Sudan University of Science and Technology

College of Graduate Studies

Some of the Limit Cycle Problems and Critical points for Planar Systems

بعض مسائل الدورات النهائية والنقاط الحرجة للأنظمة المستوية

A Thesis Submitted for the fulfillment of The Degree of Ph .D

In Mathematics

By

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DEDICATION

To my Parents and my Family for all the joy they bring to my life.

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Abstract

This study is an applied analytical one that helps in solving problems of the limit cycle and critical points for Planar systems.

We introduced the classification of stable and unstable critical points of linear and nonlinear systems. The study found that the linear systems do not have a limit cycle. The study dealt with isolated limit cycle with its different patterns in an analytical and applied manner in the differential Planar systems of the second degree. The study investigated the problems related to the system limit cycle from Liénard type, and the researcher cited many examples and applications in this field.

We discussed the problems of the limit cycle from the system other than Liénard and the method of converting it into the system from type of Liénard by applying some different techniques such as some nonlinear integrations, methods of comparison and some conversion techniques, and we supported this field with appropriate examples and applications.

From these we concluded the application of some functions, equations and theories such as Dulac, Vander pol and Poincare, respectively. And that some of the systems do not have limit cycle, some of them have a single and stable limit cycle (Liénard), and some of them have many limit points.

الملخص

هذه الدراسة تحليلية تطبيقية تساعد في حل مسائل الدوارةِ النهائية والنقاط الحرجةِ للأنظمةِ المستويةِ,قدمنا تصنيفا للنقط الحرجة المستقرة وغير المستقرة للأنظمة الخطية واللاخطية وتوصلت الدراسة إلى أن الأنظمة الخطية ليست لها دوارة نهائية,وتناولنا الدوارة النهائية المعزولة بأنماطها المختلفة بصورة تحليلية تطبيقية في الأنظمة التفاضلية المستوية من الدرجة الثانية , أيضاً تناولنا دراسة المسائل التي تتعلق بالدوارة النهائية للنظام من النوع لينارد وأوردنا العديد من الأمثلة والتطبيقات في هذا المجال , وكذلك تناولت الدراسة مسائل الدوارة النهائية من النظام غير لينارد وطرق تحويلها للنظام من النوع لينارد وذلك بتطبيق بعض التقنيات المختلفة مثل بعض التكاملات اللاخطية وطرق المقارنة وبعض تقنيات التحويل , ودعّمنا هذا المجال بالأمثلة والتطبيقات المناسبة. ومنها توصلنا إلى أن بعض الأنظمة ليست لها دوارة نهائية ولبعضها دوارة نهائية وحيدة ومستقرة (لينارد) ولبعضها العديد من الدوارات النهائية وذلك بتطبيق بعض الدوال والمعادلات و النظريات مثل دولاك وفاندربول وبونكري- بنديكسن على التوالي .

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Chapter one

Introduction

Since 1940s, many mathematical models from physics, engineering, chemistry, biology, economics, etc., have been displayed as autonomous planar systems.
A wide class of autonomous planar systems can be transformed into Liénard-type systems. concerning the maximum number of limit cycles of all quadratic differential systems (the second part of Hilbert's 16th problem), the study of the qualitative behavior of the solutions of autonomous planar systems of Liénard-type has become more and more important and has attracted the attention of many pure and applied mathematicians.
The purpose of this thesis is to develop the qualitative theory of autonomous planar systems of Liénard-type. More explicitly, we give conditions for global asymptotic stability, existence of local centers and global centers, existence of oscillatory solutions, existence and nonexistence of periodic solutions, and also existence and uniqueness of limit cycles for some autonomous planar systems of generalized Liénard-type. Moreover, in case of having uniqueness of limit cycles, the hyperbolicity of the limit cycle is relevant.
We apply different techniques for different types of systems. The main tools used are some nonlinear integral inequalities, methods of comparison and some transformation techniques .
Furthermore, some powerful methods for Liénard systems .
We apply the criteria for existence, uniqueness and hyperbolicity of limit cycles, existence of centers, existence of oscillatory solutions, and global asymptotic stability of an uniqueness positive equilibrium systems found in the literature.
So conjectured that the number of limit cycles is at most two for these systems.
We show the number of periodic orbits are construct an example with at least two limit cycles.

the nontrivial periodic coexistence does happen even invasion from either of the other species in this case, new amenable conditions are given on the coefficients under which the system has no nontrivial periodic coexistence.

These conditions imply that the positive equilibrium, if it exists, is globally asymptotically stable.

Have found that orbits cannot cross, can be attracted to (fixed points), etc. One other possibility is limit cycle.

ODE is 'well behaved' i.e. all derivatives exist and are continuous –

Therefore, all orbits smoothly follow neighbours in phase space.

One other possibility only:

limit cycle

we have defined the stability of limit cycles for a system of equations

$$\frac{dx}{dt} = p(x,y), \qquad \frac{dy}{dt} = Q(x,y) \qquad (1.1)$$

In applications only a stable limit cycle has practical significance, since every spiral sufficiently close to a limit cycle can approximately represent an oscillation of constant amplitude independent of initial conditions; and an unstable limit cycle similar to an unstable equilibrium position in mechanics, which in reality does not exist; hence, how to distinguish stability of limit cycles becomes a very important problem.

First of all, if the existence of a limit cycle is determined from the fixed point of a point transformation, then no matter whether P and Q on the right sides of (1.1) are continuous or not, we can, under suitable conditions, distinguish whether the limit cycle is stable, provided that the point transformation is continuous.

This is the often-used Konigs theorem in the theory of nonlinear oscillations.

1.1:Definition . Let $s = f(s)$ be a continuous point transformation which carries some line. segment l into itself, and let $s*$ be a fixed point of this transformation, i.e., $s^* = f(s^*)$.

If there exists a small neighborhood of s^* (on l) such that for any point s inside it the sequence of points

$$s_1 = f(s), \quad s_2 = f(s_1), \dots, = s_{n+1} = f(s_n), \dots$$

always converges to s^*, then s^* is *stable* under this point transformation.

Conversely, if in any small neighborhood of s^* we can find a point 8 such that the above sequence of points do not converge to s^*, then s^* is called unstable.

1.2:linear and nonlinear system:

This study deals with linear and nonlinear system of ordinary differential equations.

Many physical quantities, such as a vehicle's velocity, or electrical signals, have an upper bound.

When that upper bound is reached, linearity is lost .

The differential equations.

Governing some systems, such as some thermal, fluidic, or biological systems, are nonlinear in nature.

It is therefore advantageous to consider the nonlinearities directly while analyzing and designing controllers for such systems. Mechanical systems may be designed with backlash – this is so a very small signal will produce no output (for example , in gearboxes). In addition, many mechanical systems are subject to nonlinear friction.

Relays, which are part of many practical control systems, are inherently nonlinear.

Linear systems must verify two properties, superposition and homogeneity.

The principle of superposition states that for two different inputs, x and y, in the domain of the function f,

$$f(x + y) = f(x) + f(y) \quad .$$

The property of homogeneity states that for a given input, x, in the domain of the function f, and for any real number k,

$$f(kx) = kf(x)$$

Any function that does not satisfy superposition and homogeneity is nonlinear. It is worth noting that there is no unifying characteristic of nonlinear systems, except for not satisfying the two above-mentioned properties.

1.3: The dynamical system:

a dynamical system has a state determined by a collection of real numbers, or more generally by a set of points in an appropriate state space.

Small changes in the state of the system correspond to small changes in the numbers. The evolution rule of the dynamical system is a fixed rule that describes what future states follow from the current state.

The rule is deterministic for a given time interval only one future state follows from the current state.

The mathematical models used to describe the swinging of a clock pendulum, the flow of water in a pipe, or the number of fish each spring in a take are examples of dynamical systems.

1.3.1:Definition: Autonomous System. **[**2**]**

An autonomous differential equation is a system of ordinary differential

equations which does not depend on the independent variable it is of the form:

$$\frac{d}{dt}x(t) = F\big(x(t)\big) \qquad (1.2)$$

X(t) = F(x(t)),

where x takes values in n-dimensional Euclidean space and t is usually time.

It is distinguished from systems of differential equations of the form

$$\frac{d}{dt}\ x(t) = G(x(t), t), \qquad (1.3)$$

in which the law governing the rate of motion of a particle depends not only on the particle's location, but also on time; such systems are not autonomous.

Autonomous systems are closely related to dynamical systems.

Any autonomous system can be transformed into a dynamical system and, using very weak assumptions, a dynamical system can be transformed into an autonomous systems.

1.3.2:**In the following we consider the general linear system** [2],

$$\dot{x} = \mathrm{A}x \qquad (1.4)$$

Where x $\in R^n$, A is an n x n matrix and $\dot{x} = \frac{dx}{dt}$

It is shown that the solution of the linear system (2) together with the

initial condition $x(0) = x_0$ is given by

$$x(t) = e^{At}x_0$$

where e^{At} is an n x n matrix function defined by its Taylor series.

A good portion of this chapter is concerned with the computation of the matrix e^{At} in terms of the eigenvalues and eigenvectors of the square matrix A.

Chapter two

The Theory of stability and Classification of singular points

2.1.1: stability

If we find a fixed point, or more generally an invariant set, of a dynamical system we want to know what happens to the system under small perturbations away from the invariant set.

We also want to know which invariant sets will be approached at large times.

If in some sense the solution stays "nearby", or the set is approached after long times, then we call the set stable . [21],

There are several differing definitions of stability. We will consider stability of whole invariant sets (and not just of points in those sets).

This shortens the discussion.

Consider an invariant set A in a general (autonomous) dynamical system described by a flow $\emptyset_t$. (This could be a fixed point, periodic orbit, torus etc.)

We need a definition of points near the set A.

2.1.2: Definition (Neighbourhood of a set A).

For $\delta > 0$ the neighbourhood $N_\delta(A) = \{x : \exists\, y \in A \text{ s.t. } |x - y| < \delta\}$

We also need to define the concept of a flow trajectory 'tending to' A.

2.1.3: Definition (flow tending to A):

The flow $\emptyset_t(x) \to A$ iff $\min_{y\in A} |\emptyset_t(x) - y| \to 0$ as $t \to \infty$

2.1.4: Definition (Lyapunov stability):

The set A is Lyapunov stable if $\forall \in > 0, \exists\, \delta > 0$ s.t. $x \in N_\delta(A) \Rightarrow \emptyset_t(x) \in N_\delta(A)$ $\forall\, t \geq 0$. ("start near, stay near").

2.1.5:Definition (Quasi-asymptotic stability):

The set A is quasi-asymptotically stable if $\exists\, \delta > 0$ s.t. $x \in N_\delta(A) \Rightarrow \emptyset_t(x) \to A$ as $t \to \infty$. ("get arbitrarily close eventually").

2.1.6: Definition (Asymptotic stability):

The set A is asymptotically stable if it is both Lyapunov stable and quasi-asymptotically stable.

We discuss the stability of the equilibrium point of the non linear system

$$x' = f(x), \qquad (2.1)$$

The stability of any hyperbolic equilibrium point $x_0\, of$ (2.1) is determined by the signs of the real part of the eigenvalues of the matrix $Df(x)$.

Ahyperbolic equilibrium point x_0 is asymptotically stable iff

$$De(\lambda_j) < 0 \; for \;\; j = 1, \dots, n \;\; iff \; x_0$$

is a sink and hyperbolic equilibrium point x_0 is saddle. The stability of non-hyperbolic equilibrium point is typically more difficult to determine.

2.1.7: Definition:

let $\emptyset_t$ denoted the flow of the differential equation(2.1) defined for all t∈R .

An equilibrium point x_0 of (2.1) is stable if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in N_\delta(x_0)$ and $t \geq 0$ we have

$$\emptyset_t x \in N_\varepsilon(x_0), \qquad (2.2)$$

The equilibrium point x_0 is unstable if it is not stable.

And equilibrium point x_0 is asymptotically stable if it is stable and if there exists $a\, \delta > 0$ such that for all $x \in \lambda_\delta(x_0)$ we have

$$\lim_{t\to\infty} \emptyset_t(x) = x_{0,} \qquad (2.3)$$

Not that the above limit being satisfied for all x in some neighborhood of x_0 does not imply that x_0 is stable.

2.1.8:Theorem:

If x_0 is a sink of the nonlinear system (2.1)

and $Re(\lambda_j) < -\alpha < 0$ for all of the eigenvalues λ_j of the matrix $Df(x_0)$ then given $\varepsilon > 0$ there exists $a\, \delta > 0$ such that for all $x \in N_\delta(x_0)$ the flow $\emptyset_{(t)}(x)$ of (2.1) satisfies

$$|\emptyset_t(x) - x_0| \leq \varepsilon e^{-\alpha t}, \qquad (2.4)$$

For all $t \geq 0$

Since hyperbolic equilibrium point are either as asymptotically stable or unstable the only time that an asymptotically equilibrium point x_0 of (2.1) can be stable but not asymptotically stable is when $Df(x_0)$ has a zero eigen value or a pair of complex – conjugate.

Pure imaginary eigenvalues $\lambda = \pm ib$.

Follows from the next theorem, proved, that other eigenvalues $\lambda_j of\ Df(x_0)$must satisfy $Re(\lambda_j) \leq 0$ if x_0 stable.

2.1.9: Theorem

If x_0 is a stable equilibrium point x_0 of (2.1) an eigenvalue of $Df(x_0)$ has positive real part.

We see that stable equilibrium points which are not asymptotically stable can only occur at non hyperbolic equilibrium points.

But the question as to where anon hyperbolic equilibrium point is stable – asymptotically unstable is a delicate question.

2.1.10: Definition:

If $f\epsilon c^1(E), v\epsilon c^1(E)$ and $\emptyset_t$ is the flow of the differential equation of (2.4) them for $x \in E$ the derivative of the function $v(x)$ along the solution $\emptyset_t(x)$.

$$v'(x)\frac{d}{dt}v[\emptyset_t(x)]\big]_{t=0} = Dv(x)f(x), \qquad (2.5)$$

The last equality follows from the chain rule if $v'(x)$ is negative in E then $V(x)$ decreases along the solution $\emptyset_{(t)}(x_0)$ througt $x_0\epsilon E\ at\ t = 0$ furthermore, in $R^2\ if\ v'(x) \leq 0$ with equality only at $x = 0$ then for small positive c the family of curves $u(x) = c$ constitutes a family of closed curves enclosing the origin and the trajectories of (2.1)

Cross these curves from their exterior to their interior with increasing t, the origin of (2.1) is asymptotically stable. A function $V: R^n \to R$ satisfying the hypothese of the next theorem is called a Liapunov function.

2.1.11: Theorem:

Let E be an open subset of R^n containing (x_0) suppose that $f \in c^1(E)$ and that $f(x_0) = 0$. Suppose further that there exists areal valued $v \in c^1(E)$ satisfying $v(x_0) = 0$ and $v(x) > 0 \;\; if \; x \neq x_0$. Then:

a. $if \; v'(x) \leq 0$ for all $x \in E, x_0$ is stable;

b. $if \; v'(x) < 0$ for all $x \in E \sim \{x_0\}, x_0$ is asymptotically stable ;

c. $if \; v'(x) > 0$ for all $x \in E \sim \{x_0\}, x_0$ is unstable;

Proof: without loss of generality we shall assume that the equilibrium point $x_0 = 0$ Choose $\varepsilon > 0$ sufficiently small that the $N_\varepsilon(0) \subset E$ and let $M_\varepsilon be$ the minimum of the continuous function $V(x)$ on the compact set

$$S_\varepsilon\big\{\{x\epsilon R^n||x| = \varepsilon\}\big\}$$

2.1.12: Lyapunov functions:

We can prove much about stability of a fixed point (which, for convenience, will be taken to be at the origin) if we can find a suitable positive function V of the independent variables that is zero at the origin and decreases monotonically under the flow $\emptyset_t$. Then under certain reasonable conditions we can show that V → 0 so that the appropriately defined distance from the origin of the solution similarly tends to zero.

This is a Lyapunov function, [26], defined precisely by:

2.1.13: Definition (Lyapunov function):

Let E be a closed connected region of containing the origin.

A function V : R_n → R which is continuously differentiable except perhaps at the origin is a Lyapunov function for a flow $\emptyset$ if

(i) V(0) = 0,

(ii) V is positive definite (V(x) > 0 when $x \neq 0$), and if also

(iii) V($\emptyset_t$ (x)) ≤ V(x) ∀ x ∈ E (or equivalently if $\dot{V} \leq 0$ on trajectories).

Then we have the following theorems

Theorem (Lyapunov's first theorem) .

Suppose that a dynamical system $\dot{x}$ = f (x) has a fixed point at the origin.

If a Lyapunov function exists, as defined above, then the origin is Lyapunov stable. [26],

2.2: Equilibrium points:

Equilibrium point of dynamical system generated by an autonomous system of ordinary differential equation is solution that does not change with time for example, each motion less pendulum position corresponds to at equilibrium of the corresponding equation of motion one is stable the other one is not, geometrically equilibrium are point systems phase space.

More precisely the $x' = f(x)$ has an equilibrium solution $x(t) = x$ equilibrium are sometime called fixed points or steady states.

Most mathematicians refer to equilibria as time independent solution of (O. D. E_s) and to fixed point as time independent solution iterated maps.

$$x(t+1) = f[x(t)], \qquad (2.6)$$

The qualitative behavior of anon linear system new an equilibrium point can take one of the patterns we have seen with linear systems correspondingly the equilibrium points are classified as stable node, unstable node saddle, stable focus, unstable focus, center.

2.3:The fundamental existence – uniqueness theorem:

We establish the fundamental existence – uniqueness theorem for nonlinear autonomous system of ordinary differential equations in system (2.1).

Under the hypothesis that $f \in \overset{1}{C}(E)$ when E is an opens subset of R^n.

Picard's classical method of successive approximations is used to prove this theorem. The more modern approach based on the contraction mapping principle is relegated to the problems. [24],

The method of successive approximations not only allows us to establish the existence and uniqueness of the solution of the initial value problem associated with (2.1).

But it also allows us to establish the continuity and differentiability of the solution with respect to initial conditions and parameters.

2.3.1: Definition:

Suppose that $f \in C(E)$ there E is an open subset of R^n then $x(t)$ is a solution of the differential equation (2.1) of an interval I if $x(t)$ is differential on I and if for all $t \in I$, $x(t) \in E$ and $\dot{x}(t) = f(x(t))$ and given $x_o \in E, x(t)$ is a solution of the initial value problem.

$$\dot{x} = f(t)$$
$$x(t_o) = x_o$$

an interval I if $t_o \in I, x(t_o)$ is a solution of the differential equation (2.8) on the interval .

In order to apply the method of successive approximations to establish the existence of a solution of (2.1) we need to define the concept of Lipchitz condition.

2.3.2: Lipschtiz condition:

Let $m > 0$ be a constant and f the function define in domain D of the xy plane a Lipschtiz condition is the inequality.

$$|f(x, y_1) - f(x, y_2)| \le m|y_1 - y_2|, \qquad (2.8)$$

assume hold for all (x, y_1) and (x, y_2) in D the most common way to satisfy this conditions is to require the partial derivative $f_y(x, y_2)$ to be continuous.

2.3.3: Definition:

Let E be an open subset of R^n. A function $f : E \to R^n$ is said to satisfy a Lipschtiz condition E if there is appositive constant K such that for all $x, y \in E$.

$$|f(x) - f(y)| \le K|x - y|, \qquad (2.9)$$

The function f is said to be locally Lipschtiz on E if for each point $x_o \in E$ there is an neighborhood of x_o, $N_\varepsilon(x_o) \subset E$ and a constant $K_o > 0$ such that for all $x, y \in N_\varepsilon(x_o)$.

$$|f(x) - f(y)| \le K_o|x - y|$$

By an $\in -$Neighborhood of a point $x_o \in R^n$, we mean an open ball of positive radius $\in$.

$$N_\varepsilon(x_o) = \{ x \in R^n \,|x - x_o| < \in \} \qquad (2.10)$$

2.3.4: Lemma:

Let E be an open subset of R^n and let $f: E \to R^n$. Then if $f \in \overset{1}{C}(E)$, f is locally Lipschtiz on E.

Proof :

Since E is an open subset of R^n, given $x_o \in E$ there is an $\varepsilon > 0$ such that $N_\varepsilon(x_o) \subset E$, let $K = \max\limits_{|x-x_o| \le \frac{\varepsilon}{2}} \| Df(x) \|$.

The maximum of continuous function $Df(x)$ on the compact set $|x - x_o| \le \dfrac{\varepsilon}{2}$. Let N_o denote the $\dfrac{\varepsilon}{2}$-neighborhood of $x_o, N_{\frac{\varepsilon}{2}}(x_o)$ then for $x, y \in N_o$, set $u = y - x$.

It follows that $x + su \in N_o$ for $0 \le S \le 1$ since N_o is a convex set.

Define the functions $f: [0,1] \to R^n$ by $f(x) = f(x+su)$then by the chain rule.

$$F'(S) = Df(x+su)u$$

And therefore

$$f(y) - f(x) = F(1) - F(0)$$

$$= \int_0^1 F'(s)\, ds = \int_0^1 D f(x+su) u\, dy$$

From known:

$$|f(y) - f(x)| \le \int_0^1 |D f(x+su)| u ds$$

$$\le \int_0^1 \| D f(x+su) \| |u| ds$$

$$\le K|u| = K|y - x|$$

And this proves the lemma.

2.4: Picard's method of successive approximation: [36],

Is based on the fact that $x(t)$is a solution of the initial value problem.

$$\frac{dy}{dt}=f(t,y),y(t_o)=y_o, \qquad (2.11)$$

Any solution to equation (1.5) also to be solution to

$$\phi(t)=y_o+\int_0^t f(s,\phi(t))ds, \qquad (2.12)$$

The successive approximations a real based on the integral equation (2.12) as follows.

$$\phi(t)=y_o, \qquad (2.13)$$

$\phi_o,\phi_1,...\phi_k$ converges to some function $\phi(t)$that satisfies equation(2.12).

2.4.1: Definition:

A flow in R^2 is a mapping $\pi:R^2\to R^2$ such that:

- π is continuous.
- $\pi(x,0)=x$ for all $x\in R^2$.
- $\pi(\pi(x,t_1),t_2)=\pi(x,t_1+t_2)$

2.4.2: Definition:

Suppose that I_x is the max interval existence the trajectory (or orbit) through x is define as $y(x)=\{\pi(x,t):t\in I_x\}$, the positive semi orbit is defined as $\gamma^+(x)=\{\pi(x,t):t>0\}$ the negative semi orbit is defined as $\gamma(x)=\{\pi(x,t):t>0\}$.

2.4.3: Definition: The positive limit set of point x

is defined as $A^+(x)=\{y:\text{there exists a sequence}\, t_n\to\infty \,\text{such that}\, \pi(x,t)\to y\}$.

The negative limit set of point x is defined as

$A^-(x)=\{y:t_n\to\infty s\,\pi(x,)\to y\}$.

In the phase plane, trajectory tend to critical a closed orbit, or infinity.

2.4.4:**Theorem** [24], Let δ= det A and τ = trace A and consider the linear system

$$\dot{x} = Ax$$

i. If $\delta < 0$ then (1.3) has a saddle at the origin.

ii. If $\delta > 0$ and $\tau^2 - 4\delta \geq 0$ then (1.13) has a node at the origin; it is stable if τ< 0 and unstable if τ> 0.

iii. If $\delta > 0, \tau^2 - 4\delta < 0 and$ $\tau \neq 0$ then (1.13) has a focus at the origin; it is stable if τ < 0 and unstable if $\tau > 0$.

iv. If $\delta > 0$ and $\tau = 0$ then (1.13)has a center at the origin.

Proof: The eigenvalues of the matrix A are given by

$$\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2}.$$

Thus i. if $\delta < 0$ there are two real eigenvalues of opposite sign.

ii. If $\delta > 0$ and $\tau^2 - 4\delta \geq 0$ then there are two real eigenvalues of the same sign as τ ;

iii. If $\delta > 0, \tau^2 - 4\delta < 0 and$ $\tau \neq 0$ then there are two complex conjugate eigenvalues λ = a ± ib.

iv. If $\delta > 0$ and $\tau = 0$ then there are two pure imaginary complex conjugate eigenvalues.

2.4.5:**Definition**[24],

A stable node or focus (i) is called a sink of the linear system and an unstable node or focus(i) is called a source of the linear system.

We describe the phase plane of linear system, besides allowing us to visually observe the motion patters of linear system, this will also help the development of nonlinear system analysis in the next section, be case a nonlinear system behaves similarly to a linear system avowed each equilibrium point, the general form of a linear system (second-order) is:

$$\dot{x}_1 = a x_1 + b x_2$$
$$\dot{x}_2 = c x_1 + d x_2 \quad (2.2)$$

Leads to

$$\ddot{x}_1 = (a+d)\dot{x}_1 + (cb-ad)x_1$$

Therefore we will simply consider the second-order linear system described by

$$\ddot{x} + a\dot{x} + bx = 0 \qquad (2.14)$$

To obtain the phase portrait of this linear system, we first solve for the time history.

$$x(t) = K_1 e^{h1} + k_2 e^{h_2} \quad for\ \lambda_1 \neq \lambda_2 \qquad (2.15)$$

$$x(t) = K_1 e^{h1} + k_2 e^{h_2} \text{ for } \lambda_1 = \lambda_2 \qquad (2.16)$$

Where the constants λ_1 and λ_2 are the solution of the characteristic equation.

$$S^2 + as + b = (S-\lambda_1)(S-\lambda_2) = 0$$

The roots λ_1 and λ_2 can be explicitly represented as

$$\lambda_1 = \left(-a + \sqrt{a^2 - 4b}\right)/2$$

$$\lambda_2 = \frac{\left(-a - \sqrt{a^2 - 4b}\right)}{2}$$

For linear system described by $\ddot{x} + a\dot{x} + bx = 0$

There is only one singular point $(b \neq 0)$ namely the origin. However, the trajectories in the vicinity of this singularity point can display quite-different characteristics depending on the values off hand. The following cases can occur.

1. λ_1 and λ_2 are both real and have the same sing.
2. λ_1 and λ_2 are both real and have the opposite signs
3. λ_1 and λ_2 are complex conjugates with real part
4. λ_1 and λ_2 are complex conjugates with non zero real part.

We discuss each of the above four cases.

2.4.6: **Stable or unstable Node** [24],

The first case corresponds to anode.

Anode can be stable or uunstable .If the eigenvalues are negative the singular point is called a stable order because both $x(t)$ and $\dot{x}(t)$ converge to zero.

If both eigenvalues are positive the singular point is called an unstable node because both $x(t)$ and $\dot{x}(t)$diverge from zero exponentially.

Since the eigenvalues are real there is no oscillation in the trajectories.

2.4.7:**Saddle point** [24],

The second case say ($\lambda_1 < 0$ and $\lambda_2 > 0$) corresponds to a saddle point the phase protract of the system has the interesting saddle. Because of the unstable pole λ_2 almost all of the system trajectories diverge to infinity. The diverging line corresponds to inline conditions which make K_2 equal zero. The converging straight line corresponds to initial condition which make K_1 equal zero.

2.4.8:**Stable and unstable focus:**[24],

The third case corresponds to a focus a stable focus occurs when the real part of the Eigenvalues negative which implies that $x(t)$ and $\dot{x}(t)$ both converge to zero. The system trajectories encircle the origin one or move times before converging to it unlike the situation for stable node. If the real part of the eigenvalues is positive then $x(t)$ and $\dot{x}(t)$ both diverge to infinite, and the singular point is called an unstable focus.

The trajectories corresponding to an unstable focus.

2.4.9:**Center point:** [24],

The last case corresponds to a center point .

The name comes from the fact that all trajectories are ellipses and the singularity point it the center of these ellipses. The phase portrait of the undammed mass-spring system be longs to this category.

2.5: Classification of singular points of linear systems:[46],

Recall that a point x_0 is called ordinary if p (x) and Q(x) from the equation $Y'' + P(x)y' + Q(x)y = 0$ are both analytic at x_0. Apoint x_0 which is not ordinary is called singular point.

In this portion we discuss the various phase portraits that are possible for the linear system.

$$\dot{x} = Ax$$

When $x \in R^2$ and A is 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$T = a + d \ , \quad D = ad - bc \ , P(\lambda) = \lambda^2 - T\lambda + D$$

Case A

$$T^2 - 4D > 0$$

There are sub cases of case A

1. $if\ D < 0$, saddle solely

2. $D > 0$ there are four cases

i. $T > 0$ source

ii. $T < 0$ sink

iii. $T^2 > 4D$ node

iv. $T^2 < 4D$ spiral (focus)

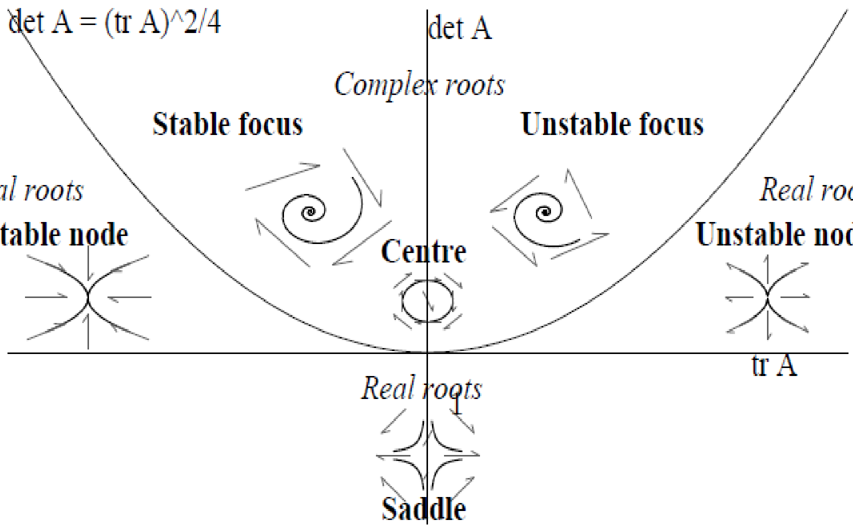
Border line case

i. $if\ T = 0\ and\ D > 0$ center.

ii. $if\ T \neq 0\ and\ T^2 > 4D$ saddle – node

iii. $if\ T > 0$ unstable

iv. $if\ T$ <0 stable



2.5.1: Examples:

Determine the linear system $Ax = x'$ has saddle – node, center at the origin and determine the stability of each node or focus.

$A = \begin{pmatrix} 8 & 5 \\ -10 & -7 \end{pmatrix}$, $D = -6 < 0$ the system is saddle at the origin.

$A = \begin{pmatrix} -2 & 0 \\ 1 & -1 \end{pmatrix}$, $D = 2 > 0$, $T = -3 < 0$ the system is node sink at the origin

$A = \begin{pmatrix} -10 & -25 \\ 5 & 10 \end{pmatrix}$, $D = 25 > 0$, $T = 0$

Center and counter clock wise direction of rotation.

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, D = 8 > 0, T = 6 > 0$$

The system unstable node at the origin.

Case A

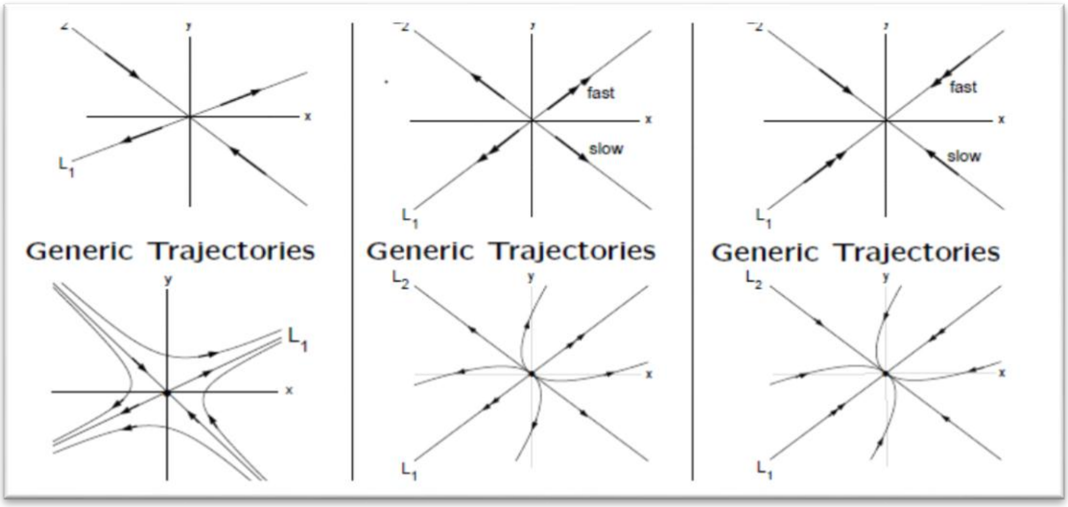
$T^2 - 4D > 0$. gives the real distinct eigenvalues

λ_1 and $\lambda_2 = \frac{T \pm \sqrt{T^2 - 4D}}{2}$

Subcases of case A

$\lambda_1 > 0 > \lambda_2$, saddle$\lambda_1 \geq \lambda_2 > 0$ node Soule$\lambda_1 \leq \lambda_2 < 0$ $sink$

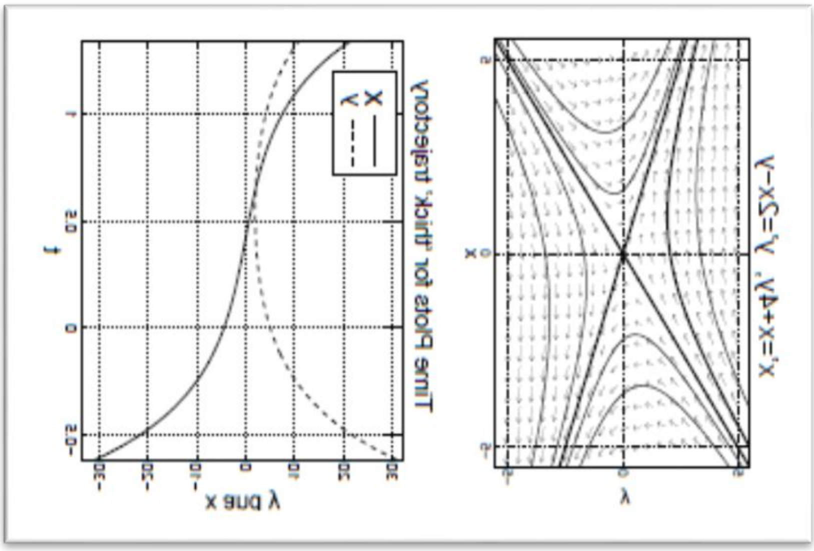
Half line trajectories half line trajectories half line trajectories



2.5.2: **Examples:**

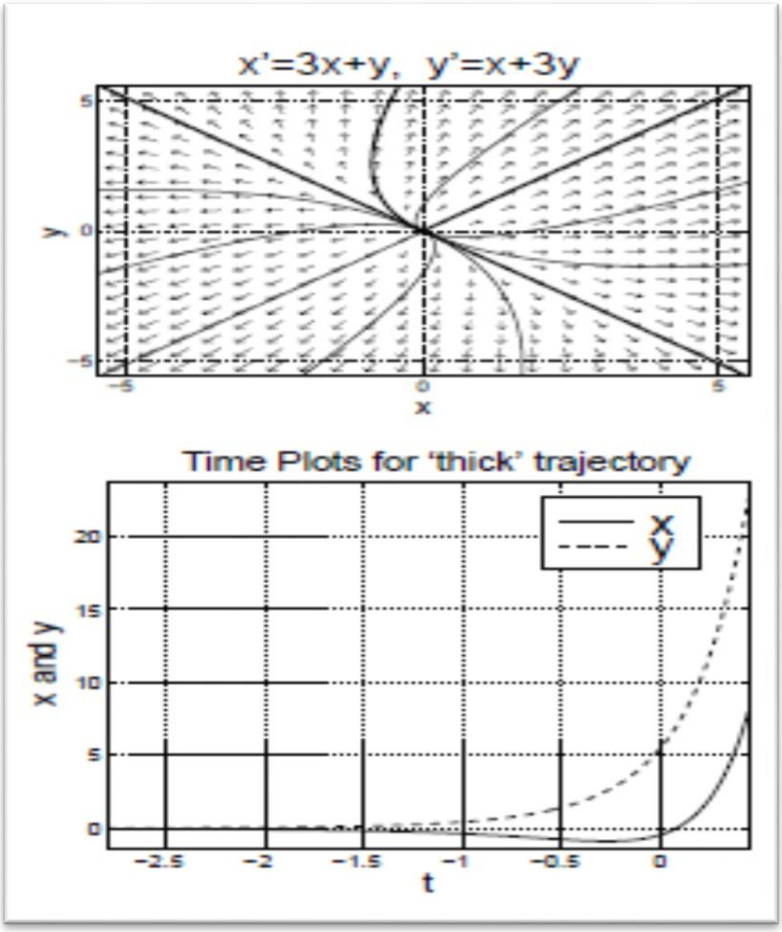
Solve the following linear system and draw the phase portrait

1. $A = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}$, $\lambda_1 = 3, v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\lambda_2 = -3, v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$



$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$, $\lambda_1 = 4, v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\lambda_2 = 2, v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

The system nodal – source



Case B

$$T^2 - 4D < 0\,, \lambda = \alpha + \beta i\,, \alpha = \frac{T}{2}$$

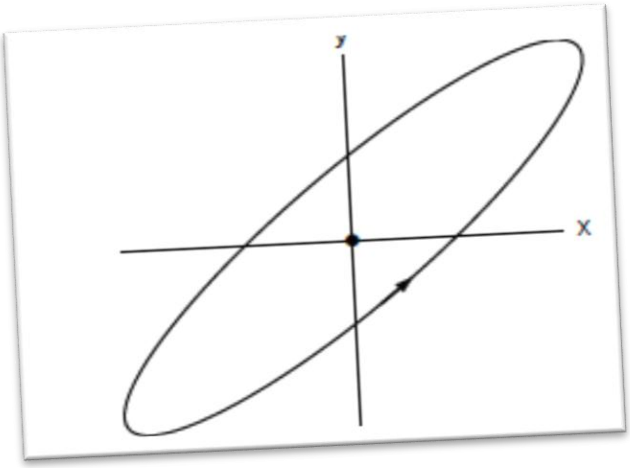
$$\beta = \frac{\sqrt{4D - T^2}}{2}$$

λ complex number , eigenvector$v = u + iw$ complex and general solution

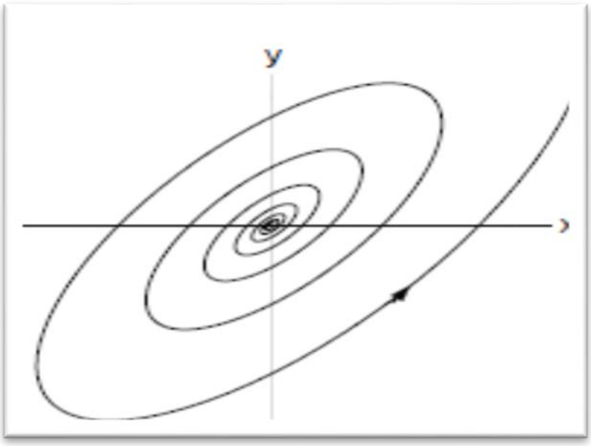
$x(t) = e^{\alpha t}[c_1(ucos\beta t - wsin\beta t) + (ucos\beta t + wsin\beta t)]$

Sub cases of B

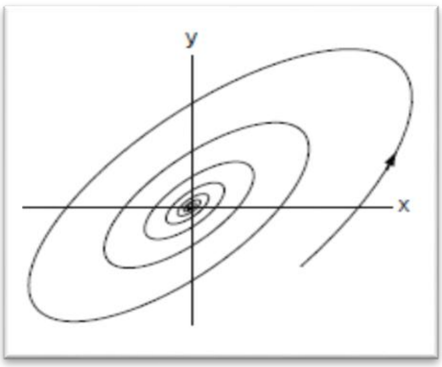
i. $\alpha = 0$, center, $x(t)$ periodic trajectories are closed curves



ii. $\alpha > 0$, spiral source, growing oscillations trajectories are spiral the direction of rotation, counter clockwise



iii. $\alpha < 0$, spiral sink, decaying oscillations trajectories are ingoing spirals The direction of rotation clockwise

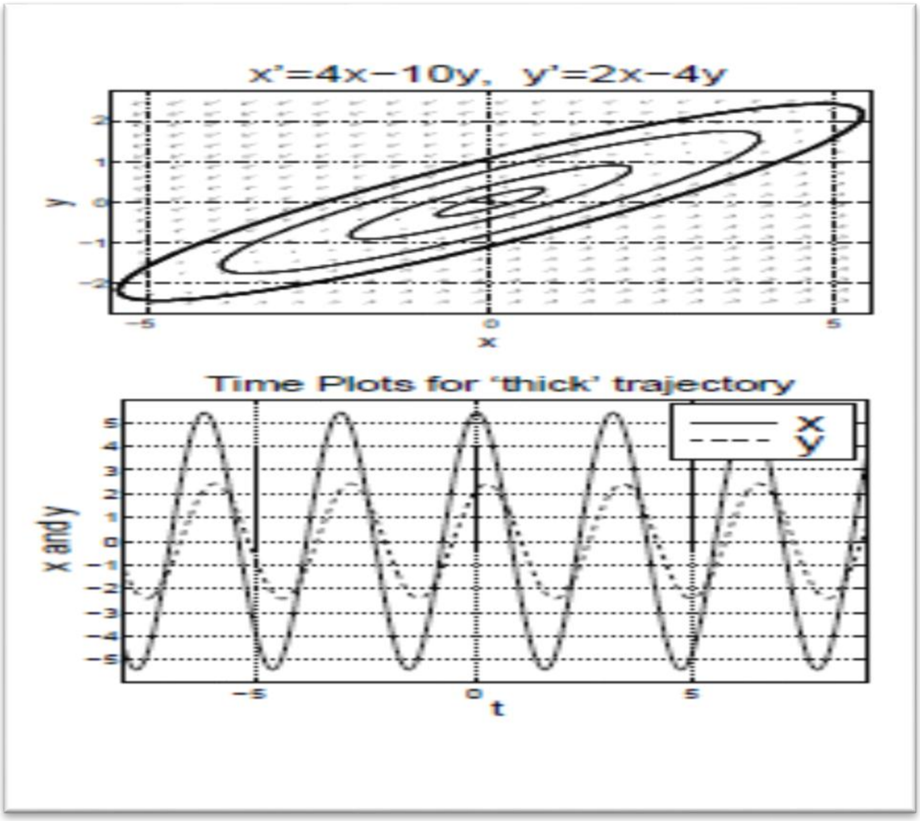


2.5.3: Examples

Solve the initial value problem for

1. $A = \begin{pmatrix} 4 & -10 \\ 2 & -4 \end{pmatrix}, \lambda = 2i, v = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$

The matrix A has the complex eigenvalues and his center.

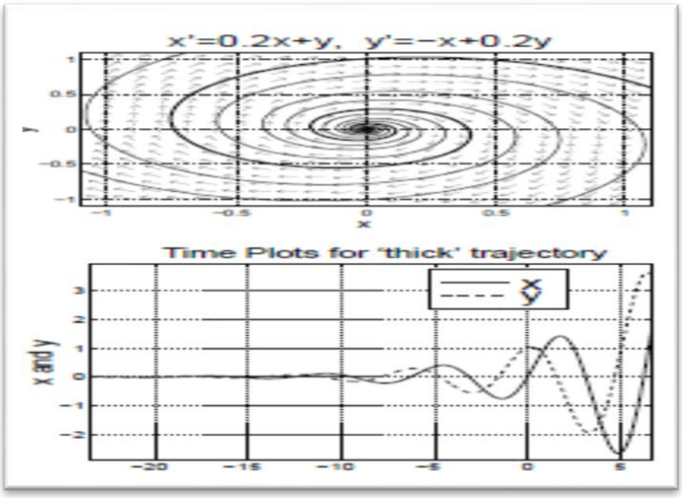


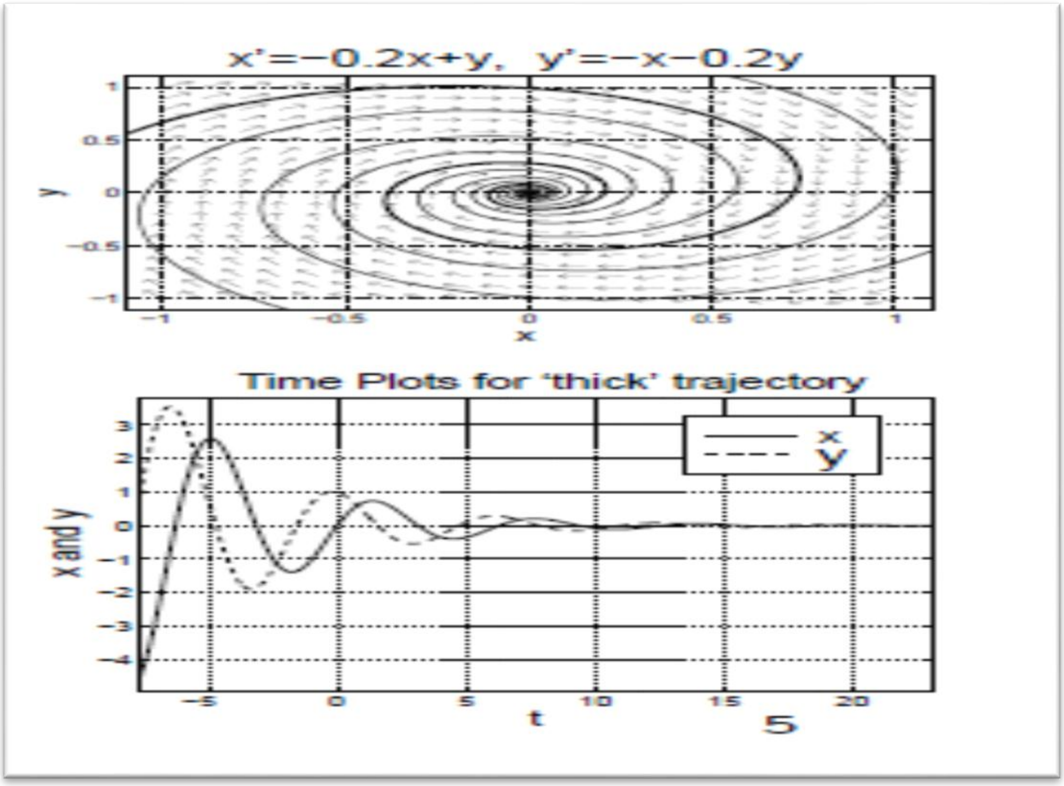
2. $A = \begin{pmatrix} 0.2 & 1 \\ -1 & 0.2 \end{pmatrix}, \alpha = 0.2 > 0$ spiral source

$\lambda = 0.2 + i \leftrightarrow v = \begin{pmatrix} 1 \\ i \end{pmatrix}$

2. $A = \begin{pmatrix} 0.2 & 1 \\ -1 & -0.2 \end{pmatrix}, \alpha = -0.2 < 0$ spiral sink

And $\lambda = -0.2 + i, v = \begin{pmatrix} 1 \\ i \end{pmatrix}$





Border ling case spiral-Node

$T^2 - 4D = 0$,single eigenvalue $\lambda = \frac{T}{2}$

Generic case$(A - \lambda I) \neq 0$, single eigenvector v

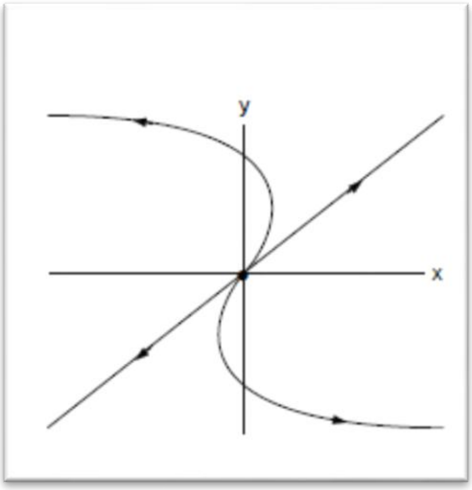
$$(A - \lambda I)w = v,$$

$$\text{general solution } x(t) = c_1 e^{\lambda t} + c_2 e^{\lambda t}(w + tv)$$

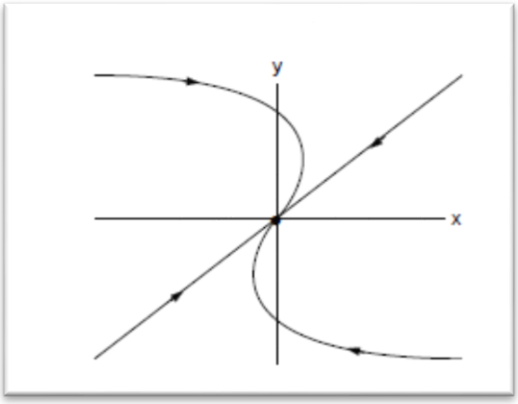
only two half line solutions on straight line generated by v

$T > 0$ nodal - source

Border line case $[^{nodal}_{spiral} - source$



$T < 0$ node border line case $[{}^{nodi}_{spiral} sink$



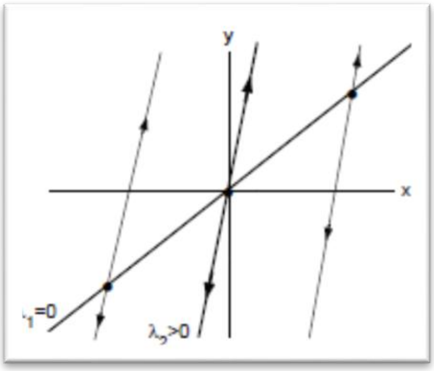
Border line case saddle - node

$D = 0$, $T \neq 0$, eigenvalue $\lambda_1 = 0$, $\lambda_2 = T$,let v_1 , v_2 be the eigenvectors

general solution $x(t) = c_1 v_1 + c_2 e^{\lambda_2} v_2$

Line of equilibrium points generated by $v_1$infinitely many half line solution on straight lines parallel to line generated by v_2

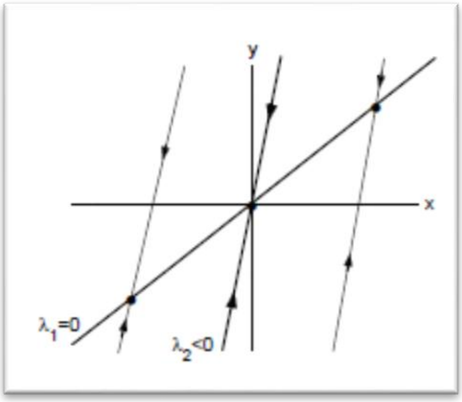
$T > 0$ unstable saddle - node



border line case $\left[\begin{matrix} nodal\ source \\ saddle \end{matrix}\right.$ t

$T < 0$ **stable saddle - node**

border line case $\left[\begin{matrix} nodesink \\ saddle \end{matrix}\right.$



2.5.4: By using the theorem lets consider the following linear system:

$$\dot{x} = Ax$$

has a saddle, node, focus or center at the origin and determine the stability of each node or focus:

i. $$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \ , T > 0 \ , \quad D < 0$$

the system is saddle at the origin.

ii. $$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \quad , \quad T^2 - 4D = 0$$

The system single eigenvalue spiral– node and general solution

$x(t) = c_1 e^{3t} + c_2 e^{3t}(w + tv)$

iii. $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, $T^2 - 4D < 0$, $\alpha > 0$

the system focus (spiral source),

$$\lambda = 3 + 2i$$

general solution = $x(t) = e^{3t}[c_1(ucos2t - wsin2t) + (ucos2t + wsin2t)]$

2.6: Singular Points of nonlinear Systems [2],

Consider a nonlinear system of differential equations:

$$\dot{x} = F(x, y)$$

$$(2.17)\dot{y} = G(x, y)$$

Where F and G are functions of two variables: $x = x(t)$ and $y = y(t)$; and such that F and G are not both linear functions of x and y.

Unlike a linear system, a nonlinear system could have none, one, two, three, or any number of critical points. Like a linear system, however, the critical points are found by setting $x' = y' = 0$, and solve the resulting system

$$F(x, y) = 0$$

$$G(x, y) = 0 \qquad (2.18)$$

Any and every solution of this system of algebraic equations is a critical point of the given system of differential equations.

Since there might be multiple critical points present on the phase portrait, each trajectory could be influenced byrdr more than one critical point.

This results in a much more chaotic appearance of the phase portrait consequently, the type and stability of each critical point need to be determined locally (in a small neighborhood on the phase plane around the critical point in question) on a case-by-case basis. Without detailed calculation, we could estimate (meaning, the result is not necessarily 100% accurate) the type and stability by a little bit of multi-variable calculus.

We will approximate the behavior of the nearby trajectories using the linearization (i.e. the tangent approximations) of F and G about each critical point.

This converts the nonlinear system into a linear system whose phase portrait approximates the local behavior of the original nonlinear system near the critical point. To start with the linearization of *F* and *G* (recall that such a linearization is just the three lowest order terms in the Taylor series expansion of each function) about the critical point $(x, y) = (\alpha, \beta)$

$\dot{x} = F(x, y) = F(\alpha, \beta) + F_x(\alpha, \beta)(x - \alpha) + F_y(\alpha, \beta)(y - \beta)$

$\dot{y} = G(x, y) = G(\alpha, \beta) + G_x(\alpha, \beta)(x - \alpha) + G_y(\alpha, \beta)(y - \beta)$

But since (α, β) is a critical point, so $F(\alpha, \beta) = 0 = G(\alpha, \beta)$, the above Linearization become

$$\dot{x} = F_x(\alpha, \beta)(x - \alpha) + F_y(\alpha, \beta)(y - \beta)$$

$$\dot{y} = G_x(\alpha, \beta)(x - \alpha) + G_y(\alpha, \beta)(y - \beta)$$

As before, the critical point could be translated to (0, 0) and still retains it type and stability, using the substitutions $\chi = (x - \alpha)$ and $\gamma = (y - \beta)$. After the translation, the approximated system becomes

$$(2.19)\dot{x} = F_x(\alpha, \beta)\chi + F_y(\alpha, \beta)\gamma$$

$$\dot{y} = G_x(\alpha, \beta)\chi + G_y(\alpha, \beta)\gamma,$$

It is now a homogeneous linear system with a coefficient matrix

$$\boldsymbol{A} = \begin{bmatrix} F_x(\alpha, \beta) & F_y(\alpha, \beta) \\ G_x(\alpha, \beta) & G_y(\alpha, \beta) \end{bmatrix}$$

That is, it is a matrix calculate by plugging in $x = \alpha$ and $y = \beta$ into the matrix of first partial derivative

$$J = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix}$$

This matrix of first partial derivatives, ***J***, is often called the Jacobian matrix. It just needs to be calculated once for each nonlinear system. For each critical point of the system, all we need to do is to compute the coefficient matrix of the linearized system about the given critical point

$$,(x, y) = (\alpha, \beta)$$

and then use its eigenvalues to determine the (approximated) type and stability

2.6.1:Examples:

$$x' = x - y$$

$$y' = x^2 + y^2 - 2, \qquad (2.20)$$

The critical points are at (1, 1) and (−1, −1). The Jacobian matrix is

$$\boldsymbol{J} = \begin{bmatrix} 1 & -1 \\ 2x & 2y \end{bmatrix}$$

At (1, 1), the linearized system has coefficient matrix:

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$$

The eigenvalues are

$$r = \frac{3 \pm \sqrt{7}i}{2}$$

The critical point is an unstable spiral point.

At (−1, −1), the linearized system has coefficient matrix

$$A = \begin{bmatrix} 1 & -1 \\ -2 & -2 \end{bmatrix}$$

The eigenvalues are

$$r = \frac{-1 \pm \sqrt{17}}{2}.$$

The critical point is an unstable saddle point.

The phase portrait is shown on the next .

$$x' = x - xy$$

$$y' = y + 2\,xy\,, \qquad (2.21)$$

The critical points are at (0, 0) and (−1/2, 1). The Jacobian matrix is

$$\boldsymbol{J} = \begin{bmatrix} 1 - y & -x \\ 2y & 1 + 2x \end{bmatrix}$$

At (0, 0), the linearized system has coefficient matrix:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

There is a repeated eigenvalue $r = 1$.

A linear system would normally have an unstable proper node (star point) here.

But as a nonlinear system it actually has an unstable node. (Didn't I say that this approximation using linearization is not always 100% accurate?)

At (−1/2, 1), the linearized system has coefficient matrix:

$$\boldsymbol{A} = \begin{bmatrix} 0 & \frac{1}{2} \\ 2 & 0 \end{bmatrix}$$

The eigenvalues are $r = 1$ and −1. Thus, the critical point is an unstable saddle point.

The phase portrait is shown on the next .

2.6.2: Find all the Singular points of nonlinear system and stability of each Singular point.

$$1.\ x' = xy + 3y$$
$$y' = xy - 3x, \qquad (2.22)$$

solution

Critical points are (0, 0) and (−3, 3). (0, 0) is a stable center, and (−3, 3) is an unstable saddle point.

$$2.\ x' = x^2 + y^2 - 13$$
$$y' = xy - 2x - 2y + 4, \qquad (2.23)$$

solution

Critical points are (2, 3), (2, −3), (3, 2), and (−3, 2). (2, 3) is an unstable saddle point, (2, −3) is an unstable saddle point, (3, 2) is an unstable node, and (−3, 2) is an asymptotically stable node.

$$5.\ x' = x^2y + 3xy - 10y$$
$$y' = xy - 4x, \qquad (2.24)$$

solution

Critical points are (0, 0), (2, 4), and (−5, 4). (0, 0) is an unstable saddle point, (2, 4) is an unstable node, and (−5, 4) is an asymptotically stable node.

Chapter Three

Limit Cycles Problems

We investigate a limit cycle**:**

3.1:Limit Cycles [10],

A limit cycle is an isolated periodic solution limit cycles in planar differential systems commonly occur when modeling both the technological and natural sciences. Most of the early history in the theory of limit cycles in the plane was stimulated by practical problems.

For example the differential equation derived by Rayleigh, related to the oscillation of a violin string, is given by:

$$\ddot{x}+\in\left(\frac{1}{3}(\dot{x})^2-1\right)(\dot{x})+x=0 \tag{3.1}$$

Where $\ddot{x}=\dfrac{d^2x}{dt^2}$ and $\dot{x}=\dfrac{dx}{dt}$

Let $\dot{x}=y$ then this differential equation can be written as a system of first order autonomous differential initial equations in the plane.

$$\dot{x}=y \qquad , \qquad \dot{x}=-x-\in\left(\frac{y^2}{3}-1\right)y \tag{3.2}$$

Periodic behavior in the Rayleigh system (3.2) when $\in=1$following the invention of the triode vacuum tube, which was able to produce following differential equation to describe this phenomena

$$\ddot{x}=\varepsilon(x^2-1)\dot{x}+x=0 \tag{3.3}$$

Which can be written as planar system of the form (3.2) periodic behavior for system (3.3) when $\in=5$

Class of differential equation that generalize (3.3) are those first investigated Lienard equation

$$\ddot{x}+f(x)\dot{x}+g\left(x\right)=0,\ \text{or in the phase plane}$$

$$\dot{x} = y \, , \, \dot{y} = -g(x) - f(x) y \qquad (3.4)$$

This system can be used to model mechanical system, where $f(x)$ is known as the damping term and $g(x)$ is called the restoring force or stiffness equation (3.4) is also used to model resistor inductor.

Capacitor circuits with nonlinear circuit elements, limit cycle of Lienard systems possible physical interpretations for the limit cycle behavior of certain dynamical systems limit cycles are common solution for all types times it becomes necessary to prove the existence and uniqueness.

3.2: Limit cycles in Phase plane nonlinear system[2]

Nonlinear systems can display much more complicated patterns in the phase plane such as multiple equilibrium points and limit cycles, we now discuss these points in more detail.

In the phase plane of non linear system Vander pol equation, observes that the system has an unstable node at the origin. Furthermore there is a closed curve in the phase portrait trajectories inside the curve and those outside the curve all tend to this curve, while a motion started on this curve will stay on it forever circling periodically around the origin.

This curve is an instance of the so-called limit cycle phenomenon limit cycle are unique features of nonlinear system.

In the phase plane a limit cycle is defined as isolated closed curve the trajectory has to be both closed indicating the periodic nature of the motion, and isolated, in dictating the limiting cycle. While there many closed curves in the plane portraits depending on the motion patterns of the trajectory in the vicinity of the limit cycle one can distorting wish three kinds of limit cycle one can distinguish three kinds of limit cycles.

1. Stable limit cycles all trajectories in the vicinity of the limit cycle converge to it as $t \to \infty$.

2. Unstable limit cycles. All trajectories vicinity of the limit cycle converge to it as $t \to \infty$
3. Semi – stable limit cycles : some of the trajectories in the vicinity converge to it while the others diverge from it as $t \to \infty$.

The limit cycle of the Vander pol equation is clearly stable.

Let us consider some additional examples of stable unstable, and semi-stable limit cycles.

3.2.1:Examples:

Consider the following nonlinear systems

a. $\dot{x}_1 = x_2 - x_1\left(x_1^2 + x_2^2 - 1\right)$, $\dot{x}_2 = -x_1 - x_2\left(x_1^2 + x_2^2 - 1\right)$.

b. $\dot{x}_1 = x_2 + x_1\left(x_1^2 + x_2^2 - 1\right)$, $\dot{x}_2 = -x_1 + x_2\left(x_1^2 + x_2^2 - 1\right)$.

c. $\dot{x}_1 = x_2 - x_1\left(x_1^2 + x_2^2 - 1\right)$, $\dot{x}_2 = -x_1 - x_2\left(x_1^2 + x_2^2 - 1\right)^2$**.**

3.2.2 :Existence of limit cycles[9]

We state three simple classical the ovens to that effect. These the ovens are easy to understand and apply the first the oven to be presented reveals a simple relationship between the existence of limit cycle and the number of singular points it encloses. In the statement of the theorem, we use N to represent the number of nodes, sinters and foci enclosed by a limit cycle, and to represent the number of enclosed saddle points.

Theorem (Poincare): if a limit cycle exists in the second-order autonomous system. Then $N = s + 1$.

This theorem is sometimes called the index theorem. Its proof is mathematically involved one simple inference from this theorem is that a limit cycle must enclose at least one equilibrium point, the theorem's result can be verified easily.

The second theorem is concerned with the asymptotic properties of the trajectories of second-order systems.

3.2.3: **Theorem** If trajectory of the second-order autonomous system remains in a finite region Ω then one of the following is true.

a. The trajectory goes to a equilibrium point.

b. The trajectory tends to an asymptotically stable limit cycle.
c. The trajectory is itself a limit cycle.

The third theorem provides a sufficient condition for non-existence of limit cycles .

3.2.4: **theorem** for the nonlinear system.

$$\dot{x_1} = f_1(x_1, x_2)$$

$$\dot{x_2} = f_1(x_1, x_2) \qquad (3.5)$$

No limit cycle can exist in a region Ω of the phase plane in which of $\partial f_1 / dx_1 + \partial f_2 / dx_2$ dose not finished and does not change sign proof. Let us prove this theorem by contradiction first note that the equation $f_2\, dx_1 - f_1\, dx_2 = 0$ is satisfied for any system trajectories including a limit cycle.

Thus a long the closed curve of a limit cycle we have

$$\int_L \left(f_1\, dx_2 - f_2\, dx_1 \right) = 0$$

Using stokes theorem in calculus we have

$$\int_L \left(f_1\, dx_2 - f_2\, dx_1 \right) = \iint \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) dx_1 dx_2 = 0$$

Where the integratial the right – hand side is carried out on the area enclosed by equation.

$$\int_L \left(f_1\, dx_2 - f_2\, dx_1 \right) = 0$$

The left – hand side must equal zero this however contradicts the fact that the right hand side cannot equal (zero because by hypothesis $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ doesn't vanish and does not change sing let us illustrate the result on an example.

3.2.5: Example consider the nonlinear system

$\dot{x}_1 = g(x_2) + 4x_1x_2^2$, $\dot{x}_2 = h(x_1) + 4x_1^2 x_2$ since $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 4\left(x_1^2 + x_2^2\right)$.

Which is always positive the system does not have any limit cycles any where in the phase plane.

3.3:Existence and uniqueness of limit cycles in the plane [2], [25]

To understand the existence and uniqueness theorem it is necessary to define some features of phase portraits assume that the existence and uniqueness theorem holds for all solutions considered there

Periodic orbits in the plane are special in that they divide the plane in to a region in side the orbit and a region outside it this makes it possible to obtain criteria for detecting the presence or absence of periodic orbits for second order systems, which have no generalizations to higher order systems theorem (Poincare – Bendixson criterion).

consider the system:

$$\dot{x} = f(x)$$

and let M be a closed bounded subset of the plane such that M contains an equilibrium points, or contains only one equilibrium point such that Jacobean matrix $[\partial f / \partial x]$ at this points has eigenvalues with positive real parts. Every trajectory starting in M remains in M for all future time. Then M contains a periodic orbit of

$$\dot{x} = f(x)$$

3.3.1: **Theorem** (Negative Poincare – Bendixson)

If on simply connected region D of the plane, the expression $\partial f_1 / \partial x_1 + \partial f_2 / dx_2$ is not identically zero and dose not change sigh then the system.

$$\dot{x} = f(x)$$

has no periodic orbits lying entirely in D.

Proof.

On any orbit $\chi^{\cdot} = (f)$, we have $dx_1 / dx_2 = f_2 / f_1$ I therefore on any closed orbit γ , we have

$$\int_{\gamma} f_2(x_1, x_2) dx_1 - f_1(x_1, x_2) dx_2 = 0$$

This implies, by Green's theorem that

$$\iint_s \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) \partial x_1 \, \partial x_2 = 0$$

Where is the interior of y . if $\frac{\partial f_1}{\partial x_1}+\frac{\partial f_2}{\partial x_2}>0$ or (<0) on D then we can not find a region S⊂D such that the last equality holds hence there can be no closed orbits entirely in D.

3.3.2: **Definition**

A limit cycle, say Γ is

1. A stable limit cycle if $A^{+}(x)=\Gamma$ for all x in some neighborhood this implies that nearby trajectories are attracted to the limit cycle.
2. An unstable limit cycle $if\ A^{-}(x)=\Gamma$ for all x some neighborhood this implies that nearby trajectories are repelled away from the limit cycle.
3. A semi stable limit cycle if it is attracting on one side and repelling on the other.

The stability of limit cycle can also be deduced analytically using the each of definition before and one below

3.3.3: **Definition**

The periodic say T of a limit cycle is given by $x(t)=x(t+T)$ when T is the minimum period the period can be found for example be low.

3.3.4: **Example**

Describe some of the features for the following set of polar differential equation.

$$r^{\cdot}=r(1-r)(2-r)(3-r),\theta=-1 \qquad (3.6)$$

Solution

There is a unique critical point at the origin since $\theta^{\cdot}$ is nonzero.

They are three limit cycle that may be determined from the equation $\dot{r}=0$ they are the circles of radii one, two and three, all centered at the origin.

Let Γ_i denote the limit cycle of radius $r=i$. There is one critical point at the origin. If trajectory stated at this point.

It remain there forever. A trajectory starting at (1,0) will reach the point (-1,0) when $t_i=\pi$

And the motion is clockwise. Continuing on this path for another time interval$t_2 = \pi$the orbit returns to (1,0).Using definition(3.3.1)part(3)

One can write $\pi\left(\pi(1,0),t_2\right)=\pi\left(1,0),2\pi\right)$ since the time limit cycles is of period 2π.

On the limit cycle Γ_1,both the positive and negative semi orbits lie on Γ_1.

Suppose that $P\left(\frac{1}{2},0\right)$ and $Q=(4,0)$ are two points in the plane.

The limit sets are given by $A^+(P)=\Gamma_1, A^-(P)=(0,0), A^+(Q)=\Gamma_3$, and $A^-(Q)=\infty$ the annulus $A_1=\{r\in R^2: 0<r<1\}$ is positively

, and the annulus $A_2\{r\in R^2: 1<r<2\}$is negatively invariant.

If $0<r<1$ then $r^{\cdot}>0$ and the vertical point at the origin is unstable.

If $1<r<2$ then $r^{\cdot}<0$ Γ_1 is stable limit.

If $2<r<3$ then $r^{\cdot}>0$ and Γ_2 is an unstable limit cycle. Finally if $r>3$then $r^{\cdot}<0$and $\Gamma_3$is a stable limit cycle.

Integral both sides of $\dot{\theta}=-1$ with the respect to time to show that the period of all of the limit cycles are 2π.

3.3.5: **Theorem**

Suppose that γ^+ is contained in a bounded region in which there are finitely many critical points.

Then $A^+(\gamma)$ is either.

- A single critical point.
- A single closed orbit.
- A graphic. critical points joined by heteroclinic orbit.

Corollary

Let D be abounded closed set containing critical points and suppose that D is positively in variant. Then there exists.

A limit cycle contained in D .

3.3.6: **Example**:

By considering the flow across the rectangle with corners at (-1,2) , (1,2) ,(1,-2) , (-1, -2)

Proof

That the flowing system has one limit

$$\dot{x} = y - 8x^3 \quad , \dot{y} = 2y - 4x - 2y^3 \qquad (3.7)$$

Solution

The critical points are found by solving the equation

$$y^{\cdot} = x^{\cdot} = 0 \text{ , let } \quad y = 8x^3$$

Then $y^{\cdot} = 0$ if $x(1 - 4x^2 + 256x^8) = 0$

The graph of the function

$$y = 1- 4x^2 + 256x^8 .$$

Linearze at the origin in the usual way it is not difficult to show that the origin is an unstable focus, consider the flow on the sides of the given rectangle on

$$y = 2 \quad , \qquad |x| \le 1 \quad , \qquad y^{\cdot} = -4x - 12 < 0$$

$$y = -2 \quad , \qquad |x| \le 1 \qquad , y^{\cdot} = -4x + 12 > 0$$

$$x = 1 \quad , \qquad |y| \le 2 \quad , \quad x^{\cdot} = y - 8 < 0$$

$$x = -1 \quad , \quad |y| \le -2 \quad , x^{\cdot} = y + 8 > 0$$

The rectangle is positive and there are no critical points outer than the origin ,which is unstable consider a small deleted neighborhood , say N_ε around this critical point .

3.4: Limit cycles and Bifurcation[10]

Oscillation are one of the most important phenomena that occur in dynamical systems .A system oscillation when it has a nontrivial periodic solution.

$$x(t+T)=x(t), \forall t \ge 0$$

For some $T>0$. In a phase portrait an oscillation or periodic solution, like a closed curve.

3.4.1: **Example**

Vander pol oscillator.

$$\dot{x_1} = x_2$$

$$N_\varepsilon(x_o)=\{x\in R^n \,|x-x_o|<\in\}$$

In the case $\in=0$ we have a continuum of periodic solutions, while in the second case $\in\neq 0$ there is only one.

An isolated periodic orbit is called a limit cycle.

3.5: Limit cycle in Bounded quadratic systems[2], [17]

are trajectories remain bounded for $t\geq 0$.

The research work in this area began with Dickson and perko, who studied autonomous quadratic bounded differential systems in dimensional space.

Then turned to deep and detailed study of plane quadratic systems.

$$\frac{dx}{dt}=p_{10}x+p_{01}y+x_2(x,y),\frac{dy}{dt}=q_{10}x+q_{01}y+y_2(x,y) \quad (3.8)$$

Where $P_{10}, P_{01}, q_{10}, \mathrm{q}_{01}$ are constants and $X_2(x,y), Y_2(x,y)$ are Homogeneous quadratic polynomials.

We study the bounded of the quadratic system (3.8) which the properties of the corresponding homogeneous quadratic system.

$$\frac{dx}{dt}=X_2(x,y),\frac{dy}{dt}=Y_2(\mathrm{x},\mathrm{y})(3.9)$$

Have a definite relationship. Changing - (3.9) into polar coordinates, we get

$$\frac{dr}{dt}=r^2[X_2\cos\theta,\sin\theta]\cos\theta+Y_2(\cos\theta\sin\theta)\sin\theta=rG(\theta)(3.10)$$

Let $\theta = \theta_o$ be a root of G $(\theta) = 0$

And $f(\theta_o)\neq 0$

since $G(\theta_o+\pi)\equiv 0$, we may as well assume $F(\theta_o)>0$.

Then the solution of (3.10) which satisfies the initial conditions $r(t_0)=r_0$ and $\theta(t_0)=\theta_0$ is

$$r(t)=\frac{1}{F(\theta_0)t_0}+\frac{1}{(1/\mathrm{r}_0 F(\theta_0))-t}\ ,\ \theta=\theta_0 \text{(3.11)}$$

It is a ray in the (x,y) - plane, and $r(t)\to+\infty$ as $t\to t_0+\frac{1}{r_0F(\theta_0)}$

3.5.1: **Definition**

Suppose $r=r(t)$ is a solution of the system (3.8) such that approaches a finite value as $v\to+\infty$. Then $v(t)$ is a solution possessing finite escape time for (3.10) if there exists a θ_0 such that $G(\theta_0)=0$ and $F(\theta_0)\neq 0$,(3.11) is a solution possessing finites cape time then (3.11) is called ray solution.

3.5.2: **Theorem**

If (3.9) has are solution, then (3.8) has an unbounded solution possessing finite a scape time.

Proof

Apply the transformation $x=rcos\theta, y=r\sin\theta, dr/dt=r$.

Then (3.8) become

$$\frac{dr}{dr}=p_{10}\cos^2\theta+(p_{01}+q_{10})\cos\theta\sin\theta+q_{01}\sin^2\theta$$

$$+rF(\theta)=P_1(\theta)+rF(\theta)$$

$$\frac{d\theta}{dr}=\frac{1}{r}\left[q_{10}\cos^2\theta+(q_{01}-P_{10})\cos\theta\sin\theta-P_{01}\sin^2\theta\right]+G(\theta)=$$

$$\frac{1}{r}\theta_1(\theta)+G(\theta)\,(3.12)$$

From the hypothesis, we know there exists a $\theta=\theta_0$ such that $G(\theta_0)=0$ and $F(\theta_0)\neq 0$

We may assume $F(\theta_0)=\lambda>0$. let $\sigma=r^{-1}$ then (2=3.12) become.

$\dot{\sigma}=-\sigma F(\theta)-\sigma^2 P_1(\theta), \theta^{\cdot}=\sigma G(\theta)+G(\theta)$ which has a singular point $(\sigma=0, \theta=\theta_0)$ and the first equation has a negative characteristic root λ. Hence from the Classical theorem of Lyapunov we know that the system of equation has a trajectory approaching to $(\sigma = 0, \theta = \theta_0)$, corves ploddingly system (3.12) has trajectory $(r(r), \theta(x)) \rightarrow (+\infty, \theta_0)$.

Applying the formulary of variation of constants to the first equation of (3.12) we see that as $r(r) \rightarrow \infty$ we must have $r \rightarrow +\infty$. finally, form

$$t = \int_0^r \frac{ds}{r(s)}$$

We can see that as $T \rightarrow +\infty$ we have $T > \infty$ the proof is complete.

From the necessary an sufficient condition for a quadratic system to be bounded.

3.5.3: **Theorem**[2]

All the trajectories of quadratic system (3.12) are bounded when $T \geq 0$ if and only if there exists a linear transformation which changes (3.12) into one of the following types.

(A)

$$\dot{x} = a_{11}x + a_{12}y + y^2, \dot{y} = a_{21}x + a_{22}y - xy + cy^2$$

Where $|c| < 2$ and the other coefficients satisfy one of the following groups of condition.

(i) $a_{11} < 0$

(ii) $a_{11} = a_{21} = 0$

(iii) $a_{11} = 0, a_{21} = -a_{12} \neq 0, a_{21} + a_{22} \leq 0;$

$$\dot{x} = a_{11}x, \dot{y} = a_{21}x + a_{22}y + xy$$

Where $a_{11} < 0$ and $a_{22} \leq 0$

(B)

$$\dot{x} = a_{11}x + a_{21}y + y^2, \dot{y} = a_{22}y$$

Where $a_{11} \leq 0$ and $a_{22} \leq 0$, $a_{11} + a_{22} < 0$

(C)

$a_{11} < 0$, $a_{22} < 0$ and $2a_{22} - a_{11} > 0$.

3.5.4:Remark.

It is clean that under that condition(ii) . B and C we can write the equation of the family of the trajectories of (A) and using this we can prove it is bounded system and does not have a limit cycle.

Hence we are only interested in the case when (A) satisfies (i) or (iii).

Our familiar form of equation of class(iii)without loss of generality we can assume (0,0) is an elementary singular point of (A) of index +1, hence $a_{11} a_{22} - a_{21} a_{12} = \Delta > 0$ now in (A) we apply the change of variables $v = a_{11} x - a_{21} y, u = x$ or $y = (a_{11} u - v)/a_{21}, x = u$ then we get.

$$\dot{u} = -v + (a_{11} + a_{22})u - u(a_{11}u - v)/a_{21} + cu^2 \qquad (3.13)$$

$$\dot{v} = u\left[(a_{11}a_{22} - a_{21}a_{12}) - a_{11}(a_{11}u - v)/a_{21} + (ca_{11} - a_{21})u \right]$$

Again let

$$= (a_{11}a_{22} - a_{21}a_{12})^{\frac{1}{2}} u = \Delta^{\frac{1}{2}} u, T = \Delta^{\frac{1}{2}} t, \bar{y} = v$$

Then the above system of equation is hanged to

$$\frac{d\bar{x}}{d\bar{r}} = -\bar{y} + \Delta^{\frac{1}{2}}(a_{11} + a_{22})\bar{x} + \bar{x}\bar{y} + \Delta^{\frac{1}{2}}/a_{21} + (c a_{21} - a_{11}) x^{-2} \Delta^{-1}/a_{21}$$

$$\frac{d\bar{x}}{dt} = \bar{x} + \left[a_{21}(ca_{11} - a_{21}) - a_{11}^2\right]\Delta^{\frac{-3}{2}} \bar{x}^2/a_{21} + a_{11}\bar{x}\bar{y}\Delta^{-1}/a_{21}$$

It already possesses the form:

$$\frac{d\bar{x}}{dt} = -\bar{y} + \delta\bar{x} + L\bar{x}^2 + m\bar{x}\bar{y} + n\bar{y}^2, \frac{d\bar{y}}{dt} = \bar{x}\left(1 + 1\bar{x} + b\bar{y}\right), \qquad (3.14)$$

here we have

$$\delta = (a_{11} + a_{22})\Delta^{\frac{-1}{2}}, L = (ca_{21} - a_{11})\Delta^{-1}/a_{21}$$

$$m = \Delta^{-1}/a_{21}, \; n = 0, \; a = \left[a_{21}(c a_{11} - a_{21}) - a_{11}^2\right]\Delta^{\frac{-3}{2}}/a_{21}$$

$$b = a_{11}\Delta^{-1}/a_{21}$$

No le now that

$$ma+(b-L)^2=(c^2-4)\,\Delta^{-2}\ \ , mt=a_{11}\ \Delta^{\frac{-3}{2}}/a_{21}^2$$

Hence condition (i) of theorem **3.5.3**: and the inequality $|c|<2$ can be rewritten as

$$n=0\ ,\ (b-L)^2+4ma<0\ \ ,\ \ mb<0\,,$$

And condition (ii) and $|c|<2$ log ether can be rewritten as:

$$n=0\ ,\ (b-L)^2+4ma<0\ \ ,\ b=m+a=0$$

$$m(L+m\delta)\leq 0$$

In the above two conditions we have $n=0$. Hence we may rewrite (3.14) as

$$r(t)=\frac{1}{F(\theta_0)t_0}+\frac{1}{(1/\mathrm{r}_0\,F(\theta_0))-t}\ \ ,\theta=\theta_0\ (3.15)$$

3.6:two limit cycles[27]

We first look at simpler case suppose that in equations:

$$\frac{dx}{dt}=-y+\delta x+Lx^2+mxy+ny^2\,,\ \frac{dy}{dt}=x(1+ax)\ (3.16)$$

Two of the coefficients of the terms of second degree on the right side of the system (3.16) are zero.

Then we have:

$$\frac{dx}{dt}=-y+\delta x+ny^2\ ,\ \frac{dy}{at}=x(1+ax);\ \ (3.17)$$

$$\frac{dx}{dt}=-y+\delta x+mxy\ ,\ \frac{dy}{at}=x(1+ax);\ (3.18)$$

$$\frac{dx}{dt}=-y+\delta x+Lx^2\ ,\ \ \frac{dy}{at}=x(1+ax);(3.19)$$

It is easy to see that equations (3.17) and (3.18) can be integrate when $\delta=0$ and they take $(0,0)$ as their center; and when $\delta\neq 0$ they have no limit cycles.

Because the divergence of (3.17) is a constant δ, and (3.18) can be proved to have no limit cycles by the Dulac function $(1-mx)^{-1}$ for (3.19), the situation is not the same. We may as well assume $L>0$ and $a>0$; and it is not difficult to use the well – known method to prove that when $\delta\leq 0$ or $\delta\geq\frac{L}{a}$, (3.19) dose not have a limit cycles, but when $\delta al>0$ and δ lies in some interval $(0,\delta^{*})$, (3.19) has a unique limit cycle. In the following we study mainly the case where the coefficients of the quadratic terms on the right side of the first equation have only one zero.

$L=0$. In this case we have system

$$\frac{dx}{dt}=-y+\delta x+mxy-y^{2}\ ,\ \frac{dy}{dt}=x(1+ax)\quad(3.20)$$

First we prove a useful theorem for nonexistence of a closed trajectory and singular closed trajectory.

Theorem system (3.20) cannot have a closed trajectory or singular closed trajectory passing a saddle point in either of the following cases

1) $m\delta\leq 0,|m|+|\delta|\neq 0$
2) $\delta(m-\delta)\leq 0,|m|+|\delta|\neq 0$

Proof

When the first group of conditions holds, it is easy to see that any closed trajectory or any singular closed trajectory.

Passing a saddle point of (3.20) cannot intersect the line $1-mx=0$

Now we take the Dulack function to be $B=\frac{1}{1-mx}$; then we have

$$\frac{\partial}{\partial x}(Bp)+\frac{\partial}{\partial y}(BQ)=\frac{\delta-my^{2}}{(1-mx)^{2}}$$

The right side of this formula always keep a constant sign on any side of the line $1-mx=0$; hence the theorem is proved.

Now we suppose the second group of conditions holds.

We translate the x-ax is to the line $y=-\delta/m$ (the case of $m=0$ has been in (3.17) hence we may as well assume $m\neq 0$) and keep the y-axis unchanged. Then (3.20) becomes

$$\frac{dx}{dt}=\frac{\delta}{m}\left(1-\frac{\delta}{m}\right)+\left(1+\frac{2\delta}{m}\right)y+mxy-y^2,\frac{dy}{dt}=x(1+ax) \quad (3.21)$$

The system of equation whose vector field is symmetric to that of (3.21) with respect to the new x-axis is

$$\frac{dx}{dt}=\frac{\delta}{m}\left(1-\frac{\delta}{m}\right)-\left(1+\frac{2\delta}{m}\right)y-mxy-y^2,$$

$$\frac{dy}{dt}=x(1+ax) \quad (3.22)$$

The locus of points of contact of the trajectories of these two systems is easily seen to be

$x=0$,$1+ax=0$ and

$$\frac{\delta}{m}\left(1-\frac{\delta}{m}\right)-y^2=0 \quad (3.23)$$

since the divergence of (3.21) is only zero one the $x-axis$, and closed or singular closed trajectory Γ must intersect the $x-axis$.

We know that if Γ appears in the vicinity of (0,0) then it cannot meet $1+ax=0$.

More over, for $\delta(m-\delta)<0$ the last equation of (3.23) does not have a real locus, and for $\delta(m-\delta)=0$ its locus it the $x-axis$.

From this we can see that the curve symmetric to Γ^+ (the part of Γ above the $x-axis$) with respect to the $x-axis$, and Γ^- (the part of Γ below the $x-axis$, do not have a common point except on the $x-axis$, that is, the curve symmetric.

To Γ^+ lies entirely above (or below) Γ^- thus, for any closed or singular closed trajectory Γ of (6) we must have

$$\iint_{\text{int}\,\Gamma}\left(\frac{\partial p}{\partial x}+\frac{\partial Q}{\partial y}\right)dx\,dy=\iint_{\text{int}\,\Gamma} my\,dx\,dy\neq 0$$

This contradiction shows that Γ does not exist. Similarly, we know that the vicinity of one there singular point with index +1 on $1+ax=0$ cannot contain a closed or singular closed trajectory.

The theorem is completely proved.

The following transformation of coordinates in useful for our later discussion.

Note the two singular points on the line $1+ax=0$ are

$$R\left(-\frac{1}{a},\frac{1}{2}\left[-1-\frac{m}{a}-\sqrt{\left(1+\frac{m}{a}\right)^{2}-\frac{4\delta}{a}}\right]\right)\text{, of index + 1}$$

$$N\left(-\frac{1}{a},\frac{1}{2}\left[-1-\frac{m}{1}+\sqrt{\left(1+\frac{m}{a}\right)^{2}-\frac{4\delta}{a}}\right]\right)\text{, of index – 1}$$

Moving the origin to R we get

$$\frac{dx}{dt}=(\delta+my_2)x-\left(1+\frac{m}{a}+2y_2\right)y+mxy-y^3$$

$$\frac{dy}{dt}=-x+ax^2\ ,\ \ (3.24)$$

Where y_2 represents the ordinate of R.

Now we apply the transformation

$$x=-\left[\left(1+\frac{m}{a}\right)^{2}-\frac{4\delta}{a}\right]^{\frac{3}{4}}x',\ y=-\left[\left(1+\frac{m}{a}\right)^{2}-\frac{4\delta}{a}\right]^{\frac{1}{2}}y'$$

$$\frac{1}{r}\theta_1(\theta)+G(\theta)\ (3.25)$$

To(3.24)and get

$$\frac{dx'}{dt'}=-y'+\delta'x'+m'x'y'-y'^2,\frac{dy'}{dt'}=x'(1+a'x')\ (3.26)$$

Where

$\delta'=-(\delta+my_2)$

$$\delta' = -\frac{(\delta + m y_2)}{\left[\left(1+\frac{m}{a}\right)^2 - \frac{4\delta}{a}\right]^{\frac{1}{4}}} \quad , \quad a' + a\left[\left(1+\frac{m}{a}\right)^2 - \frac{4\delta}{a}\right]^{\frac{3}{4}}$$

$$m' = m\left[\left(1+\frac{m}{a}\right)^2 - \frac{4\delta}{m}\right]^{\frac{1}{4}} \quad (3.27)$$

According to theorem (3.6.3) it is only possible for system (3.16) to have limit cycles near the two singular points of index+1 then

$$m\delta > 0 \text{ but } |\delta| < |m| \quad (3.28)$$

In the following we want to explain whether limit cycles can coexist near both singular points of index + 1.

This problem is closely related to the order between m and a.

1. $m > -a > 0$. For $\delta = 0, 0$ is stable R is unstable, m and n are saddle points, let δ increase from zero, Then origin becomes unstable.

On the other hand, the two singular points on the line $1 + ax = 0$, move for apart. Now we prove that the stability of R' is different from R, since hear R' there also appears a unique unstable cycle hence we note that for $m > -a > 0$ and $|\delta|$ sufficiently small,

$$\delta + m y_2 = \delta - \frac{m}{2}\left[1 + \frac{m}{a} + \sqrt{\left(1+\frac{m}{a}\right)^2 - 4\frac{\delta}{a}}\right]$$

1. $$= \delta - \frac{m}{2}\left[1 + \frac{m}{a} - \left(1+\frac{m}{a}\right)\left(1 - \frac{2\delta}{a\left(1+\frac{m}{2}\right)^2}\right)\right] + O\left(\delta^2\right)$$

$$= \frac{a\delta}{a+m} + o\left(\delta^2\right) < 0,$$

Where y_2 is the ordinate of R' - Hence form (3.26) and (3.27) we know that R' is a stable focus of (3.24).

Hence when $m > -a > 0$ and $0 < \delta << 1$ limit cycles coexist near the singular points origin and R', but their stability is different.

When δ continuously increases, it is not certain whether these limit cycles disappear simultaneously.

Now suppose near origin the limit cycle, it exists, is always unique, and at $\delta = \delta'$ it expands and becomes.

A separatrix cycle passing through m forms a family of generalized rotated vector fields on each side of the line $1 + ax = 0$, hence when δ increases, thus δ' should clearly be a function $\delta' = f(m, a)$ of m and a , hence from (3.26) we know that the value δ'^1 of δ^1 which makes the limit cycle near R' expend and become a sepratrix cycle through N' must be the same function of M' and a'; that is, $\overline{\delta}^1 = f(m', a')$ whether this function f can be determined is a problem worthy of our consideration of course, the uniqueness of a limit cycle in the vicinity of every singular point has to be proved.

2. $O < m \le -a$ first we examine the case then $O < m < -a$. When $\delta = 0$, $N(\frac{-1}{a}, 0)$ is a saddle point and $R\left(-\frac{1}{a}, -\frac{(a+m)}{a}\right)$ below N is a focus. If δ increases from 0, when N' and R' move for a part. Since for $\delta = 0$

$$\delta + my_2 = -m\left(1 + \frac{m}{a}\right) < 0,$$

Initially R' rernais unstable.

Now we ask = does the stability of R' change as δ continuously increases?

When does happen? Clearly we can see that a necessary condition for R' to change its stability is $\delta' = 0$, that is,

$$\delta + my_2 = \delta - \frac{m}{2}\left[1 + \frac{m}{a} + \sqrt{\left(1 + \frac{m}{a}\right)^2 - \frac{4\delta}{a}}\right] = 0$$

Solving this equation, we get $\delta = m$ and

Hence when δ increases from, δ' decreases from O , that is R' as a singular point of (3.26).

$$\left.\frac{\partial\bar{\delta}}{\partial\delta}\right|_{\delta=m}=-\left[\frac{a-3\delta+\frac{3m}{2}+\frac{m^2}{2a}+\left(\frac{m}{2}\right)\sqrt{\left(1+\frac{m}{a}\right)^2-\frac{4\delta}{a}}}{a\left[\left(1+\frac{m}{a}\right)^2-4\frac{\delta}{a}\right]^{\frac{3}{2}}}\right]_{\delta=m}=\frac{-1}{\left(1-\frac{m}{a}\right)^{\frac{3}{2}}}<0 \qquad (3.29)$$

Change from stable to unstable, and so as the singular point of the original L system (3.20), it melts change stable to unstable, however, according to theorem (3.6.3) we know that for $\delta \geq m$ there is no closed trajectory in the vicinity of R'.

(i) It is generated from a separatist cycle passing through m and N' surrounding R'.

(ii) It is generated from a aspartic cycle passing through m and N' surrounding R'.

(iii)It is generated by splitting a semi stable cycle which suddenly appears in the vicinity of R'.

Which case it belongs to depends on the order relation between m and $\frac{1}{a}-a$, the order relation between m and $1+a$. In fact, when $\delta=m=1+a$ the coordinates of R' are $\left(-\frac{1}{a},-1\right)$, and at this time the coefficients in (3.26) are

$$m'=m\sqrt{\frac{-1}{a}}=(1+a)\sqrt{\frac{-1}{a}},a'=a\left(\frac{-1}{a}\right)^{\frac{3}{2}}=-\sqrt{\frac{-1}{a}}.$$

From this we can see that $m'=\frac{1}{a'}-a'$ that is, in this case $\overline{MN}$ has become an integral line, and when $\delta=m$, from $m'<\frac{1}{a'}-a'$ we can deduce that $m>1+a$, and from $m'>\frac{1}{a'}-\dot{a}$ we can deduce that $m<1+a$.

For $0<m<-a$, which correspond to $m=\delta<1+a, m=\delta=1+a$ and $m=\delta>1+a$ respectively.

3.7: Classification of quadratic systems limit cycle [2], [42]

$$\frac{dx}{dt}=P_2(x,y) \quad , \quad \frac{dy}{dt}=Q_2(x,y) \_\_\_\_\_ (3.30)$$

Where P_2 and Q_2 are general quadratic polynomials.

Hence we now first introduce a method of classification, that is we apply some simple trans formations to system (3.30).

Which may have a limit cycle, to bring it into one of three.

We may assume $P_2(x,y), Q_2(x,y)$ do not have a common factor for otherwise (3.30) can be simplified to a linear system, which obviously does not have a limit cycle.

From the theory of quadratic curves, we know there exists at least one real λ making the equation.

$$\lambda P_2(x,y)+Q_2(x,y)=0 \qquad (3.31)$$

Into a degenerate quadratic curve when this a degenerate curve represents a point or does not have a real locus, the system obtained from (3.30) by the transformation

$$y'=\lambda x+y \ , \ x'=x \quad \text{is}$$

$$\frac{dy'}{dt}=\lambda P_2+Q_2=Q_2{}'(x',y')$$

$$\frac{dx\prime}{dt}=\dot{p}_2(\dot{x},\dot{y}) \qquad (3.32)$$

Hence we nay as well assume that

$$\lambda P_2+Q_2=R_1R_2 \ , \qquad where \ R_i(i=1,2)$$

Is a real polynomial with degrees of x and y not higher then one, one of them is not a constant.

If $i=1,2$ the determinant of the transformation

$$y'=\lambda x+y \ , \qquad x'=R_i \qquad (3.33)$$

Hence we only have to discuss the case when the determinant of transformation (3.11) in not zero for $i=1$ or 2 . in this case the system (3.30) under this transformation becomes

$$\frac{dx'}{dt} = P_2'^{(x',y')} \quad , \frac{dy'}{dt} = x'(ax\dot{} + b\dot{y} + c) \qquad (3.34)$$

We still write x', y' as x, y and depending on the value of a, b, and c , we can divide the system (2.12) into three classes

i. $a = b = 0$, $c \neq 0$

$$\frac{dx}{dt} = K + \delta x + ey + lx^2 + mxy + ny^2$$

$$\frac{dy}{dt} = cx \qquad (3.35)$$

ii. $a \neq 0$, $b = 0$, $c \neq 0$

$$\frac{dx}{dt} = K + \delta x + ey + lx^2 + mxy + ny^2$$

$$\frac{dy}{dt} = x(ax + c) \qquad (3.36)$$

iii. $b \neq 0$,

$$\frac{dx}{dt} = K + \delta x + ey + lx^2 + mxy + ny^2$$

$$\frac{dy}{dt} = x(ax + by + c) \qquad (3.37)$$

If system (3.35 – 3.37) has a closed trajectory, then its interior must contain a unique focus or center with index + 1.

Then apply a suitable transformation

$x = \mu x'$, $y = vy'$, $t = \lambda t'$ to change (3.35 – 3.37)

Into

i. $$\frac{dx}{dt} = -y + \delta x + ey + lx^2 + mxy + ny^2$$

$$\frac{dy}{dt} = x$$

ii. $$\frac{dx}{dt} = -y + \delta x + ey + lx^2 + mxy + ny^2$$

$$\frac{dy}{dt} = x(1 + ax), a \neq 0$$

iii. $$\frac{dx}{dt} = -y + \delta x + ey + lx^2 + mxy + ny^2$$

$$\frac{dy}{dt} = x(1 + ax + by)\,, b \neq 0$$

Divided the system (2.30) into two classes

$$A\left[\begin{array}{c}\dfrac{dx}{dt} = b_{00} + xy \\ \dfrac{dy}{dt} = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 = Q_2(x,y)\end{array}\right.$$

$$B\left[\frac{dx}{dt} = b_{20}x^2 + y \quad , \quad \frac{dy}{dt} = Q_2(x,y)\right.$$

For system of classes ii and iii , under certain conditions cycle.

Changing the system into (A) or (B) the formula for the coefficients of (iii)

3.7.1: Examples :

As follows:

1. $n \neq 0$ let K denote anon zero root of the equation.

$$a + (b - l)k - mk^2 - nk^3 = 0, \qquad (3.38)$$

Where $\alpha = k^3 - \delta k^2 + k$ and $\beta = kl - a - nk^3$

When $\beta = 0$ and $\propto = 0$ (iii) can be changed to B,

Where

$a_{00} = 0,\ a_{10} = -k\alpha + k^2 - \delta k^3 = -k^4,$

$$a_{20} = -\frac{a\alpha}{nk^3} - \frac{\delta k^2 - k}{nk^3}\left[2a + bk - nk^3 + \frac{1}{\alpha}(a + bk)(\delta k^2 - k)\right],$$

$$a_{01} = \delta k^2, \qquad a_{0=11} = \frac{1}{nk^3}\left[2a + bk + \frac{2}{\alpha}(a + bk)(\delta k^2 - k)\right],$$

$$a_{02} = -\frac{1}{nk^3\alpha}[a + bk],$$

When $\beta \neq 0$, (iii) can be changed to (A), where

$$b_{00} = \alpha\left[\frac{\alpha}{\beta} - (a - kl) + k - \delta k^2\right],$$

$$a_{00} = \frac{1}{\beta}\left[k\alpha\beta + a\alpha^2 - b_{10}\big(k\beta + \alpha(2a + bk)\big) + b^2{}_{10}\,(a + bk) + b_{00}(a - kl)\right],$$

$$a_{10} = \frac{1}{\beta}\left[-(k\beta + 2a\alpha) - b_{20}\big(k\beta + \alpha(2a + bk)\big) + b_{10}\,(2a + bk) + 2b_{10}b_{20}(a + bk)\right],$$

$$a_{20} = \frac{1}{\beta}\left[a + b_{20}\left(2a + bk\right) + (a + bk)\right)b^2{}_{20}\,]$$

$$a_{01} = \frac{1}{\beta}[k\beta + \alpha(2a + bk) - 2b_{10}(a + bk)],$$

$$a_{11} = \frac{-1}{\beta}[a + kl + bk + 2(a + bk)b_{20}],$$

$$a_{02} = \frac{1}{\beta}[(a + bk)] \quad \text{in which}$$

$$b_{10} = {[\delta k^2 - k - 2\alpha(a - kl)]}\Big/{\beta} \text{and}$$

$$b_{20} = \frac{1}{\beta}[(a - kl)]$$

Since for this case (iii) cannot have a limit cycle.

(2) $n = 0$ when $m \neq 0$ we can prove that (iii) can be changed to (A)where $b_{00} = L + m\delta$, $a_{00} = m^2 + ma - b(2L + m\delta) + L(L + m\delta)$,

$$a_{10} = m^2 + 2ma - b(3L + m\delta) \quad , \quad a_{01} = b,$$

$$a_{20} = ma - bL \ , \quad a_{11} = b + L \ ,$$

Chapter Four

Limit Cycle Problems for a Liénard systems:

We provide a new contribution to this subject which can be also applied to Li´enard differential systems with some kind of discontinuities.

We consider for x ∈ [a, b], where −∞ < a < 0 < b < ∞, the Li´enard differential equation

$$x'' - f(x)x' + g(x) = 0, \qquad (4.1)$$

$$f(x) = \begin{cases} f_1(x) & \text{if} \quad x < 0, \\ f_2((x) & \text{if} \quad x > 0, \end{cases} \qquad \begin{matrix} g_1(x) & \text{if} \quad x < 0, \\ g_2((x) & \text{if} \quad x > 0, \end{matrix} \qquad (4.2)$$

being f_1, g_1 continuously differentiable in [a, 0], and f_2 , g_2 continuously differentiable in [0, b].

Note that the functions f and g are not defined at x = 0 so that, if we eventually define f(0) and g(0), they are allowed to have a jump discontinuity at the origin.

By using the classical Li´enard plane we can obtain the equivalent differential system

$$x' = F(x) - y, \qquad y' = g(x), \quad \text{where} \quad F(x) = \int_0^x f(s)ds, \quad (4.3\ )$$

and it is understood that F(0) = 0, while g(0) is not defined by now.

This system has associated the vector field

$$X(x) = \begin{cases} X_1(x) & \text{if} \quad x < 0, \\ X_2\ (x) & \text{if} \quad x > 0, \end{cases} \quad \text{where} \ X_i(x) \begin{cases} F(x) - y, \\ g_i\ (x) \end{cases}, \qquad (4.4)$$

with $x = (x, y)^t$ and standing i = 1 for x ≤ 0, and i = 2 for x ≥ 0.

The ambiguity in the definition of X(x) on x = 0 will be clarified later on.

Since the system can be discontinuous we must adopt some criterion in order to define solutions starting at or passing through the allowed discontinuity line x = 0. Typically this is done by using the so called Filippov approach, see for instance [10]. However here only the vertical component of the vector field (4.4) could be discontinuous at the y-axis, while its horizontal component turns out to be continuous.

In fact, we have x′ = −y on x = 0.

Thus if we consider for instance orbits starting at points with x < 0, then these orbits are well defined whenever they do not touch the y-axis but they can arrive at this straight line (obviously only at points (0, y) with y ≤ 0) by extending g(x) as if g(0) were equal to $g_1(0)$.

Now starting from the point (0, y) with y < 0 we assume that g(0) = g_2 (0) and we continue the orbit inside x > 0 using system (4.3).

From the above paragraph and using the standard terminology of planar Filippov systems [10], the crossing set of the discontinuity line of system (4.4) includes the negative y-axis. Similar arguments for x > 0 imply that the crossing set is the y-axis without the origin.

In [10] the origin is then called a singular isolated sliding point.

In short, except orbits arriving at the origin and assuming that the system is actually discontinuous, it is natural to allow concatenation of solutions in an obvious way so that the system has no sliding (Filippov) solutions.

The only possible singular point may be the origin, where each vector field can

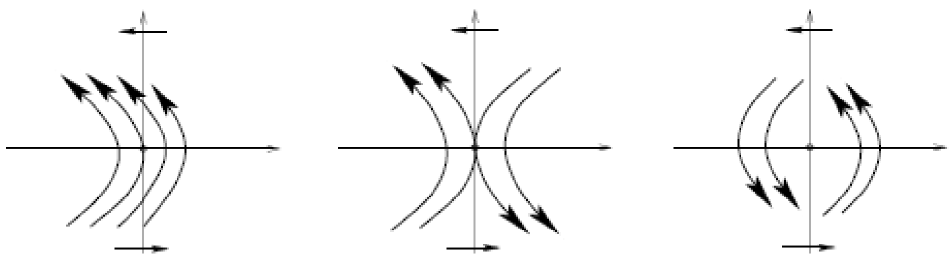


Figure 1.

The three main cases for the local phase plane at the origin when it is not a boundary equilibrium point, regular point, pseudo-saddle and pseudo-focus.

Either vanish or have a tangency with the y-axis.

If at least one vector field vanishes at the origin we say that it is a boundary equilibrium point.

If both vector fields are not zero at the origin we still can have a pseudo-equilibrium point when both vector fields are anti-collinear (i.e. g_1 (0) g_2 (0) < 0).

Then it behaves as an equilibrium point that may be reached in finite time.

Its stability and local phase portrait will be determined by studying its nearby orbits, see Figure 1.

4.1: Propositions

Proposition4.1 1: For system (4.3) the following statements hold.

(a) If g_1 (0) g_2 (0) > 0 then the origin can be thought of a regular point.

(b) If g_1 (0) g_2 (0) = 0 then the origin is a boundary equilibrium point.

(c) If g_1 (0) g_2 (0) < 0 then the origin is a pseudo-equilibrium point, being of saddle type if g_1 (0) > 0 and g_2 (0) < 0, and of focus type if g_1 (0) < 0 and g_2(0) > 0.

4.1 2: **Proposition**

From the point of view of practical engineering problems the most interesting case corresponds to the existence of a pseudo-equilibrium point or a proper equilibrium point of focus type at the origin, because then it is possible that the system behaves locally or even globally as an oscillator.

Thus we will be mainly interested in possible periodic orbits.

to assure that there is no more singular points the following hypothesis is assumed.

(H1) The function g satisfy xg(x) > 0 for $x \neq 0$.

We will require that the divergence of the vector field does not change its sign in each side of the discontinuity line, i.e.

(H2) The function f satisfy xf(x) > 0 for $x \neq 0$.

Under the last hypothesis we have positive divergence for x > 0 and negative divergence for x < 0.

Then in order to have some periodic orbit surrounding the origin, there must be some balance between the x-positive and x-negative parts of the interior of the bounded region limited by the periodic orbit.

This idea will be precisely stated below in the same spirit of comparing the x-positive and x-negative half-planes and following [1], it will be useful to introduce some auxiliary functions as follows.

Under Hypothesis H2 and recalling the definition of F in (4.3), we define a variable

$$p = p(x) = F(x).$$

As $p'(x) = f(x)$, then $p(x) \geq 0$ for all x, and $\operatorname{sgn}(p'(x)) = \operatorname{sgn}(x)$ for $x \neq 0$

We deduce that the function p(x) has inverse functions both for $x \leq 0$ and for $x \geq 0$, namely the non-positive decreasing function

$$F_1: [0, F(a)] \to [a, 0], \text{such that } F(x_1(p)) = p, \quad (4.5)$$

and the non-negative increasing function

$$x_2 : [0, F(b)] \to [0, b], \text{ such that } F(x_2(p)) = p. \quad (4.6)$$

Hence for $x \neq 0$ we have that both systems (3.3) and (3.4) are equivalent to the two differential equations

$$dy\left(\frac{x_{i(p)}}{dp}\right) - \frac{1}{yf(x_{i(p))}} = \frac{1}{p-y}\,\frac{g(x_{i(p)})}{yf(x_{i(p)})}, \qquad (4.7)$$

where i = 1, 2, according to x < 0 or x > 0 respectively, and these new differential equations are both meaningful only for p > 0.

Now by considering the functions

$$h_i(p) = \frac{g(x_{i(p)})}{f(x_{i(p)})}, \qquad (4.8)$$

equations (4.7) can be written in the more compact form

$$dy\left(\frac{x_{i(p)}}{dp}\right) = R)$$

Note that hi(p) > 0 for p > 0 and i = 1, 2, and that the effect of considering equations (4.9) instead of the original systems (4.3.) or (4.4) can be thought of as if the plane (x, y) had been folded into the half-plane (p, y) with p > 0.

When h_1(p) = h_2(p) for p sufficiently small and the origin is a topological focus it is not difficult to show that we have indeed a center, see for instance.

4.1.3: **Theorem** in [8].

We add a third hypothesis precluding such possibility.

It is written in a dual way to facilitate the checking of its validity in the applications.

(H3) Assume that there exist the two limits.

$$\lim_{x\to 0-}\frac{g(x)}{f(x)} = \lim_{p\to 0+} h_1(p) = l_1, \qquad \lim_{x\to 0+}\frac{g(x)}{f(x)} = \lim_{p\to 0+} h_2(p) = l_2,$$

satisfying

$$0 \le l_2 \le l_1 \le \infty,$$

and if $l_2 = l_1$ then $h_2(p) < h_1(p)$ for p > 0 and sufficiently small (when $l_2 < l_1$ this last requirement is always fulfilled).

It is worth mentioning that this hypothesis implies that the origin is topologically an unstable focus when $l_2 > 0$

Next result states a necessary condition for the existence of periodic orbits under the above hypotheses.

4.1. 4:Theorem. Let f and g be the functions defined in (4.2) such that f_i and are of class C^1 in [a, 0] and [0, b] for i = 1, 2, respectively, where $-\infty < g_i$ a < 0 < b < ∞.

Let F and h_i be the functions defined in (4.3) and (4.8) and assume that hypotheses H_2-H_3 are fulfilled.

If system (3.3) has a periodic orbit contained in the band a < x < b, then the system

$$(\mathrm{F}(x_1) = \mathrm{F}(x_2) \quad , \quad \frac{\mathrm{g}(x_1)}{\mathrm{f}(x_1)}, \quad \frac{\mathrm{g}(x_2)}{\mathrm{f}(x_2)}, \qquad (4.10)$$

has at least one solution $(x_1 , x_2) = (s_1, s_2)$ with a < s_1) < 0 < s_2) < b, or equivalently there exists at least one solution ˆp ∈ (0, F(a))∩(0, F(b)) for the equation h_1) (p) = h_2) (p).

Now we give a result on uniqueness of limit cycles for Li´enard equations where discontinuities are allowed at x = 0.

4.1.5:Theorem . Under the same conditions of Theorem 4.1 2, assume that system (4.10) has exactly one solution $(x_1 , x_2) = (s_1, s_2)$ with a <s_1 < s_1 < 0 < b, or equivalently there exists exactly one solution ˆp ∈ (0, F(a)) ∩ (0, F(b)) for the equation h_1 (p) =h_1(p).

The following statement holds.

If the positive function

$$\alpha(x) = \frac{g(x)}{f(x)F(x)}, \qquad (4.11)$$

is increasing for x ∈ (a, 0), or equivalently the positive function

$$\frac{h_1(p)}{p}, \qquad (4.12)$$

is decreasing for p ∈ (0, F(a)), then system (4.3) has at most one periodic orbit contained in the band a < x < b, and if it exists has a negative characteristic exponent.

Although our main motivation is the case of discontinuous systems, it should be noted that the above results can be useful also for continuous differential equations. For instance we can state the following result.

4.1.6: **Proposition**

The following Li´enard system

$$\dot{x} = \alpha x^2 + \beta x^2 + \dot{x}^4 - y, \qquad (4.13)$$
$$\dot{y} = x$$

where $\beta > 0$ and $9\beta^2 - 32\alpha < 0$ has no limit cycles in the plane.

4.1.7: Proposition

We finish by considering an application of the above results to discontinuous piecewise linear differential systems.

This class is increasingly used in engineering and applied sciences to model a large variety of technological devices and physical systems [2, 15].

Similar differential systems had been considered before in [6] but under the assumption of continuity for the corresponding vector field.

4.1. 8:Theorem. Consider the Li´enard piecewise linear differential system

$$\begin{cases} \dot{x} = t_1 - y, \\ \dot{y} = d_1 + a_{1,} \end{cases} \quad if \quad x < 0\,, \quad \begin{cases} \dot{x} = t_2 - y, \\ \dot{y} = d_2 + a_{2,} \end{cases} \quad if \quad x \ \geq 0, \qquad (3.14)$$

where it is assumed

$$t_1 < 0\,, d_1 < 0, a_1 < 0, t_2 > 0\,, d_2 > 0\,, a_2 > 0\,.$$

Then the following statements hold.

(a) If $\frac{a_2}{t_2} < \frac{a_1}{t_1}$ then a necessary condition for the existence of periodic orbits is

$$\frac{d_2}{{t_2}^2} > \frac{d_1}{{t^2}_1}.$$

If the system has periodic orbits, then it has a unique periodic orbit which is a stable limit cycle.

(b) If $\frac{a_1}{t_1} < \frac{a_2}{t_2}$ then a necessary condition for the existence of periodic orbits is

$$\frac{d_1}{{t^2}_1} > \frac{d_2}{{t_2}^2}.$$

If the system has periodic orbits, then it has a unique periodic orbit which is an unstable limit cycle.

(c) If $\frac{a_2}{t_2} = \frac{a_1}{t_1}$ then either the system has no periodic orbits when is

$$\frac{d_1}{{t^2}_1} \neq \frac{d_2}{{t_2}^2},$$

or it has a center at the origin when

$$.\frac{d_1}{{t^2}_1} = \frac{d_2}{{t_2}^2},$$

Observe that statement (c) of Theorem 5 when

$$0 < \frac{a_2}{t_2} = \frac{a_1}{t_1}, \text{ and } \frac{d_1}{{t^2}_1} = \frac{d_2}{{t_2}^2}.$$

says that the origin is a center even when the dynamics of the linear differential system in each half–plane could be of node type.

This situation happens when

$$\frac{d_i}{{t^2}_i} \leq \frac{1}{4} \quad \text{for i = 1, 2.}$$

When both dynamics are of focus type and we are under the assumptions of statements (a) and (b) of **Theorem 4.1.8** the necessary condition for the existence of limit cycles is also sufficient, as stated in our last main result.

4.2: generalized Li´enard system:

$$\frac{dx}{dt} = \frac{1}{a(x)}[h(y) - F(x)],$$

$$\frac{dy}{dt} = -a(x)g(x), \qquad (4.15)$$

where $F(x)$, $g(x)$, $a(x)$ and $h(y)$ are continuous real functions defined on **R** satisfying:

(A_0) $F(0) = 0$, $a(x) > 0$ for $x \in \mathbf{R}$, $xg(x) > 0$ for $x \neq 0$;

$y\,h(y) > 0$ for $y \neq 0$, $h(y)$ is strictly increasing and $h(\pm\infty) = \pm\infty$.(A_1)

These assumptions guarantee that the origin is the only critical point of (4.15). We also assume that the initial value problem always has a unique solution.

We call the curve $h(y) = F(x)$ the characteristic curve of system (4.15).

We write $\gamma+(P)$ (resp., $\gamma-(P)$) the positive (resp., negative) semi orbit of (4.15) starting at a point $P \in R^2$. For the sake of convenience, we denote

$$D1 = \{(x, y) : x \geq 0,\, h(y) > F(x)\},\, C$$

$$D2 = \{(x, y) : x > 0,\, h(y) \leq F(x)\},$$

$$D3 = \{(x, y) : x \leq 0,\, h(y) < F(x)\},$$

$$D4 = \{(x, y) : x < 0,\, h(y) \geq F(x)\}.$$

$(x) = \max\{0, F(x)\}$, $F_-(x) = \max\{0, -F(x)\}$,$F_+$

$$\Gamma_+(x) = \int_0^x a^2(s)g(s)\,(1 + F_+(s))^{-1}ds\ ,\ \Gamma_- = \int_0^x a^2(s)g(s)\,(1 + F_-(s))^{-1}ds\ .$$

$Y+ = \{(0, y) : y > 0\}$, $Y- = \{(0, y) : y < 0\}$,

$$C+ = \{(x, y) : x > 0,\, h(y) = F(x)\},\ G(x) = \int_0^x a^2(s)g(s)ds.$$

Then by (A_0), $G(x)$ is strictly increasing, and therefore, the inverse function $(G)^{-1}(w)$ of $w = G(x)$ exists.

Throughout this Section we shall suppose that the following conditions Hold.

$$(A_2)\ \int_0^{\infty} a^2(s)g(s)ds = \int_0^{-\infty} a^2(s)g(s)ds.$$

(A_3) $F(G)^{-1}(-w)) = F\,(G)^{-1}(w))$ for $0 < w < M$,

where $M = \min\{G(\infty), G(-\infty)\}$ (M may be ∞).

If $F(x)$ and $a^2(x)g(x)$ are even and odd functions, respectively, then it is obvious that ($A2$) and ($A3$) are satisfied and that all the orbits of (4.15) have mirror symmetry about the y-axis in the phase space.

Moreover, for example, if $F(x) = 3x$, $a(x) = 1$ and $g(x) = 2x$ for $x \geq 0$, and $F(x) = -3\sqrt{2x}$, $a(x) = 1$ and $g(x) = 4x$ for $x \leq 0$, then ($A2$) and ($A3$) are also satisfied. Firstly, employing an argument similar to that in [9, 22], we show that under the conditions ($A2$) and ($A3$), the orbits of (3) have deformed mirror symmetry about the y-axis.

4.2.1:Lemma. Suppose that the conditions (A2) and (A3) are satisfied.

If an orbit of (4.15) starting from a point A(0, yA) with yA > 0 passes through a point B(0, yB) with yB < 0, then it reaches the point A again.

proof. Consider an orbit of (4. 15) which starts from a point $A(0, yA)$ with $yA > 0$ and passes through a point $B(0, yB)$ with $yB < 0$.

We denote this orbit by $T(x, y)$ and write $T_1(x, y) = \{(x, y) \in T : x \geq 0\}$ and $T_2(x, y) = \{(x, y) \in T : x < 0\}$.

Let $K = \int_0^\infty a^2(x)g(x)dx$ (K may be ∞) and let the mapping $\phi : (x, y) \to (u, v)$ defined by

$$u = \begin{cases} \sqrt{2G(x)} for \;\; x \geq 0, \\ -\sqrt{2G(x)} \;\; for \; x < 0, \end{cases}$$

$$v = y.$$

Then we can see that the image $\phi T_1(u, v)$ of $T_1(x, y)$ is an orbit of the system

$$h(v) - F^*(u), u' =$$

$$= -u, v'$$

defined on $(-\sqrt{2}K, \sqrt{2}K) \times \mathbf{R}$, where

$$F^*(u = \begin{cases} F\left((G)^{-1}(\frac{1}{2}u^2)\right) & for \;\; 0 \leq u < \sqrt{2K} \\ F\left((G)^{-1}(-\frac{1}{2}u^2)\right) & for - \sqrt{2K} < u < 0. \end{cases}$$

In fact, for any point $(u, v) \in \phi\, T_1$,

$$\frac{du}{dv} = \frac{a^2(x)g(x)}{\sqrt{2G(x)}} \cdot \frac{h(y) - F(x)}{-a^2(x)g(x)} = \frac{h(v) - F^*(u)}{-u}.$$

Note that the curve $\phi T1(u, v)$ contains the points A and B.

It follows from ($A2$) and ($A3$) that$F^*(u)$ is an even function on $(-\sqrt{2}K, \sqrt{2}K)$. Hence, the curve $\phi T1(-u, v)$ is also an orbit of (4.15) which contains the points A and B.

Let $T_3(x, y)$ be the inverse image of $\phi T_1(-u, v)$ under the mapping ϕ.

Then for any point $(x, y) \in T_3$,

$$\frac{dx}{dy} = \frac{\sqrt{-2G(x)}}{a^2(x)g(x)} \cdot \frac{h(y) - F\left(G^{-1}\left(\frac{u^2}{2}\right)\right)}{-u} = \frac{h(v) - F(x)}{-a^2(x)g(x)}.$$

Thus, $T_3(x, y)$ is an orbit of (4.15)which starts from the point B and arrives at the point A.

Since the solutions of (4.15) are unique, $T_2(x, y)$ and T_3 (x, y) coincide, and hence the orbit $T(x, y)$ reaches the point A again.

This completes the proof.

4.2.2Remark 1. If the condition (A3) holds for $w > 0$ sufficiently small, then all the orbits of (4.15) near the origin have deformed mirror symmetry with respect to the y-axis.

4.2. 3: Lemma. Consider the functions f and g defined as in (4.2).

If system (4.3) has a periodic orbit Γ and the interior of the bounded region limited by includes the origin and it is denoted by Δ , then Γ crosses the y-axis in two points different from the origin, and the function f satisfies the condition

$$= 0, \iint_{\Delta} f(x)dxdy$$

Proof. Since x′ = −y on x = 0 and the origin is in Δ, it follows that Γ intersects the y-axis in two points M = (0, yM) and N = (0, yN) with yM < 0 < yN.

We define Δ_1, Γ_1 , and Δ_2 , Γ_2 to be the parts of Δ and Γ contained in x < 0 and x > 0 respectively.

We denote by A the oriented segment on the y-axis from the point M to the point N while the same segment with the opposite orientation is denoted by −A.

Then by applying the Green's Theorem w

$$\iint_{\Delta} f(x)dxdy = \iint_{\Delta_1} f(x)dxdy + \iint_{\Delta_2} f(x)dxdy = \int_{\Gamma_1} [F(x) - y]\, dy - g(x)dx + \int_{\mathrm{A}} [F(x) - y]\, dy - g(x)dx + \int_{\Gamma_2} [F(x) - y]\, dy - g(x)dx + \int_{-\mathrm{A}} [F(x) - y]\, dy - g(x)dx = 0 + \int_{yM}^{yN} (-y)dy + 0 + \int_{yN}^{yM} (-y)dy = 0$$

and the conclusion follows.

4.2.4: Theorem. Assume f and g are smooth functions such that $g(x) > 0$ for $x > 0$ and such that f has exactly three zeros (0,a,-a) with $f'(0) < 0$ and $f'(x) \geq 0$ for $x >$

a and $f(x) \to \infty$ for x then the corresponding lienard system has exactly one limit cycle and this cycle is stable

proof.

Drawing xy – plane the graph of the function $x \to f(x)$. On his graph the vector field is vertical it is called a null cline is for $x > 0$ we have $\frac{dy}{dx} < 0$

The y – axes the vector field is horizontal because $g(0) = 0$

The y–axes the also a nullcline consider an orbit which starts at $(0\,,y_0)$ on the positive y-axes. It goes to the right be cause $g(x) > 0$ for $x \geq 0$ because $g(x) > 0$ the orbit also moves down.

It has to hit the graph off.

It intersects that nullcline at appoint $(x_1\,,0)$ with positive vertical velocity and enters the vegan.

Where $\frac{dy}{dx} < 0$it must then go the left and hit again some where the y- axes horizon tally in some point $(0\,,y_0) = [0, -S(y_0)]$ we can analyze the fate of the orbit on the left half plane in the some way as on the right plane.

Limit cycle if the map $y_0 \to S(y_0)$ was affixed point.

Alternatively we can express this that the energy.

$H(x,y) = \frac{y^2}{2} + G(x)$Is the same at$(0\,,y_0)$ and $(0\,,y_1)$.

The idea of the proof is to determine the energy gain along the orbit and to see that only for one single orbit, the energy is conserved compute.

$$\frac{d}{dt}H(x,y) = y\frac{dy}{dt} + g(x)\frac{dx}{dt} = -F(x)g(x) \qquad (4.16)$$

If $F[x^{\cdot}(t)]$ were positive on the inter trajectory from $(0\,,y_0)$ to $(0\,,y_1)$ then $H(0\,,y_1)$ - $H(0\,,y_0)$ is positive.

It must there fore cross the graph of f at a point , where $f(x) > 0$.

The theorem is proven if we can show the following statement a but the energy difference.

$\Delta(y_0) = H(0\,,S\,y_0)$ - $H(0\,,y_0)$.

Depending on the intersection point $x_1, f(x_1)$ with the null cline.

$$if\ x_1 < a\ , than\ \Delta(y_0) > 0. for\ y_0\ such\ that\ x_1 > a$$

$\Delta(y_0)$ is a monotonically decreasing function for y_0 and $\Delta(y_0) \to -\infty\ for\ \ y_0 \to \infty$

As consequence, there exist then exactly one point y_0 belongs to a limit cycle.

The vest of the proof is devoted to the verification of the above claim.

(i) $\Delta(y) > 0\ \ if\ y_0$is such that $x_1 \leq a$.

That $f(x)$ is negative in the interval $[0, a]$

If $x_1 \leq a$ then $x(t) \leq a$ until we hit the y- axes again. But since then $F(x(t)) < a$ and $g(x) > 0$ for $x > 0$ we have $\frac{d}{dt}H(x,y) = -F(x)g(x) > 0$ the energy gain is positive.

(ii) The monotonic claim for $x_1 \geq a$ let $A(y_0) be\ the\ path\ [x(0), y(0)] = (0, y_0)$ And

$[x(T), y(T)] = (0, y_0)$ from $\frac{d}{dt}H(x,y) = -F(x)g(x)$ we obtain.

$$\Delta(H)(y_0) = \int_A -(F(x(t))g(x(t))dt = \int_A F(x(y))dy = \int_A \frac{-F(x)g(x)}{y - F(x)}.$$

Split the path a in to a path A_1 from $(0\ , y_0)$ to $x(t) = a\ , a\ path\ A_2$ which is the continuation un til $x(t) = a$ again and into a path A_3 we can parameterize the curve by x instead of t, along A_2we can use the parameter y.

We see that increasing y_0 increases $y(t)$ and so decreases integral

$$\Delta_1(H)(y_0) = \int_0^a \frac{-F(x)g(x)}{y-F(x)}dx, \text{ along } A_1.$$

On A_3 increases y_0 decreases$y(t)$ which decreases the integral

$$\Delta_3(H)(y_0) = \int_0^a \frac{-F(x)g(x)}{y-F(x)}dx \text{ along } A_3.$$

Along A_2 , usey as variable.

Increasing y_0 pushes the path $A_2$to the right so that $F(x(t))$ is increasing and the integral

$$\Delta_2(\mathrm{H})(\mathrm{y}_0) = -\int_{\mathrm{y}_2}^{\mathrm{y}_3} \mathrm{F}(\mathrm{x}(\mathrm{y}))\mathrm{dy}, \text{ is decreasing.}$$

The sum $\Delta(\mathrm{H})(\mathrm{y}_0) = \Delta_1(\mathrm{y}_0) + \Delta_2(\mathrm{y}_0) + \Delta_3(\mathrm{y}_0)$ is decreasing in y_0

(iii) The limit $\mathrm{y}_0 \to \infty$

To see that $\Delta(y_0)$ goes to $\rightarrow \infty$ for $y_0 \rightarrow \infty$ we split an orbit in to paths B_1 , B_2 , B_3in the same way as A_1 , A_2 , A_3 but that the value of a has been replaced by (a+1).

The integrals along $B_1$and B_2 are bounded by aconstant independent of y_0.

While the integral along B_2 is bigger or equal to F(a+1) times they differences of the two points, where $x(t) = a + 1$.

This differences goes to $\rightarrow \infty$ for $y_0 \rightarrow \infty$ so the energy gain along the sum of the paths B_1 , B_2 , B_3 goes to $\rightarrow \infty$ for $y_0 \rightarrow \infty$.

In the previous we so that the Poincare Bendixson theorem could be used to establish the existence of limit cycles for certain planar systems.

It is afar more Delieale equation[2], [3] to determine the exact number of limit cycle of a certain system or class of systems depending on parameters.

In this study we present a proof of a classical result on the uniqueness of the limit cycle for systems of the form.

$$\begin{aligned} x' &= y - f(x) \\ y' &= g(x) \end{aligned} \qquad (4.17)$$

Under certain conditions on the functions f and g.

This result was first established by the French physicist a Liénard and the system is referred to as a Liénard system.

Liénard studied this system in the different but equivalent from.

$$x'' + f(x)x' + g(x) = 0$$

Where $F(x) = f'(x)$ in on sustained oscillations this second order differential equation includes the famous Vander pol equation.

$$x'' + \mu(x^2 - 1)x' + x = 0\,, \qquad (4.18)$$

Of vacuum tube circuit theory as a special case we present serval other interesting resolution the number of the limit cycle of Liénard systems and polynomial systems.

In the proof of Liénard's theorem and in the statements of some of the other theorems in this section it will be useful to define the functions f.

$$F(x) = \int_0^x f(s)ds \quad and \quad g(x) = \int_0^x g(s)ds,$$

And the energy function

$$\mathrm{u}(\mathrm{x}, \mathrm{y}) = \frac{y^2}{2} + G(x), \qquad (4.19)$$

3.2.5: Theorem[24],

under the assumptions that , f , $g \in C'(R) f$ and g odd functions of x , $xg(x) > 0$ for

$$x \neq 0 \ , \ f(0) = 0,$$

$f'(0) < 0$, f has single positive zero at $x = a$ and f increase monotonically to infinity for $x \geq a$ as $x \to \infty$, it follows that the Lienard system. Has exactly one limit cycle and it is stable.

The proof of this theorem makes use of the diagram below where the points p_j have coordinates $(x_j , y_j)$for j= 0,1,2,3,4 and Γ is a trajectory of the Liénard system (4.17)

The function $f(x)$ which satisfy the hypotheses of theorem 3.2.5.

Before presenting the proof of this theorem we first of all make same simple observations under the assumptions the above theorem. The origin is the only critical point . The flow on the positive y – axis is horizontal and to the right and the flow on the negative y – axis is horizontal and to the left the flow of the curve $y = f(x)$ is vertical downward for $x > 0$ and upward for $x < 0$ the system (4.17).

is invariant under $(x, y) \to (-x, -y)$ and therefore if $(x(t), y(t))$describes a trajectory of (4.11). . So dose $(-x(t), -y(t))$ is follows.

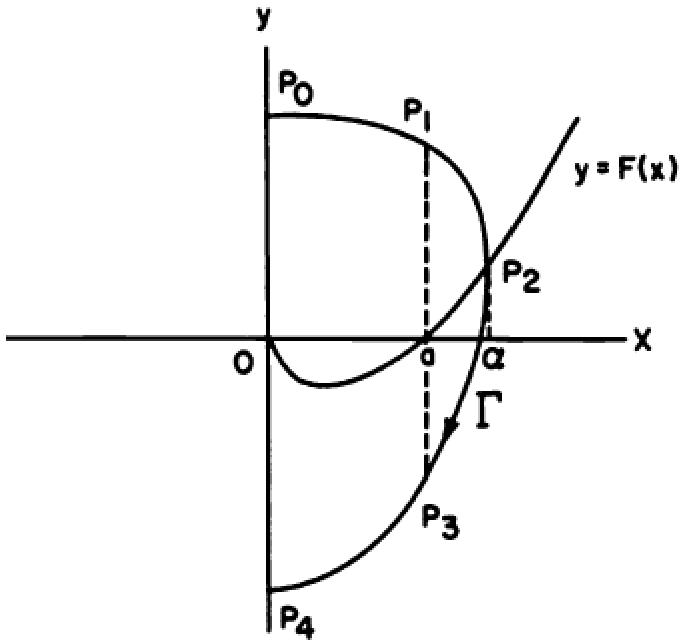


Figure 1

That if Γ is closed trajectory of (4.17) . a periodic orbit of (4.17).

Then Γ is symmetric with respect to the origin.

Proof.

Due to the nature of the flow on the y – axis and on the curve $y = f(x)$ any trajectory Γ starting at appoint $p_0$on the positive y – axis crosses the curve $y = f(x)$ vertically at point p_2 and they it crosses the negative y – axis crosses horizon tally at the point p_4.

Due to the symmetry of the equation. If follows that Γ is a closed trajectory.

If and only if y_4= - y_0 and for

$$u(x, y) = \frac{y^2}{2} + G(x), \qquad (4.20)$$

This is equivalent to $u(0, y_4) = u(0, y_0)$

Now let A be the arc $\overline{p_0 p_4}$ of that trajectory Γand consider the function $\emptyset(\alpha)$ define by the line integral

$$\emptyset(\alpha) = \int_A du = u(0, y_4) - u(0, y_0),$$

Where $\alpha = x_2$, the abscissa of the point p_2 it follows that Γ is a closed trajectory of (4.17), if and only if $\emptyset(\alpha) = 0$.

We shall show that the function $\emptyset(\alpha)$ has exactly one zero $\alpha = \alpha_0$ and that $\alpha_0 > a$.

First of all, note that along the trajectory

$$du = g(x)dx + ydy = f(x)dy,$$

And if $\alpha \leq a$ then both $f(x) < 0$ and

$$dy = -g(x)dt < 0$$

There fore $\emptyset(\alpha) > 0, u(0, y_4) > u(0, y_0)$

Hence any trajectory Γ which crosses the curve $y = f(x)$ at the point p_2 with $0 < x_2 = \alpha \leq a$ is not closed.

4.2.6: Lemma[20], for $\alpha \geq a, \emptyset(\alpha)$ is a monotone decreasing function which decreases from the positive value $\emptyset(\alpha) to \to \alpha$ as α increasing in the interval (a, α).

For $\alpha > a,$ as in fig1 we split the arc Ain to three parts $A_1 = \overline{p_0 p} A_2 = \overline{p_1 p_3}$ and $A_3 = \overline{p_3 p_4}$ define the functions

$\emptyset_1(\alpha) = \int_A du_1 , \emptyset_2(\alpha) = \int_{A_2} du \; and \; \emptyset_3(\alpha) = \int_{A_3} du$t

If follows that $\emptyset(\alpha) = \emptyset_1(\alpha) + \emptyset_2(\alpha) + \emptyset_3(\alpha)$

Along we have

$$du = \left[g(x) + y\frac{dy}{dx}\right]dx = \left[g(x) - \frac{yg(x)}{y - f(x)}\right]dx = \frac{-f(x)g(x)}{y - f(x)}dx$$

Along the arc, A_1 and A_3 we have

$$f(x) < 0, g(x) > 0 \; and \; {}^{dx}\!/_{y - f(x)} = dt > 0$$

There fore $\emptyset_2(\alpha) < 0$ since trajectories.

do not cross.

It follows that increasing α raises the arc $A_1$and lowers the arc A_3.

Along the x limits of integration remain fixed at $= x_0 = 0 \; and \; x = x_1 = a$, and for each fixed x in (0,a). increasing α raises A_1 which increasing which in turn decreasing the above integrand and therefore decreases $\emptyset_1(\alpha)$. Along A_3, the x limits of integration remain fixed at $x_3 = a$ and $x_4 = 0$; and for each fixed $x \in [0, a]$, increasing α lowers A_3 which decreases y which in turn decreases the magnitude of the above integrand and therefore decreases $\emptyset_3(\alpha)$ since

$$\emptyset_2(\alpha) = \int_a^0 \frac{-f(x)g(x)}{y - f(x)} dx = \int_0^a \left|\frac{f(x)g(x)}{y - f(x)}\right| dx$$

Long the arc A_2 of Γ we can write $du = f(x)dy$ and since trajectory of. Do not cross , it follows that increasing α causes the arc A_2 to move to the right. Along A_2 the y – limits of integration remain fixed at $y = y_1 and\ y = y_3$; and for each fixed $y \in (y_3, y_1)$ increasing x increases $f(x)$ and since.

$$\emptyset_3(x) = -\int_{y_3}^{y_1} f(x)dy$$

This in turn decreases $\emptyset_2(x)$. Hence for $\alpha \geq a$ anontone decreasing function of α. It remains to show that

$$\emptyset(\alpha) \to -\infty\ as\ \alpha \to \infty$$

If suffices to show that

$$\emptyset_2(x) \to -\infty\ as\ \alpha \to \infty$$

But along A_2.

$$du = f(x)dy = -f(x)g(x)dt < 0$$

And therefore any sufficiently small $\varepsilon > 0$

$$|\emptyset_2(\alpha)| = -\int_{y_1}^{y_3} f(x)\, dy =$$

$$\int_{y_3}^{y_1} f(x)\, dy > \int_{y_3+\varepsilon}^{y_1-\varepsilon} f(x)\, dy > f(\varepsilon) \int_{y_3+\varepsilon}^{y_1-\varepsilon} dy$$

$$= f(\varepsilon)[y_1 - y_3 - 2\varepsilon] > f(\varepsilon)[y_1 - 2\varepsilon]$$

But $y_1 > y_2\ and\ y_2 \to \infty\ as\ x_2 = \propto \to \infty$

Therefore $|\emptyset_2(\alpha)| \to \infty\ as \propto \to \infty, \emptyset_2(\alpha) \to -\infty\ as \propto \to \infty$

Finally, since the continuous function $\emptyset(\propto)$ decreases monotonically from the positive value $\emptyset(a) = 0$ at exactly one value of $\propto$ say $\propto = \propto_0$ in $(a, \propto)$.

Thus (4.17) has exactly on closed trajectory Γ_0 which goes through the point.

$(\propto_0) = f(\propto_0)$ furthermore, since $\emptyset(\propto) < 0$ for $\propto > \propto_0$, it follows from the symmetry of the system (4.17). That for $\propto \neq \propto_0$ successive points of inter section of trajectory Γ through the point $(\propto, f(\propto))$ with the y – axis approach Γ_0, Γ_0 is stable limit cycle of (4.17). this completes the proof of Lienard theorem.

4.2.7:Corollary:

For $\mu > 0$ Vander pol's equation has a unique limit cycle and it is stable

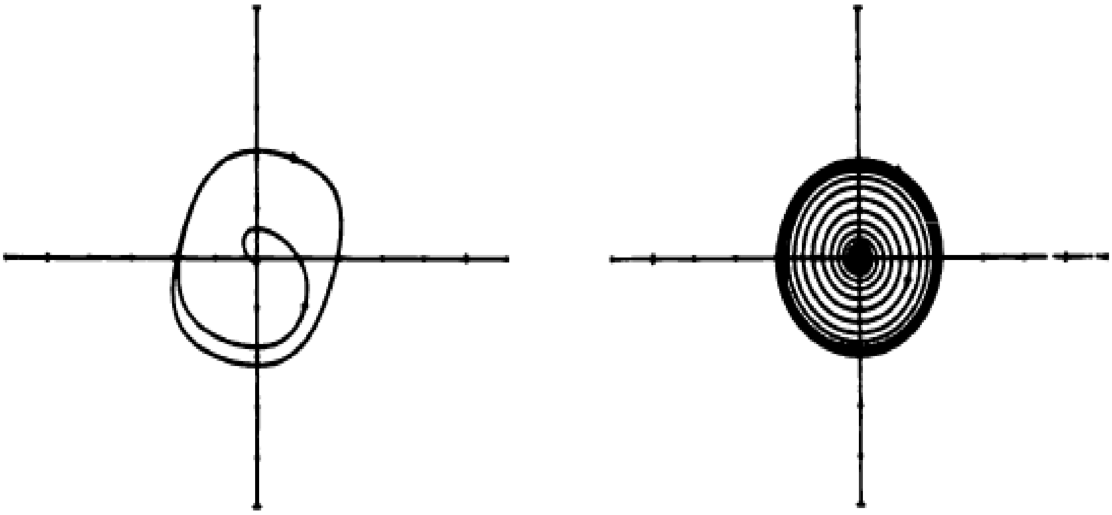


Figure2The limit cycle for the Vander pol equation for $\mu = 1$ *and* $\mu = -1$

Figure (2) shows the limit cycle for the Vander pol equation (4.18) with $\mu = 1$ *and* $\mu = 0.1$ it can be shown that the limit cycle of (4.18) is asymptotic to the circle of radius (4.18) centered at the origin as $\mu \to 0$

4.2.8: Example The function

$$f(x) = \frac{(x^3 - x)}{(x^2+1)} \, and \; g(x) = x \qquad (4.21)$$

Satisfy the hypotheses of Liénard theorem it therefore follows that the system (4.17) with these functions has exactly one limit cycle which is stable.

These limit cycle is shown in figure (3)

ZhandZhifer the Chinese mathematical who proved the following useful result . Which complements Lienard's theorem.

4.2.9: Theorem .

Under the assumptions that $a < 0 < b$, $f, g\epsilon C'(a,b), xg(x) > 0$ for $x \neq 0$

$$G(x) \to \infty \; as \; x \to a \; if \; a = -\infty$$

$$and \;\; G(x) \to \infty \; as \; x \to b \; if \; b = \infty \; , f(x)g(x)$$

Is monotone increasing on $(a,0) \cap (0,b)$ and is not constant in any neighborhood of $x = 0$, it follows that the system (3.17) has at most one limit in the regain $a < x < b$ and if it exists it is stable

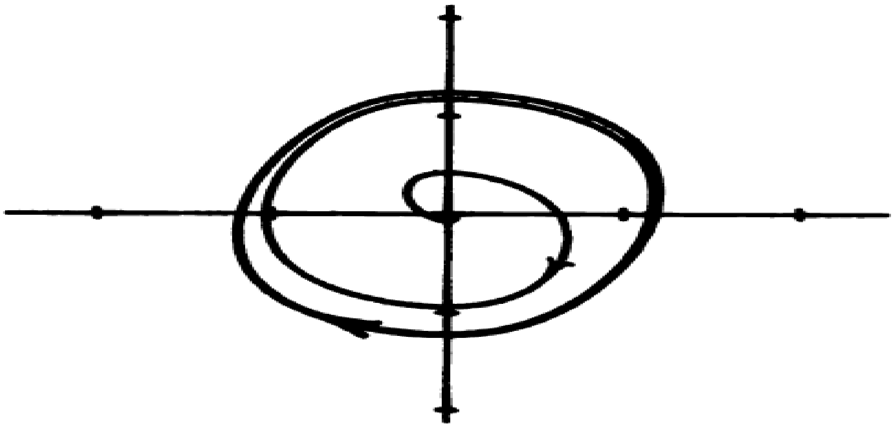


Figure 3. The limit cycle for the Lienard system in Example 1.

The limit cycle for the Liénard system in example2

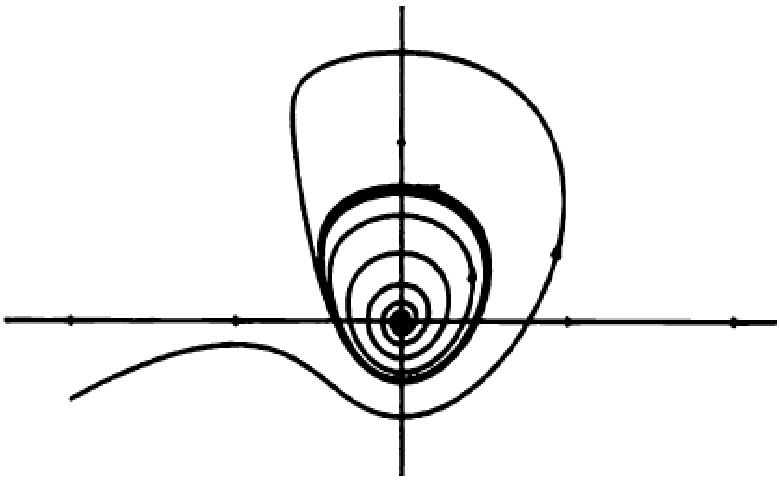


figure4

The limit cycle for the Liénard system in the example with $\propto = 0.02$

4.2.10: Example:

We used this theorem to show that for $\propto\in (0.1)$ the quadratic system.

$$x' = -y(1+x) + \propto x + (\propto +1)x^2$$
$$y' = x(1+x) \quad , \qquad (4.22)$$

Has exactly one limit cycle and it is stable. It is easy to see that the follow is horizontal and to the right on the line $= -1$, therefore any closed trajectory lies in the region $x > -1$

If we define anew independent variable T by $dT = -(1+x)dt$ along trajectories $x = x(t)$ of this system it then takes the form a Liénard system

$$\frac{dx}{dt} = y - \frac{\propto x + (\propto +1)x^2}{1+x}$$
$$\frac{dy}{dt} = -x$$

Even though the hypotheses of the Liénard theorem are not satisfied it can be shown that the hypotheses of Zhang's theorem are satisfied.

Therefore this system as exactly one limit cycle and it is stable.

The limit cycle for this system with $\propto = 0.02$ is shown in figure (4).

And her interesting theorem concerning the number of limit proved by Zhang[43] ,[24] ,[44] of the Liénard system

3.2.11: Theorem

Under the assumptions that

$$g(x) = x\,, \qquad f \in C'(R)$$

$f(x)$ is an even function with exactly tow positive zero

$a_1 < a_2$ $with$ $f(a_1) > 0$ and $f(a_2) < 0$ and $f(x)$ is monotone increasing.

3.3 : Generalized Liénard equation. [24], [27] [29],

In this section consider the following equation:

$$x'' + f(x)x' + g(x) = e(t)\,, \qquad (4.23)$$

We assume the following condition .

(i) F and G are continuous function for all real x

(ii) e is a section continuous function for real t , $e(t) \neq zero$ almost every there, e is periodic with smallest positive periodic t and

$$\int_{c}^{c+t} e(t)dt = o, for\ all\ real\ c$$

(iii) $f(x) > 0\ \ for\ all\ x$

(iv) $xg(x) > 0\ \ for\ all\ x \neq 0$

We shall determine necessary and sufficient conditions for bounded of solution, for existences of periodic solutions and for oscillation of solutions.

We define the functions F,G and E_c by

$$F(x) = \int_{0}^{x} f(s)ds\ \ , G(x) = \int_{0}^{0} g(s)ds\ \ ,\ \ E_C(t) = \int_{c}^{t} e(s)ds.$$

Several systems of equations equivalent to (3.23) will be consider

One such system is defined as follows

Let $x' = y - f(x) + E_c(t)$ then

$$x" = y' - f(x)x' + e(t)\ or\ \ y' = -g(x).$$

We thus have the system

$$x' = y - f(x) + E_C(t)\ \ \text{(c real)}$$

$$y' = -g(x), \qquad \text{(4.24)}$$

We note that a solution $x(t)$ of (1) is bounded if and only if the x – component of a corresponding solution $[x(t)\ , y(t)]$ of (4.24) is bounded, likewise a solution $x(t) of$ (4.23) $and\ it\ is\ derivalive\ x^{\cdot}(t)$ is bounded if and only if the corresponding solution$[x(t)\ , y(t)]$ of (4.24) is bounded .

A solution $[x(t)\ , y(t)]$ of (4.24) is periodic if and only of $x(t)$ is a periodic solution of (4.23).

Conditions (i-iv) imply that for any real $x_0\ , y_0$and to there exists a solution $[x(t_0)\ , y(t_0)] = (x_0, y_0)$ further the solution $[x(t)\ , y(t)]$ is defined for all $> t_0\ x$.

4.4:Uniqueness of limit cycles for a class of Liénard system:[22],

We shall given three criteria for the uniqueness of limit cycles of system of Liénard type.

$$x' = \alpha(y) - \beta(y)F(x)$$

$$y' = -g(x) \quad , \qquad (4.25)$$

Where the functions in (4.25) are assumed to be continuous such that uniqueness of solution for initial value problems in guaranteed. If we define as usual

$$G(x) = \int_0^x g(x)ds \ , A(y) = \int_0^y \alpha(r)dr$$

Then we assume that the following conditions hold.

i. $\alpha(0)=0, \alpha(y)$ is strictly increasing and $\alpha(\pm\infty)=\pm\infty$; .

ii. $xg(x)>0$ when $x \neq 0$ and $G(\pm\infty)=\infty$.

iii. $\beta(y)>0$ for $y \in R$ is decreasing function.

iv. There exist constant x_1, x_2 with $x_1 < 0 < x_2$ such that $F(x_1)=F(0)=F(x_2)=0$ and $xF(x)<0$ for $x \in (x_1, x_2)/\{0\}$.

v. Where exist constant $m>0, k_1 k_0$ with $K>k_1$, such that $F(x)>k$ for $x \geq m$ and $F(x)<k_0$ for $x \langle -M$.

vi. One of the following.

1. $G(x_1)=G(x_2)$ or
2. $G(-x) \geq G(x)$ for $x>0$.

Furthermore, we assume that first equation (3.25) define implicitly a function $y=h(x)$ such that $h:(-m,m) \to R$ and

- $m>0$.
- $h(0)=0$.
- $\alpha(h(x)) - \beta(h(x))F(x)=0, x \in (-m,m),$
- Sgn h$(x)=sng\,F(x)$ when $x \neq 0$.

4.4.1:Lemma if there exist some positive constants N and M such that:
$\left|F(x)\right|\leq N, x\forall\in R$ and $\beta(y)\leq M\ \forall\, y\in R$.

Then $h(x)$ is bounded and $m=+\infty$ considering $W(x)=\int_0^{h(x)}\alpha(y)dy$, where h is the above function, we have

1. If $x_2\leq|x_1|$ then $\max\limits_{0\leq x\leq x_2}\{G(x)+w(x)\}\geq G(x_1)$.
2. If $0<|x_1|<x_2$ then $\max\limits_{x_1\leq x\leq 0}\{G(x)+w(x)\}\geq G(x_2)$.

The system(1) is the classical Lienard differential equation
$x''+f(x)x'+x=0$ when $\alpha(y)=y$,$\beta(y)\equiv 1$, $f'(x)=f(x)$ and $g(x)=x$.

The following facts are we know

a. Condition $((i)\rightarrow(1v))$ imply that system – (4.25) has a unique singularity, which will be an unstable focus or node. [1].
b. If v. holds then there exists a closed curve Γ such that every trajectory interesting it crosses it in the exterior-to-interior direction, hence implying the existences of at least one stable limit cycle, by he Poincare.
c. Bendixon theorem [3], [4] ,[5].
d. Condition vi. Assures that all closed trajectory of system (4.25) have intersecting both $x=x_1$ and $x=x_2$ [24] , [25].
e. We proved that under conditions (i-iv) all solution of (4.25) are containable.

Some attempts [46] have been made to find sufficient conditions for existence and uniqueness of limit cycles of some particular cases of system (4.25) under the condition $f(\pm\infty)=\pm\infty$.

In this study we obtain sufficient condition for uniqueness of limit cycles.

of (4.25) without make use of above condition.

These criteria are refinements of early results so we consider that the following condition is added:

$F(x)$ is no decreasing for $x\in(-\infty,x_1)\cup(x_2,\infty)$, (4.26)

If (i-vi) and (4.26) do hold we will give a short proof that system (4.25) has exactly one limit cycle, not by using a comparison method but estimating the divergence of system (4.25) integrated a long a limit cycle.

By this we can show that the limit cycle is hyperbolic.

A limit cycle is hyperbolic or simple.

If for any arbitrarily small analytic perturbation of the system there is not other limit cycle in a sufficiently small neighbor of the limit cycle.

Let X be vectorial field plane and γ a closed trajectory of X with period T.

The number $C(\gamma) = \int_0^T divx(\gamma)dt$

Is called "characteristic exponent of γ .

4.4.2:We consider the Lienard system :

$$x^{\cdot} = y - f(x)\,, \quad y^{\cdot} = -g(x)\,, \qquad (4.27)$$

The above system equivalent form:

$$x^{:} + f(x)x^{\cdot} + g(x) = 0,$$

Where $f(x)$ and $g(x)$, are continuous functions .

This second order if equation includes the famous Van der Pol,s equation

$$\ddot{x} + \mu(x^2 - 1)x^{\cdot} + x = 0 \qquad (4.28)$$

defined the functions

$$F(x) = \int_0^x f(s)ds \quad and \quad G(x) = \int_0^x g(s)ds\,,$$

4.4.3:Theorem Assume $f(x)\ even\,,\ \ g(x)\ odd,\ and\ xg(x) > 0, for\ x \neq 0, If$

i. $\mathrm{f}(0) = 0, F^{\cdot}(0) < 0, F\ has$ single positive zero at $x = 0 = a$

ii. $F\ increases\ for\ x \geq a\ ,\quad as\ x \to \infty,$

iii. $G(x) = F(+\infty) = +\infty.$

Then the system (3.21) has exactly one limit cycle and it is stable.

4.5: We consider the following Lienard system:

$$\dot{x} = y \quad , \quad \dot{y} = -x - (ax^2 - c)y, \qquad (4.29)$$

And apply the theorem **4.4.3:** to prove the existence and uniqueness of limit cycles

In this portion we discuss the various plane portraits that are possible for the linear system $\dot{x} = Ax, \ where \ x \in R^2$, A is a 2×2 matrix and A=$\begin{bmatrix} c & 1 \\ -1 & 0 \end{bmatrix}$

If we let D=det(A) ,T = trac(A) , we get $T^2 < 4D \rightarrow T^2 - 4D = C^2 - 4$, the result from , the linear system in the origin has focus .

If $T^2 - 4D = c^2 - 4 < 0$, also has spiral focus see [46] limit cycle can appear only around focus .

Thus we solely need $c^2 - 4 < 0$,and we consider the case $0 < c < 2,$ othrwise no limit cycle .

The three vocal values are ,

$$W_1 = c \ \ , \quad W_2 = \frac{1}{3}a \ , \quad W_3 = 0.$$

If $c = 0$ the origin is a center weak , and for $a < 0$ the origin is stable focus ,when c increasing from zero has one limit cycle appear .We apply

Theorem4.4 3**,**to the system (4.23) in system (4.23) we see $f(x) and g(x)$ are even and odd respectively .

Its possible we test that $xg(x) = x^2 > 0 \ , since$

$$, 0 < c < 2 \ , \ when \, we \, have \ (x^2 - 4) > 0$$

and we deduced that $xg(x) > 0, for \ x \neq 0 \ \ we \ \ let \ F(x) = \frac{1}{3}x^3 + cx, and \ f(x) < 0,$

Thus for $x_0 = \sqrt{\frac{c}{a}} > 0, we \ have \ F(x) < 0, in \ \left(0, \sqrt{\frac{c}{a}}\right), in \ F(x) > 0x = \pm\sqrt{\frac{c}{a}}.$

$$, inceasing \ in \ \left(\sqrt{\frac{c}{a}}, +\infty\right),$$

Thus two condition holds , also we can check that $F(+\infty) = +\infty, since \ \ xg(x) > 0, for \ x \neq 0, then \ we \ can \ holds \ that$

$$G(+\infty) = +\infty, thus$$

Condition(iii) satisfies.

Hence theorem **4.4.3** has been with satisfy , and system (4.29) has exactly one limit cycle .

4.5.1:Remark for $\mu > 0$, Van der Pol's equation (4.28) has a unique limit cycle and it is stable.

Chapter Five

Limit Cycle Problems for a non-Liénard Systems

This chapter is devoted to the investigation of the qualitative behavior of the solutions of the autonomous system of two differential equations

$$\dot{x} = p_2(y)q_2(x)y, \quad \dot{y} = p_3(y)q_3(x)x + p_4(y)q_4(x)y, \qquad (5.1)$$

where $p_i(y)$ and $q_i(x)$ ($i = 2, 3, 4$) are continuous real functions defined on **R** $= (-\infty, +\infty)$.

Krechetov [8] studied the global asymptotic behavior of solutions of system (5.1), described the configurations of the domains of stability (when there is no global asymptotic stability) and constructed estimates of the boundaries of these domains.

In the study of stability for (5.1), the most important condition given by Krechetov [17] is

$$q_2(x)\, q_4(x) > 0 \text{ for all } x \in \mathbf{R}, \quad (5.2)$$

by using the Lyapunov function method, he gave necessary and sufficient conditions for the zero solution of (5.1) to be globally asymptotically stable under some additional assumptions.

Recently, Yan and Jiang [34] first introduced the transformation techniques to investigate the global asymptotic stability of the following system (5.3),

$$\dot{x} = p_2(y)\, q_2(x)y, \quad \dot{y} = p_3(y)\, q_3(x)x + p_4(y)\, q_4(x)y, \qquad (5.3)$$

With out the assumption (5.2).

Under the following conditions $p_2(y) > 0$, $p_3(y) > 0$ for all y,

$$q_2(x) > 0,\ q_3(x) < 0 \text{ for all } x, \quad (5.4)$$

they transformed system (4.3) into the following generalized Liénard system

$$\dot{x} = \emptyset\big(z - F(x)\big), \qquad \dot{z} = -g(x), \qquad (5.5)$$

and obtained necessary and sufficient conditions for the zero solution of (5.3) (respectively (5.5)) to be globally asymptotically stable.

Such system (5.5) with $\varphi(u) \equiv u$ arises in several different settings, modelling phenomena appearing in the study of physical, as well as biological, chemical, and economical systems, it naturally has been studied by a number of authors [1, 18, 22,41 ,43,46].

The main problem connected to the study of such models consists of giving a complete description of the behavior of solutions as $t \to +\infty$.

In general, this is not possible, due to the complexity of the equations and the phenomena involved.

The aim of the qualitative theory is to give an approximate description of the behavior of the system, by identifying suitable regions of the phase space, where the solutions behave in a similar way.

We shall investigate the qualitative behavior of system (5.1) without the assumption (5.2).

Especially, we shall pay our attention to the oscillation, center, existence and uniqueness of nontrivial periodic solutions of system (5.1) (respectively (5.5)).

no restriction on the sign of q_4 *(x)* is required, we only assume that

$$(y) > 0,\ p_3(y) > 0,\ p_4(y) > 0 \text{ for all } y,$$

$$(x) < 0,\ q_3(x) > 0\ (\text{or } q_2(x) > 0,\ q_3(x) < 0) \text{ for all } x, q_2$$

$$\rho(y) \in C^1(\mathbf{R}),\ \dot{\rho}(y) > 0 \text{ for all } y,\ \rho(\pm\infty) = \pm\infty,$$

$$\text{where} \qquad \rho(y) = \frac{y p_4(y)}{p_3(y)}, \qquad (5.6)$$

.

If $p_3(y) \equiv p_4(y)$, one case of assumption (5.6) reduces to (5.4).

Under assumption (5.6), we shall prove that system (5.1) is equivalent to a form of system (5.5) which is a Liénard like system, the investigation of the qualitative behavior of solutions of system (5.5) has independent interest and value.

For example, applying the results , the following system and equation have a unique nontrivial periodic solution,

$$\begin{cases} \dot{x} = x^3 - 3x^5 + 3x^7 - x^9 + 3(x^6 - 2x^4 + x^2)y + 3(- + x^3)yy^2 + y^3, \\ \dot{y} = -x, \end{cases} \qquad (5.7)$$

And

$$\ddot{x} + 3(x^2 - 1)\dot{x}^{5/3} + 3x\dot{x}^{2/3} = 0. \qquad (5.8)$$

Transformation [1]), and the methods for Liénard systems, especially those developed by Villari and Zanolin [18], Hara and Sugie [40]

We introduce suitable transformations which change (5.1) into the form of (5.5), present assumptions and some auxiliary lemmas which will be essential to our proofs.

We give the necessary and sufficient conditions for all solutions of 5.5) to be oscillatory and for the origin to be a global center.

We give the theorems of existence and uniqueness of nontrivial periodic solutions of (5.5).

5.1:Transformation for the system (5.1) and auxiliary lemmas :

We first transform system (5.1) into a Liénard-like system, and then state some results which will be useful .

We transform system (5.1), suppose that the assumption (5.6) is satisfied, we only discuss

the case $q_2(x) < 0$, $q_3(x) > 0$ for all x, the other case (i.e., $q_2(x) > 0$, $q_3(x) < 0$ for all x) can be considered in a similar way.

By using the substitution $u = \rho(y)$, where $\rho(y)$ is given in (5.6), from (5.1), we have

$$\dot{y} = p_3(y)\, q_3(x)x + p_3(y)\, q_4(x)u,$$

$$\frac{d}{du}[\rho^{-1}(u)]\dot{u} = p_{3(\rho^{-1}(u))} q_3(x)x + p_{3(\rho^{-1}(u))} q_4(x)u,$$

we change system (5.1) into

$$\dot{x} = [\rho^{-1}(u)]p_2[\rho^{-1}(u)]q_2(x), \qquad \dot{u} = \rho'[\rho^{-1}(u)]p_3[\rho^{-1}(u)]q_3(x)x + \rho'[\rho^{-1}(u)]p_3[\rho^{-1}(u)]q_4(x)u\,, \tag{5.9}$$

by assumption (5.6), $\rho^{-1}(u)p_2\left(\rho^{-1}(u)\right)q_2(x)$ and $-u$ have the same sign, it is easy to see that the qualitative behavior of (5.9) is identical to that of the system

$$\dot{x} = -u, \qquad \dot{u} = \frac{\rho'[\rho^{-1}(u)]p_3[\rho^{-1}(u)]q_3(x)}{\rho_1(u)p_2\left(\rho^{-1}(u)\right)q_2(x)}x - \frac{\rho'[\rho^{-1}(u)]p_3[\rho^{-1}(u)]q_4(x)}{\rho_1(u)p_2\left(\rho^{-1}(u)\right)q_2(x)} \tag{5.10}$$

where $\rho_{1(u)} = \frac{\rho^{-1}(u)}{u}$ for $u \neq 0$, $\qquad \rho_{1(0)} = \lim_{u\to 0} \frac{\rho^{-1}(u)}{u}$.

From (5.10), we get

$$\frac{\ddot{x}+\rho^{\prime}\left(\rho^{-1}(-\dot{x})\right)p_3\left(\rho^{-1\beta}(-\dot{x})\right)q_4(x)}{\rho_1(-\dot{x})p_2(\rho^{-1})\left(\rho^{-1}(-\dot{x})\right)q_2(x)}x-\frac{\ddot{x}+\rho^{\prime}\left(\rho^{-1}(-\dot{x})\right)p_3\left(\rho^{-1}(-\dot{x})\right)q_3(x)}{\rho_1(-\dot{x})p_2(\rho^{-1})\left(\rho^{-1}(-\dot{x})\right)q_2(x)}$$

$$= 0 \qquad (5.11)$$

It follows from (5.11) that

$$= 0.\frac{d}{dt}\left[\int_0^x \frac{q_4(s)}{q_2(s)}ds-\int_0^{-\dot{x}}\frac{\rho_1(s)p_{2\left(\rho^{-1}(s)\right)}}{\rho^{\prime}\left(\rho^{-1}(s)\right)p_3\left(\rho^{-1}(s)\right)q_3}ds\right]-\frac{q_3(x)}{q_2(x)}x$$

Letting

$$\psi(y)=\int_0^y \frac{\rho_1(s)p_{2\left(\rho^{-1}(s)\right)}}{\rho^{\prime}\left(\rho^{-1}(s)\right)p_3\left(\rho^{-1}(s)\right)q_3}ds$$

and introducing the substitution

$$z=-\psi(-\dot{x})+\int_0^x \frac{q_4(s)}{q_2(s)}ds$$

We change system (5.11) into

$$\dot{x}=-\psi^{-1}\left(\int_0^x \frac{q_4(s)}{q_2(s)}ds-z\right), \qquad \dot{z}=\frac{q_3(x)}{q_2(x)}x\,, \qquad (5.12)$$

If we let ϕ denote ψ^{-1} and replace *x* and *z* by –*x* and –*z*, respectively, then we obtain

$$\dot{x}=\phi(z-F(x)), \qquad \dot{z}=-g(x), \qquad (5.13)$$

where

$$F(x)=-\int_0^{-x}\frac{q_4(s)}{q_2(s)}ds, \qquad g(x)=-\frac{q_3(-x)}{q_2-(x)}x\,.$$

5.1.1:Lemma

Under the assumption (5.6), the qualitative behavior of (5.1) is the same as that of (5.13).

In the following, we shall present the basic assumptions and auxiliary lemmas.

We assume that

*(C1) F(x)*and *g(x)* are continuous on **R** with *F(0)* = 0 and *xg(x)* > 0 for $x \neq 0$ and $\emptyset(u)$ is continuous differentiable and strictly increasing with

$$\emptyset(0)=0 \text{ and } \emptyset(\pm\infty)=,\pm\infty\,.$$

(C2) For any fixed number *k* > 0, there exists *M(k)* > 0 with *M(k)* ≡ *k* for 0 < *k* ≤1

such that $\emptyset(ku)$ $M(k)$

$$|\emptyset(uk)| \leq M(k)\emptyset(|u|) \text{ for all } u.$$

Sometimes, we only need the condition

For any fixed $k \in (0, 1]$ and $u \in$ **R**,$(C_2{}')$

$|\emptyset(uk)| \leq (k)\emptyset(|u|)$.

5.1.2:Lemma (see [Proposition 1]).

If (C1) is satisfied, then for any initial point $\mathrm{p}(\mathrm{x}_0, \mathrm{z}_0)$, (5.13) has a unique orbit passing through p.

We call the curve *L*: *z* = *F(x)* the characteristic curve of (5.13), we denote

$$L^+ = \{x, F(x): > 0\} \text{ and } L^- = \{x, F(x): < 0\}$$

Let $G(x) = \int_0^x g(s)ds$.

If $x > 0$, then we set $u = u_1(x) = G(x)$, $u \in (0, G(\pm\infty))$, (5.14)

the inverse function of which is denoted by $x = x_1(u)$.

Replacing ($x > 0$) in *F(x)* by *x*

(u), we havex_1

$$F_1(u) = F_{(x_1(u))}, \ (0, G(+\infty)). \qquad (5.15)$$

Similarly, if $x < 0$, then we write

$$u = u_2(x) = G(x), u \in (0, G(-\infty)), \qquad (5.16)$$

whos inverse function is given by $x = x_2(u)$.

Thus, substituting $x = x_2(u)$ in *F(x)* if $x < 0$, we obtain

$$F_2(u) = F(x_2, u), u \in (0, G(-\infty)) \qquad (5.17)$$

Therefore, Eqs. (5.13) in the cases $x > 0$ and $x < 0$ are equivalent to the following two equations, respectively:

$$\frac{du}{dz} = -\emptyset(z - F_1(u)), u \in (0, G(+\infty)), \qquad (5.18)$$

$$\frac{du}{dz} = -\emptyset(z - F_2(u)), u \in (0, G(-\infty)). \qquad (5.19)$$

Now we introduce the condition *(C*3*)*.

The system (5.13) is called to satisfy the condition *(C*3*)* if the following condition hold.

$$F_1(u) \equiv F_2(u) \text{ for } u \in (0, \min\{G(+\infty), G(-\infty)\}),$$

where $F_1((u)$ and $F_2((u)$ are given in (5.18) and (5.19).

If the condition *(C3)* is true, then Eqs. (5.19) and (5.19) are identical in $(0,\min\{G(+\infty),G(-\infty)\})$, employing an argument similar to that in [20,40], we have the following lemma which shows that the orbit of (5.13) have deformed mirror symmetry about the *z*-axis.

5.1.3: Lemma

Suppose that the conditions (C1) and (C3) are satisfied,

$G(+\infty) = G(-\infty)$.

If an orbit of (4.13) starting from A = (0, zA) (zA > 0) passes through a point B = (0, zB) (zB < 0), then it reaches the point A again.

5.2:The oscillation and the global center for system (5.13)

First, we give the result on the oscillation of all solutions for (5.13).

A solution *(x(t), z(t))* of (5.13) is oscillatory if there are two sequences $\{t_n\}$ and $\{\tau_n\}$ tending monotonically to $+\infty$ such that $x(t_n) = 0$ and $z(\tau_n) = 0$ for every $n \geq 1$.

As is usual in the investigation of oscillation properties, by solution, we mean those which are defined in the future. Some attempts have been made to find necessary as well as sufficient conditions on F, φ and g for solutions of (5.13) to be continued in the future [25].

The system (5.13) is said to satisfy $(C_4{}^+)$ if one of the following conditions holds: $(C_4{}^+)1$ there exists a positive decreasing sequence $\{x_n\}$ such that $x_n \to 0$ as $n \to +\infty$ and $F(x_n) > 0$ for each n; $(C_4{}^+)$ 2 there exist constants $a > 0$ and $\beta > \frac{1}{4}$ such that $F(x) > 0$ for $0 < x \leq a$ and

$$\int_0^x \frac{g(x)}{\emptyset(F(s))} ds \geq \beta F(x) \;\; for \; 0 < x \leq a.$$

The system (5.13) is said to satisfy *(C4)* if both ; $(C_4{}^+)$ and ; $(C_4{}^-)$ hold

Comparing with the most results of limit cycle problems are related to the Liédnard type systems, it is interesting to study the non-Liénard type system in the following form:

5.3: the non Liénard perturbation system:

$$\dot{x} = -ax + y \ , \ \ \dot{y} = \frac{x}{1+cx^2} - bx^2y \qquad (5.20)$$

Where $c \in R$, a , b are positive real numbers.

System (5.20) has a unique equilibrium point , and the uniqueness of solutions of initial value problems for the system is guaranteed.

The system (5.20) it has been given the unique equilibrium point for the case $a = 0$, is a global attractor but unstable .

the result that system (5.20) has the special orbit called "a Homoclinic" has been announced by the method of non standard.

Our aim is to classify the orbits of system (5.20) completely by the values of the parameters.

5.3.1:Theorem :

The unique equilibrium point (0, 0) for system (is globally asymptotic stable if and only if one of the following is satisfied.

$$a < 0 \ , \qquad a = 0, \qquad or \ \ a \geq 0$$

We shall see that System (5.23) is transformed to a usual Liénard system (see System (5.25)) with the unique equilibrium point at the origin.

The existence of the homoclinic orbit of the system will be discussed by using the method in [3].

In virtue of this result, the interesting fact that both the limit cycle and the homoclinic orbit of the system cannot coexist is given.

If $a > 0$, are sufficiently small, it has been well-known by E. Benoît (13]) that System has the orbit changes to the homoclinic orbit for the system as $a = 0$.

So the orbit has Homoclinic" ([31]).

The fact that the system has at most one limit cycle will be proved by using the method of [5].

When $0 < a < 1$, the orbit has spirals to a unique limit cycle of the system.

We call the orbit Limit Cycl.

Finally, a phase portrait of System with respect to Theorem **5.4.1** will be presented .

5.3.2: Lemma:

If there exists a constant $m \geq 0$such that $\dot{F}(x)G(x) - \mathrm{mF(x)g(x)} \geq 0 for\ x \neq 0$, the system has at most one limit cycle.

5.4: Transformation to a Liénard System

changed the system (5.23) to the following Liénard system

By using the transformation:

$$\dot{x} \rightarrow z = -ax + y \ , \quad y = z + ax \quad (5.21)$$

$$\dot{z} \rightarrow -a\,\dot{x} + \dot{y} \rightarrow -az + \frac{x}{1+cx^2} - bx^2y \quad (5.22)$$

$$\dot{z} = -az + \frac{x}{1+cx^2} - bx^2z - abx^3 \quad (5.23)$$

Then let $c = 0$

the System (5.23) is transformed to the system the (5.24)

$$\dot{z} = -az + x - bx^2z - abx^3 \quad (5.24)$$

$$\dot{z} = -(a + bx^2)z - (abx^3 - x) \quad (5.25)$$

The system (5.25) has a unique equilibrium point (0, 0) and the uniqueness of solutions of initial value problems is also guaranteed.

The main form of the Liénard System it is:

$$\dot{x} = y - F(x) \ , \dot{y} = -g(x) \quad (5.26)$$

$$f(x) = a + bx^2 \quad \rightarrow \quad F(x) = ax + \frac{1}{3}\mathrm{bx}^3 \quad (5.27)$$

$$-g(x) = -(abx^3 - x)$$

$$g(x) = (abx^3 - x) \rightarrow G(x) = \frac{1}{4}abx^4 - \frac{1}{2}x^2$$

The vocal values of the system (5.27) are

$$W_1 = \mathrm{a} \ , \quad W_2 = 0 \quad , W_3 = -b$$

5.4.1:Theorem :

We shall assume the conditions

i. $a = 0$

Then we can easily check that System (5.25) has at least one limit cycle.

In facts, the unique equilibrium point, O(0, 0) is a unstable weak focus of order one by $f(x) < 0$, and the all orbits are uniformly ultimately bounded (for the details see [20]).

Thus, by the well-known Poincaré-Bendixson theorem, the system has a limit cycle (for instance see [32]).

The following is a useful method ([45]) in order to guarantee that a Liénard system has at most one limit cycle.

ii. $W_3 > 0$, that means, in this case origin O (0,0) is stable and as decreasing from zero became unstable so unique limit cycle appear for Hop Bifurcation

5.4.2: **Proof of Lemma** The system (5.25) has at most one limit cycle.

We have

$$\dot{F}(x)G(x) - \mathrm{mF(x)g(x)} \geq 0$$

$$f(x) = a + bx^2$$

$$F(x) = ax + \frac{1}{3}\mathrm{bx}^3$$

$$g(x) = (abx^3 - x)$$

$$G(x) = \frac{1}{4}abx^4 - \frac{1}{2}x^2$$

$$f(x)\ G(x) = (\ a + bx^2)\left(\frac{1}{4}abx^4 - \frac{1}{2}x^2\right)$$

$$\mathrm{m}F(x)g(x) = m\left[(ax + \frac{1}{3}\mathrm{bx}^3)(abx^3 - x)\right]$$

$\dot{F}(x)G(x) - \mathrm{mF(x)g(x)} \geq 0 for\ x \neq 0$,

$$(\ a + bx^2)\left(\frac{1}{4}abx^4 - \frac{1}{2}x^2\right) - m\left[(ax + \frac{1}{3}\mathrm{bx}^3)(abx^3 - x)\right] \geq 0$$

$$\emptyset = (x, m, a, b) = (\ a + \mathrm{bx}^2)\left(\frac{1}{4}abx^4 - \frac{1}{2}x^2\right) - m\left[(ax + \frac{1}{3}\mathrm{bx}^3)(abx^3 - x)\right]$$

$let\ a = 0\ , m = 3\ , and\ b = -1\ , \quad in\ \emptyset = (x, m, a, b).$

Then we have

$$\emptyset(x, a, m, b) = \frac{1}{2}x^4 - x^4 > 0\ , x \neq 0$$

$$-\frac{1}{2}x^4 > 0 \quad , x^4 > 0\ \rightarrow x \geq 0\ ,$$

Thus, we see from Lemma **5.4.2**: that System (5.25) has at most one limit cycle.

So we conclude that System (5.25) has a unique limit cycle.

Conversely, suppose that System (5.20) has a limit cycle.

Then if System (5.25) doesn't satisfy the condition $-1 < \alpha < 0$, this contradicts to the existence of the limit cycle by Theorem: **5.4.1**.

5.4.3:Remark. If there exists a constant $m \geq 0$ such that:

$$\dot{F}(x)G(x) - \mathrm{mF(x)g(x)} \geq 0$$

the system has mast one limit cycle.

Thus, a unique equilibrium point of System (5.20) unstable weak focus.

5.4.4:Conclusion

We have transformed the non Liénard system to a Liénard system by using method of non standard and theorems, Lemma .

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