بسم الله الرحمن الرحيم

Sudan University of Science & Technology

College of Graduate Studies & Scientific Research

Solution of Nonlinear Boussinesq Equations By Modified Adomian Decomposition Method

حل معادلات بوزنسك غير الخطية بواسطة طريقة تفكيك أدوميان المعدلة

A Thesis Submitted in Fulfillment for the Degree of PhD *in* Mathematics.

By

Abulfida Mohamed Ahmed Mohamed

Supervisor

Prof. Osman Mohammed El Mekki

Dedication

I would like to dedicate my work to my parents

Acknowledgements

I would like to thank my supervisor Prof. Osman Mohamed EL Mekki for making this an enjoyable period of study on a very interesting project.

I would also like to thank Dr. Emad Eldeen A'Allah Abdelrahim who have supported this work throughout.

I am grateful to the Ministry of higher education and scientific research for supporting this work.

Abstract

The application of Adomian's decomposition method and its modifications to partial differential equations, when the exact solution is not reached, demands the use of truncated series. But the solution's series may have small convergence radius and the truncated series may be inaccurate in many regions. In order to enlarge the convergence domain of the truncated series, Pad´e approximants technique is applied to partial differential equations, particularly to Boussinesq equations to find explicit and travelling waves solutions. Graphical illustrations were used to show that this technique can enlarge the domain of convergence of Adomian's solution. It is also showed that the solution accuracy can be improved by increasing the order of the Pad´e approximants. In this thesis, besides graphical illustrations, also numerical results are presented to show that this technique can not only enlarge the domain of convergence of the solution but also improves its accuracy even when the actual solution cannot be expressed as the ratio of two polynomials.

الخلاصة

يهدف هذا البحث الى عرض طريقة أدوميان وتحديثاتها على المعادلات التفاضلية الجزئية الغير خطية (معادلات بوزنسك).

طريقة أدوميان تعطى الحل في شكل متسلسلة وعندما يكون الحل غير منتظم نلجأ الى إقتطاع المتسلسلة وفي هذه الحالة يكون التقارب الى الحل غير دقيق في كثير من الحالات, ومن أجل زيادة دقة الحل إستخدمنا طريقة بادي التقريبية لإيجاد تقريب منتظم للحل, خلال هذا البحث تم الحصول على معادلات بوزنسك بعدد من الطرق , ومن ثم حلها بطريقة أدوميان و أدوميان المحسنة , كما إستخدمنا تحويلات لابلاس وبعض التحويلات الأخرى. النتائج أظهرت أن إستخدام تقنية بادي أدت الى تحسين طريقة أدوميان و أعطت الحل المضبوط بإختيار عدد محدود من حدود المتسلسلة.

Abbreviations

BTMs	Boussinesq-type Models
NSWEs	Non-linear Shallow Water Equations
IBq	Improved Boussinesq
AP`s	Adomian Polynomials
ADM	Adomian Decomposition Method
MADM	Modified Adomian Decomposition Method
IADM	Improved Adomian Decomposition Method
LADM	Laplace Adomian Decomposition Method
STM	Sumudu Transform Method

Introduction

Nonlinear partial differential equations are encountered in various fields of mathematics, physics, chemistry, and biology, and numerous applications.

One of the crucial problems in the theory of partial differential equations (PDEs) at its early stages in the eighteenth and nineteenth century was finding and studying classes of important equations that were integrable in closed form and, in particular, possessed explicit solutions. It seems that the first general type of explicit solutions were traveling waves in d'Alembert's formula for the linear wave equation. Many famous mathematicians, such as Euler, Lagrange, Liouville, Sturm, Laplace, Boussinesq, and others developed various techniques for obtaining explicit solutions of a variety of linear and nonlinear models from physics and mechanics. Their methods included a number of particular transformations, symmetries, expansions, separation of variables, etc.

The Boussinesq equations described here are model equations for propagation of long waves. The Boussinesq equations are named after the French scientist J. Boussinesq who derived a version of the equations to find solutions for solitary waves on a water surface. Later, Boussinesq equations have been presented in many different versions, and there is no strict assent concerning the use of "Boussinesq equations" in relation to a single set of equations or group of equations.

Table of Contents

List of figures

List of Tables

Chapter 1

Literature Review

1.1 Introduction

Over the last years, Boussinesq-type models (BTMs) have enjoyed increasing favor within the coastal engineering community. The main reasons for this are the optimal blend of physical adequacy (i.e. their ability to represent all main physical phenomena) and computational ease (i.e. their numerical cheapness). Hence, Boussinesq- type equations (BTEs), beating the competition of non-linear shallow-water equations (NSWEs), have become the most favored approximations of Navier–Stokes equations for coastal-type computations [2].

Such computations are essential tools for any design activity in which water waves play a significant role. Examples of such activities can be found in the field of off shore engineering (design of off shore platforms, pipelines, underwater cables, etc.), near-shore engineering (design of harbors, coastal defense structures, etc.), and environmental management (evaluation of morphodynamic evolution, assessment of pollutant and nutrients flows, etc.). Applications are now also increasing in the field of riverine engineering.

1.2. Fundamentals of BTMs

The fundamentals of BTMs are proposed here using a schematic approach, which aims at highlighting the fundamental principles of the modelling and the initial applications of the models [2].

1.2.1 The Modelling

Any BTM to be used for predictive purposes is built by:

- proper account of the physics to be described (e.g. wave propagation)
- choice of the most suited model equations (e.g. best description of nature or idealized dynamics);
- use of appropriate boundary conditions (BCs)
- use of the most suited numerical methods for the chosen equations and BCs

1.2.2 The Approach

The above fundamental ingredients of any BTM can be balanced as functions of the chosen modelling approach. The following two main approaches can be singled out, which largely depend on the scientific community.

- Mathematicians use model equations for inspection of mathematical behaviour (integrability, symmetries, etc.)
- Physicists/engineers use equations derived from physical arguments and for practical applications. In this case, many fundamental physical phenomena are to be accounted for and mathematical properties are subordinate to physical requirements.

1.2.3 The Model Equations

We use a Cartesian coordinate system with the x^* and z^* axes lying on the still-water level and, respectively, pointing seawards and upwards, i.e. $x^* = 0$ gives the still-water shoreline (figure 1.1). In the following, asterisks are used to label dimensional quantities, whereas plain variables give dimensionless quantities.

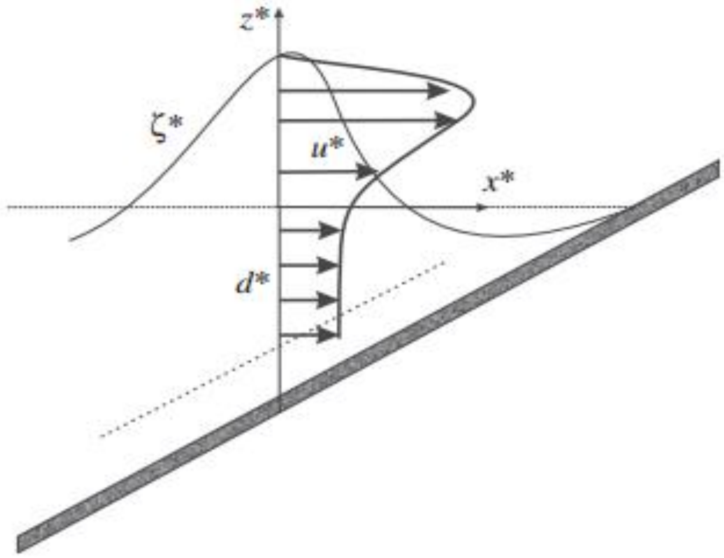

Figure 1-1. sketch of the flow over the bottom boundary layer (dotted line)

Neglecting boundary-layer dynamics (i.e. modelling the flow above the dotted line of figure 1.1), a vast majority of BTEs of the above-mentioned classes can be formally written as follows:

$$d_t + \nabla_H.[\bar{u}\, d] = 0 \qquad (1.1)$$

$$[\bar{u}d]_{,t} + d\nabla_H + \varepsilon\nabla_H[\bar{u}^2 d + \text{rotational contributions}]$$

$$+\mu^2[\xi(\varepsilon^{2n}, \bar{u}, h, \xi) + \text{rotational and turbulence contributions}] = 0, \qquad (1.2)$$

Closure equations for vorticity and turbulence (1.3)

$$\varepsilon \equiv \frac{a}{h}, \mu \equiv kh, d \equiv h + \varepsilon\xi, \nabla_H^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \nabla^2 \equiv \nabla_H^2 + \frac{1}{\mu^2}\frac{\partial^2}{\partial z^2}, \qquad (1.4)$$

a is a reference wave amplitude, k is the wave number and the comma denotes partial differentiation. The continuity equation (1.1) is exact, whereas the momentum equation (1.2) is given in conservation form. Equations (1.3) and (1.4) are solved for the total water depth (d) and for a reference velocity ($\bar{u}$)

once closures (1.3) are given. Because these may be of many different forms, they are given here by a generic formal representation. As for all BTEs, solution of the system (1.1)–(1.4) is a function of the size of the wave nonlinearity (ε) and frequency dispersion (μ) parameters.

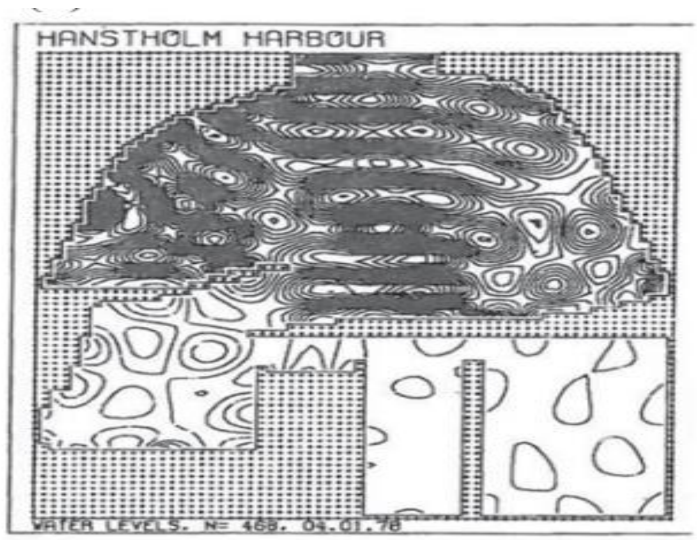

Figure 1.2 example of a contour map of harbor water oscillations

1.2.4. Early Applications: Harbor Waves 1

The first applications of BTMs to real-life problems date back to the 1970s and 1980s, and were related to the description of harbor free-surface oscillations. Such oscillations may cause excessive ship motion and negatively interfere with loading and unloading operations at port facilities. Hence, predictions of such oscillations (e.g. figure 1.2), of the related ship motions and mooring forces were made by means of weakly nonlinear and weakly dispersive BTMs

1.3. Applications

The advances and progress in BTMs can be described not only through the analysis of the physical, mathematical and numerical ingredients that are used to build a specific model, but also on the basis of the actual applications of the models. The most important are described here through some examples whose illustration have been made available.

1.3.1 Harbor Waves 2

Harbor waves and ship motions have been the object of BTMs because their initial application is still being studied by means of BTMs. Figure 1.3 illustrates the harbor wave fields induced by wind waves.

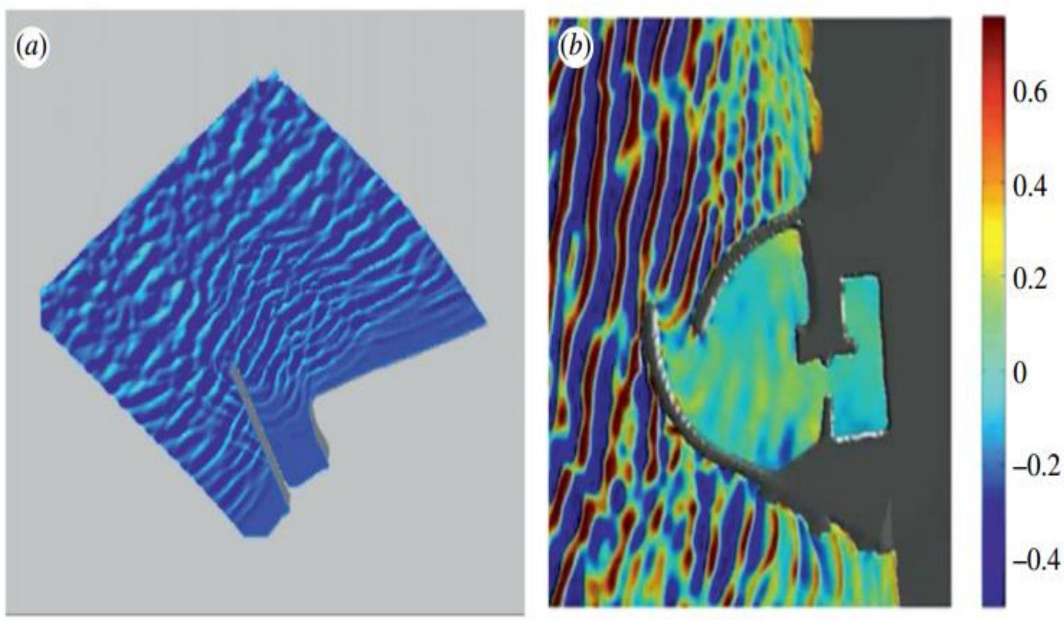

Figure 1.3 Harbor waves. (a) Funwave wave simulation in Ponce de Leon Inlet (courtesy of J. T. Kirby & F. Shi) and (b) finite volume modelling of random directional waves into a harbor (courtesy of P. Lynett).

1.3.2 Complex Bathymetries

Much more testing are applications associated with the propagation of waves over complex bathymetries. For these computations, most of the characteristics of the models are put under serious scrutiny, such as, for example dispersive properties, shoaling behavior and interaction with possibly steep slopes. Other properties of importance for this sort of computations are the wave–wave interaction (e.g. at the lee side of an obstacle) and the existence of multiple shorelines. The latter mechanisms are visible in figure 1.4

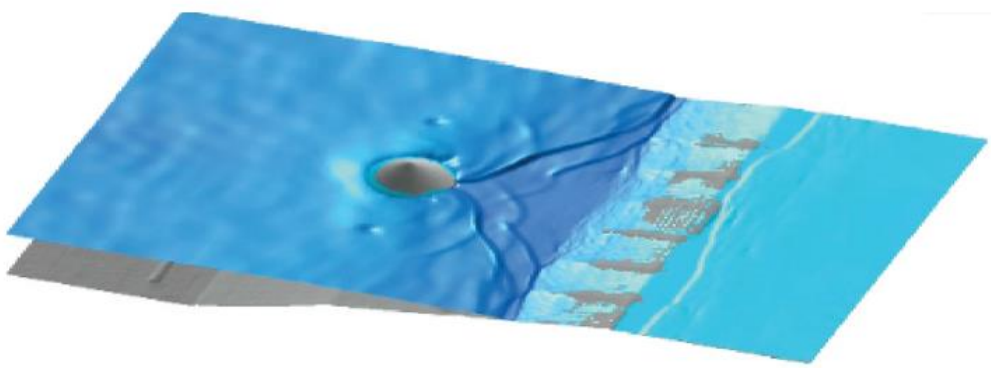

Figure 1.4 Finite-volume modelling of a plane wave interacting with a conical island (courtesy of V. Roeber, K. F. Cheung and M. H. Kobayashi).

1.3.3 Tsunamis

Because of the growing societal impact of tsunami waves and in virtue of the modelling improvements made over the last 30 years, applications of BTMs to the description of the evolution of tsunamis are growing. Although, theoretically appropriate only for the far-field modelling of such long waves (i.e. far from the generation point of the tsunami wave), BTMs are practically being used to describe tsunamis over their entire lifespan. Figure 1.5 illustrates the computations of laboratory solitary waves.

1.3.4 Flow Mixing

Another recent application of BTMs is in the field of study of shallow-water turbulence and water quality management. BTMs are being used to run numerical experiments whose outputs are used both to analyses Lagrangian statistics of a large number of water particles and to assess the main characteristics of the horizontal flow mixing.

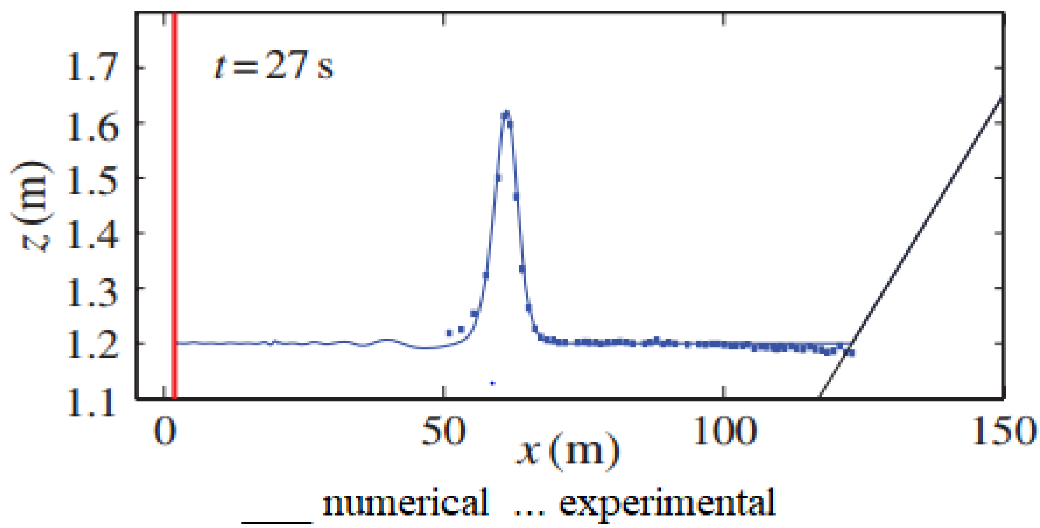

Figure 1.5 BTE NSWE, finite-difference modelling of solitary wave run-up (courtesy of A. G. L. Borthwick, J. Orszaghova and P. H. Taylor

1.3.5 Morphodynamics

One fundamental field of application of BTMs is near-shore morpho-dynamics used in conjunction with simplified morphological evolution equations (i.e. Exner equations) and simplified sediment transport closures, BTMs have been used to model many specific features of the near-shore sediment transport, morphodynamics and morphology. Erosion and accretion of a sandy beach (figure 1.6 *a* , *b*) as well as water infiltration/

exfiltration at a porous beach (figure 1.6 *c* , *d*) are among the most typical morphodynamic computations made with BTMs.

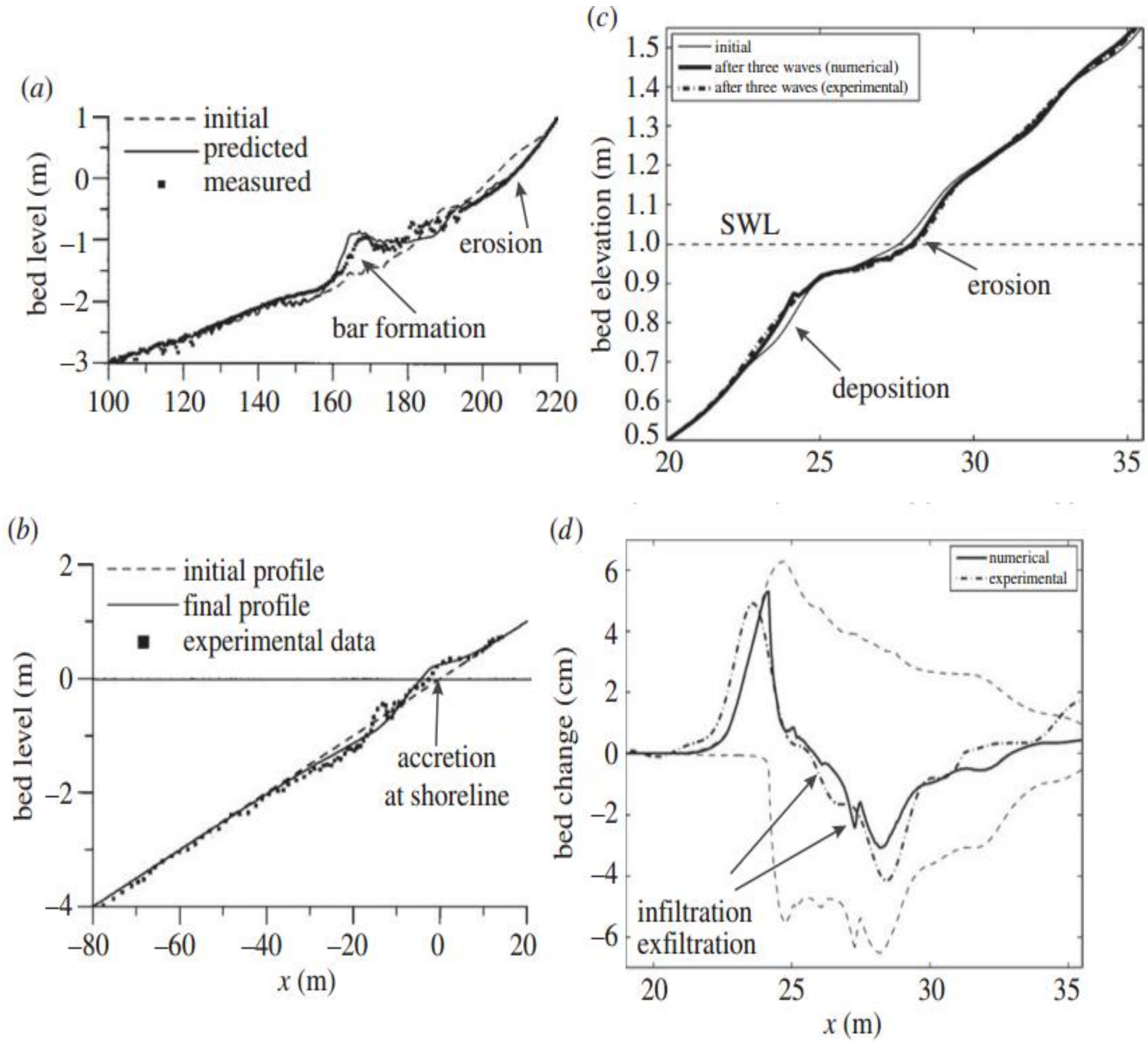

Figure 1.6 Morphodynamic modelling. (*a*,*b*) Application of Karambas & Koutitas's model for erosive (*a*) and accretive (*b*) conditions (adapted from Karambas& Koutitas) .(*c*,*d*) Evolution of a porous beach by the model of Xiao *etal.* Seabed profiles at initial stages (*c*) and corresponding morphological changes (*d*). Adapted from Xiao*eta*. SWL: still water level.

1.4 Numerics

Once the physics of interest and the mathematics in use have been described, the final aspect to be analyzed is the numerics used to carry out suitable BTM computations. This aspect, which was a specific target of the original international Conference on Coastal Engineering talk, is discussed in some detail.

1.4.1 Early Approaches

A brief overview is here given of the numerical approaches used before the 2000s, when finite volumes (FVs) became the dominating method.

(a) Finite differences (FDs)

This approach has found much favor because of the wealth of specific literature available and because of the ease of application, as it is a rather intuitive approach for writing difference equations. The best-known FD scheme used in BTMs is that initially proposed by Abbott *et al.*, and subsequently used by many authors. In brief, the differential equations were discretized using a time-centred implicit scheme with variables defined on a space-staggered rectangular grid. The method is based on the alternating direction implicit algorithm.

(b) Finite elements (FEs)

This method has found less favor than the FD method because it is relatively more difficult to implement. However, its excellent performances in dealing with complex geometries suggested its use in the case of confined water bodies (e.g. harbors). The early FE BTMs relied on the Galerkin-weighted residual method and used linear shape functions for interpolation.

1.4.2 Finite Volumes

A more detailed analysis is dedicated to the description of the use of FVs in BTMs because such an approach, which makes the most of the wealth of knowledge in the integration of hyperbolic equations

In the majority of currently available FV BTMs:

i. *Weakly nonlinear BTEs are integrated*. Examples of exceptions are the models of Kim *et al.* and Shi *et al*, which integrate fully nonlinear equations.

ii. FV methods are used for the conservative (left-hand side) part of the equations, while FD schemes are used for the source terms *S*B and *S*D: *hybrid FV–FD models*. Examples of exceptions are the models of Roeber *et al.*, Roeber & Cheung and Dutykh *et al*, which use fully FV methods.

iii. The data reconstruction (averages) is performed by means of the *fourth-order monotone upstream-centred schemes for conservation laws* (*MUSCL*) *method*. Examples of exceptions are the models of Cienfuegos *et al.*(Padè interpolation) and Roeber *et al* (multidimensional limiting process).

iv. The flux computation is performed by using the *approximate Harten, Lax and Van Leer finite volume scheme* (*HLL*) *and the Harten, Lax and Van Leer finite volume scheme for contact waves* (*HLLC*). Examples of exceptions are the models of Erduran *et al* (Roe scheme), Roeber *et al.*and Dutykh *et al* (various schemes).

v. The time stepping is performed by either the *fourth-order predictor-corrector Adams– Bashforth–Moulton* (ABM) scheme or the *Runge–Kutta scheme* Examples of exceptions are the models of Ning *et al.*and

Shiach & Mingham (second-order MUSCL–Hancock, claiming second order is enough for time stepping).

1.4.3 Finite elements

Although still lagging behind FV methods, FE methods are increasingly being applied to modern BTMs in virtue of the power and flexibility of the discontinuous Galerkin (DG) approach. For the sake of simplicity, the derivation of the scheme is briefly given starting from a simple, generic, scalar conservation law,

$$\bar{u}_t + A_x(\bar{u})_x = S, x \in \Omega, \bar{u} \in V$$

The main steps can summarized as follows.

a. Define the FE subdivision of the domain

$$\Omega = \bigcup_{i=1}^{I} \Omega_i, V_h \subset V, V_h = \text{Subspace such that } \bar{u}_i = \bar{u}(x_i, t)$$

b. Use of local basis functions. The DG approach makes use of elemental basis functions.

$$\overset{i}{\psi} = \overset{i}{\psi}_{1,2}(x; h, p), p = \text{ power of the polynomial functions}$$

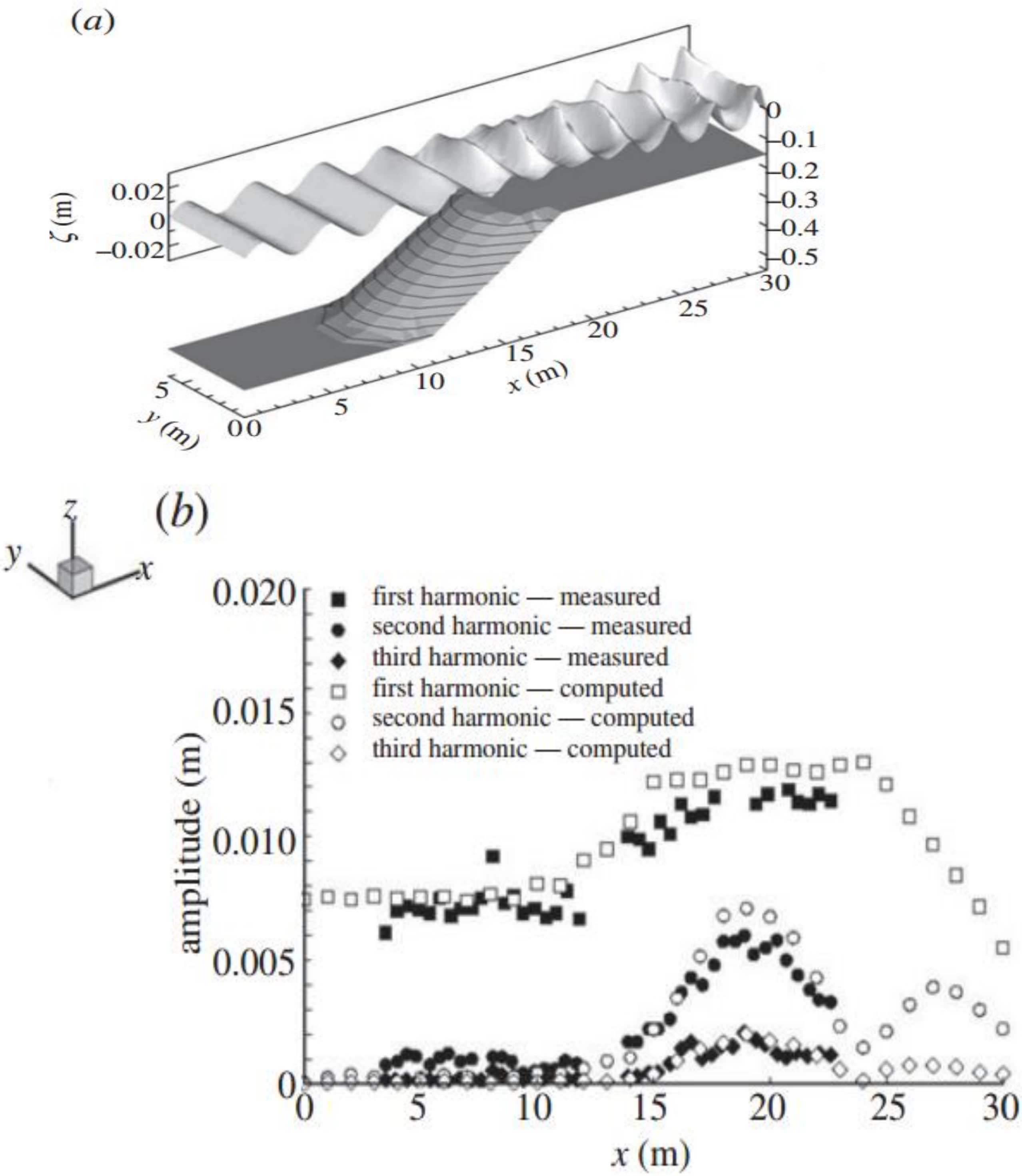

Figure 1.7 Propagation of waves over a semicircular shoal by the model of Eskilsson & Sherwin. (*a*) Snapshot of surface elevation. (*b*) Wave amplitude for first, second and third harmonics along the centre line.Adapted from Eskilsson & Sherwin.

Among the important models and applications, the following can be mentioned.

a. The model of Eskilsson & Sherwin. This solves the equations by Madsen & Sørensen using the HLLC scheme for the IF. The model performances are assessed here on the basis of the propagation of waves over a semicircular shoal (figure 1.7*a*). In particular, the amplitudes of the first three harmonics obtained from the model are compared with those from the laboratory experiments of Whalin to show a fairly good match (figure 1.7*b*).
b. The model of Engsig-Karup *et al*. This model solves the equations of Madsen *et al.* through a Lax–Friedrich scheme for the IF. Application of the model to the scattering of linear waves about a vertical cylinder in open water and comparison with the analytical solution owing to McCamy & Fuchs reveal the good performance of the model.

Chapter 2

The Boussinesq Equations

2.1 Indroduction

The Boussinesq equations are named after the French scientist J. Boussinesq who derived a version of the equations to find solutions for solitary waves on a water surface. Later, Boussinesq equations have been presented in many different versions, and there is no strict consensus concerning the use of “Boussinesq equations” in relation to a single set of equations or group of equations.

There are several basic assumptions for simplified descriptions for waves, such as small amplitude, slowly varying medium, and large wavelength. For surface gravity waves, the simplest form of long-wave equations is the shallow water equations which require that the waves are much longer than the depth. This implies that the pressure distribution is hydrostatic. The Boussinesq equations extend the shallow water equations by including the leading correction to hydrostatic pressure due to the vertical acceleration, while nonlinearity is retained, either approximately or fully.

2.2 The Mathematical Model

The starting point for modelling water waves is the incompressible Navier-Stokes equations. However the numerical solution of these equations, a three-dimensional problem with a free-surface boundary, is extremely complex and it is usual to make some simplifying assumptions before the numerical solution is attempted [3].

The Boussinesq equations describing shallow water flow are derived from the incompressible, irrotational Euler equations for simplicity of presentation only two spatial dimensions are considered; (x, z) being horizontal and vertical respectively. The z coordinate varies between the free surface $\eta(x,t)$ and the sea bed $-h(x)$ with the origin taken at the still-water depth. The frame of reference is illustrated in Figure 2.1

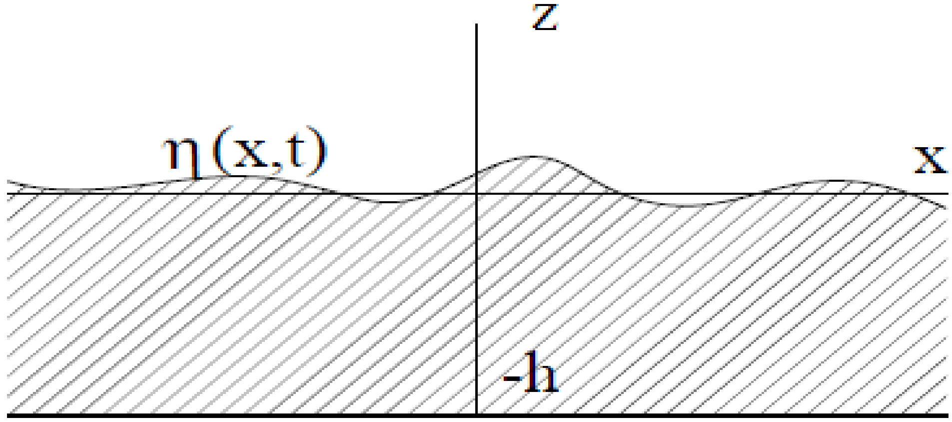

Figure *2.1* .The frame of reference.

2.3 Derivations of the Boussinesq Equations

Boussinesq equations can be derived in a number of ways: One classical approach (followed by e.g. Peregrine, 1967) is to use depth-integration of the continuity and Euler equations, a procedure which involves the determination of the pressure by integration of the vertical Euler equation. Another classical approach (followed by e.g. Boussinesq, 1872; Mei and Mehaute, 1966; Svendsen, 1974; Mei, 1983) is to use the Laplace equation combined with the dynamic and kinematic free surface boundary conditions formulated in terms of the velocity potential.

2.3.1 Boussinesq's (1872) Derivation

We consider the original method of derivation for horizontal bottom, due the Boussinesq (1872).The Boussinesq equation is built from the following system of equations [4]:

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}=0 \qquad \mathit{for} \qquad -h\le z\le\eta(x,t) \tag{2.1}$$

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right]+gz=0 \qquad \text{at } z=\eta(x,t) \tag{2.2}$$

$$\frac{\partial\eta}{\partial t}+\frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x}=\frac{\partial\phi}{\partial z} \qquad \text{at } z=\eta(x,t) \tag{2.3}$$

$$\frac{\partial\phi}{\partial x}\frac{\partial h}{\partial x}+\frac{\partial\phi}{\partial z}=0 \qquad \text{at } z=-h \tag{2.4}$$

These express the Laplace equation, dynamic and kinematic free surface condition and the boundary condition on $z=-h(x)$ respectively.

Integrating of the Laplace equation (2.1) twice respect to z yields (use the kinematic bottom condition $\frac{\partial\phi}{\partial z}=0$ at $z=-h$)

$$\phi(x,z,t)=\varphi_0(x,t)-\int_{-h}^{z}dz`\int_{-h}^{z`}\frac{\partial^2\phi}{\partial x^2}dz \tag{2.5}$$

Where $\varphi_0=\phi(x,-h,t)$ is the value of the potential at the bottom. For the shallow water the velocity $\frac{\partial\phi}{\partial x}$ is not much different from the velocity at the bottom, $\frac{\partial \varphi_0}{\partial x}$.Taking also the horizontal acceleration to be nearly equal to its value at the bottom i.e setting $\phi \cong \varphi_0$ in the integral, one obtain from (2.5)

$$\phi(x,z,t) \cong \varphi_0(x,t) - \frac{(z+h)^2}{2}\frac{\partial^2 \varphi_0}{\partial x^2} \tag{2.6}$$

By substituting this approximation back into the integral (2.5) one obtain next

$$\phi(x,z,t) = \phi_0(x,t) - \frac{(z+h)^2}{2!}\frac{\partial^2 \varphi_0}{\partial x^2} + \frac{(z+h)^4}{4!}\frac{\partial^4 \varphi_0}{\partial x^4} \tag{2.7}$$

Substitute (2.7) in the dynamic and kinematic free surface condition (2.2) and (2.3), from which the following two equation in $\eta(x,t)$ and $u_0(x,t) = \frac{\partial \varphi_0}{\partial x}$ when the quadratic terms in η and u are kept,

$$\frac{\partial u_0}{\partial t} + u_0 \frac{\partial u_0}{\partial x} + g\frac{\partial \eta}{\partial x} = \frac{h^2}{2}\frac{\partial^3 u_0}{\partial x^2 \partial t} \tag{2.8}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial((h+\eta)u_0)}{\partial x} = \frac{h^3}{6}\frac{\partial^3 u_0}{\partial x^3} \tag{2.9}$$

Equation (2.8) (2.9) were obtained by Boussinesq (1872). Without the right hand side the classical shallow water equation are obtained

$$\frac{\partial u_0}{\partial t} + u_0 \frac{\partial u_0}{\partial x} + \frac{\partial \eta}{\partial x} = 0 \tag{2.10}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial((h+\eta)u_0)}{\partial x} = 0 \tag{2.11}$$

2.3.2 Derivation by Expansion of the Velocity Potential

Equations (2.1)-(2.4) are written in dimensionless form with x, z, ζ, t and h made dimensionless in the same way as in (2.31) the velocity potential ϕ is scaled with $\lambda\sqrt{gh_0}$ ($\lambda \equiv$ a typical wave length). After introduction also the scaling $\eta = \varepsilon\zeta$ and $\phi = \varepsilon\varphi$ with $\varepsilon = a / h_0 \ll 1$ ($\varepsilon \equiv$ ratios) the dimensionless equations are (dropping the distinction between dimensional and dimensionless variables) [4]:

$$\mu \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \qquad for \qquad -h(x) \le z \le \varepsilon\eta(x,t) \tag{2.12}$$

$$\mu\left[\frac{\partial \varphi}{\partial t} + \frac{1}{2}\varepsilon\left(\frac{\partial \varphi}{\partial x}\right)^2 + \frac{1}{\varepsilon}z\right] + \frac{1}{2}\varepsilon\left(\frac{\partial \varphi}{\partial z}\right)^2 \quad at\ z = \varepsilon\eta(x,t) \tag{2.13}$$

$$\mu\left[\frac{\partial \eta}{\partial t} + \varepsilon\frac{\partial \varphi}{\partial x}\frac{\partial \eta}{\partial x}\right] = \frac{\partial \varphi}{\partial z} \quad at \qquad z = \varepsilon\eta(x,t) \tag{2.14}$$

$$\mu\frac{\partial \varphi}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial \varphi}{\partial z} = 0 \qquad at\ z = -h(x) \tag{2.15}$$

Where $\mu = \delta^2 = \left(\frac{h_0}{\lambda}\right)^2 \ll 1$

Notice that a factor μ appears in the continuity equation. This shows that differentiation of φ to z necessarily introduce an $O(\delta)$ effect.

The velocity potential φ is expanded as

$$\varphi(x,z,t) = \sum_{n=0}^{\infty} \delta^n (z+h)^n \varphi_n(x,t) \tag{2.16}$$

It is noted that $\varphi_0 = O(1)$ due the normalization .The derivatives of φ directly follow from (2.16) one has

$$\frac{\partial \varphi}{\partial t} = \sum_{n=0} \delta^n (z + h)^n \frac{\partial \varphi_n}{\partial t}$$

$$\frac{\partial \varphi}{\partial z} = \sum_{n=0} \delta^n (z + h)^n (n+1)\delta \varphi_{n+1}$$

$$\frac{\partial \varphi}{\partial x} = \sum_{n=0} \delta^n (z + h)^n \left[(n+1)\delta h_x \varphi_{n+1} + \frac{\partial \varphi_n}{\partial x} \right]$$

$$\frac{\partial^2 \varphi}{\partial z^2} = \sum_{n=0} \delta^n (z + h)^n (n+1)(n+2)\delta^2 \varphi_{n+2}$$

$$\frac{\partial^2 \varphi}{\partial x^2} = \sum_{n=0} \delta^n (z + h)^n [\frac{\partial^2 \varphi_n}{\partial x^2} + (n+1)\delta h_{xx} \varphi_{n+1} + 2(n+1)\delta h_x \frac{\partial \varphi_{n+1}}{\partial x} + (n+1)(n+2)\delta^2 h_x{}^2 \varphi_{n+1}]$$

It can be seen that in the expressions for $\frac{\partial \varphi}{\partial x}$ and $\frac{\partial^2 \varphi}{\partial x^2}$ the extra factors δ always occur in the combination δh_x *or* δh_{xx} , where as in those for $\frac{\partial \varphi}{\partial z}$ and $\frac{\partial^2 \varphi}{\partial z^2}$ they occur in the combination $\delta \varphi_{n+1}$ and $\delta^2 \varphi_{n+2}$.This clear shows that $\frac{\partial \varphi}{\partial z}$ is not of $O(\varphi)\,(in\ \delta)$.In present normalization we have $h_x = O(1)$ and therefore the combination δh_x always occurs because the physical bottom slope is $O(\varepsilon)$.

Substituting the expansion (2.16) in the continuity equation (2.12) and collecting equal powers of $(z + h)^n$ leads to the following recursion formula

$$\varphi_{n+2} = \frac{\frac{\partial^2 \varphi_n}{\partial x^2} + (n+1)\delta h_{xx} \varphi_{n+1} + 2(n+1)\delta h_x \frac{\partial \varphi_{n+1}}{\partial x}}{(n+1)(n+2)(1 + (\delta h_x)^2)}, \quad n = 0,1,2,... \qquad (2.17)$$

From the bottom condition is obtained

$$\varphi_1 = -\frac{\delta h_x}{1 + \mu h_x^2} \frac{\partial \varphi_0}{\partial x} \qquad (2.18)$$

It follows from (2.17) that $\varphi_{n+2} = O(\varphi_n)$; from (2.18) it can be seen that $\varphi_1 = O(\delta)$. Because we have $\varphi_0 = O(1)$ we thus obtain the order estimates

$$\varphi_{2n} = O(1) \, and \; \varphi_{2n+1} = O(\delta) \quad for \; n = 0,1,2,... \tag{2.19}$$

Substitution of the expansion (2.16) in the dynamic and kinematic free surface conditions (2.13) and (2.14) yields after neglection of $O(\varepsilon\mu, \mu^2)$ terms the following two equations

$$\begin{aligned} &\frac{\partial \varphi_0}{\partial t} + \delta h \frac{\partial \varphi_1}{\partial t} + \mu h^2 \frac{\partial \varphi_2}{\partial t} + \frac{1}{2}\varepsilon \left(\frac{\partial \varphi_0}{\partial x}\right)^2 + \eta + O(\varepsilon\mu, \mu^2) = 0 \\ &\frac{\partial \eta}{\partial t} + \varepsilon \frac{\partial \eta}{\partial x}\frac{\partial \varphi_0}{\partial x} = \frac{1}{\mu}\varphi_1 + 2(h + \varepsilon\eta)\varphi_2 + 3\delta h^2 \varphi_3 + 4\mu h^3 \varphi_4 + O(\varepsilon\mu, \mu^2) \end{aligned} \tag{2.20}$$

Using the recursion formula (2.17) and (2.18) it is possible to express φ_2, φ_3 and φ_4 in terms of φ_0 .To the necessary order of magnitude is obtained.

$$\frac{1}{\delta}\varphi_1 = -\frac{h_x}{1 + \mu h_x^2}\frac{\partial \varphi_0}{\partial x}$$

$$\varphi_2 = -\frac{1}{2}\frac{1 - \mu h_x^2}{(1 + \mu h_x^2)^2}\frac{\partial^2 \varphi_0}{\partial x^2} + \frac{3}{2}\frac{\mu h_x h_{xx}}{(1 + \mu h_x^2)^3}\frac{\partial \varphi_0}{\partial x} + O(\mu^2)$$

$$\delta\varphi_3 = \frac{1}{2}\frac{\mu h_x}{(1 + \mu h_x^2)^2}\frac{\partial^3 \varphi_0}{\partial x^3} + \frac{1}{2}\frac{\mu h_{xxx}}{(1 + \mu h_x^2)^3}\frac{\partial^2 \varphi_0}{\partial x^2} + \frac{1}{6}\frac{\mu h_{xxx}}{(1 + \mu h_x^2)^4}\frac{\partial \varphi_0}{\partial x} + O(\mu^2)$$

$$\mu\varphi_4 = \frac{1}{24}\mu\frac{1}{(1 + \mu h_x^2)^4}\frac{\partial^4 \varphi_0}{\partial x^4} + O(\mu^2)$$

After expansion of the denominators, the above results for φ_1 to φ_4 are substituted into Eqs. (2.20). After differentiating the first equation to x and introducing the notation $u_m = \frac{\partial \varphi_0}{\partial x}$ the following Boussinesq-like set of equations results:

$$\frac{\partial u_m}{\partial t}+\frac{\partial \eta}{\partial x}+\varepsilon u_m\frac{\partial u_m}{\partial x}-\frac{1}{2}\mu h^2\frac{\partial^3 u_m}{\partial t\,\partial x^2}$$
$$=\mu(h_x^2+hh_{xx})\frac{\partial u_m}{\partial t}+2\mu hh_x\frac{\partial^2 u_m}{\partial t\,\partial x}+O(\varepsilon\mu,\mu^2) \quad (2.21)$$

$$\frac{\partial \eta}{\partial t}+\frac{\partial}{\partial x}\left[(h+\varepsilon\eta)u_m\right]-\frac{1}{6}\mu h^3\frac{\partial^3 u_m}{\partial x^3}$$
$$=\mu(h_x^3+\frac{3}{2}h^2h_{xx})\frac{\partial u_m}{\partial x}+\frac{3}{2}\mu h^2h_x\frac{\partial^2 u_m}{\partial x^2}+O(\varepsilon\mu,\mu^2)$$

This set of equations has been derived for a flat horizontal bed; that is the mean depth h is a constant independent of position x . (2.21) reduce to

$$\frac{\partial u_m}{\partial t}+\frac{\partial \eta}{\partial x}+\varepsilon u_m\frac{\partial u_m}{\partial x}=\frac{1}{2}\mu h^2\frac{\partial^3 u_m}{\partial t\,\partial x^2}+O(\varepsilon\mu,\mu^2) \quad (2.22)$$

$$\frac{\partial \eta}{\partial t}+\frac{\partial}{\partial x}\left[(h+\varepsilon\eta)u_m\right]=\frac{1}{6}\mu h^3\frac{\partial^3 u_m}{\partial x^3}+O(\varepsilon\mu,\mu^2) \quad (2.23)$$

2.3.3 Peregrine Derivations

The governing equation system is comprised of horizontal momentum, vertical momentum, incompressibility and irrotationality equations respectively. These equations are written in terms of the two dimensional velocity field (u,w) pressure p , constant density ρ and acceleration due to gravity g with respect to the coordinate system (x,z) and time t [3]

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}+\frac{1}{\rho}\frac{\partial P}{\partial x}=0 \quad (2.24)$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}+\frac{1}{\rho}\frac{\partial P}{\partial x}+g=0 \quad (2.25)$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{2.26}$$

$$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0 \tag{2.27}$$

At the free surface $z = \eta(x, t)$ there are a kinematic boundary condition:

$$w - u\frac{\partial \eta}{\partial x} - \frac{\partial \eta}{\partial t} = 0 \tag{2.28}$$

and

$$p = 0 \tag{2.29}$$

The kinematic boundary condition on $z = -h(x)$ is now

$$w + u\frac{\partial h}{\partial x} = 0 \tag{2.30}$$

The parameters ρ, g and H are chosen to non-dimensionalise the system variables, where H is a typical water depth eg. the average depth.

$$\begin{aligned} (x, z, h, \eta) &= H(\tilde{x}, \tilde{z}, \tilde{h}, \tilde{\eta}) \\ (u, w) &= \sqrt{gH}(\tilde{u}, \tilde{w}) \\ p = \rho g H \tilde{p}, t &= \sqrt{H / g}\,\tilde{t} \end{aligned} \tag{2.31}$$

Variables are scaled in order to make their magnitude explicit. An (*) is used here to denote non-dimensional scaled variables.

$$t^* = \sigma \tilde{t}, x^* = \sigma \tilde{x}, \eta^* = \frac{\tilde{\eta}}{\varepsilon},$$

$$u^* = \frac{\tilde{u}}{\varepsilon}, w^* = \frac{\tilde{w}}{\varepsilon \sigma} \tag{2.32}$$

All other variables are assumed to have scaling factors of one.

Inserting these new variables into the equations (2.24)-(2.27) gives,

$$\varepsilon \frac{\partial u}{\partial t^*} + \varepsilon^2 u^* \frac{\partial u^*}{\partial x^*} + \varepsilon^2 w^* \frac{\partial u^*}{\partial z^*} + \frac{\partial \tilde{P}}{\partial x} = 0 \qquad (2.33)$$

$$\varepsilon\sigma^2 \frac{\partial w^*}{\partial t^*} + \varepsilon^2\sigma^2 u^* \frac{\partial w^*}{\partial x^*} + \varepsilon^2\sigma^2 w^* \frac{\partial w^*}{\partial \tilde{z}} + \frac{\partial \tilde{P}}{\partial \tilde{z}} + 1 = 0 \qquad (2.34)$$

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial w^*}{\partial \tilde{z}} = 0 \qquad (2.35)$$

$$\frac{\partial u^*}{\partial \tilde{z}} - \sigma^2 \frac{\partial w^*}{\partial x^*} = 0 \qquad (2.36)$$

At the free surface $\tilde{z} = \varepsilon\eta^*$ the boundary conditions (2.28) and (2.29) become

$$w^* - \varepsilon u^* \frac{\partial \eta^*}{\partial x^*} - \frac{\partial \eta^*}{\partial t^*} = 0 \qquad (2.37)$$

and

$$\tilde{p} = 0 \qquad (2.38)$$

At the bed $\tilde{z} = -\tilde{h}$ the boundary condition (2.28) becomes,

$$w^* + u^* \frac{\partial \tilde{h}}{\partial x^*} = 0 \qquad (2.39)$$

Peregrine was derived the Boussinesq equations by expanding equations (2.33)-(2.36) in terms of the small parameters ε and σ.

Integrating equation (2.35) with respect to $\tilde{z}$ and applying Liebnitz' Rule gives

$$\int_{-\tilde{h}}^{\tilde{z}} \frac{\partial w^*}{\partial z} d\tilde{z} = -\int_{-\tilde{h}}^{\tilde{z}} \frac{\partial u^*}{\partial x^*} d\tilde{z}$$

$$w^*|_{\tilde{z}} - w^*|_{-\tilde{h}} = -\frac{\partial}{\partial x^*}\int_{-\tilde{h}}^{\tilde{z}} u^* d\tilde{z} - u^*|_{-\tilde{h}} \frac{\partial(-\tilde{h})}{\partial x^*} + u^*|_{\tilde{z}} \frac{\partial \tilde{z}}{\partial x^*}$$

Using the boundary condition (2.39) at $\tilde{z} = -\tilde{h}$, and the fact that the coordinates x^* and $\tilde{z}$ are independent gives,

$$w^* = -\frac{\partial}{\partial x^*}\int_{-\tilde{h}}^{\tilde{z}} u^* d\tilde{z} \tag{2.40}$$

The vertical velocity $w^*|_{\tilde{z}}$ is simply denoted w^* Substituting equation (2.40) in equation (2.36), and integrating with respect to $\tilde{z}$

$$\frac{\partial u^*}{\partial z} = \sigma^2 \frac{\partial w^*}{\partial x^*}$$

$$= \sigma^2 \frac{\partial^2}{\partial x^{*2}}\int_{-\tilde{h}}^{\tilde{z}} u^* d\tilde{z} \tag{2.41}$$

$$u^* = u_0^*(x^*, t^*) + O(\sigma^2) \tag{2.42}$$

Where $u_0^*(x^*, t^*)$ an arbitrary function of x^* and t^* introduced by the integration. Equation (2.42) therefore implies that u^* is independent of $\tilde{z}$ to $O(\sigma^2)$ Substituting equation (2.42) in equation (2.40) and using the independence of u^* from $\tilde{z}$ to evaluate the integral leads to an expression for the vertical velocity

$$W^* = -\frac{\partial}{\partial x^*}\int_{-\tilde{h}}^{\tilde{z}} u_0^*(x^*, t^*) d\tilde{z} + O(\sigma^2)$$

$$= -\frac{\partial}{\partial x^*}((\tilde{z} + \tilde{h})u_0^*) + O(\sigma^2)$$

$$= -\tilde{z}\,\frac{\partial u_0^*}{\partial x^*} - \frac{\partial(\tilde{h}u_0^*)}{\partial x^*} + O(\sigma^2) \tag{2.43}$$

Substituting equation (2.43) in equation (2.36) and integrating with respect to $\tilde{z}$ obtains an expression for U^*

$$\frac{\partial u^*}{\partial \tilde{z}} = -\sigma^2\left(\tilde{z}\,\frac{\partial^2 u_0^*}{\partial x^{*2}} + \frac{\partial^2(\tilde{h}u_0^*)}{\partial x^{*2}}\right) + O(\sigma^4) \tag{2.44}$$

$$u^* = u_0^*(x^*, t^*) - \sigma^2\left(\frac{\tilde{z}^2}{2}\frac{\partial^2 u_0^*}{\partial x^{*2}} + \tilde{z}\,\frac{\partial^2(\tilde{h}u_0^*)}{\partial x^{*2}}\right) + O(\sigma^4) \tag{2.45}$$

Equations (2.34) and (2.43) are now used to obtain an expression for the pressure.

$$\begin{aligned} -\frac{\partial \tilde{p}}{\partial \tilde{z}} &= \varepsilon\sigma^2\,\frac{\partial w^*}{\partial t^*} + 1 + O(\varepsilon^2\sigma^2) \\ &= -\varepsilon\sigma^2\left(\tilde{z}\,\frac{\partial^2 u_0^*}{\partial x^*\partial t^*} + \frac{\partial^2(\tilde{h}u_0^*)}{\partial x^*\partial t^*}\right) + 1 + O(\varepsilon^2\sigma^2, \varepsilon\sigma^4) \end{aligned}$$

This is integrated with respect to $\tilde{z}$ from an arbitrary depth $\tilde{z}$ to the free surface $\varepsilon\eta^*$

$$-\int_{\tilde{z}}^{\varepsilon\eta^*} \frac{\partial \tilde{p}}{\partial \tilde{z}} d\tilde{z} = -\varepsilon\sigma^2\left[\frac{\tilde{z}^2}{2}\frac{\partial^2 u_0^*}{\partial x^*\partial t^*} + \tilde{z}\,\frac{\partial^2(\tilde{h}u_0^*)}{\partial x^*\partial t^*}\right]_{\tilde{z}}^{\varepsilon\eta^*} + \left[\tilde{z}\right]_{\tilde{z}}^{\varepsilon\eta^*} + O(\varepsilon^2\sigma^2, \varepsilon\sigma^4) \tag{2.46}$$

The boundary condition (2.38) gives $\tilde{p}\,|_{\varepsilon\eta^*} = 0$. Denoting $\tilde{p}\,|_{\tilde{z}}$ simply as $\tilde{p}$ and expanding the right hand side terms, noting that evaluation at $\varepsilon\eta^*$ introduces only $O(\varepsilon^2\sigma^2)$ terms, gives,

$$\tilde{p} = \varepsilon\sigma^2\left(\frac{\tilde{z}^2}{2}\frac{\partial^2 u_0^*}{\partial x^*\partial t^*} + \tilde{z}\,\frac{\partial^2(\tilde{h}u_0^*)}{\partial x^*\partial t^*}\right) + \varepsilon\eta^* - \tilde{z} + O(\varepsilon^2\sigma^2, \varepsilon\sigma^4) \qquad (2.47)$$

The expressions for w^*, u^* and $\tilde{p}$ are substituted in the horizontal momentum equation (2.33) rearranging the terms and collecting the high order terms in ε and σ on the right hand side,

$$\frac{\partial u_0^*}{\partial t^*} + \varepsilon u_0^*\frac{\partial u_0^*}{\partial x^*} + \frac{\partial \eta^*}{\partial x^*} = O(\varepsilon\sigma^2, \sigma^4) \qquad (2.48)$$

Now define the depth averaged velocity, $\bar{u}^*(x^*, t^*)$, in terms of the velocity field $u^*(x^*, \tilde{z}, t^*)$

$$\bar{u}^* = \frac{1}{\tilde{h} + \varepsilon\eta^*}\int_{-\tilde{h}}^{\varepsilon\eta^*} u^* d\tilde{z} \qquad (2.49)$$

Substituting from equation (2.45) and rearranging the terms to separate out higher order terms in ε and σ

$$\frac{1}{\tilde{h} + \varepsilon\eta^*}\left(u_0^*(\tilde{h} + \varepsilon\eta^*) - \sigma^2\left(\left(\frac{(\varepsilon\eta^*)^3 + \tilde{h}^3}{6}\right)\frac{\partial^2 u_0^*}{\partial x^{*2}} + \left(\frac{(\varepsilon\eta^*)^2 - \tilde{h}^2}{2}\right)\frac{\partial^2(\tilde{h}u_0^*)}{\partial x^{*2}}\right) + O(\sigma^4)\right)$$

$$= u_0^* - \frac{\sigma^2}{\tilde{h}\left(1 + \varepsilon\dfrac{\eta^*}{\tilde{h}}\right)}\left(\frac{\tilde{h}^3}{6}\frac{\partial^2 u_0^*}{\partial x^{*2}} - \frac{\tilde{h}^2}{2}\frac{\partial^2(\tilde{h}u_0^*)}{\partial x^{*2}} + O(\varepsilon\sigma^2, \sigma^4)\right)$$

$$= u_0^* - \sigma^2\left(\frac{\tilde{h}^2}{6}\frac{\partial^2 u_0^*}{\partial x^{*2}} - \frac{\tilde{h}}{2}\frac{\partial^2(\tilde{h}u_0^*)}{\partial x^{*2}} + O(\varepsilon\sigma^2, \sigma^4)\right) \qquad (2.50)$$

from equation (2.39) $u_0^* = \bar{u}^* + O(\sigma^2)$

$$u_0^* = \bar{u}^* + \sigma^2 \left(\frac{\tilde{h}^2}{6} \frac{\partial^2 u_0^*}{\partial x^{*2}} - \frac{\tilde{h}}{2} \frac{\partial^2 (\tilde{h} u_0^*)}{\partial x^{*2}} \right) + O(\varepsilon\sigma^2, \sigma^4) \tag{2.51}$$

Substituting this in equation (2.48) obtains the momentum equation in the Boussinesq system

$$\frac{\partial \bar{u}^*}{\partial t^*} + \varepsilon \bar{u}^* \frac{\partial \bar{u}^*}{\partial x^*} + \frac{\partial \eta^*}{\partial x^*} + \sigma^2 \left(\frac{\tilde{h}^2}{6} \frac{\partial^2}{\partial x^{*2}} \left(\frac{\partial \bar{u}^*}{\partial t^*} \right) - \frac{\tilde{h}}{2} \frac{\partial^2}{\partial x^{*2}} \left(\tilde{h} \frac{\partial \bar{u}^*}{\partial t^*} \right) \right) = O(\varepsilon\sigma^2, \sigma^4) \tag{2.52}$$

To obtain the second equation of the Boussinesq system equation (2.35) is integrated through the depth.

$$\int_{-\tilde{h}}^{\varepsilon\eta^*} \left(\frac{\partial u^*}{\partial x^*} + \frac{\partial w^*}{\partial \tilde{z}} \right) d\tilde{z} = 0 \tag{2.53}$$

Using Liebnitz' Rule, the boundary conditions (2.37) and (2.42) and the definition of the depth averaged velocity (2.49),

$$\frac{\partial}{\partial x^*} \int_{-\tilde{h}}^{\varepsilon\eta^*} u^* d\tilde{z} - u^* |_{\varepsilon\eta^*} \frac{\partial(\varepsilon\eta^*)}{\partial x^*} + u^* |_{-h} \frac{\partial(-\tilde{h})}{\partial x^*} + w^* |_{\varepsilon\eta^*} - w^* |_{-h} = 0$$

$$\frac{\partial}{\partial x^*} ((\tilde{h} + \varepsilon\eta^*) u^*) + \frac{\partial \eta^*}{\partial t^*} = 0 \tag{2.54}$$

Returning the variables to dimensional, unscaled form via the transformations defined before, gives the Boussinesq equation system,

$$\frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} + \frac{\partial \eta}{\partial x} + \frac{h^2}{6} \frac{\partial^3 \bar{u}}{\partial x^2 \partial t} - \frac{h}{2} \frac{\partial^3 (h\bar{u})}{\partial x^2 \partial t} = 0 \tag{2.55}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial ((h + \eta) u)}{\partial x} = 0 \tag{2.56}$$

(2.55) and (2.56) can be reduced to a single partial differential equation for the free surface elevation $\eta(x,t)$ by depth h scales to1.differentiating the (2.55) to x and (2.56) to t, a subsequent subtraction yields.

$$\frac{\partial^2\eta}{\partial t^2}-\frac{\partial^2\eta}{\partial x^2}=-\frac{\partial^2(u\,\eta)}{\partial t\,\partial x}+\frac{1}{2}\frac{\partial^2(u)^2}{\partial x^2}-\frac{1}{3}\frac{\partial^4(u)}{\partial t\,\partial^3 x} \tag{2.57}$$

To obtain an equation is either u *or* η alone, assume $u=\eta+O(\varepsilon)$

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}=-\frac{\partial^2(u^2)}{\partial t\,\partial x}+\frac{1}{2}\frac{\partial^2(u^2)}{\partial x^2}-\frac{1}{3}\frac{\partial^4(u)}{\partial t\,\partial^3 x} \tag{2.58}$$

by applying $\partial_t=-\partial_x$ by using $u=A\cos t\cos x+B\sin t\sin x$ A,B constants in the last (nonlinear) term in the right-hand side we get

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}=-\frac{\partial^2}{\partial x^2}\left[\frac{1}{2}u^2+\frac{1}{3}\frac{\partial^2 u}{\partial x^2}\right] \tag{2.59}$$

And the generalized form given by

$$u_{tt}-\alpha u_{xx}+\beta(f(u))_{xx}+\gamma u_{xxxx}=0 \tag{2.60}$$

where $|\gamma|=1$ is a real parameter. Setting $\gamma=-1$ gives the good Boussinesq equation (GB), while by setting $\gamma=1$, we get the bad Boussinesq equation.

2.4 The improved Boussinesq Equation

From (2.47) by applying in the first (nonlinear) term in the right-hand side and applying $\partial_x=-\partial_t$ once in the last term. Then the term u_{xx}, is replaced with u_{tt} this gives the so called improved Boussinesq (iBq) equation[18,19]:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2}{\partial x^2}\left[\frac{3}{2}u^2 + \frac{1}{3}\frac{\partial^2 u}{\partial t^2}\right] \qquad (2.61)$$

And the generalized form given by [20]

$$u_{tt} - \alpha u_{xx} + \beta (f(u))_{xx} + u_{xxtt} = 0 \qquad (2.62)$$

Similarily, using an analogous characterization used for Boussinesq equation (2.48), the IBq equation for $\gamma = -1$ will give the good (GIBq), while for $\gamma = 1$ the bad (BIBq) equation [54,55,56].

Chapter 3

Adomian Decomposition Method (ADM)

3.1 Introduction

In the beginning of the 1980's an American mathematician named George Adomian presented a powerful decomposition methodology for practical solution of linear or nonlinear, including ordinary differential equations, partial differential equations, integral equations, integro-differential equations, etc. The Adomian decomposition method or in short ADM is a powerful technique, which provides efficient algorithms for analytic approximate solutions and numeric simulations for real-world applications in the applied sciences and engineering. It permits us to solve both nonlinear initial value problems (IVPs) and boundary value problems (BVPs) without unphysical restrictive assumptions such as required by linearization, perturbation. The accuracy of the analytic approximate solutions obtained can be verified by direct substitution. A key notion is the Adomian polynomials, which are tailored to the particular nonlinearity to solve nonlinear operator equations.

3.1.1 Advantages of the Method

Advantages of the Adomian's decomposition method applied to many linear and nonlinear problems are emphasized by many authors. Common analytical procedures, to solve nonlinear differential equations, linearize the system or assume that the nonlinearities are relatively small, transforming the physical problem into a purely mathematical one with an available solution. This procedure may change the real solution of the mathematical model which represents the physical reality. Generally, the numerical methods are based on discretization techniques, and permit only to calculate the approximate solutions for some values of time and space variables, which has the disadvantage of causing overlooking for some important phenomena occurring in very small time and space intervals, such as chaos and bifurcations. Perturbation methods may only be applied when nonlinear effects are very small [6].

Adomian's method does not require discretization of the variables. Hence, the solution is not affected by computation roundoff errors and the necessity of large computer memory. Moreover it does not require linearization or perturbation, and, therefore it does not need any modification of the actual model that could change the actual solution, being very efficient on determining an approximate or even exact solution in a closed form, on both linear and nonlinear problems, minimizing in many cases the computational work.

Another advantage of Adomian's decomposition technique is that it provides a fast accurate convergent series, being this the reason why it is only necessary a small number of terms to obtain an approximate solution with high accuracy.

3.2 Adomian Decomposition Method in Mathematical Form

A brief description of the ADM follows along with a list of the necessary Adomian Polynomials, an essential component of the method [6].

Definition 3.1 (Decomposition scheme) Let $\sum C_K(X_0,\ldots,X_K)$ be a strongly convergent decomposition series. The decomposition scheme associated with

$\sum C_K$ is the recurrent $u_0 = 0, u_{n+1} = C_n(u_0,..u_n)$ which constructs a series $\sum C_n$ in a Banach space E.

Definition 3.2 (Decomposition Method)

The decomposition method is the method consisting of constructing the solution of an equation with a decomposition scheme.

3.2.1 Analysis of Adomian Decomposition Method

The Adomian decomposition method consists of decomposition the unknown function $u(x,t)$ of any equation into sum of infinite number of components defined by [1,7,8]

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{3.1}$$

Where $u_n(x,t), n \geq 0$ are to be determined in a recursive manner.

The ADM concerns itself by finding the component $u_0, u_1, u_2, \ldots$ individually. This technique is very simple in an abstract formulation but the difficulty arises in calculating the Adomian polynomials and proving convergence of the series of the function.

For a clear overview of the ADM, we consider a differential equation

$$F\,u(t) = g(t)$$

Where represent a general nonlinear ordinary or partial differential operator comprising of both linear and nonlinear terms. Linear terms are decomposed into $L + R$, where L is invertible and is taken as the highest order derivative, and R is the remainde of the linear operator thus the equation may be written as

$$Lu + Ru + Nu = g \qquad (3.2)$$

Where Nu represents nonlinear terms. Solving for Lu, we obtain

$$Lu = g - Ru - Nu \qquad (3.3)$$

L is invertible and L^{-1} is a twofold integration operator and is defined as a define integration.

$$L^{-1} = \int_0^t \int_0^t (\,.\,) dt\, dt \qquad (3.4)$$

For the operator $L = \frac{\partial^2}{\partial t^2}$, we have

$$L^{-1} Lu = u(x,t) - u(x,0) - t u_t(x,0) \qquad (3.5)$$

Operating on both side of equation (3.3) with L^{-1} we have,

$$L^{-1} Lu = L^{-1} g - L^{-1} Ru - L^{-1} Nu \qquad (3.6)$$

Combining equation (3.5) and (3.6) yields,

$$u(x,t) = u(x,0) + t u_t(x,0) + L^{-1} g - L^{-1} Ru - L^{-1} Nu \qquad (3.7)$$

The decomposition method represent the solution $u(x,t)$ as a series of this form (3.1)

The non-linear term Nu is decomposed as

$$Nu = \sum_{n=0}^{\infty} A_n \qquad (3.8)$$

Substitute equation (3.1) and (3.8) into equation (3.7)

$$\sum_{n=0}^{\infty} u_n(x,t) = u_0 - L^{-1} R \sum_{n=0}^{\infty} u_n - L^{-1} \sum_{n=0}^{\infty} A_n \qquad (3.9)$$

Where

$$u_0 = f \ ,$$

and f represents the term arising from integrating the source term g and from given condition all are assumed to be prescribed.

Consequently, we can write

$$u_1 = -L^{-1}Ru_0 - L^{-1}A_0,$$

$$u_2 = -L^{-1}Ru_1 - L^{-1}A_1,$$

$$\cdots$$

$$\cdots$$

$$\cdots$$

$$u_{n+1} = -L^{-1}Ru_n - L^{-1}A_n,$$

where the $A_n, n \geq 0$, are the Adomian polynomials generated for each nonlinearity so that A_0 depends only on u_0, A_1 depends only on u_0 and u_1, and so on.

3.3 Adomian Polynomial

The main part of Adomian decomposition method is calculating Adomian polynomials for nonlinear polynomials [9].

To compute A_n take $N(u) = f(u)$ to be a nonlinear function in u , where $u = (x, t)$, and consider the Taylor series expansion of $f(u)$ around u_0

$$f(u) = f(u_0) + f`(u_0)(u - u_0) + \frac{1}{2!}f``(u_0)(u - u_0)^2 + \frac{1}{3!}f```(u_0)(u - u_0)^3 + ...$$

But $u = u_0 + u_1 + u_2 + ...$

Then

$$f(u) = f(u_0) + f`(u_0)(u_1 + u_2 + ...) + \frac{1}{2!} f``(u_0)(u_1 + u_2 + ...)^2$$
$$+ \frac{1}{3!} f```(u_0)(u_1 + u_2 + ...)^3 + ...$$

by expanding all terms we get

$$f(u) = f(u_0) + f`(u_0)u_1 + f`(u_0)u_2 + f`(u_0)u_3 + ... + \frac{1}{2!} f``(u_0)u_1^2 + \frac{1}{2!} f``(u_0)(u_1 u_2)$$
$$+ \frac{1}{2!} f``(u_0)(u_1 u_3) + ... + \frac{1}{3!} f```(u_0)(u_1)^3 + \frac{1}{3!} f```(u_0)u_1^2 u_2 + \frac{1}{3!} f```(u_0)u_1^2 u_3 + ...$$

now, let $l(i)$ be the order of $u_l^{\,i}$ and $l(i) + m(j)$ be the order of $u_l^{\,i} + u_m^{\,j}$

then A_n consists of all terms of order n, so we have

$$A_0 = f(u_0)$$

$$A_1 = f^{'}(u_0)u_1, \; A_2 = f^{'}(u_0)u_2 + f^{''}(u_0)\frac{u_1^{\,2}}{2!}$$

$$A_3 = f^{'}(u_0)u_3 + f^{''}(u_0)u_1 u_2 + f^{(3)}(u_0)\frac{u_1^{\,3}}{3!}$$

$\vdots$

Hene

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F(\sum_{i=0}^{\infty}\lambda^i u_i\right]_{\lambda=0} \quad , n = 0,1,2,.... \qquad (3.10)$$

where λ , is a grouping parameter of convenience.

There are different methods to calculate Adomian polynomials [10,14,24]. Here we present another method and we show by some examples that this method is very easy for computing Adomian polynomials [27].

This formula given by

$$A_n = f(S_n) - \sum_{j=0}^{n-1} A_j \quad (3.11)$$

where the partial sum is $S_n = \sum_{i=0}^{n} u_i(t)$

The Adomian polynomials can be generated using formula (3.10) or formula (3.11). Formula (3.11) is programmable and the Adomian series solution can be converged faster when using it. For example, if $f(u) = u^2$ the first four polynomials using formulas (3.10) and (3.11) are computed to be:

Using formula (3.10):

$A_0 = u_0^2$

$A_1 = 2u_0 u_1$

$A_2 = u_1^2 + 2u_0 u_1$

$A_3 = 2u_1 u_2 + 2u_0 u_3$

$A_4 = u_2^2 + 2u_1 u_3 + 2u_0 u_4$

Using formula (3.11):

$A_0 = u_0^2$

$A_1 = 2u_0 u_1 + u_1^2$

$A_2 = u_2^2 + 2u_0 u_2 + 2u_1 u_2$

$A_3 = 2u_0 u_3 + 2u_1 u_3 + 2u_2 u_3 + u_3^2$

$A_4 = 2u_0 u_4 + 2u_1 u_4 + 2u_2 u_4 + 2u_3 u_4 + u_4^2$

Clearly, the first four polynomials computed using formula (3.11) include the first four polynomials computed using formula (3.10) in addition to other terms which should appear in $A_5, A_6, A_7, \ldots$ using formula (3.10). Thus, the solution using formula (3.11) forces many terms to be entered into the calculation processes earlier, yielding a faster convergence.

So , the practical solution for the n-term approximation is ,

$$\varphi_n(x,t)=\sum_{i=0}^{n-1}u_i(x,t) \tag{3.12}$$

Where

$$u(x,t)=\lim_{n\to\infty}\varphi_n(x,t)=\sum_{i=0}^{\infty}u_i(x,t) \tag{3.13}$$

3.3.1 Modified Adomian Polynomials

A new class of the Adomian Polynomials is defined [13], denoted by $\overline{A_n}$ several studies have been proposed to modifiy the regular Adomian polynomials A_n a rapidly converging approximant to the solution u denoted by

$$\varphi_m[u]=\sum_{n=0}^{m-1}u_n \text{ then } u(x,t)=\lim_{m\to\infty}\varphi_m[u]=\lim_{m\to\infty}\sum_{n=0}^{m-1}u_n=u$$

Where u_n are components to be determined such that we have convergence to u.

Now we make an analogous definition that just as $\varphi_m[u]$ or simply φ_m approximates $u, i.e,\ \varphi_m[u]=\sum_{n=0}^{m-1}u_n$

$\phi_m[f(u)]$ similarly approximates $f(u)$ or

$$\phi_m[f(u)]=\sum_{n=0}^{m-1}\overline{A_n}$$

The $\overline{A_n}$ represent a new class of the Adomian polynomials and the

$\lim_{m\to\infty}\left(\phi_m[f(u)]=\sum_{n=0}^{m-1}\overline{A_n}\right)=f(u)$ thus we view $\phi_m=f(u)$ and $\varphi_m[u]=u$ as truncated representations of $f(u)$ and u . The $\overline{A_n}$, can now be defined by:

$$\overline{A_m}=\phi_{m+1}-\phi_m \tag{3.14}$$

just as $u_m = \varphi_{m+1} - \varphi_m$

from $\phi_m = \sum_{n=0}^{m-1} \overline{A_n}$, we see that $\phi_1 = \overline{A_0}$, for $m \geq 1$

$$\overline{A_m} = \varphi_{m+1}[f\ (u)] - \varphi_m[f\ (u)]$$

Thus

$$\overline{A_1} = \varphi_2[f\ (u)] - \varphi_1[f\ (u)]$$
$$\overline{A_2} = \varphi_3[f\ (u)] - \varphi_2[f\ (u)]$$
$$\overline{A_3} = \varphi_4[f\ (u)] - \varphi_3[f\ (u)]$$
$$\vdots$$

we can also write from (3.14)

$$\overline{A_1} = \phi_2 - \phi_1$$
$$\overline{A_2} = \phi_3 - \phi_2$$
$$\overline{A_3} = \phi_4 - \phi_3$$
$$\vdots$$

The ϕ_m are conveniently defined as:

$$\phi_m = \sum_{n=0}^{m-1} \left(\frac{\varphi_{m-n+1} - u_0}{n!} \right)^n f^{\ n}(u_0)$$

Hence

$$\phi_1 = f\ (u_0)$$
$$\phi_2 = f\ (u_0) + u_1 f^{\ 1}(u_0)$$
$$\phi_3 = f\ (u_0) + (\varphi_3 - u_0) f^{\ 1}(u_0) + \frac{(\varphi_2 - u_0)^2}{2} f^{\ 2}(u_0)$$
$$\vdots$$

from which

$$\overline{A_0} = f\ (u_0)$$

$$\overline{A_1} = \phi_2 - \phi_1 = f\ (u_0) + u_1 f^{\ 1}(u_0)$$

$$\overline{A_2} = \phi_3 - \phi_2 = f\ (u_0) + (\varphi_3 - u_0) f^{\ 1}(u_0) + \frac{(\varphi_2 - u_0)^2}{2} f^{\ 2}(u_0)$$

$$\vdots$$

which so far, are identical to the classical or original A_0, A_1, A_2, respectively. For $m \geq 3, \overline{A_m} = A_{m,}$ To see this, we calculate ϕ_4 and A_3.

$$\phi_4 = f\ (u_0) + (\varphi_4 - u_0) f^{\ 1}(u_0) + \frac{(\varphi_3 - u_0)^2}{2!} f^{\ 2}(u_0) + \frac{(\varphi_2 - u_0)^3}{3!} f^{\ 3}(u_0),$$

Since

$$\overline{A_3} = \phi_4 - \phi_3 = u_3 f^{\ 1}(u_0) + (\frac{u_2^{\ 2}}{2!} + u_1 u_2) f^{\ 2}(u_0) + \frac{u_1^{\ 3}}{3!} f^{\ 3}(u_0),$$

But

$$A_3 = u_3 f^{\ '}(u_0) + f^{\ ''}(u_0) u_1 u_2 + f^{\ (3)}(u_0) \frac{u_1^{\ 3}}{3!}$$

clearly, then the decomposition components u_n of the solution u of a differential equation using the A_n for nonlinearities are equal to the components using the $\overline{A_n}$ for $u_0, u_1, u_2, u_3,$ but not for $u_4, u_5, ...$

3.3.2 Properties of Adomian polynomials

The polynomial A_n $, n > 0$, possess the following properties [17]

i. *Property 1.* A_n depends by construction only on the vector $(u_1, u_2, ..u_n)$ and does not depend on u_n with $m > n$;

ii. *Property 2.* A_n is the sum of terms of the form

$$A_n = \sum_{k=1}^{n} Z_{m,k} (u_1, u_2, ..., u_n) F^{(k)}(u_0)$$

Where $Z_{m,k}$ are called here the reduced polynomial and depend on $(u_1,u_2,..u_n)$.

iii. Property 3. In each reduced polynomial $Z_{m,k}$, the components of the vector $(u_1,u_2,..u_n)$ appear in such a way that the sum of their subscripts is equal to m a consequence of this property is that each reduced polynomial $Z_{m,k}$ depends only on few components of the vector $(u_1,u_2,..u_n)$ this is obvious since the sum of the subscripts of the components of the vector $(u_1,u_2,..u_n)$ exceeds m

3.4 The Noise Terms Phenomenon

A particular phenomenon noticed by Adomian and Rach, is the existence of the "self-canceling noise terms" for a decomposition series solution, where the summation vanishes at the limit. The noise terms are the identical terms with opposite signs that arise in the components. It was found that by canceling the noise terms between $u_0(x)$ and $u_1(x)$, even though $u_1(x)$ contains other terms, the remaining non canceled terms of $u_0(x)$ may give the exact solution of the equation[12].

3.4.1 Necessary Condition for the Appearance of Noise Terms

We consider the nonhomogeneous differential equation [15]

$$Lu+Ru=g \tag{3.15}$$

With the initial conditions $u(0)=A$, $u'(0)=B$

Where L is a second order operator .Applying L^{-1} to both side yields

$$u(x)=A+Bx+L^{-1}(g(x))-L^{-1}(R(u)) \tag{3.16}$$

The first two components of $u(x)$ are

$$u_0(x) = A + Bx + L^{-1}(g(x)) \qquad (3.17)$$

and

$$u_1(x) = -L^{-1}(R\,u_0) \qquad (3.18)$$

For the differential Equation (3.15) we obtain

$$u_0(x) = f(x) \qquad (3.19)$$

The solution $u(x)$ of (3.15) must exist in $A + B\,x + L^{-1}(g(x))$ of (3.17)

Assuming that $u(x)$ appears in the zeroth component u_0, and this occurs for a specific style of nonhomogeneous problems, then based on this assumption, we may rewrite (3.16) as

$$u_0(x) = u(x) + T(x) \qquad (3.20)$$

Where $T(x)$ are other terms of u_0 that were called the noise terms.

Using (3.1) into (3.19) yields

$$u_1(x) = -T(x) - (u_2 + u_3 + u_4 + ...) \qquad (3.21)$$

And immediate consequence of (3.20) and (3.21) is that if u is included in u_0 , then the noise terms $T(x)$ of u_0 must appear in u_1 with opposite signs .It now seems reasonable to call these terms the effective noise terms so that by canceling these terms[15,16] , the exact solution follows immediately .one further important point to be noted in this analysis :Because u is not known ,the remaining noncanceled terms of u_0 must be substituted into the equation to verify that these terms provide the exact solution , and the canceled terms are the effective noise terms[13].

3.4.2 The necessary conditions for the appearance of the effective noise terms between u_0 and u_1.

1. Effective noise terms appear for specific style of nonhomogeneous differential and integral equations.
2. Effective noise terms appear if the exact solution u appears in u_0 explicitly
3. Verification that the remaining noncanceled terms of u_0 provide the exact solution is derived from the substitution of these terms into the equation.

3.5 The Convergence Analysis of ADM

The convergence of Adomian's method has been subject of investigation by several authors.

Following Cherruault [18] has given the first proof of convergence of the ADM and he used fixed point theorems for abstract functional equations. We give the proof of convergence of the Adomian decomposition method Consider the general functional equation [15]

$$u - Nu = f, \quad u \in H \tag{3.22}$$

where H is the Hilbert space and N is the nonlinear operator $N : H \to H$ and $f = L^{-1}g$ is also in H .

Substituting (3.3) and (3.4) in (3.22) yields

$$\sum_{n=0}^{\infty} u_n - \sum_{n=0}^{\infty} A_n = f$$

then the recursive terms are got from this algorithm

$$u_0 = f$$
$$u_{n+1} = A_n(u_0, u_1, u_2, ..., u_n)$$

The Adomian decomposition method is equivalent to find the sequence

$S_n = u_1 + u_2 + u_3 + \cdots + u_n$ by using iterative scheme

$$S_0 = 0$$
$$S_{n+1} = N(S_n + u_0)$$
$$\text{where } N(S_n + u_0) = \sum_{k=0}^{n} A_k$$

If this limit exist $S = \lim_{n \to \infty} S_n$ in a Hilbert space, then S is a solution of the fixed point functional equation $S = N(S_n + u_0)$ in H

Theorem 3.1.

Let N be a nonlinear operator from a Hilbert space H where $N : H \to H$ and u be the exact solution of (3.22). The decomposition series $\sum_{n=0}^{\infty} u_n$ of u converges to u when

$$\exists \alpha < 1, \|u_n + 1\| \le \alpha \|u_n\|, \forall n \in N \cup \{0\}$$

Proof. We have the sequence

$$S_n = u_1 + u_2 + u_3 + \cdots + u_n$$

We need to show that this sequence is a Cauchy sequence in the Hilbert space H.

$$\|S_{n+1} - S_n\| = \|u_n + 1\| \le \alpha \|u_n\| \le \alpha^2 \|u_{n-1}\| \le \ldots \le \alpha^{n+1} \|u_0\|$$

In order to prove that S_n is Cauchy sequence

$$\|S_m - S_n\| = \|(S_m - S_{m-1}) + (S_{m-1} - S_{m-2}) + ... + (S_{n+1} - S_n)\|$$
$$\leq \|S_m - S_{m-1}\| + \|S_{m-1} - S_{m-2}\| + ... + \|S_{n+1} - S_n\|$$
$$\leq \alpha^m \|u_0\| + \alpha^{m-1}\|u_0\| + ... + \alpha^{n+1}\|u_0\|$$
$$= (\alpha^m + \alpha^{m-1} + ... + \alpha^{n+1})\|u_0\|$$
$$\leq (\alpha^{n+1} + \alpha^{n+1} + ...)\|u_0\|$$

Then $\|S_m - S_n\| = \frac{\alpha^{n+1}}{1-\alpha}\|u_0\|$, for $n, m \in N, m \geq n$ (3.23)

Since $\alpha < 1$. From definition in [15] ,the sequence $\{S_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Hilbert space. Hence,

$$\lim_{n\to\infty} S_n = S \text{ for } S \in H$$

where $S = \sum_{n=0}^{\infty} u$ solving (3.21) is the same as solving the functional equation $N(S + u_0) = S$; by assuming that N is a continuous operator we get

$$N(S + u_0) = N(\lim_{n\to\infty}(S_n + u_0)) = \lim_{n\to\infty} N(S_n + u_0) = \lim_{n\to\infty} S_{n+1} = s$$

So S is the solution of (3.21)

3.6 Pade` Approximation

Often time, power series don't give a good approximation to a function except the radius of convergence is sufficiently large to contain the domain $[a,b]$ over which the function is approximated. In order to make the maximum error as small as possible, a rational (Pade') approximation method which has a smaller error on $[a,b]$ than a polynomial approximation can be constructed [17].

Pade´ approximant extrapolation technique consists in approximating a truncated series by a rational function. In general, the latter function has the advantage of extending the range of validity of the initial polynomial. Pade´ approximants have thus far been successfully applied to physical problems.

3.6.1 Pade´ approximants

The main concept of Padé approximants is to replace a series function [22]

$$f(x)=\sum_{k=0}^{\infty} c_k x^k \text{ by}$$

$$f(L,M)=\frac{P_L(x)}{Q_M(x)}=\frac{a_0+a_1x+a_2x^2+\ldots+a_Lx^L}{1+b_1x+b_2x^2+\ldots+b_mx^m} \tag{3.24}$$

Given $[L/m]$ called the Padé approximants.

For a function

$$f(x)=c_0+c_1x+c_2x^2+\ldots \tag{3.25}$$

the coefficients $a_0,a_1,\ldots a_L,b_0,b_1,\ldots b_M$ can be calculated by matching the series coefficients.

Therefore, one can acquire the following sequence [118]

$$[1/1]=\frac{a_0+a_1x}{1+b_1x} \quad ,\lim_{x\to\infty}[1/1]=\frac{a_1}{b_1}$$

$$[2/2]=\frac{a_0+a_1x+a_2x^2}{1+b_1x+b_2x} \quad ,\lim_{x\to\infty}[2/2]=\frac{a_2}{b_2} \tag{3.26}$$

$$\ldots$$

$$\lim_{x\to\infty}[m/m]=\frac{a_m}{b_m}$$

For a fixed value of $L+M$ the error in the approximation is smallest when $P_L(x)$ and $Q_M(x)$ have the same degree or when $P_L(x)$ has degree one higher than $Q_M(x)$.

Illustrative example 3.6.1

Establish the Pad´e approximants [2/2] for $f(x)=e^{-x}$ [118]

Solution

The Taylor expansion for the exponential function is

$$e^{-x}=1-x+\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{24}x^4-\frac{1}{120}x^5...$$

The [2/2] approximant is defined by

$$\left[2/2\right]=\frac{a_0+a_1x+a_2x^2}{1+b_1x+b_2x}$$

Set

$$\frac{a_0+a_1x+a_2x^2}{1+b_1x+b_2x}=1-x+\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{24}x^4$$

Cross multiplying yields

$$1+(b_1-1)x+(-b_1+b_2+\frac{1}{2})x^2+(\frac{1}{2}b_1-b_2-\frac{1}{6})x^3+$$

$$(-\frac{1}{6}b_1+\frac{1}{2}b_2+\frac{1}{24})x^4=a_0+a_1x+a_2x^2$$

Equating powers of x leads to

coefficient of x^0: $1=a_0$

coefficient of x^1: $b_1-1=a_1$

coefficient of x^2: $-b_1+b_2+\frac{1}{2}=a_2$

coefficient of x^3: $\frac{1}{2}b_1-b_2-\frac{1}{6}=0$

coefficient of x^4: $-\frac{1}{6}b_1+\frac{1}{2}b_2+\frac{1}{24}=0$

This system of equations gives

$$a_0 = 1, a_1 = -\frac{1}{2}, a_2 = \frac{1}{12}, b_1 = \frac{1}{2}, b_2 = \frac{1}{12}$$

so that the Pad´e approximant is

$$[2/2] = \frac{1 - \frac{1}{2}x + \frac{1}{12}x^2}{1 + \frac{1}{2}x + \frac{1}{12}x}$$

3.6.2 The Used Algorithm

Decomposition method is used first to obtain the approximate truncated series solution $u(x,t)$. Then the Padé approximants is used to obtain an equivalent rational function approximation or the closed form solution [20] (see Figure 3.1)

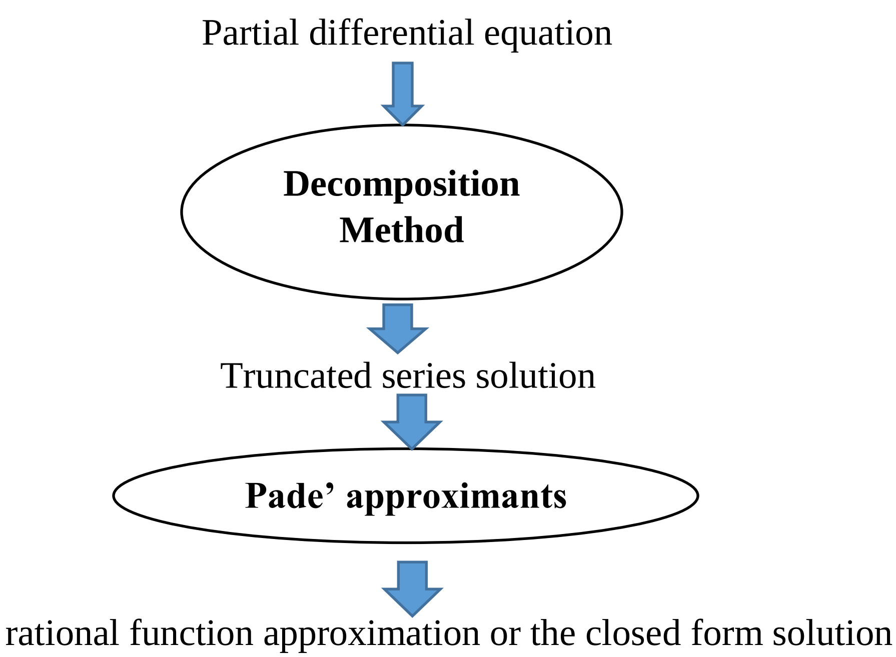

Figure 3.1 The used Algorithm

3.7. Applications

We will demonstrate the Adomian decomposition methd and Pade\` decomposition method on the well-known Boussinesq type equations that was derived in chapter two.

Example 3.7.1: Consider the nonlinear Boussinesq equation [37,39]:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - 6\frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^4 u}{\partial x^4} \tag{3.27}$$

subject to the initial conditions:

$$u(x,0) = \frac{1}{x^2}, \quad \frac{\partial u}{\partial t}(x,0) = -\frac{2}{x^2} \tag{3.28}$$

Applying the ADM on Eq. (3.27) we have

$$L_t u = \frac{\partial^2 u}{\partial x^2} - 6\frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^4 u}{\partial x^2} \tag{3.29}$$

Where $L_t = \frac{\partial^2}{\partial t^2}$, $L_L^{-1}(.) = \int_0^t \int_0^t (.)d\,t\,d\,t$

Operating with L^{-1} on the both sides of Eq. (3.29) and using the initial conditions (3.28) gives

$$u(x,t) = \frac{1}{x^2} - \frac{2}{x^2}t + L^{-1}\left(\frac{\partial^2 u}{\partial x^2} - 6\frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^4 u}{\partial x^4}\right) \tag{3.30}$$

Using Eq. (3.1) − (3.8) in equation (3.30) we get

$$\sum_{n=0}^{\infty} u_n = \frac{1}{x^2} - \frac{2}{x^2}t + L^{-1}\left(\sum_{n=0}^{\infty}\frac{\partial^2 u_n}{\partial x^2} - 6\sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty}\frac{\partial^4 u_n}{\partial x^4}\right)$$

leads to the following results

$$u_0 = \frac{1}{x^2} - \frac{2t}{x^2}$$

$$u_{n+1} = L^{-1}\left(\frac{\partial^2 u_n}{\partial x^2} - 6A_n + \frac{\partial^4 u_n}{\partial x^2}\right)$$

Then

$$A_0 = (u_0^{\ 2})_{xx} = \frac{20}{x^6} - \frac{80t}{x^6} + \frac{80t^2}{x^6}$$

$$u_1 = L^{-1}\left(\frac{\partial^2 u_0}{\partial x^2} - 6A_0 + \frac{\partial^4 u_0}{\partial x^2}\right) = \frac{3t^2}{x^4} + \frac{40t^3}{x^6} - \frac{2t^3}{x^4} - \frac{40t^4}{x^6}$$

$$A_1 = (2u_0 u_1)_{xx} = -\frac{17280t^4}{x^{10}} - \frac{672t^3}{x^8} + \frac{5760t^3}{x^{10}} + \frac{252t^2}{x^8} + \frac{11520t^5}{x^{10}} + \frac{336t^4}{x^8}$$

$$u_2 = L^{-1}\left(\frac{\partial^2 u_1}{\partial x^2} - 6A_1 + \frac{\partial^4 u_1}{\partial x^4}\right)$$

$$= \frac{5t^4}{x^6} + \frac{84t^4}{x^8} - \frac{2t^5}{x^6} + \frac{4320t^5}{x^{10}} + \frac{1008}{5}\frac{t^5}{x^8} - \frac{576t^6}{x^{10}} - \frac{616}{5}\frac{t^6}{x^8} - \frac{11520}{7}\frac{t^7}{x^{10}}$$

$$A_2 = (u_1^{\ 2} + 2u_0 u_2)_{xx} = \frac{57280}{7}\frac{t^8}{x^{12}} + \frac{3264}{5}\frac{t^7}{x^{10}} - \frac{29312}{7}\frac{t^7}{x^{12}} - \frac{7264}{5}\frac{t^6}{x^{10}} + \frac{12t^6}{x^8} - \frac{36t^5}{x^8} - \frac{16832t^6}{x^{12}} + \frac{1536}{5}\frac{t^5}{x^{10}} + \frac{19t^4}{x^8} + \frac{8640t^5}{x^{12}} + \frac{168t^4}{x^{10}}$$

$$u_3 = L^{-1}\left(\frac{\partial^2 u_2}{\partial x^2} - 6A_2 + \frac{\partial^4 u_2}{\partial x^4}\right)$$

$$\frac{672t^6}{x^{10}} + \frac{16}{5}\frac{t^6}{x^8} + \frac{22176t^6}{x^{12}} + \frac{22}{7}\frac{t^7}{x^8} + \frac{1104}{7}\frac{t^7}{x^{10}} + \frac{48096t^7}{x^{12}} + \frac{12355200}{7}\frac{t^7}{x^{14}} - \frac{9}{7}\frac{t^8}{x^8} - \frac{96}{35}\frac{t^8}{x^{10}}$$
$$-\frac{16752t^8}{x^{12}} - \frac{1235520}{7}\frac{t^8}{x^{14}} - \frac{2745600}{7}\frac{t^9}{x^{14}} - \frac{6496}{3}\frac{t^9}{x^{12}} - \frac{272}{5}\frac{t^9}{x^{10}} - \frac{11456}{21}\frac{t^{10}}{x^{12}}$$

$\vdots$

Which in closed form gives exact solution

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t)=\frac{1}{x^2}-\frac{2t}{x^2}+\frac{3t^2}{x^4}+\frac{40t^3}{x^6}-\frac{2t^3}{x^4}-\frac{40t^4}{x^6}\frac{5t^4}{x^6}+\frac{84t^4}{x^8}-\frac{2t^5}{x^6}+\dots$$

Thus

$$u(x,t)=\frac{1-2t}{x^2}$$

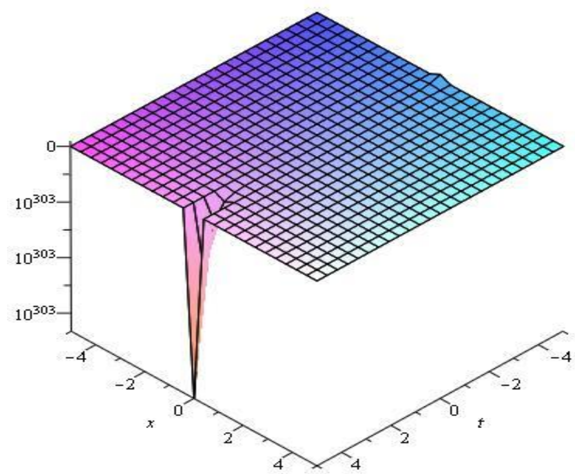

Figure 3.2 Example 3.7.1: Three-dimensional plot for the solution obtained by Adomian's method for n = 3.

Pade`[2,2]=

$$\frac{4x^6-89x^4+\left(-8x^6+102x^4+288x^2\right)t+\left(160x^4-788x^2-1348\right)t^2}{4x^8-89x^6+\left(-76x^6+288x^4\right)t+\left(-4x^6+55x^4-1348x^2\right)t^2}$$

And pade`[3,3]=

$$\frac{1}{5}(105450x^{10}-382200x^8-12944000x^6+(-205500x^{10}+1026675x^8+30004350x^6$$
$$+16056000x^4)t+(-10800x^{10}-416250x^8-10359340x^6+66789600x^4$$
$$+1241280000x^2)t^2+(5417580x^6-241741660x^4+216000x^8-2468813000x^2+$$
$$1639392000)t^3)/21090x^{12}-76440x^{10}-2588800x^8+(1080x^{12}+52455x^{10}$$
$$+823270x^8+3211200x^6)t+(-41610x^{10}-196008x^8+27546720x^6+248256000x^4)t^2$$
$$+(-1080x^{10}-462345x^8+2155298x^6+96667800x^4+327878400x^2)t^3)$$

(a) (b)

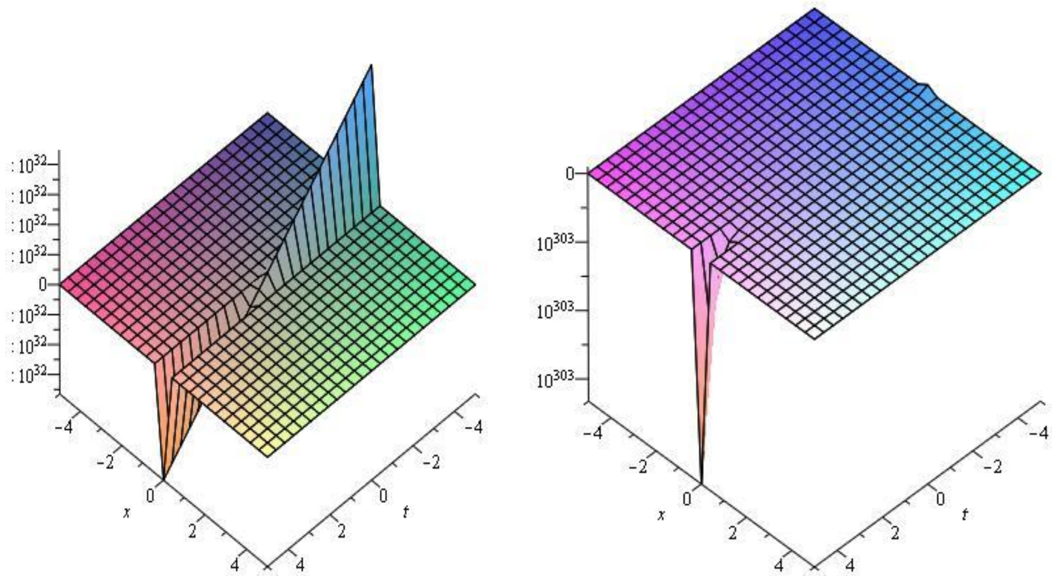

Figure 3.3 Example 3.7.1: (a) Three-dimensional plot for the exact solution. (b) Three-dimensional plot for the solution obtained by Pad’ Adomian method for n = 3.

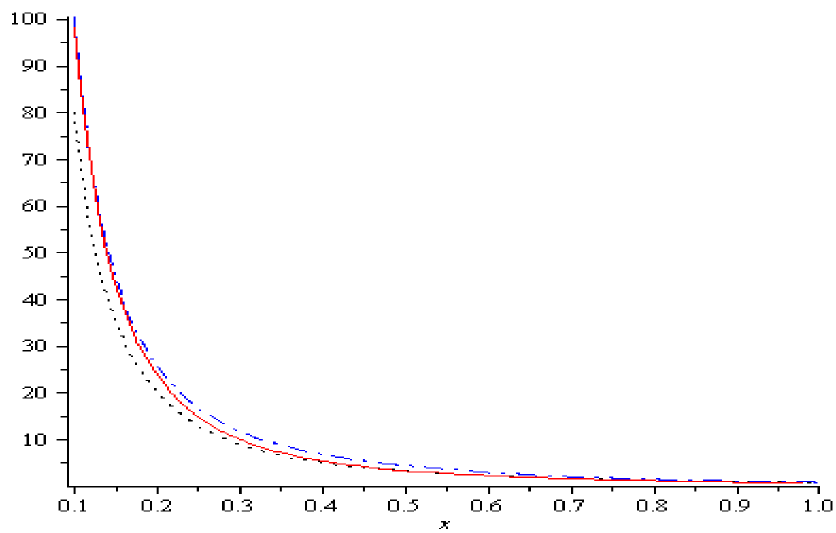

Figure 3.4 Example 3.7.1: Results obtained for the exact solution (dashed line) and by application of a Pad´e approximant [2,2] (dash dot)and [3,3] (solid line) to Adomian’s series solution.

Table 3.1: Example 1: Results obtained for the the exact solution compared with ADM and by application of a Pad´e approximant [2,2] and [3/3] to Adomian's series solution when $t = 0.1$.

x	ADM	Pade` [2,2]	Pade` [3,3]	Exact solution
0.9000	10.63443547	1.101364904	0.9512080464	0.9876543208
1.1000	1.012937449	0.7025608046	1.161750005	0.6611570248
1.3000	0.5126145033	0.4913522013	0.4998843046	0.4733727810
1.5000	0.3671548149	0.3645201245	0.3659588214	0.3555555555
1.7000	0.2822855326	0.2817742412	0.2821321816	0.2768166090
1.9000	0.2247073916	0.2245721139	0.2246820308	0.2216066482
2.1000	0.1833347437	0.1832908087	0.1833296461	0.1814058957
2.3000	0.1525022411	0.1524858831	0.1525010596	0.1512287334
2.5000	0.1288784239	0.1288717434	0.1288781247	0.1280000000
2.7000	0.1103664671	0.1103635686	0.1103663907	0.1097393690
2.9000	0.09558527158	0.09558397349	0.09558525610	0.09512485136
3.1000	0.08359269479	0.08359211576	0.08359269734	0.08324661808
3.3000	0.07372727844	0.07372703758	0.07372728862	0.07346189164

Example 3.7.2. Consider now a higher nonlinear Boussinesq equation with the initial conditions [40,41]

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u^3}{\partial x^2} + \frac{\partial^4 u}{\partial x^4} \tag{3.31}$$

$$\begin{aligned} u(x,0) &= \sqrt{2}\,\mathrm{sech}(x) \\ u_t(x,0) &= \sqrt{2}\,\mathrm{sech}(x)\tanh(x) \end{aligned} \tag{3.32}$$

The exact solution is given by

$$u(x,t) = \sqrt{2}\,\mathrm{sech}(x - t)$$

Using the Adomian decomposition method, we obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = \sqrt{2}\,\text{sech}(x) + \sqrt{2}\,\text{sech}(x)\tanh(x)$$

$$t + L^{-1}\left(\sum_{n=0}^{\infty} A_n(x,t) + \sum_{n=0}^{\infty} \frac{\partial^4 u_n}{\partial x^4}\right) \tag{3.33}$$

Being $L^{-1}{}_{xx} = \int_0^{x`} dx \int_0^{x``} (.)dx``$

The a view nonlinear terms are given by

$A_0 = u_0^3, \quad ,A_1 = 3u_0{}^2u_1 \quad A_2 = 3u_0{}^2u_2 + 3u_0\,u_1^2$

Then (3.34) leads to the following results

$u_0 = \sqrt{2}\,\text{sech}(x) + \sqrt{2}\,\text{sech}(x)\tanh(x)\,t$

$$u_1 = L^{-1}\left(A_0 + \frac{\partial^4 u_0}{\partial x^4}\right) = (12\sqrt{2}\text{sech}(x)\tanh^4(x) - 14\sqrt{2}\text{sech}(x)\tanh^2(x)$$

$$+\frac{5}{2}\sqrt{2}\text{sech}(x) + 12\sqrt{2}\text{sech}^3(x)\tanh^2(x) - 3\sqrt{2}\text{sech}^3(x))\,t^2 + (20\sqrt{2}\text{sech}(x)\tanh^5(x)$$

$$-30\sqrt{2}\text{sech}(x)\tanh^3(x) + \frac{61}{6}\sqrt{2}\text{sech}(x)\tanh(x) + 20\sqrt{2}\text{sech}^3(x)\tanh^3(x) -$$

$$11\sqrt{2}\text{sech}^3(x)\tanh(x))\,t^3 + (15\sqrt{2}\text{sech}^3(x)\tanh^4(x) - \frac{23}{2}\sqrt{2}\text{sech}^3(x)\tanh^2(x)$$

$$+\sqrt{2}\text{sech}^3(x))\,t^4 + (\frac{21}{5}\sqrt{2}\text{sech}^3(x)\tanh^5(x) - \frac{39}{10}\sqrt{2}\text{sech}^3(x)\tanh^3(x)$$

$$+\frac{3}{5}\sqrt{2}\text{sech}^3(x)\tanh(x))\,t^5$$

$\vdots$

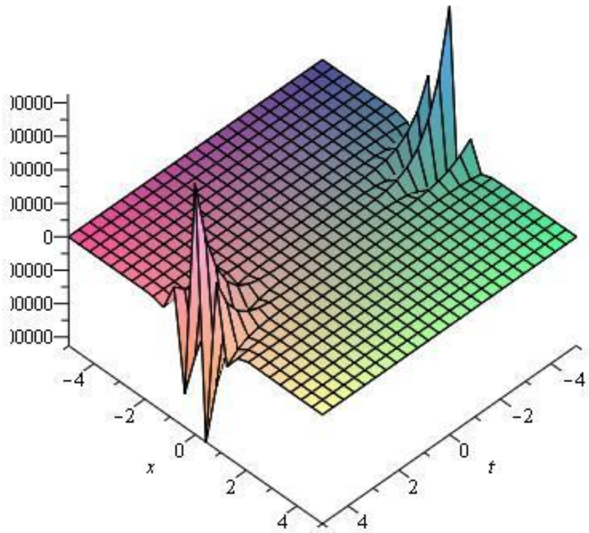

Figure 3.5 Example 3.7. 2: Three-dimensional plot for the solution obtained by Adomian's method for n = 2.

(a) (b)

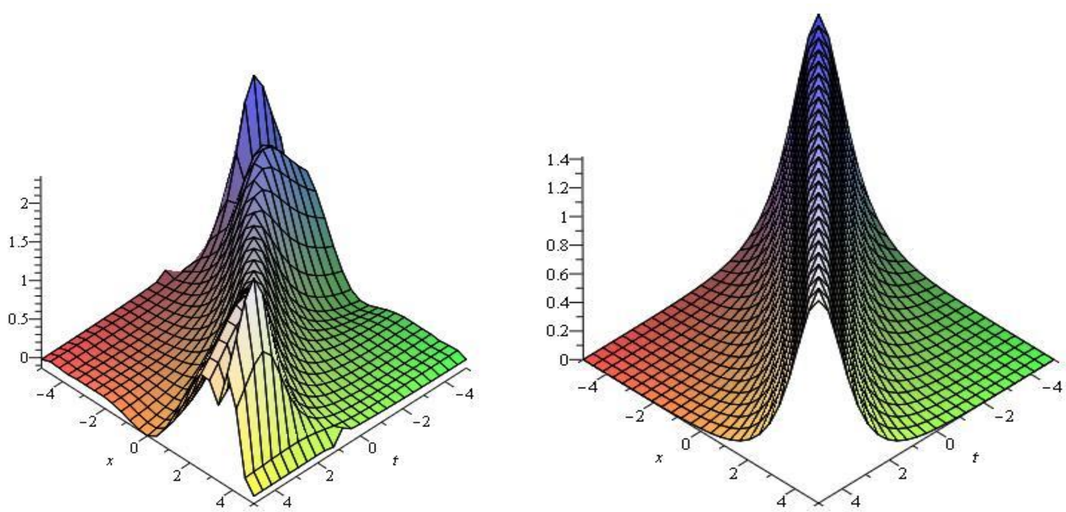

Figure 3.6 Example 3.7.2: (a) Three-dimensional plot for the solution obtained by Pad' Adomian method for n = 2. (b) Three-dimensional plot for the exact solution.

Example 3.7.3 Cconsider now a linear example [42]

$$u_{tt} = (\frac{1}{2}u^2)_{xx} + u_{xxxx} + 2x^3 - 15x^4t^4 \qquad (3.34)$$

$$u(x,0) = 0 \quad ,u_t(x,0) = 0$$

Using the Adomian decomposition method, we obtain

$$\sum_{n=0}^{\infty} u_{n+1}(x,t) = x^3t^2 - \frac{x^4t^6}{2} + L^{-1}\left[\frac{1}{2}\sum_{n=0}^{\infty}\frac{\partial^2 A_n(x,t)}{\partial x^2} + \sum_{n=0}^{\infty}\frac{\partial^4 u_n}{\partial x^4}\right]$$

leads to the following results

$$u_0 = x^3t^2 - \frac{x^4t^6}{2}$$

$$A_0 = (u_0^2)_{xx} = 30x^4t^4 - 42x^5t^8 + 14x^6t^{12}$$

$$u_1 = L^{-1}\left[\frac{\partial^2 A_0}{\partial x^2} + \frac{\partial^4 u_0}{\partial x^4}\right] = -\frac{3}{14}t^8 + \frac{1}{2}x^4t^6 - \frac{7}{30}t^{10}x^5 + \frac{1}{26}t^{14}x^6$$

$$A_1 = (2u_0u_1)_{xx} = 42x^5t^8 - \frac{812}{15}x^6t^{12} + \frac{1452}{65}t^{16}x^7 - \frac{45}{13}t^{20}x^8 - \frac{18}{7}t^{10}x + \frac{18}{7}t^{14}x^2$$

$$u_2 = L^{-1}\left[\frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^4 u_1}{\partial x^4}\right] =$$

$$\frac{3}{14}t^8 - \frac{205}{924}t^{12}x + \frac{459}{7280}t^{16}x^2 + \frac{7}{30}t^{10}x^5 - \frac{29}{195}t^{14}x^6 + \frac{121}{3315}t^{18}x^7 - \frac{15}{4004}t^{22}x^8$$

$$A_2 = (u_1^2 + 2u_0u_2)_{xx} = \frac{18}{7}t^{10}x - \frac{806}{77}t^{14}x^2 + \frac{7687}{858}t^{18}x^3 - \frac{1737}{728}t^{22}x^4 + \frac{602}{15}x^6t^{12}$$
$$- \frac{3576}{65}t^{16}x^7 + \frac{62579}{2210}t^{20}x^8 - \frac{31622}{4641}t^{24}x^9 + \frac{816}{1183}t^{28}x^{10}$$

$$u_3 = L^{-1}\left[\frac{\partial^2 A_2}{\partial x^2}+\frac{\partial^4 u_2}{\partial x^4}\right]=\frac{43}{390}t^{14}x^6-\frac{298}{3315}t^{18}x^7+\frac{5689}{185640}t^{22}x^8$$
$$-\frac{15811}{3016650}t^{26}x^9+\frac{68}{171535}t^{30}x^{10}+\frac{205}{924}t^{12}x-\frac{58831}{240240}t^{16}x^2+\frac{1025111}{11085360}t^{20}x^3$$
$$-\frac{39969}{2946944}t^{24}x^4$$
$$\vdots$$

The self-canceling “noise” terms appear, looking into the first and third term of u_1 and first and third term of u_2, are the self-canceling “noise” terms, and so on.

$u_0+u_1+u_2+\ldots$ gives the solution $u(x,t)$ in a series form and in a closed form by $u(x,t)=x^3t^2$

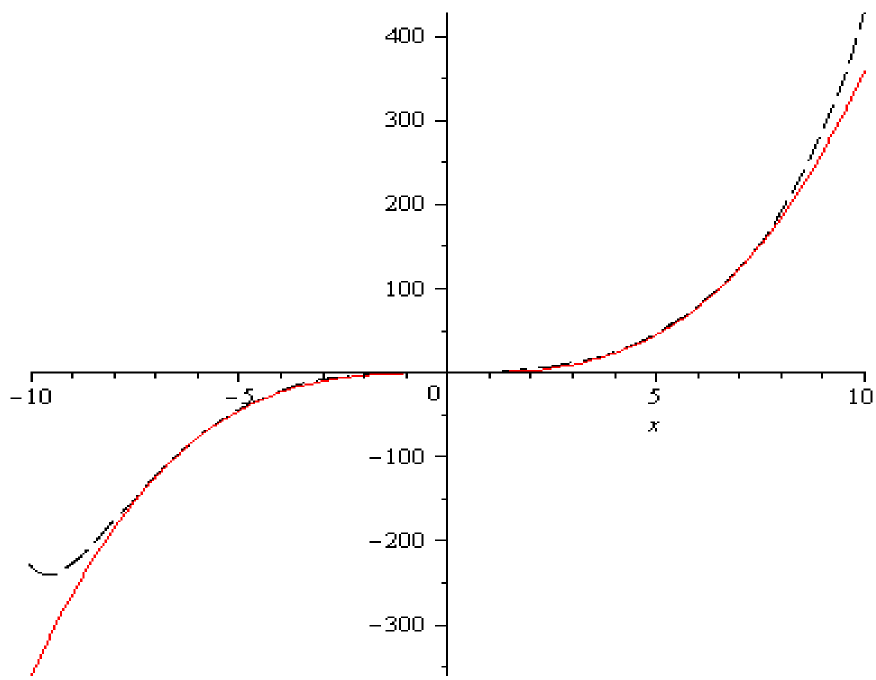

Figure 3.7 Example 3.7.3: Results obtained for the exact solution (line) and by application Adomian (dash line) when t=0.6.

Calculating the [3,3] Pade'approximant about t by MAPLE, give the exact solution

$$\text{pade}'[\varphi_5(x,t)] = x^3 t^2$$

Table 3.2 Resuts obtained for exact solution and the one obtained by application of Pade [3, 3] to ADM solution φ_5 and exact solution.t=0.6

x	ADM solution	Pade[3,3] solution	Exact solution
0	0	0	0
0.1	0.0003575364834	0.00036	0.00036
0.2	0.002874103650	0.00288	0.00288
0.3	0.009709812595	0.00972	0.00972
0.4	0.02302454636	0.02304	0.02304
0.5	0.04497791747	0.04500	0.04500
0.6	0.07772906599	0.07776	0.07776
0.7	0.1234364078	0.12348	0.12348
0.8	0.1842573379	0.18432	0.18432
0.9	0.2623478892	0.26244	0.26244
1	0.3598623597	0.36000	0.36000

Chapter 4

Adomian's Method Modifications

4.1. Introduction

A modification of Adomian method is proposed to overcome the computational difficulties arising when obtaining the solution of differential equations, particularly when the initial approximation is not a constant. Here we proposed to kinds modified and improved Adomian decompositions method [27].

4.2 Modified Adomian Decomposition Method

The modified decomposition method was introduced by Wazwaz [43] in order to simplify the calculations and accelerate the rapid convergence of the series solution. The modified forms was established on the assumption that the function $f(x,t)$ can be divided into two parts, namely $f_1(x,t), f_2(x,t)$. in order to obviate the appearance of the so-called noise terms occurring in special cases.Under this assumption we set

$$f(x,t) = f_1(x,t) + f_2(x,t) \tag{4.1}$$

Under this assumption, we propose a slight variation only in the components u_0, u_1 . The variation we propose is that only the part $f_1(x,t)$ be assigned to the u_0, whereas the remaining part $f_2(x,t)$ be combined with the other terms given in Eq. **(**3.10**)** to define u_1. In view of these suggestion, we formulate the modified recursive algorithm as follows [44]:

$$u_0 = f_1(x,t)$$

$$u_1(x,t)=f_2(x,t)-L^{-1}\left[Ru_0+A_0\right] \tag{4.2}$$

$$u_{n+1}=-L^{-1}\left[R\sum_{n=0}^{\infty}u_n+\sum_{n=0}^{\infty}A_n\right],\quad n\geq 1$$

The solution through the modified Adomian decomposition method is highly depend upon the choice of $f_1(x,t)$ and $f_2(x,t)$.

4.3 Improved Adomian Decomposition Method

Consider the following general nonlinear non-homogenous (3.2) with the initial problem

$$u(x,t_0)=f_0(x)$$

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0}=f_1(x) \tag{4.3}$$

$$\vdots$$

$$\left.\frac{\partial^{s-1}u(x,t)}{\partial t^{s-1}}\right|_{t=0}=(s-1)!f_{s-1}(x)$$

Where $L=\frac{\partial^s}{\partial t^s}, s=1,2,3,...$ is the highest partial derivative with respect to t.

Applying L^{-1} to Eg. (3.2) and using the initial condition (4.3) lead to $u(x,t)=f_0(x)+f_1(x)t+...+f_{s-1}(x)t^{s-1}-$

$$L^{-1}(N(u(x,t))+Ru(x,t)-g(x,t)) \tag{4.4}$$

Where the component A_n is called the Adomian polynomial and it is redefined by Abassy in the form [17, 36, 41]

$$A_n=\frac{1}{ns!}\frac{d^{ns}}{d\lambda^{ns}}\left[N(\sum_{i=0}^{\infty}\lambda^i f_i t^i\right]_{\lambda=0}+\frac{1}{(ns+1)!}\frac{d^{(ns+1)}}{d\lambda^{(ns+1)}}\left[N(\sum_{i=0}^{\infty}\lambda^i f_i t^i\right]_{\lambda=0}+....$$

$$\frac{1}{(n\,s+s-1)!}\frac{d^{(n\,s+s-1)}}{d\lambda^{(n\,s+s-1)}}\left[N\,(\sum_{i=0}^{\infty}\lambda^i f_i t^i\right]_{\lambda=0}, \quad n=0,1,2,3,.. \tag{4.5}$$

Where f_n is the coefficient of t^n in $u_n(x,t)$ components .

The non-homogenous term $g(x,t)$ is decomposed using Taylor series in the form

$$g(x,t)=\sum_{n=0}^{\infty} g_n(x,t) \tag{4.6}$$

Where g_n equals

$$g_n=\frac{1}{n\,s!}\left(\frac{d^{n\,s}}{dt^{n\,s}}\,g(x,t)\right)_{t=0} t^{n\,s}+\frac{1}{(n\,s+1)!}\left(\frac{d^{(n\,s+1)}}{dt^{(n\,s+1)}}\,g(x,t)\right)_{t=0} t^{n\,s+1}+...$$

$$\frac{1}{(n\,s+s-1)!}\left(\frac{d^{(n\,s+s-1)}}{dt^{(n\,s+s-1)}}\,g(x,t)\right)_{t=0} t^{n\,s+s-1} \quad ,n=0,1,2,3,...$$

Substituting (3.8) and (4.5) into Eq. (4.4) gives

$$\sum_{n=0}^{\infty}u_n(x,t)=f_0(x)+f_1(x)t+...+f_{s-1}(x)t^{s-1}-$$

$$L^{-1}\left(\sum_{n=0}^{\infty}A_n+R\sum_{n=0}^{\infty}u_n(x,t)-\sum_{n=0}^{\infty}g_n(x,t)\right). \tag{4.7}$$

The component of $u_n(x,t)$ follows immediately upon setting

$u_0(x,t)=f_0(x)+f_1(x)t+...+f_{s-1}(x)t^{s-1},$

$u_{n+1}(x,t)=-L^{-1}(A_n+Ru_n-g_n), \quad n\geq 0.$

4.4 Applications

We will demonstrate the modified Adomian decomposition methd, Improved Adomian Decomposition method and Pade`decomposition method on the well-known Boussinesq type equations that was derived in chapter two.

Example 4.4.1: Consider the nonlinear Boussinesq equation [39]:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - 6\frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^4 u}{\partial x^4} \tag{4.8}$$

subject to the initial conditions:

$$u(x,0) = \frac{1}{x^2}, \quad \frac{\partial u}{\partial t}(x,0) = -\frac{2}{x^2} \tag{4.9}$$

*Using Modified Adomian Decomposition Method (**MADM**)*

Let $f_1 = \dfrac{1}{x^2}$ and $f_2 = \dfrac{-2t}{x^2}$

By using the recursive (4.2) we obtain

$$u_0 = \frac{1}{x^2}$$

$$u_1(x,t) = \frac{-2t}{x^2} + L^{-1}\left(\frac{\partial^2 u_0}{\partial x^2} - 6A_0 + \frac{\partial^4 u_0}{\partial x^2}\right)$$

$$u_{n+1} = L^{-1}\left(\frac{\partial^2 u_n}{\partial x^2} - 6A_n + \frac{\partial^4 u_n}{\partial x^2}\right)$$

This lead to

$$A_0 = (u_0{}^2)_{xx} = \frac{20}{x^6}$$

$$u_1 = \frac{-2t}{x^2} + L^{-1}\left(\frac{\partial^2 u_0}{\partial x^2} - 6A_0 + \frac{\partial^4 u_0}{\partial x^2}\right) = -\frac{2t}{x^2} + \frac{3t^2}{x^4}$$

$$A_1 = (2u_0 u_1)_{xx} = -\frac{80t}{x^6} + \frac{252t^2}{x^8}$$

$$u_2 = L^{-1}\left(\frac{\partial^2 u_1}{\partial x^2} - 6A_1 + \frac{\partial^4 u_1}{\partial x^4}\right) = \frac{5t^4}{x^6} + \frac{84t^4}{x^8} - \frac{2t^3}{x^4} + \frac{40t^3}{x^6}$$

$$A_2 = (u_1^2 + 2u_0 u_2)_{xx} = \frac{4t^2}{x^4} - \frac{16t^3}{x^6} + \frac{19t^4}{x^8} + \frac{168t^4}{x^{10}} + \frac{80t^3}{x^8}$$

$$u_3 = L^{-1}\left(\frac{\partial^2 u_2}{\partial x^2} - 6A_2 + \frac{\partial^4 u_2}{\partial x^4}\right) =$$

$$\frac{672t^6}{x^{10}} + \frac{22176t^6}{x^{12}} + \frac{16}{5}\frac{t^6}{x^8} + \frac{14}{5}\frac{t^5}{x^6} + \frac{6048t^5}{x^{10}} - \frac{24t^5}{x^8} - \frac{2t^4}{x^4}$$

$$\vdots$$

Which in closed form gives exact solution

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \frac{1}{x^2} - \frac{2t}{x^2} + \frac{3t^2}{x^4} + \frac{5t^4}{x^6} + \frac{84t^4}{x^8} - \frac{2t^3}{x^4} + \frac{40t^3}{x^6} \ldots$$

By *Improved Adomian Decomposition Method (IADM)*

$$u_0(x,t) = f_0(x) + f_1(x)t = \frac{1}{x^2} - \frac{2}{x^2}t$$

$$u_{n+1}(x,t) = L^{-1}\left(\sum_{n=0}^{\infty}\frac{\partial^2 u_n}{\partial x^2} - 6\sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty}\frac{\partial^4 u_n}{\partial x^4}\right)$$

Where A_n are the Adomian Polynomial which represent the nonlinear term $u^2{}_{xx}$ and are defined by (4.5) where $S = 2, A_n$ takes the form

$$A_n = \frac{1}{ns!}\frac{d^{ns}}{d\lambda^{ns}}\left[N\left(\sum_{i=0}^{\infty}\lambda^i f_i t^i\right)\right]_{\lambda=0} + \frac{1}{(ns+1)!}\frac{d^{(ns+1)}}{d\lambda^{(ns+1)}}\left[N\left(\sum_{i=0}^{\infty}\lambda^i f_i t^i\right)\right]_{\lambda=0}$$

The following result are obtain

$$f_0=\frac{1}{x^2} \quad ,f_1=\frac{-2}{x^2}$$

$$A_0=\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(f_0{}^2\right)+2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(f_0 f_1 t\right)=\frac{20}{x^6}-\frac{80t}{x^6}$$

$$u_1=L^{-1}\left(\frac{\partial^2 u_0}{\partial x^2}-6A_0+\frac{\partial^4 u_0}{\partial x^4}\right)=-\frac{2t^3}{x^4}+\frac{40t^3}{x^6}+\frac{3t^2}{x^4}$$

$$f_2=\frac{3}{x^4} \quad ,f_3=-\frac{2}{x^4}+\frac{40}{x^6}$$

$$A_1=\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(f_1{}^2+2f_0 f_2\right)t^2+\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(2f_1 f_2+2f_0 f_3\right)t^3$$

$$=\frac{80t^2}{x^6}+\frac{252t^2}{x^8}-\frac{672t^3}{x^8}+\frac{5760t^3}{x^{10}}$$

$$u_2=L^{-1}\left(\frac{\partial^2 u_1}{\partial x^2}-6A_1+\frac{\partial^4 u_1}{\partial x^4}\right)=\frac{1008}{5}\frac{t^5}{x^8}+\frac{4320t^5}{x^{10}}-\frac{2t^5}{x^6}-\frac{35t^4}{x^6}+\frac{84t^4}{x^8}$$

$$f_4=-\frac{35}{x^6}+\frac{84}{x^8} \quad ,f_5=\frac{1008}{5x^8}+\frac{4320}{x^{10}}-\frac{2}{x^6}$$

$$A_2=\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(f_2{}^2+2f_1 f_3+2f_0 f_4\right)t^4+\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(2f_3 f_2+2f_0 f_5+2f_1 f_4\right)t^5$$

$$=-\frac{15912t^4}{x^{10}}+\frac{336t^4}{x^8}+\frac{18480t^4}{x^{12}}+\frac{8928t^5}{x^{10}}+\frac{33792t^5}{x^{12}}+\frac{1347840t^5}{x^{14}}$$

$$u_3 = L^{-1}\left(\frac{\partial^2 u_2}{\partial x^2} - 6A_2 + \frac{\partial^4 u_2}{\partial x^4}\right) = -\frac{2t^7}{x^8} - \frac{37584}{35}\frac{t^7}{x^{10}} + \frac{311520}{7}\frac{t^7}{x^{12}} +$$

$$\frac{1572480t^7}{x^{14}} - \frac{144t^6}{x^{10}} - \frac{581}{5}\frac{t^6}{x^8} + \frac{18480t^6}{x^{12}}$$

$\vdots$

And so on

Considering these components, the solution can be approximate as:

$$u(x,t) \approx \varphi_n(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$$

$$\varphi_1 = \frac{1}{x^2} - \frac{2t}{x^2} - \frac{2t^3}{x^4} + \frac{40t^3}{x^6} + \frac{3t^2}{x^4}$$

$$\varphi_2 = \frac{1}{x^2} - \frac{2t}{x^2} - \frac{2t^3}{x^4} + \frac{40t^3}{x^6} + \frac{3t^2}{x^4} + \frac{1008}{5}\frac{t^5}{x^8} + \frac{4320t^5}{x^{10}} - \frac{2t^5}{x^6} - \frac{35t^4}{x^6} + \frac{84t^4}{x^8}$$

φ_n is an approximate power series expansion in which successive converges to Eq.(4.8) closed form solution

$$u(x,t) = \frac{1-2t}{x^2}$$

(a) (b)

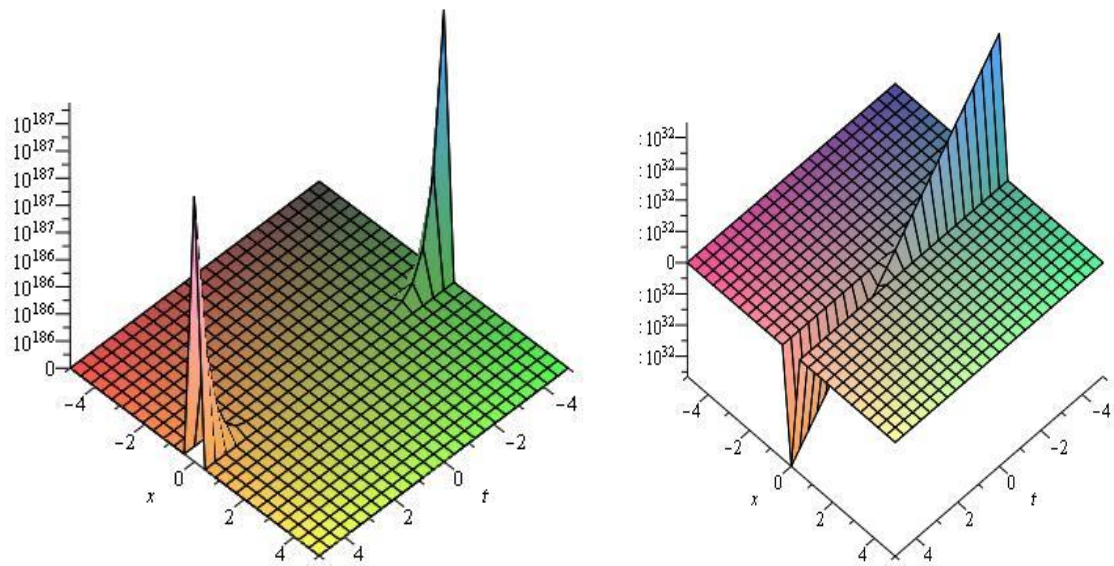

Figure 4.1 Example 4.4.1(a) Three-dimensional plot for the solution obtained by Modified Adomian's method for n = 5 (b) Three-dimensional plot for the exact solution

using 4 terms of an improved Adomian power series, Padé approximant size of [1/1],[2,2] we obtain the following expression of the Padé approximant for this case respectively

$$Pade_3`[1,1] = \frac{2x^2 + \left(-4x^2 + 3\right)t}{2x^4 + 3tx^2}$$

$$Pade_3`[2,2] = \frac{4x^6 - 89x^4 + \left(-8x^6 + 102x^4 + 288x^2\right)t + \left(160x^4 - 788x^2 - 1348\right)t^2}{4x^8 - 89x^6 + \left(-76x^6 + 288x^4\right)t + \left(-4x^6 + 55x^4 - 1348x^2\right)t^2}$$

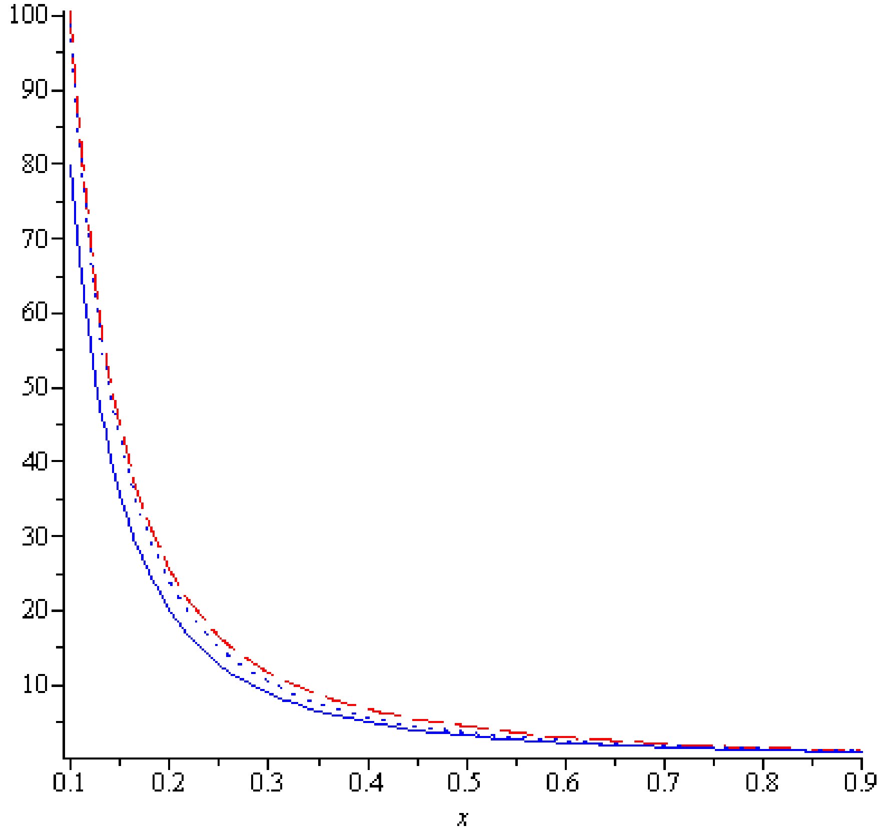

Figure 4.2 The behavior of the exact solution (solid line) and Pade[1,1] (dot) and Pade [2,2] (dash) of $\varphi_3(x,t)$ in case $t=0.1$.

Example 4.4.2. Consider now a higher nonlinear example [53],

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u^3}{\partial x^2}+\frac{\partial^4 u}{\partial x^2 \partial t^2} \quad ,-\infty<x<\infty, t\geq 0 \tag{4.10}$$

$$u(x,0)=-x/\sqrt{3} \quad , \quad u_t(x,0)=-x/\sqrt{3}$$

By using the recursive (4.2) we obtain

$$u_0=-\frac{1}{\sqrt{3}}x$$

$$u_1(x,t)=-\frac{1}{\sqrt{3}}xt+L^{-1}\left(A_0+\frac{\partial^4 u_0}{\partial x^2 \partial t^2}\right) \tag{4.11}$$

$$u_{n+1}=L^{-1}\left(A_n+\frac{\partial^4 u_n}{\partial x^2 \partial t^2}\right)$$

leads to the following results

$$A_0 = \frac{d}{dx^2} u_0^3 \quad = \quad -\frac{1}{9} x^3 \sqrt{3}$$

$$u_1 = -\frac{1}{\sqrt{3}} x + \int_0^t \int_0^t \left(A_0 + \frac{d^4}{dx^2 dt^2} u_0 \right) = -\frac{1}{\sqrt{3}} xt - \frac{1}{\sqrt{3}} xt^2$$

$$A_1 = \frac{d}{dx^2} 3u_0^2 u_1 = -2\sqrt{3}t\,(1+t)\,x$$

$$u_2 = \int_0^t \int_0^t \left(A_1 + \frac{d^4}{dx^2 dt^2} u_1 \right) = -\frac{1}{2\sqrt{3}} xt^4 - \frac{1}{\sqrt{3}} xt^3$$

$$A_2 = \frac{d}{dx^2} \left(3u_0^2 u_2 + 3u_0 u_1^2 \right) = -3x\sqrt{3}t^4 - 6x\sqrt{3}t^3 - 2x\sqrt{3}t^2$$

$$u_3 = \int_0^t \int_0^t \left(A_2 + \frac{d^4}{dx^2 dt^2} u_2 \right) = -\frac{3}{10\sqrt{3}} xt^6 - \frac{9}{10\sqrt{3}} xt^5 - \frac{1}{2\sqrt{3}} xt^4$$

$$\vdots$$

(a) (b)

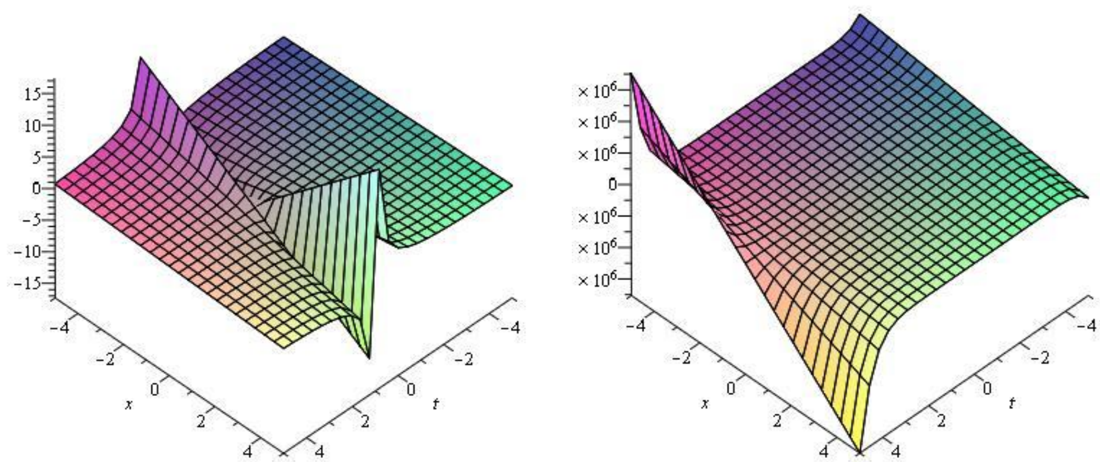

Figure. 4.3: Example 4.4.2 (a) Three-dimensional plot for the exact solution (b) Three-dimensional plot for the solution obtained by Modified Adomian's method for n = 5

using 5 terms of modified Adomian power series, Padé approximant size of [2,2] we obtain the exact solution

$$\text{Pade}_5`[2,2] = \frac{-x}{\sqrt{3}(t-1)}$$

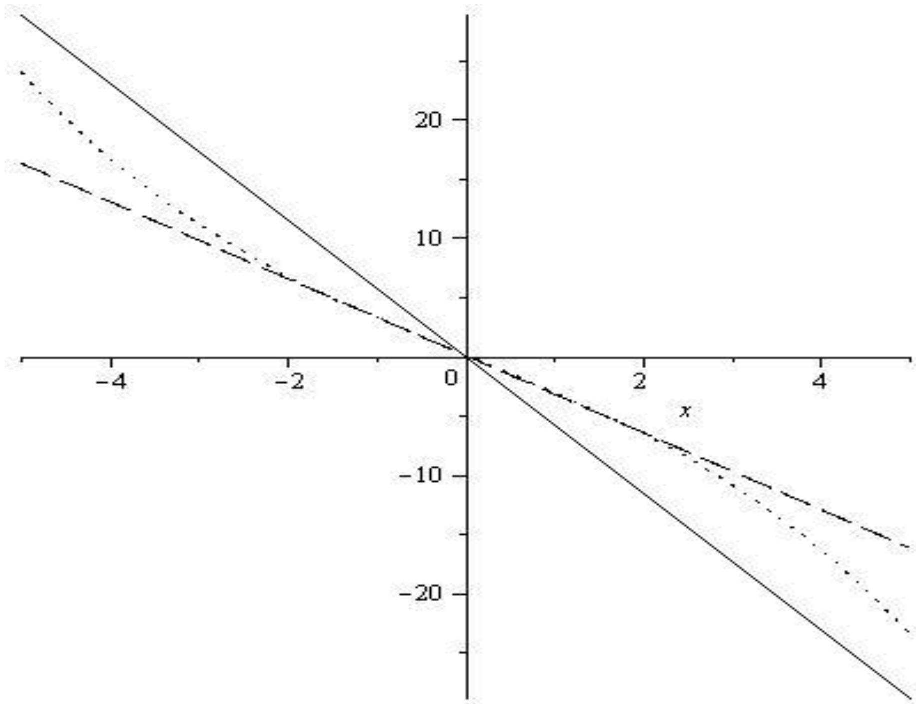

Figure. 4.4. The behavior of the exact solution (solid line) and MADM (dot) and IADM (dash) of $\varphi_4(x,t)$ in case $t = 0.9$.

Using the Improved Adomian decomposition method, *we obtain*

$$u_0(x,t) = f_0(x) + f_1(x)t = -\frac{1}{\sqrt{3}}x - \frac{1}{\sqrt{3}}x\,t$$

$$u_{n+1}(x,t) = L^{-1}(A_n + \frac{\partial^4 u_n}{\partial x^2 \partial t^2})$$

Where A_n are the Adomian Polynomial which represent the nonlinear term u^2_{xx} and are defined by (3.18) where $S = 2$, A_n takes the form

$$A_n=\frac{1}{n\,s\,!}\frac{d^{n\,s}}{d\,\lambda^{n\,s}}\left[N\,(\sum_{i=0}^{\infty}\lambda^i f_i t^i\right]_{\lambda=0}+\frac{1}{(n\,s+1)!}\frac{d^{(n\,s+1)}}{d\,\lambda^{(n\,s+1)}}\left[N\,(\sum_{i=0}^{\infty}\lambda^i f_i t^i\right]_{\lambda=0}$$

leads to the following results

$$f_0=-\frac{1}{3}x\sqrt{3}\quad ,f_1=-\frac{1}{3}x\sqrt{3}$$

$$A_0=\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(f_0{}^3\right)+3\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(f_0{}^2 f_1 t\right)=-\frac{2}{3}x\sqrt{3}-2x\sqrt{3}t$$

$$u_1=\int_0^t\int_0^t\left(A_0+\frac{d^4}{dx^2dt^2}u_0\right)=-\frac{1}{\sqrt{3}}xt^2-\frac{1}{\sqrt{3}}xt^3$$

$$f_2=-\frac{1}{3}x\sqrt{3}\quad ,\quad f_3=-\frac{1}{3}x\sqrt{3}$$

$$A_1=\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(3f_2 f_0{}^2+3f_0 f_1{}^2\right)t^2+\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(f_1{}^3+6f_0 f_1+3f_0{}^2 f_3\right)t^3$$

$$=-4x\sqrt{3}t^2-\frac{8}{3}x\sqrt{3}t^3+4t^3$$

$$u_2=\int_0^t\int_0^t\left(A_1+\frac{d^4}{dx^2dt^2}u_1\right)\quad=-\frac{2}{5\sqrt{3}}t^5x+\frac{1}{5}t^5-\frac{1}{\sqrt{3}}xt^4$$

$$f_4=-\frac{35}{x^6}+\frac{84}{x^8}\quad ,f_5=\frac{1008}{5x^8}+\frac{4320}{x^{10}}-\frac{2}{x^6}$$

$$A_2=\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(3f_2f_1{}^2+3f_0f_2{}^2+6f_0f_1f_3+3f_2f_0{}^2\right)t^4$$
$$+\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(3f_2{}^2f_1+3f_1{}^2f_3+6f_0f_2f_3+6f_0f_1f_4+3f_5f_0{}^2\right)t^5$$

$$= -10x\sqrt{3}t^4 - \frac{64}{5}t^5 x\sqrt{3} + \frac{2}{5}t^5$$

$$u_3 = u_2 = \int_0^t\int_0^t \left(A_1 + \frac{d^4}{dx^2 dt^2}u_1\right) = -\frac{32}{35\sqrt{3}}t^7 x + \frac{1}{105}t^7 - \frac{1}{\sqrt{3}}xt^6$$

…

…

And so on

Considering these components, the solution can be approximate as:

$$u(x,t) \approx \varphi_n(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$$

$$\varphi_1 = -\frac{1}{\sqrt{3}}x - \frac{1}{\sqrt{3}}x\ t - \frac{1}{\sqrt{3}}xt^2 - \frac{1}{\sqrt{3}}xt^3$$

$$\varphi_2 = -\frac{1}{\sqrt{3}}x - \frac{1}{\sqrt{3}}x\ t - \frac{1}{\sqrt{3}}xt^2 - \frac{1}{\sqrt{3}}xt^3 - \frac{1}{\sqrt{3}}xt^4 - \frac{2}{5\sqrt{3}}t^5 x + \frac{1}{5}t^5$$

⋮

φ_n is an approximate power series expansion in which successive converges to Eq.(3.32) closed form solution $u(x,t) = \frac{-x}{\sqrt{3}(t-1)}$

Example 4.4.3. Consider the cubic modified Boussinesq equation [39,45],

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^t u}{\partial x^2 \partial t} + \frac{2}{9}\frac{\partial^4 u}{\partial x^4} - \frac{\partial^2 u^3}{\partial x^2} = 0 \qquad (4.12)$$

$$u(x,0) = 1 + \tanh\left(\frac{3}{2}x\right), \quad u_t(x,0) = -3\,\text{sech}^2\left(\frac{3}{2}x\right)$$

By using the recursive (4.2) we obtain

$$u_0 = -3\operatorname{sech}^2\left(\frac{3}{2}x\right)$$

$$u_1(x,t) = \left(1+\tanh\left(\frac{3}{2}x\right)\right)t + L^{-1}\left(A_0 - \frac{\partial^3 u_0}{\partial x^2 \partial t} - \frac{2}{9}\frac{\partial^4 u_0}{\partial x^4}\right) \quad (4.11)$$

$$u_{n+1} = L^{-1}\left(A_n - \frac{\partial^3 u_n}{\partial x^2 \partial t} - \frac{2}{9}\frac{\partial^4 u_n}{\partial x^4}\right)$$

leads to the following results

$$A_0 = \frac{d}{dx^2}u_0^3 = -2187t^3\operatorname{sech}^6\left(\frac{3}{2}x\right)\tanh^2\left(\frac{3}{2}x\right) + 243t^3\operatorname{sech}^6\left(\frac{3}{2}x\right)\left(\frac{3}{2} - \frac{3}{2}\tanh^2\left(\frac{3}{2}x\right)\right)$$

$$u_1 = -3\operatorname{sech}^2\left(\frac{3}{2}x\right)t + L^{-1}\left(A_0 - \frac{\partial^3 u_0}{\partial x^2 \partial t} - \frac{2}{9}\frac{\partial^4 u_0}{\partial x^4}\right)$$

$$1+\tanh\left(\frac{3}{2}x\right) - \frac{5103}{40}t^5\operatorname{sech}^6\left(\frac{3}{2}x\right)\tanh^2\left(\frac{3}{2}x\right) + \frac{729}{40}t^5\operatorname{sech}^6\left(\frac{3}{2}x\right) +$$

$$9t^3\operatorname{sech}^2\left(\frac{3}{2}x\right) - \frac{135}{2}t^3\operatorname{sech}^2\left(\frac{3}{2}x\right)\tanh^2\left(\frac{3}{2}x\right) + \frac{135}{2}t^3\operatorname{sech}^2\left(\frac{3}{2}x\right)\tanh\left(\frac{3}{2}x\right)^4$$

$$-\frac{27}{4}t^2\operatorname{sech}^2\left(\frac{3}{2}x\right) + \frac{81}{4}t^2\operatorname{sech}^2\left(\frac{3}{2}x\right)\tanh^2\left(\frac{3}{2}x\right)$$

$$u_2 = L^{-1}\left(A_1 - \frac{\partial^3 u_1}{\partial x^2 \partial t} - \frac{2}{9}\frac{\partial^4 u_1}{\partial x^4}\right) = -9t^2\tanh\left(\frac{3}{2}x\right) - \frac{27}{2}t^2\tanh^5\left(\frac{3}{2}x\right)$$

$$+\frac{45}{2}t^2\tanh^3\left(\frac{3}{2}x\right) - \frac{2511}{10}t^5\operatorname{sech}^2\left(\frac{3}{2}x\right) + \frac{1377}{8}t^4\operatorname{sech}^2\left(\frac{3}{2}x\right)$$

$$-\frac{81}{4}t^4\operatorname{sech}^4\left(\frac{3}{2}x\right) + \frac{1215}{4}t^3\operatorname{sech}^2\left(\frac{3}{2}x\right)\tanh^2\left(\frac{3}{2}x\right) - \frac{81}{2}t^3\operatorname{sech}^2\left(\frac{3}{2}x\right)$$

$$-\frac{59049}{160}t^9\operatorname{sech}^{10}\left(\frac{3}{2}x\right) - \frac{177147}{280}t^7\operatorname{sech}^6\left(\frac{3}{2}x\right) + \frac{24057}{80}t^6\operatorname{sech}^6\left(\frac{3}{2}x\right)\ldots$$

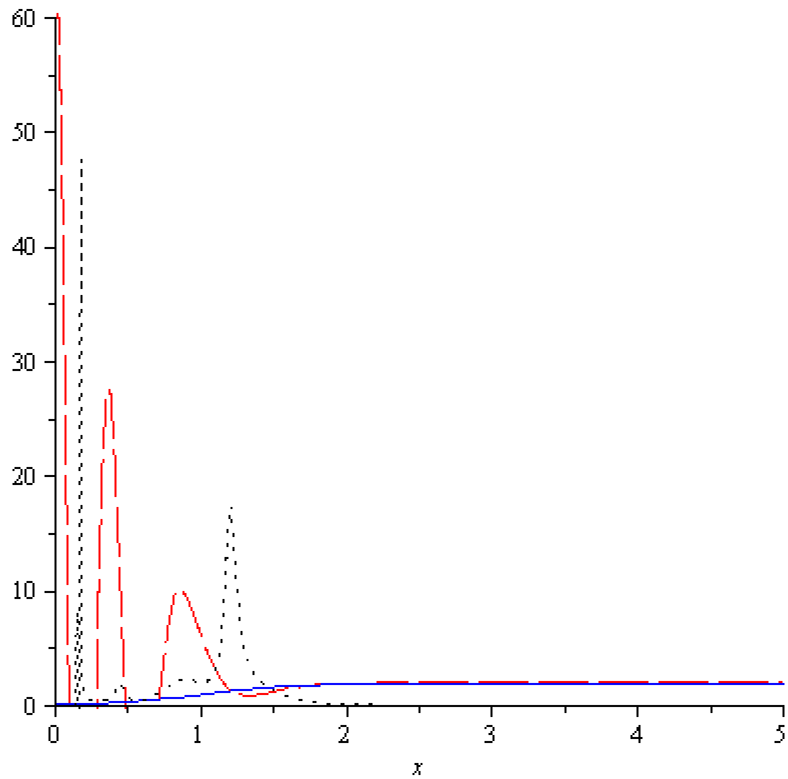

Figure. 4.5. The behavior of the exact solution (solid line) and MADM (dot) and pade[2,4] (dash) of $\varphi_3(x,t)$ in case $t=0.5$.

Using the Improved Adomian decomposition method, we obtain

$$u_0(x,t)=f_0(x)+f_1(x)t=1+\tanh\left(\frac{3}{2}x\right)-3\operatorname{sech}^2\left(\frac{3}{2}x\right)t$$

$$u_{n+1}=L^{-1}\left(A_n-\frac{\partial^3 u_n}{\partial x^2\partial t}-\frac{2}{9}\frac{\partial^4 u_n}{\partial x^4}\right)$$

Where A_n are the Adomian Polynomial which represent the nonlinear term u^3_{xx} and are defined by (3.18) where $S=2$, A_n takes the form

$$A_n=\frac{1}{ns!}\frac{d^{ns}}{d\lambda^{ns}}\left[N(\sum_{i=0}^{\infty}\lambda^i f_i t^i\right]_{\lambda=0}+\frac{1}{(ns+1)!}\frac{d^{(ns+1)}}{d\lambda^{(ns+1)}}\left[N(\sum_{i=0}^{\infty}\lambda^i f_i t^i\right]_{\lambda=0}$$

leads to the following results

$$f_0 = 1 + \tanh\left(\frac{3}{2}x\right) \quad , f_1 = -3\operatorname{sech}^2\left(\frac{3}{2}x\right)$$

$$A_0 = \frac{d^2}{dx^2}\left(f_0^{\,3}\right) + 3\frac{d^2}{dx^2}\left(f_0^{\,2} f_1 t\right)$$

$$u_1 = L^{-1}\left(A_0 - \frac{\partial^3 u_0}{\partial x^2 \partial t} - \frac{2}{9}\frac{\partial^4 u_0}{\partial x^4}\right)$$

$$\begin{aligned}
&= \Big(\frac{27}{4} - 27\tanh^2\left(\frac{3}{2}x\right) + \frac{81}{4}\tanh^4\left(\frac{3}{2}x\right) - 9\tanh\left(\frac{3}{2}x\right) + 9\tanh^3\left(\frac{3}{2}x\right) \\
&+ \frac{81}{4}\operatorname{sech}^2\left(\frac{3}{2}x\right)\tanh^2\left(\frac{3}{2}x\right) - \frac{27}{4}\operatorname{sech}^2\left(\frac{3}{2}x\right)\Big)t^2 + \Big(9\operatorname{sech}^2\left(\frac{3}{2}x\right) \\
&- 27\operatorname{sech}^2\left(\frac{3}{2}x\right)\tanh^2\left(\frac{3}{2}x\right) + 54\operatorname{sech}^2\left(\frac{3}{2}x\right)\tanh\left(\frac{3}{2}x\right) - 81\operatorname{sech}^2\left(\frac{3}{2}x\right)\tanh^3\left(\frac{3}{2}x\right)
\end{aligned}$$

$$\begin{aligned}
f_2 &= \frac{27}{4} - 27\tanh^2\left(\frac{3}{2}x\right) + \frac{81}{4}\tanh^4\left(\frac{3}{2}x\right) - 9\tanh\left(\frac{3}{2}x\right) + 9\tanh^3\left(\frac{3}{2}x\right) \\
&+ \frac{81}{4}\operatorname{sech}^2\left(\frac{3}{2}x\right)\tanh^2\left(\frac{3}{2}x\right) - \frac{27}{4}\operatorname{sech}^2\left(\frac{3}{2}x\right)
\end{aligned}$$

$$\begin{aligned}
f_3 &= \Big(9\operatorname{sech}^2\left(\frac{3}{2}x\right) - 27\operatorname{sech}^2\left(\frac{3}{2}x\right)\tanh^2\left(\frac{3}{2}x\right) + 54\operatorname{sech}^2\left(\frac{3}{2}x\right)\tanh\left(\frac{3}{2}x\right) \\
&- 81\operatorname{sech}^2\left(\frac{3}{2}x\right)\tanh^3\left(\frac{3}{2}x\right)
\end{aligned}$$

$$A_1 = \frac{d^2}{dx^2}\left(3f_2 f_0^{\,2} + 3f_0 f_1^{\,2}\right)t^2 + \frac{d^2}{dx^2}\left(f_1^{\,3} + 6f_0 f_1 + 3f_0^{\,2} f_3\right)t^3$$

$$\vdots$$

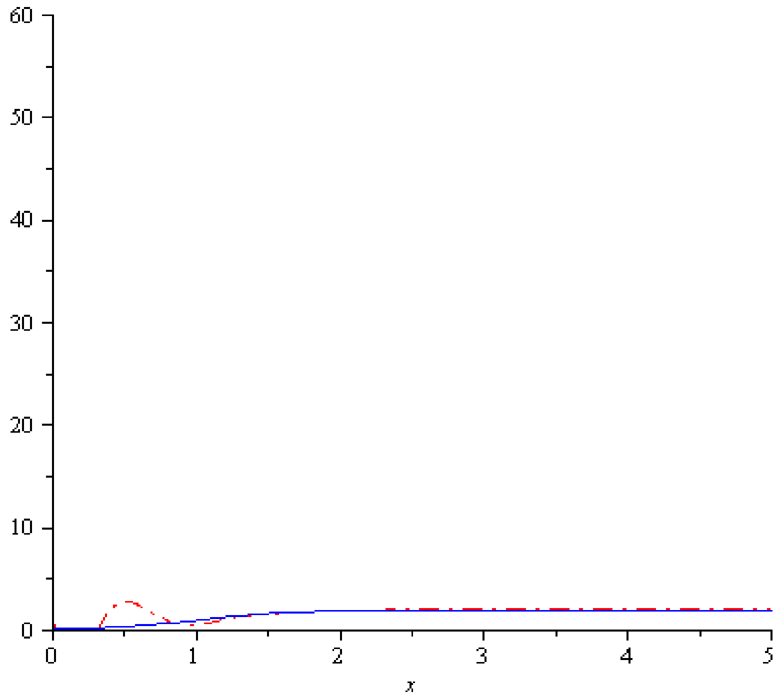

Figure. 4.6. The behavior of the exact solution (solid line) and IADM (dash line) of $\varphi_2(x,t)$ in case $t = 0.5$.

Chapter 5

Integral Transformation Method

5.1. Introduction

In modern time, integral transforms (Laplace transform, Sumudu transform, etc.) have very useful role in Mathematics, Physics, Chemistry, Social Science, Biology, Radio Physics, Nuclear Science, Electrical and Mechanical Engineering for solving the advanced problems of these fields.

5.2 Laplace Decomposition Method

Laplace Adomian Decomposition Method (LADM) is a combination of Adomian decomposition method (ADM) and Laplace Transform. It is an approximate analytical method, which can be adapted to solve nonlinear partial differential equations [46,66].

We will see how to use the ADM in combination with the Laplace transform.

Given a partial (or ordinary) differential equation

$$Fu(x,t)=h(x,t) \tag{5.1}$$

With initial condition

$$u(x,0)=f(x), \quad u_t(x,0)=g(x) \tag{5.2}$$

Where F is a differential operator that could, in general, be nonlinear and therefore includes some linear and nonlinear terms.

In general, equation (5.1) could be written as

$$Lu(x,t)+Ru(x,t)+Nu(x,t)=h(x,t) \tag{5.3}$$

where L is second order differential operator, R is the is remaining linear operator, Nu represents a general non-linear differential operator and $h(x,t)$ is source term.

The Laplace transform $\mathcal{L}$ is an integral transform discovered by Pierre-Simon Laplace and is a powerful and very useful technique for solving ordinary and partial differential Equations, which transforms the original differential equation into an elementary algebraic equation. Before using the Laplace transform combined with Adomian decomposition method we review some basic definitions and results on it.

Definition 5.1 Given a function $f(t)$ defined for all $t \geq 0$, the Laplace transform of f is the function F defined by[97,112]

$$F(s)=\mathcal{L}\{f(t)\}=\int_0^\infty f(t)e^{-st}dt \tag{5.4}$$

for all values of s for which the improper integral converges.

In particular $\mathcal{L}\{t^n\}=\dfrac{n!}{s^{n+1}}$.

Definition 5.2 Given a continuous function $f(t)$ if $F(s)=\mathcal{L}\{f(t)\}$, then $f(t)$ is called the inverse Laplace transform of $F(s)$ and denoted $f(t)=\mathcal{L}^{-1}\{F(s)\}$.

The methodology consists of applying Laplace transform first on both sides of Eq. (5.3)

$$\mathcal{L}[Lu(x,t)]+\mathcal{L}[Ru(x,t)]+\mathcal{L}[Nu(x,t)]=\mathcal{L}[h(x,t)]$$

Using the differentiation property of Laplace transform we get

$$s^2 \mathcal{L}[u(x,t)] - s\,f(x) - g(x) + \mathcal{L}[R\,u(x,t)] + \mathcal{L}[N\,u(x,t)] = \mathcal{L}[h(x,t)]$$

and so:

$$\mathcal{L}[u(x,t)] = \frac{f(x)}{s} + \frac{g(x)}{s^2} - \frac{1}{s^2}\mathcal{L}[R\,u(x,t)] - \frac{1}{s^2}\mathcal{L}[N\,u(x,t)] + \frac{1}{s^2}\mathcal{L}[h(x,t)] \qquad (5.5)$$

The next step in Laplace decomposition method is representing the solution as an infinite series given below

$$\boldsymbol{u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)} \qquad (5.6)$$

The nonlinear operator is decomposed as

$$Nu = \sum_{n=0}^{\infty} A_n \qquad (5.7)$$

Using (4.6), (4.7) and (4.5) we get

$$\sum_{n=0}^{\infty} \mathcal{L}[u(x,t)] = \frac{f(x)}{s} + \frac{g(x)}{s^2} - \frac{1}{s^2}\mathcal{L}[R\,u(x,t)] - \frac{1}{s^2}\sum_{n=0}^{\infty}\mathcal{L}[A_n] + \frac{1}{s^2}\mathcal{L}[h(x,t)] \qquad (5.8)$$

On comparing both sides of the Eq. (4.8) we have

$$\mathcal{L}[u_0(x,t)] = \frac{f(x)}{s} + \frac{g(x)}{s^2} + \frac{1}{s^2}\mathcal{L}[h(x,t)] = k(x,s) \qquad (5.9)$$

$$\mathcal{L}[u_1(x,t)] = -\frac{1}{s^2}\mathcal{L}[R\,u_0(x,t)] - \frac{1}{s^2}\sum_{n=0}^{\infty}\mathcal{L}[A_0] \qquad (5.10)$$

$$\mathcal{L}[u_2(x,t)] = -\frac{1}{s^2}\mathcal{L}[R\,u_1(x,t)] - \frac{1}{s^2}\sum_{n=0}^{\infty}\mathcal{L}[A_1] \qquad (5.11)$$

$$\vdots$$

In general, the recursive relation is given by

$$\mathcal{L}[u_{n+1}(x,t)] = -\frac{1}{s^2}\mathcal{L}[R\,u_n(x,t)] - \frac{1}{s^2}\sum_{n=0}^{\infty}\mathcal{L}[A_n], \quad n \geq 1 \qquad (5.12)$$

Applying inverse Laplace transform to Eq. (5.9) − (5.12), so our required recursive relation is given below

$$u_0(x,t) = k(x,t)$$

$$u_{n+1}(x,t) = -\mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[Ru_n(x,t)] + \frac{1}{s^2}\sum_{n=0}^{\infty}\mathcal{L}[A_n]\right], n \geq 1 \qquad (5.13)$$

Where $k(x,t)$ represent the term arising from source term and prescribe initial conditions.

5.3 Modified Laplace Decomposition Method

To apply this modification, we assume that $k(x,t)$ can be divided into the sum of two parts namely $k_1(x,t), k_2(x,t)$ therefore we get [71]

$$k(x,t) = k_1(x,t) + k_2(x,t) \qquad (5.14)$$

In view of these suggestion, we formulate the modified recursive algorithm as follows:

$$u_0 = k_1(x,t)$$

$$u_1(x,t) = k_2(x,t) - \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[Ru_0(x,t)] + \frac{1}{s^2}\sum_{n=0}^{\infty}\mathcal{L}[A_0]\right]$$

$$u_{n+1}(x,t) = -\mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[Ru_n(x,t)] + \frac{1}{s^2}\sum_{n=0}^{\infty}\mathcal{L}[A_n]\right], n \geq 1 \qquad (5.15)$$

5.4 Double Laplace Transforms

Let $f(x,t)$ be a function of two variables x and t ,The double Laplace transform of the function $f(x,t)$ is defined by [113]

$$\mathcal{L}_x\mathcal{L}_t f\{(x,t)\} = F(p,s) = \int_0^{\infty} e^{-px}\int_0^{\infty} e^{-st} f(x,t)\,dt\,dx$$

Whenever that integral exist . Here p and s are complex numbers.

The invers double Laplace transform $\mathcal{L}_x^{-1}\mathcal{L}_t^{-1}\left[F(p,s)\right]=f(x,t)$ is defined by the complex double integral formula.

$$\mathcal{L}_x^{-1}\mathcal{L}_t^{-1}\left[F(p,s)\right]=f(x,t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{px}dp\,\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty}e^{st}F(p,s)ds$$

Where $F(p,s)$ must be an analytic function for all p and s in the region defined by the inequalities $\operatorname{Re} p\geq c$ and $\operatorname{Re} s\geq d$, where c and d are real constants to be chosen suitably .

The double Laplace transform for second partial derivative with respect to x and t are defined as follows

$$\mathcal{L}_x\mathcal{L}_t\left[\frac{\partial^2 f}{\partial x^2}\right]=p^2F(p,s)-pF(0,s)-\frac{\partial F(0,s)}{\partial x}$$

$$\mathcal{L}_x\mathcal{L}_t\left[\frac{\partial^2 f}{\partial t^2}\right]=s^2F(p,s)-pF(p,0)-\frac{\partial F(p,0)}{\partial t}$$

5.5 Sumudu Transform

Sumudu transform was introduced in 1993 by Watugala to solve differential equations and control engineering problems. Since then, several authors have studied its properties, application to solving various problems and its relationship with some other common integral transforms such as Laplace transform.

The Sumudu transform of a function $f(t)$, defined for all real numbers $t\geq 0$, is the function $F(u)$ defined by [93]

$$S(f(t))=F(v)=\int_0^\infty \frac{1}{v}e^{\frac{-v}{t}}f(t)dt \qquad (5.16)$$

where the symbol S denotes the Sumudu transform.

If c_1, c_2 are non-negative constants, $f(t)$ and $g(t)$ are functions having Sumudu transform $F(v)$ and $G(v)$, respectively, then

1. Linearity Property

   $S[c_1 f(t) + c_2 g(t)] = c_1 S[f(t)] + c_2 S[g(t)]$

2. Differentiation Property

   $S\left[f^{(n)}(t)\right] = v^{-n}\left[F(u) - \sum_{k=0}^{n-1} u^k f^{(k)}(0)\right]$

5.5.1 Analysis of Adomain Sumudu Transform Method

Taking Sumudu transform on both sides of Eq. (5.3), to get:

$$S\left[Lu(x,t)\right] + S\left[Ru(x,t)\right] + S\left[Nu(x,t)\right] = S\left[h(x,t)\right]$$

Applying the differentiation property of Sumudu transform, we have:

$$S\left[u(x,t)\right] = v^2\left[h(x,t)\right] + vf(x) + V^2 g(x) - v^2 S\left[Ru(x,t) + Nu(x,t)\right] \quad (5.17)$$

Taking the inverse Sumudu on both sides of equation (3.5), we have

$$u(x,t) = G(x,t) - S^{-1}\left[\frac{1}{v^2} S\left[Ru(x,t) + Nu(x,t)\right]\right] \quad (5.18)$$

Then (5.18) becomes

$$\sum_{n=0}^{\infty} u_n(x,t) = G(x,t) - S^{-1}\left[\frac{1}{v^2} S\left[R\sum_{n=0}^{\infty} u_n(x,t) + \sum_{n=0}^{\infty} A_n\right]\right] \quad (5.19)$$

Then the recursive relation is given by

$$u_0(x,t) = G(x,t)$$

$$u_{n+1}(x,t) = -S^{-1}\left[\frac{1}{v^2} S\left[R\sum_{n=0}^{\infty} u_n(x,t) + \sum_{n=0}^{\infty} A_n\right]\right], n \geq 1 \quad (5.20)$$

Applying Sumudu transform of the right hand side Eq.(5.20) and then taking inverse Sumudu transform, we get $u_0, u_1, u_2, \ldots$

5.6 Applications

To demonstrate the applicability of the above-presented method, for nonlinear partial differential equations, we now consider some examples.

Example 5.6.1. We consider now a linear example [35]

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^4 u}{\partial x^2 \partial t^2} \tag{5.21}$$

$$u(x,0) = e^{x/\sqrt{2}} \quad u_t(x,0) = e^{x/\sqrt{2}}$$

using double Laplace transform method we obtain

$$U(p,s) = \frac{1}{s\left(p - \frac{1}{\sqrt{2}}\right)} + \frac{1}{s^2\left(p - \frac{1}{\sqrt{2}}\right)} + \frac{1}{s^2} \mathcal{L}_x \mathcal{L}_t \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^4 u}{\partial x^2 \partial t^2} \right)$$

taking double inverse Laplace transform, we have

$$u(x,t) = e^{x/\sqrt{2}} + t\, e^{x/\sqrt{2}} + \mathcal{L}_x^{-1} \mathcal{L}_t^{-1} \left[\frac{1}{s^2} \mathcal{L}_x \mathcal{L}_t \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^4 u}{\partial x^2 \partial t^2} \right) \right]$$

Then

$$\sum_{n=0}^{\infty} u_n(x,t) = e^{x/\sqrt{2}} + t\, e^{x/\sqrt{2}} + \mathcal{L}_x^{-1} \mathcal{L}_t^{-1} \left[\frac{1}{s^2} \mathcal{L}_x \mathcal{L}_t \left(\frac{\partial^2 u_n}{\partial x^2} + \frac{\partial^4 u_n}{\partial x^2 \partial t^2} \right) \right] \tag{5.22}$$

By using (5.21) and (5.22) we get

$$u_0 = e^{x/\sqrt{2}} + t\, e^{x/\sqrt{2}}$$

$$u_{n+1}(x,t) = \mathcal{L}_x^{-1}\,\mathcal{L}_t^{-1}\left[\frac{1}{s^2}\mathcal{L}_x\mathcal{L}_t\left(\frac{\partial^2 u_n}{\partial x^2} + \frac{\partial^4 u_n}{\partial x^2 \partial t^2}\right)\right]$$

we obtain

$$u_1 = \mathcal{L}_x^{-1}\,\mathcal{L}_t^{-1}\left[\frac{1}{s^2}\mathcal{L}_x\mathcal{L}_t\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^4 u_0}{\partial x^2 \partial t^2}\right)\right]$$

$$= \mathcal{L}_x^{-1}\,\mathcal{L}_t^{-1}\left[\frac{1}{s^2}\mathcal{L}_x\mathcal{L}_t\left(\frac{1}{2}e^{\frac{x}{\sqrt{2}}}(1+t)\right)\right]$$

$$= \mathcal{L}_x^{-1}\,\mathcal{L}_t^{-1}\left[\frac{1}{2s^3\left(p-\frac{1}{\sqrt{2}}\right)} + \frac{1}{2s^4\left(p-\frac{1}{\sqrt{2}}\right)}\right]$$

$$= \frac{1}{2}e^{x/\sqrt{2}}\left(\frac{t^2}{2!} + \frac{t^3}{3!}\right)$$

And

$$u_2 = \mathcal{L}_x^{-1}\,\mathcal{L}_t^{-1}{}_t\left[\frac{1}{s^2}\mathcal{L}_x\mathcal{L}_t\left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^4 u_1}{\partial x^2 \partial t^2}\right)\right]$$

$$= \mathcal{L}_x^{-1}\,\mathcal{L}_t^{-1}\left[\frac{1}{s^2}\mathcal{L}_x\mathcal{L}_t\left(\frac{1}{4}e^{\frac{x}{\sqrt{2}}}\left(1+t+\frac{t^2}{2}+\frac{t^3}{6}\right)\right)\right]$$

$$= \mathcal{L}_x^{-1}\,\mathcal{L}_t^{-1}\left[\frac{1}{4s^3\left(p-\frac{1}{\sqrt{2}}\right)} + \frac{1}{4s^4\left(p-\frac{1}{\sqrt{2}}\right)} + \frac{1}{4s^5\left(p-\frac{1}{\sqrt{2}}\right)} + \frac{1}{4s^6\left(p-\frac{1}{\sqrt{2}}\right)}\right]$$

$$= \frac{1}{4}e^{x/\sqrt{2}}\left(\frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!}\right)$$

And so on

Considering these components, the solution can be approximate as:

$$u(x,t) \approx \varphi_n(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$$

$$\varphi_0 = e^{x/\sqrt{2}}(1+t)$$

$$\varphi_1 = e^{x/\sqrt{2}}(1+t) + e^{x/\sqrt{2}} \frac{1}{2}\left(\frac{t^2}{2!} + \frac{t^3}{3!}\right)$$

$$\varphi_2 = e^{x/\sqrt{2}}(1+t) + e^{x/\sqrt{2}}\left(\frac{1}{2} + \frac{1}{4}\right)\left(\frac{t^2}{2!} + \frac{t^3}{3!}\right) + e^{x/\sqrt{2}}\left(\frac{t^4}{4!} + \frac{t^5}{5!}\right)$$

$\vdots$

As $N \to \infty \;\; \varphi_N = e^{x/\sqrt{2}}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \dots\right) = e^{t + x/\sqrt{2}}$

Which is exact solution.

Example 5.6.2 we consider a nonlinear example, with no source, with inhomogeneous initial conditions, namely [35]

$$u_{tt} = u_{xx} - (u^2/2)_{xx} + u_{xxtt} \quad , -\infty < x < \infty, t \ge 0 \tag{5.23}$$

$$u(x,0) = -x^2 + 2x - 3, u_t(x,0) = -2x^2 + 4x - 8 \quad -\infty < x < \infty \tag{5.24}$$

Eq. (5.18) is the improved Boussinesq equation for the longitudinal strain $u(x,t)$ of the acoustic waves in elastic rods with circular cross section. The analytical solution of (5.18) and (5.19) is given by

$$u(x,t) = 1 - \frac{x^2 - 2x + 4}{(t-1)^2} \quad , -\infty < x < \infty, t \ge 0 \tag{5.25}$$

which obviously blows up as $t \to 1$. The purpose of example is to illustrate this phenomenon using the LADM.

Applying the Laplace transform and by using the initial condition we have

$$S^2 U(x,S) = S(-x^2 + 2x - 3) + -2x^2 + 4x - 8 + \mathcal{L}\left[u_{xx} - (u^2/2)_{xx} + u_{xxtt}\right]$$

Applying the inverse Laplace transform we get

$$u(x,t) = -x^2 + 2x - 3 + (-2x^2 + 4x - 8)t + \mathcal{L}^{-1}\left(\frac{1}{S^2}\mathcal{L}\left[u_{xx} - (u^2/2)_{xx} + u_{xxtt}\right]\right) \tag{5.26}$$

We decompose the solution as an infinite sum given below

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{5.27}$$

Using (5.26) on (5.27) we get

$$u_n(x,t) = -x^2 + 2x - 3 + (-2x^2 + 4x - 8)t + \mathcal{L}^{-1}\left(\frac{1}{S^2}\mathcal{L}\left[u_{n_{xx}} - A_{n_{xx}} + u_{n_{xxtt}}\right]\right)$$

In which $A_n = \frac{u^2}{2}$ the recursive relation is given below

Now by using equation (5.27) through the LADM method, recursively obtain

$$u_0 = -x^2 + 2x - 3 + (-2x^2 + 4x - 8)t, \quad u_1 = \mathcal{L}^{-1}\left(\frac{1}{S^2}\mathcal{L}\left[u_{0_{xx}} - A_{0_{xx}} + u_{0_{xxtt}}\right]\right)$$

$$u_{n+1} = \mathcal{L}^{-1}\left(\frac{1}{S^2}\mathcal{L}\left[u_{n_{xx}} - A_{n_{xx}} + u_{n_{xxtt}}\right]\right) \quad n \geq 1,$$

Then

$$u_0 = -x^2 + 2x - 3 + \ -2x^2 + 4x - 8\ t$$

$$u_1 = -t^2\left(2 + x^2 - 2x\right)\left(3 + 2t^2 + 4t\right)$$

$$u_2 = -6t^2 - 6t^5x^2 - 10t^5 - 3t^4x^2 + 6t^4x - 9t^4 - \frac{8}{7}x^2t^7 + \frac{16}{7}xt^7 - \frac{40}{21}t^7 + 12t^5x - 4t^6x^2 + 8t^6x - \frac{20}{3}t^6 - 8t^3$$

$$u_3 = \frac{1}{15120}t^6(-63x^4 - 60tx^4 - 15t^2x^4 - 504x - 1296tx - 1008t^2x + 756 + 2304t + 1944t^2 - 224t^3x + 448t^3)$$

$$u_4 = \frac{1}{19958400} t^5 (-11975040x + 53460t^3 + 35840t^7 + 202752t^6 + 237600t^4$$
$$+ 355872t^5 - 208896t^6 x + 194304t^5 x^2 - 7700t^5 x^4 - 393888t^5 x - 13305600tx$$
$$+ 720t^6 x^5 + 112860t^3 x^2 - 106920t^3 x - 3801600t^2 x + 8960x^2 t^7 - 35840xt^7$$
$$-1440t^6 x^4 + 69888t^6 x^2 + 3960t^5 x^5 - 12540t^4 x^4 + 241560t^4 x^2 - 311520t^4 x$$
$$- 4455t^3 x^4 + 6820t^4 x^5 + 2970t^3 x^5)$$

Considering these components, the solution can be approximated as:

$$\varphi_n = \sum_{n=0}^{\infty} u_n \tag{5.28}$$

Calculating the [4/4] Pade' approximant about t by MAPLE,

$$\text{pade}'[\varphi_4] = \frac{-2 - \frac{2564266754}{484474673}t - \frac{275470121237}{40695872532}t^2 - \frac{242229193357}{30521904399}t^3 - \frac{34989810074}{16956613555}t^4}{1 - \frac{171290642}{484474673}t - \frac{234855157}{4283776056}t^2 - \frac{67487229821}{244175235192}t^3 + \frac{186368768021}{813917450640}t^4}$$

Figure. 5.1 show the 4th order of approximation of $u_n(t)$ and [4/4] Adomian–Padé approximation, comparing with the exact solution

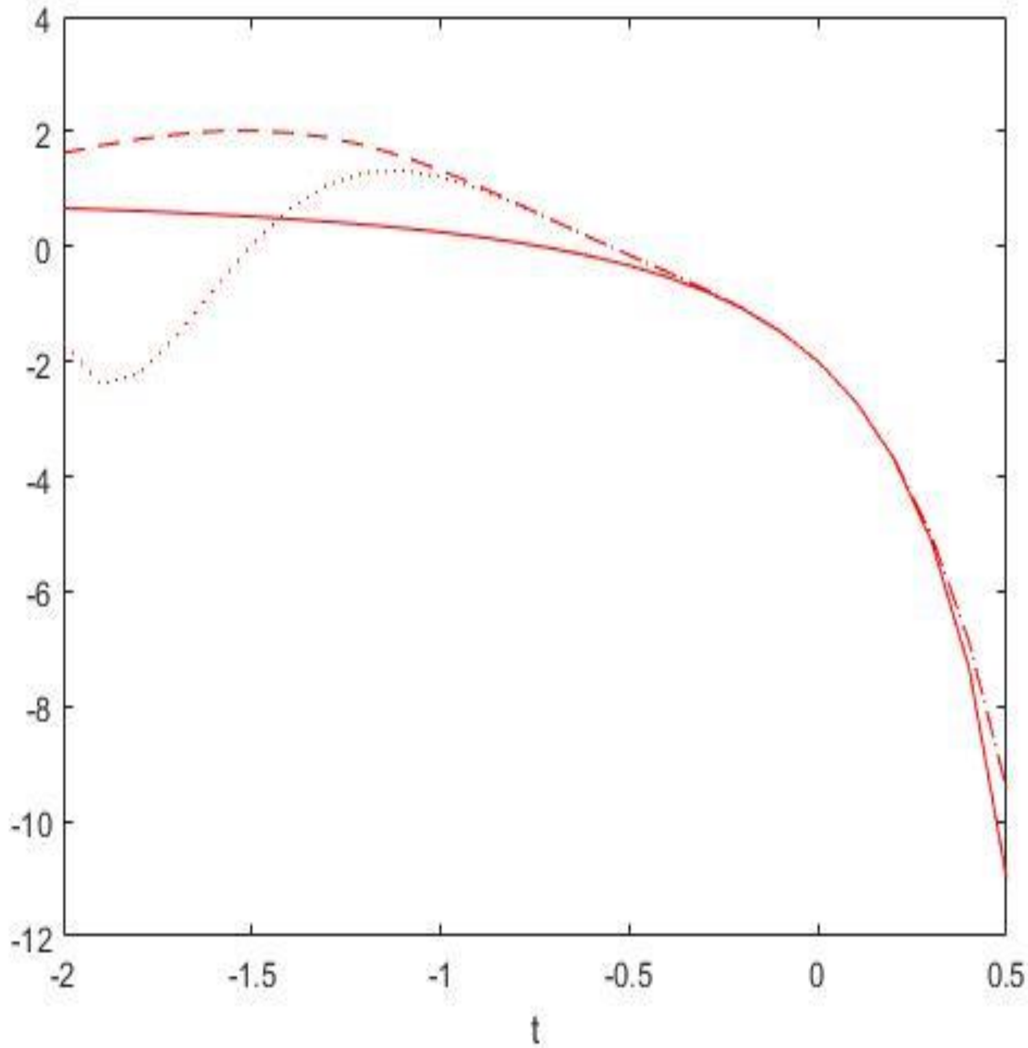

Figure. 5.1. The curves of the approximant solution $\varphi_4(1,t)$ (dashed line) and [4/4] Adomian–Padé approximation (-) and exact solution (solid line)

*Using Modified Laplace Adomian Decomposition Method (**MLADM**)*

Let $f_1 = -x^2 + 2x - 3$ and $f_2 = (-2x^2 + 4x - 8)t$

By using the recursive (5.15) we obtain

$$u_0 = -x^2 + 2x - 3$$

$$u_1(x,t) = (-2x^2 + 4x - 8)t + \mathcal{L}^{-1}\left(\frac{1}{S^2}\mathcal{L}\left(\frac{\partial^2 u_0}{\partial x^2} - A_0 + \frac{\partial^4 u_0}{\partial x^2 t^2}\right)\right)$$

$$u_{n+1} = \mathcal{L}^{-1}\left(\frac{1}{S^2}\mathcal{L}\left(\frac{\partial^2 u_n}{\partial x^2} - A_n + \frac{\partial^4 u_n}{\partial x^2 t^2}\right)\right)$$

This lead to

$$A_0 = (u_0^2)_{xx} = 24x - 24$$

$$u_1 = (-2x^2 + 4x - 8)t + \mathcal{L}^{-1}\left(\frac{1}{S^2}\mathcal{L}\left(\frac{\partial^2 u_0}{\partial x^2} - A_0 + \frac{\partial^4 u_0}{\partial x^2 t^2}\right)\right)$$

$$= \frac{1}{3}t^2\left(-tx^2 + 2tx - 4t + 18 - 18x\right)$$

$$A_1 = (2u_0 u_1)_{xx} = 16t^3 x - 16t^3 + 72t^2$$

$$u_2 = \mathcal{L}^{-1}\left(\frac{1}{S^2}\mathcal{L}\left(\frac{\partial^2 u_1}{\partial x^2} - A_1 + \frac{\partial^4 u_1}{\partial x^2 t^2}\right)\right) = \quad \frac{1}{5}t^4\left(-15 - 2tx + 2t\right)$$

$$A_2 = (u_1^2 + 2u_0 u_2)_{xx} = \quad \frac{1}{45}t^4(2430 + 80t^2 - 80t^2 x + 60t^2 x^2 - 648tx^2 +$$
$$1260tx + 5t^2 x^4 - 20t^2 x^3 + 216tx^3 - 828t - 3780x + 1890x^2)$$

$$u_3 = \mathcal{L}^{-1}\left(\frac{1}{S^2}\mathcal{L}\left(\frac{\partial^2 u_2}{\partial x^2} - A_2 + \frac{\partial^4 u_2}{\partial x^2 t^2}\right)\right) =$$
$$\frac{1}{5040}t^6\begin{pmatrix} -4536 + 1104t + 864tx^2 - 1680tx + 80t^2 x - 80t^2 - \\ 5t^2 x^4 + 20t^2 x^3 - 60t^2 x^2 - 288tx^3 + 7056x - 3528x^2 \end{pmatrix}$$

$\vdots$

Which in closed form gives exact solution

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = -3 + 6t^2 - 3t^4 - 6t^2 x + \frac{2}{5}t^5 - \frac{25}{126}t^8 x - \frac{17}{1260}t^8 x^4 +$$
$$\frac{17}{315}t^8 x^3 - \frac{97}{840}t^8 x^2 + \frac{1}{252}t^9 x^4 \ldots$$

Calculating the [4/4] Pade' approximant about t by MAPLE,

$$\text{pade}'[\varphi_4(1,t)] = \frac{2738610 + \frac{5310765}{8}t - \frac{15932295}{8}t^2 + \frac{1788237}{8}t^3 + \frac{98418645}{16}t^4}{-1369305 - \frac{5310765}{16}t + \frac{15932295}{16}t^2 + \frac{9166203}{16}t^3 - \frac{3422655}{4}t^4}$$

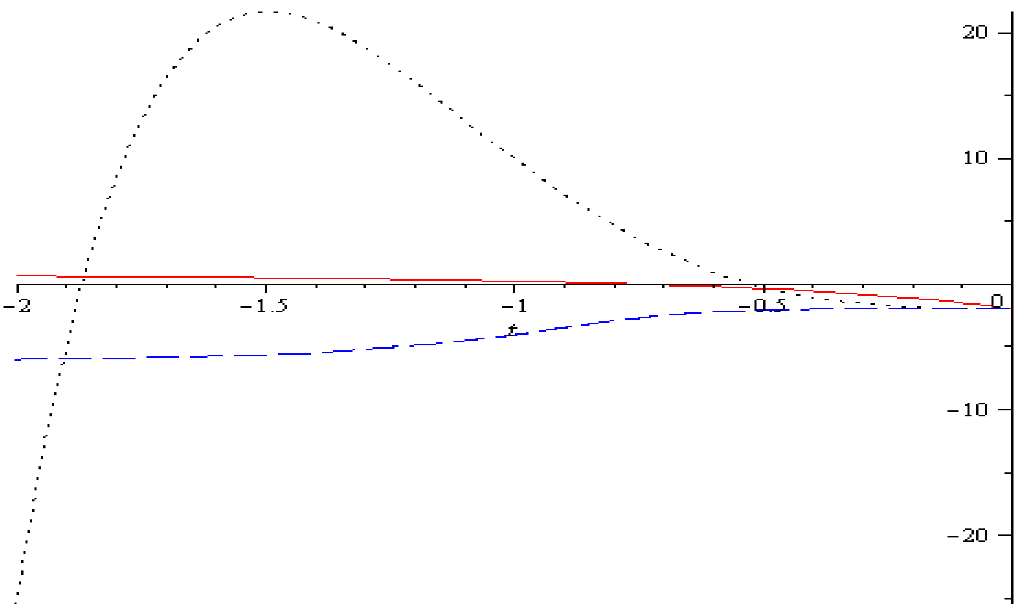

Figure. 5.2. The curves of the MLADM solution $\varphi_4(1,t)$ (dashed line) and [4/4] Adomian–Padé approximation (-) and exact solution (solid line)

Table 5.1 : Example 5.6.2: Results obtained for the the exact solution compared with LADM,MLADM and by application of a Pad´e approximant [4,4] to Adomian's series solution when $x = 1$.

t	LADM	MLADM	Pade' (LADM)	Pade' (MLADM)	Exact solution
-1	1.2147	-6.5698	1.3284	-4.0886	0.25
-0.9	1.006547704	-4.610046625	1.053432958	-3.540382531	0.1689750694
-0.8	0.7387807031	-3.394860506	0.7553963911	-3.006484971	0.0740740741
-0.7	0.4438117030	-2.682245930	0.4486961972	-2.567408200	-0.038062284
-0.6	0.1437411340	-2.293908845	0.1448698028	-2.267255833	-0.171875000
-0.5	-0.1509768497	-2.103097292	-0.1507888831	-2.098497065	-0.333333333
-0.4	-0.4418039898	-2.023447982	-0.4417846882	-2.022913122	-0.530612245
-0.3	-0.7418445386	-1.999195542	-0.7418436685	-1.999161294	-0.775147929
-02	-1.075528994	-1.996966418	-1.075528990	-1.996965664	-1.083333333
-0.1	-1.478757232	-1.999302600	-1.478757231	-1.999302598	-1.479338843
0	-2.	-2.000000000	-2.000000000	-2.000000000	-2.000000000
0.1	-2.702849410	-2.001302600	-2.702849410	-2.001302602	-2.703703703
0.2	-3.670495745	-2.012966465	-3.670495637	-2.012966827	-3.687500000
0.3	-5.012588772	-2.053197370	-5.012582944	-2.053208259	-5.122448981

Example 5.6.3. The other equation we are considering is

$$\frac{\partial^2 u}{\partial t^2} = 3\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 \ln u}{\partial x^2} + \frac{\partial^4 u}{\partial x^4} \tag{5.29}$$

With initial conditions

$$\begin{aligned} u(x,0) &= e^x \\ u_t(x,0) &= 2e^x \end{aligned} \tag{5.30}$$

Using the Sumudu Adomian decomposition method, we obtain

$$\sum_{n=0}^{\infty} u_{n+1}(x,t) = e^{x} + 2e^{x}\,t + \mathcal{S}^{-1}\left[\frac{1}{v^{2}}\mathcal{L}(3\frac{\partial^{2}u_{n}}{\partial x^{2}} + \frac{\partial^{2}A_{n}}{\partial x^{2}} + \frac{\partial^{4}u_{n}}{\partial x^{4}})\right]$$

leads to the following results

$$u_0 = e^{x} + 2e^{x}\,t$$

$$A_0(u_0) = [\ln u_0]_{xx}$$

$$u_1 = \mathcal{L}^{-1}\left[\frac{1}{S^{2}}\mathcal{L}(3\frac{\partial^{2}u_{0}}{\partial x^{2}} + \frac{\partial^{2}A_{0}}{\partial x^{2}} + \frac{\partial^{4}u_{0}}{\partial x^{4}})\right] = u_1 = e^{x}\left(\frac{(2t)^{2}}{2!}\right) + e^{x}\left(\frac{(2t)^{3}}{3!}\right)$$

$$A_1(u_1,u_0) = \left[\frac{u_1}{u_0}\right]_{xx}$$

$$u_2 = \mathcal{S}^{-1}\left[\frac{1}{v^{2}}\mathcal{L}(3\frac{\partial^{2}u_{1}}{\partial x^{2}} + \frac{\partial^{2}A_{1}}{\partial x^{2}} + \frac{\partial^{4}u_{1}}{\partial x^{4}})\right] = e^{x}\left(\frac{(2t)^{4}}{4!}\right) + e^{x}\left(\frac{(2t)^{5}}{5!}\right)$$

$$A_2(u_2,u_1,u_0) = \frac{1}{2}\left[\frac{u_2}{u_0} - \frac{u_1^{2}}{u_0^{2}}\right]_{xx}$$

$$u_3 = \mathcal{S}^{-1}\left[\frac{1}{v^{2}}\mathcal{L}(3\frac{\partial^{2}u_{2}}{\partial x^{2}} + \frac{\partial^{2}A_{2}}{\partial x^{2}} + \frac{\partial^{4}u_{2}}{\partial x^{4}})\right] = e^{x}\left(\frac{(2t)^{6}}{6!}\right) + e^{x}\left(\frac{(2t)^{7}}{7!}\right)$$

$$\vdots$$

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = e^{x}\left[1 + 2t + \frac{(2t)^{2}}{2!} + \frac{(2t)^{3}}{3!} + \frac{(2t)^{4}}{4!} + \frac{(2t)^{5}}{5!} + \frac{(2t)^{6}}{6!} + \ldots\right]$$

$$= e^{(x+2t)}$$

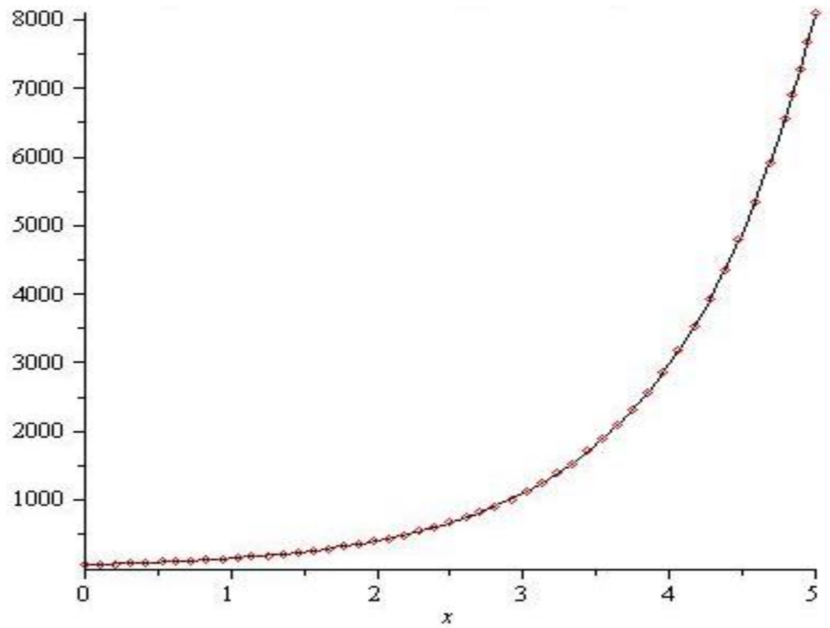

Figure. 5.3. The behavior of the exact solution and SADM solution of $u(x,t)$ in case $t=2$.

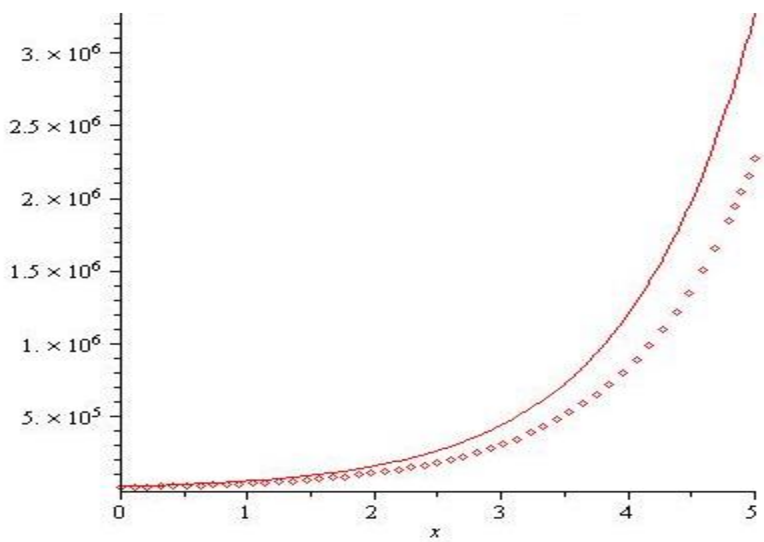

Figure. 5.4. The behavior of the exact solution and SADM solution of $u(x,t)$ in case $t=5$.

Example 5.6.4. We consider now a linear example

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2 \partial t^2}$$

$$u(x,0) = e^{x/\sqrt{2}} \quad u_t(x,0) = e^{x/\sqrt{2}} \tag{5.31}$$

Using the Adomian decomposition method [78], we obtain

$$\sum_{n=0}^{\infty} u_{n+1}(x,t) = e^{x/\sqrt{2}} + e^{x/\sqrt{2}}\, t + K^{-1}\left[v^2 K\left(\frac{\partial^2 u_n}{\partial x^2} + \frac{\partial^2 u_n}{\partial x^2 \partial t^2}\right)\right]$$

Leads to the following results

$$u_0 = e^{x/\sqrt{2}} + e^{x/\sqrt{2}}\, t$$

$$u_1 = K^{-1}\left[v^2 K\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial x^2 \partial t^2}\right)\right] = K^{-1}\left[v^2 \left(\frac{v}{2} e^{x/\sqrt{2}} + \frac{v^2}{2} e^{x/\sqrt{2}}\right)\right]$$

$$= \frac{1}{2} e^{x/\sqrt{2}} \left(\frac{t^2}{2!} + \frac{t^3}{3!}\right)$$

$$u_2 = K^{-1}\left[v^2 K\left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial x^2 \partial t^2}\right)\right]$$

$$= K^{-1}\left[v^2 \left(\frac{v^3}{4} e^{x/\sqrt{2}} + \frac{v^4}{12} e^{x/\sqrt{2}} + \frac{v}{4} e^{x/\sqrt{2}} + \frac{v^2}{6} e^{x/\sqrt{2}}\right)\right]$$

$$= \frac{1}{4} e^{x/\sqrt{2}} \left(\frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!}\right)$$

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = u_0 + u_1 + u_2 + u_3 + \ldots$$

$$u(x,t) = e^{x/\sqrt{2}} \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{2t^4}{4!} + \frac{2t^5}{5!} + \ldots\right) = e^{t + x/\sqrt{2}}$$

By MDM

Using the Adomian decomposition method [77,79], we obtain

$$\sum_{n=0}^{\infty} u_{n+1}(x,t) = e^{x/\sqrt{2}} + e^{x/\sqrt{2}}\, t + M^{-1}\left[\frac{1}{v^2} M\left(\frac{\partial^2 u_n}{\partial x^2} + \frac{\partial^2 u_n}{\partial x^2 \partial t^2}\right)\right]$$

leads to the following results

$$u_0 = e^{x/\sqrt{2}} + e^{x/\sqrt{2}}\, t$$

$$\begin{aligned} u_1 &= M^{-1}\left[\frac{1}{v^2} M\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial x^2 \partial t^2}\right)\right] \\ &= M^{-1}\left[\frac{1}{v^2}\left(\frac{1}{2} e^{x/\sqrt{2}} + \frac{1}{2v} e^{x/\sqrt{2}}\right)\right] \\ &= \frac{1}{2} e^{x/\sqrt{2}}\left(\frac{t^2}{2!} + \frac{t^3}{3!}\right) \end{aligned}$$

$$\begin{aligned} u_2 &= K^{-1}\left[v^2 K\left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial x^2 \partial t^2}\right)\right] \\ &= K^{-1}\left[\frac{1}{v^2}\left(\frac{1}{4v^2} e^{x/\sqrt{2}} + \frac{1}{4v^3} e^{x/\sqrt{2}} + \frac{1}{4} e^{x/\sqrt{2}} + \frac{1}{4v} e^{x/\sqrt{2}}\right)\right] \\ &= \frac{1}{4} e^{x/\sqrt{2}}\left(\frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!}\right) \end{aligned}$$

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = u_0 + u_1 + u_2 + u_3 + \ldots$$

$$e^{x/\sqrt{2}}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{2t^4}{4!} + \frac{2t^5}{5!} + \ldots\right) = e^{t + x/\sqrt{2}}$$

Example 5.6.5. We consider now a nonlinear example

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u^2}{\partial x^2} = 0 \tag{5.32}$$

$$\text{With } u(x,0) = \frac{6}{x^2} \quad u_t(x,0) = -\frac{12}{x^3} \tag{5.33}$$

Taking transform [38] of "Eq. (6.24)", we have

$$M\left[\frac{\partial^2 u_2}{\partial t^2}\right] = M\left[\frac{\partial^2 u}{\partial x^2} - \frac{\partial^4 u}{\partial x^4} - \frac{\partial^2 u^2}{\partial x^2}\right] \tag{5.34}$$

Now using the property, transform [38] of the derivatives of the function, in "Eq. (6.26)", we have

$$v^2 R(v) - v^3 F(0) - v^2 F^{`}(0) = M\left[\frac{\partial^2 u}{\partial x^2} - \frac{\partial^4 u}{\partial x^4} - \frac{\partial^2 u^2}{\partial x^2}\right] \tag{5.35}$$

Using "Eq. (5.34)" in "Eq. (5.35)", we have

$$u(x,v) = v\,\frac{6}{x^2} - \frac{12}{x^3} + \frac{1}{v^2} M\left[\frac{\partial^2 u}{\partial x^2} - \frac{\partial^4 u}{\partial x^4} - \frac{\partial^2 u^2}{\partial x^2}\right] \tag{5.36}$$

Now taking inverse transform [38] of "Eq. (5.36)", we have

$$u(x,t) = \frac{6}{x^2} - \frac{12}{x^3} t + M^{-1}\left(\frac{1}{v^2} M\left[\frac{\partial^2 u}{\partial x^2} - \frac{\partial^4 u}{\partial x^4} - \frac{\partial^2 u^2}{\partial x^2}\right]\right) \tag{5.37}$$

Eq (5.37) can decompose as

$$\sum_{n=0}^{n} u(x,t) = \frac{6}{x^2} - \frac{12}{x^3} t + M^{-1}\left(\frac{1}{v^2} M\left[\sum_{n=0}^{n} \frac{\partial^2 u_n}{\partial x^2} + \sum_{n=0}^{n} \frac{\partial^4 u_n}{\partial x^4} - \sum_{n=0}^{n} \frac{\partial^2 A_n}{\partial x^2}\right]\right)$$

$$u_0 = \frac{6}{x^2} - \frac{12}{x^3} t$$

$$A_0 = \frac{d^2(u_0^{\,2})}{dx^2} = \frac{720}{x^6} - \frac{4320t}{x^7} + \frac{6048t^2}{x^8}$$

$$u_1 = M^{-1}\left(\frac{1}{v^2} M\left[\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^4 u_0}{\partial x^4} - A_0\right]\right) = \frac{18t^2}{x^4} - \frac{24t^3}{x^5} - \frac{504t^4}{x^8}$$

$$A_1 = \frac{d^2(2u_0 u_1)}{dx^2} = -\frac{665280t^4}{x^{12}} - \frac{40320t^3}{x^9} + \frac{9072t^2}{x^8} + \frac{1596672t^5}{x^{13}} + \frac{41472t^4}{x^{10}}$$

$$u_2 = M^{-1}\left(\frac{1}{v^2} M\left[\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^4 u_1}{\partial x^4} - A_1\right]\right) = \frac{30t^4}{x^6} - \frac{36t^5}{x^7} + \frac{504t^4}{x^8} - \frac{2592t^6}{x^{10}}$$

$$- \frac{110880t^6}{x^{12}} - \frac{38016t^7}{x^{13}}$$

$$A_2 = \frac{d^2(u_1^2 + 2u_0 u_2)}{dx^2} = -\frac{1166400t^8}{x^{16}} + \frac{86400t^7}{x^{13}} - \frac{49248t^6}{x^{12}} + \frac{1440t^6}{x^{10}}$$

$$- \frac{2016t^5}{x^9} + \frac{684t^4}{x^8} + \frac{2204928t^7}{x^{15}} - \frac{1330560t^6}{x^{14}} + \frac{6048t^4}{x^{10}} - \frac{12096t^5}{x^{11}}$$

$$u_3 = M^{-1}\left(\frac{1}{v^2} M\left[\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^4 u_2}{\partial x^4} - A_2\right]\right) = -\frac{12960t^{10}}{x^{16}} - \frac{23063040t^9}{x^{17}} - \frac{126720t^9}{x^{15}}$$

$$-\frac{1200t^9}{x^{13}} - \frac{7555680}{7}\frac{t^8}{x^{14}} - \frac{64864800t^8}{x^{16}} - \frac{4212t^8}{x^{12}} - \frac{180}{7}\frac{t^8}{x^{10}} - \frac{4032t^7}{x^{11}} + \frac{4032t^6}{x^{10}}$$

$$+ \frac{96}{5}\frac{t^6}{x^8} + \frac{133056t^6}{x^{12}}$$

$\vdots$

Then the series solution expression can be written in the form,

$$u(x,t) = u_0 + u_1 + u_2 + \ldots = \frac{6}{x^2} - \frac{12t}{x^3} + \frac{18t^2}{x^4} - \frac{24t^3}{x^5} - \frac{504t^4}{x^8} + \ldots$$

This will, in the limit of infinitely many terms, yields the closed-form solution

$$\frac{6}{(x+t)^2}$$

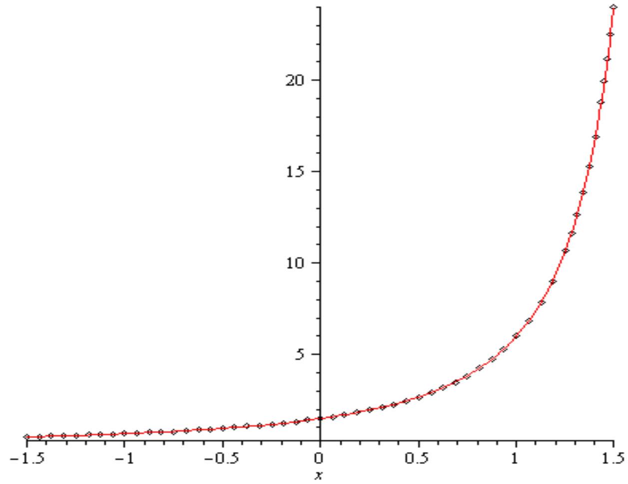

Figure. 5.5. The behavior of the exact solution and Pade[3,3] SADM solution of $u(x,t)$ in case $t=-2$.

Calculating the [3,3] Pade' approximant about t by MAPLE,

$$\text{pade}'[\varphi_3(x,t)] = \frac{6}{(x+t)^2}$$

Which is exact solution

Conclusions

Nonlinear phenomena is present in every science field and therefore, it is of fundamental importance to develop efficient methods to solve nonlinear differential equations. Unfortunately, in most cases, only numerical solutions can are obtainable. This makes evident the importance of analytical techniques, such Adomian's decomposition method, since it searches for solutions under a series form, not requiring any discretization or assumption for a small parameter to be present in the problem, which, in fact, may not exist at all. The application of this method to partial differential equations poses some obstacles, as the computational effort that is heavier than the one required to solve ordinary differential equations and the possible convergence to a solution that does not satisfy all the boundary conditions. Anyway, as the exact analytical solution is probably not recognized from the solution series, truncated series must be used to represent the solution. A disadvantage of such approach is that the truncated series may have a small convergence radius. To overcome this drawback when applying Adomian's method to ordinary differential equations, some authors have used an aftertreatment with Pad´e approximants. This technique was applied to partial differential equations where graphical illustrations were used to show that the domain of convergence of Adomian's solution was improved by the application of this technique. In this thesis, it is shown not only graphically but also numerically, that this technique can enlarge the domain of convergence of the solution and that the accuracy is generally also improved inside the domain of convergence of Adomian's series solution as the numerical results show.

To increase the accuracy of the solution, some authors proposed the choice of a smaller time interval and/or to add more terms to the solution series.

References

[1] Adomian, G., Solving Frontier Problems of Physics: The Decomposition Method. 1994. Klumer, Boston.

[2] Brocchini, M., 2013. A reasoned overview on Boussinesq-type models: the interplay between physics, mathematics and numerics. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 469(2160), p.20130496.

[3] Walkley, M.A., 1999. A numerical method for extended Boussinesq shallow-water wave equations (Doctoral dissertation, University of Leeds).

[4] Dingemans, M.W., 1994. Water wave propagation over uneven bottoms.

[5] Madsen, P.A. and SCHÄFFER, H.A., 1999. A review of Boussinesq-type equations for surface gravity waves. In Advances in coastal and ocean engineering (pp. 1-94).

[6] S. S. Chauhan, Vijay Saini,2018. Modification of the adomian decomposition method, 1724053108

[7] Adomian, G., 1988. A review of the decomposition method in applied mathematics. Journal of mathematical analysis and applications, 135(2), pp.501-544.

[8] Dita, P. and Grama, N., 1997. On Adomian's decomposition method for solving differential equations. arXiv preprint solv-int/9705008.

[9] Fatoorehchi, H. and Abolghasemi, H., 2011. On calculation of Adomian polynomials by MATLAB. J. Appl. Comput. Sci. Math, 5, pp.85-88.

[10] Duan, J.S., 2011. New recurrence algorithms for the nonclassic Adomian polynomials. Computers & Mathematics with Applications, 62(8), pp.2961-2977.

[11] . El-Kalla, I.L., 2008. Convergence of the Adomian method applied to a class of nonlinear integral equations. Applied Mathematics Letters, 21(4), pp.372-376.

[12] Wazwaz, A.M., 1997. Necessary conditions for the appearance of noise terms in decomposition solution series. Applied Mathematics and Computation, 81(2-3), pp.265-274.

[13] Adomian, G. and Rach, R., 1992. Noise terms in decomposition solution series. Computers & Mathematics with Applications, 24(11), pp.61-64. Abassy, T.A., 2010.

[14] Azreg-Aïnou, M., 2009. A developed new algorithm for evaluating Adomian polynomials. arXiv preprint arXiv:0906.0244.

[15] Eman Al awawdah,2016, The Adomian Decomposition Method For Solving Partial Differential Equations

[16] Turkyilmazoglu, M., 2019. Accelerating the convergence of Adomian decomposition method (ADM). Journal of Computational Science, 31, pp.54-59.

[17] Abassy, T.A., El-Tawil, M.A. and Saleh, H.K., 2004. The solution of KdV and mKdV equations using Adomian Pade approximation. International Journal of Nonlinear Sciences and Numerical Simulation, 5(4), pp.327-340.

[18] Abdou, M.A., Soliman, A.A. and El-Basyony, S.T., 2007. New application of Exp-function method for improved Boussinesq equation. Physics Letters A, 369(5-6), pp.469-475.

[19] Iskandar, L. and Jain, P.C., 1980, September. Numerical solutions of the improved Boussinesq equation. In Proceedings of the Indian Academy of Sciences-Mathematical Sciences (Vol. 89, No. 3, pp. 171-181). Springer India.

[20] Maulén, C. and Muñoz, C., 2019. Decay in the one dimensional generalized Improved Boussinesq equation. arXiv preprint arXiv:1904.10129.

[21] Pei, Y., Yong, C. and Zhi-Bin, L., 2008. Adomian decomposition method and Padé approximants for solving the Blaszak–Marciniak lattice. Chinese Physics B, 17(11), p.3953.

[22] Wang, Z., Zou, L. and Zong, Z., 2011. Adomian decomposition and Padé approximate for solving differential-difference equation. Applied Mathematics and Computation, 218(4), pp.1371-1378.

[23] Chrysos, M., Sanchez, F. and Cherruault, Y., 2002. Improvement of convergence of Adomian's method using Pade approximants. Kybernetes, 31(6), pp.884-895.

[24] Bellman, N.D. and Adomian, G., 1984. Partial differential equations: new methods for their treatment and solution (Vol. 15). Springer Science & Business Media.

[25] Duan, J.S., 2011. Convenient analytic recurrence algorithms for the Adomian polynomials. Applied Mathematics and Computation, 217(13), pp.6337-6348.

[26] Duan, J.S., Rach, R., Baleanu, D. and Wazwaz, A.M., 2012. A review of the Adomian decomposition method and its applications to fractional differential equations. Communications in Fractional Calculus, 3(2), pp.73-99.

[27] Duan, J.S., Rach, R., Wazwaz, A.M., Chaolu, T. and Wang, Z., 2013. A new modified Adomian decomposition method and its multistage form for solving nonlinear boundary value problems with Robin boundary conditions. Applied Mathematical Modelling, 37(20-21), pp.8687-8708.

[28] Hosseini, M.M., 2006. Adomian decomposition method with Chebyshev polynomials. Applied Mathematics and Computation, 175(2), pp.1685-1693.

[29] Acan, O. and Baleanu, D., 2017. A new numerical technique for solving fractional partial differential equations. arXiv preprint arXiv:1704.02575.

[30] Molabahrami, A., 2015. A practical review of the Adomian decomposition method: computer implementation aspects. Iranian Journal of Numerical Analysis and Optimization, 5(2), pp.29-43.

[31] Kataria, K.K. and Vellaisamy, P., 2016. Simple parametrization methods for generating Adomian polynomials. Appl. Anal. Discrete Math, 10(1), pp.168-185.

[32] Adomian, G. and Rach, R., 1992. Modified decomposition solution of nonlinear partial differential equations. Applied Mathematics Letters, 5(6), pp.29-30.

[33] Araghi, M.F. and Fallahzadeh, A., 2012. Explicit series solution of Boussinesq equation by homotopy analysis method. Journal of American Science, 8(11), pp.448-452.

[34] Behzadi, S.S., 2011. Numerical solution of Boussinesq equation using modified Adomian decomposition and homotopy analysis methods. International Journal of Mathematical Modelling and Computations, 1(1), pp.45-58.

[35] Kaya, D., 2001. Explicit solutions of generalized nonlinear Boussinesq equations. Journal of Applied Mathematics, 1(1), pp.29-37.

[36] Abassy, T.A., El-Tawil, M.A. and El-Zoheiry, H., 2007. Modified variational iteration method for Boussinesq equation. Computers & Mathematics with Applications, 54(7-8), pp.955-965.

[37] Abd-el-Malek, M.B., Badran, N.A., Hassan, H.S. and Abbas, H.H., 2015. New solutions for solving Boussinesq equation via potential symmetries method. Applied Mathematics and Computation, 251, pp.225-232.

[38] Aggarwal, S. and Chauhan, R., 2019. A comparative study of Mohand and Aboodh transforms. International Journal of Research in Advent Technology, 7(1), pp.520-529.

[39] Anco, S.C., Gandarias, M.L. and Recio, E., 2018. Conservation laws, symmetries, and line soliton solutions of generalized KP and Boussinesq equations with p-power nonlinearities in two dimensions. Theoretical and Mathematical Physics, 197(1), pp.1393-1411.

[40] Wazwaz, A.M. and El-Sayed, S.M., 2001. A new modification of the Adomian decomposition method for linear and nonlinear operators. Applied Mathematics and Computation, 122(3), pp.393-405.

[41] Abassy, T.A., 2010. Improved Adomian decomposition method. Computers & mathematics with applications, 59(1), pp.42-54.

[42] Song, L. and Wang, W., 2013. A new improved Adomian decomposition method and its application to fractional differential equations. Applied Mathematical Modelling, 37(3), pp.1590-1598.

[43] Babolian, E. and Biazar, J., 2002. Solution of nonlinear equations by modified Adomian decomposition method. Applied Mathematics and Computation, 132(1), pp.167-172.

[44] Mahmoudi, Y., 2014. A new modified Adomian decomposition method for solving a class of hypersingular integral equations of second kind. Journal of Computational and Applied Mathematics, 255, pp.737-742.

[45] Mohamed M. Mousa, and Aidarkhan Kaltayev. Constructing Approximate and Exact Solutions for Boussinesq Equations using Homotopy Perturbation Padé Technique. International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering Vol:3, No:2, 2009

[46] Anjum, N. and He, J.H., 2019. Laplace transform: making the variational iteration method easier. Applied Mathematics Letters, 92, pp.134-138.

[47] Lesnic, D., 2006. Blow-up solutions obtained using the decomposition method. Chaos, Solitons & Fractals, 28(3), pp.776-787.

[48] Antuono, M., Liapidevskii, V. and Brocchini, M., 2009. Dispersive Nonlinear Shallow-Water Equations. Studies in Applied Mathematics, 122(1), pp.1-28.

[49] Behzadi, S.S., 2013. Convergence of Iterative Methods applied to Boussinesq equation.

[50] Bhatnagar, P.L., 1979. Nonlinear Waves in One-Dimensional Dispersive Systems, Clarendon.

[51] Bogolubsky, I.L., 1977. Some examples of inelastic soliton interaction. Computer Physics Communications, 13(3), pp.149-155.

[52] Bona, J.L. and Sachs, R.L., 1988. Global existence of smooth solutions and stability of solitary waves for a generalized Boussinesq equation. Communications in Mathematical Physics, 118(1), pp.15-29.

[53] Bona, J.L. and Tzvetkov, N., 2009. Sharp well-posedness results for the BBM equation. Discrete Contin. Dyn. Syst, 23(1241-1252), p.7.

[54] Bona, J.L., Chen, M. and Saut, J.C., 2002. Boussinesq Equations and Other Systems for Small-Amplitude Long Waves in Nonlinear Dispersive Media. I: Derivation and Linear Theory. Journal of Nonlinear Science, 12(4).

[55] Bona, J.L., Chen, M. and Saut, J.C., 2002. Boussinesq Equations and Other Systems for Small-Amplitude Long Waves in Nonlinear Dispersive Media. I: Derivation and Linear Theory. Journal of Nonlinear Science, 12(4).

[56] Brugnano, L., Gurioli, G. and Zhang, C., 2019. Spectrally accurate energy-preserving methods for the numerical solution of the "good" Boussinesq equation. Numerical Methods for Partial Differential Equations, 35(4), pp.1343-1362.

[57] Kishimoto, N., 2013. Sharp local well-posedness for the "good" Boussinesq equation. Journal of Differential Equations, 254(6), pp.2393-2433.

[58] Ostermann, A. and Su, C., 2019. Two exponential-type integrators for the" good" Boussinesq equation. arXiv preprint arXiv:1902.07478.

[59] Chen, Q., 2006. Fully nonlinear Boussinesq-type equations for waves and currents over porous beds. Journal of Engineering Mechanics, 132(2), pp.220-230.

[60] Daripa, P., 2006. Higher-order Boussinesq equations for two-way propagation of shallow water waves. European Journal of Mechanics-B/Fluids, 25(6), pp.1008-1021.

[61] Dingemans, M.W., 1997. Water wave propagation over uneven bottoms (Vol. 13). World Scientific.

[62] Ebadi, G., Johnson, S., Zerrad, E. and Biswas, A., 2012. Solitons and other nonlinear waves for the perturbed Boussinesq equation with

power law nonlinearity. Journal of King Saud University-Science, 24(3), pp.237-241.

[63] Engelbrecht, J., Tamm, K. and Peets, T., 2017. On solutions of a Boussinesq-type equation with displacement-dependent nonlinearities: the case of biomembranes. Philosophical Magazine, 97(12), pp.967-987.

[64] Fang, Y.F. and Grillakis, M.G., 1996. Existance and Uniqueness for Boussinesq type equations on a circle. Communications in Partial Differential Equations, 21(7-8), pp.1253-1277.

[65] Gandarias, M.L. and Bruzon, M.S., 1998. Classical and nonclassical symmetries of a generalized Boussinesq equation. Journal of Nonlinear Mathematical Physics, 5(1), pp.8-12.

[66] González-Gaxiola, O., 2017. The Laplace-Adomian Decomposition Method Applied to the Kundu-Eckhaus Equation. arXiv preprint arXiv:1704.07730.

[67] Griffiths, G. and Schiesser, W.E., 2010. Traveling wave analysis of partial differential equations: numerical and analytical methods with Matlab and Maple. Academic Press.

[68] Guo, B. and Shen, L., 1990. The global solution of initial value problem for nonlinear Schrödinger-Boussinesq equation in 3-dimensions. Acta Mathematicae Applicatae Sinica, 6(1), pp.11-21.

[69] Haldar, K., 2015. Decomposition Analysis Method in Linear and Nonlinear Differential Equations. Chapman and Hall/CRC.

[70] Hamdi, S., Enright, W.H., Ouellet, Y. and Schiesser, W.E., 2004. Exact solutions of extended Boussinesq equations. Numerical Algorithms, 37(1-4), pp.165-175.

[71] Heris, J.M., 2012. Solving the Integro-Differential Equations Using the Modified Laplace Adomian Decomposition Method. Journal of Mathematical Extension.

[72] Hosseini, K., Ansari, R. and Gholamin, A.P., 2012. Exact solutions of some nonlinear systems of partial differential equations by using the first integral method. Journal of Mathematical Analysis and Applications, 387(2), pp.807-814.

[73] Hussain, M. and Khan, M., 2010. Modified Laplace decomposition method. Appl. Math. Scie, 4, pp.1769-1783.

[74] Javidi, M. and Jalilian, Y., 2008. Exact solitary wave solution of Boussinesq equation by VIM. Chaos, Solitons & Fractals, 36(5), pp.1256-1260.

[75] Jia-Qi, M., 2009. A variational iteration solving method for a class of generalized Boussinesq equations. Chinese Physics Letters, 26(6), p.060202.

[76] Kaptsov, O.V., 1998. Construction of exact solutions of the Boussinesq equation. Journal of applied mechanics and technical physics, 39(3), pp.389-392.

[77] Khandelwal, R., Khandelwal, Y. and Chanchal, P., 2018. Mahgoub Deterioration Method and its Application in Solving Duo-combination of Nonlinear PDE's. Mathematical Journal of Interdisciplinary Sciences, 7(1), pp.37-44.

[78] Khandelwal, R., Kumawat, P. and Khandelwal, Y., 2019. Solution of the Blasius Equation by Using Adomian Kamal Transform. International Journal of Applied and Computational Mathematics, 5(1), p.20.

[79] Khandelwal, Y. and Umar, A.B., 2018. Solution of the Blasius Equation by using Adomain Mahgoub transform. International Journal of Mathematics Trends and Technology, 56(5), pp.303-306.

[80] Khusnutdinova, K.R. and Tranter, M.R., 2019. Weakly-nonlinear solution of coupled Boussinesq equations and radiating solitary waves. In Dynamical Processes in Generalized Continua and Structures (pp. 321-343). Springer, Cham.

[81] Kirby, J.T., 2003. Boussinesq models and applications to nearshore wave propagation, surf zone processes and wave-induced currents. In Elsevier Oceanography Series (Vol. 67, pp. 1-41). Elsevier.

[82] Li, S., 2014. Application of the bifurcation method to the modified Boussinesq equation. Electronic Journal of Qualitative Theory of Differential Equations, 2014(42), pp.1-14.

[83] Li, S., Chen, M. and Zhang, B.Y., 2018. Exact controllability and stability of the Sixth Order Boussinesq equation. arXiv preprint arXiv:1811.05943.

[84] Liang, Z., Li-Feng, Z. and Chong-Yin, L., 2008. Some new exact solutions of Jacobian elliptic function about the generalized Boussinesq equation and Boussinesq–Burgers equation. Chinese Physics B, 17(2), p.403.

[85] Linares, F., 2006. Notes on Boussinesq Equation. Inst. de Matemática Pura e Aplicada.

[86] Madsen, P.A. and Schäffer, H.A., 1998. Higher–order Boussinesq–type equations for surface gravity waves: derivation and analysis. Philosophical Transactions of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences, 356(1749), pp.3123-3181.

[87] Madsen, P.A., Bingham, H.B. and Schäffer, H.A., 2003. Boussinesq-type formulations for fully nonlinear and extremely dispersive water waves: derivation and analysis. Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences, 459(2033), pp.1075-1104.

[88] Madsen, P.A., Murray, R. and Sørensen, O.R., 1991. A new form of the Boussinesq equations with improved linear dispersion characteristics. Coastal engineering, 15(4), pp.371-388.

[89] Makhankov, V.G., 1978. Dynamics of classical solitons (in non-integrable systems). Physics reports, 35(1), pp.1-128.

[90] McKean, H.P., 1981. Boussinesq's equation on the circle. Communications on Pure and Applied Mathematics, 34(5), pp.599-691.

[91] Meyer, R.E. ed., 2013. Waves on Beaches and Resulting Sediment Transport: Proceedings of an Advanced Seminar, Conducted by the Mathematics Research Center, the University of Wisconsin, and the Coastal Engineering Research Center, US Army, at Madison, October 11–13, 1971 (Vol. 28). Elsevier.

[92] Natsis, D.G., 2007. Solitary wave solutions of the one-dimensional Boussinesq equations. Numerical Algorithms, 44(3), pp.281-289.

[93] Ogunlaran, O.M. and Sagay-Yusuf, H., 2016. Adomain Sumudu transform method for the Blasius equation. British Journal of Mathematics & Computer Science, 14(3), p.1.

[94] Ozawa, T. and Tsutsumi, Y., 2004. Lectures on nonlinear dispersive equations I.Hokkaido University technical report series in mathematics, 85, p.1.

[95] Palais, R.S., 2000. An introduction to wave equations and solitons. The Morningside Center of Mathematics, Chinese Academy of Sciences, Beijing.

[96] Parlange, J.Y., Hogarth, W.L., Govindaraju, R.S., Parlange, M.B. and Lockington, D., 2000. On an exact analytical solution of the Boussinesq equation. Transport in Porous Media, 39(3), pp.339-345.

[97] Patel, H.S. and Meher, R., 2015. Application of Laplace Adomian Decomposition Method for the soliton solutions of Boussinesq-Burger equations. Int. J. Adv. Appl. Math. and Mech, 3(2), pp.50-58.

[98] Pava, J.A., 2009. Nonlinear dispersive equations: existence and stability of solitary and periodic travelling wave solutions (No. 156). American Mathematical Soc.

[99] Peets, T., Tamm, K., Simson, P. and Engelbrecht, J., 2019. On solutions of a Boussinesq-type equation with displacement-dependent nonlinearity: a soliton doublet. Wave Motion, 85, pp.10-17.

[100] Polyanin, A.D. and Nazaikinskii, V.E., 2015. Handbook of linear partial differential equations for engineers and scientists. Chapman and hall/crc.

[101] Polyanin, A.D. and Zaitsev, V.F., 2016. Handbook of nonlinear partial differential equations. Chapman and Hall/CRC.

[102] Quispel, G.R.W., Nijhoff, F.W. and Capel, H.W., 1982. Linearization of the Boussinesq equation and the modified Boussinesq equation. Physics Letters A, 91(4), pp.143-145.

[103] Rafei, M., Ganji, D.D., Daniali, H.M. and Pashaei, H., 2007. Application of homotopy perturbation method to the RLW and generalized modified Boussinesq equations. Physics Letters A, 364(1), pp.1-6. Babolian, E. and Javadi, S., 2004. New method for calculating

Adomian polynomials. Applied Mathematics and Computation, 153(1), pp.253-259.

[104] Rafei, M., Ganji, D.D., Daniali, H.M. and Pashaei, H., 2007. Application of homotopy perturbation method to the RLW and generalized modified Boussinesq equations. Physics Letters A, 364(1), pp.1-6.

[105] Schaffer, H.A. and Madsen, P.A., 1970. A new formulation of higher order Boussinesq equations. WIT Transactions on the Built Environment, 10.

[106] Steinstraesser, J.G.C., Kemlin, G. and Rousseau, A., 2018. A domain decomposition method for linearized Boussinesq-type equations.

[107] Sulaiman, T.A., Bulut, H., Yokus, A. and Baskonus, H.M., 2019. On the exact and numerical solutions to the coupled Boussinesq equation arising in ocean engineering. Indian Journal of Physics, 93(5), pp.647-656.

[108] Veeresha, P., Prakasha, D.G., Qurashi, M.A. and Baleanu, D., 2019. An efficient technique for fractional modified Boussinesq and approximate long wave equations. arXiv preprint arXiv:1904.01230.

[109] Wang, S. and Xue, H., 2008. Global solution for a generalized Boussinesq equation. Applied Mathematics and Computation, 204(1), pp.130-136.

[110] Wazwaz, A.M., 2001. Construction of soliton solutions and periodic solutions of the Boussinesq equation by the modified decomposition method. Chaos, Solitons & Fractals, 12(8), pp.1549-1556.

[111] Wazwaz, A.M., 2007. The variational iteration method for rational solutions for KdV, K (2, 2), Burgers, and cubic Boussinesq equations. Journal of Computational and Applied Mathematics, 207(1), pp.18-23.

[112] Wazwaz, A.M., 2010. The combined Laplace transform–Adomian decomposition method for handling nonlinear Volterra integro–differential equations. Applied Mathematics and Computation, 216(4), pp.1304-1309.

[113] Ranjit R. Dhunde and Govind L. Waghmare.2019. Double Laplace iterative method for solving nonlinear partial differential equations. NTMSCI 7, No. 2, 138-149

[114] Wazwaz, A.M., 2015. A first course in integral equations. World Scientific Publishing Company.

[115] Whitham, G.B., 1975. Linear and nonlinear waves. Modern Book Incorporated.

[116] Yanovsky, I., 2005. Partial differential equations: Graduate level problems and solutions.

[117] Zill, D.G., 2012. Differential equations with boundary-value problems. Nelson Education.

[118] Abdul-Majid Wazwaz.2009. Partial Differential Equations and Solitary Waves Theory. Springer Dordrecht Heidelberg London New York

[119] Zou, Z.L., 1999. Higher order Boussinesq equations. Ocean Engineering, 26(8), pp.767-792.

[120] Zwillinger, D., 1998. Handbook of differential equations (Vol. 1). Gulf Professional Publishing.