Introduction

A $C^*$-algebra is called primitive if it admits a faithful and irreducible $*$-representation. We show that if $A_1$ and $A_2$ are separable, unital, residually finite dimensional $C^*$-algebras satisfying $(\dim(A_1) - 1)(\dim(A_2) - 1) \geq 2$, then the unital $C^*$-algebra full free product, $A = A_1 * A_2$, is primitive. It follows that $A$ is antiliminal, it has an uncountable family of pairwise in equivalent irreducible faithful $*$-representations and the set of pure states is $L^*$-dense in the state space. We prove the following: Suppose that $\phi, \psi : A \to B$ are unital $*$-monomorphisms. There exists a sequence of unitaries $\{u_n\} \subset B$ such that $\lim_{n \to \infty} u_n^\varepsilon(\phi(a))u_n = \psi(a)$ for all $a \in A$ if and only if $[\phi] = [\psi]$ in $KL(A,B)$, $\phi_\# = \psi_\#$ and $\phi^\dagger = \psi^\dagger$, where $\phi_\#, \psi_\# : \text{Aff}(T(A)) \to \text{Aff}(T(B))$ and $\phi^\dagger, \psi^\dagger : U(A)/CU(A) \to U(B)/CU(B)$ are the induced maps (where $T(A)$ and $T(B)$ are the tracial state spaces of $A$ and $B$, and $CU(A)$ and $CU(B)$ are the closures of the commute subgroups of the unitary groups of $A$ and $B$, respectively).

Also show that there is a unital homomorphism $\phi : A \to B$ so that $([\phi], \phi_\#, \phi^\dagger) = (\kappa, \lambda, \gamma)$, at least in the case that $K_1(A)$ $K_1(A)$ is a free group.

Let $A$ be unital separable simple $Z$-stable $C^*$-algebra which has rational tracial rank almost one and let $u \in U_0(A)$, where $U_0(A)$ is the connected component of the unitary group of $A$ containing the identity. We show that, for any $\epsilon > 0$, there exists an element $h \in A$ such that $\|u - \exp(ih)\| < \epsilon$. But there is no control of $\|h\|$ in general. For the Jiang–Su algebra $Z$, we show that, if $u \in U_0(Z)$ and $\epsilon > 0$, there exists a real number $-\pi < t \leq \pi$ and aself-adjoint element $h \in Z$ with $\|h\| \leq \pi$ such that $\|e^{it}u - \exp(ih)\| < \epsilon$. Also we show $\phi$ and $\psi$ are approximately unitarily equivalent if and only if $[\phi] = [\psi]$ in $KL(C,A)$, $\tau \circ \phi = \tau \circ \psi$ for all tracial states of $A$ and $\phi^\dagger = \psi^\dagger$, where $\phi^\dagger$ and $\psi^\dagger$ are homomorphisms from $U(C)/CU(C) \to U(A)/CU(A)$ induced by $\phi$ and $\psi$, respectively, and where $CU(C)$ and $CU(A)$ are closures of the subgroup generated by commutators of the unitary groups of $C$ and $B$.

A more practical but approximate version of the above is also presented.

Let $\epsilon > 0$ be a positive number. Is there a number $\delta > 0$ satisfying the following. Given any pair of unitaries $u$ and $v$ in a unital simple $C^*$-algebra $A$ with $[v] = 0$ in $K_1(A)$ for which $\|uv - vu\| < \delta$, there is a continuous path of unitaries $\{v(t) : t \in [0,1]\} \subset A$ such that $v(0) = v, v(1) = 1$ and $\|uv(t) - v(t)u\| < \epsilon$ for all $t \in [0,1]$. An answer is given to this question when $A$ is assumed to be a unital simple $C^*$-algebra with tracial rank no more than one. Also we study the case that $A$ is no longer assumed to have real rank zero, or tracial rank zero.

We give a classification theorem for unital separable nuclear $C^*$-algebras with tracial rank no more than one. Let $A$ and $B$ be two unital separable simple nuclear $C^*$-algebras with $TR(A), TR(B) \leq 1$ which satisfy the universal coefficient theorem.