NOTE ON THE SOLUTIONS OF COUPLED ELLIPTIC EQUATIONS
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ABSTRACT
A theorem is firstly proved which governs separation of a system of two coupled equations. This separation leads to study boundary value problems for elliptic equations. The existence and uniqueness of the generalized solution is proved Sobolev space.

INTRODUCTION
Coupled partial differential equations are frequent in many different problems, such as in the study of temperature distribution within a composite heat conduction (¹), in diffusion problems (²), biochemistry (³), armament models (⁴), analysis of pollutant migration through soil, elastic and inelastic contact problems of solids (⁵), etc.

Let Ω be a bounded domain in Rⁿ of x = (x₁, x₂, ..., xₙ), with sufficiently smooth boundary Γ = ∂Ω. Considering the following system of coupled partial differential equations:

\[
\begin{align*}
-\Delta u₁ + a₁(x)u₁ &= b(x)u₂ + g₁(x) \\
-\Delta u₂ + a₂(x)u₂ &= b(x)u₁ + g₂(x), \\
u₁ = u₂ &= 0, \text{on } Γ = ∂Ω
\end{align*}
\]

where aᵢ(x), b(x) are assumed to be analytic functions and gᵢ ∈ L₂(Ω), (i=1,2), and Δ denotes the Laplacian operator in Rⁿ.

The theorem on the separation of the equation of this system states the following:

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**Theorem 1**

The system (P) can always be decoupled without increase of the order of the differential equation if and only if $b(x)$ is proportional to the difference $a_1(x) - a_2(x)$.

**Proof**

In matrix notation let

$$(Z + A) U = B U + G$$

where $U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $B = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$, $G = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$,

$$Z + A = \begin{pmatrix} -\Delta + a_1 & 0 \\ 0 & -\Delta + a_2 \end{pmatrix}$$

Considering a transformation $N$ defined by

$$N = \begin{pmatrix} 1 - \lambda & 1 + \lambda \\ - (1 + \lambda) & 1 - \lambda \end{pmatrix}$$

(1-2)

Where $\lambda$ may be any function of $x$. Note that

$$N = X_1 X_2$$

where $X_1 = X_1 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $X_2 = \begin{pmatrix} 1 & \lambda \\ -\lambda & 1 \end{pmatrix}$

The form (1-2) may be used, to diagonals the matrix equation

$$N(Z + A) N^{-1} U = N B N^{-1} U + N G$$

(1-3)

It may be shown that equation (1-3) is separated if and only if, the following conditions are satisfied simultaneously:

(i) $[Z, \lambda] = 0$, (where $[\ ]$ is a commutator)

(ii) $(-a_1 \cdot a_2) \lambda^2 - 4b \lambda - (a_1 \cdot a_2) = 0$.

Condition (i) means that $\lambda$ must be independent of, $x$, while (ii) connects this quantity with $b(x), a_1(x), a_2(x)$.

Solving this last equation, are comes to the conclusion that the ratio $\frac{b}{a_1 - a_2}$ Must be independent of $x$.

when defining $V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, then $V = NU$,

And (1-3) is separated in the forms:

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\[
\begin{align*}
\text{(P_1)} & \quad \begin{cases}
- \Delta v_1 + \Phi_1(x)v_1 = f_1(x) \\
v_1 = 0, \text{ on } \Gamma = \partial \Omega
\end{cases}, \\
\text{(P_2)} & \quad \begin{cases}
- \Delta v_2 + \Phi_1(x)v_2 = f_2(x) \\
v_2 = 0, \text{ on } \Gamma = \partial \Omega
\end{cases}
\end{align*}
\]

Where \( \Phi_1(x) = -\frac{1}{2} \left( a_1(x) + a_2(x) \right) + \frac{1}{2} \left[ (a_1(x) - a_2(x))^2 + 4b^2 \right]^\frac{1}{2} \)

\[
\Phi_2(x) = -\frac{1}{2} \left( a_1(x) + a_2(x) \right) - \frac{1}{2} \left[ (a_1(x) - a_2(x))^2 + 4b^2 \right]^\frac{1}{2} , \quad \left( \frac{f_1}{f_2} \right) = 2(1 + \lambda^2)NG
\]

**EXISTENCE AND UNIQUENESS THEOREM**

Here the existence and uniqueness of solutions of problems \((P_1)\) (resp. \((P_2)\)) is proved

**Theorem 2**

If the conditions of theorem 1 are satisfied and supposing that \( \Phi_1(x) \) is defined and \( f_i, i = 1,2 \), there is a unique solution \( v_i \in H^1_0(\Omega) [\text{resp. } v_2 \in H^1_0(\Omega)] \) of the problem \((P_1)\) (resp. \((P_2)\)).

**Proof**

Form the theorem of Lax–Milgram follows immediately existence and uniqueness of solution of problem \((P_1)\) (resp. \((P_2)\)) in \( H^1_0(\Omega) \).

**COROLLARY**

Under the Hypothesis of theorem 1 and 2. Then problem \((P)\) has a unique generalized solution \( \{u_1,u_2\} \) in \( H^1_0(\Omega) \times H^1_0(\Omega) \).

**REFERENCES**