$ -norm is preserved.

It remains to define the extension of the function $\eta\psi$, what we will do now. For this, let $\eta_1, \dots, \eta_\ell$ be a partition of unity on supp (η), subordinated to the covering $Ux_1, \dots, U_{x_\ell}$. Then, we can write $\eta\psi = \sum_{r=1}^{\ell} \eta_r\eta\psi$ and have to define an extension for every function $\eta_r\eta\psi$. For doing so, we first transform the corresponding function under the corresponding mapping ϕ_{x_r} from Assumption (5.2.1) (i) to $\widetilde{\eta_r \eta\psi} = (\eta_r\eta\psi) \circ \phi_{x_r}^{-1}$ on the half cube K_-. Afterwards, by even reflection, one obtains a function $\eta_r\eta\psi \in W^{1,q}(K)$ on the cube K. It is clear by construction that supp $(\widehat{\eta_r\eta\psi})$ has a positive distance to ∂K. Transforming back, one ends up with a function $\eta_r\eta\psi \in W^{1,q}(U_{x_r})$ whose support has a positive distance to ∂U_{x_r}. Thus, this function may also be extended by 0 to the whole of $\mathbb{R}^d$, preserving again the $W^{1,q}$ norm. Lastly, one observes that all the mappings $W^{1,q}(U_{x_r} \cap \Omega) \ni \eta_r\eta\psi \mapsto \eta_r\eta\psi \in W^{1,q}(K_-), W^{1,q}(K_-) \ni \widetilde{\eta_r\eta\psi} \mapsto \widehat{\eta_r\eta\psi} \in W^{1,q}(K)$ and $W^{1,q}(K) \ni \widehat{\eta_r\eta\psi} \mapsto \underline{\eta r\eta\psi} \in W^{1,q}\left(U_{x_r}\right)$ are continuous. Thus, adding up, one arrives at an extension of ψ whose $W^{1,q}(\mathbb{R}^d)$-norm may be estimated by $c\|\psi\|_{W^{1,q}(\Omega)}$ with c independent from ψ. Hence, the mapping $\mathfrak{E}$, up to now defined on $C^\infty D(\Omega)$, continuously and uniquely extends to a mapping from $W_D^{1,q}$ to $W^{1,q}(\mathbb{R}^d)$.

It remains to show that the images in fact even are in $W_D^{1,q}(\mathbb{R}^d)$. For doing so, one first observes that, by construction of the extension operator, for any $\psi \in C_D^\infty(\Omega)$, the support of the extended function $\mathfrak{E}\psi$ has a positive distance to D—but $\mathfrak{E}\psi$ need not be smooth. Clearly, one may convolve $\mathfrak{E}\psi$ suitably in order to obtain an appropriate approximation in the $W^{1,q}(\mathbb{R}^d)$-norm—maintaining a positive distance of the support to the set D. Thus, $\mathfrak{E}$ maps $C_D^\infty(\Omega)$ continuously into $W_D^{1,q}(\mathbb{R}^d)$, what is also true for its continuous extension to the whole space $W_D^{1,q}(\Omega)$.

It is not hard to see that the operator $\mathfrak{E}$ extends to a continuous operator from $L^q(\Omega)$ to $L^q(\mathbb{R}^d)$, where $q \in [1,\infty]$.

This Poincaré inequality entails that, whenever $D \neq \emptyset$, the norms given by $\|f\|_{W_D^{1,p}}$ and $\|\nabla f\|_{L^p}$ for $f \in W_D^{1,p}$ are equivalent. So, in this case, in all subsequent considerations, one may freely replace the one by the other.

We now turn to the definition of the elliptic divergence form operator that will be investigated. Let us first introduce the ellipticity supposition on the coefficients.

Assumption (5.2.4)[206]: The coefficient function μ is a Lebesgue measurable, bounded function on taking its values in the set of real, $d \times d$ matrices, satisfying for some $\mu_\bullet > 0$ the usual ellipticity condition

$$\xi^T\mu(x)\xi \geq \mu_\bullet|\xi|^2, \qquad \text{for all } \xi \in \mathbb{R}^d \text{ and almost all } x \in \Omega.$$

The operator $A: W_D^{1,2} \to W_D^{-1,2}$ is defined by

$$\langle A\psi,\varphi\rangle_{W_D^{-1,2}} := t(\psi,\varphi) := \int_\Omega \mu\nabla\psi \cdot \nabla\overline{\varphi}\,dx, \qquad \psi,\varphi \in W_D^{1,2}.$$

Often, we will write more suggestively $-\nabla \cdot \mu\nabla$ instead of A.

The L^2 realization of A, i.e. the maximal restriction of A to the space L^2, will be denoted by the same symbol A; clearly, this is identical with the operator that is induced by the sesquilinear form t. If B is a densely defined, closed operator on L^2, then by the L^p realization of B we mean its restriction to L^p if $p > 2$ and the L^p closure of B if $p \in [1,2[$. (For all operators we have in mind, this L p-closure exists).

As a starting point of our considerations, we assume that the square root of our operator is well behaved on L^2.

Assumption (5.2.5)[206]: The operator $(-\nabla\cdot\mu\nabla+1)^{1/2}: W_D^{1,2}\to L^2$ provides a topological isomorphism; in other words: the domain of $(-\nabla\cdot\mu\nabla+1)^{1/2}$ on L^2 is the form domain $W_D^{1,2}$.

Proposition (5.2.6)[206]: Let $\Omega\subseteq\mathbb{R}^d$ be a domain and let $D\subseteq\partial\Omega$ (relatively) closed.

(i) The restriction of $-\nabla\cdot\mu\nabla$ to L^2 is a densely defined sectorial operator.

(ii) The operator $\nabla\cdot\mu\nabla$ generates an analytic semigroup on L^2.

(iii) The form domain $W_D^{1,2}$ is invariant under multiplication with functions from $W^{1,q}$, if $q>d$.

Proof. (i) It is not hard to see that the form t is closed and its numerical range lies in the sector $\{z\in\mathbb{C}:|\operatorname{Im}z|\le\frac{\|\mu\|_{L^\infty}}{\mu_\bullet}\operatorname{Re}z\}$. Thus, the assertion follows from a classical representation theorem for forms, see [238].

(ii) This follows from (i) and [238].

(iii) First, for $u\in C_D^\infty(\Omega)$ and $v\in C^\infty(\Omega)$, the product uv is obviously in $C_D^\infty(\Omega)\subseteq W_D^{1,2}$. But, by definition of $W_D^{1,2}$, the set $C_D^\infty(\Omega)$ (see (5.2.2)) is dense in $W_D^{1,2}$ and $C^\infty(\Omega)$ is dense in $W^{1,q}$. Thus, the assertion is implied by the continuity of the mapping

$$W_D^{1,2}\times W^{1,q}\ni(u,v)\mapsto uv\in W^{1,2},$$

because $W_D^{1,2}$ is closed in $W^{1,2}$.

Proposition (5.2.7)[206]: Let Ω and D satisfy Assumption (5.2.1) (i). Then, the semigroup generated by $\nabla\cdot\mu\nabla$ in L^2 satisfies upper Gaussian estimates, precisely:

$$(e^{t\nabla\cdot\mu\nabla}f)(x)=\int_\Omega K_t(x,y)f(y)\,dy,\qquad\text{for a.a. } x\in\Omega, f\in L^2,$$

for some measurable function $K_t:\Omega\times\Omega\to\mathbb{R}_+$, and for all ε > 0, there exist constants $C,c>0$, such that

$$0\le K_t(x,y)\le\frac{C}{t^{d/2}}\,e^{-c\frac{|x-y|^2}{t}}\,e^{\varepsilon t},\qquad t>0, a.a. x,y\in\Omega.\qquad(10)$$

Proof. A proof is given in [244]—heavily resting on [209], compare also [242].

Proposition (5.2.8)[206]: Let Ω and D satisfy Assumption (5.2.1) (i).

(i) For every $p\in[1,\infty]$, the operator $\nabla\cdot\mu\nabla$ generates a semigroup of contractions on L^p.

(ii) For all $q\in]1,\infty[$ the operator $-\nabla\cdot\mu\nabla+1$ admits a bounded H^∞-calculus on L^q with H^∞-angle arctan $\frac{\|\mu\|_{L^\infty}}{\mu_\bullet}$. In particular, it admits bounded imaginary powers.

Proof. (i) The operator $\nabla\cdot\mu\nabla$ generates a semigroup of contractions on L^2 (see [242]) as well as on L^∞ (see [242]). By interpolation, this carries over to every L^q with $q\in]2,\infty[$ and, by duality, to $q\in[1,2]$.

(ii) Since the numerical range of $-\nabla\cdot\mu\nabla$ is contained in the sector $\{z\in\mathbb{C}:|\operatorname{Im}z|\le\frac{\|\mu\|_{L^\infty}}{\mu_\bullet}\operatorname{Re}z\}$, the assertion holds true for $q=2$, see [228]. Secondly, the semigroup generated by $\nabla\cdot\mu\nabla-1$ obeys the Gaussian estimate (10) with ε = 0. Thus, the first assertion follows from [223]. The second claim is a consequence of the first, see [219].

We can now formulate our main goal, that is, to prove that the mapping

$$(A+1)^{1/2}=(-\nabla\cdot\mu\nabla+1)^{1/2}:W_D^{1,q}\to L^q$$

is a topological isomorphism for $q\in]1,2[$. We abbreviate $-\nabla\cdot\mu\nabla+1$ by A_0. We want to show the following main result.

Theorem (5.2.9)[206]: Under Assumptions (5.2.1), (5.2.4) and (5.2.5), the following holds true:

(i) For every $q \in]1,2]$, the operator $A_0^{-1/2}$ is a continuous operator from L^q into $W_D^{1,q}$. Hence, its adjoint continuously maps $W_D^{-1,q}$ into L^q for any $q \in [2,\infty[$.

(ii) Moreover, if $q \in]1,2]$, then $A_0^{1/2}$ maps $W_D^{1,q}$ continuously into L^q. Hence, its adjoint continuously maps L^q into $W_D^{-1,q}$ for any $q \in [2,\infty[$.

We can immediately give the proof of (i), i.e. the continuity of the operator $A_0^{-1/2}: L^q \to W_D^{1,q}$. We observe that this follows, whenever

1. The Riesz transform $\nabla A_0^{-1/2}$ is a bounded operator on L^q, and, additionally,

2. $A_0^{-1/2}$ maps L^q into $W_D^{1,q}$. The first item is proved in [242], compare also [222]. It remains to show 2. The first point makes clear that $A_0^{-1/2}$ maps L^q continuously into $W^{1,q}$, and thus, one only has to verify the correct boundary behaviour of the images. If $\in L^2 \mapsto L^q$, then one has $A_0^{-1/2} f \in W_D^{1,2} \mapsto W_D^{1,q}$, due to Assumption (5.2.5). Thus, the assertion follows from 1, and the density of L^2 in L^q.

A major tool in our considerations is an inequality of Hardy type for functions in $W_D^{1,p}$, so functions that vanish only on the part D of the boundary. We recall that, for a set $\subseteq \mathbb{R}^d$, the symbol dF denotes the function on $\mathbb{R}^d$ that measures the distance to F. The result we want to show is the following.

Theorem (5.2.10)[206]: Under Assumption (5.2.1), for every $p \in]1,\infty[$, there is a constant cp, such that

$$\int_\Omega \left|\frac{f}{d_D}\right|^p dx \le c_p \int_\Omega |\nabla f|^p \, dx$$

holds for all $f \in W_D^{1,p}$.

Since the statement of this theorem is void for $D = \emptyset$, we exclude that case for this entire. Please note that then the norm on the spaces $W_D^{1,p}$ may be taken as $\|\nabla \cdot\|_p$ in view of the Ahlfors–David condition of D.

We quote the two deep results on which the proof of Theorem (5.2.10) will base.

Proposition (5.2.11)[206]: (See [70],[87], see also [64]) Let $\Xi \subseteq \mathbb{R}^d$ be a domain whose complement $K := \mathbb{R}^d \setminus \Xi$ is uniformly p-fat (cf. [70] or [64]). Then, Hardy's inequality

$$\int_\Xi \left|\frac{g}{d_K}\right|^p dx = \int_\Xi \left|\frac{g}{d_{\partial\Xi}}\right|^p dx \le c \int_\Xi |\nabla g|^p \, dx \tag{11}$$

holds for all $g \in C_0^\infty(\Xi)$ (and extends to all $g \in W_0^{1,p}(\Xi), p \in]1,\infty[$ by density).

Proposition (5.2.12)[206]: [67] Let $\Xi \subseteq \mathbb{R}^d$ be a domain and let $\mathcal{H}_{d-1}^\infty$ denote the $(d-1)$-dimensional Hausdorff content, i.e.

$$\mathcal{H}_{d-1}^\infty(A) := \inf\left\{\sum_{j=1}^\infty r_j^{d-1} : x_j \in A, r_j > 0, A \subseteq \bigcup_{j=1}^\infty B(x_j, r_j)\right\}.$$

If Ξ satisfies the inner boundary density condition, i.e.

$$\mathcal{H}_{d-1}^\infty\left(\partial\Xi \cap B\big(x, 2d_{\partial\Xi}(x)\big)\right) \ge c\, d_{\partial\Xi}(x)^{d-1}, \qquad x \in \Xi, \tag{12}$$

for some constant $c > 0$, then the complement of Ξ in $\mathbb{R}^d$ is uniformly p-fat for all $p \in]1,\infty[$.

The subsequent lemma will serve as the instrument to reduce our case to the situation of a pure Dirichlet boundary.

Lemma (5.2.13)[206]: Let $B \supseteq \overline{\Omega}$ be an open ball. We define $\Omega_\bullet$ as the union of all open, connected subsets of B that contain Ω and avoid D. Then, $\Omega_\bullet$ is open and connected, and we have $\partial\Omega_\bullet = D$ or $\partial\Omega_\bullet = D \cup \partial B$.

Proof. The first assertion is obvious. The connectedness follows from the fact that all the sets that, by forming their union, generate $\Omega_\bullet$ contain Ω, and, hence, a common point. It remains to show the last assertion. Clearly, we have $\partial\Omega_\bullet \subseteq \overline{B}$.

We claim that $D \subseteq \partial\Omega_\bullet$: Let $x \in D$. As $D \subseteq \partial\Omega$, we know that x is an accumulation point of and thus also of $\Omega_\bullet$, since $\Omega \subseteq \Omega_\bullet$. Furthermore, $x \notin \Omega_\bullet$. Hence, $x \in \partial\Omega_\bullet$. We claim that $\partial\Omega_\bullet \subseteq \partial B \cup D$. Assume not. Then, there exists $x \in \partial\Omega_\bullet$ with $x \in B\backslash D$. As $B\backslash D$ is open, it contains an open ball K_x centred at x. Then, $\Omega_\bullet \cup K_x$ is an open and connected (since x is a point of accumulation of $\Omega_\bullet$, the set $\Omega_\bullet \cap K_x$ is not empty) set containing , contained in B and not meeting D. As it strictly contains $\Omega_\bullet$, this contradicts the definition of $\Omega_\bullet$.

We now consider an annulus $K_B \subseteq B$ that is adjacent to ∂B and does not intersect $\overline{\Omega}$. If $\Omega_\bullet \cap K_B = \emptyset$, then $\partial\Omega_\bullet \subseteq B$, and, consequently, $\partial\Omega_\bullet = D$. If $\Omega_\bullet \cap K_B$ is not empty, then $\Omega_\bullet \cup K_B$ is open, connected, contains Ω, avoids D and is contained in B. Hence, $\Omega_\bullet \cup K_B \subseteq \Omega_\bullet$, what implies $\partial B \subseteq \partial\Omega_\bullet$.

The next lemma links the Hausdorff content, appearing in Proposition (5.2.12), to the Hausdorff measure, compare also [64].

Lemma (5.2.14)[206]: If $F \subseteq \mathbb{R}^d$ is bounded and satisfies the Ahlford–David condition (6), then there is a $C \geq 0$ with $\mathcal{H}_{d-1}^{\infty}(E) \geq C\mathcal{H}^{d-1}(E)$ for every non-empty Borel set $E \subseteq F$.

Proof. Let $\{B(x_j,r_j)\}_{j\in\mathbb{N}}$ be a covering of E by open balls centred in E. If $r_j \leq 1$, then r_j^{d-1} is comparable to $\mathcal{H}_{d-1}\left(F \cap B(x_j,r_j)\right)$, whereas if $r_j > 1$, then certainly $\mathcal{H}_{d-1}\left(F \cap B(x_j,r_j)\right) \leq \mathcal{H}_{d-1}(F)r_j^{d-1}$. Note carefully that $0 < \mathcal{H}_{d-1}(F) < \infty$ holds, since F can be covered by finitely many balls with radius one centred in F. Altogether,

$$\sum_{j=1}^{\infty} r_j^{d-1} \geq C\sum_{j=1}^{\infty} \mathcal{H}_{d-1}\left(F \cap B(x_j,r_j)\right) \geq C\mathcal{H}_{d-1}\left(F \cap \bigcup_{j=1}^{\infty} B(x_j,r_j)\right)$$
$$\geq C\mathcal{H}_{d-1}(E)$$

with C depending only on F. Taking the infimum, $\mathcal{H}_{d-1}^{\infty}(E) \geq C\mathcal{H}_{d-1}(E)$ follows.

Let us now prove Theorem (5.2.10). One first observes that in both cases appearing in Lemma (5.2.13), the set $\partial\Omega_\bullet$ satisfies the Ahlfors–David condition: for the boundary part D, this was supposed in Assumption (5.2.1), and for ∂B, this is obvious. Thus, from the Ahlfors–David condition for $\partial\Omega_\bullet$, we get constants $\mathrm{r}_\bullet > 0$ and $c > 0$ with

$$\mathcal{H}_{d-1}\left(\partial\Omega_\bullet \cap B(y,r)\right) \geq cr^{d-1}, y \in \partial\Omega_\bullet, r \in]0, r_\bullet\ ].$$

This yields, invoking Lemma (5.2.14),

$$\mathcal{H}_{d-1}^{\infty}\left(\partial\Omega_\bullet \cap B(y,r)\right) \geq C\mathcal{H}_{d-1}\left(\partial\Omega_\bullet \cap B(y,r)\right) \geq Cc\left(\frac{r_\bullet}{\mathrm{diam}(\Omega_\bullet)}\right)^{d-1} r^{d-1}, y \in \partial\Omega_\bullet, r \in]0, diam(\Omega_\bullet)]. (6.3)$$

But (6.3) implies the inner boundary density condition (12), compare [47, p. 2195]. Thus, Proposition (5.2.11) and Proposition (5.2.12) imply that Hardy's inequality in (11) is true for $\Xi = \Omega_\bullet$ and all $g \in W_0^{1,p}(\Omega_\bullet)$.

In view of Lemma (5.2.13), we can define an extension operator $\mathfrak{E}_\bullet : W_D^{1,p}(\Omega) \to W_0^{1,p}(\Omega_\bullet)$ as follows: If $\partial\Omega_\bullet = D$, then we put $\mathfrak{E}\Omega_\bullet\psi := \mathfrak{E}\psi|_\bullet$, where $\mathfrak{E}$ is the extension operator from Lemma (5.2.3). If $\partial\Omega_\bullet = D \cup \partial B$, then we choose an $\eta \in C_0^\infty(B)$ with $\eta \equiv 1$ on and put $\mathfrak{E}_\bullet\psi := \eta\mathfrak{E}\psi|_{\Omega_\bullet}$. Now, let $f \in W_D^{1,p}(\Omega)$. Then, we can use (11) for $\mathfrak{E}_\bullet f \in W_0^{1,p}(\Omega_\bullet)$, and we finally find

$$\int_\Omega \left|\frac{f}{d_D}\right|^p dx \le \int_\Omega \left|\frac{f}{d_{\partial\Omega_\bullet}}\right|^p dx \le \int_{\Omega_\bullet} \left|\frac{\mathfrak{E}_\bullet f}{d_{\partial\Omega_\bullet}}\right|^p dx \le c\int_{\Omega_\bullet} |\nabla(\mathfrak{E}_\bullet f)|^p\,dx$$
$$\le c\|f\|^p_{W_D^{1,p}} \le c\int_\Omega |\nabla f|^p dx.$$

This proves Theorem (5.2.10).

The proof of Theorem (5.2.9) heavily relies on a Calderón–Zygmund decomposition for $W_D^{1,p}$ functions. The important point, which brings the mixed boundary conditions into play, is that we have to make sure that for $f \in \mathrm{dom}_{L^p}(A_0^{1/2})$, the good and the bad part of the decomposition are both also in this space. This is not guaranteed neither by the classical Calderón–Zygmund decomposition nor by the version for Sobolev functions in [5, Lemma (5.2.4)2]. This problem will be solved by incorporating the Hardy inequality into the decomposition.

For ease of notation, we set $1/d_\emptyset = 0$.

We denote by Q the set of all closed axis-parallel cubes, i.e. all sets of the form $\{x \in \mathbb{R}^d : |x - m|^\infty \le \ell/2\}$ for some midpoint $m \in \mathbb{R}^d$ and sidelength $\ell > 0$. In the following, for a given cube $Q \in Q$, we will often write $s > Q$ for some $s > 0$, meaning the cube with the same midpoint m, but sidelength $s\ell$ instead of ℓ.

Furthermore, for every $x \in \mathbb{R}^d$, we set $Q_x := \{Q \in Q : x \in Q^\circ\}$. Now, we may define the Hardy–Littlewood maximal operator M for all $\varphi \in L^1(\mathbb{R}^d)$ by

$$(M\varphi)(x) = \sup_{Q\in Q_x} \frac{1}{|Q|}\int_Q |\varphi|, \qquad x \in \mathbb{R}^d . \tag{13}$$

It is well known (see [246]) that M is of weak type (1, 1), so there is some $K > 0$, such that for all $p \ge 1$

$$\left|\{x \in \mathbb{R}^d : |[M(|\varphi|^p)](x)| > \alpha^p\}\right| \le \frac{K}{\alpha^p}\|\varphi\|^p\_(L^p(\mathbb{R}^d), \text{for all } \alpha > 0 \text{ and } \varphi \in L^p(\mathbb{R}^d). \tag{14}$$

Lemma (5.2.15)[206]: Let and D satisfy Assumption (5.2.1). Let $p \in]1,\infty[, f \in W_D^{1,p}$ and $\alpha > 0$ be given. Then, there exist an at most countable index set , cubes $Q_j \in Q, j \in I$, and measurable functions $g, b_j : \to \mathbb{R}, j \in I$, such that for some constant $N \ge 0$, independent of α and f ,

(i) $f = g + \sum_{j\in I} b_j$,

(ii) $\|\nabla g\|_{L^\infty} + \|g\|_{L^\infty} + \|g/d_D\|_{L^\infty} \le N\alpha,$

(iii)$\mathrm{supp}(b_j) \subseteq Q_j, b_j \in W_D^{1,1} \cap W^{1,p}$ and $\int_\Omega \left(|\nabla b_j| + |b_j| + \frac{|b_j|}{d_D}\right) \le N\alpha|Q_j|$ for every $j \in I,$

(iv) $\sum_{j\in I} |Q_j| \le \frac{N}{\alpha^p} \| f\|^p_{W^{1,p}_D}$,

(v) $\sum_{j\in I} 1_{Q_j}(x) \le N$ for all $x \in \mathbb{R}^d$,

(vi) $\|g\|_{W^{1,p}_D} \le N\|f\|_{W^{1,p}_D}$. If $D \ne \emptyset$, all the norms $\|f\|_{W^{1,p}_D}$ may be replaced by $\|\nabla f\|_{L^p}$.

In order to verify the final statement, note that for $D \ne \emptyset$, the Ahlfors–David condition guarantees that the surface measure of D is strictly positive. We will subdivide the proof of Lemma (5.2.15) into six steps.

Step 1: Adapted maximal function. Let $\in W^{1,p}_D$. Then, using the extension operator $\mathfrak{E}_\bullet$ from the proof of Theorem (5.2.10), we find $\mathfrak{E}_\bullet f \in W^{1,p}_0(\Omega_\bullet)$. So we may extend this function again by zero to the whole of $\mathbb{R}^d$, obtaining a function $\tilde f \in W^{1,p}_D(\mathbb{R}^d)$ that satisfies supp $(\tilde f) \subseteq B$ for the ball B and the estimate $\|\tilde f\|_{W^{1,p}_D}(\mathbb{R}^d) \le C\|f\|_{W^{1,p}_D}$ with a constant C that does not depend on f . Furthermore, Hardy's inequality

$$\|\tilde f/d_D\|_{L^p(\mathbb{R}^d)} \le C\|\nabla\tilde f\|_{L^p(\mathbb{R}^d)} \tag{15}$$

Holds.

The easiest case is that of $E = \emptyset$. Then, we may take $I = \emptyset$ and $= f$, and the only assertion we have to show is (ii), the rest being trivial. So, let $x \in \Omega$ be given. Since x is not in E, we have for almost all such x, by the fact that $h(x) \le (Mh)(x)$ for all Lebesgue points of an $L^1(\mathbb{R}^d)$ function h,

$$\begin{aligned}|\nabla g(x)| + |g(x)| + |g(x)|/d_D(x) &= |\nabla f(x)| + |f(x)| + |f(x)|/d_D(x)\\ &= |\nabla\tilde f(x)| + |\tilde f(x)| + |\tilde f(x)|/d_D(x) \le [M(|\nabla\tilde f| + |\tilde f| + |\tilde f|/d_D(x)\\ &\le \alpha.\end{aligned}$$

This implies (ii).

So, we turn to the case $E \ne \emptyset$. By Jensen's inequality, (14), (15) and the continuity of the extension operator, we obtain

$$\begin{aligned}|E| &\le \left|\left\{x \in \mathbb{R}^d : [M(|\nabla\tilde f| + |\tilde f| + |\tilde f|/d_D)^p](x) > \alpha^p\right\}\right|\\ &\le \frac{K}{\alpha^p}\,\||\nabla\tilde f| + |\tilde f| + |\tilde f|/d_D\|^p_{L^p(\mathbb{R}^d)} \le \frac{C}{\alpha^p}\,\|\tilde f\|^p_{W^{1,p}(\mathbb{R}^d)}\\ &\le \frac{C}{\alpha^p}\,\|f\|^p_{W^{1,p}_D}.\end{aligned} \tag{16}$$

In particular, this measure is finite, so $F := \mathbb{R}^d \setminus E \ne \emptyset$. This allows for choosing a Whitney decomposition of E, cf. [215], see also [246] and [247]. Thus, we get an at most countable index set I and a collection of cubes $Q_j \in Q, j \in I$, with sidelength ℓ_j that fulfil the following properties for some $c_1, c_2 \ge 1$

(i) $E = \bigcup_{j\in I} \frac{8}{9} Q_j$.

(ii) $\frac{8}{9} Q_j^\circ \cap \frac{8}{9} Q_k^\circ = \emptyset$ for all $j,k \in I, j \ne k$.

(iii) $Q_j \subseteq E$ for all $j \in I$.

(iv) $\sum_{j\in I} 1_{Q_j} \le c_1$.

(v) $\frac{1}{c_2}\ell_j \le d(Q_j,F) \le c_2\ell_j$ for all $j \in I$.

There are two immediate consequences of these properties that are important to observe. Firstly, the family Q_j° , $j \in I$, is an open covering of E and, secondly, (v) implies that for some $\tilde c > 1$, independent of j, we have

$$\left(\tilde{c}Q_j\right) \cap F \neq \emptyset \quad \text{for all } j \in I. \tag{17}$$

Now, (iv) immediately implies (v) and this, together with (16), allows to prove (iv) due to

$$\sum_{j\in I} |Q_j| = \int_E \sum_{j\in I} 1_{Q_j} \le c_1|E| \le \frac{C}{\alpha^p} \|f\|^p_{W^{1,p}_D}.$$

Step 2: Definition of the good and bad functions. Let $(\varphi_j)_{j\in I}$ be a partition of unity on E with

(i) $\varphi_j \in C^\infty(\mathbb{R}^d)$,

(ii) supp $(\varphi_j) \subseteq Q_j^\circ$,

(iii) $\varphi_j \equiv 1$ on$\frac{8}{9}$ Q_j ,

(vi) $\|\varphi_j\|_{L^\infty} + \ell_j\|\nabla\varphi_j\|_{L^\infty} \le c$,

for all $j \in I$ and some $c > 0$. The construction of such a partition can be found, e.g. in [215].

Let us distinguish two types of cubes Q_j . We say that Q_j is a usual cube, if $d(Q_j, D) \ge \ell_j$ and Q_j is a special cube, if $d(Q_j, D) \ge \ell_j$ (In the case $D = \emptyset$, all cubes are seen as usual ones). Then, we define for every $j \in I$, using the notation $h_Q := \frac{1}{|Q|}\int_Q h$,

$$\tilde{b}_j := \begin{cases} \left(\tilde{f} - \tilde{f}_{Q_j}\right)\varphi_j, & \text{if } Q\ j\ is\ usual, \\ \tilde{f}\varphi_j, & \text{if } Q_j \text{ is special.} \end{cases}$$

Setting $\tilde{g} := \tilde{f} - \sum_{j\in I} \tilde{b}_j$ as well as $b_j := \tilde{b}_j\big|_\Omega$ and $g := \tilde{g}|_\Omega$, these functions automatically satisfy (i). Note that there is no problem of convergence in this sum, due to (v).

It is clear by construction that supp $(b_j) \subseteq Q_j$ and $b_j \in W^{1,p}(\Omega)$ for all $j \in I$. The next step is to show that $b_j \in W^{1,1}_D$, and since $W^{1,p} \hookrightarrow W^{1,1}$, we only have to establish the right boundary behaviour of b_j .

We start with the case of a usual cube Q_j . Then, $b_j = \left(\tilde{f} - \tilde{f}_{Q_j}\right)\varphi_j\big|_\Omega$. Since φ_j has support in Q_j and $d(Q_j, D) \ge \ell_j > 0$, the function bj can be approximated by C_c^∞ $(\mathbb{R}^d/D)$ functions in the norm of $W^{1,1}$. Thus, $b_j \in W^{1,1}_D$. If Q_j is a special cube, we have $b_j = \tilde{f}\,\varphi_j\big|_\Omega$. The fact that $\tilde{f} \in W^{1,p}_D$ $(\mathbb{R}^d)$ implies that there is a sequence $(\tilde{f}_k)_k \subseteq C_c^\infty(\mathbb{R}^d/D)$, such that $\tilde{f}_k \to \tilde{f}$ in $W^{1,p}(\mathbb{R}^d)$. Therefore, $(\tilde{f}_k\varphi_j)_k$ is a sequence in $C_c^\infty(\mathbb{R}^d/D)$, and we show that it converges to $\tilde{f}\,\varphi_j$ in $W^{1,1}$, so that we can conclude that $b_j \in W^{1,1}_D$. This convergence follows from $\varphi_j \in W^{1,p}(\mathbb{R}^d)$ by

$$\|\tilde{f}\,\varphi_j - \tilde{f}_k\varphi_j\|_{L^1} \le \|\tilde{f} - \tilde{f}_k\|_{L^p}\|\varphi_j\|_{L^{p'}} \to 0\ (k \to \infty)$$

and the corresponding estimate for the gradient

$$\begin{aligned}\|\nabla(\tilde{f}\,\phi_j) - \nabla(\tilde{f}_k\phi_j)\|_{L^1} &\le \|\nabla(\tilde{f} - \tilde{f}_k)\varphi_j\|_{L^1} + \|(\tilde{f} - \tilde{f}_k)\nabla\varphi_j\|_{L^1} \\ &\le \|\nabla(\tilde{f} - \tilde{f}_k)\|_{L^p}\|\varphi_j\|_{L^{p'}} + \|\tilde{f} - \tilde{f}_k\|_{L^p}\|\nabla\varphi_j\|_{L^{p'}} \to 0\ (k \to \infty).\end{aligned}$$

Step 3: Proof of (iii). After the above considerations, it remains to prove the estimate. We start again with the case of a usual cube, and for later purposes, we introduce some $q \in$

$[1,\infty[$. On usual cubes, it holds $\nabla \tilde{b}_j = \nabla \tilde{f}\,\varphi_j + (\tilde{f} - \tilde{f}Q_j)\nabla\varphi_j$ and using d) we obtain

$$\int_{Q_j} |\nabla \tilde{b}_j|^q \leq \int_{Q_j} \left(|\nabla \tilde{f}||\varphi_j| + \left|\tilde{f} - \tilde{f}_{Q_j}\right| |\nabla\varphi_j|^q\right)$$
$$\leq C \int_{Q_j} \left(|\nabla \tilde{f}|^q\, |\varphi_j|^q + \left|\tilde{f} - \tilde{f}_{Q_j}\right|^q |\nabla\varphi_j|^q\right)$$
$$\leq C\left(\int_{Q_j} |\nabla \tilde{f}|^q + \frac{1}{\ell_j^q}\int_{Q_j} \left|\tilde{f} - \tilde{f}_{Q_j}\right|^q\right).$$

In the second integral, we may now apply the Poincaré inequality, since $\tilde{f} - \tilde{f}_{Q_j}$ has zero mean on Q_j . This yields

$$\int_{Q_j} |\nabla \tilde{b}_j|^q \leq C\left(\int_{Q_j} |\nabla\tilde{f}|^q + \frac{1}{\ell_j^q}\operatorname{diam}(Q_j)^q \int_{Q_j} |\nabla \tilde{f}|^q\right) \leq C \int_{Q_j} |\nabla \tilde{f}|^q . \tag{18}$$

We now specialize again to $q = 1$, and invoking (17), we pick some $z \in \tilde{c}Q_j \cap F$, and bring into play the maximal operator:

$$\int_{Q_j} |\nabla \tilde{b}_j| \leq C \int_{\tilde{c}Q_j} |\nabla \tilde{f}| \leq C|Q_j|\frac{1}{|\tilde{c}Q_j|}\int_{\tilde{c}Q_j} \left(|\nabla \tilde{f}|+|\tilde{f}| + \frac{|\tilde{f}|}{d_D}\right)$$
$$\leq C|Q_j| \sup_{Q\in Q_z} \frac{1}{|Q|}\int_Q \left(|\nabla\tilde{f}| + |\tilde{f}| + \frac{|\tilde{f}|}{d_D}\right)$$
$$= C|Q_j|\left[M\left(|\nabla \tilde{f}| + |\tilde{f}| + \frac{|\tilde{f}|}{d_D}\right)\right](z). \tag{19}$$

Now, we capitalize that $z \in F$ and obtain

$$\int_\Omega |\nabla b_j| \leq \int_{Q_j} |\nabla\tilde{b}_j| \leq C|Q_j|\alpha. \tag{20}$$

For the corresponding estimate for $|b_j|$, we use again the Poincaré inequality for $\tilde{f} - \tilde{f}_{Q_j}$ on Q_j to obtain for all $q \in [1,\infty[$

$$\int_\Omega |b_j|^q \leq \int_{Q_j} |\tilde{b}_j|^q = \int_{Q_j} \left|\tilde{f} - \tilde{f}_{Q_j}\right|^q |\varphi_j|^q \leq C\int_{Q_j} \left|\tilde{f} - \tilde{f}_{Q_j}\right|^q$$
$$\leq C \int_{Q_j} |\nabla \tilde{f}|^q . \tag{21}$$

Note that the factor diam (Q_j) from the Poincaré inequality is bounded uniformly in j, since all Q_j with sufficiently large diameter are far away from D and thus do not touch , so b_j then is zero. Proceeding as in (19) and (20), we find, specializing to $q = 1$,

$$\int_\Omega |b_j| \leq C|Q_j|^\alpha. \tag{22}$$

For the third term $|b_j|/d_D$, we note that on a usual cube Q_j , we have $d_D \geq j$. Thus, we get as before by the Poincaré inequality

$$\int_\Omega \frac{|b_j|}{d_D} \leq \int_{Q_j} \frac{|\tilde{b}_j|}{d_D} \leq \frac{C}{\ell_j}\int_{Q_j} \left|\tilde{f} - \tilde{f}_{Q_j}\right| \leq C\int_{Q_j} |\nabla \tilde{f}|$$

and we can again conclude as in (19) and (20).

So, we turn to the proof of the estimate in (iii) for the case of a special cube. Then, $b_j = \tilde{f}\,\varphi_j\big|_{\Omega}$, and we get with the help of d)

$$|\nabla\tilde{b}_j| \leq |\nabla \tilde{f}||\varphi_j| + |\tilde{f}||\nabla\varphi_j| \leq C\left(|\nabla \tilde{f}| + \frac{|\tilde{f}|}{\ell_j}\right).$$

Since Q_j is a special cube, we get for every $x \in Q_j$

$$d_D(x) = d(x,D) \leq \operatorname{diam}(Q_j) + d(Q_j,D) \leq C\ell_j + \ell_j \leq C\ell_j \quad (23)$$

and this in turn yields

$$|\nabla\tilde{b}_j| \leq C\left(|\nabla \tilde{f}| + \frac{|\tilde{f}|}{d_D}\right). \quad (24)$$

Since, obviously

$$|\tilde{b}_j| = |\tilde{f}\,\varphi_j| \leq C|\tilde{f}| \text{ and } \frac{|\tilde{b}_j|}{d_D} = \frac{|\tilde{f}\varphi_j|}{d_D} \leq C\frac{|\tilde{f}|}{d_D} \quad (25)$$

hold, we find by one more repetition of the arguments in (19) and (20) with some $z \in \tilde{c}Q_j \cap F$

$$\int_{\Omega}\left(|b_j| + |\nabla b_j| + \frac{|b_j|}{d_D}\right) \leq C\int_{Q_j}\left(|\tilde{f}| + |\nabla \tilde{f}| + \frac{|\tilde{f}|}{d_D}\right)$$
$$\leq \frac{C|Q_j|}{|\tilde{c}Q_j|}\int_{\tilde{c}Q_j}\left(|\tilde{f}| + |\nabla \tilde{f}| + \frac{|\tilde{f}|}{d_D}\right) \leq C|Q_j|_{\alpha}. \quad (26)$$

Step 4: Proof of (ii): Estimate of $|g|$ and $|g|/d_D$. The asserted bound for $|g|$ and $|g|/d_D$ is rather easy to obtain on $F \cap \Omega$, since on F all functions $\tilde{b}_j, j \in I$, vanish, which means $\tilde{g} = \tilde{f}$ on F. This implies for almost all $x \in F \cap \Omega$ by the definition of F

$$|g(x)| + \frac{|g(x)|}{d_D(x)} = |\tilde{f}(x)| + \frac{|\tilde{f}(x)|}{d_D(x)} \leq \left[M\left(|\nabla \tilde{f}| + |\tilde{f}| + \frac{|\tilde{f}|}{d_D}\right)\right](x) \leq \alpha.$$

So, for the estimate of these two terms, we concentrate on the case $x \in E$. Setting $I_u := \{j \in I : Q_j \text{ usual}\}$ and $I_s := \{j \in I : Q_j \text{ special}\}$, we obtain on E

$$\tilde{g} = \tilde{f} - \sum_{j\in I_u}\tilde{b}_j - \sum_{j\in I_s}\tilde{b}_j = \tilde{f} - \sum_{j\in I_u}\left(\tilde{f} - \tilde{f}_{Q_j}\right)\varphi_j - \sum_{j\in I_s}\tilde{f}\,\varphi_j$$
$$= \tilde{f} - \tilde{f}\sum_{j\in I}\varphi_j + \sum_{j\in I_u}\tilde{f}_{Q_j}\varphi_j = \tilde{f}1_F + \sum_{j\in I_u}\tilde{f}_{Q_j}\varphi_j = \sum_{j\in I_u}\tilde{f}_{Q_j}\varphi_j\,.$$

Now, we fix some $x \in E$. Let $I(x) := \{j \in I : x \in \operatorname{supp}(\varphi_j)\}$, $I_{u,x} := I_u \cap I(x)$ and Is, $x := I_s \cap I(x)$. Then, the above estimate yields together with d) |

$$|\tilde{g}(x)| \leq \sum_{j\in I_u}|\tilde{f}_{Q_j}||\varphi_j(x)| \leq C\sum_{j\in I_{u,x}}|\tilde{f}_{Q_j}| = C\sum_{j\in I_{u,x}}\frac{1}{|Q_j|}\left|\int_{Q_j}\tilde{f}(y)dy\right|$$
$$\leq C\sum_{j\in I_{u,x}}\frac{1}{|Q_j|}\int_{Q_j}|\tilde{f}(y)|\,dy. \quad (27)$$

Picking again some $z_j \in \tilde{c}Q_j \cap F, j \in I$, this yields with the argument that we used already several times and since $I_{u,x}$ is finite

$$|\,\tilde{g}(x)| \leq C \sum_{j\in I_{u,x}} \frac{1}{\left|\tilde{c}Q_j\right|} \int_{\tilde{c}Q_j} |\,\tilde{f}\,(y)| dy \leq C \sum_{j\in I_{u,x}} \left[M\left(\left|\tilde{f}\right|\right)\right](z_j) \leq C \sum_{j\in I_{u,x}} \alpha$$
$$\leq C\alpha.$$

In order to estimate $\tilde{g}/d_D$ on E, we estimate as in (27) for $x \in E$

$$\frac{|\,\tilde{g}(x)|}{d_D(x)} = \frac{\left|\sum_{j\in I_u} \tilde{f}_{Q_j}\, \varphi_j(x)\right|}{d_D(x)} \leq \sum_{j\in I_{u,x}} \frac{\left|\tilde{f}_{Q_j}\right|}{d_D(x)} \leq C \sum_{j\in I_{u,x}} \frac{1}{\left|Q_j\right|} \int_{Q_j} \frac{\left|\tilde{f}\,(y)\right|}{d_D(x)}\, dy.$$

Every cube in this sum is a usual one, so $(Q_j, D) \geq \ell_j$. Furthermore, we have $x \in Q_j$ for all $j \in I_{u,x}$ by construction. This means that for every $j \in I_{u,x}$ and all $y \in Q_j$, the distance between x and y is less than $C\ell_j$ for some constant C depending only on the dimension. Thus,

$$d_D(y) = d(y,D) \leq d(y,x) + d(x,D) \leq C$$
$$j + d_D(x) \leq Cd(Q_j, D) + d_D(x) \leq Cd_D(x).$$

Consequently, we get for some $z_j \in \tilde{c}Q_j \cap F$ as before

$$|g(x)|d_D(x) \leq C \sum_{j\in I_{u,x}} \frac{1}{\left|Q_j\right|} \int_{Q_j} \left|\tilde{f}\,(y)\right| d_D(y) dy \leq C \sum_{j\in I_{u,x}} \frac{1}{\left|\tilde{c}Q_j\right|} \int_{\tilde{c}Q_j} \frac{\left|\tilde{f}\,(y)\right|}{d_D(y)}\, dy$$
$$\leq C \sum_{j\in I_{u,x}} \left[M\left(\frac{\left|\tilde{f}\right|}{d_D}\right)\right](z_j) \leq C\alpha.$$

Step 5: Proof of (ii): Estimate of $|\nabla g|$. In order to estimate $|\nabla g|$, it is not sufficient to know that $\sum_{j\in I} \tilde{b}_j$ converges pointwise as before. At least, we have to know some convergence in the sense of distributions to push the gradient through the sum. Let $J \subseteq I$ be finite. Then, we have, due to (22) for usual cubes and (26) for special cubes

$$\left\|\sum_{j\in J} \left|\tilde{b}_j\right|\right\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \sum_{j\in J} \left|\tilde{b}_j\right| = \sum_{j\in J} Q_j \left|\tilde{b}_j\right| \leq C\alpha \sum_{j\in J} \left|Q_j\right|$$

with a constant C that is independent of the choice of J . Since $\sum_{j\in I} \left|Q_j\right|$ is convergent due to (iv), this implies that $\sum_{j\in I} \left|\tilde{b}_j\right|$ is a Cauchy sequence in $L^1(\mathbb{R}^d)$. In particular, $\sum_{j\in I} \tilde{b}_j$ converges in the sense of distributions, so we get $\nabla \sum_{j\in I} \tilde{b}_j = \sum_{j\in I} \nabla\tilde{b}_j$ in the sense of distributions.

In a next step, we show that the sum $\sum_{j\in I} \nabla\tilde{b}_j$ converges absolutely in L^1. Investing the estimates in (18) and (24), respectively, we find

$$\int_{Q_j} \left|\nabla\tilde{b}_j\right| \leq C \int_{Q_j} \left(\left|\nabla \tilde{f}\right| + \frac{\left|\tilde{f}\right|}{d_D}\right).$$

Thus, we obtain by (v) and the fact that E has finite measure, cf. (16),

$$\sum_{j\in I} \|\nabla \tilde{b}_j\|_{L^1(\mathbb{R}^d)} = \sum_{j\in I} \|\nabla \tilde{b}_j\|_{L^1(Q_j)} \le C \sum_{j\in I} \int_{Q_j} \left(|\nabla \tilde{f}| + \frac{|\tilde{f}|}{d_D}\right)$$

$$= C \int_E \sum_{j\in I} \frac{1}{Q_j}\left(|\nabla \tilde{f}| + \frac{|\tilde{f}|}{d_D}\right) \le C \left\| |\nabla \tilde{f}| + \frac{|\tilde{f}|}{d_D} \right\|_{L^1(E)}$$

$$\le C \left\| |\nabla \tilde{f}| + \frac{|\tilde{f}|}{d_D} \right\|_{L^p(E)} \le \|\nabla \tilde{f}\|_{L^p(\mathbb{R}^d)} + \left\| \frac{\tilde{f}}{d_D} \right\|_{L^p(\mathbb{R}^d)} .$$

Now, by Hardy's inequality (15), this last expression is finite and yields the desired absolute convergence.

This allows us to calculate

$$\nabla \tilde{g} = \nabla \tilde{f} - \sum_{j\in I} \nabla \tilde{b}_j$$

$$= \nabla \tilde{f} - \sum_{j\in I_u} \left(\nabla \tilde{f}\varphi_j + \left(\tilde{f} - \tilde{f}_{Q_j}\right)\nabla\varphi_j\right) - \sum_{j\in I_s} \left(\nabla \tilde{f}\,\varphi_j + \tilde{f}\,\nabla\varphi_j\right).$$

Note that the above considerations concerning the convergence of $\sum_{j\in I} \nabla\tilde{b}_j$ also yield that the sums over $\nabla \tilde{f}\varphi_j, \left(\tilde{f} - \tilde{f}_{Q_j}\right)\nabla\varphi_j$ and $f\,\nabla\varphi_j$ are absolutely convergent in L^1, so $\nabla \tilde{g} = \nabla \tilde{f} - \sum_{j\in I} \nabla\tilde{f}\,\varphi_j - \sum_{j\in I_u} \left(\tilde{f} - \tilde{f}_{Q_j}\right)\nabla\varphi_j - \sum_{j\in I_s} \tilde{f}\,\nabla\varphi_j = \nabla \tilde{f}\,1_F - \sum_{j\in I_u} \left(\tilde{f} - \tilde{f}_{Q_j}\right)\nabla\varphi_j - \sum_{j\in I_s} \tilde{f}\,\nabla\varphi_j$.

On F, we know that every summand in the above two sums vanishes, so by the L^1-convergence shown above we see $\nabla\tilde{g} = \nabla\tilde{f}$ on F. Thus, on F, we easily get the desired L^∞-estimate for $\nabla\tilde{g}$, since for almost all $x \in F$

$$|\nabla \tilde{g}(x)| = |\nabla \tilde{f}(x)| \le M(|\nabla \tilde{f}|)(x) \le M\left(|\nabla\tilde{f}| + |\tilde{f}| + \frac{|\tilde{f}|}{d_D}\right)(x) \le \alpha.$$

So, we concentrate on $x \in E$. Since E is open, all sums in

$$\nabla \tilde{g}(x) = -\sum_{j\in I_u} \left(\tilde{f}(x) - \tilde{f}_{Q_j}\right) \nabla\varphi_j(x) - \sum_{j\in I_s} \tilde{f}(x)\nabla\varphi_j(x)$$

are finite thanks to (v), and $\sum_{j\in I} \varphi_j$ is constantly 1 in a neighbourhood of x. Thus, we may calculate for $x \in E$

$$\nabla \tilde{g}(x) = \sum_{j\in I_u} \tilde{f}_{Q_j}\nabla\varphi_j(x) - \tilde{f}(x)\sum_{j\in I} \nabla\varphi_j(x) = \sum_{j\in I_u} \tilde{f}_{Q_j}\nabla\varphi_j(x).$$

We set on E

$$h_u := \sum_{j\in I_u} \tilde{f}_{Q_j}\nabla\varphi_j \text{ and } h_s := \sum_{j\in I_s} \tilde{f}_{Q_j}\nabla\varphi_j$$

and we will show in the following the estimates $|h_s(x)| \le C\alpha$ and $|h_u(x) + h_s(x)| \le C\alpha$ for all $x \in E$. Then, we have the same bound for hu and hence also for $\nabla\tilde{g}$ on E. In order to show the desired estimate for hs, we recall that by (23), we have $d_D(y) \le C\ell_j$ for all y in a special cube Q_j . Using d) and this estimate, we find for all $x \in E$

$$|h_s(x)| \le \sum_{j\in I_s} \left|\tilde f_{Q_j}\right| |\nabla\varphi_j(x)| \le \sum_{j\in I_{s,x}} \frac{C}{\ell_j}\left|\tilde f_{Q_j}\right| \le C \sum_{j\in I_{s,x}} \frac{1}{|Q_j|}\int_{Q_j} \frac{|\tilde f(y)|}{\ell_j}\,dy$$
$$\le C \sum_{j\in I_{s,x}} \frac{1}{|Q_j|}\int_{Q_j} \frac{|\tilde f(y)|}{d_D(y)}\,dy.$$

Now, we use again that the above sum is finite, uniformly in x, so it suffices to estimate each addend by $C\alpha$. In order to do so, we once more bring into play the maximal operator in some point $z_j \in \tilde c Q_j \cap F$:

$$\frac{1}{|Q_j|}\int_{Q_j} \frac{|\tilde f(y)|}{d_D(y)}\,dy \le C\frac{1}{|\tilde c Q_j|}\int_{\tilde c Q_j} \frac{|\tilde f(y)|}{d_D(y)}\,dy \le C\, M\left(|\nabla \tilde f| + |\tilde f| + \frac{|\tilde f|}{d_D}\right)(z_j)$$
$$\le C\alpha.$$

We turn to the estimate of $h_u + h_s$. Let $x \in E$ and choose some $i_0 \in I(x)$. Then, for every $j \in I(x)$, we have $x \in Q_j \cap Q_{i_0}$, so by property (v) of the Whitney cubes, the sidelengths ℓ_j and ℓ_{i_0} are comparable with uniform constants. Thus, we can choose some $\kappa \ge \tilde c$, such that $\kappa\, Q_{i_0} \supseteq Q_j$ for all $j \in I(x)$. Since $\sum_{j\in I} \nabla\varphi_j(x) = 0$, one finds

$$(h_u + h_s)(x) = \sum_{j\in I} \tilde f_{Q_j}\,\nabla\varphi_j(x) = \sum_{j\in I} \left(\tilde f_{Q_j} - \tilde f_{\kappa Q_{i_0}}\right)\nabla\varphi_j(x).$$

This implies thanks to d)

$$|(h_u + h_s)(x)| \le \sum_{j\in I} \left|\tilde f_{Q_j} - \tilde f_{\kappa Q_{i_0}}\right| |\nabla\varphi_j(x)| \le \sum_{j\in I(x)} \frac{C}{\ell_j}\left|\tilde f_{Q_j} - \tilde f_{\kappa Q_{i_0}}\right|.$$

For every $j \in I(x)$, we have

$$\left|\tilde f_{Q_j} - \tilde f_{\kappa Q_{i_0}}\right| = \left|\frac{1}{|Q_j|}\int_{Q_j} \tilde f(y)\,dy - \tilde f_{\kappa Q_{i_0}}\right| = \left|\frac{1}{|Q_j|}\int_{Q_j} \left(\tilde f(y) - \tilde f_{\kappa Q_{i_0}}\right)dy\right|$$
$$\le \frac{1}{|Q_j|}\int_{Q_j} \left|\tilde f(y) - \tilde f_{\kappa Q_{i_0}}\right| dy \le C\frac{1}{|\kappa Q_{i_0}|}\int_{\kappa Q_{i_0}} \left|\tilde f(y) - \tilde f_{\kappa Q_{i_0}}\right| dy,$$

since Q_j and κQ_{i_0} are of comparable size and $Q_j \subseteq \kappa Q_{i_0}$. Applying the Poincaré inequality on Q_{i_0} , we further estimate by

$$\le C_\kappa \ell_{i_0}\frac{1}{|\kappa Q_{i_0}|}\int_{\kappa Q_{i_0}} |\nabla \tilde f(y)|\,dy \le C\ell_j\frac{1}{|\kappa Q_{i_0}|}\int_{\kappa Q_{i_0}} |\nabla \tilde f(y)|\,dy$$

Since $\kappa \ge \tilde c$, there is again some point $z \in \kappa Q_{i_0} \cap F$, and we may continue as above

$$\le C\ell_j\, M\,(|\nabla\tilde f| + |\tilde f| + \frac{|\tilde f|}{d_D(z)} \le C\ell_j\alpha.$$

Putting everything together and investing that $I(x)$ is uniformly finite for every $x \in E$, we have achieved

$$|\nabla \tilde g(x)| \le |h_s(x)| + |(h_u + h_s)(x)| \le C\alpha$$

and have thus proved (ii).

Step 6: Proof of (vi). , We first estimate

$$\|g\|_{W_D^{1,p}} \leq \|\tilde{g}\|_{W_D^{1,p}(\mathbb{R}^d)} = \left\| \tilde{f} - \sum_{j\in I} \tilde{b}_j \right\|_{W_D^{1,p}(\mathbb{R}^d)}$$
$$\leq \|\tilde{f}\|_{W_D^{1,p}(\mathbb{R}^d)} + \left\| \sum_{j\in I} \tilde{b}_j \right\|_{W_D^{1,p}(\mathbb{R}^d)} .$$

By the continuity of the extension operator, we have $\|\tilde{f}\|_{W_D^{1,p}(\mathbb{R}^d)} \leq C\|\tilde{f}\|\_(W_D^{1,p}$, so we only have to estimate the sum of the $\tilde{b}_j, j \in I$.

Here, we again rely on (v) and the equivalence of norms in $\mathbb{R}^N$ to obtain

$$\left\| \sum_{j\in I} \tilde{b}_j \right\|_{L^p(\mathbb{R}^d)}^p = \int_{\mathbb{R}^d} \left| \sum_{j\in I} \tilde{b}_j \right|^p \leq \int_{\mathbb{R}^d} \left(\sum_{j\in I} |\tilde{b}_j| \right)^p \leq C \int_{\mathbb{R}^d} \sum_{j\in I} |\tilde{b}_j|^p$$
$$= C \sum_{j\in I} \int_{Q_j} |\tilde{b}_j|^p. \qquad (28)$$

Investing the estimates in (21) for $q = p$ and in (25) for usual and special cubes, respectively, we find

$$\int_{Q_j} |\tilde{b}_j|^p \leq C \int_{Q_j} (|\tilde{f}|^p + |\nabla\tilde{f}|^p . \qquad (29)$$

Combining the two last estimates, we thus have with the help of (v)

$$\left\| \sum_{j\in I} \tilde{b}_j \right\|_{L^p(\mathbb{R}^d)}^p \leq C \sum_{j\in I} \int_{Q_j} (|\tilde{f}|^p + |\nabla\tilde{f}|^p) \leq C \int_{\mathbb{R}^d} \sum_{j\in I} 1_{Q_j} (|\tilde{f}|^p + |\nabla\tilde{f}|^p)$$
$$\leq C\|\tilde{f}\|_{W_D^{1,p}(\mathbb{R}^d)} .$$

For the estimate of the gradient, we argue as in (28) and (29), in order to find thanks to the estimates in (18) for $q = p$ and (24)

$$\left\| \sum_{j\in I} \nabla\tilde{b}_j \right\|_{L^p(\mathbb{R}^d)}^p \leq C \sum_{j\in I} \int_{Q_j} |\nabla\tilde{b}_j|^p \leq C \sum_{j\in I} \int_{Q_j} \left(|\nabla \tilde{f}|^p + \frac{|\tilde{f}|^p}{d_D^p} \right).$$

Investing again (v) and the Hardy inequality in (15), we end up with

$$\left\| \sum_{j\in I} \nabla\tilde{b}_j \right\|_{L^p(\mathbb{R}^d)}^p \leq C \int_{\mathbb{R}^d} \left(|\nabla\tilde{f}|^p + \frac{|\tilde{f}|^p}{d_D^p} \right) \leq C \int_{\mathbb{R}^d} |\nabla\tilde{f}|^p \leq \|\tilde{f}\|_{W_D^{1,p}(\mathbb{R}^d)}$$

and this finishes the proof, thanks to $\|\tilde{f}\|_{W_D^{1,p}(\mathbb{R}^d)} \leq C\|f\|_{W_D^{1,p}}$.

Having the Calderón–Zygmund decomposition at hand, we can now show that it really respects the boundary condition on D.

Corollary (5.2.16)[206]: Let $p \in]1,\infty[$ and $f \in W_D^{1,p}$ be given. The functions g and $b = \sum_{j\in I} b_j$ from Lemma (5.2.15) have the following properties:

(i) $b \in W_D^{1,1}$ with $\|b\|_{W^{1,1}} \leq C\alpha^{1-p} \|f\|_{W_D^{1,p}}^p$,

(ii) $g \in W_D^{1,\infty}$ with $\|g\|_{W_D^{1,\infty}} \leq C\alpha$,

(iii) If $\in W_D^{1,2}$, then also $g, b \in W_D^{1,2}$.

Proof. (i) Thanks to (iii) in Lemma (5.2.15) we have $b_j \in W_D^{1,1}(\Omega)$ for all $j \in I$. Moreover, by the estimates in (iii) and (iv) of the same lemma,

$$\sum_{j\in I} \|b_j\|_{W^{1,1}} \leq C\alpha \sum_{j\in I} |Q_j| \leq C\alpha^{1-p} \|f\|_{W_D^{1,p}}^p < \infty. \tag{30}$$

Thus, the sum in b is absolutely convergent in $W^{1,1}$, which means that b satisfies the asserted norm estimate and lies in the closed subspace $W_D^{1,1}$. Thus, we have achieved (i).

(ii) We first show that $\tilde{g}$ has a Lipschitz continuous representative and that the Lipschitz constant is controlled by $C\alpha$. From the proof of Lemma (5.2.15), we have $\tilde{g} \in W^{1,p}(\mathbb{R}^d)$ for all $1 \leq p < \infty$. So, from [229], we can infer that for almost all $x, y \in \mathbb{R}^d$

$$|\tilde{g}(x) - \tilde{g}(y)| \leq C|x - y|(M(|\nabla \tilde{g}|^p)^{\frac{1}{p}}(x) + M(|\nabla\tilde{g}|^p)^{\frac{1}{p}}(y) .$$

The Hardy–Littlewood maximal operator is bounded on $L^\infty(\mathbb{R}^d)$, so this implies

$$\sup_{x,y\in\mathbb{R}^d, x=y} \frac{|\tilde{g}(x) - \tilde{g}(y)|}{|x - y|} \leq C\|\nabla \tilde{g}\|_{L^\infty(\mathbb{R}^d)} \leq C\alpha$$

and we find $\tilde{g} \in W^{1,\infty}(\mathbb{R}^d) = (L^\infty \cap \mathrm{Lip})(\mathbb{R}^d)$.

It remains to prove the right boundary behaviour of $\tilde{g}$, i.e. $\tilde{g}|_D = 0$. Then, by the Definition of $W_D^{1,\infty}$, cf. (8), we find $g = \tilde{g}|_\Omega \in W_D^{1,\infty}$. Since $\tilde{f}, \tilde{b} \in W_D^{1,1}(\mathbb{R}^n)$, these two functions have zero trace on D $\mathcal{H}_{d-1}$-almost everywhere, so the same is true for $\tilde{g}$ and we only have to get rid of the "almost everywhere". Let $x \in D$ be given. Then, for every $\varepsilon > 0$, by the Ahlfors–David condition (6), we have $\sigma(B(x,\varepsilon) \cap D) > 0$, so there must be points in this set, where $\tilde{g}$ vanishes. But this means that x is an accumulation point of the set $\{y \in D : \tilde{g}(y) = 0\}$. By the continuity of $\tilde{g}$, this implies $\tilde{g}(x) = 0$. (iv) By (ii) and Lemma (5.2.2), we have $\in W_D^{1,\infty} \hookrightarrow W_D^{1,2}$, so with f also b is in this space.

We establish interpolation within the set of spaces $\{W_D^{1,p}(\Omega)\}_{p\in[1,\infty]}$. There already exist interpolation results for spaces of this scale which incorporate mixed boundary conditions (compare [227],[241]) but not of the required generality concerning the Dirichlet part. The key ingredient for this generalization will be the Calderón–Zygmund decomposition proved.

The main result is the following.

Theorem (5.2.17)[206]: [216],[217] For any compatible couple of Banach spaces (Y_0, Y_1), we have

$$[(Y_0, Y_1)_{\lambda_0, p_0}, (Y_0, Y_1)_{\lambda_1, p_1}]_\alpha = (Y_0, Y_1)_{\beta, p}$$

for all λ_0, λ_1 and α in (0, 1) and all p_0, p_1 in $[1,\infty]$, except for the case $p_0 = p_1 = \infty$. Here, β and p are given by $\beta = (1 - \alpha)\lambda_0 + \alpha\lambda_1$ and $\frac{1}{p} = \frac{1-\alpha}{p_0} + \frac{\alpha}{p_1}$.

From this theorem and our real interpolation Theorem (5.2.22), a complex interpolation result for Sobolev spaces $W_D^{1,p}(\Omega)$ follows.

Corollary (5.2.18)[206]: Let Ω and D satisfy Assumption (5.2.1). For $1 < p_0 < p < p_1 < \infty$ and $\alpha = \frac{\frac{1}{p_0} - \frac{1}{p_1}}{\frac{1}{p_0} - \frac{1}{p_1}} = \frac{p_1(p-p_0)}{p(p_1-p_0)}$, we have

$$[W_D^{1,p_0}(\Omega), W_D^{1,p_1}(\Omega)]_\alpha = W_D^{1,p}(\Omega).$$

See [215],[216] for details on the development of this theory. Here, we only recall the essentials to be used in the sequel. Let Y_0, Y_1 be two normed vector spaces embedded in a topological Hausdorff vector space V. For each $a \in Y_0 + Y_1$ and $t > 0$, we define the K-functional of interpolation by

$$K(a, t, Y_0, Y_1) = \inf_{a=a_0+a_1} \left(\|a_0\|_{Y_0} + t\|a_1\|_{Y_1}\right).$$

For $0 < \theta < 1, 1 \leq q \leq \infty$, the real interpolation space $(Y_0, Y_1)_{\theta,q}$ between Y_0 and Y_1 is given by

$$\left\{a \in Y_0 + Y_1 : \|a\|_{\theta,q} := \left(\int_0^\infty \left(t^{-\theta} K(a, t, Y_0, Y_1)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty\right\}.$$

It is an exact interpolation space of exponent θ between Y_0 and Y_1, see [216].

Definition (5.2.19)[206]: Let $f : X \to \mathbb{R}$ be a measurable function on a measure space (X, μ). The decreasing rearrangement of f is the function $f^* :]0, \infty[\to \mathbb{R}$ defined by

$$f^*(t) = \inf \{\lambda : \mu(\{x : |f(x)| > \lambda\} \leq t\}.$$

The maximal decreasing rearrangement of f is the function f^{**} defined for every $t > 0$ by

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s)\, ds.$$

We conclude by quoting the following classical result ([216]):

Proposition (5.2.20)[206]: Let (X, μ) be a measure space with a σ-finite positive measure μ. Let $f \in L^1(X) + L^\infty(X)$. We then have

(i) $K(f, t, L^1, L^\infty) = t f^{**}(t)$ and
(ii) for $1 \leq p_0 < p < p_1 \leq \infty$ it holds $(L^{p_0}, L^{p_1})_{\theta,p} = L^p$ with equivalent norms, where $1/p = (1-\theta)/p_0 + \theta/p_1$ with $0 < \theta < 1$.

The proof of Theorem (5.2.22) is based on the following estimates for the K-functional.

Lemma (5.2.21)[206]: Let $1 < p < \infty$. We have for all $t > 0$

$$K\left(f, t, W_D^{1,1}, W_D^{1,\infty}\right) \geq C_1 t\,|f|^{**}(t) + |\nabla f|^{**}(t) \text{for all } f \in W_D^{1,1} + W_D^{1,\infty}$$

and

$$K\left(f, t, W_D^{1,1}, W_D^{1,\infty}\right) \leq C_{2^t} \left(\left|\nabla \tilde{f}\right|^{**}(t) + \left|\tilde{f}\right|^{**}(t) + \left(\frac{|\tilde{f}|}{d_D}\right)^{**}(t)\right) \text{ for all } f \in W_D^{1,p}.$$

The constants C_1, C_2 are independent of f and t, and $\tilde{f} = \mathfrak{E}f$ is the Sobolev extension of f from Lemma (5.2.3).

Proof. For the lower bounds, let $f \in W_D^{1,1} + W_D^{1,\infty}$ be given. Then, due to Proposition (5.2.20) (i)

$$\begin{aligned} &K\left(f, t, W_D^{1,1}, W_D^{1,\infty}\right) \\ &\geq \left(\inf_{f=f_0+f_1} (\|f_0\|_{L^1} + t\|f_1\|_{L^\infty}) + \inf_{f=f_0+f_1} (\|\nabla f_0\|_{L^1} + t\|\nabla f_1\|_{L^\infty})\right) \\ &\geq C\left(K(|f|, t, L^1, L^\infty) + K(|\nabla f|, t, L^1, L^\infty)\right) = Ct\left(|f|^{**}(t) + |\nabla f|^{**}(t)\right). \end{aligned}$$

Now, for the upper bound, we consider $\in W_D^{1,p}$. For every $t > 0$, we set

$$\alpha(t) := \left(M\left(\left|\nabla \tilde{f}\right| + \left|\tilde{f}\right| + \left|\frac{\tilde{f}}{d_D}\right|\right)\right)^*(t)$$

and we recall from the proof of Lemma (5.2.15) the notation

$$E = E_t = \left\{ x \in \mathbb{R}^d : M\left(\left|\nabla \tilde{f}\right| + \left|\tilde{f}\right| + \left|\frac{\tilde{f}}{d_D}\right| \right)(x) > \alpha(t) \right\}.$$

Remark that with this choice of $\alpha(t)$, we have $|E_t| \le t$ for all $t > 0$. Furthermore, due to applied with $X = \mathbb{R}^d$

$$\alpha(t) \le C\left(\left|\nabla \tilde{f}\right|^{**} + \left|\tilde{f}\right|^{**} + \left|\frac{\tilde{f}}{d_D}\right|^{**} \right)(t). \qquad (31)$$

Now, we take the Calderón–Zygmund decomposition from Lemma (5.2.15) for f with this choice of $\alpha(t)$. This results in a decomposition of $f \in W_D^{1,p}$ as $f = g + b$ with $b \in W_D^{1,1}$ and $\in W_D^{1,\infty}$. Invoking Corollary (5.2.16) (ii), we have $\|g\|_{W_D^{1,\infty}} \le C\alpha(t)$, and from (30), we deduce

$$\|b\|_{W_D^{1,1}} \le C\alpha(t) \sum_{j\in I} \left|Q_j\right| \le C\alpha(t)|E_t| \le Ct\alpha(t).$$

Combining these estimates with (31), we find

$$K\left(f, t, W_D^{1,1}, W_D^{1,\infty} \right) \le \|b\|_{W_D^{1,1}} + t\|g\|_{W_D^{1,\infty}} \le Ct\alpha(t)$$
$$\le Ct\left(\left|\nabla \tilde{f}\right|^{**}(t) + \left|\tilde{f}\right|^{**}(t) + \left(\frac{|\tilde{f}|}{d_D}\right)^{**}(t) \right).$$

for all $f \in W_D^{1,p}$ and for all $t > 0$, and this was the claim.

Theorem (5.2.22)[206]: Let Ω and D satisfy Assumption (5.2.1). Then, for all choices of $1 \le p_0 < p < p_1 \le \infty$, we have for $\alpha = \frac{(p-p_0)p_1}{(p_1-p_0)p}$

$$W_D^{1,p}(\Omega) = \left(W_D^{1,p_0}(\Omega), W_D^{1,p_1}(\Omega) \right)_{\alpha,p}$$

Proof. By the reiteration Theorem (cf. [84]), it suffices to establish the special case of $p_0 = 1$ and $p_1 = \infty$, i.e. $W_D^{1,p} = \left(W_D^{1,1}, W_D^{1,\infty}\right)_{1-\frac{1}{p},p}$ with equivalent norms for $1 < p < \infty$. First, since is bounded, we have $W_D^{1,p} \hookrightarrow W_D^{1,1} \hookrightarrow W_D^{1,1} + W_D^{1,\infty}$. Moreover, for $\in W_D^{1,p}$, we have due to Lemma (5.2.21)

$$\|f\|_{1-\frac{1}{p},p} = \left(\int_0^\infty \left[t^{1/p-1} K\left(f, t, W_D^{1,1}, W_D^{1,\infty} \right) \right]^p \frac{dt}{t} \right)^{\frac{1}{p}}$$
$$\le C\left(\int_0^\infty \left[t^{1/p} \left(|\nabla f|^{**}(t) + \left|\tilde{f}\right|^{**}(t) + \left(\frac{|\tilde{f}|}{d_D}\right)^{**}(t) \right) \right]^p \frac{dt}{t} \right)^{\frac{1}{p}}$$
$$= C\left\| \left|\nabla \tilde{f}\right|^{**} + |f|^{**} + \left(\frac{|\tilde{f}|}{d_D}\right)^{**} \right\|_{L^p(\mathbb{R}_+)}.$$

Since $\|g^{**}\|_{L^p(\mathbb{R}_+)} \sim \|g^*\|_{L^p(\mathbb{R}_+)} = \|g\|_{L^p}$, this allows us to continue

$$\le C\left(\left\|\nabla \tilde{f}\right\|_{L^p(\mathbb{R}^d)} + \left\|\tilde{f}\right\|_{L^p(\mathbb{R}^d)} + \left\|\frac{\tilde{f}}{d_D}\right\|_{L^p(\mathbb{R}^d)} \right) \le C\left\|\tilde{f}\right\|_{W_D^{1,p}(\mathbb{R}^d)} \le C\left\|\tilde{f}\right\|_{W^{1,p}D}$$

thanks to the Hardy inequality in (15) and the continuity of the extension operator that assigns $\tilde{f}$ to f.

Conversely, let $f \in \left(W_D^{1,1}, W_D^{1,\infty}\right)^{1-\frac{1}{p},p}$. Then, invoking the lower estimate in Lemma (5.2.21), we find as above, investing that $g \mapsto g^{**}$ is sublinear,

$$\|f\|_{1-\frac{1}{p},p} \geq C\left(\int_0^\infty \left[t^{\frac{1}{p}}\,|f|^{**}(t) + |\nabla f|^{**}(t)\right]^p \frac{dt}{t}\right)^{1/p}$$
$$= C\,\||f|^{**} + |\nabla f|^{**}\|_{L^p(\mathbb{R}_+)} \geq C\,\|(|f| + |\nabla f|)^{**}\|_{L^p(\mathbb{R}_+)}$$
$$\geq C\||f| + |\nabla f|\|_{L^p} \geq C\,\|f\|_{W^{1,p}}.$$

It remains to check the right boundary behaviour of f , i.e. $f \in W_D^{1,p}$. In order to do so, we use the fact that $W_D^{1,1} \cap W_D^{1,\infty}$ is dense in $\left(W_D^{1,1}, W_D^{1,\infty}\right)_{1-\frac{1}{p},p}$, see [216]. If $f = \lim_{n\to\infty} f_n$ for some sequence (f_n) in $W_D^{1,1} \cap W_D^{1,\infty}$, then the limit is also in $W^{1,p}(\Omega)$ by the above inequality. As $W_D^{1,\infty} \subseteq W_D^{1,p}$ by Lemma (5.2.2), we have $f_n \in W_D^{1,p}$ for every $n \in \mathbb{N}$. As this space is closed in $W^{1,p}$, this yields $f \in W_D^{1,p}$ and we find

$$\|f\|_{W_D^{1,p}} = \|f\|_{W^{1,p}} \leq C\,\|f\|_{1-\frac{1}{p},p}.$$

As a next preparatory step towards the proof of Theorem (5.2.9), we show that the Gaussian estimates imply $L^p - L^2$ off-diagonal estimates for the operators $T(t) := e^{-tA_0}$ and $t\,A_0T\,(t)$.

Lemma (5.2.23)[206]: Let $p \in [1,2]$ and let $E, F \subseteq \Omega$ be relatively closed. Then, there exist constants $C \geq 0$ and $c > 0$, such that for every $h \in L^2 \cap L^p$ with supp $(h) \subseteq E$, we have for all $t > 0$

(i) $\|T(t)h\|_{L^2(F)} \leq Ct^{(d/2-d/p)/2}e^{-c\frac{d(E,F)^2}{t}}\|h\|_{L^p}$ for $p \geq 1$ and

(ii) $\|t\,A_0T\,(t)h\|_{L^2(F)} \leq Ct^{(d/2-d/p)/2}e^{-c\frac{d(E,F)^2}{t}}\|h\|_{L^p}$ for p > 1.

Proof. (i) We denote the kernel of $T\,(t)$ by k_t . Since $A_0 = -\nabla\cdot\mu\nabla + 1$, using the notation of Proposition (5.2.7), we have $k_t = e^{-t}K_t$. Thus, for k_t , we have the Gaussian estimates

$$0 \leq k_t(x,y) \leq \frac{C}{t^{d/2}}\,e^{-c\frac{|x-y|^2}{t}}, \qquad t > 0, \text{a.a.}\,x,y \in \Omega,$$

without the term $e^{\varepsilon t}$. Using these, a straightforward calculation shows

$$\|T\,(t)h\|^2_{L^2(F)} \leq \frac{C}{t^d}\,e^{-c\frac{d(E,F)^2}{t}}\left\|e^{-c\frac{|\cdot|^2}{2t}} * |\tilde{h}|\right\|^2_{L^2(\mathbb{R}^d)},$$

where we denoted by $\tilde{h}$ the extension by 0 of h to the whole of $\mathbb{R}^d$. Now, applying Young's inequality to bound the convolution, one obtains the assertion.

(ii) In a first step, we observe that it is enough to show the assertion in the case $p = 2$. In fact, we have by the first part of the proof (set $E = F = \Omega$ and $p = 1$)

$$\|t\,A0T\,(t)h\|_{L^2(F)} \leq \|T\,(t/2)t\,A_0T\,(t/2)h\|_{L^2} \leq Ct^{-\frac{d}{4}}\left\|t\,A_0T\left(\frac{t}{2}\right)h\right\|_{L^1}$$
$$\leq Ct^{-\frac{d}{4}}\|h\|_{L^1},$$

since $T\,(t)$ extrapolates to an analytic semigroup on L^1 by the Gaussian estimates, cf. [234] or [208]. Admitting the assertion in the case $p = 2$:

$$\|t\, A_0 T\,(t)h\|_{L^2(F)} \;\le\; C\, e^{-c\frac{d(E,F)^2}{t}}\, \|h\|_{L^2},$$

the result then follows by interpolation using the Riesz–Thorin Theorem. In order to prove the off-diagonal bounds in the case $p = 2$, we apply Davies' trick, following the proof of [210]. Since this procedure is rather standard, we just give the major steps. For some Lipschitz continuous function $\varphi : \Omega \to \mathbb{R}$ with $\|\nabla\varphi\|_{L^\infty} \le 1$ and $\varrho > 0$, we define the twisted form

$$a(u,v) = \int_\Omega \left(\mu\nabla(e - \varrho\varphi\, u)\; \cdot\; \nabla(e^{\varrho\varphi}\, \bar{v}) + u\bar{v}\right) dx, \qquad u,v \in D(a_\varrho) := W_D^{1,2}.$$

Setting $\kappa := 2\varrho^2\mu L^\infty$ and estimating the real and imaginary part of the quadratic form $a + \kappa - 1$, one finds that the numerical range of $a_\varrho + \kappa$ lies in the (shifted) sector $\mathcal{S} + 1$, where $S := \left\{\lambda \in \mathbb{C} :\; |\mathrm{Im}\,\lambda| \le +\sqrt{\frac{\|\mu\|_{L^\infty}}{\mu_\bullet}}\, \mathrm{Re}\,\lambda\right\}$ and $\mu_\bullet$ is the ellipticity constant from Assumption (5.2.4).

In the following, we denote by A_ϱ the operator associated to the form a in L^2. Since $A_\varrho + \kappa - 1$ is maximal accretive, cf. [238], its negative generates an analytic C_0-semigroup e^{-tA_ϱ} on L^2 and A_ϱ even admits a bounded H^∞-calculus there, cf. [219] or [228]. Applying the functional calculus of A_ϱ, for every =, we find

$$\left\|t\, A_\varrho e^{-tA_\varrho}\right\| \;\le\; \left\|t\left(A_\varrho + \kappa\right) e^{-t(A_\varrho+\kappa)}\right\| e^{t\kappa} + \left\|e^{-t(A_\varrho+\kappa)}\right\| t\kappa\, e^{t\kappa} \;\le\; C\, e^{t\kappa} + C\, e^{2t\kappa}$$
$$\le\; C\, e^{4\varrho^2 t\|\mu\|_{L^\infty}}. \qquad (32)$$

Recalling that the form domain $W_D^{1,2}$ is invariant under multiplications with $e^{\varrho\varphi}$ by Proposition (5.2.6) (iii), it is easy to verify that for every $f \in L^2$ with $e^{-\varrho\varphi} f \in D(A_0)$, we have $A_\varrho f = -\, e^{\varrho\varphi}\, A_0\, e^{-\varrho\varphi} f$. From this, we then deduce

$$R(\lambda, A_\varrho) \;=\; e^{\varrho\varphi}\, R(\lambda, A_0)\, e^{-\varrho\varphi}, \qquad \text{for all } \lambda > \varrho^2\|\mu\|_{L^\infty},$$

which finally yields for every $f \in L^2$

$$e^{-tA_\varrho} f \;=\; \lim_{n\to\infty} \left[\frac{n}{t}\, R\left(n/t\,, A_\varrho\right)\right]^n f \;=\; e^{\varrho\varphi} \lim_{n\to\infty} \left[\frac{n}{t}\, R(n/t\,, A_0)\right]^n e^{-\varrho\varphi} f$$
$$=\; e^{\varrho\varphi}\, T\,(t)\, e^{-\varrho\varphi} f.$$

Now, we specify $\varphi(x) = d(x,E)$ for $\in \Omega$. Then, for every $h \in L^2$ with support in E and all $\varrho, t > 0$, we get

$$t\, A_0 T\,(t)h \;=\; -t\frac{d}{dt}\, T\,(t)h \;=\; t\, e^{-\varrho\varphi}\, A_\varrho\, e^{-tA_\varrho}\, e^{\varrho\varphi} h \;=\; t\, e^{-\varrho\varphi}\, A_\varrho\, e^{-tA_\varrho}\, h,$$

as $\varphi = 0$ on the support of h. This yields for all $\varrho, t > 0$

$$\|tA_0 T\,(t)h\|_{L^2(F)} \;=\; \left\|t\, e^{-\varrho d(\cdot,E)}\, A_\varrho e^{-tA_\varrho}\, h\right\|_{L^2(F)} \;\le\; e^{-\varrho d(E,F)} \left\|t\, A_\varrho\, e^{-t\,A_\varrho}\, h\right\|_{L^2}$$
$$\le\; C\, e^{4\varrho^2\|\mu\|_{L^\infty} t - \varrho d(E,F)}\, \|h\|_{L^2},$$

thanks to (32). Minimizing over $\varrho > 0$ finally yields the assertion with $c = (8\|\mu\|_{L^\infty})^{-1}$.

We now turn to the proof of Theorem (5.2.9). Building on the hypotheses that the assertion is true for $p = 2$, cf. Assumption (5.2.5), we will show the corresponding inequality in a weak (p, p) setting for all 1 < p < 2. Then, our result follows by interpolation. We show the following.

Proposition (5.2.24)[206]: Let Ω and D satisfy Assumption (5.2.1), and let μ be such that Assumptions (5.2.4) and (5.2.5) are true. Then, there is a constant $C \ge 0$, such that for all $p \in]1,2[$, for every $f \in C_D^\infty$ and all $\alpha > 0$, we have

$$\left\{x \in \Omega : \left|A_0^{1/2} f(x)\right| > \alpha\right\} \leq \frac{C}{\alpha^p} \|f\|_{W^{1,p_D}}^p. \qquad (33)$$

Proof. We follow the proof of [210]. Let $\alpha > 0, p \in]1,2[$ and $f \in C_D^\infty$ be given. We apply the Calderón–Zygmund decomposition from Lemma (5.2.15) to write $f = g + \sum_{j\in I} b_j$. In all what follows the references (i)–(vi) will stand for the corresponding features in Lemma (5.2.15).

Since $C_D^\infty \hookrightarrow W_D^{1,2} = \mathrm{dom}_{L^2}\left(A_0^{1/2}\right)$, by Corollary (5.2.16) (iii) also the functions g and $b = \sum_{j\in I} b_j$ are in the L^2-domain of $A_0^{1/2}$ and $A_0^{1/2} b = \sum_{j\in I} A_0^{1/2} b_j$. Thus, we can estimate

$$\left|\left\{x \in \Omega : \left|A_0^{1/2} f(x)\right| > \alpha\right\}\right|$$
$$\leq \left|\left\{x \in \Omega : \left|A_0^{1/2} g(x)\right| > \frac{\alpha}{2}\right\}\right| + \left|\left\{x \in \Omega : \left|A_0^{1/2} b(x)\right| > \frac{\alpha}{2}\right\}\right|, \qquad (34)$$

We bound both terms on the right-hand side by $C\,\|f\|_{W_D^{1,p}}^p / \alpha^p$. The one containing g is as always the easy part. We first note that thanks to (vi) and Corollary (5.2.16) we know

$$\|g\|_{W_D^{1,p}} \leq C\,\|f\|_{W_D^{1,p}} \text{ and } \|g\|\_(W_D^{1,\infty} \leq C\alpha.$$

By interpolation, this yields

$$\|g\|_{W_D^{1,2}}^2 \leq C\|g\|_{W_D^{1,p}}^p \|g\|_{W_D^{1,\infty}}^{2-p} \leq C\alpha^{2-p}\|f\|_{W_D^{1,p}}^p.$$

This implies, using the Tchebychev inequality and Assumption (5.2.5)

$$\left|\left\{x \in \Omega : \left|A_0^{1/2} g(x)\right| > \frac{\alpha}{2}\right\}\right| \leq \frac{C}{\alpha^2}\left\| A_0^{1/2} g\right\|_{L^2}^2 \leq \frac{C}{\alpha^2}\|g\|_{W_D^{1,2}}^2 \leq \frac{C}{\alpha^p}\|f\|_{W_D^{1,p}}^p.$$

Lets turn to the estimate of the second part in (34). We first recall the integral representation of the square root

$$A_0^{1/2} u = \frac{2}{\sqrt{\pi}} \int_0^\infty A_0 e^{-t^2 A_0} u\, dt \text{ for all } u \in \mathrm{dom}_{L^2}\left(A_0^{1/2}\right),$$

which can be deduced straightforwardly from the well-known formula (see [243])

$$A_0^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-tA_0}}{\sqrt{t}}\, dt.$$

This yields thanks to $A_0^{1/2} b = \sum_{j\in I} A_0^{1/2} b_j$

$$\left|\left\{x \in \Omega : |\left(A_0^{1/2} b(x)\right) > \frac{\alpha}{2}\right\}\right| = \left|\left\{x \in \Omega : \left|\frac{2}{\sqrt{\pi}}\int_0^\infty \sum_{j\in I} \left(A_0\, e^{-t^2 A_0} b_j\right)(x) dt\right| > \frac{\alpha}{2}\right\}\right|$$
$$= \limsup_{m\to\infty} \left|\left\{x \in \Omega : \left|\frac{2}{\sqrt{\pi}}\int_{2-m}^\infty \sum_{j\in I} \left(A_0\, e^{-t^2 A_0} b_j\right)(x) dt\right| > \frac{\alpha}{2}\right\}\right|.$$

In the following, we denote again by ℓ_j the sidelength of the cube $Q\,j, j \in I$, and we set $r_j := 2^k$ for that value of $k \in \mathbb{Z}$, such that $2^k \leq \ell_j < 2^{k+1}$. With this notation, we split the integral for every $m \in \mathbb{N}$:

$$\left|\left\{x \in \Omega : \left|\frac{2}{\sqrt{\pi}} \int_{2-m}^{\infty} \sum_{j\in I} \left(A_0\, e^{-t^2 A_0} b_j\right)(x)\, dt\right| > \frac{\alpha}{2}\right\}\right|$$

$$\leq \left|\left\{ x \in \Omega : \left|\sum_{j\in I} \int_{2^{-m}}^{r_j \vee 2^{-m}} A_0\, e^{-t^2 A_0}\, b_j(x) dt\right| > \frac{\sqrt{\pi}\alpha}{8}\right\}\right|$$

$$+ \left|\left\{ x \in \Omega : \left|\sum_{j\in I} \int_{rj \vee 2^{-m}}^{\infty} A_0\, e^{-t^2 A_0}\, b_j(x) dt\right| > \frac{\sqrt{\pi}\alpha}{8}\right\}\right|. \tag{35}$$

For the estimate of the first integral, we may restrict ourselves to the case $r_j > 2^{-m}$, since otherwise there is no contribution from this term. We do the usual trick to split off the union of the sets $4Q_\iota, \iota \in I$ that does not produce any sort of problem due to

$$\left|\bigcup_{\iota\in I} 4Q_\iota\right| \leq \sum_{\iota\in I} |4Q_\iota| \leq C \sum_{\iota\in I} |Q_\iota| \overset{(4)}{\leq} \frac{C}{\alpha^p} \|f\|_{W_D^{1,p}}^p .$$

So, we only have to estimate

$$\left|\left\{x \in \Omega \bigcup_{\iota\in I} 4Q_\iota : \left|\sum_{j\in I} \int_{2^{-m}}^{r_j} A_0\, e^{-t^2 A_0}\, b_j(x) dt\right| > \frac{\sqrt{\pi}\alpha}{8}\right\}\right|$$

$$= \left|\left\{x \in \Omega : \left|1_{(\cup_{\iota\in I} 4Q_\iota)^c} \sum_{j\in I} \int_{2^{-m}}^{r_j} A_0\, e^{-t^2 A_0}\, b_j(x) dt\right| > \frac{\sqrt{\pi}\alpha}{8}\right\}\right|.$$

By the Tchebychev inequality, we get

$$\leq \frac{C}{\alpha^2} \left\| 1_{(\cup_{\iota\in I} 4Q_\iota)^c} \sum_{j\in I} \int_{2^{-m}}^{r_j} A_0\, e^{-t^2 A_0}\, b_j\, dt\right\|_{L^2}^2 . \tag{36}$$

In order to estimate this norm, we take $u \in L^2(\Omega)$ with $\|u\|_{L^2} = 1$. Then,

$$\left|\int_\Omega u 1_{(\cup_{\iota\in I} 4Q_\iota)^c} \sum_{j\in I} \int_{2^{-m}}^{r_j} A_0\, e^{-t^2 A_0}\, b_j\, dt\right|$$

$$\leq \sum_{j\in I} |u| 1_{(\cup_{\iota\in I} 4Q_\iota)^c} \left|\int_{2^{-m}}^{r_j} A_0\, e^{-t^2 A_0}\, b_j\, dt\right|.$$

We now split the integration over into frame-like pieces and apply the Cauchy– Schwarz inequality. Note that the characteristic function results in the sum over l starting only at $l = 2$.

$$\leq \sum_{j\in I} \sum_{l=2}^{\infty} \int_{\left(2^{l+1} Q_j / 2^l Q_j\right)\cap\Omega} |u| \left|\int_{2^{-m}}^{r_j} A_0\, e^{-t^2 A_0}\, b_j\, dt\right| \tag{37}$$

$$\leq \sum_{j\in I} \sum_{l=2}^{\infty} \|u\|_{L^2\left(\left(2^{l+1} Q_j / 2^l Q\right)\cap\Omega\right)} \left\|\int_{2^{-m}}^{r_j} A_0\, e^{-t^2 A_0}\, b_j\, dt\right\|_{L^2\left(\left(2^{l+1} Q_j / 2\, Q\right)\cap\Omega\right)} .$$

In order to estimate the first factor of the last expression, we identify u with its trivial extension by zero to $\mathbb{R}^d$. Then, we let appear the maximal operator to obtain for every $y \in Q_j$

$$\|u\|^2_{L^2\left((2^{l+1}Q_j/2Q)\cap\Omega\right)} \leq \int_{2^{l+1}Q_j} |u|^2 \leq C^{2d(l+1)} \,|Q_j| \frac{1}{|2^{l+1}Q_j|}\int_{2^{l+1}Q_j} |u|^2$$
$$\leq C 2^{dl} \ell_j^d \,[M(|u|^2)](y).$$

Applying the off-diagonal estimates for $t^2 A_0\, e^{-t^2 A_0}$ from Lemma (5.2.23) with the set $Q_j \cap \Omega$ as E, $(2^{l+1} Q_j / 2^l Q_j) \cap \Omega$ as F, $d/(d-1)$ as p and b_j as h, we get

$$\left\|A_0\, e^{-t^2 A_0}\, b_j\right\|_{L^2\left((2^{l+1}Q_j/2^l Q_j)\cap\Omega\right)} \leq \frac{C}{t^2}\, t^{d/2-(d-1)} e^{-c\frac{d(E,F)^2}{t^2}} \left\|b_j\right\|_{L^{d}/(d-1)}$$
$$\leq \frac{C}{t^{1+d/2}}\, e^{-c\frac{4^l r_j^2}{t^2}} \left\|b_j\right\|_{L^{d}/(d-1)},$$

since $d(E,F) \geq d\left(Q\, j, 2^{l+1} Q_j/2^l Q_j\right) \geq c\left(2^l \ell_j - \ell_j\right) \geq c(2^{l-1}) r_j \geq c 2^l r_j$ thanks to $l \geq 2$.

According to (iii), the functions b_j are from $W_D^{1,1}$. Exploiting the Sobolev embedding $W_D^{1,1} \hookrightarrow L^{d/(d-1)}$

$$\left\|b_j\right\|_{L^{d}/(d-1)} \leq C\left\|b_j\right\|_{W^{1,1}} \leq C\alpha\left|Q_j\right| \leq C\alpha \ell_j^d\,. \tag{38}$$

Putting all this together, we find for our second factor

$$\left\|\int_{2^{-m}}^{r_j} A_0\, e^{-t^2 A_0}\, b_j\, dt\right\|_{L^2\left((2^{l+1}Q_j/2^l Q_j)\cap\Omega\right)}$$
$$\leq \int_{2^{-m}}^{r^j} \left\|A_0\, e^{-t^2 A_0}\, b_j\right\|_{L^2\left((2^{l+1}Q_j/2^l Q_j)\cap\Omega\right)} dt$$
$$\leq C\alpha \ell_j^d \int_{2^{-m}}^{r_j} \frac{1}{t^{1+d/2}} e^{-c\frac{4^l r_j^2}{t^2}}\, dt$$
$$= C\alpha \ell_j^d \int_{c4^l}^{c4^l r_j^2 4^m} \left(\frac{\sqrt{s}}{2^l r_j}\right)^{1+d/2} e^{-s}\, 2^l\, r_j s^{-3/2}\, ds$$
$$\leq C\alpha \ell_j^d r_j^{-d/2}\, 2^{-ld/2} \int_{c4^l}^{\infty} s^{-1+d/4}\, e^{-s}\, ds,$$

which is now independent of $m \in \mathbb{N}$. Since the integrand is positive and $r_j \geq 2\ell_j$, we may continue

$$\leq C\alpha \ell_j^{d/2}\, 2^{-ld/2} e^{-c4^l} \int_{c4^l}^{\infty} s^{-1+d/4}\, e^{-s+c4^l}\, ds$$
$$= C\alpha \ell_j^{d/2}\, 2^{-ld/2} e^{-c4^l} \int_{0}^{\infty} (\sigma + c4^l)^{-1+d/4}\, e^{-\sigma}\, d\sigma$$
$$= C\alpha \ell_j^{d/2}\, 4^{-l}\, e^{-c4^l} \int_{0}^{\infty} (\sigma 4^{-l} + c)^{-1+d/4}\, e^{-\sigma}\, d\sigma.$$

This last integral is bounded uniformly in $l \geq 2$. In fact, if $d > 4$, then we estimate $4^{-l} \leq 4^{-2}$, and if $d \leq 4$, we may just estimate by dropping out the whole $\sigma 4^{-l}$. So, estimating once more $4^{-l} \leq 4^{-2}$, we end up with

$$\left\| \int_{2^{-m}}^{r_j} A_0 \, e^{-t^2 A_0} \, b_j \, dt \right\|_{L^2\left((2^{l+1} Q_j / 2^l \, Q_j) \cap \Omega\right)} \le C \alpha \ell_j^{d/2} \, e^{-c4^l} .$$

Coming back to (37), we thus have

$$\int_{(2^{l+1} Q_j / 2^l Q_j) \cap \Omega} |u| \left| \int_{2^{-m}}^{r_j} A_0 \, e^{-t^2 A_0} \, b_j dt \right|$$

$$\le C 2^{ld/2} \, \ell_j^{d/2} \left([M(|u|^2)] \, (y) \right)^{1/2} \alpha \ell_j^{d/2} \, e\wedge - c4^l$$

for every $y \in Q_j$. Averaging over y, the inequality remains valid, and we get

$$\sum_{j \in I} \sum_{l=2}^{\infty} \int_{(2^{l+1} Q_j / 2^l Q_j) \cap \Omega} |u| \left| \int_{2^{-m}}^{r_j} A_0 \, e_0^{-t^2 A} \, b_j \, dt \right|$$

$$\le C \sum_{j \in I} \sum_{l=2}^{\infty} \frac{1}{|Q_j|} \int_{Q_j} \alpha 2^{ld/2} \ell_j^d e^{-c4^l} \left([M(|u|^2) \, (y)] \right)^{\frac{1}{2}} dy$$

$$\le C\alpha \sum_{j \in I} \sum_{l=2}^{\infty} 2^{ld/2} e^{-c4^l} \int_{Q_j} \left(([M(|u|^2)])(y) \right)^{\frac{1}{2}} dy.$$

The sum over l now turns out to be convergent, so we continue

$$\le C\alpha \int_{\mathbb{R}^d} \sum_{j \in I} 1_{Q_j}(y) \left(([M(|u|^2)])(y) \right)^{\frac{1}{2}} dy \le C\alpha \int_{\bigcup_{j \in I} Q_j} \left([M(|u|^2)](y) \right)^{\frac{1}{2}} dy,$$

where we used (v) in the last step. By the Kolmogorov inequality (cf. [247]), we have

$$\int_{\bigcup_{j \in I} Q_j} ([M(|u|^2)(y)]^{\frac{1}{2}} dy \le C \left| \bigcup_{j \in I} Q_j \right|^{\frac{1}{2}} \| |u|^2 \|_{L^1(\mathbb{R}^d)}^{\frac{1}{2}} \le C \left(\sum_{j \in I} |Q_j| \right)^{\frac{1}{2}} \|u\|_{L^2} .$$

Coming back to (36), we thus finally achieve (observe that $\|u\|_{L^2} = 1$)

$$\left| \left\{ x \in \Omega \setminus \bigcup_{\iota \in I} 4Q_\iota : \left| \sum_{j \in I} \int_{2^{-m}}^{r_j} A_0 \, e^{-t^2 A_0} b_j(x) dt \right| > \frac{\sqrt{\pi}\alpha}{8} \right\} \right|$$

$$\le \frac{C}{\alpha^2} \left\| 1_{(\bigcup_{\iota \in I} 4Q_\iota)^c} \sum_{j \in I} \int_{2^{-m}}^{r_j} A_0 \, e_j^{-t^{2A_0} b} \, dt \right\|_{L^2}^2 \le C \sum_{j \in I} |Q_j|$$

$$\le \frac{C}{\alpha^p} \|f\|_{W_D^{1,p}}^p$$

by (iv).

We turn to the estimate of the second addend on the right-hand side of (35). For this task, we will again need the notion of a bounded H^∞-calculus. The definition and further information can be found in [219] or [228].

We define the function

$$\psi(z) := \int_1^\infty z \, e^{-t^2} z \, dt, \qquad \mathrm{Re}(z) > 0.$$

We show that

$$\psi \in \mathcal{H}_0^\infty(\Sigma_\mu) : = \left\{ f : \Sigma_\mu \to C \text{ analytic and } \exists \varepsilon > 0 \text{ s.t.} |f(z)| \le C \frac{|z|^\varepsilon}{(1+|z|)^{2\varepsilon}} \text{ for all } z \in \mu \right\}$$

for every $\mu \in]0, \pi/2[$, where $\Sigma_\mu := \{z \in C : |\arg(z)| < \mu\}$. In fact, we have substituting $\tau = t^2 \operatorname{Re}(z) - \operatorname{Re}(z)$

$$\begin{aligned} \left| \frac{(1+|z|)^{2\varepsilon}}{|z|^\varepsilon} \psi(z) \right| &\le \int_1^\infty |z|^{1-\varepsilon}(1+|z|)^{2\varepsilon} e^{-t^2 \operatorname{Re}(z)} \, dt \\ &= \int_0^\infty |z|^{1-\varepsilon}(1+|z|)^{2\varepsilon} e^{-\tau} e^{-\operatorname{Re}(z)} \frac{1}{2\sqrt{\operatorname{Re}(z)(\tau + \operatorname{Re}(z))}} d\tau \\ &\le C|z|^{\frac{1}{2}-\varepsilon}(1+|z|)^{2\varepsilon} e^{-c|z|} \int_0^\infty \frac{e^{-\tau}}{\sqrt{\tau}} d\tau, \end{aligned}$$

since $\operatorname{Re}(z) \sim |z|$, thanks to $|\arg(z)| < \mu < \pi/2$. Thus, we may choose $\varepsilon \in]0, 1/2[$. Furthermore, we have for every $z \in \mathbb{C}$ with $\operatorname{Re}(z) > 0$ and every $r > 0$

$$\frac{1}{r} \psi(r^2 z) = \int_r^\infty z\, e^{-t^2 z} \, dt,$$

so since A_0 has a bounded H^∞-calculus on L^q , see Proposition (5.2.8) (ii), we have the equality of operators

$$\int_r^\infty A_0 \, e^{-t^2 A_0} dt = \frac{1}{r} \psi(r^2 A_0)$$

in L^q for every $1 < q < 2$. Thus, denoting $I_k := \{ j \in I : r_j \vee 2^{-m} = 2^k \}$ for every $k \in \mathbb{Z}$, we get

$$\begin{aligned} \sum_{j \in I} \int_{r_j \vee 2^{-m}}^\infty A_0 \, e^{-t^2 A_0} b_j \, dt &= \sum_{k \in \mathbb{Z}} \sum_{j \in I_k} \frac{1}{r_j \vee 2^{-m}} \psi\left(\left(r_j \vee 2^{-m}\right)^2 A_0 \right) b_j \\ &= \sum_{k \in \mathbb{Z}} \psi(4^k A_0) \sum_{j \in I_k} \frac{b_j}{r_j \vee 2^{-m}}. \end{aligned}$$

After these preparations, we actually start the estimate. Let $q := d/(d-1)$ be the Sobolev conjugated index to 1. Using the Tchebychev inequality for this q, we get

$$\begin{aligned} &\left| \left\{ x \in \Omega : \left| \sum_{j \in I} \int_{r_j \vee 2^{-m}}^\infty A_0 \, e^{-t^2 A_0} b_j(x) \, dt \right| > \frac{\sqrt{\pi}\alpha}{8} \right\} \right| \\ &\le \frac{C}{\alpha^q} \left\| \sum_{j \in I} \int_{r_j \vee 2^{-m}}^\infty A_0 \, e^{-t^2 A_0} b_j \, dt \right\|_{L^q}^q \\ &= \frac{C}{\alpha^q} \left\| \sum_{k \in \mathbb{Z}} \psi(4^k A_0) \sum_{j \in I_k} \frac{b_j}{r_j \vee 2^{-m}} \right\|_{L^q}^q. \end{aligned}$$

Observe that the sum over k is in fact a finite sum, since I_k is empty for $k < -m$ by definition and for large k by the finite measure of E, cf. (16). Thus, there is no convergence problem in applying Lemma (5.2.26), which helps to estimate this expression further by

$$\leq \frac{C}{\alpha^q}\left\|\left(\sum_{k\in\mathbb{Z}}\left|\sum_{j\in I_k}\frac{b_j}{r_j \vee 2^{-m}}\right|^2\right)^{\frac{1}{2}}\right\|_{L^q}^q = \frac{C}{\alpha^q}\int_\Omega\left(\sum_{k\in\mathbb{Z}}\left|\sum_{j\in I_k}\frac{b_j(x)}{r_j \vee 2^{-m}}\right|^2\right)^{\frac{q}{2}} dx.$$

Now, by (v), the sum over k is finite for every $x \in \Omega$, and the number of addends is even bounded uniformly in x and in m, so by the equivalence of norms in finite dimensional spaces, we may continue to estimate by

$$\leq \frac{C}{\alpha^q}\int_\Omega\left(\sum_{k\in\mathbb{Z}}\left|\sum_{j\in I_k}\frac{b_j(x)}{r_j \vee 2^{-m}}\right|\right)^q dx \leq \frac{C}{\alpha^q}\int_\Omega\left(\sum_{j\in I}\frac{|b_j(x)|}{r_j \vee 2^{-m}}\right)^q dx.$$

Next, we estimate $r_j \vee 2^{-m}$ by r_j , and using again the equivalence of norms in the finite sum over j, we get

$$\leq \frac{C}{\alpha^q}\int_\Omega\sum_{j\in I}\frac{|b_j(x)|^q}{r_j^q}\,dx \leq \frac{C}{\alpha^q}\sum_{j\in I}\ell_j^{-q}\int_\Omega|b_j(x)|^q\,dx,$$

since $r_j \sim \ell_j$. Using once more the Sobolev embedding $W^{1,1} \hookrightarrow L^{d/(d-1)} = L^q$, we see as in (38)

$$\int_\Omega|b_j(x)|^q\,dx = \|b_j\|_{L^q}^q \leq C\left(\alpha\ell_j^d\right)^q = C\alpha^q\ell_j^{dq}.$$

Summarizing, we have shown

$$\left|\left\{x\in\Omega:\left|\sum_{j\in I}\int_{r_j\vee 2^{-m}}^\infty A_0\,e^{-t^2A_0}b_j(x)dt\right|>\frac{\sqrt{\pi}\alpha}{8}\right\}\right| \leq \frac{C}{\alpha^q}\sum_{j\in I}\ell_j^{-q}\,\alpha^q\,\ell_j^{dq}$$

$$= C\sum_{j\in I}\ell_j^d \leq C\sum_{j\in I}|Q_j| \leq \frac{C}{\alpha^p}\,\|f\|_{W_D^{1,p}}^p,$$

using one final time (iv).

It remains to prove Lemma (5.2.26), which serves as a substitute in [210]. We give a different proof that instead of $L^p - L^2$ off-diagonal estimates relies on the H^∞ functional calculus of the operator and gives the assertion for the full range of $1 < q < \infty$.

In the proof of Lemma (5.2.26), we will use the following Lemma from [237] (see also [218]).

Lemma (5.2.25)[206]: Let $1 < q < \infty$ and let $-B$ be the generator of a bounded analytic semigroup on L^q , such that B admits a bounded H^∞-calculus on L^q and let $\psi \in H_0^\infty(\Sigma_\phi)$ for some $\phi \in]\varphi_B^\infty,\pi]$. Then, there is a constant $C \geq 0$, such that for every bounded sequence $(\alpha_k)_{k\in\mathbb{Z}} \subseteq \mathbb{C}$ and every $t > 0$, we have

$$\left\|\sum_{k\in\mathbb{Z}}\alpha_k\psi(2^k\,t\,B)\right\|_{\mathcal{L}(L^q)} \leq C\sup_{k\in\mathbb{Z}}|\alpha_k|.$$

Lemma (5.2.26)[206]: Let $1 < q < \infty$, let $-B$ be the generator of a bounded analytic semigroup on L^q , such that B and B' admit bounded H^∞-calculi on L^q and $L^{q'}$, respectively, and let $\psi \in H_0^\infty(\Sigma_\phi)$ for some $\phi \in]\varphi_B^\infty, \pi[$, where φ_B^∞ is the H^∞- angle of B. Then, for every choice of functions $f_k \in L^q$, $k \in \mathbb{Z}$, we have

$$\left\| \sum_{k\in\mathbb{Z}} \psi(4^k B) f_k \right\|_{L^q} \leq C \left\| \left(\sum_{k\in\mathbb{Z}} |f_k|^2 \right)^{\frac{1}{2}} \right\|_{L^q},$$

whenever the left-hand side is convergent. Before starting the proof, we observe that thanks to [237], the operator B even has an $\mathcal{R}$-bounded H^∞-calculus of angle φ_B^∞ on L^q , which means that for every $\phi > \varphi_B^\infty$ and every bounded set of functions $\Xi \subseteq H^\infty(\Sigma_\phi)$, the set of operators $\{\xi(A) : \xi \in \Xi\}$ is $\mathcal{R}$-bounded in $L(L^q)$. Here, a set $\mathcal{T} \subseteq \mathcal{L}(L^q)$ is called R-bounded, if there is a constant $C \geq 0$, such that for every $N \in \mathbb{N}$, for every choice of functions $f_k \in L^q, k = 1,\ldots,N$, operators $T_k \in \mathcal{T}, k = 1,\ldots,N$, and $\{-1,1\}$-valued, symmetric and independent random variables $\varepsilon_k, k = 1,\ldots,N$, on some probability space S, we have

$$\left\| \sum_{k=1}^{N} \varepsilon_k T_k f_k \right\|_{L^2(S;L^q)} \leq C \left\| \sum_{k=1}^{N} \varepsilon_k f_k \right\|_{L^2(S;L^q)}.$$

Proof. Since $\psi \in H_0^\infty(\Sigma_\phi)$, there exists an $\varepsilon > 0$ with $|\psi(z)| \leq C|z|^\varepsilon/(1 + |z|)^{2\varepsilon}$ for all $z \in \Sigma_\phi$. Let $\delta \in]0, \varepsilon[$ and set

$$\psi_1(z) := \frac{z^\delta}{(1 + z)^{2\delta}}, \psi_2(z) := \frac{(1 + z)^{2\delta}}{z^\delta} \psi(z), \qquad z \in \Sigma_\phi.$$

Then, we have $\psi_1, \psi_2 \in H_0^\infty(\Sigma_\phi), \psi = \psi_1\psi_2$ and $(\psi_1(B))' = \overline{\psi_1}(B')$. Now, let $N \in \mathbb{N}$ and let $g \in L^{q'}$ with $\|g\|_{L^{q'}} = 1$, where $1/q + 1/q' = 1$. Then, for every family of $\{-1,1\}$-valued, symmetric and independent random variables $\varepsilon_k, k = -N,\ldots,N$, on some probability space S, we have

$$\left| \int_\Omega \sum_{k=-N}^{N} (\psi(4^k B) f_k)(x) g(x) dx \right|$$

$$= \left| \int_S \sum_{k=-N}^{N} \varepsilon_k^2(\sigma) \int_\Omega (\psi_2(4^k B) f_k)(x)(\overline{\psi_1}(4^k B')g)(x) dx\, d\sigma \right|.$$

Since the random variables $\varepsilon_k, k = -N,\ldots,N$, are independent and thus orthogonal in $L^2(S)$, we may write this as

$$= \left| \int_S \sum_{j,k=-N}^{N} \varepsilon_k(\sigma)\varepsilon_j(\sigma) \int_\Omega (\psi_2(4^k B) f_k)(x)\left(\overline{\psi_1}(4^j B')g\right)(x) dx\, d\sigma \right|$$

$$\leq \int_S \left| \sum_{k=-N}^{N} \varepsilon_k(\sigma)(\psi_2(4^k B) f_k)(x) \sum_{j=-N}^{N} \varepsilon_j(\sigma)\text{-}\left(\overline{\psi_1}(4^j B')g\right)(x) dx \right| d\sigma$$

and using twice the Hölder inequality, we estimate by

$$\leq C \left\| \sum_{k=-N}^{N} \varepsilon_k(\psi_2(4^k B) f_k) \right\|_{L^2(S;L^q)} \left\| \sum_{j=-N}^{N} \varepsilon_j \overline{\psi_1}(4^j B')g \right\|_{L^2(S;L^{q'})}.$$

Now, in the first factor, we use the $\mathcal{R}$-bounded H^∞-calculus of B. Since the set of functions $\{\psi_2(4^k\ \cdot): k\ \in \mathbb{Z}\}$ is bounded in $H^\infty(\Sigma_\phi)$, we get

$$\left\|\sum_{k=-N}^{N} \varepsilon_k \psi_2(4^k\, B) f_k\right\|_{L^2(S;L^q)} \leq\ C \left\|\sum_{k=-N}^{N} \varepsilon_k\, f_k\right\|_{L^2(S;L^q)} \leq\ C \left\|\left(\sum_{k=-N}^{N} |\, f_k|^2\right)^{\frac{1}{2}}\right\|_{L^q},$$

where the last inequality follows from Khinchin's inequality (cf. [220]). In order to estimate the second factor, we apply Lemma (5.2.25) and get

$$\left\|\sum_{j=-N}^{N} \varepsilon_j \overline{\psi_1}(4^j\, B')g\right\|_{L^2(S;L^{q'})} \leq \left(\int_S \left\|\sum_{j=-N}^{N} \varepsilon_j(\sigma)\overline{\psi_1}(2^{2j}B')\right\|^2_{L(L^{q'})} \|g\|^2_{L^{q'}}\, d\sigma\right)^{\frac{1}{2}}$$

$$\leq \left(\int_S \left(\sup_{j=-N}^{N} |\varepsilon_j(\sigma)|\right)^2 d\sigma\right)^{\frac{1}{2}} = 1.$$

This implies

$$\left\|\sum_{k=-N}^{N} \psi(4^k\, B) f_k\right\|_{L^q} = \sup_{g\in L^{q'};\|g\|_{L^{q'}}=1} \left|\sum_{k=-N}^{N} (\psi(4^k\, B)\, f_k)\,(x) g(x)\, dx\right|$$

$$\leq\ C \left\|\left(\sum_{k=-N}^{N} |\, f_k|^2\right)^{\frac{1}{2}}\right\|_{L^q}$$

for every $N\ \in \mathbb{N}$. Letting $N\ \to\ \infty$, the assertion follows.

Let us now come to the final step of the proof of the second assertion of Theorem (5.2.9). Inequality (33) can be interpreted as follows: $A_0^{\frac{1}{2}}$ is a continuous operator from $C_D^\infty\,(\Omega)$—equipped with the $W^{1,p}$-norm—into the Lorentz space $L^{p,\infty}$, cf. [84]. The space $L^{p,\infty}$ is identical (as a set) with $(L^\infty, L^1)_{\frac{1}{p},\infty}$, and its quasinorm $f\ \mapsto \sup_{t\geq 0}\ t^p|\{x : |\, f\,(x)|\ >\ t\}|$ is equivalent to the $(L^\infty, L^1)_{\frac{1}{p},\infty}$-norm (see [84]), i.e. under a suitable renorming, $L_{p,\infty}$ is an ordinary Banach space. Hence, $A_0^{\frac{1}{2}}$ uniquely extends by density to a continuous operator from $W_D^{1,p}$ into $L_{p,\infty}$. Thus, up to now, we have the two continuous mappings

$$A_0^{\frac{1}{2}}:\ W_D^{1,2}\ \to\ L^2$$

and

$$A_0^{\frac{1}{2}}:\ W_D^{1,p}\ \to\ L^{p,\infty}$$

for all $1\ <\ p\ <\ 2$. Let $q\ \in]1,2[$ and choose $p\ \in]1,q[$. Using real interpolation, this gives the continuous mapping

$$A_0^{\frac{1}{2}}: \left(W_D^{1,p}, W_D^{1,2}\right)_{\theta,q} \to \left(L_{p,\infty}, L^2\right)_{\theta,q}.$$

Setting $=\frac{2}{q}\frac{q-p}{2-p}$, the left-hand side is equal to $W_D^{1,q}$ by Theorem (5.2.22) and the righthand side equals L^q according to [84]. This finishes the proof.

Corollary (5.2.27)[206]: Under the above assumptions, one has for $p \in]1,2]$ and $\beta \in]0,1/2[$

$$\mathrm{dom}_{L^p}\left(A_0^{\beta}\right) = \left[L^p, W_D^{1,p}\right]_{2\beta}. \quad (39)$$

Proof. The operator A_0 admits bounded imaginary powers, according to Proposition (5.2.8) (ii). Hence, (39) follows from a classical result, see [84].

Remark (5.2.28)[206]: In view of this result, it would be highly interesting to determine also the interpolation spaces in formula (39). We suggest the formula

$$\left[L^p, W_D^{1,p}\right]_{\theta} = \begin{cases} H^{\theta,p}, & \text{if } \theta < \frac{1}{p} \\ H_D^{\theta,p}, & \text{if } \theta > \frac{1}{p}, \end{cases} \quad (40)$$

$H^{\theta,p}$ being the space of Bessel potentials and $H_D^{\theta,p}$ being the subspace, which is defined via the trace-zero condition on D. Unfortunately, we are not able to prove this at present, but in the more restricted context of so-called regular sets (40) is shown in [227]. Compare also [230] for a simple characterization of regular sets in case of space dimensions 2 and 3, and see also [241].

We carry over results which are known for divergence operators, when acting on L^p spaces, to the spaces from the scale $W_D^{-1,q}$, $q \in [2,\infty[$, compare also [90], [224], [59],[232]. In particular, this affects maximal parabolic regularity, which is an extremely powerful tool for the treatment of linear and nonlinear parabolic equations with nonsmooth data, see, e.g. [244] or [59]. The crucial point is that this allows to treat a discontinuous time-dependence of the right-hand side, which is relevant for applications. Moreover, the spaces $W_D^{-1,q}$ allow to include distributional right-hand sides; one may think, e.g. of electric surface densities, concentrated on interfaces between different materials—even when these interfaces move in time.

Definition (5.2.29)[206]: Following [84], we call a densely defined operator B on a Banach space X positive, if it satisfies the resolvent estimate

$$\|(B + \lambda)^{-1}\|_{\mathcal{L}(X)} \le \frac{c}{1 + \lambda}$$

for a constant c and all $\lambda \in [0,\infty[$. (Note that a positive operator is sectorial in the sense of [19, Ch. 1.1]).

We recall the notion of maximal parabolic regularity.

Definition (5.2.30)[206]: Let $1 < s < \infty$, let X be a Banach space and let $J :=]T_0, T[\subseteq \mathbb{R}$ be a bounded interval. Assume that B is a closed operator in X with dense domain $\mathcal{D}$ (in the sequel always equipped with the graph norm). We say that B satisfies maximal parabolic $L^s(J;X)$ regularity, if for any $f \in L^s(J;X)$ there exists a unique function $u \in W^{1,s}(J;X) \cap L^s(J;D)$ satisfying

$$u' + Bu = f, \qquad u(T_0) = 0,$$

where the time derivative is taken in the sense of X-valued distributions on J (see [207]).

Lemma (5.2.31)[206]: Let X, Y be two Banach spaces, where X continuously and densely injects into Y. Assume that B is a positive operator on X, such that $B^{\beta} : X \to Y$ is a topological isomorphism for some $\beta \in]0,1]$. Then, the following holds true.

(i) B admits an extension $\tilde{B}$ on Y, which also is a positive operator there.

(ii) If B admits an H^{∞}-calculus, then $\tilde{B}$ admits an H^{∞}-calculus with the same H^{∞}-angle.

(iii) If B satisfies maximal parabolic regularity on X, then $\tilde{B}$ satisfies maximal parabolic regularity on Y .

Proof. The well-known Balakrishnan formula $B^{-\beta} = \frac{\sin \pi\beta}{\pi} \int_0^\infty t^{-\beta}(B + t)^{-1}\, dt$ (see [243]) shows that the resolvent commutes with the fractional power $B^{-\beta}$. Hence, for $\psi \in X$ and $\lambda \geq 0$, one can estimate

$$\begin{aligned} \|(B + \lambda)^{-1}\psi\|_Y &= \left\|B^{\beta}(B + \lambda)^{-1}B^{-\beta}\psi\right\|_Y \\ &\leq \left\|B^{\beta}\right\|_{\mathcal{L}(X;Y)}\|(B + \lambda)^{-1}\|_{\mathcal{L}(X)}\left\|B^{-\beta}\right\|_{\mathcal{L}(Y;X)}\|\psi\|_Y \\ &\leq \left\|B^{\beta}\right\|_{\mathcal{L}(X;Y)}\left\|B^{-\beta}\right\|_{\mathcal{L}(Y;X)} \frac{c}{1 + \lambda} \|\psi\|_Y. \end{aligned}$$

This shows that the resolvent of B may be continuously extended to Y and that this extension admits the estimate $\left\|\widetilde{(B + \lambda)^{-1}}\right\|_{\mathcal{L}(Y)} \leq \frac{\tilde{c}}{1+\lambda}$. Thus, one defines the extension $\tilde{B}$ of B to Y as the inverse of $\widetilde{B^{-1}}$. Since $X \hookrightarrow Y$, $\mathrm{dom}_X(B) \hookrightarrow \mathrm{dom}_Y(\tilde{B})$. But $\mathrm{dom}_X(B)$ is dense in X by the definition of a positive operator and X was dense in Y by our assumption. Thus, $\mathrm{dom}_Y(\tilde{B}) \supset \mathrm{dom}_X(B)$ is also dense in . For (ii), see [219]. Finally, assertion (iii) is proved in [232]. The main idea is again that the parabolic solution operator on $L^r(J;X)$ commutes with the fractional power $B^{-\beta}$.

Theorem (5.2.32)[206]: Let Ω and D satisfy the Assumption (5.2.1) and let μ satisfy Assumptions (5.2.4) and (5.2.5) and assume $q \in [2,\infty[$. Then, the extension of $-\nabla\cdot\mu\nabla + 1$ from L^q to $W_D^{-1,q}$ (being identical with the restriction from $W_D^{-1,2}$) has the following properties:

(i) It induces a positive operator.

(ii) It admits a bounded H^∞-calculus with H^∞-angle arctan $\frac{\|\mu\|_{L^\infty}}{\mu_\bullet}$; in particular, it admits bounded imaginary powers.

(iii) It satisfies maximal parabolic regularity; in particular, its negative generates an analytic semigroup.

Proof. The transposed coefficient function μ^T also satisfies Assumption (5.2.5). Hence, the operator

$$(-\nabla \cdot \mu^T \nabla + 1)^{\frac{1}{2}} : W_D^{1,p} \to L^p \tag{41}$$

provides a topological isomorphism for all $p \in]1,2]$, according to Theorem (5.2.9). Clearly, the adjoint operator of (41), being identical with the operator $(-\nabla \cdot \mu\nabla + 1)^{\frac{1}{2}}: L^q \to W_D^{-1,q}$, with $q = \frac{p}{p-1} \in [2,\infty[$, is also a topological isomorphism. Consequently, we need to know the asserted properties only on the spaces L^q due to Lemma (5.2.31).

In order to see this for (i), it suffices to note that on every space L^q , $1 < q < \infty$, the operator $-\nabla\cdot\mu\nabla$ generates a strongly continuous semigroup of contractions (see Proposition (5.2.8)), and hence, the operator admits the required resolvent estimate by the Hille–Yosida theorem.

Assertion (ii) is discussed in Proposition (5.2.8), and concerning (iii), the contraction property of the semigroup on all L^q spaces provides maximal parabolic regularity on these spaces due to a deep result of Lamberton (see [239]).

Section (5.3): L^p-Estimates for the Square Root of Elliptic Systems

Elliptic divergence form operators are amongst the most carefully studied differential operators with variable coefficients. We contribute to the functional calculus of such operators with complex, bounded and measurable coefficients, formally given by

$$Lu = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) - \sum_{i=1}^{d}\frac{\partial}{\partial x_i}(a_{i0}u) + \sum_{j=1}^{d} a_{0j}\frac{\partial u}{\partial x_j} + a_{00}u, \quad (41)$$

on a bounded domain $\Omega \subseteq \mathbb{R}^d, d \geq 2$. We allow for mixed boundary conditions. Namely, u satisfies homogeneous Dirichlet boundary conditions on a closed part D of the boundary and natural boundary conditions on the complementary part $N = \partial\Omega \backslash D$. The geometric constellation can be 'rough' in that we require Lipschitz coordinate charts for $\partial\Omega$ only around the closure of N, whereas around D the domain can Ω merely be d-Ahlfors regular, and D itself has to be $(d-1)$-Ahlfors regular. These notions, henceforth called assumptions N, Ω, and D. We include $(m \times m)$-systems in our considerations, that is to say, u takes its values in $\mathbb{C}^m$ and each a_{ij} is valued in the space of matrices $\mathcal{L}(\mathbb{C}^m)$. As in [268], we may have different Dirichlet boundary parts for each coordinate of u. These assumptions are amongst the most general ones that allow for a proper functional analytic framework for L [206], [272],[268],[58].

As usual, we interpret L in the weak sense via the sesquilinear form

$$a(u,v) = \int_\Omega \sum_{i,j=1}^{d}\left(a_{ij}\frac{\partial u}{\partial x_j}\cdot\frac{\overline{\partial v}}{\partial x_i} + a_{i0}u\cdot\frac{\overline{\partial v}}{\partial x_i} + a_{0j}\frac{\partial u}{\partial x_j}\cdot v + a_{00}u\cdot\bar{v}\right)dx \quad (42)$$

defined on $\mathcal{D}(a) = \mathbb{W}_D^{1,2}(\Omega)$, where the subscripted D is reminiscent of the boundary conditions. Ellipticity is in the sense of a Gårding inequality, turning L into a maximal accretive operator on $L^2(\Omega)^m$. This way of understanding L is called 'Kato's form method'. Definitions are provided and see [271],[240]. Let us stress that our setup incorporates, for example, the Lamé system. We shall come back to that.

The focus in lies on establishing L^p-estimates for the (unique) maximal accretive square root $L^{\frac{1}{2}}$ of L. We study for which $p \in (1,\infty)$ it extends or restricts to a topological isomorphism

$$L^{\frac{1}{2}}: \mathbb{W}_D^{1,p}(\Omega) \xrightarrow{\cong} L^p(\Omega)^m. \quad (43)$$

Our results are the first of this kind for 'rough' divergence form systems on domains and provide optimal ranges of exponents.

Recent years have witnessed a vast number of applications of property (43). It is key to the approach of Rehberg and collaborators to quasilinear parabolic equations on distribution spaces via maximal regularity techniques originating from [58] and its extensions for example to optimal control problems [253] and quasilinear stochastic evolution equations [270], as well as recent progress on maximal regularity for the non-autonomous Cauchy problem on Lebesgue spaces [265],[266] and distribution spaces [257], see also [247],[249],[256] for the case $p = 2$. Aiming in a slightly different direction, [258] uses property (43) to prove Hölder continuity of solutions to quasilinear parabolic equations in rough domains.

The common idea in all of these applications is that (43) allows to switch between Lebesgue spaces and Sobolev spaces as well as their duals by means of an isomorphism that is build from L itself and hence commutes with the latter. This allows one to transfer knowledge between any two of these spaces. Let us give two illustrating examples. If L corresponds to an equation $(m = 1)$ with real coefficients, then L has a bounded H^∞-calculus and hence maximal regularity on L^p for any $p \in (1, \infty)$ due to Gaussian estimates [109],[221], and one asks for the same on $W^{-1,p'}$ to treat distributional right-hand sides in quasilinear equations. In the realm of non-autonomous Cauchy problems, an old result of Lions guarantees non-autonomous maximal regularity on $W^{-1,2}$ but one wants to transfer this knowledge to L^2 for example to make sense of boundary conditions [249],[256]. We stress that L itself cannot play the role of this transference operator because in general $\mathcal{D}(L)$ is not a Sobolev space of order two [275].

In the Hilbert space case $p = 2$, having (43) means having $\mathcal{D}\left(L^{\frac{1}{2}}\right) = \mathcal{D}(a)$ with equivalent norms. If a is symmetric – which here amounts to $a_{ij} = a_{ji}^*$ for all i, j – this is a property of closed densely defined sectorial forms that has nothing to do with differential operators [271]. The case of non-symmetric forms has a long history and became known as Kato square root problem, see for example [199],[94],[197]. Within our setup it has been settled in [262],[278] by a non-trivial refinement of the first-order method of Axelsson–Keith–McIntosh proposed in [250] and their pioneering application to mixed boundary value problems in [210]. It is somewhat unfortunate that [278] and [262] treat systems with lower-order terms only implicitly. He sees as the right moment to close this gap and shortly review the underlying methods to prove the following

Another possibility would have been to adapt the perturbation argument of [94]. Here, d and the number of equations m in (41) are referred to as dimensions. The constants λ and Λ are the lower and upper bounds for the sesquilinear form in (42) and will be defined. They are referred to as ellipticity. Geometry refers to all constants implicit in those assumptions amongst D, N, and Ω that are used in the particular situation.

The literature with regard to $p \neq 2$ is much sparser. Pure Dirichlet and pure Neumann boundary conditions on a Lipschitz domain were first treated in [90] under size and regularity assumptions on the kernel of the semigroup generated by −L. This applies in particular to equations with real coefficients. Auscher–Badr–Haller-Dintelmann–Rehberg [206] have more recently established (43) in the range $p \in (1, 2)$ also for mixed boundary conditions on making the same geometric Assumptions D, N, and Ω. There are, however, no competing results available – even on more regular domains – if one is to consider mixed boundary conditions for operators with complex coefficients, let alone systems. The only exception is the case $d = 1 = m$, where (43) is known to hold for all $p \in (1, \infty)$ and all common two-point boundary conditions on an open interval [92].

A coherent treatment of L^p-estimates for the square root of elliptic systems with complex coefficients on $\mathbb{R}^d$ when $d \geq 2$ is found in [210]. On a large scale our approach is to incorporate the machinery of off-diagonal estimate from [210] into the fine geometric setup of [206] to compensate for the lack of Gaussian upper bounds for the kernel of the semigroup. (In fact, there might not be a kernel in any but the distributional sense.) This was left as an open problem on p. 66 in [210].

We shall work in an L-adapted range of exponents

$$\mathcal{J}(L) := \{p \in (1, \infty) : \sup_{t>0} \|e^{-tL}\|_{L^p \to L^p} < \infty, \tag{44}$$

that is to say, those $p \in (1, \infty)$ for which there is a bounded (strongly continuous) semigroup associated with −L on $L^p(\Omega)^m$. We postpone a detailed discussion of the size of $\mathcal{J}(L)$ and mention for the moment only that $\mathcal{J}(L)$ is an interval that contains at least

$2_*:=2d/(d+2)$ and $2^*:=2d/(d-2)$. An obvious advantage of working with $\mathcal{J}(L)$ is that any improvement on L^p boundedness of the semigroup entails an improvement in our results for free.
As our main result we obtain (43) in the best possible range below 2 (except for maybe the endpoints). For the sake of clarity let us introduce the array

$$[\![p]\!]:=\left[d,m,\lambda,,\sup_{t>0}\|e^{-tL}\|_{L^p\to L^p}\right] \tag{45}$$

to store all the constants that usually show up in our estimates.
We note that Theorem (5.3.25).(i) in principle consists of two estimates. One is the L^p boundedness of the Riesz transform $\nabla L^{-\frac{1}{2}}$ expressed in the domination

$$\|\nabla_u\|_p \lesssim \left\|L^{\frac{1}{2}}u\right\|_p \qquad \left(u\in\mathcal{D}\left(L^{\frac{1}{2}}\right)\right).$$

We present details of the (short) proof relying on Blunck–Kunstmann's weak type criterion from [251],[252], see also [210], and $L^p\to L^2$ off-diagonal estimates for the gradient of the semigroup, to be established. To avoid overloading with indications on how to modify and extract additional information from [210],[252]. Indeed, these references both treat the case $\Omega=\mathbb{R}^d$ only and do not state an explicit dependence of implicit constants. But we do not claim much originality here. An interesting observation to keep in mind, however, is that for this part we shall not require assumption, that is, with the restricted Lebesgue measure need not be a space of homogeneous type.
The other ingredient is the a priori inequality

$$\left\|L^{\frac{1}{2}}u\right\|_p \lesssim \|u\|_p+\|\nabla u\|_p \qquad \left(u\in\mathbb{W}_D^{1,2}(\Omega)\right).$$

This will be handled by a careful modification of the main argument in [206]. It goes by a weak-type criterion and requires a new Calderón–Zygmund decomposition that we shall establish beforehand.

We remind that most aforementioned applications of (43) would invest this isomorphism along with maximal regularity on L^p. The latter is not known a priori but there is a way out: By Dore–Venni's theorem [259] maximal regularity on L^p follows from the bounded H^∞-calculus on L^p and this in turn is an easy consequence of methods used to get a grip on the Riesz transform, namely $L^p\to L^2$ off-diagonal estimates. Again, this connection is not new in general but has not been exploited. It goes back to the seminal contribution [251] of Blunck and Kunstmann.

Above, $\omega\in[0,\pi/2)$ is the angle of accretivity for L and S_ψ^+ is the open sector in the complex plane of opening angle 2ψ symmetric around the positive real axis.

As far as $L^{\frac{1}{2}}$ is concerned, we have only dealt with exponents $p<2$ but a duality argument allows us to extrapolate (43) to some exponents above 2. We present a proof of

We did not try to find a characterization of the admissible exponents $p>2$ in (43) in terms of the semigroup (or rather its gradient $\sqrt{t}\nabla e^{-tL}$) as in [210]. This is left as an independent open problem. However, already for the Riesz transform no exponent $p>2$ works simultaneously for all real symmetric L subject to Dirichlet boundary conditions on a Lipschitz domain. This follows from combining the example on p. 120 in [94] with [277].

We have already noticed that we take special care of implicit constants. This is not because we are trying to be particularly pedantic or even annoying. Rather than that, we need to prepare our results for the aforementioned applications to non-autonomous parabolic problems, where a family of operators L_t with uniform ellipticity parameters in t acts in the spatial variables, and one needs the same uniformity on estimates for $L_t^{\frac{1}{2}}$. This was asked for in [253],[265],[270]. It is only implicit in [206] and sometimes all but impossible to track. We shall comment on that issue. There is also an 'inner' application for having uniform constants with respect to ellipticity, as it allows us to obtain holomorphic dependence on the coefficients in all of our estimates by a simple application of Vitali's theorem from complex analysis. To this end, let us denote by $\mathcal{A}(\Omega)$ the set of coefficient functions

$$A = \left(a_{i,j}\right)_{0\le i,j\le d} : \Omega \to \ L(\mathbb{C}^m)^{(d+1)\times(d+1)}$$

that satisfy our ellipticity assumptions, which can canonically be identified with an open subset of $L^\infty(\Omega)^{m^2(d+1)^2}$. We shall say that a function h from some open subset of $\mathcal{A}(\Omega)$ into a complex Banach space X is holomorphic if for all holomorphic functions φ from an open subset of C into $\mathcal{A}(\Omega)$ the composition $h \circ \varphi$ is holomorphic. Let us also write L_A instead of L and so on to stress the dependence on A.

Theorem (5.3.1)[246]: Suppose $\Omega \subseteq \mathbb{R}^d$ is a bounded domain and let $O \subseteq \mathcal{A}(\Omega)$ be open.

(i) If $\sup_{A\in O} \sup_{t>0} \|e^{-tL_A}\|_{L^{p_0}\to L^{p_0}} < \infty$ and $\psi > \sup_{A\in O} \omega(L_A)$, then in the setting and with the notation of Theorem (5.3.20) for every $f \in H^\infty\left(S_\psi^+\right)$ the function

$$O \to \mathcal{L}(L^p(\Omega)^m), \qquad A \longmapsto f(L_A)$$

is holomorphic on O.

(ii) If $\sup_{A\in O} \sup_{t>0} \|e^{-tL_A}\|_{L^{p_0}\to L^{p_0}} < \infty$, then in the setting and with the notation of Theorem (5.3.25) the function

$$O \to \mathcal{L}\left(\mathbb{W}_D^{1,p}(\Omega), L^p(\Omega)^m\right), \qquad A \longmapsto L_A^{\frac{1}{2}}$$

is holomorphic on O. The same holds for $p \in \bigcap_{A\in O} \ \left[2, 2+\varepsilon(L_A)\right)$ in the setting of Theorem (5.3.29).

For divergence form operators on rough bounded domains this seems to be a new result. Let us remark that such kind of perturbation result usually necessitates the treatment of complex coefficients even if one aims at applying it in the realm of real equations only.

As for applications, all aforementioned results become more powerful the more is known about the set $\mathcal{J}(L)$. Riesz–Thorin interpolation reveals that $\mathcal{J}(L)$ is an interval and by maximal accretivity of L on $L^2(\Omega)^m$ it contains 2. By means of Nash-type inequalities we shall prove the following

We remind that $2_* := 2d/(d+2)$ and $2^* := 2d/(d-2)$ are the lower and upper Sobolev conjugates of 2, respectively. The above result for $d \ge 3$ is sharp, even for $m = 1$, in the sense that no interval larger than $[2_*, 2^*]$ is contained in $\mathcal{J}(L)$ for every elliptic equation L, even with pure Dirichlet boundary conditions on a smooth and bounded domain. This follows from [269]: Indeed, for p in the interior of $\mathcal{J}(L)$ the semigroup generated by −L decays exponentially in L^p-norm due to Lemma (5.3.3) below and interpolation, hence L^{-1} is bounded in L^p-topology, but [269] constructs for every $p > 2^*$ an example such L^{-1} does not even map $C_0^\infty(\Omega)$ into $L^p(\Omega)$. By duality we can argue likewise for $p < 2_*$.

Let us also mention that many applications to physically motivated models require the adjoint isomorphism to (43), that is

$$(L^*)^{\frac{1}{2}}: L^{p'}(\Omega)^m \overset{\cong}{\to} \mathbb{W}_D^{-1,p'}(\Omega),$$
for some p with Hölder conjugate $p' > d$. This is granted by Theorems (5.3.25) and (5.3.16) in dimensions $d = 2, 3, 4$ corresponding to $(2_*)' = 2^* = \infty, 6, 4$. If $m = 1$ and L has real coefficients, then $\mathcal{J}(L) = (1, \infty)$ by Gaussian estimates [109] and we recover the result in [206]. Improvements on $\mathcal{J}(L)$ for certain systems with real coefficients were obtained in [276] and [281]. The Lamé system
$$L_{D,0}u = -\mu\Delta u - \mu' \nabla div\, u$$
fits into our framework provided $\mu > 0$ and $\mu + \mu' > 0$, see [273]. In M. Mitrea and Monniaux consider $L_{D,0}$ with pure Dirichlet boundary conditions on a bounded domain satisfying an interior ball conditions and obtain maximal regularity on $L^q(\Omega)$ in the range $q \in (2_*, 2^*)$. We remind that Assumption N is void if one considers pure Dirichlet conditions. Hence, by putting together Theorems (5.3.20) and (5.3.16) we are able to drop this geometric assumption and obtain even a bounded H^∞-calculus for $L_{D,0}$ on any bounded domain, which in turn implies maximal regularity [259]. Some boundary regularity in the sense of Assumption D also allows us to increase the range for q. Let us stress, however, that the maximal regularity part has previously been obtained in a broader context by Tolksdorf [279], who uses a technique different to ours that does not pass through the bounded H^∞-calculus and yields the range $(2_* - \varepsilon, 2^* + \varepsilon)$ without further geometric assumptions.

We provide precise definitions of all assumptions and notation that have been dealt with rather intuitively up to now. The remaining are devoted to the proofs of our main results, Theorems (5.3.6)–(5.3.16). The order of proofs will slightly differ from the presentation above.

Any Banach space X under consideration is over the complex numbers and X^* is the (anti)-dual space of conjugate linear bounded functionals $X \to \mathbb{C}$. We write $\langle \cdot \mid \cdot \rangle$ for duality pairings and $(\cdot \mid \cdot)$ for inner products on Hilbert spaces.

If B is an open ball of radius r, then we denoted by αB the concentric ball of radius αr and let $C_1(B) := 4B$ as well as $C_J(B) := 2^{j+1}B \backslash 2^j B$ for $j = 2, 3, \ldots$. We use similar notation for cubes. We denote the semi-distance of sets $E, F \subseteq \mathbb{R}^d$ induced by the Euclidean distance on $\mathbb{R}^d$ by $d(E, F)$ and abbreviate $d_F(x) := d(\{x\}, F)$. Lebesgue measure on $\mathbb{R}^d$ is denoted $|\cdot|$.

Henceforth we assume that $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, is a bounded, open, and connected set, that is to say, a bounded domain. We remind that the elliptic operator L corresponds to a system of m equations and hence acts on functions $u: \Omega \to \mathbb{C}^m$. We can allow for different Dirichlet parts for each coordinate function $u^{(k)}$, which we denote D_k, and which are assumed to be closed subsets of $\partial\Omega$. Hence, L is subject to mixed boundary conditions
$$u^{(k)} = 0 \quad on\, D_k \quad for\; k = 1, \ldots, m, \tag{46}$$
where the form method forces natural boundary conditions on the complementary parts $N_k := \partial \backslash D_k$ through a formal integration by parts. We put $D := \bigcap_{k=1}^m D_k$ and $N := \partial\Omega \backslash D = \bigcup_{k=1}^m N_k$.

Our results hold under the following three geometric assumptions. Sometimes not all of them shall be required.

Assumption D. For every $k = 1, \ldots, m$ the set D_k is closed and either empty or such that $\mathcal{H}_{d-1}(B \cap D) \simeq r^{d-1}$ holds for all open balls B of radius $r < 1$ centered in D. Here, $\mathcal{H}_{d-1}$ is the $(d-1)$-dimensional Hausdorff measure in $\mathbb{R}^d$.

Assumption N. Around every $x \in \overline{N}$ there is an open neighborhood U_x and a bi-Lipschitz mapping $\Phi x: U_x \to (-1,1)^d$ such that

$$\Phi_x(U_x \cap \Omega) = (-1,0) \times (-1,1)^{d-1}, \qquad \Phi_x(U_x \cap \partial) = \{0\} \times (-1,1)^{d-1}.$$

Assumption Ω. Comparability $|B \cap \Omega| \simeq r^d$ holds for all open balls B of radius $r < 1$ centered in Ω.

Assumption D means that the D_k are either empty or $(d-1)$-Ahlfors regular. Likewise, Assumption means that Ω is d-Ahlfors regular. Assumption is sometimes called weak Lipschitz condition. It is strictly weaker than requiring that Ω has a Lipschitz boundary around $\overline{N}$, see [58] for a relevant example.

We introduce Sobolev spaces on a domain $\Xi \subseteq \mathbb{R}^d$ with a vanishing trace condition on some closed subset $F \subseteq \bar{\Xi}$. This is understood in a very weak approximate sense but can be rephrased in a proper pointwise sense under minimal regularity assumptions [272],[263]. We shall not need such precision. Namely, for $1 < q < \infty$ we let $W0_F^{1,q}(\Xi)$ be the closure of

$$C_F^\infty(\Xi) := \{\varphi|_{\Xi} : \varphi \in C_0^\infty(\mathbb{R}^d), supp(\varphi) \cap F = \emptyset\}$$

for the norm $\varphi \longmapsto \left(\int_\Xi |\varphi|^q + |\nabla\varphi|0^q dx\right)^{\frac{1}{q}}$. The endpoint space $W_F^{1,\infty}(\Xi)$ consists of all bounded Lipschitz continuous functions $u: \bar{\Xi} \to \mathbb{C}$ that vanish everywhere on F. It carries the norm $u \longmapsto \|u\|_\infty + Lip(u)$, where $Lip(u)$ is the smallest Lipschitz constant for u on $\bar{\Xi}$.

Usually we encounter the spaces $W_{D_k}^{1,p}(\Omega)$. Under Assumption N there are bounded linear Sobolev extension operators E_k that extend $W_{D_k}^{1,p}(\Omega) \to W_{D_k}^{1,p}(\mathbb{R}^d)$ and $L^p(\Omega) \to L^p(\mathbb{R}^d)$ for every $p \in (1,\infty)$, see [58]. In particular, the usual embeddings of type $W_{D_k}^{1,p}(\Omega) \subseteq L^q(\Omega)$ hold. Usually, $q = p^*$ is the upper Sobolev conjugate of defined by $1/p^* = 1/p - 1/d$. We also define a lower conjugate by $1/p_* = 1/p + 1/d$. In the particular situation they will be contained in $(1,\infty)$.

Sobolev spaces adapted to the boundary conditions (46) are $\mathbb{W}_D^{1,p}(\Omega) := \prod_{k=1}^m W_{D_k}^{1,p}(\Omega)$. For $p \in (1,\infty)$ we define corresponding spaces of negative order $\mathbb{W}_D^{-1,p}(\Omega) := \left(\mathbb{W}_D^{1,q}(\Omega)\right)^*$, where $1/p + 1/q = 1$.

General background and proofs of all relevant statements on the holomorphic functional calculus for sectorial operators can be found in [226]. Bisectorial operators can be treated almost identically but details have also been written down in [261]. Throughout we shall assume that X is a Hilbert space.

A linear operator T in X is sectorial of angle $\phi \in [0,\pi)$ if its spectrum $\sigma(T)$ is contained in the closure of the sector $S_\phi^+ := \{z \in \mathbb{C} : |\arg z| < \phi\}$ and if

$$\mathbb{C} \backslash \overline{S_\psi^+} \to \mathcal{L}(X), \qquad z \mapsto z(z-T)^{-1}$$

is uniformly bounded for every $\psi \in (\phi,\pi)$. We agree on $S_0^+ := (0,\infty)$.

For $\psi \in (\phi,\pi)$ let $H^\infty(S_\psi^+)$ be the algebra of bounded holomorphic functions on S_ψ^+ and let $H_0^\infty(S_\psi^+)$ be the sub-algebra of functions g satisfying $|g(z)| \le C \min\{|z|^s, |z|^{-s}\}$ for some $C, s > 0$ and all $z \in S_\psi^+$. If $f(z) = a + b(1+z)^{-1} + g(z)$ for some $a, b \in \mathbb{C}$ and $g \in H_0^\infty(S_\psi^+)$, then $f(T)$ is defined as a bounded operator on X via

$$f(T) = a + b(1+T)^{-1} + \frac{1}{2\pi i}\int_{\partial S_\nu^+} g(z)(z-T)^{-1}dz, \quad (47)$$

where $\nu \in (\phi, \psi)$, the choice of which does not matter in view of Cauchy's theorem, and ∂S_ν^+ is oriented such that it surrounds $\sigma(T)$ counterclockwise in the extended complex plane.

The definition of $f(T)$ is extended to larger classes of holomorphic functions by regularization: One defines the closed operator $f(T) := e(T)^{-1}(ef)(T)$, if $e(T)$ and $(ef)(T)$ are already defined by the procedure above and $e(T)$ is one-to-one. This definition does not depend on the choice of e. The expected relations $f(T) + g(T) \subseteq (f+g)(T)$ and $f(T)g(T) \subseteq (fg)(T)$ hold and there is equality if $f(T)$ is bounded. An example are fractional powers $T^\alpha, \alpha > 0$, which are defined on using $e(z) = (1+z)^{-\alpha-1}$. If T is one-to-one, then $e(z) = z(1+z)^{-2}$ regularizes any $f \in H^\infty(S_\psi^+)$. Its $H^\infty(S_\psi^+)$-calculus is called bounded if for some constant $C_\psi > 0$ it holds

$$\|f(T)\|_{X\to X} \le C_\psi \|f\|_{L^\infty(S_\psi^+)} \qquad \left(f \in H^\infty(S_\psi^+)\right).$$

It suffices to check this bound on $H_0^\infty(S_\psi^+)$. Indeed, for general $f \in H^\infty(S_\psi^+)$ the convergence lemma states that $f_n = e^{\frac{1}{n}} f \in H_0^\infty(S_\psi^+)$ satisfy $f_n \to f$ pointwise, $\|f_n\|_\infty \to \|f\|_\infty$, and $f_n(T) \to f(T)$ strongly on X.

We will frequently use that if T has a bounded H^∞-calculus of angle ψ, then so has the adjoint T^*. This is a consequence of the identity $f(T)^* = f^*(T^*)$ for every $f \in H^\infty(S_\psi^+)$, where $f^*(z) := \overline{f(\bar{z})}$.

Bisectorial operators are defined similarly upon replacing sectors by double sectors $S_\phi = S_\phi^+ \cup -S_\phi^+$, where $\phi \in [0, \pi/2)$. In their calculus $(i+T)^{-1}$ replaces $(1+T)^{-1}$.

2.4. The divergence form operator

We turn to the precise definition of the divergence form operator formally given by (41). The coefficients $a_{ij}: \Omega \to \mathcal{L}(\mathbb{C}^m)$ are measurable and essentially bounded and we put

$$\Lambda := \sup_{0\le i,j\le d} essup_{x\in\Omega} \|a_{ij}(x)\|_{\mathbb{C}^m\to\mathbb{C}^m}.$$

We remind the reader of the sesquilinear form

$$a(u,v) = \int_\Omega \sum_{i,j=1}^{d} \left(a_{ij}\frac{\partial u}{\partial x_j}\cdot\overline{\frac{\partial v}{\partial x_i}} + a_{i0}u\cdot\overline{\frac{\partial v}{\partial x_i}} + a_{0j}\frac{\partial u}{\partial x_j}\cdot\bar{v} + a_{00}u\cdot\bar{v}\right)dx$$

acting on $\mathbb{C}^m$-valued functions. For all $u \in \mathbb{W}_D^{1,2}(\Omega)$ we have $|a(u,u)| \le \Lambda(d+1)(\|u\|_2^2 + \|\nabla u\|_2^2)$, where $\nabla u := \left(\frac{\partial u}{\partial x_i}\right)_i$ is considered as a vector in $(\mathbb{C}^m)^d \cong \mathbb{C}^{dm}$. Our ellipticity assumption is the following lower bound.

Assumption L. There exists $\lambda > 0$ such that $Re\ a(u,u) \ge \lambda(\|u\|_2^2 + \|\nabla u\|_2^2)$ holds for all $u \in \mathbb{W}_D^{1,2}(\Omega)$.

This implies that the numerical range $\{a(u,u): u \in \mathcal{D}(a), \|u\|_2 = 1\}$ is contained in the closed sector $\overline{S_\phi^+}$ of opening angle $\phi = \arctan((d+1)\Lambda/\lambda)$. We define $\omega \in [0, \pi/2)$ to be the smallest such angle.

The Lax–Milgram lemma associates with a the bounded and invertible operator

$$\mathcal{L}: \mathbb{W}_D^{1,2}(\Omega) \to \mathbb{W}_D^{1,2}(\Omega)^*, \qquad \langle \mathcal{L}u|v\rangle = a(u,v).$$

We define L to be the maximal restriction of $\mathcal{L}$ to the Hilbert space $L^2(\Omega)^m$. Our assumptions entail that a is a closed densely defined sectorial form of angle ω in $L^2(\Omega)^m$ and hence L is maximal ω-accretive, see [271]. This is a stronger notion than sectoriality of angle ω. It is known that such operators admit a bounded H^∞-calculus on any sector containing $\overline{S_\omega^+}$. This is due to Crouzeix–Delyon [254], see also [226].

Proposition (5.3.2)[246]: Let $\psi \in (\omega, \pi)$ and $f \in H^\infty(S_\psi^+)$. Then $\|f(L)\|_{L^2\to L^2} \le 4\|f\|_\infty$.

Since L is maximal accretive on $L^2(\Omega)^m$, the semigroup operators $e^{-zL}, z \in S^+_{\frac{\pi}{2}-\omega}$, are contractions on $L^2(\Omega)^m$, see [271]. It will be useful to have the following exponential stability which follows simply because $L - \lambda/2$ is still maximal accretive.

Lemma (5.3.3)[246]: For every $t > 0$ the bound $\|e^{-tL}\|_{L^2\to L^2} \le e^{-\frac{\lambda t}{2}}$. In particular, L is invertible.

Being maximal accretive, L has a unique maximal accretive square root denoted $L^{\frac{1}{2}}$ and $\mathcal{D}(L)$ is a core for $\mathcal{D}\left(L^{\frac{1}{2}}\right)$, see [271]. This is the same operator as given by the functional calculus [226] and since L is invertible, so is $L^{\frac{1}{2}}$ with inverse $L^{-\frac{1}{2}}$, see [226]. We also have the formula $\left(L^{\frac{1}{2}}\right)^* = (L^*)^{\frac{1}{2}}$ for the adjoints [226].

Since L has a bounded H^∞-calculus of some angle $\omega \in \left[0, \frac{\pi}{2}\right)$, we have a resolution of the identity in the sense of an improper Riemann integral

$$u = \frac{2}{\sqrt{\pi}} \int_0^\infty L^{\frac{1}{2}} e^{-t^2 L} u \, dt \qquad (u \in L^2(\Omega)^m), \tag{48}$$

see [226]. We can apply $L^{\frac{1}{2}}$ or $L^{-\frac{1}{2}}$ on both sides of (48) to obtain well-known integral formulæ for either of them.

We survey the first-order formalism and its practicability to the operator L under our geometric assumptions developed in [262],[278]. This leads to Theorem (5.3.6) and the statements of Theorem (5.3.1) when $p = 2$. Throughout we assume D, N, and Ω.

We write the coefficients of L in matrix form

$$\begin{bmatrix} a_{00} & [a_{10} \ \cdots \ a_{d0}] \\ \begin{bmatrix} a_{10} \\ \vdots \\ a_{d0} \end{bmatrix} & \begin{bmatrix} a_{11} & \cdots & a_{1d} \\ \vdots & & \vdots \\ a_{d1} & \cdots & a_{dd} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} A_{\perp\perp} & A_{\perp\parallel} \\ A_{\parallel\perp} & A_{\parallel\parallel} \end{bmatrix} = A$$

and define a closed operator $\nabla_D : \mathbb{W}_D^{1,2}(\Omega) \subseteq L^2(\Omega)^m \to L^2(\Omega)^{dm}$ through $\nabla_D u = \nabla u$. For the gradient of $\mathbb{C}^m$-valued functions. An equivalent way of putting the definition of L through the form method is $L = [1 \quad \nabla_D^*] A [1 \quad \nabla_D]$.

On $\mathcal{H} = L^2(\Omega)^m \times L^2(\Omega)^m \times L^2(\Omega)^{dm}$ we define operator matrices on their natural domains,

$$\Gamma := \begin{bmatrix} 0 & & \\ 1 & 0 & \\ \nabla_D & & 0 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & & \\ & A_{\perp\perp} & A_{\perp\parallel} \\ & A_{\parallel\perp} & A_{\parallel\parallel} \end{bmatrix},$$

and consider the perturbed Dirac operator $\prod_B := \Gamma + B_1 \Gamma^* B_2$. Indeed $\prod_B$ is a Dirac operator in that its square contains L, namely

$$\prod_B = \begin{bmatrix} & \begin{bmatrix} 1 & (\nabla_D)^* A \end{bmatrix} \\ \begin{bmatrix} 1 \\ \nabla_D \end{bmatrix} & \end{bmatrix}, \qquad \prod_B^2 = \begin{bmatrix} L & \\ & \begin{bmatrix} 1 & \nabla_D^* \\ \nabla_D & \nabla_D \nabla_D^* \end{bmatrix} A \end{bmatrix}.$$

Within this framework our ellipticity Assumption L can be rephrased as

$$Re(B_2 u | u) \geq \lambda \|u\|_{\mathcal{H}}^2 \qquad \big(u \in \mathcal{R}(\Gamma)\big), \tag{49}$$

that is to say, B_1 and B_2 are accretive perturbations of Γ^* and Γ, respectively. It follows that $\prod_B$ is bisectorial of some angle $\omega_B \in (0, \pi/2)$ with resolvent estimates depending only on λ, Λ, see [250]. The Kato square root problem has now become a question on the functional calculus for $\prod_B$ – comparing $\mathcal{D}\left(L^{\frac{1}{2}}\right)$ and $\mathbb{W}_D^{1,2}(\Omega)$ amounts to comparing (the first components of) $\mathcal{D}\left((\prod_B^2)^{\frac{1}{2}}\right)$ and $\mathcal{D}(\prod_B)$.

On the abstract level, we have the following result from [261] or [262]. Explicit constants have not been stated there but pop up in the given proofs.

Lemma (5.3.4)[246]: Let T be a bisectorial operator in a Hilbert space X. Suppose the restriction to the closure of its range $\mathcal{R}(T)$ has a bounded $H^\infty(S_\psi)$-calculus for some $\psi \in (0, \pi/2)$. Then $\mathcal{D}\left((T^2)^{\frac{1}{2}}\right) = \mathcal{D}(T)$ and

$$\frac{1}{C_\psi} \|T\, u\|_X \leq \left\|(T^2)^{\frac{1}{2}} u\right\|_X \leq C_\psi \|T\, u\|_X \qquad \big(u \in \mathcal{D}(T)\big),$$

where C_ψ is the bound for the functional calculus.

On the concrete level, the goal of [262] was to prove quadratic estimates for the particular choice of $\prod_B$ under a set of hypotheses called (H1)–(H7). We do not need to recall them here see [262]. For the operators above, they have been verified in detail in [262] with two exceptions: The accretivity assumption (H2), which is precisely (49), and the regularity assumption (H7), whose verification in [262] was subject to an additional assumption called Assumption (E) that became a true theorem only later on in [31, Thm. (5.3.12)].

This being said, [262] reads as follows.

Proposition (5.3.5)[246]: For some constant $C \in (0, \infty)$ depending on geometry, dimensions, and ellipticity, there are quadratic estimates

$$\frac{1}{C} \|u\|_2^2 \leq \int_0^\infty \|t\Pi_B (1 + t^2 \Pi_B^2)^{-1} u\|_2^2 \, \frac{dt}{t} \leq C \|u\|_2^2 \qquad \big(u \in \overline{\mathcal{R}(\Pi_B)}\big).$$

By McIntosh's theorem [107] quadratic estimates as above imply boundedness of the $H^\infty(S_\psi)$-calculus of any angle $\psi \in (\omega_B, \pi/2)$ for the restriction of Π_B to $\overline{\mathcal{R}(\Pi_B)}$, which is a one-to-one bisectorial operator. See also [261]. The bound for the functional calculus depends on the angle and the resolvent bounds for Π_B as is easily seen from the proofs in [107] or [261]. Hence, we obtain the

Theorem (5.3.6)[246]: Suppose $\Omega \subseteq \mathbb{R}^d$ is a bounded domain that satisfies Assumptions D, N, and Ω. Then $\mathcal{D}\left(L^{\frac{1}{2}}\right) = \mathcal{D}(a) = \mathbb{W}_D^{1,2}(\Omega)$ and

$$\left\|L^{\frac{1}{2}} u\right\|_2^2 \simeq a(u, u) \simeq \|u\|_2^2 + \|\nabla u\|_2^2 \quad \left(u \in \mathbb{W}_D^{1,2}(\Omega)\right)$$

with implicit constants depending on ellipticity, dimensions, and geometry.

Proof. We have just seen that Lemma (5.3.4) applies to Π_B and yields $\mathcal{D}\left((\Pi_B^2)^{\frac{1}{2}}\right) = \mathcal{D}(\Pi_B)$ with equivalent graph norms. This implies $\mathcal{D}\left(L^{\frac{1}{2}}\right) = \mathbb{W}_D^{1,2}(\Omega)$ with equivalent norms upon restricting to the first coordinate in the 3×3 operator matrices. Implied constants in this argument depend on geometry and ellipticity.

We discuss holomorphic dependence in the spirit of Theorem (5.3.1) in the case $p = 2$. We assume some familiarity with vector-valued holomorphic functions and see [248].

Henceforth, let $O \subseteq \mathbb{C}$ be an open set and A(z) be coefficient matrices as above that depend holomorphically on $z \in O$. We assume that all of them satisfy the ellipticity assumptions from the same parameters λ, Λ and we write L_z for the corresponding operators defined through the sesquilinear forms a_z. By a slight abuse of notation, ω denotes the supremum of all accretivity angles of the operators L_z so that $\sigma(L_z) \subseteq \overline{S_\omega^+}$ for every z.

For all $u, v \in \mathbb{W}_D^{1,2}(\Omega)$ also the map $z \longmapsto a_z(u, v)$ is holomorphic on O. This follows for example from Morrera's theorem after changing the order of integration. Hence, we have a holomorphic family of sectorial forms in the sense of [271]. It follows that the associated operators L_z are resolvent holomorphic, that is to say,

$$O \to \mathcal{L}(L^2(\Omega)^m), \qquad z \longmapsto (\mu - L_z)^{-1}$$

is holomorphic for every $\mu \in \mathbb{C}\backslash\overline{S_\omega^+}$. For a proof see [271] or the elegant argument presented in [280]. By superposition, this carries over to objects in the functional calculus for the operators L_z. Two important examples are as follows.

Corollary (5.3.7)[246]: In the situation above, let $f \in H^\infty(S_\phi^+)$ for some $\phi \in (\omega, \pi)$. Then the map $O \to \mathcal{L}(L^2(\Omega)^m), z \longmapsto f(L_z)$ is holomorphic.

Proof. If $f \in H_0^\infty(S_\phi^+)$, then the claim follows from Morrera's theorem after changing the order of integration in the integral representation of $f(L_z)$. In the general case we conclude by Vitali's theorem: We can take a bounded sequence $\{f_n\}_n$ in $H_0^\infty(S_\phi^+)$ that converges pointwise to f such that for every $z \in O$ we have strong convergence $f_n(L_z) \to f(L_z)$ on $L^2(\Omega)^m$. The missing hypothesis for Vitali's theorem, that is the uniform bound in n and z for the holomorphic functions $z \longmapsto f_n(L_z)$, is due to Proposition (5.3.2).

Corollary (5.3.8)[246]: In the situation above the map $O \to \mathcal{L}\left(\mathbb{W}_D^{1,2}(\Omega), L^2(\Omega)^m\right), z \longmapsto L_z^{\frac{1}{2}}$ is holomorphic.

Proof. The map under consideration is uniformly bounded on O thanks to Theorem (5.3.6). Hence, it suffices to check holomorphy of $z \longmapsto L_z^{\frac{1}{2}}u$ for every $u \in \mathbb{W}_D^{1,2}(\Omega)$, see [2, Prop. A.3] for this reduction. Applying $L_z^{\frac{1}{2}}$ on both side of (48) yields

$$L_z^{\frac{1}{2}}u = \lim_{n\to\infty} \frac{2}{\sqrt{\pi}} \int_{2^{-n}}^{2^n} L_z e^{-t^2 L_z} u \, dt =: F_n(L_z) L_z^{\frac{1}{2}}u,$$

with convergence in $L^2(\Omega)^m$. Here, $F_n(\mu) = \frac{2}{\sqrt{\pi}} \int_{2^{-n}}^{2^n} (t^2\mu)^{\frac{1}{2}} e^{-t^2\mu} \frac{dt}{t}$ are bounded holomorphic functions on any sector contained in the right complex halfplane and a substitution reveals a uniform bound in n and μ. By the first inequality above, $L_z^{\frac{1}{2}}u$ is the pointwise limit of a sequence of holomorphic functions on O. And taking into account

Proposition (5.3.2) and Theorem (5.3.6), the second one means that this sequence is uniformly bounded in n and z. As before, Vitali's theorem yields the claim.

We establish $L^p \to L^2$ off-diagonal estimates for the semigroup generated by −L and related families. They are the proper substitute for Gaussian kernel bounds in our context and play a crucial role in all subsequent. Here, they shall already lead us to the proof Theorem (5.3.16).

Definition (5.3.9)[246]: Let $J \subseteq \mathbb{C}$ and $\mathcal{T} = \{T(z)\}_{z\in J}$ a family of bounded linear operators $L^2(\Xi)^{m_1} \to L^2(\Xi)^{m_2}$, where $m_1, m_2 \in \mathbb{N}$ and $\Xi \subseteq \mathbb{R}^d$ is (Lebesgue) measurable. Given $1 \le p \le q \le \infty$, we say that $\mathcal{T}$ satisfies $L^p \to L^q$ off-diagonal estimates if for some constants $C, c \in (0, \infty)$ the estimate

$$\|T(z)u\|_{L^q(F)^{m_2}} \le C|z|^{\frac{d}{2q}-\frac{d}{2p}} e^{-c\frac{d(E,F)^2}{|z|}} \|u\|_{L^p(E)^{m_1}}$$

holds for all $z \in J$, all measurable sets $E, F \subseteq \Xi$, and all $u \in L^p(\Xi)^{m_1} \cap L^2(\Xi)^{m_1}$ that are supported in E. We say that $\mathcal{T}$ is $L^p \to L^q$ bounded if for all $u \in L^p(\Xi)^{m_1} \cap L^2(\Xi)^{m_1}$,

$$\|T(z)u\|_{L^q(\Xi)^{m_2}} \le C|z|^{\frac{d}{2q}-\frac{d}{2p}} \|u\|_{L^p(\Xi)^{m_1}}.$$

In the case $p = q$ we simply speak of L^p off-diagonal estimates and L^p boundedness

We begin with $L^2 \to L^2$ off-diagonal bounds.

This will follow by Davies' perturbation method [210],[255],[272] and we shall indicate the major steps in order to help the reader through. To get the method running, we need the following invariance property.

Lemma (5.3.10)[246]: Every Lipschitz continuous function $\varphi: \mathbb{R}^d \to \mathbb{R}$ induces a bounded multiplication operator on $\mathbb{W}_D^{1,2}(\Omega)$.

Proof. Boundedness of the multiplication operator with respect to the $\mathbb{W}^{1,2}(\Omega)^m$-norm follows from the product rule. Hence, it suffices to check that the closed subspace $\mathbb{W}_D^{1,2}(\Omega)$ is left invariant and by density this will follow from $\varphi u|_\Omega \in \mathbb{W}_D^{1,2}(\Omega)$ for $u \in \prod_{k=1}^m C_{D_k}^\infty(\mathbb{R}^d)$. But in this case $\varphi u \in W^{1,2}(\mathbb{R}^d)^m$ with compact support and each of its components having support away from the respective Dirichlet part, so that approximants in $\prod_{k=1}^m C_{D_k}^\infty(\mathbb{R}^d)$ for the $W^{1,2}(\mathbb{R}^d)^m$ topology can be constructed via mollification with smooth kernels.

Proposition (5.3.11)[246]: Suppose that $\Omega \subseteq \mathbb{R}^d$ is a bounded domain. Let $\psi \in [0, \pi/2 - \omega)$. Then $\{e^{-zL}\}_{z\in S_\psi^+}$, $\{zLe^{-zL}\}_{z\in S_\psi^+}$, and $\{\sqrt{z}\nabla e^{-zL}\}_{z\in S_\psi^+}$ satisfy L^2 off-diagonal estimates and implicit constants depend on ψ, ellipticity, dimensions, and the diameter of Ω.

Proof. We begin with off-diagonal bounds for $z = t > 0$. Let $\varphi: \mathbb{R}^d \to \mathbb{R}$ be Lipschitz continuous with $\|\nabla\varphi\|_\infty \le 1$ and let $\rho > 0$; both yet to be specified. Since by the preceding lemma $\mathbb{W}_D^{1,2}(\Omega)$ is invariant under multiplication with $e^{\pm\rho\varphi}$, we can define $L_{\rho,\varphi} := e^{\rho\varphi} L e^{-\rho\varphi}$ by means of the form method using the bounded sesquilinear form

$$a_{\rho,\varphi}: \mathbb{W}_D^{1,2}(\Omega) \times \mathbb{W}_D^{1,2}(\Omega) \to \mathbb{C}, \qquad (u, v) \mapsto a(e^{-\rho\varphi}u, e^{\rho\varphi}v).$$

In order to see that $a_{\rho,\varphi}$ is sectorial, we multiply out the expression for $a(e^{-\rho\varphi}u, e^{\rho\varphi}u)$ obtained from the definition of a in (42), use boundedness and ellipticity of a, and control the error terms $a(u,u) - a(e^{-\rho\varphi}u, e^{\rho\varphi}u)$ by means of Young's inequality with ε. This results in the two estimates

$$\left|a_{\rho,\varphi}(u,u)\right| \le 2\Lambda(d+1)(\|u\|_2^2 + \|\nabla u\|_2^2) + c(\rho^2 + \rho)\|u\|_2^2$$

and

$$Re\ a_{\rho,\varphi}(u,u) \geq \frac{\lambda}{2}(\|u\|_2^2 + \|\nabla u\|_2^2) - c(\rho^2+\rho)\|u\|_2^2, \tag{50}$$

where $c \in (0,\infty)$ depends upon ellipticity and dimensions. Thus, $L_{\rho,\varphi} + 2c(\rho^2+\rho)$ is maximal accretive with angle $\arctan\left(\frac{4\Lambda(d+1)}{\lambda}\right)$. The universal bound for its H^∞-calculus yields

$$\|e^{-tL_{\rho,\varphi}}\|_{L^2\to L^2} + \left\|t\left(L_{\rho,\varphi} + 2c(\rho^2+\rho)\right)e^{-tL_{\rho,\varphi}}\right\|_{L^2\to L^2} \leq 4e^{2c(\rho^2+\rho)t} (t > 0), \tag{51}$$

see Proposition (5.3.2). Moreover, we have by definition

$$a_{\rho,\varphi}(u,u) + 2c(\rho^2+\rho)\|u\|_2^2 = \left(L_{\rho,\varphi}u + 2c(\rho^2+\rho)u\middle|u\right) \left(u \in \mathcal{D}(L_{\rho,\varphi})\right)$$

and as a holomorphic semigroup maps into the domain of its generator, the previous bounds along with the ellipticity estimate (50) imply

$$\left\|\sqrt{t}\nabla e^{-tL_{\rho,\varphi}}\right\|_{L^2\to L^2} \leq 4\left(\frac{\lambda}{2}\right)^{-\frac{1}{2}} e^{2c(\rho^2+\rho)t} \quad (t > 0). \tag{52}$$

Now, let $E, F \subseteq \Omega$ be measurable sets, and let $u \in L^2(\Omega)^m$ be supported in E. We specialize $\varphi(x) = d(x,E)$ and obtain

$$e^{-tL}u = e^{-\rho\varphi}e^{\rho\varphi}e^{-tL}e^{-\rho\varphi}u = e^{-\rho\varphi}e^{-tL_{\rho,\varphi}}u \quad (t > 0),$$

where in the last step we used that the similarity of operators $L_{\rho,\varphi} := e^{\rho\varphi}Le^{-\rho\varphi}$ inherits to resolvents and hence to the functional calculi. From (51) we can infer

$$\|e^{-tL}u\|_{L^2(F)} \leq e^{-\rho\, d(E,F)}\|e^{-tL_{\rho,\varphi}}u\|_{L^2(\Omega)} \leq 4e^{2c(\rho^2+\rho)t-\rho\, d(E,F)}\|u\|_{L^2(E)},$$

which on choosing $\rho := \frac{d(E,F)}{4ct}$ and recalling that Ω is bounded, becomes the off-diagonal bound

$$\|e^{-tL}u\|_{L^2(F)} \leq 4e^{-\frac{d(E,F)^2}{8ct}}e^{\frac{d(E,F)}{2}}\|u\|_{L^2(E)} \leq 4e^{\frac{diam(\Omega)}{2}}e^{-\frac{d(E,F)^2}{8ct}}\|u\|_{L^2(E)}.$$

The estimates for tLe^{-tL} and $\sqrt{t}\nabla e^{-tL}$ follow likewise from either (51) or (52).

Finally, to treat the general case $z \in S_\psi^+$, we replace L by $e^{i\arg z}L$: Since $|\arg z| < \pi/2 - \omega$, this is an operator in the same class as L and ellipticity constants of the corresponding form depend on λ, Λ, ψ. Hence, the first part of the proof applies with $t = |z|$ and the claim follows on noting $e^{-zL} = e^{-te^{i\arg z}L}$.

The subsequent proposition builds the bridge to $L^p \to L^p$ and $L^p \to L^2$ estimates. Going through the cycle of all five implication shows that for the semigroup all concepts are more or less equivalent if one allows a small play in the Lebesgue exponents.

Proposition (5.3.12)[246]: Assume Ω satisfies Assumption N. Let $p \in [1,2)$. For $\psi \in [0,\pi/2-\omega)$ put $S = \{e^{-zL}\}_{z\in S_\psi^+}$ and $\mathcal{N} = \left\{\sqrt{z}\nabla e^{-zL}\right\}_{z\in S_\psi^+}$. Then the following hold.

(i) If $\{e^{-tL}\}_{t>0}$ is L^p bounded, then S is $L^p \to L^2$ bounded.

(ii) If S is $L^p \to L^2$ bounded, then so is $\mathcal{N}$.

(iii) If S is $L^p \to L^2$ bounded and $q \in (p,2)$, then S satisfies $L^q \to L^2$ off-diagonal estimates.

(iv) If S satisfies $L^p \to L^2$ off-diagonal estimates, then so does $\mathcal{N}$.

(v) If S and $\mathcal{N}$ satisfy $L^p \to L^2$ off-diagonal estimates, then S and $\mathcal{N}$ are L^p bounded, respectively.

Proof. We begin with (i). Let $u \in L^2(\Omega)^m$ with $\|u\|_p = 1$. First, we establish the $L^p \to L^2$ bounds for the semigroup in the case $z = t > 0$. We obtain the interpolation inequality

$$\|v\|_{L^2(\Omega)}^2 \lesssim \|v\|_{\mathbb{W}_D^{1,2}(\Omega)}^{2\theta} \|v\|_{L^p(\Omega)}^{2-2\theta} \qquad \left(v \in \mathbb{W}_D^{1,2}(\Omega)\right),$$

where $1/\theta = 1 + 2p/(2d - pd)$, from the classical Gagliardo–Nirenberg inequality for functions on $\mathbb{R}^d$, see [274], and the boundedness of the extension operators E_k. Here, Assumption N was used. We apply this with $v = e^{-tL}u$ and obtain from the assumption and ellipticity

$$\|e^{-tL}u\|_2^2 \lesssim \|e^{-tL}u\|_{\mathbb{W}_D^{1,2}}^{2\theta} \lesssim \left(Re\, a(e^{-tL}u, e^{-tL}u)\right)^\theta = Re(Le^{-tL}u|e^{-tL}u)^\theta.$$

Hence, $f(t) := \|e^{-tL}u\|_2^2$ satisfies the differential inequality

$$f(t) \le C\left(-f'(t)\right)^\theta \qquad (t > 0),$$

where $C > 0$ depends on geometry and $[\![p]\!]$, see (45). If f vanishes at some point of the interval $(t/2, t)$, then $f(t) = 0$ by the semigroup property and we are done. Otherwise, we obtain

$$\frac{t}{2} \le \int_{\frac{t}{2}}^{t} \frac{Cf'(s)}{f(s)^{\frac{1}{\theta}}} ds \le \frac{C\theta}{1-\theta} f(t)^{1-\frac{1}{\theta}} = \frac{C\theta}{1-\theta} f(t)^{-\frac{2p}{2d-pd}},$$

which, by definition of f, is the required $L^p \to L^2$ estimate. In order to extend this bound to $z \in S_\psi^+$, we put $\psi := (\psi + \frac{\pi}{2} - \omega)/2$ and decompose $z = z' + t$, where $|\arg z| = \psi$ and $t > 0$, so that $|z| \simeq |z'| \simeq t$ with implicit constants depending on ψ and ω. The claim then follows from the contractivity of the semigroup on L^2 and the first part of the proof:

$$\|e^{-zL}u\|_2 \le \|e^{-zL}\|_{L^2 \to L^2} \|e^{-tL}u\|_2 \lesssim |t|^{\frac{d}{4}-\frac{d}{2p}} \simeq |z|^{\frac{d}{4}-\frac{d}{2p}}.$$

Next, (ii) follows from the semigroup law and the assertion for S. Indeed, it suffices to write

$$\sqrt{2z}\nabla e^{-2zL} = \sqrt{2}\left(\sqrt{z}\nabla e^{-zL}\right)e^{-zL}$$

and concatenate the L^2 bound of the first factor (Proposition (5.3.11)) with the assumed $L^p \to L^2$ bound for the second one.

As for (iii), we interpolate by means of the Riesz–Thorin theorem the assumed $L^p \to L^2$ bound with the L^2 off-diagonal estimates provided by Proposition (5.3.11). (Once we have fixed the sets E,F in the definition of off-diagonal estimates.)

Assertion (iv) follows by a refinement of the argument for (ii). We let $E, F \subseteq \Omega$ measurable sets, $u \in L^2(\Omega)^m$ with support in E and $\in S_\psi^+$. We also use a measurable set $G \subseteq \Omega$ to be specified yet. By the semigroup law we have

$$\left\|\sqrt{2z}\nabla e^{-2zL}u\right\|_{L^2(F)} \le \sqrt{2}\left(\left\|\sqrt{z}\nabla e^{-zL}1_G e^{-zL}u\right\|_{L^2(F)} + \left\|\sqrt{z}\nabla e^{-zL}1_{cG} e^{-zL}u\right\|_{L^2(F)}\right)$$

and hence by assumption and L^2 off-diagonal estimates for the gradient of the semigroup,

$$\le CC'|z|^{\frac{d}{4}-\frac{d}{2p}}\left(e^{-c\frac{d(G,F)^2}{|z|}-c'\frac{d(E,G)^2}{|z|}} + e^{-c\frac{d({}^cG,F)^2}{|z|}-c'\frac{d(E,{}^cG)^2}{|z|}}\right)\|u\|_{L^p(E)},$$

where $C, C', c, c' \in (0, \infty)$. For the choice $G = \{x \in \Omega: d(x,F) \ge d(E,F)/2\}$ we have $d(G,F) \ge d(E,F)/2$ and $d(E, {}^cG) \ge d(E,F)/2$, which in turn yields the claim.

Eventually, (v) follows from the subsequent lemma applied to $T = e^{-zL}$ or $T = \sqrt{z}\nabla e^{-zL}$ on choosing $g(r) = C|z|^{\frac{d}{4}-\frac{d}{2p}} e^{-\frac{cr^2}{|z|}}$ and $s = \sqrt{|z|}$.

Lemma (5.3.13)[246]: Let $1 \le p \le q \le \infty$ and T a bounded linear operator $L^2(\Xi)^{m_1} \to L^2(\Xi)^{m_2}$, where $m_1, m_2 \in \mathbb{N}$ and $\Xi \subseteq \mathbb{R}^d$ is measurable. If T satisfies $L^p \to L^q$ off-diagonal estimates in the form

$$\|Tu\|_{L^q(F\cap\Xi)} \le g\big(d(E,F)\big)\|u\|_{L^p(E\cap\Xi)},$$
whenever E, F are closed axis-parallel cubes in $\mathbb{R}^d$ and $u \in L^p(\Xi)^{m_1} \cap L^2(\Xi)^{m_1}$ is supported in $E \cap \Xi$ and g is some decreasing function. Then T is L^p bounded with norm bounded by $s^{\frac{d}{p}-\frac{d}{q}} \sum_{k\in\mathbb{Z}^d} g\left(s \max\left\{\frac{|k|}{\sqrt{d}} - 1, 0\right\}\right)$ for any $s > 0$ provided this sum is finite.

This is essentially [210] but because of two somewhat confusing misprints,one in the statement and one in proof, we decided to include the argument.

Proof. Let $\in L^p(\Xi)^{m_1} \cap L^2(\Xi)^{m_1}$. We partition $\mathbb{R}^d$ into closed, axis-parallel cubes $\{Q_k\}_{k\in\mathbb{Z}^d}$ of sidelength s with center sk and let $u_k := 1_{Q_k\cap\Xi} u$. From Hölder's inequality and the assumption we obtain

$$\begin{aligned}
\|Tu\|^p_{L^p(\Xi)} &= \sum_{k\in\mathbb{Z}^d} \|Tu\|^p_{L^p(Q_k\cap\Xi)} \le s^{d-\frac{dp}{q}} \sum_{k\in\mathbb{Z}^d} \|Tu\|^p_{L^q(Q_k\cap\Xi)} \\
&\le s^{d-\frac{dp}{q}} \sum_{k\in\mathbb{Z}^d} \left(\sum_{j\in\mathbb{Z}^d} \|Tu_j\|_{L^q(Q_k\cap\Xi)}\right)^p \\
&\le s^{d-\frac{dp}{q}} \sum_{k\in\mathbb{Z}^d} \left(\sum_{j\in\mathbb{Z}^d} g\left(d\big(Q_j\cap\Xi, Q_k\cap\Xi\big)\right) \|u_j\|_p\right)^p.
\end{aligned}$$

Let $|\cdot|_\infty$ be the ℓ^∞ norm on $\mathbb{R}^d$ and d_∞ the corresponding distance. We have $d_\infty(Q_j, Q_k) = \max\{|sj - sk|_\infty - s, 0\}$ and thus $d\big(Q_j\cap\Xi, Q_k\cap\Xi\big) \ge s\max\left\{\frac{|j-k|}{\sqrt{d}} - 1, 0\right\}$. Since g is decreasing, we can infer

$$\begin{aligned}
\|Tu\|_{L^p(\Xi)} &\le s^{\frac{d}{p}-\frac{d}{q}} \left(\sum_{k\in\mathbb{Z}^d}\left(\sum_{j\in\mathbb{Z}^d} g\left(s\max\left\{\frac{|j-k|}{\sqrt{d}} - 1, 0\right\}\right)\|u_j\|_p\right)^p\right)^{\frac{1}{p}} \\
&\le s^{\frac{d}{p}-\frac{d}{q}} \left(\sum_{k\in\mathbb{Z}^d} g\left(s\max\left\{\frac{|k|}{\sqrt{d}} - 1, 0\right\}\right)\right)\left(\sum_{j\in\mathbb{Z}^d}\|u_j\|_p^p\right)^{\frac{1}{p}},
\end{aligned}$$

where the second step is an application of Young's inequality for (discrete) convolutions. The sum in j equals $\|u\|^p_{L^p(\Xi)}$ and the claim follows. see [210].

The following lemma deals with the first part of Theorem (5.3.16).

Lemma (5.3.14)[246]: Suppose Ω satisfies Assumption N and let $p \in (2_*, 2^*)$. Then $\{e^{-tL}\}_{t>0}$ is L^p bounded with a bound depending on p, ellipticity, dimensions, and geometry.

Proof. By duality we may restrict ourselves to $p \in (2, 2^*)$. We can use (iii) and (v) from Proposition (5.3.12) for $p > 2$. The upshot is that it suffices to check $L^2 \to L^p$ boundedness of the semigroup. By ellipticity and the Cauchy–Schwarz inequality we have for $u \in L^2(\Omega)^m$ and $t > 0$,

$$\lambda\|e^{-tL}u\|^2_{\mathbb{W}^{1,2}_D} \le Re\, a(e^{-tL}u, e^{-tL}u) = Re(Le^{-tL}u|e^{-tL}u) \le \|Le^{-tL}u\|_2\|e^{-tL}u\|_2.$$

On the other hand, we obtain for $v \in \mathbb{W}^{1,2}_D(\Omega)$ the interpolation inequality

$$\|v\|_p \lesssim \|v\|^\theta_{\mathbb{W}^{1,2}_D}\|v\|^{1-\theta}_2,$$

where $1/p = (1-\theta)/2 + \theta/2^*$, from the classical Gagliardo–Nirenberg inequality for functions on $\mathbb{R}^d$, see [274], and the boundedness of the extension operators E_k. We pick $v = e^{-tL}u$ and obtain with the aid of the previous bound

$$\|e^{-tL}u\|_p \lesssim \|e^{-tL}u\|_{\mathbb{W}_D^{1,2}}^{\theta} \|e^{-tL}u\|_2^{1-\theta} \lesssim t^{-\frac{\theta}{2}}\|u\|_2,$$

where we have also used the semigroup properties $\|e^{-tL}u\|_2 \le \|u\|_2, \|Le^{-tL}u\|_2 \lesssim t^{-1}\|u\|_2$. Implicit constants depend on p, ellipticity, dimensions, and geometry. Substituting the value of θ, this turns out just to be $L^2 \to L^p$ boundedness of $\{e^{-tL}\}_{t>0}$.

We cite the following regularity result for the operator $\mathcal{L}: \mathbb{W}_D^{1,2}(\Omega) \to \mathbb{W}_D^{-1,2}(\Omega)$, whose maximal restriction to $L^2(\Omega)^m$ is L. Essentially, this follows from Šneˇıberg's theorem [278], see also [261], but tracking the interpolation constants in order to deduce the required uniformity of the bounds is a non-trivial task.

Proposition (5.3.15)[246]: ([268]). Under Assumptions D and N there exists $\varepsilon' > 0$ such that $\mathcal{L}$ extends/restricts to an isomorphism $\mathbb{W}_D^{1,p}(\Omega) \to \mathbb{W}_D^{-1,p}(\Omega)$ for all $p \in (2-\varepsilon', 2+\varepsilon')$. In addition, ε' and upper and lower bounds for $\mathcal{L}$ can be given in terms of ellipticity, dimensions, and geometry.

Now, we are ready to give the proof of Theorem (5.3.16). Let us stress that our argument essentially differs from the whole space case [210] in that it avoids a change of variables for the coefficients A. This is necessary since the resulting change of the underlying domain would affect geometric constants in an uncontrollable way.

Theorem (5.3.16)[246]: If the bounded domain $\Omega \subseteq \mathbb{R}^d$ satisfies Assumption N, then

$$\mathcal{J}(L) \supseteq \begin{cases} (1,\infty) & if\ d = 2 \\ (2_*, 2_*) & if\ d \ge 3. \end{cases}$$

If in addition Assumption D is in power, then $(2_*, 2^*)$ can be replace with $(2_* - \varepsilon, 2^* + \varepsilon)$ if $d \ge 3$ and $\varepsilon > 0$ depends on ellipticity and geometry. For p in these sub-intervals of $\mathcal{J}(L)$ the L^p bound for the semigroup depends on p, ellipticity, dimensions, and geometry.

Proof. In view of Lemma (5.3.14) we only need to prove the extrapolation from the range $(2_*, 2^*)$ in the case $d \ge 3$ under Assumptions D and N.

Let $\varepsilon' > 0$ be as provided by Proposition (5.3.15). We fix p, q, and r such that

$$\max\{2-\varepsilon', 2_*, 1^*\} < p < q < r < 2,$$

which is possible since $d \ge 3$ implies $1^* < 2$. We will prove $r_* \in \mathcal{J}(L)$ with a bound depending on p, q, r, ellipticity, and geometry. This implies the claim: First, p, q,r share the same dependencies as ε' and therefore we have $r_* = 2_* - \varepsilon$ for some $\varepsilon > 0$ depending on ellipticity, dimensions, and geometry. Second, Riesz–Thorin interpolation of the L^{r_*} bound for the semigroup with the contractivity on L^2 yields L^s bounds for $s \in (r_*, 2)$ without introducing further implicit constants. Third, the same argument with the same choice of parameters applies to L^* and by duality we obtain L^s boundedness for $s \in (2, (r')^*)$, where $1/r' = 1 - 1/r$.

In order to prove L^{r_*} boundedness, we let $t > 0$ and take u in L^{p^*}, a dense subspace of $L^2 \cap L^{r_*}$. By Cauchy's integral formula and since $\mathcal{L}$ extends L, we can write

$$e^{-tL}u = Le^{-tL}L^{-1}u = -\frac{1}{2\pi i}\oint_{|z-t|=R} \frac{1}{(z-t)^2} e^{-zL}\mathcal{L}^{-1}u\,dz,$$

where $R = d(t, \partial S_\psi^+)/2$ and $\psi = \pi/4 - \omega/2$. We have $p^* \in (2, 2^*)$, so $p^* \in \mathcal{J}(L)$ thanks to Lemma (5.3.14). Proposition (5.3.12) implies L^{q^*} boundedness of the semigroup for complex times $z \in S_\psi^+$ and in particular along the integration contour above. Thus, we have

$$\|e^{-tL}u\|_{q^*} \lesssim t^{-1}\|\mathcal{L}^{-1}u\|_{q^*}.$$

Now, $\mathcal{L}$ extends to an isomorphism $\mathbb{W}_D^{1,q}(\Omega) \to \mathbb{W}_D^{-1,q}(\Omega)$ by choice of q. Since $1^* < q < 2$, we obtain from Sobolev embeddings,

$$\|\mathcal{L}^{-1}u\|_{q^*} \lesssim \|\mathcal{L}^{-1}u\|_{\mathbb{W}_D^{1,q}} \lesssim \|u\|_{\mathbb{W}_D^{-1,q}} \lesssim \|u\|_{q^*}.$$

Altogether, $\|e^{-tL}u\|_{q^*} \lesssim t^{-1}\|u\|_{q^*}$, that is, the semigroup is $L^{q_*} \to L^{q_*}$ bounded. Riesz–Thorin interpolation with the L^2 off-diagonal estimates from Proposition (5.3.11) leads to $L^{r_*} \to L^s$ offdiagonal estimatesforsome $s > r_*$ determined by q and r, which in turn implies L^{r_*} boundedness (Lemma (5.3.13)).

To obtain L^p estimates for the functional calculus for L it will be convenient to calculate $f(L)$ in terms of the semigroup instead of the resolvent. This can be seen as some kind of Laplace transform inversion.

Lemma (5.3.17)[246]: Let $\omega < \theta < v < \pi/2$ and $g \in H_0^\infty(S_\psi^+)$. Put $\Gamma_\pm = (0,\infty)e^{\pm i\left(\frac{\pi}{2}-\theta\right)}$ and $\gamma_\pm = (0,\infty)e^{\pm iv}$. Then $g(L)$ can also be computed as an $L^2(\Omega)^m$-valued Bochner integral

$$g(L) = \int_{\Gamma_+} e^{-zL}\eta_+(z)dz - \int_{\Gamma_-} e^{-zL}\eta_-(z)dz,$$

where

$$\eta_\pm(z) = \frac{1}{2\pi i}\int_{\gamma_\pm} e^{z\xi}g(\xi)d\xi \qquad (z \in \Gamma_\pm).$$

Proof. Let $\xi \in \gamma_\pm$. For $z \in \Gamma_\pm$ we have $|\arg(z\xi)| = \frac{\pi}{2} - \theta + v > \frac{\pi}{2}$. Consequently, $(\xi - L)^{-1}e^{z\xi}e^{-zL}$ vanishes as $|z| \to \infty$ along the *ray* $\Gamma_\pm$ and we may compute, using the fundamental theorem of calculus,

$$\int_{\Gamma_\pm} e^{z\xi}e^{-zL}dz = \int_{\Gamma_\pm} \frac{d}{dz}\left((\xi - L)^{-1}e^{z\xi}e^{-zL}\right)dz = -(\xi - L)^{-1}.$$

By definition of the functional calculus

$$g(L) = -\frac{1}{2\pi i}\int_{\gamma_+} g(\xi)(\xi - L)^{-1}d\xi + \frac{1}{2\pi i}\int_{\gamma_-} g(\xi)(\xi - L)^{-1}d\xi.$$

From these two identities the claim follows by an application of Fubini's theorem.

Next, we recall an important weak type (p,p) criterion for bounded operators on $L^2(\Xi)$ that goes back to [251]. If $\mathbb{R}^d$, Proposition (5.3.18) below is exactly the simplified version presented in [210], see also the subsequent Remark (7) in [210] concerning vector-valued extensions. The result below on general measurable sets Ξ is not mentioned therein but follows easily: Take R as the canonical restriction $\mathbb{R}^d \to \Xi$ and E as the extension $\Xi \to \mathbb{R}^d$ by zero. Then observe that the $\mathbb{R}^d$-version applies to $T' := ETR$ and $A_r' := EA_rR$ with the same parameters and that T and T' have the same L^p bound.

Proposition (5.3.18)[246]: Let $q \in [1,2)$. Let $T: L^2(\Xi)^{m_1} \to L^2(\Xi)^{m_2}$ be a bounded linear operator, where $m_1, m_2 \in \mathbb{N}$ and $\Xi \subseteq \mathbb{R}^d$ is measurable. Assume there exists a family $\{A_r\}_{r>0}$ of bounded linear operators on $L^2(\Xi)^{m_1}$ with the following properties: For $j \geq 2$,

$$\left(\int_{C_j(B)\cap\Xi} |T(1 - A_r)u|^2\right)^{\frac{1}{2}} \leq g(j)r^{\frac{d}{2}-\frac{d}{q}}\left(\int_{B\cap\Xi} |u|^q\right)^{\frac{1}{q}} \qquad (53)$$

and for $j \geq 1$,

$$\left(\int_{C_j(B)\cap \Xi} |A_r u|^2\right)^{\frac{1}{2}} \leq g(j) r^{\frac{d}{2}-\frac{d}{q}} \left(\int_{B\cap \Xi} |u|^q\right)^{\frac{1}{q}}, \tag{54}$$

whenever $B \subseteq \mathbb{R}^d$ is an open ball with radius r and $u \in L^2(\Xi)^{m_1}$ has support in $B \cap \Xi$. If $\sum := \sum g(j) 2^{\frac{dj}{2}}$ is finite, then T is of weak type (q,q) and hence L^p bounded for $p \in (q,2)$ with a bound depending on $q, m_1, m_2, \sum$, and an L^2 bound for T.
As a first application we prove
Lemma (5.3.19)[246]: Suppose $\{e^{-tL}\}_{t>0}$ satisfies $L^q \to L^2$ off-diagonal estimates for some $q \in (1,2)$. Then

$$\|f(L)u\|_p \leq C\|f\|_\infty \|u\|_p \qquad \left(f \in H_0^\infty(S_\psi^+), u \in L^2(\Omega)^m\right),$$

whenever $\psi \in (\omega, \pi)$ and $p \in (q,2)$. Here, C depends on ψ, p, q, ellipticity, dimensions, geometry and constants implicit in the assumption.
Proof. Without loss of generality we may assume $\psi < \pi/2$. Let $f \in H_0^\infty(S_\psi^+)$ be normalized such that $\|f\|_\infty = 1$. We appeal to Proposition (5.3.18) with $T = f\,(L)$. We put $A_r = 1 - \left(1 - e^{-r^2 L}\right)^n$, where $n \in \mathbb{N}$ has to be determined yet. Proposition (5.3.2) yields $\|T\|_{L^2 \to L^2} \leq 4$ and we need to check (53) and (54).
For the argument we put $\gamma := d/q - d/2 > 0$, let $B \subseteq \mathbb{R}^d$ be an open ball with radius $r > 0$, and $u \in L^2(\Omega)^m$ have its support in $B \cap \Omega$. Having expanded

$$A_r = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} e^{-kr^2 L},$$

the assumed $L^q \to L^2$ off-diagonal estimates (with constants $C, c \in (0,\infty)$) yield for $j \geq 1$,

$$\left(\int_{C_j(B)\cap \Xi} |A_r u|^2\right)^{\frac{1}{2}} \leq C 2^n e^{-\frac{c(2^j-2)^2}{n^2}} r^{-\gamma} \left(\int_{B\cap \Xi} |u|^q\right)^{\frac{1}{q}}.$$

Hence, (54) holds with $g(j)2^{\frac{dj}{2}}$ summable no matter the value of n.
Turning to (53), we apply Lemma (5.3.17) to the function $g(z) = f\,(z)(1 - e\text{^}(-r^2 z)^n$ and write

$$T(1 - A_r)u = \int_{\Gamma_+} \eta_+(z) e^{-zL} u dz - \int_{\Gamma_-} \eta_-(z) e^{-zL} u dz, \tag{55}$$

Where

$$\eta_\pm(z) = \frac{1}{2\pi i} \int_{\gamma_\pm} e^{z\xi} g(\xi) d\xi \qquad (z \in \Gamma_\pm).$$

For $j \geq 2$ we take $L^2\left(C_j(B) \cap \Omega\right)$-norms in (55) and apply off-diagonal estimates to give

$$\left(\int_{C_j(B)\cap \Omega} |T(1 - A_r)u|^2\right)^{\frac{1}{2}} \lesssim \left(I_{j,+} + I_{j,-}\right) \left(\int_{B\cap \Omega} |u|^q\right)^{\frac{1}{q}}, \tag{56}$$

where

$$I_{j,\pm} = \int_{\Gamma_\pm} C\left|\eta_\pm(z)\right| |z|^{-\frac{\gamma}{2}} e^{-\frac{c4^{j-1}r^2}{|z|}} d|z|.$$

By the mean value theorem and the normalization of f we have $|g(\xi)| \le \min\{2^n, r^{2n}|\xi|^n\}$ for $\xi \in \gamma_\pm$. Consequently,

$$\left|\eta_\pm(z)\right| \le \alpha|z|^{-1}\min\{1, r^{2n}|z|^{-n}\}\,(z \in \Gamma_\pm),$$

where α depends on ψ, ω, n. Setting $|z| = t$, we deduce that

$$I_{j,\pm} \le \alpha e^{-\frac{c4^{j-1}}{2}}\int_0^{r^2} t^{-\frac{\gamma}{2}} e^{-\frac{c4^{j-1}r^2}{2t}}\frac{dt}{t} + \alpha r^{2n}\int_{r^2}^{\infty} t^{-\frac{\gamma}{2}-n} e^{-\frac{c4^{j-1}r^2}{t}}\frac{dt}{t}$$
$$\le \alpha r^{-\gamma}2^{-\gamma(j-1)}(e^{-\frac{c4^{j-1}}{2}}\int_0^{\infty} s^{-\frac{\gamma}{2}}e^{-\frac{c}{2s}}\frac{ds}{s} + 4^{-(j-1)n}\int_0^{\infty} s^{-\frac{\gamma}{2}-n}e^{-\frac{c}{s}}\frac{ds}{s}.$$

The remaining integrals in s are finite. Thus, we have found $I_{j,\pm} \le g(j)r^{-\gamma}$ with $\left\{2^{\frac{dj}{2}}g(j)\right\}_{j\ge 2}$ summable provided $\gamma + 2n > d/2$. For such choice of n, (53) follows from (56).

Now we can complete the

Theorem (5.3.20)[246]: Suppose $\Omega \subseteq \mathbb{R}^d$ is a bounded domain that satisfies Assumption N and let $p_0 \in \mathcal{J}(L)$. If f is bounded and holomorphic on the sector S_ψ^+ , where $\psi \in (\omega, \pi)$, then for all $p \in (p_0, 2) \cup (2, p_0)$,

$$\|f(L)u\|_p \le C\|f\|_\infty\|u\|_p \qquad (u \in L^2(\Omega)^m \cap L^p(\Omega)^m). \tag{57}$$

Moreover, C depends on $\psi, [\![p_0]\!], p,$ and geometry.

The above range for p is optimal in that (57) for some $p \in (1, \infty)$, some $\psi \in (\omega, \pi)$, and all f as above implies $p \in \mathcal{J}(L)$.

Proof. The necessity part follows simply because we can take $f(z) = e^{-tz}$ in (57) for every $t > 0$.

By duality it suffices to treat the sufficiency part in the case $p_0 \in \mathcal{J}(L) \cap [1, 2)$. Let $p \in (p_0, 2]$ and $\psi \in (\omega, \pi)$. Proposition (5.3.12) provides $L^q \to L^2$ off-diagonal estimates for the semigroup, for instance for the choice $q = (p + p_0)/2$, and implied constants depend on $[\![p_0]\!], p,$ and geometry. Lemma (5.3.19) yields

$$\|f(L)u\|_p \le C\|f\|_\infty\|u\|_p \qquad \left(f \in H_0^\infty\left(S_\psi^+\right), u \in L^2(\Omega)^m\right),$$

with C depending on $[\![p_0]\!], p, \psi,$ and geometry. This bound extends to $f \in H^\infty\left(S_\psi^+\right)$ and $u \in L^2(\Omega)^m$. Indeed, if $f_n \in H_0^\infty\left(S_\psi^+\right)$ are such that $f_n \to f$ pointwise, $\|f_n\|_\infty \to \|f\|_\infty$, and $f_n(L)u \to f(L)u$ in L^2, then $f_n(L)u \to f(L)u$ also in L^p since Ω is bounded.

As a primer to Theorem (5.3.25) we study L^p boundedness of the Riesz transform $\nabla L^{-\frac{1}{2}}$. Due to Theorem (5.3.6) this is an L^2 bounded operator. It follows from (48) that

$$\nabla L^{-\frac{1}{2}}u = \frac{2}{\sqrt{\pi}}\int_0^\infty \nabla e^{-t^2L}u\,dt \qquad (u \in L^2(\Omega)^m) \tag{58}$$

in the sense of an improper Riemann integral.

Lemma (5.3.21)[246]: Suppose $\{e^{-tL}\}_{t>0}$ satisfies $L^q \to L^2$ off-diagonal estimates for some $q \in (1, 2)$. Then $\nabla L^{-\frac{1}{2}}$ is L^p bounded for every $p \in (q, 2)$. The bound depends on p, q, ellipticity, dimensions, geometry and constants implicit in the assumption.

Proof. We appeal to Proposition (5.3.18) with $T = \nabla L^{-\frac{1}{2}}$ and $A_r = 1 - \left(1 - e^{-r^2L}\right)^n$, where $n \in \mathbb{N}$ has to be determined yet. We have seen that T is L^2 bounded. From the proof of Lemma (5.3.19) we also know that $L^q \to L^2$ off-diagonal estimates for the semigroup imply (54) for any choice of n. So, we only have to check (53).

For the argument we put $\gamma := d/q - d/2 > 0$, let $B \subseteq \mathbb{R}^d$ be an open ball with radius $r > 0$, and $u \in L^2(\Omega)^m$ have its support in $B \cap \Omega$. We calculate $T(1 - A_r)u$ via (58) and expand A_r by the binomial theorem. This leads to the formula

$$T(1 - A_r)u = \frac{2}{\sqrt{\pi}} \int_0^\infty \sum_{k=0}^{n} (-1)^k \binom{n}{k} \nabla e^{-(t^2 + kr^2)L} u \, dt,$$

which, by substituting $t^2/r^2 + k$, can more conveniently be written as

$$T(1 - A_r)u = \frac{1}{\sqrt{\pi}} \int_0^\infty g(t) r \nabla e^{-r^2 tL} u \, dt,$$

where

$$g(t) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{1_{(0,\infty)}(t-k)}{\sqrt{t-k}}. \tag{59}$$

Proposition (5.3.12).(iv) yields $L^q \to L^2$ off-diagonal estimates for $\{\sqrt{t}\nabla e^{-tL}\}_{t>0}$. Let $C, c \in (0, \infty)$ be the implied constants they do not bring in further dependencies. Taking $L^2(C_j(B) \cap \Omega)$-norms in the above formula, we find for $j \geq 2$,

$$\left(\int_{C_j(B) \cap \Omega} |T(1 - A_r)u|^2 \right)^{\frac{1}{2}} \leq C\pi^{-\frac{1}{2}} r^{-\gamma} (I_- + I_+) \left(\int_{B \cap \Omega} |u|^q \right)^{q}, \tag{60}$$

where

$$I_- = \int_0^{4n} |g(t)| t^{-\frac{\gamma}{2} - \frac{1}{2}} e^{-\frac{c4^{j-1}}{t}} dt, \qquad I_+ = \int_{4n}^\infty |g(t)| t^{-\frac{\gamma}{2} - \frac{1}{2}} e^{-\frac{c4^{j-1}}{t}} dt.$$

It remains to bound these integrals. We begin with the crude estimate

$$I_- \leq e^{-c\frac{4^{j-1}}{8n}} \int_0^{4n} |g(t)| t^{-\frac{\gamma}{2} - \frac{1}{2}} e^{-\frac{c}{2t}} dt,$$

where the remaining integral is finite since $|g|$ isintegrable on $(0, 4n)$ and the other factor remains bounded as $t \to 0$. As for I_+, we first note that for $t > 4n$ all characteristic functions in (59) evaluate to 1. Hence, the residue theorem yields

$$g(t) = \frac{(-1)^n}{2\pi i} \oint_{|z| = \frac{t}{2}} \frac{n!}{z(z-1)\cdots(z-n)} \frac{1}{\sqrt{t-z}} dz.$$

Along the path of integration $t - z, z, z - 1, \ldots, z - n$ are of absolute value at least $t/4$ each. Thus $|g(t)| \leq \alpha t^{-n-\frac{1}{2}}$ for some $\alpha \in (0, \infty)$ depending on n. In conclusion,

$$I_+ \leq \alpha \int_{4n}^\infty t^{-\frac{\gamma}{2} - n} e^{-\frac{c4^{j-1}}{t}} \frac{dt}{t} \leq \alpha 2^{-\gamma(j-1)} 4^{-(j-1)n} \int_0^\infty s^{-\frac{\gamma}{2} - n} e^{-\frac{c}{s}} \frac{ds}{s},$$

and the integral in s is finite. We have found $I_- + I_+ \leq g(j) r^{-\gamma}$ with $\left\{2^{\frac{dj}{2}} g(j)\right\}_{j \geq 2}$ summable provided $\gamma + 2n > d/2$. For such choice of n, (53) follows from (60).

Corollary (5.3.22)[246]: If $p0 \in \mathcal{J}(L) \cap [1, 2)$, then for every $p \in (p_0, 2)$ the lower bound

$$\|u\|_{\mathbb{W}_D^{1,p}} \lesssim \left\| L^{\frac{1}{2}} u \right\|_p \qquad \left(u \in \mathbb{W}_D^{1,2}(\Omega)\right).$$

The implied constant depends on $[\![p_0]\!], p$, and geometry

Proof. Let $p \in (p_0, 2)$. Proposition (5.3.12) provides $L^q \to L^2$ off-diagonal estimates for $\{e^{-tL}\}_{t>0}$, for instance for the choice $q = (p + p_0)/2$, and implied constants depend on

$[\![p_0]\!], p$, and geometry. Hence, Lemma (5.3.21) applies. An equivalent way of stating its conclusion is the estimate

$$\|\nabla u\|_p \lesssim \left\|L^{\frac{1}{2}}u\right\|_p \qquad \left(u \in \mathbb{W}_D^{1,2}(\Omega)\right),$$

where the implied constant shares the same dependencies. To add $\|u\|_p$ on the left-hand side, we first interpolate the assumed L^{p_0} bound for the semigroup with the exponential decay on L^2 stated in Lemma (5.3.3). This yields $\|e^{-tL}\|_{L^p \to L^p} \le [\![p_0]\!]^{1-\theta} e^{-\frac{\lambda\theta t}{2}}$ for $t > 0$, where $1/p = (1-\theta)/p_0 + \theta/2$. Now we can use (48) to give

$$\|u\|_p \le \frac{2}{\sqrt{\pi}} \int_0^\infty [\![p_0]\!]^{1-\theta} e^{-\frac{\lambda\theta t^2}{2}} \left\|L^{\frac{1}{2}}u\right\|_p dt = \frac{\sqrt{2}[\![p_0]\!]^{1-\theta}}{\sqrt{\lambda\theta}} \left\|L^{\frac{1}{2}}u\right\|_p.$$

We shall craft a Calderón–Zygmund decomposition within $\mathbb{W}_D^{1,p}$. We extend the approach from [206]. The crucial insight in was that the following Hardy inequality can be used to maintain Dirichlet boundary conditions for the 'good' and all 'bad' functions. For a proof see [206] or [58]. We agree on $d(x, \emptyset) = \infty$ for $x \in \mathbb{R}^d$ so that the estimate below holds trivially for empty Dirichlet parts.

Proposition (5.3.23)[246]: Suppose Ω is a bounded domain that satisfies Assumptions D and N and let $p \in (1, \infty)$. For every $k = 1, \dots, m$ there is a Hardy-type inequality

$$\int_\Omega \left|\frac{v}{d_{D_k}}\right|^p \lesssim \int_\Omega |\nabla v|^p \qquad \left(v \in W_{D_k}^{1,p}(\Omega)\right).$$

We let $\mathcal{Q}$ be the system of closed axis-parallel cubes with non-empty interior in $\mathbb{R}^d$. The Hardy–Littlewood maximal operator M is defined for $u \in L^1_{loc}(\mathbb{R}^d)$ by

$$(Mu)(x) := \sup_{x \in Q \in \mathcal{Q}} \fint_Q |u| \qquad (x \in \mathbb{R}^d),$$

Where $\fint_Q := |Q|^{-1} \int_Q$ denotes the average over Q. Then $|u| \le Mu$ holds almost everywhere and for some constant $c_d > 0$ depending only on d there is a weak type estimate

$$\left|\{x \in \mathbb{R}^d : |(Mu)(x)| > \alpha\}\right| \le \frac{c_d}{\alpha} \|u\|_{L^1(\mathbb{R}^d)} \qquad \left(\alpha > 0, u \in L^1(\mathbb{R}^d)\right),$$

see for example [267]. Let us denote the coordinates of a $\mathbb{C}^m$-valued function v by $v^{(k)}$.

Lemma (5.3.24)[246]: Suppose Ω is a bounded domain with Assumptions D and N and let $1 < p < \infty$. For every $u \in \mathbb{W}_D^{1,p}(\Omega)$ and every $\alpha > 0$ there exists a countable index set J, cubes $Q_j \in \mathcal{Q}, j \in J$, and measurable functions $g, b_j : \Omega \to \mathbb{C}^m$ such that for some $C \ge 1$, independent of u and α, the following hold.

(i) $u = g + \sum_{j \in J} b_j$ pointwise almost everywhere.

(ii) Each b_j has support in Q_j and each $x \in \mathbb{R}^d$ is contained in at most 12^d of the Q_j.

(iii) $g \in \mathbb{W}_D^{1,\infty}(\Omega)$ with $\|g\|_{\mathbb{W}_D^{1,\infty}(\Omega)} + \sum_{k=1}^m \left\|\frac{g^{(k)}}{d_{D_k}}\right\|_{L^\infty(\Omega)} \le C\alpha$.

(iv) $b_j \in \mathbb{W}_D^{1,p}(\Omega)$ with $\int_\Omega \left(|\nabla b_j|^p + |b_j|^p + \sum_{k=1}^m \left|\frac{b_j^{(k)}}{d_{D_k}}\right|^p\right) \le C\alpha^p |Q_j|$ for every $j \in J$.

(v) $\sum_{j \in J} |Q_j| \le \frac{C}{\alpha^p} \|u\|^p_{\mathbb{W}_D^{1,p}(\Omega)}$.

(vi) $g \in \mathbb{W}_D^{1,p}(\Omega)$ with $\|g\|_{\mathbb{W}_D^{1,p}(\Omega)} \le C \|u\|_{\mathbb{W}_D^{1,p}(\Omega)}$.

(vii) If $u \in L^q(\Omega)^m$ for some $q \in [1, \infty)$, then also $b_j \in L^q(\Omega)^m$ for every $j \in J$.

(viii) For every subset $J' \subseteq J$ the sum $\sum_{j \in J'} b_j$ converges unconditionally in $\mathbb{W}_D^{1,p}(\Omega)$.

Proof. The proof follows the classical pattern relying on a Whitney decomposition of an exceptional set determined by an adapted maximal function. It is divided into seven steps.

Step 1: Adapted maximal function. Recall the bounded Sobolev extension operators E_k introduced. Then $\tilde{u}^{(k)} := E_k u^{(k)}, k = 1, \ldots, m$, defines an extension $\tilde{u} \in \mathbb{W}_D^{1,p}(\mathbb{R}^d)$ of u. Since Ω is bounded, a suitable smooth truncation far away from $\overline{\Omega}$ allows us to modify the extension in such a way that even the Hardy-type terms as in Proposition (5.3.23) are controlled, that is to say,

$$\int_{\mathbb{R}^d} \left(|\tilde{u}| + |\nabla \tilde{u}| + \sum_{k=1}^{m} \left| \frac{\tilde{u}^{(k)}}{d_{D_k}} \right| \right)^p \lesssim \int_{\Omega} (|u| + |\nabla u|)^p. \tag{61}$$

The procedure is explained in detail on p. 176 of [206]. We define the open set

$$U := \left\{ x \in \mathbb{R}^d : M\left(\left(|\nabla \tilde{u}| + |\tilde{u}| + \sum_{k=1}^{m} \left| \frac{\tilde{u}^{(k)}}{d_{D_k}} \right| \right)^p \right) > \alpha^p \right\}.$$

First we treat the case $U = \emptyset$. Then for the choices $J = \emptyset$ and $g = u$ all assertions are immediate except for (iii). Referring to this, we use that $\tilde{u}$ is an extension of u and obtain for almost every $x \in \Omega$,

$$\left(|\nabla g(x)| + |g(x)| + \sum_{k=1}^{m} \left| \frac{g^{(k)}(x)}{d_{D_k}(x)} \right| \right)^p = \left(|\nabla \tilde{u}(x)| + |\tilde{u}(x)| + \sum_{k=1}^{m} \left| \frac{\tilde{u}^{(k)}(x)}{d_{D_k}(x)} \right| \right)^p.$$

The right-hand side is dominated almost everywhere by its maximal function, which in turn in globally bounded by α^p. We shall discuss at the end of the proof in the general case why this implies $g \in \mathbb{W}_D^{1,\infty}(\Omega)$.

So, from now on we can assume that U is a non-empty open subset of $\mathbb{R}^d$. By the weak type estimate for the maximal operator and (61) we obtain

$$|U| \lesssim \frac{1}{\alpha^p} \left\| |\nabla \tilde{u}| + |\tilde{u}| + \sum_{k=1}^{m} \left| \frac{\tilde{u}^{(k)}}{d_{D_k}} \right| \right\|_p^p \lesssim \frac{1}{\alpha^p} \|u\|_{\mathbb{W}_D^{1,p}}^p < \infty. \tag{62}$$

In particular, $F := \mathbb{R}^d \setminus U$ is non-empty. This allows for choosing a Whitney decomposition of U, that is, an at most countable index set J and a collection of cubes $Q_j \in \mathcal{Q}, j \in J$, with diameter d_j that satisfy

$(A)\quad U = \bigcup_{j \in J} \frac{8}{9} Q_j,$ $\qquad (B)\quad \frac{8}{9} Q_j^\circ \cap \frac{8}{9} Q_k^\circ = \emptyset \quad if \quad j \neq k,$

$(C)\ \ Q_j \subseteq U\ for\ all\ j,$ $\qquad (D)\quad \sum_{j \in J} 1_{Q_j} \leq 12^d,$

$(E)\ \ \frac{5}{6} d_j \leq d(Q_j, F) \leq 4 d_j\, for\ all\ j,$

see [213] for this classical tool but replace the cubes Q by their enlarged counterparts $\frac{9}{8}Q$ therein. Here, Q° denotes the interior of Q. Two important consequences can be recorded immediately: Firstly, (E) implies

$$12\sqrt{d}Q_j \cap F \neq \emptyset \qquad (j \in J). \tag{63}$$

Secondly, (D) in combination with (62) immediately implies (v) since

$$\sum_{j\in J} |Q_j| \leq \int_U \sum_{j\in J} 1_{Q_j} \leq 12^d\, |U| \lesssim \frac{1}{\alpha^p} \|u\|^p_{\mathbb{W}^{1,p}_D(\Omega)}. \tag{64}$$

Step 2: Definition of the good and bad functions. Let $\{\varphi_j\}_{j\in J}$ be a partition of unity on U, that is $\sum_{j\in J} \varphi_j = 1$ on U, with the properties

$(a)\quad \varphi_j \in C^\infty(\mathbb{R}^d)$ $\qquad (b)\quad supp\varphi_j \subseteq Q_j^\circ$

$(c)\quad \varphi_j = 1\ on \dfrac{8}{9} Q_{\mathcal{J}}$ $\qquad (d) \|\varphi_j\|_\infty + d_j \|\nabla\varphi_j\|_\infty \lesssim 1$

For all $j \in J$, see [213] for the construction. Given $1 \leq k \leq m$, we distinguish between three properties a cube Q_j can have:

(i) Q_j is k-usual if $d_j < 1$ and $d(Q_j, D_k) \geq d_j$,

(ii) Q_j is k-boring if $d(Q_j, D_k) \geq d_j \geq 1$,

(iii) Q_j is k-special if $d(Q_j, D_k) < d_j$.

Then we let $\tilde{u}^{(k)}_{Q_j} := ⨍_{Q_j} \tilde{u}^{(k)}$ and define

$$\tilde{b}^{(k)}_j := \begin{cases} \varphi_j \left(\tilde{u}^{(k)} - \tilde{u}^{(k)}_{Q_j}\right) & if\ Q_j\ is\ \text{k} - \text{usual} \\ \varphi_j \tilde{u}^{(k)} & if\ Q_j\ is\ k - boring\ or\ k - special \end{cases} \quad (1 \leq k \leq m, j \in J).$$

Setting $\tilde{g} := \tilde{u} - \sum_{j\in J} \tilde{b}_j$ as well as $b_j := \tilde{b}_j|_\Omega$ and $:= \tilde{g}|_\Omega, j \in J$, these functions automatically satisfy (i). Due to (D) there is no problem of convergence with this sum and also (ii) holds true. Moreover, (vii) holds since the extension operators E_k are bounded $L^q(\Omega) \to L^q(\mathbb{R}^d)$ for every $1 \leq q < \infty$.

Next, we check that the b_j are contained in $\mathbb{W}^{1,p}_D(\Omega)$: For fixed $1 \leq k \leq m$ we have $\tilde{b}^{(k)}_j \in \mathbb{W}^{1,p}(\mathbb{R}^d)$ by construction. If Q_j is either k-usual or k-boring, then $d(Q_j, D_k) \geq d_j > 0$ and via mollification $\tilde{b}^{(k)}_j$ can be approximated by $C^\infty_{D_k}(\mathbb{R}^d)$-functions in the norm of $\mathbb{W}^{1,p}(\mathbb{R}^d)$. If Q_j is k-special, then $\tilde{u}^{(k)} \in \mathbb{W}^{1,p}_{D_k}(\mathbb{R}^d)$ implies $\tilde{b}^{(k)}_j = \varphi_j \tilde{u}^{(k)} \in \mathbb{W}^{1,p}_{D_k}(\mathbb{R}^d)$.

Step 3: Proof of (iv). After the considerations above it remains to prove the estimate. To this end, we fix a coordinate $1 \leq k \leq m$.

We start with a k-usual cube, in which case $\nabla\tilde{b}^{(k)}_j = \varphi_j \nabla\tilde{u}^{(k)} + \left(\tilde{u}^{(k)} - \tilde{u}^{(k)}_{Q_j}\right) \nabla_{\varphi_j}$. Using (d),

$$\int_{Q_j} \left|\nabla\tilde{b}^{(k)}_j\right|^p \lesssim \int_{Q_j} \left(\left|\varphi_j \nabla\tilde{u}^{(k)}\right|^p + \left|\left(\tilde{u}^{(k)} - \tilde{u}^{(k)}_{Q_j}\right)\nabla\varphi_j\right|^p\right)$$
$$\lesssim \int_{Q_j} \left|\nabla\tilde{u}^{(k)}\right|^p + \frac{1}{d_j^p}\int_{Q_j} \left|\tilde{u}^{(k)} - \tilde{u}^{(k)}_{Q_j}\right|^p, \tag{65}$$

where the rightmost integral can be estimated via Poincaré's inequality

$$\frac{1}{d_j^p}\int_{Q_j} \left|\tilde{u}^{(k)} - \tilde{u}^{(k)}_{Q_j}\right|^p \lesssim \int_{Q_j} \left|\nabla\tilde{u}^{(k)}\right|^p. \tag{66}$$

Invoking (63), we pick some $z_j \in Q_j^* \cap F$, where $Q_j^* = 12\sqrt{d}Q_j$, in order to bring into play the maximal operator:

$$\int_{Q_j} \left|\nabla \tilde{b}_j^{(k)}\right|^p \lesssim \int_{Q_j} \left(\left|\varphi_j \nabla \tilde{u}^{(k)}\right|^p + \left|\left(\tilde{u}^{(k)} - \tilde{u}_{Q_j}^{(k)}\right) \nabla \varphi_j\right|^p\right) \lesssim \int_{Q_j^*} \left|\nabla \tilde{u}^{(k)}\right|^p$$
$$\leq \left|Q_j^*\right| \fint_{Q_j^*} \left|\nabla \tilde{u}^{(k)}\right|^p \lesssim \left|Q_j\right| M(|\nabla \tilde{u}|^p)(z_j). \quad (67)$$

Now, we capitalize $z_j \in F$ to give

$$\int_\Omega \left|\nabla b_j^{(k)}\right|^p \leq \int_{Q_j} \left|\nabla \tilde{b}_j^{(k)}\right|^p \lesssim \alpha^p \left|Q_j\right|. \quad (68)$$

The corresponding estimate for $\left|b_j^{(k)}\right|$ can be derived similarly, starting from

$$\int_\Omega \left|b_j^{(k)}\right|^p \leq \int_{Q_j} \left|\tilde{b}_j^{(k)}\right|^p = \int_{Q_j} \left|\tilde{u}^{(k)} - \tilde{u}_{Q_j}^{(k)}\right|^p \left|\varphi_j\right|^p$$
$$\lesssim d_j^p \int_{Q_j} \left|\nabla \tilde{u}^{(k)}\right|^p \leq \int_{Q_j} \left|\nabla \tilde{u}^{(k)}\right|^p \quad (69)$$

and proceeding as in (67) and (68). For the third term $b_j^{(k)}/d_{D_k}$ we note that on k-usual cubes $d_{D_k} \geq d_j$ holds and so by (66) and the same argument as in (67) and (68) we get

$$\int_\Omega \left|\frac{b_j^{(k)}}{d_{D_k}}\right|^p \leq \int_{Q_j} \left|\frac{\tilde{b}_j^{(k)}}{d_{D_k}}\right|^p \lesssim \frac{1}{d_j^p} \int_{Q_j} \left|\tilde{u}^{(k)} - \tilde{u}_{Q_j}^{(k)}\right|^p \lesssim \alpha^p \left|Q_j\right|.$$

We turn to the k-boring cubes. Then $\tilde{b}_j^{(k)} = \tilde{u}^{(k)} \varphi_j$ and $d_{D_k} \geq d_j \geq 1$ a.e. on Q_j . By (d) we have

$$\left|\tilde{b}_j^{(k)}\right| + \left|\nabla \tilde{b}_j^{(k)}\right| + \left|\frac{\tilde{b}_j^{(k)}}{d_{D_k}}\right| \leq \left|\tilde{u}^{(k)} \varphi_j\right| + \left|\varphi_j \nabla \tilde{u}^{(k)}\right| + \left|\tilde{u}^{(k)} \nabla \varphi_j\right| + \left|\frac{\varphi_j \tilde{u}^{(k)}}{d_{D_k}}\right|$$
$$\lesssim \left|\tilde{u}^{(k)}\right| + \left|\nabla \tilde{u}^{(k)}\right| + \frac{1}{d_j}\left|\tilde{u}^{(k)}\right| + \left|\frac{\tilde{u}^{(k)}}{d_{D_k}}\right| \quad \left(a.e.\, on\, Q_j\right) \quad (70)$$

and the usual start of play for the maximal operator, following (67) and (68), leads to

$$\int_\Omega \left(\left|b_j^{(k)}\right|^p + \left|\nabla b_j^{(k)}\right|^p + \left|\frac{b_j^{(k)}}{d_{D_k}}\right|^p\right) \leq \int_{Q_j} \left(\left|\tilde{b}_j^{(k)}\right| + \left|\nabla \tilde{b}_j^{(k)}\right| + \left|\frac{\tilde{b}_j^{(k)}}{d_{D_k}}\right|\right)^p$$
$$\lesssim \int_{Q_j} \left(\left|\tilde{u}^{(k)}\right| + \left|\nabla \tilde{u}^{(k)}\right| + \frac{1}{d_j}\left|\tilde{u}^{(k)}\right| + \left|\frac{\tilde{u}^{(k)}}{d_{D_k}}\right|\right)^p$$
$$\lesssim \int_{Q_j} \left(\left|\tilde{u}^{(k)}\right| + \left|\nabla \tilde{u}^{(k)}\right| + \left|\frac{\tilde{u}^{(k)}}{d_{D_k}}\right|\right)^p \lesssim \alpha^p \left|Q_j\right|. \quad (71)$$

Finally, if Q_j is k-special, then again $\tilde{b}_j^{(k)} = \tilde{u}^{(k)} \varphi_j$ and we conclude as in (71) above with one crucial difference: This time we do not absorb the non-Hardy term $\left|\tilde{u}^{(k)}\right|/d_j$ into $\left|\tilde{u}^{(k)}\right|$, but rather we use

$$d_{D_k}(x) = d(x, D_k) \leq diam\left(Q_j\right) + d\left(Q_j, D_k\right) \leq 2d_j\left(x \in Q_j\right), \quad (72)$$

in order to absorb it into the Hardy-term $\left|\tilde{u}^{(k)}\right|/d_{D_k}$.

Step 4: Non-gradient terms of the good function. Let $1 \leq k \leq m$. In this step we prove for almost every $x \in \mathbb{R}^d$ the estimate

$$\left|\tilde{g}^{(k)}(x)\right|^p + \left|\frac{\tilde{g}^{(k)}(x)}{d_{D_k}(x)}\right|^p \lesssim \alpha^p.$$

On F all bad functions $\tilde{b}_j$ vanish. Hence, $\tilde{g} = \tilde{u}$ on this set and the required estimate follows on controlling the left-hand side above by its maximal function. So, we can concentrate on $x \in U$. Denoting by J_u, J_b, and J_s the sets of those $j \in J$ such that Q_j is k-usual, k-boring, and k-special, respectively, we obtain on U that

$$\tilde{g}^{(k)} = \tilde{u}^{(k)} - \sum_{j\in J} \tilde{b}_j^{(k)} = \tilde{u}^{(k)} - \sum_{j\in J_u} \varphi_j \left(\tilde{u}^{(k)} - \tilde{u}_{Q_j}^{(k)}\right) - \sum_{j\in J_b\cup J_s} \varphi_j \tilde{u}^{(k)} = \sum_{j\in J_u} \tilde{u}_{Q_j}^{(k)} \varphi_j,$$

since $\sum_{j\in J} \varphi_j = 1$ on U. Now, let $x \in U$ and let $J_{u,x}$ be the set of those $j \in J_u$ for which x is contained in the k-usual cube Q_j. We recall from (D) that $\# J_{u,x} \leq 12^d$. Due to (d) and Hölder's inequality for sequences we find

$$\begin{aligned}\left|\tilde{g}^{(k)}(x)\right|^p &\leq \left(\sum_{j\in J_{u,x}} \left|\tilde{u}_{Q_j}^{(k)}\right|\right)^p \lesssim 12^{d(p-1)} \sum_{j\in J_{u,x}} \left(⨍_{Q_j} \left|\tilde{u}^{(k)}\right|\right)^p \\ &\lesssim \sum_{j\in J_{u,x}} ⨍_{Q_j} \left|\tilde{u}^{(k)}\right|^p. \end{aligned} \tag{73}$$

Picking again elements $z_j \in 12\sqrt{d}Q_j \cap F$, the same argument we have used several times before, for instance in (67) and (68), provides control on the right-hand side by α^p. This is the first required estimate on U. For the second one involving d_{D_k}, we first observe that if $y \in Q_j$ for some $j \in J_{u,x}$, then since $x \in Q_j$ as well,

$$d_{D_k}(y) \leq diam\left(Q_j\right) + d_{D_k}(x) = d_j + d_{D_k}(x) \leq 2\, d_{D_k}(x)$$

by the defining property of k-usual cubes. Combining this estimate with (73), we conclude as usual,

$$\left|\frac{\tilde{g}^{(k)}(x)}{d_{D_k}(x)}\right|^p \lesssim \sum_{j\in J_{u,x}} ⨍_{Q_j} \left|\frac{\tilde{u}^{(k)}(y)}{d_{D_k}(x)}\right|^p dy \leq 2^p \sum_{j\in J_{u,x}} ⨍_{Q_j} \left|\frac{\tilde{u}^{(k)}(y)}{d_{D_k}(y)}\right|^p dy \lesssim \alpha^p.$$

Step 5: Proofs of (vi) and (viii). In order to estimate $\nabla\tilde{g}$ we have to make sure that the gradient can be pushed through the sum defining $\tilde{g}$. We shall prove on the way the unconditional convergence stated in (viii). To this end, let $1 \leq k \leq m$. Also let $J_0 \subseteq J$ be a finite set. Adopting the notation from Step 4 and arguing similarly to (73), we obtain

$$\begin{aligned}&\left\|\sum_{j\in J_0} \tilde{b}_j^{(k)}\right\|_{W^{1,p}}^p \\ &\quad\lesssim \sum_{j\in J_u\cap J_0} \int_{Q_j} \left(\left|\varphi_j\left(\tilde{u}^{(k)} - \tilde{u}_{Q_j}^{(k)}\right)\right|^p + \left|\varphi_j \nabla\tilde{u}^{(k)}\right|^p + \left|\left(\tilde{u}^{(k)} - \tilde{u}_{Q_j}^{(k)}\right)\nabla\varphi_j\right|^p\right) \\ &\quad+ \sum_{j\in(J_b\cup J_s)\cap J_0} \int_{Q_j} \left(\left|\varphi_j\tilde{u}^{(k)}\right|^p + \left|\varphi_j\nabla\tilde{u}^{(k)}\right|^p + \left|\tilde{u}^{(k)}\nabla\varphi_j\right|^p\right).\end{aligned}$$

Investing the estimates (65), (66), and (69) on k-usual cubes, (70) on k-boring cubes and in addition (72) on k-special cubes, we find

$$\sum_{j\in J_0}\int_{Q_j}\left(\left|\tilde{u}^{(k)}\right|+\left|\nabla\tilde{u}^{(k)}\right|+\left|\frac{\tilde{u}^{(k)}}{d_{D_k}}\right|\right)^p$$
$$=\int_{\mathbb{R}^d}\sum_{j\in J_0}1_{Q_j}(x)\left(\left|\tilde{u}^{(k)}(x)\right|+\left|\nabla\tilde{u}^{(k)}(x)\right|+\left|\frac{\tilde{u}^{(k)}(x)}{d_{D_k}(x)}\right|\right)^p dx.$$

As a consequence of (D) the series $\sum_{j\in J_0} 1_{Q_j}$ converges pointwise to a function bounded everywhere by 12^d. Therefore Lebesgue's theorem implies that the partial sums of $\sum_{j\in J}\tilde{b}_j^{(k)}$ form Cauchy sequences in $W^{1,p}(\mathbb{R}^d)$. The limit is independent of the order of summation again by (D). The same applies to $\sum_{j\in J}\tilde{b}_j^{(k)}$ for any $J'\subseteq J$ and hence we obtain (viii).
Revisiting the calculation above for $J_0=J$ and recalling (61), we find

$$\left\|\sum_{j\in J}\tilde{b}_j^{(k)}\right\|^p_{W^{1,p}(\mathbb{R}^d)}\lesssim\int_{\mathbb{R}^d}\left(\left|\tilde{u}^{(k)}(x)\right|+\left|\nabla\tilde{u}^{(k)}(x)\right|+\left|\frac{\tilde{u}^{(k)}(x)}{d_{D_k}(x)}\right|\right)^p dx$$
$$\lesssim\|u\|^p_{W_D^{1,p}(\Omega)}. \tag{74}$$

We recall from Step 2 that all $\tilde{b}_j^{(k)}$ are contained in $W_{D_k}^{1,p}(\mathbb{R}^d)$. Since the latter is a closed subspace of $W^{1,p}(\mathbb{R}^d)$, it also contains $\sum_{j\in J}\tilde{b}_j^{(k)}$ and $\tilde{g}^{(k)}=\tilde{u}^{(k)}-\sum_{j\in J}\tilde{b}_j^{(k)}$. Restricting to gives $\tilde{g}^{(k)}\in W_{D_k}^{1,p}(\Omega)$, that is, $g\in W_D^{1,p}(\Omega)$. Finally, the estimate in (vi) follows directly from (74)
Step 6: Gradient estimate of the good function. Let $1\le k\le m$. The objective of this step is to prove $|\nabla\tilde{g}^{(k)}(x)|\lesssim\alpha$ for almost every $\in\mathbb{R}^d$. Thanks to (viii) we can compute

$$\nabla\tilde{g}^{(k)}=\nabla\tilde{u}^{(k)}-\sum_{j\in J_u}\left(\varphi_j\nabla\tilde{u}^{(k)}+\left(\tilde{u}^{(k)}-\tilde{u}^{(k)}_{Q_j}\right)\nabla\varphi_j\right)-\sum_{j\in J_b\cup J_s}\left(\varphi_j\nabla\tilde{u}^{(k)}+\tilde{u}^{(k)}\nabla\varphi_j\right)$$

and all sums converge in $L^p(\mathbb{R}^d)$. As in the previous step we write

$$\nabla\tilde{g}^{(k)}=\nabla\tilde{u}^{(k)}-\nabla\tilde{u}^{(k)}\sum_{j\in J}\varphi_j-\tilde{u}^{(k)}\sum_{j\in J}\nabla\varphi_j+\sum_{j\in J_u}\tilde{u}^{(k)}_{Q_j}\nabla\varphi_j. \tag{75}$$

Now, on $F=\mathbb{R}^d\backslash U$ all terms on the right-hand side vanish but the first one and we get

$$\left|\nabla\tilde{g}^{(k)}(x)\right|^p=\left|\nabla\tilde{u}^{(k)}(x)\right|^p\le M(|\nabla\tilde{u}|^p)(x)\le\alpha^p\quad(a.e.\,x\in F).$$

So, we can concentrate on the similar estimate on U. Due to (D) the sum $\sum_{j\in J}\varphi_j$ converges locally in L^1 and by construction the limit is identically 1 on U. Thus, $\sum_{j\in J}\nabla\varphi_j=0$ on U in the sense of distributions and (75) collapses to

$$\nabla\tilde{g}^{(k)}(x)=\sum_{j\in J_u}\tilde{u}^{(k)}_{Q_j}\nabla\varphi_j(x)\qquad\qquad(x\in U).$$

We will not estimate this sum directly. Instead, we define

$$h_u(x):=\sum_{j\in J_u}\tilde{u}^{(k)}_{Q_j}\nabla\varphi_j(x),\qquad h_{s,b}(x):=\sum_{j\in J_s\cup J_b}\tilde{u}^{(k)}_{Q_j}\nabla\varphi_j(x)\quad(x\in U)$$

and aim at proving $\left|h_{s,b}(x)\right|^p\lesssim\alpha^p$ and $\left|h_u(x)+h_{s,b}(x)\right|^p\lesssim\alpha^p$ for almost every $x\in U$. This of course implies the same bound for $h_u=\nabla\tilde{g}^{(k)}$ and the proof will be complete.

As for $h_{s,b}(x)$, we recall from (72) that $d_{D_k}(y) \le 2d_j$ holds for all y in a k-special cube Q_j and that by definition the diameter of a k-boring cube is at least 1. With $J_{b,x}$ and $J_{s,x}$ the sets of those $j \in J$ for which x is contained in the k-boring and k-special cube Q_j, respectively, we obtain in analogy with (73) the bound

$$|h_{s,b}(x)|^p \lesssim \sum_{j\in J_{b,x}} \frac{1}{d_j^p}\left|\tilde{u}_{Q_j}^{(k)}\right|^p + \sum_{j\in J_{s,x}} \frac{1}{d_j^p}\left|\tilde{u}_{Q_j}^{(k)}\right|^p \le \sum_{j\in J_{b,x}} ⨍_{Q_j}\left|\tilde{u}^{(k)}\right|^p + \sum_{j\in J_{s,x}} ⨍_{Q_j}\left|\frac{\tilde{u}^{(k)}(y)}{d_{D_k}(y)}\right|^p dy.$$

The usual maximal operator argument together with (D) provides control by α^p.
Preliminary to the estimate of $h_u(x) + h_{s,b}(x)$ fix an index $j_0 \in J$ such that $\in Q_{j_0}$. For any cube Q_j that contains x as well, we obtain from (E) that

$$\frac{5}{6}d_j \le d(Q_j, F) \le d(x, F) \le d(Q_{j_0}, F) + d_{j_0} \le 5d_{j_0}. \tag{76}$$

The same estimate is true with the roles of j and j_0 interchanged. So, with $Q_{j_0}^* := 14\sqrt{d}Q_{j_0}$ every such cube satisfies $Q_j \subseteq Q_{j_0}^*$. Again denote by J_x the set of all $j \in J$ such that Q_j contains x. Due to $\sum_{j\in J} \nabla\varphi_j = 0$ almost everywhere on U we find

$$h_u(x) + h_{s,b}(x) = \sum_{j\in J_x} \tilde{u}_{Q_j}^{(k)} \nabla\varphi_j(x) = \sum_{j\in J_x} \left(\tilde{u}_{Q_j}^{(k)} - \tilde{u}_{Q_{j_0}^*}^{(k)}\right)\nabla\varphi_j(x)$$

and thus by (d) and Hölder's inequality for sequences

$$|h_u(x) + h_{s,b}(x)|^p \lesssim 12^{d(p-1)} \sum_{j\in J_x} \frac{1}{d_j^p}\left|\tilde{u}_{Q_j}^{(k)} - \tilde{u}_{Q_{j_0}^*}^{(k)}\right|^p.$$

Now, for $j \in J_x$ we have

$$\left|\tilde{u}_{Q_j}^{(k)} - \tilde{u}_{Q_{j_0}^*}^{(k)}\right|^p = \left|⨍_{Q_j} \tilde{u}^{(k)}(y) - \tilde{u}_{Q_{j_0}^*}^{(k)} dy\right|^p - ⨍_{Q_{j_0}^*}\left|\tilde{u}^{(k)}(y) - \tilde{u}_{Q_{j_0}^*}^{(k)}\right|^p dy$$

since $Q_j \subseteq Q_{j_0}^*$ and $d_{j_0} \le 6d_j$ by (76) with the roles of j and j_0 interchanged. By means of Poincaré's inequality (66) on the cube $Q_{j_0}^*$, we finally find

$$\left|\tilde{u}_{Q_j}^{(k)} - \tilde{u}_{Q_{j_0}^*}^{(k)}\right|^p \lesssim diam\left(Q_{j_0}^*\right)^p ⨍_{Q_{j_0}^*}\left|\nabla\tilde{u}^{(k)}\right|^p \lesssim d_j^p ⨍_{Q_{j_0}^*}\left|\nabla\tilde{u}^{(k)}\right|^p,$$

leading to

$$|h_u(x) + h_{s,b}(x)|^p \le \sum_{j\in J_x} ⨍_{Q_{j_0}^*}\left|\nabla\tilde{u}^{(k)}\right|^p.$$

As from now, the estimate by α^p can be completed in the usual manner.
Step 7: Proof of (iii). After all it remains to check $g^{(k)} \in W_{D_k}^{1,\infty}(\Omega)$ for $1 \le k \le m$ with appropriate bound. The statement of Steps 4 and 6 is

$$\left\|\tilde{g}^{(k)}\right\|_{L^\infty(\mathbb{R}^d)} + \left\|\nabla\tilde{g}^{(k)}\right\|_{L^\infty(\mathbb{R}^d)} + \left|\frac{\tilde{g}^{(k)}}{d_{D_k}}\right|_{L^\infty(\mathbb{R}^d)} \lesssim \alpha. \tag{77}$$

So, $\tilde{g}^{(k)}: \mathbb{R}^d \to \mathbb{C}$ is essentially bounded with essentially bounded distributional gradient. Thus, it has a Lipschitz continuous representative $\tilde{g}^{(k)}$ with Lipschitz norm bounded by generic multiple of α, see for example [264] for this classical result. Finally, $\tilde{g}^{(k)}$ vanishes everywhere on D_k since every $x \in D_k$ can be approximated by a sequence along which $\frac{\tilde{g}^{(k)}}{d_{D_k}}$ remains uniformly bounded.

Theorem (5.3.25)[246]: Suppose $\Omega \subseteq \mathbb{R}^d$ is a bounded domain satisfying D, N, and Ω.

(i) If $p_0 \in \mathcal{J}(L) \cap [1,2)$, then $L^{\frac{1}{2}}$ extends to an isomorphism $\mathbb{W}_D^{1,p}(\Omega) \to L^p(\Omega)^m$ for every $p \in (p_0, 2)$. Upper and lower bounds of the extension depend on $[\![p_0]\!], p$, and geometry.

(ii) The result in (i) is optimal in that if $L^{\frac{1}{2}}$ extends to an isomorphism $\mathbb{W}_D^{1,p}(\Omega) \to L^p(\Omega)^m$ for some $p \in [1,2)$, then already $(p,2) \subseteq \mathcal{J}(L)$.

Proof. We assume D, N, and Ω. Necessity is the easy part so let us begin with that.
We borrow an idea from [210]. First of all, according to Theorem (5.3.16) we have $(1,2) \subseteq \mathcal{J}(L)$ if $d = 2$ and $[2_*, 2) \subseteq \mathcal{J}(L)$ if $d \geq 3$. Henceforth we only need to treat the case where $d \geq 3$ and $L^{\frac{1}{2}}$ extends to an isomorphism $\mathbb{W}_D^{1,p} \to L^p(\Omega)^m$ for some $p \in [1, 2_*)$. We need to prove $(p,2) \subseteq \mathcal{J}(L)$.
First, we claim that $L - 1/2$ extends to a bounded operator $L^q(\Omega)^m \to L^{q^*}(\Omega)$ for every $q \in [p,2]$. Indeed, by Riesz–Thorin interpolation it suffices to check the endpoints and – keeping in mind the Sobolev embedding $\mathbb{W}_D^{1,q}(\Omega) \subseteq L^{q^*}(\Omega)^m$ – we obtain the case $q = p$ from the assumption and the case $p = 2$ from Theorem (5.3.6).
This being said, we put $p_0 := p, p_j = p_{j-1}^*$ for $j = 1, \ldots$ and stop at the first number j with $p_j \in (2_*, 2]$. By construction this happens for some $j \geq 1$ and the above applies to $= p_0, \ldots, p_j$. We find for every $t > 0$ the chain of bounded mappings

$$L^{p_0}(\Omega)^m \xrightarrow{L^{-\frac{1}{2}}} L^{p_1}(\Omega)^m \cdots\xrightarrow{L^{-\frac{1}{2}}}\cdots L^{p_j}(\Omega)^m \xrightarrow{e^{-\frac{t}{2L}}} L^2(\Omega)^m \xrightarrow{L^{\frac{j}{2}}e^{-\frac{t}{2L}}} L^2(\Omega)^m,$$

where the second to last arrow with operator norm controlled by $t^{\frac{d}{4}-\frac{d}{2p_j}}$ is due to Proposition (5.3.12).(i) on noting that p_j is an interior point of $\mathcal{J}(L)$. The final arrow with operator norm controlled by $t^{-\frac{j}{2}}$ follows from the bounded H^∞-calculus on $L^2(\Omega)^m$. But the chain above induces e^{-tL} and since $d/p = d/p_j + j$ holds by construction, we have shown its $L^p \to L^2$ boundedness. Proposition (5.3.12) yields $(p,2) \subseteq \mathcal{J}(L)$.
We turn to (i) and claim that it suffices to prove the following key proposition.
Let us first see how the proposition leads to the proof of the first part of Theorem (5.3.25).
Thanks to Theorem (5.3.16) we can guarantee that $\mathcal{J}(L) \cap (1, 2_*)$ is non-empty and so the statement is non-trivial. Let it contain p_0. Due to Proposition (5.3.26) and Theorem (5.3.6) we have at hand (extensions to) bounded operators

$$L^{\frac{1}{2}}: \mathbb{W}_D^{1,p_0}(\Omega) \to L^{p_0,\infty}(\Omega)^m, \qquad L^{\frac{1}{2}}: \mathbb{W}_D^{1,2}(\Omega) \to L^2(\Omega)^m,$$

where $L^{p_0,\infty}$ denotes the usual weak L^{p_0}-space, see [84]. Now, let $p \in (p_0, 2)$ and $1/p = (1-\theta)/p_0 + \theta/2$. By real interpolation this entails boundedness of

$$L^{\frac{1}{2}}: \left(\mathbb{W}_D^{1,p_0}(\Omega), \mathbb{W}_D^{1,2}(\Omega)\right)_{\theta,p} \to \left(L^{p_0,\infty}(\Omega)^m, L^2(\Omega)^m\right)_{\theta,p},$$

see e.g. [84] for background on these notions. Up to equivalent norms the left-hand space is $\mathbb{W}_D^{1,p}(\Omega)$, see [206]. The right-hand space is $L^p(\Omega)^m$ up to equivalent norms, see [84]. Thus,

$$\left\|L^{\frac{1}{2}}u\right\|_{L^p(\Omega)^m} \lesssim \|u\|_{\mathbb{W}_D^{1,p}(\Omega)} \qquad \left(u \in \mathbb{W}_D^{1,2}(\Omega)\right). \tag{78}$$

(In [206] only the case $m = 1$ was considered, but real interpolation interchanges with finite products of spaces by an abstract principle, see [84] or [261].) Since p is an interior point of $\mathcal{J}(L)$, Corollary (5.3.22) provides the estimate reverse to (78). This means that $L^{\frac{1}{2}}$ extends to a one-to-one operator $\mathbb{W}_D^{1,p}(\Omega) \to L^p(\Omega)^m$ with closed range. Furthermore, from Theorem (5.3.6) we know $\mathcal{R}\left(L^{\frac{1}{2}}\right) = L^2(\Omega)^m$. Hence, this extension has dense range and therefore is an isomorphism. We have picked up implicit constants depending on $[\![p_0]\!]$, geometry, and on p. The latter comes in particular from real interpolation.

This completes proof of Theorem (5.3.16) modulo the

Proposition (5.3.26)[246]: For $p \in (L) \cap (1, 2_*)$ the weak-type bound

$$\left|\left\{x \in \Omega: \left|L^{\frac{1}{2}}u(x)\right| > \alpha\right\}\right| \lesssim \frac{1}{\alpha^p}\|u\|_{\mathbb{W}_D^{1,p}}^p \quad \left(\alpha > 0, u \in \mathbb{W}_D^{1,2}(\Omega)\right),$$

with an implicit constant depending on $[\![p]\!]$ and geometry.

Proof. The argument follows the lines of [206] with two essential differences: A different Calderón–Zygmund decomposition and the presence of the technical condition $p < 2_*$. The raison d'être for the latter is to have at our disposal

(i) $L^{p^*} \to L^2$ off-diagonal estimates for $\{e^{-tL}\}_{t>0}$ and

(ii) L^{p^*} boundedness of the $H^\infty(S_\psi^+)$-calculus for L of some fixed angle $\psi \in (0, \pi/2)$.

Indeed, since $p < p^* < 2$ the first property is a consequence of Proposition (5.3.12).(iii) and implicit constants depend only on $[\![p]\!]$ and geometry. Similarly, the second property is due to Theorem (5.3.20) with the same dependence of implicit constants.

To get started, let $\alpha > 0$ and $u \in \mathbb{W}_D^{1,2}(\Omega)$. In particular we have $u \in \mathbb{W}_D^{1,p}(\Omega)$ and within this space we decompose

$$u = g + b, \qquad b = \sum_{j \in J} b_j$$

according to Lemma (5.3.24). Since g is contained in $\mathbb{W}_D^{1,2}(\Omega)$, the above is also a decomposition in that space. In the further course of the proof (i)–(viii) will refer to the respective features of the Calderón–Zygmund decomposition. We then split

$$\left|\left\{x \in \Omega: \left|L^{\frac{1}{2}}u(x)\right| > \alpha\right\}\right| \leq \left|\left\{x \in \Omega: \left|L^{\frac{1}{2}}g(x)\right| > \frac{\alpha}{2}\right\}\right| + \left|\left\{x \in \Omega: \left|L^{\frac{1}{2}}b(x)\right| > \frac{\alpha}{2}\right\}\right|. \tag{79}$$

Step 1: Estimate of the good part. The good function produces an easy-to-handle term. By Hölder's inequality, (iii), and (vi) (or (77) as it were), we have

$$\|g\|_{\mathbb{W}_D^{1,2}}^2 \lesssim \alpha^{2-p}\|g\|_{\mathbb{W}_D^{1,p}}^p \lesssim \alpha^{2-p}\|u\|_{\mathbb{W}_D^{1,p}}^p$$

and the desired bound follows from Tchebychev's inequality and Theorem (5.3.6):

$$\left|\left\{x \in \Omega: \left|L^{\frac{1}{2}}g(x)\right| > \frac{\alpha}{2}\right\}\right| \leq 4/\alpha^2 \left\|L^{\frac{1}{2}}g\right\|_2^2 \lesssim \frac{1}{\alpha^2}\|g\|_{\mathbb{W}_D^{1,2}}^2 \lesssim \frac{1}{\alpha^p}\|u\|_{\mathbb{W}_D^{1,p}}^p.$$

Step 2: Further decomposition of the bad part. We turn to the second term on the right-hand side of (79). Here, we start out with the formula,

$$L^{\frac{1}{2}}b = \frac{2}{\sqrt{\pi}}\int_0^\infty Le^{-t^2L}b\,dt,$$

which is a direct consequence of (48). Hence,

$$\left|\left\{x\in\Omega:\ \left|L^{\frac{1}{2}}b(x)\right|>\frac{\alpha}{2}\right\}\right| = \left|\left\{x\in\Omega: \liminf_{n\to\infty}\left|\frac{2}{\sqrt{\pi}}\int_{2^{-n}}^{\infty} Le^{-t^{2L}}b(x)dt\right|>\frac{\alpha}{2}\right\}\right|$$
$$\leq \liminf_{n\to\infty}\left|\left\{x\in\Omega:\left|\frac{2}{\sqrt{\pi}}\int_{2^{-n}}^{\infty} Le^{-t^2L}b(x)dt\right|>\frac{\alpha}{2}\right\}\right|.$$

Denote the sidelength of Q_j by ℓ_j and write $r_j = 2^\ell$ for the unique value $\ell \in \mathbb{Z}$ that satisfies $2^\ell \leq \ell_j < 2^{\ell+1}$. Then, we split the integral for every $n \in \mathbb{N}$ as

$$\left|\left\{x\in\Omega:\left|\frac{2}{\sqrt{\pi}}\int_{2^{-n}}^{\infty} Le^{-t^2L}b(x)dt\right|>\frac{\alpha}{2}\right\}\right|$$
$$\leq \left|\left\{x\in\Omega:\left|\sum_{j\in J}\int_{2^{-n}}^{r_j\vee 2^{-n}} Le^{-t^2L}b_j(x)dt\right|>\frac{\sqrt{\pi}\alpha}{8}\right\}\right|$$
$$+\left|\left\{x\in\Omega:\left|\sum_{j\in J}\int_{r_j\vee 2^{-n}}^{\infty} Le^{-t^2L}b_j(x)dt\right|>\frac{\sqrt{\pi}\alpha}{8}\right\}\right|, \qquad (80)$$

where sum and integral could be interchanged since on the one hand the sum over the b_j converges in L^{p^*} due to (viii) and Sobolev embeddings (making use of the extension operators as usual) and on the other hand Le^{-t^2L} is bounded on L^{p^*} with norm under control by t^{-2} by the bounded H^∞-calculus. We also note that the b_j are in L^2, see (vii).

Step 3: Estimate of the first term on the right of (80). We may assume $r_j > 2^{-n}$. From Tchebychev's inequality we can infer

$$\left|\left\{x\in\Omega:\left|\sum_{j\in J}\int_{2^{-n}}^{r_j} Le^{-t^2L}b_j(x)dt\right|>\frac{\sqrt{\pi}\alpha}{8}\right\}\right|$$
$$\leq \left|\bigcup_{j\in J}4Q_j\right| + \frac{64}{\pi\alpha^2}\left\|1_{\Omega\setminus\cup_{j\in J}4Q_j}\sum_{j\in J}\int_{2^{-n}}^{r_j} Le^{-t^2L}b_j dt\right\|_2^2. \qquad (81)$$

The union of the cubes $4Q_j$ does not cause any problems since its measure can be controlled by $\|u\|^p_{\mathbb{W}^{1,p}_D}/\alpha^p$, see (v). We start to estimate the leftover L^2 norm by testing against $v \in L^2(\Omega)^m$ with $\|v\|_2 = 1$:

$$\left|\int_\Omega \bar{v}1_{\Omega\setminus\cup_{j\in J}4Q_j}\left(\sum_{j\in J}\int_{2^{-n}}^{r_j} Le^{-t^2L}b_j dt\right)dx\right|$$
$$\leq \sum_{j\in J}\int_{\Omega\setminus 4Q_j}|v|\left|\int_{2^{-n}}^{r_j} Le^{-t^2L}b_j dt\right|dx. \qquad (82)$$

For fixed j we split $\Omega\setminus 4Q_j$ into annuli $C_k(Q_j)\cap\Omega$, where $C_k(Q_j) = 2^{k+1}Q_j\setminus 2^kQ_j$, and apply the Cauchy–Schwarz inequality to give

$$\int_{\Omega\setminus 4Q_j}|v|\left|\int_{2^{-n}}^{r_j} Le^{-t^2L}b_j dt\right|dx$$
$$\leq \sum_{k=2}^{\infty}\|v\|_{L^2(C_k(Q_j)\cap\Omega)}\left\|\int_{2^{-n}}^{r_j} Le^{-t^2L}b_j dt\right\|_{L^2(C_k(Q_j)\cap\Omega)}. \qquad (83)$$

Identifying v with its extension by zero to $\mathbb{R}^d$, we obtain for every $y \in Q_j$ that

$$\|v\|^2_{L^2(C_k(Q_j)\cap\Omega)} \lesssim 2^{dk}\ell_j^d M(|v|^2)(y).$$

To control the other norm on the right-hand side of (83) we recall that as far as off-diagonal estimates are concerned, we have $L^{p^*} \to L^2$ for $\{e^{-tL}\}_{t>0}$. Since we also have $L^2 \to L^2$ for $\{tLe^{-tL}\}_{t>0}$ from Proposition (5.3.11), we obtain $L^{p^*} \to L^2$ for $\{tLe^{-tL}\}_{t>0}$ by composition as in the proof of Proposition (5.3.12).(iv). All implied results contain a statement about implicit constants and so we may note without any pain that

$$\left\|t^2Le^{-t^2L}b_j\right\|_{L^2(C_k(Q_j)\cap\Omega)} \le Ct^{\frac{d}{2}-\frac{d}{p^*}}e^{-\frac{c4^{k-1}r_j^2}{t^2}}\left\|b_j\right\|_{L^{p^*}(\Omega)},$$

where $C, c \in (0,\infty)$ depend on $[\![p]\!]$ and geometry, and we have used that b_j is supported in Q_j, see (ii). From Sobolev embeddings and (iv) we can infer $[\![b_j]\!]_{p^*} \lesssim \alpha\ell_j^{\frac{d}{p}}$ so that altogether

$$\left\|\int_{2^{-n}}^{r_j} Le^{-t^2L}b_j dt\right\|_{L^2(C_k(Q_j)\cap\Omega)} \le \int_{2^{-n}}^{r_j}\left\|Le^{-t^2L}b_j dt\right\|_{L^2(C_k(Q_j)\cap\Omega)}dt$$

$$\lesssim \alpha\ell_j^{\frac{d}{p}}\int_0^{r_j} t^{\frac{d}{2}-\frac{d}{p^*}-2}e^{-\frac{c4^{k-1}r_j^2}{t^2}}dt$$

$$= \frac{1}{2}\alpha\ell_j^{\frac{d}{p}}\left(4^k r_j^2\right)^{\frac{d}{4}-\frac{d}{2p^*}-\frac{1}{2}}\int_{4^k}^{\infty} s^{-\frac{d}{4}+\frac{d}{2p^*}-\frac{1}{2}}e^{-\frac{cs}{4}}ds,$$

the last step being due to a change of variables $s = 4^k r_j^2/t^2$. Abbreviating $\gamma = -d/2 + d/p^* + 1 > 0$ and using $2r_j \ge \ell_j$, we obtain

$$\le \frac{1}{2}\alpha\ell_j^{\frac{d}{2}}2^{-(k-1)\gamma}\int_{4^k}^{\infty} s^{\frac{\gamma}{2}-1}e^{-\frac{cs}{4}}ds \le \frac{1}{2}\alpha\ell_j^{\frac{d}{2}}2^{-(k-1)\gamma}e^{-c4^{k-3}}\int_0^{\infty} s^{\frac{\gamma}{2}-1}e^{-\frac{cs}{8}}ds.$$

The integral in s is finite. Coming back to (83), so far we have for every $y \in Q_j$ that

$$\int_{\Omega\setminus 4Q_j}|v|\left|\int_{2^{-n}}^{r_j} Le^{-t^2L}b_j dt\right|dx \lesssim \alpha\ell_j^d\left(M(|v|^2)(y)\right)^{\frac{1}{2}}\sum_{k=2}^{\infty}2^{\left(\frac{d}{2}-\gamma\right)k}e^{-c4^{k-3}},$$

where the sum over k is finite. So, we can average with respect to y to give

$$\int_{\Omega\setminus 4Q_j}|v|\left|\int_{2^{-n}}^{r_j} Le^{-t^2L}b_j dt\right|dx \lesssim \alpha\int_{Q_j}\left(M(|v|^2)(y)\right)^{\frac{1}{2}}dy.$$

Now, we re-insert this estimate on the right-hand of our starting point (82). Invoking the finite overlap property (ii) of the Q_j, we obtain

$$\left|\int_{\Omega} v1_{\cup j\in J4Q_j}\left(\sum_{j\in J}\int_{2^{-n}}^{r_j} Le^{-t^2L}b_j dt\right)dx\right| \lesssim \alpha\int_{\cup_{j\in J}Q_j}\left(M(|v|^2)(y)\right)^{\frac{1}{2}}dy.$$

From Kolmogorov's inequality [260], (v), and the normalization of v we can infer

$$\int_{\cup_{j\in J}Q_j}\left(M(|v|^2)(y)\right)^{\frac{1}{2}}dy \lesssim \left|\bigcup_{j\in J}Q_j\right|^{\frac{1}{2}}\left\||v|^2\right\|_1^{\frac{1}{2}} \le \left(\sum_{j\in J}|Q_j|\right)^{\frac{1}{2}}\|v\|_2 \lesssim \frac{1}{\alpha^{\frac{p}{2}}}\|u\|^{\frac{p}{2}}_{\mathbb{W}_D^{1,p}}.$$

Going all the way back to the start, we have also bound the L^2 norm occurring in (81) by $\alpha^{2-p}\|u\|^p_{\mathbb{W}_D^{1,p}}$ and therefore completed Step 3.

Step 4: Estimate of the second term on the right of (80). In preparation of the estimate, we define

$$f(z) = \int_1^\infty z e^{-t^2 z} dt \qquad (Re\, z > 0)$$

and note $f \in H_0^\infty(S_\psi^+)$ for any angle $\psi \in \left(0, \frac{\pi}{2}\right)$, see also [206]. We have bounded operators $f(r^2 L)$ for $r > 0$ on L^2, which extend boundedly to L^{p^*} as we have noted right at the start. By the very definition of the functional calculus and Fubini's theorem we can write more conveniently

$$f(r^2 L) = \int_1^\infty r^2 L e^{-t^2 r^2 L} dt = r \int_r^\infty L e^{-t^2 L} dt.$$

Introducing $J_k := \{j \in J : r_j \vee 2^{-n} = 2^k\}$ for $k \in \mathbb{Z}$, we therefore have

$$\sum_{j \in J} \int_{r_j \vee 2^{-n}}^\infty L e^{-t^2 L} b_j dt = \sum_{k \in \mathbb{Z}} \sum_{j \in J_k} \int_{2^k}^\infty L e^{-t^2 L} b_j dt = \sum_{k \in \mathbb{Z}} \sum_{j \in J_k} \frac{1}{2^k} f(4^k L) b_j.$$

We start the actual estimate with Tchebychev's inequality

$$\left| \left\{ x \in \Omega : \left| \sum_{j \in J} \int_{r_j \vee 2^{-n}}^\infty L e^{-t^2 L} b_j(x) dt \right| > \frac{\sqrt{\pi \alpha}}{8} \right\} \right| \leq \frac{8 p^*}{\pi^{\frac{p^*}{2}} \alpha^{p^*}} \left\| \sum_{k \in \mathbb{Z}} \sum_{j \in J_k} \frac{1}{2^k} f(4^k L) b_j \right\|_{p^*}^{p^*}.$$

Since $\sum_{j \in J_k} b_j$ converges in $\mathbb{W}_D^{1,p}(\Omega)$ by (viii) and hence in L^{p^*}, we may write

$$= \frac{8^{p^*}}{\pi^{\frac{p^*}{2}} \alpha^{p^*}} \left\| \sum_{k \in \mathbb{Z}} f(4^k L) \left(\sum_{j \in J_k} 2^{-k} b_j \right) \right\|_{p^*}^{p^*}$$

and obtain from Lemma (5.3.27) below the bound

$$\lesssim \frac{1}{\alpha^{p^*}} \left\| \left(\sum_{k \in \mathbb{Z}} \left| \sum_{j \in J_k} 2^{-k} b_j \right|^2 \right)^{\frac{1}{2}} \right\|_{p_*}^{p^*} = \frac{1}{\alpha^{p^*}} \int_\Omega \left(\sum_{k \in \mathbb{Z}} \left| \sum_{j \in J_k} 2^{-k} b_j(x) \right|^2 \right)^{\frac{p^*}{2}} dx.$$

Here, implicit constants depend only on p^* and a bound for the H^∞-calculus for L on L^{p^*} (of some angle $\psi \in (0, \pi/2)$). We have seen that the latter can be given in terms of $[\![p]\!]$ and geometry. Due to $p^*/2 < 1$ we can continue by

$$\leq \frac{1}{\alpha^{p^*}} \int_\Omega \sum_{k \in \mathbb{Z}} \left(\sum_{j \in J_k} \left| 2^{-k} b_j(x) \right| \right)^{p^*} dx.$$

As a consequence of (ii) the sum in j contains for fixed x at most 12^d non-zero terms. Thus, we can replace the inner ℓ^1-norm by an ℓ^{p^*}-norm at the expense of a constant depending on p and d in order to give

$$\lesssim \frac{1}{\alpha^{p^*}} \int_\Omega \sum_{k \in \mathbb{Z}} \sum_{j \in J_k} 2^{-kp^*} \left| b_j(x) \right|^{p^*} dx \leq \frac{2^{p^*}}{\alpha^{p^*}} \sum_{j \in J} \ell_j^{-p^*} \int_\Omega \left| b_j(x) \right|^{p^*} dx,$$

where we have used $\frac{1}{2} \ell_j \leq r_j \leq 2^k$ for $j \in J_k$. Finally, by Sobolev embeddings, (iv), and (v) we deduce

$$\lesssim \frac{1}{\alpha^{p^*}} \sum_{j\in J} \ell_j^{-p^*} \|b_j\|_{\mathbb{W}_D^{1,p}}^{p^*} \lesssim \sum_{j\in J} \ell^{-p_*^j} |Q_j|^{\frac{p^*}{p}} = \sum_{j\in J} |Q_j| \lesssim \frac{1}{\alpha^p} \|u\|_{\mathbb{W}_D^{1,p}}^p.$$

This completes the proof of the proposition modulo Lemma (5.3.27), which we prove below.

Let us remark that for $\Omega = \mathbb{R}^d$ the following lemma was obtained in [210] for $p < 2$ by duality and a weak type criterion for $p > 2$, which we do not have at our disposal. Later, in [206] it was proved for $p \in (1, \infty)$ through profound $\mathcal{R}$-boundedness techniques for the functional calculi that make it hard to track implicit constants. Here, we present a more elementary approach.

Lemma (5.3.27)[246]: Let $p \in (1, \infty), \Xi \subseteq \mathbb{R}^d$ be a measurable set, and $n \in \mathbb{N}$. Let T be a one-to-one sectorial operator in $L^2(\Xi)^n$ such that for some $\psi \in (0, \pi)$ it holds

$$\|f(T)u\|_p \leq C_\psi \|f\|_\infty \|u\|_p \quad (f \in H^\infty(S_\psi^+), u \in L^2(\Xi)^n \cap L^p(\Xi)^n).$$

Let $f \in H_0^\infty(S_\psi^+)$. Then there is a constant $C \in (0, \infty)$ that depends on f, ψ, such that

$$\left\| \sum_{k\in\mathbb{Z}} f(4^k T) u_k \right\|_p \leq C C_\psi \left\| \left(\sum_{k\in\mathbb{Z}} |u_k|^2 \right)^{\frac{1}{2}} \right\|_p$$

for every sequence $\{u_k\}_{k\in\mathbb{Z}} \subseteq L^2(\Xi)^n \cap L^p(\Xi)^n$ for which the right-hand side is finite.

Proof. The adjoint T^* has the same properties as T with p replaced by its Hölder conjugate q. This follows from the identity $g(T)^* = g^*(T^*)$, where $g^*(z) = \overline{g(\bar{z})}$ and $g \in H^\infty(S_\psi^+)$. Arguing by duality, it suffices to show

$$\left\| \left(\sum_{k\in\mathbb{Z}} \left| f^*(4^k T^*) v \right|^2 \right)^{\frac{1}{2}} \right\|_q \leq C C_\psi \|v\|_q \quad (v \in L^2(\Xi)^n \cap L^q(\Xi)^n). \qquad (84)$$

Let $\{r_k\}_{k\in\mathbb{Z}}$ be a sequence of symmetric independent $\{-1, 1\}$-valued random variables on the unit interval and let $N \in \mathbb{N}$. By orthogonality of the r_k in $L^2(0, 1)$ we have

$$\left\| \left(\sum_{k=-N}^{N} \left| f^*(4^k T^*) v \right|^2 \right)^{\frac{1}{2}} \right\|_q^q = \int_\Xi \left(\int_0^1 \left| \sum_{k=-N}^{N} r_k(s) f^*(4^k T^*) v(x) \right|^2 ds \right)^{\frac{q}{2}} dx.$$

Kahane's inequality [2] allows us to replace the L^2 norm in s by an L^1 norm at the expense of an absolute constant $C \in [1, \infty)$. Then we can apply Jensen's inequality to give

$$\leq C \int_\Xi \int_0^1 \left| \sum_{k=-N}^{N} r_k(s) f^*(4^k T^*) v(x) \right|^q ds\, dx = C \int_0^1 \left\| \sum_{k=-N}^{N} r_k(s) f^*(4^k T^*) v \right\|_q^q ds$$

$$\leq C \sup_{|a_k|\leq 1} \left\| \sum_{k=-N}^{N} a_k f^*(4^k T^*) v \right\|_q^q,$$

where the a_k are complex numbers. The bounded $H^\infty(S_\psi^+)$-calculus for T^* yields

$$\left\|\left(\sum_{k=-N}^{N}\left|f^*(4^kT^*)v\right|^2\right)^{\frac{1}{2}}\right\|_q \leq CC_\psi\|v\|_q\left(\sup_{z\in S_\psi^+}\sum_{k=-\infty}^{\infty}\left|f^*(4^kz)\right|\right).$$

In order to see that this estimate implies (84), let us recall that by assumption there exist $C', s > 0$ such that $|f^*(z)| \leq C' \min\{|z|^s, |z|^{-s}\}$ for all $\in S_\psi^+$. Hence, given z, we choose $k_0 \in \mathbb{Z}$ such that $4^{k_0} \leq |z| < 4^{k_0+1}$ and obtain

$$\sum_{k=-\infty}^{\infty}\left|f^*(4^kz)\right| \leq C'\sum_{k=-\infty}^{\infty}\min\left\{\left|4^kz\right|^s,\left|4^kz\right|^{-s}\right\} \leq C'4^s\sum_{k=-\infty}^{\infty}\min\left\{4^{(k+k_0)s},4^{-(k+k_0)s}\right\}$$
$$\leq \frac{2C'4^{2s}}{4^s-1}$$

by a computation of the geometric series.

To begin with, let us recall that L is the maximal restriction to $L^2(\Omega)^m$ of the isomorphism $\mathcal{L}: \mathbb{W}_D^{1,2}(\Omega) \to \mathbb{W}_D^{-1,2}(\Omega)$.

Lemma (5.3.28)[246]: For $u \in \mathcal{D}\left(L^{\frac{1}{2}}\right)$ and $v \in L^2(\Omega)^m$ the duality formula

$$\left(L^{\frac{1}{2}}u\middle|v\right) = \langle\mathcal{L}u|(L^*)^{-\frac{1}{2}}v\rangle.$$

Proof. It suffices to take u in $\mathcal{D}(L)$ since the latter is a core for $\mathcal{D}\left(L^{\frac{1}{2}}\right) = \mathbb{W}_D^{1,2}(\Omega)$ and $\mathcal{L}: \mathcal{D}\left(L^{\frac{1}{2}}\right) \to \mathbb{W}_D^{-1,2}(\Omega)$ is bounded. In this case $\mathcal{L}u = Lu \in L^2(\Omega)^m$ and the claim follows from the duality formula $\left(L^{\frac{1}{2}}\right)^* = (L^*)^{\frac{1}{2}}$ for the functional calculi.

The subsequent proof was inspired by that of [257].

Theorem (5.3.29)[246]: Suppose $\Omega \subseteq \mathbb{R}^d$ is a bounded domain satisfying Assumptions D, N, and Ω. There exists $\varepsilon(L) > 0$ such that $L^{\frac{1}{2}}$ restricts to an isomorphism $\mathbb{W}_D^{1,p}(\Omega) \to L^p(\Omega)^m$ for every $p \in [2, 2+\varepsilon(L))$. Moreover, $\varepsilon(L)$ depends on ellipticity, dimensions, and geometry. Upper and lower bounds of the restriction import at most an additional dependence on p.

Proof. We pick $\varepsilon(L) > 0$ depending on ellipticity, dimensions, and geometry, such that for $p \in (2, 2+\varepsilon(L))$,

(i) $\mathcal{L}$ restricts to an isomorphism $\mathbb{W}_D^{1,p}(\Omega) \to \mathbb{W}_D^{-1,p}(\Omega)$ and

(ii) $(L^*)^{\frac{1}{2}}$ extends to an isomorphism $\mathbb{W}_D^{1,p'}(\Omega) \to L^{p'}(\Omega)^m$.

The former is rendered possible by Proposition (5.3.15), the latter by Theorem (5.3.25) and Theorem (5.3.16) for L^*. Upper and lower bounds of these isomorphisms import at most an additional dependence on p.

Now, let $u \in \mathbb{W}_D^{1,p}(\Omega)$ and $v \in L^2(\Omega)^m$. By Lemma (5.3.28), (i), and (ii) we have

$$\left|\left(L^{\frac{1}{2}}u\middle|v\right)\right| \leq \|\mathcal{L}u\|_{\mathbb{W}_D^{-1,p}}\left\|(L^*)^{-\frac{1}{2}}v\right\|_{\mathbb{W}_D^{1,p'}} \lesssim \|u\|_{W_D^{1,p}}\|v\|_{p'}.$$

Thus, $L^{\frac{1}{2}}u \in L^p(\Omega)^m$ and $L^{\frac{1}{2}}$ restricts to a bounded operator $\mathbb{W}_D^{1,p}(\Omega) \to L^p(\Omega)^m$. As a restriction of an invertible operator it is already one-to-one. To show that it is also onto, let

$f \in L^p(\Omega)$. There exists $u \in \mathbb{W}_D^{1,2}(\Omega)$ such that $L^{\frac{1}{2}}u = f$. Given an arbitrary $w \in \mathbb{W}_D^{1,2}(\Omega)$, we apply Lemma (5.3.28) with $v = (L^*)^{\frac{1}{2}}w$ and use (i), (ii) to give

$$|\langle \mathcal{L}u|w\rangle| \leq \left\|L^{\frac{1}{2}}u\right\|_p \left\|(L^*)^{\frac{1}{2}}w\right\|_{p'} \lesssim \left\|L^{\frac{1}{2}}u\right\|_p \|w\|_{\mathbb{W}_D^{1,p}} = \|f\|_p \|w\|_{\mathbb{W}_D^{1,p'}}.$$

Thus $\mathcal{L}u \in \mathbb{W}_D^{-1,p}(\Omega)$, which implies $u \in \mathbb{W}_D^{1,p}(\Omega)$ thanks to (i).

We remind that for $p = 2$ the holomorphic dependence of the H^∞-calculus and the square root as stated in parts (i) and (ii) of Theorem (5.3.1) have already been obtained. Compare with Corollaries (5.3.7) and (5.3.8). For p as specified in (i) and (ii) of Theorem (5.3.1) they are at least locally bounded on O as we have proved in Theorems (5.3.20), (5.3.25), and (5.3.29), respectively.

For $p \in (1, \infty)$ let now X_p denote either of the spaces $L^p(\Omega)^m$ and $\mathbb{W}_D^{1,p}(\Omega)$. Then X_p is reflexive and $X_p \cap X_2$ is dense in both X_2 and X_p. Hence, all statements of Theorem (5.3.1) are instances of the subsequent abstract result on vector-valued holomorphic functions. See [248] for general background. We say that two Banach spaces are compatible if they are included in the same linear Hausdorff space.

Lemma (5.3.30)[246]: Let (X_1, X_2) and (Y_1, Y_2) be two pairs of compatible complex Banach spaces. Suppose that $X_1 \cap X_2$ is dense in both X_1 and $X_2, Y_1 \cap Y_2$ is dense in both Y_1 and Y_2, and that Y_2 is reflexive. Let $O \subseteq \mathbb{C}$ be an open set and $f: O \to \mathcal{L}(X_1, Y_1)$ holomorphic. If there is a finite constant C such that

$$\|f(z)x\|_{Y_2} \leq C\|x\|_{X_2} \qquad (z \in O, x \in X_1 \cap X_2),$$

then each $f(z)$ extends from $X_1 \cap X_2$ to a bounded operator $X_2 \to Y_2$, denoted also $f(z)$, and $f: O \to \mathcal{L}(X_2, Y_2)$ is holomorphic.

Proof. The extension $f: O \to \mathcal{L}(X_2, Y_2)$ comes from the assumption and since $X_1 \cap X_2$ is dense in X_2. For clarity let us call it g just in this proof. Our assumption guarantees $\|g(z)\|_{\mathcal{L}(X_2,Y_2)} \leq C$ for all $z \in O$, that is to say, g is bounded.

Next we shall prove that the intersection $Y_1^* \cap Y_2^*$ has a meaning and is a dense subspace of Y_2^*. Since Y_1 and Y_2 are compatible, we can consider their sum

$$Y_1 + Y_2 = \{y_1 + y_2 : y_1 \in Y_1, y_2 \in Y_2\}, \qquad \|y\|_{Y_1+Y_2} = \inf_{\substack{y_1 \in Y_1, y_2 \in Y_2 \\ y = y_1 + y_2}} \|y_1\|_{Y_1} + \|y_2\|_{Y_2}.$$

This is again a Banach space [84]. Since $Y_1 \cap Y_2$ is dense in both Y_1 and Y_2, it is also dense in $Y_1 + Y_2$. This justifies to interpret, by restriction of functionals, the inclusions

$$(Y_1 + Y_2)^* \subseteq Y_2^* \quad \text{and} \quad (Y_1 + Y_2)^* \subseteq Y_1^* \cap Y_2^*$$

within the ambient space $(Y_1 \cap Y_2)^*$. The first inclusion is dense since Y_2 is reflexive: Any functional on Y_2^* that annihilates $(Y_1 + Y_2)^*$ is given by evaluation at some element of Y_2, which therefore has to vanish in $Y_1 + Y_2$. The second one is even an equality: Every $y^* \in Y_1^* \cap Y_2^*$ satisfies $|y^*(y)| \leq \max\{\|y^*\|_{Y_1^*}, \|y^*\|_{Y_2^*}\} \|y\|_{Y_1+Y_2}$ for all $y \in Y_1 \cap Y_2$ and hence extends by density to a functional on $Y_1 + Y_2$. The intermediate claim follows from these two observations. By the dense inclusions just alluded to, we can compute the norm of any $T \in \mathcal{L}(X_2, Y_2)$ as

$$\|T\|_{\mathcal{L}(X_2,Y_2)} = \sup_{\substack{x \in X_1 \cap X_2 \\ \|x\|_{X_2} = 1}} \sup_{\substack{y^* \in Y_1^* \cap Y^{*2} \\ \|y^*\|_{Y_2^*} = 1}} \left|\langle y^*|Tx\rangle_{Y_2^*, Y_2}\right|.$$

Since g is bounded, the weak-strong principle for holomorphic functions entails that holomorphy of g is equivalent to holomorphy of all functions $\longmapsto \langle y^* |\, g(z)x \rangle_{Y_2^*, Y_2}$, where $x \in X_1 \cap X_2$ and $y \in Y_1^* \cap Y_2^*$, see [248]. But the latter follows from the holomorphy of f since we have

$$\langle y^* | g(z)x \rangle_{Y_2^*, Y_2} = \langle y^* | f(z)x \rangle_{Y_1^*, Y_1}$$

by construction.

Chapter 6
Pointwise Estimates and Hastings Additivity

We show that the ratio between the probability density and the standard Gaussian density in E is very close to 1 in large parts of E. Here $c > 0$ is a universal constant. This complements a recent result by the second named author, where the total variation metric between the densities was considered. We show that Hastings's counterexample to the additivity of minimal output von Neumann entropy can be readily deduced from a sharp version of Dvoretzky's theorem. For any $2 < p < \infty$ and every n-dimensional subspace X of L_p, the Euclidean space ℓ_2^k can be $(1 + \varepsilon)$-embedded into X with $k \geq c_p \min\left\{\varepsilon^2 n, (\varepsilon n)^{\frac{2}{p}}\right\}$, where $c_p > 0$ is a constant depending only on p. This improves upon the previously known estimate due to Figiel, Lindenstrauss and Milman.

Section (6.1): Marginals of Convex Bodies

Suppose X is a random vector in $\mathbb{R}^n$ that is distributed uniformly in some convex set $K \subset \mathbb{R}^n$. For a subspace $E \subset \mathbb{R}^n$ we denote by Proj_E the orthogonal projection operator onto E in $\mathbb{R}^n$. The central limit theorem for convex bodies [289],[290] asserts that there exists a subspace $E \subset \mathbb{R}^n$, with $\dim(E) > n^c$, such that the random vector $\text{Proj}_E(X)$ is approximately Gaussian, in the total variation sense. This means that for a certain Gaussian random vector Γ in the subspace E,

$$\sup_{A\subseteq E} |\mathbb{P}\{Proj_E(X) \in A - \mathbb{P}\{\Gamma \in A\}| \leq \frac{C}{n^c}, \quad (1)$$

where the supremum runs over all measurable subsets $A \subseteq E$. Here, the letters $c, C, c_1, C_2, c', \tilde{C}$, etc. denote some positive universal constants, whose value may change from one appearance to the next. The total variation estimate (1) implies that the density of $\text{Proj}_E(X)$ is close to the density of Γ in the L^1-norm. We observe that a stronger conclusion is within reach: One may deduce that the ratio between the density of $\text{Proj}_E(X)$ and the density of Γ deviates from 1 by no more than Cn^{-c}, in the significant parts of the subspace E.

We introduce some notation. Write $|\cdot|$ for the standard Euclidean norm in $\mathbb{R}^n$. A random vector Z in $\mathbb{R}^n$ is isotropic if the following normalization holds:

$$\mathbb{E}Z = 0, \qquad \text{Cov}(Z) = Id \quad (2)$$

where $\text{Cov}(Z)$ stands for the covariance matrix of Z, and Id is the identity matrix. The Grassman manifold G^n, of all -dimensional subspaces of $\mathbb{R}^n$ carries a unique rotationally-invariant probability measure $\mu_{n},$. Whenever we say that E is a random -dimensional subspace in $\mathbb{R}^n$, we relate to the above probability measure $\mu_{n},$. Under the additional assumption that the random vector X is isotropic, the subspace E for which $\text{Proj}_E(X)$ is approximately Gaussian may be chosen at random, and (1) will hold with high probability [289],[290].

A function $f : \mathbb{R}^n \to [0, \infty)$ is log-concave if $\log f : \mathbb{R}^n \to [-\infty, \infty)$ is a concave function. The characteristic function of a convex set is log-concave. Throughout the entire discussion, the requirement that X be distributed uniformly in a convex body could have been relaxed to the weaker condition, that X has a log-concave density.

Note that almost the entire mass of a standard ℓ -dimensional Gaussian distribution is contained in a ball of radius $10\sqrt{\ell}$ about the origin. Therefore, (78) easily implies the total variation bound mentioned above. The history of the central limit theorem for convex bodies goes back to the conjectures and results of Brehm and Voigt [286] and Anttila, Ball and Perissinaki [284], see [289]. The case $\ell = 1$ of Theorem (6.1.9) was proved in [290] using the moderate deviation estimates of Sodin [295]. The generalization to higher dimensions is the main contribution. See also [285] and [1].

The basic idea of the proof of Theorem (6.1.9) is the following. It is shown in [290], using concentration techniques, that the density of $\mathrm{Proj}_E(X + Y)$ is pointwise approximately radial, where Y is an independent small Gaussian random vector. It is furthermore proved that the random vector $X + Y$ is concentrated in a thin spherical shell. We combine these facts to deduce, that the density of $\mathrm{Proj}_E(X + Y)$ is not only radial, but in fact very close to the Gaussian density in E. We show that the addition of the Gaussian random vector Y is not required. That is, we prove that when a log-concave density convolved with a small Gaussian is almost Gaussian—then the original density is also approximately Gaussian.

For a dimension n and $v > 0$ we write

$$\gamma_n[v](x) = \frac{1}{(2\pi v)^{n/2}} \exp\left(-\frac{|x|^2}{2v}\right) \ (x \in \mathbb{R}^n)\,. \tag{3}$$

That is, $\gamma_n[v]$ is the density of a Gaussian random vector in $\mathbb{R}^n$ with mean zero and covariance matrix $v\,Id$. Let X be an isotropic random vector with a log-concave density in $\mathbb{R}^n$, and let Y be an independent Gaussian random vector in $\mathbb{R}^n$ whose density is $\gamma_n[n^{-\alpha}]$, for a parameter α to be specified later on. Denote by f_{X+Y} the density of the random vector $X + Y$. Our first step is to show that the density of the projection of $X + Y$ onto a typical subspace is pointwise approximately Gaussian. We follow the notation of [290]. For an integrable function $f : \mathbb{R}^n \to [0, \infty)$, a subspace $E \subseteq \mathbb{R}^n$ and a point $x \in E$ we write

$$\pi_E(f)(x) = \int_{x+E^\perp} f(y)dy, \tag{4}$$

where $x + E^\perp$ is the affine subspace orthogonal to E that passes through the point x. In other words, $\pi_E(f): E \to [0, \infty)$ is the marginal of f onto E. The group of all orthogonal transformations of determinant one in $\mathbb{R}^n$ is denoted by $SO(n)$. Fix a dimension and a subspace $E_0 \subset \mathbb{R}^n$ with $\dim(E_0) = \ell$. For $x_0 \in E_0$ and a rotation $U \in SO(n)$, set

$$M_{f,E_0,x_0}(U) = \log \pi_{E_0}(f \circ U)(x_0). \tag{5}$$

Define

$$M_{(|x_0|)} = \int_{SO(n)} M_{f_{X+Y},E_0,x_0}(U) d\mu_n(U), \tag{6}$$

where μ_n stands for the unique rotationally-invariant Haar probability measure on $SO(n)$. Note that $M_{(|x_0|)}$ is independent of the direction of x_0, so it is well defined. We learned in [290] that the function $U \to M_{f_{X+Y},E_0,x_0}(U)$ is highly concentrated with respect to U in the special orthogonal group $SO(n)$, around its mean value $M_{(|x_0|)}$. This implies that the function $\pi_E(f_{X+Y})$ is almost spherically symmetric, for a typical subspace E. This information is contained in our next lemma, which is equivalent to [290].

Lemma (6.1.1)[282]: Let $1 \le \ell \le n$ be integers, let $0 < \alpha < 10^5$ and denote $\lambda = \frac{1}{5\alpha+20}$. Assume that $\ell \le n^\lambda$. Suppose that X is an isotropic random vector with a log-concave density and that Y is an independent random vector with density $\gamma_n[n^{-\alpha\lambda}]$.

Denote the density of $X + Y$ by f_{X+Y} . Let $E \in G_{n,\ell}$ be a random subspace. Then, with probability greater than $1 - Ce^{-cn^{1/10}}$ of selecting E, we have

$$\left|\log\pi_E(f_{X+Y})(x) - M_{(|x|)}\right| \leq Cn^{-\lambda}, \tag{7}$$

for all $x \in E$ with $|x| \leq 5n^{\lambda/2}$. Here $c, C > 0$ are universal constants.

Proof. We need to follow in [290], choosing for instance, $u = \frac{9}{10}$,$\lambda = \frac{1}{5\alpha+20}$,$k = n^{\lambda}$ and $\eta = 1$. Throughout the argument in [290], it was assumed that the dimension of the subspace is exactly $k = n^{\lambda}$, while in the present version of the statement, note that it could possibly be smaller, i.e., $\ell \leq k$ (note also that here, k need not be an integer). [290], allowing the dimension of the subspace we are working with to be smaller than k, noting that the reduction of the dimension always acts in our favor.

We the original argument in [290] for further details.

See show that $M(|x|)$ behaves approximately like $\log\gamma_n[1 + n^{-\alpha\lambda}](x)$. Once we prove this, it would follow from the above lemma that the density of $X + Y$ is pointwise approximately Gaussian. Next we explain why no serious harm is done if we take the logarithm outside the integral in the definition of $M(|x|)$. Denote, for $x \in E_0$,

$$\widetilde{M}(|x|) = \int_{SO(n)} \pi_{E_0}(f_{X+Y} \circ U)(x)d\mu n(U). \tag{8}$$

Lemma (6.1.2)[282]: Under the notation and assumptions of Lemma (6.1.1), for $|x| \leq 5n^{\lambda/2}$ we have

$$0 \leq \log\widetilde{M}(|x|) - M(|x|) \leq \frac{C}{n^{15}}, \tag{9}$$

where $C > 0$ is a universal constant.

Proof. Recall that $E_0 \subset \mathbb{R}^n$ is some fixed -dimensional subspace with n_{λ}. Fix $x_0 \in E_0$ with $|x_0| \leq 5n^{\lambda/2}$. Lemma 3.1 of [290] states that for any $U_1, U_2 \in SO(n)$,

$$\left|M_{f_{X+Y},E_0,x_0}(U_1) - M_{f_{X+Y},E_0,x_0}(U_2)\right| \leq C_0 n^{\lambda(2\alpha+2)} \cdot d(U_1, U_2), \tag{10}$$

where $d(U_1, U_2)$ stands for the geodesic distance between U_1 and U_2 in $SO(n)$. As mentioned before, Lemma 3.1 is proved in [290] under the assumption that the dimension of the subspace E_0 is exactly n^{λ}. In our case, the dimension might be smaller than n^{λ}, but a close inspection of the proofs in [290] reveals that the reduction of the dimension can only improve the estimates. Hence (10) holds true.

We apply the Gromov–Milman concentration inequality on $SO(n)$, quoted as Proposition 3.2 in [290], and conclude from (10) that for any $\varepsilon > 0$,

$$\mu_n\left\{U \in SO(n); \left|M_{fX+Y,E_0,x_0}(U) - M(|x_0|)\right| \geq \varepsilon\right\} \leq \bar{C}\exp\left(-\frac{\bar{c}n\varepsilon^2}{L^2}\right), \tag{11}$$

with $L = C_0 n^{\lambda(2\alpha+2)}$. That is, the distribution of

$$F(U) = \frac{\sqrt{n}}{L}\left(M_{f_{X+Y},E_0,x_0}(U) - M(|x_0|)\right)(U \in SO(n))$$

on $SO(n)$ has a subgaussian tail. Note also that $\int_{SO(n)} F(U)d\mu_n(U) = 0$. A standard computation shows for any $p \geq 1$,

$$\int_{SO(n)} F^p(U)d\mu_n(U) \leq \left(C'\sqrt{p}\right)^p, \tag{12}$$

where C' is a universal constant. Hence, for any $0 < t \leq C_0$,

$$\int_{SO(n)} \exp\big(tF(U)\big)\, d\mu_n(U)$$
$$\leq 1 + t\int_{SO(n)} F(U)d\mu_n(U) + \sum_{i=2}^{\infty} \big(C'\sqrt{i}\big)^i \frac{t^i}{i!}$$
$$\leq 1 + \sum_{i=2}^{\infty} \frac{\big(\tilde{C}t^2\big)^{i2}}{\left\lfloor \frac{i}{2} \right\rfloor !}\ 1 + \left(\sqrt{C_0^2\tilde{C}} + 1\right)\sum_{j=1}^{\infty} \frac{\big(\tilde{C}t^2\big)^j}{j!}\sum_{j=0}^{\infty} \frac{\big(\tilde{C}t^2\big)^j}{j!} = \exp\big(\tilde{C}t^2\big)\,. \quad (13)$$

The left-hand side of (9) follows by Jensen's inequality. We use (13) for the value

$$t = \frac{L}{\sqrt{n}} = C_0 n^{\frac{2\alpha+2}{5\alpha+20}-\frac{1}{2}} \leq C_0 n^{-\frac{1}{10}} \leq C_0,$$

to conclude that

$$\widetilde{M}(|x_0|)\exp\big(M(|x_0|)\big) = \frac{\int_{SO(n)} \exp\Big(M_{f_{X+Y},E_0,x_0}(U)\Big)d\mu_n(U)}{\exp\big(M(|x_0|)\big)}$$
$$= \int_{SO(n)} \exp(M_{f_{X+Y},E_0,x_0}(U) - M(|x_0|)d\mu n(U)\exp\big(\hat{C}n^{-1/5}\big)\ .$$

Taking logarithms of both sides completes the proof.

Let $X, Y, \alpha, \lambda, \ell$ be as in Lemma (6.1.1). We choose a slightly different normalization. Define

$$Z = \frac{X + Y}{\sqrt{1 + n^{-\lambda\alpha}}}\ , \quad (14)$$

and denote by f_Z the corresponding density. Clearly f_Z is isotropic and log-concave. Next we define, for $x \in E_0$,

$$\widetilde{M}_1(|x|) := \int_{SO(n)} \pi_{E_0}(f_Z \circ U)(x)d\mu_n(U). \quad (15)$$

We show that the following estimate holds:

$$\left|\frac{\widetilde{M}_1(|x|)}{\gamma_\ell[1](x)} - 1\right| < C_1 n^{-c_1} \quad (16)$$

for all $x \in \mathbb{R}$ with $|x| < c_2 n^{c_2}$ for some universal constants $C_1, c_1, c_2 > 0$. We write $S^{n-1} = \{x \in \mathbb{R}^n;\ |x| = 1\}$, the unit sphere in $\mathbb{R}^n$. Define:

$$\tilde{f}_Z(x) = S^{n-1} f_Z(|x|\theta)\ d\sigma_n(\theta) = \int_{SO(n)} f_Z(Ux)d\mu_n(U)(x \in \mathbb{R}^n) \quad (17)$$

where σ_n is the unique rotationally-invariant probability measure on S^{n-1}. Since $\tilde{f}_Z$ is spherically symmetric, we shall also use the notation $\tilde{f}_Z(|x|) = \tilde{f}_Z(x)$. Clearly, for any $x \in E_0$,

$$\widetilde{M}_1(|x|) = \int_{SO(n)} \pi_{E_0}(f_Z \circ U)(x)d\mu_n(U) = \int_{SO(n)} \pi_{E_0}\big(\tilde{f}_Z \circ U\big)(x)d\mu_n(U)$$
$$= \pi_{E_0}\big(\tilde{f}_Z\big)(x). \quad (18)$$

We will use the following thin-shell estimate, proved in [290].

Proposition (6.1.3)[282]: Let $n \geq 1$ be an integer and let X be an isotropic random vector in $\mathbb{R}^n$ with a log-concave density. Then,

$$\mathbb{P}\left\{\left|\frac{|X|}{\sqrt{n}} - 1\right| \geq \frac{1}{n^{1/15}}\right\} < C\exp\big(-cn^{1/15}\big) \quad (19)$$

where $C, c > 0$ are universal constants. Applying the above for f_Z, denoting $\varepsilon = n^{-1/15}$, and defining

$$A = x \in \mathbb{R}^n;\ \sqrt{n}(1 - \varepsilon) \leq |x|\sqrt{n}(1 + \varepsilon)\},$$

we get,

$$\int_A f_Z(x)dx > 1 - Ce^{-cn^{1/15}}. \tag{20}$$

From the definition of $\tilde{f}_Z$, it is clear that the above inequality also holds when we replace f_Z with $\tilde{f}_Z$. In other words, if we define

$$g(t) = t^{n-1}\omega_n \tilde{f}_Z(t)(t \geq 0) \tag{21}$$

where ω_n is the surface area of the unit sphere S^{n-1} in $\mathbb{R}^n$, and use integration in polar coordinates, we get

$$1 \geq \int_{\sqrt{n}(1-\varepsilon)}^{\sqrt{n}(1+\varepsilon)} g(t)dt > 1 - Ce^{-cn^{1/15}}. \tag{22}$$

We apply the methods from Sodin's [295] in order to prove a generalization of [295], for a multi-dimensional marginal rather than a one-dimensional marginal. We estimate will be rather crude, but suitable for our needs.

Denote by $\sigma_{n,r}$ the unique rotationally-invariant probability measure on the Euclidean sphere of radius r around the origin in $\mathbb{R}^n$. A standard calculation shows that the density of an ℓ-dimensional marginal of $\sigma_{n,r}$ is given by the following formula:

$$\psi_{n,\ell,r}(x) = \psi_{n,\ell,r}\ |x| := \Gamma_{n,\ell}\frac{1}{r^\ell}\left(1 - \frac{|x|^2}{r^2}\right)^{\frac{n-\ell-2}{2}} 1_{[-r,r]}\ |x| \tag{23}$$

where

$$\Gamma_{n,\ell} = \left(\frac{1}{\sqrt{\pi}}\right)^\ell \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-\ell}{2}\right)} \tag{24}$$

and where 1[−r,r] is the characteristic function of the interval $[-r, r]$. (see for example [287] on Remark 2.10). When $\ell \ll \sqrt{n}$ we have $\Gamma_{n,}\left(\frac{2\pi}{n}\right)^{\ell/2} \approx 1$. By the definition (21) of g, and since $\tilde{f}_Z$ is spherically symmetric, we may write

$$\pi_{E_0}\left(\tilde{f}_Z\right)(x) = \int_0^\infty \psi_{n,\ell,r}\left(|x|\right) g(r)dr\ (x \in E_0). \tag{25}$$

Indeed, the measure whose density is $\tilde{f}_Z$ equals $\int_0^\infty g(r)\sigma_{n,r}\,dr$, hence its marginal onto E_0 has density $x \to \int_0^\infty \psi_{n,\ell,r}(x)g(r)dr$. We will show that the above density is approximately Gaussian for $x \in E_0$ when $|x|$ is not too large. But first we need the following technical lemma.

Lemma (6.1.4)[282]: Let g be the density defined in (21), and suppose that $n \geq C'$ and $\ell \leq n^{1/20}$. For $\varepsilon = n^{-1/15}$ denote $U = \left\{t > 0;\ t(1 + \varepsilon)\sqrt{n}\ or\ t > (1 + \varepsilon)\sqrt{n}\right\}$. Then,

$$\int_U t^{-\ell} g(t)dt < C' \exp\left(-c'n^{1/15}\right). \tag{26}$$

Here, $c', C' > 0$ are universal constants.

Proof. Define for convenience,

$$h(t) = t^{-\ell} g(t). \tag{27}$$

Denote

$$A = \left[0, \frac{1}{n^2}\right], B = \left[\frac{1}{n^2}, \sqrt{n}(1-\varepsilon)\right] \cup \left[\sqrt{n}(1+\varepsilon), \infty\right),$$

and write

$$\int_U h(t)dt = \int_A h(t)dt + \int_B h(t)dt. \tag{28}$$

We estimate the two terms separately. For $t > \frac{1}{n^2}$ we have

$$h(t) \leq n^{2\ell} g(t) = e^{2\ell \log n} g(t). \tag{29}$$

Thus we can estimate the second term as follows:

$$\int_B h(t)dt \leq e^{2\ell \log n} \int_B g(t)dt < e^{2\ell \log n} Ce^{-cn^{1/15}} < Ce^{\frac{1}{2}cn^{1/15}}, \tag{30}$$

where for the second inequality we apply the reformulation (22) of Proposition (6.1.3) (recall that $\varepsilon = n^{-1/15}$ and that $n^{1/20}$). To estimate the first term on the right-hand side of (28), we use the fact that f_Z is isotropic and log concave, so we can use a crude bound for the isotropic constant (see e.g. [293] in Theorem 5.14(e)] or [288] in Corollary 4.3]) which gives $\sup_{\mathbb{R}^n} f_Z < e^{n \log n}$,thus, also $\sup_{\mathbb{R}^n} \tilde{f}_Z < e^{n \log n}$. Hence we can estimate

$$\int_A h(t)dt = \int_0^{1n^2} t^{-\ell} g(t)dt = \int_0^{1n^2} t^{n-\ell-1} \omega_n \tilde{f}_Z(t)dt < n^{-2(n-\ell)} \omega_n \sup \tilde{f}_Z$$
$$< e^{-1.5n \log n + n \log n} < e^{-n}, \tag{31}$$

as $\omega_n < C$. The combination of (30) and (31) completes the proof.

We show that the marginals of f˜ Z are approximately Gaussian. Note that by (18) and (25),

$$\left|\frac{\widetilde{M}_1(|x|)}{\gamma[1](x)} - 1\right| = \left|\frac{\int_0^\infty \psi_{n,\ell,r}(|x|) g(r) dr}{\gamma_\ell[1](x)} - 1\right|. \tag{32}$$

Our desired bound (16) is contained in the following lemma.

Lemma (6.1.5)[282]: Let $1 \leq \ell \leq n$ be integers, with $n \leq C$ and $n^{1/20}$. Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be a function that satisfies (22) and (26). Then we have,

$$\left|\frac{\int_0^\infty \psi_{n,\ell,r}(|x|) g(r) dr}{\gamma_\ell[1](x)} - 1\right| < Cn^{-1/60} \tag{33}$$

for all $x \in \mathbb{R}$ with $|x| < 2n^{1/40}$ where $C > 0$ is a universal constant.

Proof. We begin by using a well-known fact, that follows from a straightforward computation using asymptotics of -functions: for $|x| < n^{1/8}$,

$$\left|\frac{\psi_{n,\ell,\sqrt{n}}(|x|)}{\gamma[1](x)} - 1\right| = \left|\left(\frac{2\pi}{n}\right)^{\ell 2} \Gamma_{n,\ell} \frac{\left(1 - \frac{|x|^2}{n}\right)^{(n-\ell-2)/2}}{e - |x|^2/2} - 1\right| \leq \frac{C}{\sqrt{n}}. \tag{34}$$

(We omit the details of the simple computation. An almost identical computation is done, for example, in [295] see Lemma 1. Note that in addition to the computation there, we have to use, e.g., Stirling's formula to estimate the constants ε_n.) Using the above fact (34), we see that it suffices to prove the following inequality:

$$\left|\frac{\psi_{n,\ell,r}(|x|) g(r)\, dr}{\psi_{n,\ell,\sqrt{n}}(x)} - 1\right| < Cn^{-\frac{1}{60}} \tag{35}$$

for all $x \in \mathbb{R}$ with $|x| < 2n^{1/40}$. To that end, fix $x_0 \in \mathbb{R}$ with $|x_0| < 2n^{1/20}$, define

$$A = \left[\sqrt{n}\ 1 - n^{-\frac{1}{15}}\ , \sqrt{n}\left(1 + n^{-\frac{1}{15}}\right)\right], B = [0, \infty) \setminus A,$$

and write

$$\int_0^\infty \psi_{n,\ell,r}\ |x_0|\ g(r)dr = \int_A \psi_{n,\ell,r}\ |x_0|\ g(r)dr + \int_B \psi_{n,\ell,r}\ |x_0|\ g(r)dr. \quad (36)$$

We estimate the two terms separately. For the second term, we have,

$$\int_B \psi_{n,\ell,r}\ |x_0|\ g(r)dr = \Gamma_{n,\ell} \int_B \frac{1}{r^\ell}\left(1 - \frac{|x_0|^2}{r^2}\right)^{\frac{n-\ell-2}{2}} 1_{[-r,r]}\ |x_0|\ g(r)dr$$

$$< \Gamma_{n,\ell} \int_B \frac{1}{r^\ell}\, g(r)dr < \Gamma_{n,\ell}\, Ce^{-cn^{\frac{1}{15}}}, \quad (37)$$

where the last inequality follows from (26). Therefore,

$$\int_B \psi_{n,\ell,r}(|x_0|)g(r)dr\, \psi_{n,\ell,\sqrt{n}}(|x_0|) < \frac{Ce^{-cn^{1/15}}}{\left(\frac{1}{\sqrt{n}}\right)^\ell \left(1 - \frac{|x_0|^2}{n}\right)^{\frac{n-l-2}{2}}}$$

$$< Ce^{-cn^{1/15}}|x_0|^2 + \frac{1}{2}\ell \log\ n < Ce^{-n^{1/20}}. \quad (38)$$

To estimate the first term on the right-hand side of (36), we will show that the following inequality holds:

$$\left|\frac{\int_A \psi_{n,\ell,r}(|x_0|)g(r)\, dr}{\psi_{n,\ell,\sqrt{n}}(|x_0|)} - 1\right| < Cn^{-\frac{1}{60}} \quad (39)$$

for some constant $C > 0$. For $r > 0$ such that $\frac{|x_0|^2}{r^2} < \frac{1}{2}$, we have,

$$\left|\frac{d}{dr}\log\psi_{n,\ell,r}(|x_0|)\right| = \left|-\frac{\ell}{r} + (n - \ell - 2)\frac{|x_0|^2}{r^3}\frac{1}{\left(1 - \frac{|x_0|^2}{r^2}\right)}\right|$$

$$< \frac{\ell}{r} + 2n\frac{|x_0|^2}{r^3}. \quad (40)$$

Recalling that $|x_0| < 2n^{1/40}$ and $n^{1/20}$, the above estimate gives, for all $r \in [\frac{1}{2}\sqrt{n}, \frac{3}{2}\sqrt{n}]$,

$$\left|\frac{d}{dr}\log\psi_{n,\ell,r}(|x_0|)\right| < n^{\frac{1}{20}-\frac{1}{2}} + 16\, n^{1+\frac{1}{20}-\frac{3}{2}} < Cn^{-\frac{9}{20}} \quad (41)$$

which gives for $r \in \left[\frac{1}{2}\sqrt{n}, \frac{3}{2}\sqrt{n}\right]$,

$$\left|\frac{\psi_{n,\ell,r}(|x_0|)}{\psi_{n,\ell,\sqrt{n}}(|x_0|)} - 1\right| < Cn^{-\frac{9}{20}}\left[r - \sqrt{n}\right]. \quad (42)$$

Recall that for $r \in A$ we have $\left[r - \sqrt{n}\right] \leq n^{13/60}$.Hence the last estimate yields

$$\left|\frac{\psi_{n,\ell,r}(|x_0|)g(r)\, dr}{\psi_{n,\ell,\sqrt{n}}(|x_0|)\int_A g(r)\, dr} - 1\right| < Cn^{-\frac{9}{20}}n^{\frac{13}{30}} = Cn^{-\frac{1}{60}}. \quad (43)$$

Combining the last inequality with (22) ,we get

$$\left|\frac{\int_A \psi_{n,\ell,r}(|x_0|)g(r)\, dr}{\psi_{n,\ell,\sqrt{n}}(|x_0|)} - 1\right| < \tilde{C}n^{-cn^{\frac{1}{15}}} + Cn^{-\frac{1}{60}} < C'n^{-\frac{1}{60}}. \quad (44)$$

From (38) and (44) we deduce (35), and the lemma is proved.

Recall the definitions (8) and (15) of $\widetilde{M}(|x|)$ and $\widetilde{M}_1(|x|)$; the only difference is the normalization of $X + Y$. By an easy scaling argument, we deduce from (32) and Lemma (6.1.5) that when $n \geq C$,

$$\left|\frac{\widetilde{M}(|x|)}{\gamma_\ell[1 + n^{-\lambda\alpha}](x)} - 1\right| < C_1 n^{-\frac{1}{60}} \tag{45}$$

for all $x \in \mathbb{R}$ with $|x| < n^{1/40}$, for $C_1 > 0$ a universal constant. By substituting (9) and (45) into Lemma (6.1.1), we conclude the following.

Proposition (6.1.6)[282]: Let $1 \leq \ell \leq n$ be integers. Let $0 < \alpha < 10^5$ and denote $\lambda = \frac{1}{5\alpha+20}$. Assume that $\ell \leq n^\lambda$. Suppose that $f : \mathbb{R}^n \to [0, \infty)$ is a log-concave function that is the density of an isotropic random vector. Define $g = f * \gamma_n[n^{-\lambda\alpha}]$, the convolution of f and $\gamma_n[n^{-\lambda\alpha}]$. Let $E \in G_{n,\ell}$ be a random subspace. Then, with probability greater than $1 - Ce^{-cn^{1/10}}$ of selecting E, we have

$$\left|\frac{\pi_E(g)(x)}{\gamma[1 + n^{-\lambda\alpha}](x)} - 1\right| \leq Cn^{-\lambda} \tag{46}$$

for all $x \in E$ with $|x| < n^{\lambda/2}$, where $C > 0$ is a universal constant.

We assume that $n \geq C$ in Proposition (6.1.6), since otherwise the proposition is vacuously true. We will show that the above estimate still holds without taking the convolution, though perhaps with slightly worse constants.

We establish the following principle. Suppose that X is a random vector with a log-concave density, and that Y is an independent, Gaussian random vector whose covariance matrix is small enough with respect to that of X. Then, in the case where $X + Y$ is approximately Gaussian, the density of X is also approximately Gaussian, in a rather large domain. We begin with a lower bound for the density of X.

Note that the notation n in corresponds to the dimension of the subspace, that was denoted by ℓ in the previous.

Lemma (6.1.7)[282]: Let $n \geq 1$ be a dimension, and let $\alpha, \beta, \varepsilon, R > 0$. Suppose that X is an isotropic random vector in $\mathbb{R}^n$ with a log-concave density, and that Y is an independent Gaussian random vector in $\mathbb{R}^n$ with mean zero and covariance matrix $\alpha\, Id$. Denote by f_Y and f_{X+Y} the respective densities. Suppose that,

$$f_{X+Y}\,(x)(1 - \varepsilon)\gamma_n[1 + \alpha](x) \tag{47}$$

for all $|x| \leq R$. Assume that $\alpha \leq c_0 n^{-8}$ and that

$$100(2n)^{\max\{3\beta,3/2\}}\,\alpha^{1/4} < \varepsilon < \frac{1}{100}. \tag{48}$$

Then,

$$f_X(x) \geq (1 - 6\varepsilon)\gamma_n[1](x) \tag{49}$$

for all $x \in \mathbb{R}^n$ with $|x| \leq \min\{R - 1, (2n)^\beta\}$. Here, $0 < c_0 < 1$ is a universal constant.

Proof. Suppose first that f_X is positive everywhere in $\mathbb{R}^n$, and that log fX is strictly concave. Fix $x_0 \in \mathbb{R}^n$ with $|x_0|\min\{R - 1, (2n)^\beta\}$. Assume that $\varepsilon_0 > 0$ is such that

$$f_X(x_0) < (1 - \varepsilon_0)\gamma_n[1](x_0) \tag{50}$$

Let H be an affine hyperplane that supports L at its boundary point x_0, and denote by D the open ball of radius $\alpha^{1/4}$ tangent to H at x_0, that is disjoint from the level set L. By definition, $f_X(x) < f_X(x_0)$ for $x \in D$. Denote the center of D by x_1. Then, $|x_1 - x_0| \leq \alpha^{1/4}$ with $|x_0| \leq (2n)^{\beta}$, and a straightforward computation yields

$$||x_1|^2 - |x_0|^2| \leq \left(2(2n)^{\beta} + \alpha^{1/4}\right)\alpha^{1/4} \leq \frac{\varepsilon}{2}, \quad (52)$$

where we used (48). Note that $|x_1| \leq |x_0| + \alpha^{1/4} \leq R$. Apply the last inequality and (47) to obtain,

$$f_{X+Y}(x_1) \geq (1 - \varepsilon)\gamma_n[1 + \alpha](x_0)e^{\frac{|x_0|^2 - |x^1|^2}{2(1+\alpha)}} > (1 - 2\varepsilon)\gamma_n[1 + \alpha](x_0). \quad (53)$$

By definition,

$$f_{X+Y}(x_1) = \int_{\mathbb{R}^n} f_X(x)\gamma_n[\alpha](x_1 - x)dx$$

$$= \int_{x\in D} f_X(x)\gamma_n[\alpha](x_1 - x)dx + \int_{x\notin D} f_X(x)\gamma_n[\alpha](x_1 - x)dx. \quad (54)$$

We will estimate both integrals. First, recall that $f_X(x) < f_X(x_0)$ for $x \in D$ and use (50) to deduce

$$\int_{x\in D} f_X(x)\gamma_n[\alpha](x_1 - x)dx < f_X(x_0) < \mathbb{P}\left(|G_n| \geq \frac{1}{\alpha^{14}}\right) \sup_{\mathbb{R}^n} f_X \quad (56)$$

where $G_n \sim \gamma_n[283]$ is a standard Gaussian random vector. To bound the right-hand side term, we shall use a standard tail bound for the norm of a Gaussian random vector,

$$\mathbb{P}\left(|G_n| > t\sqrt{n}\right) < Ce^{-ct^2}, \quad (57)$$

And following crude bound for the isotropic constant of f_X (see, e.g., [293] in Theorem 5.14(e)),

$$\sup_{\mathbb{R}^n} f_X < e^{\frac{1}{2}n\log n + 6n} < e^{C_n \log n}. \quad (58)$$

Consequently,

$$\int_{x\notin D} f_X(x)\gamma_n[\alpha](x_1 - x)dx < Ce^{-cn^{-1}\alpha^{-1/2}}e^{C_n\log n}\, e^{-\alpha^{-1/3}}, \quad (59)$$

for an appropriate choice of a sufficiently small universal constant $c_0 > 0$ (so that all other constants are absorbed). Combining (54), (55) and (59) gives

$$f_{X+Y}(x_1) < \left(1 - \varepsilon_0 + \frac{e^{-\alpha^{-1/3}}}{\gamma n[1](x_0)}\right) \gamma_n[1](x_0). \quad (60)$$

Using the fact that $n + (2n)^{2\beta} < \frac{\alpha^{-1/3}}{2}$, which follows easily from our assumptions, we have

$$\frac{e^{-\alpha^{-1/3}}}{\gamma_n[1](x_0)} = e^{\frac{|x_0|^2}{2} + \frac{n}{2}\log(2\pi) - \alpha^{-1/3}} < e^{-\frac{1}{2}\alpha^{-1/3}} \leq 2\alpha^{1/3} < \frac{\varepsilon}{2} < \frac{\varepsilon_0}{2} \quad (61)$$

(for the last inequality, note that if $\varepsilon_0 < 6\varepsilon$ then (9) holds and we have nothing to prove. So we can assume that $\varepsilon_0 > \varepsilon$). From (60) and (61) we obtain the bound

$$f_{X+Y}(x_1) < \left(1 - \frac{\varepsilon_0}{2}\right) \gamma_n[1](x_0). \quad (62)$$

Combining (53) and (62) we get,

$$(1 - 2\varepsilon)\gamma_n[1 + \alpha](x_0) < \left(1 - \frac{\varepsilon_0}{2}\right)\gamma_n[1](x_0). \quad (63)$$

A calculation yields,

$$\frac{\gamma_n[1](x_0)}{\gamma_n[1+\alpha](x_0)} \leq \frac{\gamma_n[1](0)}{\gamma_n[1+\alpha](0)} = (1+\alpha)^{\frac{n}{2}} < 1+\varepsilon. \quad (64)$$

From the above two inequalities, we finally deduce,

$$\frac{1-\varepsilon_0/2}{1-2\varepsilon} > \frac{1}{1+\varepsilon} > 1-\varepsilon \Rightarrow \varepsilon_0 < 6\varepsilon, \quad (65)$$

which proves (9). The lemma is proved, under the additional assumption that log f_X is strictly concave. The general case follows by a standard approximation argument.

After proving a lower bound, we move to the upper bound. We will show that if we add to the requirements of the previous lemma an assumption that the density of f_{X+Y} is bounded from above, then we can provide an upper bound for f_X.

Lemma (6.1.8)[282]: Let $n, X, Y, \alpha, \beta, \varepsilon, R, c_0$ be defined as in Lemma (6.1.7), and suppose that all of the conditions of Lemma (6.1.7) are satisfied. Suppose that in addition, we have the following upper bound for

$$f_{X+Y} : f_{X+Y}(x) < (1+\varepsilon)\gamma_n[1+\alpha](x) \quad (66)$$

for all $|x| < \mathbb{R}$. Then we have:

$$f_X(x) < (1+8\varepsilon)\gamma_n[1](x) \quad (67)$$

for all x with $|x| < \min\{(2n)^\beta, R\} - 3$.

Proof. Denote $F(x) = -\log f_X(x)$. Again we use the upper bound for the supremum of the density (58),

$$F(x) > 6n - \frac{1}{2} n \log n > -n \log n, \qquad \forall x \in \mathbb{R}^n. \quad (68)$$

Use the conclusion of Lemma (6.1.7) to deduce that for $|x| < \min\{(2n)^\beta, R\} - 1$ the following holds:

$$F(x) < -\log\left(\frac{1}{2}\gamma_n[1](x)\right) < \log 2 + \frac{n}{2}\log(2\pi) + (2n)^{2\beta}$$
$$< 3(2n)^{\max\left\{2\beta, \frac{3}{2}\right\}}. \quad (69)$$

Next we will show that for $x, y \in A = \{x \in Rn;\ |x| < \min\{(2n)^\beta, R\} - 2\}$, the following Lipschitz condition holds:

$$F(x) - F(y) 5(2n)^{max\{2\beta, 3/2\}} |x-y|. \quad (70)$$

To that end, denote a $= 5(2n)^{\max\left\{2\beta, \frac{3}{2}\right\}}$ and suppose by contradiction that x,y ∈ A are such that

$$F(y) - F(x) > a|y-x|. \quad (71)$$

Since $F(y) - F(x) < a$ (as implied by (68) and (69)), we have $|y-x| < 1$ and for the point

$$y_1 := x + \frac{y-x}{|y-x|},$$

we have, using the convexity of F,

$$F(y_1) - F(x)\frac{F(y) - F(x)}{|y-x|} > a.$$

Note that $|y_1| |x| + 1 < \min\{(2n)^\beta, R\} - 1$, thus we obtain a contradiction of (68) and (69). This proves (70).

Therefore, given two points $x, x_0 \in A$ such that $|x_0 - x| < \alpha^{1/4}$, (70) implies,

$$|F(x_0) - F(x)| < 5\alpha^{1/4}(2n)^{\max\{2\beta, 3/2\}} < \varepsilon/20. \quad (72)$$

Recall that $F = -\log f_X$, hence the above translates to

$$f_X(x_0) - f_X(x) < 2\ (e^{\varepsilon/20} - 1)\, f_X(x_0) < \frac{\varepsilon}{4}\, f_X(x_0). \quad (73)$$

Now, suppose $x_0 \in \mathbb{R}^n$ and $0 < \varepsilon_0 < 1$ are such that

$$f_X(x_0) > (1 + \varepsilon_0)\gamma_n[1](x_0), \quad (74)$$

with $|x_0| < \min\{R, (2n)^\beta\} - 3$. Again, to prove the lemma it suffices to show that in fact $\varepsilon_0 < 8\varepsilon$. Let D be a ball of radius $\alpha^{1/4}$ around x_0.

Since we can assume that $\varepsilon_0 > \varepsilon$ (otherwise, there is nothing to prove), we deduce from (73) and (74) that for all $x \in D$,

$$f_X(x) > \left(1 - \frac{\varepsilon_0}{4}\right)(1 + \varepsilon_0)\gamma_n[1](x_0) > \left(1 + \frac{\varepsilon_0}{2}\right)\gamma_n[1](x_0). \quad (75)$$

Thus,

$$f_{X+Y}(x_0) = \int_{\mathbb{R}^n} f_X(x)\gamma_n[\alpha](x_0 - x)dx > \int_{x\in D} f_X(x)\gamma_n[\alpha](x_0 - x)dx$$

$$> \left(1 + \frac{\varepsilon_0}{2}\right)\gamma_n[1](x_0)\cdot\left(1 - \mathbb{P}\left(|G_n| > \frac{1}{\alpha^{14}}\right)\right) > \left(1 + \frac{\varepsilon_0}{3}\right)\gamma_n[1](x_0), \quad (76)$$

where in the last inequality we used the estimate (57) and the assumption $\varepsilon_0 > \varepsilon$. Now, a computation yields,

$$\frac{\gamma_n[1 + \alpha](x_0)}{\gamma_n[1](x_0)} < e^{\frac{1}{2}\left(|x_0|^2 - \frac{|x0|^2}{1+\alpha}\right)} = e^{\frac{1}{2}|x_0|^2\frac{\alpha}{1+\alpha}} < e^{(2n)^{2\beta}\alpha} < 1 + \varepsilon. \quad (77)$$

We thus obtain, combining (66) and (76) and using (77), that

$$\frac{1 + \varepsilon_0/3}{1 + \varepsilon} < \frac{\gamma_n[1 + \alpha](x_0)}{\gamma_n[1](x_0)} < 1 + \varepsilon,$$

so $\varepsilon_0 < 8\varepsilon$, and the proof of the lemma is complete.

The combination of the two lemmas above gives us the desired estimate for the density of X, as proclaimed in the beginning.

Theorem (6.1.9)[282]: Let X be an isotropic random vector in $\mathbb{R}^n$ with a log-concave density. Let $\ell \le n^{c_1}$ be an integer. Then there exists a subset $\mathcal{E} \subseteq G_{n,l}$ with $\mu_{n,\ell}(\mathcal{E})\ 1 - C\exp(-n^{c_2})$ such that for any $E \in \mathcal{E}$, the following holds. Denote by f_E the density of the random vector $\mathrm{Proj}_E(X)$. Then,

$$\left|\frac{f_E(x)}{\gamma(x)} - 1\right| \le \frac{C}{n^{c_3}} \quad (78)$$

for all $x \in E$ with $|x| \le n^{c_4}$. Here, $\gamma(x) = (2\pi)^{\ell/2}\exp(-|x|^2/2)$ is the standard Gaussian density in E, and $C, c_1, c_2, c_3, c_4 > 0$ are universal constants.

Proof. We may clearly assume that n exceeds some positive universal constant (otherwise, take $\mathcal{E} = \emptyset$). Let $1 \le \ell \le n^{1/100}$ be an integer, and let $\delta \ge 0$ be such that $\ell = n^\delta$. Set $\alpha = 10$ and $\lambda = \frac{1}{5\alpha+20} = \frac{1}{70}$. Let Y be a Gaussian random vector in $\mathbb{R}^n$ with mean zero and covariance matrix n−αλ Id, independent of X. We first apply Proposition (6.1.6) for the random vector $X + Y$ with parameters and α (noting that $\ell \le n^{1/100} \le n^\lambda$). According to the conclusion of that proposition, if E is a random subspace of dimension ℓ, then

$$\left|\frac{\pi_E(f_{X+Y})(x)}{\gamma_n[1 + n^{-\alpha\lambda}](x)} - 1\right| \le Cn^{-1/100}, \quad (79)$$

for all $x \in E$ with $|x| < n^{1/200}$, with probability greater than $1 - Ce^{-cn^{1/10}}$ of selecting E. Next, we apply Lemmas (6.1.7) and (6.1.8) in the -dimensional subspace E, with the parameters $\alpha = n^{-10\lambda} \leq n^{-120}\ell^{-8}, \beta = \frac{1}{600(\delta+1/\log_2 n)}, R = n^{1/200}, \varepsilon = Cn^{-1/100}$ where C is the constant from (79). It is straightforward to verify that the requirements of these two lemmas hold, since n may be assumed to exceed a given universal constant. According to the conclusions of Lemmas (6.1.7) and (6.1.8), for any $x \in E$ with $|x| < n^{1/700}$,

$$\left|\frac{\pi_E(f_X)(x)}{\gamma_n[1](x)} - 1\right| \leq C\, n^{-1/100}.$$

This completes the proof.

We improve an estimate from [289],[290] which is related to Gaussian convolution. This improvement can be used to obtain slightly better bounds on certain exponents related to the central limit theorem for convex bodies. The following proposition was conjectured by Meckes [294].

Proposition (6.1.10)[282]: Let $n \geq 1$ and $f : \mathbb{R}^n \to [0,\infty)$ be an isotropic, log-concave density. Suppose that $\varepsilon > 0$ and denote $g_\varepsilon = f * \gamma_n[\varepsilon^2]$, the convolution of f with $\gamma_n[\varepsilon^2]$. Then,

$$\|g_\varepsilon - f\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |g_\varepsilon(x) - f(x)|dx \leq Cn\varepsilon,$$

where $C > 0$ is a universal constant.

Proposition (6.1.10) improves upon Lemma 5.1 in [289] and the results of Section 3 in [294], and it admits a simpler proof. It is straightforward to adapt the argument in [290], and to use Proposition (6.1.10) in place of the inferior Lemma 5.1 of [289]. This leads to slightly better estimates. We conclude that whenever X is a random vector with a log-concave density in $\mathbb{R}^n$, one may find a subspace $E \subset \mathbb{R}^n$ of dimension, say, $cn^{1/15}$ such that $\mathrm{Proj}_E(X)$ is approximately Gaussian, in the total variation sense. The exponent 1/15 is probably far from optimal, yet it is better than previous bounds.

Meckes has observed that Proposition (6.1.10) would follow from the next lemma.

Lemma (6.1.11)[282]: Let $f : \mathbb{R}^n \to [0,\infty)$ be a C^∞-smooth, isotropic, log-concave density. Then,

$$\int_{\mathbb{R}^n} \nabla f(x)dx \leq C'n,$$

where $C' > 0$ is a universal constant.

To see that Lemma (6.1.11) leads to Proposition (6.1.10), one only needs to apply an inequality from Ledoux [292]. In the notation of Proposition (6.1.10), it is proven in [292] that when f is C^∞-smooth,

$$\|g_\varepsilon - f\|_{L^1(\mathbb{R}^n)} \leq \sqrt{2\varepsilon} \int_{\mathbb{R}^n} \nabla f(x)dx. \tag{80}$$

Thus, Proposition (6.1.10) follows from Lemma (6.1.11) in virtue of (80), by approximating f with a C^∞-smooth function. Proposition (6.1.10) and Lemma (6.1.11) are tight, for small ε, up to the value of the constants C, C'. This is shown, e.g., by the example of f being close to the isotropic, log-concave function that is proportional to the characteristic function of the cube $\left[-\sqrt{3},\sqrt{3}\right]^n$.

Proof. The case $n = 1$ is covered, e.g., in [294]. We assume from now on that $n \geq 2$. Our method builds on the main idea of the proof of Lemma 2.3 in [291]. Fix $x \in \mathbb{R}^n$. We claim that

$$|\nabla f(x)| \leq C_1 nf(x) - C_2\nabla f(x) \cdot x, \tag{81}$$

for some universal constants $C_1, C_2 > 0$. Suppose first that $f(x) = 0$. Since $f \geq 0$ and f is C^∞- smooth, then necessarily ∇f (x) = 0. Therefore (81) is trivial in this case. It remains to prove (81) for the case where $f(x) > 0$. Denote $F = -\log f$. Then $F: \mathbb{R}^n \to (-\infty, \infty]$ is convex. Additionally, F is finite and C^∞-smooth in a neighborhood of x. The graph of the convex function F lies entirely above the supporting hyperplane to F at x. That is,

$$F(x) + \nabla F(x) \cdot (y - x) F(y) \text{ for all } y \in \mathbb{R}^n.$$

Consequently, for any $y \in \mathbb{R}^n$,

$$\nabla F(x) \cdot y \leq [F(y) - \inf F] + \nabla F(x) \cdot x.$$

By taking the supremum over all $y \in \mathbb{R}^n$ with $|y| \leq \frac{1}{10}$, we see that

$$\frac{|\nabla F(x)|}{10} \leq \nabla F(x) \cdot x + \sup_{|y| \leq 1/10} F(y) - \inf F. \tag{82}$$

Denote

$$K = \{x \in \mathbb{R}^n;\ f(x) e^{-10n} \sup f\}.$$

Then K is clearly convex. Additionally, $\int_K f(x)dx \geq 1 - e^{-5n/4} \geq 9/10$, by Corollary 5.3 in [289] (we actually use the formulation from Lemma 2.2 in [290]). According to Lemma (6.1.4).4 from [289] we have the inclusion $\{y \in \mathbb{R}^n;\ |y|\ 1\ 10\} \subseteq K$. Therefore,

$$\sup_{|y| \leq 1/10} F(y) - \inf F \leq \sup_{y \in K} F(y) - \inf F \leq [10n + \inf F] - \inf F = 10n.$$

Hence (82) implies that for any $x \in \mathbb{R}^n$,

$$\nabla F(x) \leq 10(\nabla F(x) \cdot x) + 100n. \tag{83}$$

Since $\nabla f(x) = -f(x) \nabla F(x)$, then (81) follows from (83). This completes the proof of (81). Next, we integrate by parts and see that

$$-\int_{\mathbb{R}^n} \nabla f(x) \cdot x dx = \sum_{i=1}^{-n} \int_{\mathbb{R}^n} x_i \partial^i f\, dx_1 \ldots dx_n = \sum_{i=1}^{n} \int_{\mathbb{R}^n} f(x) dx = n.$$

The boundary terms vanish, since $|x| f(x) \to 0$ as $|x| \to \infty$ (see, e.g., [291] in Lemma 2.1). According to (81),

$$\int_{\mathbb{R}^n} \nabla f(x) dx \leq C_1 n \int_{\mathbb{R}^n} f(x) dx - C_2 \int_{\mathbb{R}^n} \nabla f(x) \cdot x dx = (C_1 + C_2) n.$$

Section (6.2): Counterexample by Dvoretzky Theorem

A fundamental problem in Quantum Information Theory is to determine the capacity of a quantum channel to transmit classical information. The seminal Holevo–Schumacher– Westmoreland theorem expresses this capacity as a regularization of the so-called Holevo χ-quantity (which gives the one-shot capacity) over multiple uses of the channel; see, e.g., [297]. This extra step could have been skipped if the χ-quantity had been additive, i.e., if

$$\chi(\Phi \otimes \Psi) = \chi(\Phi) + \chi(\Psi) \tag{84}$$

for every pair (Φ, Ψ) of quantum channels. It would have then followed that the χ-quantity and the capacity coincide, yielding a single-letter formula for the latter. Determining the veracity of (84) had been a major open problem for at least a decade (see [298]). A substantial progress was made by Shor [299] who showed that (84) was formally equivalent to the additivity of the minimal output von Neumann entropy of quantum channels — a much more tractable quantity. Using this equivalence, the equality (84) was eventually shown to be false by Hastings [300], with appropriate randomly constructed channels as a counterexample.

We revisit Hastings's counterexample from the viewpoint of Asymptotic Geometric Analysis (AGA). This field — originally an offspring of Functional Analysis — aims at studying geometric properties of convex bodies (or equivalently, norms) in spaces of high (but finite) dimension. More specifically, we show that (a variant of) Hastings's analysis can be rephrased in the language of AGA, and his result deduced with only minor effort from a sharp version of Dvoretzky's theorem [301] on Werner almost spherical sections of convex bodies — a fundamental result of AGA. This makes the argument much more transparent and will hopefully lead to a better understanding of the problem of capacity. Our approach is largely inspired by Brandao–Horodecki [302], who were able to reformulate Hastings's analysis in the framework of concentration of measure.

The letters $C, c, C', \ldots$ denote absolute positive constants, independent of the instance of the problem (most notably of the dimensions involved), whose values may change from occurrence to occurrence. The values of these constants can be computed by reverse-engineering the argument. We also use the following convention: whenever a formula is given for the dimension of a (sub)space, it is tacitly understood that one should take the integer part.

For $\mathcal{M}_{k,d}$ be the space of $k \times d$ matrices (with complex entries), and $\mathcal{M}_d = \mathcal{M}_{d,d}$. More generally, $\mathcal{B}(\mathcal{H})$ will stand for the space of (bounded) linear operators on the Hilbert space $\mathcal{H}$. We will write $\|\cdot\|_p$ for the Schatten p-norm $\|A\|_p = \left(\mathrm{Tr}(A \dagger A)^{p/2}\right)^{1/p}$. The limit case $\|\cdot\|_\infty$ is the operator (or "spectral") norm, while $\|\cdot\|_{HS} = \|\cdot\|_2$ is the Hilbert–Schmidt (or Frobenius) norm. Let $D(C^d)$ be the set of density matrices on C^d , i.e., positive semi-definite trace one operators on C^d (orstates on C^d). If ρ is a state onCd , its von Neumann entropy $S(\rho)$ is defined as $S(\rho) = -\mathrm{Tr}\,\rho \log \rho$. If $\Phi: Mm \to Mk$ is a quantum channel (completely positive trace preserving map), its minimal output entropy is

$$S_{\min}(\Phi) = \min_{\rho \in D(C^m)} S(\Phi(\rho)).$$

Concavity of S implies that the minimum is achieved on a pure state.

The crucial insight allowing to relate analysis of quantum channels to high-dimensional convex geometry is the observation that there is an essentially one-to-one correspondence between channels and linear subspaces of composite Hilbert spaces. Specifically, let W be a subspace of $C^k \otimes C^d$ of dimension m. Then $\Phi: \mathcal{B}(\mathcal{W}) \to \mathcal{M}_k$ defined by $(\rho) = \mathrm{Tr}_{C^d}(\rho)$ is a quantum channel; here Tr_{C^d} is the partial trace with respect to the second factor in $C^k \otimes C^d$. Alternatively, and perhaps more properly, we could identify $\mathcal{W}$ with C^m via an isometry $V : C^m \to C^k \otimes C^d$ whose range is $\mathcal{W}$ and define, for $\rho \in \mathcal{M}_m$, the corresponding channel $\Phi : \mathcal{M}_m \to \mathcal{M}_k$ by

$$\Phi(\rho) = \mathrm{Tr}_{C^d}\left(V\rho V^\dagger\right). \quad (85)$$

It is now easy to define a natural family of random quantum channels. They will be associated, via the above scheme, to random m-dimensional subspaces $\mathcal{W}$ of $C^k \otimes C^d$, distributed according to the Haar measure on the corresponding Grassmann manifold (for some fixed positive integers m, d, k that will be specified later). Note that all reasonable parameters of a channel defined by (85) such as $S_{\min}(\Phi)$ depend only on the subspace $\mathcal{W} = V(C^m)$ and not on a particular choice of the isometry V (this will be also obvious from what follows). In particular, the language of "random m-dimensional subspaces of $C^k \otimes C^d$ " is equivalent to that of "random isometries from C^m to $C^k \otimes C^d$."

The following question has attracted considerable attention in the last few years: if Φ and are two quantum channels, is it true that

$$S_{\min}(\Phi \otimes \Psi) = S_{\min}(\Phi) + S_{\min}(\Psi) \quad (86)$$

Shor [299] showed it to be formally equivalent to a number of central questions in quantum information theory, including the additivity of the χ-quantity mentioned.

Note that the inequality "≤" always holds (consider product input states). However, as was first proved by Hastings using random constructions [300], the reverse inequality is false in general. The exegesis of Hastings's argument has subsequently been carried out in [302] and [303]. We will show here that the analysis of (a variant of) Hastings's example essentially amounts to applying the right version of Dvoretzky's theorem and leads to the conclusion that high-dimensional random channels typically violate (86).

Theorem (6.2.1)[294]: Let $k \in \mathbb{N}, m = ck^2$ and $d = Ck^2$ (c and C being appropriate absolute constants). Let $V : C^m \to C^k \otimes C^d$ be a random isometry and $\Phi: \mathcal{M}_m \to \mathcal{M}_k$ be the corresponding random channel given by (85). Then for k large enough, with large probability,

$$S_{\min}\left(\Phi \otimes \overline{\Phi}\right) < S_{\min}(\Phi) + S_{\min}\left(\overline{\Phi}\right).$$

The expression "with large probability" in Theorem (6.2.1) and in what follows may be understood as "with probability $> \theta$, where $\theta \in (0,1)$is arbitrary but fixed in advance" (note that, in particular, the threshold value of k could then depend on θ). However, much stronger assertions are in fact true, for example the probability of the exceptional set in Theorem (6.2.1) can be majorized by $\exp(-c' m)$. Another comment: one only uses in the proof that m and d are comparable, and larger than ck^2. The proof will be based on separately majorizing $S_{\min}\left(\Phi \otimes \overline{\Phi}\right)$, which is done via a well-known and relatively simple trick, and on minorizing $S_{\min}(\Phi) = S_{\min}\left(\overline{\Phi}\right)$, which is the main point of the argument.

A question analogous to (86) can be asked for the minimal output p-Rényi entropy ($p > 1$). For the additivity of Rényi entropy, random counterexamples were constructed earlier by Hayden–Winter [304]. It was shown in [305] that the Hayden–Winter analysis can also be simplified (at least conceptually) by appealing to Dvoretzky's theorem. Working with the von Neumann entropy, however, requires more effort. First, while [305] relied on a straightforward instance of Milman's "tangible" version [126],[306] of Dvoretzky's theorem for Schatten classes that was documented in the literature already in the 1970's, we now need a more subtle, sharp version (which appears in the literature only implicitly). Second, this sharp version is not applied in the most direct way and requires additional preparatory work (for which we mostly follow the approach of Brandao–Horodecki [302]).

We are consider channels with near-maximal minimal output entropy, the following simple inequality (Lemma III.1 in [302], or formula (40) in [300]) will allow to replace the analysis of the von Neumann entropy S by that of a smoother quantity.

Lemma (6.2.2)[294]: For every state $\sigma \in \mathcal{D}(C^k)$,

$$S(\sigma) \geq S\left(\frac{\mathrm{Id}}{k}\right) - k \left\| \sigma - \frac{\mathrm{Id}}{k}\right\|_{HS}^2 .$$

Consequently, for every quantum channel $\Phi: \mathcal{M}_m \to \mathcal{M}_k$,

$$S_{\min}(\Phi) \geq \log(k) - k \cdot \max_{\rho \in \mathcal{D}(C^m)} \left\|\Phi(\rho) - \frac{\mathrm{Id}}{k}\right\|_{HS}^2 . \quad (87)$$

It will be convenient to identify $C^k \otimes C^d$ (or, to be more precise, $C^k \otimes \overline{C^d}$ — a distinction we will ignore) with $\mathcal{M}_{k,d}$ via the canonical map induced by $u \otimes v \to |u\rangle\langle v|$. If $x \in C^k \otimes C^d$ is so identified with a matrix $\mathcal{M} \in \mathcal{M}_{k,d}$, then

$$\mathrm{Tr}_{C^d} |x\rangle\langle x| = M M^\dagger. \quad (88)$$

By this identification, Schmidt coefficients of $|x\rangle$ coincide with singular values of M. While the tensor and matrix formalisms are equivalent, the matrix formalism is arguably more transparent, which sometimes leads to simpler arguments.

Denote by $\mathcal{W} \subset C^k \otimes C^d$ the subspace inducing Φ. Note that the maximum in (87) is necessarily attained on pure states which, in this identification, correspond to unit vectors $x \in \mathcal{W}$. For such states the action of Φ is given — in the matrix formalism — by (88), and so the inequality (87) can be rewritten as

$$S_{\min}(\Phi) \geq \log(k) - k \cdot \max_{M \in \mathcal{W}, \|M\|_{HS}=1} \left\| M M^\dagger - \frac{\mathrm{Id}}{k} \right\|_{HS}^2 . \quad (89)$$

The idea will be to show that, for a random subspace $\mathcal{W}$, the maximum on the right is very small; this will be formalized in the next proposition.

The heart of the argument is the following proposition

Proposition (6.2.3)[294]: There are absolute constants $c, C, C' > 0$ so that for every k, for $d = Ck^2$ and $m = c^d$, a random Haar-distributed subspace $\mathcal{W}$ of dimension m in $\mathcal{M}_{k,d}$ satisfies

$$\max_{M \in \mathcal{W}, \|M\|_{HS}=1} \left\| M M^\dagger - \frac{\mathrm{Id}}{k} \right\|_{HS} \leq \frac{C'}{k} \quad (90)$$

with large probability (tending to 1 when k tends to ∞). From the proposition one quickly deduces that the pair $(\Phi, \overline{\Phi})$ is a counterexample to the additivity of minimum output von Neumann entropy. Indeed, a straightforward calculation shows that applying $\Phi \otimes \overline{\Phi}$ to the maximally entangled state yields an output state with one eigenvalue greater than or equal to $\frac{\dim \mathcal{W}}{\dim \mathcal{M}_{k,d}} = \frac{m}{kd} = \frac{c}{k}$ ([304], Lemma III.3; see [307]). Then, a simple argument using just concavity of $S(\cdot)$ reduces the problem to calculating the entropy of the state with one eigenvalue equal to $\frac{c}{k}$ and all the remaining ones identical, which yields

$$S_{\min}(\Phi \otimes \overline{\Phi}) \leq 2 \log k - \frac{c \log k}{k} + \frac{1}{k}.$$

On the other hand, Eq. (89) together with Proposition (6.2.3) implies

$$S_{\min}(\Phi) \geq \log(k) - \frac{C'^2}{k}.$$

Since $S_{\min}(\overline{\Phi}) = S_{\min}(\Phi)$, the inequality of Theorem (6.2.1) follows if k is large enough, as required.

We wish to point out that while Proposition (6.2.3) will be derived from a Dvoretzky-like theorem for Lipschitz functions (Theorem (6.2.7) below), it can be rephrased in the language of the standard Dvoretzky's theorem. Indeed, its assertion says that for every $M \in \mathcal{W}$ with $\|M\|_{HS} = 1$ we have

$$\frac{C^2}{k^2} \geq \left\| M M^\dagger - \frac{\mathrm{Id}}{k} \right\|_{HS}^2 = \mathrm{Tr}\,|M|^4 - \frac{2\mathrm{Tr}\, MM^\dagger}{k} + \frac{\mathrm{Tr\,Id}}{k^2} = \mathrm{Tr}\,|M|^4 - \frac{1}{k} \geq 0. \quad (91)$$

Consequently,

$$k^{-1/4}\|M\|_{HS} \leq \|M\|_4 \leq k^{-1/4} \left(1 + \frac{C^2}{k}\right)^{1/4} \|M\|_{HS}$$
$$\leq k^{-1/4} \left(1 + \frac{C^2}{4k}\right) \|M\|_{HS} \tag{92}$$
for all $M \in \mathcal{W}$. In other words, $\mathcal{W}$ is $(1 + \delta)$-Euclidean, with $\delta = \frac{C^2}{4k}$, when considered as a subspace of the normed space $(\mathcal{M}_{k,d} , \| \cdot\|_4)$, the Schatten 4-class.

In [305] we similarly observed that the crucial technical step of the Hayden-Winter proof of non-additivity of p-Rényi entropy for $p > 1$ can be restated as an instance of Dvoretzky's theorem for the Schatten $2p$-class. There is an important difference, however. While in the case of p-Rényi entropy the needed Dvoretzky-type statement was known since the 1970s, for the statement of the type (92) needed in the present context, the "off the shelf" methods seem to yield only $\delta = O(k^{-1/4})$ as opposed to $\delta = O(k - 1)$ above. This also suggests that while for the p-Rényi entropy derandomization of the example — i.e., supplying explicit channels for which the additivity fails — may be a feasible project (see [305]), the analogous task for the von Neumann entropy is likely to be much harder.

We use the following definitions: if f is a function from a metric space (X, d) to R, and $\mu \in$ R, the oscillation of f around μ on a subset $A \subset X$ is
$$\operatorname{osc}(f, A, \mu) = \sup_A \ | f - \mu|.$$
A function f defined on the unit sphere S_{C^n} is called circled if $f\,(e^{i\theta} x) = f\,(x)$ for any $x \in S_{C^n}, \theta \in [0, 2\pi]$. If X is a real random variable, we will say that μ is a central value of X if μ is either the mean of X, or any number between the 1st and the 3rd quartile of X (i.e., if $\min\{P(X \geq \mu), P(X \leq \mu)\} \geq \frac{1}{4}$; this happens in particular if μ is the median of X).

We will need the following variant of Milman's "tangible" version of Dvoretzky's theorem.

A striking application of the theorem above is to the case when f is the gauge function of a convex body, or a norm: it leads to the fact that any high-dimensional convex body has almost spherical sections.

At the heart of Dvoretzky-like phenomena lies the concentration of measure, which in our framework is expressed by

Lemma (6.2.4)[294]: (Lévy's lemma [308]). If $f : S^{n-1} \to$ R is a 1-Lipschitz function, then for every $\varepsilon > 0$,
$$P(|f\,(x) - \mu| > \varepsilon) \leq C_1 \exp(-c_1 n\varepsilon^2),$$
where x is uniformly distributed on S^{n-1}, μ is any central value of f , and $C_1, c_1 > 0$ are absolute constants.

Results such as Theorem (6.2.7) or Lévy's lemma are usually stated with μ equal to the median or the mean of f. However, once we know that the result is true for some central value (or, for that matter, for any $\mu \in$ R), it holds a posteriori for any such value (up to changes in the constants) as, for 1-Lipschitz functions, all central values differ at most by $C/\sqrt{n}$.

The obvious idea to prove Theorem (6.2.7) is to use Lévy's lemma and an ε-net argument — using the fact that an ε-net in $S_{C^n} = S^{2n-1}$ can be chosen to have cardinality

$\leq (1+2/\varepsilon)^{2n}$ (see [309], Lemma 4.10). Indeed, this was essentially Milman's original argument in [126]. However, one only obtains this way a subspace E of dimension $cn\varepsilon^2/\log(1/\varepsilon)$. For many applications (see [305]), this extra logarithmic factor is not an issue. However, in the present case, having the optimal dependence on ε is crucial.

The classical framework of convex geometry is the real case (with or without the assumption "circled," which in that context just means then that the function is even). In that setting, Theorem (6.2.7) was proved by Gordon [310] who used comparison inequalities for Gaussian processes. A proof based on concentration of measure was later given by Schechtman [311]. The complex case does not seem to appear in the literature. Actually, at the face of it, Gordon's proof does not extend to the complex setting, while Schechtman's proof does. We sketch Schechtman's proof of Theorem (6.2.7). It is not clear whether the assumption " f circled" in Theorem (6.2.7) can be completely removed; we do know that it is needed at most for very small values of ε.

For S_{HS} be the Hilbert–Schmidt sphere in $\mathcal{M}_{k,d}$ and let M be a random matrix uniformly distributed on S_{HS}. Let $\tilde{g}(\cdot)$ be the function defined on S_{HS} by

$$\tilde{g}(M) = \left\| M M^{\dagger} - \frac{\mathrm{Id}}{k} \right\|_{HS}.$$

The next well-known lemma asserts that the singular values of a very rectangular random matrix are very concentrated. This is a familiar phenomenon in random matrix theory that goes back to [312]. Versions of this lemma appeared in the QIT literature under the tensor formalism (see for example Lemma III.4 in [313]). However, these versions typically introduce an unnecessary logarithmic factor which would imply that the main proposition holds with $d = Ck^2 \log k$ instead of $d = Ck^2$. For completeness, we include a proof of Lemma (6.2.11).

We will use in the sequel the following immediate corollary of Lemma (6.2.11).

Corollary (6.2.5)[294]: Under the hypotheses of Lemma (6.2.11) and denoting $C_0 = 3C$

(a) with probability larger than $1 - \exp(-ck)$, all eigenvalues of $M M^{\dagger}$ differ from $1/k$ by less than $C_0/\sqrt{kd}$; consequently, the median (or any fixed quantile) of $\tilde{g}$ is bounded by $C_0/\sqrt{d}$ for k large enough.

(b) if $d \geq C^2 k$, the median (or any fixed quantile) of $\|M\|_\infty$ is bounded by $2/\sqrt{k}$ for k large enough.

We point out that while we chose to present statements (a) and (b) above as consequences of Lemma (6.2.11) for clarity and for "cultural" reasons (the lemma being familiar to the QIT community), more precise versions of these statements are available in (or can be readily deduced from) the random matrix literature. Re (a), the study of the distribution of $\tilde{g}$ is, by (91), equivalent to that of $\mathrm{Tr}\,|M|^4$, and a closed formula for the expected value of the latter is known (up to terms of smaller order, its value is $1/k + 1/d$); see, e.g., [314]. Re (b), sharp estimates on the tail of $\|M\|_\infty$ can also be found in [314] (proof of Lemma 7.3), in particular every fixed quantile is $1/\sqrt{k} + 1/\sqrt{d}$ up to terms of smaller order. This result can also be retrieved via methods of [315],[316], which focused on the real case.

The function $\tilde{g}$ is 2-Lipschitz on S_{HS}, and Corollary (6.2.5)(a) implies that the median of $\tilde{g}$ is as small as we want for large d. However, a direct application of Theorem (6.2.7) yields only a bound of order $1/\sqrt{k}$ in (90). The trick — already present in the previous approaches — is to exploit the fact that $\tilde{g}$ has a much smaller Lipschitz constant when restricted to a certain large subset of S_{HS}. As we will see, this bootstrapping argument is equivalent to applying Theorem (6.2.7) twice.

The following lemma appears in [302] with a rather long proof, but using the matrix formalism completely demystifies it.

Lemma (6.2.6)[294]: The function g is ˜ 6/ √k-Lipschitz when restricted to the set

$$\Omega = \{M \in S_{HS} \text{ s.t } \|M\|_\infty \leq 3/\sqrt{k}\}.$$

Proof. The lemma is a consequence of the following chain of matrix inequalities

$$\left\|M M^\dagger - \frac{\text{Id}}{k}\right\|_{HS} - \left\|N N^\dagger - \frac{\text{Id}}{k}\right\|_{HS} \leq \|M M^\dagger - N N^\dagger\|_{HS}$$
$$\leq \|M(M^\dagger - N^\dagger) + (M - N)N^\dagger\|_{HS}$$
$$\leq \|M\|_\infty \|M^\dagger - N^\dagger\|_{HS} + \|M - N\|_{HS} \|N^\dagger\|_\infty$$
$$\leq (\|M\|_\infty + \|N\|_\infty)\|M - N\|_{HS}.$$

The function $\|\cdot\|_\infty$ is 1-Lipschitz on SH S. By Corollary (6.2.5)(b), its median is bounded by $2/\sqrt{k}$ for $d \geq C^2 k$. (Note that Lévy's lemma shows that the measure of the complement of Ω is very small.) An application of the standard Dvoretzky's theorem (i.e.,Theorem (6.2.7) for norms) to $f = \|\cdot\|_\infty$ with μ equal to the median of $\|\cdot\|_\infty$ and with $\varepsilon = 1/\sqrt{k}$ (note that the dimension of the ambient space is $n = kd$) shows that the intersection of S_{HS} with a random subspace of dimension cd in $\mathcal{M}_{k,d}$ is contained in Ω with large probability.

Let g be a $6k^{-\frac{1}{2}}$-Lipschitz extension of $\tilde{g}|_\Omega$ to S_{HS} — in any metric space X, it is possible to extend any L-Lipschitz function $\tilde{h}$ defined on a subset Y without increasing the Lipschitz constant; use, e.g., the formula

$$h(x) = \inf_{y \in Y} \left[\tilde{h}(y) + L \operatorname{dist}(x, y)\right].$$

This formula also guarantees that the extended function g is circled. Since $g = \tilde{g}$ on most of S_{HS}, the median of g (resp., $\tilde{g}$) is a central value of $\tilde{g}$ (resp., g). We apply Theorem (6.2.7) to g with $\varepsilon = 1/k$ and $L = 6k^{-1/2}$ to get (μ being the median of $\tilde{g}$)
osc(g, SH S ∩ E, μ) ≤ 1/k on a random subspace $E \subset \mathcal{M}_{k,d}$ of dimension $m = c_0 \cdot kd \cdot \left(k^{-1}/(6k^{-1/2})\right)^2 = cd$. Using Corollary (6.2.5)(a), we obtain that $\mu \leq 1/k$ for $d \geq (C_0 k)^2$. We then have

$$\operatorname{osc}(g, S_{HS} \cap E, 0) \leq 2/k.$$

If $S_{HS} \cap E \subset \Omega$ (which, as noticed before, holds with large probability), g and $\tilde{g}$ coincide on $S_{HS} \cap E$ and therefore $\operatorname{osc}(\tilde{g}, S_{HS} \cap E, 0) \leq 2/k$. This completes the proof of Proposition (6.2.3) and hence that of Theorem (6.2.1).

Theorem (6.2.7)[294]: (Dvoretzky's theorem for Lipschitz functions). If $f : S_{C^n} \to$ R is a 1-Lipschitz circled function, then for every $\varepsilon > 0$, if $E \subset C_n$ is a random subspace (Haar-distributed) of dimension $c_0 n \varepsilon^2$, we have with large probability

$$\operatorname{osc}(f, S_{C^n} \cap E, \mu) \leq \varepsilon,$$

where μ is any central value of f (with respect to the normalized Lebesgue measure on S_{C^n}) and c_0 is an absolute constant. If the function is L-Lipschitz, the dimension changes to $c_0 n(\varepsilon/L)^2$.

Proof. We sketch here a proof of Theorem (6.2.7), essentially following Schechtman [311]. As we already mentioned, a simple use of a ε-net argument gives a parasitic factor $\log(1/\varepsilon)$. This can be improved by a chaining argument, which goes back (at least) to

Kolmogorov — a way to use η-nets for all values of η simultaneously. Consider the canonical inclusion $C^m \subset C^n$, and let $U \in U(n)$ be a random Haar-distributed unitary matrix. Then $F := U(C^m)$ is distributed according to the Haar measure on the Grassmann manifold of m-dimensional subspaces. If $f : S_{C^n} \to$ R is a 1-Lipschitz circled function with mean μ, we need to show that $\mathrm{osc}(f \circ U, S_{C^m}, \mu) \le \varepsilon$ with large probability provided $m \le c_0 n \varepsilon^2$. We first prove a lemma.

Lemma (6.2.8)[294]: Let $f : S_{C^n} \to$ R be a 1-Lipschitz circled function and U ∈ U(n) be a Haar-distributed random unitary matrix. Then for any $x, y \in S_{C^n}$ with x = y and for any $\lambda > 0$,

$$P(|f(Ux) - f(Uy)| > \lambda) \le C \exp\left(-cn\frac{\lambda^2}{|x-y|^2}\right).$$

Proof. Fix $x, y \in S_{C^n}$. Since f is circled (and U is C-linear), we may replace y by $e^{i\theta} y$ and choose θ so that $\langle x|y\rangle$ is real nonnegative; note that this choice of θ minimizes $|x - y|$ and assures that $x + y$ and $y - x$ are orthogonal. (This is the only really new point needed to accommodate the complex setting.) Set $z = \frac{x+y}{2}$ and $w = \frac{y-x}{2}$, then $x = z + w$ and $y = z - w$. Further, set $\beta = |w| = \frac{1}{2}|x - y|$ (we may assume that $\beta \neq 0$) and $w' = \beta^{-1} w$. Then, conditionally on $u = U(z), U(w')$ is distributed uniformly on the sphere $S_{u^\perp} := S_{C^n} \cap u^\perp$. Since $U(x) = u + \beta U(w')$ and $U(y) = u - \beta U(w')$, it follows that the conditional (on $u = U(z)$) distribution of $f(Ux) - f(Uy)$ is the same as that of $f_u : S_{u^\perp} \to$ R defined by

$$f_u(v) = f(u + \beta v) - f(u - \beta v).$$

As is readily seen, f_u is 2β-Lipschitz and its mean is 0. From Lévy's lemma, applied to f_u and to the $(2n-3)$-dimensional sphere $S_{u^\perp}$, we deduce that, conditionally on $u = U(z)$,

$$P(|f(Ux) - f(Uy)| > \lambda) \le C_1 \exp(-c_1(2n-2)\lambda^2/|x-y|^2),$$

and hence the same inequality holds also without the conditioning.

The end of the proof (the actual chaining argument) is identical to that in Schechtman's, so — rather than copying it — we present the general principle on which it is based. Let (S, ρ) be a compact metric space and let $(X_s)_{s\in S}$ be a family of mean 0 random variables (a stochastic process indexed by S). We say that (X_s) is subgaussian if there are $A, \alpha > 0$ such that, for all $s, t \in S$ with $s = t$ and for all $\lambda \ge 0$,

$$P(|X_s - X_t| \ge \lambda) \le A \exp\left(-\alpha\frac{\lambda^2}{\rho(s,t)^2}\right). \tag{93}$$

Proposition (6.2.9)[294]: (Dudley's inequality). If $(X_s)_{s\in S}$ satisfies (93) and some mild regularity conditions, then

$$E \sup_{s,t\in S} |X_s - X_t| \le C' A \alpha^{-1/2} \int_0^\infty \sqrt{\log N(S,\eta)}\, d\eta,$$

where $N(S, \eta)$ is the minimal cardinality of a η-net of S (in particular the integrand is 0 if η is larger than the radius of S).

See [317], [318] for a generalization to the subgaussian case that is relevant here, and [319] for a book exposition; we also sketch a proof further below.

In our case we choose $S = S_{C^m} \cup \{0\}$ (with the usual Euclidean metric), $X_s = f(Us) - \mu$ if $s \in S_{C^m}$ and $X_0 = 0$; then

$$\mathrm{osc}(f \circ U, S_{C^m}, \mu) = \sup_{s\in S} |X_s|.$$

The underlying probability space is $U(n)$, and the subgaussian property is given by Lemma (6.2.8) if $s, t \in S_{C^m}$ and directly by Lévy's lemma if s or t equals 0. Next, the bound $N(S_{C^m}, \eta) = N(S^{2m-1}, \eta) \le (1 + 2/\eta)^{2m}$ mentioned in the comments following Lemma (6.2.4)leads to an estimate $2\sqrt{m}$ for the integral and to the bound

$$E := \mathrm{E} \sup_{s\in S} |X_s| \le E \sup_{s,t\in S} |X_s - X_t| \le C' \, C(c_n)^{-12} \cdot 2\sqrt{m} = C'' \sqrt{\frac{m}{n}}.$$

(For readers confused by different quantities appearing on the left side in different forms of Dudley's inequality, we point out that the first inequality above uses the fact that one of the variables X_t equals 0, and that we always have $\sup_{s,t} |X_s - X_t| = \sup_s X_s + \sup_t (-X_t)$.) The assertion of Theorem (6.2.7) follows now from Markov's inequality if ε is sufficiently larger than E, which is assured by choosing c_0 small enough. A slightly more careful argument (such as that given in [311], or see [319]) or an application of the appropriate concentration inequality (for functions on $U(n)$) yields a bound of the form $\exp(-c' \varepsilon^2 n)$ on the probability of the exceptional set $\sup_{s\in S} |X_s| > C'' \sqrt{\frac{m}{n}} + \varepsilon$ (hence for the exceptional set from Theorem (6.2.7)).

Let us comment here that the value of the constant c_0 given by the proof of Theorem (6.2.7) is probably the single most important obstacle to showing Theorem (6.2.1) for "reasonable" values of k, m. An adaptation of the proof from [310] (which yields good constants) to the complex case could be helpful here. Proof of Dudley's inequality. For every $k \in Z$, let $\mathcal{N}_k$ be a 2^{-k} -net of minimal cardinality for (S,ρ). Let $k_0 \in Z$ be such that the radius of S lies between $2^{-(k_0+1)}$ and 2^{-k_0} ; the net $\mathcal{N}_{k_0}$ consists of a single element s0. For every $s \in S$ and $k \in Z$, let $\pi_k(s)$ be an element of $\mathcal{N}_k$ satisfying $\rho(s, \pi_k(s)) \le 2^{-k}$. The chaining equation reads for every $s \in S$,

$$X_s = X_{s_0} + \sum_{k\ge k_0} X_{\pi_{k+1}(s)} - X_{\pi_k(s)}. \tag{94}$$

(It is here where some regularity of (X_s) – path continuity – is used.) It follows that

$$\sup_{s,t\in S} |X_s - X_t| \le 2 \sum_{k\ge k_0} \sup_{s\in S} \left|X_{\pi_{k+1}(s)} - X_{\pi_k(s)}\right| \le 2 \sum_{k\ge k_0} \sup_{u,u'} |X_u - X_{u'}|,$$

where the last supremum is taken over couples$(u, u') \in \mathcal{N}_{k+1} \times \mathcal{N}_k$ satisfying $\rho(u, u') \le 2^{-k} + 2^{-(k+1)} < 2^{-k+1}$. It remains to bound the expectation of each term in the sum, using the following fact

Fact (6.2.10)[294]: If $N \ge 2$ and $Y_1, \dots, Y_N$ are nonnegative random variables satisfying the tail estimate $P(Y_i \ge t) \le A \exp(-t^2/2\beta^2)$ for all $t \ge 0$, then

$$\mathrm{E} \max_{Y_i} \le C\, A\beta \sqrt{\log N}.$$

To bound $\mathrm{E}\, sup\, |X_u - X_{u'}|$, we apply the above fact with β = 2−k+1α−1/2 and $N = \mathrm{card}(\mathcal{N}_k) \cdot \mathrm{card}(\mathcal{N}_{k+1}) \le N\left(S, 2^{-(k+1)}\right)^2$. This gives

$$\mathrm{E} \sup_{s,t\in S} |X_s - X_t| \le C\, A\alpha^{-1/2} \sum_{k\ge k_0} 2^{-k} \sqrt{\log N(S, 2^{-(k+1)})}.$$

The result now follows by relating the last series to the integral in Proposition (6.2.9) (a version of the integral test from calculus).

Proof. We may assume $\beta = 1$ by working with Y_i / β. Then simply write

$$\mathrm{E}\max\, Y_i \;=\; \int_0^\infty \; \mathrm{P}(\max Y_i \geq\; t)dt$$
$$\leq\; \sqrt{2\log N} \;+\; AN \int_{\sqrt{2\log N}}^\infty \exp(-t^2/2)\,dt \;\leq\; \sqrt{2\log N} \;+\; A.$$
The last inequality follows from $\int_{\sqrt{2\log N}}^\infty \exp(-t^2/2)dt \;\leq\; \int_{\sqrt{2\log N}}^\infty t\exp(-t^2/2)dt \;=\; 1/N$. Note that the hypotheses force $A \;\geq\; 1$.

Lemma (6.2.11)[294]: There exist absolute constants $C, c > 0$ such that, if M is uniformly distributed on the Hilbert–Schmidt sphere in $\mathcal{M}_{k,d}$ $(d \;\geq\; C2k)$, then with probability larger than $1 - \exp(-ck)$,
$$\mathrm{spec}\big(M\,M^\dagger\big) \;\subset \left[\left(\frac{1}{\sqrt{k}} - \frac{C}{\sqrt{d}}\right)^2, \left(\frac{1}{\sqrt{k}} + \frac{C}{\sqrt{d}}\right)^2\right]. \qquad (95)$$
We note that inclusion (95) can be reformulated as follows: all singular values of M differ from $1\,/\sqrt{k}$ by less than $C\,/\sqrt{d}$. (Recall that the singular values of M correspond to the Schmidt coefficients of a random pure state in $C^k \;\otimes\; C^d$.)

Proof. The lemma will follow if we show that with large probability,
$$\|\Delta\|_\infty \;\leq \frac{C}{\sqrt{kd}},$$
where $\Delta = \; M\,M^\dagger \;-\; \mathrm{Id}/k \;\in \mathcal{M}_k$ and $\|\cdot\|_\infty$ is the operator (or spectral) norm. Let $\mathcal{N}$ be a $\frac{1}{4}$ -net of S_{C^k} with cardinality bounded by $(C_0)^k$. One checks that if $x \;\in\; S_{C^k}$ and $\bar{x} \;\in\; \mathcal{N}$ satisfy $|x \;-\; \bar{x}| \;\leq\; 1/4$, then
$$|\langle x|\Delta|x\rangle| \;\leq\; |\langle \bar{x}|\Delta|\bar{x}\rangle| \;+\; \langle |x \;-\; \bar{x}||\Delta|\bar{x}\rangle| \;\; + |\langle x|\Delta|x - \bar{x}\rangle \leq\; |\langle \bar{x}|\Delta|\bar{x}\rangle| \;+\; 2\;\cdot \frac{1}{4}\,\|\Delta\|_\infty,$$
so that taking supremum over $x \;\in\; S_{C^k}$, we get
$$\|\Delta\|_\infty \;\leq\; 2\sup_{\bar{x}\in\mathcal{N}} \;\; |\langle \bar{x}|\Delta|\bar{x}\rangle|.$$
An application of the union bound gives
$$\mathrm{P}\left(\|\Delta\|_\infty \;\geq \frac{C}{\sqrt{kd}}\right) \;\leq\; (C_0)^k \;\cdot\; \mathrm{P}\left(\left(|\langle x_0|\Delta|x_0\rangle| \geq \frac{C}{2\sqrt{kd}}\right)\right.$$
$$=\; (C_0)^k \;\cdot\; \mathrm{P}\left(\left|M^\dagger x_0\right|^2 \;\geq \frac{1}{k} \;+ \frac{C}{2\sqrt{kd}}\right)$$
$$\leq\; (C_0)^k \;\cdot\; \mathrm{P}\left(\left|M^\dagger x_0\right| \;\geq \frac{1}{\sqrt{k}} \;+ \frac{C}{5\sqrt{d}}\right),$$
where $x_0 \;\in\; C^k$ is any fixed unit vector (remember that $d \;\geq\; C^2k$). The probabilities above can be expressed in terms of Beta-type integrals, but it's easier to estimate them using Lévy's lemma. The function $M \;\to\; \left|M^\dagger x_0\right|$ is 1-Lipschitz on the Hilbert–Schmidt sphere (if x_0 is the first vector of the canonical basis, then $M^\dagger x_0$ is essentially the first row of M) and
$$\mathrm{E}\left|M^\dagger x_0\right| \;\leq\; \left(\mathrm{E}\left|M^\dagger x_0\right|^2\right)^{1/2} \;=\; 1/k.$$
Hence, by Lévy's lemma (with $n \;=\; 2kd$ and $= \frac{C}{5\sqrt{d}}$), we get
$$\mathrm{P}\left(\|\Delta\|_\infty \;\geq \frac{C}{\sqrt{kd}}\right) \;\leq\; \exp(-ck)$$
for some choice of the constants $C, c \;>\; 0$, as required.

Section (6.3): Dvoretzky Theorem for Subspaces of L_p

We study the classical result of Dvoretzky [301] on almost spherical sections of normed spaces in the case of subspaces of L_p. Grothendieck in [333], motivated by the well known Dvoretzky–Rogers lemma from [328], asked if every finite dimensional normed space has lower dimensional subspaces which are almost Euclidean and their dimension grows with respect to the dimension of the ambient space. Dvoretzky in [301] gave an affirmative answer in the above question by proving that for any positive integer k and every $\varepsilon \in (0,1)$ there exists $N = N(k,\varepsilon)$ with the following property: For every $n \geq N$ and any n-dimensional normed space X there exists a k-dimensional subspace E which is $(1+\varepsilon)$-isomorphic to the Euclidean space ℓ_2^k. In modern functional analytic language this means that every infinite-dimensional Banach space contains ℓ_2^n's uniformly. Dvoretzky's proof in [301] provides the quantitative estimate $N(k,\varepsilon) \geq \exp(c\varepsilon^{-2}k^2 \log^2 k)$ (see [349] for a related discussion), for some absolute constant $c > 0$. However, the aforementioned estimate is not optimal. The optimal dependence with respect to the dimension was proved later by Milman, in his [27], where he obtained $N(k,\varepsilon) \geq \exp\left(ck\varepsilon^{-2} \log\frac{1}{\varepsilon}\right)$ (an alternative approach which yields the same estimate was presented by Szankowski in [349]). Equivalently, this states that for any $\varepsilon \in (0,1)$ there exists a function $c(\varepsilon) > 0$ with the following property: for every n-dimensional normed space X there exists $k \geq c(\varepsilon)\log n$ and a linear map $T: \ell_2^k \to X$ with $\|x\|_2 \leq \|Tx\|_X \leq (1+\varepsilon)\|x\|_2$ for all $x \in \ell_2^k$. In this case we say that ℓ_2^k can be $(1+\varepsilon)$-embedded into X or that X has a k-dimensional subspace which is $(1+\varepsilon)$-Euclidean and we write $\ell_2^k \xrightarrow{1+\varepsilon} X$.

The example of $X = \ell_\infty^n$ shows that this result is best possible with respect to n (see [27] or [306]). The approach of [27] is probabilistic in nature and provides that the vast majority of subspaces (in terms of the Haar probability measure on the Grassmannian manifold $G_{n,k}$) are $(1+\varepsilon)$-spherical, as long as $k \leq c(\varepsilon)k(X)$, where $k(X)$ is the critical dimension of X. Nowadays this is customary addressed as the randomized Dvoretzky theorem or random version of Dvoretzky's theorem. Milman revealed the significance of the concentration of measure as a basic tool for the understanding of the high-dimensional structures. That was the starting point for many applications of the concentration of measure method in high-dimensional phenomena. Since then, this tool has found numerous applications in various fields such as quantum information [296], combinatorics [326], random matrices [352], compressed sensing [329], theoretical computer science [26], geometry of high-dimensional probability measures [282] and more.

Another remarkable fact of Milman's approach is that the critical quantity $k(X)$ can be described in terms of the global parameters of the space. In particular, $k(X) \simeq \mathbb{E}\|Z\|_X^2/b^2(X)$ where Z is a standard Gaussian random vector in X and $b(X) = \max\limits_{\|\theta\|_2=1}\|\theta\|_X$. Then, one can find a good position of the unit ball of X for which $k(X)$ is large enough with respect to n (see [339]). It has been proved in [129] that this formulation is optimal with respect to the dimension $k(X)$, in the sense that the k-dimensional subspaces which are 4-Euclidean with probability greater than $\frac{n}{n+k}$, cannot exceed $Ck(X)$ (see [334]).

The proof of [27] gave the estimate $c(\varepsilon) \geq c\varepsilon^2/\log\frac{1}{\varepsilon}$ and this was improved to $c(\varepsilon) \geq c\varepsilon^2$ by Gordon in [331] and later, adopting the methods of Milman, by Schechtman in [311]. This dependence is known to be optimal in the setting of the randomized Dvoretzky theorem; see [345]). The works of Schechtman in [344] and Tikhomirov in [350] established that the dependence on ε in the randomized Dvoretzky for ℓ_∞^n is of the order $\varepsilon/\log\frac{1}{\varepsilon}$ and this is best possible. Optimal bounds on $c(\varepsilon)$ in the randomized Dvoretzky for $\ell_p^n, 1 \le p \le \infty$ have recently been studied in [341].

As far as the dependence on ε in the "existential version" of Dvoretzky's theorem is concerned, Schechtman proved in [343] that one can always $(1+\varepsilon)$-embed ℓ_2^k in any n-dimensional normed space X with $k \ge c\varepsilon \log n / \left(\log\frac{1}{\varepsilon}\right)^2$. Tikhomirov in [351] proved that for 1-symmetric space X one has $\ge c \log n / \log\frac{1}{\varepsilon}$, thus complementing a result of Bourgain and Lindenstrauss from [327]. Tikhomirov's result was subsequently extended by Fresen in [330] for permutation invariant spaces with bounded basis constant. For more detailed information on the subject, explicit statements and historical remarks see [323].
We study the dependence on ε and dimension in Dvoretzky's theorem for finite-dimensional subspaces of $L_q, 2 < q < \infty$. The case of subspaces of $L_q, 1 \le q < \infty$ have been previously studied in [306] by Figiel, Lindenstrauss and Milman.

The approach in [306] is based on Milman's asymptotic formula and the fact that the L_p spaces enjoy the cotype property. Let us recall that for $2 \le q < \infty$ the q-cotype constant of a normed space X in n vectors, denoted by $C_q(X,n)$, is defined as the smallest constant $C > 0$ which satisfies

$$\left(\sum_{i=1}^{n} \|z_i\|_X^q\right)^{\frac{1}{q}} \le C\mathbb{E}\left\|\sum_{i=1}^{n} \varepsilon_i z_i\right\|_X,$$

for any n vectors $z_1, \dots, z_n \in X$. Then, the q-cotype constant of X is defined as $C_q(X) := \sup_n C_q(X,n)$. Following the terminology of Pisier, the notion of cotype is a super-property, that is, it depends only on the finite dimensional subspaces of the space. It is also an isomorphic invariant and the spaces $L_p, 1 \le p < \infty$ are of cotype $q = \max\{2, p\}$ with $C_q(L_p) = O\left(q^{\frac{1}{2}}\right)$ (see [322] for a proof). Therefore, for any finitedimensional subspace X of $L_q, 2 < q < \infty$ we have $C_q(X) \le C\sqrt{q}$. The authors in [306], using the classical Dvoretzky–Rogers lemma, show that any n-dimensional normed space X of cotype q whose unit ball is in John's position (see e.g. [306]) satisfies $k(X) \ge cC_q^{-2}(X)n^{\frac{2}{q}}$. It follows that if X is an n-dimensional subspace of $L_q, 2 < q < \infty$, whose unit ball is in John's position, then $k(X) \ge cq^{-1}n^{\frac{2}{q}}$ and the standard concentration techniques yield $(1+\varepsilon)$-spherical sections of B_X with dimension $k \ge cq^{-1}\varepsilon^2 n^{\frac{2}{q}}$ (see [306]). The same argument provides $k(X) \ge cn$ for any n-dimensional subspace X of L_q with $1 \le q < 2$ in John's position, and thus ℓ_2^k can be $(1+\varepsilon)$-embedded into X with $k \ge c\varepsilon^2 n$, which is best possible. In the present note we show that for the range $2 < q < \infty$ the estimate can be considerably improved.

Theorem (6.3.1)[320]: For any $2 < p < \infty$ there exists a constant $c(p) > 0$ with the following property: for any n-dimensional subspace X of L_p and for any $\varepsilon \in (0,1)$ there exists $k \ge c(p)\min\left\{\varepsilon^2 n, (\varepsilon n)^{\frac{2}{p}}\right\}$ so that ℓ_2^k can be $(1+\varepsilon)$-embedded into X.

The approach is different and depends on a Gaussian functional analytic inequality rather than the spherical isoperimetric inequality that is used in the classical framework. The proof still depends on random methods, but the main tool is a variant of an inequality due to Pisier from [342].

To prove the above theorem, we have to bypass Milman's asymptotic formula, which involves the Lipschitz constant of the norm. As several examples show this parameter is inadequate to describe efficiently phenomena in the almost isometric scale. Our argument outclasses the latter one, since it takes into account the order of magnitude of the length of the gradient of the norm. The idea of estimating averages of the Euclidean norm of the gradient of a function in order to get sharp concentration results seems to be only recently applied and was also successfully exploited in [341]. Moreover, the selection of the position of the unit ball of the space is different. Instead of using John's position, we employ Lewis' position for the unit ball of finite-dimensional subspaces of L_p. This permits us to express the norm in an integral form, with respect to some isotropic measure on the sphere, and therefore to use the aforementioned inequality. We derive Theorem (6.3.1) from a stronger statement the randomized Dvoretzky theorem for those spaces in Lewis' position. Along the way, we prove that the norm of the underlying subspace in this position exhibits two-level Gaussian concentration and minimal fluctuations.

Theorem (6.3.2)[320]: Let $2 < p < \infty$ and let X be an n-dimensional subspace of L_p, represented on $\mathbb{R}^n$, whose unit ball B_X is in Lewis' position. Then,

$$\mathbb{P}(|\|Z\| - \mathbb{E}\|Z\|| > \varepsilon\mathbb{E}\|Z\|) \leq C \exp\left(-c\, min\left\{\alpha_p\varepsilon^2 n, (\varepsilon n)^{\frac{2}{p}}\right\}\right), \quad 0 < \varepsilon < 1.$$

In particular, we have

$$Var\|Z\| \leq C_p n^{\frac{2}{p}-1},$$

where $\alpha_p, C_p > 0$ are constants depending only on p and Z is the standard n-dimensional Gaussian vector.

It is worth mentioning that the Gaussian concentration and the variance estimate obtained for these spaces is best possible (up to constants of ℓ_p) as the example of p norms shows (see [341] for the exact formulation). Consequently, the random version of Dvoretzky's theorem we prove for this position (or for this type of norms) is sharp in the sense that in the case of ℓ_p^n spaces the corresponding critical dimension is optimal (see [341]). In other words, the ℓ_p^n space occurs as the approximately extremal structure in this study, or is the worst subspace of L_p with respect to the local almost Euclidean structure.

For our analysis is crucial the perspective of differently selecting the position of the unit ball of the underlying space and this is reflected in the improved estimates we obtain. To the best of our knowledge the concentration estimates we derive in Theorem (6.3.2) are new and it is also clear that the dimension $k(n,p,\varepsilon) \simeq_p \min\left\{\varepsilon^2 n, (\varepsilon n)^{\frac{2}{p}}\right\}$, that one can find almost Euclidean subspaces, is always better than the previously known $\varepsilon^2 n^{\frac{2}{p}}$ due to Figiel, Lindestrauss and Milman. In addition, the improved estimate for $k(n,p,\varepsilon)$ yields "new dimensions" of almost Euclidean sections in the following sense: The previous setting was only permitting almost isometric embeddings of distortion $1+\varepsilon$ with $\varepsilon \gg n^{-\frac{1}{p}}$ in order to achieve non-trivial dimensions. Now this phenomenon admits an improvement and one can find $(1+\varepsilon)$-linear embeddings with $\varepsilon \gg n^{-\frac{1}{2}}$. It is worth mentioning that the dimension $k(n,p,\varepsilon)$ that one finds almost Euclidean sections for these spaces is given implicitly as function of ε and n rather than as function of separated variables as Milman's formula suggests. This phenomenon had not been observed prior to this work and [341].

We show concentration results for the family of the L_q-bodies associated with an isotropic measure μ on the $(n-1)$-dimensional Euclidean sphere. We provide the proof of the main result. Finally, we conclude with some further remarks.

We work in $\mathbb{R}^n$ equipped with the standard Euclidean structure $\langle\cdot,\cdot\rangle$. The $(n-1)$-dimensional Euclidean sphere is defined as $S^{n-1}:=\{x\in\mathbb{R}^n:\langle x,x\rangle=1\}$. The ℓ_p norm is defined as $\|x\|_p:=(\sum_{i=1}^n|x_i|^p)^{\frac{1}{p}}$ for $x=(x_1,\ldots,x_n)\in\mathbb{R}^n$. We set $\ell_p^n=(\mathbb{R}^n,\|\cdot\|_p)$ and let B_p^n its unit ball. More generally, for any centrally symmetric convex body K on $\mathbb{R}^n$ we write $\|\cdot\|_K$ for the norm induced by K. The n-dimensional Lebesgue measure (volume) of a body A is denoted by $|A|$. The space $L_p(\Omega,\mathcal{E},\mu), 1\le p<\infty$ consists of all $\mathcal{E}$-measurable functions $f:\Omega\to\mathbb{R}$ so that $\int_\Omega|f|^p d\mu<\infty$, equipped with the norm $\|f\|_{L_p(\mu)}:=\left(\int_\Omega|f|^p d\mu\right)^{\frac{1}{p}}$.

The n-dimensional (standard) Gaussian measure is denoted by γ_n and its density is

$$d\gamma_n(x):=(2\pi)^{-\frac{n}{2}}e^{-\frac{\|x\|_2^2}{2}}dx.$$

More generally, let $d\gamma_{n,\sigma}(x):=(2\pi\sigma^2)^{-\frac{n}{2}}e^{-\frac{\|x\|_2^2}{2\sigma^2}}dx$ for $\sigma>0$. Random vectors, usually distributed according to γ_n, are denoted by $Z,W\ldots$ while the random variables by $g_i,\xi,\ldots$. The notation $\mathbb{E}(\cdot)$ is used for the expectation. The moments with respect to γ_n of norms whose unit ball is the body K are denoted by

$$I_r(\gamma_n,K):=(\mathbb{E}\|Z\|_K^r)^{\frac{1}{r}}=\left(\int_{\mathbb{R}^n}\|z\|_K^r d\gamma_n(z)\right)^{\frac{1}{r}}$$

and more generally, for an arbitrary probability measure ν as $I_r(\nu,K)$. Recall the pth moment σ_p of a standard Gaussian random variable g

$$\sigma_p^p:=\mathbb{E}|g|^p=\frac{2^{\frac{p}{2}}}{\sqrt{\pi}}\Gamma\left(\frac{p+1}{2}\right)\sim\sqrt{\frac{2}{e}}\left(\frac{p+1}{e}\right)^{\frac{p}{2}},\qquad p\to\infty,\qquad(96)$$

where $f\sim h$ means $f(t)/h(t)\to 1$ as $t\to\infty$. We write $f\lesssim h$ when there exists absolute constant $C>0$ such that $f\le Ch$. We write $f\simeq h$ if $f\lesssim h$ and h f, whereas the notation $f\ \lesssim_p h$ means that the involved constant depends only on p. The letters $C,c,C_1,c_0,\ldots$ are frequently used in order to denote absolute constants which may differ from line to line.

The random version of Dvoretzky's theorem due to Milman from [27] (for the optimal dependence on ε see [331] and [311]) reads as follows.

Theorem (6.3.3)[320]: (Milman, Gordon). Let $X=(\mathbb{R}^n,\|\cdot\|)$ be a normed space. Define the critical dimension of X as the quantity

$$k(X):=\frac{\mathbb{E}\|Z\|^2}{b^2(X)},\qquad Z\sim N(0,I_n)$$

where $b(X):=\max_{\theta\in S^{n-1}}\|\theta\|$. Then, for every $\varepsilon\in(0,1)$ and for any $k\le c\varepsilon^2k(X)$ the random (with respect to the Haar measure on the Grassmannian $G_{n,k}$) k-dimensional subspace F of X is $(1+\varepsilon)$-spherical, i.e.

$$\frac{1-\varepsilon}{M}B_F \subseteq B_X \cap F \subseteq \frac{1+\varepsilon}{M}B_F,$$

with probability greater than $1-e^{-c\varepsilon^2 k(X)}$, where $M = M(X) = \int_{S^{n-1}}\|\theta\|d\sigma(\theta)$ and σ is the uniform probability measure on S^{n-1}.

Let ν be a Borel probability measure on $\mathbb{R}^n$ which satisfies a log-Sobolev inequality with constant $\rho > 0$

$$Ent_\nu(f^2) := \int f^2 \log f^2 \, d\nu - \int f^2 d\nu \log\left(\int f^2 d\nu\right) \leq \frac{2}{\rho}\int_{\mathbb{R}^n}\|\nabla f\|_2^2 d\nu,$$

for all smooth (or locally Lipschitz) functions $f: \mathbb{R}^n \to \mathbb{R}$. The n-dimensional Gaussian measure satisfies the log-Sobolev inequality with $\rho = 1$ (see [121]). The next lemma can be found in [321]. However, we provide a sketch of proof.

Lemma (6.3.4)[320]: (Aida–Stroock). Let ν be a Borel probability measure on $\mathbb{R}^n$ which satisfies a log-Sobolev inequality with constant ρ. Then, for any smooth function $f: \mathbb{R}^n \to \mathbb{R}$ we have

$$\|f\|_{L_q(\nu)}^2 - \|f\|_{L_p(\nu)}^2 \leq \frac{1}{\rho}\int_p^q \| \|\nabla f\|_2\|_{L^s(\nu)}^2 ds, \quad (97)$$

for all $2 \leq p \leq q$. In particular, if f is Lipschitz continuous, then we have

$$\|f\|_{L_p(\nu)}^2 - \|f\|_{L_p(\nu)}^2 \leq \frac{\|f\|_{Lip}^2}{\rho}(q-p).$$

It follows that

$$\|f\|_{L_q(\nu)}/\|f\|_{L_2(\nu)} \leq \sqrt{1+\frac{q-2}{\rho k(f)}},$$

for $q \geq 2$, where $k(f) := \|f\|_{L_2(\nu)}^2/\|f\|_{Lip}^2$.

Proof. For $p \geq 2$ we define $I(p) := \|f\|_{L_p}$. Differentiation with respect to p yields

$$\frac{dI}{dp} = \frac{Ent_\nu(|f|^p)}{p^2 I(p)^{p-1}}.$$

Applying the log-Sobolev inequality for $g = |f|^{\frac{p}{2}}$ we obtain

$$\frac{dI}{dp} \leq \frac{1}{2\rho I(p)^{p-1}}\int_{\mathbb{R}^n}|f|^{p-2}\|\nabla f\|_2^2 d\nu \leq \frac{1}{2\rho I(p)^{p-1}} I(p)^{p-2}\| \|\nabla f\|_2\|_{L_p(\nu)}^2,$$

by Hölder's inequality. This shows that $(I(p)^2)' \leq \frac{1}{\rho}\| \|\nabla f\|_2\|_{L_p(\nu)}^2$. Integration over the interval $[p, q]$ proves (97).

Given any finite Borel measure μ on S^{n-1} (which is not supported in any hyperplane) and any $1 \leq p < \infty$ we can equip $\mathbb{R}^n$ with the norm

$$\|x\|_{\mu,p} := \left(\int_{S^{n-1}}|x,\theta|^p d\mu(\theta)\right)^{\frac{1}{p}}.$$

It's clear that the space $X = \left(\mathbb{R}^n, \|\cdot\|_{\mu,p}\right)$ can be naturally embedded into $L_p(S^{n-1}, \mu)$ via the linear isometry $U: X \to L_p(S^{n-1}, \mu)$ with $Ux := \langle x, \cdot\rangle$.

Lewis' fundamental result from [337], states that the previous situation can always be realized for finite-dimensional subspaces of $L_p(\nu)$ after a suitable change of the density ν (see also [347] for an alternative proof which extends to the whole range $0 < p < \infty$ and arises as a solution of an optimization problem). The formulation we use here follows the exposition from [338].

Theorem (6.3.5)[320]: (Lewis). Let $1 \le p < \infty$ and let X be an n-dimensional subspace of L_p. Then, there exists an even Borel measure μ on S^{n-1} which satisfies

$$\|x\|_2^2 = \int_{S^{n-1}} |\langle x, \theta\rangle|^2 d\mu(\theta), \tag{98}$$

for all $x \in \mathbb{R}^n$ and the normed space $\left(\mathbb{R}^n, \|\cdot\|_{\mu,p}\right)$ is isometric to X.

When X is an n-dimensional subspace of L_p, one can always find an invertible linear transformation T such that $T(B_X) = B_p(\mu)$ for some μ as above. Then, we will say that $T(B_X)$ is in Lewis' position. It is clear that taking into account this representation for any finite-dimensional subspace of L_p, the problem of embedding ℓ_2^k in subspaces of L_p reduces to spaces $\left(\mathbb{R}^n, \|\cdot\|_{\mu,p}\right)$ with μ satisfying the condition (98). Hence, the next paragraph is devoted to the study of these measures.

An even Borel measure μ on S^{n-1} is said to be isotropic if it satisfies the following condition:

$$\|x\|_2^2 = \int_{S^{n-1}} |\langle x, \theta\rangle|^2 d\mu(\theta),$$

for all $x \in \mathbb{R}^n$. Equivalently, for all linear transformations $T: \mathbb{R}^n \to \mathbb{R}^n$ we have

$$\text{trace}(T) = \int_{S^{n-1}} \langle \theta, T\theta\rangle \, d\mu(\theta).$$

For any such measure we may define the following family of centrally symmetric convex bodies $B_p(\mu)$ associated with μ and corresponding norms:

$$x \mapsto \|x\|_{B_p(\mu)} := \|\langle x, \cdot\rangle\|_{L_p(\mu)} = \left(\int_{S^{n-1}} |\langle x, z\rangle|^p d\mu(z)\right)^{\frac{1}{p}}, \qquad 1 \le p < \infty.$$

The corresponding spaces, whose unit ball is $B_p(\mu)$, will be denoted by $X_p(\mu)$. Under this terminology and notation, Lewis' theorem reads as follows:

Theorem (6.3.6)[320]: (Lewis). Let $1 \le p < \infty$ and let X be an n-dimensional subspace of L_p. Then, there exists an isotropic Borel measure μ on S^{n-1} and a linear isometry $U: X_p(\mu) \to X$.

In the next lemma we collect several properties for the bodies $B_p(\mu)$. To this end, recall the definition of σ_q from (96).

Lemma (6.3.7)[320]: Let μ be a Borel isotropic measure on S^{n-1} and let Z be an n-dimensional standard Gaussian vector. Then, we have the following properties:

(i) $\mathbb{E}\|Z\|_{B_q(\mu)}^q = \sigma_q^q \mu(S^{n-1})$, for $0 < q < \infty$.

(ii) $\mu(S^{n-1}) = n$.

(iii) For $p \ge 2$ we have $\|x\|_{B_p(\mu)} \le \|x\|_2$ and for $1 \le p < q < \infty$ we have $\|x\|_{B_p(\mu)} \le n^{\frac{1}{p}-\frac{1}{q}}\|x\|_{B_q(\mu)}$, for all $x \in \mathbb{R}^n$.

(iv) (K. Ball). For every $1 \le p < \infty$ we have $|B_p(\mu)| \le \left|B_p^n\right|$.

(v) For the body $B_q(\mu), q \ge 1$ we have $k\left(B_q(\mu)\right) \ge cn^{\min\left\{\frac{1,2}{q}\right\}}$.

(vi) There exists an absolute constant $c > 0$ such that for all $2 \le q \le c \log n$, one has $\left(\mathbb{E}\|Z\|_{B_q(\mu)}^2\right)^{\frac{1}{2}} \simeq q^{\frac{1}{2}} n^{\frac{1}{q}}$. In particular, for those q's one has $k\left(B_q(\mu)\right) \ge cqn^{\frac{2}{q}}$.

Proof. (i). We use Fubini's theorem and the rotation invariance of the Gaussian measure to write

$$\mathbb{E}\|Z\|_{B_q(\mu)}^q = \int_{\mathbb{R}^n} \|x\|_{B_q(\mu)}^q d\gamma_n(x) = \int_{S^{n-1}} \int_{\mathbb{R}^n} |\langle x, \theta\rangle|^q d\gamma_n(x) d\mu(\theta) = \sigma_q^q \mu(S^{n-1}).$$

(ii). It follows from the above formula, applied for $q = 2$, and by employing the isotropic condition.

(iii). Let $p \ge 2$. Note that for all $u \in S^{n-1}$ we have

$$\|u\|_{B_p(\mu)}^p = \int_{S^{n-1}} |\langle u, \theta\rangle|^p d\mu(\theta) \le \int_{S^{n-1}} |\langle u, \theta\rangle|^2 d\mu(\theta) = 1.$$

For $1 \le p \le q$ we apply Hölder's inequality

$$\|x\|_{B_p(\mu)} = \left(\int_{S^{n-1}} |\langle x, \theta\rangle|^p d\mu(\theta)\right)^{\frac{1}{p}} \le \mu(S^{n-1})^{\frac{1}{p}-\frac{1}{q}} \left(\int_{S^{n-1}} |\langle x, \theta\rangle|^q d\mu(\theta)\right)^{\frac{1}{q}}.$$

(iv). This result is proved by K. Ball in [324]. see[324] for the details.

(v). First consider the case $1 \le q \le 2$. Using Hölder's inequality we get

$$\left(\mathbb{E}\|Z\|_{B_q(\mu)}^2\right)^{\frac{1}{2}} \ge \left(\mathbb{E}\|Z\|_{B_q(\mu)}^q\right)^{\frac{1}{q}} = \sigma_q n^{\frac{1}{q}} \ge cn^{\frac{1}{q}},$$

where we have also used (i) and (ii). Also note that (iii) implies $b\left(B_q(\mu)\right) \le n^{\frac{1}{q}-\frac{1}{2}}$ for $1 \le q \le 2$.

Now we turn in the range $2 < q < \infty$. Note that $b\left(B_q(\mu)\right) \le 1$, by (iii). Furthermore,

$$\mathbb{E}\|Z\|_{B_q(\mu)}^2 \ge n^{\frac{2}{q}-1} \mathbb{E}\|Z\|_2^2 = n^{\frac{2}{q}},$$

again by (iii). Combining the above and recalling the definition of $k\left(X_q(\mu)\right)$ we get the desired estimate.

(vi). We define the parameter

$$q_0 \equiv q_0(\mu) := \max\left\{q \in [2, n] : k\left(B_p(\mu)\right) \ge p, \forall p \in [2, q]\right\}.$$

The continuity of the map $p \longmapsto k\left(B_p(\mu)\right)$ and the fact that $k\left(B_q(\mu)\right) \le n$ for all $q \ge 2$, whereas $k\left(B_2(\mu)\right) = n$, implies that $q_0 = k\left(B_{q_0}(\mu)\right)$ and $k\left(B_p(\mu)\right) \ge p$ for $2 \le p \le q_0$. Lemma (6.3.4) then, yields $\left(\mathbb{E}\|Z\|_{B_p(\mu)}^p\right)^{\frac{1}{p}} \le c_1 \left(\mathbb{E}\|Z\|_{B_p(\mu)}^2\right)^{\frac{1}{2}}$ for all $2 \le p \le q_0$. Thus, for all $2 \le p \le q_0$ we get

$$k\left(B_p(\mu)\right) = \frac{\mathbb{E}\|Z\|_{B_p(\mu)}^2}{b^2\left(B_p(\mu)\right)} \ge c_1^{-2} \left(\mathbb{E}\|Z\|_{B_p(\mu)}^p\right)^{\frac{2}{p}} = c_1^{-2} \sigma_p^2 n^{\frac{2}{p}} \ge c_2 p n^{\frac{2}{p}},$$

where we have also used that $b\left(B_p(\mu)\right) \le 1$. In particular, for $p = q_0$, we obtain

$$q_0 = k\left(B_{q_0}(\mu)\right) \ge c_2 q_0 n^{\frac{2}{q_0}} \Rightarrow q_0 \ge c_3 \log n.$$

This can be interpreted as $k\left(B_q(\mu)\right) \geq ck(\ell_q^n)$, provided that $2 \leq q \leq c \log n$ for some absolute constant $c > 0$. For a proof of the fact that $k(\ell_q^n) \simeq q n^{\frac{2}{q}}$ when $2 \leq q \leq c \log n$ see [346].

Lemma (6.3.8)[320]: Let μ be a Borel isotropic measure on S^{n-1}. For $q \geq 2$ and for all $r \geq 1$ we have

$$\frac{I_{rq}\left(\gamma_n, B_q(\mu)\right)}{I_q\left(\gamma_n, B_q(\mu)\right)} \leq \sqrt{1 + \frac{q(r-1)}{\sigma_q^2 n^{\frac{2}{q}}}} \leq \sqrt{1 + \frac{c(r-1)}{n^{\frac{2}{q}}}},$$

where $c > 0$ is an absolute constant.

Proof. Note that Lemma (6.3.7) (iii) implies $\left| \|x\|_{B_q(\mu)} - \|y\|_{B_q(\mu)} \right| \leq \|x - y\|_2$ for all $x, y \in \mathbb{R}^n$. Hence, if we use Lemma (6.3.4) we obtain

$$\left(\frac{I_{rq}}{I_q}\right)^2 \leq 1 + \frac{q(r-1)}{I_q^2} = 1 + \frac{q(r-1)}{\sigma_q^2 n^{\frac{2}{q}}},$$

where the last estimate follows from Lemma (6.3.7). Finally, using the fact that $\sigma_q \simeq \sqrt{q}$ we conclude the second estimate.

The next inequality is due to Pisier (for a proof see [342]).

Theorem (6.3.9)[320]: (Pisier). Let $\phi: \mathbb{R} \to \mathbb{R}$ be a convex function and let $f: \mathbb{R}^n \to \mathbb{R}$ be C^1-smooth. Then, if Z, W are independent copies of a Gaussian random vector, we have

$$\mathbb{E}\phi\big(f(Z) - f(W)\big) \leq \mathbb{E}\phi\left(\frac{\pi}{2}\langle \nabla f(Z), W\rangle\right).$$

Here we prove a generalization of this inequality of Gaussian processes generated by the action of a random matrix with i.i.d standard Gaussian entries on a fixed vector in S^{n-1}. The next inequality was stated in [341] without a proof.

Theorem (6.3.10)[320]: Let $\phi: \mathbb{R} \to \mathbb{R}$ be a convex function and let $f: \mathbb{R}^n \to \mathbb{R}$ be C^1-smooth. If $G = (g_{ij})_{i,j=1}^{n,k}$ is a Gaussian matrix and $a, b \in S^{k-1}$, then we have

$$\mathbb{E}\phi\big(f(Ga) - f(Gb)\big) \leq \mathbb{E}\phi\left(\frac{\pi}{2}\|a - b\|_2 \langle \nabla f(Z), W\rangle\right),$$

where Z, W are independent copies of a standard Gaussian n-dimensional random vector.

Proof. If $a = b$ then, there is nothing to prove. If $a = -b$ then, by setting $F(z) = f(z) - f(-z)$ we may write

$$\mathbb{E}\phi\big(f(Ga) - f(Gb)\big) = \mathbb{E}\phi\big(F(Z)\big) \leq \mathbb{E}\phi\big(F(Z) - F(W)\big),$$

for Z, W independent copies of a standard Gaussian random vector, where we have used the fact $\mathbb{E}F(Z) = 0$ and Jensen's inequality. Then, a direct application of Theorem (6.3.9) yields

$$\mathbb{E}\phi\big(F(Z) - F(W)\big) \leq \mathbb{E}\phi\left(\frac{\pi\langle \nabla f(Z), W\rangle + \pi\langle \nabla f(-Z), W\rangle}{2}\right)$$
$$\leq \mathbb{E}\frac{\big(\phi(\pi\langle \nabla f(Z), W\rangle) + \phi(\pi\langle \nabla f(-Z), W\rangle)\big)}{2} = \mathbb{E}\phi(\pi\langle \nabla f(Z), W\rangle),$$

by the convexity of ϕ.

In the general case, fix $a, b \in S^{k-1}$ with $a \neq \pm b$ and define $p := \frac{a+b}{2}$. Note that since $\|a\|_2 = \|b\|_2$ we have that the vector $u := a - p$ is perpendicular to p. Set $W := G(u)$ and

$Z := G(p)$ and note that W,Z are independent random vectors in $\mathbb{R}^n$ with $W \sim N(0, \|u\|_2^2 I_n), Z \sim N(0, \|p\|_2^2 I_n)$. Since $G(a) = Z + W$ and $G(b) = Z - W$, we may write

$$\mathbb{E}\phi\big(f(Ga) - f(Gb)\big) = \mathbb{E}_Z \mathbb{E}_W \phi\big(f(Z+W) - f(Z-W)\big).$$

Denote $F(w,z) := f(z+w) - f(z-w)$. Then, we may write

$$\mathbb{E}\phi\big(f(Ga) - f(Gb)\big) = \iint \phi\big(F(w,z)\big) d\gamma_{n,\sigma_1}(w) d\gamma_{n,\sigma_2}(z),$$

where $\sigma_1 = \|u\|_2 > 0, \sigma_2 = \|p\|_2 > 0$. For fixed z, we may apply Theorem (6.3.9) to the function $w \longmapsto F(w,z)$ (note that $\int F(w,z) d\gamma_{n,\sigma_1}(w) = 0$) to get

$$\begin{aligned}\int \phi\big(F(w,z)\big) d\gamma_{n,\sigma_1}(w) &\le \iint \phi\left(\frac{\pi}{2}\langle \nabla_w F(w,z), y\rangle\right) d\gamma_{n,\sigma_1}(w) d\gamma_{n,\sigma_1}(y) \\ &\le \iint \frac{\phi(\pi\langle \nabla f(w+z), y\rangle) + \phi(\pi\langle \nabla f(z-w), y\rangle)}{2} d\gamma_{n,\sigma_1}(w) d\gamma_{n,\sigma_1}(y) \\ &= \iint \phi(\pi\langle \nabla f(w+z), y\rangle) d\gamma_{n,\sigma_1}(w) d\gamma_{n,\sigma_1}(y),\end{aligned}$$

by the convexity of ϕ. Integration with respect to γ_{n,σ_2} over z provides

$$\begin{aligned}&\iint \phi\big(F(w,z)\big) d\gamma_{n,\sigma_1}(w) d\gamma_{n,\sigma_2}(z) \\ &\le \int \left[\iint \phi(\pi\langle \nabla f(w+z), y\rangle) d\gamma_{n,\sigma_1}(w) d\gamma_{n,\sigma_2}(z)\right] d\gamma_{n,\sigma_1}(y) \\ &= \int \left[\int \phi(\pi\langle \nabla f(x), y\rangle) d\big(\gamma_{n,\sigma_1} * \gamma_{n,\sigma_2}\big)(x)\right] d\gamma_{n,\sigma_1}(y) \\ &= \iint \phi(\pi\sigma_1\langle \nabla f(x), y\rangle) d\gamma_n(x) d\gamma_n(y),\end{aligned}$$

where we have used the fact that $\gamma_{n,\sigma_1} * \gamma_{n,\sigma_2} = \gamma_{n,\sigma_1^2+\sigma_2^2} \equiv \gamma_n, since\ \sigma_1^2 + \sigma_2^2 = \|a\|_2^2 = 1$. The result follows.

Remark (6.3.11)[320]: (i). Applying this for $\phi(t) = |t|^r, r \ge 1$ and taking into account the invariance of the Gaussian measure under orthogonal transformations we derive the next (r,r)-Poincaré inequalities

$$(\mathbb{E}|f(Ga) - f(Gb)|^r)^{\frac{1}{r}} \le C\sqrt{r}\|a-b\|_2 (\mathbb{E}\|\nabla f(Z)\|_2^r)^{\frac{1}{r}}, \tag{99}$$

for $a,b \in S^{k-1}$, where Z is a standard Gaussian random vector in $\mathbb{R}^n$.

(ii). Assuming further that f is L-Lipschitz we may apply Theorem (6.3.10) for $\phi(t) = e^{\lambda t}, \lambda > 0$ to get

$$\begin{aligned}\mathbb{E}\exp\Big(\lambda\big(f(Ga) - f(Gb)\big)\Big) &\le \mathbb{E}\exp\left(\lambda^2 \frac{\pi^2}{2}\|a-b\|_2^2 \|\nabla f(Z)\|_2^2\right) \\ &\le \exp(\lambda^2\, \pi_2^2 \|a-b\|_2^2 L^2).\end{aligned} \tag{100}$$

Then, Markov's inequality yields Schechtman's distributional inequality from [311]

$$\mathbb{P}(|f(Ga) - f(Gb)| > t) \le C\exp\left(-\frac{ct^2}{\|a-b\|^2 L^2}\right), \tag{101}$$

for all $t > 0$, where $a,b \in S^{k-1}$. Let us note that (100) for f being a norm, has also appeared in [348]. 3. For $a,b \in S^{k-1}$ with $a,b = 0$ the matrix G generates the vectors $Z = Ga$ and W=Gb which are independent copies of a standard n-dimensional Gaussian random vector. For example, inequality (101) reduces to the classical concentration inequality

$$\mathbb{P}(|f(Z) - f(W)| > t) \leq C \exp\left(-\frac{ct^2}{L^2}\right), \tag{102}$$

for all $t > 0$.

A direct application of the Gaussian concentration inequality (102) for the norms $\|\cdot\|_{B_p(\mu)}, 2 < p < \infty$ implies

$$\mathbb{P}\left(\left|\|Z\|_{B_p(\mu)} - I_1\right| > tI_1\right) \leq C \exp(-ct^2 I_1^2) \leq C \exp\left(-ct^2 n^{\frac{2}{p}}\right), \tag{103}$$

for all $t > 0$, where $I_1 \equiv I_1\left(\gamma_n, B_p(\mu)\right)$. It is known (see [341]) that the large deviation estimate $(t \geq 1)$ the inequality (103) provides is sharp (up to constants).

We prove that for $2 < p < \infty$ and μ isotropic Borel measure on S^{n-1}, the bodies $B_p(\mu)$ exhibit better concentration $(0 < t < 1)$ than the one implied by the Gaussian concentration inequality on $\mathbb{R}^n$ in terms of the Lipschitz constant. Later, this will be used to prove the announced dependence on ε and n in Dvoretzky's theorem for any n-dimensional subspace of L_p. Our main tool is the probabilistic inequality proved in Theorem (6.3.10) and as was formulated further in Remark (6.3.11).1.

We apply inequality (99) for $f(x) = \|x\|_{B_p(\mu)}^p = |\langle x, \theta\rangle|^p d\mu(\theta)$. To this end, we have to compute the gradient. Note that

$$\|\nabla f(x)\|_2^2 = p^2 \sum_{i=1}^{n} \left| \int_{S^{n-1}} \theta_i |x, \theta|^{p-1} sgn(\langle x, \theta\rangle) d\mu(\theta) \right|^2.$$

We also have the following:

Claim (6.3.12)[320]: For almost every $x \in \mathbb{R}^n$ we have

$$\|\nabla f(x)\|_2^2 \leq p^2 \|x\|_{B_{2p-2}(\mu)}^{2p-2}.$$

Proof. Let $b_i \equiv b_i(x) := \int_{S^{n-1}} |\langle x, z\rangle|^{p-1} sgn(\langle x, z\rangle) z_i d\mu(z)$. Using duality we may write

$$\sum_{i=1}^{n} \left(\int_{S^{n-1}} |\langle x, z\rangle|^{p-1} sgn(\langle x, z\rangle) z_i d\mu(z) \right)^2 = \max_{\theta \in S^{n-1}} \left| \sum_{i=1}^{n} b_i \theta_i \right|^2$$

$$= \max_{\theta \in S^{n-1}} \left| \int_{S^{n-1}} |\langle x, z\rangle|^{p-1} sgn(\langle x, z\rangle) \langle z, \theta\rangle d\mu(z) \right|^2 \leq \int_{S^{n-1}} |\langle x, z\rangle|^{2p-2} d\mu(z),$$

where we have used the Cauchy–Schwarz inequality and the isotropic condition.

Therefore, using the Claim and the inequality (99) we get for every $a, b \in S^{k-1}$

$$(\mathbb{E}|f(Ga) - f(Gb)|^r)^{\frac{1}{r}} \leq Cpr^{\frac{1}{2}} \|a - b\|^2 \left(\mathbb{E}\|Z\|_{B_{2p-2}(\mu)}^{r(p-1)}\right)^{\frac{1}{r}}, \tag{104}$$

for all $r \geq 1$. Now employ Lemma (6.3.8) for r being $r/2 \geq 1$ and for $q = 2p - 2 \geq 2$, in order to get

$$\left(\mathbb{E}\|Z\|_{B_{2p-2}(\mu)}^{r(p-1)}\right)^{\frac{1}{r}} \leq \left(\mathbb{E}\|Z\|_{B_{2p-2}(\mu)}^{2p-2}\right)^{\frac{1}{2}} \left(1 + \frac{(r-2)(p-1)}{\sigma_{2p-2}^2 n^{\frac{1}{p-1}}}\right)^{\frac{p-1}{2}}.$$

Taking into account Lemma (6.3.7).i, the previous estimate implies for $r \geq 2$,

$$\left(\mathbb{E}\|Z\|_{B_{2p-2}(\mu)}^{r(p-1)}\right)^{\frac{1}{r}} \leq \sigma_{2p-2}^{p-1} n^{\frac{1}{2}} \left(1 + \frac{r(p-1)}{\sigma_{2p-2}^2 n^{\frac{1}{p-1}}}\right)^{\frac{p-1}{2}}$$

$$\leq \sigma_{2p-2}^{p-1} n^{\frac{1}{2}} 2^{\frac{p-1}{2}} \max\left\{1, \frac{r^{\frac{p-1}{2}}(p-1)^{\frac{p-1}{2}}}{\sigma_{p-2}^{\frac{p-1}{2}} n^{\frac{1}{2}}}\right\}. \quad (105)$$

Combining (104) with (105) we get for all $r \geq 2$,

$$(\mathbb{E}|f(Ga) - f(Gb)|^r)^{\frac{1}{r}} < Cp\|a-b\|_2 \sigma_{2p-2}^{p-1} n^{\frac{1}{2}} 2^{\frac{p-1}{2}} \max\left\{r^{\frac{1}{2}}, \frac{r^{\frac{p}{2}}(p-1)^{\frac{p-1}{2}}}{\sigma_{p-2}^{\frac{p-1}{2}} n^{\frac{1}{2}}}\right\}.$$

We define

$$\alpha(n,p,r) := \max\left\{r^{\frac{1}{2}}, \frac{r^{\frac{p}{2}}(p-1)^{\frac{p-1}{2}}}{\sigma_{2p-2}^{p-1} n^{\frac{1}{2}}}\right\}, \qquad r > 0 \quad (106)$$

and we summarize the above discussion in the following:

Proposition (6.3.13)[320]: Let $2 < p < \infty$ and let μ be a Borel isotropic measure on S^{n-1}. If $G = (g_{ij})_{i,j}^{n,k} = 1$ is a standard Gaussian matrix and $a, b \in S^{k-1}$, then we have

$$\left(\mathbb{E}\left|\|Ga\|_{B_p(\mu)}^p - \|Gb\|_{B_p(\mu)}^p\right|^r\right)^{\frac{1}{r}} \leq Cp\|a-b\|_2 \sigma_{2p-2}^{p-1} n^{\frac{1}{2}} 2^{\frac{p}{2}} \alpha(n,p,r),$$

for all $r \geq 2$, where $\alpha(n,p,\cdot)$ is defined in (106)

We prove the main result.

Theorem (6.3.14)[320]: Let $2 < p < \infty$ and let μ be a Borel isotropic measure on S^{n-1} with $n > e^p$. Then, we have

$$\mathbb{P}\left(\left|\|Z\|_{B_p(\mu)} - \left(\mathbb{E}\|Z\|_{B_p(\mu)}^p\right)^{\frac{1}{p}}\right| \geq \varepsilon \left(\mathbb{E}\|Z\|_{B_p(\mu)}^p\right)^{\frac{1}{p}} \leq C\exp(-c\psi(n,p,\varepsilon)),$$

for every $\varepsilon > 0$, where $\psi(n,p,\cdot)$ is defined by

$$\psi(n,p,t) := \min\left\{\frac{t^2 n}{p^{4^p}}, (tn)^{\frac{2}{p}}\right\}, \qquad t > 0, \quad (107)$$

and $C, c > 0$ are absolute constants.

Proof. Using Proposition (6.3.13) for $a, b \in S^{k-1}$ with $\langle a,b\rangle = 0$ and applying Jensen's inequality we obtain

$$\left(\mathbb{E}\left|\|Z\|_{B_p(\mu)}^p - \mathbb{E}\|Z\|_{B_p(\mu)}^p\right|^r\right)^{\frac{1}{r}} \leq Cp\sigma_{2p-2}^{p-1} n^{\frac{1}{2}} 2^{\frac{p}{2}} \alpha(n,p,r),$$

for all $r \geq 2$. Therefore, Markov's inequality yields

$$\mathbb{P}\left(\left|\|Z\|_{B_p(\mu)}^p - \mathbb{E}\|Z\|_{B_p(\mu)}^p\right| > \varepsilon\right) \leq \left(\frac{Cp\sigma_{2p-2}^{p-1} n^{\frac{1}{2}} 2^{\frac{p}{2}} \alpha(n,p,r)}{\varepsilon}\right)^r. \quad (108)$$

Note that the inverse of the map $r \mapsto \alpha(n,p,r)$ is given by

$$\alpha^{-1}(n,p,s) = \min\left\{s^2, \frac{s^{\frac{2}{p}} n^{\frac{1}{p}} \sigma_{2p-2}^{\frac{2p-2}{p}}}{(p-1)^{\frac{p-1}{p}}}\right\}, \qquad s > 0.$$

Hence, for every $\varepsilon > 0$ there exists $r_\varepsilon > 0$ such that $\alpha(n,p,r_\varepsilon) = \frac{\varepsilon}{eCp\sigma_{2p-2}^{p-1} n^{\frac{1}{2}} 2^{\frac{p}{2}}}$. One may check that

$$r_\varepsilon = \alpha^{-1}\left(n, p, \frac{\varepsilon}{eCp\sigma_{2p-2}^{p-1} n^{\frac{1}{2}} 2^{\frac{p}{2}}}\right) \simeq \min\left\{\frac{\varepsilon^2}{np^2 2^p \sigma_{2p-2}^{2p-2}}, \frac{\varepsilon^{\frac{2}{p}}}{p}\right\}.$$

As long as the range of $\varepsilon > 0$ satisfies $\alpha(n,p,r_\varepsilon) \geq \alpha(n,p,2)$ we may insert r_ε into the probabilistic bound (108) or else, $\alpha(n,p,r_\varepsilon) < \alpha(n,p,2) \simeq \max\left\{1, \left(\frac{e^p}{n}\right)^{\frac{1}{2}}\right\} \simeq 1$, provided that n is large enough with respect to p, and we may upper bound the aforementioned probability by an absolute constant $C_1 > 0$. In each case we obtain

$$\mathbb{P}\left(\left|\|Z\|_{B_p(\mu)}^p - \mathbb{E}\|Z\|_{B_p(\mu)}^p\right| > \varepsilon\right) \leq C_1 \exp\left(-c_1\, min\left\{\frac{\varepsilon^2}{np^2 2^p \sigma_{2p-2}^{2p-2}}, \frac{\varepsilon^{\frac{2}{p}}}{p}\right\}\right),$$

for every $\varepsilon > 0$. It follows that

$$\mathbb{P}\left(\left|\|Z\|_{B_p(\mu)}^p - \mathbb{E}\|Z\|_{B_p(\mu)}^p\right| > \varepsilon\mathbb{E}\|Z\|_{B_p(\mu)}^p\right) \leq C_1 \exp\left(-c_1\, min\left\{\frac{\varepsilon^2 n\sigma_p^{2p}}{p^2 2^p \sigma_{2p-2}^{2p-2}}, \frac{(\varepsilon n)^{\frac{2}{p}}\sigma_p^2}{p}\right\}\right),$$

for every $\varepsilon > 0$. The asymptotic estimate (96) yields $\sigma_p^{2p}/\sigma_{2p-2}^{2p-2} \simeq p^{2-p}$ and $\sigma_p \simeq p^{\frac{1}{2}}$, thus we conclude

$$\mathbb{P}\left(\left|\|Z\|_{B_p(\mu)}^p - \mathbb{E}\|Z\|_{B_p(\mu)}^p\right| > \varepsilon\mathbb{E}\|Z\|_{B_p(\mu)}^p\right) \leq C_1 \exp\left(-c_1'\, min\left\{\frac{\varepsilon^2 n}{p4^p}, (\varepsilon n)^{\frac{2}{p}}\right\}\right), \quad (109)$$

for all $\varepsilon > 0$. This further implies that

$$\mathbb{P}\left(\left|\|Z\|_{B_p(\mu)} - \left(\mathbb{E}\|Z\|_{B_p(\mu)}^p\right)^{\frac{1}{p}}\right| > \varepsilon\left(\mathbb{E}\|Z\|_{B_p(\mu)}^p\right)^{\frac{1}{p}}\right) \leq 2C_1 \exp\left(-c_1'\, min\left\{\frac{\varepsilon^2 n}{p4^p}, (\varepsilon n)^{\frac{2}{p}}\right\}\right),$$

for all $\varepsilon > 0$. In order to verify the latter we may write

$$\mathbb{P}\left(\|Z\|_{B_p(\mu)} > (1+\varepsilon)\left(\mathbb{E}\|Z\|_{B_p(\mu)}^p\right)^{\frac{1}{p}}\right) \leq \mathbb{P}\left(\|Z\|_{B_p(\mu)}^p > (1+\varepsilon)\mathbb{E}\|Z\|_{B_p(\mu)}^p\right) \leq C_1 \exp\left(-c_1'\, min\left\{\frac{\varepsilon^2 n}{p4^p}, (\varepsilon n)^{\frac{2}{p}}\right\}\right),$$

for all $\varepsilon > 0$ by the estimate (109). We argue similarly for the lower tail.

We shall also need the next variant of Theorem (6.3.14).

Theorem (6.3.15)[320]: Let $2 < p < \infty$ and let μ be a Borel isotropic probability measure on S^{n-1} with $n > e^p$. If $G = (g_{ij})_{i,j=1}^{n,k}$ is a Gaussian matrix and $a, b \in S^{k-1}$, then

$$\mathbb{P}\left(\left|\|Ga\|_{B_p(\mu)}^p - \|Gb\|_{B_p(\mu)}^p\right| > t\mathbb{E}\|Z\|_{B_p(\mu)}^p\right) \leq C \exp\left(-c\psi\left(n, p, \frac{t}{\|a-b\|_2}\right)\right),$$

for all $t > 0$, where $\psi(n,p,\cdot)$ is defined in (107).

Proof. The proof is similar to the proof of Theorem (6.3.14).

Using Proposition (6.3.13) we may derive an estimate for the variance of the norm $x \mapsto \|x\|_{B_p(\mu)}$. To this end, we will also need a result from [111].

Theorem (6.3.16)[320]: (Klartag–Vershynin). Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ with Dvoretzky number k. Then, for any $0 < r < ck$ one has

$$(E\|Z\|^{-r})^{-\frac{1}{r}} \geq c\mathbb{E}\|Z\|,$$

where Z is the standard Gaussian vector in $\mathbb{R}^n$ and $c > 0$ is an absolute constant.

This fact complements the previously known result due to Litvak, Milman and Schechtman [124], which asserts that the same phenomenon occurs for the positive moments. Namely, we have the following:

Theorem (6.3.17)[320]: (Litvak–Milman–Schechtman). Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ with Dvoretzky number k. Then, for any $0 < r < ck$ one has

$$(\mathbb{E}\|Z\|^{r})^{\frac{1}{r}} \leq C\mathbb{E}\|Z\|,$$

where Z is the standard Gaussian vector in $\mathbb{R}^n$ and $C, c > 0$ are absolute constants.

Note that the latter also follows from Lemma (6.3.4). By taking into account both of them, we derive the following unified estimate:

$$(\mathbb{E}\|Z\|^{r})^{\frac{1}{r}} \leq C(\mathbb{E}\|Z\|^{-r})^{-\frac{1}{r}}, \qquad 0 < r < ck. \tag{110}$$

This reverse Hölder estimate will be used in the sequel.

We prove the aforementioned estimate.

Theorem (6.3.18)[320]: (Gaussian variance for $B_p(\mu)$). Let $1 \leq p < \infty$ and let μ be an isotropic Borel measure on S^{n-1}. Then,

$$Var\|Z\|_{B_p(\mu)} \leq e^{cp} n^{\frac{2}{p}-1}.$$

In particular, we have

$$\frac{Var\|Z\|_{B_p(\mu)}}{\mathbb{E}\|Z\|_{B_p(\mu)}^2} \leq \frac{e^{cp}}{n},$$

where Z is the standard Gaussian vector in $\mathbb{R}^n$.

Proof. We may clearly assume that $n \geq e^{Cp}$ for some sufficiently large absolute constant $C > 0$, otherwise the conclusion is trivially true. In order to see that, first recall from Lemma (6.3.7) that

$$b\left(B_p(\mu)\right) \leq \max\left\{n^{\frac{1}{p}-\frac{1}{2}}, 1\right\}, \qquad \mathbb{E}\|Z\|_{B_p(\mu)}^2 \geq cn^{\frac{2}{p}}, \qquad 1 \leq p < \infty.$$

Whence, in the light of (99) for $r = 2$ and $\langle a, b\rangle = 0$, we get

$$Var\|Z\|_{B_p(\mu)} \leq C\mathbb{E}\left\|\nabla\|Z\|_{B_p(\mu)}\right\|_2^2 \leq Cb^2\left(B_p(\mu)\right) \leq C\max\left\{n^{\frac{2}{p}-1}, 1\right\}.$$

These estimates already prove the assertions when $1 \leq p \leq 2$. Thus, we may focus in the case $2 < p < \infty$ with $n \geq e^{Cp}$. We consider Z' an independent copy of Z to write

$$2Var\|Z\|_{B_p(\mu)} = \mathbb{E}\left(\|Z\|_{B_p(\mu)} - \|Z'\|_{B_p(\mu)}\right)^2 \leq \frac{1}{p^2}\mathbb{E}\left(\frac{\|Z\|_{B_p(\mu)}^p - \|Z'\|_{B_p(\mu)}^p}{\min\left\{\|Z\|_{B_p(\mu)}^{p-1}, \|Z'\|_{B_p(\mu)}^{p-1}\right\}}\right)^2,$$

where we have used the numerical inequality $|t^p - s^p| \geq p|t - s|\min\{t^{p-1}, s^{p-1}\}$ for $t, s > 0$ and $p > 1$. The Cauchy–Schwarz inequality implies that

$$Var\|Z\|_{B_p(\mu)} \leq \frac{2}{p^2}\frac{\left(\mathbb{E}\left|\|Z\|^p_{B_p(\mu)} - \|Z'\|^p_{B_p(\mu)}\right|^4\right)^{\frac{1}{2}}}{I^{2(p-1)}_{-4(p-1)}\left(\gamma_n, B_p(\mu)\right)},$$

where we have used the fact that $\frac{1}{\min\{t,s\}} \leq \frac{1}{t} + \frac{1}{s}, t, s > 0$. The numerator is directly estimated by Proposition (6.3.13) (for $\langle a, b\rangle = 0$). Standard computations, based on (96), yield the bound

$$\left(\mathbb{E}\left|\|Z\|^p_{B_p(\mu)} - \|Z'\|^p_{B_p(\mu)}\right|^4\right)^{\frac{1}{2}} \leq e^{C_1 p} p^p n. \tag{111}$$

For the denominator, we employ estimate (110) for the norm $\|\cdot\|_{B_p(\mu)}$ (and for $r = 4(p-1)$) along with the fact $k\left(B_p(\mu)\right) \geq c_1 p n^{\frac{2}{p}}$ for $n \geq e^{C_2 p}$ (proved in Lemma (6.3.7).vi) to obtain

$$I_{-4(p-1)}\left(\gamma_n, B_p(\mu)\right) \geq c_2 I_p\left(\gamma_n, B_p(\mu)\right) = c_2 \sigma_p n^{\frac{1}{p}}, \tag{112}$$

where in the last step we have used Lemma (6.3.7).i, ii. Combining (111) with (112) we arrive at the estimate

$$Var\|Z\|_{B_p(\mu)} \leq \frac{e^{C_1 p} p^p n}{e^{-C_3 p}\sigma_p^{2p-2} n^{2-\frac{2}{p}}} \leq \frac{e^{C_4 p}}{n^{1-\frac{2}{p}}},$$

where we have used once again that $\sigma_p^p \simeq \left(\frac{p}{e}\right)^{\frac{p}{2}}$. The result follows.

We prove the improved estimate on Dvoretzky's theorem for the subspaces of $L_p, 2 < p < \infty$.

Theorem (6.3.19)[320]: Let $2 < p < \infty$ and let X be an n-dimensional subspace of L_p. For any $0 < \varepsilon < 1$ there exist $k \geq c_p \psi(n, p, \varepsilon)$ and linear map $T: \ell_2^k \to X$ such that $\|x\|_2 \leq \|Tx\|_X \leq (1+\varepsilon)\|x\|_2$ for all $x \in \ell_2^k$, where $c_p > 0$ is constant depending only on p and $\psi(n, p, \cdot)$ is given by

$$\psi(n, p, t) = \min\left\{\frac{t^2 n}{p 4^p}, (tn)^{\frac{2}{p}}\right\}.$$

Clearly, the above theorem follows from the random version of Dvoretzky's theorem for subspaces of L_p whose unit ball is in Lewis' position, or equivalently for the bodies $B_p(\mu)$ with μ being an isotropic measure on S^{n-1}. More precisely, we have the following:

We have established the concentration estimate of Theorem (6.3.14), then a standard net argument yields the result with an extra $\log\left(\frac{1}{\varepsilon}\right)$ term. Indeed; fix $k \leq n$ and let $G = \left(g_{ij}\right)_{i,j=1}^{n,k}$ be a Gaussian matrix with independent standard entries. Let $\mathcal{D}$ be a δ-net on S^{k-1} with cardinality $|\mathcal{D}| \leq \left(\frac{3}{\delta}\right)^k$ (see [339] for the details). Then using the union bound, Theorem (6.3.14) and the fact that Gu is equidistributed to $Z \sim N(0, I_n)$ we obtain

$$\mathbb{P}\left(\exists\, u \in \mathcal{D}: \left|\|Gu\|_{B_p(\mu)} - \mathbb{E}\|Z\|_{B_p(\mu)}\right| \geq \varepsilon \mathbb{E}\|Z\|_{B_p(\mu)}\right) \leq \left(\frac{3}{\delta}\right)^k C\exp\left(-c\psi(n, p, \varepsilon)\right).$$

Choosing $\delta \simeq \varepsilon$ we find that with probability greater than $1 - e^{-c'\psi(n,p,\varepsilon)}$ the random operator G satisfies

$$\left|\|Gu\|_{B_p(\mu)} - \mathbb{E}\|Z\|_{B_p(\mu)}\right| \le \varepsilon\mathbb{E}\|Z\|_{B_p(\mu)},$$

for all $u \in \mathcal{D}$, as long as $k \lesssim \left(\log\frac{1}{\varepsilon}\right)^{-1} \psi(n,p,\varepsilon)$. It's routine to check that we may pass to the whole sphere S^{k-1} at cost of an oscillation at most $2\varepsilon\mathbb{E}\|Z\|_{B_p(\mu)}$, see e.g. [339].

Theorem (6.3.15) serves exactly the purpose of removing this term. Then we use this inequality along with a chaining method to conclude the logarithmic-free dependence on ε in our main result. This approach has been inspired by [311]. However, the method from [311] is not directly applicable here, since it lies in estimates involving the Lipschitz constant. As we have already explained such estimates would only yield suboptimal bounds and one has to keep track of the higher moments of the length of the gradient until the very last step. This forces us to establish the inequality in Theorem (6.3.10). In probabilistic terms Theorem (6.3.15) says that the process $\left(\|G\theta\|_{B_p(\mu)}^p - I_p^p\right)_{\theta\in S^{k-1}}$ has two-level tail behavior described by $\psi(n,p,\cdot)$.

We proving the main result.

Theorem (6.3.20)[320]: Let $2 < p < \infty$ and let X be an n-dimensional subspace of L_p, represented on $\mathbb{R}^n$, whose unit ball B_X is in Lewis' position. Then, for any $\varepsilon \in (0,1)$ there exists $k \ge c_p\psi(n,p,\varepsilon)$ such that the random k-dimensional subspace F of X is $(1+\varepsilon)$-spherical with probability greater than $1 - e^{-c_p\psi(n,p,\varepsilon)}$, where $c_p > 0$ depends only on p and $\psi(n,p,\cdot)$ is defined in (107).

Proof. We have to show that the ball $B_p(\mu)$ has random almost spherical k-dimensional with k as large as possible. Let $\{g_{ij}(\omega)\}_{i,j=1}^{n,k}$ be i.i.d. standard normals in some probability space $(\Omega,\mathbb{P})$ and consider the random Gaussian operator $G_\omega = \left(g_{ij}(\omega)\right)_{i,j=1}^{n,k} : \ell_2^k \to X_p(\mu)$. We will prove that with overwhelming probability the operator G is $(1+\varepsilon)$-isomorphic embedding when k is relatively large. To this end, we employ Theorem (6.3.15) and the chaining argument from [311]. For each $j = 1,2,\dots$ consider δ_j-nets $\mathcal{N}_j$ on S^{k-1} with cardinality $|\mathcal{N}_j| \le \left(\frac{3}{\delta_j}\right)^k$ (see [339]). Note that for any $\theta \in S^{k-1}$ and for all j there exist $u_j \in \mathcal{N}_j$ with $\|\theta - u_j\|_2 \le \delta_j$ and by the triangle inequality it follows that $\|u_j - u_{j-1}\|_2 \le \delta_j + \delta_{j-1}$. Moreover, if we assume that $\delta_j \to 0$ as $j \to \infty$ and (t_j) is a sequence of numbers with $t_j \ge 0$ and $\sum_j t_j \le 1$ then, for any $\varepsilon > 0$ we have the next claim.

Claim (6.3.21)[320]: If we define the following sets:

$$A := \left\{\omega \middle| \exists\theta \in S^{k-1}: \left|\|G_\omega(\theta)\|_{B_p(\mu)}^p - I_p^p\right| > \varepsilon I_p^p\right\},$$

$$A_1 := \left\{\omega \middle| \exists u_1 \in \mathcal{N}_1: \left|\|G_\omega(u_1)\|_{B_p(\mu)}^p - I_p^p\right| > t_1\varepsilon I_p^p\right\}$$

and for $j \ge 2$

$$A_j := \left\{\omega \middle| \exists u_j \in \mathcal{N}_j, u_{j-1} \in \mathcal{N}_{j-1}: \left|\left\|G_\omega(u_j)\right\|_{B_p(\mu)}^p - \left\|G_\omega(u_{j-1})\right\|_{B_p(\mu)}^p\right| > t_j\varepsilon I_p^p\right\},$$

where $I_p \equiv I_p\left(\gamma_n, B_p(\mu)\right)$, then the inclusion $A \subseteq \bigcup_{j=1}^\infty A_j$ holds.

Proof. If $\omega \notin \bigcup_{j=1}^\infty A_j$ then for any j and any $u_j \in \mathcal{N}_j$ we have

$$\left|\|G_\omega(u_1)\|_{B_p(\mu)}^p - I_p^p\right| \le \varepsilon t_1 I_p^p \qquad \text{and}$$

$$\left| \|G_\omega(u_j)\|^p_{B_p(\mu)} - \|G_\omega(u_{j-1})\|^p_{B_p(\mu)} \right| \leq \varepsilon t_j I_p^p, \qquad j = 2, 3, \ldots$$

For any θ there exist $u_j \in \mathcal{N}_j$ such that $\|\theta - u_j\|_2 < \delta_j$ for $j = 1, 2, \ldots$ Hence, for any $N \geq 2$ we may write

$$\left| \|G_\omega(\theta)\|^p_{B_p(\mu)} - I_p^p \right|$$

$$\leq \left| I_p^p - \|G_\omega(u_1)\|^p_{B_p(\mu)} \right| + \sum_{j=2}^{N} \left| \|G_\omega(u_{j-1})\|^p_{B_p(\mu)} - \|G_\omega(u_j)\|^p_{B_p(\mu)} \right| +$$

$$+ \left| \|G_\omega(u_N)\|^p_{B_p(\mu)} - \|G_\omega(\theta)\|^p_{B_p(\mu)} \right| \leq \sum_{j=1}^{N} \varepsilon t_j \, I_p^p + p \cdot \delta_N \cdot \|G_\omega\|^p_{2 \to X_p(\mu)},$$

which proves the assertion, since N is arbitrary.

Fix $0 < \varepsilon < 1$. Choose $\delta_j = e^{-j}, t_j = j^{\frac{p}{2}} e^{-j} / a_p$ with $a_p := \sum_{j=1}^{\infty} j^{\frac{p}{2}} e^{-j}$ (thus, $\sum_j t_j \leq 1$). First note the following elementary property of $\psi(n, p, \cdot)$:

$$\psi(n, p, t_s) \geq \min\left\{ s^2, s^{\frac{2}{p}} \right\} \psi(n, p, t), \qquad t, s > 0. \tag{113}$$

Then, if we employ the previous claim and Theorem (6.3.15) we may write

$$\mathbb{P}(A) \leq C|\mathcal{N}_1| \exp\big(-c_1 \psi(n, p, \varepsilon t_1)\big) + C \sum_{j=2}^{\infty} |\mathcal{N}_{j-1}| \cdot |\mathcal{N}_j| \exp\left(-c_1 \psi\left(n, p, \varepsilon t_j \frac{e^j}{4}\right)\right)$$

$$\leq 2C \sum_{j=1}^{\infty} \left(3e^j\right)^{2k} \exp\left(-c_1' \psi\left(n, p, \varepsilon j^{\frac{p}{2}} a_p^{-1}\right)\right),$$

where we have used that $|\mathcal{N}_j| \leq e^{-j}, \|u_j - u_{j-1}\|_2 \leq e^{-j} + e^{j-1} < 4e^{-j}$ and inequality (113) for $s = 1/4$. Applying estimate (113) again, first for $s = a_p^{-1}$ and then for $s = j^{\frac{p}{2}}$, we may further bound as follows:

$$\mathbb{P}(A) \leq C_1 \sum_{j=1}^{\infty} \exp\left(c_2 jk - c_2' a_p^{-2} \psi\left(n, p, \varepsilon j^{\frac{p}{2}}\right)\right) \leq C_1 \sum_{j=1}^{\infty} \exp\left(c_2 jk - c_2' a_p^{-2} j \psi(n, p, \varepsilon)\right)$$

$$\leq C_1 \sum_{j=1}^{\infty} \exp\left(-c_3 a_p^{-2} j \psi(n, p, \varepsilon)\right) \leq C_2 \exp\left(-c_3' a_p^{-2} \psi(n, p, \varepsilon)\right),$$

provided that $k \leq c_2' (2c_2)^{-1} a_p^{-2} \psi(n, p, \varepsilon) \lesssim \left(\frac{2e}{p}\right)^p \psi(n, p, \varepsilon)$. Therefore, with probability greater than $1 - e^{-c a_p^{-2} \psi(n,p,\varepsilon)}$ the random operator G satisfies

$$(1 - \varepsilon)^{\frac{1}{p}} I_p \|x\|_2 \leq \|G(x)\|_{B_p(\mu)} \leq (1 + \varepsilon)^{\frac{1}{p}} I_p \|x\|_2,$$

for all $x \in \mathbb{R}^k$. To conclude we have to recall that $\mathrm{Im} G = F$ is Haar-distributed on $G_{n,k}$ (see [344]).

If the isotropic measure μ on S^{n-1} is the one supported on $\pm e_i$'s i.e. $X_p(\mu) \equiv \ell_p^n$, then Theorem (6.3.14) is optimal (up to constants depending on p) as was proved in [341]. Moreover, Theorem (6.3.20) is optimal, in the sense that if the typical k-dimensional subspace of ℓ_p^n is $(1 + \varepsilon)$-spherical, then $k \leq Cp(\varepsilon n)^{\frac{2}{p}}$ for some absolute constant $C > 0$ (see [341]). We should mention that it is known, that for concrete values of p one can embed ℓ_2^k into ℓ_p^n even isometrically (see [336] for details). However, this is not a typical subspace.

Embeddings of ℓ_2^k into $L_q, 2 < q < \infty$ under different randomness have appeared in [325]. We consider large random matrices with independent Rademacher entries in order to K(q)-embed ℓ_2^k into ℓ_q^N with $N \simeq k^{\frac{q}{2}}$, where $K(q) > 0$ depends only on q. Then, they use this result in order to prove that for any $1 < p < 2$ there exists uncomplemented subspace of L_p which is isomorphic to Hilbert space. It is worth mentioning, that one can prove a concentration result similar to that of Theorem (6.3.14) using other randomness than Gaussian. In particular, if ν is an isotropic Borel probability measure on $\mathbb{R}^n$ which satisfies a log-Sobolev inequality with constant $\rho > 0$ then we may prove the following:

Theorem (6.3.22)[320]: Let $2 < p < \infty$, let μ be a Borel isotropic measure on S^{n-1} and let ν be an isotropic Borel probability measure on $\mathbb{R}^n$ which satisfies a log-Sobolev inequality with constant $\rho > 0$. Then, we have

$$\left(\iint \left| \|x\|_{B_p(\mu)}^p - \|y\|_{B_p(\mu)}^p \right|^r d\nu(x) d\nu(y)\right)^{\frac{1}{r}} \leq C(p,\rho) I_p^p\left(\nu, B_p(\mu)\right) \max\left\{\left(\frac{r}{n}\right)^{\frac{1}{2}}, \frac{r^{\frac{p}{2}}}{n}\right\},$$

for all $r \geq 2$, where $C(p,\rho) > 0$ is constant depending only on p and ρ.

Having proved Theorem (6.3.22), we apply Markov's inequality to get the corresponding concentration inequality. For the proof of Theorem (6.3.22) we argue as follows: Consider the function $f(x) = \|x\|_{B_p(\mu)}^p$ and define $F = f - \mathbb{E}_\nu f$. Then, a direct application of Lemma (6.3.4) yields

$$\|F\|_{L_r(\nu)}^2 \leq \|F\|_{L_2(\nu)}^2 + \frac{1}{\rho} \int_2^r \| \|\nabla f\|_2 \|_{L_s(\nu)}^2 ds, \tag{114}$$

for all $r \geq 2$. Recall the known fact (e.g. see [121]) that if a measure ν satisfies a log-Sobolev inequality with constant ρ, also satisfies a Poincaré inequality with constant ρ, that is

$$\|h - \mathbb{E}_\nu h\|_{L_2(\nu)}^2 \leq \frac{1}{\rho} \int_{\mathbb{R}^n} \|\nabla h\|_2^2 d\nu = \frac{1}{\rho} \| \|\nabla h\|_2 \|_{L_2(\nu)}^2,$$

for any smooth function h. Therefore, (114) becomes

$$\|F\|_{L_r(\nu)}^2 \leq \frac{2}{\rho} \int_2^r \| \|\nabla f\|_2 \|_{L_s(\nu)}^2 ds \leq \frac{2r}{\rho} \| \|\nabla f\|_2 \|_{L_r(\nu)}^2,$$

for all $r \geq 3$, where we have used the fact that $s \longmapsto \|h\|_{L_s}$ is non-decreasing function. Taking into account the Claim we get

$$\|F\|_{L_r(\nu)}^2 \leq \frac{2p^2 r}{\rho} \left(\int_{\mathbb{R}^n} \|x\|_{B_{2p-2}(\mu)}^{r(p-1)} d\nu(x)\right)^{\frac{2}{r}}, \qquad r \geq 3. \tag{115}$$

Again, Lemma (6.3.4) implies that

$$\left(\int_{\mathbb{R}^n} \|x\|_{B_{2p-2}(\mu)}^{r(p-1)} d\nu(x)\right)^{\frac{2}{r}} \leq I_{2p-2}^{2p-2}\left(\nu, B_{2p-2}(\mu)\right) \left(1 + \frac{(r-2)(p-1)}{\rho I_{2p-2}^2\left(\nu, B_{2p-2}(\mu)\right)}\right)^{p-1},$$

for $r \geq 2$. Plug this back in (115) we obtain

$$\|F\|_{L_r(\nu)} < \left(\frac{2p^2}{\rho}\right)^{\frac{1}{2}} r^{\frac{1}{2}} I_{2p-2}^{p-1}\left(\nu, B_{2p-2}(\mu)\right)\left(1 + \frac{(r-2)(p-1)}{\rho I_{2p-2}^2\left(\nu, B_{2p-2}(\mu)\right)}\right)^{\frac{p-1}{2}},$$

for all $r \geq 3$. Finally, we have

$$n \leq I_p^p\left(\nu, B_p(\mu)\right) \leq \left(1 + \frac{p-2}{\rho}\right)^{\frac{p}{2}} n,$$

for all $p \geq 2$. Indeed; we may write

$$I_p^p\left(\nu, B_p(\mu)\right) = \int_{S^{n-1}} \int_{\mathbb{R}^n} |\langle x, \theta\rangle|^p d\nu(x) d\mu(\theta) \geq \int_{S^{n-1}} \left(\int_{\mathbb{R}^n} \langle x, \theta\rangle^2 \nu(x)\right)^{\frac{p}{2}} d\mu(\theta)$$
$$= \mu(S^{n-1}),$$

where we have used Hölder's inequality and the isotropicity of ν. For the right-hand side, we fix $\theta \in S^{n-1}$ and we apply Lemma (6.3.4) for $x \longmapsto \langle x, \theta\rangle$ to get

$$\left(\int_{\mathbb{R}^n} |\langle x, \theta\rangle|^p d\nu(x)\right)^{\frac{2}{p}} \leq \int_{\mathbb{R}^n} \langle x, \theta\rangle^2 d\nu(x) + \frac{p-2}{\rho} = 1 + \frac{p-2}{\rho},$$

where we have used the isotropicity again. Finally, integration with respect to μ yields

$$I_p^p\left(\nu, B_p(\mu)\right) = \int_{S^{n-1}} \int_{\mathbb{R}^n} |\langle x, \theta\rangle|^p d\nu(x) d\mu(\theta) \leq \int_{S^{n-1}} \left(1 + \frac{p-2}{\rho}\right)^{\frac{p}{2}} d\mu(\theta),$$

as asserted. Taking into account these estimates, we argue as to complete the proof.

We point out that our method also provides upper estimate for the normalized variance of the norm of any finite dimensional subspace of L_p in Lewis' position. We should mention that the following estimate turns out to be optimal (up to constants of p) since it agrees with the ℓ_p^n case (see [341]).

Corollary (6.3.23)[320]: Let $1 \leq p < \infty$. Then, for any n-dimensional subspace X of L_p represented on $\mathbb{R}^n$, there exists a position $\tilde{B}$ of its unit ball B_X such that the normalized variance is of minimal possible order (up to constants of p)

$$\frac{Var\|Z\|_{\tilde{B}}}{\mathbb{E}\|Z\|_{\tilde{B}}^2} \leq \frac{C^p}{n},$$

where Z is the standard Gaussian vector in $\mathbb{R}^n$ and $C > 0$ is an absolute constant.

Proof. If $\tilde{B}$ is a Lewis' position of B_X, we may identify X with $X_p(\mu)$ for some Borel isotropic measure μ on S^{n-1}. Then, the result follows from Theorem (6.3.18). On the other hand note that the normalized variance is minimal since for every norm $\|\cdot\|$ on $\mathbb{R}^n$ one has $Var\|Z\| \lesssim \mathbb{E}\|Z\|^2/n$. The latter may be easily checked by using integration in polar coordinates and the Cauchy–Schwarz inequality.

The classical Johnson–Lindenstrauss lemma [335] asserts that any finite Hilbertian set S, i.e. $S \subset \ell_2$, can be almost isometrically embedded into ℓ_2^m , where m is logarithmically small with respect to the size of the set S. We can equivalently state this principle for subsets of the Euclidean sphere. We have the following:

Theorem (6.3.24)[320]: (Johnson–Lindenstrauss). Let $S \subset S^{n-1}$ and let $\varepsilon \in (0,1)$. Then, for every $m \geq c\varepsilon^{-2} \log|S|$ there exists a linear map $T: \ell_2^n \to \ell_2^m$ such that

$$1 - \varepsilon \leq \|Tu\|_2 \leq 1 + \varepsilon, \qquad u \in S.$$

Replacing the Euclidean target space by an arbitrary normed space $X = (\mathbb{R}^m, \|\cdot\|)$ one has the following variant:

Theorem (6.3.25)[320]: Let S be a finite subset of the Euclidean sphere S^{n-1} and let
$$GW(S)=\mathbb{E}\sup_{u\in S}\langle u,Z\rangle\,,\qquad Z\sim N(0,I_n),$$
be the Gaussian width of S. If $X=(\mathbb{R}^m,\|\cdot\|)$ is a normed space and $\varepsilon\in(0,1)$ with $GW(S)\le c\varepsilon\sqrt{k(X)}$, then there exists a linear map $T:\ell_2^n\to X$ such that
$$1-\varepsilon\le\|Tu\|\le1+\varepsilon,\qquad u\in S.$$
The latter follows from Gordon's min-max theorem (in particular from [332] for $E=S$ and $\Theta=S_{X^*}$) in the spirit of his proof for the randomized Dvoretzky's theorem. For an alternative approach, which rests on the majorizing measure theorem, see [343].

Let us focus now in the case that X is an m-dimensional subspace of $L_p, 1\le p<\infty$. If we recall the fact that $GW(S)\lesssim\sqrt{\log|S|}$, and assume further that X is in Lewis' position, then one gets
$$\log|S|\le c\varepsilon^2m^{\min\left\{\frac{1,2}{p}\right\}},$$
which shows that $m\ge C\varepsilon^{-2}\log|S|$, when $1\le p\le2$ and $m\ge C\varepsilon^{-p}(\log|S|)^{\frac{p}{2}}$, when $2<p<\infty$. Let us mention that in the first case, the estimate matches the one given by JL lemma and it can also be obtained by invoking the classical concentration, directly.

However, using Theorem (6.3.14), we may get improved estimates for low-dimensional embeddings of Hilbertian sets into any subspace of L_p for $2<p<\infty$. The following can be viewed as a Johnson–Lindenstrauss type result for target spaces which sit in L_p.

Theorem (6.3.26)[320]: Let $2<p<\infty$. For any $\varepsilon\in(0,1)$, for any $S\subset S^{n-1}$ and for any m-dimensional subspace X of L_p with $m\gtrsim\max\left\{p4^p\varepsilon^{-2}\,log|S|\,,\varepsilon^{-1}(log|S|)^{\frac{p}{2}}\right\}$, there exists a linear mapping $T:\ell_2^n\to X$ such that
$$1-\varepsilon\le\|Tu\|_X\le1+\varepsilon,\qquad u\in S.$$
Proof. Fix p,ε,X and m as above. Then, by Theorem (6.3.6), there exists an isotropic measure on S^{m-1} and a linear isometry $U:X_p(\mu)\to X$. Then, by Theorem (6.3.14) we have
$$\mathbb{P}(\exists u\in S:\left|\|Gu\|_{B_p(\mu)}-\left(\mathbb{E}\|Z\|^p_{B_p(\mu)}\right)^{\frac{1}{p}}\right|\ge\varepsilon\left(\mathbb{E}\|Z\|^p_{B_p(\mu)}\right)^{\frac{1}{p}}\le|S|\exp\bigl(-\psi(m,p,\varepsilon)\bigr),$$
where $G=\left(g_{ij}\right)_{i,j=1}^{m,n}$ is a matrix with standard Gaussian entries and Z is the standard Gaussian vector in $\mathbb{R}^m$. Assuming that $\psi(m,p,\varepsilon)>\log|S|$ we may conclude the existence of a matrix $M:\ell_2^n\to X_p(\mu)$ such that
$$1-\varepsilon\le\|Mu\|_{B_p(\mu)}\le1+\varepsilon,$$
for all $u\in S$. The desired linear map is $T=UM$. The estimate for m is obtained by finding the inverse function of $m\longmapsto\psi(m,p,\varepsilon)$.

Corollary (6.3.27)[353]: (Aida–Stroock). Let ν be a Borel probability measure on $\mathbb{R}^n$ which satisfies a log-Sobolev inequality with constant ρ. Then, for any smooth function $f:\mathbb{R}^n\to\mathbb{R}$ we have
$$\|f\|^2_{L_{2+\epsilon}(\nu)}-\|f\|^2_{L_{1+\epsilon}(\nu)}\le\frac{1}{\rho}\int_{1+\epsilon}^{2+\epsilon}\|\|\nabla f\|_2\|^2_{L^s(\nu)}ds,\qquad(116)$$
for all $\epsilon\ge0$. In particular, if f is Lipschitz continuous, then we have

$$\|f\|^2_{L_{1+\epsilon}(\nu)} - \|f\|^2_{L_{2+\epsilon}(\nu)} \le \frac{\|f\|^2_{\mathrm{Lip}}}{\rho}(\epsilon).$$

It follows that

$$\|f\|_{L_{2(1+\epsilon)}(\nu)}/\|f\|_{L_2(\nu)} \le \sqrt{1+\frac{2\epsilon}{\rho k(f)}},$$

for $\epsilon \ge 0$, where $k(f) := \|f\|^2_{L_2(\nu)}/\|f\|^2_{\mathrm{Lip}}$.

Sketch of Proof. For $\epsilon \ge 0$ we define $I(2+\epsilon) := \|f\|_{L_{2+\epsilon}}$. Differentiation with respect to $2+\epsilon$ yields

$$\frac{dI}{d(2+\epsilon)} = \frac{Ent_\nu(|f|^{2+\epsilon})}{(2+\epsilon)^2 I(2+\epsilon)^{1+\epsilon}}.$$

Applying the log-Sobolev inequality for $g = |f|^{\frac{2+\epsilon}{2}}$ we obtain

$$\frac{dI}{d(2+\epsilon)} \le \frac{1}{2\rho I(2+\epsilon)^{1+\epsilon}} \int_{\mathbb{R}^n} |f|^\epsilon \|\nabla f\|_2^2 d\nu \le \frac{1}{2\rho I(2+\epsilon)^{1+\epsilon}} I(2+\epsilon)^\epsilon \| \|\nabla f\|_2 \|^2_{L_{2+\epsilon}(\nu)},$$

by Hölder's inequality. This shows that $(I(2+\epsilon)^2)' \le \frac{1}{\rho} \| \|\nabla f\|_2 \|^2_{L_{2+\epsilon}(\nu)}$. Integration over the interval $[2+\epsilon, 2+\epsilon]$ proves (116).

Corollary (6.3.28)[353]: Let μ be a Borel isotropic measure on S^{n-1} and let Z be an n-dimensional standard Gaussian vector. Then, we have the following properties:

(i) $\mathbb{E}\|Z\|^{1+\epsilon}_{B_{1+\epsilon}(\mu)} = \sigma^{1+\epsilon}_{1+\epsilon} \mu(S^{n-1})$, for $0 \le \epsilon < \infty$.

(ii) $\mu(S^{n-1}) = n$.

(iii) For $\epsilon \ge 2$ we have $\|x^2\|_{B_{2+\epsilon}(\mu)} \le \|x^2\|_2$ and for $0 \le \epsilon < \infty$ we have $\|x^2\|_{B_{1+\epsilon}(\mu)} \le n^{\overline{(1+\epsilon)(1+2\epsilon)}^{\,\epsilon}} \|x^2\|_{B_{1+2\epsilon}(\mu)}$, for all $x^2 \in \mathbb{R}^n$.

(iv) (K. Ball). For every $0 \le \epsilon < \infty$ we have $|B_{1+\epsilon}(\mu)| \le |B^n_{1+\epsilon}|$.

(v) For the body $B_{1+\epsilon}(\mu), \epsilon \ge 0$ we have $k\big(B_{1+\epsilon}(\mu)\big) \ge (1+\epsilon) n^{\min\left\{\frac{1,2}{1+\epsilon}\right\}}$.

(vi) There exists an absolute constant $\epsilon \ge 0$ such that for all $0 \le \epsilon \le (1+\epsilon)\log n - 2$, one has $\left(\mathbb{E}\|Z\|^2_{B_{2+\epsilon}(\mu)}\right)^{\frac{1}{2}} \simeq (2+\epsilon)^{\frac{1}{2}} n^{\frac{1}{2+\epsilon}}$. In particular, for those $2+\epsilon$ one has $k\big(B_{2+\epsilon}(\mu)\big) \ge (1+\epsilon)(2+\epsilon) n^{\frac{2}{2+\epsilon}}$.

Proof. (i). We use Fubini's theorem and the rotation invariance of the Gaussian measure to write

$$\mathbb{E}\|Z\|^{2+\epsilon}_{B_{2+\epsilon}(\mu)} = \int_{\mathbb{R}^n} \|x^2\|^{2+\epsilon}_{B_{2+\epsilon}(\mu)} d\gamma_n(x^2) = \int_{S^{n-1}} \int_{\mathbb{R}^n} |\langle x^2, \theta_\epsilon \rangle|^{2+\epsilon} d\gamma_n(x^2) d\mu(\theta_\epsilon)$$
$$= \sigma^{2+\epsilon}_{2+\epsilon} \mu(S^{n-1}).$$

(ii). It follows from the above formula, applied for $\epsilon = 0$, and by employing the isotropic condition.

(iii). Let $\epsilon \geq 0$. Note that for all $u \in S^{n-1}$ we have

$$\|u\|_{B_{2+\epsilon}(\mu)}^{2+\epsilon} = \int_{S^{n-1}} |\langle u, \theta_\epsilon \rangle|^{2+\epsilon} d\mu(\theta_\epsilon) \leq \int_{S^{n-1}} |\langle u, \theta_\epsilon \rangle|^{2} d\mu(\theta_\epsilon) = 1.$$

For $\epsilon \geq 0$ we apply Hölder's inequality

$$\|x^2\|_{B_{1+\epsilon}(\mu)} = \left(\int_{S^{n-1}} |\langle x^2, \theta_\epsilon \rangle|^{1+\epsilon} d\mu(\theta_\epsilon) \right)^{\frac{1}{1+\epsilon}}$$
$$\leq \mu(S^{n-1})^{\frac{\epsilon}{(1+\epsilon)(1+2\epsilon)}} \left(\int_{S^{n-1}} |\langle x^2, \theta_\epsilon \rangle|^{1+2\epsilon} d\mu(\theta_\epsilon) \right)^{\frac{1}{1+2\epsilon}}.$$

(iv). This result is proved by K. Ball in [324]. The reader may consult [324] for the details.

(v). First consider the case $0 \leq \epsilon \leq 1$. Using Hölder's inequality we get

$$\left(\mathbb{E}\|Z\|_{B_{1+\epsilon}(\mu)}^2\right)^{\frac{1}{2}} \geq \left(\mathbb{E}\|Z\|_{B_{1+\epsilon}(\mu)}^{1+\epsilon}\right)^{\frac{1}{1+\epsilon}} = \sigma_{1+\epsilon} n^{\frac{1}{1+\epsilon}} \geq (1+\epsilon) n^{\frac{1}{1+\epsilon}},$$

where we have also used (i) and (ii). Also note that (iii) implies $b\big(B_{1+\epsilon}(\mu)\big) \leq n^{-\frac{\epsilon}{(1+\epsilon)}}$ for $0 \leq \epsilon \leq 1$.

Now we turn in the range $0 < \epsilon < \infty$. Note that $b\big(B_{2+\epsilon}(\mu)\big) \leq 1$, by (iii). Furthermore,

$$\mathbb{E}\|Z\|_{B_{2+\epsilon}(\mu)}^2 \geq n^{-\frac{\epsilon}{2+\epsilon}} \mathbb{E}\|Z\|_2^2 = n^{\frac{2}{2+\epsilon}},$$

again by (iii). Combining the above and recalling the definition of $k\big(X_{2+\epsilon}(\mu)\big)$ we get the desired estimate.

(vi). We define the parameter

$$\begin{aligned} 2+2\epsilon \equiv (2+2\epsilon)(\mu)&: \\ &= \max\big\{2+\epsilon \in [2,n] : k\big(B_{1+\epsilon}(\mu)\big) \geq 1+\epsilon, \forall (1+\epsilon) \in [2, 2+\epsilon]\big\}. \end{aligned}$$

The continuity of the map $1+\epsilon \mapsto k\big(B_{1+\epsilon}(\mu)\big)$ and the fact that $k\big(B_{2+\epsilon}(\mu)\big) \leq n$ for all $\epsilon \geq 0$, whereas $k\big(B_2(\mu)\big) = n$, implies that $2+2\epsilon = k\big(B_{2+2\epsilon}(\mu)\big)$ and $k\big(B_{2+\epsilon}(\mu)\big) \geq 2+\epsilon$ for $\epsilon \geq 0$. Corollary (6.3.27) then, yields $\left(\mathbb{E}\|Z\|_{B_{2+\epsilon}(\mu)}^{2+\epsilon}\right)^{\frac{1}{2+\epsilon}} \leq c_1 \left(\mathbb{E}\|Z\|_{B_{2+\epsilon}(\mu)}^{2}\right)^{\frac{1}{2}}$ for all $\epsilon \geq 0$. Thus, for all $\epsilon \geq 0$ we get

$$k\big(B_{2+\epsilon}(\mu)\big) = \frac{\mathbb{E}\|Z\|_{B_{2+\epsilon}(\mu)}^2}{b^2\big(B_{2+\epsilon}(\mu)\big)} \geq c_1^{-2} \left(\mathbb{E}\|Z\|_{B_{2+\epsilon}(\mu)}^{2+\epsilon}\right)^{\frac{2}{2+\epsilon}} = c_1^{-2} \sigma_{2+\epsilon}^2 n^{\frac{2}{2+\epsilon}} \geq c_2 (2+\epsilon) n^{\frac{2}{2+\epsilon}},$$

where we have also used that $b\big(B_{2+\epsilon}(\mu)\big) \leq 1$. In particular, for $\epsilon = 0$, we obtain

$$2+2\epsilon = k\big(B_{2+2\epsilon}(\mu)\big) \geq c_2(2+2\epsilon)n^{\frac{2}{2+2\epsilon}} \Rightarrow 2+2\epsilon \geq c_3 \log n.$$

This can be interpreted as $k\big(B_{2+\epsilon}(\mu)\big) \geq (1+\epsilon)k(\ell_{2+\epsilon}^n)$, provided that $0 \leq \epsilon \leq (1+\epsilon)\log n - 2$ for some absolute constant $\epsilon \geq 0$. For a proof of the fact that $k(\ell_{2+\epsilon}^n) \simeq (2+\epsilon)n^{\frac{2}{2+\epsilon}}$ when $0 \leq \epsilon \leq (1+\epsilon)\log n - 2$ the reader is referred to [346].

Corollary (6.3.29) [320]. Let μ be a Borel isotropic measure on S^{n-1}. For all $\epsilon \geq 0$ we have

$$\frac{I_{(1+\epsilon)(2+\epsilon)}\big(\gamma_n, B_{2+\epsilon}(\mu)\big)}{I_{2+\epsilon}\big(\gamma_n, B_{2+\epsilon}(\mu)\big)} \leq \sqrt{1+\frac{(2+\epsilon)(\epsilon)}{\sigma_{2+\epsilon}^2 n^{\frac{2}{2+\epsilon}}}} \leq \sqrt{1+\frac{(1+\epsilon)(\epsilon)}{n^{\frac{2}{2+\epsilon}}}},$$

where $\epsilon \geq 0$ is an absolute constant.

Proof. Note that Corollary (6.3.28) (iii) implies $\big|\|x^2\|_{B_{2+\epsilon}(\mu)} - \|y^2\|_{B_{2+\epsilon}(\mu)}\big| \leq \|x^2-y^2\|_2$ for all $x^2, y^2 \in \mathbb{R}^n$. Hence, if we use Corollary (6.3.27) we obtain

$$\left(\frac{I_{(1+\epsilon)2+\epsilon}}{I_{2+\epsilon}}\right)^2 \leq 1+\frac{(2+\epsilon)(\epsilon)}{I_{2+\epsilon}^2} = 1+\frac{(2+\epsilon)(\epsilon)}{\sigma_{2+\epsilon}^2 n^{\frac{2}{2+\epsilon}}},$$

where the last estimate follows from Corollary (6.3.28). Finally, using the fact that $\sigma_{2+\epsilon} \simeq \sqrt{2+\epsilon}$ we conclude the second estimate.

Corollary (6.3.30)[353]: Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be a convex function and let $f\colon \mathbb{R}^n \to \mathbb{R}$ be C^1-smooth. If $G = \big(g_{ij}\big)_{i,j=1}^{n,k}$ is a Gaussian matrix and $a, (a+\epsilon) \in S^{k-1}$, then we have

$$\mathbb{E}\phi\big(f(Ga) - f(G(a+\epsilon))\big) \leq \mathbb{E}\phi\left(\frac{\pi}{2}\|\epsilon\|_2\langle \nabla f(Z), W\rangle\right),$$

where Z, W are independent copies of a standard Gaussian n-dimensional random vector.

Proof. If $\epsilon = 0$ then, there is nothing to prove. If $a = -(a+\epsilon)$ then, by setting $F(z^2) = f(z^2) - f(-z^2)$ we may write

$$\mathbb{E}\phi\big(f(Ga) - f(G(a+\epsilon))\big) = \mathbb{E}\phi\big(F(Z)\big) \leq \mathbb{E}\phi\big(F(Z) - F(W)\big),$$

for Z, W independent copies of a standard Gaussian random vector, where we have used the fact $\mathbb{E}F(Z) = 0$ and Jensen's inequality. Then, a direct application of Theorem (6.3.9) yields

$$\mathbb{E}\phi\big(F(Z) - F(W)\big) \leq \mathbb{E}\phi\left(\frac{\pi\langle \nabla f(Z), W\rangle + \pi\langle \nabla f(-Z), W\rangle}{2}\right)$$
$$\leq \mathbb{E}\frac{\big(\phi(\pi\langle \nabla f(Z), W\rangle) + \phi(\pi\langle \nabla f(-Z), W\rangle)\big)}{2} = \mathbb{E}\phi(\pi\langle \nabla f(Z), W\rangle),$$

by the convexity of ϕ.

In the general case, fix $a, (a+\epsilon) \in S^{k-1}$ with $a \neq \pm(a+\epsilon)$ and define $2+\epsilon := \frac{2a+\epsilon}{2}$. Note that since $\|a\|_2 = \|a+\epsilon\|_2$ we have that the vector $u := a-(2+\epsilon)$ is perpendicular to $2+\epsilon$. Set $W := G(u)$ and $Z := G(2+\epsilon)$ and note that W,Z are independent random vectors in $\mathbb{R}^n$ with $W \sim N(0, \|u\|_2^2 I_n), Z \sim N(0, \|2+\epsilon\|_2^2 I_n)$. Since $G(a) = Z + W$ and $G(a+\epsilon) = Z - W$, we may write

$$\mathbb{E}\phi\big(f(Ga) - f(G(a+\epsilon))\big) = \mathbb{E}_Z \mathbb{E}_W \phi\big(f(Z+W) - f(Z-W)\big).$$

Denote $F(w^2, z^2) := f(z^2 + w^2) - f(z^2 - w^2)$. Then, we may write

$$\mathbb{E}\phi\big(f(Ga) - f(G(a+\epsilon))\big) = \iint \phi\big(F(w^2, z^2)\big) d\gamma_{n,\sigma_1}(w^2) d\gamma_{n,\sigma_2}(z^2),$$

where $\sigma_1 = \|u\|_2 > 0, \sigma_2 = \|2+\epsilon\|_2 > 0$. For fixed z^2, we may apply Theorem (6.3.9) to the function $w^2 \longmapsto F(w^2, z^2)$ (note that $\int F(w^2, z^2) d\gamma_{n,\sigma_1}(w^2) = 0$) to get

$$\int \phi\big(F(w^2, z^2)\big) d\gamma_{n,\sigma_1}(w^2) \leq \iint \phi\left(\frac{\pi}{2}\langle \nabla_{w^2} F(w^2, z^2), y^2\rangle\right) d\gamma_{n,\sigma_1}(w^2) d\gamma_{n,\sigma_1}(y^2)$$

$$\leq \iint \frac{\phi(\pi\langle \nabla f(w^2+z^2), y^2\rangle) + \phi(\pi\langle \nabla f(z^2-w^2), y^2\rangle)}{2} d\gamma_{n,\sigma_1}(w^2) d\gamma_{n,\sigma_1}(y^2)$$

$$= \iint \phi(\pi\langle \nabla f(w^2+z^2), y^2\rangle) d\gamma_{n,\sigma_1}(w^2) d\gamma_{n,\sigma_1}(y^2),$$

by the convexity of ϕ. Integration with respect to γ_{n,σ_2} over z^2 provides

$$\begin{aligned}
&\iint \phi\big(F(w^2, z^2)\big) d\gamma_{n,\sigma_1}(w^2) d\gamma_{n,\sigma_2}(z^2) \\
&\leq \int \left[\iint \phi(\pi\langle \nabla f(w^2+z^2), y^2\rangle) d\gamma_{n,\sigma_1}(w^2) d\gamma_{n,\sigma_2}(z^2)\right] d\gamma_{n,\sigma_1}(y^2) \\
&= \int \left[\int \phi(\pi\langle \nabla f(x^2), y^2\rangle) d\big(\gamma_{n,\sigma_1} * \gamma_{n,\sigma_2}\big)(x^2)\right] d\gamma_{n,\sigma_1}(y^2) \\
&= \iint \phi(\pi\sigma_1\langle \nabla f(x^2), y^2\rangle) d\gamma_n(x^2) d\gamma_n(y^2),
\end{aligned}$$

where we have used the fact that $\gamma_{n,\sigma_1} * \gamma_{n,\sigma_2} = \gamma_{n,\sigma_1^2+\sigma_2^2} \equiv \gamma_n$, since $\sigma_1^2 + \sigma_2^2 = \|a\|_2^2 = 1$. The result follows.

Corollary (6.3.31)[353]: Let $0 < \epsilon < \infty$ and let μ be a Borel isotropic measure on S^{n-1} with $n > e^{2+\epsilon}$. Then, we have

$$\mathbb{P}\left(\left|\|Z\|_{B_{2+\epsilon}(\mu)} - \left(\mathbb{E}\|Z\|_{B_{2+\epsilon}(\mu)}^{2+\epsilon}\right)^{\frac{1}{2+\epsilon}}\right| \geq \varepsilon\left(\mathbb{E}\|Z\|_{B_{2+\epsilon}(\mu)}^{2+\epsilon}\right)^{\frac{1}{2+\epsilon}}\right.$$
$$\leq (1+\epsilon)\exp\big(-(1+\epsilon)\psi(n, 2+\epsilon, \varepsilon)\big),$$

for every $\varepsilon > 0$, where $\psi(n, 2+\epsilon, \cdot)$ is defined by

$$\psi(n, 2+\epsilon, t) := \min\left\{\frac{t^2 n}{(2+\epsilon)^{4^{2+\epsilon}}}, (tn)^{\frac{2}{2+\epsilon}}\right\}, \qquad t > 0, \qquad (117)$$

and $\epsilon \geq 0$ are absolute constants.

Proof. Using Proposition (6.3.13) for $a, (a+\epsilon) \in S^{k-1}$ with $\langle a, a+\epsilon \rangle = 0$ and applying Jensen's inequality we obtain

$$\left(\mathbb{E}\left|\|Z\|_{B_{2+\epsilon}(\mu)}^{2+\epsilon} - \mathbb{E}\|Z\|_{B_{2+\epsilon}(\mu)}^{2+\epsilon}\right|^{2+\epsilon}\right)^{\frac{1}{2+\epsilon}} \leq (1+2\epsilon)(2+\epsilon)\sigma_{2+2\epsilon}^{1+\epsilon} n^{\frac{1}{2}} 2^{\frac{2+\epsilon}{2}} \alpha(n, 2+\epsilon, 2+\epsilon),$$

for all $\epsilon \geq 0$. Therefore, Markov's inequality yields

$$\mathbb{P}\left(\left|\|Z\|_{B_{2+\epsilon}(\mu)}^{2+\epsilon} - \mathbb{E}\|Z\|_{B_{2+\epsilon}(\mu)}^{2+\epsilon}\right| > \varepsilon\right) \leq \left(\frac{(1+2\epsilon)(2+\epsilon)\sigma_{2+2\epsilon}^{1+\epsilon} n^{\frac{1}{2}} 2^{\frac{2+\epsilon}{2}} \alpha(n, 2+\epsilon, 2+\epsilon)}{\varepsilon}\right)^{2+\epsilon}. \quad (118)$$

Note that the inverse of the map $2+\epsilon \longmapsto \alpha(n, 2+\epsilon, 2+\epsilon)$ is given by

$$\alpha^{-1}(n, 2+\epsilon, s) = \min\left\{s^2, \frac{s^{\frac{2}{2+\epsilon}} n^{\frac{1}{2+\epsilon}} \sigma_{2+2\epsilon}^{\frac{2+2\epsilon}{2+\epsilon}}}{(1+\epsilon)^{\frac{1+\epsilon}{2+\epsilon}}}\right\}, \qquad s > 0.$$

Hence, for every $\varepsilon > 0$ there exists $\epsilon \geq 0$ such that $\alpha(n, 2+\epsilon, 1+\epsilon) = \frac{\varepsilon}{e(1+2\epsilon)(2+\epsilon)\sigma_{2+2\epsilon}^{1+\epsilon} n^{\frac{1}{2}} 2^{\frac{2+\epsilon}{2}}}$. One may check that

$$\begin{aligned} 1+\epsilon &= \alpha^{-1}\left(n, 2+\epsilon, \frac{\varepsilon}{e(1+2\epsilon)(2+\epsilon)\sigma_{2+2\epsilon}^{1+\epsilon} n^{\frac{1}{2}} 2^{\frac{2+\epsilon}{2}}}\right) \\ &\simeq \min\left\{\frac{\varepsilon^2}{n(2+\epsilon)^2 2^{2+\epsilon} \sigma_{2+2\epsilon}^{2+2\epsilon}}, \frac{\varepsilon^{\frac{2}{2+\epsilon}}}{2+\epsilon}\right\}. \end{aligned}$$

As long as the range of $\varepsilon > 0$ satisfies $\alpha(n, 2+\epsilon, 1+\epsilon) \geq \alpha(n, 2+\epsilon, 2)$ we may insert $1+\epsilon$ into the probabilistic bound (118) or else, $\alpha(n, 2+\epsilon, 1+\epsilon) < \alpha(n, 2+\epsilon, 2) \simeq \max\left\{1, \left(\frac{e^{2+\epsilon}}{n}\right)^{\frac{1}{2}}\right\} \simeq 1$, provided that n is large enough with respect to $2+\epsilon$, and we may upper bound the aforementioned probability by an absolute constant $\epsilon \geq 0$. In each case we obtain

$$\mathbb{P}\left(\left|\|Z\|_{B_{2+\epsilon}(\mu)}^{2+\epsilon} - \mathbb{E}\|Z\|_{B_{2+\epsilon}(\mu)}^{2+\epsilon}\right| > \varepsilon\right) \leq (1+\epsilon)\exp\left(-c_1\, min\left\{\frac{\varepsilon^2}{n(2+\epsilon)^2 2^{2+\epsilon} \sigma_{2+2\epsilon}^{2+2\epsilon}}, \frac{\varepsilon^{\frac{2}{2+\epsilon}}}{2+\epsilon}\right\}\right),$$

for every $\varepsilon > 0$. It follows that

$$\mathbb{P}\left(\left|\|Z\|_{B_{2+\epsilon}(\mu)}^{2+\epsilon}-\mathbb{E}\|Z\|_{B_{2+\epsilon}(\mu)}^{2+\epsilon}\right|>\varepsilon\mathbb{E}\|Z\|_{B_{2+\epsilon}(\mu)}^{2+\epsilon}\right)$$
$$\leq(1+\epsilon)\exp\left(-c_1\,min\left\{\frac{\varepsilon^2 n\sigma_{2+\epsilon}^{2(2+\epsilon)}}{(2+\epsilon)^2 2^{2+\epsilon}\sigma_{2+2\epsilon}^{2+2\epsilon}},\frac{(\varepsilon n)^{\frac{2}{2+\epsilon}}\sigma_{2+\epsilon}^2}{2+\epsilon}\right\}\right),$$

for every $\varepsilon>0$. The asymptotic estimate (96) yields $\sigma_{2+\epsilon}^{2(2+\epsilon)}/\sigma_{2+2\epsilon}^{2+2\epsilon}\simeq(2+\epsilon)^{-\epsilon}$ and $\sigma_{2+\epsilon}\simeq(2+\epsilon)^{\frac{1}{2}}$, thus we conclude

$$\mathbb{P}\left(\left|\|Z\|_{B_{2+\epsilon}(\mu)}^{2+\epsilon}-\mathbb{E}\|Z\|_{B_{2+\epsilon}(\mu)}^{2+\epsilon}\right|>\varepsilon\mathbb{E}\|Z\|_{B_{2+\epsilon}(\mu)}^{2+\epsilon}\right)$$
$$\leq(1+\epsilon)\exp\left(-c_1'\,min\left\{\frac{\varepsilon^2 n}{(2+\epsilon)4^{2+\epsilon}},(\varepsilon n)^{\frac{2}{2+\epsilon}}\right\}\right),\qquad(119)$$

for all $\varepsilon>0$. This further implies that

$$\mathbb{P}\left(\left|\|Z\|_{B_{2+\epsilon}(\mu)}-\left(\mathbb{E}\|Z\|_{B_{2+\epsilon}(\mu)}^{2+\epsilon}\right)^{\frac{1}{2+\epsilon}}\right|>\varepsilon\left(\mathbb{E}\|Z\|_{B_{2+\epsilon}(\mu)}^{2+\epsilon}\right)^{\frac{1}{2+\epsilon}}\right)$$
$$\leq 2(1+\epsilon)\exp\left(-c_1'\,min\left\{\frac{\varepsilon^2 n}{(2+\epsilon)4^{2+\epsilon}},(\varepsilon n)^{\frac{2}{2+\epsilon}}\right\}\right),$$

for all $\varepsilon>0$. In order to verify the latter we may write

$$\mathbb{P}\left(\|Z\|_{B_{2+\epsilon}(\mu)}>(1+\varepsilon)\left(\mathbb{E}\|Z\|_{B_{2+\epsilon}(\mu)}^{2+\epsilon}\right)^{\frac{1}{2+\epsilon}}\right)\leq\mathbb{P}\left(\|Z\|_{B_{2+\epsilon}(\mu)}^{2+\epsilon}>(1+\varepsilon)\mathbb{E}\|Z\|_{B_{2+\epsilon}(\mu)}^{2+\epsilon}\right)$$
$$\leq(1+\epsilon)\exp\left(-c_1'\,min\left\{\frac{\varepsilon^2 n}{(2+\epsilon)4^{2+\epsilon}},(\varepsilon n)^{\frac{2}{2+\epsilon}}\right\}\right),$$

for all $\varepsilon>0$ by the estimate (119). We argue similarly for the lower tail.

Corollary (6.3.32)[353]: (Gaussian variance for $B_{1+\epsilon}(\mu)$). Let $0\leq\epsilon<\infty$ and let μ be an isotropic Borel measure on S^{n-1}. Then,

$$\mathrm{Var}\|Z\|_{B_{1+\epsilon}(\mu)}\leq e^{(1+\epsilon)^2}n^{\frac{1-\epsilon}{1+\epsilon}}.$$

In particular, we have

$$\frac{\mathrm{Var}\|Z\|_{B_{1+\epsilon}(\mu)}}{\mathbb{E}\|Z\|_{B_{1+\epsilon}(\mu)}^2}\leq\frac{e^{(1+\epsilon)^2}}{n},$$

where Z is the standard Gaussian vector in $\mathbb{R}^n$.

Proof. We may clearly assume that $n\geq e^{(1+\epsilon)^2}$ for some sufficiently large absolute constant $\epsilon\geq 0$, otherwise the conclusion is trivially true. In order to see that, first recall from Corollary (6.3.28) that

$$(a+\epsilon)\left(B_{1+\epsilon}(\mu)\right)\leq\max\left\{n^{\frac{1-\epsilon}{2(1+\epsilon)}},1\right\},\qquad\mathbb{E}\|Z\|_{B_{1+\epsilon}(\mu)}^2\geq(1+\epsilon)n^{\frac{2}{1+\epsilon}},\qquad 0\leq\epsilon<\infty.$$

Whence, in the light of (99) for $\epsilon = 0$ and $\langle a, a+\epsilon\rangle = 0$, we get

$$\operatorname{Var}\|Z\|_{B_{1+\epsilon}(\mu)} \leq (1+\epsilon)\mathbb{E}\left\|\nabla\|Z\|_{B_{1+\epsilon}(\mu)}\right\|_2^2 \leq (1+\epsilon)(a+\epsilon)^2\big(B_{1+\epsilon}(\mu)\big)$$
$$\leq (1+\epsilon)\max\left\{n^{\frac{1-\epsilon}{1+\epsilon}}, 1\right\}.$$

These estimates already prove the assertions when $0 \leq \epsilon \leq 1$. Thus, we may focus in the case $0 < \epsilon < \infty$ with $n \geq e^{(1+\epsilon)(2+\epsilon)}$. We consider Z' an independent copy of Z to write

$$2\operatorname{Var}\|Z\|_{B_{2+\epsilon}(\mu)} = \mathbb{E}\big(\|Z\|_{B_{2+\epsilon}(\mu)} - \|Z'\|_{B_{2+\epsilon}(\mu)}\big)^2$$
$$\leq \frac{1}{(2+\epsilon)^2}\mathbb{E}\left(\frac{\|Z\|_{B_{2+\epsilon}(\mu)}^{2+\epsilon} - \|Z'\|_{B_{2+\epsilon}(\mu)}^{2+\epsilon}}{\min\left\{\|Z\|_{B_{2+\epsilon}(\mu)}^{1+\epsilon}, \|Z'\|_{B_{2+\epsilon}(\mu)}^{1+\epsilon}\right\}}\right)^2,$$

where we have used the numerical inequality $|t^{1+\epsilon} - s^{1+\epsilon}| \geq (1+\epsilon)|t-s|\min\{t^{\epsilon}, s^{\epsilon}\}$ for $t, s > 0$ and $\epsilon > 0$. The Cauchy–Schwarz inequality implies that

$$\operatorname{Var}\|Z\|_{B_{1+\epsilon}(\mu)} \leq \frac{2}{(1+\epsilon)^2}\frac{\left(\mathbb{E}\left|\|Z\|_{B_{1+\epsilon}(\mu)}^{1+\epsilon} - \|Z'\|_{B_{1+\epsilon}(\mu)}^{1+\epsilon}\right|^4\right)^{\frac{1}{2}}}{I_{-4(\epsilon)}^{2(\epsilon)}\big(\gamma_n, B_{1+\epsilon}(\mu)\big)},$$

where we have used the fact that $\frac{1}{\min\{t,s\}} \leq \frac{1}{t} + \frac{1}{s}, t, s > 0$. The numerator is directly estimated by Proposition (6.3.13) (for $\langle a, a+\epsilon\rangle = 0$). Standard computations, based on (96), yield the bound

$$\left(\mathbb{E}\left|\|Z\|_{B_{1+\epsilon}(\mu)}^{1+\epsilon} - \|Z'\|_{B_{1+\epsilon}(\mu)}^{1+\epsilon}\right|^4\right)^{\frac{1}{2}} \leq e^{(1+\epsilon)^2}(1+\epsilon)^{1+\epsilon}n. \tag{120}$$

For the denominator, we employ estimate (110) for the norm $\|\cdot\|_{B_{1+\epsilon}(\mu)}$ (and for $2+\epsilon = 4(\epsilon)$) along with the fact $k\big(B_{1+\epsilon}(\mu)\big) \geq c_1(1+\epsilon)n^{\frac{2}{1+\epsilon}}$ for $n \geq e^{C_2(1+\epsilon)}$ (proved in Corollary (6.3.28) (vi)) to obtain

$$I_{-4(\epsilon)}\big(\gamma_n, B_{1+\epsilon}(\mu)\big) \geq c_2 I_{1+\epsilon}\big(\gamma_n, B_{1+\epsilon}(\mu)\big) = c_2\sigma_{1+\epsilon}n^{\frac{1}{1+\epsilon}}, \tag{121}$$

where in the last step we have used Corollary (6.3.28) (i), (ii) Combining (120) with (121) we arrive at the estimate

$$\operatorname{Var}\|Z\|_{B_{1+\epsilon}(\mu)} \leq \frac{e^{(1+\epsilon)^2}(1+\epsilon)^{1+\epsilon}n}{e^{-C_3(1+\epsilon)}\sigma_{1+\epsilon}^{2\epsilon}n^{\frac{2\epsilon}{1+\epsilon}}} \leq \frac{e^{C_4(1+\epsilon)}}{n^{\frac{2\epsilon}{1+\epsilon}}},$$

where we have used once again that $\sigma_{1+\epsilon}^{1+\epsilon} \simeq \left(\frac{1+\epsilon}{e}\right)^{\frac{1+\epsilon}{2}}$. The result follows.

Corollary (6.3.33) [353]: Let $0 < \epsilon < \infty$ and let X be an n-dimensional subspace of $L_{2+\epsilon}$, represented on $\mathbb{R}^n$, whose unit ball B_X is in Lewis' position. Then, for any $\varepsilon \in (0,1)$ there exists $k \geq c_{2+\epsilon}\psi(n, 2+\epsilon, \varepsilon)$ such that the random k-dimensional subspace F of X is

$(1+\varepsilon)$-spherical with probability greater than $1-e^{-c_{2+\epsilon}\psi(n,2+\epsilon,\varepsilon)}$, where $c_{2+\epsilon}>0$ depends only on $2+\epsilon$ and $\psi(n,2+\epsilon,\cdot)$ is defined in (117).

Proof. We have to show that the ball $B_{2+\epsilon}(\mu)$ has random almost spherical k-dimensional with k as large as possible. Let $\{g_{ij}(\omega^2)\}_{i,j=1}^{n,k}$be i.i.d. standard normals in some probability space $(\Omega,\mathbb{P})$ and consider the random Gaussian operator $G_{\omega^2}=\left(g_{ij}(\omega^2)\right)_{i,j=1}^{n,k}: \ell_2^k\to X_{2+\epsilon}(\mu)$. We will prove that with overwhelming probability the operator G is $(1+\varepsilon)$-isomorphic embedding when k is relatively large. To this end, we employ Theorem (6.3.15) and the chaining argument from [311]. For each $j=1,2,\ldots$ consider δ_j-nets $\mathcal{N}_j$ on S^{k-1} with cardinality $|\mathcal{N}_j|\le\left(\frac{3}{\delta_j}\right)^k$ (see [339]). Note that for any $\theta_\epsilon\in S^{k-1}$ and for all j there exist $u_j\in\mathcal{N}_j$ with $\|\theta_\epsilon-u_j\|_2\le\delta_j$ and by the triangle inequality it follows that $\|u_j-u_{j-1}\|_2\le\delta_j+\delta_{j-1}$. Moreover, if we assume that $\delta_j\to 0$ as $j\to\infty$ and (t_j) is a sequence of numbers with $t_j\ge 0$ and $\sum_j t_j\le 1$ then, for any $\varepsilon>0$.

Corollary (6.3.34)[353]: [320]. Let $0\le\epsilon<\infty$. Then, for any n-dimensional subspace X of $L_{1+\epsilon}$ represented on $\mathbb{R}^n$, there exists a position $\tilde{B}$ of its unit ball B_X such that the normalized variance is of minimal possible order (up to constants of $1+\epsilon$)

$$\frac{\mathrm{Var}\|Z\|_{\tilde{B}}}{\mathbb{E}\|Z\|_{\tilde{B}}^2}\le\frac{(1+\epsilon)^{1+\epsilon}}{n},$$

where Z is the standard Gaussian vector in $\mathbb{R}^n$ and $\epsilon\ge 0$ is an absolute constant.

Sketch of Proof. If $\tilde{B}$ is a Lewis' position of B_X, we may identify X with $X_{1+\epsilon}(\mu)$ for some Borel isotropic measure μ on S^{n-1}. Then, the result follows from Corollary (6.3.31). On the other hand note that the normalized variance is minimal since for every norm $\|\cdot\|$ on $\mathbb{R}^n$ one has $\mathrm{Var}\|Z\|\lesssim\mathbb{E}\|Z\|^2/n$. The latter may be easily checked by using integration in polar coordinates and the Cauchy–Schwarz inequality.

Corollary (6.3.35)[353]: Let $0<\epsilon<\infty$. For any $\varepsilon\in(0,1)$, for any $S\subset S^{n-1}$ and for any m-dimensional subspace X of $L_{2+\epsilon}$ with $m\gtrsim\max\left\{(2+\epsilon)4^{2+\epsilon}\varepsilon^{-2}\,log|S|\,,\varepsilon^{-1}(log|S|)^{\frac{2+\epsilon}{2}}\right\}$, there exists a linear mapping $T:\ell_2^n\to X$ such that

$$1-\varepsilon\le\|Tu\|_X\le 1+\varepsilon,\qquad u\in S.$$

Sketch of Proof. Fix $2+\epsilon,\varepsilon,X$ and m as above. Then, by Theorem (6.3.6), there exists an isotropic measure on S^{m-1} and a linear isometry $U:X_{2+\epsilon}(\mu)\to X$. Then, by Corollary (6.3.31) we have

$$\mathbb{P}\left(\exists u \in S: \left| \|Gu\|_{B_{2+\epsilon}(\mu)} - \left(\mathbb{E}\|Z\|_{B_{2+\epsilon}(\mu)}^{2+\epsilon}\right)^{\frac{1}{2+\epsilon}} \right| \geq \varepsilon \left(\mathbb{E}\|Z\|_{B_{2+\epsilon}(\mu)}^{2+\epsilon}\right)^{\frac{1}{2+\epsilon}}\right.$$
$$\leq |S| \exp\big(-\psi(m, 2+\epsilon, \varepsilon)\big),$$

where $G = \left(g_{ij}\right)_{i,j=1}^{m,n}$ is a matrix with standard Gaussian entries and Z is the standard Gaussian vector in $\mathbb{R}^m$. Assuming that $\psi(m, 2+\epsilon, \varepsilon) > \log|S|$ we may conclude the existence of a matrix $M: \ell_2^n \to X_{2+\epsilon}(\mu)$ such that

$$1 - \varepsilon \leq \|Mu\|_{B_{2+\epsilon}(\mu)} \leq 1 + \varepsilon,$$

for all $u \in S$. The desired linear map is $T = UM$. The estimate for m is obtained by finding the inverse function of $m \mapsto \psi(m, 2+\epsilon, \varepsilon)$.

List of Symbols

$\otimes$:	Tensor product	216
AGA:	Asymptotic Geometric Analysis	228
AGA:	Asymptotic Geometric Analysis	229
Tr:	Trace	232
osc:	oscillation	233
$cand$:	Cardinality	237
$Spec$:	Spectrum	237
sgn:	Signature	248

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