Chapter 1

WCG Banach Spaces

We show that if the Banach space is strongly generated by a convex super weak compact set, then there is an equivalent norm on X such that its restriction to any reflexive subspace of X is both uniformly convex and uniformly Frechet smooth.

Section (1.1): Super Weak Compactness

The notion of weakly compactly generated Banach space (WCG) is the first and most remarkable attempt to generalize separable Banach spaces keeping quite a few good structural, geometrical and topological properties. Recall that a Banach space X is WCG if there exists a weakly compact $K \subset X$ such that $\text{span}(K) = X$. The deep impact of WCG spaces in Banach space theory began, and nowadays the amount of material is overwhelming, for an account of properties of WCG Banach spaces in the frame of nonseparable Banach space theory. We are dealing only with real Banach spaces and any operator here is always linear and bounded. As usual, if X is a Banach space, then B_X and S_X denote its unit ball and its unit sphere respectively.

Theorem (1.1.1)[1]. For a Banach space X the following are equivalent:

(a) X is weakly compactly generated;

(b) there exists a weakly compact operator $T: Z \to X$ with dense range;

(c) there exists a reflexive space Z and operator $T: Z \to X$ with dense range.

In other words, WCG is the same as "weakly compact operator generated" or "reflexive generated". Recent results depend on the possibility of changing reflexivity in (c) by a stronger condition, as super reflexivity or Hilbert, leading to the classes of super reflexive-generated Banach spaces and Hilbert-generated Banach spaces. In particular, where several particular classes of "space-generated" properties are involved with smoothness conditions on equivalent renormings. We shall need the following notions.

Definition (1.1.2)[1]. The norm of the Banach space $(X, \|\cdot\|)$is said to be uniformly Gâteaux (UG) smooth if for every $h \in X$

$$\sup\{\|x + th\| + \|x - th\| - 2 : x \in S_X\} = o(t) \text{ when } t \to 0.$$

Given a bounded set $H \subset X$, the norm is said to be $H - UG$ smooth if

$$\sup\{\|x + th\| + \|x - th\| - 2 : x \in S_X, h \in H\} = o(t) \text{ when } t \to 0.$$

Finally, the norm is said to be strongly UG smooth if it is $H - UG$ smooth for some bounded and linearly dense subset $H \subset X$.

The following result showing different classes of space generation and its relationships.

Theorem (1.1.3)[1]. For a Banach space Xconsider the assertions:

(i) X is Hilbert-generated.

(ii) X is super reflexive-generated.

(iii) X is generated by the ℓ_2-sum of super reflexive spaces.

(iv) X admits an equivalent strongly UG smooth norm.

(v) X is WCG and admits an equivalent UG smooth norm.

(vi) X is a subspace of a Hilbert-generated space.

Then (i) ⇒(ii) ⇒(iii) ⇒(iv) ⇒(v) ⇒(vi). Moreover, no one of these implications can be reversed in general.

We study the suitable "ideal-generated" or "subset-generated" version of super reflexive-generated Banach spaces, that is, in the spirit of (b) or (a) from Theorem (1.1.1). For this purpose we shall need the operator version of super reflexivity. Among several equivalent definitions, we shall give one based on ultrapowers. An operator $T: X \to Y$ induces an operator between the ultrapowers of the spaces $T^{\mathcal{U}}: S^{\mathcal{U}} \to Y^{\mathcal{U}}$ for a free ultrafilter $\mathcal{U}$ on an index set.

Definition (1.1.4)[1]. An operator $T: X \to Y$ is said to be super weakly compact if $T^{\mathcal{U}}$ is weakly compact for any ultrafilter $\mathcal{U}$(equivalently, a free ultrafilter on N). The class of all super weakly compact operators will be denoted $\mathfrak{W}^{\text{super}}$.

The class $\mathfrak{W}^{\text{super}}$ is an operator ideal that was first studied, under the name of uniformly convexifying operators, being the link between the alternative definitions a sort of Enflo's renorming theorem for operators. Note that $\mathfrak{W}^{\text{super}}$ lies strictly between the compact and the weakly compact operators, and like them, it is a symmetric ideal as well, that is, $T \in \mathfrak{W}^{\text{super}}$ if and only if $T^* \in \mathfrak{W}^{\text{super}}$.

Clearly, the identity map of a super reflexive Banach space is a natural example of super weakly compact operator. We can state now the following definition.

Definition (1.1.5)[1]. A Banach space X is said to be super weakly compactly generated (super WCG for short) if there exist a Banach space Z and a super weakly compact operator $T: Z \to X$ such that $T(Z)$is dense in X.

We show that an operator $T: Z \to X$ is super weakly compact if and only if $T(B_X)$ is finitely dentable (see Definition (1.1.9)). Moreover, if $K \subset X$ is a finitely dentable bounded closed convex subset, then there exists a reflexive Banach space Z and an operator $T: Z \to X$ such that $K \subset T(B_Z)$. The suggestive notion of super weakly compact set which is, for bounded closed convex sets, equivalent to being finitely dentable. Therefore, The results yield that X is super WCG if and only if there is $K \subset X$ convex super weakly compact such that span $(K) = X$. In other words, in this setting "ideal-generated" and "subset-generated" essentially coincide.

The ideal $\mathfrak{W}^{\text{super}}$ does not have the factorization property, see also Example (1.2.10). In particular, that means that there are super WCG Banach spaces which are not super reflexive-generated. It is natural to wonder how the class super WCG is related to the six classes in Theorem (1.1.3). The answer is the following.

Theorem (1.1.6)[1]. A Banach space X is super WCG if and only if it admits an equivalent strongly UG smooth norm.

Bearing in mind that X admits an equivalent UG smooth norm if and only if B_{X^*} is uniform Eberlein for the weak*, Theorem(1.1.6) improves the previous result where we first dealt with Banach spaces generated by bounded closed convex finitely dentable subsets. A key ingredient for the proof is the symmetry of the ideal $\mathfrak{W}^{\text{super}}$.

A stronger notion of generation for Banach spaces is necessary in order to transfer properties from a super weakly compact generator to all the weakly compact subsets of the space.

Proof. If X is generated by an absolutely convex super weakly compact K, Theorem (1.2.7) implies that X has an equivalent K-UG smooth renorming. Suppose now that X is strongly UG smooth, so there is $K \subset X$ total such that the norm is K-UG smooth. Lemma (1.2.5) implies that we may suppose K to be absolutely convex and closed. K is weakly compact. Use the Davis–Figiel–Johnson–Pelczynski interpolation theorem to find a reflexive Banach space Z and an operator $T: Z \to X$ such that $T(B_Z) \subset 2^n K + 2^{-n} B_X$ for every $n \in \mathbb{N}$. Note that $T(B_Z)$ is strongly generated by K, and so X is $T(B_Z)$-UG smooth by Lemma (1.2.6). Now Lemma (1.2.5) yields that

$$0 = \lim_n p_T(B_Z)(x_n^* - y_n^*) = \lim_n \sup\{\left|(x_n^* - y_n^*)\big(T(z)\big)\right| : z \in B_Z\}$$
$$= \lim_n \sup\{|T^*(x_n^* - y_n^*)(z)| : z \in B_Z\} = \lim_n \|T^*(x_n^*) - T^*(y_n^*)\|$$

whenever $(x_n^*), (y_n^*) \subset S_{X^*}$ are such that $\lim_n \|x_n^* + y_n^*\| = 2$, that is, T^* is uniformly convex. Therefore T is a super weakly compact operator and so X is super WCG.

Definition (1.1.7)[1]. A Banach space X is said to be strongly generated by a subset $K \subset X$ if for any weakly compact $H \subset X\ and\ \varepsilon > 0$ there is $n \in N$ such that $H \subset n K + \varepsilon BX.$

This definition admits “space-generated” and “ideal-generated” variations in a quite obvious way. Banach spaces strongly generated by a weakly compact subset are called strongly WCG Banach spaces, and denoted $SWCG$ (or WCG). Their interesting properties have been studied by G. For instance. if X is $SWCG$ then it is weakly sequentially complete, and so the subspaces of X either contain ℓ are reflexive, by Rosenthal’s theorem. Here we shall consider Banach spaces strongly generated by a convex super weakly compact subset.

Definition (1.1.8)[1]. A Banach space X is said to be strongly super weakly compactly generated (S^2WCG for short) if there is a convex super weakly compact set $K \subset X$ that strongly generates X.

In spite of the length of the name, the notion of S^2WCG has very natural examples. It is well known that $L_1(\mu)$for a finite measure μ is strongly Hilbert-generated and so it is $S^2WCG.$ Moreover, if X is super reflexive then the Lebesgue–Bochner space $L_1(\mu, X)$is strongly super reflexive generated. Indeed, we may assume that μ is a probability. If $H \subset L_1(\mu, X)$ is weakly compact, then it is uniformly integrable, that is, the sequence defined by

$$a_n = sup\{ \int_{\|f\| \geq n} \|f\|\ d\mu : f \in H\}$$

converges to 0. The decomposition $1_{\|f\| \geq n} f + 1_{\|f\| < n} f$ for $f \in H$shows that $H \subset a_n B_{L(\mu,X)} + n B_{L_2(\mu,X)}$ where $L_2(\mu, X)$is identified with a subset of X by its continuous injection into $L_1(\mu, X)$. That finishes the proof since $L_2(\mu, X)$.

Definition (1.1.9)[1]. The subset $C \subset X$ is said to be ρ-finitely dentable if for every $\varepsilon > 0$ there is $n \in N$ such that $[C]_\varepsilon^n = \emptyset$, where the set derivation is made with respect to ρ. If

ρ is the norm metric, then we simply say that C is finitely dentable. The first $n \in N$ such that $[C]_\varepsilon^n = \emptyset$ is called the index of ε-dentability and it is denoted $D_Z\ (C, \varepsilon)$.

X is super reflexive if and only if B_X is finitely dentable. Moreover, if X is uniformly convex then $D_Z(B_X, \varepsilon) \leq 1 + \delta X(\varepsilon)^{-1}$ where δ_X is the modulus of convexity of X. Note that Pisier's celebrated renorming result implies that for a super reflexive space X there exists $c > 0$ and $p \geq 2$ such that $D_Z\ (B_X, \varepsilon) \leq c\ \varepsilon^{-p}$ for every $\varepsilon \in (0, 1]$. Devoted to the study of the properties of finitely dentable sets in Banach spaces. The most relevant properties are that convex finitely dentable sets are weakly compact and uniform Eberlein with respect to the weak topology. Another interesting fact is that they characterize the super weak compactness of operators.

Proposition (1.1.10)[1]. A linear operator $T: X \to Y$ is super weakly compact (equivalently, uniformly convexifying) if and only if $T(B_X)$ is finitely dentable.

Definition (1.1.11)[1]. A subset $K \subset X$ is said to be super weakly compact if $K^{\mathcal{U}}$ is a weakly compact subset of $X^{\mathcal{U}}$ for any free ultrafilter $\mathcal{U}$.

The relation of equivalence here is the same as in the case of Banach spaces, that is, $(X_i)_{i \in I} \sim (y_i)_{i \in I}$ if and only if $\lim_{i,\mathcal{U}} \|x_i - y_i\| = 0$ where $\mathcal{U}$ is a free ultrafilter on a set I. Note that it is enough to consider free ultrafilters on $\mathbb{N}$ since the weak compactness is separably determined. We shall need some assorted definitions. A convex set $C \subset X$ is said to have the finite tree property if there exists $\varepsilon > 0$ such that Ccontains ε-separated dyadic trees of arbitrary height. Recall that a dyadic tree of height $n \in \mathbb{N}$ is a set of the form $\{x_s: |s| \leq n\}$, indexed by finite sequences $s \in \bigcup_{k=0}^n \{0, 1\}^k$ of length $|s| \leq n$, such that $x_s = 2^{-1}(x_{s\frown 0} - x_{s\frown 1})$ for every $|s| < n$, where $\{0, 1\}^0 := \{\emptyset\}$ indexes the root $x_\emptyset$ and "_" denotes concatenation. We say that a dyadic tree $\{x_s: |s| \leq n\}$is ε-separated if $\|x_{s\frown 0} - x_{s\frown 1}\| \geq \varepsilon$for every $|s| < n$. A function $f: C \to \mathbb{R}$ defined on a convex subset $C \subset X$ is said uniformly convex if for every $\varepsilon > 0$ there is $\delta > 0$such that $\|x - y\| < \varepsilon$ when ever $x,\ y \in C$ are such that

$$\frac{f(x) + f(y)}{2} - f\left(\frac{x + y}{2}\right) < \delta$$

The most typical convex function on a Banach space, its norm $\|\cdot\|$, cannot be an uniformly convex function (neither a strictly convex function), so we shall modify the definition for norms. We say that a norm $\|\cdot\|$is uniformly convex on some bounded convex set $K \subset X$ if $f(x) = \|x\|^2$ is a uniformly convex function on K.

Note that a space X is uniformly convex if and only if its norm is uniformly convex (in the previous sense) on B_X, equivalently on any bounded convex subset $K \subset X$.

As we announced, super weak compactness coincide with finite dentability for bounded closed convex subsets of a Banach space. The following result establishes the equivalence between both properties and several others studied.

Proposition (1.1.12)[1]. Let X be a Banach space and $K \subset X$ a bounded closed convex subset. The following conditions are equivalent:

(i) K is super weakly compact;

(ii) K is finitely dentable;

(iii) K does not have the finite tree property;

(iv) there is a reflexive Banach space Z and a super weakly compact operator $T: Z \to X$ such that $K \subset T(B_Z)$;

(v) there is a bounded uniformly convex function $f: K \to R$;

(vi) there is an equivalent norm $|||\cdot|||$on X which is uniformly convex on K.

Proof. The equivalences (ii)⇔(v)⇔(vi) applied to the identity map on K. On the other hand, (iv)⇒(ii) is consequence of Proposition (1.1.10), and (ii)⇒(iv) is contained. Note that if K contains a ε-separated dyadic tree of height n, then $Dz(K, \varepsilon/2) > n,$ following that (ii)⇒(iii). The equivalence (i)⇔(iii) . In order to close the loop, assume (ii). Then $H = K - K$ is finitely dentable (see also Proposition (1.1.15)). Clearly, if the norm$|||\cdot|||$ of X is uniformly convex on H, then $f_{x_0}(x) := |||x - x_0|||^2$ is uniformly convex on K for every $x_0 \in K$, and thus K is uniformly convexifiable. We get that K is super weakly compact and so (ii)⇒(i), which completes the proof

Note that if a bounded closed convex subset $K \subset X$ has the finite tree property, then there is $\varepsilon > 0$ such that $K^{\mathcal{U}}$ contains an infinite ε −separated dyadic tree for any free ultrafilter υ on ℕ, and therefore $K^{\mathcal{U}}$is not weakly compact. That means that super weak compactness for closed convex sets can be checked just by one free ultrafilter on ℕ.

Property (vi) suggests to reproduce arguments involving uniformly convex norms for super weakly compact sets. For the next two examples we shall need a couple of definitions. It is said that a subset K has the Banach–Saks property if every sequence $(x_n) \subset K$ has a subsequence (x_{n_k})such that its Cesàro averages $k^{-1}\sum_{j=1}^{k} x_{k_j}$ are norm converging to some point of X. A bounded convex set $K \subset X$ is said to have normal structure if every nonsingleton convex subset $H \subset K$ has a nondiametral point $x \in H$, that is, sup$\{\| y - x \| : y \in H\} <$ diam$(H)\}$.

Proposition (1.1.13)[1]. Convex super weakly compact sets of Banach spaces have the Banach–Saks property

Proof. It is possible to adapt the proof of Kakutani's theorem as presented, but it is easier to use that a super weakly compact operator has the Banach–Saks property.

Proposition (1.1.14)[1]. If $K \subset X$ is a convex super weakly compact set, then there is a renorming of X such that K has normal structure.

Proof. By Proposition (1.1.15) the set $H = K - K$ is a convex super weakly compact set. By Proposition (1.1.12) (vi) there is a norm$\|\cdot\|$of X which is uniformly convex on H. Let $S \subset K$ be a convex subset containing at least two different points $u, v \in S$. Take

$d = diam(S)$ and for $\varepsilon = \|u - v\|$ find $\delta > 0$ witnessing the uniform convexity of $\|\cdot\|^2$ on H. Put $x = (u + v)/2$ and observe that for any $z \in S$ we have

$$\delta \le \frac{\|u - z\|^2 + \|v - z\|^2}{2} - \|x - z\|^2 \le d^2 - \|x - z\|^2$$

since $u-z, v-z, x-z \in H$ and the uniform convexity of $\|\cdot\|^2$. Therefore $\|x-z\| \leq \sqrt{d^2-\delta} < d$ for every $z \in S$ and thus x is a nondiametral point of S.

Note that the normal structure implies the fixed point property for nonexpansive mappings.

Observe that the Proposition (1.1.12) requires the hypothesis of convexity, and implicitly. The results Theorem (1.1.6) and Theorem (1.2.8) too. Actually, we do not know if the closed convex hull of a super weakly compact is again super weakly compact, that is, a sort of Krein's theorem. We know that the answer is negative for finite dentability and it is also negative for the somehow related property.

The estimations of the dentability indices are implicit in the proofs.

Proposition (1.1.15)[1]. "Stability properties of convex super weakly compact sets".

(i) A closed convex subset $H \subset K$ of a convex super weakly compact is again super weakly compact. Moreover, $Dz(H, \varepsilon) \leq Dz(K, \varepsilon)$.

(ii) The image of a convex super weakly compact K set through an operator T is again super weakly compact. Moreover, $Dz(T(K), \varepsilon) \leq Dz(K, \varepsilon/2\|T\|)$.

(iii) The product of convex super weakly compact sets in a finite direct sum of Banach spaces is super weakly compact.

(iv) The sum and the convex hull of two convex super weakly compact sets areagain super weakly compact. In particular, the absolute convex hull of a convex super weakly compact is super weakly compact.

(v) Let $K \subset X$ be such that for every $\varepsilon > 0$ there is a convex super weakly compact set H_ε such that $K \subset H_\varepsilon + \varepsilon B_X$. Then Kis super weakly compact.

Section (1.2): Renormings in Super WCG Spaces

The most remarkable result in renorming of WCG spaces that ensures the existence of equivalent locally uniformly convex norms. As super weakly compact sets are exactly the sets supporting uniformly convex functions, we may expect that renormings for super WCG should be "more uniform". Actually, the uniform convexity of the norm given by Proposition (1.1.12) (vi) only extends to certain family of weakly compact sets which satisfy a local version of Definition (1.1.7).

Definition (1.2.1)[1]. Let $K, H \subset X$ be subsets and suppose moreover that K is absolutely convex. The set H is said to be strongly generated by K if for every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $H \subset n_K + \varepsilon B_X$.

This definition is necessary where the strongly generated subsets are known. For instance, consider a SWCG Banach X space and a probability measure space (Ω, Σ, μ). Says that there exists a symmetric weakly compact $K \subset L_1(\mu, X)$that strongly generates any weakly compact decomposable setof $L_1(\mu, X)$. A set $H \subset L_1(\mu, X)$ is called decomposable if $\mathbf{1}_A f + \mathbf{1}_{\Omega \setminus A} g \in H$ whenever $f, g \in H$ and $A \in \Sigma$.

We shall begin with an improvement of statement (vi) of Proposition (1.1.12).

Theorem (1.2.2)[1]. Let $K \subset X$ be an absolutely convex super weakly compact. There is an equivalent norm $|||\cdot|||$ on X such that its restriction to convex sets strongly generated by K is uniformly convex.

Proof. Without loss of generality we may assume that $K = T(B_Z)$ where $T: Z \to X$ is an uniformly convex operator and Z is reflexive by Proposition (1.1.12) (iv). Consider the sequence of equivalent norms on X

$$\|x\|_k^2 = \inf\{\|x - T(z)\|^2 + k^{-1}\|z\|^2 : z \in Z\}.$$

Note that the infimum is actually attained since Z is reflexive. Fix H a subset strongly generated by K. Note that $\lim_k \|x\|_k = 0$ uniformly on H. We claim that the norm $|||\cdot|||$on X defined by $|||\cdot|||^2 = \sum_{k-1}^{\infty} 2^{-k} \|\cdot\|_k^2$ has the desired property. Fix $\varepsilon > 0$ and suppose that $(x_n), (y_n) \subset H$ are such that

$$\lim_n (2|||x_n|||^2 + 2|||y_n|||^2 - |||x_n + y_n|||^2) = 0.$$

An standard convexity argument yields that

$$\lim_n (2\|x_n\|_k^2 + 2\|y_n\|_k^2 - \|x_n + y_n\|_k^2) = 0$$

for any $k \in \mathbb{N}$. Fix $k \in \mathbb{N}$ such that $\|x\|_k < \varepsilon$ for every $x \in H$ and find $(z_n), (w_n) \subset Z$ such that

$$\|x_n\|_k^2 = \|x_n - T(z_n)\|^2 + k^{-1}\|z_n\|^2, \|y_n\|_k^2 = \|y_n - T(w_n)\|^2 + k^{-1}\|w_n\|^2.$$

Note that the sequences $(z_n), (w_n)$ are bounded. For the sum of the points we have

$$\|x_n + y_n\|_k^2 \le \|x_n + y_n - T(z_n + w_n)\|^2 + k^{-1}\|z_n + w_n\|^2$$

and so, using the convexity of the squared norm, we obtain that

$$k^{-1}(2\|z_n\|^2 + 2\|w_n\|^2 - \|z_n + w_n\|^2) \le 2\|x_n\|_k^2 + 2\|y_n\|_k^2 - \|x_n + y_n\|_k^2$$

Therefore

$$\lim_n (2\|z_n\|^2 + 2\|w_n\|^2 - \|z_n + w_n\|^2) = 0$$

which implies that $\lim_n \|T(z_n) - T(w_n)\| = 0$ by the uniform convexity of T. Take an $N \in \mathbb{N}$ such that $\|T(z_n) - T(w_n)\| < \varepsilon$ if $n \ge N$. Then we have

$$\|x_n - y_n\| \le \|x_n - T(z_n)\| + \|T(z_n) - T(w_n)\| + \|y_n - T(w_n)\| < 3\varepsilon$$

for $n \ge N$. That shows $\lim_n \|x_n - y_n\| = 0$ as we wanted.

Proposition (1.2.3)[1]. Let X be a dual Banach space generated by a super weakly compact convex set K. There is an equivalent dual norm $|||\cdot|||$ on X such that its restriction to convex sets strongly generated by K is uniformly convex.

Proof. Let $\|\cdot\|$ be a dual norm on X. We construct $|||\cdot|||$ as in Theorem (1.2.2). We only need to check that it is w^*-lower semicontinuous which is easy using the fact that the infimum in the definition of $\|\cdot\|_k$ is attained.

This is another observation about dual renormings and, actually.

Lemma (1.2.4)[1]. Suppose that X is a dual Banach space and $T: X \to Y$ is super weakly compact and $w^* - w$-continuous. Then there is an equivalent dual norm on X such that T becomes uniformly convex.

Proof. Suppose that X is endowed with a (nondual) norm such that T is uniformly convex. We claim that the norm $|||.|||$ on X having $\overline{B_X}^{w^*}$ as the unit ball makes T also uniformly convex. Given $\varepsilon > 0$ there is $\delta > 0$ such that $x,\ y \in B_X$ and $\left\|\frac{x+y}{2}\right\| > 1 - \delta$ implies $\|T(x) - T(y)\| < \varepsilon$. As a consequence, if H is a half space such that $H \cap (1-\delta)B_X = \emptyset$ then diam $(T(H \cap B_X)) \le \varepsilon$. Take $x, y \in X$ with $|||x||| = |||y||| = 1$ and $|||x + y||| > 2 - 2\delta$, that is, $x, y \in \overline{B_X^{w^*}}$ and $\frac{x+y}{2} \notin (1-\delta)\overline{B_X^{w^*}}$. Take H a w^*-open halfspace with $\frac{x+y}{2} \in H$ and $H \cap (1-\delta)\overline{B_X^{w^*}} = \emptyset$. Observe that $\|x - y\| \le 2$ diam$(H \cap \overline{B_X^{w^*}})$. Now, by the $w^* - w$-continuity of T we have

$$T(H \cap \overline{B_X^{w^*}}) \subset \overline{T(H \cap BX)}^{w} = \overline{T(H \cap BX)}.$$

As diam$\overline{(T(H \cap B_X}))$ =diam$(T(H \cap B_X)) \le \varepsilon$, we get that $\|T(x) - T(y)\| \le 2\varepsilon$ and so the uniform convexity of T with respect to $|||.|||$.

Given $H \subset X$, the seminorm on X^* of uniform convergence on H is denoted p_H, that is, $p_H(x^*) = \sup\{|x^*(x)| : x \in H\}$. The following Šmulyian's criterion for H-UG.

Lemma (1.2.5)[1]. Let X be a Banach space and $H \subset X$ a bounded subset. The norm on X is H-UG smooth if and only if $p_H(x_n^* - y_n^*) = 0$ whenever $(x_n^*), (y_n^*) \subset S_{X^*}$ are such that $\lim_n \|x_n^* + y_n^*\| = 2$.

Lemma (1.2.6)[1]. Suppose that X is K-UG smooth and H is strongly generated by K, then X is H-UG smooth as well.

Proof. Let $(x_n^*), (y_n^*) \subset S_{X^*}$ such that $\lim_n \|x_n^* + y_n^*\| = 2$. Fix $\varepsilon > 0$ and find $m \in \mathbb{N}$ such that $H \subset mK + \varepsilon B_X$. By Lemma (1.2.5), take $N \in \mathbb{N}$ such that $p_K(x_n^* - y_n^*) < \varepsilon/m$ for $n \ge N$. It is easy to see that $p_H(x_n^* - y_n^*) < 3\varepsilon$ for $n \ge N$, and thus the norm of X is H-UG smooth, again by Lemma (1.2.5).

Let us recall that it is a natural consequence of the symmetry of $\mathfrak{W}^{super}$.

Theorem (1.2.7)[1]. Let $K \subset X$ be an absolutely convex super weakly compact set. There is an equivalent norm $\|\cdot\|$ on X such that it is H-UG smooth for any $H \subset X$ bounded and strongly generated by K.

Proof. Take $T : Z \to X$ a super weakly compact operator such that $K \subset T(B_Z)$ where Z is reflexive (see Proposition (1.1.12) (iv)). Then the adjoint $T^* : X^* \to Z^*$ is super weakly compact as well. By Lemma (1.2.4) we may renorm X^* with a dual norm $\|\cdot\|$ such that T^* is uniformly convex. Moreover, we may assume that X is endowed with the induced predual norm. We claim that this norm is K-UG smooth. Indeed, applying Lemma (1.2.5), take $(x_n^*), (y_n^*) \subset S_{X^*}$ such that $\lim_n \|x_n^* + y_n^*\| = 2$. Since T^* is uniformly convex, we have

$$\begin{aligned} 0 &= \lim_n \|T^*(x_n^*) - T^*(y_n^*)\| = \lim_n \sup\{|T^*(x_n^* - y_n^*)(z)| : z \in B_Z\} \\ &= \lim_n \sup\{\left|(x_n^* - y_n^*)\big(T(z)\big)\right| : z \in B_Z\} \ge \lim_n p_K(x_n^* - y_n^*). \end{aligned}$$

Therefore, $\lim_{n} p_K(x_n^* - y_n^*) = 0$ and the norm on X is K-UG smooth as desired. Now, by Lemma (1.2.6) the norm $\|\cdot\|$ built on X is H-UG smooth for every $H \subset X$ strongly generated by K.

Theorem (1.2.8)[1]. Let X be a S^2WCG Banach space. Then there is an equivalent norm on X such that its restriction to any reflexive subspace of X is both uniformly convex and uniformly Fréchet smooth.

This result extends qualitatively renorming results done for the spaces $L_1(\mu)$ and $L_1(\mu, X)$with X super reflexive. Note that it is established that for a strongly super reflexive generated Banach space there is a renorming which is uniformly Fréchet smooth on its reflexive subspaces. Example (1.2.9) shows that the class of S^2WCG Banach spaces is strictly larger than the class of strongly super reflexive generated Banach spaces.

The structure is a survey on super weak compactness which includes the main equivalences in the convex case and an account of the properties which are relevant for the results. We also describe the relationships with the uniformly convexifying operators of Beauzamy which are extremely useful for the proofs. We devoted to super WCG Banach spaces and their renormings, including the proof of the main results and two examples.

The interesting example of a super property is the super reflexivity, introduced. A Banach space X is super reflexive if any ultrapower $X^{\mathcal{U}}$ is reflexive for $\mathcal{U}$ any free ultrafilter. States that X is super reflexive if and only if it has an equivalent uniformly convex renorming. Extended the notion of super reflexivity to operators. An operator $T: Z \to X$ is uniformly convexif given $\varepsilon > 0$ there is $\delta > 0$ such that $\|T(x) - T(y)\| < \varepsilon$ when ever $\|x\| = \|y\| = 1$ and $\|x + y\| > 2 - \delta$. An operator $T: Z \to X$ is said uniformly convexifying if it becomes uniformly convex after a suitable renorming of Z. Of course, uniformly convexifying operators coincide with the super weakly compact operators.

A localized version of super reflexivity for subsets was introduced. Let C be a bounded closed convex set of a Banach space X and let ρ be a metric defined on X (the norm metric, for instance). We say that C is ρ-dentable if for any nonempty closed convex subset $D \subset C$ and $\varepsilon > 0$ it is possible to find an open half space H intersecting D such that ρ-diam$(D \cap H) < \varepsilon$. If C is ρ-dentable we may consider the following "derivation"

$$[D]'_\varepsilon = \{x \in D : \rho - diam(D \cap H) > \varepsilon, \forall H \in \mathbb{H}, x \in H\}.$$

Here H denotes the set of all the open half spaces of X. Clearly, $[D]'_\varepsilon$ is what remains of D after removing all the slices of ρ-diameter less or equal than ε. Consider the sequence of sets defined by $[C]_0^\varepsilon = C$ and, for every $n \in \mathbb{N}$, inductively by

$$[C]_n^\varepsilon = [[C]_\varepsilon^{n-1}]'_\varepsilon$$

Such a process can be extended to transfinite ordinal numbers in a quite natural way, and for any dentable set the process finishes at the empty set. However, we are only interested in sets for which the iteration process is finite.

Proof. By Theorem (1.2.2) we know that there is an equivalent norm $\|\cdot\|_1$ on X such that its restriction to any reflexive subspace of X is uniformly convex. On the other hand, by Theorem (1.2.7) there is an equivalent norm $\|\cdot\|_2$ on X such that given a reflexive subspace $Y \subset X$, then $\|\cdot\|_2$ is B_Y -UG smooth. In particular, the restriction of $\|\cdot\|_2$ to Y is uniformly

Fréchet smooth. Our aim is to show that the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ can be "averaged". Let P denote the set of equivalent norms on X endowed with the distance $\rho(p,q) = \sup\{|p(x) - q(x)| : \|x\| = 1\}$. The metric space (P,ρ) is a Baire space. We claim that the set of norms sharing the property of $\|\cdot\|_1$ is a residual subset of (P,ρ), that is, it contains a dense $\mathcal{G}_\delta$-set. For any $p \in P$ consider the set

$$G(p,j) = \{q \in P : \sup\{|p(x)^2 + j^{-1}\|x\|_1^2 - q(x)^2| : \|x\| = 1\} < j^{-2}\}.$$

By construction $G(p,j)$ is open in (P,ρ) and $G_k = \bigcup_{p\in P}\bigcup_{j\geq k} G(p,k)$ is dense. We will show that any $q \in \bigcap_{k=1}^{\infty} G_k$ is uniformly convex restricted to any $Y \subset X$ reflexive. Suppose that (x_n),$(y_n) \subset B_Y$ are such that

$$\lim_n (2q(x_n)^2 + 2q(y_n)^2 - q(x_n + y_n)^2) = 0.$$

Given $k \in \mathbb{N}$ then $q \in G(p,j)$ for some $p \in P$ and some $j \geq k$. Using convexity we deduce that

$$j^{-1}(2\|x_n\|_1^2 + 2\|y_n\|_1^2 - \|x_n + y_n\|_1^2) < 8j^{-2} + 2q(x_n)^2 + 2q(y_n)^2 - q(x_n + y_n)^2.$$

Taking limits as $n \to \infty$ we have

$$j^{-1} \limsup_n (2\|x_n\|_1^2 + 2\|y_n\|_1^2 - \|x_n + y_n\|_1^2) \leq 8j^{-2}.$$

That is, $\limsup_n (2\|x_n\|_1^2 + 2\|y_n\|_1^2 - \|x_n + y_n\|_1^2) \leq 8j^{-2} \leq 8k^{-1}$. Since $k \in \mathbb{N}$ was arbitrary, we have $\lim_n (2\|x_n\|_1^2 + 2\|y_n\|_1^2 - \|x_n + y_n\|_1^2) = 0$, and the uniform convexity of $\|\cdot\|_1$ implies that $\lim_n \|x_n - y_n\| = 0$. Therefore q is uniformly convex on Y as desired.

In order to show that the set of norms sharing the property of $\|\cdot\|_2$ is a residual subset of (P,ρ) too it is enough to work on the set of equivalent dual norms on X^* because the duality map is a homeomorphism.

By Lemma (1.2.5) it is enough to show that there is a residual set of equivalent dual norms $|||\cdot|||$ on X^* such that $\lim_n pB_Y(x_n^* - y_n^*) = 0$ whenever $Y \subset X$ is reflexive and $(x_n^*), (y_n^*) \subset B_{X^*}$ are such that

$$\lim_n (2|||x_n^*|||^2 + 2|||y_n^*|||^2 - |||x_n^* + y_n^*|||^2) = 0.$$

It is obvious that the same argument as the one used before for $\|\cdot\|_1$ will give the desired result. Now, the intersection of two residual sets in the Baire space (P,ρ) is nonempty, thus there exist norms sharing the properties of $\|\cdot\|_1$ and $\|\cdot\|_2$.

The following example shows that there exist S^2WCG Banach spaces which are not super reflexive generated. Note that such spaces cannot be reflexive because a reflexive S^2WCG Banach space is super reflexive, and they must be nonseparable since separable Banach spaces are Hilbert generated.

Example (1.2.9)[1]. Let (p_k) be an enumeration of $(1, 2] \cap \mathbb{Q}$. Then the space

$$X = \left(\sum_{k=1}^{\infty} \ell_{p_k}(\omega_1) \right)_1$$

is S^2WCG, but X is not super reflexive generated.

Proof. We claim that $K = \prod_{k=1}^{\infty} 2^{-k} B_{\ell_{p_k}(\omega_1)}$ is super weakly compact. Indeed, observe that $K \subset \prod_{k=1}^{n} 2^{-k} B_{\ell_{p_k}(\omega_1)} + 2^{-n} B_X$ for every $n \in \mathbb{N}$. Since $\prod_{k=1}^{n} 2^{-k} B_{\ell_{p_k}(\omega_1)}$ is a finite sum of convex super weakly compact subsets it is again super weakly compact by Proposition (1.1.15). Now Proposition (1.1.15) implies that K is super weakly compact.

Let $H \subset X$ be weakly compact, and let π_k be the projection on the k-th summand of X. We claim that for every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that

$$\sup\left\{ \sum_{k=n+1}^{\infty} \|\pi_k(x)\| : x \in H \right\} \leq \varepsilon. \qquad (1)$$

Indeed, assume that for some $\varepsilon > 0$ the property does not hold. Then we can find $x_1 \in H, n_1 \in \mathbb{N}$ and $w_k \in p_k(\omega_1)^*$ with $\|w_k\| = 1$ for $k \leq n_1$ such that

$$\sum_{k=1}^{n_1} w_k\big(\pi_k(x_1)\big) > \varepsilon \quad \text{and} \quad \sum_{k=n_1+1}^{\infty} \|\pi_k(x_1)\| < \varepsilon/2.$$

Find now $x_2 \in H, n_2 > n_1$ and $w_k \in p_k(\omega_1)^*$ with $\|w_k\| = 1$ for $n_1 < k \leq n_2$ such that

$$\sum_{k=n_1+1}^{n_2} w_k\big(\pi_k(x_2)\big) > \varepsilon \quad \text{and} \quad \sum_{k=n_2+1}^{\infty} \|\pi_k(x_2)\| < \varepsilon/2.$$

Repeating inductively this argument we get sequences $(x_k) \subset H, (n_k) \subset N$ and norm one functionals $w_k \in \ell_{p_k}(\omega_1)^*$ satisfying analogous estimations. Consider the operator $T : X \to \ell_1$ defined by $T(x) = \big(w_k(\pi_k(x))\big)_{k=1}^{\infty}$. Since ℓ_1 has the Schur property, we have that T(H) is a norm compact subset of ℓ_1. On the other hand, by the previous construction we have $\|T(x_k) - T(x_j)\| > \varepsilon/2$ for $k \neq j$, and thus $T(H)$ cannot be norm compact. This contradiction proves the claim.

Now we are ready to show that K strongly generates X. Let $H \subset X$ be a weakly compact subset and $\varepsilon > 0$. Take $n \in \mathbb{N}$ such that inequality is satisfied. If $m > 0$ is such that $\pi_k(H) \subset m2^{-k} B_{\ell_{p_k}}(\omega_1)$ for every $k \leq n$, then $H \subset mK + \varepsilon B_X$.

In order to prove the second statement, consider the space

$$Y = \left(\sum_{k=1}^{\infty} \ell_{p_k}(\omega_1) \right)_2 .$$

The identity map $J : X \to Y$ is an operator with dense range. If X were super reflexive generated, then Y would be super reflexive generated too. But Y is not super reflexive generated since this space is the example showing the nonreversibility of (ii)⇒(iii) in Theorem (1.1.3).

The construction given in Example (1.2.9) easily implies that $\mathfrak{W}^{\text{super}}$ does not have the factorization property, but it has the disadvantage of being nonseparable. The following is an example of separable convex super weakly compact set that cannot be interpolated by a super reflexive space.

Example (1.2.10)[1]. Consider $X = (\sum_{k=2}^{\infty} \ell_k)_2$ and $= \prod_{k=2}^{\infty} 2^{-k} B_{\ell_k}$. Then X is reflexive and separable, K is a super weakly compact set that generates X and

$$\sup\{\varepsilon^p Dz(K,\varepsilon) : \varepsilon \in (0,1)\} = +\infty$$

for every $p > 1$. In particular, we have $K \not\subset T(B_Z)$ for any super reflexive space Z and any operator $T : Z \to X$.

Proof. Some of the statements can be easily checked and the super weak compactness of K follows by the same proof as in the previous example. Only the estimation of the growth of $Dz(K,\varepsilon)$ needs a proof. Fix $k \in \mathbb{N}$ and take $\varepsilon \in (0, 2^{-k})$. A simple homogeneity argument gives that

$$Dz(K,\varepsilon) \geq Dz(B_{\ell_k}, 2^k\varepsilon) \geq (2^k\varepsilon)^{-k} = 2^{-k^2}\varepsilon^{-k}$$

where we are using that $Dz(B_{\ell_p}, \varepsilon) \geq \varepsilon^{-p}$. Such an estimation is obtained as follows. Start a 2ε-separated dyadic tree in ℓ_p with root 0. Set the first level as $(\varepsilon, 0, 0, \dots)$ and $(-\varepsilon, 0, 0, \dots)$, the second level as $(\varepsilon, \varepsilon, 0, \dots)$, $(\varepsilon, -\varepsilon, 0, \dots)$, $(-\varepsilon, \varepsilon, 0, \dots)$, $(-\varepsilon, -\varepsilon, 0, \dots)$, and so on until the n-th level. If $n\varepsilon^p \leq 1$, then that tree is contained in the unit ball and, in such a case, $Dz(B_{\ell_p}, \varepsilon) > n$. Taking n as the integer part of ε^{-p} we get the desired bound.

We finish with a reflection on an alternative meaning for the sentence "super WCG": What are the Banach spaces X such that their ultraproducts $X^{\mathcal{U}}$ are WCG? Such class of Banach spaces might be very restrictive as the next partial result hints.

Proposition (1.2.11)[1]. Let X be a Banach space, let K be a convex weakly compact set and let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Assume that $K^{\mathcal{U}}$ is weakly compact and generates $X^{\mathcal{U}}$, then X is super reflexive.

Proof. First note that K is a super weakly compact and by Proposition (1.1.15) we may assume that K is absolutely convex taking $\mathrm{con}v(K \cup (-K))$. We claim that B_X is strongly generated by K. Assume the contrary, so there is $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ we can find $x_n \in B_X \setminus (nK + \varepsilon B_X)$. By construction, the element $(x_n) \in X^{\mathcal{U}}$ satisfies $\|(x_n) - (y_n)\| \geq \varepsilon$ for every $(y_n) \in \bigcup_{m=1}^{\infty} mK^{\mathcal{U}}$ which is a contradiction. Now B_X is weakly compact since it is strongly generated by a weak compact. Moreover B_X is super weakly compact by Proposition (1.1.15), and thus X is super reflexive.

Definition (1.2.12)[5]. A Banach space X is WCG if X admits $M.2R$ norm for som bounded

Definition (1.2.13)[6]. A Banach space X is said to be super. Reflexive if every Banach space Y which an be finitely repre sentable in X refixive.

Chapter 2

p-Compact Sets of p-Compact Linear Operators

After obtaining a factorization of p-compact linear operators via universal Banach spaces, and using the lifting property of quotient maps for p-compact sets we prove a factorization result for relatively r-compact subsets of p-compact operators, where $r \geq 2, 1 \leq p \leq r < \infty$. To apply our results to homogeneous polynomials.

Section (2.1): Preliminaries

W.B. Johnson proved that an operator in the closure of finite rank operators can be factorized through a universal Banach space. T. Figiel proved that compact operators can be factorized through a closed subspace of Johnson's universal Banach space. D.J. Randtke, T. Terzioglu, and J. Dazord factorized compact operators defined on some certain Banach spaces, such as $\mathcal{L}_1, \mathcal{L}_\infty$. Then W. H. Graves and W. M. Ruess, extended these works to simultaneous factorization of operators belonging to compact subsets of compact operators. But the (uniform) factorization of compact subsets of compact operators on arbitrary Banach spaces was studied by R. Aron, M. Lindstrom, W.M. Ruess, R. Ryan. Showing that the universal Banach space established by W.B. Johnson and T. Figiel also serves as a uniform factorization space for factorization of operators belonging to the space of compact weak*-weak continuous operators, they obtain a factorization of relative compact subsets of compact operators defined on an arbitrary Banach space.

As a stronger form of compactness D.P. Sinha and A.K. Karn introdueced pcompactness notion, which was motivated by the well-known characterization of compact sets duee to A. Grothendieck.

We study simultaneous factorization of operators belonging to a p-compact subset of p-compact operators, basing on R. Aron, M. Lindstr¨om, W.M. Ruess, R. Ryan. We firstly consider factorization of p-compact operators via universal Banach spaces, then we study factorization of relatively r-compact subsets of the Banach space of all p-compact operators. We get a factorization of p-compact operators through a universal Banach space. We study uniform factorization of relatively r-compact subsets of p-compact operators.

The characterization of relatively p-compact sets in the projective tensor produect of Banach spaces, strengthen a result given, and then making a careful modification with quantitative strengthening of a method given and show that every p-compact operator in certain relatively r-compact subsets of the Banach space of p-compact operators with $r \geq 2$ and $1 \leq p \leq r < \infty$, can be factorized simultaneously through a universal Banach space. It should be pointed out that we do not use any selection principal in our proof, rather we use the lifting property of quotient maps for p-compact sets. Finally, we prove partial p-compact versions of a result of E. Toma for homogeneous polynomials. We show that for any $p \geq 1$ every relatively p-compact subset of a Banach space of p-compact operators is collectively p-compact.

The letters X and Y will always represent complex Banach spaces. The symbol B_X represents the closed unit ball of X, S_X represents the unit sphere of X. For any topology τ on X, $\bar{M}^\tau$ will denote the τ -closure of a set M in X. The space of all bounded linear operators from X to Y will be denoted by $L(X,Y)$. The subspace of all compact (respectively, finite rank) operators of $L(X,Y)$ is denoted by $K(X,Y)$ (respectively, $F(X,Y)$). If X is a Banach space, and $1 \leq p \leq \infty$ with the conjugate index p^* given by $\frac{1}{p} + \frac{1}{p^*} = 1$ (where $p^* = 1$ if $p = \infty$), we let $\ell_p(X)$ $(1 \leq p < \infty)$ $\left(\text{resp.}, \ell_\infty (X)\right)$ denote the set of all sequences $(x_n)_{n=1}^\infty$ in X such that $\sum_{n=1}^\infty \|x_n\|^p < \infty$ (resp., $(x_n)_{n=1}^\infty$ is bounded), and we let $c_0(X)$ denote the set of all sequences $(x_n)_{n=1}^\infty$ in X such that $x_n \to 0$ in X. A set $K \subset X$ is said to be relatively p-compact if there is a sequence $(x_n)_{n=1}^\infty$ in $\ell_p(X)$ $((x_n)_{n=1}^\infty$ in $c_0(X) \subset \ell_\infty(X)$ if $p = \infty)$ such that $K \subset \left\{\sum_{n=1}^\infty a_{nx_n}: (a_n)_{n=1}^\infty \in B_{\ell_{p^*}}\right\}$. A relatively p-compact and closed set will be called p-compact. We denote this last set by $p - co\{(x_n)_{n=1}^\infty\}$ and we will call it a fundamental p-compact set since these sets form a fundamental system of pcompact sets of X. From Grothendieck's characterization of compact sets, a subset K of a Banach space X is relatively compact if and only if there is a sequence $(x_n)_{n=1}^\infty$ in $c_0(X)$ such that $K \subset \left\{\sum_{n=1}^\infty a_n x_n : (a_n)_{n=1}^\infty \in B_{\ell_1}\right\}$.

Thus, by the above definition one can consider compact sets as ∞-compact. Also, note that p-compact sets are q-compact if $1 \leq p < \infty$, For $1 \leq p \leq \infty$ an operator $T \in L(X,Y)$ is said to be p-compact if $T(B_X)$ is a relatively p-compact set in Y. The subspace of all p-compact operators of $L(X,Y)$ will be denoted by $K_p(X,Y)$.

If T belongs to $K_p(X,Y)$, we define

$$k_p(T) = \inf\left\{\|(y_n)_{n=1}^\infty\|_p : (y_n)_{n=1}^\infty \in l_p(Y) \ and \ T(B_X) \subset p - co\{(y_n)_{n=1}^\infty\}\right\}$$

where k_∞ coincides with the supremum norm. It is easy to see that k_p is a norm on $K_p(X,Y)$, and that $\left(K_p, k_p\right)$ is a Banach ideal.

We recall that a mapping $P: X \to Y$ is an n-homogeneous polynomial if there

is an n-linear mapping $A: X \times \ldots \times X \to Y$ such that $P(x) = A(x, \ldots, x)$ for all $x \in X$. Let $P(^nX,Y)$ denote the space of all continuous n-homogeneous polynomials from X to Y, endowed with the usual sup norm. Given a polynomial $P \in P(^nX,Y)$, there is a unique symmetric continuous n-linear mapping $\check{P} \in L(^nX,Y)$ such that $P(x) = \check{P}\,(\underbrace{x, \ldots, x}_{n\ times})$. It is well known that the correspondence $\check{P} \leftrightarrow P$ is a topological isomorphism between $L^s(^n X,Y)$, the space of all symmetric continuous n-linear mappings from X to Y, and $P(^nX,Y)$. The space of n-homogeneous polynomials that are weakly uniformly continuous on bounded subsets of X is denoted $P_{wu}(^nX,Y)$ and the corresponding space of symmetric n-linear mappings is denoted by $L^s_{wu}(^nX,Y)$. When $Y = C$, instead of $P_{wu}(^nX,Y)$, $L^s(^nX,Y)$ and $L^s_{wu}(^nX,Y)$ we will shortly write $P_{wu}(^nX)$, $L^s(^nX)$ and $L^s_{wu}(^nX)$, respectively. For each n-homogeneous polynomial P there is a linear operator $T_P: X \to L^s(^{n-1}X)$, defined by $T_P(x_1)(x_2, \ldots, x_n) = A(x_1, x_2, \ldots, x_n)$. It is known that P belongs to $P_{wu}(^nX)$ if and only if the operator T_P is compact.

Following R.M. Aron, M. Maestre and P. Rueda we say that an n-homogenous polynomial is p-compact if for each $x \in X$ there is a neighborhood V_x of x such that $P(V_x)$ is relatively p-compact in Y. We denote by $PK_p(^nX,Y)$ the space of p-compact n-homogeneous polynomials. By an n-homogenous polynomial P is p-compact if and only if $P(B_X)$ is relatively p-compact in Y. On $PK_p(^nX,Y)$ we define

$$k_p(P) = \inf\left\{\|(x_n)_{n=1}^{\infty}\|_p: (x_n)_{n=1}^{\infty} \in l_p(Y), P(B_X) \subset p - co\{(x_n)_{n=1}^{\infty}\}\right\}$$

which is a norm satisfying that $\|P\| \le k_p(P)$ for any p-compact homogeneous polynomial P. Also, $(PK_p(^nX,Y), k_p)$ is a Banach space.

$X \otimes_\pi Y$ denotes the tensor produect of X and Y endowed with the projective norm π, which is defined as $\pi(u) = \inf\{\sum_{n=1}^{n} \|x_n\|\|y_n\|: n \in \mathbb{N}, u = \sum_{n=1}^{n} x_n \otimes y_n\}$ for $x \in X \otimes_\pi Y$. $\widehat{\otimes}_{\pi_s}^{n,s} X$ will denote the completed n-fold symmetric tensor produect of X endowed with the projective s-tensor norm π_s, which is defined as $\pi_s(z) = \inf\left\{\sum_{j=1}^{\infty} |\lambda_j|\|x_j\|^n: z = \sum_{j=1}^{\infty} \lambda_j \otimes^n x_j\right\}$ for $z \in \widehat{\otimes}_{\pi_s}^{n,s} X$, where $\otimes^n x := \underbrace{x \otimes \ldots \otimes x}_{n-times}$.

Finally, $l_{p^*} = c_0$ if $p = 1$ and the l_p-sum (of Banach spaces), as usual, will stand for the c_0-sum if $p = \infty$.

D. Galicer, S. Lassalle and P. Turco showed that a linear operator is p-compact if and only if it is quotiented in l_{p^*}. To be more precise, their proof can be sketched as follows: Given $T \in K_p(X,Y)$ there is a $z = (z_n)_{n=1}^{\infty} \in l_p(Y)$ such that $T(B_X) \subset \{\sum_{n=1}^{\infty} \alpha_n z_n : (\alpha_n)_{n=1}^{\infty} \in L\}$, where L is a compact set in $B_{\ell_{p^*}}$. Then, define two linear mappings $\theta_z: l_{p^*} \to Y$ by $\theta_z(\alpha) = \sum_{n=1}^{\infty} \alpha_n z_n, \alpha = (\alpha_n)_{n=1}^{\infty} \in l_{p^*}$, and $\hat{\theta}_z: l_{p^*}/ker\theta_z \to Y$ by $\hat{\theta}_z([\alpha]) = \theta_z(\alpha)$. And define a linear operator $R: X \to l_{p^*}/ker\theta_z$ by $R(x) = [(\alpha_n)_{n=1}^{\infty}]$, where $(\alpha_n)_{n=1}^{\infty} \in L$ is a sequence satisfying that $T(x) = \sum_{n=1}^{\infty} \alpha_n z_n$.

Then one can easily see that $T = \hat{\theta}_z oR$. Here, we notice that $\hat{\theta}_z$ is p-compact and R is compact. We get the following factorization of p-compact operators through a universal Banach space.

Theorem (2.1.1)[2]. Let $1 \le p < \infty$, let X and Y be Banach spaces. Then there is a universal Banach space $Z^{(p)}$ such that every operator $T \in K_p(X,Y)$ can be factored as $T = B \text{ o } A$, where $A \in K(X, Z^{(p)})$ and $B \in K_p(Z^{(p)}, Y)$. In particular, for $1 \le q \le \infty, Z^{(p)}$ can be chosen as $Z^{(p)} = \left(\sum_{W^{(p)}} W^{(p)}\right)_q$, for a fixed q, where $W^{(p)}$ runs through the quotient spaces of l_{p^*} which are Banach spaces.

Proof. Given $T \in K_p(X,Y)$, there exist $z = (z_n)_{n=1}^{\infty} \in l_p(Y), R \in K(X, l_{p^*}/ker\theta_z)$ and $\tilde{\theta}z \in K_p(l_{p^*}/ker\theta_z, Y)$ such that $T = \tilde{\theta}_z \circ R$. Let $I_{l_{p^*}/ker\theta_z} : l_{p^*}/ker\theta_z \to Z^{(p)}$ denote the natural norm one embedding and let $P_{l_{p^*}}/ker\theta_z : Z^{(p)} \to l_{p^*}/ker\theta_z$ denote the natural norm one projection. If we define the mappings $A := I_{l_{p^*/ker\theta_z}} \circ R$ and $:= \tilde{\theta}_z \circ P_{l_{p^*}/ker\theta_z}$, then $A \in K(X, Z^{(p)}), B \in K_p(Z^{(p)}, Y)$ and $T = B \circ A$.

On the other hand, we know by results of T. Figiel and W.B. Johnson, combined with a result of S. Banach and S. Mazur, that compact operators between Banach spaces can be factored compactly through a quotient space of l_1. We note that by a slight modification we recover this result easily as follows, which we include here for the sake of completeness.

Proposition (2.1.2)[2]. Let X and Y be Banach spaces and let $T \in K(X,Y)$. Then there exist $(z_n)_{n=1}^{\infty} \in c_0(Y), R \in K(X, l_1/ker\theta_z)$ and $\tilde{\theta}_z \in K(l_1/ker\theta_z, Y)$ such that $T = \tilde{\theta}_z \circ R$.

Proof. Let $(z_n)_{n=1}^{\infty} \in c_0(Y)$ be such that $T(B_X) \subset \{\sum_{n=1}^{\infty} \alpha_n z_n : (\alpha_n)_{n=1}^{\infty} \in B_{l_1}$. Choosing a sequence $(\lambda_n)_{n=1}^{\infty}$ with, $\lambda_n \geq 1$ and $\lambda_n \to \infty$ such that $(\lambda_n z_n)_{n=1}^{\infty} \in c_0(Y)$, and defining $(y_n)_{n=1}^{\infty} := (\lambda_n z_n)_{n=1}^{\infty}$, we get $T(B_X) \subset \{\sum_{n=1} \beta_n y_n : (\beta_n)_{n=1}^{\infty} \in L\}$, where L is a relatively compact set in B_{l_1} . Now following the lines of the proof one can get the required factorization.

As a consequence of Proposition (2.1.2), we obtain the following $p = \infty$ case of Theorem (2.1.1), which is nothing more than a factorization of compact operators through a universal Banach space, and is well known as we already mentioned above.

Theorem (2.1.3)[2]. Let X and Y be Banach spaces. Then there is a universal Banach space $Z^{(\infty)}$ such that a compact operator $T \in K(X,Y)$ can be factored as $T = B \circ A$, where $A \in K(X, Z^{(\infty)})$ and $B \in K(Z^{(\infty)}, Y)$. In particular, for $1 \leq q \leq \infty$ $Z^{(\infty)}$ can be chosen as $Z^{(\infty)} = (\sum_W W)_q$ for a fixed q, where W runs through the quotient spaces of l_1 which are Banach spaces.

The above factorization results will be used in the next section. It should be pointed out that factorization results for operators are quite useful when working with approximation properties of Banach spaces, since in many cases they have a crucial role for determining whether or not certain (classes of) Banach spaces have certain type of approximation properties. For instance, we consider the approximation and the k_p-approximation properties. We recall that a Banach space X is said to have the approximation property (AP for short) if for every compact set K in X and every $\varepsilon > 0$, there exists a $T \in F(X;X)$ such that $\sup_{x \in K} \|Tx - x\| \leq \varepsilon$, and a Banach space X is said to have the k_p-approximation property (k_p-AP for short) if for every Banach space $\overline{Y, F(Y,X)}^{k_p} = K_p(Y,X)$. It is known that there are quotient spaces of l_q for $1 < q < 2$, which does not have the AP. But using the factorization for p-compact operators given in one gets the same result at once without any effort. By using standard methods, we get easily another known result asserting that if $1 \leq p < \infty, p \neq 2$, then there are quotient spaces of l_1 which does not have the k_p-AP.

Section (2.2): The Results

R. Aron, M. Lindström, W. Ruess and R. Ryan obtained (uniform) factorizations of compact subsets of compact operators between Banach spaces. Here by a suitable and careful modification of their method we obtain, speaking roughly, (uniform) factorizations of r-compact subsets of p-compact operators between Banach spaces. In order to obtain this result we need some preparation. We will start with a sequence of lemmas.

Lemma (2.2.1)[2]. (the lifting property of quotient maps for p-compact sets) Let X be Banach space and let $1 \leq p < \infty$. Let N be a closed subspace of X and let $Q_N^X : X \to X/N$ be the quotient mapping. If K is a relatively p-compact subset of X/N, then there is a relatively p-compact subset C of X such that $K \subset Q_N^X(C)$.

Proof. If K is a relatively p-compact subset of $\frac{X}{N}$, there exists $(X_n)_n^{\infty} \in l_p\left(\frac{X}{N}\right)$ such that $K \subset \left\{\sum_{n=1}^{\infty} \alpha_n X_n : (\alpha_n)_{n=1}^{\infty} \in B_{p^*}\right\}$. For each $n \in \mathbb{N}$, choose $x_n \in X_n$ such that $\|x_n\| < \|X_n\| + \frac{1}{n^2}$, so that $(x_n)_{n=1}^{\infty} \in l_p(X)$. Taking $C := p - co\{(x_n)_{n=1}^{\infty})\}$ ends up the proof.

The following lemma says that any p-compact subset in the range of a surjective continuous linear operator is always contained in the image of a p-compact set by the operator.

Lemma (2.2.2)[2]. Let X and Y be Banach spaces, let $1 \leq p < \infty$ and let $T \in L(X,Y)$ be onto. If H is a relatively p-compact subset of , then there exists a relatively p-compact subset A of X such that $H \subset T(A)$.

Proof. If $T \in L(X,Y)$ is onto, then it admits a factorization $T = T_0 \circ Q$, where $T_0: X/N(T) \to Y$ is an isomorphism and $Q: X \to X/N(T)$ is the quotient map. Letting $C := T_0^{-1}(H)$ and applying Lemma (2.2.1) we get the conclusion.

A result similar to the above lemma, replacing quotient maps by continuous surjective linear maps, can be stated for fundamental p-compact sets as follows.

Lemma (2.2.3)[2]. Let X and Y be Banach spaces and let $T \in L(X,Y)$ be onto.

a) If $p = 1, \alpha > 1$ and $H \subset p - co\{(a_k)_{k=1}^{\infty}\}$ with $(k^{\alpha} a_k)_{k=1}^{\infty} \in l_p(Y)$, then there exist a sequence $(\tau_k)_{k=1}^{\infty} \subset X$ with $(k^{\alpha}\tau_k)_{k=1}^{\infty} \in l_p(X)$ such that for $L := p - co\{(\tau_k)_{k=1}^{\infty}\}$ we have $H \subset T(L)$.

b) If $1 < p < \infty$ and $H \subset p - co\{(a_k)_{k=1}^{\infty}\}$ with $(k a_k)_{k=1}^{\infty} \in l_p(Y)$, then there exist a sequence $(\tau_k)_{k=1}^{\infty} \subset X$ with $(k\tau_k)_{k=1}^{\infty} \in l_p(X)$ such that for $L := p - co\{(\tau_k)_{k=1}^{\infty}\}$ we have $H \subset T(L)$.

Proof. We give a proof for the case $p = 1$ since the proof for the case $1 < p < \infty$ is similar. Since $T \in L(X,Y)$ is onto, as in the proof of Lemma (2.2.2), we can write $T = T_0 \circ Q$, where $T_0: X/N(T) \to Y$ is an isomorphism and $Q: X \to X/N(T)$ is the quotient map. If $y \in H$, then there exists $(\alpha_k)_{k=1}^{\infty} \in B_{p^*}$ such that $y = \sum_{k=1}^{\infty} \alpha_k a_k$. For each $k \in \mathbb{N}$, there is a $\tau_k \in T_0^{-1}(a_k) := [\tau_k] \in X/N(T)$ such that $\|\tau_k\| < \|[\tau_k]\| + \frac{1}{k^{2\alpha}}$. Therefore, since $\sum_{k=1}^{\infty} \|k^{\alpha}\tau_k\| < \|T_0^{-1}\| \sum_{k=1}^{\infty} \|k^{\alpha} a_k\| + \sum_{k=1}^{\infty} \frac{1}{k^{\alpha}} < \infty$, letting $L := p - co\{(\tau_k)_{k=1}^{\infty}\}$, we get that $H \subset T(L)$. By strengthening a result we obtain the following lemma, which relies on a result of A. Grothendieck characterizing tensor products. This lemma will be the key result since the main result Theorem (2.2.5) will be based on the tensor representation provided therein.

Lemma (2.2.4)[2]. Let X and Y be Banach spaces.

a) Let $2 \leq p < \infty$. If $L \subset p - co\{(\tau_k)_{k=1}^{\infty}\}$ with $(k\tau_k)_{k=1}^{\infty} \in l_p(X \widehat{\otimes}_\pi Y)$, then there exist sequences $(r_k)_{k=1}^{\infty} \in c_0(X)$, $(s_k)_{k=1}^{\infty} \in l_p(Y)$ and a subset K of $B_{L_{p^*}}$ with $K \subset p^* - co\{(t_k)_{k=1}^{\infty}\}, (t_k)_{k=1}^{\infty} \in l_{p^*}\left(B_{\ell_{p^*}}\right)$, such that $L \subset \{\sum_{k=1}^{\infty} \lambda_k r_k \otimes s_k : (\lambda_k)_{k=1}^{\infty} \in K\}$.

b) Let $1 < p < \infty$. If $L \subset p - co\{(\tau_k)_{k=1}^{\infty}\}$ with $(\tau_k)_{k=1}^{\infty} \in l_p(X \widehat{\otimes}_\pi Y)$, then there exist sequences $(r_k)_{k=1}^{\infty} \in c_0(X), (s_k)_{k=1}^{\infty} \in l_p(Y)$ and a compact subset K of $B_{\ell_{p^*}}$, such that $L \subset \{\sum_{k=1}^{\infty} \lambda_k r_k \otimes s_k : (\lambda_k)_{k=1}^{\infty} \in K\}$.

Proof. a) If $L \subset p - co\{(\tau_k)_{k=1}^{\infty}\}$ with $(k\tau_k)_{k=1}^{\infty} \in l_p(X \widehat{\otimes}_\pi Y)$, then for any $\tau \in L$, there exists $(\lambda_k^{\tau})_{k=1}^{\infty} \in B_{\ell_{p^*}}$ such that $\tau = \sum_{k=1}^{\infty} \lambda_k^{\tau} \tau_k$. Since $\tau_k \in X \widehat{\otimes}_\pi Y$, it follows from that $\tau_k = \sum_{i=1}^{\infty} \lambda_{k,i} r_{k,i} \otimes s_{k,i}$, where for every $i \in \mathbb{N}, r_{k,i} \in S_X, s_{k,i} \in S_Y$ and $\sum_{i=1}^{\infty} |\lambda_{k,i}| < \infty$ with $\mu_k := \sum_{i=1}^{\infty} |\lambda_{k,i}| < \pi(\tau_k) + \frac{1}{2^k k}$. Thus we get that

$$\tau = \sum_{k=1}^{\infty} \lambda_k^{\tau} \tau_k = \sum_{k=1}^{\infty} \lambda_k^{\tau} \sum_{i=1}^{\infty} \lambda_{k,i} r_{k,i} \otimes s_{k,i}$$

$$= \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \frac{1}{k} \lambda_k^{\tau} \left(\frac{\lambda_{k,i}}{\mu_k}\right)^{1/p^*} r_{k,i} \otimes k \left(\frac{\lambda_{k,i}}{\mu_k}\right)^{1/p} \mu_k s_{k,i}.$$

Since the series $\sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \frac{1}{k} \lambda_k^{\tau} \left(\frac{\lambda_{k,i}}{\mu_k}\right)^{1/p^*} r_{k,i} \otimes k \left(\frac{\lambda_{k,i}}{\mu_k}\right)^{1/p} \mu_k s_{k,i}$ converges absolutely in $X \widehat{\otimes}_{\pi} Y$, and since $\sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \left|\frac{1}{k} \lambda_k^{\tau} \left(\frac{\lambda_{k,i}}{\mu_k}\right)^{1/p^*}\right|^{p^*} = \sum_{k=1}^{\infty} \frac{1}{k^{p^*}} |\lambda_k^{\tau}|^{p^*} \leq 1$ and $\sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \left\| k \left(\frac{\lambda_{k,i}}{\mu_k}\right)^{1/p} \mu_k s_{k,i} \right\|^p \leq 2^p \left(\sum_{k=1}^{\infty} \left(\pi(k\tau_k)\right)^p + \frac{1}{2^p k} \right) < \infty$, by choosing a specific order one can write

$$(\gamma_l^{\tau})_{l=1}^{\infty} := \left(\frac{1}{k} \lambda_k^{\tau} \left(\frac{\lambda_{k,i}}{\mu_k}\right)^{1/p^*}\right)_{(k,i)\in\mathbb{N}\times\mathbb{N}} \in B_{\ell_p},$$

$$(x_l)_{l=1}^{\infty} := \left(r_{k,i}\right)_{(k,i)\in\mathbb{N}\times\mathbb{N}} \in l_{\infty}(X),$$

$$(y_l)_{l=1}^{\infty} := \left(k \left(\frac{\lambda_{k,i}}{\mu_k}\right)^{1/p} \mu_k s_{k,i}\right)_{(k,i)\in\mathbb{N}\times\mathbb{N}} \in l_p(Y),$$

so that we obtain a representation $\tau = \sum_{l=1}^{\infty} \gamma_l^{\tau} x_l \otimes y_l$. Moreover, since $\sum_{l=1}^{\infty} \|y_l\|^p < \infty$, we may choose a positive increasing sequence $(c_l)_{l=1}^{\infty}$, diverging to infinity, such that $\sum_{l=1}^{\infty} \|y_l\|^p c_l < \infty$. Thus, writing $\tau = \sum_{l=1}^{\infty} \gamma_l^{\tau} x_l \frac{1}{c_l^{1/p}} \otimes c_l^{1/p} y_l$, and letting $r_l := x_l \frac{1}{c_l^{1/p}}$ and $s_l := c_l^{1/p} y_l$ for each L, we get that $\tau = \sum_{l=1}^{\infty} \gamma_l^{\tau} r_l \otimes s_l$, where $(r_l)_{l=1}^{\infty} \in c_0(X)$ and $(s_l)_{l=1}^{\infty} \in l_p(Y)$.

Now, we claim that the sequences $(\gamma_l^{\tau})_{l=1}^{\infty}$, $(\tau \in L)$, range over a relatively p^*-compact subset K of $B_{\ell_{p^*}}$. Indeed, according to the order chosen above, we can write

$$(\gamma_l^{\tau})_{l=1}^{\infty} = \left(\frac{1}{k} \lambda_k^{\tau} \left(\frac{\lambda_{k,i}}{\mu_k}\right)^{\frac{1}{p^*}}\right)_{(k,i)\in\mathbb{N}\times\mathbb{N}} = \sum_{k=1}^{\infty} \lambda_k^{\tau} \sum_{i=1}^{\infty} \frac{1}{k} \left(\frac{\lambda_{k,i}}{\mu_k}\right)^{\frac{1}{p^*}} e_i^k$$

where the vectors e_i^k appeared in the double indexed set $\left(e_i^k\right)_{(k,i)\in\mathbb{N}\times\mathbb{N}}$ are the standard basis vectors e_l of l_{p^*} ordered as above. Next, for each $k \in \mathbb{N}$, we define $t_k := \frac{1}{k} \sum_{i=1}^{\infty} \left(\frac{\lambda_{k,i}}{\mu_k}\right)^{1/p^*} e_i^k$. Hence, for every $k \in \mathbb{N}$ we have that $t_k \in B_{\ell_{p^*}}$, and, since $1 < p^* \leq 2$, we have $\sum_{k=1}^{\infty} \|t_k\|_{p^*}^{p^*} = \sum_{k=1}^{\infty} \frac{1}{kp^*} \sum_{i=1}^{\infty} \left|\frac{\lambda_{k,i}}{\mu_k}\right| \left\|e_i^k\right\|^{p^*} < \infty$. Thus, $(t_k)_{k=1}^{\infty} \in l_{p^*}(B_{\ell_{p^*}}) \subset l_{p^*}\left(l_{p^*}\right)$. On the other hand since $p^* \leq p$, we have $(\lambda_k^{\tau})_{k=1}^{\infty} \in$

B_{l_p} . Therefore, since $(\gamma_l^\tau)_{l=1}^\infty = \sum_{k=1}^\infty \lambda_k^\tau t_k$ with $(\lambda_k^\tau)_{k=1}^\infty \in B_{\ell_p}$, if we take K as the set $\{(\gamma_l^\tau)_{l=1}^\infty : \tau \in L\}$, then $K \subset p^* - co\{(t_k)_{k=1}^\infty\}$, and the proof of part a) is complete. b) If $L \subset p - co\{(\tau_k)_{k=1}^\infty\}$ with $(\tau_k)_{k=1}^\infty \in l_p(X \widehat{\otimes}_\pi Y)$, then by a similar argument as in (a) any $\tau \in L$ can be written as $\tau = \sum_{i=1}^\infty \lambda_i^\tau r_i \otimes t_i$ with $(\lambda_i^\tau)_{i=1}^\infty \in B_{l_{p^*}}$ where $(r_i)_{i=1}^\infty \in c_0(X)$ and $(t_i)_{i=1}^\infty \in l_p(Y)$. Since $(t_i)_{i=1}^\infty \in l_p(Y)$, we may choose $\beta = (\beta_i)_{i=1}^\infty \in B_{c_0}$ such that $\left(\frac{t_i}{\beta_i}\right)_{i=1}^\infty \in l_p(Y)$. Accordingly we write $\tau = \sum_{i=1}^\infty \lambda_i^\tau r_i \otimes t_i = \sum_{i=1}^\infty \beta_i \lambda_i^\tau r_i \otimes \frac{t_i}{\beta_i}$, where $(\lambda_i^\tau)_{i=1}^\infty \in B_{l_{p^*}}$. If for every $i \in \mathbb{N}$ we let $\theta_i^\tau := \beta_i \lambda_i^\tau$ and $s_i := \frac{t_i}{\beta_i}$, then,

$$\tau = \sum_{i=1}^\infty \theta_i^\tau r_i \otimes s_i, (\theta_i^\tau)_{i=1}^\infty \in B_{l_{p^*}}$$

with $(r_i)_{i=1}^\infty \in c_0(X), (s_i)_{i=1}^\infty \in l_p(Y)$. To see that the sequences $(\theta_i^\tau)_{i=1}^\infty$ range over a compact subset of $B_{l_{p^*}}$, note that the set $K := \{(\beta_i \gamma_i)_{i=1}^\infty : (\gamma_i)_{i=1}^\infty \in B_{l_{p^*}}\}$ is a compact subset of $B_{l_{p^*}}$, so the proof of the claim is complete.

As a final step towards our main result, let $1 \leq p < \infty$, and let $Z^{(p)}$ be the universal Banach space given in Theorem (2.1.1). Given Banach spaces X and , according to Theorem (2.1.1), the continuous bilinear map

$$\tau : K(X, Z^{(p)}) \times (K_p(Z^{(p)}, Y), k_p) \to (K_p(X, Y), k_p), \tau(u, v) = v \circ u,$$

is onto. The linearization of τ , $\hat{\tau} : K(X, Z^{(p)}) \widehat{\otimes}_\pi (K_p(Z^{(p)}, Y), k_p) \to (K_p(X, Y), k_p)$, defined by $\hat{\tau}(u \otimes v) = \tau(u, v) = v \circ u$, is a continuous linear map which is onto. For the proof we will carefully modify a method. It should be emphasized that in our proof we do not use any selection principal as it is done in the first method, instead we use the lifting of p-compact sets (given by Lemma (2.2.1)) which is already pointed out at the end of the compact case.

Theorem (2.2.5)[2]. Let X and Y be Banach spaces, let $r \geq 2$ and let $1 \leq p \leq r < \infty$. For every (balanced and convex) relatively r-compact subset H of $(K_p(X, Y), k_p)$ such that $H \subset r - co\{(a_k)_{k=1}^\infty\}$ with $(ka_k)_{k=1}^\infty \in l_r(K_p(X, Y), k_p)$, there exist an operator $u \in K(X, Z_{FJ})$, a (resp. balanced and convex) relatively r^*-compact subset $\{B_T : T \in H\}$of $K(Z_{FJ}, Z^{(r)})$ and an operator $v \in K_r(Z^{(r)}, Y)$ such that $T = v \circ B_T \circ u$ for all $T \in H$, where Z_{FJ} denotes a universal factorization space of Figiel and Johnson, and $Z^{(r)}$ is the universal Banach space given in Theorem (2.1.1).

Proof. Since $\hat{\tau}$ is a continuous linear onto map and $H \subset r - co\{(a_k)_{k=1}^\infty\}$ with $(ka_k)_{k=1}^\infty \in l_r(K_p(X, Y), k_p)$, by Lemma (2.2.3) b), there exists $(\tau_k)_{k=1}^\infty$ in $K(X, Z^{(p)}) \widehat{\otimes}_\pi (K_p(Z^{(p)}, Y), k_p)$ with $(k\tau_k)_{k=1}^\infty \in l_r(K(X, Z^{(p)}) \widehat{\otimes}_\pi (K_p(Z^{(p)}, Y), k_p))$ such that for $L := r - co\{(\tau_k)_{k=1}^\infty\}$ we have $H \subset \hat{\tau}(L)$. Thus, for every $T \in H$ there exist $\tau_T \in L$ such that $T = \hat{\tau}(\tau_T)$. By Lemma (2.2.4) a) we have a representation $\tau_T = \sum_{i=1}^\infty \lambda_i^{\tau_T} r_i \otimes s_i$ with $\left(\lambda_i^{\tau_T}\right)_{i=1}^\infty \in K$, where $(r_i)_{i=1}^\infty \in$

$c_0\Big(K\big(X, Z^{(p)}\big)\Big), (s_i)_{i=1}^{\infty} \in l_r\big(K_p\big(Z^{(p)}\big), Y\big), k_p\big)$ and $K \subset l_{r^*}$ is a relatively r^*-compact subset. Now, define $r : X \to c_0\big(Z^{(p)}\big) by\, r(x) := \big(r_i(x)\big)_{i=1}^{\infty}$. Then $r \in K\Big(X, c_0\big(Z^{(p)}\big)\Big)$. Next, for each $T \in H$ define $A_T : c_0\big(Z^{(p)}\big) \to l_{r^*}\big(Z^{(p)}\big)$ by $A_T(z) = \big(\lambda_i^{\tau_T} z_i\big)_{i=1}^{\infty}, z = (z_i)_{i=1}^{\infty} \in c_0\big(Z^{(p)}\big)$. Since

$$\sum_{i=1}^{\infty} \|\lambda_i^{\tau_T} z_i\|^{r^*} \leq \sum_{i=1}^{\infty} |\lambda_i^{\tau_T}|^{r^*} \|z_i\|^{r^*} \leq \Big(\sup_{i\in\mathbb{N}} z_i\Big)^{r^*} \sum_{i=1}^{\infty} |\lambda_i^{\tau_T}|^{r^*} < \infty,$$

A_T is well defined and that $A_T \in L\Big(c_0\big(Z^{(p)}\big), l_{r^*}\big(Z^{(p)}\big)\Big)$. Now we consider the continuous linear map $A : l_{r^*} \to L(c_0\big(Z^{(p)}\big), l_{r^*}\big(Z^{(p)}\big)$ defined by $A(\lambda)z := (\lambda_i z_i)_{i=1}^{\infty}, \lambda = (\lambda_i)_{i=1}^{\infty}, z = (z_i)_{i=1}^{\infty}$. Since $\{A_T : T \in H\} \subset A(K)$and K is a relatively r^*-compact subset in l_{r^*}, it follows that the subset $\{AT : T \in H\} of\, L\Big(c_0\big(Z^{(p)}\big), l_{r^*}\big(Z^{(p)}\big)\Big)$ is relatively r^*-compact. Finally we define $s : l_{r^*}\big(Z^{(p)}\big) \to Y$ by $s(w) := \sum_{i=1}^{\infty} s_i(w_i), w = (w_i)_{i=1}^{\infty} \in l_{r^*}\big(Z^{(p)}\big)$. Since

$$\sum_{i=1}^{\infty} \|s_i(w_i)\| \leq \sum_{i=1}^{\infty} k_p(s_i)\|w_i\| \leq \left(\sum_{i=1}^{\infty} \big(k_p(s_i)\big)^r\right)^{\frac{1}{r}} \left(\sum_{i=1}^{\infty} \|w_i\|^{r^*}\right)^{\frac{1}{r^*}} < \infty,$$

s is well defined, and since $\|s\| \leq \Big\|\big(k_p(s_i)\big)_{i=1}^{\infty}\Big\|_r$, s is a continuous operator. Now we show that s is, in fact, r-compact. For every $i \in \mathbb{N}$, since $s_i \in K_p\big(Z^{(p)}, Y\big)$ and $p \leq r$, then $s_i \in K_r(Z^{(p)}, Y)$ and $k_r(s_i) \leq k_p(s_i)$. Hence, since $(s_i)_{i=1}^{\infty} \in l_r\big(K_p\big(Z^{(p)}, Y\big), k_p\big))$, then $\sum_{i=1}^{\infty} \big(k_r(s_i)\big)^r < \infty$. Now, for every $i \in \mathbb{N}$, choose a sequence $\big(c_n^i\big)_{n=1}^{\infty} \in l_r(Y)$ such that $\Big\|\big(c_n^i\big)_{n=1}^{\infty}\Big\|_r < k_r(s_i) + \frac{1}{2^i}$ with $s_i(B_{Z^{(p)}}) \subset r - co\Big\{\big(c_n^i\big)_{n=1}^{\infty}\Big\}$. Since $\sum_{i=1}^{\infty} \Big\|\big(c_n^i\big)_{n=1}^{\infty}\Big\|_r^r < 2^r \sum_{i=1}^{\infty} \big(k_r(s_i)\big)^r + \sum_{i=1}^{\infty} \frac{1}{2^{ir}} < \infty$, we have $\sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \big\|c_n^i\big\|^r = \sum_{i=1}^{\infty} \Big\|\big(c_n^i\big)_{n=1}^{\infty}\Big\|_r^r < \infty$. Next, let $w = (w_i)_{i=1}^{\infty} \in B_{l_{r^*}}\big(Z^{(p)}\big)$ (without loss of generality we can assume that $w_i \neq 0$ for each $i \in \mathbb{N}$). Now, one can write $s(w) = \sum_{i=1}^{\infty} \|w_i\| \sum_{n=1}^{\infty} \alpha_n^{w_i} c_n^i$ with $\big(\alpha_n^{w_i}\big)_{n=1}^{\infty} \in B_{l_{r^*}}$. Note that

$$\sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \|w_i\|\alpha_n^{w_i} c_n^i \leq \sum_{i=1}^{\infty} \left[\left(\sum_{n=1}^{\infty} \big(w_i|\alpha_n^{w_i}|\big)^{r^*}\right)^{1/r^*} \left(\sum_{n=1}^{\infty} \big\|c_n^i\big\|^r\right)^{1/r}\right]$$

$$\leq \left(\sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \|w_i\|^{r^*} \big|\alpha_n^{w_i}\big|^{r^*}\right)^{1/r^*} \left(\sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \big\|c_n^i\big\|^r\right)^{1/r} < \infty,$$

and since $\sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \big\|c_n^i\big\|^r < \infty$ and $\sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \big(w_i\big|\alpha_n^{w_i}\big|\big)^{r^*} \leq 1$, choosing a specific order for these double series and writing $(\lambda_l)_{l\in\mathbb{N}} := \big(w_i \alpha_n^{w_i}\big)_{(i,n)\in\mathbb{N}\times\mathbb{N}} \in$

$B_{l_{r^*}}$ and $(z_l)_{l\in\mathbb{N}} := \left(c_n^i\right)_{(i,n)\in\mathbb{N}\times\mathbb{N}} \in l_r(Y)$, we obtain a representation $s(w) = \sum_{l=1}^{\infty} \lambda_l z_l$, which shows that $s \in K_r\left(l_{r^*}\left(Z^{(p)}\right), Y\right)$.

Now for $T \in H$, where $T = \hat{\tau}(\tau_T) = \sum_{i=1}^{\infty} \lambda_i^{\tau_T} s_i \circ r_i$, we have $T = s \circ A_T \circ r$. Finally we factor r and s through Z_{FJ} and (r) , respectively. That is, there exist operators $u \in K(X, Z_{FJ}), \alpha \in K\left(Z_{FJ}, c_0\left(Z^{(p)}\right)\right), \beta \in K\left(l_{r^*}\left(Z^{(p)}\right), Z^{(r)}\right)$ and $v \in K_r\left(Z^{(r)}, Y\right)$ such that $r = \alpha \circ u$ and $s = v \circ \beta$. For each $T \in H$, let $B_T := \beta \circ A_T \circ \alpha$. Then it can be easily seen that $\{B_T : T \in H\}$ is a relatively r^*-compact subset of $K(Z_{FJ}, Z^{(r)})$ and $T = v \circ B_T \circ u$ for every $T \in H$.

In addition if we assume that H is convex and balanced, then one can readily see that $\{B_T : T \in H\}$ is also convex and balanced, with which the proof is complete.

If we relax the hypothesis of the previous theorem by removing the factor "k" in the sequence $(ka_k)_{k=1}^{\infty}$, as compared to Theorem (2.2.5), we obtain the following weaker result.

Proposition (2.2.6)[2]. Let X and Y be Banach spaces, let $1 \le p \le r < \infty$ with $r > 1$. For every (balanced and convex) relatively r-compact subset H of $\left(K_p(X,Y), k_p\right)$, there exist an operator $u \in K\left(X, Z_{FJ}\right)$, a (resp. balanced and convex) relatively compact subset $\{B_T : T \in H\}$of $K\left(Z_{FJ}, Z^{(r)}\right)$ and an operator $v \in K_r\left(Z^{(r)}, Y\right)$ such that $T = v \circ B_T \circ u$ for all $T \in H$.

Proof. Let H be a relatively r-compact subset of $\left(K_p(X,Y), k_p\right)$. By Lemma (2.2.2) there exist a relatively r-compact subset L of $K\left(X, Z^{(p)}\right) \widehat{\otimes}_\pi \left(K_p\left(Z^{(p)}, Y\right), k_p\right)$ such that $H \subset \hat{\tau}(L)$. Now by Lemma (2.2.4) b) any $\tau_T \in L$ has a representation $\tau_T = \sum_{i=1}^{\infty} \theta_i^{\tau_T} r_i \otimes s_i$ with $\left(\theta_i^{\tau_T}\right)_{i=1}^{\infty} \in K$, where $(r_i)_{i=1}^{\infty} \in c_0\left(K\left(X, Z^{(p)}\right)\right), (s_i)_{i=1}^{\infty} \in l_r(K_p(Z^{(p)}, Y), k_p)$, and K is a compact subset of Blr∗ . Now the set $\{A_T : T \in H\}$ obtained in Theorem (2.2.5) is a relatively compact subset of $L(c_0(Z^{(p)}), l_{r^*}\left(Z^{(p)}\right)$ and so is the corresponding set $\{B_T : T \in H\}$. Finally if H is balanced and convex then, one can see that the set $\{B_T : T \in H\}$ has the same properties. Thus, we have the proof.

We can improve Theorem (2.2.5) and Proposition (2.2.6) a little bit more, since in the factorizations given in these results one of the spaces through which the p-compact operators factorize depends on the number $r \ge 2$.

Let $1 \le p, q \le \infty$ and let $Z = \left(1 \le p \le \infty\, Z^{(p)}\right)_q$ for a fixed q, where $Z^{(p)}$ is the universal Banach space given in Theorems (2.1.1) and (2.1.3). Thus by Theorem (2.1.1) and Theorem (2.1.3), it can be easily seen that Z is a universal Banach space for the factorization of all pcompact operators between arbitrary Banach spaces, which is independent of p. That is, given Banach spaces X and Y , and any $1 \le p \le \infty$ and any $T \in K_p(X,Y)$, we can write $T = vou, u \in K(X,Z), v \in K_p(Z,Y)$. As a consequence we obtain the following strengthening of Theorem (2.2.5) and Proposition (2.2.6), respectively, in which the corresponding factorizations are obtained through a universal Banach space which does not depend on the number $r \ge 2$.

Corollary (2.2.7)[2]. Let X and Y be Banach spaces, let $r \geq 2$ and let $1 \leq p \leq r < \infty$. For every (balanced and convex) relatively r-compact subset H of $(K_p(X, Y), k_p)$ such that $H \subset r - co\{(a_k)_{k=1}^{\infty}\}$ with $(ka_k)_{k=1}^{\infty} \in l_r(K_p(X, Y), k_p)$, there exist an operator $u \in K(X, Z)$, a (resp. balanced and convex) relatively r∗-compact subset $\{B_T : T \in H\}$ of $K(Z, Z)$ and an operator $v \in K_r(Z, Y)$ such that $T = v \circ B_T \circ u$ for all $T \in H$.

Corollary (2.2.8)[2]. Let X and Y be Banach spaces, let $r > 1$ and $1 \leq p \leq r < \infty$. For every (balanced and convex) relatively r-compact subset H of $(K_p(X, Y), k_p)$ there exist an operator $u \in K(X, Z)$, a (resp. balanced and convex) relatively compact subset $\{B_T : T \in H\}$ of $K(Z, Z)$ and an operator $v \in Kr(Z, Y)$ such that $T = v \circ B_T \circ u$ for all $T \in H$.

We now look at the use of uniform factorization result given in Corollary (2.2.7). The motivation is that whether or not compact sets can be replaced by p-compact sets in a result of E. Toma which gives a characterization of scalar valued homogeneous polynomials that are weakly uniformly continuous on the unit ball. It is worth saying that in the p-compact case the situation is quite complicated due to the nature of p-compact sets.

We will begin by defining collectively p-compact set, which is the natural extension of notion of collectively compactness.

Definition (2.2.9)[2]. Let X and Y be Banach spaces, let $p \geq 1$. A subset A of $L(X, Y)$ is said to be collectively p-compact if $A(B_X) = \{T x : T \in A, x \in B_X\}$ is a relatively p-compact set.

We obtain the following result, which is of independent interest, and will be needed in the proof of the next theorem.

Proposition (2.2.10)[2]. Let $1 \leq p < \infty$. Every relatively p-compact subset K of $(K_p(X, Y), k_p)$ is collectively p-compact.

Proof. We give a proof for the case $1 < p < \infty$ since the proof for the case $p = 1$ is similar. Let K be a relatively p-compact subset of $(K_p(X, Y), k_p)$. Thus, for a given $T \in K$ there exist $(\alpha_n^T)_{n=1}^{\infty} \in B_{l_{p^*}}$ and $(T_n)_{n=1}^{\infty} \in l_p(K_p(X, Y), k_p))$ such that $T = \sum_{n=1}^{\infty} \alpha_n^T T_n$. For every $n \in \mathbb{N}$ we choose a sequence $(z_k^n)_{k=1}^{\infty} \in l_p(Y)$ such that $\|(z_k^n)_{k=1}^{\infty}\|_p < k_p(T_n) + \frac{1}{2^n}$ and $T_n(B_X) \subset p - co\{(z_k^n)_{k=1}^{\infty}\}$. Hence, for $T \in K$ and $x \in B_X$ we have that $T(x) = \sum_{n=1}^{\infty} \alpha_n^T T_n(x) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \alpha_n^T \lambda_k^{n,x} z_k^n$, where $(\alpha_n^T)_{n=1}^{\infty}, (\lambda_k^{n,x})_{k=1}^{\infty} \in B_{l_{p^*}}$. Since $\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\alpha_n^T \lambda_k^{n,x}|^{p^*} \leq \sum_{n=1}^{\infty} |\alpha_n^T|^{p^*} \leq 1$ and

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \|z_k^n\|^p = \sum_{n=1}^{\infty} \|(z_k^n)_{k=1}^{\infty}\|_p^p < 2^p \left(\sum_{n=1}^{\infty} k_p^p(T_n) + \frac{1}{2^{np}} \right) < \infty,$$

by choosing a specific order one can write $(\gamma_l^{T,x})_{l=1}^{\infty} := (\alpha_n^T \lambda_k^{n,x})_{(n,k)\in\mathbb{N}\times\mathbb{N}} \in B_{\ell_{p^*}}$ *and* $(s_l)_{l=1}^{\infty} := (z_k^n)_{(n,k)\in\mathbb{N}\times\mathbb{N}} \in l_p(Y)$, so that we obtain $T(x) = \sum_{l=1}^{\infty} \gamma_l^{T,x} s_l$. Thus, $K(B_X) = \{T x : T \in K, x \in B_X\} \subset p - co\{(s_l)_{l=1}^{\infty}\}$.

Theorem (2.2.11)[2]. Let X be a Banach space with X' having the AP, and let $r \geq 2, 1 \leq p \leq r < \infty$. Let H be a relatively r-compact subset of $(K_p(X, X'), k_p)$ such that $H \subset$

$r - co\{(a_k)_{k=1}^{\infty}\}$ with $(ka_k)_{k=1}^{\infty} \in l_r\big(K_p(X, X'), k_p\big)$. Then for every $\varepsilon > 0$ there exists an r-compact subset K'_ε of X' such that for every $T \in H$ and $x \in X$

$$|T(x)(x)| \leq \varepsilon \, \|x\| \sup_{k' \in K'_\varepsilon} |k'(x)| + \sup_{k' \in K'_\varepsilon} |k(x)|^2.$$

Proof. By Corollary (2.2.7), there are a Banach space Z, a relatively r^*-compact subset $\{L_T : T \in H\}$ of $K(X, Z)$, and an operator $v \in K_r(Z, X')$ such that $T = v \circ L_T$ for all $T \in H$. Thus, for each $x \in X \subset X''$ and for each $T \in H$, we have $|T(x)(x)| = |v \circ L_T(x)(x)| \leq \|v'(x)\| \|L_T(x)\|$, where v' is the adjoint of v. Note that $\|v'(x)\| = \sup_{z \in B_Z} |v(z)(x)| \leq \sup_{k' \in K'_1} |k'(x)|$, where $K'_1 := \overline{v(B_Z)} \subset X'$, which is an r-compact set. Furthermore,

$$\|L_T(x)\| = \sup_{z' \in B_{Z'}} \left|z'\left(L_T(x)\right)\right| = \sup_{z' \in B_{Z'}} |(L'_T z')(x)|.$$

Let $K := \{L_T : T \in H\}$ and let $K^* := \{L_{T'} : T \in H\}$. Since K^* is relatively r^*- compact subset of $K(Z', X')$, there exists $(S_n)_{n=1}^{\infty} \in l_{r^*}\big(K(Z, X)\big)$ such that $K^* \subset r^* - co\{(S'_n)_{n=1}^{\infty}\}$. Hence, for any $\varepsilon > 0$ there is $N = N(\varepsilon) \in \mathbb{N}$ such that $\sum_{n=N+1}^{\infty} \|S'_n\|^{r^*} \leq \left(\frac{\varepsilon}{2}\right)^{r^*}$. Since X' has the AP, for every $n \in \mathbb{N}$ there is an $S_n^F \in F(Z', X')$ such that $\|S'_n - S_n^F\| < \frac{\varepsilon}{2n^2}\left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{-1}$. So, if we define a sequence $(S_n^*)_{n=1}^{\infty}$ in $F(Z', X')$ by $S_n^* := S_n^F$ for $n = 1, 2, \ldots, N$, and $S_n^* := 0$ for $n > N$, and consequently a set by $K_{F,\varepsilon}^* := \left\{\sum_{n=1}^{\infty} \alpha_n S_n^* : (\alpha_n)_{n=1}^{\infty} \in B_{l_r} \text{ with } \sum_{n=1}^{\infty} \alpha_n S'_n \in K^*\right\}$ then, $K_{F,\varepsilon}^*$ is a relatively r^*-compact subset of $(F(Z', X'), k_{r^*})$. Now, given any $L'_T = \sum_{n=1}^{\infty} \alpha_n^T S'_n \in K^*$, let $L_T^* := \sum_{n=1}^{\infty} \alpha_n^T S_n^*$. Thus, we have that

$$\|L_T - L_T^*\| \leq \sum_{n=1}^{N} \|S'_n - S_n^F\| + \left(\sum_{n=N+1}^{\infty} |\alpha_n^T|^r\right)^{1/r} \sum_{n=N+1}^{\infty} \|S'_n\|^{r^*})^{1/r^*} < \varepsilon.$$

Hence, we have shown that for any $L_T \in K$ there is $L_T^* \in K_{F,\varepsilon}^*$ such that $L_T - L_T^* < \varepsilon$. Therefore, by (1), for every $x \in X$ we get that

$$\|L_T(x)\| \leq \|L'_T - L_T^*\| \sup_{z' \in B_{Z'}} \|z'\| \, \|x\| + \sup_{z' \in B_{Z'}} |L_T^* z(x)| < \varepsilon \|x\| + \sup_{z' \in B_{Z'}} |L_T^* z'(x)|.$$

Since $K_{F,\varepsilon}^*$ is a relatively r^*-compact subset of $(K_{r^*}(Z, X), k_{r^*})$, thus by Proposition (2.2.10) the set $K_{F,\varepsilon}^*$ is collectively r^*-compact in $L(Z', X')$, so that the set $K'_2 := \overline{\{L_T^*(z) : L_T^* \in K_{F,\varepsilon}^*, z' \in B_{Z'}\}}$ is an r^*-compact, hence r-compact, subset of X. Therefore, $\|L_T(x)\| < \varepsilon \|x\| + \sup_{z' \in B_{Z'}} |L_T^* z'(x)| \leq \varepsilon \|x\| + \sup_{k' \in K'_2} |k'(x)|$. Finally, letting $K'_\varepsilon := K'_1 \cup K'_2$, which is also r-compact, for all $T \in H$ and $x \in X$ we obtain $|T(x)(x)| \leq \varepsilon \| x \| \sup_{k' \in K'_\varepsilon} |k'(x)| + \sup_{k' \in K'_\varepsilon} |k'(x)|^2$.

Now as an application of Theorem (2.2.11) we get the following partial p-compact version of a result of E. Toma for 2-homogenous polynomials.

Corollary (2.2.12)[2]. Let X be a Banach space with X having the AP and let $2 \leq r < \infty$. Then for a $P \in P_{wu}(^2X)$ with TP being r-compact, given any $\varepsilon > 0$ there exists an r compact subset K'_ε of X such that $|P(x)| \leq \varepsilon \|x\| \sup_{k' \in K'_\varepsilon} |k'(x)| + \sup_{k' \in K'_\varepsilon} |k'(x)|^2$ for all $x \in X$.

Proof. Let $P \in P_{wu}(^2X)$ with T_P being r-compact, $2 \leq r < \infty$. Then taking $H := \{T_P\}$ and applying Theorem (2.2.11) we obtain the desired inequality.

Recall that a polynomial $P \in P(^nX, Y)$ is of finite type if it can be written as a linear combination of functions $\phi^n \otimes y \, (n \in \mathbb{N}, \phi \in X', y \in Y)$, where $\phi^n \otimes y(x) = \phi^n(x)y$ for each $x \in X$. Note that if a polynomial P is of finite type then the corresponding operator is also of finite type, hence, is r-compact for any $r \geq 2$.

We do not know if the reverse implication in Corollary (2.2.12) is true. If that would be the case, Corollary (2.2.12) would be an improvement for the case $n = 2$, since the compact sets are replaced by r-compact sets. Motivated and Corollary (2.2.12), a result for vector-valued p-compact n-homogeneous polynomials can be stated in a similar fashion. Therefore, as a consequence of Theorem (2.2.11) we prove the following interesting result concerning p-compact polynomials with values in $\left(\widehat{\otimes}_{\pi_s}^{n,s} X\right)'$.

Corollary (2.2.13)[2]. Let X be a Banach space such that $\left(\widehat{\otimes}_{\pi_s}^{n,s} X\right)$ has the AP. Let $r \geq 2, 1 \leq p \leq r < \infty$, and let H_n be a relatively r-compact subset of $\left(P_{k_p}\left(^nX, \left(\widehat{\otimes}_{\pi_s}^{n,s} X\right), k_p\right)\right)$ such that $H_n \subset r - co\{(a_k^n)_{k=1}^\infty\}$ with $(ka_k^n)_{k=1}^\infty \in l_r\left(P_{k_p}\left(^nX, \left(\widehat{\otimes}_{\pi_s}^{n,s} X\right)'\right), k_p\right)$. Then for every $\varepsilon > 0$ there exists an r-compact subset K'_ε of $\left(\widehat{\otimes}_{\pi_s}^{n,s} X\right)$ such that for all $P \in H_n$ and all $x \in X, |P(x)(\otimes^n x)| \leq \sup_{k' \in K'_\varepsilon} |k(\otimes^n x)| \left(\varepsilon \|x\|^n + \sup_{k' \in K'_\varepsilon} |k(\otimes^n x)|\right)$.

Proof. Since $\left(P_{k_p}\left(^nX\left(\widehat{\otimes}_{\pi_s}^{n,s} X\right)'\right), k_p\right)$ and $\left(K_p\left(\widehat{\otimes}_{\pi_s}^{n,s} X, \left(\widehat{\otimes}_{\pi_s}^{n,s} X\right)'\right), k_p\right)$ are isometrically isomorphic, there is a sequence $(T_k^n)_{k=1}^\infty \subset K_p\left(\widehat{\otimes}_{\pi_s}^{n,s} X, \left(\widehat{\otimes}_{\pi_s}^{n,s} X\right)'\right)$ such that $(kT_k^n)_{k=1}^\infty \in l_r\left(K_p\left(\widehat{\otimes}_{\pi_s}^{n,s} X, \left(\widehat{\otimes}_{\pi_s}^{n,s} X\right)'\right), k_p\right)$ and $C_n := \{P^L : P \in H_n\} \subset r - co\{(T_k^n)_{k=1}^\infty\}$, where the mapping $P^L : \widehat{\otimes}_{\pi_s}^{n,s} X \to Y$, defined by $P^L(\otimes^n x) = P(x)$ is the linearization of P. Now since $\left(\widehat{\otimes}_{\pi_s}^{n,s} X\right)'$ has the AP hence, by Theorem (2.2.11), given any $\varepsilon > 0$, there exits an r-compact subset K'_ε of $(\widehat{\otimes}_{\pi_s}^{n,s} X)$ such that for all $P^L \in C_n$ and for all $x \in X$, we have

$|P^L(\otimes^n x)(\otimes^n x)| \leq \sup_{k' \in K'_\varepsilon} |k'(\otimes^n x)| \left(\varepsilon \| \otimes^n x\| + \sup_{k' \in K'_\varepsilon} |k'(\otimes^n x)|\right)$, from which we get the conclusion.

Note that in Corollary (2.2.13) taking $n = 1$ one gets exactly Theorem (2.2.11). In this sense it is a generalization of Theorem (2.2.11).

Definition (2.2.14)[7]. Let X be a Bauch space and $k \subset X$. Then k is said to be realatively weakly p-compact $1 \leq p \leq \infty$ if there is $X \in l_p^w(X)$, such that $k \subset E_X(l_{p'})(l_{p'}\colon (l_p)^*)$, and k is said to be relatively weakly ∞-compact if there is $x \in C_0^w(X)$ such that $k \subset E_X(\text{ball } (l_1))$.

Chapter 3

Almost over Total Sequences in Banach Spaces

We show information about the structure of such sequences. In particular it can happen that, an $AOC\langle$resp. $AOT\rangle$given sequence admits countably many not nested subsequences such that the only subspace contained in the closed linear span of every of such subsequences is the trivial one $<$ resp. the closure of the linear span of the union of the annihilators in X of such subsequences is the whole $X >$. Moreover, any AOC sequence $\{x_n\}_{n\in\mathbb{N}}$ contains some subsequence $\left\{x_{n_j}\right\}_{j\in\mathbb{N}}$ that is OC in $\left[\left\{x_{n_j}\right\}_{j\in\mathbb{N}}\right]$; any AOT sequence $\{f_n\}_{n\in\mathbb{N}}$contains some subsequence $\left\{f_{n_j}\right\}_{j\in\mathbb{N}}$ that is OT on any subspace of X complemented to $\left\{f_{n_j}\right\}_{j\in\mathbb{N}}^{T}$.

Section (3.1): Almost over Complete Sequences

– [S] stands for the closure of the linear span of the set S;

– the annihilator in X^* of a subset Γ of the Banach space X is the subspace $\Gamma^{\perp} \subset X^*$ whose members are the bounded linear functionals on X that vanish on Γ;

– the annihilator in X of a subset Γ of the dual space X^* is the subspace $\Gamma^{\top} \subset X$, $\Gamma^{\top} = \cap_{f\in\Gamma} \ker f$;

– a set $\Gamma \subset X^*$ is called total over X whenever $\Gamma^{\top} = \{0\}$.

Recall that a sequence in a Banach space X is called overcomplete (OC in short) in X whenever the linear span of each of its subsequences is dense in X. It is a well-known fact that overcomplete sequences exist in any separable Banach space.

– A sequence in a Banach space X is called almost overcomplete (AOC in short) whenever the closed linear span of each of its subsequences has finite codimension in X.

– A sequence in the dual space X^* of the Banach space X is called overtotal on X (OT in short) whenever each of its subsequences is total over X.

– A sequence in the dual space X^* of the Banach space X is called almost overtotal (AOT in short) on X whenever the annihilator (in X) of each of its subsequences has finite dimension. Some applications have been shown to support the usefulness of these notions.

For instance, the fact that bounded AOC as well as AOT sequences must be strongly relatively compact makes it possible to answer quickly in the positive the following questions.

– Must any infinite-dimensional closed subspace of l_∞ contain infinitely many linearly independent elements with infinitely many zero-coordinates?

– Let $X \subset C(K)$ be an infinite-dimensional subspace of $C(K)$ where K is metric compact. Must an (infinite) sequence $\{t_k\}_{k\in\mathbb{N}}$ exist in K such that $x(t_k) = 0$ for infinitely many linearly independent $x \in X$?

It provide information about the structure of AOC and AOT sequences. For any separable Banach space X the following questions seem to be of interest.

– Does an AOC sequence exist in X that admits countably many subsequences such that the intersection of their closed linear spans is the origin?

– Does an AOT sequence exist on X that admits countably many subsequences such that the closure of the linear span of the union of their annihilators in X is the whole X?

We answer in the positive both of them, respectively. It is a remarkable fact that, in both cases, the involved subsequences cannot be nested (Propositions (3.1.4) and (3.1.2). The second aim is to give a possible explanation for the following fact. As a consequence, by using strong relative compactness of bounded AOT sequences we get e.g., as a special case, that any infinite-dimensional closed subspace of l_p contains infinitely many elements with infinitely many zero-coordinates not only when $p = \infty$, as we mentioned at the beginning, but for any $p \geq 1$. However, the case $p < \infty$ looks much more complicated to be handled than the case $p = \infty$. We provide an example to show one possible reason for that.

We refer for general information about AOC and AOT sequences. Here we point out only the evident fact that, if $\{(x_n, x_n^*)\}$ is a countable biorthogonal system, then neither $\{x_n\}$ can be almost overcomplete in $[\{x_n\}]$, nor $\{x_n^*\}$ can be almost overtotal on $[\{x_n\}]$.

Almost overcomplete and to totatal sequeuces

We start by recalling a simple method, to get an overcomplete sequence in any separable Banach space X. We will use it in the proof of Proposition (3.1.2).

Fact (3.1.1)[3]. Let $\{e_k\}_{k\in\mathbb{N}}$ be any bounded sequence such that $[\{e_k\}_{k\in\mathbb{N}}] = X$. Then the sequence

$$\{y_m\}_{m=2}^{\infty} = \left\{\sum_{k=1}^{\infty} e_k m^{-k}\right\}_{m=2}^{\infty}$$

is OC in X.

Proof. Let $\left\{y_{m_j}\right\}_{j=1}^{\infty}$ be any subsequence of $\{y_m\}_{m=2}^{\infty} = \{\sum_{k=1}^{\infty} e_k m^{-k}\}_{m=2}^{\infty}$, let

$$f \in X^* \cap \left\{y_{m_j}\right\}^{\perp} \tag{1}$$

and let D be the open unit disk in the complex field. Since the complex function $\phi : D \to \mathbb{C}$ defined by $\phi(t) = \sum_{k=1}^{\infty} f(e_k)t^k$ is holomorphic, from $f\left(y_{m_j}\right) = \phi(1/m_j) = 0$ for $=$ $1, 2, \ldots$, it follows $\phi \equiv 0$ that forces $f(e_k) = 0$ for every $k \in \mathbb{N}$. Since f in (1) was arbitrarily chosen, it follows $[\{y_{m_j}\}] = X$.

Proposition (3.1.2)[3]. Any (infinite-dimensional) separable Banach space X contains an AOC sequence $\{x_n\}_{n\in\mathbb{N}}$ with the following property: for each $i \in \mathbb{N}$, $\{x_n\}_{n\in\mathbb{N}}$ admits a subsequence, that we denote by $\left\{x_j^i\right\}_{j\in\mathbb{N}}$ to lighten notation, such that both the following conditions are satisfied

a) $\operatorname{codim}_X \left[\left\{x_j^i\right\}_{j\in\mathbb{N}}\right] = i$;

b) $\bigcap_{i\in\mathbb{N}} \left[\left\{x_j^i\right\}_{j\in\mathbb{N}}\right] = \{0\}$.

Proof. Let the biorthogonal system $\{e_k, e_k^*\}_{k\in\mathbb{N}} \subset X \times X^*$ provide a normalized M-basis for X. We recall that, by definition, the sequence $\{e_k^*\}_{k\in\mathbb{N}}$ must be total on X. Moreover, it is a well-known fact that, at least when A is a finite subset of N, a (topological) complement in X to the subspace $[\{e_k\}_{k\in A}]$ is the subspace $[\{e_k\}_{k\in\mathbb{N}/A}]$. For $i = 1, 2, \ldots$ put

$$Y_i = [\{e_k\}_{k\notin\{i,i+1,i+2,\ldots,2i-1\}}] \tag{2}$$

so $\operatorname{codim}_X Y_i = i$. For each integer $i \in \mathbb{N}$, Y_i is a Banach space itself so, by Fact (3.1.1), the sequence $\left\{y_m^i\right\}_{m\geq 2} \subset Y_i$ defined by.

$$y_m^i = \sum_{k=1,k\notin\{i,i+1,i+2,\ldots,2i-1\}}^{\infty} m^{-ik} e_k \quad i = 1,2,\ldots, m = 2,3,\ldots \tag{3}$$

provides an OC sequence in Y_i.

Order in any way the countable set $\bigcup_{i\in\mathbb{N},m\geq 2} \left\{y_m^i\right\}$ as a sequence $\{x_n\}_{n\in\mathbb{N}}$. For each i, select a subsequence $\left\{x_p^i\right\}_{p\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ whose terms belong to $\left\{y_m^i\right\}_{m\geq 2}$: this last sequence being OC in Y_i, we have $\operatorname{codim}_X[\left\{x_p^i\right\}_{p\in\mathbb{N}}] = \operatorname{codim}_X Y_i = i$. Moreover, since the sequence $\{e_k^*\}_{k\in\mathbb{N}}$ is total on X, it is clear that $\bigcap_{i=1}^{\infty} Y_i = \{0\}$, so $\bigcap_{i=1}^{\infty} [\left\{x_p^i\right\}_{p\in\mathbb{N}}] = \{0\}$ too.

It remains to show that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is AOC in X. Let $\left\{x_{n_j}\right\}_{j\in\mathbb{N}}$ be any of its subsequences. Two cases are possible.

A) For some i, $\left\{x_{n_j}\right\}_{j\in\mathbb{N}}$ contains infinitely many terms from $\left\{y_m^i\right\}_{m\geq 2}$: being $\left\{y_m^i\right\}_{m\geq 2}$ OC in Y_i, we have $\operatorname{codim}_X \left[\left\{x_{n_j}\right\}_{j\in\mathbb{N}}\right] \leq \operatorname{codim}_X Y_i = i$ and we are done.

B) For each i, $\left\{x_{n_j}\right\}_{j\in\mathbb{N}}$ contains at most finitely many terms from $\left\{y_m^i\right\}_{m\geq 2}$. Take any

$$f \in \left\{x_{n_j}\right\}_{j\in\mathbb{N}}^{\perp}. \tag{4}$$

We prove that $f(e_k) = 0$ for every $k \in \mathbb{N}$: it implies $f = 0$, that means that $\left\{x_{n_j}\right\}_{j\in\mathbb{N}}$ is complete in X. Suppose by contradiction that $f(e_{\bar{k}}) \neq 0$ for some index $\bar{k}$: without loss of generality we may assume that $\bar{k}$ is the first of such indexes. For $j \in \mathbb{N}$, let

$$y_{m(j)}^{i(j)} = x_{n_j};$$

Put

$$A = \{i : i = i(j), j \in \mathbb{N}, i(j) > k\}.$$

Under our assumption $i(j)$ goes to infinity with j, so A is infinite and we have $e_{\bar{k}} \in Y_i$ for every $i \in A$. For $i \in A$, put

$$m_i = \min\{m(j) : i(j) = i, y_{m(j)}^{i(j)} \in \{y_m^i\}_{m\geq 2}\}.$$

From (4) it follows that, for each $i \in A$, we have

$$f(e_{\bar{k}}) = -m_i^{i\bar{k}} \sum_{k>\bar{k}, k\notin\{i,i+1,i+2,\dots,2i-1\}}^{\infty} m_i^{-ik} f(e_k) \qquad (5)$$

hence

$$|f(e_{\bar{k}})| \leq m_i^{i\bar{k}} \|f\| \sum_{k>\bar{k}, k\notin\{i,i+1,i+2,\dots,2-1\}}^{\infty} m_i^{-ik} \leq$$

$$\leq \|f\| \sum_{k=\bar{k}+1}^{\infty} m_i^{i(\bar{k}-k)} \leq 2\|f\| m_i^{-i} \to 0 \quad \text{as} \quad i \to \infty \qquad (6)$$

that forces $f(e_k) = 0$, so contradicting our assumption. We are done.

Our construction above can be modified by replacing (2) with

$$Y_i = [\{e_k\}_{k\neq i}] \qquad (7)$$

and modifying (3), (5) and (6) according to that. In this case it is still true that $\bigcap[\{x_{n_j}\}_{j\in\mathbb{N}}] = \{0\}$ as $\{x_{n_j}\}_{j\in\mathbb{N}}$ ranges among all possible subsequences of the AOC sequence $\{x_n\}_{n\in\mathbb{N}}$, but actually the codimension of the closure of the linear span of any subsequence is at most 1. In other words, the following alternative version to Proposition (3.1.2) holds.

Proposition (3.1.3)[3]. Any (infinite-dimensional) separable Banach space X contains an AOC sequence $\{x_n\}_{n\in\mathbb{N}}$ with the following property: $\{x_n\}_{n\in\mathbb{N}}$ admits countably many subsequences $\{x_j^i\}_{j\in\mathbb{N}}, i = 1, 2, \dots$, such that both the following conditions are satisfied

a) $\operatorname{codim}_X[\{x_j^i\}_{j\in\mathbb{N}}] = 1$ for each i;

b) $\bigcap_{i\in\mathbb{N}} [\{x_j^i\}_{j\in\mathbb{N}}] = \{0\}$.

By the previous proposition, it is matter of evidence that actually the conclusion $\bigcap_{i\in\mathbb{N}} [\{x_j^i\}_{j\in\mathbb{N}}] = \{0\}$ is due to the fact that infinitely many pairwise "skew" subsequences can be found of $\{x_n\}_{n\in\mathbb{N}}$. This consideration is stressed by the following proposition.

Proposition (3.1.4)[3]. Let $\{x_n\}_{n\in\mathbb{N}}$ be any AOC sequence in any (infinite-dimensional) separable Banach space X and let $\{x_j^1\}_{j\in\mathbb{N}} \supset \{x_j^2\}_{j\in\mathbb{N}} \supset \{x_j^3\}_{j\in\mathbb{N}} \supset \cdots$ be any countable family of nested subsequences of $\{x_n\}_{n\in\mathbb{N}}$. Then the increasing sequence of integers $\left\{\operatorname{codim}_X\left[\{x_j^i\}_{j\in\mathbb{N}}\right]\right\}_{i\in\mathbb{N}}$ is finite (so eventually constant).

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be an AOC not OC sequence in X and let $\{x_j^1\}_{j\in\mathbb{N}}$ be any of its subsequences whose linear span is not dense in X. Put

$$X_1 = \left[\{x_j^1\}_{j\in\mathbb{N}}\right], \qquad p_1 = \operatorname{codim}_X X_1 \geq 1.$$

If $\{x_j^1\}_{j\in\mathbb{N}}$ is OC in X_1 we are done; otherwise, let $\{x_{j_k}^1\}_{k\in\mathbb{N}}$ be any of its subsequences whose linear span is not dense in X_1. Put

$$\{x_{j_k}^1\}_{k\in\mathbb{N}} = \{x_j^2\}_{j\in\mathbb{N}}, \qquad X_2 = \left[\{x_j^2\}_{j\in\mathbb{N}}\right], \qquad p_2 = \operatorname{codim}_X X_2 > p_1.$$

Now we can continue in this way. Let us prove that this process must stop after finitely many steps. Assume the contrary, i.e. that a nested infinite family

$$\{x_j^1\}_{j\in\mathbb{N}} \supset \{x_j^2\}_{j\in\mathbb{N}} \supset \cdots \supset \{x_j^i\}_{j\in\mathbb{N}} \supset \cdots$$

of subsequences of $\{x_n\}_{n\in\mathbb{N}}$ can be found such that $p_i \uparrow \infty$ as $i \uparrow \infty$, where $p_i = \operatorname{codim}_X X_i$ with

$$X_i = [\{x_j^i\}_{j\in\mathbb{N}}].$$

Under this assumption, we can construct a linearly independent sequence $\{f_i\}_{i=1}^{\infty} \subset X^*$ such that, for each i, $f_i \in X_{i+1}^{\perp} \setminus X_i^{\perp}$. For each i, let y_i be an element of the sequence $\{x_j^i\}_{j\in\mathbb{N}}$ not belonging to the sequence $\{x_j^{i+1}\}_{j\in\mathbb{N}}$ such that $f_i(y_i) \neq 0$ (of course such an element must exist): because of our construction we have $f_k(y_i) = 0$ for each $k < i$. Without loss of generality we may assume $f_i(y_i) = 1$.

Now, following a standard procedure due to Markushevich, put

$$g_1 = f_1, \qquad g_2 = f_2 - f_2(y_1)g_1, \qquad g_3 = f_3 - f_3(y_1)g_1 - f_3(y_2)g_2, \ldots$$

$$\ldots, g_k = f_k - \sum_{i=1}^{k-1} f_k(y_i)g_i \ldots$$

Clearly we have $g_k(y_i) = \delta_{k,i}$ for each $k, i \in \mathbb{N}$, so actually $\{y_k, g_k\}_{k\in\mathbb{N}}$ is a biorthogonal system with $\{y_k\}_{k\in\mathbb{N}} \subset \{x_n\}_{n\in\mathbb{N}}$. This is a contradiction since $\{x_n\}_{n\in\mathbb{N}}$ was an AOC sequence.

As an immediate consequence of Proposition (3.1.4) we get the following

Corollary (3.1.5)[3]. Any AOC sequence $\{x_n\}_{n\in\mathbb{N}}$ in a separable Banach space X contains some subsequence $\{x_{n_j}\}_{j\in\mathbb{N}}$ that is OC in $[\{x_{n_j}\}_{j\in\mathbb{N}}]$ (with, of course, $[\{x_{n_j}\}_{j\in\mathbb{N}}]$of finite codimension in X).

Section (3.2): Almost over Total Sequences

The results shown about AOC sequences have a dual restatement for AOT sequences.

Proposition (3.2.1)[3]. Let X be any (infinite-dimensional) separable Banach space. Then there is a sequence $\{f_n\}_{n\in\mathbb{N}} \subset X^*$ that is AOT on X and, for each $i \in N$, admits a subsequence $\{f_j^i\}_{j\in\mathbb{N}}$ such that both the following conditions are satisfied

a) $\dim \{f_j^i\}_{j\in\mathbb{N}}^{\top} = i$;

b) $[\cup_{i\in\mathbb{N}} \{f_j^i\}_{j\in\mathbb{N}}^{\top}] = X$.

Proof. The idea for the proof is the same as for the proof of Proposition (3.1.2), so we confine ourselves to sketch the fundamental steps.

Let the biorthogonal system $\{e_k, e_k^*\}_{k\in\mathbb{N}} \subset X \times X^*$ provide an M-basis for X with $\{e_k^*\}_{k\in\mathbb{N}}$ a norm-one sequence in X^*. For $i = 1, 2, \ldots$ put

$$Z_i = \left[\{e_k\}_{k=i}^{2i-1}\right], \qquad Y_i = [\{e_k\} k \notin \{i, i+1, i+2, \ldots, 2i-1\}],$$
$$^*\mathrm{Y_i} = [\{\mathrm{e}_k^*\}\mathrm{k} \notin \{\mathrm{i}, \mathrm{i}+1, \mathrm{i}+2, \ldots, 2\mathrm{i}-1\}].$$

Clearly $X = Z_i \oplus Y_i$ and ${}^*Y_i^{\top} = Z_i$, so $\dim {}^*Y_i^{\top} = i$ for $i = 1, 2, \ldots$. For each integer $i \in N$, the sequence $\{y_m^{*i}\}_{m\geq 2} \subset {}^*Y_i$ defined by

$$y_m^{*i} = \sum_{k=1, k\notin\{i, i+1, i+2, \ldots, 2i-1\}}^{\infty} m^{-ik} e_k^* \qquad i = 1, 2, \ldots, m = 2, 3, \ldots$$

being over complete in the Banach space ${}^*Y_i$, is overtotal on Y_i.

Order in any way the countable set $\cup_{i\in\mathbb{N}, m\geq 2} \{y_m^{*i}\}$as a sequence $\{f_n\}_{n\in\mathbb{N}}$. For each i, select a subsequence $\{f_p^i\}_{p\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}}$ whose terms belong to $\{y_m^{*i}\}_{m\geq 2}$: since this last sequence is overtotal on Y_i, we have $\{f_p^i\}_{p\in\mathbb{N}}^{\top} = Z_i$ too, so $\dim\{f_p^i\}_{p\in\mathbb{N}}^{\top} = i$. Moreover, since the sequence $\{e_k\}_{k\in\mathbb{N}}$ is complete in X, we have $[\cup_{i=1}^{\infty} Z_i] = X$.

It remains to show that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is AOT on X. Let $\{f_{n_j}\}_{j\in\mathbb{N}}$ be any of its subsequences. Two cases are possible.

A) For some i, $\{f_{n_j}\}_{j\in\mathbb{N}}$ contains infinitely many terms from $\{y_m^{*i}\}_{m\geq 2}$: being $\{y_m^{*i}\}_{m\geq 2}$ OT on Y_i, we have $\{f_{n_j}\}_{j\in\mathbb{N}}^{\top} \subset Z_i$, $\dim\{f_{n_j}\}_{j\in N}^{\top} \leq i$ and we are done.

B) For each i, $\{f_{n_j}\}_{j\in\mathbb{N}}$ contains at most finitely many terms from $\{y_m^{*i}\}_{m\geq 2}$. Take any $x \in \{f_{n_j}\}_{j\in\mathbb{N}}^{\top}$: by proceeding exactly as in B) of the proof of Proposition (3.1.2), just interchanging the roles of points and functionals, we get $e_k^*(x) = 0$ for every $k \in \mathbb{N}$.

$\{e_k^*\}_{k\in\mathbb{N}}$ being total on X, it follows $x = 0$. It means that $\left\{f_{n_j}\right\}_{j\in\mathbb{N}}$ too is total on X and again we are done. The proof is complete.

As we did for AOC sequences, with obvious modifications in the previous proof we can obtain for AOT sequences the following alternative version to Proposition (3.2.1): it is the dual version to Proposition (3.1.3).

Proposition (3.2.2)[3]. Let X be any (infinite-dimensional) separable Banach space. Then there is a sequence $\{f_n\}_{n\in\mathbb{N}} \subset X^*$ that is AOT on X and admits countably many subsequences $\{f_j^i\}_{j\in\mathbb{N}}$, $i = 1, 2, \ldots$, such that both the following conditions are satisfied

a) $\dim\{f_j^i\}_{j\in\mathbb{N}}^{\top} = 1$ for each i;

b) $[\bigcup_{i\in\mathbb{N}} \ \{f_j^i\}_{j\in N}^{\top}] = X$.

We point out that, though the existence of an AOT sequence on a Banach space X does not imply X to be separable (one of the significant applications of this concept was to the space l_∞), the results we have shown in Propositions (3.2.1) and (3.2.2), as they have been stated, must concern only separable spaces. In fact, the annihilator of any subsequence of any AOT sequence being finite-dimensional, the closed linear span of the union of countably many of such annihilators must be separable too.

Finally we notice that also Proposition (3.1.4) has its dual version that shows that the countably many subsequences in the statement of Proposition (3.2.2) cannot be assumed to be nested. The proof can be carried on exactly like the proof of Proposition (3.1.4), just interchanging the roles of points and functionals.

Proposition (3.2.3)[3]. Let $\{f_n\}_{n\in\mathbb{N}}$ be any sequence AOT on any (infinite-dimensional) Banach space Xand let $\{f_j^1\}_{j\in\mathbb{N}} \supset \{f_j^2\}_{j\in\mathbb{N}} \supset \{f_j^3\}_{j\in\mathbb{N}} \supset \cdots$ be any countable family of nested subsequences of $\{f_n\}_{n\in\mathbb{N}}$. Then the increasing sequence of integers $\left\{\dim\{f_j^i\}_{j\in\mathbb{N}}^{\top}\right\}_{i\in\mathbb{N}}$ is finite (so eventually constant).

As an immediate consequence of Proposition (3.2.3) we get the following

Corollary (3.2.4)[3]. Any AOT sequence $\{f_n\}_{n\in\mathbb{N}}$ on a Banach space X contains some subsequence $\left\{f_{n_j}\right\}_{j\in\mathbb{N}}$ that is OT on any subspace of X complemented to $\left\{f_{n_j}\right\}_{j\in\mathbb{N}}^{\top}$ (with, of course, $\left\{f_{n_j}\right\}_{j\in\mathbb{N}}^{\top}$ of finite dimension).

We provide an example that may be of interest in Operator theory. That any infinite-dimensional closed subspace of l_p contains infinitely many elements with infinitely many zero-coordinates not only when $p = \infty$, as we mentioned at the beginning, but for any $p \geq 1$. In fact the following much more general results have been proved there.

Theorem (3.2.5)[3]. Let X be a separable infinite-dimensional Banach space and $T: X \to l_\infty$ be a one-to-one bounded non-compact linear operator. Then there exist an infinite-dimensional subspace $Y \subset X$ and a strictly increasing sequence $\{n_k\}$ of integers such that $e_{n_k}(Ty) = 0$ for any $y \in Y$ and for any k (enthe "n-coordinate functional" on l_∞).

Theorem (3.2.6)[3]. Let X, Y be infinite-dimensional Banach spaces. Let Y have an unconditional basis $\{u_i\}_{i=1}^{\infty}$ with $\{e_i\}_{i=1}^{\infty}$ as the sequence of the associated coordinate functionals. Let $T: X \to Y$ be a one-to-one bounded non-compact linear operator. Then there exist an infinite-dimensional subspace $Z \subset X$ and a strictly increasing sequence $\{k_l\}$ of integers such that $e_{k_l}(Tz) = 0$ for any $z \in Z$ and any $l \in \mathbb{N}$.

To show both the theorems, the fundamental tool was the fact that bounded AOT sequences are strongly relatively compact. However, despite Theorem (3.2.5) was then obtained as a quite easy consequence of the Ascoli–Arzelà Theorem, the proof of Theorem (3.2.6) has required some additional delicate tools. One could expect that Theorem (3.2.6) should be proved in a simple way by the following argument.

"Under notation as in the statement of Theorem (3.2.6), assume by contradiction that for each sequence of integers $\{i_j\}$ we have $\dim\left(\left\{T^* e_{i_j}\right\}^{\top}\right) < \infty$. Then the sequence $\{T^* e_i\} \subset X^*$ is almost over total on X, so $\{T^* e_i\}$ is relatively norm-compact in X^*. $\{e_i\}$ being the sequence of the coordinate functionals associated to the (unconditional) basis $\{u_i\}$ of Y, that forces T to be a compact operator, contradicting our assumption."

In fact this argument does not work since the last conclusion T being forced to be compact is false, as the following example shows.

Example (3.2.7)[3]. There exist a Banach space Y with an unconditional basis $\{u_i\}_{i\in\mathbb{N}}$, $\{e_i\}_{i\in\mathbb{N}}$ being the sequence of the associated coordinate functionals, and a non-compact operator $T: c_0 \to Y$ such that $T^* e_i \to 0$ as $i \to \infty$ (so the sequence $\{T^* e_i\}$ is relatively norm compact).

Proof. Let $\{u_i^k\}_{i=1}^{k}$ be the natural (algebraic) basis of $\mathbb{R}^k$. For $k \in \mathbb{N}$, define $T_k: \mathbb{R}^k \to \mathbb{R}^k$ in the following way

$$T_k\left(\sum_{i=1}^{k} a_i u_i^k\right) = \sum_{i=1}^{k} a_i\, u_i^k / k, \qquad a_i \in \mathbb{R} \text{ for } i = 1, \dots, k.$$

Let l_∞^k ⟨resp. l_1^k⟩ be the k-dimensional space $\mathbb{R}^k$ endowed with the max-norm ⟨resp. the $1-$ norm⟩. If we consider $T^k: l_\infty^k \to l_1^k$, we easily get $\|T_k\| = 1$ for every $k \in \mathbb{N}$.

For a sequence $\left\{X_k, \|\cdot\|_{X_k}\right\}_{k=1}^{\infty}$ of Banach spaces, consider the Banach space $\left(\bigoplus_{k=1}^{\infty} X_k\right)_{c_0}$ (the linear space, under the usual algebraic operations, whose elements are the sequences $\{x_k\}_{k=1}^{\infty}$, $x_k \in X_k$ for each k, such that $\|x_k\|_{X_k} \to 0$ as $k \to \infty$, endowed with the norm $\|\{x_k\}_{k=1}^{\infty}\| = \max_k \|x_k\|_{X_k}$).

Clearly we have

$$c_0 = \left(\bigoplus_{k=1}^{\infty} l_\infty^k\right)_{c_0}. \tag{8}$$

Put

$$Y = \left(\bigoplus_{k=1}^{\infty} l_1^k\right)_{c_0}.$$

Order the set $\cup_{k=1}^{\infty} \{u_i^k\}_{i=1}^{k}$ in the natural way and rename it as

$$\{u_1^1, u_1^2, u_2^2, \dots, u_1^k, \dots, u_k^k, \dots\} = \{u_1, u_2, u_3, \dots\}. \tag{9}$$

Of course $\{u_i\}_{i=1}^{\infty}$ is an unconditional basis both for c_0 and for Y. Call P_k the natural norm-one projection of c_0 on to l_∞^k suggested by (8) and define $T: c_0 \to Y$ in the following way

$$Tx = \sum_{i=0}^{\infty} T_k P_k x, \qquad x \in c_0.$$

T is a (linear) non-compact operator, since $\left\|T\left(\sum_{i=1}^{k} u_i^k\right)\right\| = 1$ and $\sum_{i=1}^{k} u_i^k$ is weakly null as $k \to \infty$. However, if we denote by $\{e_i\}_{i=1}^{\infty}$ the sequence of the coordinate functionals associated to the basis $\{u_i\}_{i=1}^{\infty}$ of Y, it is true that $T^* e_i \to 0$ in X^* as $i \to \infty$. In fact, for $x = \sum_{k=1}^{\infty} \sum_{j=1}^{k} x_j^k u_j^k \in B_{c_0}$ the following holds

$$\left|x_j^k\right| \le 1 \quad 1 \le j \le k, \qquad k = 1, 2, \dots$$

so, if we denote by $u_{j_i}^{k_i}$ the element u_i as identified by (9), we have

$$|(T^* e_i)(x)| = \left|e^i(Tx)\right| = \left|e_i\left(\sum_{k=1}^{\infty} \sum_{j=1}^{k} x_j^k u_j^k / k\right)\right| = \left|x_{j_i}^{k_i}\right| / k_i \le 1/k_i.$$

Since $k_i \to \infty$ with i, we are done.

Theorem (3.2.8)[8]. Each almost over complete bounded sequence in a Banach space is relatively norm-compact.

Proof. Let $\{x_n\}$ be an almost over complete bounded sequence is (separable) Banach space $(X, \|\cdot\|)$ without loss of generality we may assume, possibly passing to an equivalent norm that the norm $\|\cdot\|$ is locally Uniformly rotund (LUR) and that $\{x_n\}$ is normalized under that norm.

First note that $\{x_n\}$ is relatively weakly compact other wise, it is known that it should admit som subsequence that is a basic sequence, a contradiction. Hence by the Eberlein-Smulyan theorem states that the three are equivalent on a Banach space. While this equivalence is truein general for (metric space), the weak topology is not metrizable in infinite dimensional vector spaces, and so the Eberlein-Smulyan theorem is needed [9] that $\{x_n\}$ admits som subsequence $\{x_{n_k}\}$ that weakly converges to some point $x_0 \in B_X$. Two coses must now be considered:

(i) $\|x_0\| < 1$. Form $\left\|x_{n_k} - x_0\right\| \ge 1$. $\|x_0\| > 0$, according to a well known we sult, it follows that sequence: hence codim $\left\{x_{n_{k_i}} - x_0\right\}$ is a basic sequence: hence codim $\left[\left\{x_{n_{k_{2i}}} - x_0\right\}\right] = \text{codim}\left[\left\{x_{n_{k_{2i}}}\right\}, 0\right] = \text{codim}\left[\left\{x_{n_{k_{2i}}}\right\}\right] = \infty$ a contradiction.

$\|x_0\| = 1$ sine we are working with a LUR norm, the subsequence $\{x_{n_k}\}$ a ctually converges to x_0 in the norm too and we are done.

Chapter 4
Banach Spaces and Superprojectivity

We show that the class of superprojective spaces is stable under finite products, certain unconditional sums, certain tensor products, and other operations, providing new examples.

Section (4.1): Some Properties of Superprojective Spaces

A Banach space X is called subprojective if every (closed) infinite-dimensional subspace of X contains an infinite-dimensional subspace complemented in X, and X is called superprojective if every infinite- codimensional subspace of X is contained in an infinite-codimensional subspace complemented. These two classes of Banach spaces were introduced by Whitley in order to find conditions for the conjugate of an operator to be strictly singular or strictly cosingular. More recently, they have been used to obtain some positive solutions to the perturbation classes problem for semi-Fredholm operators. This problem has a negative solution in generae, but there are some positive answers when one of the spaces is subprojective.

There are many examples of subprojective spaces, like ℓ_p for $1 \leq p < \infty, L_p$ $(0, 1)$ for $2 \leq p < \infty, C(K)$ with K a scattered compact and some Lorentz and Orlicz spaces. It is not difficult to show that subspaces of subprojective spaces are subprojective, and quotients of superprojective spaces are superprojective (Proposition (4.1.4)) and, as a consequence of the duality relations between subspaces and quotients, a reflexive space is sub- projective (superprojective) if and only if its dual space is superprojective (subprojective), which provides many examples of reflexive superprojective spaces. However, the only examples of non- reflexive superprojective spaces previously known are the $C(K)$ spaces with K a scattered compact and their infinite-dimensional quotients.

Some of the duality relations between subprojective and superprojective spaces are known to fail in general:

(a) X being subprojective does not imply that X^* is superprojective, for instance for $X = c_0$ and $X^* = \ell_1$.

(b) X^* being subprojective does not imply that X is superprojective, for instance for the hereditarily indecomposable space obtained whose dual is isomorphic to ℓ_1.

However we do not know if the remaining relations are valid:

(a') Does X being superprojective imply that X^* is subprojective?

(b') Does X^* being superprojective imply that X is subprojective?

The answer to these two questions is likely negative, but we know of few examples of non-reflexive super- projective spaces to check, and none of them is a dual space.

Studied the stability properties of subprojective spaces under vector sums, tensor products and other operations, obtaining plenty of new examples of subprojective spaces.

We will begin with some auxiliary results and show some properties of subprojective and superprojective spaces, such as the fact that superprojective spaces cannot contain copies of 1, which restricts the search for non-reflexive examples of these spaces, and we also characterise the superprojectivity of some projective tensor products. Following the scheme, we show several stability results for the class of superprojective spaces under finite products, certain unconditional sums and certain tensor products, and we provide new examples of superprojective spaces.

The dual space of a Banach space X is X^*, and the action of $x^* \in X^*$ on $x \in X$ is written as $\langle x^*, x\rangle$. Given a subset M of a Banach space X, its annihilator in X^* will be denoted by $M_\perp$; if M is a subset of X^*, its annihilator in X will be denoted by $M_\perp$. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X, then $[x_n : n \in \mathbb{N}]$ will denote the closed linear span of $(x_n)_{n\in\mathbb{N}}$ in X. The injective and projective tensor products of X and Y are respectively denoted by $X \widehat{\otimes}_\pi$
Operators will always be bounded. The identity operator on X is denoted by I_X.
Given an operator $T : X \to Y$, $N(T)$ and $R(T)$ denote the kernel and the range of T, and $T^* : Y^* \to X^*$ denotes its conjugate operator. An operator $T : X \to Y$ is strictly singular if $T|_M$ is an isomorphism only if M is finite-dimensional; and T is strictly cosingular if there is no operator $Q : Y \to Z$ with Z infinite-dimensional such that QT is surjective or, equivalently, if there is no infinite-codimensional (closed) subspace N of Y such that $R(T) + N = Y$.

The way that superprojective Banach spaces are defined means that we will be dealing with infinite- codimensional subspaces and their induced quotients often, so we will adopt the following definition.

Definition (4.1.1)[4]. We will say that an operator $T : X \to Y$ is a surjection if T is surjective and Y is infinite- dimensional.

The following results will be useful when dealing with complemented subspaces, subjections and super projective spaces.

Proposition (4.1.2)[4]. For a Banach space X, the following are equivalent:

(i) X is superprojective;

(ii) for any surjection $T : X\, Y$, there exists another surjection $S : Y \longrightarrow Z$ *such* that $N(ST)$ is complemented in X.

Proof. For the direct implication, let $T : X \longrightarrow Y$ be a surjection, so that $N(T)$ is infinite-codimensional in X. By the superprojectivity of X, $N(T)$ is contained in a complemented, infinite-codimensional subspace M of X, and clearly $T(M)$ is closed in Y. Thus the quotient map Q from Y onto $Y/T(M)$ is a surjection such that $N(QT) = M$ is complemented in X.

For the converse implication, let M be an infinite-codimensional subspace of X, so that $QM : X \longrightarrow X/M$ is a surjection. Then there exists another surjection $S : X/M \longrightarrow Z$ such that $N(SQM)$ is infinite- codimensional and complemented in X, and contains M.

The next result allows to push the complementation of a subspace through an operator under certain conditions.

Proposition (4.1.3)[4]. Let X, Y and Z be Banach spaces and let $T : X \longrightarrow Y$ and $S : Y \longrightarrow Z$ be operators such that ST is a surjection and $N(ST)$ is complemented in X. Then $N(S)$ is complemented in Y.

Proof. Let H be a subspace of X such that $X = N(ST) \oplus H$. Since $ST : X \longrightarrow Z$ is a surjection, $ST|_H$ must be an isomorphism onto Z; in particular, $T|_H$ is an isomorphism and $Y = N(S) \oplus T(H)$, as proved by the projection $T(ST|_H)^{-1} S : Y \longrightarrow Y$.

A simple consequence of Propositions (4.1.2) and (4.1.3) is the fact that the class of superprojective spaces is stable under quotients.

Proposition (4.1.4)[4]. Let X be a superprojective Banach space and let $T : X \longrightarrow Y$ be a surjection. Then Y is superprojective

Proof. Let $S : Y \longrightarrow Z$ be a surjection; then ST is a surjection and, by Proposition (4.1.2), there exists an-other surjection $R : Z \longrightarrow W$ such that $N(RST)$ is complemented in X. By

Proposition (4.1.3), $N\ (RS)$ is complemented in Y, which means, again by Proposition (4.1.2), that Y is superprojective.
Finally, we will state a technical observation on the behaviour of surjections on spaces that have a complemented superprojective subspace.
Proposition (4.1.5)[4]. Let X be a Banach space, let $P : X \longrightarrow X$ be a projection with $P(X)$ superprojective and let $S : X \longrightarrow Y$ be a surjection such that SP is not strictly cosingular. Then there exists another surjection $R : Y \longrightarrow Z$ such that $N(RS)$ is complemented in X.
Proof. Let $J : P\ (X\ ) \longrightarrow X$ be the natural inclusion; then $SP = SJP$ is not strictly cosingular, so neither is $SJ : P\ (X\ ) \longrightarrow Y$. Therefore, there exists a quotient map $Q : Y \longrightarrow W$ such that QSJ is a surjection, and Proposition (4.1.2) provides another surjection $R : W \longrightarrow Z$ such that $N(RQSJ)$ is complemented in $P(X\ )$; by Proposition (4.1.3), $N(RQS)$ is complemented in X, where RQ: $Y \longrightarrow Z$ is a surjection.

The following results gives some simple but useful necessary conditions for Bananch space X to be subprojectiv or superprojective.
Proposition (4.1.6)[4]. Let X and Z be infinite-dimensional Banach spaces.
(i) If $J : Z \longrightarrow X$ is a strictly cosingular embedding, then X is not subprojective.
(ii) If $Q : X \longrightarrow Z$ is a strictly singular surjection, then X is not superprojective.
Proof. (i) If $X = M \oplus H$ with $M \subseteq J(Z\ )$, then $Q_H J$ is surjective. Since J is strictly cosingular, H is finite-codimensional and M is finite-dimensional.
(ii) If $X = M \oplus H$ with $N(Q) \subseteq M$, then $Q|_H$ is an embedding. Since Q is strictly singular, H is finite-dimensional.
In spite of its simplicity, Proposition (4.1.6) has several straight forward consequences. Proposition (4.1.7) for subprojective spaces with the same example but a different argument. Here we extend it to superprojective spaces. Recall that a class C of Banach spaces satisfies the three-space property if a Banach space X belongs to C whenever M and X/M belong to C for some subspace M of X.
Proposition (4.1.7)[4]. The classes of subprojective and superprojective spaces do not satisfy the three-space prop- erty.
Proof. Let $1 < p < \infty$ and recall that $\ell_p$is both subprojective and superprojective. Let Z_p be the Kalton– Peck space introduced. Then there exists an exact sequence

$$0 \longrightarrow \ell_{\mathrm{p}} \longrightarrow^{i} \longrightarrow^{q} \ell_{\mathrm{p}} \longrightarrow 0$$

In which i is strictly cosingular and q is strictly singular. By Proposition (4.1.6), Z_p is neither subprojective nor superprojective.
Since ℓ_1 is subprojective, the following result suggests that the class of non-reflexive superprojective spaces is smaller than that of non-reflexive subprojective spaces.
Proposition (4.1.8)[4]. Let X be a Banach space containing a subspace isomorphic to ℓ_1. Then X is not superpro- jective and X^* is not subprojective.
Proof. If X contains a subspace isomorphic to ℓ_1, then there exists a surjective operator $Q : X \longrightarrow 2$ which is 2-summing, therefore weakly compact and completely continuous, therefore strictly singular: Indeed, if $Q|_M$ is an isomorphism, then M is reflexive and weakly convergent sequences in M are convergent, so M is finite-dimensional. By Proposition (4.1.6), X is not superprojective. For the second part, observe that
$Q^{**} : X^{**} \longrightarrow \ell_2$ is also 2-summing. Then Q^{**} is strictly singular, hence $Q^* : \ell_2 \longrightarrow X^*$ is a strictly cosingular embedding. Proposition (4.1.8) allows to fully characterise the

superprojectivity of $C(K)$ spaces. Recall that a compact space is called scattered if each of its non-empty subsets has an isolated point.

Corollary (4.1.9)[4]. Let K be a compact set. Then $C(K)$ is superprojective if and only if K is scattered.

Proof. If K is scattered, then $C(K)$ is superprojective. On the other hand, if K is not scattered, then $C(K)$ contains a copy of ℓ_1 and cannot be superprojective by Proposition (4.1.8).

It also follows immediately that certain tensor products cannot be superprojective.

Corollary (4.1.10)[4]. Let X and Y be Banach spaces and suppose that X admits an unconditional finite-dimensional decomposition and $L(X, Y^*) \neq K(X, Y^*)$. Then $X \widehat{\oplus}_\pi Y$ is not superprojective

Proof. Note that $(X \widehat{\oplus}_\pi Y)^* \equiv L(X, Y^*)$. Since $L(X, Y^*) \neq K(X, Y^*)$, we have that $L(X, Y^*)$ contains ℓ_∞, hence $X \widehat{\oplus}_\pi$ Ycontains a (complemented) copy of ℓ_1.

Since the spaces ℓ_p phave an unconditional basis and are subprojective and superprojective for $1 < p < \infty$, we can now characterise the superprojectivity of the tensor products $\ell_p \widehat{\oplus}_\pi \ell_q$.

Corollary (4.1.11)[4]. Let $1 < p, q < \infty$. Then the following are equivalent:

(i) $\ell_p \widehat{\oplus}_\pi \ell_q$ is superprojective;

(ii) $\ell_p \widehat{\oplus}_\pi \ell_q$ is reflexive;

(iii) $L(\ell_p, \ell_q^*) = K(\ell_p, \ell_q^*)$;

(iv) $p > q/(q-1)$.

Proof. We have that $\ell_p \widehat{\oplus}_\pi \ell_q$is reflexive if and only if $L(\ell_p, \ell_q^*) = K(\ell_p, \ell_q^*)$ if and only if $p > q/(q-1)$. If $L(\ell_p, \ell_q^*) = K(\ell_p, \ell_q^*)$, then $\ell_p \widehat{\oplus}_\pi \ell_q$ is not superprojective by Corollary (4.1.10); otherwise, $\ell_p \widehat{\oplus}_\pi \ell_q$ is reflexive and $\ell_p \widehat{\oplus}_\pi \ell_q = \left(\ell_p^* \widehat{\oplus}_\varepsilon \ell_q^*\right)^*$, so $\ell_p^* \widehat{\oplus}_\varepsilon \ell_q^*$ is reflexive and subprojective and $\ell_p \widehat{\oplus}_\pi \ell_q$ is superprojective.

Corollary (4.1.12)[4]. $\ell_p \widehat{\oplus}_\pi \ell_q$ is not superprojective for any $1 \leq p, q \leq \infty$.

Proof. If p is either 1or strictly greater than 2, then L_p itself is not superprojective, so neither is $\ell_p \widehat{\oplus}_\pi \ell_q$ and similarly for q. Thus, we are only concerned with the case $1 < p, q \leq 2$, but then both L_p and L_q contain complemented copies of ℓ_2, so $L\left(L_p \widehat{\oplus}_\pi L_q^*\right) \neq K\left(L_p, L_q^*\right)$ and $L_p \widehat{\otimes}_\pi L_q$ is not superprojective by Corollary (4.1.10).

Section (4.2): Stability Results for Superprojective Spaces

We show some stability results for the class of superprojective spaces. Our first result here, and key to subsequent ones, proves that the direct sum of two superprojective Banach spaces is again superprojective.

Proposition (4.2.1)[4]. Let X and Y be Banach spaces. Then $X \oplus Y$ is superprojective if and only if both X and Y are superprojective.

Proof. X and Y are quotients of $X \oplus Y$; if $X \oplus Y$ is superprojective, then so are X and Y by Proposition (4.1.4). Conversely, assume that X are Y are both superprojective, and define the projections $P_X: X \oplus Y \longrightarrow X \oplus Y$, with range X and kernel Y, and $P_Y: X \oplus Y \longrightarrow X \oplus Y$, with range Y and kernel X. Take any surjection $S: X \oplus Y \longrightarrow Z$. Then $S = SP_X + SP_Y$ is not strictly cosingular, so either SP_X or SP_Y is not strictly cosingular; without loss of generality, we will assume that it is SP_X. By Proposition (4.1.5), there exists another surjection $R: Z \longrightarrow W$such that $N(RS)$ is complemented in $X \oplus Y$, which finishes the proof by Proposition (4.1.2).

We will now state the result, which proves that a space is superprojective if it admits a suitable decomposition into superprojective parts. Recall that an operator $T: X \longrightarrow Y$ is upper semi-Fredholmif $N(T)$ is finite-dimensional and $R(T)$ is closed, and T is lower semi-Fredholm if $R(T)$ is finite-codimensional (hence closed). Note that T is lower semi-Fredholm if and only if T^* is upper semi-Fredholm.

Theorem (4.2.2)[4]. Let X be a Banach space, let Λ be a well-ordered set and let $(P_\lambda)_{\lambda\in\Lambda}$ and $(Q_\lambda)_{\lambda\in\Lambda}$ be bounded families of projections on X such that:

(i) $P_\lambda^* x^* \longrightarrow_\lambda x^*$ for every $x^* \in X^*$;

(ii) $P_\mu P_\nu = P_{\min\{\mu,\nu\}}$ and $Q_\mu Q_\nu = Q_{\min\{\mu,\nu\}}$ for every $\mu, \nu \in \Lambda$;

(iii) $Q_\mu P_\nu = P_\nu Q_\mu$ for every $\mu, \nu \in \Lambda$, and $Q_\mu P_\nu = P_\nu$ if $\mu \geq \nu$;

(iv) $Q_\lambda(X)$ is superprojective for every $\lambda \in \Lambda$;

(v) for every unbounded strictly increasing sequence $(\lambda_k)_{k\in\mathbb{N}}$ of elements in Λ and every sequence $(x_k^*)_{k\in\mathbb{N}}$ of non-null elements in X^* such that $x_1^* \in R(P_{\lambda_1}^*)$ and $x_k^* \in R(P_{\lambda_k}^*(I - Q_{\lambda_{k-1}}^*))$ for $k > 1$, the subspace $[x_k^*: k \in \mathbb{N}]_\perp$ is contained in a complemented infinite-codimensional subspace of X. Then X is superprojective.

Then X is superprojective.

Here, an unbounded sequence in Λ is one that does not have an upper bound within Λ. Also, this result is only really interesting if Λ does not have a maximum element; otherwise, if λ is the maximum of Λ, then $P_\lambda = I_X$ by condition (i) and $Q_\lambda = Q_\lambda P_\lambda = P_\lambda = I_X$ by condition (iii), so $X = Q_\lambda(X)$ is already superprojective by condition (iv).

Proof. Let M be an infinite-codimensional subspace of X and let us denote its natural quotient map by $S: X \longrightarrow X/M$. If there exists $\lambda \in \Lambda$ such that SQ_λ is not strictly cosingular, then Proposition (4.1.5) provides another surjection $R: X/M \longrightarrow Z$ such that $N(RS)$ is complemented in X. Since $N(RS)$ is infinite-codimensional and contains Mwe are done.

Otherwise, assume that SQ_λ is strictly cosingular for every $\lambda \in \Lambda$. Let $C \geq 1$ be such that $\|P_\lambda\| \leq C$ and $\|Q_\lambda\| \leq C$ for every $\lambda \in \Lambda$, and let $\varepsilon = 1/8C^3 > 0$. We will construct a strictly increasing sequence $\lambda_1 < \lambda_2 < \ldots$ of elements in Λ and a sequence $(x_n^*)_{n\in\mathbb{N}}$ of norm-one elements in $M^\perp \subseteq X^*$ such that $\|Q_{\lambda k-1}^* x_k^*\| < 2^{-k}\varepsilon$ and $\|P_{\lambda k}^* x_k^* - x_k^*\| < 2^{-k}\varepsilon$ for every $k \in \mathbb{N}$, where we write $Q_{\lambda 0} = 0$ for convenience. To this end, let $k \in \mathbb{N}$, and assume that λ_{k-1}1has already been obtained. By hypothesis, $Q_{\lambda_{k-1}}^* S^* = (SQ_{\lambda_{k-1}})^*$is not an isomorphism, where $S^*: (X/M)^* \longrightarrow X^*$ is an isometric embedding with range $M^\perp$, so there exists $x_k^* \in M^\perp$ such that $\|x_k^*\| = 1$ and $\|Q_{\lambda_{k-1}}^* x_k^*\| < 2^{-k}\varepsilon$, and then there is $\lambda_k > \lambda_{k-1}$ such that $\|P_{\lambda_k}^* x_k^* - x_k^*\| < 2^{-k}\,\varepsilon$ by condition (i), which finishes the inductive construction process. Let $H = [x_k^*: k \in \mathbb{N}] \subseteq X^*$; then $H_\perp$ is infinite-codimensional and contains M.

It is easy to check that the operators $T_k := (I - Q_{\lambda k-1})P_{\lambda_k}$ are projections with norm $\|Tk\| \leq (1 + C)C \leq 2C^2$, and that $T_i T_j = 0$ if $i \neq j$.

Let now $z_k^* = T_k^*(x_k^*) = P_{\lambda_k}^*(I - Q_{\lambda_k}^* - 1)\, x^*k$for each $k \in \mathbb{N}$; then

$$\|z_k^* - x_k^*\| \leq \|P_{\lambda_k}^* x_k^* - x_k^*\| + \|P_{\lambda_k}^* Q_{\lambda_{k-1}}^* x_k^*\| < 2^{-k}\varepsilon + 2^{-k}\varepsilon C \leq 2^{1-k}\varepsilon C < 1/2,$$

So $1/2 < \|z_k^*\| < 3/2$for every $k \in \mathbb{N}$. If we take $x_k \in X$ such that $\|x_k\| < 2$ and $\langle z_k^* k, x_k\rangle = 1$ for each $k \in \mathbb{N}$, and define $z_k = T_k x_k$, it follows that

$$\langle z_k^*, z_k\rangle = \langle z_k^*, T_k x_k\rangle = \langle T_k^* z_k^*, x_k\rangle = \langle z_k^*, x_k\rangle = 1$$

For every $k \in \mathbb{N}$ and

$$\langle z_i^*, z_j\rangle = \langle T_i^* z_i^*, T_j z_j\rangle = \langle z_i^*, T_i T_j z_j\rangle = 0$$

if $i \neq j$, which makes $(z_k^*, z_k)_{n\in\mathbb{N}}$ a biorthogonal sequence in (X^*, X). In the spirit of the principle of small perturbations, let $K: X \longrightarrow X$be the operator defined as $K(x) = \sum_{n=1}^{\infty}\langle x_k^* - z_n^*, x\rangle$; then

$$\sum_{n=1}^{\infty}\|x_n^* - z_n^*\| \, \|z_n\| < \sum_{n=1}^{\infty}(2^{1-n}\varepsilon C)(4C^2) = \sum_{n=1}^{\infty} 2^{-n} = 1,$$

So K is well defined and $U = I + K$ is an isomorphism on X. Moreover, $K^*: X^* \longrightarrow X^*$ is defined as $K^*(x^*) = \sum_{n=1}^{\infty}\langle x^*, z_n\rangle \ (x_n^* - z_k^*)$, so $K^*(z_k^*) = x_k^* - z_k^*$ and $U^*(z_k^*) \ = x_k^*$for every $k \in \mathbb{N}$. Let $Z = [z_k^*: k \in \mathbb{N}]$; then $U^*(Z) = H$ and $U(H_\perp) \ = Z_\perp$.

Next we will show that Z is weak* closed in X^*. Note first that $T_j P_{\lambda_i} = \left(I - Q_{\lambda_{j-1}}\right) P_{\lambda_j} P_{\lambda_i} = \left(I - Q_{\lambda_{j-1}}\right) P_{\lambda_j} = T_j$ if $i \ \geq j$, and $T_j P_{\lambda_i} = \left(I - Q_{\lambda_{j-1}}\right) P_{\lambda_j} P_{\lambda_i} = \left(I - Q_{\lambda_j - 1}\right) P_{\lambda_i} = 0$ otherwise. Given that $z_k^* \in R(T_k^*)$ for every $k \in N$, this means that $P_{\lambda_i}^* z_j^* = z_j^* \ if \ i \ \geq j$ and $P_{\lambda_i}^* z_j^* = 0$ otherwise, so $P_{\lambda_k}^*(Z) \ = [z_1^*, \ldots, z_k^* \ ]$, which is finite-dimensional, for every $k \in N$. Let x^* be a weak*cluster point of Z; then $P_{\lambda_k}^* x^* \in P_{\lambda_k}^*(Z) \ \subseteq Z$ and $P_{\lambda_k}^* x^* \longrightarrow_k x^*$ by condition (i), so $x^* \in Z$ and Z is indeed weak*closed. The fact that $H = U^*(Z)$ implies that H is weak*closed, as well.

This means, in turn, that no Q_λ^* can be an isomorphism on H for any $\lambda \ \in \Lambda$. To see this, consider the natural quotient $Q_{H_\perp}: X \longrightarrow X/H_\perp$, where $X/H_\perp$is infinite-dimensional. Since $M \subseteq H_\perp$, the operator $QH_\perp$ factors through $S = Q_M: X \longrightarrow X/M$and, since SQ_λ is strictly cosingular for every $\lambda \ \in \Lambda$ by our initial hypothesis, it follows that $Q_{H_\perp} Q_\lambda$ cannot be surjective for any $\lambda \ \in \Lambda$, or even lower semi-Fredholm; equivalently, Q_λ^* cannot be upper semi-Fredholm on $H_\perp^\perp$for any $\lambda \ \in \Lambda$, where $H_\perp^\perp = H$ because His weak*closed.

Finally, we will check that the sequence $(\lambda_k)_{k\in\mathbb{N}}$is unbounded. Assume, to the contrary, that there existed some $\lambda \ \in \Lambda$ such that $\lambda_k \leq \lambda$ for every $k \in \mathbb{N}$. Then, for every $k \in \mathbb{N}$, we would have $T_k Q_\lambda = \left(I - Q_{\lambda_{k-1}}\right) P_{\lambda_k} Q_\lambda = \left(I - Q_{\lambda_{k-1}}\right) P_{\lambda_k} = T_k$, so $Q_\lambda^* z_k^* = z_k^*$ and Q_λ^* would be an isomorphism on Z. But then $Q_\lambda^* U^{-1*}$ would be an isomorphism on H, where $U^{-1} = I - U^{-1} \ K$ is a compact perturbation of the identity, so Q_λ^* would be upper semi-Fredholm on H, a contradiction.

Now that the sequence $(\lambda_k)_{k\in\mathbb{N}}$ is known to be unbounded, condition (v) states that $Z_\perp$ is contained in a complemented infinite-codimensional subspace of X, and then so is $H_\perp = U^{-1}(Z_\perp)$.

Note that any sequence $(P_n)_{n\in\mathbb{N}}$ of projections in X satisfying the conditions of Theorem (4.2.2) effectively defines a Schauder decomposition for X, where the components are the ranges of each operator $P_n(I - P_{n-1}) \ = P_n - P_{n-1}$; equivalently, each Pnis the projection onto the sum of the first ncomponents. For the purposes of Theorem (4.2.2), these components need not be finite-dimensional.

Regarding condition (v), a further remark is in order. It may very well be the case that there are no unbounded strictly increasing sequences in Λ, for instance if $\Lambda \ = [0, \omega_1)$, where ω_1 is the first uncountable ordinal, in which case condition (v) is trivially satisfied and does not impose any additional restriction on X or the projections. In terms of the proof of Theorem (4.2.2), this means that SQ_λ must be eventually not strictly cosingular for some λ ∈Λ, and this is so because Q_λ^* is an isomorphism on Z for any λ greater than the supremum of $(\lambda_k)_{k\in\mathbb{N}}$, so Q_λ^* is upper semi-Fredholm on H and SQ_λ is not strictly cosingular.

Theorem (4.2.3)[4]. Let X be a Banach space, let Λ be a well-ordered set and let $(P_\lambda)_{\lambda\in\Lambda}$ be a bounded family of projections on X such that:
(i) $P_\lambda^* x^* \longrightarrow_\lambda x^*$ for every $x^* \in X^*$;
(ii) $P_\mu P_\nu = P_{\min\{\mu,\nu\}}$ for every $\mu, \nu \in \Lambda$;
(iii) $P_\lambda(X)$ *is* superprojective for every $\lambda \in \Lambda$;
(iv) for every unbounded strictly increasing sequence $(\lambda_k)_{k\in\mathbb{N}}$ of elements in Λ and every sequence $(x_k^*)_{k\in\mathbb{N}}$ of non-null elements in X^* such that $x_1^* \in R\left(P_{\lambda_1}^*\right)$ and $x_k^* \in R(P_{\lambda_k}^* - P_{\lambda_{k-1}}^*)$ for $k > 1$, the subspace $[x_k^*: k \in \mathbb{N}]_\perp$ is contained in a complemented infinite-codimensional subspace of X. Then X is superprojective.

Our first use of Theorems (4.2.2) and (4.2.3) will be to prove that the (infinite) sum of superprojective spaces, such as $\ell_p(X_n)$ *or* $c_0(X_n)$, is also superprojective, if the sum is done in a "superprojective" way.

Definition (4.2.4)[4]. We will say that a Banach space $E \subseteq \mathbb{R}^{\mathbb{N}}$ is a solid sequence space if, for every $(\alpha_n)_{n\in\mathbb{N}} \in E$ and $(\beta_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ with $|\beta_n| \le |\alpha_n|$ for every $n \in \mathbb{N}$, it holds that $(\beta_n)_{n\in\mathbb{N}} \in E$ and $\|(\beta_n)_{n\in\mathbb{N}}\| \le \|(\alpha_n)_{n\in\mathbb{N}}\|$.
We will say that E is an unconditional sequence space if it is a solid sequence space and the sequence of canonical vectors $(e_i)_i \in \mathbb{N}$ is a normalised basis for E, where $e_i = \left(\delta_{ij}\right)_{j\in\mathbb{N}}$.

If E is an unconditional sequence space, then its canonical basis $(e_n)_{n\in\mathbb{N}}$ is actually 1-unconditional, and its conjugate E^* can be identified with a solid sequence space itself in the usual way, where the action of $\beta = (\beta_n)_{n\in\mathbb{N}} \in E^*$on $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in E$ *is* $\langle\beta,\alpha\rangle = \sum_{n=1}^{\infty} \beta_n\alpha_n$. If the canonical basis $(e_n)_{n\in\mathbb{N}}$ is shrinking in E, then E^* is additionally unconditional (the coordinate functionals are a basis for E^*).
Solid sequence spaces will play a central role in some of our results because of the following construction.

Definition (4.2.5)[4]. Let E be a solid sequence space and let $(X_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces. We will write $E(X_n)$ for the Banach space of all sequences $(x_n)_{n\in\mathbb{N}} \in n \in \prod_{n\in\mathbb{N}} X_n$ for which $(\|x_n\|)_{n\in\mathbb{N}} \in E$, with the norm $\|(x_n)_{n\in\mathbb{N}}\| = \|(x_n)_{n\in\mathbb{N}}\|_E$.
The identification of the dual of an unconditional sequence space with another solid sequence space can be carried up to the sum of spaces.

Proposition (4.2.6)[4]. Let E be an unconditional sequence space and let $(X_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces. Then $E(X_n)^* \equiv E^*(X_n^*)$.

Proof. Each $(x_n^*)_{n\in\mathbb{N}} \in E^*(X_n^*)$ clearly defines an element of $E(X_n)^*$, so we only have to show the converse identification.
Let $z^* \in E(X_n)^*$, let $J_n: X_n \longrightarrow E(X_n)$ be the canonical inclusion of X_n into $E(X_n)$ for each $n \in \mathbb{N}$ and let $x_n^* = J_n^*(z^*) \in X_n^*$ for each $n \in N$; we will prove that $z^* = (x_n^*)_{n\in\mathbb{N}} \in E^*(X_n^*)$.

To prove that $(x_n^*)_{n\in\mathbb{N}} \in E^*(X_n^*)$. choose $x_n \in X_n$ such that $\|x_n\| = 1$ and $\langle x_n^*, x_n\rangle$ for each $n \in N$, and take any non-negative $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in E$. By the definition of $E(X_n)$, we have that $(\alpha_n x_n)_{n\in\mathbb{N}} \in E(X_n)$, so

$$\sum_{n=1}^{\infty} \|x_n^*\|_{\alpha_n} \le \sum_{n=1}^{\infty} 2\langle x_n^*, x_n\rangle\alpha_n = 2\sum_{n=1}^{\infty}\langle x_n^*, \alpha_n x_n\rangle =$$

$$2\sum_{n=1}^{\infty}\langle J_n^*(z^*), \alpha_n x_n\rangle = 2\sum_{n=1}^{\infty}\langle z^*, J_n(\alpha_n x_n)\rangle$$

$$= 2\langle z^*, (\alpha_n x_n)_{n\in\mathbb{N}}\rangle \le 2\|z^*\|\|(\alpha_n x_n)_{n\in\mathbb{N}}\| = 2\|z^*\|\|\alpha\|.$$

This proves that $(\|x_n^*\|)$ and, as a consequence, $(x_n^*)_{n\in\mathbb{N}} \in E^*(X_n^*)$

Now, given $i \in \mathbb{N}$ and $x_i \in X_i$, we have $\langle (x_n^*)_{n\in\mathbb{N}}, J_i(x_i)\rangle = \langle x_i^*, x_i\rangle = \langle z^*, J_i(x_i)\rangle$, so (x_n^*) and z^* coincide over the finitely non-null sequences of $E(X_n)$ and therefore $z^* = (x_n^*)_{n\in\mathbb{N}}$.

We will show that the sum of superprojective spaces is also superprojective, if the sum is done in a superprojective way, which translates to the requirement that the space E governing the sum must be superprojective itself. This excludes ℓ_1 and, more generally, imposes that any unconditional basis in E be shrinking, for the same reasons that ℓ_1 is not superprojective, or precisely because of this.

Proposition (4.2.7)[4]. Let X be a superprojective Banach space and let $(x_n)_{n\in\mathbb{N}}$be an unconditional basis of X. Then $(x_n)_{n\in\mathbb{N}}$ is shrinking.

Proof. If $(x_n)_{n\in\mathbb{N}}$ is unconditional but not shrinking, then X contains a (complemented) copy of ℓ_1 and cannot be superprojective by Proposition(4.1.8).

Theorem (4.2.8)[4]. Let E be an unconditional sequence space and let $(X_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces. Then $E(X_n)$ is superprojective if and only if all of E and X_n are superprojective.

Proof. (i) Let $X = E(X_n)$. All of E and X_n are quotients of X; if X is superprojective, then so are Eand each X_n. Assume now that E and each X_n are superprojective, and define the projections $P_n: X \longrightarrow X$ as $P_n((x_n)_{n\in\mathbb{N}}) = (x_1, \ldots, x_n, 0, \ldots)$for each $n \in \mathbb{N}$. We will prove that the sequence $(P_n)_{n\in\mathbb{N}}$ meets the criteria of Theorem (4.2.3). The fact that $(P_n)_{n\in\mathbb{N}}$ is associated with the natural Schauder decomposition of $X = E(X_n)$ is enough for condition (ii) to hold. For condition (iii), note that $P_n(X)$ is isometric to $\oplus_{i=1}^{n}$which is superprojective by Proposition (4.2.1). As for condition (i), E is superprojective and its canonical basis $(e_n)_{n\in\mathbb{N}}$ is unconditional, therefore shrinking by Proposition (4.2.7), so E^* is unconditional and $(P_n^*)_{n\in\mathbb{N}}$ is the sequence of projections associated with the natural Schauder decomposition of $E(X_n)^* \equiv E^*(X_n^*)$.

To prove condition (iv), let $(n_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence of integers, let $T_1 = P_{n_1}$and $T_k = P_{n_k} - P_{n_{k-1}}$for $k > 1$, and let $x_k^* \in R(T_k^*)$ be non-null for each $k \in N$, as in Theorem (4.2.3). Define $M = [x_k^*: k \in \mathbb{N}]^\perp$, which is infinite-codimensional. Then $x_k^* \in X^* \equiv E^*(X_n^*)$, so

$$x_k^*\big(0, \ldots, 0, z_{n_{k-1}+1}^*, \ldots, z_{n_k}^*, 0, \ldots\big),$$

Where $z_i^* \in X_i^*$. Pick a normalised $z_i \in X_i$ such that $\langle z_i^*, z_i\rangle \geq \|z_i^*\|/2$ for each $i \in \mathbb{N}$, and consider the operator $J: E \longrightarrow X$ defined as $J(\alpha_n)_{n\in\mathbb{N}} = (\alpha_n z_n)_{n\in\mathbb{N}}$ which is an isometric embedding by the definition of $X = E(X_n)$.

We claim that $Q_M J: E \longrightarrow X/M$ is a surjection. Indeed, given $x = (x_n)_{n\in\mathbb{N}} \in X$, with each $x_n \in X_n$, let $a_n = \langle z_n^*, x_n\rangle / \langle z_n^*\rangle$ if $z_n^* \neq 0$, for each $n \in \mathbb{N}$, and define $\alpha = (\alpha_n)_{n\in\mathbb{N}}$. Then $|a_n| \leq 2|x_n|$ for every $n \in \mathbb{N}$, so $\alpha \in E$, and $\langle x_k^*, x - J(\alpha)\rangle = \sum_{i=n_k-1}^{n_k} \langle z_i^*, x_i - \alpha_i z_i\rangle = 0$ for every $k \in \mathbb{N}$, so $x - J(\alpha) \in M$ and $Q_M(x) = Q_M J(\alpha) \in R(Q_M J)$.

Now, by the superprojectivity of E and Proposition (4.1.2), there exists another surjection $S: X/M \longrightarrow Z$ such that $N(SQ_M J)$ is complemented inE; by Proposition (4.1.3), $N(SQ_M)$ is complemented in X, where $M \subseteq N(SQ_M)$ and $R(SQ_M) = Z$, which is infinite-dimensional.

The following result will help us check for the last condition in Theorems (4.2.2) and (4.2.3).

Lemma (4.2.9)[4]. Let X be a Banach space, let E be an unconditional sequence space and let $T, (T_k)_{k\in\mathbb{N}}$ be projections in X such that

(i) $T_i T_j = 0 \; if \; i \neq j$;

(ii) $T_k T = T T_k = T_k \; for \; every \; k \in N$;

(iii) $R(T)$ embeds into $E\big(R(Tk)\big)$via the mapping that takes

$x \in R(T) to \big(T_k(x)\big)_{k\in\mathbb{N}}$.

Let $x_k^* \in R\ (T_k^*)$ be non-null for each $k \in N$. Then $[x_k^*: k \in N]_\perp$ is complemented in X.

Proof. We will assume without loss of generality that $\|x_k^*\| = 1$ for every $k \in N$. Let $Z = E(R(T_k))$ and let $U: R(T) \longrightarrow Z$ be the isomorphism into Z defined as $U(x) = \big(T_k(x)\big)_{k\in\mathbb{N}}$.

Note that, in fact, $\big(T_k(x)\big)_{k\in\mathbb{N}} = \big(T_k(T_x)\big)_{k\in\mathbb{N}} = U\big(T(x)\big) \in Z$ for every $x \in X$, so $(\|T_k(x)\|)_{k\in\mathbb{N}} \in E$ and $\|(\|T_k(x)\|)_{k\in\mathbb{N}}\|_E = \big\|U\big(T(x)\big)\big\|_Z$ for every $x \in X$. Define $Q: X \longrightarrow E$ as $Q(x) = (\langle x_k^*, x\rangle)_{k\in\mathbb{N}}$; then

$$|\langle x_k^*, x\rangle| = |\langle T_k^*(x_k^*), x\rangle| = |\langle x_k^*, T_k(x)\rangle| \leq \|T_k(x)\|$$

For every $x \in X$, so Q is well defined and $\|Q\| \leq \|UT\|$. Also, $(\|T_k(x)\|)_{k\in\mathbb{N}} \in E$ implies that $T_k x \longrightarrow_k 0$ for every $x \in X$, so there exist a constant C such that $\|T_k\| \leq C$ for every $k \in \mathbb{N}$.

Take now $x_k \in X$ such that $\langle x_k^*, x_k\rangle = 1$ and $\|x_k\|$ for each $k \in \mathbb{N}$, so that $\langle x_i^*, T_j x_j\rangle = \langle T_j^* x_i^*, x_j\rangle = \delta_{ij}$ every $i, j \in \mathbb{N}$, and define $J: E \longrightarrow X$ as $J((\alpha_n)_{n\in\mathbb{N}}) = \sum_{n=1}^{\infty} \alpha_n T_n(x_n)$. Then $U\big(J((\alpha_n)_{n\in\mathbb{N}})\big) = \big(\alpha_k T_k(x_k)\big)_{k\in\mathbb{N}}$, as seen by considering the action of $U\,J$ over the finitely non-null sequences of E, where $1 \leq \|T_k(x_k)\| \leq C$ for every $k \in \mathbb{N}$, so $U\,J: E \longrightarrow Z$ is an isomorphism, and so must be J. Finally,

$$QJ((\alpha_n)_{n\in\mathbb{N}}) = \left(\langle x_k^* \sum_{n=1}^{\infty} \alpha_n T_n(x_n)\rangle\right)_{k\in\mathbb{N}} = (\alpha_k)_{k\in\mathbb{N}},$$

So $Q_J = I_E$ and J_Q is a projection in X with kernel $[x_k^*: k \in \mathbb{N}]_\perp$.

Theorem (4.2.10)[4]. Let X and Y be c_0 or ℓ_p for $1 < p < \infty$. Then $X \widehat{\otimes}_\varepsilon Y$ is superprojective.

Proof. Let $R_n: X \longrightarrow X$ be the projection given by $R_n(\alpha_k)_{k\in\mathbb{N}} = (\alpha_1, \ldots, \alpha_n, 0, \ldots)$ for each $n \in N$, and similarly for Y. (We are abusing the notation here for the sake of simplicity in that R_n is really a different operator on each of X and Y unless they are the same space.) Define the projections

$$P_n = R_n \otimes R_n$$
$$Q_n = I_{X \widehat{\otimes}_\varepsilon Y} - (I_X - R_n) \otimes (I_Y - R_n)$$
$$= R_n \otimes R_n + (I_X - R_n) \otimes R_n + R_n \otimes (I_Y - R_n)$$

We will prove that the sequences $(P_n)_{n\in\mathbb{N}}$ and $(Q_n)_{n\in\mathbb{N}}$ meet the criteria of Theorem (4.2.2). Conditions (ii) and (iii) are readily satisfied, because they clearly hold for the elementary tensors $e_i \otimes e_j$. For condition (i), both X^* and Y^* are ℓ_q spaces for some $1 \leq q < \infty$, so $R_n^*(x^*) \longrightarrow x^*$ for every $x^* \in X^*$, and similarly for Y^*, so $P_n^*(z^*) = (R_n^* \otimes R_n^*)(z^*) \longrightarrow_n$ for every $z^* \in (X \widehat{\otimes}_\varepsilon)^* = X^* \widehat{\otimes}_\pi Y^*$, again because it holds for the elementary tensors. For condition(iv), note that the range of Q_n is the direct sum of the ranges of $R_n \otimes R_n$, $(I_X - R_n) \otimes R_n$and $R_n \otimes (I_Y - R_n)$, where the first one is finite-dimensional and the other two are the sum of finitely many copies of $N(R_n)$ in X and Y, respectively, which are finite-codimensional subspaces of X and Y, respectively, hence superprojective.

To prove condition (v), let $(n_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence of integers and let $T_1 = P_{n_1}$ and $T_k = \big(I - Q_{n_{k-1}}\big)P_{n_k}$ for $k > 1$, as in Theorem(4.2.2). Note that, for $k > 1$, T_k is the projection $T_k = (R_{n_k} - R_{n_{k-1}}) \otimes (R_{n_k} - R_{n_{k-1}})$, so $T_i T_j = 0$ if $i \neq j$. The operator

$T = \sum_{k=1}^{\infty} T_k$ is a norm-one projection in $X \widehat{\otimes}_{\varepsilon} Y$, with $T_k T = T T_k = T_k$ for every $k \in \mathbb{N}$, and $R(T)$ embeds into $c_0(R(T_k))$ or $\ell_s(R(T_k))$ for suitable $1 < s < \infty$, so Lemma (4.2.9) ensures that $[x^*k : k \in \mathbb{N}]_{\perp}$is complemented in $X \widehat{\otimes}_{\varepsilon} Y$ for any choice of non-null elements $x^*k \in R(T_k^*)$.

Theorem (4.2.10) can actually be extended to injective tensor products of finitely many copies of c_0 and $\ell_p (1 < p < \infty)$ inductively in the obvious way with only minor modifications.

Lastly, we will show that $C(K, X)$ is superprojective whenever so is X at least if K is an interval of ordinals, which includes the case where K.

Theorem (4.2.11)[4]. Let X be a superprojective Banach space and let λ be an ordinal. Then $C_0([0, \lambda], X)$ and $C([0, \lambda], X)$ are superprojective.

Proof. The proof will proceed by induction in λ . Assume that $C_0([0, \mu], X)$ and $C([0, \mu], X)$ are indeed superprojective for all $\mu < \lambda$; we will first prove that $C_0([0, \lambda], X)$ is superprojective too. If λ is not a limit ordinal, then $\lambda = \mu + 1$ for some μ and $C_0([0, \lambda], X) \equiv C([0, \mu], X)$, which is superprojective by the induction hypothesis.
Otherwise, if λ is a limit ordinal, define the projections

$$P_{\mu} : C_0([0, \lambda], X) \longrightarrow C_0([0, \lambda], X)$$

as $P_{\mu}(f) = f \chi_{[0,\mu]}$for each $\mu < \lambda$. We will prove that the family $(P_{\mu})\mu < \lambda$ meets the criteria of Theorem (4.2.3). Condition (ii) is immediate to check. For condition (iii), $P_{\mu}(C_0([0, \lambda], X))$ is isometric to $C([0, \mu], X)$, which is superprojective by the induction hypothesis.

For condition (i), we have $C_0([0, \lambda])^* = \ell_1([0, \lambda))$ and $C_0([0, \lambda], X)^* = (C_0([0, \lambda]) \widehat{\otimes}_{\varepsilon} X)^* = C_0([0, \lambda])^* \widehat{\otimes}_{\pi} X^*$, so $C_0([0, \lambda], X)^* = \ell_1([0, \lambda)) \widehat{\otimes}_{\pi} X^* = \ell_1([0, \lambda), X^*)$ and $P_{\mu}^*(z) = z \chi_{[0,\mu]} \rightarrow \mu_z$ for every $z \in \ell_1([0, \lambda), X^*)$.

As for condition (iv), let $(\lambda_k)_{k \in \mathbb{N}}$ be an unbounded strictly increasing sequence of elements in $[0, \lambda)$, should it exist, and let $T_1 = P_{\lambda_1}$ and $T_k = P_{\lambda_k} - P_{\lambda_{k-1}}$ for $k > 1$, as in Theorem (4.2.3). Then T_k is the projection given by $T_k(f) = f \chi_{[\lambda_{k-1}+1, \lambda_k]}$ for $k > 1$, so $T_i T_j = 0$ if $i \neq j$. Since $(\lambda_k)_{k \in \mathbb{N}}$ is unbounded in $[0, \lambda)$, its supremum must be λ itself, so $C_0([0, \lambda], X) = c_0(R(T_k)) = c_0(C([\lambda_{k-1} + 1, \lambda_k], X))$ and Lemma (4.2.9), with $T = I$, ensures that $[x^*k : k \in \mathbb{N}]_{\perp}$ is complemented in $C_0([0, \lambda]$, X) for any choice of non-null elements $x^*k \in R(T_k^*)$.
Finally, $C([0, \lambda], X) = C_0([0, \lambda], X) \oplus X$, which is superprojective by Proposition (4.2.1).

Again, note that unbounded strictly increasing sequences in $[0, \lambda)$ may not exist for certain λ, in which case the remark after Theorem (4.2.2) applies and P_{μ} cannot be strictly cosingular for all $\mu < \lambda$.

List of Symbols

Symbol		Page
WCG:	Weakly compactly generated	1
Sup:	Supremum	1
ℓ_2:	Hilberlt space of sequences	1
$L_1(M, x)$:	Lebesgue-Bochner space	3
L_2:	Hilberlt space	3
diam:	diameter	5
inf:	infimum	7
$\mathcal{L}_1$:	Banach space	14
$\mathcal{L}_\infty$	Banach space	14
ℓ_p:	Lebesgue space	15
ℓ_∞:	essential Lebesgue space	15
co:	Compact	15
$\otimes$:	tensor product	16
ker:	kernel	16
AOC:	Almost over complete	27
OT:	Over total	27
AOT:	Almost over total	27
codim:	codimention	29
dim:	dimention	32
$\oplus$	dineet sum	34
LUR:	Locally uniform rotund	35
$\mathrm{L_p}$:	Lebesgue space	37
z_p	Kalton-peck space	39
ℓ_q:	Dual Lebesgue space	40
L_q:	Dual Lebesgue space	41
min:	Minimum	41

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