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Plasma Trajectories on Gravitational Fields via Exterior Differential Forms

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Abstract: In this paper we show that Einstein field equations can be reduced to Newton's law of gravitation at the study of charged particles trajectories on electromagnetic fields by using a generalization of the fundamental Newtonian equation for the general physical or engineering systems in term of exterior differential forms.

Keywords*:* Exterior differential system; Riemannian configuration manifold; covariant force law; Geodesic spray.

INTRODUCTION

Recall that the curve y at a point m of an n manifold M is a C^0-map from an open interval $I \subset \mathbb{R}$ into M such that $0 \in I$ and $y(0) = m$. For such a curve y we assign a tangent vector at each point $y(t)$, $t \in I$, by $y(t) = T_t y(I)$.

So an integral curve (or flow line) of the tangent vector–field X is a paramet- erized curve $y : I \to M$ satisfying the following condition

$$y(t) = X(y(t)) \qquad for\ all\ t \in I \tag{1}$$

Where the maximal integral curve of X through $m \in M$ is a parameterized curve satisfying [8]:

(i) $y(0) = m$

(ii) If $\beta : \tilde{I} \to M$ is any other parameterized curve in M satisfying (i) and (1), then $\tilde{I} \subset I$ and $\beta(t) = y(t)$ for all $t \in \tilde{I}$.

The velocity y' of the parameterized curve $y(t)$ is a vector–field along y defined by $y'(t) = (y(t), x_1'(t), \ldots x_n'(t))$, and its speed defined by $|y'|(t) = |y'(t)|$ for all $t \in I$.

Now each vector–field X along y is of the form

$$X(t) = (y(t), X_1(t), \ldots, X_n(t)) \tag{2}$$

where each component X_i is a function along y. X is smooth if each $X_i : I \to M$ is smooth. The derivative of a smooth vector–field X along a curve $y(t)$ is the vector–field $X^{\cdot}$ along y defined by

$$X'(t) = (y(t), X_1'(t), \ldots X_n'(t)) \tag{3}$$

Here $X'(t)$ measures the rate of change of the vector part $(X_1(t), \ldots X_n(t))$ of $X(t)$ along y. Thus, the acceleration $y''(t)$ of a parameterized curve $y(t)$ is the vector–field along y get by differentiating the velocity field $y'(t)$.

A parameterized curve $y : I \to M$ is a geodesic of M if and only if its acceleration $y''(t)$ is everywhere perpendicular to M, i.e., if and only if $y''(t)$ is a multiple of the orientation $N(y(t))$ for all $t \in I$, i.e., $y''(t) = g(t)N(y(t))$, where $g : I \to \mathbb{R}$. Taking the scalar product of both sides of this equation with $N(y(t))$ we find $g = -y'N'(y(t))$. Thus $y : I \to M$ is geodesic if and only if it satisfies the differential equation

$$y''(t) + N'(y(t))\, N(y(t)) = 0. \tag{4}$$

This vector equation represents the system of second–order component ODEs

$$x_i'' + N_i(x+1, \ldots, x_n)\frac{\partial N_j}{\partial x_k}(x+1, \ldots, x_n)x_j'x_k' = 0$$

If we substitute $u_i = x_i'$ we reduce this second–order differential system to the first–order differential system

$$x_i' = u_i,\ u_i' = N_i(x+1, \ldots, x_n)\frac{\partial N_j}{\partial x_k}(x+1, \ldots, x_n)x_j'x_k' \tag{5}$$

This first–order system is just the differential equation for the integral curves of the vector–field (geodesic spray) X in $U \times \mathbb{R}$ where U is an open chart in M.

Now if the parameter t represents time, and when an integral curve $y(t)$ is the path a mechanical system Ξ follows, i.e., the solution of the equations of motion, it gives a trajectory.

Here the motion of the system Ξ on its configuration manifold described by

$$y_i(t) = X_i(y(t)), \quad for\ all\ t \in I \subseteq \mathbb{R}$$ (6)

If $X(m)$ is C^0 the existence of a local solution is guaranteed, and a Lipschitz condition would imply that it is unique. Therefore, exactly one integral curve passes through every point, and different integral curves can never cross.

Generalization of Newtonian 3D equation Now we are going to study charged particles motion, we need to generalize Newtonian 3D equation, $F = ma$ to thought of as covariant force law:

Force 1-form-field =
Mass distribution × Acceleration vector-field

In other words, the field or, family of force one–forms F_i, acting in all movable joints (with constrained rotations on T^n and very restricted translations on I^n), causes both rotational and translational accelerations of all body segments, within the mass distribution mg_{ij}, along the flow– lines of the vector–field a_j.

First consider the linear and homogenous transformation of velocities,

$$x_i' = \frac{\partial x_i}{\partial y^e} y'^e \qquad (7)$$

our internal velocity vector–field is defined by the set of ODEs (7), at each representative point $x_i = x_i(t)$ of the system's configuration manifold $M = T^n \times I^n$, where T^n is n-torus, and $I^n \subset \mathbb{R}^n$ is hypercube, as $v_i \equiv v_i(x_i, t) := x_i'(x_i, t)$

Note that in general, a vector–field represents a field of vectors defined at every point x_i within some region U of the total configuration manifold M. Analytically, vector–field is defined as a set of autonomous ODEs (7). Its solution gives the flow, consisting of integral curves of the vector–field, such that all the vectors from the vector–field are tangent to integral curves at different representative points $x_i \in U$. In this way, through every representative point $x_i \in U$ passes both a curve from the flow and its tangent vector from the vector–field. Geometrically, vector– field is defined as a cross–section of the tangent bundle TM. Its geometrical dual is the 1–form–field, which represents a field of one – forms, defined at the same representative points $x_i \in U$. Analytically, 1–form–field is defined as an exterior differential system [8], an algebraic dual to the autonomous set of ODEs. Geometrically, it is defined as a cross–section of the cotangent bundle T^*M. Together, the vector–field and its corresponding 1–form–field define the scalar potential field at the same movable region $U \subset M$.

According to Newton, acceleration is a rate–of–change of velocity, and since $a_i \neq v_i'$ we have:

$$a_i := \bar{v}_i' = v_i' + \Gamma^i_{jk} v_j v_k = x_i'' + \Gamma^i_{jk} x_j' x_k'. \qquad (8)$$

Once we have the internal acceleration vector–field $a_i \equiv a_i(x_i, x_i', t)$, defined by the set of ODEs (8), where Γ^i_{jk} is Levi–Civita connections of the Riemannian configuration manifold M, we can finally define the internal force 1–form field, $F_i = F_i(x_i, x_i', t)$, as a family of force one–forms, half of them rotational and half translational, acting in all movable joints,

$$F_i := mg_{ij} a_j = mg_{ij}(v_j' + \Gamma^j_{ik} v_i v_k) = mg_{ij}(x_j'' + \Gamma^j_{ik} x_i' x_k'),$$

where we have used the simplified material metric tensor, mg_{ij}, for the system defined by its Riemannian kinetic energy form

$$T = \frac{1}{2} m g_{ij} v_i v_j$$

Equation $F_i = m g_{ij} a_j$, defined properly by (9) at every representative point x_i of the system's configuration manifold M, formulates the sought for covariant force law, that generalizes the fundamental Newtonian equation, $F = ma$, for the generic physical or engineering system. Its meaning is:

Force 1-form-field = Mass distribution × Acceleration vector-field.

Newton's law and Gravitational force fields

Recall that the gravitational field is a vector field that describes the gravitational force. In general relativity the gravitational field is determined as the solution of Einstein's field equations (EFEs). These equations are dependent on the distrib- ution of matter and energy in a region of space, which would be applied on an object in any given point in space per unit mass [1]. It is actually equal to the gravitational acceleration at that point. So it is a generalization of the vector form, which becomes particularly useful if more than two objects are involved.

Note that Gravitational fields are also conservative [3], that is, the work done by gravity from one position to another is path-independent. This has the consequence that there exists a gravitational potential field. For point mass or the mass of a sphere with homogeneous mass distribution mg_{ij}, the force field g(r) outside the sphere is isotropic, i.e., depends only on the distance r from the center of the sphere. Now Dieudonne, J.A. [4] show the following theorem about trajectories separation:

Theorem 1: Given a vector–field $X \in \chi(M)$, for all points $p \in M$, there exist $\eta > 0$, a neighborhood V of p, and a function

$$y : (-\eta,\eta) \times V \to M,\ (t, x_i(0)) \mapsto y(t, x_i(0))$$ such that

$$y' = X \circ y, \quad y(0, x_i(0)) = x_i(0) \quad for\ all\ x_i(0) \in V \subseteq M$$

for all $|t| < \eta$, the map $x_i(0) \mapsto y(t, x_i(0))$ is a diffeomorphism f_t^X between V and some open set of M.

This Theorem states that trajectories that are near neighbors cannot suddenly be separated. There is a well–known estimate (see [4]) according to which points cannot diverge faster than exponentially in time if the derivative of X is uniformly bounded.

Next according to theorem 1, it showing that, where the gravitational field is weak the Einstein field equations (EFEs) can be reduced to Newton's law of gravitation by using both the weak-field approximation and the slow-motion approximation. In fact, the constant appearing in the EFEs is determined by making these two approximations.

Newtonian gravitation can be written as the theory of a scalar field Φ, which is the gravitational potential

$$\nabla^2 \Phi[\vec{x}, t] = 4\pi G\ \rho[\vec{x}, t]$$

where ρ is the mass density. The orbit of a free-moving particle satisfies

$$\vec{x}''[t] = -\nabla \Phi[\vec{x}[t], t]$$

or in tensor notation we write

$$\Phi_{ii} = 4\pi G\ \rho$$

$$\frac{d^2 x_i}{dt^2} = -\Phi$$

which written by the Einstein field equations in the trace-reversed form

$$R_{\mu\nu} = K\ (T_{\mu\nu} - \frac{1}{2} T\ g_{\mu\nu})$$

for some constant K, and the geodesic equation

$$x''^{\alpha} = -\Gamma^{\alpha}_{\beta\gamma} x'^{\beta} x'^{\gamma}$$

or

$$\frac{d^2 x^{\alpha}}{dT^2} = -\Gamma^{\alpha}_{\beta\gamma} \frac{dx^{\beta}}{dT} \frac{dx^{\gamma}}{dT}.$$

where the Christoffel symbols are

$$\Gamma_{\alpha\beta\gamma} = (g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\gamma\beta,\alpha}) / 2$$

the time component of the metric is

$$g_{00} = K^{-2} \approx 1 + 2V$$

which is the weak field limit.

Now we assume that the test particle's velocity is approximately zero

$$\frac{dx^{\beta}}{dT} \approx (\frac{dt}{dT}, 0, 0, 0)$$

and thus $\frac{d}{dt}(\frac{dt}{dT}) \approx 0$

and that the metric and its derivatives are approximately static and that the squares of deviations from the Minkowski metric are negligible. Applying these simplifying assumptions to the spatial components of the geodesic equation gives

$$\frac{d^2 x^i}{dt^2} \approx -\Gamma^i_{00}$$

where two factors of $\frac{dt}{dT}$ have been divided out.

This will reduce to its Newtonian counterpart, provided

$$\Phi_{,i} \approx \Gamma^i_{00} = \frac{1}{2} g^{i\alpha} (g_{\alpha 0,0} + g_{0\alpha,0} - g_{00,\alpha})$$

Our assumptions force α=i and the time (0) derivatives to be zero. So this simplifies to

$$2\Phi_{,i} \approx g^{ij}(-g_{00,j}) \approx -g_{00,i}$$

which is satisfied by letting

$$g_{00} \approx -c^2 - 2\Phi$$

Turning to the Einstein equations, we only need the time-time component

$$R_{00} = K\ (T_{00} - \frac{1}{2} T\ g_{00})$$

the low speed and static field assumptions imply that

$$T_{\mu\nu} \approx \mathrm{diag}(T_{00}, 0, 0, 0) \approx \mathrm{diag}(\rho c^4, 0, 0, 0)$$

So

$$T = g^{\alpha\beta} T_{\alpha\beta} \approx g^{00} T_{00} \approx \frac{-1}{c^2} \rho c^4 = -\rho c^2$$

and thus

$$K\ (T_{00} - \frac{1}{2} T\ g_{00}) \approx K\ (\rho c^4 - \frac{1}{2}(-\rho c^2)(-c^2)) = \frac{1}{2} K\ \rho c^4$$

From the definition of the Ricci tensor

$$R_{00} = \Gamma^{\rho}_{00,\rho} - \Gamma^{\rho}_{\rho 0,0} + \Gamma^{\rho}_{\rho\lambda} \Gamma^{\lambda}_{00} - \Gamma^{\rho}_{0\lambda} \Gamma^{\lambda}_{0\rho}$$

our simplifying assumptions make the squares of Γ disappear together with the time derivatives

$$R_{00} \approx \Gamma^{\rho}_{00,i}$$

Combining the above equations together

$$\Phi_{,ii} \approx \Gamma^i_{00,i} \approx R_{00} = K\ (T_{00} - \frac{1}{2} T\ g_{00}) \approx \frac{1}{2} K\ \rho c^4$$

which reduces to the Newtonian field equation provided

$$\frac{1}{2} K\ \rho c^4 = 4\pi G\ \rho\ ;$$

This will occur if

$$K = \frac{8\pi G}{c^4}.$$

Which simplifying the study of motion of a charged particle**,** as subatomic particles, or a collection of charged particles which is called plasma, in electromagnetic fields, since charges structures made up by Eikonal surfaces and wave fronts, and so on are examples of physical structures. The emergence of physical structures in evolutionary process reveals in material system as an advent of certain observable formations, which develop spontaneously. Such formations and their manifestations are fluctuations, turbulent pulsations, waves, vortices, and others. It appears that structures of physical fields and the formations of material systems observed are a manifestation of the same phenomena [8]. The light is an example of such a duality. The light manifests itself in the form of a massless particle, like photons, and of a wave.

Now we give our main result:

Theorem 2: Trajectories of free particles on a smooth manifold are geodesics with exterior differential forms.

Proof: Let M be smooth manifold then a geodesic with an affine connection ∇ is defined as a curve $y(t)$ such that parallel transport along the curve preserves the tangent vector to the curve, so

$$\nabla_{y'} y' = 0 \qquad (10)$$

at each point along the curve, where y' is the derivative with respect to t. to define the covariant derivative of y' it is necessary first to extend y' to a continuously differentiable vector field in an open set. However, the resulting value of (10) is independent of the choice of extension.

Now we can write the geodesic equation as:

$$\frac{d^2 x^\alpha}{dt^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{dt} \frac{dx^\gamma}{dt} = 0 \qquad (11)$$

where $x^\beta(t)$ are the coordinates of the curve $y(t)$ on M and $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols of the connection ∇. This is just an ordinary differential equation for the coordinates. It has a unique solution, given an initial position and an initial velocity. Therefore, from the point of view of classical mechanics, geodesics can be thought of as trajectories of free particles in a manifold. Indeed, the equation $\nabla_{y'} y' = 0$ means that the acceleration of the curve has no components in the direction of the surface (and therefore it is perpendicular to the tangent plane of the surface at each point of the curve). So, the motion is completely determinded by the bending of the surface. This is also the idea of the general relativity where particles move on geodesics and the bending is caused by the gravitation.

Gravitational energy is the potential energy associated with gravitational force. If an object move from point A to point B inside a gravitational field, the force of gravity will do positive work on the object and the gravitational potential energy will decrease by the same amount.

So when electromagnetic fields are determined using charges and currents via Maxwell's equations [10], the EFEs are used to determine the spacetime geometry resulting from the presence of mass-energy and linear momentum, that is, they determine the metric tensor of spacetime [5], for a given arrangement of stress-energy in the spacetime [11]. The relationship between the metric tensor and the Einstein tensor allows the EFE to be written as a set of non-linear partial differential equations (and so an exterior differential system [8]), so the solutions of the EFE are the components of the metric tensor [6], when used in this way, the solutions are the integral manifold of exterior differential system.

REFERENCES

1. **Aczel, Amir D., (1999).** God's Equation: Einstein, Relativity, and the Expanding Universe. Delta Science. A popular account.
2. **Brown, Harvey (2005),** Physical Relativity, Oxford University Press, p. 164.
3. **Charles Misner, Kip Thorne, and John Wheeler, (1973).** Gravitation. W H Freeman.
4. **Dieudonne, J.A. (1969).** Foundations of Modern Analysis (in four volumes). Academic ress, New York.
5. **Dimitrijevic, M., Meyer, F., M¨oller, L., Wess, J. (2004).** Gauge theories on the kapa–Minkowski spacetime. Eur. Phys. J. C36, 117.
6. **Dragovic, V., Gajic, B. (2003).** The Wagner Curvature Tensor in Nonholonomic Mechanics.
7. **Einstein, Albert (1916).** "The Foundation of the General Theory of Relativity"(PDF). Annalen der Physik. http://www.alberteinstein.info/gallery/gtext3.html.
8. **Logman. A.S, (2009).** "The Theory of Exterior Differential Systems and its Applications", PhD, Sudan University of Science and Technology (SUST).
9. **Stephani, Hans; D. Kramer, M. Mac Callum, C. Hoenselaers and E. Herlt (2003).** Exact Solutions of Einstein's Field Equations. Cambridge University Press. ISBN 0-521-46136-7.
10. **Trautman, Andrzej (1977).** "Solutions of the Maxwell and Yang-Mills Equations associated with hopf fibrings", International Journal of Theoretical Physics
11. **Turner, Michael S. (May, 2001).** "A Spacetime Odyssey". Int.J.Mod.Phys. A17S1: 180–196.