Chapter (1)

The Sine-Cosine Function Method for Exact Solutions of Nonlinear Partial Differential Equations

Sec (1.1) Introduction:

Nonlinear evolution equations have a major role in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, solid state physics, chemical physics and geochemistry. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations [1]. In recent years, quite a few methods for obtaining explicit solitary wave solutions of nonlinear evolution equations have been proposed. A variety of powerful methods, such as, tanh-sech method [2,3,4], extended tanh method [5,6,7], hyperbolic function method [8,9], Jacobi elliptic function expansion method [10], F-expansion method [11], and the First Integral method [12,13]. The sine-cosine method [14, 15] has been used to solve different types of nonlinear systems of PDEs.

This chapter contains two parts. The first part explains the proposed method, while the second part contains the applications. The aim of this chapter is to find new exact solutions of the K(n + 1, n + 1) equation, Schrödinger-Hirota equation, Gardner equation, modified Korteweg–de Vries equation (KdV) equation, perturbed Burgers equation, general Burger's-Fisher equation, and Cubic modified Boussinesq equation by the sine-cosine method.

Sec (1.2) The Sine-Cosine Function Method

Consider the nonlinear partial differential equation in the form:

$$\mathrm{F}(u\,, u_t\,, u_x\,, u_y\,, u_{tt}\,, u_{xx}\,, u_{xy,} u_{yy,} \ldots \ldots \ldots \ldots) = 0 \qquad (1.1)$$

Where $u(x, y, t)$ is a traveling wave solution of nonlinear partial differential equation Eq. (1.1). We use the transformations.

$$u(x, y, t) = f(\xi) \tag{1.2}$$

Where $\xi = x + y - \lambda t$ this enables us to use the following changes:

$$\frac{\partial}{\partial t}(.) = -\lambda \frac{d}{d\xi}(.)\,,\ \frac{\partial}{\partial x}(.) = \frac{d}{d\xi}(.)\,,\ \frac{\partial}{\partial y}(.) = \frac{d}{d\xi}(.) \tag{1.3}$$

Using Eq. (1.3) to transfer the nonlinear partial differential equation Eq. (1.1) to nonlinear ordinary differential equation.

$$Q(f, f', f'', f''', \ldots\ \ldots\ \ldots\ \ldots.) = 0 \tag{1.4}$$

The ordinary differential equation (1.4) is then integrated as long as all terms contain derivatives, where we neglect the integration constants. The solutions of many nonlinear equations can be expressed in the form [16, 17]:

$$f(\xi) = \alpha \sin^{\beta}(\mu\xi), \qquad |\xi| \le \frac{\pi}{2\mu}$$

Or in the form (1.5)

$$f(\xi) = \alpha \cos^{\beta}(\mu\xi), \qquad |\xi| \le \frac{\pi}{2\mu}$$

Where, α, μ and β are parameters to be determined, μ and ξ are the wave number and the wave speed, respectively [15, 18]. We use:

$$f(\xi) = \alpha \sin^{\beta}(\mu\xi)$$

$$f'(\xi) = \alpha \beta\mu \sin^{\beta-1}(\mu\xi) \cos(\mu\xi)$$

$$f''(\xi) = \alpha \beta(\beta - 1)\mu^2 \sin^{\beta-2}(\mu\xi) - \alpha\beta^2\mu^2\sin^{\beta}(\mu\xi) \tag{1.6}$$

And their derivatives, are use:

$$f(\xi) = \alpha \cos^{\beta}(\mu\xi)$$

$$f'(\xi) = -\alpha \beta\mu \cos^{\beta-1}(\mu\xi) \sin(\mu\xi)$$

$$f''(\xi) = -\alpha \beta(\beta - 1)(\beta - 2)\mu^3 \cos^{\beta-3}(\mu\xi) \sin(\mu\xi) + \alpha\beta^3\mu^3\cos^{\beta-1}(\mu\xi) \sin(\mu\xi) \tag{1.7}$$

And so on. We substitute Eq.(1.6) or Eq.(1.7) into the reduced equation Eq.(1.4), balance the terms of the sine functions when Eq.(1.6) are used, or balance the terms of the cosine functions when Eq.(1.7) are used, and solve the resulting system of algebraic equations by using computerized symbolic packages. We next collect all terms with the same power in $\sin^{k}(\mu\xi)$ or

$\cos^{k}(\mu\xi)$ and set to zero their coefficients to get a system of algebraic equations among the unknown's, and solve the subsequent system.

Sec (1.3) Applications

Problem (1):-

The K (n + 1, n + 1) equation

Let us consider the following K (n + 1, n + 1) equation [19]:

$$u_t + a(u^{n+1})_x + b(u(u^n)_{xx})_x = 0 \qquad (1.8)$$

$$u_t = -k\,\lambda\,u'$$

$$(u^{n+1})\,x = (n+1)\,k\,u^n\,u'$$

$$(u^n)x = kn\,u^{n-1}u'$$

$$(u^n)xx = n\,k^2u^{n-1}u'' + n(n-1)\,k^2u^{n-2}(u')^2$$

$$u(u^n)xx = n\,k^2u^nu'' + n(n-1)\,k^2u^{n-1}(u')^2$$

$$(u(u^n)xx)x = n\,k^3u^nu''' + n^2\,k^3u^{n-1}u'' + 2n(n-1)\,k^3u^{n-1}u'u'' \qquad (1.9)$$

Where a and b are nonzero constants. We introduce the transformation $\xi = k\,(x-\lambda t)$, where k and λ are real constants. The traveling wave variable permits us converting Eq. (1.8) into the following ODE:

$$-\lambda\,ku' + a(n+1)k\,u^nu' + bnk^3u^nu''' + bn^2k^3u^{n-2}u''$$
$$+2bn(n-1)k^3u^{n-1}u'u'' + bn(n-1)^2k^3u^{n-2}(u')^3 = 0$$

Multiplying by $k^{-1}u^{2-n}$, we have:

$$-\lambda u'u^{2-n} + a(n+1)u^2u' + bnk^2u^2u''' + bn^2k^2uu'u'' -$$
$$+2bn(n-1)k^2uu'u'' + bn(n-1)^2k^2(u')^3 = 0$$

$$-\lambda\,u'u^{2-n} + a(n+1)u^2u' + bnk^2u^2u''' + bk^2n[3n-2]$$
$$uu'u'' + bn(n-1)^2k^2(u')^3 = 0 \qquad (1.10)$$

Seeking the solution in Eq.(1.7):

$\lambda\,\beta\,\mu\,\alpha^{3-n}cos^{(3-n)\beta-1}(\mu\xi)\,sin\,(\mu\xi) -$

$a(\text{n}+1)\,\alpha^3\,\beta\,\mu\,cos^{3\beta-1}(\mu\xi)\,sin(\mu\xi)$ – b n $k^2\alpha^3\,\beta(\beta-1)$

$(\beta-2)\,\mu^3\,cos^{3\beta-3}\,(\mu\xi)\,sin\,(\mu\xi) +$

$b\, n\, k^2 \alpha^3\ \beta^3\ \mu^3\ cos^{3\beta-1}(\mu\xi)\ sin\ (\mu\xi)$ – $b\ k^2\ n$ (3n –2) $\alpha^3\ \beta^2$
$(\beta-1)\ \mu^3\ cos^{3\beta-3}(\mu\xi)\ sin\ (\mu\xi) + b\ k^2\ n$ (3n –2)
$\alpha^3\ \beta^3 \mu^3\ \ cos^{3\beta-1}(\mu\xi)\ sin\ (\mu\xi)$ – $b\ n\ (n-1)^2$
$k^2 \beta^3 \mu^3 \alpha^3\ \ cos^{3\beta-3}(\mu\xi)\ sin\ (\mu\xi) = 0$ (1.11)

From Eq.(1.10), equating exponents $(3-n)\ \beta - 1$ and $3\beta - 3$ yield:

$(3-n)\ \beta - 1 = 3\ \beta - 3$ (1.12)

So that

$\beta = \frac{2}{n}$ (1.13)

Thus setting coefficients of Eq.(1.11) to zero yields the following system of equations:

$$\lambda\ \beta\ \mu\ \alpha^{3-n} - bnk^2\ \alpha^3\ \beta(\beta-1)\ (\beta-2)\ \mu^3 - bk^2 n(3n-2)$$
$$\alpha^3 \beta^2 (\beta-1) \mu^3 - bn(n-1)^2 k^2 \beta^3 \mu^3 \alpha^3 = 0$$

Apply $\beta = \frac{2}{n}$

$$\frac{2\lambda}{n}\ \alpha^{3-n} - 2bk^2\ \alpha^3 \left(\frac{2-n}{n}\right)\left(\frac{2-2n}{n}\right) \mu^3 - bk^2 n(3n-2)\alpha^3 . \frac{4}{n^2}\left(\frac{2-n}{n}\right) \mu^3$$

$$-bn(n-1)^2 k^2 . \frac{8}{n^3} \mu^3 \alpha^3 = 0$$

Multiplying by $\alpha^{-3}\mu$:

$$\frac{2\lambda}{n}\ \alpha^{-n} - 2bk^2 \left(\frac{2-n}{n}\right)\left(\frac{2-2n}{n}\right) \mu^2 - \frac{4b}{n} k^2 (3n-2)\left(\frac{2-n}{n}\right)$$

$$-\frac{8b}{n^2}(n-1)^2 k^2 \mu^2 = 0$$

$$\frac{2\lambda}{n}\ \alpha^{-n} - 2bk^2 \mu^2 [\left(\frac{2}{n}-1\right)\left(\frac{2}{n}-2\right) - \frac{2}{n}(3n-2)\left(\frac{2}{n}-1\right)$$

$$-\frac{4}{n^2}(n-1)^2] = 0$$

$$\frac{2\lambda}{n}\ \alpha^{-n} = 2bk^2 [\frac{an(n+1)}{4bk^2(3n-1)}][\ \left(\frac{2}{n}-1\right)\left(\frac{2}{n}-2\right) - \frac{2}{n}(3n-2)\left(\frac{2}{n}-1\right)$$

$$-\frac{4}{n^2}(n-1)^2]$$

$$\alpha^{-n} = \frac{a(n+1)(2n+1)(n-4)}{2(3n-1)}\ \lambda$$

$$\alpha = \ [\frac{a(3n-1)}{a(3n+1)(2n+1)(n-4)}\ \lambda]^{\frac{1}{n}}$$

$$-a(n+1)\alpha^3\ \beta\ \mu + bnk^2\alpha^3\beta^3\mu^3 + bk^2n(3n-2)$$
$$\alpha^3\beta^3\mu^3 = 0$$
$$-\frac{2a}{n}(n+1)\alpha^3\mu + bk^2\alpha^3\mu^3\left[\frac{8}{n^2}(3n-1)\right] = 0$$

$$\frac{8b}{n}k^2\mu^2(3n-n) = 2a(n+1)$$

$$\mu^2 = \frac{2a(n+1)}{\frac{8b}{n}k^2(3n-1)} \qquad = \qquad \frac{an(n+1)}{4bk^2(3n-1)} \tag{1.14}$$

By solving the algebraic system (1.14), we get:

$$\alpha = \{\frac{2(3n-1)}{a(n+1)(2n+1)(n-4)}\lambda\}^{\frac{1}{n}}\ \ ,\ \mu = \frac{\sqrt{\frac{an(n+1)}{b(3n-1)}}}{2k} \tag{1.15}$$

Then by substituting Eq. (1.15) into Eq. (1.7), the exact soliton solution of Eq.(1.8) can be written in the form:

$$u(x,t)\ = [\frac{2(3n-1)}{a(n+1)(2n+1)(n-4)}\lambda\ cos^2\ (\sqrt{\frac{an(n+1)}{4b(3n-1)}}(x-\lambda t)]^{\frac{1}{n}} \tag{1.16}$$

Problem (2):-

Schrödinger-Hirota equation

Consider the nonlinear Schrödinger-Hirota Equation which governs the propagation of optical soliton in a dispersive optical fiber:

$$i\ q_t + \frac{1}{2}q_{xx} + |q|^2\ q + i\lambda q_{xxx} = 0 \tag{1.17}$$

$$q_t = -2\alpha k_0 e^{i\theta}u' + iwe^{i\theta}u$$

$$q_x = e^{i\theta} k_0 u' + i\,\alpha e^{i\theta} u$$

$$q_{xx} = {k_0}^2 e^{i\theta} u'' + 2i\,\alpha k_0 e^{i\theta} u' - \alpha^2 e^{i\theta} u$$

$$q_{xxx} = {k_0}^3 e^{i\theta} u''' + 3i\,\alpha {k_0}^2 e^{i\theta} u'' - 3\alpha^2 k_0 e^{i\theta} u' - i\alpha^3 e^{i\theta} u$$

$$|q|^2 q = e^{i\theta} u^3$$

$$-2i\,\alpha k_0 e^{i\theta} u' - w e^{i\theta} u^3 + \frac{1}{2} {k_0}^2 e^{i\theta} u'' + i\,\alpha k_0 e^{i\theta} u' - \frac{1}{2}\alpha^2 e^{i\theta} u$$

$$+e^{i\theta} u^3 + i\lambda k^3 e^{i\theta} u''' - 3\,\alpha\lambda {k_0}^2 e^{i\theta} u'' - 3i\alpha^2 \lambda k_0 e^{i\theta} u'$$

$$=> (-i\,\alpha k_0 - 3i\alpha^2 \lambda k_0) e^{i\theta} u' + (\frac{1}{2} {k_0}^2 - 3\,\alpha\lambda {k_0}^2)\, e^{i\theta} u''$$

$$+i\,\lambda {k_0}^3 e^{i\theta} u''' + \left(-w - \frac{1}{2}\alpha^2 + \alpha^3\,\lambda\right) e^{i\theta} u + u^3 = 0$$

Multiplying by $e^{-i\theta}$ and take real part:

$$\left(\frac{1}{2} {k_0}^2 - 3\,\alpha\lambda {k_0}^2\right) u'' + \left(-w - \frac{1}{2}\alpha^2 + \alpha^3\,\lambda\right) u + u^3 = 0$$

$$-i\,\alpha k_0 - 3i\alpha^2 \lambda k_0 = 0$$

$$-\alpha = 3\lambda\alpha^2 \quad => \quad \alpha = -\frac{1}{3\lambda}$$

$$\left(\frac{1}{2} {k_0}^2 - 3(-\frac{1}{3\lambda})\,\lambda {k_0}^2\right) u'' + \left(-w - \frac{1}{2}(-\frac{1}{3\lambda})^2 + (-\frac{1}{3\lambda})^3\,\lambda\right) u + u^3 = 0$$

$$\left(\frac{1}{2} {k_0}^2 + {k_0}^2\right) u'' + \left(-w - \frac{1}{18\lambda^2} - \frac{1}{27\lambda^2}\right) u + u^3 = 0$$

$$\frac{3}{2} {k_0}^2 u'' + \left(-w + \frac{-3-2}{54\lambda^2}\right) u + u^3 = 0 \tag{1.18}$$

This equation studied by Biswas *et al* [20] by the ansatz method for bright and dark 1-soliton solution. The power law nonlinearity was assumed. The equation was solved also by using the $tanh$ method.

Introduce the transformations:

$$q(x,t) = e^{i\theta}.u(\xi) \;,\; \theta = \alpha x + \omega t + \epsilon_0 \;,\;\; \xi = k_0(x - 2at + X) \qquad (1.19)$$

Where , ω , ϵ_0 , k_0 and X are real constants. Substituting Eq. (1.19) into Eq. (1.17) we obtain that $\alpha = \frac{-1}{3\lambda}$ and $u(\xi)$ satisfy into the ODE:

$$-\left(\frac{5}{54\lambda^2} + \omega\right) u(\xi) + \frac{3}{2}{k_0}^2 u''(\xi) + (u(\xi))^3 = 0$$

Multiplying by $\frac{3}{2}{\mathrm{k}_0}^{-2}$ we have:

$$u'' - \left(\frac{w + \frac{5}{54\lambda^2}}{{}^3/_2{k_0}^2}\right) u + \frac{1}{{}^3/_2{k_0}^2} u^3 = 0 \qquad (1.20)$$

Then we can write the following equation:

$$u'' + k_1 u^3 - k_2\, u = 0 \qquad (1.21)$$

Where:

$$k_1 = \frac{1}{\frac{3}{2}{k_0}^2} \;,\; k_2 = \frac{\left(\frac{5}{54\lambda^2} + \omega\right)}{\frac{3}{2}{k_0}^2} \qquad (1.22)$$

Seeking solutions of the form Eq.(1.6) we get:

$$\alpha\, \beta(\beta-1)\, \mu^2\, sin^{\beta-2}(\mu\xi) - \alpha\, \beta^2 \mu^2\, sin^{\beta}(\mu\xi) +$$

$$k_1\, \alpha^3\, sin^{3\beta}(\mu\xi) - k_2\, \alpha\, sin^{\beta}(\mu\xi) = 0 \qquad (1.23)$$

Equating the exponents and the coefficients of each pair of the cosine functions we find the following algebraic system:

$$\beta - 2 = 3\,\beta$$

$$\beta = -1$$

$$\alpha\, \beta(\beta-1)\, \mu^2 + k_1\, \alpha^3 = 0$$

$$2\alpha\mu^2 + k_1\alpha^3 = 0$$

$$-\alpha\, \beta^2 \mu^2 - k_2\, \alpha = 0$$

$$-\alpha\mu^2 - k_2\,\alpha = 0$$

$$\mu^2 = -k_2$$

$$\alpha^2 = -\frac{2\mu^2}{k_1} \tag{1.24}$$

By solving the algebraic system (1.24), we get:

$$\beta = -1\ ,\ \ \mu = \pm i\sqrt{k_2}\quad ,\quad \alpha = \pm\frac{\sqrt{2k_2}}{k_1} \tag{1.25}$$

Then by substituting Eq. (1.25) into Eq. (1.6), the exact soliton solution of equation (1.19) can be written in the form:

$$u(\xi) = \pm\sqrt{\frac{5}{27\lambda^2} + 2\beta}\, csc\,(\pm i\sqrt{k_2}\xi) \tag{1.26-a}$$

Or

$$u(\xi) = \mp\sqrt{\frac{5}{27\lambda^2} + 2\beta}\, csch\,(\sqrt{k_2}\xi) \tag{1.26-b}$$

$$u(x,y,t) = \pm\sqrt{\left(\frac{5}{27\lambda^2} + 2\omega\right)}$$

$$csch\left(\sqrt{\frac{\left(\frac{5}{54\lambda^2}+\omega\right)}{\frac{3}{2}k_0{}^2}}\ k_0\left(x + \frac{2}{3\lambda}t + x\right)\right)e^{i\left(\frac{-1}{3\lambda}x+\omega t+\epsilon_0\right)} \tag{1-27}$$

For $\alpha = \omega = k_0 = 1$, $\epsilon_0 = x = 0$, $= \frac{-1}{3}$, then (1.27) become:

$$u(x,y,t) = \pm\sqrt{\frac{11}{3}}\, csch\left(\sqrt{\frac{11}{3}}(x - 2t)\right)e^{i(x+t)} \tag{1.28}$$

Problem (3):-

Gardner equation

Let us consider the Gardner equation [21, 22].

$$u_t - 6(u + \varepsilon^2 u^2)u_x + u_{xxx} = 0 \tag{1.29}$$

$u_t = -u'$ $\qquad\qquad$ $u_x = ku'$

$u_{xx} = k^2 u''$ $\qquad\qquad$ $u_{xxx} = k^3 u'''$

$$-\lambda u'k - 6(u + \varepsilon^2 u^2)ku' + k^3 u''' = 0$$

$$-\lambda ku' - 6ku'u - 6k\varepsilon^2 u^2 u' + k^3 u''' = 0 \qquad (1.30)$$

This equation known as the mixed KdV- mKdV equation is very widely studied in various areas of Physics that includes Plasma Physics, Fluid Dynamics, Quantum Field Theory, Solid State Physics and others [22].

Give $\xi = k(x - \lambda t)$ we introduce the transformation, where k and λ are real constants. Equation (1.27) transforms to the ODE:

$$-k\lambda u' - 3k(u^2)' - 3k\varepsilon^2 (u^3)' + k^3 u''' = 0 \qquad (1.31)$$

Integrating Eq. (1.31) once with zero constant to get the following ordinary differential equation and multiply by $-k^{-1}$:

$$\lambda u + 3u^2 + 2\varepsilon^2 u^3 - k^2 u'' = 0 \qquad (1.32)$$

Seeking the solution in Eq.(1.7):

$$\lambda\, \alpha cos^{\beta}(\mu\xi) + 3\alpha^2 cos^{2\beta}(\mu\xi) + 2\,\boldsymbol{\varepsilon}^{2}\alpha^3 cos^{3\beta}(\mu\xi) -$$
$$\alpha\, \beta(\beta - 1)k^2\mu^2 cos^{\beta-2}(\mu\xi) + \alpha\beta^2\mu^2 k^2 cos^{\beta}(\mu\xi) = 0 \qquad (1.33)$$

Equating the exponents and the coefficients of each pair of the cosine functions we find the following algebraic system:

$$\beta(\beta - 1)(\beta - 2) \neq 0$$

$$3\beta = \beta - 2 \; \rightarrow \; \beta = -1 \qquad (1.34)$$

Substituting Eq. (1.34) into Eq. (1.33) to get:

$$\lambda\, \alpha cos^{-1}(\mu\xi) + 3\alpha^2 cos^{-2}(\mu\xi) + 2\,\boldsymbol{\varepsilon}^{2}\alpha^3 cos^{-3}(\mu\xi) -$$
$$2\alpha\, k^2\mu^2 cos^{-3}(\mu\xi) + \alpha\mu^2 k^2 cos^{-1}(\mu\xi) = 0 \qquad (1.35)$$

Equating the exponents and the coefficients of each pair of the cosine function, we obtain a system of algebraic equations:

$$cos^{-3}(\mu\xi)\; : 2\varepsilon^2\alpha^3 - 2\,\alpha k^2 \mu^2 = 0$$

$$\alpha^3 = \frac{k^2\mu^2}{\varepsilon^2}$$

$$cos^{-2}(\mu\xi)\; : 3\alpha^3 = 0$$

$cos^{-1}(\mu\xi)\ : \lambda\alpha + \ \alpha\mu^2 k^2 = 0$

$$\lambda = -k^2\mu^2 \tag{1.36}$$

By solving the algebraic system (1.36), we get:

$$\beta = -1 \ \ , \ \lambda = -k^2\mu^2 \ \ , \ \ \alpha = \pm\frac{k\mu}{\varepsilon} \tag{1.37}$$

Then by substituting Eq. (1.37) into Eq. (1.7), the exact soliton solution of Eq.(1.29)be in the form:

$$u(x,t) = \mp\frac{k\mu}{\varepsilon} sec\big(\mu k(x+\mu^2k^2t)\big), 0 < \mu k(x+\mu^2k^2t) < \pi \tag{1.38}$$

For,$\mu = k = \varepsilon = 1$, then Eq.(1.35) becomes:

$$u(x,t) = sec(x+t) \quad , 0 < (x+t) < \pi \tag{1.39}$$

Problem (4):-

Dispersive equation

Consider the (1+1)-dimensional nonlinear dispersive equation [23].

$$u_t - \delta u^2 u_x + u_{xxx} = 0 \tag{1.40}$$

$$u_t = -\lambda k u' \qquad\qquad u_x = ku'$$

$$u_{xx} = k^2 u'' \qquad\qquad u_{xxx} = k^3 u''' \tag{1.41}$$

Where δ is a nonzero positive constant. This equation is called the modified KdV equation Elsayed et al [23], which arises in the process of understanding the role of nonlinear dispersion and in the formation of structures like liquid drops, and it exhibits compaction solitons with compact support. To find the traveling wave solutions of Eq. (1.40), He et al [24] used the Exp-function method, and [23] used expansion method.

Let us now solve Eq. (1.40) by the proposed method. We introduce the transformation $\xi = (x - \lambda t)$, where k and λ are real constants. Equation (1.40) transforms to the ODE:

$$-\lambda k u' - \frac{\delta}{3}k(u^3)' + k^3u''' = 0 \tag{1.42}$$

Integrating Eq. (1.42) once with zero constant to get the following ordinary differential equation and multiply by $-k^{-1}$:

$$\lambda u + \frac{\delta}{3} u^3 - k^2 u'' = 0 \tag{1.43}$$

Seeking the solution in Eq.(1.7):

$$\lambda\, \alpha\cos^{\beta}(\mu\xi) + \frac{\delta}{3}\, \alpha^3 \cos^{3\beta}(\mu\xi) -$$

$$\alpha\, \beta(\beta-1)k^2\mu^2\cos^{\beta-2}(\mu\xi) + \alpha\beta^2\mu^2k^2\cos^{\beta}(\mu\xi) = 0 \tag{1.44}$$

Equating the exponents and the coefficients of each pair of the cosine functions we find the following algebraic system:

$$3\beta = \beta - 2 \rightarrow \beta = -1$$

$$cos^{-3}(\mu\xi) : \ \frac{\delta}{3}\alpha^3 - 2\alpha k^2\mu^2 = 0$$

$$\alpha^2 = \frac{6k^2\mu^2}{\delta}$$

$$cos^{-1}(\mu\xi) : \ \lambda\alpha + \alpha\mu^2k^2 = 0$$

$$\lambda = -k^2\mu^2 \tag{1.45}$$

By solving the algebraic system (1.45), we get:

$$\beta = -1 \ , \ \lambda = -k^2\mu^2 \ , \ \alpha = \pm\sqrt{\frac{6}{\delta}}\ \mu k \tag{1.46}$$

Then by substituting Eq. (1.46) into Eq. (1.7), the exact soliton solution of Eq.(1.40) can be written in the form:

$$u(x,t) = \mp\sqrt{\frac{6}{\delta}}\ k\mu \quad \sec(\mu k(x + \mu^2k^2t))$$

$$0 < \mu k(x + \mu^2k^2t) < \pi \tag{1.47}$$

Problem (5):-

Perturbed Burgers equation

The study is going to be focused on the perturbed Burgers equation [25]. The solitary wave ansatz method will be adopted to obtain the exact 1-soliton solution of the Burgers equation in (1+1) dimensions. The search is going to be for a topological 1-soliton solution. The perturbed Burgers equation that is given by the following form [25]:

$$u_t + auu_x + bu_{xx} = cu^2u_x + duu_{xx} + \gamma(u_x)^2 + \delta u_{xxx} \qquad (1.48)$$

$$u_t = -\lambda ku' \qquad u_x = ku'$$

$$u_{xx} = k^2u'' \qquad u_{xxx} = k^3u''' \qquad (1.49)$$

Eq.(1.48) appears in the study of gas dynamics and also in free surface motion of waves in heated fluids. The perturbation terms are obtained from long-wave perturbation theory. Eq.(1.48) shows up in the long-wave small-amplitude limit of extended systems dominated by dissipation, where dispersion is also present at a higher order [25].

To solve Eq.(1.48) by the proposed method. We introduce the transformation $\xi = k(x - \lambda t)$, where k and λ are real constants. Equation (1.48) transforms to the ODE:

$$-\lambda ku' + akuu' + bk^2u'' = cku^2u' + dk^2uu''$$

$$+\gamma k^2(u')^2 + \delta\, k^3u''' \qquad (1.50)$$

Seeking the solution in Eq. (1.7)

$\lambda\, \alpha\beta\mu\, cos^{\beta-1}(\mu\xi)sin(\mu\xi) -$

$a\alpha^2\beta\mu\, cos^{2\beta-1}(\mu\xi)\, sin(\mu\xi) +$

$bk\alpha\, \beta(\beta-1)\mu^2cos^{\beta-2}(\mu\xi) - bk\alpha\beta^2\mu^2cos^{\beta}(\mu\xi) +$

$c\alpha^3\beta\mu\, cos^{3\beta-1}(\mu\xi)\, sin(\mu\xi) -$

$dk\alpha^2\beta(\beta-1)\mu^2cos^{2\beta-2}(\mu\xi) + dk\alpha^2\beta^2\mu^2cos^{2\beta}(\mu\xi) -$

$\gamma k\alpha^2\beta^2\mu^2 cos^{2\beta-2}(\mu\xi)+\gamma k\alpha^2\beta^2\mu^2 cos^{2\beta}(\mu\xi)+$

$\alpha\,\beta(\beta-1)(\beta-2)\mu^3\delta k^2 cos^{\beta-3}(\mu\xi)\,sin(\mu\xi)-$

$\alpha\beta^3\mu^3\,\delta\,k^2 cos^{\beta-1}(\mu\xi)\,sin(\mu\xi)=0$ (1.51)

From (1.51), equating exponents $2\beta-2$ and $3\beta-1$ yield:

$$2\beta-2=\ 3\beta-1 \tag{1.52}$$

So that

$$\beta=-1 \tag{1.53}$$

It needs to be noted that the same value of β is obtained when the exponent pairs are equated. Thus setting their coefficients to zero yields:

$$-dk\alpha^2\,\beta(\beta-1)\mu^2-\gamma k\alpha^2\beta^2\mu^2+\alpha\beta(\beta-1)(\beta-2)\mu^3\delta k^2=0$$

$$bk\alpha\,\beta(\beta-1)\mu^2-a\alpha^2\beta\mu=0$$

$$(dk+\gamma k)\alpha\,\beta\mu+\lambda-\beta^2\mu^2\delta k^2=0 \tag{1.54}$$

By solving the algebraic system (1.54) we get:

$$\delta=\frac{(2d+\gamma)b}{3a}\quad,\qquad \alpha=-\frac{2bk}{a}\,\mu\quad,$$

$$\lambda=[4d-5\gamma]\frac{b}{3a}\quad k^2\mu^2 \tag{1.55}$$

Then by substituting Eq. (1.55) into Eq. (1.7), the exact soliton solution of equation (1.48) can be written in the form:

$$u(x,t)=-\frac{2bk}{a}\,\mu\ \ \sec[\,\mu k(x-[4d-5y]\,\frac{b}{3a}k^2\mu^2 t)] \tag{1.56}$$

Problem (6):-

The general Burgers-Fisher equation

Consider the following general Burger's-Fisher equation [26].

$$u_t-au^n u_x+bu_{xx}+cu(1-u^n)=0 \tag{1.57}$$

$$u_t=k\lambda u' \qquad\qquad u_x=ku'$$

$$u_{xx}=k^2u'' \tag{1.58}$$

Where a, b and c are nonzero constants. We introduce the transformation $\xi=k(x-\lambda t)$, where k and λ are real constants.

The traveling wave variable permits us converting Eq. (1.57) into the following ODE:

$$-k\lambda u' + aku^n u' + bk^2 u'' + cu - cu^{n+1} = 0 \qquad (1.59)$$

Seeking the solution in Eq. (1.7):

$\lambda k\ \alpha\beta\mu\ cos^{\beta-1}(\mu\xi)\ sin(\mu\xi) -$
$ak\alpha^{n+1}\beta\mu\ \ cos^{(n+1)\beta-1}(\mu\xi)\ sin(\mu\xi) + bk^2\alpha\beta(\beta-1)$
$\mu^2\ cos^{\beta-2}(\mu\xi) - [bk^2\alpha\ \beta^2\mu^2 - c\alpha]\ cos^{\beta}(\mu\xi) -$
$c\alpha^{n+1}\ \ cos^{(n+1)\beta}(\mu\xi) = 0 \qquad (1.60)$

From Eq. (1.60), equating exponents $(\ n+1)\beta$ and $\beta - 1$ yield:

$$(\ n+1)\beta = \ \beta - 1 \qquad (1.61)$$

So that

$$\beta \ = \frac{-\mathbf{1}}{\boldsymbol{n}} \qquad (1.62)$$

When the exponent pair $(\ \mathrm{n}+1)\beta - 1 = \ \beta - 2$, is equated gave the same value of $\beta = \frac{-\mathbf{1}}{\boldsymbol{n}}$, Thus setting their coefficients to zero yields:

$$c\alpha^{n+1}\lambda k\ \alpha\beta\mu = 0$$

$$c\alpha^{n+1} - \frac{\lambda k\alpha\mu}{n} = 0$$

Multiplying by α^{-1}:

$$\lambda = \frac{c\alpha^n n}{k\mu} \qquad => \ \lambda = \frac{cn}{k\mu}\ .\ \frac{k\mu(1+n)}{na}$$

$$\lambda = \frac{cb(1+n)}{a}$$

$$bk^2\alpha\beta(\beta-1)\mu^2 - ak\alpha^{n+1}\beta\mu = 0$$

$$bk^2\alpha(1+n)\mu^2 - \frac{ak\alpha^{n+1}}{n}\ \mu = 0$$

Multiplying by α^{-1}:

$$\alpha^n = \frac{bk^2\mu^2(1+n)}{n^2} \quad . \quad \frac{n}{ak\mu} \quad => \alpha^n = \frac{bk\mu(1+n)}{na} \tag{1.63}$$

By solving the algebraic system (1.54), we get:

$$\lambda = -\frac{bc(n+1)}{a} \quad , \quad \alpha = (\frac{b(n+1)}{an}k\mu)^{\frac{1}{n}} \tag{1.64}$$

Then by substituting Eq. (1.64) into Eq. (1.7), the exact soliton solution of equation (1.57) can be written in the form:

$$u(x,t) = [\frac{b(n+1)}{an}k\mu \; sec \; (\; \mu k(x + \frac{bc(n+1)}{a}t))]^{\frac{1}{n}} \tag{1.65}$$

Problem (7):-

Cubic modified Boussinesq equation

Consider the cubic modified Boussinesq equation [27].

$$u_{tt} + u_{xxt} + \frac{2}{9}u_{xxx} - (u^3)_{xx} = 0 \tag{1.66}$$

$$u_t = -\lambda u' \qquad u_{tt} = \lambda^2 u''$$

$$u_x = ku' \qquad u_{xx} = k^2 u''$$

$$u_{xxx} = k^3 u''' \qquad u_{xxxx} = k^4 u^{(4)}$$

$$u_{xxt} = -\lambda k^2 u''' \qquad (u^3)^4{}_{xx} = 3(u^2u')' \tag{1.67}$$

To solve Eq.(1.66) by applied the Homotopy Perturbation method and Padé approximants. Eq.(1.66) has an exact solution[27].

$$u(x,t) = 1 + \tanh\frac{3}{2}(x - 2t) \tag{1.68}$$

The traveling wave hypothesis as given by:

$$\xi = kx - \lambda t \tag{1.69}$$

The nonlinear partial differential equation (1.68) is carried to an ordinary differential equation.

$$\lambda^2 U'' - \lambda k^2 U''' + \frac{2}{9}k^4 U^{(4)} - 3k^2(U^2U')' = 0 \tag{1.70}$$

Integrating Eq.(1.70) twice with zero constant, Eq.(1.70) reduces to:

$$\lambda^2 U - k^2 \lambda U' + \frac{2}{9} k^4 U'' - k^2 U^3 = 0 \qquad (1.71)$$

Applying Sine-cosine method to solve Eq.(1.71), and seeking the solution in Eq. (1.7) then:

$$\lambda^2 \alpha \cos^{\beta}(\mu\xi) + k^2 \lambda\, \alpha\beta\mu \cos^{\beta-1}(\mu\xi) \sin(\mu\xi) +$$
$$\frac{2}{9} k^4 \alpha\beta(\beta-1)\mu^2 \cos^{\beta-2}(\mu\xi) - \frac{2}{9} k^4 \alpha\beta^2\mu^2 \cos^{\beta}(\mu\xi) -$$
$$k^2\, \alpha^3 \cos^{3\beta}(\mu\xi) = 0 \qquad (1.72)$$

Equating the exponents and the coefficients of each pair of the cosine functions we find the following algebraic system.

$\beta - 2 = 3\beta$, then $\beta = -1$

$\lambda^2 \alpha - \frac{2}{9} k^4 \alpha\, \beta^2 \mu^2 = 0$

$\lambda^2 = \frac{2}{9} k^4 \mu^2 \qquad => \qquad \lambda = \pm\sqrt{\frac{2}{9}}\, k^2 \mu$

$\frac{2}{9} k^4 \alpha\ \beta(\beta-1)\mu^2 - k^2 \alpha^3 = 0$

$$\frac{4}{9} k^2 \mu^2 = \alpha^2 \qquad => \qquad \alpha = \pm\frac{2}{3} k\mu \qquad (1.73)$$

By solving the algebraic system (1.72), we get:

$$\lambda = \pm\sqrt{\frac{2}{9}}\, k^2\mu\ ,\ \alpha = \pm\frac{2}{3} k\mu \qquad (1.74)$$

Then by substituting Eq.(1.74) into Eq. (1.7) then, the exact soliton solution of equation (1.66) can be written in the form:

$$u_{1.2}(x,t) = \frac{2}{3} k\ \mu\ \ sec(\,\mu(kx \mp \sqrt{\frac{2}{9}}\ k^2\mu t)) \qquad (1.75)$$

For $k = \frac{2}{3}\ k$, $\mu = 1$, then:

$$u_{1.2}(x,t) = \sec\left(\frac{3}{2}x \mp t\right) \qquad (1.76)$$

Figure (1) represents the soliatry of the solution.

$u_2(x\,,t)$ = $Sec\,(\frac{2}{3}x - t\,)$ at -10 < x < 10, and $0\ <\ t\ < 1$.

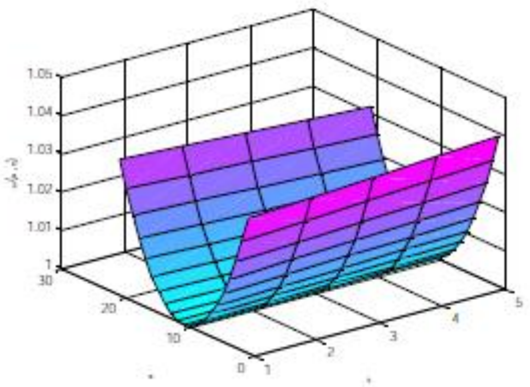


Fig. (1)

Problem (8):-

Cubic modified Boussinesq equation

Consider the cubic modified Boussinesq equation [27].

$$u_{tt} - u_{xxxx} - (u^3)_{xx} = 0 \tag{1.77}$$

$u_t = -\lambda u'$ $\qquad\qquad$ $u_{tt} = \lambda^2 u''$

$u_x = ku'$ $\qquad\qquad$ $u_{xx} = k^2 u''$

$u_{xxx} = k^3 u'''$ $\qquad\qquad$ $u_{xxxx} = k^4 u^{(4)}$

$$(u^3)_{xx} = 3(u^2 u')' \tag{1.78}$$

To solve Eq.(1.77) by applied the Homotopy Perturbation method and Padé approximants. The exact solution of Eq.(1.77) is:

$$u(\,x,t)\ =\ \sqrt{2}\,\mathrm{sech}(\,x - t) \tag{1.79}$$

The nonlinear partial differential equation (1.77) is carried to an ordinary differential equation using the transformation:

$$\xi\ = kx - \lambda t \tag{1.80}$$

Then

$\lambda^2 U'' - \ k^4 U'''' - 3\, k^2 (U^2 U')' = 0$ (1.81)

Integrating Eq.(1.81) twice and assuming the constant of integration equal to zero, then

$\lambda^2 U \ - \ k^4 U'' - k^2 U^3 = 0$ (1.82)

By applying Sine-cosine method to solve Eq.(1.82), and seeking the solution in Eq. (1.7) then

$- \lambda^2 \alpha\beta\mu \cos^{\beta-1}(\mu\xi) \sin(\mu\xi) + k^4[\alpha\beta(\beta-1)(\beta-2)$

$\mu^3 \cos^{\beta-3}(\mu\xi) \sin(\mu\xi) -$

$\alpha\beta^3\mu^3 \cos^{\beta-1}(\mu\xi) \sin(\mu\xi)] +$

$3k^2\ \alpha^3\beta\mu \cos^{3\beta-1}(\mu\xi) \sin(\mu\xi) = 0$ (1.83)

Equating the exponents and the coefficients of each pair of the sine functions we find the following algebraic system:

$\beta - 3 \ = \ 3\beta - 1, then\ \beta = 1$

$-\lambda^2\alpha\beta\mu - \alpha\, \beta^3\mu^3 k^4 = 0$

Multiplying by $\mu^{-1}\ \alpha^{-1}$:

$\lambda^2 = -\mu^2 k^4 \qquad => \qquad \lambda\ = \mp i\mu k^2$

$k^4\ \alpha\ \beta\ (\beta -1)\ (\beta -2)\ \mu^3 + 3k^2\alpha^3\ \beta\ \mu = 0$

$- 6k^4\alpha\mu^3 - 3k^2\alpha^3\mu = 0$

Multiplying by $\mu^{-1}\ \alpha^{-1}$:

$\alpha^2 = -\frac{6}{3} k^4\mu^2 \qquad => \qquad \alpha = \mp i\sqrt{2}\ \mu k^2$ (1.84)

By solving the algebraic system (1.84), we get:

$\lambda = \pm\, \mathrm{i}k^2\mu\,, \quad \alpha = \pm\, \mathrm{i}\sqrt{2}\mathrm{k}\,\mu$ (1.85)

Then by substituting Eq.(1.85) into Eq. (1.7) then, the exact soliton solution of equation (1.77) can be written in the form:

$u\,(x\,, t\,) = \ \pm\, \mathrm{i}\,\sqrt{2}\mathrm{k}\,\mu\ \ sec\,(\mu\, k\ (x\ \pm\ \mathrm{i}\ k\ \mu\ \mathrm{t}))$ (1.86)

Or

$u\,(x\,,t\,) = \pm\sqrt{2}\mathrm{k}\,\mu\;\; sec\,h\,(\mu\,k\,(ix\;\pm\;k\,\mu\,\mathrm{t}))$ (1.87)

For k = μ= 1, Eq.(1.78) becomes:

$u\,(x\,,t\,) = \pm\sqrt{2}\,sec\,h\,(ix\;\pm\;\mathrm{t}\,)$ (1.88)

Chapter (2)

application of Sine-Cosine Method for the Generalized (2+1) - Dimensional Nonlinear Evolution Equations

Sec (2.1) Introduction

We will study the generalized (2+1)-dimensional nonlinear evolution equations.

$$u_{xt} + a u_x u_{xy} + b u_{xx} u_y + u_{xxxy} = 0 \tag{2.1}$$

Where a and b are parameters. For example, namely the (2+1)-dimensional Calogero - Bogoyavlenskii - Schiff (CBS) Equation for which a = 4 and b = 2

$$u_{xt} + 4u_x u_{xy} + 2u_{xx} u_y + u_{xxxy} = 0 \tag{2.2}$$

And the (2+1)-dimensional breaking soliton equation.

For which a = -4 and b = -2

$$u_{xt} - 4u_x u_{xy} - 2u_{xx} u_y + u_{xxxy} = 0 \tag{2.3}$$

and the (2+1)-dimensional Bogoyavlenskii's Breaking Soliton equation for which a = 4 and b = 4,

$$u_{xt} + 4u_x u_{xy} + 4u_{xx} u_y + u_{xxxy} = 0 \tag{2.4}$$

We solve equation (2.1) by the sine-cosine method and obtain some exact and new solutions for (2.2), (2.3) and (2.4).

Sec (2.2) The Sine-Cosine Method

1. We introduce the wave variable ξ = x – ct into the PDE.

$$\mathrm{P}(u\,, u_t\,, u_x\,, u_{tt}\,, u_{xx}\,, u_{tx}\,, \ldots) \tag{2.5}$$

Where $u(x\,,t)$ is traveling wave solution. This enables us to use the following changes:

$$\frac{\partial}{\partial t} = -c\frac{\partial}{\partial \xi}\,, \frac{\partial^2}{\partial t^2} = c^2\frac{\partial^2}{\partial \xi^2}\,, \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\,, \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \xi^2}\,, \ldots. \tag{2.6}$$

One can immediately reduce the nonlinear PDE (2.5) into a nonlinear ODE

$$Q\,(u\,, u_{\xi}, u_{\xi\xi}\,, u_{\xi\xi\xi}\,,, \dots) \;=\; 0 \tag{2.7}$$

The ordinary differential equation (2.7) is then integrated as long as all terms contain derivatives, where we neglect integration constants.

2. The solutions of many nonlinear equations can be expressed in the form [28]:

$$u\,(x,t) = \begin{cases} \lambda\;sin^{\beta}(\mu\xi)\,, & |\xi| \le \frac{\pi}{\mu}\,, \\ 0 & \text{otherwise,} \end{cases} \tag{2.8}$$

Or in the form:

$$u\,(x,t) = \begin{cases} \lambda\;cos^{\beta}(\mu\xi)\,, & |\xi| \le \frac{\pi}{2\mu}\,, \\ 0 & \text{otherwise,} \end{cases} \tag{2.9}$$

Where λ *and* $\beta \neq 0$ are parameters that will be determined, μ and c are the wave number and the wave speed respectively. We use:

$$u(\xi) \;=\; \lambda\;\sin^{\beta}(\mu\xi)$$

$$u^{n}\,(\xi) = \lambda^{n}\;\sin^{n\beta}(\mu\xi) \tag{2.10}$$

$$(u^{n})_{\xi} = -n\mu\beta\lambda^{n}\,cos\,(\mu\xi)\;sin^{n\beta-1}(\mu\xi)$$

$$(u^{n})_{\xi\xi} = -n^{2}\mu^{2}\beta^{2}\lambda^{n}\;sin^{n\beta}(\mu\xi) + n\mu^{2}\lambda^{n}\beta(n\beta-1)\;sin^{n\beta-2}(\mu\xi)$$

And the derivatives of (2.9) becomes:

$$u(\xi) = \lambda\;cos^{\beta}(\mu\xi)$$

$$u^{n}\,(\xi) = \lambda^{n}\;cos^{n\beta}(\mu\xi) \tag{2.11}$$

$$(u^{n})_{\xi} = -n\mu\beta\lambda^{n}\,sin\,(\mu\xi)\;cos^{n\beta-1}(\mu\xi)$$

$$(u^{n})_{\xi\xi} = -n^{2}\mu^{2}\beta^{2}\lambda^{n}\;cos^{n\beta}(\mu\xi) + n\mu^{2}\lambda^{n}\beta(n\beta-1)\;cos^{n\beta-2}(\mu\xi)$$

And so on for other derivatives.

3. We substitute (2.10) or (2.11) into the reduced equation obtained above in(2.7), balance the terms of the cosine functions when (2.11) is used, or balance the terms of the sine functions when (2.10) is used, and solving the resulting system of algebraic equations by using the computerized symbolic calculations. We next collect all terms whith same power in $cos^k(\mu\xi)$ or $sin^k(\mu\xi)$ and set to zero their coefficients to get a system of algebraic equations among the unknowns μ , β and λ .We obtained all possible value of the parameters μ , β and λ [29].

Sec (2.3) New application Sine-Cosine Method

In this section we apply the sine-cosine method to the generalized (2+1)-dimensional nonlinear evolution equations.

$$u_{xt} + au_x u_{xy} + bu_{xx}u_y + u_{xxxy} = 0 \tag{2.12}$$

$u_x = u'$ $\qquad$ $u_{xt} = -cu'$

$u_{xy} = u''$ $\qquad$ $u_{xx} = u''$

$u_{xxx} = u'''$ $\qquad$ $u_{xxxy} = u^{(4)}$

$$u_y = u' \tag{2.13}$$

We use the wave transformation:

$$u(\xi) = u(x, y, t), \quad \xi = x + y - ct \tag{2.14}$$

Where c is constant to be determined later. Substituting (2.14) into system (2.12), we obtain an ordinary differential equation:

$$-cu'' + au'u'' + bu'u'' + u^{(4)} = 0, \tag{2.15}$$

Or equivalently

$$-cu'' + (a+b)u'u'' + u^{(4)} = 0, \tag{2.16}$$

Where prime denotes the differential with respect to ξ. Integrating (2.16) with respect to ξ and taking the integration constant as zero yields:

$$-cu'+\frac{(a+b)}{2}(u')^2+u^{(3)} = 0 \qquad (2.17)$$

Setting $u'(\xi) = v(\xi)$, Eq. (2.17) becomes:

$$-c\,v+\frac{(a+b)}{2}\,v^2+v'' = 0 \qquad (2.18)$$

Substituting (2.8) into (2.18) gives:

$$-\,c\lambda\,sin^{\beta}(\mu\xi)-\mu^2\beta^2\lambda\,sin^{\beta}(\mu\xi)$$

$$+\,\mu^2\,\lambda\,\beta(\beta-1)\,sin^{\beta-2}(\mu\xi)+\frac{(a+b)}{2}\lambda^2 sin^{2\beta}(\mu\xi) = 0 \qquad (2.19)$$

Equating the exponents and the coefficients of each pair of the sine functions we find the following system of algebraic equations:

$$(\beta-1)\neq 0$$

$$\beta-2=2\,\beta$$

$$-c\lambda-\mu^2\beta^2\lambda=0 \qquad (2.20\text{-a})$$

$$-c\lambda+4\mu^2\lambda=0 \qquad => \qquad \mu^2=\frac{c}{4}$$

$$\mu=\pm\frac{1}{2}\sqrt{c}$$

$$\lambda\,\mu^2\,\beta(\beta-1)+\frac{(a+b)}{2}\lambda^2=0 \qquad (2.20\text{-b})$$

Multiplying by λ^{-1}:

$$6\mu^2+\frac{(a+b)}{2}\lambda=0$$

$$\lambda= \quad -\frac{12\mu^2}{a+b}$$

Solving the system (2.20) yields:

$$\beta=-2 \quad , \quad \mu=\frac{1}{2}\sqrt{-c} \quad , \quad \lambda=\frac{3c}{a+b}\ , \qquad (2.21)$$

Where c is a free parameter. Hence, for $c < 0$, the following periodic solutions.

$$v_1(\xi) = -\frac{3c}{a+b}\csc^2\left[\frac{\sqrt{-c}}{2}\xi\right] \qquad (2.22)$$

Where $0 < \frac{1}{2}\sqrt{-c}\,|\xi < \pi$, and

$$v_2(\xi) = -\frac{3c}{a+b}\sec^2\left[\frac{\sqrt{-c}}{2}\xi\right] \qquad (2.23)$$

Where $|< \frac{1}{2}\sqrt{-c}\,\xi| < \frac{\pi}{2}$, in view of these results, and recall

that $u'(\xi) = v(\xi)$, integrating (2.22) and (2.23) with respect to ξ and considering the zero constants for integration we obtain :

$$u_1(\xi) = -\frac{6c}{\sqrt{-c}(a+b)}\cot\left[\frac{\sqrt{-c}}{2}\xi\right],$$

$$u_2(\xi) = -\frac{6c}{\sqrt{-c}(a+b)}\tan\left[\frac{\sqrt{-c}}{2}\xi\right], \qquad (2.24)$$

Using $u(x, y, t) = u(\xi)$ and $\xi = x + y - c\,t$ we get:

$$u_1(x, y, t) = -\frac{6c}{\sqrt{-c}(a+b)}\cot\left[\frac{\sqrt{-c}}{2}(x + y - c\,t)\right]$$

$$u_2(x, y, t) = -\frac{6c}{\sqrt{-c}(a+b)}\tan\left[\frac{\sqrt{-c}}{2}(x + y - c\,t)\right] \qquad (2.25)$$

Exact solution of (2+1)-dimensional CBS equation

We investigate explicit formula of solutions of the following (2+1)-dimensional Calogero -Bogoyavlenskii-Schiff (CBS) equation given in [30].

$$u_{xt} + 4u_x u_{xy} + 2u_{xx}u_y + u_{xxxy} = 0 \qquad (2.26)$$

By using section (2.3), we have the following exact solutions:

Exact solution I:

$$u_1(x, y, t) = -\frac{c}{\sqrt{-c}}\cot\left[\frac{\sqrt{-c}}{2}(x + y - c\,t)\right] \qquad (2.27)$$

Where $0 < \frac{1}{2}\sqrt{-c}\,(x + y - c\,t) < \pi$.

Exact solution II:

$$u_2(x, y, t) = -\frac{c}{\sqrt{-c}}\tan\left[\frac{\sqrt{-c}}{2}(x + y - c\,t)\right] \qquad (2.28)$$

Where $|\frac{1}{2}\sqrt{-c}\,(x + y - c\,t)| < \frac{\pi}{2}$.

Exact solution of (2+1)-dimensional Breaking soliton equation

We investigate explicit formula of solutions of the following (2+1)-dimensional Breaking soliton equation given in [31].

$$u_{xt} + 4u_x u_{xy} + 2u_{xx}u_y + u_{xxxy} = 0 \qquad (2.29)$$

by using section (2.3), we have the following exact solutions:

Exact solution I:

$$u_1(x, y, t) = \frac{c}{\sqrt{-c}} \cot\left[\frac{\sqrt{-c}}{2}(x + y - c\,t)\right] \qquad (2.30)$$

Where $0 < \frac{1}{2}\sqrt{-c}\,(x + y - c\,t) < \pi$,

Exact solution II:

$$u_2(x, y, t) = -\frac{c}{\sqrt{-c}} \tan\left[\frac{\sqrt{-c}}{2}(x + y - c\,t)\right] \qquad (2.31)$$

Where $\left|\frac{1}{2}\sqrt{-c}(x + y - c\,t)\right| < \frac{\pi}{2}$.

Exact solution of (2+1)-dimensional Bogoyavlenskii's Breaking soliton equation

Now we investigate explicit formula of solutions of the following (2+1)-dimensional Bogoyavlenskii's Breaking soliton equation given in [32].

$$u_{xt} + 4u_x u_{xy} + 4u_{xx}u_y + u_{xxxy} = 0 \qquad (2.32)$$

By using section (2.3), we have the following exact solutions:

Exact solution I:

$$u_1(x, y, t) = \frac{3c}{4\sqrt{-c}} \cot\left[\frac{\sqrt{-c}}{2}(x + y - c\,t)\right] \qquad (2.33)$$

Where $0 < \frac{1}{2}\sqrt{-c}\,(x + y - c\,t) < \pi$.

Exact solution II:

$$u_2(x, y, t) = -\frac{3c}{4\sqrt{-c}} \tan\left[\frac{\sqrt{-c}}{2}(x + y - c\,t)\right] \qquad (2.34)$$

Where $\left|\frac{1}{2}\sqrt{-c}(x + y - c\,t)\right| < \frac{\pi}{2}$.

Chapter (3)

Traveling Wave Solutions of ZK-BBM Equation Sine–Cosine Method

Sec (3.1) Introduction

There has been an unprecedented development in nonlinear sciences during the last two decades. In the similar context, several numerical and analytical techniques including Homotopy Analysis (HAM), Perturbation, Modified Domain's Decomposition, Variational iteration (VIM), Variation of Parameters, Finite difference, Finite volume, Backlund transformation, inverse scattering, Jacobi elliptic function expansion, tanh function have been developed to solve such equations. Most of these techniques have their inbuilt deficiencies including evaluation of the so-called Domain's polynomials, divergent results, successive applications of the integral operator, un-realistic assumptions, non -compatibility with the nonlinearity of physical problem and very lengthy calculations. Inspired and motivated by the ongoing research in this area, we apply a relatively new technique which is called Sine–Cosine method to find travelling wave solutions of
ZK-BBM equations. It is worth mentioning that Wazwaz [33, 34] made a detailed study for Compact and non -compact physical structures for the ZK–BBM equation and also calculated exact solutions of compact and
non -compact structures for the Kadomtsov-Petviashivilli-Benjamin-Bona-Mahony (KP–BBM) equation. It is to be highlighted that such equation arises frequently in various branches of physics, applied and engineering sciences. The proposed scheme is fully compatible with the complexity of such problems and is very user-friendly. Numerical results are very encouraging.

Sec (3.2) Sine–Cosine Method for ZK-BBM Equation

The main steps for using sine–cosine method, as following:

1. We introduce the wave variable $\xi = x - ct$ into the PDE, we get:

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, u_{xxx}, \ldots) = 0 \tag{3.1}$$

Where $u(x,t)$ is traveling wave solution. This enables us to use the following changes.

$$\frac{\partial}{\partial t} = -c\frac{\partial}{\partial \xi}, \frac{\partial^2}{\partial t^2} = c^2\frac{\partial^2}{\partial \xi^2}, \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}, \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \xi^2}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial \xi}, \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial \xi^2}, \frac{\partial}{\partial z} = \frac{\partial}{\partial \xi}, \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial \xi^2}, \tag{3.2}$$

One can immediately reduce the nonlinear PDE (3.1) into a nonlinear ODE:

$$Q(u, u_\xi, u_{\xi\xi}, u_{\xi\xi\xi}, \ldots) = 0 \tag{3.3}$$

The ordinary differential equation (3.3) is then integrated as long as all terms contain derivatives, where we neglect integration constants.

2. The solutions of many nonlinear equations can be expressed in the form:

$$u(x,t) = \{\lambda \sin^\beta(\mu\xi), \quad |\xi| \le \frac{\pi}{\mu} \tag{3.4}$$

Or in the form:

$$u(x,t) = \{\lambda \cos^\beta(\mu\xi), \quad |\xi| \le \frac{\pi}{2\mu} \tag{3.5}$$

Where λ, μ and β are parameters that will be determined, μ and c are the wave number and the wave speed, respectively, we use:

$$u(\xi) = \lambda \sin^\beta(\mu\xi)$$

$$u^n(\xi) = \lambda^n \sin^{n\beta}(\mu\xi) \tag{3.6}$$

$$(u^n)_\xi = n\mu\beta\lambda^n \cos(\mu\xi) \sin^{n\beta-1}(\mu\xi)$$

$(u^n)_{\xi\xi} = -n^2\mu^2\beta^2\lambda^n \ sin^{n\beta}(\mu\xi), + n\mu^2\lambda^n\beta(n\beta-1) \ sin^{n\beta-2}(\mu\xi)$

And the derivatives of (3.5) become:

$u(\xi) = \lambda \ cos^{\beta}(\mu\xi)$

$$u^n(\xi) = \lambda^n \ cos^{n\beta}(\mu\xi) \tag{3.7}$$

$(u^n)_{\xi} = -n\mu\beta\lambda^n \ sin(\mu\xi) \ cos^{n\beta-1}(\mu\xi)$

$(u^n)_{\xi\xi} = -n^2\mu^2\beta^2\lambda^n \ cos^{n\beta}(\mu\xi) + n\mu^2\lambda^n\beta(n\beta-1) \ cos^{n\beta-2}(\mu\xi)$

And so on for the other derivatives.

3. We substitute (3.6) or (3.7) into the reduced equation obtained above in (3.3), balance the terms of the sine functions when (3.6) is used, or balance the terms of the cosine functions when (3.7) is used, and solving the resulting system of algebraic equations by using**.** the computerized symbolic calculations. We next collect all terms with same power in cos^k (μξ) or sin^k (μξ) and set to zero their coefficients to get a system of Algebraic equations among the unknowns λ , μ and β . We obtained all possible value of the parameters λ , μ and β .

Sec (3.3) Solution Procedure

Let us consider the ZK-BBM equation.

$$u_t + u_x - a(u^2)_x - (bu_{xt} + ku_{yt})_x = 0 \tag{3.8}$$

$u_t = -cu'$ $\qquad$ $u_x = u'$

$(u^2)_x = (u^2)'$ $\qquad$ $u_{xt} = -cu''$

$$u_y = u' \qquad u_{yt} = -cu'' \tag{3.9}$$

We now employ the sine–cosine method. Using the wave variable

$\xi = x + y - ct$, carries (3-8) into ODE:

$$-cu' + u' - a(u^2)' - (-bcu'' - cku'')' = 0$$

$$(1-c)u' - a(u^2)' + ((b+k)u'cu'')' = 0 \qquad (3.10)$$

Integrating (3.10) gives and by considering the constant of integration to be zero, we get:

$$(1-c)u - a\,u^2 + (b+k)cu'' = 0 \qquad (3.11)$$

Substituting (3.6) into (3.11) gives:

$$(1-\ c)\lambda\, sin^{\beta}(\mu\xi) - a\lambda^2\ sin^{2\beta}(\mu\xi) - (b+k)c\,\lambda\,\mu^2\beta^2 sin^{\beta}(\mu\xi)$$

$$+(b+k)c\,\lambda\,\mu^2\beta(-1)sin^{\beta-2}(\mu\xi) = 0 \qquad (3.12)$$

Equating the exponents and the coefficients of each pair of the sine functions, we find the following system of algebraic equations:

$$\beta - 1 \neq 0$$

$$2\,\beta = \beta - 2$$

$$-(b+k)c\,\lambda\,\mu^2\beta^2 + (1-c)\,\lambda = 0 \qquad (3.13\text{-a})$$

Multiplying by λ^{-1}:

$$-4(b+k)c\mu^2 + (1-c) = 0$$

$$\mu^2 = -\frac{1-c}{4c(b+k)} \qquad , \qquad \mu = \pm\frac{1}{2}\sqrt{\frac{1-c}{c(b+k)}}$$

$$(b+k)c\,\lambda\,\mu^2\beta(\beta-1) - a\lambda^2 = 0 \qquad (3.13\text{-b})$$

$$\lambda = \frac{6c\mu^2(b+k)}{a} \qquad => \qquad \lambda = \frac{6c(1-c)(b+k)}{a.4c(b+k)}$$

$$\lambda = \frac{3(1-c)}{2a}$$

Solving the system (3.13) yields:

$$\beta = -2$$

$$\lambda = \frac{3(1-c)}{2a}$$

$$\mu = \frac{1}{2}\sqrt{\frac{1-c}{c(b+k)}} \tag{3.14}$$

The result (3.14) can be easily obtained if we also use the cosine method (3.7) Consequently, following periodic solutions for $\frac{1-c}{c(b+k)} > 0$.

$$u_1(x,y,t) = \frac{3(1-c)}{2a} \sec^2[\frac{1}{2}\sqrt{\frac{1-c}{c(b+k)}}(x + y - \mathrm{c\,t})] \;, |\mu\,\xi| < \frac{\pi}{2} \tag{3.15}$$

$$u_2(x,y,t) = \frac{3(1-c)}{2a} \sec^2[\frac{1}{2}\sqrt{\frac{1-c}{c(b+k)}}(x + y - \mathrm{c\,t})] \;, 0 < \mu\xi < \frac{\pi}{2} \tag{3.16}$$

However, for $\frac{1-c}{c(b+k)} > 0$ we obtain the soliton solution:

$$u_3(x,y,t) = \frac{3(1-c)}{2a} \operatorname{sech}^2[\frac{1}{2}\sqrt{\frac{c-1}{c(b+k)}}(x + y - \mathrm{c\,t})] \tag{3.17}$$

$$u_4(x,y,t) = -\frac{3(1-c)}{2a} \operatorname{sech}^2[\frac{1}{2}\sqrt{\frac{c-1}{c(b+k)}}(x + y - \mathrm{c\,t})] \tag{3.18}$$

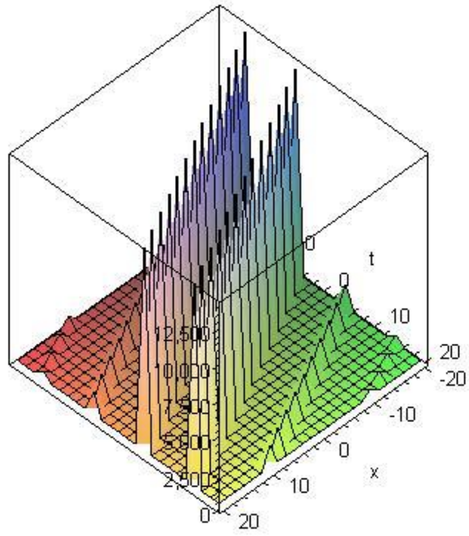


Figure 2: Periodic solution corresponding to $u_1(x, y, t)$ for $c = 2, a = -1, b = -3$, $k = -2$.

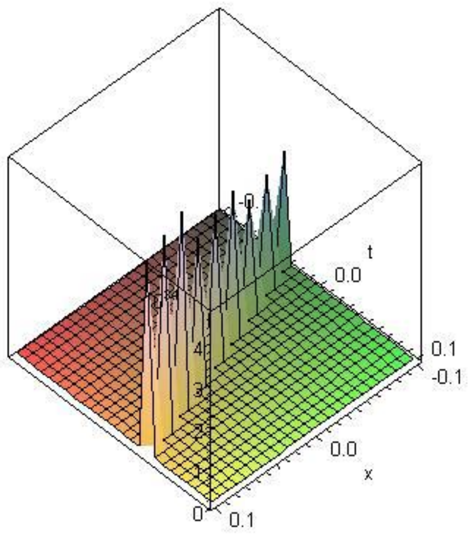


Figure 3: Periodic solution corresponding to $u_2(x, y, t)$ for $c = 3, a = -2, b = -5$, $k = -4$.

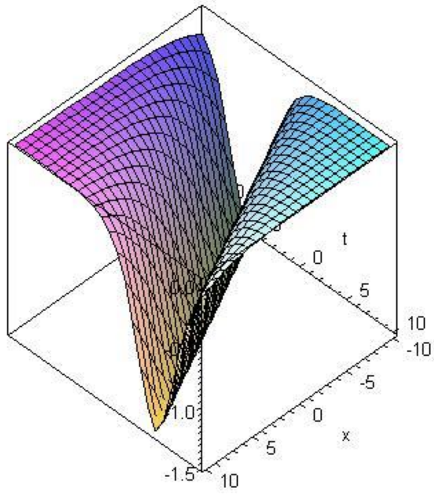


Figure 4: soliton solution corresponding to $u_3(x, y, t)$ for $\mathrm{c} = 2$, $\mathrm{a} = 1$, $\mathrm{b} = 3$, $\mathrm{k} = 1$.

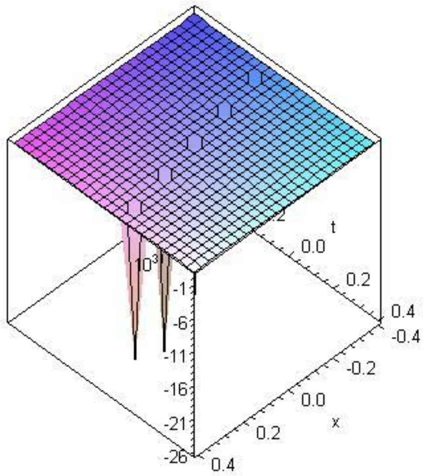


Figure 5: soliton solution corresponding to $u_4(x, y, t)$ for $\mathrm{c} = 5$, $\mathrm{a} = 7$, $\mathrm{b} = 5$, $\mathrm{k} = 3$.

Chapter (4)

The Sine-Cosine Function Method for the Davey-Stewartson Equations

Sec (4.1) Introduction

The sine-cosine method has been used to solve different types of nonlinear systems of PDEs. The higher-dimensional nonlinear wave fields have richer phenomena than one-dimensional ones, since various localized solitons may be considered in higher-dimensional space.

The Davey-Stewartson equation (DSE) was introduced in [38] to describe the evolution of a three-dimensional wave-packet on water of finite depth. It is a system of partial differential equations for a complex (wave-amplitude) field q (t, x, y) and a real(mean-flow) field ϕ(t, x, y).

$$iq_t + \frac{1}{2}\sigma^2\left(q_{xx} + \sigma^2 q_{yy}\right) + \lambda|q|^2 q - \phi_x q = 0 \qquad (4.1)$$

$$q_t = ik_3 e^{i\theta}\, u(\xi) - c e^{i\theta} u'$$

$$q_x = ik_1 e^{i\theta}\, u + e^{i\theta} u'$$

$$q_{xx} = 2ik_1 e^{i\theta} u' - {k_1}^2 e^{i\theta}\, u + e^{i\theta} u''$$

$$q_y = ik_2 e^{i\theta}\, u + e^{i\theta} u'$$

$$q_{yy} = ik_2 e^{i\theta} u' - {k_2}^2 e^{i\theta}\, u + e^{i\theta} u'' + ik_2 e^{i\theta} u'$$

$$q_{yy} = 2ik_2 e^{i\theta} u' - {k_2}^2 e^{i\theta}\, u + e^{i\theta} u''$$

$$|q|^2 q = e^{i\theta} u^3 \qquad (4.2)$$

$$\phi_{xx} - \sigma^2\phi_{yy} - 2\lambda(|q|^2)_x = 0 \qquad (4.3)$$

$$\emptyset_x = V' \qquad \emptyset_x q = e^{i\theta} u\, V'$$

$\emptyset_{xx} = V''$ $\qquad \emptyset_y = V'$

$\emptyset_{yy} = V''$

$|q|^2 = u^2$ $\qquad (|q|^2)_x = (u^2)'$

$$-k_3 e^{i\theta} u - ice^{i\theta} u' + \frac{1}{2} b^2 (2ik_1 e^{i\theta} u' - {k_1}^2 e^{i\theta}\, u + e^{i\theta} u''$$

$$+b^2\left(2ik_2 e^{i\theta} u' - {k_2}^2 e^{i\theta}\, u + e^{i\theta} u''\right) + \lambda e^{i\theta} u^3 - e^{i\theta} u\, V' = 0$$

$$-k_3 u + \frac{1}{2} b^2 \left(-{k_1}^2 u + u'' + b^2\left(-{k_2}^2 u + u''\right)\right) + \lambda u^3 - u\, V' = 0$$

$$-2k_3 u - b^2 {k_1}^2 u + b^2 u'' - b^4 {k_2}^2 u + b^4 u'' + 2\lambda u^3 - 2u\, V' = 0$$

$$b^2(1 + b^2)u'' - \left(2k_3 + b^2\left({k_1}^2 + b^2 {k_2}^2\right) + 2\, V'\right)u + 2\lambda u^3 = 0$$

$$V'' - b^2 V'' - 2\lambda (u^2)' = 0$$

$$(1 - b^2)V'' - 2\lambda (u^2)' = 0 \tag{4.4}$$

By take imagine part:

$$-cu' + b^2 k_1 + b^4 k_2$$

$$c = b^2(k_1 + b^2 k_2) \tag{4.5}$$

Where λ = ±1 and σ^2 = ±1. The case σ = 1 is called the DSI equation, while the case σ = i is called the DSII equation. The parameter λ characterizes the focusing or defocusing case [35]. The DS equation has four kinds of soliton solutions: the conventional line, algebraic, periodic and lattice solitons. The conventional line soliton has an essentially one-dimensional structure. On the other hand, the algebraic, periodic and lattice solitons have a two-dimensional localized structure.

The DSI and DSII equations are two well-known examples of integrable equations in two special dimensions, which arise as higher dimensional generalizations of the nonlinear Davey-Stewartson equation, as well as from physical considerations [36-37]. Indeed.

They appear in many applications, for example in the description of gravity-capillarity surface wave packets in the limit of shallow water. Therefore it is of interests to derive explicit solutions of the DS equation.

During the past decades, quite a few methods for obtaining explicit traveling and solitary wave solutions of nonlinear evolution equations have been proposed, such as the inverse scattering method, bilinear transformation, the tanh-sech method, extended tanh method and homogeneous balance method.

Concepts like solitons, peakons, kinks, breathers, cusps and compactions are being thoroughly investigated in the scientific literature.

Sec (4.2) Sine-Cosine Method

We introduce the wave variable $\xi = x - ct$ into the PDE:

$$P(u, ut, ux, , utt, uxx, \dots) = 0 \qquad (4.6)$$

Where $u(x,t)$ is a traveling wave solution. This enables us to use the following changes of variables:

$$\frac{\partial}{\partial t} = -c\frac{\partial}{\partial \xi}, \frac{\partial^2}{\partial t^2} = c^2\frac{\partial^2}{\partial \xi^2}, \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial^2}{\partial \xi^2}, \dots. \qquad (4.7)$$

One can immediately reduce the nonlinear PDE (4.6) into a nonlinear ODE:

$$Q(u, u_\xi, u_{\xi\xi}, , u_{\xi\xi\xi}, \dots) = 0 \qquad (4.8)$$

The ordinary differential equation (4.8) is then integrated as long as all terms contain derivatives, where we neglect the integration constants.

The solutions of many nonlinear equations can be expressed in the form:

$$u(x,t) = \begin{cases} \lambda_1 \sin^\beta(\mu\xi), & |\xi| \le \frac{\pi}{\mu} \\ 0, & \text{otherwise} \end{cases}, \qquad (4.9)$$

Or in the form:

$$u(x,t) = \begin{cases} \lambda_1 \cos^{\beta}(\mu\xi), & |\xi| \le \frac{\pi}{2\mu}, \\ 0, & \text{otherwise} \end{cases} \tag{4.10}$$

Whereλ, μ and β are parameters to be determined, μ and c are the wave number and the wave speed, respectively [29]. We use:

$$u(\xi) = \lambda_1 \sin^{\beta}(\mu\xi) \tag{4.11}$$

$$u^n(\xi) = \lambda_1{}^n \sin^{n\beta}(\mu\xi) \tag{4.12}$$

$$(u^n)_\xi = n\mu\,\beta\, \lambda_1{}^n \cos(\mu\xi) \sin^{n\beta-1}(\mu\xi) \tag{4.13}$$

$$(u^n)_{\xi\xi} = -n^2\mu^2\beta^2 \lambda_1{}^n \sin^{n\beta}(\mu\xi) + n\mu^2 \lambda_1{}^n \beta(n\beta-1)\sin^{n\beta-2}(\mu\xi) \tag{4.14}$$

And their derivatives.

$$u(\xi) = \lambda_1 \cos^{\beta}(\mu\xi) \tag{4.15}$$

$$u^n(\xi) = \lambda_1{}^n \cos^{n\beta}(\mu\xi) \tag{4.16}$$

$$(u^n)_\xi = -n\mu\beta\, \lambda_1{}^n \sin(\mu\xi) \cos^{n\beta-1}(\mu\xi) \tag{4.17}$$

$$(u^n)_{\xi\xi} = -n^2\mu^2\beta^2 \lambda_1{}^n \cos^{n\beta}(\mu\xi) + n\mu^2 \lambda_1{}^n \beta(n\beta-1)\cos^{n\beta-2}(\mu\xi) \tag{4.18}$$

And so on. We substitute (4.11) - (4.14) or (4.15) - (4.18) into the reduced equation (4.8), balance the terms of the cosine functions when (4.11) - (4.14) are used, or balance the terms of the sine functions when (4.15) - (4.18) are used, and solve the resulting system of algebraic equations by using computerized symbolic packages. We next collect all terms with the same power in $\cos^k(\mu\xi)$ or $\sin^k(\mu\xi)$ and set to zero their coefficients to get a system of algebraic equations among the unknowns λ , μ and β, and solve the subsequent system.

Sec (4.3) The Davey-Stewartson Equation

We deal with the Davey–Stewartson equation (4.1). Take the following transformations of (4.1).

$$q(x,y,t) = U(\xi)e^{i\theta}, \ \phi(x,y,t) = V(\xi) \tag{4.19}$$

$$\xi = x + y - c\,t, \ \Theta = K_1 x + K_2 y + K_3 t \tag{4.20}$$

Where K_1, K_2 and K_3 are real constants [11]. It is easy to derive from (4.19), (4.20) and (4.1) that.

$$c = \sigma^2 K_1 + K_2 \tag{4.21}$$

$$\sigma^2\{(1+\sigma^2)U''\} - [2k_3 + \sigma^2(k_1^2 + \sigma^2 k_2^2) + 2V']U + 2\lambda U^3 = 0 \tag{4.22}$$

$$(1+\sigma^2)V'' - 2\lambda(U^2)' = 0 \tag{4.23}$$

Integrating (4.23) with respect to ξ and setting the constant of integration to zero, we find:

$$V' = \frac{2\lambda}{1+\sigma^2} U^2 \tag{4.24}$$

Substituting (4.24) into (4.22) gives:

$$\sigma^2\{(1+\sigma^2)U''\} - [2k_3 + \sigma^2(k_1^2 + \sigma^2 k_2^2)]U + 2\lambda\left[\frac{-2}{1-\sigma^2} + 1\right]U^3 = 0 \tag{4.25}$$

Seeking solutions of the form (4.9), we get

$$\sigma^2(1+\sigma^2)\left[-\mu^2\beta^2\lambda_1 \sin^\beta(\mu\xi) + \mu^2\beta^2\lambda_1\beta(\beta-1)\sin^{\beta-2}(\mu\xi)\right] -$$

$$[2k_3 + \sigma^2(k_1^2 + \sigma^2 k_2^2)]\lambda_1 \sin^\beta(\mu\xi) + 2\lambda\left[\frac{-2}{1-\sigma^2} + 1\right]\lambda_1^3 \sin^{3\beta} = 0 \tag{4.26}$$

Equating the exponents and the coefficients of each pair of the sine functions we find the following algebraic system:

$$3\beta = \beta - 2$$

$$\beta - 1 \neq 0$$

$$\sigma^2(1+\sigma^2)\mu^2\lambda_1\beta(\beta-1) + 2\lambda\left[1 - \frac{2}{1-\sigma^2}\right]\lambda_1^3 = 0 \tag{4.27-a}$$

Multiplying by $\lambda_1{}^{-1}$:

$$\lambda_1{}^2 = \frac{2k_3 + \sigma^2\left(k_1{}^2 + \sigma^2 k_2{}^2\right)}{\lambda\left[1 - \frac{2}{1-\sigma^2}\right]}$$

$$-\sigma^2(1+\sigma^2)\mu^2\lambda_1 - [2k_3 + \sigma^2(k_1^2 + \sigma^2 k_2^2)]\lambda_1 = 0 \tag{4.27-b}$$

$$\mu^2 = \frac{-2k_3 + \sigma^2\left(k_1{}^2 + \sigma^2 k_2{}^2\right)}{\sigma^2(1+\sigma^2)}$$

By solving the algebraic system (4.27), we get, when $\frac{2k_3+\sigma^2 k_1^2\sigma^2+k_2^2}{\sigma^2}<0.$

$$\beta=-1$$

$$\mu=\pm\sqrt{-\frac{2k_3+\sigma^2 k_1^2\,\sigma^2+k_2^2}{\sigma^2}} \tag{4.28}$$

$$\lambda_1=\pm\sqrt{-\frac{2k_3+\sigma^2 k_1^2\,\sigma^2+k_2^2}{\lambda}}$$

In view of (4.9), (4.19), (4.20) and (4.28), for $\frac{2k_3+\sigma^2 k_1^2\sigma^2+k_2^2}{\sigma^2}<0$,

we obtain the periodic solutions.

$$q(x,y,t)=\pm=\pm\sqrt{-\frac{2k_3+k_1^2\,\sigma^2+k_2^2}{\lambda},e^{i\,(k_1x+}k_2y+k_3t)}$$

$$CSC\left[\pm\sqrt{-\frac{2k_3+k_1^2\,\sigma^2+k_2^2}{\sigma^2},(x+y-ct)}\right] \tag{4.29}$$

Where

$$0<\left[\pm\sqrt{-\frac{2k_3+k_1^2\,\sigma^2+k_2^2}{\sigma^2},(x+y-ct)}\right]<\pi$$

And

$$q(x,y,t)=\pm\sqrt{-\frac{2k_3+k_1^2\,\sigma^2+k_2^2}{\lambda},e^{i\,(k_1x+}k_2y+k_3t)}$$

$$sec\left[\pm\sqrt{-\frac{2k_3+k_1^2\,\sigma^2+k_2^2}{\sigma^2},(x+y-ct)}\right] \tag{4.30}$$

Where

$$\left[\left|\pm\sqrt{-\frac{2k_3+k_1^2\,\sigma^2+k_2^2}{\sigma^2},(x+y-ct)}\right|\right]<\frac{\pi}{2}$$

And for $\frac{2k_3+\sigma^2 k_1^2\sigma^2+k_2^2}{\sigma^2}>0$, the following solitary solutions

$$q(x,y,t)\ =\ \pm\sqrt{-\frac{2k_3+k_1^2\,\sigma^2+k_2^2}{\lambda},e^{i\,(k_1x+}k_2y+k_3t)}$$

$$CSCh\left[\pm\sqrt{-\frac{2k_3+k_1^2\,\sigma^2+k_2^2}{\sigma^2},(x+y-ct)}\right] \tag{4.31}$$

And

$$q(x,y,t) = \pm\sqrt{-\frac{2k_3+k_1^2\,\sigma^2+k_2^2}{\lambda}},e^{i\,(k_1x+}k_2y + k_3t)$$

$$sech\left[\pm\sqrt{-\frac{2k_3+k_1^2\,\sigma^2+k_2^2}{\sigma^2}},(x+y-ct)\right] \qquad (4.32)$$

Where

$c = \sigma^2\ k_1 + k_2$

To find the solutions $\emptyset(x, y, t)$, according to (4.24), we have:

$$V\,(\xi) = \frac{2\lambda}{1+\sigma^2}\int U^2(\xi)\,d\xi \qquad (4.33)$$

By means of the equations (4.19), (4.20), (4.9) and (4.33).and using equation (4.28), we have the following periodic solutions for $\emptyset\ (x, y, t)$:

When for $\frac{2k_3+\sigma^2k_1^2\sigma^2+\ k_2^2}{\sigma^2} > 0$, we get:

$$\emptyset\,(x,y,t) = -\frac{-2\,\sigma\,\sqrt{\lambda}}{1-\sigma^2}\cot\left[\sqrt{-\frac{2k_3+k_1^2\,\sigma^2+k_2^2}{\sigma^2}},(x+y-ct)\right] \qquad (4.34)$$

And

$$\emptyset\,(x,y,t) = -\frac{-2\,\sigma\,\sqrt{\lambda}}{1-\sigma^2}\tan\left[\sqrt{-\frac{2k_3+k_1^2\,\sigma^2+k_2^2}{\sigma^2}},(x+y-ct)\right] \qquad (4.35)$$

When $\boldsymbol{\frac{2k_3+\sigma^2k_1^2\sigma^2+\ k_2^2}{\sigma^2}} > 0$, we get the following solitary wave solutions**:**

$$\emptyset\,(x,y,t) = -\frac{-2\,\sigma\,\sqrt{\lambda}}{1-\sigma^2}\coth\left[\sqrt{-\frac{2k_3+k_1^2\,\sigma^2+k_2^2}{\sigma^2}},(x+y-ct)\right] \qquad (4.36)$$

And

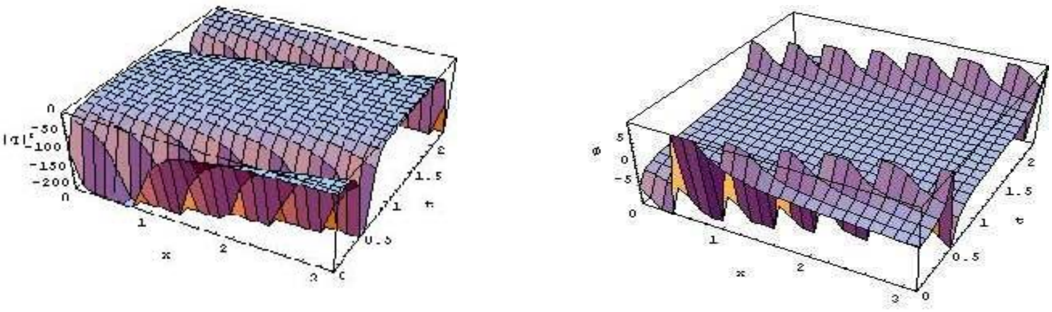
$$\emptyset\,(x,y,t) = -\frac{-2\,\sigma\,\sqrt{\lambda}}{1-\sigma^2}\,tan\left[\sqrt{-\frac{2k_3+k_1^2\,\sigma^2+k_2^2}{\sigma^2}},(x+y-ct)\right] \qquad (4.37)$$

Where

$c = \sigma^2\ k_1 + k_2.$

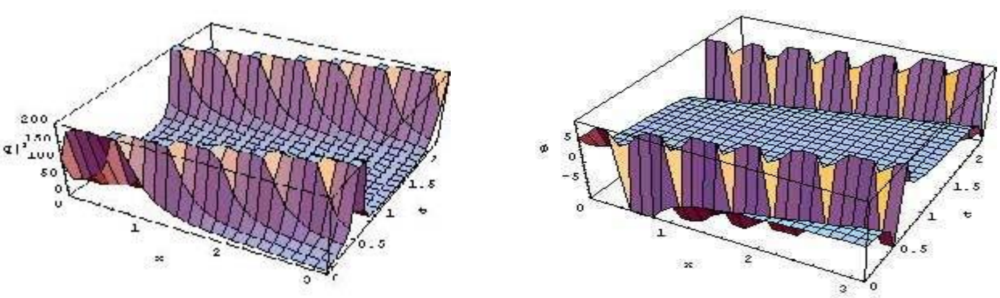
Sec (4.4) Illustrations

We now plot a few solutions found in our previous discussions.



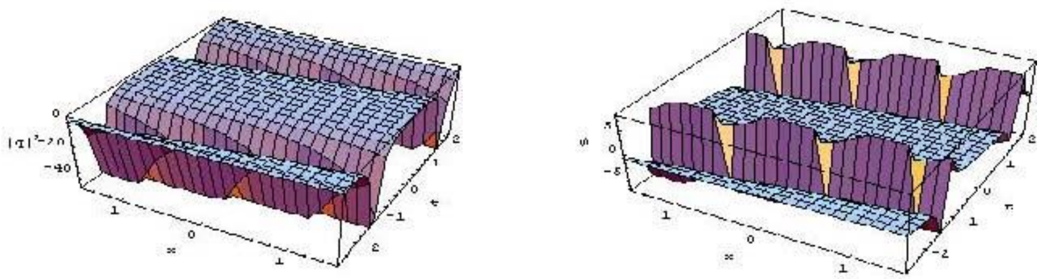
(a) **(b)**

Figure 6: q(x, y, t) in (4-29) and ϕ (*x*, *y*, *t*) in (4-34) where y = 0.1, k_1 = 0.3, k_2 = 0.5, k_3= 1.5, σ = I, λ = 1.



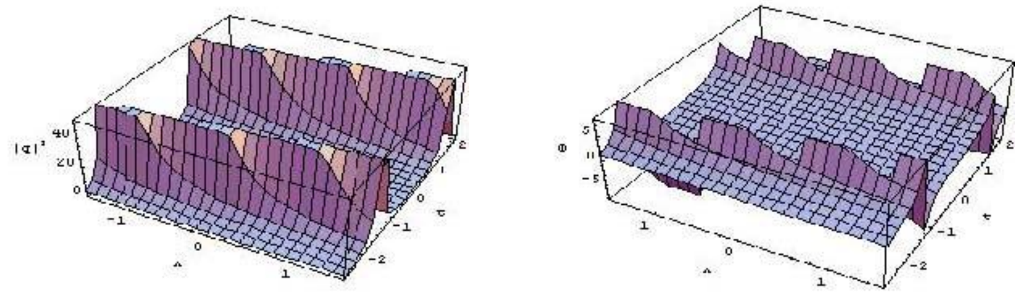
(a) **(b)**

Figure 7: q(x, y, t) in (4-29) and ϕ (x, y, t) in (4-34) where y = 0.1, k_1 = 0.3, k_2 = 0.5, k_3= 1.5, σ = I, $\lambda = -1$.



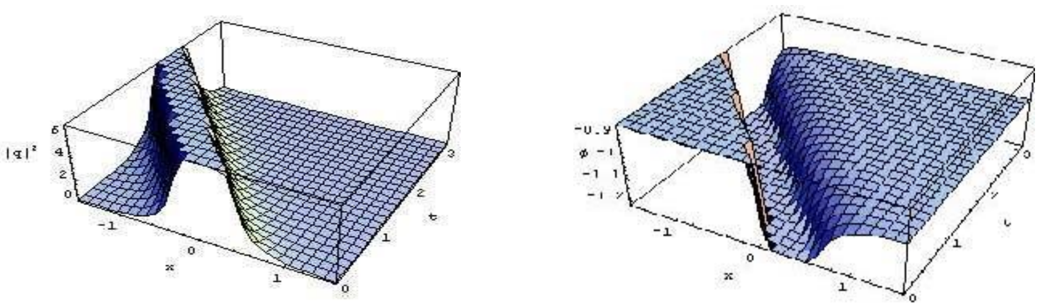
(a) **(b)**

Figure 8: q(x, y, t) in (4-30) and ϕ (x, y, t) in (4-35) where y = 0.3, k_1 = 1.3, k_2 = 1.5, k_3= 0.3, σ = I, λ =1.



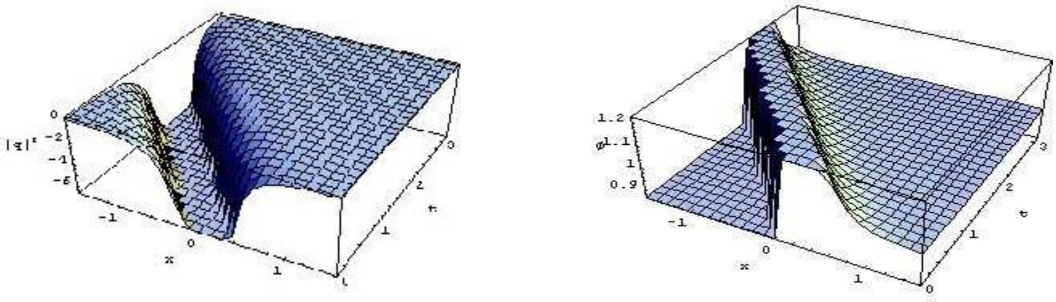
(a) **(b)**

Figure 9: q(x, y, t) in (4-30) and ϕ (x, y, t) in (4-35) where y = 0.3, k_1 = 1.3, k_2 = 1.5, k_3= 0.3, σ = I, λ =-1.



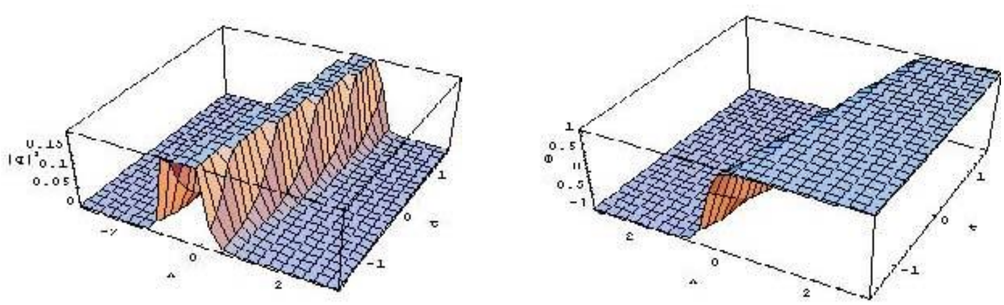
(a) **(b)**

Figure 10: q(x, y, t) in (4-31) and ϕ (x, y, t) in (4-36) where y = -0.1, k_1 = 1.5, k_2 =0.3, k_3=-0.6 , σ = I, λ =1.



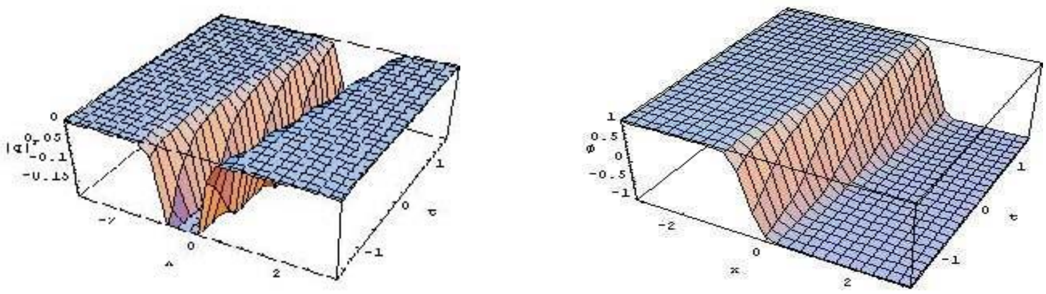
(a) **(b)**

Figure 11: q(x, y, t) in (4-31) and ϕ (x, y, t) in (4-36) where y = -0.1, k_1 = 1.5, k_2 =0.3 k_3=-0.6 , σ = I, λ =-1.



(a) **(b)**

Figure 12: q(x, y, t) in (4.32) and ϕ (x, y, t) in (4.37) where y = 0.6, k_1 =-2, k_2 =0.7, k_3=0.5 , σ = I, λ = 1.



(a) **(b)**

Figure 13: q(x, y, t) in (4.32) and ϕ (x, y, t) in (4.37) where y = 0.6, k_1 =-2, k_2 = 0.7, k_3=0.5 , σ = I, λ =-1.

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