Structures of Inner Functions on Hardy Spaces via the Least Harmonic Majorants

Emad Aldeen A. A. Rahim

Sudan University of Science and Technology - College of Science-Department of Mathematics

ABSTRACT: In this work we showed that the inner functions on Hardy space can be written in the canonical factorization form as exponential of least harmonic majorants.

KEYWORDS: Hardy space, Inner functions, Harmonic and least Harmonic majorant.

Definition of \( H^p \)

For \( 1 \leq p < \infty \) the Hardy space \( H^p \) is defined as the space of all analytic functions \( \varphi \) in the unit disk \( \mathbb{D} = \{ z \in \mathbb{C}; |z| < 1 \} \)

\[
\| \varphi \|_p = \sup_{r < 1} \left[ \frac{1}{2\pi} \int_0^{2\pi} |\varphi(re^{it})|^p \right]^{\frac{1}{p}}
\]  

(1)

is finite. The space \( H^\infty \) (Banach space) consist of all bounded analytic functions \( \varphi \) on the disk, and the norm is now

\[
\| \varphi \| = \sup_{|z| < 1} |\varphi(z)|
\]  

(2)

For function \( \varphi \) in \( H^p \), for \( 1 \leq p < \infty \), the radial limit

\[
\phi(e^{it}) = \lim_{r \to 1^-} \varphi(re^{it})
\]  

(3)

exists almost everywhere in \( t \) (Fatou’s Theorem), and needed , \( \hat{\varphi} \in L^p(T) \), where \( T \) denotes the unit circle which we equip with normalized Lebesque measure; moreover:

\[
\| \varphi \|_H = \| \hat{\varphi} \|_L^p.
\]

We normally identify \( \varphi \) with \( \hat{\varphi} \), and can thus regard \( H^p \) as a closed subspace of \( L^p(T) \). It is also possible to start by defining \( H^p \) directly as the subspace of those \( L^p(T) \) functions for which the negative Fourier coefficients vanish, that is:

\[
\frac{1}{2\pi} \int_0^{2\pi} \phi(e^{it})e^{-int} dt = 0
\]

(4)

for all \( n < 0 \).
It is classical that $\varphi \in H^\infty$ has the factorizations

$$\varphi = BSO$$

(5)

where $B$ is a Blaschke product, $S$ is a singular function, and $O$ is an outer function. Specifically, these factors are.\(^{(1)}\)

$$B(z) = z^m \prod \lambda_k \frac{z_k - z}{1 - z_k z}$$

(6)

where $m$ is the order of the zeros of $\varphi$ at the origin and $z_1, z_2, \ldots$ are the zeros of $\varphi$ in $D \setminus \{0\}$;

$$S(z) = \exp \left\{ -\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \, d\nu(t) \right\}$$

(7)

where $\nu$ is a non-negative measure singular with respect to Lebesgue measure, and

$$O(z) = \lambda \exp \left\{ \frac{1}{2\pi i} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} k(t) \, dt \right\}$$

(8)

where $\lambda$ is a unimodular constant and $k$ is real-valued integrable function. Also $\varphi$ has the factorization (canonical factorization)

$$\varphi = IO$$

(9)

where $I$ is an inner function (has a unit modulus a.e on $D$), and $O$ as in Eq.(5).\(^{(1,2)}\)

It is well-known that if $\varphi$ is an inner function, then $\frac{v - \varphi(z)}{1 - \varphi(z)v}$ is a Blaschke product for all $w \in D$ with the exception at most of a set of capacity zero\(^{(3)}\).

**Theorem A\(^{(3)}\):**

Let $N_*$ denote the set of all analytic functions $f$ on the unit disk such that the functions $\log^+ |f_n|$ have uniformly absolutely continuous integrals, and let $\varphi \in N_*$, then the set of points $w$ for which $\varphi(z) - w$ has non-trivial singular inner factor has logarithmic capacity zero. Conversely, given any compact set $E$ of logarithmic capacity zero, there is a bounded analytic function $F$ such that $\varphi(z) - w$ has a non-trivial singular inner factor if and only if $w \in E$.

The converse statement is well-known. Let $E$ be a compact set of capacity zero in $D$, the covering map $F$ of the domain $D \setminus E$ is an inner function since $E$ has capacity zero. For each $w \in E$, $\frac{F(z) - w}{1 - \overline{w}F(z)}$ is a non-vanishing inner function and so is singular. Thus since $1 - \overline{w}F(z)$ is an outer function, $F(z) - w$ is a function with non-trivial singular inner factor for all $w \in E$.

Note that for mutually prime inner functions $u$ and $v$ which have no zero in common and that there is singular inner function $S$ with $u = Su_i$ and $v = Sv_i$ for inner functions $u_i$ and $v_i$, and $\rho > 0$, the function $\frac{u(z) + \rho e^{it} v(z)}{1 - \rho e^{it} \varphi(z)}$ has a trivial singular inner factor for almost all (w.r.to Lebesgue measure) real $t$. 
The generalization of the above concept is given in the following Theorem.

**Theorem B:** Let \( f, g \in H^p \), \( 0 < p < \infty \), have mutually prime singular inner factors. Then the set of points \( w \) for which \( f(z) - w g(z) \) has a non-trivial singular inner factor has logarithmic capacity zero \(^{(3)}\).

In Theorem B above, we see that if \( g \) is an outer function, then the lack of a singular factor in \( f(z) - w g(z) \) is equivalent to the lack of a singular factor in the decomposition of the function \( \frac{f(z)}{g(z)} - w \) in \( N_* \), and is thus covered in Theorem A.

Now let \( \varphi \in H^2 \), then there exist a harmonic function \( h \) in \( D \), such that

\[
|\varphi(z)|^2 \leq h(z) \quad (10)
\]

and \( h \) is called the harmonic majorant of \( \varphi \) \(^{(4-6)}\). It is well known that if \( \varphi \) has harmonic majorant in \( D \), then there exist a least harmonic majorant, hence there exist a harmonic function \( h_\varphi \) in \( D \), such that:

\[
|\varphi(z)|^2 \leq h_\varphi(z) \quad (11)
\]

and such that \( h_\varphi \leq h \). Also if \( \varphi \in H^2 \), then for a fixed \( z_0 \in D \) there is a norm on \( H^2 \) defined by:

\[
\|\varphi\| = \inf \left\{ h(z)^{\frac{1}{2}} : h \text{ is a harmonic majorant of } |\varphi|^2 \right\}
\]

Now, one can make the following definitions:

**Definition 1:** for \( \varphi \in H^2 \), we say that \( h \) is the harmonic majorant of \( \varphi \) in \( D \) if \( h \) is harmonic function in \( D \), such that \( \varphi \leq h \).

**Definition 2:** let \( \varphi \in H^2 \), and \( h \) is the harmonic majorant of \( \varphi \) in \( D \), we say that \( h_\varphi \) is least harmonic majorant of \( \varphi \) in \( D \) if \( h_\varphi \) is harmonic function in \( D \), such that \( \varphi \leq h_\varphi \), and such that \( h_\varphi \leq h \).

Now let \( \varphi \) and \( \varphi' \) be two functions in \( H^2 \). We say that \( \varphi \) divides \( \varphi' \) (or \( \varphi|\varphi' \)), if \( \varphi' \) can be written as \( \varphi' = \varphi u \), for some \( u \in H^2 \).

Now we need the following lemma.

**Lemma C\(^{(3)}\):**

If \( \varphi_1 \) and \( \varphi_2 \) are inner functions without a common factor, then:

\[
\lim_{r \to 1} \int_{\Gamma} \log \left( \max \left\{ |\varphi_1(re^{i\theta})|, |\varphi_2(re^{i\theta})| \right\} \right) d\sigma(\theta) = 0
\]
where \( d\sigma \) is the normalized Lebesgue measure on the unit circle \( T \).

**proof:** The limit on the left side is the value at the origin of the least harmonic majorant in \( D \) of the subharmonic function \( \max\{\log|\varphi_1|, \log|\varphi_2|\} \). So it remains to show that this least harmonic majorant is the constant function \( 0 \). Let \( h \) denote this least harmonic majorant. Then

\[
\log|\varphi_1| \leq h \leq 0.
\]

This implies that \( h \) has radial limits \( 0 \) almost everywhere on \( T \).

So, if \( h \) is not identically zero, then \( h \) is the Poisson integral of a negative singular measure on \( T \). Hence \( \varphi = e^{h+i\theta} \) is a singular inner function (here \( \tilde{h} \) denotes the harmonic conjugate of \( h \) in \( D \)).

Since \( |\varphi_1| \leq |e^{h+i\theta}| \), the inner function \( \varphi \) divides \( \varphi_1 \). But \( |\varphi_2| \leq |e^{h+i\theta}| \) implies that \( \varphi \) also divides \( \varphi_2 \), contradicting our assumption about \( \varphi_1 \) and \( \varphi_2 \). Thus \( h \equiv 0 \). This completes the proof of the lemma.

**RESULTS**

Now we arrive to the following theorem which gives the principle result of this paper.

**THEOREM 1:** If \( \varphi_1 \) and \( \varphi_2 \) are inner functions with a common factor, say \( \varphi \), then \( \varphi = e^{h+i\theta} \), where \( \tilde{h} \) denotes the harmonic conjugate of \( h \) in \( D \), and \( h \) is the least harmonic majorant of the subharmonic function \( \max\{\log|\varphi_1|, \log|\varphi_2|\} \).

**Proof:**

By lemma C above, if \( h \) denotes the least harmonic majorant of the subharmonic function \( \max\{\log|\varphi_1|, \log|\varphi_2|\} \), then \( \log|\varphi_1| \leq h \leq 0 \). This implies that \( h \) has radial limits \( 0 \) almost everywhere on \( T \).

So, if \( h \) is not identically zero, then \( h \) is the Poisson integral of a negative singular measure on \( T \). Hence \( \varphi = e^{h+i\theta} \) is a singular inner function.

Since \( |\varphi_1| \leq |e^{h+i\theta}| \), the inner function \( \varphi \) divides \( \varphi_1 \). But \( |\varphi_2| \leq |e^{h+i\theta}| \) implies that \( \varphi \) also divides \( \varphi_2 \), morever this representation of inner function \( \varphi \) is similar to that one in Eq. (9), i.e. \( \varphi = e^{h+i\theta} = e^h.e^{i\theta} \), where \( e^h \) determine the inner factor, and \( e^{i\theta} \) determine the outer factor, and the theorem is proved.

**REFERENCES:**


