**Sudan University of Science and Technology**

**College of Graduate Studies**


# Geometrical Fourier Transform and its Applications to Engineering Problems

# تحويل فوريير الهندسي وتطبيقاته على المسائل الهندسية

**A thesis submitted for fulfillments of the requirement of the degree of Ph.D. in mathematic**


**Submitted by**

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# Dedication

To my Family…

Kids…

And you…

# Acknowledgement

First, I would like to thank God, then I thank Professor Mohammed Ali Bashir for his supervision and his great efforts in completing this research, and Dr. BelgissAbdelaziz for her great assistance.

# Abstract

In this research we studied Fourier transform and Fourier Analysis. We first introduced an analytical formulation using Hilbert space. We utilized the principle of uniform boundedness and the open mapping theorem to establish the convergence of Fourier series and the existence of Fourier transform. Here the geometry of Hilbert space has been involved. Then we applied Fourier transform to Engineering problems, these include Motion group, Robotics, Statistical mechanics, Mass density, and frame density.

# خلاصة البحث

درسنا في هذا البحث تحويلات فورير وتحليل فورير. في البداية تحدثنا عن تحليل فورير باستخدام فضاء هيلبرت . وقد استخدمنا نظريتي المحدودية المنتظمة والتطبيق المفتوح لإنشاء تقارب متسلسلة فورير، ووجود تكامل فورير. حيث ادخلنا هندسة فضاء هيلبرت ، ومن ثم طبقنا تحويل فورير على الهندسة في عدة مجالات منها زمرة الحركة، الروبوتات ، الميكانيكا الإحصائية للجزيئات الكبيرة،كثافة الكتلة واطار الكثافة.

## Introduction:

The Fourier transform method, named after Joseph Fourier, a French mathematicianin 1801,to explainthe flow of heat around an anchor ring. It has become a powerful tool in diverse fields of science and engineering. It can provide a means of solving unwieldy equations that describe dynamic responses to electricity, heat or light. In some cases, it can also identify the regular contributions to a fluctuation signal. Fourier transform has become indispensable in the numerical calculations needed to design electrical circuits, to analyze mechanical vibrations, and to studywave propagation.

In mathematics, Fourier analysis is the study of the way general functions may be represented or approximated by sums of simpler trigonometric functions. Fourier analysis grew from the study ofFourier series, that showed representing a function as a sum of trigonometric functions greatly simplifies the study of heat propagation.

the subject of Fourier analysis encompasses a vast spectrum of mathematics. In the sciences and engineering, the process of decomposing a function into simpler pieces is often called Fourier analysis, while the operation of rebuilding the function from these pieces is known as Fourier synthesis. In mathematics, the term Fourier analysis often refers to the study of both operations.

The decomposition process itself is called a Fourier transform. The transform is often given a more specific name which depends upon the domain and other properties of the function being transformed. Moreover, the original concept of Fourier analysis has been extended over time to apply to more and more abstract and general situations, and the general field is often known as harmonic analysis.

Each transform used for analysis has a correspondinginverse transform that can be used for synthesis

Fourier transform techniques have been widely used to solve problems involving Simi-infinite or totally infinite range of the variables or unbounded regions. In order to deal with such problems, it is necessary to generalize Fourier series to include infinite intervals and to introduce the concept of Fourier integral, with many applications in physics and engineering. We review a number of engineering problems that can be posed or solved using Fourier transforms for the groups of rigid-body motions of the plane or three-dimensional space. Mathematically and computationally these problems can be divided into two classes: (1) physical problems that are described as degenerate difusions on motion groups; (2) enumeration problems in which fast Fourier transforms are used to efciently compute motion-group convolutions. We examine engineering problems including the analysis of noise in optical communication systems, the allowable positions and orientations reachable with a robot arm, and the statistical mechanics of polymer chains. In all of these cases, concepts from non-commutative harmonic analysis are put to use in addressing real-world problems, thus rendering them tractable.

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# Chapter One

# Introduction to Fourier Transform

## Introduction (1.1):

The motivation for the Fourier transform comes from the study ofFourier series. In the study of Fourier series, complicated but periodic functions are written as the sum of simple waves mathematically represented by sins and cosines. The Fourier transform is an extension of the Fourier series that results when the period of the represented function is lengthened and allowed to approach infinity.

Due to the properties of sine and cosine, it is possible to recover the amplitude of each wave in a Fourier series using an integral. In many cases it is desirable to use Euler's formula, which states that $e^{2\pi i\theta} = cos(2\pi\theta) + i\ sin(2\pi\theta)$, to write Fourier series in terms of the basic waves $e^{2\pi i\theta}$ This has the advantage of simplifying many of the formulas involved, and provides a formulation for Fourier series that more closely resembles the definition followed in this research. Re-writing sins and cosines as complex exponentials makes it necessary for the Fourier coefficients to be complex valued. The usual interpretation of this complex number is that it gives both the amplitude (or size) of the wave present in the function and the phase (or the initial angle) of the wave. These complex exponentials sometimes contain negative "frequencies". If θ is measured in seconds, then the waves $e^{2\pi i\theta}$ and $e^{-2\pi i\theta}$ both complete one cycle per second, but they represent different frequencies in the Fourier transform. Hence, frequency no longer measures the number of cycles per unit time, but is still closely related.

With the assumption that there exists a series expansion of the type

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \qquad (1)$$

$$a_n = \int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\, d \qquad n= 0,1,2,\ldots$$

$$b_n = \int_{-L}^{L} \sin\frac{n\pi t}{L}\, dt \qquad n= 1,2,\ldots$$

Valid in the interval $-L \leq x \leq L$ it is a simple matter to determine the coefficients, $a_n$ and $b_n$.Indeed, disregarding the question of validity of interchange of order of summation and integration, we proceed as follows.

Multiply each term of equation (1) by sin $\frac{t\pi x}{L}$ dx, where t is a positive integer. And then integrate each term from –L to +L thus arriving at

$$\int_{-L}^{L} f(x) \sin\frac{t\pi x}{L}\, dx = \frac{1}{2}a_0 \int_{-L}^{L} \sin\frac{t\pi x}{L}\, dx + \sum_{n=1}^{\infty}\left[a_n \int_{-L}^{L} cos\frac{n\pi x}{L}\ sin\frac{t\pi x}{L}\, dx + b_n \int_{-L}^{L} sin\frac{n\pi x}{L}\ sin\frac{t\pi x}{L}\, dx\right] (2)$$

As seen earlier,

$$\int_{-L}^{L} cos\frac{n\pi x}{L}\ sin\frac{t\pi x}{L}\, dx = 0 \qquad \text{for all t and n.}(3)$$

And

$$\int_{-L}^{L} sin\frac{n\pi x}{L}\ sin\frac{t\pi x}{L}\, dx = 0 \qquad \text{for } t \neq n; t, n = 1,2,3, \ldots(4)$$

Therefore each term on the right-hand side of equation (2) is zero except for the term n = t thus equation (2) reduces to

$$\int_{-L}^{L} f(x)\ \ sin\frac{t\pi x}{L}\, dx = b_t \int_{-L}^{L} sin^2\frac{t\pi x}{L}\, dx \qquad (5)$$

Since

$$\int_{-L}^{L} sin^2 \frac{t\pi x}{L}\ dx\ = \text{L}$$

We have

$b_t = \frac{1}{L}\int_{-L}^{L} f(x)\ \ sin\frac{t\pi x}{L}\ dx$ t= 1, 2, 3,…

From which the coefficients $b_n$ in equation (1) follow by mere replacement of t with n; that is

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\ \ sin\frac{n\pi x}{L}\ dx \qquad \text{n= 1, 2, 3,…(6)}$$

Let us obtain the $a_n$ in a like manner. Using the multiplier cos ($\frac{t\pi x}{L}$) dx throughout

$$\int_{-L}^{L} f(x)\ \ cos\frac{t\pi x}{L}\ dx = \frac{1}{2}a_0 \int_{-L}^{L} \cos\frac{t\pi x}{L}\ \text{dx} + \sum_{n=1}^{\infty}\left[a_n \int_{-L}^{L} cos\frac{n\pi x}{L}\ \ cos\frac{t\pi x}{L}\ dx + b_n \int_{-L}^{L} sin\frac{n\pi x}{L}\ \ cos\frac{t\pi x}{L}\ dx\right] \quad (7)$$

To coefficient $b_n$ in (7) is zero for all n and t. if t $\neq 0$ we know that

$$\int_{-L}^{L} cos\frac{n\pi x}{L}\ \ cos\frac{t\pi x}{L}\ dx = 0 \qquad \text{for n} \neq \text{t}$$

$= \text{L}$ for n $=$ t

And also the coefficient of $\frac{1}{2}a_0$ is zero. Such that for t ≠0, equation (7) reduces to

$$\int_{-\text{L}}^{\text{L}} \text{f(x)} \cos\frac{t\pi x}{L}\ \text{dx} = a_t \int_{-\text{L}}^{\text{L}} \cos^2\frac{t\pi x}{L}\ \text{dx}$$

From which $a_t$, and therefore $a_n$ , can be found in the way $b_t$ was determined

Thus we get

$$\frac{1}{L}\int_{-\text{L}}^{\text{L}} \text{f(x)} \cos\frac{n\pi x}{L}\ \text{dx}, \qquad \text{n = 1, 2, 3,} \qquad (8)$$

Next let us determine $a_0$. Suppose t = 0 in equation (7) so we have the equation

$$\int_{-L}^{L} f(x)\ dx = \frac{1}{2}a_0 \int_{-L}^{L} dx + \sum_{n=1}^{\infty}\left[a_n \int_{-L}^{L} cos\frac{n\pi x}{L}\ \ dx + \ b_n \int_{-L}^{L} sin\frac{n\pi x}{L}\ \ dx\right]$$

The term involving n ≥ 1are each zero. Hence

$$\int_{-L}^{L} f(x)\ dx = \frac{1}{2}a_0\ (2\text{L})$$

From which we obtain

$$a_0 = \int_{-L}^{L} f(x)\ dx \qquad (9)$$

Equation (9) fits in with equation (8) as the special case n = 0 had the factor $\frac{1}{2}$ not been inserted in equation (1), a separate formula would have been needed. As it is, we may write the formal expansion as follows:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \qquad (10)$$

With

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi X}{L}\ dx \qquad n= 0, 1, 2, \quad (11)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} F(x)\sin\frac{n\pi X}{L}\ dx \qquad n= 1, 2, \quad (12)$$

Before proceeding to specific examples and applications it behooves us to state conditions under which the equality in (10) makes sense.

When $a_n$ and $b_n$ are given by (11) and (12) above, then the right-hand member of equation (10) is called the Fourier series, over the interval $-L \le x \le L$ for the function f(x). A statement of condition sufficient to insure that the Fourier series in (10) represent the function f(x) in a reasonably meaningful manner follows.

Let f(x) be continuous and differentiable at every point in the interval

$-L \le x \le L$ Except for, at most, a finite number of points, and at those points let $f(x)$ and $f'$ (x) have right-hand and left-hand limits.[39]

**Theorem (1.2):**

Under the stipulations of the preceding paragraph, the Fourier series $f(x)$, namely the series on the right in equation (10) with coefficients given by equation (1) and (12), converges to the value $f(x)$ at each point of continuityof $f(x)$; at each point of discontinuity of $f(x)$ the Fourier series converges to arithmetic mean of the value approached by $f(x)$ from the right and the left.

Since the Fourier series for $f(x)$ may not converge to the value $f(x)$ everywhere (for instance, at discontinuities of the function), it is customary to replace the equals sine in equation (10) by the symbol$\sim$, which may be read "has for its Fourier series." We write

$$f(x)\sim\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right)$$

With $a_n$ and $b_n$ given by equation (11) and (12).

An interesting fact and one often useful as a check in numerical problems is that $\frac{1}{2}a_0$ is the average value of $f(x)$ over the interval $-L<x<L$ .

The sine and cosine functions are periodic with period $2\pi$, so the term in the Fourier series (12) for $f(x)$ are periodic with period $2L$ therefore the series represents a function that is as described above for the interval $-L<x<L$ and repeats that structure over and over outside that interval, the corresponding Fourier series would converge to the periodic function. Not the convergence to the average value at discontinuities, the periodicity, and the way in which the two together determine the value to which the series converges at $x=L$ and$x=-L$.

We have an expansion of a function $f(x)$ in a series involving only sine functions, the expansion to represent, the original $f(x)$ in an interval $0 < x < L$ with the notation we have been using, the Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

Will reduce to a series with each term containing a sine function if somehow the $a_n$,where n = 1,2,3,… , can be made to be zero. Examining the formula for $a_n$, reveals that the $a_n$ will vanish if the function being expanded is an odd function over the interval $-L < x < L$.

Therefore, to get a sine series for $f(x)$ we introduce a new function $g(x)$ defined to equal $f(x)$ in the interval $0 < x < L$ and to be the odd extension of that function in the remaining interval, $-L < x < 0$. That is, we define $g(x)$ by

$$g(x) = f(x), \qquad 0 < x < L$$

$$= -f(-x) \qquad -L < x < 0$$

Then g(x) is an odd function over the interval $-L < x < L$. Hence from

$$\mathrm{g}(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

it follows that

$$a_n = \frac{1}{\mathrm{L}}\int_{-\mathrm{L}}^{\mathrm{L}} \mathrm{g}(x) \cos\frac{\mathrm{n\pi x}}{\mathrm{L}}\,\mathrm{dx} = 0 \qquad \mathrm{n}= 0, 1, 2,$$

$$b_n = \frac{1}{\mathrm{L}}\int_{-\mathrm{L}}^{\mathrm{L}} \mathrm{g}(x) \sin\frac{n\pi x}{L}\,\mathrm{dx} = \frac{2}{\mathrm{L}}\int_{0}^{\mathrm{L}} \mathrm{f}(x) \sin\frac{n\pi x}{L}\,\mathrm{dx}$$

The resultant series represent f(x) in the interval 0<x<L, because g(x) and f(x)are identical over that portion of the whole interval.

Thus we have

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \qquad 0<x<L \quad (13)$$

In which

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi X}{L}\,dx, \qquad n= 1, 2, 3, \quad (14)$$

The representation (13)is called the Fourier sine series for f(x) over the interval $0 < x < L$.

It should be realized that the device of introducing the function g(x) was a tool for arriving at (13) and (14). Those we handle by direct use of (13) and (14) above.

**Example (1):**

Expand $f(x) = x^2$ in a Fourier sine series over the interval$0 < x < 1$.

At once we may write. For $0 < x < 1$,

$$x^2 \sim \sum_{n=1}^{\infty} b_n \sin n\pi x, \qquad (15)$$

In which

$$b_n = 2\int_0^1 x^2 \sin n\pi x \,dx$$

$$= 2\left[-\frac{x^2\cos n\pi x}{n\pi} + \frac{2x\sin n\pi x}{(n\pi)^2} + \frac{2\cos n\pi x}{(n\pi)^3}\right]_0^1$$

$$= 2\left[-\frac{\cos n\pi}{n\pi} + \frac{2\cos n\pi}{n^3\pi^3} - \frac{2}{n^3\pi^3}\right] \qquad (16)$$

Hence the Fourier sine series, over 0<x<1, for $x^2$ is

$$x^2 \sim 2\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{n\pi} - \frac{2[1-(-1)^n]}{n^3\pi^3}\right]\sin n\pi x, \qquad (17)$$

The series on the right in (17) this function being called the odd periodic extension, with period 2, of the function

$$f(x) = x^2, \qquad 0 < x < 1$$

In a manner entirely to that used to obtain the Fourier sine series, it is possible to obtain a series of cosine terms, including a constant term, for a function defined over the interval$0 < x < L$. Indeed, given $f(x)$ defined over the interval

$0 < x < L$,We may define an auxiliary function $h(x)$by

$$h(x) = f(x), \qquad 0 < x < L$$

$$= f(-x) \qquad -L < x < 0$$

Then $h(x)$ is an even function of x and, of course, it is equal to $f(x)$ over the interval where $f(x)$ was defined. Since $h(x)$ is even, it follows that in its ordinary Fourier expansion over the interval$-L < x < L$, the $b_n$are all zero.

$$b_n = \frac{1}{L}\int_{-L}^{L} h(x) \sin\frac{n\pi X}{L} \, dx = 0$$

Because of the oddness of the integrand. Furthermore, since $h(x)$ is even, $h(x)$ cos $(\frac{n\pi X}{L})$ is also even and

$$a_n = \frac{2}{L}\int_0^L h(x) \cos\frac{n\pi x}{L} \, dx = \frac{2}{L}\int_0^L f(x) \cos\frac{n\pi x}{L} \, dx,$$

Since h(x)and f(x) are identical over the interval$0 < x < L$, we may write what is customarily called the Fourier cosine series for $f(x)$ over that interval, namely

$$F(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \qquad 0<x<L, \qquad (18)$$

In which

$$a_n = \frac{2}{L}\int_0^L f(x) \cos\frac{n\pi x}{L} \, dx, \qquad (19)$$

**Example (2):**

Find the Fourier cosine series over the interval 0<x<L for the function

$$f(x) = x.$$

At once we have

$$F(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

In which

$$a_n = \frac{2}{L} \int_0^L x \cos \frac{n\pi x}{L} \, dx$$

For n$\neq 0$, the $a_n$ may be evaluated as follows:

$$a_n = \frac{2}{L} \left[ \frac{L}{n\pi} x \sin \frac{n\pi x}{L} + \left( \frac{L}{n\pi} \right)^2 \cos \frac{n\pi x}{L} \right]_0^L$$

$$= \frac{2}{L} \left[ \left( \frac{L}{n\pi} \right)^2 \cos n\pi - \left( \frac{L}{n\pi} \right)^2 \right]$$

$$= - \frac{2L}{n^2\pi^2} (1 - \cos n\pi), \qquad n \neq 0$$

The remaining coefficient $a_0$ is readily obtained:

$a_0 = \frac{2}{L} \int_0^L x \, dx = \frac{2}{L} . \frac{L^2}{2} = L.$

Thus the Fourier cosine series over the interval 0<x<L for the function f(x) = x is

$$F(x) \sim \frac{1}{2} L - \frac{2L}{\pi^2} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^2} \cos \frac{n\pi x}{L}$$

Which may also be written in the form

$$f(x) \sim \frac{1}{2} L - \frac{4L}{\pi^2} \sum_{t=0}^{\infty} \frac{\cos [(2t+1)\pi x / L]}{(2t+1)^2} \qquad (20)$$

The infinite series on the right in (3) converges to a function which is often called the even periodic extension of the function $f(x) = x$.

The Fourier transformis very commonly used it transforms a mathematical function of time, f $(t)$ into a new function, sometimes denoted by $f^{\wedge}$ or F, whose argument is frequency with units of cycles or radians per second. The new function is then known as the Fourier transform and/or the frequency spectrum of the functionf TheFourier transform is also a reversible operation. Thus, given the function $f^{\wedge}$ one can determine the original function,f (See Fourier inversion theorem.) f and $f^{\wedge}$ are also respectively known as time domain and frequency domain representations of the same "event". Most often perhaps, f is a real-valued function, and $f^{\wedge}$is complex valued, where a complex number describes both the amplitude and phase of a corresponding frequency component. In general, f is also complex, such as the analytic representation of a real-valued function. The term "Fourier transform" refers to both the transform operation and to the complex-valued function it produces.

In the case of a periodic function (for example, a continuous but not necessarily sinusoidal musical sound), the Fourier transform can be simplified to the calculation of a discrete set of complex amplitudes, called Fourier series coefficients. Also, when a time-domain function is sampled to facilitate storage or computer-processing, it is still possible to recreate a version of the original Fourier transform according to the Poisson summation formula, also known as discrete-time Fourier transform. For an overview of those and other related operations, refer to Fourier analysis or List of Fourier-related transforms.[6]

**Definition (1.3): Fourier Transform:**

There are several common conventions for defining the Fourier transform F of an integral function f:**R** → **C**. We will use the definition:

$$F(\omega) = F(f)(x) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i \omega x}\, dx, \quad \text{for every real number } \omega.$$

Then $F^{-1}(\omega) = f(x) = \int_{-\infty}^{\infty} F(\omega)e^{2\pi i\omega x}$ where $F^{-1}(\omega)$ is the inverse Fourier transform.

When the independent variable $x$ represents time (with SI unit of seconds), the transform variable $\omega$ represents frequency (in hertz). Under suitable conditions, f is determined by F via the inverse transform:

$$\mathrm{F}\,(x) = \int_{-\infty}^{\infty} \mathrm{F}(\omega)\ \ \mathrm{e}^{2\pi \mathrm{i}\omega x}\ \mathrm{d}\omega, \text{for every real number } x.$$

The statement that f can be reconstructed from F is known as the Fourier inversion theorem, and was first introduced in Fourier's Analytical Theoryof Heat although what would be considered a proof by modern standards was not given until much later. The functions f and F often are referred to as *a* Fourier integral pair or Fourier transform pair.

There is a close connection between the definition of Fourier series and the Fourier transform for functions ofwhich are zero outside of an interval. For such a function, we can calculate its Fourier series on any interval that includes the points where f is not identically zero. The Fourier transform is also defined for such a function. As we increase the length of the interval on which we calculate the Fourier series, then the Fourier series coefficients begin to look like the Fourier transform and the sum of the Fourier series of f begins to look like the inverse Fourier transform. To explain this more precisely, suppose that L is large enough so that the interval [$-L/2$,$L/2$] contains the interval on which f is not identically zero. Then the $n$-the series coefficient $c_n$ is given by**:**

$$c_n = \int_{\frac{-L}{2}}^{\frac{L}{2}} f(x)e^{-2\pi i(n/L)x}\,dx$$

Comparing this to the definition of the Fourier transform, it follows that $c_{an}$ = F($n$/$L$) since $f(x)$ is zero outside $[-L/2, L/2]$. Thus the Fourier coefficients are just the values of the Fourier transform sampled on a grid of width $1/L$. As $L$ increases the Fourier coefficients more closely represent the Fourier transform of the function.

Under appropriate conditions, the sum of the Fourier series of f will equal the function f. In other words, f can be written**:**

$$f(x) = \frac{1}{L}\sum_{n=-\infty}^{\infty} F(n/L)\, e^{2\pi i (n/L)x} = \sum_{n=-\infty}^{\infty} F(\omega_n)\, e^{2\pi i (\omega_n)x}\ \Delta\omega$$

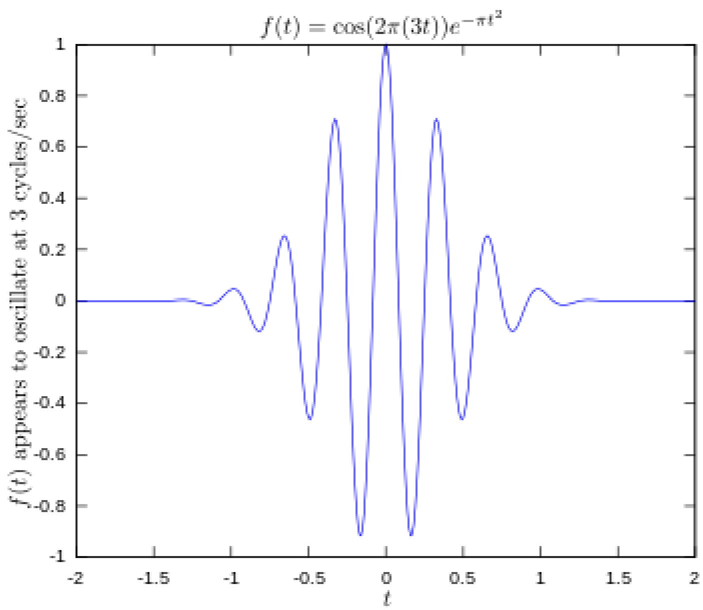
Where the last sum is simply the first sum rewritten using the definitions $\omega_n$ = $n/L$, and $\Delta\omega = (n + 1)/L - n/L = 1/L$.

This second sum is a Riemann sum, and so by letting $L \to \infty$ it will converge to the integral for the inverse Fourier transform given in the definition section. Under suitable conditions this argument may be made precise.

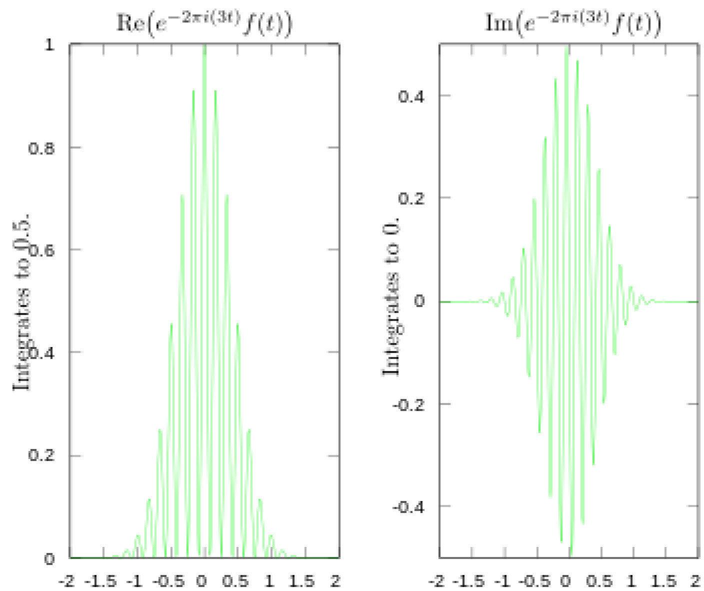
In the study of Fourier series the numbers $c_n$ could be thought of as the "amount" of the wave present in the Fourier series of f. Similarly, as seen above, the Fourier transform can be thought of as a function that measures how much of each individual frequency is present in our function f, and we can recombine these waves by using an integral (or "continuous sum") to reproduce the original function.

**Example (3):**

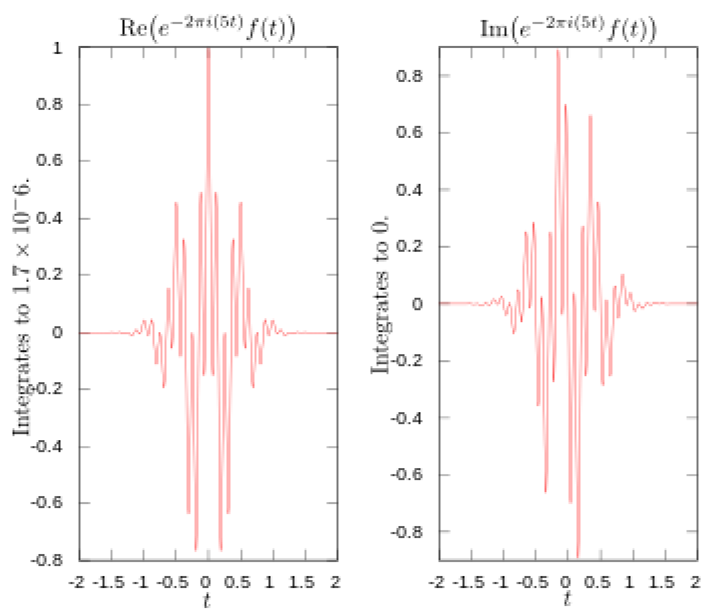
The following images provide a visual illustration of how the Fourier transform measures whether a frequency is present in a particular function. The function depicted $f(t) = cos\,(6\pi t)\; e - \pi t 2$ oscillates at 3 hertz (if $t$ measures seconds) and tends quickly to 0. (The second factor in this equation is an envelope function that shapes the continuous sinusoid into a short pulse. Its general form is a Gaussian function). This function was specially chosen to have a real Fourier transform which can easily be plotted. The first image contains its graph. In order to calculate F (3) we must integrate $e^{-2\pi I\,(3t)}\,f\,(t)$. The second image shows the plot of the real and imaginary parts of this function. The real part of the integrand is almost always positive, because when $f\,(t)$ is negative, the real part of $e^{-2\pi I\,(3t)}$ is negative as well. Because they oscillate at the same rate, when $f\,(t)$ is positive, so is the real part of $e^{-2\pi I\,(3t)}$. The result is that when you integrate the real part of the integrand you get a relatively large number (in this case 0.5). On the other hand, when you try to measure a frequency that is not present, as in the case when we look at F (5), the integrand oscillates enough so that the integral is very small. The general situation may be a bit more complicated than this, but this in spirit is how the Fourier transform measures how much of an individual frequency is present in a function $f(t)$.

- 


Original function showing oscillation 3 hertz.

- 


Real and imaginary parts of integrand for Fourier transform at 3 hertz

- 


Real and imaginary parts of integrand for Fourier transform at 5 hertz

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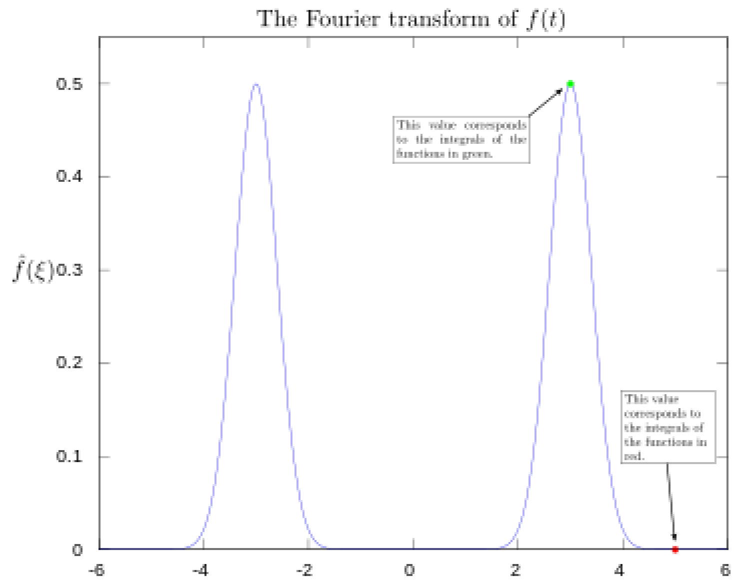



Fourier transform with 3 and 5 hertz labeled.

As would be expected, Fourier transform have many properties analogous to those of Fourier series with regard to the connection between transforms of related functions. As previously we denote FT {f (t)} by g ($\omega$). The unfamiliar last term is discussed below.

- Differentiation $FT\{f`(t)\} = i\omega g(\omega)$ (21)
- Integration $FT\{\int^t f(s)ds\} = -i\omega^{-1} g(\omega) + 2\pi c h(\omega)$(22)
- Translation $FT\{f(t + a)\} = e^{ia\omega} g(\omega)$ (23)
- Exponential multiplication $FT\{e^{\alpha t} f(t)\} = g(\omega + i\alpha)$ (24)

In (24)($\propto$) may be real, imaginary or complex. The last term $2\pi c$h ($\omega$) in (22) represents the FT of the constant of integration associated with the definition of the indefinite integral.

Here we assume$f(x)$, $g(x)$ and $h(x)$ are integral functions, are Lévesque-measurable on the real line, and satisfy:

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty.$$

We denote the Fourier transforms of these functions by F($\omega$), G($\omega$), and H($\omega$) respectively.

The Fourier transform has the following basic properties:

**Linearity:**

For any complex numbers$a$ and $b$, if $h(x) = af(x) + bg(x),$ then

$$H(\omega) = a\,.F(\omega) + b\,.G(\omega),$$

**Translation:**

For any real number$x_0$, if $h(x) = f(x - x_0)$, then

$$H(\omega) = e^{-2\pi i x_0 \omega} F(\omega)$$

**Modulation:**

For any real number$\omega_0$if h($x$) = $e^{2\pi i x \omega_0} f(x)$then

$$H(\omega) = F(\omega - \omega_0)$$

**Scaling:**

For a non-zero real number$a$, if $h(x) = f(ax)$, then H($\omega$) = $\frac{1}{|a|} F\left(\frac{\omega}{a}\right)$

The case $a$ = -1 leads to the *time-reversal* property, which states: if $h(x) = f(-x)$, then$H(\omega) = F(-\omega)$

**Conjugation:**

If $h(x) = \overline{f(x)},$ then $H(\omega) = \overline{F(-\omega)}$

In particular, if $f$ is real, then one has the *reality condition*$F(-\omega) = \overline{F(\omega)}$

And if $f$ is purely imaginary, then $F(-\omega) = -\overline{F(\omega)}$

In mathematics, the Fourier sine transform is a special case of the continuous Fourier transform, arising naturally when attempting to transform an odd function. Consider the general Fourier transform:

$$F(\omega) = F(f)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\ e^{-i\omega t}\ dt.$$

We may expand the integral by means of Euler's formula:

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\,(\cos \omega t - i \sin \omega t)\ dt,$$

Or written as the sum of two integrals:

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cos \omega t\ dt - \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \sin \omega t\ dt,$$

Now notice that if we assume f(t) is an odd function, the product f(t) cos$\omega t$ is also odd whilst the product f(t)sin $\omega t$ is an even function. Since we are integrating over an interval symmetric about the origin (i.e. -∞, +∞), the firest integral must vanish to zero, and the second may be simplified to give:

$$F(\omega) = -i\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t) \sin \omega t\ dt,$$

Which is the Fourier sine transform for odd f(t). it is clear that the transformed function F($\omega$) is also an odd function, and similar analysis of the general inverse Fourier transform yields a second sine transform, namely:

$$f(t) = i\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F(\omega) \sin \omega t\ d\omega,$$

Note that the numerical factors in the transforms are defined uniquely only their product, as discussed for general continuous Fourier transform.for this reason the imaginary units i and –i can be omitted, with the more commonly seen forms of the Fourier sine transforms being:

$$F(\omega) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t) \sin \omega t\ dt,$$

And

$$f(t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F(\omega) \sin \omega t \; d\omega,$$

In mathematics, the Fourier cosine transform is a special case of the continuous Fourier transform, arising naturally when attempting to transform an even function. Consider the general Fourier transform:

$$F(\omega) = F(f)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \; e^{-i\omega t} \; dt$$

We may expand the integral by means of Euler's formula:

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) (\cos \omega t - i \sin \omega t) \, dt,$$

Or written as the sum of two integrals:

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cos \omega t \; dt - \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \sin \omega t \; dt,$$

Now notice that if we assume f(t) is an even function, the product f(t) cos$\omega t$ is also even whilst the product f(t)sin $\omega t$ is an odd function. Since we are integrating over an interval symmetric about the origin (i.e. -∞, +∞), the second integral must vanish to zero, and the first may be simplified to give:

$$F(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos \omega t \; dt,$$

Which is the Fourier cosine transform for even f(t). it is clear that the transformed function F($\omega$) IS also an even function, and similar analysis of the integral inverse Fourier transform yields a second cosine transform, namely:

$$f(t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F(\omega) \cos \omega t \; d\omega,$$

Note that the numerical factors in the transforms are defined uniquely only their product, as discussed for general continuous Fourier transform.

The Fourier transform translates between convolution and multiplication of functions. If $f(x)$ and $g(x)$ are integrable functions with Fourier transforms $F(\omega)$, and$G(\omega)$, respectively, then the Fourier transform of the convolution is given by the product of the Fourier transforms $F(\omega)$, and$G(\omega)$, (under other conventions for the definition of the Fourier transform a constant factor may appear).

This means that if**:**

$$h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy,$$

where($*$) denotes the convolution operation, Proof the product

$$f * g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)g(x-y)dy$$

Is called the convolution of the functions f and g over the interval (-∞ , ∞). The the Fourier transform of this convolution integral yields

$$F\,[(f * g);\ \omega] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} \int_{-\infty}^{\infty} f(y)g(x-y)dy\text{dx}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\omega x}\ f(y)g(x-y)dy\ \text{dx}$$

Since f and g are absolutely integrable, the order of integration can be interchanged and, therefore,

$$F\,[(f * g);\ \omega] = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y)\ [\int_{-\infty}^{\infty} g(x-y)e^{i\omega(x-y)}e^{i\omega y}\ \text{dx}\ ]\ dy$$

Let $x - y = u$.then $dx = du$. therefore,

$$F\,[(f * g);\ \omega] = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y)\ [e^{i\omega y} \int_{-\infty}^{\infty} g(y)e^{i\omega u}\ \text{du}\ ]\ dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega y} f(y) dy \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega u} g(u) du$$

$$= F(\omega) \,.\, G(\omega)$$

Hence the theorem is proved.

then**:**

$$H(\omega) = F(\omega) \,.\, G(\omega),$$

In linear time invariant (LTI) system theory, it is common to interpret $g(x)$ as the impulse response of an LTI system with input $f(x)$ and output $h(x)$, since substituting the unit impulse for $f(x)$ yields $h(x) = g(x)$. In this case, G($\omega$)representsthefrequency response of the system.

Conversely, if $f(x)$ can be decomposed as the product of two square integrable functions $p(x)$ and $q(x)$, then the Fourier transform of $f(x)$ is given by the convolution of the respective Fourier transforms$P(\omega)$and Q($\omega$).

We can verify that $f * g = g * f$, i,e.,

$$f * g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) g(x-y) dy$$

Setting $x - y = \omega$, we have $dy = -d\,\omega. then$

$$f * g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - \omega) g(\omega) d\omega = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y) f(x - y) dy = g * f$$

Hence the convolution is commutative.

The Sine and cosine convolution integrals

1-$\int_0^\infty F_C(\omega)G_C(\omega)\ COS\ \omega x\ d\omega = \sqrt{\frac{2}{\pi}}\int_0^\infty F_C(\omega)\ COS\ \omega x\ d\omega\ \int_0^\infty g(\alpha)\ COS\alpha\omega\ d\alpha$

$= \sqrt{\frac{2}{\pi}}\int_0^\infty g(\alpha)\ d\alpha\ \int_0^\infty F_C(\omega)\ COS\ \omega x\ COS\alpha\omega\ d\omega$

$= \sqrt{\frac{1}{2\pi}}\int_0^\infty g(\alpha)\ d\alpha\ \int_0^\infty F_C(\omega)\ [COS\ |x-\ \alpha|\ \omega +\ COS\ (x+\alpha\ )\omega]\ d\omega$

But,

$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_C(\omega)\ COS\ \omega x\ d\omega\ =\ \frac{1}{2}\int_0^\infty g(\alpha)\ d\alpha\ [f(|x-\ \alpha|)\ + f(x+\alpha)]$

2- if$F_s(\omega)$ and $G_s(\omega)$ are the Fourier sine transforms of f(x) and g(x), then we can show that

$\int_0^\infty F_s(\omega)G_s(\omega)\ sin\ \omega x\ d\omega\ = \frac{1}{2}\int_0^\infty f(\alpha)\,[g(|x-\ \alpha|) - g(x+\alpha)]\ d\alpha$

3- $\int_0^\infty F_c(\omega)G_s(\omega) sin\ \omega x\ d\omega = \sqrt{\frac{2}{\pi}}\int_0^\infty F_C(\omega)\ sin\ \omega x\ d\omega\ \int_0^\infty g(\alpha) \sin\alpha\omega\ d\alpha$

$= \sqrt{\frac{2}{\pi}}\int_0^\infty g(\alpha)\ d\alpha\ \int_0^\infty F_C(\omega)\ sin\ \omega x\ \sin\alpha\omega\, d\omega$

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty g(\alpha)\ d\alpha\ \int_0^\infty F_C(\omega)\ [COS\ |x-\ \alpha|\ \omega -\ COS\ (x+\alpha\ )\omega]\ d\omega$$

$= \frac{1}{2}\int_0^\infty g(\alpha)\,[f(|x-\ \alpha|) - f(x+\alpha)]\ d\alpha$

4- $\int_0^\infty F_s(\omega)G_c(\omega)sin\,\omega x\,d\omega = \frac{1}{2}\int_0^\infty f(\alpha)\,[g(|x-\ \alpha|) - g(x+\alpha)]\,d\alpha$

## Definition (1.4): Heaviside step function:

the Heaviside step function, H, also called the unit step function, is a discontinuous function whose value is zero for negative argument and one for positive argument. It seldom matters what value is used for $H(0)$, since H is mostly used as distribution. Some common choices can be seen below.

The function is used in the mathematics of control theory and signal processing to represent a signal that switches on at a specified time and stays switches onat a specified time and stays switched on indefinitely. It was named after the English polymath Oliver Heaviside.[3]

It is the cumulative distribution function of a random variable which is almost surely 0. The Heaviside function is the integral of the Dirac delta function:

$H' = \delta$. This is sometimes written as

$$H(x) = \int_{-\infty}^{\infty} \delta(t)dt$$

Although this expansion may not hold for x = 0, depending on which formalism one uses to give meaning to integrals involving $\delta$

The Fourier transform of the Heaviside step function is a distribution. Using one choice of constants for the definition of the Fourier transform we have

$$\text{H\^{}(s)} = \int_{-\infty}^{\infty} e^{-2\pi ixs}\ H(x)dx = \frac{1}{2}\left(\delta(\text{s}) - \frac{i}{\pi s}\right)$$

Here the $\frac{1}{s}$ term must be interprteted as a distribution that takes a test function $\varphi$ to the Cauchy principal value of

$$\int_{-\infty}^{\infty} \frac{\varphi(x)}{x}$$

Using Fourier  transforms, one finds that

$$\int_{-\infty}^{\infty} 1 \,.\, e^{-2\pi i f t} \; dt$$

And therefore:

$$\int_{-\infty}^{\infty} 1 \,.\, e^{2\pi i f\,1 t}\left(e^{2\pi i f\,2 t}\right)^{*} dt = \int_{-\infty}^{\infty} \; e^{-2\pi i\,(f2-f1)t} \; dt = \delta\;(f2 - f1)$$

Which is a statement of the orthogonality property for the Fourier kernel. Equating these non-converging improper integrals to $\delta$(x)  is not mathematically rigorous. However, they behave in the same way under a definite integral. That

$$\int_{-\infty}^{\infty} F(f)(\int_{-\infty}^{\infty} \; e^{-2\pi i f\,(t)}\text{dt})\text{ df} = \text{f}(0)$$

According to the definition of the Fourier transform. Therefore, the bracketed term is considered equivalent to the Dirac delta function.[26]

# Chapter Two

# Some Classical Applications of Fourier Transform

## Introduction (2.1):

We discus some applicationsof Fourier transform to differential equationsandintegeral equations.

Consider the nth order linear ordinary differential equation with constant coefficients

$$Ly(t) = f(t) \qquad (1)$$

Where L is the nth order differential operator given by

$$L = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0$$

$a_0, a_1, \ldots, a_n$ are constants, $D = \frac{d}{dt}$, and $f \in L^1(\mathbb{R})$ or $f \in L^2(\mathbb{R})$. Application of the Fourier transform to both sides of (1) gives

$$[a_n i\omega^n + a_{n-1} i\omega^{n-1} + \cdots + a_1 i\omega + a_0]\hat{y}(\omega) = \hat{f}(\omega)$$

Or

$$\hat{p}(i\omega)\hat{y}(\omega) = \hat{f}(\omega)$$

Where $\hat{p}(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$

Thus;

$$\hat{y}(\omega) = \frac{\hat{f}(\omega)}{\hat{p}(i\omega)} = \hat{f}(\omega)\hat{g}(\omega)$$

Where

$$\hat{g}(\omega) = \frac{1}{\hat{p}(i\omega)}$$

Now the convolution theorem gives the solution

$$y(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi)\, g(t - \xi) d\xi$$

Provided $g(t) = \mathcal{F}^{-1}\{\hat{g}(\omega)\}$ is known explicity.

To give a physical interpretation of the result, we consider the differential equation associated with a sudden impulse function $f(t) = \delta(t)$.

$$LG(t) = \delta(t)$$

Application of Fourier transform to this equation yields the solution

$$G(t) = \mathcal{F}^{-1}\left\{\frac{1}{\sqrt{2\pi}}\hat{g}(\omega)\right\} = \frac{1}{\sqrt{2\pi}}g(t)$$

Now the above solution can be written as

$$y(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)\, G(t-\xi)d\xi$$

Clearly, $G(t)$ behaves like a Green's function, that is, it is the response to a unit impulse. In any physical system, $f(t)$ is usually called the input function, while y(t) is called the output obtained by the superposition principle. The Fourier transform of $\sqrt{2\pi}G(t)$ is called the admittance $\hat{g}(\omega) = (\hat{p}(i\omega))^{-1}$. To determine the response to a given input, we first find the Fourier transform of the input, multiply the result by the admittance, and then apply the inverse Fourier transform to the product.[21]

**Example (1):**

The electric current I(t) in the circuit is governed by the equation

$$L\frac{dI}{dt} + RI = E \qquad (2)$$

Where L is the inductance, R is the resistance, and E is the applied electromagnetic force. With $E(t) = E_0 e^{-|t|}$, application of the Fourier transform (with respect to t) to equation (2) gives

$$(i\omega L + R)\hat{I}(\omega) = \sqrt{\frac{2}{\pi}}\frac{E_0}{1+\omega^2}$$

Or

$$\hat{I}(\omega) = \sqrt{\frac{2}{\pi}} \frac{E_0}{(i\omega L + R)(1 + \omega^2)}$$

The inverse Fourier transform yields

$$I(t) = \frac{E_0}{\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega t}\, d\omega}{(i\omega L + R)(1 + \omega^2)}$$

This integral can readily be evaluated by the theory of residues. for$t > 0$,

$$I(t) = \frac{E_0}{\pi} 2\pi i \left( [residue\ at\ \omega = i] + [residue\ at\ \omega = \frac{iR}{L}] \right)$$

$$= 2iE_0 \left( \frac{e^{-t}}{2i(R - L)} + \frac{e^{-Rt/L}}{iL(1 - R^2/L^2)} \right)$$

$$= E_0 \left( \frac{e^{-t}}{R - L} + \frac{2Le^{-Rt/L}}{R^2 - L^2} \right)$$

Similarly, for $t < 0$, we optain

$$I(t) = -\frac{E_0}{\pi} .2\pi i [residue\ at\ \omega = -i] = \frac{E_0 e^t}{L + R}$$

At t=0, the current is continuous, hence

$$I(0) = \lim_{t \to 0} I(t) = \frac{E_0}{L + R}$$

## Wave equation (2.2):

The wave equation as an evolution equation, I.e. an ordinary differential equation describing the evolution over time of any initial distribution so it is an ordinary differential equation with values in the vector space of distributions.

The bottom line of the wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$

This is the differential equation that describes the propagation of dissipationless, dispersionless waves. This derivation will be inductive and general. For each specific case like tension waves on a string or sound waves in the air, it is also possible to give a detailed deductive derivation that applies to that physical system.

We begin with observation of waves on the hose or on the rods. We see that to a fair approximation, they have two properties. First the waves do not change size as they propagate. Actually they do slowly get smaller with time, but we can imagine doing a better job of eliminating the "friction" so that they approach the limit of propagating on indefinitely without getting smaller. This is the idealization of waves without dissipation. Second we see two other properties that are equivalent. The waves keep the same shape as they propagate, and all shapes have the same speed. Waves with this property are said to have no dispersion. (for media with dispersion, waves of different wavelength have different speeds. An example is light in glass. That is why a prism can separate the colors of the rainbow in white light.) our demonstration examples are a better approximation to the idealization of no dispersion than they are to that of no dissipation.

This means that if we know the shape of the wave at one time, then we know it at later time y just moving it over by a distance ct.

This can e expressed in equations as follows: suppose that at $t = 0$, the shape of the waves is given by a function $f(x)$ so that $u(x, 0) = f(x)$ and the function $u(x, t)$ at later time. Since it just gets moved over by ct, we have

$u(x, t) = u(x - ct, 0) = f(x - ct)$. Suppose the function $f(x)$ has peak at

$x = x_0$. Then with the form above, the peak will be where $x - ct = x_0$ or

$x = x_0 + ct$, i.e. the peak moves to the right with speed c. all this is independent of the actual form of $f(x)$.we assert that the equation we are are looking for must have the property that for any function $f(.), f(x - ct)$is a solution. But we don't want to be stinsist that for any$g(.)$, a left-moving wave of the form $g(x + ct)$ must also be a solution with this input, we can figure out what the form of the differential equation describing the waves must be

$$\frac{\partial u}{\partial x} = f^{'}(x - ct)\frac{\partial^2 u}{\partial x^2} = f^{''}(x - ct)$$

$$\frac{\partial u}{\partial t} = -cf^{'}(x - ct)\frac{\partial^2 u}{\partial t^2} = c^2 f^{''}(x - ct)$$

So that $\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$ as advertised. A similar calculation with f(x-ct) replaced by $g(x + ct)$ gives the same equation for $u(x, t)$. we generalizing the wave equation in three dimensions. a function that describes the disturbance is now a function of $x, y, z$, and t, $u(x, y, z, t)$.the three dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 [\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}]$$

The combination of partial derivatives that appears on the right hand side comes up in many places, it gets its own notation

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} .$$

Then the wave equation in three dimensions can be written more easily as

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u .$$

the wave equation in the PDE

$$\frac{\partial^2 u}{\partial t^2} = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}$$

For a function $u = u(x,t)$ of x ∈ $\mathcal{R}$. For brevity this will usually be written

$$u_{tt} = \Delta u.$$

This is usually supplemented by initial conditions, specified at t = 0. Since it is a second order equation in time the initial value and the initial time derivative are usually specified:

$$u(x,0) = f(x), \quad u_t(x,0) = \text{g(x)}$$

where$f$ and g are given function in $\mathcal{R}^n$.

Note that the wave equation is definitely not an elliptic PDE, since the sing of the second t-derivatives is opposite to the sign of the space derivatives. We shall soon see that the solutions to the wave equation have a very different character to those of elliptic equations.

The homogenous wave equation with constant coefficient can besolved y many ways such as separation of variables, the method of characteristics, Laplace transform, and Fourier transform.[2]

**Using Fourier transform to solve wave equation (2.2.1):**

If u(x, t) is the displacement from equilibrium of a string at position x and time t and if the string is undergoing small amplitude transverse vibrations, then we have seen that

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \qquad (1)$$

For a constant c. we are now going to solve this equation by multiplying both sides by $e^{-ikx}$ and integrating with respect to x. that is, we shall Fourier transform with respect to the spatial variable x. denote the Fourier transform with respect to x, for each fixed t of $u(x,t)$ by

$$\hat{u}(k,t) = \int_{-\infty}^{\infty} u(x,t)e^{-ikx}\, dx$$

We have already seen that the Fourier transform of the derivative $f'(x)$ is

$$\int_{-\infty}^{\infty} f'(x)e^{-ikx}\, dx = \text{ik}\int_{-\infty}^{\infty} f(x)e^{-ikx}\, dx = ik\hat{f}(\text{k}) \quad (2)$$

By integrating by parts with u $=e^{-ikx}$, dv = $f'(x)$ dx, du = -ik$e^{-ikx}$, v = f(x) and assuming that $f(x) \to 0$ as x $\to \pm\infty$. Applying this with f(x) = $\frac{\partial u(x,t)}{\partial x}$ and a second time with $f(x) = u(x,t)$, gives that Fourier transform of $\frac{\partial^2 u(x,t)}{\partial x^2}$is $-k^2\hat{u}(k,t)$. Computation of the Fourier transform of $\frac{\partial^2 u(x,t)}{\partial t^2}$ is even easier. For the first t-derivative,

$$\int_{-\infty}^{\infty} u_t(x,t)e^{-ikx}\, dx = \int_{-\infty}^{\infty} \lim_{h\to 0} \frac{u(x,t+h)-u(x,t)}{h} e^{-ikx}\, dx$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\int_{-\infty}^{\infty} u(x,t+h)e^{-ikx}\, dx - \int_{-\infty}^{\infty} u(x,t)e^{-ikx}\, dx\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\hat{u}(k,t+h) - \hat{u}(k,t)\right]$$

$$= \frac{\partial \hat{u}(k,t)}{\partial t} \quad (3)$$

To get two t- derivatives, we just apply this twice, with u replaced by $u_t$ the first time

$$\int_{-\infty}^{\infty} \frac{\partial^2 u(x,t)}{\partial t^2} e^{-ikx}\, dx = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} \frac{\partial}{\partial t} u(x,t)e^{-ikx}\, dx = \frac{\partial^2 \hat{u}(k,t)}{\partial t^2}$$

So applying the Fourier transform to both sides of (1) gives

$$\frac{\partial^2 \hat{u}(k,t)}{\partial t^2} = -c^2k^2\hat{u}(k,t) \quad (4)$$

This has not yet led to the solution for $u(x,t)$ or $\hat{u}(k,t)$, but it has led to considerable simplification. We now have, for each fixed k, a constant coefficient, homogeneous, second order ordinary differential equation to emphasise that each k may now be treated independently, fix any k and write $\hat{u}(k,t) = U(t)$. The differential equation (4) now is $U''(t) + c^2k^2U(t) = 0$. we know that this equation can be solved easily by trying $U(t) = e^{rt}$. since

$$U''(t) + c^2k^2U(t) = (r^2 + c^2k^2)e^{rt} = 0$$

If and only if r $= \pm ick$, the general solution to $U''(t) + c^2k^2U(t) = 0$, for any $k \neq 0$, is $U(t) = d_1e^{-ickx} + d_2e^{ickx}$. For $k = 0$, when the two values of r $= \pm ick$ are the same, the differential equation reduces to $U'' = 0$ and has general solution $U(t) = d_1 + d_2t$. We have to reject the $d_2t$ solution (i.e. we have require that $d_2 = 0$) on physical grounds small transverse oscillations certainly do not include amplitudes that grow to infinity at t goes to infinity. Recalling that $U(t) = \hat{u}(k,t)$ we conclude that the general solution to (4) is

$$\hat{u}(k,t) = \hat{F}(k)e^{-ickt} + \hat{G}(k)e^{ickt}$$

We have renamed the arbitrary constants $d_1 and\ d_2$ to $\hat{F}(k)\ and\ \hat{G}(k)$ respectively. The reason for these names will be made clear very soon. In any event, the arbitrary constants are certainly allowed to depend on k – viewed as an equation for an unknown function of t, (4) is a different equation for every different value of k. to recover $u(x,t)$ we just need to take the inverse Fourier transform

$$\hat{u}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(k,t)e^{ikx}\ dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[\hat{F}(k)e^{-ickt} + \hat{G}(k)e^{ickt}\right]e^{ikx}\ dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}(k)\ e^{ik(x-ct)}\ dk + \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{G}(k)\ e^{ik(x+ct)}\ dk$$

$$= F(x - ct) + G(x + ct)$$

The $F(x - ct)$ part of the solution represents a wave packet moving to the right with speed c. we can see this by observing that all points $(x,t)$ in space time for which $x - ct$ takes the same fixed value, $z$, have the same value of

$F(x - ct)$ namely$F(z)$. so if we move so that our position at time t is

$x = z + ct$ (i.e. move the right with speed c) we always see the same string height. The figure below illustrates this. It contains the graphs of $F(x)$,

$F(x - c) = F(x - ct)|_{t=1}$and$F(x - 2c) = F(x - ct)|_{t=2}$ for a bump shaped $F(x)$. in the figure we have chosen the location of the tick z on the x – axis so that $F(z) = max_x F(x)$.

Similarly, $G(x + ct)$ represents a wave packet moving to the left with speed c.

Suppose for example, the string starts at rest with the initial bump $u(x, 0) = p(x)$. to satisfy these initial conditions, F and G must obey

$$p(x) = u(x, 0) = F(x) + G(x)$$

And

$0 = u_t(x, 0) = -cF'(x) + cG'(x) \leftrightarrow F'(x) = G'(x) \leftrightarrow$ F(x) = G(x) + c

These equations only determine F and G up to be additive constant. This additive constant is irrelevant adding any constant to F while subtracting the same constant from G does not change the value of $F(x - ct) + G(x + ct)$ for any x or t. the functions $F(x) = G(x) = \frac{1}{2} p(x)$ do the job. So the bump resolves itself into two equal sized halves. One moves to the right with speed c and the other moves to the left with speed c. If the initial speed $u_t(x, 0) = s(x)$ is not zero, the string behaves similarly, but the left and the right moving parts need not have the same size and shape.[3]

## Heat Equation (2.3):

In a metal rode with non- uniform temperature, heat is transferred from regions of higher temperature to regions of lower temperature. Three physical principals are used here.

1. Heat energy of a body with uniform properties:

Heat energy $= \ cmu$, where m is the body mass, u is the temperature, c is the specific heat, units$[c] \ = L^2T^{-2}U^{-1}$(basic units M mass, L length, T time, U temperature). C is the energy required to raise a unit mass of the substance unit in temperature.

2. Fourier's law of heat transfer:

   Rate of heat transfer proportional to negative temperature gradient,

   Rate of heat transfer/area = - $K_0 \frac{\partial u}{\partial X}$ (5)

   Where $K_0$ is the thermal conductivity, units$[K_0]$ = $MLT^{-3}U^{-1}$. In other words, heat is transferred from areas of high temp to low temp.

3. Conservation of energy. Consider a uniform rod of length 1 with non-uniform temperature lying on the x – axis from x = 0 to x = 1. By uniform rod, we mean the density $\rho$, specific heat c, thermal conductivity $K_0$, cross sectional area A are all constants. Assume the sides of the rode are insulated and only the ends may be exposed. Also assume there is not heat source within the rod. Consider an arbitrary thin slice of the rod of width $\Delta x$ between x and x + $\Delta x$. The slice is so thin that the temperature throughout the slice is u(x, t). thus

   Heat energy of segment = c $\times \rho A \Delta x \ \times u$ = c$\rho A \Delta x \ u(x,t)$.

   By conservation of energy,

   Change of heat heat in from heat out from
   energy of segment = left boundary – right boundary
   in time $\Delta t$

   From Fourier's law (5),

   $$c\rho A\Delta x \ u(x, t+\Delta t) - c\rho A\Delta x \ u(x,t) = \Delta t A\left(-K_0 \frac{\partial u}{\partial x}\right)_x - \Delta t A\left(-K_0 \frac{\partial u}{\partial x}\right)_{x+\Delta x}$$

Rearranging yields (recall c, $\rho$, $A, K_0$ are constants),

$$\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t} = \frac{K_0}{c\rho}\left(\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x}-\left(\frac{\partial u}{\partial x}\right)_{x}}{\Delta x}\right)$$

Taking the limit $\Delta t, \Delta x \to 0$ gives the heat equation,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad (6)$$

Where

$$K = \frac{K_0}{c\rho}$$

Is called the thermal diffusivity, units[k] = $\frac{L^2}{T}$. Since the slice we chosen arbitrarily, the heat equation (6) applies through the rod.

Now consider the heat equation on a semi- infinite domain $0 \leq x < \infty$,

$$u_x = k\, u_{xx}, \qquad 0 \leq x < \infty,$$

$$u(x,0) = f(x), \qquad 0 \leq x < \infty,$$

and$f(x), u(x,t)$ approach zero fast enough as $|x| \to \infty$ so that the integral

$$\int_{-\infty}^{\infty} |u(x,t)|\, dx$$

Is finite. Also, a boundary condition is imposed at $x = 0$. Either a fixed temperature,

$$u(0,t) = 0, \qquad t > 0,$$

or insulated,

$$u_x(0,t) = 0 \qquad t > 0$$

To solve this problem, we recall the temperature distribution on the infinite domain $-\infty < x < \infty$ due to an initial temperature$\tilde{f}(x)$ was given by

$$u(x,t) = \int_{-\infty}^{\infty} k(s,x,t)\tilde{f}(s)\, ds$$

Where the heat kernel is defined as

$$k(s,x,t) = \frac{1}{\sqrt{4\pi k t}}\exp\left(-\frac{(x-s)^2}{4kt}\right)$$

Note that at $x = 0$, the heat kernel is even in s, i.e. $k(s,0,t) = k(-s,0,t)$. Also,

$$u(0,t) = \int_{-\infty}^{\infty} k(s,0,t)\tilde{f}(s)\, ds$$

Thus $u(0,t) = 0$ if $\tilde{f}(s)$ is odd. Therefore, the solution to the heat problem on the semi – infinite domain $0 \leq x < \infty$ with zero temperature at $x = 0$ $(u(0,t) = 0)$ is

$$u(x,t) = \int_{-\infty}^{\infty} k(s,x,t)\tilde{f}(s)\, ds$$

Where $\tilde{f}(s)$ is the odd extensionof $f(s)$, i.e.

$$\tilde{f}(s) = \begin{cases} f(s) & s > 0 \\ 0 & s = 0 \\ -f(s) & s < 0 \end{cases}$$

Similarly, not that the x- derivative of the heat kernel is odd at s = 0,

$$k_x(s,0,t) = \frac{s}{2kt\sqrt{4\pi kt}} \exp\left(-\frac{s^2}{4kt}\right) = -k_x(-s,0,t).$$

And hence the solution to the heat problem on the semi – infinite domain $0 \leq x < \infty$ with an insulated at $x = 0$ $(u_x(0,t) = 0)$ is

$$u(x,t) = \int_{-\infty}^{\infty} k(s,x,t)\tilde{f}(s)\, ds$$

Where $\tilde{f}(s)$ is the even extension of f(x), i.e.

$$\tilde{f}(s) = \begin{cases} f(s) & s > 0 \\ 0 & s = 0 \\ f(-s) & s < 0 \end{cases}$$

the heat equation for a function u $:\mathcal{R}_+ \times \mathcal{R}^n \to \mathbb{C}$ is the partial differential equation

$$\left(\partial_t - \frac{1}{2}\Delta\right) u = 0 \text{ with } u(0,x) = f(x) \qquad (7)$$

Where f is a given function on $\mathcal{R}^n$. By Fourier transforming equation. (7) in the x-variables only, one finds that (7) implies that

$$\left(\partial_t + \frac{1}{2}|\omega|^2\right)\hat{u}(t,\omega) = 0 \text{with} \hat{u}(0,\omega) = \hat{f}(\omega).$$

And hence that $\hat{u}(t,\omega) = e^{-t|\omega|^2/2}\hat{f}(\omega)$.Inverting the Fourier transform then shows that

$$u(x,t) = F^{-1}\left(e^{-\frac{t|\omega|^2}{2}}\hat{f}(\omega)\right)(x) = \left(F^{-1}\left(e^{-\frac{t|\omega|^2}{2}}\right) * f\right)(x) = e^{-\frac{t\Delta}{2}f(x)}.$$

From example,

$$F^{-1}\left(e^{-\frac{t|\omega|^2}{2}}\right)(x) = p_t(x) = t^{\frac{-n}{2}}e^{-\frac{1}{2t}|x|^2}$$

And therefore,

$$u(x,t) = \int_{\mathcal{R}^n} p_t(x-y)f(y)dy.$$

Now we define the Fourier transform of a function u(x, t) as an operator:

$$F(u) = \hat{u}(\omega,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)e^{i\omega x}\, dx \qquad (8)$$

Thus, the Fourier transform maps a function of (x, t) to a function of $(\omega, t)$.to transform the heat equation, we must consider how the Fourier transform maps derivatives. Note that

$$F(u_t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u_t(x,t)\, e^{i\omega x}\, dx = \frac{\partial}{\partial t}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)e^{i\omega x}\, dx\right) = \frac{\partial}{\partial t}F(u) = \frac{\partial}{\partial t}\hat{u}(\omega,t)$$

Also, integration by parts and the fact that $u(x,t) \to 0$ as $|x| \to \infty$ allow us to calculate the Fourier transform of $u_x$

$$F(u_x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u_x(x,t)\, e^{i\omega x}\, dx = \frac{1}{2\pi}\left[u(x,t)e^{i\omega x}\right]_{-\infty}^{\infty} - \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)(i\omega e^{i\omega x})\, dx$$

$$= -\, i\omega\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)e^{i\omega x}\, dx\right)$$

$$= -\, i\omega\, F(u) = -i\omega\hat{u}(\omega,t)$$

Thus, the Fourier transform of an x- derivative of a function is mapped to – $i\omega$ times the Fourier transform of the function. Hence

$$\mathrm{F}(u_{xx}) = - i\omega\, F(u_x) = -i\omega^2 F(u) = -\omega^2\, \hat{u}(\omega, t)$$

Thus the Fourier transform of the heat equation $u_x = ku_{xx}$ is

$$\frac{\partial}{\partial t}\, \hat{u}(\omega, t) = -\mathrm{k}\omega^2\, \hat{u}(\omega, t)$$

Hence the Fourier transform maps the heat equation, a PDE, to a first order ODE, integrating in time gives

$$\hat{u}(\omega, t) = C(\omega) e^{-k\omega^2 t} \qquad (9)$$

Where $C(\omega)$ is an arbitrary function, due to partial integration with respect to time. Setting t= 0 in (8) and (9) gives

$$C(\omega) = \hat{u}(\omega, 0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x, 0) e^{i\omega x}\ dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{i\omega x}\ dx$$

We could substitute $C(\omega)$ back into (9) and use the interval Fourier transform to obtain $u(x, t)$. the solution would be the same as that for separation of variables. However ,$u(x, t)$ can be obtained almost immediately using a result called the convolution theorem.

Let $u(x, t) = F(x)G(t)$then

$$FG_t = c^2 F_{xx} G\ ,$$

So

$\frac{1}{c^2}\frac{G_t}{G} = \frac{F_{xx}}{F} = A$,where A is constant.

We get a pair of ODEs:$F_{xx} = AF, and\ G_t = c^2 AG.$

By the intropy principle G must decay as t increases, so we must assume that $A <$ 0.

Let $A = -\omega^2.$ Then $F_{xx} + \omega^2 F = 0$ And $G_t + c^2\omega^2 G = 0.$

Solving both ODEs we obtain $u(x, t; \omega) = \left(C_1 \omega e^{i\omega x} + C_2 \omega e^{-i\omega x}\right) e^{-c^2\omega^2 t}$,

So the general solution is:

$$u(x,t)=\int_0^{\infty}u(x,t;\omega)d\omega=\int_0^{\infty}C_1\omega e^{-c^2\omega^2t+i\omega x}\,d\omega+\int_0^{\infty}C_2\omega e^{-c^2\omega^2t-i\omega x}\,d\omega$$

Changing variable in the second integral $\omega\to-\omega$ we obtain:

$$u(x,t)=\int_0^{\infty}C_1(\omega)e^{-c^2\omega^2t+i\omega x}\,d\omega-\int_0^{-\infty}C_2(-\omega)e^{-c^2\omega^2t+i\omega x}\,d\omega$$

$$=\int_0^{\infty}C_1(\omega)e^{-c^2\omega^2t+i\omega x}\,d\omega+\int_{-\infty}^{0}C_2(-\omega)e^{-c^2\omega^2t+i\omega x}\,d\omega$$

Recombining the two integrals we obtain:

$$u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{-c^2\omega^2t+i\omega x}\,d\omega$$

Where

$$\hat{f}(\omega)=\begin{cases}\sqrt{2\pi}\,C_1(\omega) & for\ \omega>0\\ \sqrt{2\pi}\,C_2(-\omega) & for \omega<0\end{cases}$$

[34]

## Laplace Transform (2.4):

1- the class function $\varepsilon$ consists of all piecewise continuous functions f:$[0,+\infty[\to\mathbb{C}$, for which there are constants A > 0, $B\in\mathcal{R}$ such that $|f(t)|=Ae^{Bt}$ for every t∈ $[0,+\infty[$ (10)

Using the quantors ∀, and ∃ we define

$$\rho(f)=\inf\{\,B\in\mathcal{R}\,|\,\exists\ A>0\,\forall t\geq 0:|f(t)|\leq Ae^{Bt}\}\,. \qquad (11)$$

In many simple cases concerning the Laplace transformation it suffices just to consider functions from $\varepsilon$.

2-The class of function Ƒ consists of all measurable functions

$$f:[0,+\infty[\,\to C\wedge *\ =\ C\cup\{\infty\}$$

for which there is a constant $\sigma\in\mathcal{R}$ such that

$$\int_0^{+\infty}|f(t)|e^{-\sigma t}\,dt<+\infty \qquad (12)$$

We put

$$\sigma(f) = \inf\{\sigma \in \mathcal{R}, \int_0^{+\infty} |f(t)| e^{-\sigma t}\, dt < +\infty \} \qquad (13)$$

By introducing the Lebesgue integral in (12) we see that in (13) is less complicated than (11). Furthermore, if (12) is holds for some $\sigma_0 \in \mathcal{R}$, the its clearly holds for all $\sigma \geq \sigma_0$, because $e^{-\sigma t} \leq e^{-\sigma_0 t}$, we even get that if (12) holds for some $\sigma \in \mathcal{R}$, then

$$\int_0^{+\infty} f(t) e^{-zt}\, dt, \qquad \text{for } z := \sigma + i\tau$$

Is convergent for every $\tau \in \mathcal{R}$.

Let f $\in \varepsilon$, i.e. f satisfies condition (10). for given $\varepsilon > 0$, we choose $\sigma = B + \varepsilon$, from which we get

$$\int_0^{+\infty} |f(t)| e^{-(B+\varepsilon)t}\, dt \leq A \int_0^{+\infty} |f(t)| e^{-\varepsilon t}\, dt < +\infty$$

and (12) holds for every $\sigma = B + \varepsilon, \varepsilon > 0$, from which we conclude that $\varepsilon \subset$ ƺ, and $\sigma(f) \leq \rho(f)$.

3- Let f $\in$ ƺ we define the Laplace transform $L$\{f\} of f as the complex function given by

$$L\{f\}(z) = \int_0^{+\infty} f(t) e^{-zt}\, dt\,, \qquad (14)$$

Where z is belongs to the set of the complex numbers, for which the integral on the right hand side of (14) is convergent.

**Example (2):**

Let f(t) = $t^n$, t $\geq$ 0 and n$\in \mathbb{N}$. since the exponential dominates the power function, we see that if $t^n \in \varepsilon$ with $\rho(t^n) = \sigma(t^n) = 0$. For R z > 0we get y partial integration,

$$L\{t^n\}(z) = \int_0^{+\infty} t^n e^{-zt}\, dt = \left[-\frac{1}{z} t^n e^{-zt}\right]_{t=0}^{+\infty} + \frac{n}{z} \int_0^{+\infty} t^{n-1} e^{-zt}\, dt = \frac{n}{z}\, L\{t^{n-1}\}(z),$$

So we get by recursion $L\{t^n\}(z) = \frac{n}{z} \cdot \frac{n-1}{z} \ldots \frac{1}{z} \cdot L\{1\}(z) = \frac{n!}{z^n} \cdot \frac{1}{z} = \frac{n!}{z^{n+1}}$ for R z >0, where $L\{1\}(z) = \frac{1}{z}$ .[53]

**Example (3):**

Let a∈ ℂ be a constant, and consider a function f(t) = $e^{at}$, for t ≥ 0. Then $e^{at} \in \varepsilon$ and$\rho(e^{at}) = \sigma(e^{at}) = Ra$.

We compute for R z >Ra,

$L\{t\}$(z)= $\int_0^{+\infty} f(t)e^{-zt}\,dt$ =$\int_0^{+\infty} e^{at}e^{-zt}\,dt$ =$\int_0^{+\infty} e^{-(z-a)t}\,dt$ = $\left[-\frac{1}{z-a}e^{-(z-a)t}\right]_{t=0}^{+\infty} = \frac{1}{z-a}$,

Thus the $L\{e^{at}\}$(z) = $\frac{1}{z-a}$ for R z >Ra

Then in follows for R z $> |Ra|$ that

$L\{\sinh(at)\}$(z) = $\frac{a}{z^2-a^2}$

$L\{\cosh(at)\}$(z) = $\frac{z}{z^2-a^2}$

Then put $a = ib$ so that $R\,z > R(ib)$

$Cosh\,(at) = cosh\,(ibt) = cos(bt)$and similarly

$$sinh\,(at) = sinh\,(ibt) = sin(bt)$$

Therefore, by this simple subustitution it follows from the above that

$L\{\sin(bt)\}$(z) = $\frac{b}{z^2+b^2}$

$L\{\cos(bt)\}$(z) = $\frac{z}{z^2+b^2}$

The next three formulas follows from the general property

$L\{t^n f(t)\}$= $(-1^n)F^{(n)}(z)$,

$L\{tsin\ at\}$(z) = $\frac{2az}{(z^2+a^2)^2}$

$L\{tcos\ at\}$(z) = $\frac{z^2-a^2}{(z^2+a^2)^2}$

$L\{te^{-at}\}$(z) = $\frac{1}{(z+a)^2}$ , where a > 0.

In general $L\{t^n e^{-at}\}$(z) = $\frac{n!}{(z+a)^{n+1}}$ , where a > 0.

The next formulas follows from the shift property $L\{e^{at}f(t)\}$(z) = F(z – a).

$L\{e^{-at} sin\ \beta t\}$(z) = $\frac{\beta}{(z+a)^2+\beta^2}$ , where a > 0.

$L\{e^{-at} sinh\ \beta t\}$(z) = $\frac{\beta}{(z+a)^2+\beta^2}$ , where a > 0.

$L\{e^{-at} cos\ \beta t\}$(z) = $\frac{z+a}{(z+a)^2+\beta^2}$ , where a > 0.

$L\{e^{-at} cosh\ \beta t\}$(z) = $\frac{z+a}{(z+a)^2+\beta^2}$ , where a > 0.

**Fourier Transform and Laplace (2.4.1):**

Suppose the temperature of a infinite wall is kept at f(x), for $-\infty < x < \infty$. To find the steady-state temperature I the region adjoining the wall, y > 0.the steady-state temperature satisfies Laplace's equation,

$\nabla^2 u_E = 0$, $\qquad -\infty < x < \infty$, y > 0.

The boundary conditions

$u_E = f(x)$, $\qquad -\infty < x < \infty$, y = 0,

$\lim_{y\to\infty} u_E(x, y) = 0$, $\qquad \lim_{|x|\to\infty} u_E(x, y) = 0$.

The research employ the Fourier transform in x,

$$F[g(x,y)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x,y)e^{i\omega x}\ dx$$

And define that $U_E(\omega, y)$= F[$u_E$(x, y)]. As before, we have

F[$u_{Exx}$] = -$\omega^2$F[$u_E$] = -$\omega^2 U_E(\omega, y)$, $\qquad$ F[$u_{Eyy}$] =$\frac{\partial^2}{\partial y^2}$ F[$u_E$] = $\frac{\partial^2}{\partial y^2} U_E(\omega, y)$.

Hence Laplace's equation for the steady-state temperature $u_E$(x, y) becomes

$$\frac{\partial^2}{\partial y^2} U_E(\omega, y) - \omega^2 U_E(\omega, y) = 0$$

Solving the ordinary differential equation and being careful about the fact that $\omega$ can be positive or negative, we have

$$U_E(\omega, y) = c_1(\omega)e^{-|\omega|y} + c_2(\omega)e^{|\omega|y}$$

Where $c_1(\omega),\ c_2(\omega)$ are arbitrary functions. Since the temperature must vanish as $y \to \infty$, we must have $c_2(\omega) = 0$. Thus

$$U_E(\omega, y) = c_1(\omega)e^{-|\omega|y} \qquad (15)$$

Imposing the boundary conditions at y = 0 gives

$$c_1(\omega) = U_E(\omega, 0) = F[u_E(\mathrm{x}, 0)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} u_E(\mathrm{x}, 0)e^{i\omega x}\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\, dx.$$

Where the inverse Fourier transform of $e^{-|\omega|y}$ is

$$F^{-1}\left[e^{-|\omega|y}\right] = \int_{-\infty}^{\infty} e^{-|\omega|y}e^{-i\omega x}\, d\omega = \int_{-\infty}^{0} e^{\omega(y-ix)}\, d\omega + \int_{0}^{\infty} e^{-\omega(y+ix)}\, d\omega$$

$$= \left[\frac{e^{\omega(y-ix)}}{y-ix}\right]_{-\infty}^{0} + \left[\frac{e^{-\omega(y+ix)}}{-(y+ix)}\right]_{0}^{\infty}$$

$$= \frac{1}{y-ix} + \frac{1}{y+ix} = \frac{2y}{x^2+y^2}$$

Therefore, applying the convolution theorem to (15) with $F^{-1}[c_1(\omega)]$= f(x) and $F^{-1}\left[e^{-|\omega|y}\right] = \frac{2y}{x^2+y^2}$ gives

$$u_E(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\frac{2y}{(x-s)^2+y^2}\, ds$$

[6]

## Geometric Interpretation of the Complex Fourier Transforms of a Class of Exponential Functions (2.5):

In this section we show how a class of complex Fourier transforms of exponential functions which have all their zeros on the real line may be explored from a geometric perspective. These transforms belong to the Laguerre-Pólya class, and it is proved that all the zeros are simple

Functions which map the complex plane to the complex plane provide an interesting challenge when it comes to visualisation. A full plot would require 4 dimensions, yet on paper we are obliged to draw in 2 dimensions. Here, we are going to explore ways of visualising a certain class of holomorphic entire functions defined for integer values of $n \geq 1$, namely:

$$F_{2n}(z) = \int_{-\infty}^{\infty} e^{-t^{2n}} \cdot e^{izt}\, dt = R(\sigma, w) + iI(\sigma, w), \text{ where } z = w - i\sigma,$$

and $\sigma, w, R, I \in \Re$. (1)

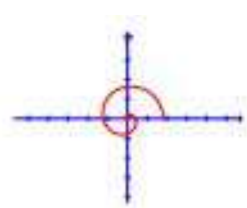
In 1923, Professor Pólya proved that all the zeros of these functions lie on the real line $m\ (z)\ =\ 0$.

**Case $n\ =\ 1$**:It is straightforward to show by contour integration that:

$$F_2(z) = \sqrt{\pi} \cdot e^{-\frac{z^2}{4}} = \sqrt{\pi} \cdot e^{\frac{(\sigma+iw)^2}{4}} = \sqrt{\pi} \cdot e^{(\sigma^2-w^2)/4} \cdot e^{(i\sigma w)/2}.$$

This is a Laguerre - Pólya function of order 2 in z . Whilst it is an elementary expression, we are going to examine some geometric aspects as they provide clues to the more involved cases which follow when $n \geq 2$.

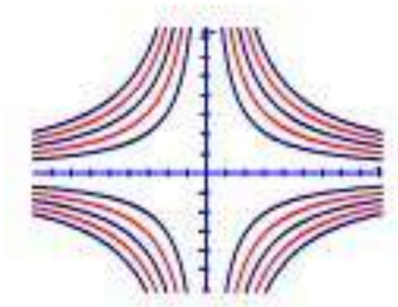
The modulus of F squared, call it $L$ (for length) squared: $L^2 = R^2 + I^2 = \pi \cdot e^{(\sigma^2-w^2)/2}$ so there are no zeros in this case. For constant $\sigma$ not equal to zero, and $w\ =\ vt$ where $v$ is a constant velocity and $t$represents time, a parametric plot of $(R, I)$ yields a spiral orbit of rapidly diminishing radius:



(Fig. 1, not to scale, $\sigma > 0$)

If a larger value of $\sigma$ is chosen, the orbit is enlarged.

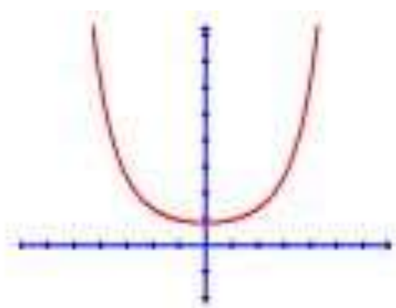
The field lines $R =\ 0$ are given by $w = \pi(1 + 2m)/\sigma$ and the field lines $I = 0$ are given by $w = \pi(2m)/\sigma$, where $m$ is any integer:



(Fig 2)

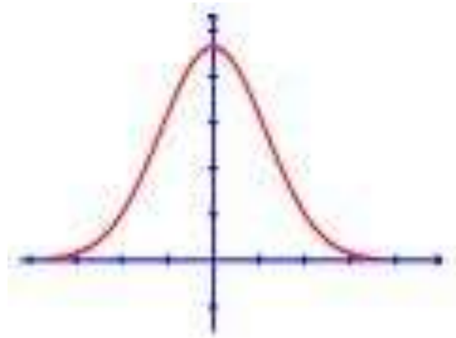
We observe that the $R = 0$ field lines never meet each other. The same is true for the $I = 0$ field lines, and also no $I = 0$ field line ever meets an$R = 0$ field line (or else there would be a zero,which would establish a contradiction). A plot of $L^2$ against $\sigma$ for constant $w$ rises steadily as

$\sigma$ moves away from 0:



(Fig 3)

Where as a plot of $L^2$ against $w$ for constant $\sigma$ falls steadily as $w$ moves away from 0:



(Fig 4)

**Case $n \geq 2$**

In the same paper previously mentioned (I), Pólya a lso proved that in this case there are an infinite number of real zeros. Pólya proved that the order of such functions is at most $2n/(2n-1) < 2$.

By Hadamard'sfactorisation theorem (see for example III), such (entire) functions have the product representation:

$F_{2n}(z) = z^m e^{G(z)} \prod_{k=1}^{\infty}(1 - z/w_k)$ where $G(z)$ is a polynomial of degree at most $d$, where $d \geq 0$ is aninteger such that $d \leq v \leq d+1$, and $v$ is the order of $F_{2n}(z)$. As $v < 2$, we see $d < 2$, and so themost general form of $F_{2n}(z)$ is given by

$F_{2n}(z) = z^m e^{(a+b\cdot z)} \cdot \prod_{k=1}^{\infty}\left(1 - \frac{z}{w_k}\right) = c \cdot z^m e^{bz} \cdot \prod_{k=1}^{\infty}\left(1 - \frac{z}{w_k}\right)$. (2).

We now observe that $z = 0 \Longrightarrow F_{2n}(z) = \int_{-\infty}^{\infty} e^{-t^{2n}}\, dt > 0$ and so

$m = 0$ and $c > 0$. As the exponential kernel in the integral

$\int_{-\infty}^{\infty} e^{-t^{2n}} \cdot e^{izt}\, dt$ is even in $t$, it follows that for every root there is another root of equal magnitude and opposite sign.

If we pair each root $w_k$ with its partner of opposite sign, then

$F_{2n}(z) = c \cdot e^{bz} \prod_{r=1}^{\infty}(1 - z^2/\alpha_r^2)$where$w_1 = \alpha_1, w_2 = -\alpha_1, w_3 = \alpha_2, w_4 = -\alpha_2$

etc, and the $\alpha_r$ are the positive roots. Owing to the kernel $e^{-t^{2n}}$ in the integral expression (equation (1)) being even in $t$, it follows that $F_{2n}(z)$ is an even function in $z$, and so $b = 0$.

Accordingly, we see that: $F_{2n}(z) = c \prod_{k=1}^{\infty}\left(1 - \frac{z}{w_k}\right)$ (3)

and also that $F_{2n}(z) = c \cdot \prod_{r=1}^{\infty}\left(1 - \frac{z^2}{\alpha_r^2}\right)$ (4)

These functions belong to the so-called Laguerre-Pólya class, which we will denote by L-P.

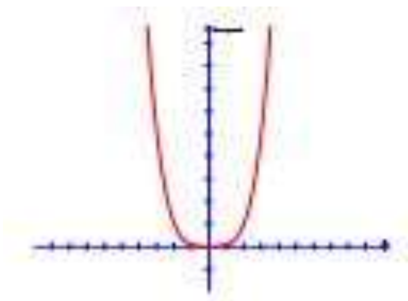
Now let us consider $L^2 = F_{2n}(z) \cdot \overline{F_{2n}(z)} = c^2 \prod_{r=1}^{\infty}\left(1 - \frac{z^2}{\alpha_r^2}\right)(1 - (z)^{-2}/\alpha_r^2)$

As $z = w - i\sigma, L^2 = c^2 \prod_{r=1}^{\infty}\big(1 - ((w - i\sigma)^2/\alpha_r)\big)\big(1 - ((w + i\sigma)^2/\alpha_r)\big)$

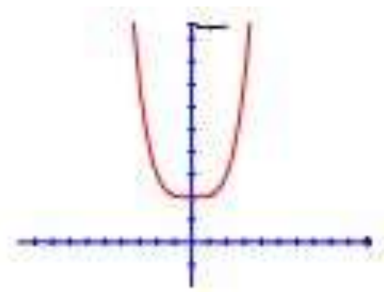
The rth term inside the product, call it $P_r = 1 - 2(w^2 - \sigma^2)/\alpha_r^2 + (w^2 - \sigma^2)^2/\alpha_r^4$ , which can be written $\left(1 - \frac{(w^2+\sigma^2)}{\alpha_r^2}\right)^2 +^4\sigma^2/\alpha_r^2$, which is always positive for positive $\sigma$. When $\sigma = 0, p_r$ reduces to

$(1 - w^2/\alpha_r^2)^2$. Now $\frac{\partial P_r}{\partial \sigma}$is $\frac{4\sigma}{\alpha_r^2} + 4\sigma(\sigma^2 + w^2)/\alpha_r^4$which is positive for positive $\sigma$.If $\sigma = 0$ and w is equal to one of the roots, then $L^2 = 0$ at that point. Since the zeros of holomorphic functions are isolated, as we increase $\sigma$ above 0 (keeping w constant) there must immediately be an interval where $L^2 > 0$, and as each $p_r$ is strictly increasing for positive $\sigma$, so $L^2$ must increase as $\sigma$ increases, and keep increasing. If on the other hand we start with $\sigma = 0$ and pick a value of w which is not a root, then $L^2$ is greater than 0 anyway, and as we increase $\sigma$above zero (with $w$ constant), $L^2$ must increase as each$p_r$ is strictly increasing for positive $\sigma$.

These arguments show that the graph of $L^2$ against $\sigma$ for constant $w$ has a similar form to the one in Fig.3 previously, except that if we start with $w = a$ root, the curve passes through



(Fig 5)$L^2$against $\sigma$, constant w(a root)



(Fig 6) $L^2$ against $\sigma$, constant w(nota root)

This reveals one geometric reason why all the zeros of $F_{2n}(z)$ as defined by equation (1) lie on the real line: for any given constant $w$, the modulus of the function rises monotonically if we travel away from the real line $\sigma = 0$.

I will now explore $L^2$ from a different angle – by looking at the integral formula which defines $F_{2n}(z)$:

$$F_{2n}(z) = \int_{-\infty}^{\infty} e^{-t^{2n}} \cdot e^{izt}\, dt \quad where\ z = w - i\sigma$$

Now$L^2 = F_{2n}(z) \cdot \overline{F_{2n}}(z)$

$$= \left( \int_{-\infty}^{\infty} e^{(-x^{2n+\sigma \cdot x})}(\cos wx + i \sin wx) \right) \cdot \left( \int_{-\infty}^{\infty} e^{(-y^{2n+\sigma \cdot y})}(\cos wy - i \sin wy)\, dy \right)$$

$$= \iint_{\Re^2} e^{(-x^{2n}-y^{2n}+\sigma\cdot(x+y)}$$

$$\cdot\big((\cos wx \cdot \cos wy + \sin wx \sin wy) + i(\sin wx \cos wy - \cos wy \sin wy)\big).\, dxdy$$

$$= \iint_{\Re^2} e^{\left(-x^{2n}-y^{2n}+\sigma\cdot(x+y)\right)} \cdot (\cos(wx - wy) + i \sin(wx - wy))\, dxdy$$

$$= \iint_{\Re^2} e^{\left(-x^{2n}-y^{2n}+\sigma\cdot(x+y)+i\cdot w(x-y)\right)}\, dxdy$$

(manipulations justified by Fubini's Theorem).

Now as $e^{\sigma(x+y)} = \sum_{m\geq 0} \sigma^m (x+y)^m / m!$we can write $L^2$as:

$\sum_{m\geq 0} \frac{\sigma^m}{m!} \iint_{\Re^2} (x+y)^m \cdot e^{-(x^{2n}-y^{2n})+iw(x-y)}\ dxdy$(justified by absolute uniform convergence) and substituting $t = x + y, X = x - y$ we see:

$$L^2 = \sum_{m\geq 0} \frac{\sigma^m}{m!} \iint_{\Re^2} t^m e^{-\left((t+X)/2\right)^{2n}+\left((t-X)/2\right)^{2n})+iwX} \left\|\frac{\partial(x,y)}{\partial(t,X)}\right\| dtdX$$

$$= \frac{1}{2} \sum_{m\geq 0} \frac{\sigma^m}{m!} \iint_{\Re^2} t^m e^{-\left((t+X)/2\right)^{2n}+\left((t-X)/2\right)^{2n})+iwX}\ dtdX$$

And as $e^{-\left((t+X)/2\right)^{2n}+\left((t-X)/2\right)^{2n})+iwX}$ is even in $t$, for odd powers of $n$ the integrals must be zero, so we need only consider the even powers and we can write:

$$L^2 = \frac{1}{2} \sum_{m\geq 0} \frac{\sigma^{2m}}{(2m)!} A_{2m,2n}(w) (5)$$

Where $A_{2m,2n}(w) = \iint_{\Re^2} t^{2m} e^{-\left(\left(\frac{(t+X)}{2}\right)^{2n} + \left(\frac{(t-X)}{2}\right)^{2n}\right) + iwX} \, dtdX$ (6)

**Lemma (2.5.1):**

$$A_{2m,2n}(w) \geq 0 \quad \forall\, m \geq 0, n \geq 0$$

**Proof :**

$$A_{2m,2n}(w) = \frac{1}{i^{2m}} \cdot \frac{\partial^{2m}}{\partial u^{2m}} \left[ \iint_{\Re^2} e^{-\left(\left(\frac{(t+X)}{2}\right)^{2n} + \left(\frac{(t-X)}{2}\right)^{2n}\right) + i(wX+ut)} \, dtdX \right]_{u=0}$$

and as the imaginary part of the exponent in the integrand can be expressed as $(w+u)(X+t)/2 \, + (u-w)(t-X)/2$, if we now reverse the substitution $t \, = \, x \, + \, y, X \, = \, x - y$ we obtain:

$$A_{2m,2n}(w) = (-1)^m \cdot \frac{\partial^{2m}}{\partial u^{2m}} \left[ \iint_{\Re^2} e^{-(x^{2n}+y^{2n}) + i((w+u)x + (u-w)y)} \, 2dxdy \right]_{u=0}$$

$$= (-1)^m \cdot \frac{\partial^{2m}}{\partial u^{2m}} \left[ \int_{-\infty}^{\infty} e^{-x^{2n} + ix(w+u)} dx \, . \int_{-\infty}^{\infty} e^{-y^{2n} + iy(u-w)} dy . \, 2 \right]_{u=0}$$

By Fubini’s theorem

$$= (-1)^m \cdot \frac{\partial^{2m}}{\partial u^{2m}} \left[ F_{2n}(u+w) \cdot F_{2n}(u-w) \right]_{u=0} (7)$$

Now let us pick one particular value of $n$. Recall equation (4), namely that $F_{2n}(z) = c \cdot \prod_{r=1}^{\infty} (1 - z^2/\alpha_r^2)$Consider the partial product

$P_N(z) = c \cdot \prod_{r=1}^{\infty}(1 - z^2/\alpha_r^2)$and suppose that for one partial value of $N, K$ say,

$T_{K,m}(w) = (-1)^m 2 \cdot \frac{\partial^{2m}}{\partial u^{2m}}[P_K(u+w) \cdot P_K(u-w)]_{u=0} \geq 0 \; \forall m \geq 0, \forall w \in \Re.$

Now let us consider $T_{K+1,m}(w) = (-1)^m 2 \cdot \frac{\partial^{2m}}{\partial u^{2m}}[P_{K+1}(u+w) \cdot P_{K+1}(u - w)]_{u=0}$

$$= (-1)^m 2 \cdot \frac{\partial^{2m}}{\partial u^{2m}}\left[\left(1 - \frac{(u+w)^2}{\alpha_{K+1}^2}\right) \cdot P_K(u+w) \cdot P_K(u-w)\right]_{u=0}$$

$$= (-1)^m 2 \cdot \frac{\partial^{2m}}{\partial u^{2m}}\left[\left(1 - 2\frac{(u+w)^2}{\alpha_{K+1}^2}\right) + (u^4 + w^4 - 2u^2w^2/\alpha_{K+1}^4) \cdot P_K(u+w) \cdot P_K(u-w)\right]_{u=0}$$

For brevity of writing, call $1 - 2(u^2 + w^2)/\alpha^2_{K+1^{+(u^4+w^4-2u^2w^2)}}/\alpha_{K+1}^4 = B$ and $P_K(u+w) \cdot P_K(u-w) = D$

If $m = 0$ then $T_{K+1,0}(w) = 2\, P_{K+1}^2(w) \geq 0.$

If $m = 0$ then $T_{K+1,1}(w) = -2(B_{uu}D + 2B_uD_u + BD_{uu})_{u=0}$where the subscripts denote partial differentiation with respect to $(wrt)$ $u$.As$B$ is an even function in $u$, the middle term is zero.

Now $B_{uu} = -\frac{4}{\alpha_{K+1}^2} + (12u^2 - 4w^2)/\alpha_{K+1}^2$, so we obtain:

$$T_{K+1,1}(w) = 4\left(\frac{1}{\alpha_{K+1}^2} + \frac{w^2}{\alpha_{K+1}^4}\right)T_{K,0}(w) + (1 - w^2/\alpha_{K+1}^2)^2 \cdot T_{K,1}(w) \geq 0$$

If $m \geq 2$ then $\frac{(-1)^m T_{K+1,m}(w)}{2} = \frac{\partial^{2m}}{\partial u^{2m}}(B,D)_{u=0} =$

$$\left(B\frac{\partial^{2m}D}{\partial u^{2m}} + {}^{2m}C_1 B_u \cdot \frac{\partial^{2m-1}D}{\partial u^{2m-1}} + {}^{2m}C_2 B_{uu} \cdot \frac{\partial^{2m-2}D}{\partial u^{2m-2}} + {}^{2m}C_3 B_{uuu} \cdot \frac{\partial^{2m-3}D}{\partial u^{2m-3}} + {}^{2m}C_4 B_{uuuu} \cdot \frac{\partial^{2m-2}D}{\partial u^{2m-2}}\right)_{u=0}$$

As before, the odd terms are zero as $B$ is an even function in $u$, and also partial derivatives of $Bw.r.t\ u$ of degree more than 4 are identically zero.

Then as additionally $B_{uuuu} = 24/\alpha_{K+1}^4$,we see $T_{K+1,m}(w) =$

$$\left(B\frac{\partial^{2m}}{\partial u^{2m}}[(-1)^m 2D] + {}^{2m}\ C_2 B_{uu}\frac{\partial^{2m-2}}{\partial u^{2m-2}}[(-1)^m 2D] + {}^{2m}\ C_4 B_{uuuu}\frac{\partial^{2m-4}}{\partial u^{2m-4}}[(-1)^m 2D]\right)_{u=0}$$

$$= \left(B\frac{\partial^{2m}}{\partial u^{2m}}[(-1)^m 2D] -{}^{2m}\ C_2 B_{uu}\frac{\partial^{2m-2}}{\partial u^{2m-2}}[(-1)^m 2D] + {}^{2m}\ C_4 B_{uuuu}\frac{\partial^{2m-4}}{\partial u^{2m-4}}[(-1)^m 2D]\right)_{u=0}$$

$$= \left(1 - \frac{w^2}{\alpha_{K+1}^2}\right)^2 \cdot T_{K,m}(w) + {}^{2m}C_2 4\left(\frac{1}{\alpha_{K+1}^2} + \frac{w^2}{\alpha_{K+1}^4}\right) \cdot T_{K,m-1}(w) + {}^{2m}C_2 \cdot 24/\alpha_{K+1}^4 \cdot T_{K,m-2}(w)$$

which is $\geq 0$.

What has been established is the inductive step, namely that $\cdot\, T_{K+1,m}(w) \geq 0$.

If we now consider the case $K = 1$ it is easy to see that $T_{1,m}(w) \geq 0\ \forall m \geq 0$:

$$T_{1,0} = 2P_1^2(w) \geq 0, T_{1,1}(w) = 8\left(\frac{1}{\alpha_1^2} + \frac{w^2}{\alpha_1^4}\right), T_{1,M}(w) \text{is 0 for M} \geq 3.$$

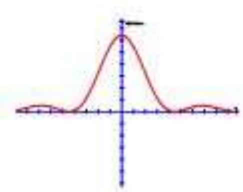
Therefore by induction $T_{N,m}(w) \geq 0$ for $N$ as large as we like, $\forall m \geq 0$.

Therefore taking the limit as $N \to \infty$ shows the expression for $A_{2m,2n}(w)$ in equation (7) is $\geq 0$.This argument holds for all integer values of $n$ from 2 upwards.

QED.

Looking again at formula for $L^2$ in (5), the graphs in Fig.5 and Fig.6 are now very clear, along with the result that there are no zeros of $F_{2n}(z)$ off the line $\sigma = 0$.

As the functions $A_{2m,2n}(w)$ are Fourier transforms and so tend to zero as $w \to \infty$, we can also see that the plot of $L^2$ against $w$ for constant $\sigma$ must also have the form of the curve in Fig.4 (perhaps with some minor undulations, as $A_{0,2n}(w) = F_{2n}^2(w) \geq 0$ has undulations because of the zeros $w_k$):



(Fig 7) plot of $A_{0,2n}(w)$ against w

Now we will consider the parametric plot of $(R, I)$ when $\sigma$ is constant and not equal to zero, and $w = v\,t$ where $v$ is a constant velocity.

The angular momentum $\underline{J} = m\,\underline{r} \wedge \underline{\dot{r}} = m \begin{vmatrix} \underline{e_1} & \underline{e_2} & \underline{e_3} \\ R & I & 0 \\ \dot{R} & \dot{I} & 0 \end{vmatrix}$ where the dots denote differentiationwith respect to time, and the vectors in the top row are a left-handed set of orthonormal vectors, with the first being in the positive $R$ direction and the second in the positive $I$ direction.

So

$$\underline{J} = m\left(R \cdot \frac{dI}{dw} \cdot \frac{dw}{dt} - I \cdot \frac{dR}{dw} \cdot \frac{dw}{dt}\right)\underline{e_3} \qquad (8)$$

As we are holding $\sigma$ fixed, we see that $\underline{J} = mv\left(R \cdot \frac{\partial I}{\partial w} - I \cdot \frac{\partial R}{\partial w}\right)\underline{e_3}$

which by the Cauchy-Riemann equations can be expressed as :

$$mv\left(R \cdot \frac{\partial R}{\partial \sigma} - I \cdot \frac{\partial I}{\partial \sigma}\right)\underline{e_3}\text{is} \quad \mathrm{m}\nu \cdot \frac{\partial}{\partial \sigma}[R^2 + I^2]\underline{e_3}(9)$$

By equation (5), we can immediately see that for positive $\sigma$ the angular momentum is also strictly positive, and as the $A_{2m.2n}(w)$ are Fourier transforms and tend to zero as $w \to \infty$, the locus of $(R, I)$ must follow a spiral orbit where the radius is never zero for any finite $w$, as in Fig.1

It is worth observing that, given any fixed value of $n$, it is not possible as m varies for all the $A_{2m.2n}(w)$ to be zero at the same time for any particular value of $w, w^*$ say, as by equation (5) this would lead to a continuous line of zeros of $F_{2n}(z)$ in the complex plane along $w = w^*$, which is impossible as holomorphic functions only have isolated zeros.

We are now going to explore the plot of the field lines $R = 0$ and $I = 0$:

Trivially, $I = 0$ when $w = 0$, because the integral in equation (1) is then real, and $I = 0$ when $\sigma = 0$, as the integral kernel is then even in $t$.

Now consider the spiral orbit discussed on the last page. This means that as we travel up the line $\sigma$ = constant at speed $v$, we are crossing the field lines $R = 0$, then $I = 0, R = 0$ etc.

The field lines with equation $R = 0$ traverse the zeros on the $w-$ axis.

If we consider a field line $R = 0$, then along this line we have:

$$R_\sigma + R_w \frac{dw}{d\sigma} = 0, \text{or alternatively } R_\sigma \frac{d\sigma}{dw} + R_w = 0 (10)$$

Where the suffixes denote the first partial differentials.

Now L-P is closed under differentiation, so it follows that all the zeros of the first derivative $F_{2n}^{(1)}(z)$ must also lie on the line $\sigma = 0$ too.

This means $R_w$ and $R_\sigma$ cannot both be zero at the same time off the line $\sigma = 0$, or else by the Cauchy-Riemann equations $F_{2n}^{(1)}(z)$ would have a zero, which would establish a contradiction.

Therefore, as we move along $R = 0$, unless $\frac{dw}{d\sigma}$ or $\frac{d\sigma}{dw}$ is zero, by (10), neither $R_w$ nor $R_\sigma$ canbe zero (if one were zero the other would also have to be zero), which means that they mustkeep the same sign, and therefore so must $\frac{dw}{d\sigma}$.

All these factors considered, we might expect the $R = 0$ field line plot to look like that in Fig.2, except with the lines bent so that they cross the $w-$axis, intercepting the zeros, with the $I = 0$ field lines in between the $R = 0$ field lines when $\sigma$ is not equal to 0.

By way of example, consider the case $n = 2$, where:

$$Re[F_4(z)] = \int_{-\infty}^{\infty} e^{-t^4+\sigma\cdot t} \cdot \cos(wt)\, dt = 0 \qquad (11)$$

Consider positive values of $\sigma$. Letting $t = T + k$, and choosing $k$ so that the coefficient of $T$ in the exponent of the integrand vanishes leads us to set $k = (\sigma/4)^{(1/3)}$ and leads to:

$$\int_{\mathfrak{R}} e^{-\left(T^4+4T^3\left(\frac{\sigma}{4}\right)^{\frac{1}{3}}+6T^2\left(\frac{\sigma}{4}\right)^{\frac{2}{3}}\right)}.\left(\cos(wT)\cos\left(w(\sigma/4)^{1/3}\right)\right)$$

$$-\sin(wT)\sin\left(w\left(\frac{\sigma}{4}\right)^{\frac{1}{3}}\right)dT = 0$$

Replacing $T$ with $u.(4/\sigma)^{1/3}$ then leads to:

$$\int_{\mathfrak{R}} e^{-\left(u^4\cdot\left(\frac{4}{\sigma}\right)^{\frac{1}{3}}+4u^3\left(\frac{4}{\sigma}\right)^{\frac{2}{3}}+6u^2\right)}.\left(\cos wu(4/\sigma)^{1/3}\cos w(4/\sigma)^{1/3}\right)$$

$$-\sin wu(4/\sigma)^{1/3}\sin w(4/\sigma)^{1/3}\, du = 0.$$

Rearrangement then yields:

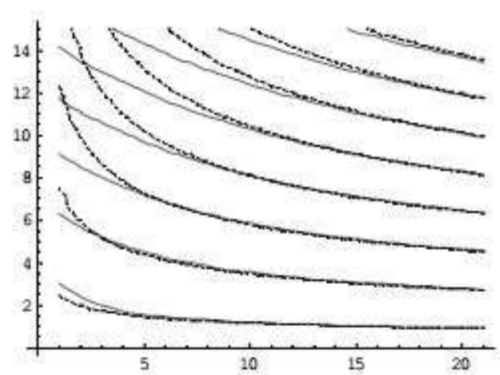
$$\cos\left(w(\sigma/4)^{1/3}\right)\int_{\mathfrak{R}} e^{-\left(u^4\cdot\left(\frac{4}{\sigma}\right)^{3/4}+4u^3\left(\frac{4}{\sigma}\right)^{\frac{2}{3}}+6u^2\right)}.\cos\left(wu(4/\sigma)^{1/3}\right)du =$$

$$\sin\left(w\left(\frac{4}{\sigma}\right)^{\frac{1}{3}}\right)\int_{\mathfrak{R}} e^{-\left(u^4\cdot\left(\frac{4}{\sigma}\right)^{\frac{3}{4}}+4u^3\left(\frac{4}{\sigma}\right)^{\frac{2}{3}}+6u^2\right)}.\sin\left(wu\left(\frac{4}{\sigma}\right)^{\frac{1}{3}}\right)du(12)$$

We can see that in the asymptotic case as $\sigma \to \infty$, the exponent in the integrand tends to $-6u^2$,so there is a family of asymptotic solutions given by $w = u(4/\sigma)^{1/3} \cdot \frac{\pi}{2} \cdot (2m+1)$, where m is any 2integer. In the asymptotic limit equation (12) then reduces to:

$0 = \text{limitas}\sigma \to \infty \left[(-1)^m \int_{\Re} e^{-(6u^2)} \sin\left(u(4/\sigma)^{1/3} \cdot \frac{\pi}{2} \cdot (2m+1)\right) du\right]$;the sine function is odd and the exponential function is even, hence the integral is zero.

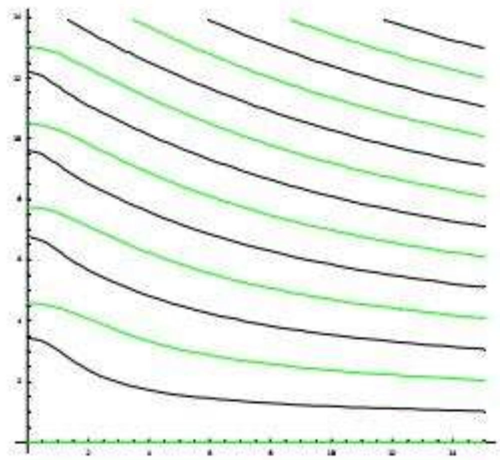
A plot of the $R = 0$ lines is shown below in Fig.8 (solid lines), along with part of the curves $w = (4/\sigma)^{1/3} \cdot \frac{\pi}{2} \cdot (2m+1)$ (dashed lines). As we can see, the asymptotic behaviour quickly becomes 2 apparent for relatively small values of $\sigma$. The horizontal axis is the$\sigma$ axis, and the vertical axis is the $w$ axis:




(Fig 8)

Finally, a plot of $R = 0$ field lines (black) and $I = 0$ field lines (green) is shown in Fig. 9 below.

The zeros of $F_4(z)$ occur only where the $R = 0$ field lines cross the $w$ axis (which is itself one of the $I = 0$ field lines), so all the zeros lie on the line $\sigma = 0$. As before, the horizontal axis is the$\sigma$ axis, and the vertical axis is the $w$ axis:



(Fig 9)

A similar analysis can be carried out in the cases $n \geq 3$, but the asymptotic convergence of the $R = 0$ field lines to the curves $w = (2n/\sigma)^{1/(2n-1)} \cdot \frac{\pi}{2} \cdot (1 + 2m)$ as $\sigma \to \infty$ is slower.

We can see that there is another geometric reason for the location of the zeros of $F_{2n}(z)$:

the only place where the $R = 0$ and $I = 0$ field lines intersect is on the $w$ axis, ie where $\sigma = 0$.

## Simplicity of zeros (2.5.2):

It will now be proved that all the zeros of $F_{2n}(w)$ are simple:

It is easy to see from equation (1) that the functions $F_{2n}(w)$ obey the differential equations

$$F_{2n}^{(2n-1)}(w) = \frac{(-1)^n}{2n} w F_{2n}(w) \qquad (13)$$

Where $F_{2n}^{(2n-1)}(w)$ denotes the $(2n-1)$th derivative with respect to $w$.

Differentiating again yields:

$F_{2n}^{(2n)}(w) = \frac{(-1)^n}{2n}\left(F_{2n}(w) + wF_{2n}^{(1)}(w)\right)$ (14)

## Lemma (2.5.3):

If $G$ is a function in L-P of the form $G(w) = c \cdot \prod_{k=1}^{\infty}(1 - w/w_k)$,

And $G(w^*) \neq 0$ and $G^{(1)}(w^*) \neq 0$,then $G^{(2)}(w^*)$ cannot be zero.

## Proof :

Differentiating the natural logarithm of the modulus of $G$ as above twice yields:

$\frac{\left(G^{(1)^2} - G^{(2)}G\right)}{G^2} = \sum_{k=1}^{\infty}(w - w_k)^{-2}$and we see that if $G(w^*) \neq 0$ (in other words, $w^* \neq w_k$for any $k$)and $G^{(1)}(w^*) = 0$, then $-G^{(2)}(w^*)G(w^*) > 0$, so $G^{(2)}(w^*) \neq 0$.

Now observe that L-P is closed under differentiation.

Set $G(w) = F_{2n}^{(2n-2)}(w)$If for any particular$F_{2n}(w)$ there were a multiple root at $w = w^*$, then by (13) and (14) $F_{2n}^{(2n-1)}(w^*) = 0$ and $F_{2n}^{(2n)}(w^*) = 0$. Therefore, by the above lemma, $G(w^*)$ must be zero, or a contradiction would be obtained.

So $F_{2n}^{(2n-2)}(w^*)$ and $F_{2n}^{(2n-1)}(w^*)$ must both be zero. By the same reasoning, $F_{2n}^{(2n-3)}(w^*), F_{2n}^{(2n-4)}(w^*), \dots \dots, F_{2n}^{(2)}(w^*)$ must all be zero, and we know $F_{2n}^{(1)}(w^*)$ and $F_{2n}(w^*)$ are both zero too. In other words, all the derivatives of $F_{2n}(w^*)$, up to and including the $(2n-1)th$, evaluated at $w = w^*$, are zero.

Successive differentiation of equation (14) shows that all the derivatives of $F_{2n}(w)$ above the $(2n-1)th$, evaluated at $w = w^*$, are zero as well.

So the Taylor expansion for $F_{2n}(z)$ about $z = w^*$ is:

$$F_{2n}(z) = F_{2n}(w^*) + F_{2n}^{(1)}(w^*)(z - w^*) + \frac{F_{2n}^{(2)}(w^*)}{2!}(z - w^*)^2 + \frac{F_{2n}^{(3)}(w^*)}{3!}(z - w^*)^3 + \cdots\cdots$$

and as all the coefficients $F_{2n}^{(r)}(w^*)$ are zero, we conclude $F_{2n}(z)$ is identically zero in some disk around $z = w^*$. This establishes a contradiction, as the zeros of holomorphic functions are isolated.

Therefore there cannot be a multiple zero of $F_{2n}(w)$, in other words all the zeros are simple.

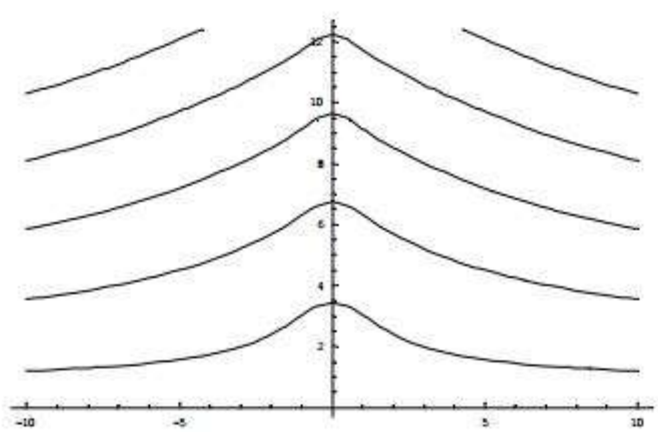
QED.

Now let us consider a particular root on the w-axis, say $w = w^*$, for a particular value of $n$.

Therefore $F_{2n}(w^*) = 0$ and by the above $F_{2n}^{(1)}(w^*)$ (in other words, $R_w$ evaluated at $w = w^*$) $\neq 0$, as all the roots are simple. We also see that at $z = w^*, R_\sigma = \int_{\Re} t \cdot e^{-t^{2n}} \cdot \cos(w^* \cdot t)dt = 0$ by symmetry,so by equation (10), $\frac{d\omega}{d\sigma} = 0$ on the $R = 0$ fieldline that crosses through $z = w^*$ when $\sigma = 0$.

The geometric significance of this result is that the $R = 0$ fieldline that crosses each zero on the w-axis does so with a zero gradient (ie perpendicular to the w-axis).

By way of example, here is part of the plot of the $R = 0$ fieldlines in the case $n = 2$:



(Fig 10)

As before, the $w-$axis is vertical, and the $\sigma$ axis is horizontal.[15]

# Chapter Three

# Functional analysis of Fourier Transform

## Introduction (3.1):

Functional analysis is a branch of mathematical analysis, the core of which is formed by the study of vector spaces endowed with some kind of limit-related structure (e.g. inner product, norm, topology, etc.) and the linear operators acting upon these spaces and respecting these structures in a suitable sense. The historical roots of functional analysis lie in the study of spaces of functions and the formulation of properties of transformation of functions such as the Fourier transform as transformation defining continuous, unitary etc. operators between functions spaces. This point of view turned out to be particularly useful for the study of differential and integral equations.

In modern introductory texts to functional analysis, the subject is seen as the study of vector spaces endowed with a topology, in particular infinite dimensional spaces. In contrast, linear algebra deals mostly with finite dimensional spaces, or does not use topology. An important part of functional analysis is the extension of the theory of measure, integration, and probability to infinite dimensional spaces, also known as infinite dimensional analysis.

The basic and historically first class of spaces studied in functional analysis are completenormed vector spaces over the real or complex numbers. Such spaces are called Banach spaces. An important example is a Hilbert space, where the norm arises from an inner product. These spaces are of fundamental importance in many areas, including the mathematical formulation of quantum mechanics.

An important object of study in functional analysis are the continuouslinear operators defined on Banach and Hilbert spaces.[12]

Hilbert spaces can be completely classified: there is a unique Hilbert space up toisomorphism for every cardinality of the orthonormal basis. Finite-dimensional Hilbert spaces are fully understood in linear algebra, and infinite-dimensional separable Hilbert spaces are isomorphic to $\ell^2(\mathcal{N}_0)$. Separability being important for applications, functional analysis of Hilbert spaces consequently mostly deals with this space. One of the open problems in functional analysis is to prove that every bounded linear operator on a Hilbert space has a proper invariant subspace. Many special cases of this invariant subspace problem have already been proven.

General Banach spaces are more complicated than Hilbert spaces, and cannot be classified in such a simple manner as those. In particular, Banach spaces lack a notion analogous to an orthonormal basis.

In Banach spaces, a large part of the study involves the dual space: the space of all continuous linear maps from the space into its underlying field, so-called functionals. A Banach space can be canonically identified with a subspace of its bidual, which is the dual of its dual space. The corresponding map is an isometry but in general not onto. A general Banach space and its bidual need not even be isometrically isomorphic in any way, contrary to the finite-dimensional situation. This is explained in the dual space article.

Also, the notion of derivative can be extended to arbitrary functions between Banach spaces.

Probably the biggest difference between analysis and algebra is that in the former one utilizes the concept of limits. With limiting notions, one can discuss such things as differentiation and integration and can deal with infinite processes in general; or, rather, one can avoid infinite processes by reducing them to a consideration of some finite process. If one looks at anything pertaining to limits of real or complex numbers one sees that they all rest on the fact that some measure of closeness can be ascribed to any pair of points. In order to wrest linear spaces from purely the realm of algebra, we now wish to introduce some sort of distance measuring device to those spaces and ultimately introduce limiting notion. Specifically, the concept of a norm on a vector space will be defined, and indeed almost all our subsequent results will involved normed spaces.[52]

## Normed space (3.2):

An inner product on X is a mapping from X×X, the Cartesian product space, into the scalar fields, which we shall denote generically by F:

$$X \times X \rightarrow F,$$

$$< X, Y > \rightarrow (X, Y)$$

That is, $(X, Y)$ denotes the inner product of the two vectors, wheareas $< X, Y >$ represents only the ordered pair X×X with the following properties:

1- Let x, y $\in X$; then (x, y) = $\overline{(y, x)}$ where the bar denotes complex conjugation;

2- If $\alpha$ and $\beta$ are scalars and x, y, and z are vectors, then

$$(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z);$$

3- $(x, x) \geq 0$for all $x \in X$ and equal to 0 if and only if $x$is the 0 vector. It is noted that by property (1) that $(x, x)$ must always be real. So this requirement always makes sense.

Some immediate consequences of the definition will now be noted, the first of which is that if a vector y has the property that $(x, y) = 0$ for all $x \in X$, then y must be the zero vector. To prove this, noting that this is true for every vector in the space. Take x = y and apply part (3) of the definition.

If x, y and z are vectors, $\alpha$ and $\beta$ are scalars, then

$$(z, \alpha x + \beta y) = \bar{\alpha}(z, x) + \bar{\beta}(z, y).$$

The mapping of X into F via $(x, x)^{1/2}$ is a norm on $X$ and will be denoted by $\|x\|$. The notion of a norm will be thoroughly elaborated upon later, and for the sake of the present discussion, it suffices to regard $\|x\|$ as merely a convenient shorthand representation for $(x, x)^{1/2}$.

We defined the notation $\|x\|$ to mean $(x, x)^{1/2}$ in an inner product space, and we would now like to note some further properties of this mapping. It is simple to verify that the following assertions about $\|x\|$ are valid:

$\| \quad \| : X \to R$,

$x \to \|x\|$ .

1- $\|x\| \geq 0$ and $= 0$ if and only if $x = 0$;

2- For a scalar $\alpha$ and a vector $x$, $\|\alpha x\| = |\alpha|\|x\|$;

3- $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality).

We shall verify here that the third assertion is indeed correct. consider

$$\|x + y\|^2 = (x + y, x + y)$$

$$= (x, x) + (x, y) + (y, x) + (y, y)$$

$$= \|x\|^2 + 2Re(x, y) + \|y\|^2.$$

But, since $2Re(x,y) \leq 2|(x,y)|$, we can say

$$\|x+y\|^2 \leq \|x\|^2 + 2|(x,y)| + \|y\|^2.$$

Appealing to the Cauchy-Schwarz inequality, we can now say

$$\|x+y\|^2 \leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2,$$

And it remains only to take the square root of both sides to complete the proof.

We now wish to view mapping of the above type abstractly; that is, we now single out the three properties cited above and define mappings of his tybe, on a real or complex space, to be norms.

To have a norm it is not necessary to have an inner product at all. Conversely, though, whenever one has an inner product, one can always define a norm right

From the inner product as we have already indicated. Real or complex spaces over which a norm is defined are called normed linear spaces (n. l. s), and we remark that there are normed linear spaces over which one cannot define an inner product that will "generate" [that is, no inner product such that $\|x\| = (x,x)^{1/2}$ ] the norm on the space. Thus the essential thing is to make the norm and inner product generate norm agree. As one might except, if further conditions are imposed on the norm, one can guarantee the existence of an inner product that will generate the norm. One such further condition is the parallelogram law. There are many other conditions that one might equally well impose upon the norm to guarantee an inner product representation but other interests prevent us from listing them all here. It is now necessary to leave normed spaces temporarily for the sake of a more general structure the metric space.[12]

## Metric spaces (3.3):

if$X$ is a normed linear space and$x, y \in X$, it easily verified that the function $d(x, y) = \|x - y\|$ satisfies the following conditions

1- $d(x, y) \geq 0$ and is equal to zero if and only if $x = y$;

2- $d(x, y) = d(y, x)$;

3- $d(x, z) \leq d(x, y) + d(y, z)$for any $x, y, z \in X$ (triangle inequality).

Again, in a manner analogous to the way the norm was defined abstractly, one can now make the following definition.[1]

# Definition (3.3.1):

$X$, an arbitrary set, will be called a metric space if there is a function

$d: X \times X \rightarrow R$satisfying properties 1-3 above. The mapping $d$ itself will be called a metric.

One thing that should be pointed out immediately is that, when one refers to metric spaces, it dose not suffice to mention just the set $X$ involved; indeed, one must speak of the pair $(X, d)$, for it is certainly possible that tow different metrics ould be defined on the same set and would, thus give rise to tow different metric spaces. To clarify these notions, several examples will now be given, nothing that normed space is, of course, a metric space.

**Example (1):**

Although the metric we shall define now is not too interesting for its own sake, it comes in quite handy in counterexample. In addition, it illustrates that one can define a metric over any set whatsoever

Let$X$ be an arbitrary set and consider the function

$$d(x,y)\begin{cases}1 & (x \neq y),\\ 0 & (x = y),\end{cases}$$

Where $xandy$ are members of$X$. It is exceedingly simple to verify that properties 1-3 are satisfied by this mapping. This particular metric is called the trivial metric.[1]

## Banach space (basic definition) (3.4):

In mathematics, more specifically in functional analysis, a Banach space is a completenormed vector space. Thus, a Banach space is a vector space with a metric that allows the computation of vector length and distance between vectors and is complete in the sense that a Cauchy sequence of vectors always converges to a well definedlimit in the space.

Banach spaces are named after the Polish mathematician Stefan Banach, who introduced and made a systematic study of them in 1920–1922 along with Hans Hahn and Eduard Helly. Banach spaces originally grew out of the study of function spaces by Hilbert, Fréchet, and Riesz earlier in the century. Banach spaces play a central role in functional analysis. In other areas of analysis, the spaces under study are often Banach spaces.

The following definition generalizes the notation of distance known from the everyday life.[13]

## Definition (3.4.1):

A metric (or distance function) d on a set $M$ is a function d :$M \times M \to \mathcal{R}_+$ from the set o pairs to non-negative real numbers such that:

1- $\mathrm{d}(\mathrm{x}, \mathrm{y}) \geq 0$for all $\mathrm{x}, \mathrm{y} \in \mathrm{M}$, $\mathrm{d}(\mathrm{x}, \mathrm{y}) = 0$ implies $\mathrm{x} = \mathrm{y}$.

2- $d(x, y) = d(y, x)$for all $\mathrm{x}, \mathrm{y} \in \mathrm{M}$.

3- $d(x, z) \leq d(x, y) + d(y, z)$for all $x, y, z \in M$ (triangle inequality).

## Definition(3.4.2):

Let $V$ be a (real complex) vector space. A norm on $V$ is a real-value function, written $\|x\|$, such that

1- $\|x\| \geq 0$ for all $x \in V$, and $\|x\| = 0$ implies $x = 0$.

2- $\|\lambda x\| = |\lambda|\|x\|$for all scalar $\lambda$ and vector $x$.

3- $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality).

A vector space with norm is called normed space.

The connection between norm and metric is as follows:

Proposition : if $\| . \|$ is a norm on $V$, then it gives a metric on $V$ by

$\mathrm{d}(\mathrm{x}, \mathrm{y}) = \|x - y\|$.

Figure1: triangle inequality in metric (a) and normed (b) space.

Proof: this is a simple exercise to derive items 1, 3 of definition 2 from corresponding items of definition 3. For example, see the figure to derive the triangle inequality

An important notations known from real analysis are limit and convergence. Particularly we usually wish to have enough limiting points for all

“ reasonable” sequences

## Definition(3.4.3):

A Banach space is a vector space$X$ over the field R of real numbers, or over the field C of complex numbers, which is equipped with a norm and which is completewith respect to that norm, that is to say, for every Cauchy sequence $\{x_n\}$ in $X$, there exists an element $x$ in $X$ such that

$$\lim_{n\to\infty} x_n = x$$

or equivalently:

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

The vector space structure allows one to relate the behavior of Cauchy sequences to that of converging series of vectors. A normed space $X$ is a Banach space if and only if each absolutely convergent series in $X$ converges,

$\sum_{n=1}^{\infty}\|v_n\|_x < \infty$implies that $\sum_{n=1}^{\infty} v_n$ converges in $X$

Completeness of a normed space is preserved if the given norm is replaced by an equivalent one.

All norms on a finite-dimensional vector space are equivalent. Every finite-dimensional normed space over R or C is a Banach space.[52]

**Example (2):**

Here is some examples of normed spaces

(i) $\ell_2^n$ is either $\mathcal{R}^2$ or $\mathbb{C}^2$ with norm defined by:

$\|(x_1 \ldots \ldots . x_n)\|_2 = \sqrt{|x_1|^2 + |x_2|^2 \ldots + |x_n|^2}$.

(ii) $\ell_1^n$ is either $\mathcal{R}^2$ or $\mathbb{C}^2$ with norm defined by:

$\|(x_1 \dots \dots . x_n)\|_1 = |x_1|^2 + |x_2|^2 \dots + |x_n|^2$

(iii) $\ell_\infty^n$ is either $\mathcal{R}^2$ or $\mathbb{C}^2$ with norm defined by:

$\|(x_1 \dots \dots . x_n)\|_\infty$ =max $(|x_1|. |x_2| \dots |x_n|)$

(iv) let $X$ be a topological space, then $C_b(X)$ is the space of continuous bounded functions : $X \to \mathbb{C}$ with norm $\|f\|_\infty = sup_X |f(X)|$.

(v) Let $X$ be any set then $\ell_\infty(X)$ is the space of all bounded (not necessarily continuous) functions $f: X \to \mathbb{C}$ with norm $\|f\|_\infty = sup_X |f(X)|$.

All these normed spaces are also complete and thus are Banach spaces.

## Hilbert space(3.5):

The mathematical concept of a Hilbertspace, named after David Hilbert, generalizes the notion of Euclidean space. It extends the methods of vector algebra and calculus from the two-dimensional Euclidean plane and three-dimensional space to spaces with any finite or infinite number of dimensions. A Hilbert space is an abstract vector space possessing the structure of an inner product that allows length and angle to be measured. Furthermore, Hilbert spaces must be complete, a property that stipulates the existence of enough limits in the space to allow the techniques of calculus to be used.

Hilbert spaces arise naturally and frequently in mathematics, physics, and engineering, typically as infinite-dimensional function spaces. The earliest Hilbert spaces were studied from this point of view in the first decade of the 20th century by David Hilbert, Erhard Schmidt, and FrigyesRiesz. They are indispensable tools in the theories of partial differential equations, quantum mechanics, Fourier analysis (which includes applications to signal processing and heat transfer) and Ergodic theory, which forms the mathematical underpinning of thermodynamics. The success of Hilbert space methods ushered in a very fruitful era for functional analysis. Apart from the classical Euclidean spaces, examples of Hilbert spaces include spaces of square-integrable functions, spaces of sequences, Sobolev spaces consisting of generalized functions, and Hardy spaces of holomorphic functions.

Geometric intuition plays an important role in many aspects of Hilbert space theory. Exact analogs of the Pythagorean theorem and parallelogram law hold in a Hilbert space. At a deeper level, perpendicular projection onto a subspace (the analog of "dropping the altitude" of a triangle) plays a significant role in optimization problems and other aspects of the theory. An element of a Hilbert space can be uniquely specified by its coordinates with respect to a set of coordinate axes (an orthonormal basis), in analogy with Cartesian coordinates in the plane. When that set of axes is countably infinite, this means that the Hilbert space can also usefully be thought of in terms of infinite sequences that are square-summable. Linear operators on a Hilbert space are likewise fairly concrete objects: in good cases, they are simply transformations that stretch the space by different factors in mutually perpendicular directions in a sense that is made precise by the study of their spectrum.[19]

## Definition(3.5.1):

A Hilbert space$H$ is a real or complexinner product space that is also a complete metric space with respect to the distance function induced by the inner product. To say that $H$ is a complex inner product space means that $H$ is a complex vector space on which there is an inner product $\langle x, y\rangle$associating a complex number to each pair of elements $x, y$ of $H$ that satisfies the following properties:

- The inner product of a pair of elements is equal to the complex conjugate of the inner product of the swapped elements:

$$\langle y, x\rangle = \overline{\langle x, y\rangle}.$$

- The inner product is linear in its first argument. For all complex numbers *a* and *b*,

$$\langle ax_1 + bx_2, y\rangle = a\langle x_1, y\rangle + b\langle x_2, y\rangle$$

- The inner product of an element with itself is positive definite:

$$\langle x, x\rangle \geq 0$$

where the case of equality holds precisely when $x = 0$.

It follows from properties 1 and 2 that a complex inner product is antilinear in its second argument, meaning that

$$\langle x, ay_1 + by_2\rangle = \bar{a}\langle x, y_1\rangle + b\langle x, y_2\rangle.$$

A real inner product space is defined in the same way, except that *H* is a real vector space and the inner product takes real values. Such an inner product will be bilinear that is, linear in each argument.

The norm is the real-valued function

$$\|x\| = \sqrt{\langle x, x\rangle},$$

and the distance *d* between two points *x*, *y* in *H* is defined in terms of the norm by

$$d(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y\rangle}.$$

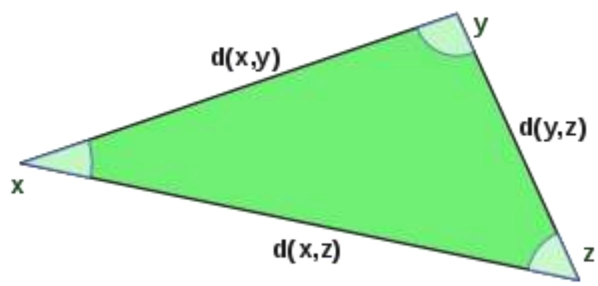
That this function is a distance function means:

(1) that it is symmetric in *x* and *y*.

(2) that the distance between $x$ and itself is zero, and otherwise the distance between $x$ and $y$ must be positive.

(3) that the triangle inequality holds, meaning that the length of one leg of a triangle *xyz* cannot exceed the sum of the lengths of the other two legs:

$$d(x,z) \leq d(x,y) + d(y,z)$$




This last property is ultimately a consequence of the more fundamental Cauchy–Schwarz inequality, which asserts

$$|\langle x,y\rangle| \leq \|x\|\|y\|$$

with equality if and only if $x$ and $y$ are linearly dependent.

Relative to a distance function defined in this way, any inner product space is a metric space, and sometimes is known as a pre-Hilbert space. Any pre-Hilbert space that is additionally also a complete space is a Hilbert space. Completeness is expressed using a form of the Cauchy criterion for sequences in $H$: a pre-Hilbert space $H$ is complete if every Cauchy sequenceconverges with respect to this norm to an element in the space. Completeness can be characterized by the following equivalent condition: if a series of vectors $\sum_{k=0}^{\infty} u_k$converges absolutely in the sense that

$$\sum_{k=0}^{\infty} \|u_k\| < \infty$$

then the series converges in $H$, in the sense that the partial sums converge to an element of H.

As a complete normed space, Hilbert spaces are by definition also Banach spaces. As such they are topological vector spaces, in which topological notions like the openness and closedness of subsets are well-defined. Of special importance is the notion of a closed linear subspace of a Hilbert space that, with the inner product induced by restriction, is also complete (being a closed set in a complete metric space) and therefore a Hilbert space in its own right.

Wewillseethat$L^2$isaHilbertspace(wealreadyreallyhaveallthe bitsofinformationweneedtoseethis)andthatinsomesensethe$L^2$-spaces(with different $\mu$’s)aretheonlyHilbertspaces.Wewillcometoseehowtheproblem thatFourierexamined,about decomposingfunctionsasinfinitesums ofother somehowmorebasicfunctions,isaproblembestphrasedandunderstoodinthe languageofabstractHilbertspaces.Oneofthetriumphsoffunctionalanalysisis totakeaveryconcreteprobleminthiscaseFourierdecompositionviewitin anabstractsetting,andusetheoreticaltoolstoobtainpowerfulresultsthatcanbe translatedbacktotheconcretesetting.Fourier’sworkcertainlyholdsanimportant spotattherootsoffunctional analysis,anditmotivated muchearlyworkinthe developmentofthefield.

Fourierbeginswithanarbitraryfunction $f$ ontheintervalfrom–$\pi$to$\pi$andstates that$i$fwecanwrite

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)$$

Thenitmustbethecasethatthecoefficients$a_k$and$b_k$aregivenbytheformulas

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)\, dx \qquad k = 0, 1, 2,$$

And

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)\, dx \qquad k = 1, 2,$$

The bigquestionisthis:whenisthisdecompositionactuallypossible?Eveniftheintegralsinvolvedmakesense,doestheseriesconverge? Ifitdoesconverge, whattypeofconvergence(pointwise,uniform,etc.)doweget?Eveniftheseriesconvergesinsomesense,doesitconvergeto $f$? Theimmediategoalistoshowyouhowtheequestionsabout Fourierseriescanbetreatedintheabstractsettingofaninnerproductspace.

Letusnowtakestockofwhatwealreadyknowbygatheringourinformation about$L^2$.First,recallthat $L^2 = L^2(\mu)$,foranyabstractmeasurespace$(X, R, \mu)$, denotesthecollectionofallmeasurablefunctions $f: X \to \mathbb{C}$ such that the integral

$$\int_x |f|^2\, d\mu$$

Isfinite.Thesefunctionsareoftencalledthe "squareintegrable"functionson X with norm

$$\|f\|_2 = \sqrt{\int_x |f|^2\, d\mu}$$

ThiscollectionoffunctionsbecomesaBanachspace.Wecandefineaninnerproduct on$L^2$ via

$$\langle f, g\rangle = \int_x f\bar{g}\, d\mu.$$

Itiseasilyseenthatthisisaninnerproduct,andthatthenormdoesindeedcomefromthisinne rproduct.Thatis,

$$\|f\|_2 = \sqrt{\langle f, f\rangle} = \sqrt{\int_x |f|^2\, d\mu}$$

ℝ: showsthat$L^2$isaHilbertspace.

Inthefollowingdefinitions,theterminologyshouldseemfamiliarfromyourexperiences with$\mathbb{R}^n$.

Let $(V, \langle .,.\rangle)$ beaninnerproductspace.Wesaythat$v$and$w$inVare orthogonal if$\langle v, w\rangle = 0$.Wesaythat$v$is normalized if $\|v\| = \sqrt{\langle v, v\rangle} = 1$. A sequence $\{v_k\}_{k=1}^{\infty}$ in$V$isan orthonormal sequenceif$\langle v_k, v_j\rangle = \delta_{kj}, 1 \leq k, j \leq \infty$. the trigonometric system

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos(nx)}{\sqrt{\pi}}, \frac{\sin(mx)}{\sqrt{\pi}}, \qquad n, m = 1,2, \ldots$$

Is an orthonormal sequence in the inner product space $L^2([-\infty, \infty], m)$Fromthis, you shouldfind itplausiblethat thegoal ofFourieranalysisisinitsgeneralsettin isthis:Givenanorthonormalsequence$\{f_k\}_{k=1}^{\infty}$ in an inner product space V and an $f \in V$, find complex numbers $c_k$ such that

$$f = \sum_{k=1}^{\infty} c_k f_k$$

Theconvergenceofthisinfinitesumisinthenorminducedbytheinnerproduct.

Further,itwouldbedesirabletobeabletodothis for all $f \in V$.Ingeneral, this cannot be done. Notice that Fourier was asserting that when $\{f_k\}_{k=1}^{\infty}$ is the trigonometric system,the coefficients are of form$\langle f, f_k \rangle$wheneverhisdecompositionworks.

Let $\{f_k\}_{k=1}^{\infty}$ be an orthonormalsequenceinV.Ifitisthecasethatfor each $f \in V$wecanfindconstants $c_k$ (dependingon$f$)suchthat

$$f = \sum_{k=1}^{\infty} c_k f_k$$

Thenwesaythatthe$\{f_k\}_{k=1}^{\infty}$isa complete orthonormalsequencein V. A complete orthonormalsequenceissometimescalledanorthonormalbasis for$V$. Thelatterterminologycancauseconfusionsinacompleteorthonormalsystem isnotabasisinthefinite-dimensionalThequestionsposedbyFourier'sworkare,tosome degree, answered by the fact that the trigonometric system doesindeedformacompleteorthonormalsequencein$L^2$. Thetrigonometricsystem iscertainlyanimportantcompleteorthonormalse- quence(fortheHilbertspace$L^2([-\pi,\pi])$).WecanusetheGram-Schmidtprocesstoconstructanorthonormalsequenceinanyinnerproductspace.

Forourfirstexample,theHilbertspaceis$L^2([-1,1])$.If one applies the Gram–Schmidtprocesstothefunctions$1, x, x^2, x^3, \ldots$ oneobtainsthecompleteorthonormalsequenceofLegender polynomials

$$\sqrt{\frac{2n+1}{2}} \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^n - 1)^n, \qquad n = 1, 2, \ldots$$

Next,considertheHilbertspace$L^2((0,\infty))$.If oneappliestheGram–Schmidtprocesstothefunctions $x^n e^{-x}, n = 0,1,..,$oneobtainsthecompleteorthonorma lsequenceof Laguerre functions.

Forourthirdexample,theHilbertspaceis$L^2(\mathbb{R})$.IfoneappliestheGram–Schmidt process other functions $x^n e^{\frac{-x^2}{2}}$, n=0, 1,2, …oneobtainsthecompleteorthonormalsequenceofHermite functions.

TheLegendre,Laguerre,andHermitefunctionsallshowupaseigenfunctions ofcertainlinearoperatorsrelatedtotheSturm–Liouvilleproblemindifferentialequations.

Thefinalfamilywediscussisthecompleteorthonormalsequenceof Haarfunctions.The Hilbertspaceis$L^2([0,1])$.Thisexampleisfundamentallydifferent fromthepreviousexamplesinthatthefunctionsinthisfamilyarenotcontinuous, andtheyarenotconnectedwithdifferential equations.[50]

## Open mapping theorem(3.6):

## Definition(3.6.1):

let$X, Y$ be metric spaces. A map T: $\mathcal{D}(T) \to Y$ with $\mathcal{D}(T) \subset X$ is called an open mapping if whenever U$\subset \mathcal{D}(T)$ is an open set, so is $T(U) \subset$ Y. that means T Takes open sets to open sets.[1]

let X, Y be Banach spaces, and T: X $\to Y$ a continuous linear map from X onto Y, ( T $\in B(X,Y)$). We shall show that T is an open mapping.

**Theorem (3.6.2):** A map T: X $\to Y$ between metric spaces is continuous if and only if whenever U$\subset$ Y is an open set, so is

$$T^{-1}(U) = \{x \in X \ / \ T(x) \in U\}$$

The preimage of U.

**Example (3):**

here are some examples of linear maps that are also open mappings.

- Identity map $\mathcal{R}^n \rightarrow \mathcal{R}^n$.This is obviously open because any open U$\subset \mathcal{R}^n$ is mapped to itself, an open set.
- Projection $\mathcal{R}^n \rightarrow \mathcal{R}^k$, defined by $(x_1, \ldots, x_n) \rightarrow (x_1, \ldots, x_k)$, Where $k < n$. Let U$\subset \mathcal{R}^n$ be an open set, and consider its image $T(U) \subset \mathcal{R}^K$. Let $y \in T(U)$. Then $y = T(x)$ for some $x \in U$. There is an open set of the form $B_\in(x) = (x_1 - \in,\ x_1 + \in) \times \ldots \times (x_n - \in,\ x_n + \in)$ where x= $(x_1, \ldots, x_n)$ and contained in U. then$T(B_\in(x))$= $(x_1 - \in,\ x_1 + \in) \times \ldots \times (x_K - \in,\ x_K + \in)$ is an open set contained in $T(U)$and containing the given point y, so $T(U)$ is open and projection is an open mapping.

## Open mapping theorem (3.7):

Let X, Y be Banach spaces. If T$\in B(X, Y)$ is surjective, then it is an open mapping. The theorem follows from the following lemma.

## Lemma(3.7.1):

let X,Y be Banach spaces. If T$\in B(X, Y)$ is surjective, then the image T($B_1$) of the open unit ball(centered at 0$\in X$) contains an open ball around 0$\in Y$.

Proof: The proof involves the Baire Category theorem. Let $B_r:=\{x \in X \,|\, \|x\| < r\}$ always denote the open ball of radius $r > 0$ centered at$0 \in X$, and $B_r(x_0)$ centered at$x_0$.

Clearly,

$$X = \bigcup_{n=1}^{\infty} B_{n/2}$$

*since* T is surjective, we have that

$$Y = \bigcup_{n=1}^{\infty} T(B_{n/2})$$

We have also assumed that Y is Banach, hence complete. The Baire Category theorem now guarantees that there must be some $n_0 \in \mathbb{N}$ such that $\overline{T(B_{n_0/2})}$ has non -empty interior. This implies that $\overline{T(B_{n_0/2})} = n_0\overline{T(B_{1/2})}$ has non -empty interior, hence, $\overline{T(B_{1/2})}$ has non -empty interior, meaning it contains some ball $B_\epsilon(y_0)$ centered at some point $y_0 \in \overline{T(B_{1/2})}$.

Suppose that X and Y are two normed spaces and T is a linear operator from X to Y. consider the equation

$$Tx=y, \qquad (1)$$

Where $y \in Y$ is given and $x \in X$ is unknown. The following are some aspects of the equation that are usually considered.

- Existence:IfTissurjective,then(1)hasatleastonesolutionx∈X for every y∈Y.

- Uniqueness:IfTisinjective,then(1)hasnotmorethanonesolutionx∈Xforeveryy∈X.

-Existenceanduniqueness:

IfTisobjective(injectiveandsubjective),then $T^{-1}$exists

and(1)haspreciselyonesolutionx=$T^{-1}y$∈Xforeveryy∈X.

Stability: Supposethat $T^{-1}$isbounded.If $y = y_e + y_\varepsilon$,where $y_e$istheexactright-hand sideand$y_\varepsilon$is the errorintheright-hand side,then

$$x = T^{-1}(y_e + y_\varepsilon) = T^{-1}y_e + T^{-1}y_\varepsilon = x_e + T^{-1}y_\varepsilon.$$

Where $x_e$istheexactsolutionto(1),whichmeansthat?

$$\|x - x_e\| = \|T^{-1}y_\varepsilon\| \le \|T^{-1}\|\|y_\varepsilon\|$$

Hence, small errors in theright-hand side of (1) will give small errors in the solution.

SupposethatXandYarenormedspacesandTisaninvertiblelinearoperatorfromX to Y. Then$T^{-1}$ is a continuous (bounded) operatorfromYtoXifandonlyif

$$(T^{-1})^{-1}(G) = T(G)$$

IsanopensubsetofYforeveryopensubsetGofX.?

Definition: SupposethatXandYarenormedspacesandT is a linear operator from X to Y. the operator T is said to be open if $T(G)$ is open for every open supset G to X.

## Theorem (3.7.2):

SupposethatXandYaretwoBanachpaces. Thenalinearoperator T∈B(X,Y)isopenifandonlyifTissubjective.

### Remark (3.7.3):

Thehardestpartoftheproofusedtoprove"sufficiency"iscontained in thefollowinglemma,whoseproofweshallomit.

## Lemma(3.7.4):

SupposethatXandYaretwoBanachspacesand $T \in B(X,Y)$.IfTissurjective, then T($B_1$(0)) contains anopenball $B_\varepsilon$(0).

**proof:**

- **Necessity:**

- SinceTisopen,T($B_1$(0))containsanopenball$B_\varepsilon$(0).
- SinceTislinear,itfollowsthat $B_R$(0)⊂T($B_{\frac{R}{\varepsilon}}$(0))foranynumberR>0.

- **Sufficiency:**
- SupposethatG⊂Xisopenandy=$T_X$∈T(G).
- Thenthereexistsanopenball$B_r$(x)⊂G.
- Thisimpliesthat $B_r$(0)⊂G−x.
- According tothelemma,thereexistsanopenball
- $B_\varepsilon$(0)⊂T($B_1$(0)).
- Butthen

$$B_{r\varepsilon}(0) \subset T\big(B_r(0)\big) \subset T(G-X) = T(G) - y,$$

So $B_{r\varepsilon}(y)$⊂ $T(G)$.

Let us consider the equation

$$F(x)=0 \qquad (2)$$

## Theorem (3.7.5):

let F be mapping of X Into Y, where X, Y are linear normed spaces. Let Z be a Banach space and f, g mapping such that $f\colon Y \to Z, g\colon Z \to X$,let y be alinear continuous mapping of Z onto Z and E a closed subset of Z. furthermore, let the following conditionsbe fulfilled:

1- for every $z_1, z_2 \in E$ the inquality

$$\big\|fF\big(g(z_1)\big) - fF\big(g(z_2)\big) - y(z_1 - z_2)\big\| \le \alpha\|z_1 - z_2\|$$

Holds, where the mapping F, f are such that $m(F) = b > 0$ on $g(E) \subset X$ and $m(f) = a > 0$ on $F\big(g(E)\big) \subset Y$,f(0)=0.

2- the closed ball$D = \{z \in Z;\ \|z \quad - z_1\| \leq r\}$ , is contained in E, where $z_1$ is defined by the equality $y_0 = y(z_1 - z_0), z_0$ being an arbitrary element of E, $y_0$ being defined by $y_0 = \ fF\big(g(z_0)\big), r \geq \beta(1-\beta)^{-1}.\ \|x_1 - x_0\|,\ \beta = \alpha m < 1,$ where m is constant . then the equation (2) has a unique solution $x^*$in $\overline{g(D)} \subset X$. the sequence$\|x_m\|$ defined by $x_m = g(z_m), y_{m-1} = y(z_m - z_{m-1}),$

$$y_m = y_{m-1} - fF\big(g(z_{m-1})\big) + fF\big(g(z_m)\big),$$

converges in the norm topology of X to $x^*$.

3- $\|x^* - x_m\| \leq \beta^m(\|y\| + \alpha)[ab(1-\beta)]^{-1}\|z_1 - z_0\|$.

## Uniform Boundedness Principle (3.8):

Let X be a Banach space and Y be a normed space. Suppose the sequence $\{T_n\} \subset\ B(X,Y)$ has the property that for every $x \in X$, the sequence $\{T_n(x)\} \subset Y$ is bounded. Then the sequence $\{\|T_n\|\}\subset\mathbb{R}$ of norms is bounded.

Remark: if X is Banach space, then pointwiseboundedness implies uniform boundedness.[18]

**Example(4):**

consider

$$\text{X}=\{p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 +\ \dots + \alpha_d x^d \big| \alpha_i \in \mathbb{F}, d \in \mathbb{N} \cup \{0\}\},$$

The set of all polynomials, and consider the norm defined by $\|p\| = max_i|\alpha_i|$ we readily see that this turns X into a normed space. Ut we claim that Xis not complete, and is therefore not Banach space : since we want to use the unifom boundedness principle to prove this, firest we show that  the conclusion of the UBP does not hold on X, and therefore X cannot be complete. In other words, we must find a sequence $\{T_n\} \subset B(X,X)$ that is bounded pointwise, but not uniformly.

We take $T_n(p)$=$\alpha_0 + \alpha_1 + \cdots + \alpha_{n-1}$. And if p=0, then T($p$). Then the linear operator $T_n$ is bounded because $\alpha_j \leq max_i|\alpha|$=$\|p\|$ for every j. $\|T_n(p)\| \leq n\|p\|$, and we indeed have $\{T_n\} \subset$ B(X,X).

Now, fixing some polynomial p(x)=$\alpha_0 + \alpha_1 x + \cdots + \alpha_d x^d$, we examine the sequence $\{\|T_n(p)\|\}_{n\in N}$. For any n, we have

$$\|T_n(p)\| = |\alpha_0 + \alpha_1 + \cdots + \alpha_{n-1}|$$
$$\leq |\alpha_0 + \alpha_1 + \cdots + \alpha_d|$$
$$\leq |\alpha_0| + |\alpha_1| + \cdots + |\alpha_d|$$
$$\leq d\|p\|,$$

And for a fixed polynomial, $\|p\|$ and d are fixed numbers, so we have shown that the sequence $\{\|T_n(p)\|\}_{n\in N}$is a bounded sequence.

Now we claim that $\{\|T_n\|\}_{n\in N}$, the sequence of norms of $T_n$, is not bounded. for every $T_n$ we choose an "inconvenient" polynomial, $p_n(x) = 1 + x + x^2 + \cdots + x^{n-1}$. then $\|p_n\| = 1$. But $\|T_n(p_n)\| = n$, which informs us that $\|T_n\| \geq n \to \infty$. so $\{T_n\}$ is not bounded as a sequence in B(X,X), contraryto the conclusion of uniform boundedness principle. So X cannot be complete.

**Example (5):**

suppose we have a sequence of complex numbers $x = (x_1, x_2, \ldots) \subset \mathbb{C}$ with the property that whenever $y = (y_1, y_2, \ldots) \subset \mathbb{C}$ is a sequence that converges to $y_n \to 0$, we have that the sum $\sum_i^\infty x_i y_i$ converges. Show that $\sum_i^\infty |x_i|$ converges.

Solution: the first step is of course to translate this problem into familiar functional analytic terms. The condition that the sequence y converges to 0 is simply the statement that y is a member of

$$c_0 = \{y \in l_\infty | y_n \to 0\} = \{\text{the set of sequence converging } 0\}$$

And we use the fact that $c_0$ is a Banach space with norm$\|.\|_\infty$. and x is just some sequence, which we would like to show is an element of $l_1$.

Since we would like to use Uniform Boundedness Principle in some way, let us start by finding a sequence $\{T_n\} \subset \mathrm{B}(c_0, y)$ where y is some normed space.

Let

$$T_n(y) = \sum_{i=1}^{n} x_i y_i, \quad \text{where} \quad y = (y_1, y_2, \dots) \in c_0$$

Be the truncated sum which we have assumed converges. It is clear that

$T_n \subset \mathrm{B}(c_0, \mathbb{C}) = (c_0)'$. Indeed, the calculation

$$|T_n(y)| = \left|\sum_{i=1}^{n} x_i y_i\right| \leq \sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|\right) \|y\|_\infty$$

Shows that the $T_n$ are bounded operators. The assumption that $\sum_{i=1}^{n} x_i y_i$ converges for any $y \in c_0$ implise that for any $y \in c_0$, the sequence $\{T_n(y)\}$ is convergent, hence bounded. So $\{T_n\}$ is a pointwise bounded sequence of operators. Therefore, by the uniform boundedness principle, it is uniformaly bounded, meaning thathere is some $M > 0$ such that $\{T_n\} \leq M$ for all $n \in \mathbb{N}$.

Now, another way to express $T_n$ is to let $x^{(n)} = (x_1, x_2, \dots, x_n, 0, \dots)$ be a truncated version of x (so that$\{x^{(n)}\}_{n \in \mathbb{N}}$) is a sequence in$l_1$), and let

$T_n(y) = \sum_{i=1}^{\infty} x_i^{(n)} y_i$ where $y = (y_1, y_2, \dots) \in c_0$.

Suppose we define $y^{(n)} = (y_1^{(n)}, y_2^{(n)}, \dots) \in c_0$ by

$$y_k^{(n)} = \begin{cases} \bar{x}_k^{(n)}, / \left| x_k^{(n)} \right| & if \;\; x_k^{(n)} = 0 \\ 0 \;\; if \;\; x_k^{(n)} = 0 \end{cases}$$

Then $T_n\left(y^{(n)}\right) = \left\| x^{(n)} \right\|_1 = \left\| x^{(n)} \right\|_1 \left\| y^{(n)} \right\|_\infty$. This implies that $\|T_n\| \geq \left\| x^{(n)} \right\|_1$, which in turn implies that $\left\| x^{(n)} \right\|_1 \leq M$ for all n. but it clear from the definition of $x^{(n)}$ that $\{\left\| x^{(n)} \right\|_1\} \subset \mathbb{R}$ is an increasing sequence of real numbers. Being bounded above by M, it must converge, hence

$$\sum_{i}^{\infty} |x_i| < \infty$$

And $x \in l_1$ is desired.[10]

## Theorem(3.8.1):

LetXbe a Banach space andYa normed space. Let E⊂L(X,Y)beanyset of bounded linear operators.suppose that for everyx∈X,thereexists$M_x \geq 0$ such that, for all T∈E,
$\|Tx\| \leq M_x$.

Then there exist M$\geq 0$ such that, for all T$\in E$,

$$\|T\| \leq M$$

Remark: Theconclusionofthetheoremstatesthattheoperators$T$in$E$areuniformly bounded.

Proof:

Foreach$x$∈$X$,let$M_x$bechosenasinthestatementofthetheorem.Foreachn,let$K_n$⊂$X$bethec losedsetdefinedby

$$K_n = \{x: \|Tx\| \le n,\ \forall T \in E\}.$$

If$n$≥$M_x$,then$x$belongsto$K_n$; sotheunionofallthese$K_n$ isthewholeof$X$.Sonotallofthe$K_n$canbenowheredense,bytheBairetheoreminitsseco ndformabove.Sowecanfindaparticular $n$sothattheclosedset$K_n$actuallycontainsaclosedball:

$$\bar{B}(x_0;\ \varepsilon) \subset K_n.$$

If$x$∈$X$,then$x_0 + \varepsilon x/\|x\|$belongstotheclosedball$\bar{B}(x_0;\ \varepsilon)$andthereforebelongstotheclo sedset$K_n$.So,forall$x$andall$T$∈$E$,

$$\left\| T\left(x_0 + \frac{\varepsilon x}{\|x\|}\right) \right\| \le n.$$

Withanapplicationofthetriangleinequality,thisyields(forall$x$and$T$∈$E$)

$$\|Tx\| \le \frac{1}{\varepsilon}(n + \|Tx_0\|)\|x\|$$

$$\le \frac{1}{\varepsilon}(n + M_{x_0})\|x\|$$

So

$$\|T\| = \le \frac{1}{\varepsilon}(n + M_{x_0})$$

For all T in E.

Wecansharpenupthestatementofthetheoremalittle.

Thelastpartoftheproofshowsthatifthereisany$n$suchthat$K_n$ containsaball,thenthefamilyofoperators$E$⊂$L(X,Y)$isuniformlybounded.Tostatethisth eotherwayround:ifthefamilyofoperators$E$isnotuniformlybounded,thenevery$K_n$isno wheredense.Thisleadstothefollowingversion:

## Corollary(3.8.2):

SupposeE ⊂ L(X,Y ) isnotuniformlybounded:thatis,

$$\sup\{\|T\|: T\in E\}=\infty$$

Thenthere xisx ∈ Xsuchthat

$$\sup\{\|Tx\|: T\in E\}=\infty$$

Indeed,thesetofpointsx ∈ Xwiththispropertyiscomeager.

**Proof:**

Thefirstpartofthiscorollaryisjustarestatementoftheoriginaltheorem. For thesecondpart,wenotedabovethatif$E$isnotuniformlybounded,thentheclosedset

$$K_n=\{x: \|Tx\|\leq n, \forall T\in E\}.$$

Isnowheredense,forall$n$.TheunionK=$U_n K_n$isthereforemeager.If$x$belongstothecomeagerset$X\backslash K$,then

$$\sup\{\|Tx\|: T\in E\}=\infty$$

Thisprovesthecorollary.[8]

## Application:non-convergenceofFourierseries(3.8.3):

LetC(T)denotetheBanachspaceofcontinuousperiodicfunctions f:ℝ→ℂwithperiod1.Foreachf∈C(T),let $s_n f$ denoteasbeforethenthe

partialsumofitsFourierseries.Inclass,weindicatedtheconstruction ofafunctionfforwhichthepartialsums($s_n f$)(0)atx=0divergedas n→∞.

Usingtheuniformboundednessprinciple,wecanestablishtheexistenceofsuchanf,ands howfurtherthatfunctionsfwiththisapparentlypathologicalbehaviorarecomeagerinC( T).

## Proposition (3.8.4):

there is acomeager set of functions f ∈C(T)for which the partial sums of the

Fourier series atx=0,($s_n f$)(0),diverge

$$sup|(s_n f)(0)| = \infty$$

Proof:Foreachn≥0,let $T_n \in L(C(T), \mathbb{C})$bethelinearfunctional

$T_n(f)$=($s_n f$)(0))

Each$T_n$ isaboundedlinearfunctional,butthecollectionoffunctions

$$E = \{T_n : n \in N\}$$

Isnotuniformlybounded.

Indeed,aswesawinclass,bytaking$f_n$ tobeacontinuousapproximationtothe

$$sign(D_n)$$

where $D_n$ istheDirichletkernel,wecanseethat

$$\|T_n\| \geq \int_0^1 |D_n|$$

Whichdivergeslikelogn.Bytheabovecorollarytotheuniformboundednesstheorem, thereexistsacomeagersetZ,offunctionsf∈C(T)suchthat

$$sup|T_n f| = \infty$$

Forall $f \in Z$.Thisstatementispreciselytheconclusionofthe Proposition. Thesameargumentapplies,ofcourse, tothevaluesof $s_n f$ atanypoint $x \in [0,1]$,notjustx=0.Becausea countable eintersectionofcomeagersetsiscomeager,wededuce:

**Corollary(3.8.5):**

Givenanycountable set of points $\{x_k\}_{k\in N}$ in$[0,1]$,wecanfinda continuous periodic function $f \in C(T)$whose Fourier series is divergent an every$x_k$.Indeed,thefunctionsf with this property are a coeager set.

So,atanygivencountablesetofpoints,convergenceofaFourierseriesistheexception, nottherule.

Thisshouldbecontrastedwiththecorollaryto Carleson'sverydifficulttheorem,whichtell susthatforanycontinuousperiodicfunctionf,thepartialsums$s_n f(x)$convergeforalmos tallx.

## Operators (3.9):

The supset D is called the Domain of definition of the operator A and denoted by Dom(A) ; the set {A(x):x∈ $D$} is called the domain of values of the operator A or its range , and is denoted by R(A).

If A is an operator from X into Y where X=Y, then A is called an operator on X.

If Dom(A)=X, then A is called an everywhere defined operators.

If $A_1, A_2$ are operators from $X_1$ into $Y_1$ and from $X_2$ into $Y_2$ with domains of definition Dom($A_1$) and Dom($A_2$), respectively, such that

Dom($A_1$) $\subset$ Dom($A_2$), and $A_1 x = A_2 x$ for all $x \in$ Dom($A_1$), then if $X_1 = X_2$, $Y_1 = Y_2$, the operator $A_1$is called a compression or restriction of the operator $A_2$, while$A_2$ is called an extension of $A_1$ ; if $X_1 \subset X_2$, $A_2$ is called an extension of $A_1$exceeding$X_1$.

Many equation in function spaces or abstract spaces can be expressed in the form Ax=y, where x$\in U$, y$\in V$ , where U, V are vector spaces; y is given, x is unknown and A is an operator from U into V. the assertion of the existence of a solution to this equation for any right-hand side y$\in V$ is equivalent to the assertion that the range of the operator A is the whole space V; the assertion that the equation Ax=y has a unique solution for any y$\in R(A)$ means that A is a one-to-one mapping from Dom(A) onto $R(A)$.

If U and V are vector spaces, Then in the set of all operators from U into V it is possible to single out the class of linear operators, the remaining operators from U into V are called non-linear operators.

If U and V are topological vector spaces, then in the set of operators from U into V the class of continuous operators can be naturally singled out. So are the class of bounded linear operators A (operators A such that the image of any bounded set in U is bounded in V) and the class of compact linear operators .

The most common kind of operator encountered are linear operators. Let U and V be vector space over field K. operator A:$U \rightarrow V$ is called linear if

$$A(\alpha x + \beta y) = \alpha Ax + \beta Ay$$

For all x, y in U and for all $\alpha, \beta$ in K.

The importance of linear operators is partially because they are morphisms between vector spaces.

In finite-dimensional case linear operators can be represented by matrices in the following way. Let K be a field, and U and V be finite-dimensional vector spaces over K. let us select a basis $u_1, u_2, \dots, u_n$, in U and $v_1, v_2, \dots, v_n$, in V. then let $x = x^i u_i$ be an arbitrary vector in U, and A:$U \to V$ be linear operator. Then

$$AX = x^i A u_i = x^i (Au_i)^j V_j.$$

Then $a_i^j := (Au_i)^j \in K$ is a matrix of the operator A in fixed bases $a_i^j$does not depend on the choiceof x, and Ax=y iff$a_i^j x^i = y^j$. Thus in fixed bases n-by-m matrices are in bijective correspondence to linear operators from U to V.[12]

If U and V are locally convex spaces, then it is natural to examine different topologies on U and V; an operator is said to be semi-continuous if it defines a continuous mapping from the space U into the space V with the weak topology (the concept of semi-continuity is mainly used in the theory of non-linear operators ); an operator is said to be strongly continuous if it is continuous as a mapping from U with the boundedly weak topology into the space V; an operator is called weakly continuous if it defines a continuous mapping from U into V where U and V have the weak topology. Compact operators are often called completely- continuous operators. Sometimes the term “completely- continuous operator” is used instead of “strongly- continuous operator”, or to denote an operator which maps any weakly-convergent sequence to a strong-convergent one; if U and V are reflexive

Banachspaces, then these conditions are equivalent to the compactness of the operator. If an operator is strongly continuous, then it is weakly continuous

The important concepts directly related to operators between finite-dimensional vector space are the ones of rank, determinant, inverse operator, and eigenspace.

Linear operators also play a great role in the infinite- dimensional case. The concept of rank and determinant cannot be extended to infinite- dimensional matrices. This is why very different techniques are employed when studying liner operators in the infinite- dimensional case. The study of linear operators in the infinite- dimensional case is known as functional analysis.

The space of sequences of real numbers, or more generally sequences of vectors in any vector space, themselves form an infinite-dimensional vector space. The most important cases are sequences of real or complex numbers, and these spaces, together with linear subspaces, are known as sequence spaces. Operators on these spaces are known as sequence transformations.

Bounded linear operators over Banach space form a Banach algebra in respect to the standard operator norm. The theory of Banach algebras develops a very general concept of spectra that elegantly generalizes the theory of Eigen spaces.

The Fourier transform is useful in applied mathematics, particularly physics and signal processing. It is knows as integral operator; it is useful mainly because it converts a function on one (temporal) domain to a function on another (frequency) domain, in a way effectively invertible. Nothing significant is lost, because there is an inverse transform operator. In the simple case of periodic functions, this result is based on the theorem that any continuous periodic function can be represented as the sum of a series of sine waves and cosine waves:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\omega n t) + b_n sin(\omega n t)$$

Coefficients *($a_0$,$a_1$,$b_1$,$a_2$,$b_2$, ...)* are in fact an element of an infinite-dimensional vector space $\ell^2$, and thus Fourier series is a linear operator.

When dealing with general function R → C, the transform takes on an integral form:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(\omega) e^{i\omega t}$$

[52]

# Chapter Four

# Engineering Interpretation of Fourier Transform

## Introduction (4.1):

The Fourier transformbyJeanBaptisteJosephFourier isanindispensabletoolfor manyfieldsofmathematics,physics,computerscienceandengineering. Especially the analysis and solution ofdifferential equations orsignaland imageprocessingcannotbeimaginedwithoutitany more. Its kernel consistsofthecomplexexponentialfunction. With thesquarerootofminusone,theimaginaryunitI,aspart of theargumentitisperiodicandthereforesuitablefortheanalysisofoscillatingsystems. William KingdonCliffordcreated the geometric algebras in1878. They usually contain continuoussubmanifolds ofgeometric square roots of minusone.Each multivectorhas anaturalgeometric interpretationso the generalization of the Fourier transform tomultivectorvaluedfunctionsin thegeometricalgebrasisveryreasonable. Ithelpstointerpret thetransform, apply itinatargetoriented waytothespecificunderlying problemandallows anewpointofviewonfieldmechanic.

Applicationoriented manydifferentdefinitionsof Fourier transformsin geometric algebras weredeveloped. For example the CliffordFourier trans-formintroducedbyJancewicz andexpanded byEblingandScheuermannandHitzer andMawardi, ortheoneestablishedbySommen, and re-establishedbyBǘlow.FurtherwehavethequaternionicFourier transformbyEll, and later byBǘlow,the space-time Fourier transformbyHitzer , the CliffordFourier transformforcolorimagesbyBatardetal. , the Cylindrical Fourier transformbyBrackxetal.thetransforms byFelsberg ,orEllandSangwine.Allthese transformshavedifferentinterestingproperties and deserve to bestudied independentlyfrom oneanother.Butthe analysis oftheirsimilarities reveals alot abouttheir qualities, too.Weconcentrateonthis matterand summarize allofthemin onegeneraldefinition.

Recently therehavebeenverysuccessfulapproachesbyDeBie,Brackx, DeSchepper andSommentoconstructCliffordFourier transformsfromoperator exponentialsanddifferential equations .Thedefinition presentedinthis research doesnotcoverallofthem,partly becausetheirclosed integral formisnotalwaysknownorhighlycomplicated,andpartly because they canbeproduced bycombinationsandfunctions ofourtransforms.

Wefocusoncontinuous geometric Fourier transformsoverflatspaces $R_{p,q}$intheir integral representation.Thatwaytheir finite, regular discrete equivalents as used in computationalsignal and image processing can be intuitively constructedanddirectapplicabilitytotheexistingpracticalissues andeasynumerical manageabilityareensured.[26]

## Definition (4.2):

Weexamine geometric algebras $C\ell_{p,q}, p+q=n \in N$over$\mathbb{R}^{p+q}$generatedbythe associative, bilinear geometric productwith neutralelement1 satisfying

$$e_j e_k + e_k e_j = \epsilon_j \delta_{jk}$$

Forall$j, k \in \{1, \dots, n\}$withtheKronecker symbol $\delta$ and

$$\epsilon_j = \begin{cases} 1 & \forall j = 1, \dots, p \\ -1 & \forall j = p+1, \dots, n \end{cases}$$

For the sake of brevity wanttorefertoarbitrarymultivectors

$$A = \sum_{k=0}^{n} \sum_{1 \le j_1 < \cdots < j_k \le n} a_{j1\ldots jk}\ e_{j1} \ldots e_{jk} \in c\ell_{pq}$$

$a_{j1\ldots jk} \in \mathbb{R}$, as

$$A = \sum_{j} a_j e_j$$

Whereeachofthe $2^n$ multi-indices $\boldsymbol{j} \subseteq \{1,\ldots,n\}$indicates abasisvector of $C\ell_{p,q}$by$e_j = e_{j1} \ldots e_{jk}, 1 \le j_1 <\ldots< j_k \le n, e_{\emptyset} = e_0 = 1$ anditsassociated coefficient$a_j = a_{j1\ldots jk} \in\mathbb{R}$.[16]

## Definition (4.3):

Theexponential functionofamultivectorA$\in C\ell_{p,q}$is defined bythepowerseries

$$e^A := \sum_{j=0}^{\infty} \frac{A^j}{j!}$$

## Lemma (4.4):

For twomultivectors$AB = BA$that commute amongst each other wehave

$$e^{A+B} = e^A e^B$$

**Proof:**

Analogoustotheexponentruleofrealmatrices.

Notation: For each geometric algebra $C\ell_{p,q}$wewillwrite $\mathfrak{I}^{p,q} = \{i \in C\ell_{p,q},$ $i^2 \in \mathbb{R}^-\}$ todenote the realmultiples ofallgeometric square roots of Minus one.Wechosethesymbol $\mathfrak{I}$tobereminiscentof the imaginary numbers.

## Definition (4.5):

Let$C\ell_{p,q}$ beageometric Algebra, A:$\mathbb{R}^m \to C\ell_{p,q}$beamultivector fieldand $\boldsymbol{x}, \boldsymbol{u} \in \mathbb{R}^m$vectors.

AGeometricFourierTransform(GFT)$\mathcal{F}_{F_1,F_2}(A)$isdefinedbytwoordered finitesets

$$F_1 = \{f_1(\boldsymbol{x},\boldsymbol{u}), \ldots, f_\mu(\boldsymbol{x},\boldsymbol{u})\}, F_2=\{f_{\mu+1}(\boldsymbol{x},\boldsymbol{u}), \ldots, f_v(\boldsymbol{x},\boldsymbol{u})\}$$

of mappings $f_k(\boldsymbol{x},\boldsymbol{u}): \mathbb{R}^m \times \mathbb{R}^m \to \mathfrak{I}^{p,q}, \forall k =$

$1, \ldots, \nu$andthecalculationrule

$$\mathcal{F}_{F_1,F_2}(A)(u) := \int_{\mathbb{R}^m} \prod_{f \in F_1} e^{-f(x,u)} \,.\, A(x) \prod_{f \in F_2} e^{-f(x,u)} \, d^m x$$

This definition combines many Fouriertransformsto asinglegeneral one.It enables ustoproofthe wellknown theorems just dependentonthe properties ofthechosenmappings.

## General Properties (4.6):

First weproofgeneralproperties validforarbitrarysets $F1, F2$.

**Theorem(4.6.1):**

Thegeometric Fouriertransformexistsforallintegrablemultivector fields $A \in L1(\mathbb{R}^n)$.

**Proof:**

Theproperty

$$f_k^2(x,u) \in \mathbb{R}^-$$

Ofthemappings $f_k$ for $k = 1, \ldots, \nu$ leadsto

$$\frac{f_k^2(x,u)}{\left|f_k^2(x,u)\right|} = -1$$

Forall$f_k(x,u) \neq 0$.Sousingthedecomposition

$$f_k(x,u) = \frac{f_k(x,u)}{|f_k(x,u)|}|f_k(x,u)|$$

Wecanwrite $\forall j \in \mathbb{N}$

$$f_k^j(x,u) = \begin{cases} (-1)^l |f_k(x,u)|^j & for\ j = 2l, l \in \mathbb{N}_0 \\ (-1)^l \frac{f_k(x,u)}{|f_k(x,u)|}|f_k(x,u)|^j & for\ j = 2l+1\ , l \in \mathbb{N}_0 \end{cases}$$

Whichresults in

$$e^{-f_k(x,u)} = \sum_{j=0}^{\infty} \frac{(-f_k(x,u))^j}{j!}$$

$$= \sum_{j=0}^{\infty} \frac{(-1)^j |f_k(x,u)|^{2j}}{(2j)!} - \frac{f_k(x,u)}{|f_k(x,u)|} \sum_{j=0}^{\infty} \frac{(-1)^j |f_k(x,u)|^{2j+1}}{(2j+1)!}$$

$$= \cos(|f_k(x,u)|) - \frac{f_k(x,u)}{|f_k(x,u)|} \sin(|f_k(x,u)|)$$

Becauseof

$$\left|e^{-f_k(x,u)}\right| = \left|\cos(|f_k(x,u)|) - \frac{f_k(x,u)}{|f_k(x,u)|} \sin(|f_k(x,u)|)\right|$$

$$\leq |\cos(|f_k(x,u)|)| + \left|\frac{f_k(x,u)}{|f_k(x,u)|}\right| \left|\sin(|f_k(x,u)|)\right| \leq 2$$

Themagnitudeoftheimproper integral

$$\left|\mathcal{F}_{F_1,F_2}(A)(u)\right| = \int_{\mathbb{R}^m} \prod_{f \in F_1} e^{-f(x,u)} \,.\, A(x) \prod_{f \in F_2} e^{-f(x,u)} \, d^m x$$

$$\leq \int_{\mathbb{R}^m} \prod_{f \in F_1} \left|e^{-f(x,u)}\right| . |A(x)| \prod_{f \in F_2} \left|e^{-f(x,u)}\right| d^m x$$

$$\leq \int_{\mathbb{R}^m} \prod_{f \in F_1} 2 \, |A(x)| \prod_{f \in F_2} 2 \, d^m x$$

$$=2^v \int_{\mathbb{R}^m} |A(x)| \, d^m x$$

IsfiniteandthereforethegeometricFouriertransformexists.

## CoorthogonalityandBases(4.7):

Wecalltwovectors$v, w$orthogonal($v \perp w$)if $v \cdot w = 0$andcolinear($v \| w$) if $v \wedge w = 0$.

## Definition(4.7.1):

Wecalltwo blades $A, B$orthogonal($A \perp B$)ifalloftheir generating vectorsaremutuallyorthogonalandcolinear($A \| B$) ifallvectors fromonebladearecollinear toallvectors intheother one.

Foravector $v$ andablade $B = b_1 \wedge \dots \wedge b_d$thefollowingequalities hold

$v \perp B \Leftrightarrow vB = v \wedge B \Leftrightarrow v \cdot B = 0 \Leftrightarrow vB = (-1)^d Bv$, $v \| B \Leftrightarrow vB = v \cdot B \Leftrightarrow v \wedge B = 0 \Leftrightarrow vB = (-1)^{d-1} Bv$.

## Definition (4.7.2):

Wecalltwoblades$A$and $B$ coorthogonal if $AB = \pm BA$.

Abladecanalternativelybewrittenasan outer productof vectors or asageometric product oforthogonalvectors. For blades$A = a_1 \wedge \dots \wedge a_\mu$ and $B = b_1 \wedge \dots \wedge b_\nu$ wewillusethenotations $span(B) := \text{span}(b_1, \dots, b_\nu)$, $A \oplus B := span(A) \oplus span(B) \subseteq \mathbb{R}^{p,q}$, $A \cap B := span(A) \cap span(B) \subseteq \mathbb{R}^{p,q}$

$\beta(A, B) := dim(A \cap B)$ and $\alpha(A, B) := dim(A \oplus B)$

$= \mu + \nu - \beta(A, B)$. For aset ofblades $B = \{B_1, \dots, B_d\}$, $d \in \mathbb{N}$weusethe notation $span(B) = \bigoplus_{k=1}^{d} span(B_k)$ and $\alpha(B) = \dim\big(span(B)\big)$.[43]

## Lemma (4.7.3):

Thebasisblades $e_k$ of $G^{p,q}$ thataregeneratedfromanorthogonal basisof $\mathbb{R}^{p,q}$ are mutually coorthogonal.

**Proof:**

Allorthogonalbasisvectorsof $\mathbb{R}^{p,q}$ satisfy

$$e_j e_k = \begin{cases} -e_k e_j & for\ j \neq k \in \mathbb{N} \\ e_k e_j & for\ j = k \end{cases}$$

Ineverygeometric algebra $G^{p,q}$.Sofortwobasisblades $e_j = e_{j_1,\dots j_\mu}, e_k = e_{k_1,\dots k_v}$, $1 < j_1 < \cdots < j_\mu \leq n, 1 \leq k_1 < \cdots < k_v \leq n$, with dimentions $\mu$, respectively $v$ we get

$$e_j e_k = (-1)^{\mu v - \beta(e_j, e_k)} e_k e_j$$

where $\beta(e_j, e_k) = |\{l \in \mathbb{N}, l \in \boldsymbol{j} and l \in \boldsymbol{k}\}|$ isthenumber ofindicesappearinginbothsets, respectively the dimension of the meetofthetwoblades.

## Lemma (4.7.4):

For twocoorthogonal blades $A$ and $B$ there isanorthonormalbasis $V = \{v_1, \dots, v_{\alpha(A,B)}\}$, of $A \oplus B \subseteq \mathbb{R}^{p,q}$ suchthat bothcanbeexpressedas real multiples ofbasis blades, that means $A = sgn(A)|A| v_{j_1}, \dots, v_{j_m}$ and $B = sgn(B)|B| v_{k_1}, \dots, v_{k_v}$, $a, b \in \mathbb{R}$ withthesignumfunction being1or−1.

**Proof:**

Weknowthateveryblade $A$ spansavectorspace $span(A)$,that thisvectorspacehasanorthonormalbasis,howitcanbeproducedandthatfor $span(B) \subset span(A)$thereisauniqueblade$A \perp B = B^{-1}$.

A Orthogonal to $B$,suchthat $span(B) \cap span(A \perp B) = \emptyset$,$span(B) \cup span(A \perp B) = span(A)$.

Thereforewecanseparatethespace$A \oplus B = (A \cap B) \cup span(A_{\perp(A \cap B)}) \cup span(B_{\perp(A \cap B)})$

into three disjointparts and immediately know that$A \cap B$isorthogonalto both $(A_{\perp(A \cap B)})$and $(B_{\perp(A \cap B)})$.Since$span(A_{\perp(A \cap B)}) \cap span(B_{\perp(A \cap B)}) = \emptyset$all basisvectors$a_1, \ldots, a_{\mu-\beta(A,B)}$of$\left(A_{\perp(A \cap B)}\right)$satisfy

$a_1, \ldots, a_{\mu-\beta(A,B)} \notin \mathrm{span}\left(B_{\perp(A \cap B)}\right) so \forall j = 1, \ldots, \mu - \beta(A,B)$: $a_j \wedge \left(B_{\perp(A \cap B)}\right) \neq 0$.Fortheproductofavectorandabladewehave

$$a_j\left(B_{\perp(A \cap B)}\right) = a_j.\left(B_{\perp(A \cap B)}\right) + a_j \wedge \left(B_{\perp(A \cap B)}\right)$$
$$= (-1)^{\upsilon-\beta(A,B)-1} B_{\perp(A \cap B)}.a_j$$
$$+ (-1)^{\upsilon-\beta(A,B)} B_{\perp(A \cap B)} \wedge a_j$$

and since $a_j \wedge B \neq 0$ necessarily $a_j.B = 0$ hastobevalid$\forall j = 1, \ldots, \mu - \beta(A,B)$inordertosatisfycoorthogonality.Thatisequivalentto $A_{\perp(A \cap B)} \perp B_{\perp(A \cap B)}$and therefore unifying the orthonormalbasesofall threeparts formanorthonormalbasis of $A \oplus B$.Let $b_1, \ldots, b_{\upsilon-\beta(A,B)}$bethe basis of $span(B)$, $c_1, \ldots, c_{\beta(A,B)}$bethebasisof $A \cap B$then the blade $A$hascanbe writtenas

$A = sgn(A)|A| c_1, \ldots, c_{\beta(A,B)}, a_1, \ldots, a_{\mu-\beta(A,B)}$and the blade as

$$B = sgn(B)|B| c_1, \ldots, c_{\beta(A,B)}, b_1, \ldots, b_{\upsilon-\beta(A,B)}$$

## Lemma(4.7.5):

Let$B = \{B_1, \ldots, B_d\}, d \in \mathbb{N}$benon-zero mutuallycoorthogonalblades.Then there isanorthonormalbasis$v_1, \ldots, v_{\alpha(B)})$ of $span(B)$such that every$B_K$,$k = 1, \ldots, d$canbewrittenasarealmultipleofabasisblade,That means $B_K = sgn(B_K)|B_K| v_{j(k)}$with
$v_{j(k)} = v_{j_1(k), \ldots, j_\mu(k)}, \mu = \mu(k) = \dim(B_K), |B_K| \in \mathbb{R}$.

**Proof:**

The set $D$isthe set ofintersectionsofthe subspaces spanned by allpossiblecombinationsofelementsof$C$.Theelements$D_K$of $D$ withminimaldimensionsatisfy$\forall D_j : D_K \cap D_j \in \{\emptyset, D_K\}$,becauseotherwise $D_K \cap D_j$ wouldhavelowerdimension than $D_K$whichisacontradiction.Inboth casesallgenerating vectorsof $D_K$canbeadded tothebasis,compare theproofofLemma(4.7.4)Sothechoiceofanyvector$c \in span(D_K)$willbesuccessful.

Onceavectorischosenthere aretwomorecasesthat already appeared inthe proofofLemma (4.7.4) Inthe caseof $c \notin span(C_K)$follows,thatcis orthogonalto $C_K$,because inthe caseof $c \in span(C_K)$themultiplicationof$c$ to $C_K$inthe algorithmalwayscreates blades $cC_K$oflowerdimension orthogonaltoc,becauseof

$$c^{-1}B = \langle c^{-1}B \rangle_{\dim(B)-1}$$

Therefore the applicationofthisoperation to all blades in $C$only leaves blades thatareorthogonalto$c$but still coorthogonal amongst each other because of

$$c^{-1}C_j c^{-1}C_k = (-1)^\mu C_j c^{-1} c^{-1} C_k$$

$$\overset{((c^{-1})^2 \in \mathbb{R})}{=} (-1)^\mu (c^{-1})^2 C_j C_k$$

$$= \pm(-1)^{\mu}(c^{-1})^2 C_k C_j$$

$$= \pm(-1)^{\mu+\nu-1} c^{-1} C_k c^{-1} C_j$$

Atthe beginning $C$ spans the wholespace $span(C) = span(B)$but as the algorithem proceeds

$$span(C) = span(B) \backslash span(Basis)$$

Suchthat $span\ (C)$and span(Basis)areorthogonal.BecauseofthatthesetBasisisorthogonalatall times.Thealgorithmstopswhen $\alpha(C\ )$vectorsareinBasis.Sofinally$\alpha(C\ )$orthogon alvectorswillbeinbasis,whichtherefore indeedisanorthogonal basisof$span(B)$, allelementsofCwillhavedimensionzeroandthealgorithm willendreturningthebasisandhowthebladescanbeconstructed.

Sotriviallyspoken,coorthogonalityofbladescanaswellbeinterpreted ascoorthogonalityofalltheir generating vectors, that meansalltheir generating vectors areeither orthogonalorcolinear.

## Theorem (4.7.6):

Afinitenumber ofbladesare coorthogonal ifand onlyifthey are realmultiplesofbasisbladesofanorthonormalbasisof$\mathbb{R}^{p,q}$.

## Proof:

Theassertion follows fromLemma( 4.7.3) and(4.7.5) together withnormalization, thebasiscompletion theorem andtheGram-Schmidtorthogonalizationprocess.

we will only deal with geometricFourier transformswhosedefiningfunctions $f_1, \dots, f_v$,aremutually coorthogonal blades, thatmeans they satisfy the property$\forall l, k = 1, \dots, v, \forall\, x, u \in \mathbb{R}^m$.

$$f_l(x,u) f_k(x,u) = \pm f_k(x,u) f_l(x,u)$$

Allowsustowrite

$$f_l(x,u) = sgn(f_l(x,u))\,|f_l(x,u)|\,e_{j_l(x,u)}$$

Forall$l = 1, \dots, v$,witharealvalued function $|f_l(x,u)|: \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$,and afunction $j_l(x,u): \mathbb{R}^m \times \mathbb{R}^m \to p(\{1, \dots, n\})$thatmaps to amulti-index indicating a basis multivectorofa certain basis. Wewillrefertoa set of functions withthispropertysimplyasasetofbasisbladefunctions.

## Theorem (4.7.7):

The geometric Fouriertransformislinear withrespect toscalar factors. Let$b, c \in \mathbb{R}$ and$A, B, C: \mathbb{R}^m \to C\ell_{p,q}$bethreemultivector fieldsthat satisfy

$$A(x) = bB(x) + cC(x), \text{then}$$

$$\mathcal{F}_{F_1,F_2}(A)(u) = b\mathcal{F}_{F_1,F_2}(B)(u) + c\mathcal{F}_{F_1,F_2}(C)(u)$$

## Proof:

Theassertion isaneasyconsequenceofthedistributivityofthe geometricproductoveraddition,thecommutativityofscalarsandthelinearity oftheintegral.

Allgeometric Fouriertransformscanalsobe expressedintermsof astronger claim.Themappings $f_1,\ldots,f_\nu$ , withthefirst μ onesleftoftheargumentfunction andthe $\nu-\mu$ others ontherightofit, areallbilinear andtherefore take theform.

$$f_k(x,u) = f_k\left(\sum_{j=1}^{m} x_j e_j, \sum_{l=1}^{m} u_l e_l\right)$$

$$= \sum_{j=1}^{m} x_j f_k(e_j, e_l) u_l = x^T M_k u$$

$\forall k = 1,\ldots,\nu$,where$M_k \in (\ell^{p,q})^{m\times m}$,$(M_k)_{jl} = f_k(e_j, e_l)$according toNotation.

1. IntheCliffordFourier transform$f_1$canbewrittenwith

$$M_1 = 2\pi i_n Id$$

2. The$\nu = m = n$ mappings $f_k$,$k = 1,\ldots,n$ofthe BülowCliffordFourier transformcanbeexpressedusing

$$(M_k)_{jl} = \begin{cases} 2\pi e_k & for\ k = l = j \\ 0 & else \end{cases}$$

3. SimilarlythequaternionicFourier transformisgeneratedusing

$$(M_1)_{li} = \begin{cases} 2\pi i & for\ l = i = 1 \\ 0 & else \end{cases}$$

$$(M_2)_{li} = \begin{cases} 2\pi j & for\ l = i = 2 \\ 0 & else \end{cases}$$

4. Wecanbuildthespacetime Fourier transformwith

$$(M_1)_{lj} = \begin{cases} e_4 & for\ l = j = 1 \\ 0 & else \end{cases}$$

$$(M_1)_{lj} = \begin{cases} \epsilon_4 i_4 e_4 & for\ l = j \in \{2,3,4\} \\ 0 & else \end{cases}$$

5. TheCliffordFourier transformforcolorimagescanbedescribed by

$$M_1 = \frac{1}{2} BId$$

$$M_2 = \frac{1}{2} iBId$$

$$M_3 = -\frac{1}{2} BId$$

$$M_4 = -\frac{1}{2} iBId$$

6. ThecylindricalFouriertransformcanalsobereproduced withmappings satisfying becausewecanwrite

$$x? = e_1 e_2 x_1 u_2 - e_1 e_2 x_2 u_1 + \cdots + e_{m-1} e_m x_{m-1} u_m - e_{m-1} e_m x_m u_{m-1}$$

And set

$$(M_1)_{lj} = \begin{cases} 0 & for\ l = j \\ e_l e_j & else \end{cases}$$

## Theorem (4.7.8):

Let $0 \neq a \in R$bea real number, $A(x) = B(ax)$

Twomultivector fieldsandall $F1, F2$bebilinear mappings thenthegeometricFouriertransformsatisfies

$$\mathcal{F}_{F_1,F_2}(A)(u) = |a|^{-m}\mathcal{F}_{F_1,F_2}(B)\left(\frac{u}{a}\right)$$

## Proof:

Achange ofcoordinatestogether with the bilinearityproves the assertion by

$$\mathcal{F}_{F_1,F_2}(A)(u) = \int_{\mathbb{R}^m} \prod_{f\in F} e^{-f(x,u)} \,.\, B(ax) \prod_{f\in B} e^{-f(x,u)}\, d^m x$$

$$\overline{\overline{ax = y}} \int_{\mathbb{R}^m} \prod_{f\in F} e^{-f\left(\frac{y}{a},u\right)} \,.\, B(y) \prod_{f\in B} e^{-f\left(\frac{y}{a},u\right)} |a|^{-m}\, d^m y$$

$$= |a|^{-m} \int_{\mathbb{R}^m} \prod_{f\in F} e^{-f\left(y,\frac{u}{a}\right)} \,.\, B(y) \prod_{f\in B} 2\, d^m x$$

$$= |a|^{-m}\mathcal{F}_{F_1,F_2}(B)\left(\frac{u}{a}\right)$$

Toobtain properties oftheGFT likelinearitywithrespect toarbitrarymultivectors orashifttheorem wewillhavetochangetheorderofmultivectorsand productsofexponentials.Sincethegeometricproductusually isneither commutativenoranticommutativethis isnottrivial. Inthissectionwe provide useful lemmathatallowaswapifat leastoneofthe factors is invertible

**Remark (4.7.9):**

Everymultiple ofasquarerootofminus one $i \in \mathcal{G}^{p,q}$isinvertible,sincefrom$i^2 = -r,\ r \in \mathbb{R}\backslash\{\mathbf{0}\}$follows $i^{-1} = \frac{-i}{r}$.Becauseofthat forall $\boldsymbol{u}, \boldsymbol{x} \in \mathbb{R}^m$afunction $f_k(x,u)\colon \mathbb{R}^m \times \mathbb{R}^m \to \mathcal{G}^{p,q}$ispointwisinvertible.

## Definition (4.7.10):

For an invertible multivector $B \in C\ell_{p,q}$ and an arbitrary multivector $A \in C\ell_{p,q}$ wedefine

$$A_{c^0(B)} = \frac{1}{2}(A + B^{-1}AB)$$

$$A_{c^1(B)} = \frac{1}{2}(A - B^{-1}AB)$$

## Lemma (4.7.11):

Let $B \in C\ell_{p,q}$ beinvertible with the unique inverse $B^{-1} = \frac{\bar{B}}{B^2}, B^2 \in \mathbb{R}\backslash\{0\}$. Every multivector A$\in C\ell_{p,q}$ can be expressed unambiguously bythesumof $A_{c^0(B)} \in C\ell_{p,q}$ thatcommutes and $A_{c^1(B)} \in C\ell_{p,q}$ thatanticommuteswithrespecttoB.Thatmeans

$$A = A_{c^0(B)} + A_{c^1(B)}$$

$$A_{c^0(B)}B = BA_{c^0(B)}$$

$$A_{c^1(B)}B = -BA_{c^1(B)}$$

## Proof:

Wewillonlyprovetheassertion for$A_{c^0(B)}$.

With Definition(3-6-11)weget

$$A_{c^0(B)} + A_{c^1(B)} = \frac{1}{2}(A + B^{-1}AB + A - B^{-1}AB)$$

$$= A$$

And considering

$$B^{-1}AB = \frac{\bar{B}AB}{B^2} = BAB^{-1}$$

We also

$$A_{c^0(B)}B = \frac{1}{2}(A + B^{-1}AB)\mathrm{B}$$

$$= \frac{1}{2}(A + BAB^{-1})\mathrm{B}$$

$$= \frac{1}{2}(AB + BA)$$

$$= B\frac{1}{2}(B^{-1}AB + A)$$

$$= BA_{c^0(B)}$$

**Uniqueness:**weget

$$A_{c^1(B)} = A - A_{c^0(B)}$$

together withthethird onethisleadsto

$$\left(A - A_{c^0(B)}\right)B = B(A - A_{c^0(B)})$$

$$AB - A_{c^0(B)}B = -BA + BA_{c^0(B)}$$

$$AB + BA = A_{c^0(B)}B + BA_{c^0(B)}$$

And from the second claim finally follows

$$AB + BA = 2BA_{c^0(B)}$$

$$\frac{1}{2}(B^{-1}AB + A) = A_{c^0(B)}$$

The derivation of the expression for $A_{c^1(B)}$ works analogously.

**Corollary (3.7.12):**

let$B \in C\ell_{p,q}$ be invertable, then $\forall A \in C\ell_{p,q}$

$$BA = (A_{c^0(B)} - A_{c^1(B)})B$$

**Corollary (4.7.13):**

let $B \in C\ell_{p,q}$ be invertable, then $\forall A \in C\ell_{p,q}$

$$BA = (A_{c^0(B)} - A_{c^1(B)})B$$

## Definition (4.7.14):

For $d \in \mathbb{N}, A \in C\ell_{p,q}$, The ordered set $B = \{B_1, \dots, B_d\}$ of invertable multivectors and any multi-index $j \in \{0,1\}^d$ we define

$$A_{c^j(\vec{B})} = ((A_{c^{j1}(B_1)})_{c^{j2}(B_2)} \dots)_{c^{jd}(B_d)}$$

$$A_{c^j(\overleftarrow{B})} = ((A_{c^{jd}(B_d)})_{c^{jd-1}(B_{d-1})} \dots)_{c^{j1}(B_1)}$$

Recursively with$c^0, c^1$ of definition (4.7.14).

**Example (1):**

Let $A = a_0 + a_1 e_1 + a_2 e_2 + a_{12} e_{12} \in G^{2,,0}$, then for example

$$A_{c^0(e_1)} = \frac{1}{2}(A + {e_1}^{-1} A e_1)$$

$$= \frac{1}{2}(A + a_0 + a_1 e_1 - a_2 e_2 - a_{12} e_{12})$$

$$= a_0 + a_1 e_1$$

And further

$$A_{c^{0,0}(\overrightarrow{e_1,e_2})} = (A_{c^0(e_1)})_{c^0(e_2)}$$

$$= (a_0 + a_1 e_1)_{c^0(e_2)} = a_0$$

The computation of the other multi-indices with d=2 works analogously and therefore

$$A = \sum_{j \in \{0,1\}^d} A_{c^j(e_1,e_2)}$$

$$= A_{c^{00}(\overrightarrow{e_1,e_2})} + A_{c^{01}(\overrightarrow{e_1,e_2})} + A_{c^{10}(\overrightarrow{e_1,e_2})} + A_{c^{11}(\overrightarrow{e_1,e_2})}$$

$$= a_0 + a_1 e_1 + a_2 e_2 + a_{12} e_{12}.$$

[26]

## GeometricConvolutionTheorem(4.8):

Wehaveseenthatcoorthogonal bladefunctionscanbeexpressedasrealmultiples ofbasis blades.

$$f(x,u) = sgn(f(x,u))e_{j(x,u)} : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathcal{G}^{p,q}$$

Forbasisbladefunctions, that during thedecomposition intocommutativeandanticommutativeparts ofa multivectornoadditionaltermsappear inthesumoverthebasisblades. Each partisarealfragmentofthemultivectoralongthebasisbladesofthe orthogonalbasisfromTheorem (4.7.6) Becauseof that anexponentialcanonly becomedecomposed intofourdifferentshapes itself,acosine,abasisblade multiplied withasineorzero.Thismotivates thegeneralization ofgeometricFourier transforms totrigonometrictransforms.

## Geometric Trigonometric Transform (4.8.1):

Let$A: \mathbb{R}^m \to G^{p,q}$be amultivectorfieldand $x, u \in \mathbb{R}^m$vectors, $F_1, F_2$twoordered finitesetsof μ,respectively $\nu - \mu$,mappings $\mathbb{R}^m \times \mathbb{R}^m \to \mathcal{I}^{p,q}$,$G_1, G_2$twoordered finite setsofμ, respectively ν−μ,mappings ($\mathbb{R}^m \times \mathbb{R}^m \to \mathcal{I}^{p,q}$)→$G^{p,q}$witheach$g_l\big(-f(x,u)\big)\forall l = 1, \ldots, \nu$, having one of shape:

$$g_l\big(-f_l(x,u)\big) = \begin{cases} e^{-f_l(x,u))} \\ \cos(|f_l(x,u))|) \\ -\dfrac{f_l(x,u)}{|f_l(x,u))|}\sin(|f_l(x,u))|) \\ 0 \end{cases}$$

The GeometricTrigonometricTransform(GTT)

$$\mathcal{F}_{G_1(F_1),G_2(F_2)}(A)(u) := \int_{\mathbb{R}^m} \prod_{l=1}^{\mu} g_l\big(-f_l(x,u)\big)\, A(x) \prod_{l=\mu+1}^{\nu} g_l\big(-f_l(x,u)\big)$$

Wehaveseenthat thedecomposition ofanexponential withrespecttobasisbladestakesthesameshapelikethefunctions$G1, G2$ ofaGTT.

Therefore forageometric Fourier transformwithbasisblade functions $F_1, F_2$,twosets ofbasis blades $B_1 = \{e_{(k)_1}, \ldots, e_{(k)_\eta}\}$, $B_2 = \{e_{(k)_{\eta+1}}, \ldots, e_{(k)_{\theta-\eta}}\}$and strictlylower and upper triangularmatrices $J \in \{0,1\}^{\mu\times\eta}$, $\mathrm{k} \in \{0,1\}^{(\nu-\mu)\times\theta}$ whose rows are $\mu$and $\nu - \mu$multi-indices$(j)_l \in \{0,1\}^\eta$ respectively$(k)_l \in \{0,1\}^\theta$, we can construct a geometricTrigonometricTransform $\mathcal{F}_{G_1(F_1),G_2(F_2)}(A)$ by setting

$g_l(-f_l(x,u))$=$e^{-f_l(x,u)}_{c^{(J)_l}}$ for $l=1,\dots,\mu$ and $g_l(-f_l(x,u))$=$e^{-f_l(x,u)}_{c^{(k)_l}}$for$l=\mu+1,\dots,v$ we refer to it shortly as

$$\mathcal{F}(F_1)_{c^J(B_1)},(F_2)_{c^K(B_2)}(A)(u) := \int_{\mathbb{R}^m} \prod_{l=1}^{\mu} e^{-f_l(x,u)}_{c^{(J)_l}(B_1)} A(x) \prod_{l=\mu+1}^{v} e^{-f_l(x,u)}_{c^{(k)_{l-\mu}}(B_2)} d^m x$$

In the case of

$\mathcal{F}(F_1)_{c^j(F_1)},(F_2)_{c^K(F_2)}$we will only write $\mathcal{F}(F_1)_{c^j},(F_2)_{c^K}$.

The geometrictrigonometrictransformisageneralizationofthe geometric Fourier transform.Wewilluseittoprovetheconvolution theorem ofthe GFT.

## Definition (4.8.2):

WecallaGFT left(right)separable,if

$$f_l = |f_l(x,u))| i_l(u)$$

$\forall l=1,\dots,\mu,(l=\mu+1,\dots,v)$, where$|f_l(x,u))|: \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is a real function and

$i_l: \mathbb{R}^m \to \mathcal{I}^{p,q}$ a function thatdoesnotdepend onx.

## Lemma(4.8.3):

Let F ={$f_1(x,u),\dots,f_d(x,u)$}bea set ofpointwiseinvertiblefunctions thentheordered product oftheir exponentialsandanarbitrarymultivector$A \in G^{p,q}$satisfies

$$\prod_{l=1}^{d} e^{-f_l(x,u))} A = \sum_{j\in\{0,1\}^d} A_{c^j\overleftarrow{(F)}}(x,u) \prod_{l=1}^{d} e^{-(-1)^{j_l} f_l(x,u))}$$

Where $A_{c^j\overleftarrow{(F)}}(x,u) \coloneqq A_{c^j\overleftarrow{(F)(x,u)}}$ is a multivector valued function $\mathbb{R}^m \times \mathbb{R}^m \to G^{p,q}$.

## Lemma (4.8.4):

Let $F = \{f_1(x,u),\dots,f_d(x,u)\}$ be a set of separable functions that are linear with respect to $x$. Further let $J \in \{0,1\}^{d\times d}$ be a strictly lower triangular matrix, that is associated column by column with a multi-index

$j \in \{0,1\}^d$ by $\forall k = 1,\dots,d : \left(\sum_{l=1}^{d} J_{l,k}\right) \bmod 2 = j_k, with\ (J)_l$ being its $l-th$ row, then

$$\prod_{l=1}^{d} e^{-f_l(x+y,u))} = \sum_{j\in\{0,1\}^d} \sum_{\substack{j\in\{0,1\}^{d\times d} \\ \sum_{l=1}^{d}(J)_l \ mod\ 2=j}} \prod_{l=1}^{d} e^{-f_l(x,u)}_{c^{(J)_l}\overleftarrow{(f_1,\dots,f_l,0,\dots,0)}} \prod_{l=1}^{d} e^{-(-1)^{j_l} f_l(y,u)}$$

Or alternatively with strictly upper triangular matrices

$$\prod_{l=1}^{d} e^{-f_l(x+y,u))} = \sum_{j\in\{0,1\}^d} \sum_{\substack{j\in\{0,1\}^{d\times d} \\ \sum_{l=1}^{d}(J)_l \ mod\ 2=j}} \prod_{l=1}^{d} e^{-(-1)^{j_l} f_l(x,u)} \prod_{l=1}^{d} e^{-f_l(y,u)}_{c^{(J)_l}\overrightarrow{(0,\dots,0,f_1,\dots,f_l)}}$$

## Definition(4.8.5):

For a set of functions $F = \{f_1(x,u),\dots,f_d(x,u)\}$ and a multi-index $j \in \{0,1\}^d$, we define the set of functions $F(j)$ by

$F(j) := \{(-1)^{j_1} f_1(x,u),\dots,(-1)^{j_d} f_d(x,u)\}$

WealsoneedageneralizationofLemma(4.8.3)thatallowsustoswaptheorderofpartia lexponentialsandmultivectors.

**Lemma (4.8.6):**

Forsetsoffunctions $F = \{f_1(x,u),\ldots,f_d(x,u)\}$,

$G=\{g_1\ldots,g_d\}$ wegetanalogously toLemma (4.8.3).

$$\prod_{l=1}^{\mu} g_l\big(-f_l(x,u)\big)A = \sum_{j\in\{0,1\}^d} A_{c^j(F)} \prod_{l=1}^{d} g_l\left(-(-1)^{j_l} f_l(x,u)\right)$$

**Proof:**

First weanalyze the interactionof $A$ with onepartialexponential $g_l\big(-f_l(x,u)\big)$.

Itcantake three differentshapes

$$g_l\big(-f_l(x,u)\big) = \begin{cases} e^{-f_l(x,u)} \\ \cos(|f_l(x,u)|) \\ -\dfrac{f_l(x,u)}{|f_l(x,u)|}\sin(|f_l(x,u)|) \\ 0 \end{cases}$$

Inthe first caselemma (4.8.3)proves the assertion

and the last oneistrivial. Assumethesecondcaseandnotethatthen $g_l\big(-f_l(x,u)\big)$equals $g_l\big(f_l(x,u)\big)$becauseofthesymmetryofthecosine.

$$g_l\big(-f_l(x,u)\big)A = COS(|f_l(x,u)|)A$$

$$= ACOS(|f_l(x,u)|)$$

$$= A_{c^0(f_l)} cos(|f_l(x,u)|) + A_{c^1(f_l)} cos(|f_l(x,u)|)$$

$$= A_{c^0(f_l)}\, g_l\big(-f_l(x,u)\big) + A_{c^1(f_l)}\, g_l\big(f_l(x,u)\big)$$

Inthethird casewehave

$$g_l\big(-f_l(x,u)\big)A = -\frac{f_l(x,u)}{|f_l(x,u)|}\sin(|f_l(x,u)|)\, A$$

$$= A_{c^0(f_l)} \frac{-f_l(x,u)}{|f_l(x,u)|}\sin(|f_l(x,u)|)$$

$$+ A_{c^1(f_l)} \frac{f_l(x,u)}{|f_l(x,u)|}\sin(|f_l(x,u)|)$$

$$= A_{c^0(f_l)}\, g_l\big(-f_l(x,u)\big) + A_{c^1(f_l)}\, g_l\big(f_l(x,u)\big)$$

Soinallcaseswehave

$$g_l\big(-f_l(x,u)\big)A = A_{c^0(f_l)}\, g_l\big(-f_l(x,u)\big) + A_{c^1(f_l)}\, g_l\big(f_l(x,u)\big)$$

$$= \sum_{j\in\{0,1\}^d} A_{c^j(f_l)} \prod_{l=1}^{d} g_l\left(-(-1)^{j_l} f_l(x,u)\right)$$

Applying itrepeatedlytothe wholeproductlikeinthe proofofLemma (4.8.3) leadstotheassertion.

## Definition (4.8.7):

Let $(x), B(x): \mathbb{R}^m \rightarrow G^{p,q}$ betwomultivectorfields.Theirconvolution $(A * B)(x)$ isdefinedas

$$(A * B)(x) = \int_{\mathbb{R}^m} A(y)B(x-y)d^m y$$

**ConvolutionTheorem (4.8.8):**

Let $A, B, C{:}\mathbb{R}^m \rightarrow G^{p,q}$ bemultivector fields with $A(x) = (C * B)(x)$ and $F_1, F_2$ becoorthogonal, separable and linearwithrespect tothefirstargument, j,$j^{'} \in \{0,1\}^{\mu}$, k,$k^{'} \in \{0,1\}^{(v-\mu)}$ and $J \in \{0,1\}^{\mu\times\mu}$ and $k \in \{0,1\}^{(v-\mu)\times(v-\mu)}$ arethestrictlylower,respectivelyupper,triangularmatriceswithrow s $(J)_l, (K)_{l-\mu}$ summing upto$(\sum_{l=1}^{\mu}(J)_l) mod 2 = j$ respectively $(\sum_{l=\mu+1}^{v}(K)_{l-\mu})\, mod 2 = k$ as in lemma(4.8.4), then the geometric FouriertransformofAsatisfies theconvolution property

$$\mathcal{F}_{F_1,F_2}(A)(u) = \sum_{j,j',\mathrm{k},k'} \sum_{J,K} (\mathcal{F}_{F_1(j),F_2(\mathrm{k}+k')}(C)(u))_{c^{j'}(F_1)} \mathcal{F}_{(F_1(j'))_{c^J},(F_2)_{c^K}} \left(\mathrm{B}^{\mathrm{K}}_{c^{k'}(F_2)}\right)(\mathrm{u})$$

**Proof:**

$\mathcal{F}_{F_1,F_2}(A)(u) =$

$$\int_{\mathbb{R}^m} \prod_{f\in F_1} e^{-f(x,u)} . (C * B)(x) \prod_{f\in F_2} e^{-f(x,u)}\, d^m x$$

$$= \int_{\mathbb{R}^m} \prod_{f\in F_1} e^{-f(x,u)} \int_{\mathbb{R}^m} C(y)B(x-y)\, d^m x \prod_{f\in F_2} e^{-f(x,u)}\, d^m x$$

$$\overline{\overline{x-y=z}} \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \prod_{f\in F_1} e^{-f(z+y,u)}\, C(y)B(z) \prod_{f\in F_2} e^{-f(z+y,u)}\, d^m x\, d^m z$$

We separate the products into parts that only depend on y and ones that only depend on z.

$$= \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \sum_{j \in \{0,1\}^\mu} \sum_{\substack{J \in \{0,1\}^{\mu \times \mu} \\ (\sum (J)_l) mod\, 2 = j}} \prod_{l=1} e^{-f_l(z,u)}_{c^{(J)_l}(F_1)} \prod_{l=1}^{\mu} e^{-(-1)^{j_l} f_l(y,u)} C(y)B(z)$$

$$\sum_{k \in \{0,1\}^v} \sum_{\substack{k \in \{0,1\}^{v-\mu \times v-\mu} \\ \sum K_l mod\, 2 = j}} \prod_{l=1+\mu}^{v} e^{-(-1)^{k_{l-\mu}} f_l(y,u)} \prod_{l=1+\mu}^{v} e^{-f_l(z,u)}_{c^{(K)_{l-\mu}}(F_2)} d^m y\, d^m z$$

$$= \sum_{j,k} \sum_{J,K} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} e^{-f_l(z,u)}_{c^{(J)_l}(F_1)} \prod_{l=1}^{\mu} e^{-(-1)^{j_l} f_l(y,u)} C(y)B(z) \prod_{l=1+\mu}^{v} e^{-(-1)^{k_{l-\mu}} f_l(y,u)}$$

$$\prod_{l=1+\mu}^{v} e^{-f_l(z,u)}_{c^{(K)_{l-\mu}}(F_2)} d^m y\, d^m z$$

Next step is to collect all parts then depend on y.

$$\sum_{j,k} \sum_{J,K} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \prod_{l=1}^{\mu} e^{-f_l(z,u)}_{c^{(J)_l}(F_1)} \prod_{l=1}^{\mu} e^{-(-1)^{j_l} f_l(y,u)} C(y)$$

$$\sum_{k' \in \{0,1\}^{v-\mu}} \prod_{l=1+\mu}^{v} e^{-(-1)^{k_{l-\mu}+k'_{l-\mu}} f_l(y,u)} B_{c^{k'}(F_2)}(z)$$

$$\prod_{l=1+\mu}^{v} e^{-f_l(z,u)}_{c^{(K)_{l-\mu}}(F_2)} d^m y\, d^m z$$

$$= \sum_{j,k,k'} \sum_{J,K} \int_{\mathbb{R}^m} \prod_{l=1}^{\mu} e^{-f_l(z,u)}_{c^{(J)_l}(F_1)} \int_{\mathbb{R}^m} \prod_{l=1}^{\mu} e^{-(-1)^{j_l} f_l(y,u)} C(y)$$

$$\prod_{l=1+\mu}^{v} e^{-(-1)^{k_{l-\mu}+k'_{l-\mu}} f_l(y,u)} B_{c^{k'}(F_2)}(z) \prod_{l=1+\mu}^{v} e^{-f_l(z,u)}_{c^{(K)_{l-\mu}}(F_2)} d^m z$$

$$= \sum_{j,k,k'} \sum_{J,K} \int_{\mathbb{R}^m} \prod_{l=1}^{\mu} e^{-f_l(z,u)}_{c^{(J)_l}(F_1)} \mathcal{F}_{F_1(j),F_2(k+k')}(C)(u) B_{c^{k'}(F_2)}(z)$$

$$\prod_{l=1+\mu}^{v} e^{-f_l(z,u)}_{c^{(K)_{l-\mu}}(F_2)} d^m z$$

Finally we gather the parts on z.

$$= \sum_{j,\mathrm{k},k'} \sum_{J,K} \int_{\mathbb{R}^m} \sum_{j' \in \{0,1\}^\mu} (\mathcal{F}_{F_1(j),F_2(k+k')}(C)(u))_{c^{j'}(F_1)}$$

$$\prod_{l=1}^{\mu} e^{-(-1)^{j_l} f_l(z,u)}_{c^{(J)_l}(F_1)} B_{c^{k'}(F_2)}(z) \prod_{l=1+\mu}^{v} e^{-f_l(z,u)}_{c^{(K)_{l-\mu}}(F_2)} \, d^m z$$

$$= \sum_{j,j',\mathrm{k},k'} \sum_{J,K} (\mathcal{F}_{F_1(j),F_2(k+k')}(C)(u))_{c^{j'}(F_1)}$$

$$\int_{\mathbb{R}^m} \prod_{l=1}^{\mu} e^{-(-1)^{j_l} f_l(z,u)}_{c^{(J)_l}(F_1)} B_{c^{k'}(F_2)}(z) \prod_{l=1+\mu}^{v} e^{-f_l(z,u)}_{c^{(K)_{l-\mu}}(F_2)} \, d^m z$$

$$= \sum_{j,j',\mathrm{k},k'} \sum_{J,K} (\mathcal{F}_{F_1(j),F_2(k+k')}(C)(u))_{c^{j'}(F_1)} \mathcal{F}_{(F_1(j'))_{c^J},(F_2)_{c^K}} (B_{c^{k'}(F_2)})(u)$$

[43]

## Engineering Applications of the Motion-Group Fourier Transform (4.9):

Noncommutative harmonic analysis is a beautiful and powerful area of pure mathematics that has connections to analysis, algebra, geometry, and the theory of algorithms. Unfortunately, it is also an area that is almost unknown to engineers. In this research, we have addressed a number of seemingly intractable “real-world” engineering problems that are easily modeled and/or solved using techniques of noncommutative harmonic analysis. In particular, we have addressed physical/mechanical problems that are described well as functions or processes on the rotation and rigid-body-motion groups. The interactions and evolution of these functions are described using group-theoretic convolutions and diffusion equations, respectively. We provide some of these applications and show how computational harmonic analysis on motion groups is used.

The group of rigid-body motions, denoted as $\mathrm{SE}(N)$ (shorthand for “special Euclidean” group in $N$-dimensional space), is a unimodularsemidirect product group, and general methods for constructing unitary representations of such Lie groups have been known for some time. In the past 40 years, the representation theory and harmonic analysis for the Euclidean groups have been developed in the pure mathematics and mathematical physics literature. The study of matrix elements of irreducible unitary representation of $\mathrm{SE}(3)$.

However, despite the considerable progress in mathematical developments of the representation theory of $\mathrm{SE}(3)$, these achievements have not yet been widely incorporated in engineering and applied fields. In work summarized here we try to fill this gap.

we review the representation theory of $\mathrm{SE}(2)$, give the matrix elements of the irreducible unitary representations and review the definition of the Fourier transform for $\mathrm{SE}(2)$. We also review operational properties of theFourier transform. We do not go into the intricate details of the Fourier transform for $\mathrm{SE}(3)$, as those are provided in the references described above and they add little to the understanding of how to apply noncommutative harmonic analysis to real-world problems. Then we devoted to application areas: coherent optical communications, robotics, and polymer statistical mechanics, respectively.[15]

## Fourier Analysis of Motion (4.10):

we review the basic definitions and properties of the Euclidean motion groups. Our emphasis is on the motion group of the plane, but most of the concepts extend in a natural way to three-dimensional space.

## Euclidean motion group (4.10.1):

The Euclidean motion group, $\mathrm{SE}(N)$, is the semidirect product of $\mathbb{R}^N$with the special orthogonal group, $\mathrm{SO}(N)$. We denote elements of $\mathrm{SE}(N)$as $g = (a, A) \in \mathrm{SE}(N)$ where $A \in \mathrm{SO}(N)$and $a \in \mathbb{R}^N$. The identity element is $e = (0, I)$ where $I$ is the $N \times N$identity matrix. For any $g = (a, A)$and $h = (r, R) \in \mathrm{SE}(N)$, the group law is written as $g \circ h = (a + Ar, AR)$;and $g^{-1} = (-A^T a, A^T)$. Any $g = (a, A) \in \mathrm{SE}(N)$ acts transitively on a position
$x \in \mathbb{R}^N$as

$$g \cdot x = Ax + a$$

That is, position vector *x* is rigidly moved by rotation followed by a translation.
Often in the engineering literature, no distinction is made between a motion,
$g$, and the result of that motion acting on the identity element. Hence, we interchangeably use the words “motion” and
“frame” when referring to elements of $\mathrm{SE}(N)$.
It is convenient to think of an element of $\mathrm{SE}(N)$ as an$(N + 1) \times (N + 1)$matrix of the form:

$$g = \begin{pmatrix} A & a \\ 0^T & 1 \end{pmatrix}$$

In the engineering literature, matrices with this kind of structure are called homogeneous transforms.
For example, each element of $\mathrm{SE}(2)$ can be parameterized using polar coordinates as:

$$g(r, \theta, \phi) = \begin{pmatrix} \cos\phi & -\sin\phi & r\cos\theta \\ \sin\phi & \cos\phi & r\sin\theta \\ 0 & 0 & 1 \end{pmatrix}$$

where $r \geq 0$ is the magnitude of translation. $\mathrm{SE}(2)$is a 3-dimensional manifold much like $\mathbb{R}^3$. We can integrate over $\mathrm{SE}(2)$ using the volume element

$d\big(g(r,\theta,\phi)\big) = (4\pi^2)^{-1} r\, dr\, d\theta\, d\phi$. This volume element is bi-invariant in the sense that it does not change under left and right shifts by any fixed element
$h \in \mathrm{SE}(2)$:

$$d(h \circ g) = d(g \circ h) = d(g).$$

Bi-invariant volume elements exist for $\mathrm{SE}(N)$ for $N = 2,3,4, \ldots$A group with bi-invariant volume element is called a unimodular group.

The Lie group $\mathrm{SE}(2)$ has an associated Lie algebra $se(2)$. Physically, elements of $\mathrm{SE}(2)$ describe finite motions in the plane, whereas elements of $se(2)$ represent infinitesimal motions. Since $\mathrm{SE}(2)$ is a three-dimensional Lie group, there are three independent directions along which any infinitesimal motion can be decomposed.
The vector space of all such motions relative to the identity element
$e \in SE(2)$together with the matrix commutator operation defines *se*(2). As with any vector space, we can choose an appropriate basis. One such basis for the Lie algebra *se*(2) consists of the following three matrices:

$$X_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}; \quad X_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The following one-parameter motions are obtained by exponentiating the above basis elements of $se(2)$:

$$g_1(t) = \exp(tX_1) = \begin{pmatrix} 1 & 0 & t \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

$$g_2(t) = \exp(tX_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix};$$

$$g_3(t) = \exp(tX_3) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For the purposes of the current discussion, we can take as a definition of $se(2)$the vector space spanned by any linear combination of $X_1$, $X_2$, and $X_3$. The exponential mapping

$$\exp\colon se(2) \to \mathrm{SE}(2)$$

is well defined for every element of *se*(2) and is invertible except at a set of measure zero in SE(2).

Any rigid-body motion in the plane can be expressed as an appropriate combination of these three basic motions. For example, $g = g_1(x)g_2(y)g_3(\phi)$.[30]

## Differential operators on SE(2)(4.10.2):

The way to take partial derivatives of a function of motion is to evaluate

$$\tilde{X}_i^R f \triangleq \frac{d}{dt} f(g \circ \exp(tX_i))|_{t=0}, \quad \tilde{X}_i^L f \triangleq \frac{d}{dt} f(\exp(tX_i) \circ g)|_{t=0}.$$

(In our notation, *R* means that the exponential appears on the right, and *L* means that it appears on the left. This means that $\tilde{X}_i^R$is invariant under leftshifts, while $\tilde{X}_i^L$is invariant under right shifts. Our notation is different thanothers in the mathematics literature where the superscript denotes the invarianceof the vector field formed by the concatenation of these derivatives.) Explicitly, we find the differential operators$\widetilde{X}_i^R$in polar coordinates to be

$$\tilde{X}_1^R = \cos(\phi - \theta)\frac{\partial}{\partial r} + \frac{\sin(\phi - \theta)}{r}\frac{\partial}{\partial \theta},$$

$$\tilde{X}_2^R = -\sin(\phi - \theta)\frac{\partial}{\partial r} + \frac{\cos(\phi - \theta)}{r}\frac{\partial}{\partial \theta},$$

$$\tilde{X}_3^R = \frac{\partial}{\partial \theta},$$

and in Cartesian coordinates to be

$$\tilde{X}_1^R = \cos\phi\frac{\partial}{\partial x} - \sin\phi\frac{\partial}{\partial y}, \quad \tilde{X}_2^R = \sin\phi\frac{\partial}{\partial x} + \cos\phi\frac{\partial}{\partial y}, \quad \tilde{X}_3^R = \frac{\partial}{\partial\phi}$$

The differential operators $\tilde{X}_i^L$in polar coordinates are

$$\tilde{X}_1^L = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}, \quad \tilde{X}_2^L = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}, \quad \tilde{X}_3^L = \frac{\partial}{\partial\phi} + \frac{\partial}{\partial\theta}.$$

## Fourier analysis onSE(2) (4.10.3):

The Fourier transform,$\mathcal{F}$, of a function of motion, $f(g)$ where $g \in \mathrm{SE}(N)$, is an infinite-dimensional matrix defined as

$$\mathcal{F}(f) = \hat{f}(p) = \int_G f(g)U(g^{-1}, p)d(g)$$

where $U(g, p)$ is an infinite dimensional matrix function with the property that $U(g_1 \circ g_1, p) = U(g_1, p)U(g_2, p)$: This kind of matrix is called a *matrix representation* of $\mathrm{SE}(N)$. It has the property that it converts convolutions on $\mathrm{SE}(N)$ intomatrix products:

$$\mathcal{F}(f_1 * f_2) = \mathcal{F}(f_2)\mathcal{F}(f_1).$$

In the case when $N = 2$, the original function is reconstructed as

$$\mathcal{F}^{-1}(\hat{f}) = f(g) = \int_0^\infty \mathrm{trace}\left(\hat{f}(p)U(g, p)\right) p\, dp,$$

and the matrix elements of $U(g, p)$ are expressed explicitly as:

$$u_{mn}(g(r,\theta,\phi),p) = j^{n-m} e^{-j[n\phi+(m-n)\theta]} J_{n-m}(pr)$$

where $J_v(x)$ is the $v^{th}$ order Bessel function and $j = \sqrt{-1}$. This inverse transform can be written in terms of elements as

$$f(g) = \sum_{m,n\in\mathbb{Z}} \int_0^\infty \hat{f}_{mn} u_{nm}(g,p) p dp. \qquad (1)$$

In analogy with the classical Fourier transform, which converts derivatives of functions of position into algebraic operations in Fourier space, there are operational properties for the motion-group Fourier transform.

By the definition of the SE(2)-Fourier transform $\mathcal{F}$ and operators $\tilde{X}_i^R$ and $\tilde{X}_i^L$, we can write the Fourier transform of the derivatives of a function of motion as

$$\mathcal{F}\left|\tilde{X}_i^R f\right| = \tilde{u}(X_i,p)\hat{f}(p), \qquad \mathcal{F}\left|\tilde{X}_i^L f\right| = -\hat{f}(p)\tilde{u}(X_i,p),$$

where

$$\tilde{u}(X_i,p) \triangleq \frac{d}{dt} U(\exp(tX_i),p)|_{t=0}$$

Explicitly,

$$u_{mn}(\exp t X_1), p = j^{n-m} J_{m-n}(pt).$$

and

We know that

$$\frac{d}{dx} J_m(x) = \frac{1}{2}[J_{m-1}(x) - J_{m+1}(x)]$$

Hence,

$$\tilde{u}_{mn}(X_1,p) = \frac{d}{dt} u_{mn} \exp(tX_1),p)|_{t=0} = -\tfrac{jp}{2}(\delta_{m,n+1} + \delta_{m,n-1})$$

Likewise,

$$u_{mn}\ \exp(tX_2)\,,p) = j^{n-m}\, e^{-j\,(n-m)}\,\pi/2\, J_{m-n}(pt) = J_{m-n}(pt),$$

and so

$$\tilde{u}_{mn}\,(X_2,p) = \frac{d}{dt} u_{mn} \exp(tX_2)\,,p)|_{t=0} = \tfrac{p}{2}\big(J_{m-n-1}(0) - J_{m-n+1}(0)\big)$$

$$= \tfrac{p}{2}\big(\delta_{m,n+1} + \delta_{m,n-1}\big).$$

Similarly, we find

$$u_{mn}\ \exp(tX_3)\,,p) = e^{-jmt}\,\delta_{m,n}$$

And

$$\tilde{u}_{mn}\,(X_3,p) = \frac{d}{dt} u_{mn} \exp(tX_3)\,,p)|_{t=0} = -jm\delta_{m,n}.$$

Fast Fourier transforms for SE(2)and SE(3).

Operational properties for SE(3) which are analogous to those presented here for SE(2). [26]

## Phase Noise in Coherent Optical Communications (4.11):

In optical communications, laser light is used to transmit information along fiber optic cables. There are several methods that are used to transmit and detect information within the light. Coherent detection is a method that has the ability to detect the phase, frequency, amplitude and polarization of the incident light signal. Therefore, information can be transmitted via phase, frequency, amplitude, or polarization modulation. However, the phase of the light emitted from a semiconductor laser exhibits random fluctuations due to spontaneous emissions in the laser cavity. This phenomenon is commonly referred to as *phase noise*. Phase noise puts strong limitations on the performance of coherent communication systems. Evaluating the influence of phase noise is essential in system design and optimization . Analytical models that describe the relationship between phase noise and the filtered signal. In particular, the Fokker–Planck approach represents the most rigorous description of phase noise effects . To better apply this approach to system design and optimization, an efficient and powerful computational tool is necessary. we describe one such tool that is based on the motion-group Fourier transform. Readers unfamiliar with the technical terms used below. The discussion in the following paragraph provides a context for this particular engineering application, but the value of noncommutative harmonic analysis in this context is solely due to its ability to solve equation (2).

Let $s(t)$ be the input signal to a bandpass filter which is corrupted by phase noise. Using the equivalent baseband representation and normalizing it to unit amplitude

$$s(t) = e^{j\phi(t)}$$

Where $\phi(t)$ is the phase noise, usually modeled as a Brownian motion process.

The function $h(t)$ is the impulse response of the bandpass filter. The output of the bandpass filter is denoted $z(t)$. Let us represent $z(t)$ through its real and imaginary parts:

$$z(t) = x(t) + jy(t) = r(t)e^{j\theta(t)}.$$

The 3-D Fokker–Planck equation defining the probability density function (pdf) of $z(t)$ is derived:

$$\frac{\partial f}{\partial t} = -h(t)cos\phi\frac{\partial f}{\partial x} - h(t)\sin\phi\frac{\partial f}{\partial y} + \frac{D}{2}\frac{\partial^2 f}{\partial \phi^2} \quad (2)$$

with initial condition $f(x, y, \phi; 0) = \delta(x)\delta(y)\delta(\phi)$, where $\delta$ being the Dirac delta function. The parameter $D$ is related to the laser line width $\Delta v$by $D = 2\pi\Delta v$.

Having an efficient method for solving equation (2) is of great importance in the design of filters.

A number of papers have attempted to solve the above equations using a variety of techniques including series expansions, numerical methods based on discretizing the domain, and analytical methods. However, all of them are based on classical partial differential equation solution techniques.

In our work, we present a new method for solving these methods using harmonic analysis on groups. These techniques reduce the above Fokker–Planck equations to systems of linear ordinary differential equations with constant or time-varying coefficients in a generalized Fourier space. The solution to this system of equations in generalized Fourier space is simply a matrix exponential for the case of constant coefficients. A usable solution is then generated via the generalized Fourier inversion formula.

Using the differential operators defined on the motion group, the 3-D Fokker–Planck equation in (3–10-1) can be rewritten as

$$\frac{\partial f}{\partial t} = \left(-h(t)\tilde{X}_2^R + \frac{D}{2}\left(\tilde{X}_3^R\right)^2\right)f. \qquad (3)$$

This equation describes a kind of process that evolves on the group of rigid-body motions SE(2). Applying the motion-group Fourier transform to (3–2), we can convert it to an infinite system of linear ordinary differential equations:

$$\frac{d\hat{f}}{dt}A(t)\hat{f}. \qquad (4)$$

For equation (3–10-2), the matrix is

$$A(t) = -h(t)\tilde{u}(X_2, p) + \frac{D}{2}(\tilde{u}(X_3, p))^2$$

And its elements are

$$A(t)_{mn} = -h(t)\frac{p}{2}\left(\delta_{m,n+1} - \delta_{m,n-1}\right) - \frac{D}{2}m^2\delta_{m,n}.$$

Numerical methods such as Runge–Kutta integration can be applied to easily solve the truncated version of this system. In the case when $h(t)$ is a constant, then $A$ is a constant matrix and the solution to the resulting linear time-invariant system can be written in closed form as

$$\hat{f}(p;t) = \exp(At)$$

With the initial condition that $\hat{f}(p;0)$ is the infinite-dimensional identity matrix.

In practice we truncate $A$ at finite dimension, then exponentiate.

Once we get the solution to (4), we can then substitute it into the Fourier inversion formula for the motion group in (1) to recover the pdf $f(g;t)$ of $z(t)$.

To get the pdf $f(r,\theta;t)$ is just an integration with respect to $\phi$as

$$f(r,\theta;t) = \frac{1}{2\pi}\int_0^{2\pi} f(g;t)d\phi = \sum_{n\in\mathbb{Z}} j^{-n} e^{-jn\theta} \int_0^{\infty} \hat{f}_{0,n} J_{-n}(pr)p\,dp\,. \quad (5)$$

Integrating equation (3–10-4) over $\theta$will give us the marginal pdf of $|z(t)|$as:

$$f(r;t) = \int_0^{\infty} \hat{f}_{0,0}(p) J_0(p\,r)p\,dp. \qquad (6)$$

Using our method, we can get a simple and compact expression for the marginal pdf for the output of the bandpass filter given in (6).

For details and numerical results generated using this approach.

## Robotics (4.12):

A robotic manipulator arm is a device used to position and orient objects in space. The set of all reachable positions and orientations is called the workspace of the arm. A robot arm that can attain only a finite number of different states is called a discretely-actuated manipulator. For such manipulators, it is a combinatorially explosive problem to enumerate by brute force all possible states for arms that have a high degree of articulation. The function that describes the relative density of reachable positions and orientations in the workspace (called a *workspace density* *function*) has been shown to be an important quantity in planning the motions of these manipulator arms. This function is denoted as

$f(g; L)$where $g \in \mathrm{SE}(N)$, and $L$ is the length of the arm.

Noncommutative harmonic analysis enters in this problem as a way to reduce this complexity. that the workspace density function $f(g; L_1 + L_2)$ for two concatenated manipulator segments with length $L_1$ and $L_2$ is the motion-group convolution

$$f(g; L_1 + L_2) = f(g; L_1) * f(g; L_2) = \int_G f(h; L_1) f(h^{-1} \circ g; L_2) dh, \quad (7)$$

where $h$ is a dummy variable of integration and $dh$ is the bi-invariant measure for $\mathrm{SE}(N)$. That is, given two short arms with known workspace densities, we can generate the workspace density of the long arm generated by stacking one short arm on the other using equation (7). In order to perform these convolutions efficiently, the concept of FFTs for the motion groups.

we discuss an alternative method for generating manipulator workspace density functions that does not explicitly compute convolutions.

Instead, it relies on the same kinds of degenerate diffusions we have seen already in the context of phase noise.

## Inspiration of the algorithm (4.12.1):

Consider a discretely-actuated serial manipulator which consists of concatenated segments called modules. Suppose that each module can reach 16 different states. The workspace of this manipulator with 2 modules, 3 modules and 4 modules can be generated by brute force enumeration because $16^2$, $16^3$, and $16^4$ are not terribly huge numbers. It is easy to imagine that the size of the workspace will spread out with the increment of modules. This enlargement of the workspace is just like the diffusion produced by a drop of ink spreading in a cup of water. Inspired by this observation, we view the workspace of a manipulator as something that grows/evolves from a single point source at the base as the length of the manipulator increases from zero. The workspace is generated after the manipulator grows to full length.[25]

## Implementation of the algorithm (4.12.2):

With this analogy, we then need to determine what kind of diffusion equation is suitable to model this process. We get such an equation by realizing that some characteristics of manipulators are similar to those of polymer chains like DNA.

During our study of conformational statistics in polymer science, we derived a diffusion-type equation defined on the motion group. This equation describes the probability density function of the position and orientation of the distal end of a stiff macromolecule chain relative to its proximal end. By involving parameters which indicate the kinematic properties of a manipulator into this equation, we can modify it to the diffusion-type equation describing the evolution of the workspace density function. It is written explicitly as

$$\frac{\partial f}{\partial L} = \left(\alpha \widetilde{X}_1^R + \beta\left(\widetilde{X}_1^R\right)^2 + \widetilde{X}_3^R + \varepsilon\left(\widetilde{X}_1^R\right)^2\right) f. \qquad (8)$$

Here$f$stands for the workspace density function, and $L$ is the manipulatorlength. The differential operators $\widetilde{X}_1^R$ and $\widetilde{X}_3^R$ are those defined on SE(2) givenearlier. Parameters $\beta$, $\varepsilon$and $\alpha$describe the kinematic properties of manipulators.

We define these kinematic properties as flexibility, extensibility and the degree of asymmetry. The parameter $\beta$ describes the flexibility of a manipulator in the sense of how much a segment of the manipulator can bend per unit length. A larger value of $\beta$ means that the manipulator can bend a lot. The parameter ″describes the extensibility of a manipulator in the sense of how much a manipulator can extend along its backbone direction. A larger value of ″means that the manipulator can extend a lot. The parameter $\alpha$ describes the asymmetry in how the manipulator bends. When $\alpha = 0$, the manipulator can reach left and right with equal ease. When $\alpha < 0$, there is a preference for bending to the left, and when $\alpha > 0$ there is a preference for bending to the right. Since $\alpha$,$\beta$, and

$\varepsilon$ are qualitative descriptions of the kinematic properties of a manipulator, they are not directly measurable.

This simple three-parameter model qualitatively captures the behavior that has been observed in numerical simulations of workspace densities of discretely-actuated variable-geometry truss manipulators. Clearly, equation (8) can be solved in the same way as the phase-noise equation.

## Statistical Mechanics of Macromolecules (4.13):

we show how certain quantities of interest in polymer physics can be generated numerically using Euclidean-group convolutions. We also show how for wormlike polymer chains, a partial differential equation governs a process that evolves on the motion group and describes the diffusion of end-to-end position and orientation. This equation can be solved using the SE(3)-Fourier transform in a manner very similar to the way the phase-noise Fokker–Planck was addressed before. This builds on classical works in polymer theory.

## Mass density, frame density, and Euclidean group convolutions (4.13.1):

In statistical mechanical theories of polymer physics, it is essential to compute ensemble properties of polymer chains averaged over all of their possible conformations.

Noncommutative harmonic analysis provides a tool for computing probability densities used in these averages.

In this subsection we review three statistical properties of macromolecular ensembles. These are:

(1) The ensemble mass density for the whole chain $\rho(x)$, which is generated by imagining that one end of the chain is held fixed and a cloud is generated by all possible conformations of the chain superimposed on each other.

(2) The ensemble tip frame density $f(g)$ (where $g$ is the frame of reference of the distal end of the chain relative to the fixed proximal end).

(3) The function $\mu(g,x)$, which is the ensemble mass density of all configurations which grow from the frame fixed to one end of the chain and terminate at the relative frame $g$ at the other end.

The functions $\rho, f$, and $\mu$ are related to each other. Given $\mu(g,x)$, the ensemble mass density is calculated by adding the contribution of each $\mu$ for each different end position and orientation:

$$\rho(x) = \int_G \mu(g,x)\, dg. \qquad (9)$$

This integration is written as being over all motions of the end of the chain, but only frames $g$ in the support of $\mu$ contribute to the integral. Here $G$ is shorthand for SE(3) and $dg$ denotes the invariant integration measure for SE(3).

In an analogous way, it is not difficult to see that integrating the $x$-dependence out of $\mu$ provides the total mass of configurations of the chain starting at frame $e$ and terminating at frame $g$. Since each chain has mass $M$, this means that the frame density $f(g)$ is related to $\mu(g,x)$ as:

$$f(g) = \frac{1}{M}\int_{\mathbb{R}^3} \mu(g,x)\, dx. \qquad (10)$$

We note the total number of frames attained by one end of the chain relative to the other is

$$F = \int_G f(g)\, dg.$$

It then follows that

$$\int_{\mathbb{R}^3} \rho(x) dx = F.M.$$

If the functions $\rho(x)$ and $f(g)$ are known for the whole chain then a number of important thermodynamic and mechanical properties of the polymer can be determined.

We can divide the chain into $P$segments that are short enough to allow brute force enumeration calculation of $\rho_i(x)$ and $f_i(g)$for $i = 1, \dots, P$, where $g$ is the relativeframe of reference of the distal end of the segment with respect to the proximal one. For a homogeneous chain, such as polyethylene, these functionsare the same for each value of $i = 1, \dots, P$

In the general case of a heterogeneous chain, we can calculate the functions $\rho_{i,i+1}(x), f_{i,i+1}(g)$, and $\mu_{i,i+1}(g, \vec{x})$ for the concatenation of segments$i$ and $i+1$ from those of segments $i$ and $i+1$ separately in the following way:

$$\rho_{i,i+1}(x) = F_{i+1}\rho_i(x) + \int_G f_i(h)\rho_{i,i+1}(h^{-1} \circ x) dh, \qquad (11)$$

$$f_{i,i+1}(g) = f_i * f_{i+1}(g) + \int_G f_i(h) f_{i+1}(h^{-1} \circ g) dh, \qquad (12)$$

And

$$\mu_{i,i+1}(g, \vec{x}) = \int_G \big(\mu_i(h, \vec{x}) f_{i+1}(h^{-1} \circ g) + f_i(h)\mu_{i+1}(h^{-1} \circ g, h^{-1} \circ \vec{x})\big)\; dh. \qquad (13)$$

In these expressions $h \in G = \mathrm{SE}(3)$ is a dummy variable of integration.

The meaning of equation (11) is that the mass density of the ensemble of all conformations of two concatenated chain segments results from two contributions. The first is the mass density of all the conformations of the lower segment (weighted by the number of different upper segments it can carry, which is $F_{i+1} = \int_G f_{i+1}\, dg$). The second contribution results from rotating and translating the mass density of the ensemble of the upper segment, and adding the contribution at each of these poses. This contribution is weighted by the number of frames that the distal end of the lower segment can attain relative to its base. Mathematically $L(h)\rho_{i+1}(x) = \rho_{i+1}(h^{-1} \circ g)$ is a left-shift operation which geometrically has the significance of rigidly translating and rotating the function $\rho_{i+1}(x)$ by the transformation$h$. The weight

$f_i(h)\, dh$ is the number of configurations of the $i^{th}$segment terminating at frame of reference $h$.

The meaning of equation (12) is that the distribution of frames of reference at the terminal end of the concatenation of segments $i$ and $i + 1$ is the group theoretical *convolution* of the frame densities of the terminal ends of each of the two segments relative to their respective bases. This equation holds for exactly the same reason why equation (9) does in the context of robot arms.

Equation (13) says that there are two contributions to $\mu_{i,i+1}\ (g, \vec{x})$. The first comes from adding up all the contributions due to each $\mu_i(h, \vec{x})$. This is weighted by the number of upper segment conformations with distal ends that reach the frame $g$ given that their base is at frame $h$. The second comes from addingup all shifted copies of $\mu_{i+1}(g, \vec{x})$, where the shifting is performed by the lower distribution, and the sum is weighted by the number of distinct configurations of the lower segment that terminate at $h$. This number is $f_i(h)\,dh$.

## Statistics of stiff molecules as solutions to PDEs on $\mathbf{SO(3)}$ and $\mathbf{SE(3)}$ (4.13.2):

Experimental measurements of the stiffness constants of DNA and other stiff macromolecules have been reported in a number of papers, as well as the statistical mechanics of such molecules.

The stiffness and chirality can be described with parameters $D_{lk}$ and $d_l$ for $l, k = 1, 2, 3$. In particular, $D_{lk}$ are the elements of the inverse of the stiffness matrix. When a force is applied, these constants determine how easily one end of the molecule deflects from the helical shape that it assumes when no forces act on it. The parameters $d_l$ describe the helical shape of an undeformed molecule with flexibility described by $D_{lk}$.

Degenerate diffusion equations describing the evolution of position and orientation of frames of reference attached to points on the chain at different values of length, $L$. These equations incorporate stiffness and chirality information and are written in terms of SE(3) differential operators as

$$\left(\frac{\partial}{\partial L} - \frac{1}{2}\sum_{k,l=1}^{3} D_{lk}\,\tilde{X}_l^R \tilde{X}_k^R - \sum_{l=1}^{3} d_l\,\tilde{X}_l^R + \tilde{X}_6^R\right) f = 0. \qquad (14)$$

The initial conditions are $f(a, A; 0) = \delta(a)\delta(A)$ where $g = (a, A)$.

This equation has been solved using the operational properties of the SE(3) Fourier transform.[30]

## References:


1-A. Lytchak, open map theorem for metric spaces, St. Petersburg, Math. J, (vol 17), 2006.

2-Adam Kihcman and Hassan Eltayeb, A Note on wave equation and convolutions, Handawi publishing corporation, (vol 2007), 2007.

3-Anders Kock and Gonzalo E. Reyes, Some calculus with extensive quantities wave equation, Mathematical subject classification, (vol 11),2003.

4- Andrew Hassell, lectures on the wave equation, AMSI Summer school,2005.

5- . A. V. Balakrishnan, Applied functional analysis, Springer, 2012.

6- BradOsgood,the Fourier transform and it's applications, Stanford university,(EE 261), 2001.

7- Carothers, Neal L, A short course on Banach space theory, London Mathematical Society Student Texts 64, Cambridge: Cambridge University Press, 2005.

8- CG Moorthy, CT Ramasamy, UniformBoundedness Principle for unbounded operators, Acta Math. Univ. Comenianae,Issn 0862-9544, (vol 83) no.2, 2014.

9- Claudia Garetto, Close graph and open mapping theorems for topological modules and applications, booksc. Org,(vol 1508 II), 2017.

10- Cong-huayan, Jin-xuan, The uniform boundedness principle in L-topological vector spaces, Elselvier science, (vol 136), 2003.

11- DanialReem, open mapping Theorem and fundamental theorems of algebra, Cornell University Library, 2008.

12- ErwinKreyszig, Introductory functional analysis with applications, Library of congress cataloging in publication, 1978.

13- Felix Cabello Sanchez and Jesus M. F. Castillo, uniform boundedness and Twisted sums of Banach spaces, Houston Journal of mathematics, (vol 30), 2004.

14- Ganes. A. moorthy and CT. Ramasamy, Uniform Boundedness Principle for unbounded operators, Acta Math. Univ. Comenianae, 2013.

15- Gregory S. Chirikjian and Yunfeng Wang, Engineering applications of the Motion-Group Fourier transform, Modern signal processing MSRI Publications, (vol 46),2003.

16- George B. Thomas, J R, Calculus and analytic geometry, California library of congress, 1974.

17- Gentili, Graziano, and others, The open mapping theorem for regular quaternionic functions, Ann alidellascuola normal superior re di pisaclasse di scienze, (vol 8) no.4, 2009.

18- Giles,J.R.: Introduction to the Analysis of Normed Linear Spaces,Cambridge University Press,2000

19- Grafakos, Loukas, Classical and Modern Fourier Analysis, Prentice-Hall, 2004.

20- Hans Wilhelm Alt, Uniform Boundedness Principle, Spriger, 2016.
21- Hutson, V., Pym, J.S., Cloud M.J.: Applications of Functional Analysis and Operator Theory, 2nd edition, Elsevier Science, 2005.
22- I. T. Efimova, Ia. S. Ufliand, A generalization of Fourier's integral theorem and it's applications, booksc.org,PMM (vol 33) no. 5, 1969.
23- Isroil A. Ikromov, Detlef Muller, Uniform Estimates for the Fourier Transform of surface carried measures in $\mathbb{R}^3$and an Application to Fourier Restriction, Springer, 2011.
24- J. M. Ball, Aversion of the fundamental theorems for young measures, Springer, (vol 344), 2005.
25- J. F, James, Astudents guide to Fourier transform with applications in physics and engineering, Cambridge university, 2007.
26- Jeremy Williams, On the geometric interpretation of the complex Fourier transforms of a class of exponential functions, Springer, 2005.
27- Jeff Boyle, An applications of Fourier transform to most significant digit problem, The American Mathematical Monthly, 1994.
28- John C-Oxtoby, Measure and Category, Springer GTM2, second addition , 1980.
29- Josef Kolomy, open mapping theorem and solution of nonlinear equations in linear normed spaces, CommentationmathematicaeuniersitatisCarolinae, (vol 6),1995.
30- K. F. Riley, Mathematical methods for the physical science, Cambridge university press, 1974.
31- Knapp, Anthony W, Basic algebra, Springer, 2006.
32- Lebedev, L.P. and Vorovich, I.I.: Functional Analysis in Mechanics, Springer-Verlag, 2002

33- M Schechter, Principles of Functional analysis, Springer, (vol 8), 1971.

34- Murray R. Spiegel, John Liu, Mathematical handbook of Formulas and tables, Schaum's outline series Me Graw-Hill,2008.

35- Prestini, Elena, The evolution of applied harmonic analysis: models of the - real world, Birkhäuser,2004.

36- P PZabreiko, E l Smirnov, Principle of uniform boundedness, Springer, Issue 2, (vol 35), 2017.29-Pinsky, Mark, Introduction to Fourier Analysis and Wavelets, Brooks/Cole, 2002

37- Pietsch, Albrecht: History of Banach spaces and linear operators, Birkhauser Boston Inc., 2007.

38- Rahman, Matiur, Applications of Fourier Transforms to Generalized Functions, WIT Press, 2011.

39- Richard Beals, Applications of Fourier series, springer, (vol 12), 2017.

40- Rajendra Bhatia, TheUniform Boundedness Principle, Springer, (vol 50), june 2017.

41- R E Edwards, Fourier series amodern introduction, Springer,

(VOL 1), 2012.

42- Rodriguez,..and others, Complex analysis: the fundamental theorem in complex function theory, Springer, (vol 245 II), 2003.

43- Roxana Bujack, GerikScheuermann and EckhardHitzer, A General geometric Fourier transform convolution theorem, springer, (vol 23), 2013.

44- Ryan, Raymond A, Introduction to Tensor Products of Banach Spaces, Springer Monographs in Mathematics, London: Springer, 2002.

45- Schechter, M.: Principles of Functional Analysis, AMS, 2nd edition, 2001.

46- Stein, Elias; Shakarchi, Rami ,Fourier Analysis: An introduction, Princeton University Press,2003.

47- T. F. Bridgland, J R, On the boundedness and uniform boundedness of solution of nonhomogeneous systems, Journal of mathematical analysis and applications, (vol 12), 1965.

48- Taneja, HC, Fourier integrals and Fourier transforms", Advanced Engineering Mathematics, Volume 2, New Delhi, India: I. K. International Pvt Ltd, 2008.

49- Tim Woo, Geometric evaluation of the Fourier Transform from the Pole – Zero plot, Springer, 2009.

50- Terri Joan Harris Mrs, Hilbert spaces and Fourier series, California state university, San Bernardino, 2015.

51- Terras, Audrey, Fourier analysis on finite groups and applications, Cambridge University Press, 1999.

52- Walter Rudin, Functional analysis, Tata Me Graw-Hill, 2008.

53- William R. Derrick and Stanley I. Grossman, A First course in differential equations with applications, U. Of Montana,1987.