Sudan University Science and Technology

College of Graduate Studies

Twistor Space and Its Applications

فضاء توستر وتطبيقاته

Thesis Submitted in Fulfillment of Degree of Ph.D. in Mathematics

Research Project by

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Under supervision

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DEDICATION

TO THE SOUL OF MY FATHER

TO MY MOTHER AND MY HUSBAND

TO MY FAMILY AND MY FRIENDS

TO MY SON SALAH ELDEEN

TO MY UNCLE ABO ARWA

With love

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First of all, I render grateful thanks to Allah for offering me assistance, health and patience to complete this work. No words would rightfully express my deep gratitude to Prof/ Mohammed Ali Bashir my supervisor. I would like to thank him also for valuable advice, guidance, and encouragement. Thanks to Dr. Emad aldeen Abdallah my co. supervisor for providing advice, and continuous support. My sincerest gratitude goes to my family and husband for unlimited support and encouragement.

Abstract

We formulated space-time in terms of twistors. In this formulation the points of space-time (events) are derived from twistors. So twistors are shown to be the primitive objects from which all concepts of space-time arise. Differential equations, describing conformal fields may be written in twistor terms. We utilized complex structure in R^3 to construct geometrical solutions for Laplace equation, wave equation and monopole equation. The complex space used is the so called mini – twistor space and the solutions in all the above cases is given by a contour integral of a twistor function over a bundle space of one–dimensional complex projective space.

المستخلص

قمنا بصياغة الزمكان حسب شروط الإلتفاف، بهذه الصيغة نستمد نقاط الزمكان من الإلتفاف وبالتالى يمكن إعتباره بأنه الكائنات البدائيه التى من خلالها تنبثق كل مفاهيم الزمكان. حيث نجد أن المعادلات التفاضليه التى تصف الحقول الإمتثاليه يمكن معالجتها بدلالة الإتفاف. إستخدمنا البناء المركب لفضاء إقليدس لإنشاء حلول هندسيه لكل من معادلات لابلاس والموجه والميغناطيس أحادى القطب. إن الفضاء المركب المستخدم هو فضاء الإلتفاف الأصغر. ولقد تمت حلول المعادلات السابقة بواسطة التكامل الكنتورى لدالة الإلتفاف على فضاء الحزمه للفضاء الإسقاطى المركب.

CONTENTS

Introduction

Twistors were introduced by Sir Roger Penrose and his associates since 1960, as a new way of describing the geometry of space-time where the ordinary space–time concepts can be translated into twistor terms. The primary geometrical object is not a point in Minkowiski space but a null straight line (a twistor) or, more generally, a twisting congruence of null lines. It turns out that twistor algebra has the same type of universality in relation to the Lorentz group. Thus, twistor theory is a applicable to quantum field theory and free fields of zero- rest- mass. It also formulates other fields such as Yang Mills fields. The original motivation was to unify general relativity and quantum mechanics in a non-local theory based on complex numbers. The application of twistor theory to differential equations and integrability has been an unexpected spin off from the twistor programme. It has been developed over the last 30 years by the Oxford school of Penrose and Atiyah with the crucial early input from Ward, Hitchin and further contributions from Lionel Mason, George Sparling, Paul Tod, Nick Woodhouse and others.

Penrose realized that using the space-time continuum picture to describe physical processes is inadequate not only at the plank scales of $10^{-33}cm$ but also at the much larger scales of elementary particles or perhaps atoms, where the quantum becomes important. He believes that space time is created out of quantum processes themselves at the sub a tom level.

The mathematical tool in field theories is not suitable for the new formulation since the field equations are based on well-behaved functions varying smoothly in space time. Thus his mathematical tool is geometry Instead of differential equations. However, space –time descriptions of the normal kind have been used at the atomic or particle level for long time with extraordinary accuracy. Thus, this new geometrical picture must, at that level, be mathematically equivalent to the normal space–time picture in the scene that some kind of mathematical Transformation must exist between the two pictures.

The initial attempt to formulate discrete space – time used spinors as the building block. The spinors is a mathematical object that is used in the quantum theory to describe the spin of the elementary particles. It is the simplest quantum object having only two possible states- spin up and spin down. It is argued that if the distinction between a spin up and spin down is to have meaning within a quantum theory set in empty space, it seems to imply the spinors actually create their own space – a sort of quantum version of the more familiar space time. The rules for putting spinors together involve pure addition and subtraction and have nothing to do with the ideas of continuity. They join together to form a spin network.

Twistor theory offers another alternative to the space-time continuum, considering that the basic objects describing the geometry of the space-time are four-dimensional complex vectors, called twistors. In this approach the points are obtained from intersections of twistors, becoming secondary objects. Twistor theory attempts to reformulate basic physics in twistor language. Similar to strings, twistors are basic objects with a dual character. They are used to replace the points as the basic geometric objects, but can also be used to describe elementary particles. Interactions between particles are explained by means of twistor diagrams. One of the many advantages of twistor theory is that it has a natural complex character, which is needed in working with quantum mechanics.

In this thesis, we discuss the twistor space and some applications for differential equations representing the non Abielian monopole equation. The structure of this paper is as follows.

In chapter one we introduced the basic concepts used in this research, such as manifold, differential manifold, fiber bundle and tensors.

In chapter two we introduced the basic concepts and techniques used in spinor and twistor theory. This is necessary in order to understand why we are interested in the topics discussed in this research. Section 2.1 presents some basic spinor theory, focusing on the properties used here. One of the main features of twistor theory is that it is conformal. In section 2.2 we see how the conformal group arises naturally in the spinorial setting. This chapter ends with the presentation in section 2.3 of some important concepts and results in twistor theory, ending with the representation of points as intersections of twistors.

In chapter three we studied the zero rest mass field equations and their twistor solutions.

Chapter four delt with the basic concepts used in this chapter, such as complex projective space CP_n and holomorphic line bundle. Section (2) dealt with a complex structure on R^3. In this section we defined the twistor space to be the space of oriented lines in R^3, it is infact the non-trivial tangent bundle $T\,S^2$. Differential equations in R^3 in terms of twistor functions have been treated in section (3). In this section we motivated Penrose transform by introducing the solution of the wave equation by a closed contour integral of a twistor function. Similarly integrating of an appropriate twistor function along a closed contour integral delivers a solution of a harmonic equation. The closed contour on both cases is in the one – dimensional complex projective space. The last section provided a twistor solution to the monopole equation. This equation is infact shown to be the itegrability conditions for linear Lax equations that were interpreted geometrically as null 2- planes that correspond to the points of the twistor space T via the incidence relation given by an equation (30) that yields two affine coordinates $(\lambda\,,\eta)$ where $\lambda = \pi_0/\pi_1$ and $\eta = \frac{\omega}{\pi^2}$ correspond to the homogenous coordinates $(\omega\,,\pi_0\,,\pi_1\,)$ on the twistor space T. Thus we constructed holomorphic vector bundle over the twistor space T.

In chapter five we study advance application of twistor theory:

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Chapter One

Manifolds and Tensors

1.1 Introduction to Manifold:

Basically an m-dimensional (topological) manifold is a topological space M which is locally homeomorphic to R^m. A more precise definition is:

1.1.1 Definition: (Topological n-Manifolds)

A topological space M is called a topological n-manifolds, $n \in \mathbb{N}$, if

(i) M is Hausdorff,

(ii) for any $p \in M$ there exists a neighborhood U of p which is homeomorphic to an open subset $V \subset R^m$, and

(iii) M has a countable bases of open sets.

1.1.2 Definition: (Coordinate Charts)

Let M be a topological n-manifold. A coordinate chart of M is a pair (U, x), where

(i) $U \subset M$ is open

(ii) $x : U \rightarrow xU \subset R^n$ is a homeomorphism, $xU \subset R^n$, open

1.1.3 Definition: (Compatible Charts)

We see that two charts (U, x) and (V, y) of a topological manifolds are C^∞ -Compatible if $U \cap V = \emptyset$ or

$$z = y \circ x^{-1}|_{x(U \cap V)} : x(U \cap V) \rightarrow y(U \cap V) \qquad (1.1)$$

is a C^∞-diffeomorphism,

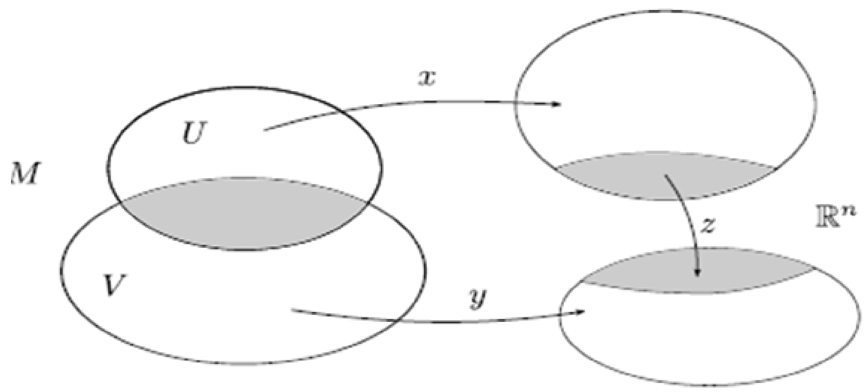


Fig (1)

1.1.4 Definition: (A C^∞-Atlas)

A C^∞-atlas, $\mathcal{A}$ or simply an atlas on a locally Euclidean space M is a set of C^∞ -compatible charts such that

$$M = \bigcup_{(U,x)\in\mathcal{A}} U \qquad (1.2)$$

An atlas $\mathcal{A}$ is said to be maximal if it is not contained in a larger atlas; in other words, if U is any other atlas containing M , then $U = M$

1.1.5 Definition: (A differentiable or (Smooth) n- Manifold)

A differentiable n-manifolds or (smooth n- manifold) is a pair $(M,\mathcal{A})$, Where M is a topological n-manifold and $\mathcal{A}$ is a maximal C^∞-atlas of M, also called a differentiable structure of M.

1.1.6 Note

We abbreviate M or M^n and say that M is a C^∞-manifold, a differentiable manifold, or a smooth manifold.

1.1.7 Definition

Let $(M^m,\mathcal{A})$ and $(N^n,\mathcal{B})$ be C^∞-manifold. We say that a mapping $f: M \to N$ is C^∞ (or smooth) if each local representation of f (with respect to $\mathcal{A}$ and $\mathcal{B}$) is C^∞. More precisely, if the composition $y \circ f \circ x^{-1}$ is smooth mapping $x(U \cap f^{-1}V) \to yV$ for every charts $(U,x) \in \mathcal{A}$ and $(V,y) \in \mathcal{B}$.We say that $f: M \to N$ is C^∞- diffeomorphism if f is C^∞ and it has an inverse f^{-1} is C^∞ , too.

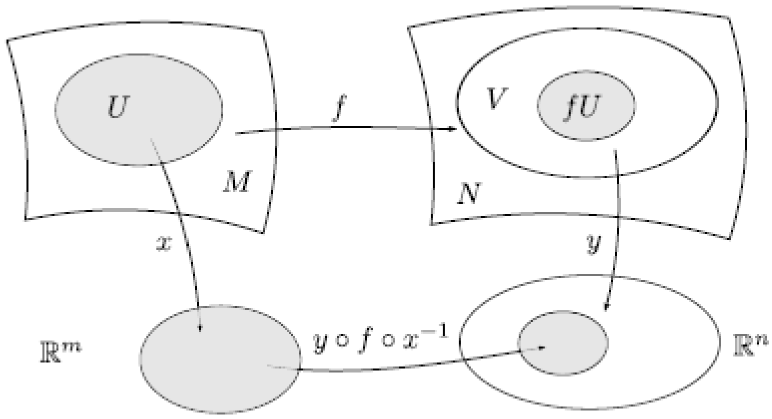


Fig (2)

1.1.8 Remark

Equivalently, $f: M \to N$ is C^∞ if, for every $p \in M$, there exist charts (U, x) in M. And (V, y) in N such that $p \in U$, $fU \subset V$, and $y \circ f \circ x^{-1}$ is $C^\infty(xU)$.

1.1.9 Example:

(i) $M = R^n$, $\mathcal{A} = \{id\}$, $\bar{\mathcal{A}} =$ a canonical structure.

(ii) If M is a differentiable manifold and $U \subset M$ is open, then U is a differentiable manifold in a natural way

(iii) Product manifolds. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be differentiable manifolds and let $p_1: M \times N \to M$ and $p_2: M \times N \to N$ be the projections.

Then

$$C = \{(U \times V, (x \circ p_1, y \circ p_2)): (U, x) \in \mathcal{A}, (V, y) \in \mathcal{B}\} \qquad (1.3)$$

is C^∞-atlas on $M \times N$. For example

(i) Cylinder $R^1 \times S^1$

(ii) Tours $S^1 \times S^1 = T^2$

1.1.10 Tangent Space

Let M be a differentiable manifold, $p \in M$ and $\gamma: I \to M$ a C^∞-path such that $\gamma(t) = p$ for some $t \in I$, where $t \in I$ is an open interval.

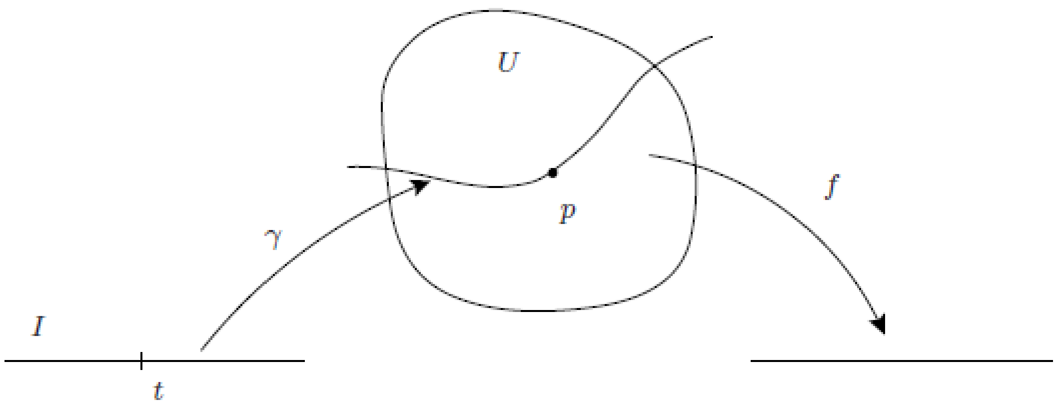


Fig (3)

Write

$$C^\infty(p) = \{f: U \to R \mid f \in C^\infty(U), U \textit{ some nieghborhood of } p\}.$$

1.1.11 Note

Here U may depend on f, therefore we write $C^\infty(p)$ instead of $C^\infty(U)$.

Now the path γ defines a mapping $\dot{\gamma}_t: C^\infty(p) \to R$,

$$\dot{\gamma}_t f = (f \circ \gamma)'(t) \tag{1.4}$$

1.1.12 Note

The real-valued function $f \circ \gamma$ is defined on some neighborhood of $t \in I$ and $(f \circ \gamma)'(t)$ is its usual derivative at t.

Interpretation: We may interprete $\dot{\gamma}_t f$ as ”a derivative of f in the direction of γ at the point p”

1.1.13 Example:($\boldsymbol{M = R^n}$)

If $\gamma = \gamma^1, \dots \dots, \gamma^n : I \to R^n$ is smooth path and $\acute{\gamma} = \acute{\gamma}^1(t), \dots \dots, \acute{\gamma}^n(t)$ is the derivative of γ at t, the

$$\dot{\gamma}_t f = (f \circ \gamma)'(t) = \acute{f}(p)\acute{\gamma}(t) = \acute{\gamma}(t) \cdot \nabla f(p) \qquad (1.5)$$

.

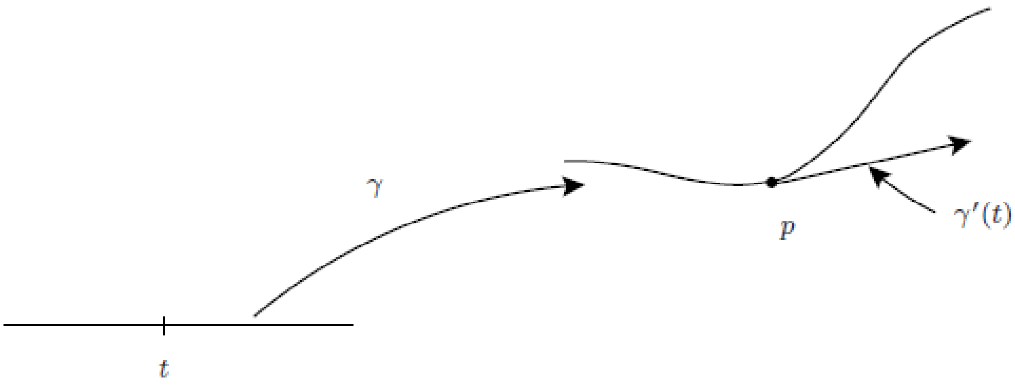


Fig (4)

In general: The mapping $\dot{\gamma}_t$ satisfies:

Suppose $f.g \in C^\infty(p)$ $f, g \in C^\infty(p)$ and $a, b \in R$. Then

(i) $\dot{\gamma}_t(af + bg) = a\dot{\gamma}_t f + b\dot{\gamma}_t g,$

(ii) $\dot{\gamma}_t(fg) = g(p)\dot{\gamma}_t f + f(p)\dot{\gamma}_t g.$

We see that $\dot{\gamma}_t$ is a derivation.

Motivated by the discussion above we define

1.1.14 Definition: (A Tangent Vector)

A tangent vector of $M, p \in M$ is a mapping $v: C^\infty(p) \to R$ that satisfies:

(i) $v(af + bg) = av(f) + bv(f), f, g \in C^\infty(p), a, b \in R$

(ii) $v(fg) = g(p)v(f) + f(p)v(f)$

1.1.15 Definition: (The Tangent Space)

The tangent space at p is (R-) linear vector of tangent vector at p, denoted by T_pM or M_p

1.1.16 Definition (Tangent Map)

Let M^m and N^n be differentiable manifolds and let $f: M \to N$ be C^∞ map. The tangent map of f at p is a linear map $f_*: T_pM \to T_{f(p)}N$ defined by

$$(f_*v)g = v(g \circ f), \forall g \epsilon C^\infty\big(f(p)\big), v \in T_pM \qquad (1.6)$$

We also write f_{*p} or T_pf

1.1.17 Remarks

It easily seen that f_{*v} is a tangent vector at $f(p)$ for all $v \in T_pM$ and that f_* is linear

1.1.18 Tangent Bundle

Let M be a differentiable manifold. We define the tangent bundle TM of M as a disjoint union of all tangent spaces of M, i.e.

$$TM = \cup_{p \in M} T_pM \qquad (1.7)$$

Points in TM are thus pairs (p, v), where $p \in M$ and $v \in T_pM$. We usually abbreviate $v = (p, v)$, because the condition $v \in T_pM$ determines $p \in M$ uniquely.

Let $\pi: TM \to M$ be the projection

$$\pi(v) = p, \quad \text{if} \quad v \in T_pM \qquad (1.8)$$

The tangent bundle TM has a canonical structure of a differentiable manifold.

1.1.19 Definition: (Sub Manifolds)

Let M and N be differentiable manifold and $F : M \to N$ be C^∞ map. We say that

(i) f is a submersion if $f_{*p} : T_pM \to Tf(p)N$ is surjective $\forall p \in M$

(ii) f is an immersion if $f_{*p} : T_pM \to Tf(p)N$ is injective $\forall p \in M$

(iii) f is an embedding if f is an $f : M \to fM$ immersion and is a homeomorphism.

If $M \subset N$ and the inclusion $i : M \to N,\ \ i(p) = p$ is an embedding, we say that M is a submanifold of N.

1.1.20 Remark

If $f : M^m \to N^n$ is an immersion, then $m \leq n$ and is the codimension of f.

1.1.21 Examples

(i) If $M_1 \ldots\ldots\ldots M_k$ are smooth manifolds, then all projections $\pi_i : M_1 \times \ldots\ldots \times M_k \to M_i$ are submersions.

(ii) $(M = R,\ N = R^2)\ \propto : R \to R^2, \alpha(t) = (t, |t|)$ is not differentiable at t = 0. This α is not an immersion

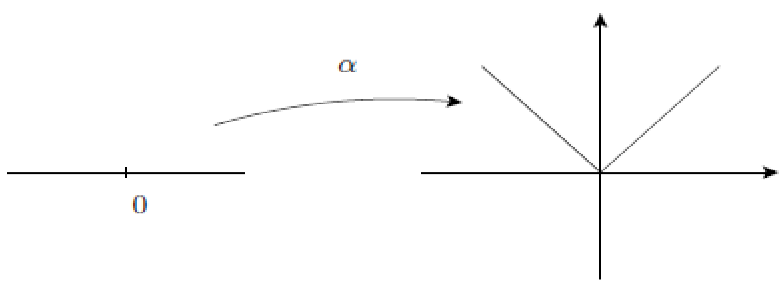


Fig (5)

(iv) $\propto : R \to R^2,\ \alpha(t) = (t^3, t^2)$ is C^∞ but not an immersion since $\acute{\propto}\ (0) = 0$.

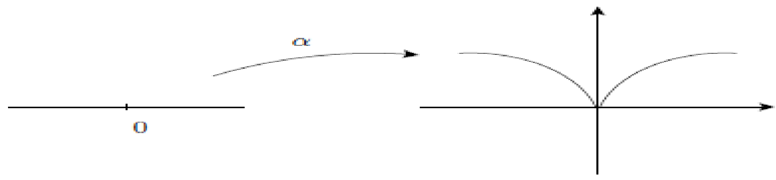


Fig (6)

(v) $\propto : R \to R^2,\ \alpha(t) = (t^3 - 4t,\ t^2 - 4)$ is C^∞ and an immersion but not an embedding $\left(\alpha(\pm 2) = (0,0)\right)$.

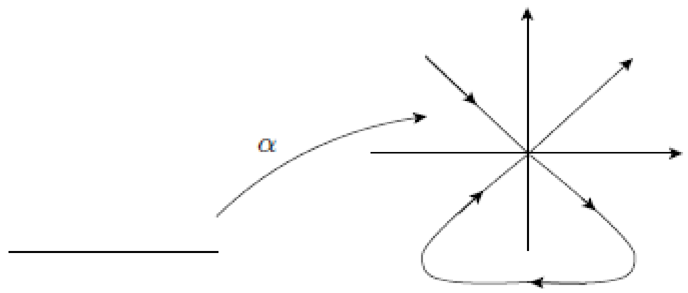


Fig (7)

(vi) The map $\propto$ (in the picture below) has an inverse but it is not an embedding since the inverse in not continuous.

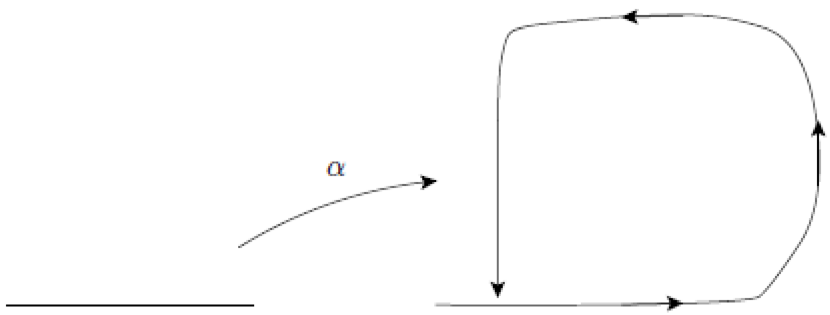


Fig (8)

(vii) The following $\propto$ is an embedding

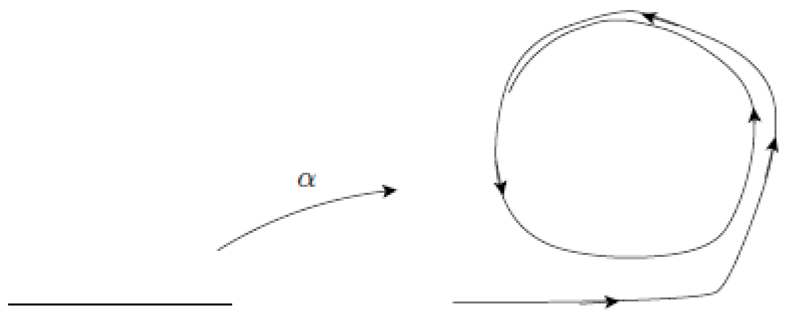


Fig (9)

1.1.22 Definition: (Orientation)

A smooth manifold M is orientation if it admits a smooth atlas $\{(U_\alpha,\ x_\alpha)\}$ such that for every α and β, with $U_\alpha \cap U_\beta = W \neq \emptyset$, the Jacobian determinant of $x_\beta \circ x_\alpha^{-1}$ is a positive at each point $q \in x_\alpha W$, i.e.

$$det\left(x_\beta \circ x_\alpha^{-1}\right)'(q) > 0, \forall q \in x_\alpha W \qquad (1.9)$$

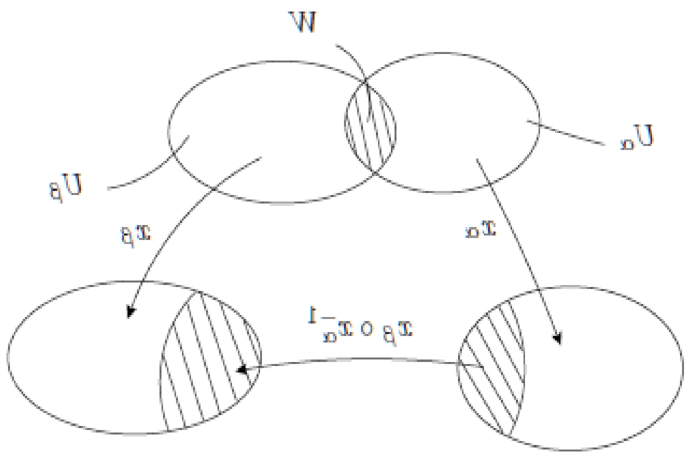


Fig (9)

In the opposite case M is non orientable. If M is orientable, then an atlas satisfying (1.9) is called an orientation of M. Furthermore, M (equipped with such atlas) is said to be oriented. We say that two atlases satisfying (1.9) determine the same orientation if their union satisfies (1.9), too.

1.2 Fibre Bundles

1.2.1 Definition: (Fibre Bundle)

A fibre bundle over a topological space X is a collection $(E,\ \pi,\ F)$ satisfying the following conditions

(i) E and F are topological spaces.

(ii) $\pi : E \to X$ is a continuous surjection.

(iii) For all $x \in X$ there is a neighbourhood $U \ni x$ and a homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ making the following diagram commute

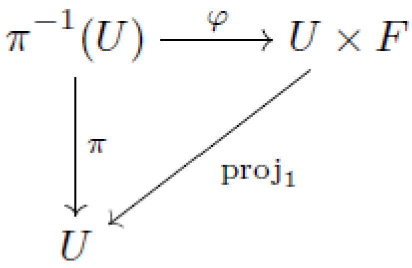


Fig (10)

We call E the total space, X the base space, π the projection, F the fibre, and $(U,\ \varphi)$ a local trivialisation.

1.2.2 Remark

Morally, a fibre bundle is a space E which is locally a direct product of spaces X and F

1.2.3 Example

The direct product $X \times F$ is called the trivial bundle with fibre F over X.

1.2.4 Defnition: (A local Section)

A local section of the fibre bundle (E, π, F, X) over an open set $U \subset X$ is a map $s : U \rightarrow E$ such that $\pi \circ s = id_X$. The space of local sections over U is denoted $\Gamma(U, E)$.

1.2.5 Remark

The sections of a fibre bundle form a sheaf on X. We abuse notation by referring to this sheaf as E, when it is convenient.

1.2.6 Definition: (Transfer Function)

Let $(\varphi_{i,} U_i)$ and $(\varphi_{j,} U_j)$ be two local trivialisations with $U_{ij} = U_i \cap U_j \neq \emptyset$. Then on $U_{ij} \times F$ we define the transfer function

$$T_{ij} = \varphi_i \circ \varphi_j^{-1} \qquad (1.10)$$

1.2.7 Remark

This is a homeomorphism by definition of φ_i and φ_j .

1.2.8 Definition: (The Transition Function)

Denote the homeomorphism group of F by Homeo(F). Define

the transition function $t_{ij} : U_{ij} \rightarrow Homeo(F)$ by

$$T_{ij}(x, f) = (x, t_{ij}(x)f) \qquad (1.11)$$

1.2.9 Remark

The transition functions for a fibre bundle tell us how to glue together the locally trivial areas on overlaps. They can be regarded as encoding the twisting of the fibre bundle. Clearly if E is the trivial bundle $X \times F$ then one can choose all transition functions such that

$$t_{ij}(x) = id_F \qquad (1.12)$$

1.2.10 Lemma

The transition functions satisfy the following relations

(i) $t_{ij}(x) = id_F$ on U_i.

(ii) $t_{ij}(x)t_{ji}(x) = id_F \, on \, Ui \cap Uj$

(iii) $t_{ij}(x)t_{jk}(x)t_{ki}(x) = id_F \, on \, Ui \cap Uj \cap Uk.$

1.2.11 Remark

Apply the language of Cech cohomology to maps $U \to Homeo(F)$ taking the abelian group operation to be pointwise multiplication. The conditions (ii) and (iii) then say that the transition functions $\{t_{ij}\}$ form a 1-cochain and a 1-cocycle respectively.

1.2.12 Theorem: (Reconstructing Fibre Bundles)

Let X be a space with open covering $\{U_i\}$. Suppose we are given a space F, a group $G \leq Homeo(F)$ and functions $t_{ij} : U_{ij} \to G$ satisfying the 1-cocycle condition. Then there exists a fibre bundle E over X with fibre F and transition functions t_{ij} .

Proof

Let $\tilde{E} = \sqcup_i (U_i \times F)$ endowed with the product topology. Define an equivalence relation on $\tilde{E}$ by

$$(x,f) \sim (y,g) \text{ iff } x = y \quad and \quad g = t_{ij}(x)f \qquad (1.13)$$

Whenever $(x,f) \in U_i \times F$ and $(y,g) \in U_i \times F$. Note that we required the cocycle condition for this to be transitive. Now we let $E = \tilde{E}/\sim$ endowed with the quotient topology.

There is a natural projection $\pi : E \to X$ given by $\pi([x,f]) = x$. Wedefine local trivialisations $\varphi_j([x,f]) = (x,f)$, which are homeomorphisms by construction of E, and clearly satisfy the required commutative diagram. Finally on U_{ij} we have

$$\varphi_i \circ \varphi_j^{-1}(x,f) = (x, t_{ij}(x)f) \qquad (1.14)$$

So the transition functions are t_{ij} .

1.2.13 Remark

We have an immediate converse to the statement in Remark 3:9, namely if we can choose all transition functions such that $t_{ij}(x) = id_F$ then the bundle is trivial.

1.2.14 Lemma

Let (E, π, F) be a fibre bundle over X with transition functions t_{ij} relative to a covering U_i of X. Suppose we are given a collection of maps $f_i : U_i \rightarrow F$ satisfying on U_{ij}

$$f_j(x) = t_{ji}(x) f_i(x) \quad (1.15)$$

Then $\{f_i\}$ determines a global section of E and all global sections arise in this way.

Proof

Let $\varphi_i : \pi^{-1}(U_i) \rightarrow U_i \times F$be the local trivialisations inducing the transition functions t_{ji} . Then f_i determines a local section $\tilde{f}_i$ of E over U_i by

$$\tilde{f}_i(x) = \varphi_i{}^{-1}\big(x, f_i(x)\big) \quad (1.16)$$

Now on U_{ij} we have

$$\begin{aligned} \tilde{f}_j(x) &= \varphi_i{}^{-1}\big(x, f_i(x)\big) = \varphi_j{}^{-1}\Big(x, t_{ji} f_i(x)\Big) \\ &= \varphi_j{}^{-1}\varphi_j\varphi_i{}^{-1}\big(x, f_i(x)\big) = \tilde{f}_i(x) \end{aligned} \quad (1.17)$$

So the local sections glue to form a global section $\tilde{f}$. Conversely if $\tilde{f}$ is a global section then by restriction we obtain local sections $\tilde{f}$ on U_i with $\tilde{f}_i = \tilde{f}_j$ on U_{ij} .

Defining $f_i(x) = proj_2 \circ \varphi_i \circ \tilde{f}(x)$ we have

$$\Big(x, f_j(x)\Big) = \varphi_j\varphi_i{}^{-1}\big(x, f_i(x)\big) = \Big(x, t_{ji} f_i(x)\Big) \quad (1.19)$$

on U_{ij} as required.

1.2.15 Definition

Let (E, π, F) be a fibre bundle over X, and G a subgroup of $Homeo(F)$. A G-atlas for (E, π, F) is a collection $(U_i; \varphi_i)$ of local trivialisations such that $X = \bigcup U_i$ and the induced transition functions are G-valued.

1.2.16 Definition

A G-bundle (E, π, F, G) is a fibre bundle with a maximal G-atlas. G is called the structure group of the bundle.

1.2.17 Remark

By definition of transition functions we consider the structure group G to have a natural left action on the fibre F. We see that for a certain class of bundles one can also define a right action of G on the total space E. This distinction is conceptually important as we develop the theory.

1.2.18 Lemma

Consider a G-bundle (E, π, F) over X. Let H be the set of transition functions at $x \in X$. Then $H = G$.

Proof

Clearly $H \subset G$. Let $g \in G$ and $h \in H$. Then there are local trivialisations φ_i and φ_j in some neighbourhood U of x such that

$$(x, h.f) = \varphi_i \circ \varphi_j(x, f) \quad \text{for all } f \in F \tag{1.20}$$

Define

$$\varphi_k = (id_U \times gh^{-1}) \circ \varphi_i : \pi^{-1}(U) \to U \times F \tag{1.21}$$

a local trivialisation. Note that φ_k must be in the G-atlas of E for it is maximal. Moreover

$$\varphi_k \circ \varphi_j(x, f) = (x, gh^{-1}h.f) = (x, g.f) \tag{1.22}$$

so $g \in H$ as required.

1.2.19 Remark

Every fibre bundle can be considered as a G-bundle by choosing $G = Homeo(F)$. More generally an H-bundle is clearly a G-bundle if $H \leq G$.

The converse is more subtle, and motivates the following definition.

1.2.20 Definition

Let E be a G-bundle, and suppose there exists a choice of local trivialisations such that the transition functions take values in $H \leq G$. Then we say that the structure group of E is reducible to H.

1.2.21 Example

A bundle is trivial iff its structure group is reducible to $\{id\}$

1.2.22 Remark

We note without proof that the reducibility of structure groups is related to spontaneous symmetry breaking in Yang-Mills theory and the identification of Riemannian metrics in differential geometry.

1.2.23 Definition

Let (E_i, π_i, F_i) be fibre bundles over X_i for $i = 1, 2$. A morphism of fibre bundles is a continuous map $\tilde{f} : E_1 \rightarrow E_2$ mapping each fibre $\pi_1^{-1}(x)$ of E_1 onto a fibre $\pi_1^{-1}(y)$ of E_2.

1.2.24 Definition (Cotangent Bundle)

We defined earlier that the differential of a function $f \in C^{\infty}(p)$ at p is a linear map $df_p: T_pM \rightarrow R$

$$df_p v = vf, v \in T_pM \tag{1.23}$$

Hence $df_p \in T_pM^*$ (= the dual of T_pM). We call T_pM^* the cotangent space of M at p .If $(U, x), x = (x_1, \dots \dots x_n)$ is a chart at p and $\left((\partial_1)_p, \dots \dots (\partial_n)_p\right)$ is the basis of T_pM consisting of coordinate vectors, then differentials $dx_i^p, i = 1, \dots \dots n$ of function x_i (at p) form the dual basis of T_pM^*.Hence the differential (at p) of function $f \in C^{\infty}(p)$can be written as

$$df_p = (\partial_i)_p f dxi^p.$$

We define the cotangent bundle of M as a disjoint union of all cotangent spaces of M

$$TM^* = \bigcup_{p\in M} T_pM^* \quad (1.24)$$

1.3 Tensors

1.3.1 Definition: (Multilinear Mapping)

Let $V_1, \dots\dots, V_k$ and W be (real) vector spaces. A mapping $F: V_1, \dots\dots, V_k \to W$ is called a multilinear (more precisely, k-linear) if it is linear in each variable, i.e.

$$F(v_1, \dots\dots, av_i + bv_i \dots\dots v_k) = aF(v_1, \dots\dots v_i, \dots v_k) + (v_1, \dots\dots v_i, \dots v_k) \quad (1.25)$$

And $a, b \in R$ $i = 1, \dots\dots\dots k$ for all

1.3.2 Definition (Dual Space)

Let V be a finite dimensional (real) vector space. A linear map $w: V \to R$ is called a covector on V and the vector space of all covectors (on V) is called the dual space of V and denoted by V^*

1.3.4 Definition: (Covariant Tensor)

A multilinear function $T: V_k \to \mathbb{R}$ is called a covariant tensor of degree k on V, and the set of all covariant tensor of degree k is denoted $T_k(V)$. if $T, S \in T_k(V)$ and $a \in \mathbb{R}$,we define

$$(S + T)(v_1, \dots v_k) = S(v_1, \dots v_k) + T(v_1, \dots v_k) \quad (1.26)$$

And

$$(aS)(v_1, \dots v_k) = aS(v_1, \dots v_k) \quad (1.27)$$

1.3.5 The Tensor Product Operation:

The Tensor Product Operation $\otimes$: $T_k(V) \times T_l(V) \to T_{k+l}(V)$ is defined by

$$(S \otimes T)(v_1, \dots, v_k, v_{k+1}, \dots, v_{k+l}) = S(v_1, \dots v_k)T(v_{k+1}, \dots v_{k+l}) \quad (1.28)$$

1.3.6 Note

It is not true in general that $S \otimes T = T \otimes S$.

1.3.7 The Identities of Tensor Product

The following of $\otimes$ are easy to establish

(i) $(S_1 + S_2) \otimes T = S_1 \otimes T + S_2 \otimes T$,

(ii) $S(T_1 + T_2) = S \otimes T_1 + S \otimes T_2$

(iii) $(aS) \otimes T = S \otimes (aT) = a(S \otimes T)$,

(iv) $(S \otimes T) \otimes U = S \otimes (T \otimes U)$.

And so both $(S \otimes T) \otimes U$, $S \otimes (T \otimes U)$ are simply written as $S \otimes T \otimes U$.

1.3.8 Note

(i) The first three identities above, indicate that $\otimes$ is bilinear, while the last indicate that $\otimes$ is associative

(ii) $T_1(V) = V^*$ (The dual of V).

1.3.9 Theorem

Let $e_1, \dots\dots, e_2$ be a basis for V, and let $e^1, \dots\dots, e^n$ be the dual basis of V^*, so that

$$e^i(e_j) = \delta_j^i = \begin{Bmatrix} 1, if\ i = j \\ 0, if\ i \neq j \end{Bmatrix} \tag{1.29}$$

Then the set of all k-fold tensor products $e^{p_1} \otimes e^{p_2} \otimes \dots \otimes e^{p_k}$, $1 \leq p_{1,}p_{2,} \dots\dots, p_k \leq n$ is a basis for $T_k(V)$, which therefore has dimension n^k.

1.3.10 Lemma

The set of all $\{p_{1,}p_{2,} \dots\dots, p_k\}$: $1 \leq p_{1,}p_{2,} \dots\dots, p_k \leq n$ is the set of all ranges of function, $\underline{p}: 1, \dots, k \to 1, \dots, n\ in\ \overline{k} \to \overline{n}$ notation, the set of all $e^{p_1} \otimes e^{p_2} \otimes \ldots \otimes e^{p_k}$ could be written $e^{\underline{p}}: \underline{p} \in \overline{n}^{k}$ where $e^{\underline{p}} = e^{p_1} \otimes e^{p_2} \otimes \ldots \otimes e^{p_k}$.

Proof

$$e^{p_1} \otimes \ldots \otimes e^{p_k}\left(e_{q_1}, \ldots, e_{q_k}\right) = e^{p_1}\left(e_{q_1}\right). e^{p_2}\left(e_{q_2}\right) \ldots e^{p_k}\left(e_{q_k}\right) =$$

$$= \delta_{q_1}^{p_1}. \delta_{q_2}^{p_2} \ldots \delta_{q_k}^{p_k}$$

$$= \begin{Bmatrix} 1, if \ \mathrm{p}_1 = q_1, \ldots, p_k = q_k \\ 0, \quad otherwise \end{Bmatrix}$$

$$= \begin{Bmatrix} 1, if \ \underline{p} = \underline{q} \\ 0, if \ \underline{p} \neq \underline{q} \end{Bmatrix} \tag{1.30}$$

Hence

$$e^{\underline{p}}\left(e_{\underline{q}}\right) = \delta_{\underline{q}}^{\underline{p}} = \begin{Bmatrix} 1, if \ \underline{p} = \underline{q} \\ 0, if \ \underline{p} \neq \underline{q} \end{Bmatrix} \tag{1.31}$$

Then, if $V_1, \ldots \ldots, V_k$ are k vectors in V with $V_i = \sum_{j=1}^{n} \mathrm{a_{ij}}\, e_j$ and $T \in T_k(V)$, then

$$\mathrm{T}(v_1, \ldots, v_k) = \sum_{q_1, \ldots, q_{k=1}}^{n} a_{1q_1} a_{2q_2} \ldots a_{kq_k} T\left(e_{q_1}, \ldots, e_{q_k}\right)$$

$$= \sum_{q_1, \ldots, q_{k=1}}^{n} T\left(e_{q_1}, \ldots, e_{q_k}\right) e^{q_1} \otimes \ldots \otimes e^{q_k}(v_1, \ldots, v_k) \tag{1.32}$$

Thus

$$T = \sum_{q_1, \ldots, q_{k=1}}^{n} T\left(e_{q_1}, \ldots, e_{q_k}\right) e^{q_1} \otimes \ldots \ldots . \otimes e^{q_k} \tag{1.33}$$

i.e

$$T = \sum_{\underline{q} \in \overline{n}^{k}} T(e_{\underline{q}})\, e^{\underline{q}} = \sum_{\underline{q} \in \overline{n}^{k}} C_{\underline{q}} e^{\underline{q}} \tag{1.34}$$

Where

$$C_{\underline{q}} = T\left(e_{\underline{q}}\right) = T\left(e_{q_1}, \dots, e_{q_k}\right) \tag{1.35}$$

And

$$T(V_1, \dots\dots, V_k) = \sum_{\underline{q}\in\overline{n}^k} C_{\underline{q}} e^{\underline{q}} (v_1, \dots, v_k) \tag{1.36}$$

Consequently

$$\{e^{\underline{p}}\}_{\underline{p}\in\overline{n}^k} \text{ span } T_k(V) \tag{1.37}$$

Suppose now that there are numbers $\{a_{\underline{p}}\}_{\underline{p}\in\overline{n}^k}$, such that

$$\sum_{\underline{p}\in\overline{n}^k} a_{\underline{p}} e^{\underline{p}} = 0 \tag{1.38}$$

Then

$$0 = \sum_{\underline{p}\in\overline{n}^k} a_{\underline{p}} e^{\underline{p}}\left(e_{\underline{q}}\right) = \sum_{\underline{p}\in\overline{n}^k} a_{\underline{p}} \delta_{\underline{q}}^{\underline{p}} = a_{\underline{q}} \tag{1.39}$$

Thus $a_{\underline{q}} = 0$, for all $\underline{q} \in \overline{n}^k$, and therefore $\{e^{\underline{p}}\}_{\underline{p}\in\overline{n}^k}$ are linearly independent, and $dim(T_k(V) = n^k)$, where $n = dim\, V$. For this reason we write

$$T_k(V) = V^* \otimes \dots\dots \otimes V^* \text{ (k factors)} = \otimes_1^k V^* \tag{1.40}$$

If $T \in T_k(V)$, and if we write

$$T = \sum_{q_1,\dots,q_k=1}^{n} T_{q_1,\dots,q_k} e^{q_1} \otimes e^{q_2} \otimes \dots \otimes e^{q_k} \tag{1.41}$$

where $T_{q_1 \dots q_k} = T\left(e_{q_1}, \dots, e_{q_k}\right)$, then the set of n^k numbers $T_{q_1,\dots,q_k}$ are called the components of T relative to the given bases $\{e_1, \dots\dots, e_n\}$ and(its dual) $\{e^1, \dots\dots, e^n\}$ of V and V^* respectively.

1.3.9 Definition: (Contravariant Tensors)

We define the space of contravariant tensors of degree s, denoted T^s as $T^s = V \otimes \dots \otimes V$ (s times). Then every contravariant tensor k of degree s can be expressed uniquely as a linear combination

$$K = \sum_{i_1,\ldots,i_s=1}^{n} K^{i_1 \ldots i_s}\, e_{i_1} \otimes \ldots \otimes e_{i_s} \qquad (1.42)$$

where $K^{i_1 \ldots i_s}$ are components (they are n^s numbers) of K with respect to the basis $\{e_1, \ldots \ldots, e_n\}$ of V.

we can write for short

$$K = \sum_{\underline{p} \in \overline{n}^{s}} K^{\underline{p}}\, e_{\underline{p}} \qquad (1.43)$$

1.3.10 Definition: (Mixed Tensor Space)

We define the (mixed) tensor space type (r, s) or tensor space of contravariant degree r and covariant degree s, as the tensor product

$$T_s^r = T^r \otimes T_s = V \otimes \ldots \otimes V \otimes \mathrm{V}^* \otimes \ldots \otimes \mathrm{V}^* \qquad (1.43)$$

(V: r times, V^*: s times) in particular $T_0^r = T^r$, $T_s^0 = T_s, T_0^0 = R$.

An element of T_s^r is called a tensor of type (r, s).

1.3.11 Remark

(i) $T^k(V)$, $T_l(V)$ and T_l^k are vector spaces in a natural way.

(ii) We make a convention that both 0-covariant and 0- contravariant tensor are real numbers, i.e. $T^0(V) = T_0(V) = R$

1.3.12 Examples

(i) Any linear map $w: V \to R$ is 1-covariant tensor. Thus $T_1(V) = V^*$. Similarly $T^1(V) = V^{**} = V$

(ii) If V is an inner product space, then any inner product on V is a 2-covariant tensor (a bilinear real -valued mapping, i.e. a bilinear form).

(iii) The determinant is an n-covariant tensor on R^n.

1.3.13 Definitions

In terms of a basis $\{e_1, \dots, e_n\}$ of V and the dual basis $\{e^1, \dots, e^n\}$ of V^*, every tensor K of type (r, s) can be expressed uniquely as:

$$K = \sum_{i_1,\dots,i_r;j_1,\dots,j_s} K^{i_1,\dots,i_r}_{j_1,\dots,j_s} = K^{i_1,\dots,i_r}_{j_1,\dots,j_s} e_{i_1} \otimes \dots \otimes e_{i_r} \otimes e^{j_1} \otimes \dots \otimes e^{j_s}$$

$$= \sum_{\underline{p}\in\overline{n}^{\bar{r}},\underline{q}\in\overline{n}^{\bar{s}}} K^{\underline{p}}_{\underline{q}}\, e_{\underline{p}} e^{\underline{q}} \qquad (1.44)$$

$K^{i_1,\dots,i_r}_{j_1,\dots,j_s}$ (there are $n^{(r+s)} of\ them$) are called the components of K with respect to the basis $\{e_1, \dots, e_n\}$ of V, and its dual $\{e^1, \dots, e^n\}$ of V^*.

1.4 Linear Transformation

If $f: V \to W$ is a linear transformer it induces a linear transformation $f^*: T_k(W) \to T_k(V)$ defined by

$$(f^*T)(v_1, \dots, v_k) = T(fv_1, \dots, fv_k) \qquad (1.45)$$

for $T \in T_k(W)$ and $v_1, \dots, v_k \in V$, it is true that $f^*(S \otimes T) = f^*(S) \otimes f^*(T)$. As an example of a covariant tensor of degree 2 on $\mathbb{R}^n$, is an inner product $<,> \in T_2(\mathbb{R}^n)$

$$< x, y > = T(x, y) = \sum_{i,j} x^i y^j\, T(e_i, e_j) = \sum_{i,j} T_{ij} x^i y^j \qquad (1.46)$$

$$T = \sum_{i,j} T_{ij} e^i \otimes e^j, \qquad (1.47)$$

where $T_{ij} = T_{ji}\ and\ \sum_{i,j} T_{ij} x^i x^j > 0\ if\ x \neq 0$.

Generally, we define an inner product on V to be a covariant tensor T of degree 2 such that T is symmetric, that is $T(v, w) = T(w, v)$, for $w, v \in V$, and such that T is positive definite: $T(v, v) \neq 0$.

$$T = \sum_{i,j} T_{ij}\, e^i \otimes e^j, \quad T_{ij} = T_{ji}, \quad \sum_{i,j} T_{ij}\, v^i v^j > 0 \text{ if } v \neq 0 \qquad (1.48)$$

It is a standard result, that if T is an inner product on V, then V has an orthonormal basis with respect to T, i,e a basis $\{e_1, \dots, e_n\}$ such that

$$T(e_i, e_j) = \delta i_j \tag{1.49}$$

Then T has the simple expansion.

$$T = e^1 \otimes e^1 + e^2 \otimes e^2 + \cdots + e^n \otimes e^n \tag{1.50}$$

And

$$T(v, w) = v_1 w_1 + v_2 w_2 + \cdots v_n w_n = \textstyle\sum_1^n v_i\, w_i \tag{1.51}$$

If $v = (v_1, \dots \dots, v_n)$, $w(w_1, \dots \dots, w_n)$ relative to the orthonormal basis

1.5 Alternating Covariant Tensors

A tensor $w \in T_r(V)$ is called alternating (or skew-symmetric) if, $w(v\sigma_{(1)}, \dots, v\sigma_{(k)}) = (sgn\, \sigma)\, w(v_1, \dots, v_n)$ for all permutations σ of $\{1, \dots, k\}$ such a tensor is called a k-form on V, the set of all alternating covariant tensors of degree k is a subspace of $T_k(V)$ denoted by $\Lambda^k(v)$.

If $T \in T_r(V)$, we define $Alt(T)$ by

$$\mathrm{Alt(T)(v_1, \dots, v_n)} = \frac{1}{\mathrm{k!}} \sum_{\sigma \in S_k} (\mathrm{sgn}\ \sigma) * \mathrm{T}(\mathrm{v}_{\sigma(1)}, \dots, \mathrm{v}_{\sigma(k)}) \tag{1.52}$$

where S_k is the set of all permutation of $\{1, 2, \dots, k\}$.

1.5.1 Lemma

(i) If $T \in T_k(V$, then $Alt(T) \in \Lambda_k(v)$.
(ii) If $w \in \Lambda_k$ (v) , then $Alt(w) = w$.
(iii) If $T \in T_k(V)$, then $Alt(Alt\ T) = Alt\ T$.

i,e $(Alt)^2 = Alt$ (idempotent) or $Alt: T_k(V) \to \Lambda_k\ (V)$ the range of Alt is a projection .

It follows that $T_k(V) = \Lambda_k(V) \oplus S_{\,k}(V)$ where $S^k(V) = N(Alt)$ (the null space of Alt) is called the space of symmetric tensor of degree.

Thus T is symmetric if $\mathrm{T}(\mathrm{v}_{\sigma(1)}, \dots, \mathrm{v}_{\sigma(\mathrm{k})}) = T(\mathrm{v}_1, \dots, \mathrm{v}_\mathrm{n})$.

Proof

(i) Noting that for a fixed permutation $\eta \to \sigma \circ \eta$ is bijection on S_k, we have

$$(\mathrm{AltT})(\mathrm{v}_{\mathrm{p}(1)}, \dots, \mathrm{v}_{\mathrm{p}(\mathrm{k})}) = \frac{1}{\mathrm{k}!}\sum_{\sigma\in \mathrm{S}_\mathrm{k}}(\mathrm{sgn}\ \sigma)\ \mathrm{T}(\mathrm{v}_{\sigma\circ\mathrm{p}(1)}, \dots, \mathrm{v}_{\sigma\circ\mathrm{p}(\mathrm{k})})$$

$$= \frac{1}{\mathrm{k}!}\sum_{\sigma\in \mathrm{S}_\mathrm{k}}(\mathrm{sgn}\ \sigma\circ \mathrm{p}^{-1})\ \mathrm{T}(\mathrm{v}_{\sigma\circ\, \mathrm{p}^{-1}\circ\mathrm{p}(1)}, \dots, \mathrm{v}_{\sigma\circ\, \mathrm{p}^{-1}\circ\mathrm{p}(\mathrm{k})})$$

$$= \frac{\mathrm{sgn\ p}}{\mathrm{k}!}\sum_{\sigma\in \mathrm{S}_\mathrm{k}}(\mathrm{sgn}\ \sigma)\ \mathrm{T}(\mathrm{v}_{\sigma(1)}, \dots, \mathrm{v}_{\sigma(\mathrm{k})}) = (\mathrm{sgn\ p})(\mathrm{Alt\ T})(\mathrm{v}_1, \dots, \mathrm{v}_\mathrm{n}).$$

$$\therefore Alt(T) \in \Lambda^\mathrm{k}(\mathrm{V}).$$

(ii) if $w \in \Lambda^\mathrm{k}(\mathrm{V})$, then

$$\mathrm{Alt}(\,\mathrm{w})(\mathrm{v}_1, \dots, \mathrm{v}_\mathrm{n})$$

$$= \frac{1}{\mathrm{k}!}\sum_{\sigma\in \mathrm{S}_\mathrm{k}}(\mathrm{sgn}\ \sigma)\ (\mathrm{v}_{\sigma(1)},\ \dots, \mathrm{v}_{\sigma(\mathrm{k})}) = \frac{1}{\mathrm{k}!}\sum_{\sigma\in \mathrm{S}_\mathrm{k}}(\mathrm{sgn}\ \sigma)^2\ \mathrm{T}(\mathrm{v}_1, \dots, \mathrm{v}_\mathrm{k})$$

$$= [\frac{1}{\mathrm{k}!}\sum_{\sigma\in \mathrm{S}_\mathrm{k}}(\mathrm{sgn}\ \sigma)^2\ ]\mathrm{w}(\mathrm{v}_1, \dots, \mathrm{v}_\mathrm{n})$$

$$= \mathrm{w}(\mathrm{v}_1, \dots, \mathrm{v}_\mathrm{n}),$$

$$\therefore Alt\ (w) = w \qquad (1.53)$$

(iii) If $T \in T_k(V)$ then $Alt\ (T) \in \Lambda^k(V)$ by (i). Hence $Alt(Alt\ T) = Alt\ T$ by (b). Now if $w \in \Lambda^\mathrm{k}(\mathrm{V})$ $\varphi \in \Lambda^\mathrm{L}(\mathrm{V})$ then $w \otimes \varphi$ is usually not in $\Lambda^{\mathrm{k+L}}(\mathrm{V})$ we there for define a new product

1.6 The Wedge (or Exterior) Product

The wedge (or exterior) product $\Lambda : \Lambda^{k}(V) \times \Lambda^{L}(V) \rightarrow \Lambda^{k+L}(V)$ by

$$w\Lambda\varphi = \frac{(k+L)!}{k!L!} Alt(w \otimes \varphi) \qquad (1.54)$$

1.6.1 Properties of Λ

The following properties of Λ are true

(i) $(w_1 + w_2)\Lambda\,\eta = w_1\Lambda\eta + w_2\Lambda\,\eta,$

(ii) $w\,\Lambda\,(\eta 1 + \eta 2) = w\,\Lambda\,\eta 1 + w\Lambda\eta 2\,,$

(iii) $(aw)\Lambda\,\eta = w\Lambda(a\eta) = a(w\,\Lambda\,\eta)\ (a \in \mathbb{R})\,,$

(iv) $w\Lambda\eta = (-1)\eta\,\Lambda\,w$ where $w \in \Lambda^{k}(V)\,, \eta \in \Lambda^{L}(V).$

(v) $f^*(w\,\Lambda\,\eta) = f^*(w)\,\Lambda\,f^*(\eta)$

(vi) $(w\,\Lambda\,\eta)\varphi = w\Lambda(\eta\,\Lambda\,\varphi)$

and so we just write $w\,\Lambda\,\eta\,\Lambda\,\varphi$ for either

$$w\,\Lambda\,\eta\,\Lambda\,\varphi = \frac{(k+L+m)!}{k!L!m!} Alt(w \otimes \eta \otimes \varphi) \qquad (1.55)$$

1.6.2 Note

Since a k-form w is alternating it follows that if $w \in \Lambda^{k}(V)$ then $w(v_1, \dots, v_n) = 0$, if any one of the k argument is repeated, it then follows that $w(v_1, \dots, v_R) = 0$, if the vectors $\{v_1, \dots, v_k\}$ are linearly dependent , and there for $\Lambda^{k} = 0$ if $k > n$, where $n = dim\,V$.

1.6.3 Theorem

The set of all $e^{p_1}\Lambda \dots \Lambda e^{p_k}, 1 \leq p_1 < p_2 < \cdots < p_k \leq n$ is a basis for $\Lambda^{k}(V)$ which there fore haes dimension $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{n!}{k!(n-k)!}$.

Proof

If $w \in \Lambda^{k}(V) \subset T_k(V)$ then we can write

$$w = \sum_{p_1,\dots,p_k=1}^{n} a_{p_1 p_2 \dots p_k}\, e^{p_1} \otimes e^{p_2} \otimes \dots \otimes e^{p_k} \qquad (1.56)$$

Thus

$$\mathrm{w} = \mathrm{Alt}\ (\mathrm{w}) = \sum_{p_1,\dots,p_k=1}^{n} a_{p_1p_2\dots p_k}\, Alt(e^{p_1} \otimes e^{p_2} \otimes \dots \otimes e^{p_k}) \quad (1.57)$$

Since each of $Alt(e^{p_1} \otimes e^{p_2} \otimes \dots \otimes e^{p_k})$ is a constant (o, $\frac{+1}{k!}, or\ \frac{-1}{k!}$) times one of $e^{p_1} \Lambda \dots \Lambda e^{p_k}, 1 \leq p_1 < p_2 < \cdots < p_k \leq n$, these elements span Λ^k (V).

$$[Alt(e^{p_1} \otimes e^{p_2} \otimes \dots \otimes e^{p_k}) = \frac{1}{k!} \delta_q^p e^{q_1} \Lambda \dots \Lambda e^{q_k}, \quad (1.58)$$

where

$$\delta_q^p = \det\left(\delta_{q_i}^{p_i}\right) = sgn\ \sigma\ , if\ p = q \circ \sigma\ ] \quad (1.59)$$

To show that $\{e^{p_1} \Lambda e^{p_2} \Lambda \dots \Lambda e^{p_k}\} 1 \leq p_1 < p_2 < \cdots < p_k \leq n,$ is independent, we note that

$$\mathrm{e}^{\mathrm{p}_1} \Lambda \mathrm{e}^{\mathrm{p}_2} \Lambda \dots \Lambda \mathrm{e}^{\mathrm{p}_k}\left(\mathrm{e}_{\mathrm{q}_1}, \dots, \mathrm{e}_{\mathrm{q}_k}\right) = \begin{cases} \mathrm{sgn}\ \sigma, \mathrm{if}\ \mathrm{p} = \mathrm{q} \circ \sigma = \delta_{\mathrm{q}}^{\mathrm{p}} \\ \quad 0\ , \quad \mathrm{otherwise} \end{cases} \quad (1.60)$$

Thus if

$$\sum_{1 \leq p_1 < p_2 < \cdots < p_k \leq n} a_{p_1p_2\dots p_k} e^{p_1} \Lambda e^{p_2} \Lambda \dots \Lambda e^{p_k} = 0 \quad (1.61)$$

Then for $1 \leq p_1 < p_2 < \cdots < p_k \leq n$, we have

$$\begin{aligned} 0 &= \sum_{1 \leq p_1 < p_2 < \cdots < p_k \leq n} a_{p_1p_2\dots p_k} e^{p_1} \Lambda e^{p_2} \Lambda \dots \Lambda e^{p_k} \left(e_{q_1}, \dots, e_{q_k}\right) \\ &= \sum_{1 \leq p_1 < p_2 < \cdots < p_k \leq n} a_{p_1p_2\dots p_k} \delta_q^p = a_{q_1q_2\dots q_k} \end{aligned} \quad (1.62)$$

If $\mathrm{w} \in \Lambda^k(V)$ then

$$w = \sum_{1 \leq p_1 < p_2 < \cdots < p_k \leq n} w_{\underline{p}} e^{p_1} \Lambda e^{p_2} \Lambda \dots \Lambda e^{p_k} \quad (1.63)$$

Where

$$w_{\underline{p}} = w_{p_1p_2\dots p_k} = w(e_{p_1}, e_{p_2}, \dots, e_{p_k}) \quad (1.64)$$

1.6.4 Note

if dim $V = n$, then $\Lambda^n V$ has dimension1, this means that n-forms on V are multiple of any non-zero one.

1.6.5 Example

The determinant function D an n-form on $\mathbb{R}^n$ the element of Λ^n $(\mathbb{R}^n)$ uniquely determined by setting $D(e_1, \dots, e_n) = 1$, and its value $D(\underline{x}^1, \dots, \underline{x}^n)$ at n-tuple $< \underline{x}^1, \dots, \underline{x}^n > \in \mathbb{R}^n \times \mathbb{R}^n \times \dots \times \mathbb{R}^n$ is the determinant of the matrix $\underline{x} = (x_{ij})$, whose j the column is $\underline{x}^j$, $j = 1, \dots, n$. Thus

$$D = D(e_1, \dots, e_n)\, e^1 \Lambda e^2 \Lambda \dots \Lambda e^n = e^1 \Lambda e^2 \Lambda \dots \Lambda e^n$$

$$\therefore D\left(\underline{x}^1, \dots, \underline{x}^n\right) = e^1 \Lambda e^2 \Lambda \dots \Lambda e^n \left(\underline{x}^1, \dots, \underline{x}^n\right)$$

$$= n!\, Alt(e^1 \otimes e^2 \otimes \dots \otimes e^n)\left(\underline{x}^1, \dots, \underline{x}^n\right)$$

$$= n! \frac{1}{n!} \sum_{\sigma \in S_n} (sgn\ \sigma)(e^1 \otimes \dots \otimes e^n)\left(\underline{x}^{\sigma(1)}, \dots, \underline{x}^{\sigma(n)}\right)$$

$$= \sum_{\sigma \in S_n} (sgn\ \sigma)\, x_{1\sigma(1)} x_{2\sigma(2)} \dots x_{n\sigma(n)} \tag{1.65}$$

1.6.6 Example:

(i) $\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} = \sum_{\sigma \in S_2} (sgn\ \sigma)\, x_{1\sigma(1)} x_{2\sigma(2)}$

$$S_2 : \binom{1,2}{1,2}\ sgn = 1, \quad \binom{1,2}{2,1}\ sgn = -1.$$

$$\therefore \sum_{\sigma \in S_2} (sgn\ \sigma)\, x_{1\sigma(1)} x_{2\sigma(2)} = x_{11} x_{22} - x_{12} x_{21}$$

(iii) $\begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} = \sum_{\sigma \in S_3} (sgn\ \sigma)\, x_{1\sigma(1)} x_{2\sigma(2)} x_{3\sigma(3)}$

$$S_3 : \binom{1,2,3}{1,2,3}\ sgn = 1, \quad \binom{1,2,3}{2,1,3}\ sgn = -1, \binom{1,2,3}{3,2,1}\ sgn = -1,$$

$$\binom{1,2,3}{1,3,2}\, sgn = -1, \binom{1,2,3}{2,3,1}\ sgn = 1,\ \ \binom{1,2,3}{3,1,2}\, sgn = 1.$$

$$\therefore \sum_{\sigma\in S_3} (sgn\,\sigma)\, x_{1\sigma(1)}x_{2\sigma(2)}x_{3\sigma(3)} = x_{11}x_{22}x_{33} + x_{12}x_{23}x_{31} +$$

$$x_{13}x_{21}x_{32} - x_{12}x_{21}x_{33} - x_{13}x_{22}x_{31} - x_{11}x_{23}x_{32}.$$

1.6.7 lemma

Let $e_1, \dots, e_n$ be a basis for V and let w $\in \Lambda^n$(V). If $v_j = \sum_{i=1}^{n} a_{ij}e_i$, $1 \le j \le n$, are n vectors in V, then

$$w(\, v_1, \dots, v_n) = det(a_{ij})w(\, e_1, \dots, e_n) \tag{1.66}$$

Proof

Define $\eta \in T_n(\mathbb{R}^n)$ by

$$\eta(a_1, \dots, a_n) = w(\textstyle\sum_{i=1}^{n} a_{i1}e_i\,, \dots, \sum_{i=1}^{n} a_{in}e_i) \tag{1.67}$$

Then $\eta \in \Lambda^n(\mathbb{R}^n)$ and so

$$\eta\ = \lambda. det\ , \lambda \in R \tag{1.68}$$

Hence $\eta\ (\delta_1, \dots, \delta_n) = \lambda 1 = w(e_1, \dots, e_n)$so that

$$\mathrm{w}(v_1, \dots, v_n) = \ \mathrm{w}(\sum_{i=1}^{n} a_{i1}e_i\,, \dots, \sum_{i=1}^{n} a_{in}e_i)$$

$$= \ \eta\ (a_1, \dots, a_n) \tag{1.69}$$

$$= w(e_1, \dots, e_n). det(a_1, \dots, a_n)\ =\ det(a_{ij})w(e_1, \dots, e_n) \tag{1.70}$$

it follows that for $1 \le p_1 < \ p_2 < \cdots < \ p_k \ \le n$,

$$(e^{q_1}\Lambda e^{q_2}\Lambda \dots \Lambda e^{q_k})(v_1, \dots, v_n) = \det(t_{ij}) \tag{1.71}$$

where $t_{ij} = e^{q_i}(v_j)$, it is true that if $f: V \to V$ is linear and $dim\, V = n$, then $f^*: \Lambda^n(V) \to \Lambda^n(V)$ is multiplication by $det\, f$.

Now let M be a C^∞-manifold with each point $p \in M$, we have associated a vector space, the tangent space M_p, thus we can perform tensor products on M_p ,and get the tensor spaces $T_k(M_p)$, space of covariant tensors of degree k on M_p, i.e

$$T_k(M_p) = M_p^* \otimes M_p^* \otimes \dots \otimes M_p^* = \otimes_1^k M_p^* \qquad (1.72)$$

Similarly $T^k(M_p)$, space of contra variant tensors of degree k on M_p:

$$T^k(M_p) = M_p \otimes M_p \otimes \dots \otimes M_p = \otimes_1^k M_p \qquad (1.73)$$

and the space of tensors of type (r, s),

$$T_s^r(M_p) = M_p \otimes \dots \otimes M_p \otimes M_p^* \otimes \dots \otimes M_p^* = T^r(M_p) \otimes T_s(M_p) \qquad (1.74)$$

Also $\Lambda^k(M_p)$ is space of alternating (skew-symmetric) covariant tensors of degree k (k-forms) over M_p.

1.7 Transformation Laws for Tensors

For a change of a basis of V, the components of tensors are subject to the following transformations.

Let $e_1, \dots, e_n$ *and* $\overline{e}_1, \dots, \overline{e}_n$, be two basis of V related by a non-singular linear transformation,

$$\overline{e}_i = \sum_j A_i^j\, e_j, i = 1, \dots, n, \qquad e_i = \sum_j B_i^j\, \overline{e}_j, i = 1, \dots, n \qquad (1.75)$$

where $B = (B_j^i)$ is the inverse matrix of the matrix $A = (A_j^i)$ (here i is a row index (the upper index), j a column index (the lower index)) so that $\sum_j A_i^j B_j^k = \delta_k^i$,the corresponding change of the dual basis in V^* is given by

$$\overline{e}^i = \sum_j B_j^i\, e^j, i = 1, \dots, n, \qquad e^i = \sum_j A_j^i\, \overline{e}^j, i = 1, \dots, n. \qquad (1.76)$$

To derive the first equation we have

$$\overline{e}^i = \sum_j \overline{e}^i(e_j)\, e^j = \sum_{j,k} \overline{e}^i\left(B_j^k \overline{e}_k\right) e^j = \sum_{j,k} B_j^k (\,\overline{e}^i\, \overline{e}_k)\, e^j, \quad (1.77)$$

$$\sum_{j,k} B_j^k \delta_k^i\, e^j = \sum_j B_j^i\, e^j \quad (1.78)$$

And for the second

$$e^i = \sum_j e^i\,(\overline{e}_j)\,\overline{e}^j = \sum_{j,k} e^i\,\left(A_j^k e_k\right)\overline{e}^j$$

$$= \sum_{j,k} A_j^k \delta_k^i\, \overline{e}^j = \sum_j A_j^i\, \overline{e}^j. \quad (1.78)$$

If k is a contravariant tensor of degree its components $K^{i_1 \dots i_r}$ and $\overline{K}^{i_1 \dots i_r}$ with respect to $e_{\,i}$ and $\overline{e}_i$ respectively, are related by

$$\overline{K}^{i_1 \dots i_r} = \sum_{j_1,\dots,j_r} B_{j_1}^{i_1} \dots B_{j_r}^{i_r} K^{j_1 \dots j_r} \quad (1.79)$$

Also

$$K^{i_1 \dots i_r} = \sum_{j_1,\dots,j_r} A_{j_1}^{i_1} \dots A_{j_r}^{i_r} \overline{K}^{j_1 \dots j_r}. \quad (1.80)$$

Similarly, the components of a covariant tensor L of degree s are related by

$$\overline{L}_{i_1 \dots i_s} = \sum_{j_1,\dots,j_s} A_{i_1}^{j_1} \dots A_{i_s}^{j_s} L_{j_1 \dots j_s} \quad (1.81)$$

and

$$L_{i_1 \dots i_s} = \sum_{j_1,\dots,j_s} B_{i_1}^{j_1} \dots B_{i_s}^{j_s} \overline{L}_{j_1 \dots j_s}, \quad (1.82)$$

For a tensor k type (r, s):

$$K = \sum_{i_1,\dots,i_r,j_1,\dots,j_s} K_{j_1,\dots,j_s}^{i_1 \dots i_r}\, e_{i_1} \otimes \dots \otimes e_{i_r} \otimes e^{j_1} \otimes \dots \otimes e^{j_s},$$

$$= \sum_{i_1,\dots,i_r,j_1,\dots,j_s} \overline{K}_{j_1,\dots,j_s}^{i_1 \dots i_r}\, \overline{e}_{i_1} \otimes \dots \otimes \overline{e}_{i_r} \otimes \overline{e}^{j_1} \otimes \dots \otimes \overline{e}^{j_s} \quad (1.83)$$

We have the following transformation of components:

$$\overline{K}_{j_1,\dots,j_s}^{i_1 \dots i_r} = \sum_{k_1,\dots,k_r,m_1,\dots,m_s} B_{k_1}^{i_1} \dots B_{k_r}^{i_r} A_{j_1}^{m_1} \dots A_{j_s}^{m_s} K_{m_1,\dots,m_s}^{k_1 \dots k_r}, \quad (1.84)$$

$$K^{i_1 \dots i_r}_{j_1,\dots,j_s} = \sum_{k_1,\dots,k_r,m_1,\dots,m_s} A^{i_1}_{k_1} \dots A^{i_r}_{k_r} B^{m_1}_{j_1} \dots B^{m_s}_{j_s} \overline{K}^{k_1 \dots k_r}_{m_1,\dots,m_s}. \qquad (1.85)$$

1.8 Tensor bundles

Let M be a smooth manifold

We define tensor bundles on M as disjoint unions

(i) k-covariant tensor bundle

$$T^k M = \bigcup_{p \in M} T^k(T_p M) \qquad (1.86)$$

(ii) l -contravariant tensor bundle

$$T_l M = \bigcup_{p \in M} T_l(T_p M) \qquad (1.87)$$

(iii) (k, l)-tensor bundle

$$T^k_l M = \bigcup_{p \in M} T^k_l(T_p M) \qquad (1.88)$$

equipped with natural C^∞-structures.

We identify

$$T^0 M = T_0 M = M \times R$$

$$T^1 M = TM^*$$

$$T^1 M = TM$$

$$T^k_0 M = T^k M$$

$$T^0_l M = T_l M \qquad (1.89)$$

1.9 Tensor Fields

1.9.1 Definition: (Tensor Fields)

A tensor field of type (r, s) on a subset N of a manifold M is an assignment of a tensor $K_p \in T^r_s(M_p)$ to each point $p \in N$. In a coordinate neighborhood U with a local coordinate system$(x^1, \dots, x^n)$, we take $\frac{\partial}{\partial x_i}$, $i = 1, \dots, n$, as a basis for each tangent space M_p, $p \in U$ and

dx^i , $i = 1, \dots, n$ as the dual basis of M_p^*. Under a change of coordinate $(\overline{x}^1, \dots, \overline{x}^n)$ these are related by transformations:

$$\frac{\partial}{\partial \overline{x}^i} = \sum_j \frac{\partial x^j}{\partial \overline{x}^i} \frac{\partial}{\partial x^j} \quad , \; A_i^j = \frac{\partial x^j}{\partial \overline{x}^i} ,$$

$$\frac{\partial}{\partial x^i} = \sum_j \frac{\partial \overline{x}_j}{\partial x^i} \frac{\partial}{\partial \overline{x}_j} \quad , B_i^j = \frac{\partial \overline{x}^j}{\partial x^i}$$

$$d\overline{x}^i = \sum_j \frac{\partial \overline{x}^i}{\partial x^j} dx^j \, ; dx^i = \sum_j \frac{\partial x^i}{\partial \overline{x}^j} d\overline{x}^j \; ; \; B_j^i = \frac{\partial \overline{x}^i}{\partial x^j} \; ; A_j^i = \frac{\partial x^i}{\partial \overline{x}^j} . \quad (1.90)$$

A tensor field K of type (r, s) defined on U is the n expressed by:

$$K = \sum_{i_1,\dots,i_r,j_1,\dots,j_s} K_{j_1,\dots,j_s}^{i_1 \dots i_r} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s} \quad (1.91)$$

where $K_{j_1,\dots,j_s}^{i_1 \dots i_r}$ are functions on U, called the components of K with respect to the local coordinate system $(x^1,\dots,x^k)$. We say that K is smooth (of class C^∞) if its components $K_{j_1,\dots,j_s}^{i_1 \dots i_r}$ are functions of class C^∞.under a change of coordinates $(\overline{x}^1, \dots, \overline{x}^n)$ the components of K transform according to

$$\overline{K}_{j_1,\dots,j_s}^{i_1 \dots i_r} = \sum_{p_1,\dots,p_r,q_1,\dots,q_s} \frac{\partial \overline{x}^{i_1}}{\partial x^{p_1}} \dots \frac{\partial \overline{x}^{i_r}}{\partial x^{p_r}} . \frac{\partial x^{q_1}}{\partial \overline{x}_{j_1}} \dots \frac{\partial x^{q_s}}{\partial \overline{x}_{j_s}} K_{q_1,\dots,q_s}^{p_1 \dots p_r} \quad (1.92)$$

$$K_{j_1,\dots,j_s}^{i_1 \dots i_r} = \sum_{p_1,\dots,p_r,q_1,\dots,q_s} \frac{\partial x^{i_1}}{\partial \overline{x}_{p_1}} \dots \frac{\partial x^{i_r}}{\partial \overline{x}_{p_r}} . \frac{\partial \overline{x}^{q_1}}{\partial x_{j_1}} \dots \frac{\partial \overline{x}^{q_s}}{\partial x_{j_s}} \overline{K}_{q_1,\dots,q_s}^{p_1 \dots p_r} . \quad (1.93)$$

1.9.2 Example:

(i) for K of type $(1, 0)$:

$$\overline{K}^i = \sum_j \frac{\partial \overline{x}^i}{\partial x^j} K^j \, ; K^i = \sum_j \frac{\partial x^i}{\partial \overline{x}^j} \overline{K}^j .$$

(iv) for K of type $(0, 1)$:

$$\overline{K}_i = \sum_j \frac{\partial x^j}{\partial \overline{x}_i} K_j;\ K_i d = \sum_j \frac{\partial \overline{x}^j}{\partial x_i} \overline{K}_j.$$

(iii) For K of type $(0, 2)$: $k \in T_2^0$

$$\overline{K}_{ij}\,\overline{K}^{ij} = \sum_{k,L} \frac{\partial x^k}{\partial \overline{x}^i} \frac{\partial x^L}{\partial \overline{x}^j} K_{kL};\ K_{ij} = \sum \frac{\partial \overline{x}^k}{\partial x^i} \frac{\partial \overline{x}^L}{\partial x^j} \overline{K}_{kL}.$$

Because

$$\overline{K}_{ij} = K\left(\frac{\partial}{\partial \overline{x}^i}, \frac{\partial}{\partial \overline{x}^j}\right) = K\left(\sum_k \frac{\partial x^k}{\partial \overline{x}^i} \frac{\partial}{\partial x^k}, \sum_L \frac{\partial x^L}{\partial \overline{x}^j} \frac{\partial}{\partial x^i}\right)$$

$$= \sum_{k,L} \frac{\partial x^k}{\partial \overline{x}^i} \frac{\partial x^L}{\partial \overline{x}^j} K_{kL}$$

Since all tensor bundles are smooth manifolds, we may consider their smooth sections.

1.9.3 Definition

We say that a section $s: M \to T_l^k M$ is a (k, l)-tensor field (recall that $\pi \circ \mathrm{s} = \mathrm{i}d_m$, and so $s(p) \in T_l^k\left(\left(T_p M\right)\right)$. A smooth -tensor field is a smooth section $M \to T_l^k$. Similarly, we define (smooth) k-covariant tensor fields and l -contravariant tensor fields. Since O-covariant and 0-contravariant tensors are real numbers, (smooth) 0-covariant tensor fields and (smooth) 0-contravariant tensor fields are (smooth) real-valued functions.

Denote

$T^k(M)$ = {smooth sections on $T^k(M)$}

= {smooth k-covariant tensor fields}

$T_l(M)$ ={smooth sections on $T_l(M)$}

= {smooth l -contavariant tensor fields}

$T_l^k(M)$ = {smooth sections on $T_l^k(M)$}

= {smooth (k,l)-tensor fields}.

If $(U,x), x = (x^1, x^2, \ldots \ldots x^n)$, is a chart and σ is a tensor field in U, we may write

$\sigma = \sigma_{i_1 \ldots\ldots i_k}\, dx^{i_1} \otimes \ldots \otimes dx^{i_k}$, if σ is a k-covariant tensor field,

$\sigma = \sigma^{j_1 \ldots\ldots\ldots j_l}\, \partial j_1 \otimes \ldots \otimes \partial j_l$, if σ is an l -contravariant tensor field, or $\sigma = \sigma_{i_1 \ldots\ldots i_k}^{j_1 \ldots\ldots\ldots j_l} dx^{i_1} \otimes \ldots \otimes dx^{i_k} \partial j_1 \otimes \ldots \otimes \partial j_l$, if σ is a (k,l)-tensor field. Functions $\sigma_{i_1 \ldots\ldots i_k}$, $\sigma^{j_1 \ldots\ldots\ldots j_l}$ and $\sigma_{i_1 \ldots\ldots i_k}^{j_1 \ldots\ldots\ldots j_l}$ are called the component functions of σ with respect to the chart (U,x). Again we have:

1.9.4 Lemma

Let σ be a (k,l)-tensor field on M. Then the following are equivalent:

(i) $\sigma \in T_l^k(M)$

(ii) The component functions of σ (with respect to any chart) are smooth;

(iv) if $U \subset M$ is open and $x_1 \ldots \ldots \ldots x_n \in \mathcal{T}(U)$ are smooth vector fields in U and $w^1 \ldots \ldots \ldots w^l \in \mathcal{T}^1(M)$ are smooth covector fields in U, then the function

$$p \mapsto \sigma\left(X_{1,} \ldots \ldots, X_k, w^1 \ldots \ldots \ldots w^l\right) \qquad (1.94)$$

is smooth.

1.10 Differential Forms

1.10.1 Definition: (A **1-form)**

Let $T^{p,q}(M) = \bigcup_{x\in M} T^{p,q}(T_xM)$. A **1-form** on M is a function $\alpha : M \to T^{0,1}(M)$ such that $\alpha_x \in T^{0,1}(T_xM)$ and (for any $Y \in \Gamma(\text{TM})$) the function $\alpha(Y)$ given by $\alpha(Y)(x) = \alpha_x(Y_x)$ is in $C^\infty(M)$.

A **tensor field** of type (p,q) on M is a function $S: M \to T^{p,q}(M)$ such that $S_x \in T^{p,q}(T_xM)$ and (for any 1-forms $\alpha_1, \dots, \alpha_p$ and vector fields $Y_1, \dots, Y_q$ on M) the function $S(\alpha_1, \dots, \alpha_p, Y_1, \dots, Y_q)$ given by

$$S\left(\alpha_1, \dots, \alpha_p, Y_1, \dots, Y_q\right)(x) = S\left(\alpha_{1x}, \dots, \alpha_{px}, Y_{1x}, \dots, Y_{qx}\right) \quad (1.95)$$

is in $C^\infty(M)$. The space of all tensor fields of type (p, q) on M is denoted by $\mathcal{J}^{p,q}(M)$.

1.10.2 Definition: (A **k-form)**

A **k-form** on M is a tensor field $\omega \in \mathcal{J}^{0,k}(M)$ such that $\omega_x \in \Lambda^k(T_xM)$. The space of k-form on M is denoted by $\Lambda^k(M)$. For $\alpha \in \Lambda^i(M)$ and $\beta \in \Lambda^j(M)$ we define $\alpha \wedge \beta \in \Lambda^{i+j}(M)$ by

$$(\alpha \wedge \beta)_x = \alpha_x \wedge \beta_x \text{ . If } \varphi: U \to \mathbb{R}^n \quad (1.96)$$

is a chart $\varphi = (x^1, \dots, x^n)(\, x^i \in C^\infty(U))$ then $dx^1, \dots, dx^n$ are defined to be those 1-forms on U with $x^i\left(\partial_j\right) = \delta^i_j$.

Any $\omega \in \Lambda^k(M)$ can be written on U as

$$\omega = \frac{1}{k!}\sum \omega_{i_1\dots i_k}\, dx^{i_1} \wedge \dots \wedge dx^{i_k} \quad (1.97)$$

Where $\omega_{i_1\dots i_k} = \omega\left(\partial_{i_1}, \dots, \partial_{i_k}\right) \in C^\infty(U)$.

1.10.3 Definition

If $f \in C^{\infty}(M)$, then $df \in \Lambda^1(M)$ is defined by $df(Y) = Y[f]$ for arbitrary $Y \in \Gamma(TM)$. For $\omega \in \Lambda^k(M)$, we define $d\omega$ to be the $(k+1)$ −form that when restricted to U is given by

$$d\omega = \frac{1}{k!}\sum d(\omega_{i_1 \dots i_k}) \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

$$= \frac{1}{k!}\sum \partial_i[\omega_{i_1 \dots i_k}]\, dx^{i_1} \wedge \dots \wedge dx^{i_k}. \tag{1.98}$$

We can prove that, as define, is independent of the choice of coordinates. In fact, $d\omega$ can be defined (without reference to coordinates) as that $(k+1)$ −form such that for any $X_1, \dots, X_{k+1} \in \Gamma(TM)$ we have

$$d\omega(X_1, \dots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} X_i\,[\omega(X_1, \dots, \hat{X}_i, \dots, X_{k+1})]$$

$$+ \sum_{1 \le i < j \le n} (-1)^{i+j}\, \omega([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{k+1}) \tag{1.99}$$

where the circumflex means that symbol beneath it is to be omitted. The operator $d: \Lambda^k(M) \to \Lambda^{k+1}(M)$ called **exterior differentiation**.

If $\alpha \in \Lambda^i(M)$ and $\beta \in \Lambda^j(M)$, then (from the coordinate definition) we easily obtain $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^i \alpha \wedge d\beta$, and

$$d^2 \equiv d \circ d = 0. \tag{1.100}$$

1.4.6 Definition

If $f: M \to N$ is a map and $\omega \in \Lambda^k(N)$, then the **pull-back** $f * \omega \in \Lambda^k(M)$ is defined by

$(f * \omega)_x(Y_1, \dots, Y_k) = \omega_{f(x)}(f_{*x}Y_1, \dots, f_{*x}Y_k)$for $Y_1, \dots, Y_k \in T_xM$. (1.101)

When

$$k = 0, f * \omega \equiv \omega^{\circ} f \in C^{\infty}(M). \tag{1.102}$$

It can be proved that

$$df * \omega = f * d\omega, f * (\alpha \wedge \beta) = (f * \alpha) \wedge (f * \beta) \qquad (1.103)$$

And

$$(f^\circ g) * \omega = g * f * \omega. \qquad (1.104)$$

1.10.4 Definition

In order to integrate forms, we introduce some topological notions. A subset $W \subset M$ is **closed** if its complement $W^c \equiv \{x \in M | x \notin W\}$ is open

1.11 Differential Calculus

Let k be a commutative ring with unit and A a commutative and associative algebra over k having 1 as its element. In Applications, k will usually be the real number field and A the algebra of differentiable functions on a manifold.

1.11.1 Definition

A *derivation* X is a map $X : A \rightarrow A$ such that

(i) $X \in Homk(A, A)$, and
(ii) $X(ab) = (Xa)b + a(Xb)$ for every $a, b \in A$

If no non-zero element in k annihilates A, k can be identified with a subalgebra of A and with this identification we have $X_x = 0$ for every $x \in k$. In fact, we have only to take $a = b = 1$ in (*ii*) to get $X_1 = 0$ and consequently $X_x = X_1 = 0$.

We shall denote the set of derivations by C. Then C is obviously an A-module with the following operations:

(i) $(X + Y)(a) = Xa + Ya$
(ii) $(aX)(b) = a(Xb)\ for\ a, b \in A\ and\ X, Y \in C.$

We have actually something more: If $X, Y, \in C$, then $[X, Y] \in C$.

1.11.2 The Properties of Bracket Product

This bracket product has the following properties:

(i) $[X_1 + X_2, Y] = [X_2, Y] + [X_2, Y]$

(ii) $[X, Y] = -[Y, X]$

(iii) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0,$

for $X, Y, Z \in C$. The bracket is not bilinear over A, but only over k. We have

$$[X, aY](b) = \{X(aY) - (aY)(X)\}(b) = (Xa)(Yb) + a[X, Y](b) \quad (1.105)$$

so that

$$[X, aY] = (Xa)Y + a[X, Y] \; for \; X, Y \in C, a \in A. \quad (1.106)$$

The skew commutativity of the bracket gives

$$[aX, Y] = -(Ya)X + a[X, Y] \quad (1.107)$$

When A is the algebra of differentiable functions on a manifold, C is the space of differentiable vector fields.

Chapter Two

Twistor Theory

2 .1 Spinors

The machinery of twistor theory is best presented in terms of spinors. These can be regarded as the square root of Minkowski geometry. Indeed they lie in the fundamental representation of $\mathbb{SL}(2,\mathbb{C})$, a double cover of the proper orthochronous Lorentz group. Much as the introduction of the imaginary unit i simplifies and clarifies elementary algebra, the language of spinors allows a unified treatment of physical theories.

We begin by demonstrating the fundamental isomorphism identifying Hermitian spinors with real vectors. This immediately extends to a dictionary between real tensors and higher valence spinors, which we use liberally. Simple algebraic properties of spinors are developed rigorously, including the definition of a covariant derivative on a spinor field.

We rewrite physical field equations in spinor language, to facilitate their solution by twistor methods.

2.1.1 Definition: (A Minkowski Space-Time M)

A Minkowski space-time M is a four-dimensional real manifold R^4 with line element given by the following expression:

$$ds^2 = \eta_{ab} dx^a dx^b = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \quad (2.1)$$

where $\eta_{ab} = diag(+1, -1, -1, -1)$ is the Minkowski metric. Here $x^0 = ct$ denotes the temporal coordinate, with c the speed of light, the remaining coordinates (x^1, x^2, x^3) represent spatial coordinates. The indices a, b assume the values 0, 1, 2, and 3 in this formula. Throughout this thesis, we will use Einstein's convention where summation is assumed on repeated indices.

2.1.2 The Light Cone Structure

Each point in Minkowski space-time can be characterized by four coordinates with respect to an arbitrary origin (x^0, x^1, x^2, x^3). Such a point is called an event in space-time. To each event we can associate a corresponding light-cone given by the vanishing of the form ds^2 in (2.1). This surface determines three regions of interest in space-time:

(i) The interior of the cone, characterized by $ds^2 > 0$. This inequality implies that the interior of the cone is causal; since the speed of propagation is less than c. Vectors joining the event E with points in the interior of the light-cone are called time-like vectors. The upper half of the cone is called future light-cone, and the lower half is called past light-cone.

(ii) The surface of the cone, characterized by $ds^2 = 0$, where the speed of propagation is equal to c. Vectors joining the event E with points on the surface of the cone are called null vectors, of length equal to zero.

(iii) The exterior of the cone, characterized by $ds^2 < 0$, this inequality implies that the exterior of the cone is acausal, due to the speed of propagation being greater than c. Vectors joining E with points outside of the cone are called space-like vectors.

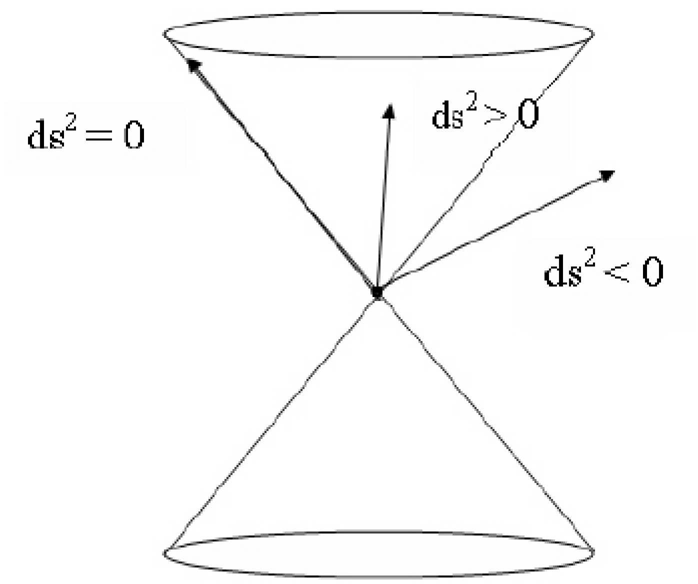


Figure 11: The light-cone associated to an event E in Minkowski space-time.

We should mention that the meaning of the inequalities defining these regions depends on the signature chosen. Here we will work with a signature (+ - - -). In a signature (- + ++), space-like vectors are characterized by $ds^2 > 0$.

2.1.3 Definition: (Lorentz Transformation)

A Lorentz transformation Λ^a_b is a linear transformation of M that preserves the metric η_{ab}:

$$\Lambda^a_c \Lambda^b_d \eta_{ab} = \eta_{cd} \tag{2.2}$$

or, in matrix notation

$$\Lambda^T \eta \Lambda = \eta \tag{2.3}$$

2.1.4 The Lorentz Group $L = \mathbb{O}(1, 3)$

The Lorentz group $L = \mathbb{O}(1\,,3)$ is the group of all such linear transformations.

2.1.5 Note

From (2.3) we have

$$(det\Lambda)^2 = 1 \text{ or } det\Lambda = \pm 1 \tag{2.4}$$

The Lorentz group is not connected, having four components. We are particularly interested in the one that contains the identity and preserves the time orientation, denoted $L^{\uparrow}_{+}$: Here + denotes the sign of the determinant preserving the overall orientation, and ↑ means that $\Lambda^0_0 > 0$, which preserves the time orientation. $L^{\uparrow}_{+}$ is doubly covered by $\mathbb{SO}(1\,,3)$

2.1.6 The Spin Space

In the Minkowski space M, consider a vector $V^a = (V^0, V^1, V^2, V^3)$ (in some orthonormal frame). We use here the abstract index notation introduced by Penrose, where the index a merely indicates the type of quantity (vector, form, etc.) rather than assuming numerical values.

To each such vector one can associate by a one-to-one correspondence a Hermitian matrix as follows:

$$f : \mathbb{M} \to \mathcal{M}_2(\mathbb{C})$$

$$f(V^a) = V^{A\acute{A}} = \frac{1}{\sqrt{2}}\begin{bmatrix} v^0 + v^3 & v^1 + iv^3 \\ v^1 - iv^2 & v^0 - v^3 \end{bmatrix} \quad (2.5)$$

where the matrix $V^{A\acute{A}}$ can be written also as:

$$V^{A\acute{A}} = \begin{pmatrix} V^{0\acute{0}} & V^{0\acute{1}} \\ V^{1\acute{0}} & V^{1\acute{1}} \end{pmatrix}: \quad (2.6)$$

The spinor indices A , $\acute{A}$ take the values 0 , 1 and $\acute{0}$, $\acute{1}$, respectively, and the prime stands for complex conjugation.

The determinant of the matrix $f(V^a)$ is half the length of the vector V^a:

$$\det f(V^a) = \frac{1}{2}[(V^0)^2 - (V^1)^2 - (V^2)^2 - (V^3)^2] = \frac{1}{2}\eta_{ab}V^aV^b \quad (2.7)$$

2.1.7 Definition

We define spinor space to be a 2-dimensional complex vector space S with elements α^A where A = 0, 1. These are called spinors acts. $\mathbb{SL}(2,\mathbb{C})$ on S in the natural way

$$\varphi: \mathbb{SL}(2,\mathbb{C}) \times S \to S$$

$$(A,\alpha\,) \longmapsto A\alpha \quad (2.8)$$

2.1.8 Definition

We define conjugate spinor space to be the 2-dimensional complex vector space $\acute{S}$ consisting of the complex conjugates of elements of S. The elements are also called spinors but are written $\beta^{\acute{A}}$ to distinguish them from elements of S. $\mathbb{SL}(2,\mathbb{C})$ acts on $\acute{S}$ according to

$$\varphi: \mathbb{SL}(2,\mathbb{C}) \times \acute{S} \to \acute{S}$$

$$(A,\beta) \longmapsto \bar{A}\alpha \quad (2.9)$$

2.1.9 Definition:

Let M_B^A be an element of $\mathbb{SL}(2,\mathbb{C})$, and $\bar{M}_{\acute{B}}^{\acute{A}}$ its Hermitian conjugate. We can define a linear transformation of the vector V^a by

$$V^{a} \longmapsto V^{A\acute{A}} \mapsto M^{A}_{B}\, V^{B\acute{B}} \overline{M}^{\acute{A}}_{\acute{B}} \tag{2.10}$$

If the vector V^{a} is null and future-pointing, the rank of $f(V^{a})$ becomes equal to one. In this case $V^{A\acute{A}}$ can be factored as:

$$V^{A\acute{A}} = \alpha^{A}\bar{\alpha}^{\acute{A}} \tag{2.11}$$

where α^{A} is a complex two-dimensional vector, and $\bar{\alpha}^{\acute{A}}$ is its complex conjugate:

$$\alpha^{A} = \begin{bmatrix}\alpha^{0}\\ \alpha^{1}\end{bmatrix} \quad \text{and} \quad \bar{\alpha}^{\acute{A}} = [\bar{\alpha}^{\acute{0}} \quad \bar{\alpha}^{\acute{1}}] \tag{2.12}$$

The vectors α^{A} determine a complex two-dimensional vector space S on which $\mathbb{SL}(2,\mathbb{C})$ acts, called spin space.

2.1.10 Definition:

The following spaces can also be defined

(i) $\bar{S} = \acute{S}$: the complex conjugate spin space with elements $\beta^{\acute{A}}$

(ii) S^{*}: the dual spin space with elements γ_{A}

(iii) $\acute{S}^{*}$: the dual of the complex conjugate spin space, with elements $\delta_{\acute{A}}$

2.1.11 Properties of Spinor

1. Note that the spinors in (2.12) have valence one. Higher valence spinors can be obtained by considering tensor products of the spin spaces defined above S , $\acute{S}$, S^{*} an $\acute{S}^{*}$:

$$\Phi^{\overbrace{A\ldots\ldots B}^{}_{k_1}\ \overbrace{A'\ldots C'}^{}_{k_2}}_{\underbrace{E\ldots F}_{k_3}\ \underbrace{E'\ldots F'}_{k_4}} \in \left(\underset{k_1}{\otimes} S\right) \otimes \left(\underset{k2}{\otimes} S'\right) \otimes \left(\underset{k_3}{\otimes} S^{*}\right) \otimes \left(\underset{k_4}{\otimes} S'^{*}\right) \tag{2.13}$$

where we used the notation $\underset{k_1}{\otimes} S$ to mean $\underbrace{S \otimes \ldots \otimes S}_{k_1}$

2. In our discussion of the five-dimensional conformal algebra we will use the concepts and properties of symmetric and antisymmetric spinors. For a spinor S of valence n we have:

$$S^{(A.....B)} = \frac{1}{n!}\sum_{\sigma} S^{\sigma(A)........\sigma(B)} \qquad (2.14)$$

and

$$S^{[A.....B]} = \frac{1}{n!}\sum_{\sigma} sign(\sigma) S^{\sigma(A)........\sigma(B)} \qquad (2.15)$$

where the sum is on all permutations σ and $sign(\sigma) = \pm 1$, depending on whether σ is an odd or an even permutation. These results hold for both primed and unprimed indices.

3. Symmetric spinors factorize into outer products of spinors of valence one:

$$S_{(A.......B)} = \propto_A \ldots . \beta_B \qquad (2.16)$$

The spinors $\propto_A \ldots . \beta_B$are called the principal null directions of the spinor $S(p, n, d, s)$. This is a significant simplification of spinor calculus. We will see shortly that antisymetric spinors simplify as well.

4. In a two-dimensional space, any completely skew quantity with more than two indices is identically equal to zero. There is thus a unique completely skew two index spinor (up to complex multiples), denoted ϵ_{AB}. This spinor is preserved by SL(2,C), much in the way the metric η_{ab} is preserved by the Lorentz transformations in (2.2):

$$M_A^B M_C^D \epsilon_{AB} = \epsilon_{AC} \qquad (2.17)$$

for any $M_B^A \in \mathbb{SL}(2, \mathbb{C})$. It follows that each spin space has such a spinor attached, and whether we mention it explicitly or not, by S we will generally mean the pair (S, ϵ_{AB}).

5. The spaces (S, ϵ_{AB}) and $(\acute{S}, \epsilon_{\acute{A}\acute{B}})$ are related by an anti-isomorphism called complex conjugation. It is usually denoted by an overbar:

$$\alpha^{A} \in S \Longrightarrow \overline{\alpha^{A}} = \bar{\alpha}^{\acute{A}} \in \acute{S};$$
$$\alpha^{\acute{A}} \in \acute{S} \Longrightarrow \overline{\alpha^{\acute{A}}} = \bar{\alpha}^{A} \in S \tag{2.18}$$

This extends to higher valence spinors as well, for example:

$$\overline{\alpha^{AB\acute{C}\acute{D}}} = \bar{\alpha}^{A'B'CD'} \tag{2.19}$$

6. We should remark here that if ϵ_{AB} is chosen such that $\epsilon_{01} = 1$ in some basis of S, we can write:

$$\epsilon_{AB} = \epsilon^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \bar{\epsilon}_{\acute{A}\acute{B}} \, \bar{\epsilon}^{\acute{A}\acute{B}} \tag{2.20}$$

7. By convention, primed and unprimed indices can be commuted:

$$T_{A'B'CD'} = T_{CA'B'} = T_{A'CB'} \; : \tag{2.21}$$

In general, the order among primed (unprimed) indices matters:

$$T_{A'B'C} \neq T_{B'A'C} \tag{2.22}$$

8. Similar to the use of the metric η_{ab} to raise and lower indices in Minkowski space, the spinor ϵ_{AB} provides an isomorphism between the spin-space S and its dual S^{*} by raising and lowering indices of spinors. Since ϵ_{AB} is skew, one must be very careful when performing these operations; the adjacent indices must be descending to the right in order to avoid introducing a sign change. For example:

$$\epsilon^{AB}\alpha_{B} = \alpha^{A},$$
$$\beta^{B}\epsilon_{AB} = -\beta^{B}\epsilon_{BA} = -\beta_{A} \tag{2.23}$$

Likewise, $\epsilon_{\acute{A}\acute{B}}$ and $\epsilon^{\acute{A}\acute{B}}$ raise and lower indices in the complex conjugate space $\acute{S}$ and its dual $\acute{S}^{*}$:

$$\epsilon^{\acute{A}\acute{B}}\gamma_{\acute{B}} = \gamma^{\acute{A}} \in \acute{S} \; ;$$
$$\rho^{\acute{B}}\,\epsilon_{\acute{A}\acute{B}} = -\rho^{\acute{B}}\,\epsilon_{\acute{A}\acute{B}} = -\rho_{\acute{A}} \in \acute{S}^{*} \tag{2.24}$$

9. Some important identities satisfied by the ϵ_{AB} spinor are:

$$\epsilon^{AB}\alpha_{CB} = \delta_C^A, \text{ and } \quad \epsilon_{AB}\epsilon^{CB} = \delta_A^C \tag{2.25}$$

where δ_A^C is the spinor Kronecker delta, satisfying:

$$\delta_A^B = \epsilon_A^B = -\epsilon_A^B \tag{2.26}$$

We also have:

$$\epsilon_{A[B^{\epsilon}CD]} = 0 \tag{2.27}$$

and

$$\epsilon_{AB}\epsilon^{CD} = \delta_A^C\delta_B^D - \delta_A^D\delta_B^C \tag{2.28}$$

These relations lead to

$$\epsilon_A^A = 2 \tag{2.29}$$

10 All spinors α^A are null with respect to ϵ_{AB}, in the sense that

$$\epsilon_{AB}\alpha^A\alpha^B = \alpha_B\alpha^B = 0: \tag{2.30}$$

The complex conjugate relation holds as well.

11. A Hermitian spinor is a spinor with equal number of primed and unprimed indices such that the spinor and its complex conjugate are the same:

$$\overline{\alpha_{AB\,C'D'}} = \bar{\alpha}_{\,A'B'CD} = \alpha_{\,A'B'CD} \tag{2.31}$$

Note

the skew spinor ϵ_{AB} is Hermitian. The Hermitian spinor ϵ_{AB}, $\epsilon_{\acute{A}\acute{B}}$corresponds in fact to the metric η_{ab}:

$$\eta_{ab} = \epsilon_{AB}\epsilon_{\acute{A}\acute{B}} \tag{2.32}$$

12. The correspondence between Hermitian spinors and tensors can be made rigorous by means of the Infeld-van der Waerden symbols, which establish a one-to-one correspondence between Hermitian spinors with n primed and n unprimed indices, and tensors of

valence n; in this process each tensor index a is replaced by a pair of spinor indices $A\acute{A}$. For example, the correspondence between a vector V^a and a spinor $V^{A\acute{A}}$ is given by

$$V^{A\acute{A}} \equiv V^a \sigma_a^{A\acute{A}}$$
$$V^a \equiv V^{A\acute{A}} \sigma_{A\acute{A}}^a : \qquad (2.33)$$

For more properties of the mixed spinor-tensor symbols $\sigma_a^{A\acute{A}}$. For simplicity, we will omit writing these symbols for the remaining of this thesis.

13. We mentioned in property (3) that antisymmetric spinors simplify. They do so with the help of the skew tensor ϵ_{AB}, as follows: a skew pair of indices can be removed as an ϵ spinor with a contraction on the removed indices:

$$S_{\ldots[AB]\ldots} = \frac{1}{2}\epsilon_{AB} S_{\ldots C} \;{}^{C}{}_{\ldots}. \qquad (2.34)$$

From this point of view, any spinor can be reduced to a combination of ϵ spinors and symmetric spinors. The same property holds for complex conjugate spinors as well. This, together with property (3), and the fact that spinor indices only take two values, shows that spinor calculus is much simpler than tensor calculus.

14. An example of interest that will be used in section 3.3 is a valence two skew tensor, S_{ab}. Such a tensor can be written as:

$$S_{ab} = S_{AA'BB'} = S_{ABA'B'} = S_{AB}\,\epsilon_{A'B'} + \bar{S}_{A'B'}\,\epsilon_{AB} \qquad (2.35)$$

where $S_{AB}$and $S_{A'B'}$ are symmetric spinors, called the anti-self-dual (a.s.d.) and self-dual (s.d.) parts of S_{ab}, respectively, satisfying:

$${}^{*}Tab = -iT_{ab} \text{ and } T_{ab} = S_{AB}\,\epsilon_{A'B'} \qquad (2.36)$$

And

$$ {}^{*}Tab = iT_{ab} \quad \text{and} \quad T_{ab} = \bar{S}_{\acute{A}\acute{B}}\epsilon_{AB} \tag{2.33}$$

In a Lorentzian space-time, S_{AB} and $\bar{S}_{\acute{A}\acute{B}}$ are related by the complex conjugation anti-isomorphism. In general, a complex space-time and a four complex-dimensional Riemannian manifold cannot be distinguished, which allows the following property to bevalid in both types of spaces. The arena for twistors, as it will be shown soon, is a complexified compactified Minkowski space-time. One can define an operation of complex conjugation in complexified space- times, but this map is not invariant under general holomorphic coordinate transformations in a complex space. In this case, a real quantity is replaced by its complex conjugate, but a pair of complex conjugate quantities $(\rho, \tilde{\rho})$ is replaced by independent complex quantities $(\rho, \tilde{\rho})$.

15. The dual of a skew two-tensor S_{ab} is given by:

$$ {}^{*}S_{ab} = \frac{1}{2}\varepsilon_{ab}{}^{cd}S_{cd} \tag{2.37}$$

where ε_{abcd} is a completely skew four-tensor. The spinor version of ε_{abcd} is:

$$\varepsilon_{abcd} = \epsilon_{A\acute{A}B\acute{B}C\acute{C}D\dot{D}} = \epsilon_{ABCD\acute{A}\acute{B}\acute{C}\dot{D}} \tag{2.38}$$

which can be simplified by using property (13) as

$$\varepsilon_{abcd} = i(\epsilon_{AC}\epsilon_{BD}\epsilon_{\acute{A}\acute{B}}\epsilon_{\acute{C}\dot{D}} - \epsilon_{AD}\epsilon_{BC}\,\epsilon_{\acute{A}\dot{D}}\epsilon_{\acute{B}\dot{C}}) \tag{2.39}$$

Raising the last two indices, we obtain:

$$\varepsilon_{ab}^{cd} = i\left(\delta_A^C\delta_B^D\delta_{\acute{A}}^{\acute{D}}\delta_{\acute{B}}^{\acute{C}} - \delta_A^D\delta_B^C\delta_{\acute{A}}^{\acute{C}}\delta_{\acute{B}}^{\acute{D}}\right) \tag{2.40}$$

which, used in (2.37), leads to:

$$ {}^{*}S_{ab} = -iS_{AB}\epsilon_{\acute{A}\acute{B}} + S_{\acute{A}\acute{B}}\epsilon_{AB} : \tag{2.41}$$

16. We end this section by introducing a brief description of the spinor connection. A spinor field α^A defines a null a.s.d. skew vector (with a sign ambiguity)

$$F_{ab} = F_{AB}\epsilon_{A'B'} + \bar{F}_{A'B'}\epsilon_{AB} \tag{2.42}$$

where F_{AB}; and $\bar{F}_{A'B'}$ are symmetric. By using property (3) we can factorize both spinors and write:

$$F_{ab} = \alpha_A\alpha_B\epsilon_{A'B'} + \bar{\alpha}_{\acute{A}}\bar{\alpha}_B\,\epsilon_{AB} \tag{2.43}$$

The Levi-Civita connection ∇_a of the Minkowski space M extends uniquely for null a.s.d. skew two-vectors to define a connection $\nabla_{A\acute{A}}$ on the spin bundles, provided:

$$\nabla_{A\acute{A}}\,\epsilon_{BC} = 0 = \nabla_{A\acute{A}}\,\epsilon_{B'C'} \tag{2.44}$$

All these properties seem to point to the fact that spinor calculus is indeed much simpler than tensor calculus.

2.2 The Conformal Group $C(1,\ 3)$

One of the main features of twistor theory is that it is a conformal theory. This section shows that the conformal character arises naturally in spinor calculus, and consequently, becomes a natural part of twistor theory.

2.2.1 The Conformal Map

A conformal map is a map of the Minkowski space- time M to itself which preserves its conformal structure, that is sends the metric g_{ab} to

$$\tilde{g}_{ab} = \Omega^2 g_{ab} \tag{2.45}$$

for some nowhere zero smooth function Ω:

We should mention here that $(\mathbb{M}, g_{ab})$ and $(\mathbb{M}, \tilde{g}_{ab})$ have identical causal structures if and only if g_{ab} and $\tilde{g}_{ab}$ are related by a conformal transformation. The conformal structure of a space-time is in fact the null cone structure of that space-time.

In addition to all the spinor quantities defined in the previous section, one natural step in constructing the spinor calculus is to find the analogue of the Lie derivative from tensor calculus, that is to find an expression for the Lie derivative of a spinor α^A in the direction of a vector field X^a.

It can be shown that this is possible only for conformal Killing vectors X which satisfy:

$$\mathcal{L}_X \, g_{ab} = k g_{ab}, \qquad (2.46)$$

for constant k, and indices a, b = 0 , 1, 2, 3. Here $\mathcal{L}_X$ denotes the Lie derivative in the direction of the vector X.
(2.46) can be written as:

$$\nabla_{(a}X_{b)} = \frac{1}{2} k g_{ab} \qquad (2.47)$$

with general solution of the form:

$$X_a = p_a - M_{ab}x^b + Dx_a + [2(q \, . x)x_a - q_a(x \, . x)] \qquad (2.48)$$

where p_a , $M_{ab} = -M_{ba}$, D and q_a are constants of integration.

The Killing vectors generate the conformal group $C(1,3)$. From (2.48) we can see that $C(1,3)$ is fifteen-dimensional, depending on the following parameters:

(i) Ten of them, p_a and M_{ab}, generate the Poincaré group which is given by the semidirect sum of the translations $p_a$and the Lorentz transformations M_{ab}:

$$x_a \longmapsto M_{ba}x^b + p_a = -M_{ab}x^b + p_a: \qquad (2.49)$$

The Lorentz transformations preserve the metric g_{ab}, the translations p_a act on x_a as:

$$x_a \longmapsto x_a + \xi_a, \tag{2.50}$$

where ξ_a is a constant.

As the full symmetry group of relativistic field theories, the representations of the Poincaré group describe all elementary particles and is therefore of major importance.

(ii) D defines a dilation, sending

$$x_a \longmapsto \rho x_a \tag{2.51}$$

for $\rho > 0$;

(iii) Four of them, q_a, define the special conformal transformations. If the meaning of p_a, M_{ab} and D is obvious, that is not the case with the special conformal transformations. To determine their significance, set all the parameters equal to zero, except q_a, in (2.48). We obtain the equation:

$$X_a = \frac{\partial x_a}{\partial s} = 2(q \cdot x)x_a - q_a(x \cdot x) \tag{2.52}$$

with solutions:

$$x_a(s) = \frac{x_a(0) - s q_a \Delta(0)}{1 - 2s\big(q \cdot x(0)\big) + s^2 (q \cdot q)\Delta(0)} \tag{2.53}$$

where $\Delta = x_a x^a = x \cdot x$.

2.2.2 Note

We obtain infinite values of X_a at the zeros of the quadratic denominator. This suggests introducing some points at infinity in Minkowski space, thus compactifying it. The role of the special conformal transformations is to interchange the points at infinity with finite points of $\mathbb{M}$.

To describe the points at infinity, one considers first a six-dimensional real manifold with a flat metric of signature $(2, 4)$ which in coordinates (t, v, w, x, y, z) has the form:

$$ds^2 = dt^2 + dv^2 - dw^2 - dx^2 - dy^2 - dz^2 \tag{2.54}$$

The $\mathbb{O}(2,\ 4)$ null cone is then given by:

$$t^{2} + v^{2} - w^{2} - x^{2} - y^{2} - z^{2} = 0: \tag{2.55}$$

The group $\mathbb{O}(2, 4)$ preserves the form (2.54) and is 2-1 isomorphic to the conformal group $C(1,\ 3)$:

The compactified Minkowski space $\mathbb{M}^{c}$ consists of $\mathbb{M}$ with a null cone at infinity, and the special conformal transformations interchange this cone with the null cone of the origin.

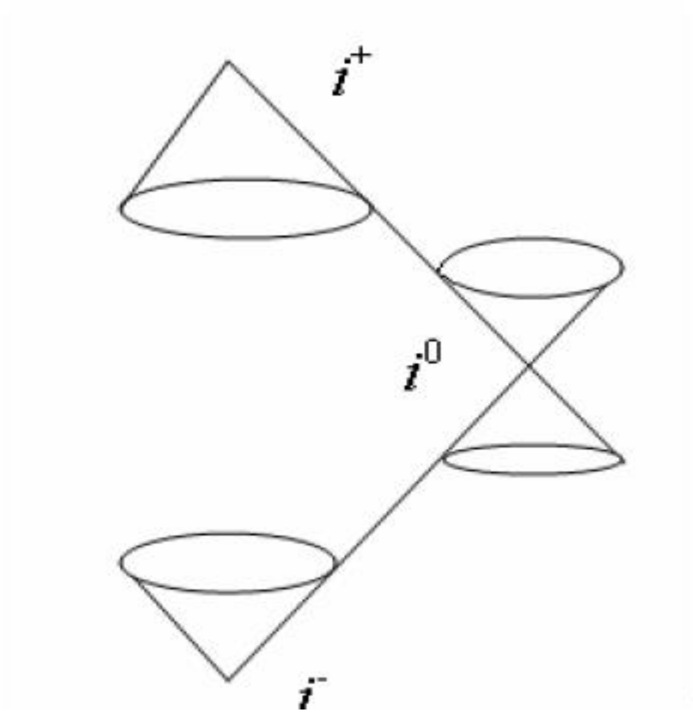


Figure 12: The null cones of the origin and infinity.

Although we started this section with the apparent goal of defining a spinor Lie derivative, the real purpose was to show that the conformal group arises naturally in spinor theory. Like the Lorentz group, the conformal group is not connected either. The component of interest is the one that contains the identity, denoted by $C_{+}^{\uparrow}(1,\ 3)$, doubly covered by $\mathbb{SO}(2,\ 4)$.

2.3 Elements of Twistor Theory

2.3.1 The Concept of A twistor

There are many ways to visualize a twistor:

(i) **Geometrically**, a (null) twistor can be described as an entire light ray (the "life" of a photon: its past, present, and future). A space-time event E will then be thought of as the family of light rays that pass through E, with an S^2 topology. This family of light rays is called a celestial sphere.

(ii) Twistors can also be defined in terms of physical quantities characterizing the classical system of zero-rest-mass, such as (null) momentum P^a, and angular momentum M_{ab}. In this approach, twistors transform in a natural way under the group $\mathbb{SU}\ (2,2)$, and in particular under the Poincaré group. Twistors can also be defined as elements of the natural representation space C^4 for $SU(2,2)$, via the following covering maps:

$$\mathbb{SU}(2,2) \xrightarrow{2:1} \mathbb{SO}(2,4) \xrightarrow{2:1} C_+^{\uparrow}(1,3) \qquad (2.60)$$

(iii) Twistors can be viewed as solutions to a diferential equation, called **twistor equation.**

(iv) From another geometric point of view, the locations of twistors can be described in terms of the geometry of a three-dimensional complex projective space, as totally null 2-surfaces, called α-planes:

2.3.2 Complexifed Minkowski Space-Time

For a complete description of twistors we will need an upgrade of the Minkowski space time, namely the complexifed compactifed Minkowski space, $\mathbb{CM}^c$. We discussed briefy the compactifcation of $\mathbb{M}$, denoted $\mathbb{M}^c$, in section (2.2)

2.3.3 Definition

$\mathbb{CM}$ is a four-dimensional complex manifold, $\mathbb{C}^4$, endowed with a non-degenerate complex bilinear form η, such that:

$$\eta(z,w) \equiv z^0w^0-z^1w^1-z^2w^2-z^3w^3=z_aw_a \qquad (2.61)$$

where z=(z^0, z^1, z^2, z^3) and $w = (w^0, w^1, w^2, w^3)$ are arbitrary four-complex dimensional vectors.

As in the real case, to each vector z^a in $\mathbb{C}^4$ we can attach a matrix $z^{A\acute{A}}$:

$$z^a \to z^{A\acute{A}} = \begin{pmatrix} z^0 + z^3 & z^1 + iz^2 \\ z^1 - iz^2 & z^0 - z^3 \end{pmatrix} \tag{2.62}$$

but this matrix is not Hermitian in general

2.4 The Twistor Equation

2.4.1 Definition The Twistor Equation

The twistor equation is a solution of a diferential equation:

$$\nabla^{\acute{A}\ (A}\zeta^{B)} = 0 \tag{2.63}$$

Here $\nabla^{\acute{A}A}$ denotes the spinor covariant derivative from equation (2.44).

Twistor theory is a conformal theory. This is derived from the fact that (2.63) is invariant under a conformal rescaling of the metric tensor, and of the epsilon spinor:

$$\tilde{g}_{ab} = \Omega^2 \mathrm{g_{ab}} \quad \text{and} \quad \tilde{\epsilon}_{AB} = \Omega\epsilon_{AB} \tag{2.64}$$

2.4.2 Solution of Twistor Equation

The general solution of (2.63), depending on the point $x \in \mathbb{CM}$, has the form:

$$\zeta^A(x) = w^A - ix^{A\acute{A}}\pi_{\acute{A}} \tag{2.65}$$

where w^A is a constant of integration, and $\pi_{\acute{A}}$ is a constant associated with this specifc solution. $\mathrm{x}^{A\acute{A}}$ is the spinor version of the position vector x^a with respect to some origin.

2.4.3 Note

The solutions ζ^A are completely determined by the four complex components of w^A and $\pi_{\acute{A}}$ in a spin-frame at the origin.

2.4.4 Difinition :The Twistor Z^α

The twistor Z^α is pair of Spinors $(w^A, \pi_{\acute{A}})$ if Z^α represented by $(w^A, \pi_{\acute{A}})$ then we can take twistor components

$$Z^\alpha = (Z^0, Z^1, Z^2, Z^3) = (w^0, w^1, \pi_{\acute{0}}, \pi_{\acute{1}}) \quad (2.66)$$

2.4.5 Difinition: a Conjugate Twistor

We define a conjugate twsitor $\bar{Z}_\alpha = (\bar{w}, \bar{\pi}^{\acute{A}})$ to have components $\bar{Z}_\alpha = (\bar{\pi}_0, \bar{\pi}_1, \bar{w}^{\acute{A}_0}, \bar{w}^{\acute{A}_1})$

2.4.6 Definition: The Twistor Space

The collection of all twistors determines a four-dimensional complex vector space, called twistor space, and denoted by T.

The four complex components of Z^α completely determine the solutions $\zeta^A(x)$. ζ^A is called the spinor field associated with the twistor Z^α.

2.4.8 Definition

A twistor is a pair of spinors related by a diferential equation, or as a nonzero four-dimensional complex vector.

2.4.9 Geometrically

The location of the twistor Z^α is given by the vanishing of the associated spinor ζ^A. This gives the equation:

$$\zeta^A(x) = 0 \implies w^A = i x^{A\acute{A}} \pi_{\acute{A}} \quad (2.67)$$

2.4.10 A Complex Conjugate Twistor Equation:

Since in spinor theory each equation is accompanied by its complex conjugate, we can also define a complex conjugate twistor equation:

$$\nabla^{A\ (A'}\varphi^{B')} = 0 \quad (2.68)$$

With solution

$$\varphi^{\acute{A}}(x) = \zeta^{\acute{A}} - ix^{\acute{A}A}\upsilon_A \quad (2.69)$$

2.4.11 Definition: The Dual Twistor Space

The pair of spinors $\left(\upsilon_A, \zeta^{\acute{A}}\right)$determines a dual twistor W_α, and the collection of all dual twistors is called the dual twistor space,T^*.

2.5 Twistor Pseudonorm

2.5.1 Definition: (The Norm of A twistor

We defne the norm of a twistor by:

$$z^\alpha \bar{z}_\alpha = \mathrm{w}^A \bar{\pi}_A + \pi_{\acute{A}} \bar{w}^{\acute{A}} = \mathrm{w}^0 \bar{\pi}_0 + \mathrm{w}^1 \bar{\pi}_1 + \pi_{\acute{0}} \bar{w}^{\acute{0}} + \pi_{\acute{1}} \bar{w}^{\acute{1}}; \quad (2.70)$$

where we used that the conjugate of z^α

2.5.2 Definition: The (pseudo) Norm

By introducing new variables $(w, x, y, z) \in z^\alpha$ via the relations

$$w^0 = w + y,\ w^1 = w + z\ ,\ \pi_{\acute{0}} = w - y, \quad \pi_{\acute{1}} = w - z \quad (2.71)$$

(2.70) becomes:

$$\tfrac{1}{2} z^\alpha \bar{z}_\alpha = w\bar{w} + x\bar{x} - y\bar{y} - z\bar{z} \quad (2.72)$$

$z^\alpha \bar{z}_\alpha$ is called the (pseudo) norm of the twistor z^α.

2.5.3 The Helicity of The Twistor z^α

The following quantity is called the helicity of the twistor z^α:

$$\sum = \frac{1}{2} z^\alpha \bar{z}_\alpha$$

2.5.4 Classification of Twistor

Based on the sign of the helicity, twistors can be classified as:

(i) Null, if $\sum = 0$. This defnes the space of null twistors, $\mathbb{N}$:
(ii) Right-handed, if $\sum > 0$. This defines the top half $\mathbb{T}^{+}$ of the twistor space $\mathbb{T}$:
(iii) Left- handed, if $\sum < 0$. This defines the bottom half $\mathbb{T}^{-}$ of $\mathbb{T}$

The case when the helicity is equal to zero is of particular interest. For a fixed twistor, w^{A} and $\pi_{\acute{A}}$ are constant spinors, equation (2.67) can then be regarded as an equation for $x^{A\acute{A}}$.

The solution of this equation is in general complex, and is given by

$$\gamma_{z}: x^{A\acute{A}} = x^{A\acute{A}}(0) + \lambda^{A}\pi^{\acute{A}} \qquad (2.73)$$

where λ^{A} is an arbitrary spinor and $x^{A\acute{A}}$ (0) is a particular solution. Since the Minkowski space is an affine space, we can adjust the origin such that the particular solution is in fact the solution at the origin.

If real solutions exist, then $x^{A\acute{A}} = \bar{x}^{A\acute{A}}$, and we obtain that:

$$z^{\alpha}\bar{z}_{\alpha} = \mathrm{w}^{\mathrm{A}}\bar{\pi}_{A} + \pi_{\acute{A}}\bar{w}^{\acute{A}} = \mathrm{i}\, \mathrm{x}^{\mathrm{A}\acute{A}}\pi_{\acute{A}}\bar{\pi}_{A} - \mathrm{i}\bar{x}^{\mathrm{A}\acute{A}}\bar{\pi}_{A}\pi_{\acute{A}} = 0 \qquad (2.74)$$

We see that real points can only exist in the region of the twistor space of zero helicity.
It can be shown that if (2.74) holds and $\pi_{\acute{A}} \neq 0$, the solution space of

(2.67) in $\mathbb{M}$ is a null geodesic for real values of r:

$$\mathrm{x}^{\mathrm{A}\acute{A}} = x^{\mathrm{A}\acute{A}}(0) + r\bar{\pi}^{A}\pi^{\acute{A}} \qquad (2.75)$$

If $\pi_{\acute{A}} = 0$, the twistor $(w^{A}, 0)$ can be regarded as a twistor at infnity, lying in the compactifcation of the Minkowski space. This twistor is denoted by $I_{\alpha\beta}$ and is represented
by the matrix:

$$I_{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon^{\acute{A}\acute{B}} \end{pmatrix} \qquad (2.78)$$

Its dual (and twistor complex conjugate) is:

$$I^{\alpha\beta} = \begin{pmatrix} \epsilon^{AB} & 0 \\ 0 & 0 \end{pmatrix} \tag{2.79}$$

This is one other way of obtaining the compactification of the complexifid Minkowski space, by adding a twistor at infinity.

The infinity twistors are objects which break the conformal invariance: the conformal group $\mathbb{SU}(2,2)$ acts on the twistor space $\approx \mathbb{C}^4/\{0\}$, but only the Poincaré group (which is a subspace of $\mathbb{SU}(2,2)$) preserves $I^{\alpha\beta}$.

2.6 $\boldsymbol{\alpha}$-Planes and $\boldsymbol{\beta}$-Planes

The locus of a twistor Z^α in $\mathbb{CM}$ is given by the region in which its associated spinor field ζ^A vanishes, leading to the equation:

$$w^A = ix^{A\acute{A}}\pi_{\acute{A}} \tag{2.80}$$

The solution of this equation is described in (2.73). Since λ^A varies, we obtain a family of vectors $x^{A\acute{A}}$ passing through $x^{A\acute{A}}$ (0). Their endpoints determine a complex two-plane with tangent vectors of the form

$$v^A = \lambda^A\pi^{\acute{A}} \tag{2.81}$$

for fixed $\pi^{\acute{A}}$ and varying λ^A.
One can easily show that these vectors are null:

$$v_a v^a = (\lambda^A\lambda_A)(\pi^{\acute{A}}\pi_{\acute{A}}) = 0 \tag{2.82}$$

And mutually orthogonal:

$$v_a w_a = (\lambda^A\mu_A)(\pi^{\acute{A}}\pi_{\acute{A}}) = 0 \tag{2.83}$$

This last relation also tells us that the metric η this complex two-plane inherits from the Minkowski space is null, since:

$$\eta(v,w) = \eta_{ab}v^a w_b = v^a w_a = 0 \tag{2.84}$$

It follows that the locus of the twistor z^{α} is a null two-plane in complexified Minkowski space. Such a plane consists of all the end points of the complex vectors $\lambda^{A}\pi^{\acute{A}}$ originating from the point $x^{A\acute{A}}$ (0), and is called an α-plane. α-planes are totally nulltwo-planes that are self-dual.

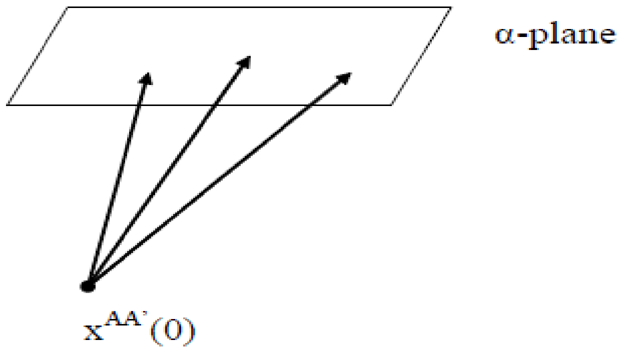


Figure 13: The α-plane is determined by the endpoints of the vectors corresponding to the solutions of the null twistor equation.

2.6.1 Solutions of the Null Twistor Equation.

Similarly, the location of a dual twistor W_{α} in $\mathbb{CM}$ is a null two-plane, called a β -plane, which has the property of being anti-self-dual. By setting $\varphi^{\acute{A}} = 0$ equal to zero, we obtain the following equation for $x^{\acute{A}A}$.

$$\zeta^{\acute{A}} = ix^{\acute{A}A}v_{A} \tag{2.85}$$

with solution

$$x^{A\acute{A}} = x_{o}{}^{A\acute{A}} + \rho^{\acute{A}}v^{A} \tag{2.86}$$

where $\rho^{\acute{A}}$ varies and v^{A} is fixed.

It is very important to note that in complex Minkowski space, there are two distinct families of totally null two-planes: the α-planes corresponding to Z^{α} twistors, and the β-planes corresponding to dual twistors W_α . This will be of interest when we discuss the interpretation of the twistor space as a quadric in CP^5.

In the case when $\pi_{\acute{A}} = 0$, there is no finite locus of the twistor Z^α. If, additionally, w^A is nonzero, then the locus of the twistor Z^α can be interpreted as a generator of the null cone at infinity.

2.7 Projective Twistor Space

We saw from equation (2.73) that a twistor $z^\alpha = (\mathrm{w}^A, \pi_{\acute{A}})$ determines an α -plane; it is obvious that a multiple of $Z^\propto$ will determine the same α -plane. Viceversa, an α –plane determines a twistor, but not uniquely, only up to a scale factor λ:

$$(\mathrm{w}^A, \pi_{\acute{A}}) \sim (\lambda \mathrm{w}^A, \lambda\pi_{\acute{A}}) \tag{2.87}$$

for $\lambda \in C^n/\{0\}$. This freedom is not a shortcoming of twistor theory, in fact it is of interest when one brings in quantum physics.

Equation (2.87) states that an α -plane is an equivalence class of twistors $[z^\propto]$, called projective twistor. The set of all such equivalence classes (α -planes) determine the projective twistor space, $\mathbb{PT}$, in which the α -planes are represented by points.

The extra information contained in the twistor space $\mathbb{T}$ compared to $\mathbb{PT}$ is the choice of scale for the spinor $\pi_{\acute{A}}$ associated to a particular α -plane.

Since the twistors $z^\propto$ are defined in $\mathbb{C}^4$ and obey the equivalence relation (2.87), it follows that the projective twistor space $\mathbb{PT}$ can be represented by a three dimensional complex projective space.

In general, we will use the notation $z^\propto$ even if we refer to the equivalence class $[z^\propto]$, but in that case the components of $z^\propto$ in (14) will be written between square brackets and referred to as "homogeneous coordinates" of the corresponding point in $\mathbb{PT}$.

Similarly , β -planes correspond to points in a dual projective twistor space, denoted $\mathbb{PT}^*$, also represented by a $\mathbb{CP}^3$.In the projective twistor space, the norm of a twistor is not well-defined any longer, but the sign of the norm can still be used to divide the projective twistor space into three regions, $\mathbb{PT}^+$, $\mathbb{PN}$ and $\mathbb{PT}^-$, corresponding to $\sum > 0$, $\sum = 0$ and $\sum < 0$ respectively.

2.8 Twistor Space and Minkowski Geometry

A twistor with $2s = Z^{\alpha}\bar{Z}_{\alpha} = 0$ represents a null real straight line (i.e. the word line of some particle of zero spin)

(i) If $S \neq 0$ there is no such real line, but there is in a certain sense a "complex line".

(ii) When $S = 0$, $Z^{\propto}$ and $\lambda Z^{\propto}$ ($\lambda \neq 0$) represent the same line so that the most directly geometrically interpretable twistor space is the space **N** of equivalence classes $\{\lambda Z^{\propto}\}$ when $S = 0$ $Z^{\propto} \neq 0$

i,e.

$$\boldsymbol{N} = \left\{\{\lambda Z^{\propto}: \lambda \neq 0, \lambda \epsilon \mathbb{C}\} \ : Z^{\propto}\bar{Z}_{\alpha} = 0,\ Z^{\propto} \neq 0\right\} \tag{36}$$

Which represents the set of null line in **M** we shall therefore consider the space $\boldsymbol{C}$ of equivalence classes of twistors, defined like **N** but without the requirement $s = 0$ (fig 14) this is complex projective three space $\mathbb{C}P(3)$ which has three complex or six real dimensions.

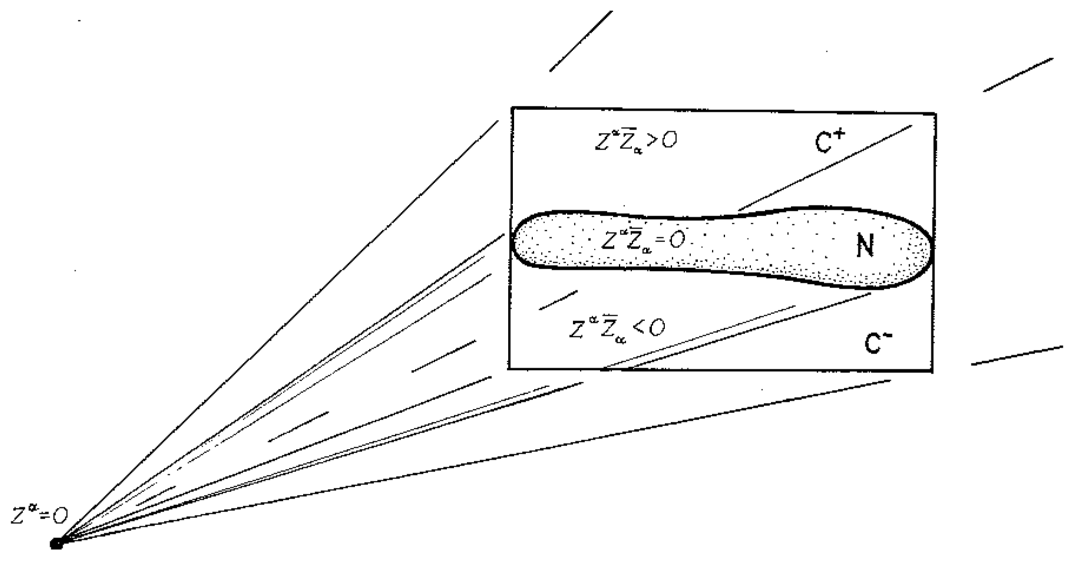


Fig (14) projection of twistor space into $\boldsymbol{C}$

It is not just the complexification of **N**, which would have ten real dominions. In fact even the complex points of $\boldsymbol{C}$ may be represented as real structure in $\boldsymbol{M}$. The conformal transformations of M correspond to proactive point transformations of $\boldsymbol{C}$ preserving $\boldsymbol{N}$.

We now consider the space of lines (projective lines) in $\mathbb{PT}$ an see the corresponding image in $\boldsymbol{M}$.

A line in $\mathbb{PT}$ is $\mathbb{CP}^1$ given by the intersection of two planes

$$Z^{\alpha}A_{\alpha} = Z^{\beta}B_{\beta} = 0 \tag{2.88}$$

Of course there is some freedom in the choice of A_{α} and B_{β}.
What is the space of these lines in $\mathbb{PT}$?

Each is determined by a skew simple (0,2) twistor $L_{\propto\beta}$. The condition for simplicity can be written

$$3L_{\propto[\beta}L_{\gamma\delta]} = L_{\propto\beta}L_{\gamma\delta} + L_{\propto\gamma}L_{\delta\beta} + L_{\propto\delta}L_{\beta\gamma} = 0 \tag{2.89}$$

which defines a quadric Q in CP^5 (called the Klein quadric). By changing the coordinates we can see that Q is actually the space of generators of the cone

$$T^2 + V^2 - W^2 - X^2 - Y^2 - Z^2 = 0 \tag{2.90}$$

In $\mathbb{C}^6$.
Here is the change of coordinates:

$$T = \frac{i}{\sqrt{2}}(L_{03} - L_{12})$$

$$V = L_{03} + \frac{1}{2}L_{01}$$

$$W = L_{23} + \frac{1}{2}L_{01}$$

$$X = \frac{i}{\sqrt{2}}(L_{02} - L_{13})$$

$$Y = \frac{-1}{\sqrt{2}}(L_{02} - L_{13})$$

$$Z = \frac{-i}{\sqrt{2}}(L_{12} - L_{02}) \tag{2.91}$$

Here is the embedding of $\mathbb{M}$ in the cone:

$$X^a \to \left(x^0, \frac{1}{2}(1 - x^b x_b), -\frac{1}{2}(1 + x^b x_b), x^1, x^2, x^3\right)$$

Thought of in R6 our cone is the $O(2,4)$ null cone of

$$ds^2 = dT^2 + dV^2 - dW^2 - dX^2 - dY^2 - dZ^2. \qquad (2.92)$$

Each of its generators (except those for which W − V = 0) meets the plane

$$W - V = 1 \qquad (2.93)$$

in a point, and the intersection of this plane and the cone is just Minkowski space $\mathbb{M}$.

So the space of generators is a compactification $\mathbb{M}^c$ of M. It is the conformal compactification: the extra generators form a null cone at infinity.

We have shown that there is a four real dimensional family of lines in $\mathbb{PT}$ corresponding to $\mathbb{M}$, but we have not so far shown how to identify them in $\mathbb{PT}$.

For a twistor Z to lie on a line L it must satisfy two linear equations. Except when the line is given by $Z^2 = Z^3 = 0$, these can be written

$$\begin{pmatrix} Z^0 \\ Z^1 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} x^0 + x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^0 - x^3 \end{pmatrix} \begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix} \qquad (2.94)$$

Where x^a is the space-time point corresponding to L. More concisely, if we write $Z^\alpha = (w^A, \pi_{\acute{A}})$ we have

$$w^A = ix^{A\acute{A}}\pi_{\acute{A}} \qquad (2.95)$$

If Z also lies on the line corresponding to $y^{A\acute{A}}$, then

$$x^{A\acute{A}}\pi_{\acute{A}} = y^{A\acute{A}}\pi_{\acute{A}} \qquad (2.96)$$

and so the matrix $x^{A\acute{A}} - y^{A\acute{A}}$ must be singular. The condition for this is that x^a and y^a are null-separated. If in

$$w^A = ix^{A\acute{A}}\pi_{\acute{A}} \tag{2.97}$$

we think of the twistor as fixed and solve for the point $x^{A\acute{A}}$ we find that

$$x^{A\acute{A}} = x_0^{A\acute{A}} + \mu^A\pi^{\acute{A}} \tag{2.98}$$

for arbitrary μ^A.

So $Z^\alpha = (w^A, \pi_{\acute{A}})$ corresponds to this alpha-plane: it is a totally null two complex dimensional plane in complex Minkowski space.

$\mathbb{PT}$		$\mathbb{CM}^c$
Complex Projective line	⟷	point
Point	⟷	alpha-plane
Intersection of lines	⟷	null-separation of points

In general an alpha-plane will have no real point, but when it does it contains a whole real null ray: if $x_0^{A\acute{A}}$ is real then so is

$$x^{A\acute{A}} = x_0^{A\acute{A}} + r\bar{\pi}^A\pi^{\acute{A}} \tag{2.99}$$

for any real r.

If Z^α is the twistor for this alpha-plane then

$$\begin{aligned} \sum(Z) &= w^A\bar{\pi}_A + \bar{w}^{\acute{A}}\pi_{\acute{A}} \\ &= Z^0\bar{Z}^2 + Z^1\bar{Z}^3 + \bar{Z}^0Z^2 + \bar{Z}^1Z^3 \\ &= 0. \end{aligned} \tag{2. 100}$$

This Hermitian form $\sum$ divides $\mathbb{PT}$ into three regions:

$\sum(Z) > 0$	⟷	$\mathbb{PT}^+$
$\sum(Z) = 0$	⟷	$\mathbb{PN}$
$\sum(Z) < 0$	⟷	$\mathbb{PT}^-$

$\mathbb{PN}$ is the space of real null rays: it is a five real dimensional manifold with a $C - R$ structure. We could imagine discovering projective twistor space this way.

If $x^{A\acute{A}}$ is real then any Z lying on the corresponding line in $\mathbb{PT}$ satisfies

$$w^A = ix^{A\acute{A}}\pi_{\acute{A}} \tag{2.101}$$

And hence has $\sum(Z) = 0$.
Thus points in real (compactified) Minkowski space correspond to lines lying entirely in $\mathbb{PN}$.
Any two twistors on a given line in $\mathbb{PN}$ represent null rays through the Corresponding point in $\mathbb{M}$.
So intrinsically the line in $\mathbb{PN}$ is the celestial sphere of the space-time point.
Lines lying entirely in $\mathbb{PT}^+$ correspond to points

$$z^{A\acute{A}} = x^{A\acute{A}} - iy^{A\acute{A}}\pi_{\acute{A}} \tag{2.102}$$

with $y^{A\acute{A}}$ timelike and future-pointing, or in other words points $z^{A\acute{A}}$ in the future tube.
This will lead later to a very elegant twistor description of positive frequency, using the fact that positive frequency fields can be characterized by having holomorphic extensions into the future tube.

2.9 Geometric Correspondences

We saw that points in $\mathbb{PT}$ correspond to α -planes, and from (2.75) we have that points in $\mathbb{PN}$ correspond to null geodesics. If an α -plane contains a real point, then it will contain the whole null geodesic given in (2.75).

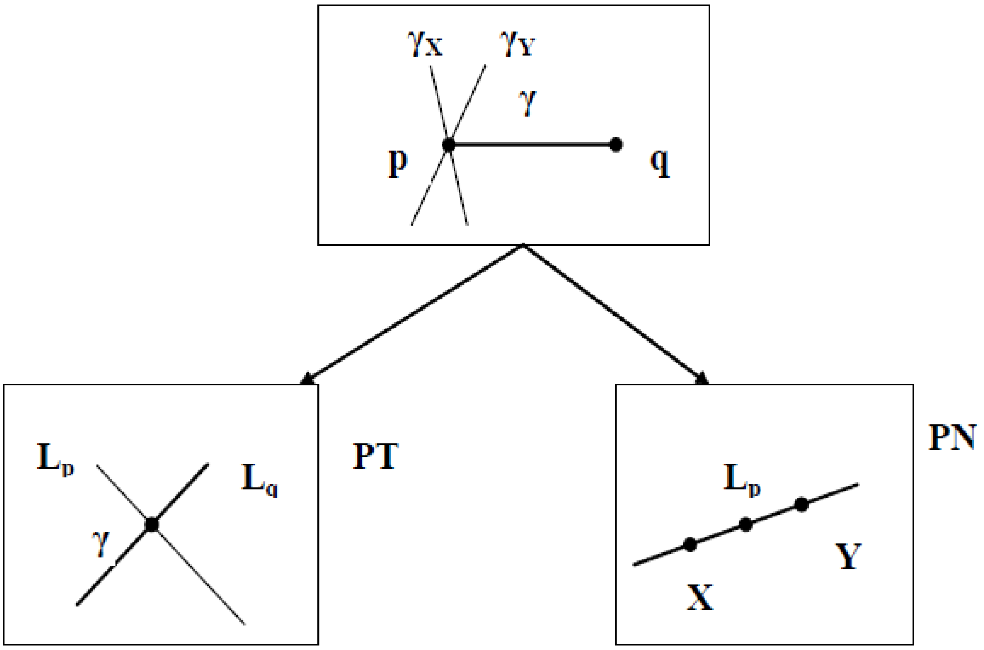


Figure 15: Geometric correspondences in the complexified Minkowskispace, PT and PN.

Figure (15) describes some of the geometric correspondences mentioned in this section: for X and Y null twistors, their corresponding null geodesics, γ_X and γ_Y, meet at the point p. The points p and q are said to be null separated if there is a null geodesic γ joining them. Each point will be represented in $\mathbb{PT}$ by a projective line (L_p and L_q), and the null geodesic γ joining p and q in $\mathbb{CM}$, becomes the intersection point of L_p and L_q in $\mathbb{PT}$. Each null twistor is represented by a point in $\mathbb{PT}$, and the point at the intersection of the null geodesics γ_X and γ_Y is represented by a line passing through the points corresponding to the two null twistors X and Y .

Other geometric correspondences can be made as follows: if we interpret (15) as an equation with $x^{A\acute{A}}$ fixed and solve for $(\mathrm{w}^{\mathrm{A}}, \pi_{\acute{\mathrm{A}}})$, we obtain that

$$w^A = ix^{A\acute{A}}\pi`_{\acute{A}} \tag{2.103}$$

with $\pi_{\acute{A}}$ arbitrary, which defines a complex two-plane.

Factorization by the equivalence relation (35) leads to a CP^1, with the two-sphere topology. The fixed space-time point x determines a Riemann sphere in $\mathbb{PT}$. If x is real, this sphere lies entirely in $\mathbb{PN}$.

We obtain that a complex space-time point corresponds to a sphere in $\mathbb{PT}$, and a real space-time point corresponds to a sphere in $\mathbb{PN}$.

2.10 Space-Time Points as Intersection of Twistors

Consider two null twistors Z_1^α and Z_2^α with their respective null geodesics, γ_{Z_1} and γ_{Z_2} defined as in (2.75). Since Z_1^α and Z_2^α are null, they satisfy

$$Z_1^\alpha \bar{Z}_{1\propto} = Z_2^\alpha \bar{Z}_{2\propto} = 0 \qquad (2.103)$$

The condition for these geodesics to meet at a point $P \epsilon$ $\mathbb{M}$ is

$$Z_1^\alpha \bar{Z}_{2\propto} = 0 \qquad (2.104)$$

This is called incidence of twistors condition.

Since real points can only exist in $\mathbb{N}$, we may define a point in the real Minkowski space $\mathbb{M}$ by the intersection of two null geodesics. From (2.103) and (2.104) it follows that any nontrivial linear combination of the null twistors Z_1^α and Z_2^α

$$Z^\alpha = \lambda Z_1^\alpha + \mu Z_1^\alpha \qquad (2.105)$$

For $(\lambda, \mu) \in C^2/(0,0)$ will also be null and will define a null geodesic, γ_z, which intersects the other two geodesics at the same intersection point, $p \in \mathbb{M}$. Since λ and μ are arbitrary, (2.105) defines a family of null geodesics intersecting at P, that is it defines the null cone of the point P.

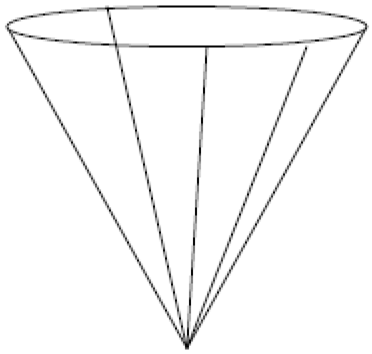


Figure 16: Points are
represented by intersections of null twistors.

This null cone is a two-dimensional subspace of the twistor space $\mathbb{T}$, lying entirely in $\mathbb{N}$, or can be thought of as a projective line L_p lying in $\mathbb{PN}$.

The family of null geodesics corresponding to the null twistor $z^{\propto}$ in (2.105), intersecting at the point P, can be interpreted as actually representing the point P.

In general, any two-dimensional subspace of $\mathbb{T}$ can be interpreted as a point in Minkowski space, but the point is not real unless Z_1^{α} and Z_2^{α} are null and orthogonal.

Consider now the lines in $\mathbb{PT}$ which do not lie entirely in $\mathbb{PN}$. An arbitrary line passing through the two points Z_1^{α} and Z_2^{α} is given by:

$$P^{\alpha\beta} = Z_1^{\alpha} Z_2^{\beta} - Z_2^{\alpha} Z_1^{\beta} \tag{2.106}$$

The point P corresponds thus (up to proportionality) to a simple skew 2-index twistor $P^{\alpha\beta}$, satisfying:

$$P^{\alpha\beta} = P^{[\alpha\beta]}\,, \qquad \text{and} \quad P^{[\alpha\beta}P^{\gamma]\delta} = 0 \tag{2.107}$$

Finally, for $P^{\alpha\beta}$ to represent a finite point of $\mathbb{M}$, it is also required that

$$P^{\alpha\beta} I_{\alpha\beta} \neq 0, \tag{2.108}$$

where $I_{\alpha\beta}$ is one of the infinity twistors defined in (2.78).

It has been shown thus that twistor geometry can be used to replace entirely the pointwise approach to the structure of space-time.

Chapter Three

Zero–Rest Mass Field Equations

3.1 The Zero-Rest Mass Equation

3.1.1 Definition: (Helicity Operator)

The helicity operator h on a particle state is defined as the projection of the spin operator s along the direction of the momentum operator p. Mathematically we write $h = (p \cdot s)/|p|$.

3.1.2 Remark.

Helicity is a good quantum number for massless fields, since we cannot boost to a frame which changes the sign of the momentum.

3.1.3 Definition: (Weyl Equations)

We define the Weyl equations for spinor field ψ_R, ψ_L on M by

$$\bar{\sigma}^{\mu}\partial_{\mu}\psi_R = 0 \quad \text{and} \quad \sigma^{\mu}\partial_{\mu}\psi_L = 0 \tag{3. 1}$$

where $\sigma^{\mu} = \left(1\,, \sigma^{i}\right)$ and $\bar{\sigma}^{\mu} = \left(1\,, -\sigma^{i}\right)$. These describe massless non-interacting fermion fields.

3.1.4 Lemma

ψ_R has helicity $+1/2$ and ψ_L has helicity $-1/2$.

Proof

Fourier transforming the first equation we obtain

$$\sigma^{i} p_i \psi_R(p) = E\psi_R(p) \tag{3.2}$$

Since $m = 0$ we have $E = |p|$ and thus

$$(\sigma \cdot p)/|p|\psi_R(p) = \psi_R(p) \tag{3.3}$$

Recall that for spin $1/2$ particles we define $S = \sigma/2$ whence

$$h\,\psi_R(p) = \frac{1}{2}\psi_R(p) \tag{3.4}$$

as required. The negative helicity case follows similarly.

3.1.5 Lemma
The Weyl equations may equivalently be written

$$\nabla_{A\acute{A}}\,\alpha^A = 0 \quad \text{and} \quad \nabla^{A\acute{A}}\beta_{\acute{A}} = 0 \tag{3.5}$$

where α^A has helicity $-1/2$ and $\beta_{\acute{A}}$ has helicity $+1/2$.

Proof
By convention we choose

$$\alpha = \psi_L \in S \quad \text{and} \quad \beta_{\acute{A}} = \psi_R \in \acute{S}^*. \tag{3.6}$$

Now recall that

$$\nabla^{A\acute{A}}\,\alpha^A = \sum \sigma^a\,\nabla^a \quad \text{and} \quad \nabla_{A\acute{A}} = \sum \sigma^a\,\nabla_a\ . \tag{3.7}$$

The result follows easily.

3.1.6 Definition: (Maxwell's Equations)
We define Maxwell's equations for a bivector field F on M by

$$dF^+ = 0 \quad \text{and} \quad dF^- = 0 \tag{3.8}$$

where F^+ is the SD and F^- the ASD part of F. These describe a masslessnon-interacting source-free electromagnetic field.

3.1.7 Remark
We note that F^+ describes a field of helicity $+1$, while F^- describes a field of helicity -1

3.1.8 Lemma.
Maxwell's equations may equivalently be written

$$\nabla^{A\acute{A}}\,\psi_{AB} = 0 \qquad \text{and} \qquad \nabla_{A\acute{A}}\,\psi_{\acute{A}\acute{B}} = 0 \tag{3.9}$$

where ψ_{AB} has helicity -1 and $\psi_{\acute{A}\acute{B}}$ has helicity +1.

Proof

An easy calculation shows that Maxwell's equations are equivalent to

$$\nabla^{a} F^{+}_{ab} = 0 \tag{3.10}$$

and

$$\nabla^{a} F^{-}_{ab} = 0 \tag{3.11}$$

Now write

$$F_{ab} = \psi_{AB} \in_{\acute{A}\acute{B}} + \psi_{\acute{A}\acute{B}} \in_{AB} \quad \text{and} \quad \nabla^{a} = \nabla^{A\acute{A}} \tag{3.12}$$

and we're done.

3.1.9 Definition: (Zero Rest Mass (ZRM) Equations)

We define the zero rest mass (ZRM) equations for symmetric valence n spinor fields $\psi_{A.....B}$ and $\psi_{\acute{A}.........\acute{B}}$ on M by

$$\begin{gathered} \nabla^{A\acute{A}}\,\psi_{A.....B} = 0 \ \text{ for helicity } -n/2 \\ \nabla^{A\acute{A}}\,\psi_{\acute{A}.........\acute{B}} = 0 \ \text{ for helicity } n/2 \\ \nabla^{A\acute{A}}\,\nabla_{A\acute{A}}\,\psi = 0 \ \text{ for helicity } 0 \end{gathered} \tag{3.13}$$

3.2 Whittaker's Formula

Ultimately we are interested in fields on space-time (solutions of some field equation -for example the wave equation) and their description as objects in the twistor space. As a first step we consider Laplace's equation in R^3 (a static solution to the wave equation), now the twistorial description is essentially a classical formula of Whittaker (1903).

The formula of Whittaker states that, up to a translation in space a (local) complex valued solution to Laplace's equation in R^3.

$$\frac{\partial^2 \varphi}{\partial {x_1}^2} + \frac{\partial^2 \varphi}{\partial {x_2}^2} + \frac{\partial^2 \varphi}{\partial {x_3}^2} = 0 \tag{3.14}$$

is given by an integral

$$\varphi(x) = \int_0^{2\pi} f(\theta, x_3 + ix_1 \cos\theta + ix_2 \ \sin\theta)d\theta \tag{3.15}$$

where $f(z,w)$ is a complex analytic function in 2-variables (with singularities away from the path of integration). Before proving this formula let us give it a different interpretation.

Set

$$q = x_1 + ix_2 \ \ , u = x_3 \ \ and \ \ \ exp\,(i\theta) = exp(i\theta) \tag{3.16}$$

to be the unit circle over which we take a contour integration. Then

$$q - 2izu + z^2\,\bar{q} = -2ie^{i\theta}(x_3 + ix_1 \cos\theta + ix_2 \sin\theta) \tag{3.17}$$

So that we may equivalently write the integral

$$\eta = \frac{1}{2}\big((x+iy) + 2z\zeta - (x+iy)\zeta^2\big) \tag{3.18}$$

(up to a modification of f) as

$$\varphi(x) = \frac{1}{2\pi i} \oint f(z, q - 2izu + z^2\,\bar{q})dz \tag{3.19}$$

We see that the 2nd argument $w = q - 2izu + z^2\bar{q}$, up to a factor of 2, is the incidence relation between a twistor (z,w) and the corresponding line in 3-space. It is therefore natural to view $f(z,w)$ as a function defined on a domain of twistor space TCP^1.

3.2.1 Note

Given a point $x \in R^3$, the set of twistors incident with x (the set of lines passing through x) form a copy of $CP^1 \subset TCP^1$ which we write $CP^1(x)$. We then consider the integration as taking place along a contour contained in $CP^1(x)$. We therefore very loosely have the correspondence:

harmonic function on a domain of $R^3 \leftrightarrow$ holomorphic function $f(z,w)$on a domain of twistor space + choice of contour.

3.2.2 Proof of Whittaker's Formula

We establish the formula (3.18). Now a solution φ to Laplace's equation $\Delta\varphi = 0$ is analytic. Let x_0 be a regular point for φ: by translation we may suppose that x_0 is the origin and we expand φ in a power series about the origin:

$$\varphi = \sum_I a_I x^I = a_0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + a_{11}(x^1)^2 + a_{12} x^1 x^2 +. \quad (3.19)$$

If we write this in homogeneous parts:

$$\varphi = Q_0 + Q_1 + Q_2 + \cdots \quad (3.20)$$

where Q_n is homogeneous of degree n, then it is easily seen that each Q_n is also harmonic.

Now in 3 variables, there are $2n + 1$ linearly independent harmonic homogeneous polynomials of degree n, e.g. $n = 1$: x ,y, $n = 2$: $xy, yz, xz, x^2 - y^2, y^2 - Z^2$. These can be generated as follows:

Consider the function of u, x^1, x^2, x^3, homogeneous of degree n in x, given by

$$(x^3 + ix^1 \cos u + ix^2 \sin u)^n = \sum_{k=0}^{n} g_k(x) \cos ku + \sum_{k=1}^{n} h_k(x) \sin ku \quad (3.21)$$

Then $g_k = g_k(x^1, x^2, x^3)$ and $h_k = h_k(x^1, x^2, x^3)$ form $(2n+1)$ linearly independent harmonic functions of degree n. By the theory of Fourier series

$$g_k(x) = \frac{1}{\pi}\int_0^{2\pi} (x^3 + ix^1 \cos u + ix^2 \sin u)^n \cos ku du \quad (3.22)$$

$$h_k(x) = \frac{1}{\pi}\int_0^{2\pi} (x^3 + ix^1 \cos u + ix^2 \sin u)^n \sin ku du \quad (3.23)$$

which gives the required form.

3.2.3 Example

Set $f(z,w) = z/w$. This function has simple poles at $z = i\,(u \pm |x|)/\bar{q}$. Evaluate the contour integral

$$\rho_P(\zeta) = \left(\tfrac{1}{2}\left((x+iy) + 2z\zeta - (x+iy)\zeta^2\right), \eta\right) \qquad (3.24)$$

along a contour surrounding the pole $i\,(u + |x|)/\bar{q}$, but not surrounding the other pole. To be more specific, take the contour $|x|$ = 2, then the above property is satisfied for $\{R^3 \colon q \neq 0, 0 > 9|x|^2 - 16u^2\}$.

Set $\{x \in R^3 \colon q \neq 0, 0 < 9|x|^2 - 16u^2\}$. Then for $x \in U$, calculating the residue, the integral (3.24) gives the harmonic function

$$\varphi(x) = \frac{u+|x|}{2\bar{q}|x|} \qquad (3.24)$$

Well-defined off the x^3-axis q = 0. Note that (3.24) only determines the harmonic function for $x \in U$, where as the function clearly extends to $R^3/\{x^3 - axis\}$.

If on the other hand we let $x \in V = \{x \,\epsilon\, R^3 : q \neq 0, 0 > 9|q|^2 - 16u^2\}$, the contour surrounds the other pole and we get a different harmonic function

$$\varphi(x) = -\frac{u-|x|}{2\bar{q}|x|} \qquad (3.25)$$

In order to describe the harmonic function φ in terms of twistor space we have to work a bit harder! We avoid discussion of twistor cohomology, but to give a flavour of what occurs, we outline the procedure to determine a global solution. Take an appropriate open cover $\{U_i\}$ of twistor space TCP^1.

3.2.4 Note

For a given $x \,\epsilon\, R^3$, the integration takes place along a contour in the corresponding Riemann sphere $CP^1(x) \subset TCP^1$ (this is where $f(z,w)$ is defined!) Suppose that $U_1 \cap U_2 \supset CP^1(x)$ and let $V_1 = U_1 \cap CP^1(x)$, $V_2 = U_2 \cap CP^1(x)$. Then we require the contour to lie in $V_1 \cap V_2$. Furthermore we require the twistor function f to be defined in a neighbourhood of this contour - in fact in $U_1 \cap U_2$and we write it as f_{12}.

More generally, with respect to the open cover $\{U_i\}$, we have a collection of twistor functions ffijg defined on the intersections $U_i \cap U_j$. These must satisfy the cocycle conditions and define an element of cohomology. In the space-time context this is the basis of the Penrose transform (an integral transform) relating sheaf cohomology on twistor space and zero-rest-mass felds on space time.

The twistor function is replaced by an element of the cohomology group and the field now becomes a function of an element of the cohomology group.

3.3 Integral Formulae

3.3.1Definition

We define the future tube of complexified Minkowski space by:

$$CM^+ = C\, g^{-1}(T^+)$$

3.3.2Remark

Recall that in quantum field theory we discard negative frequency fields, for they correspond to unphysical negative energy particles. Therefore we are most interested in solving the ZRM equations for positive frequency fields. We note a field $\varphi_{A......B}$ on Minkowski space is of positive frequency if it can be extended to the forward tube CM^+ by analytic continuation. Using hyperfunctions one may obtain the converse statement also. Motivated by this, we shall seek solutions of the ZRM equations defined on CM^+.

3.3.3 Theorem

Recall the helicity $n/2$ ZRM equations for a valence n spinor field $\psi_{\acute{A}.........\acute{B}}$, namely

$$\nabla^{A\acute{A}}\, \psi_{\acute{A}.........\acute{B}} = 0 \tag{3.26}$$

These have solutions on CM^c given by

$$\psi_{\acute{A}.........\acute{B}}(x) = \frac{1}{2\pi i} \oint \pi_{\acute{A}} \ldots\ldots \pi_{\acute{B}} p_x\, f(Z^\alpha) \pi_{\acute{C}}\, d\pi^{\acute{C}} \tag{3.27}$$

where

(i) f is homogeneous of degree $(-n - 2)$ in Z^α
(ii) $Z^\alpha = (w^A, \pi_{\acute{A}})$

(iii) p_x denotes restriction to the line $\mathbb{P}^1 \subset \mathbb{P}T$ defined by x via the twistor correspondence

(iv) $\pi_{\acute{A}}$ are homogeneous coordinates on $\mathbb{P}^1$

(v) the contour is arbitrary, provided it avoids the singularities of f and varies continuously with x

Proof

First observe that the integral is well-defined on $\mathbb{P}^1$, since the entire integrand (including the difierential) has homogeneity 0 in $\pi_{\acute{A}}$. Applying the chain rule we obtain

$$\nabla_{A\acute{A}}\, p_x f(Z^\alpha) = \frac{\partial}{\partial x^{A\acute{A}}}\, p_x f(w^A, \pi_{\acute{A}}) = p_x \frac{\partial f}{\partial w^C}\frac{\partial w^C}{\partial x^{A\acute{A}}} = i\pi_{\acute{A}} p_x \frac{\partial f}{\partial w^A} \quad (3.28)$$

Now difierentiating under the integral sign we get

$$\nabla_{C\acute{C}}\psi_{\acute{A}\ldots\ldots\ldots\acute{B}} = \frac{1}{2\pi}\oint \pi_{\acute{A}} \ldots \ldots\ \pi_{\acute{B}}\ \pi_{\acute{C}} p_x \frac{\partial f}{\partial w^C}\pi_{\acute{E}}\, d\pi^{\acute{E}} \quad (3.29)$$

which is clearly symmetric in $\acute{A} \ldots \ldots \ldots \acute{C}$ and so satisfies the ZRM equations in the form of Lemma.

3.3.4 Remark

We may regard f as a section of $O(-n-\ 2)$ on $\mathbb{P}^3$.

3.3.5 Remark

Our proof is incomplete, for we have not demonstrated that an appropriate contour exists. We see in Example 3.4.7 that this is indeed a nontrivial problem. We leave this subtle point to the rigorous methods of 3.1.3 There, we solve the problem using the fact that $\mathbb{C}M^+$ is Stein.

3.3.6 Theorem

The helicity $-n/2\ \ ZRM$ equations for a valence n spinor field $\psi_{A\ldots..B}$ have solutions on $\mathbb{C}M^+$ given by

$$\psi_{A\ldots..B}(x) = \frac{1}{2\pi i}\oint p_x \frac{\partial f}{\partial w^A} \ldots \ldots \ldots \frac{\partial f}{\partial w^B} f(Z^\alpha)\pi_{\acute{C}}\, d\pi^{\acute{C}} \quad (3.30)$$

where f is homogeneous of degree $(n - 2)$ in Z^α and all other notation is as in the previous theorem.

3.3.7 Example (Wave Equation)

The alert reader may notice that we have not explicitly verified our formulae in the case n = 0. This is not hard to check, so instead we compute an example to develop our intuition. Consider the twistor function

$$f(Z^\alpha) = \frac{1}{(A_\alpha Z^\alpha)(B_\beta Z^\beta)} \tag{3.31}$$

This has homogeneity -2 in Z^α so applying Theorem 3.4.1 should yield a solution to the wave equation. For convenience set

$$\alpha^{\acute{A}} = iA_A x^{A\acute{A}} + A^{\acute{A}} \quad \text{and} \quad \beta^{\acute{A}} = iB_A x^{A\acute{A}} + B^{\acute{A}} \tag{3.32}$$

so that the integral reads

$$\psi(x) = \frac{1}{2\pi i} \oint \frac{1}{(\alpha^{\acute{A}}\pi_{\acute{A}})(\beta^{\acute{A}}\pi_{\acute{B}})} \pi_{\acute{C}}\, d\pi^{\acute{C}} \tag{3.33}$$

Observe that an appropriate contour exists iff the poles are distinct. Indeed any choice of contour varying continuously with x and enclosing one of the poles becomes singular when the poles coincide. If we want $\psi(x)$ to be well-defined on $\mathbb{C}M^+$ we need to place some restriction on A_α and B_β.

Now A_α and B_β define a line L in $\mathbb{P}T$ and hence a point $y \epsilon M$ via the dual twistor correspondence. We see that $\psi(x))$ is singular at precisely those $x \in \mathbb{C}M$ which are complex null separated from y. We have that $\psi(x)$ is singular iff $L_x \equiv \ell(x)$ intersects L in $\mathbb{P}T$. Therefore it sufices to choose A_α and B_β such that L lies entirely in $\mathbb{P}T^-$ for φ to be well-defined on $\mathbb{C}M^+$.

We may now assume that the poles are distinct, so in particular

$$\alpha^{\acute{A}}\beta_{\acute{B}} \neq 0 \tag{3.34}$$

Let z be a coordinate on $\mathbb{P}^1$ given by

$$\pi_{\acute{A}} = \alpha_{\acute{A}} + z\beta_{\acute{B}} \tag{3.35}$$

Then the integral becomes

$$\psi(x) = \frac{1}{2\pi i} \oint \frac{dz}{(\alpha^{\acute{A}}\pi_{\acute{A}})z} = \frac{1}{\alpha^{\acute{A}}\pi_{\acute{A}}} \tag{3.36}$$

by the residue theorem. Now since A_α and B_β lie on the line defined by y we have, by the dual twistor correspondence

$$A^{\acute{A}} = -iy^{A\acute{A}} A_A \text{ and } B^{\acute{A}} = -iy^{A\acute{A}} B_A \tag{3.37}$$

Whence we obtain

$$\alpha^{\acute{A}}\beta_{\acute{A}} = A_A x^{A\acute{A}} B^B x_{B\acute{A}} - A_A y^{A\acute{A}} B^B x_{B\acute{A}} - A_A x^{A\acute{A}} B^B y_{B\acute{A}} + A_A y^{A\acute{A}} B^B y_{B\acute{A}} \tag{3.38}$$

Now using the relations

$$x^{0\acute{A}} x_{1\acute{A}} = x^{0\acute{0}} x_{1\acute{0}} + x^{0\acute{1}} x_{1\acute{1}} = x_{1\acute{1}} x_{1\acute{0}} - x_{0\acute{1}} x_{1\acute{1}} = 0 \tag{3.39}$$

$$x^{0\acute{A}} x_{0\acute{A}} = x^{1\acute{A}} x_{1\acute{A}} \tag{3.40}$$

We may conclude that

$$A_A B^B x^{A\acute{A}} x_{B\acute{A}} = \frac{1}{2} A_A B^B x^2 \tag{3.41}$$

Treating the other terms similarly we obtain

$$\psi(x) = \frac{2}{A_A B^A (x-y)^2} \tag{3.42}$$

It is now trivial to check that $\psi(x)$ satisfies the wave equation, as required.

3.3.8 Example (ASD Coulomb Field)

It is claimed that the twistor function

$$f(Z^{\alpha}) = \log \frac{Z^1 Z^2 - Z^0 Z^3}{Z^2 Z^3} \qquad (3.43)$$

Produces an ASD Coulomb field $F^{\mu v}$ where $F^{0j} \equiv E^j \equiv iB^j$ and

$$E \propto r/r^3$$

Let F be an ASD Coulomb field. Then by Theorem we may write

$$F_{ab} = F_{A\acute{A}B\acute{B}} = \varphi_{AB}\epsilon_{\acute{A}\acute{B}} \qquad (3.44)$$

In particular we have

$$\begin{aligned} E_x &= F_{01} = -\varphi_{01} \\ E_y &= F_{02} = \frac{1}{2}(\varphi_{11} - \varphi_{00}) \\ E_z &= F_{03} = -\frac{1}{2}i(\varphi_{00} - \varphi_{11}) \end{aligned} \qquad (3.45)$$

Now we calculate φ_{AB} using the contour integral formula

$$\begin{aligned} \varphi_{AB}(t,x,y,z) = \psi_{A.....B}(x) &= \frac{1}{2\pi i}\oint p_x \frac{\partial f}{\partial w^A} \ldots\ldots\ldots \frac{\partial f}{\partial w^B} f(Z^{\alpha})\pi_{\acute{E}}\, d\pi^{\acute{E}} \\ &= \frac{1}{2\pi i}\oint \frac{(\delta_A^0\pi_{\acute{1}} - \delta_A^1\pi_{\acute{0}})(\delta_B^0\pi_{\acute{1}} - \delta_B^1\pi_{\acute{0}})}{\left(x^{1\acute{A}}\pi_{\acute{A}}\pi_{\acute{0}} - x^{0\acute{A}}\pi_{\acute{A}}\pi_{\acute{1}}\right)^2}\pi_{\acute{E}}d\pi^{\acute{E}} \end{aligned} \qquad (3.46)$$

Choosing local coordinates $\pi_{\acute{E}} = (1\,, \zeta)$ and using the convention

$$\begin{pmatrix} x^{0\acute{0}} & x^{0\acute{1}} \\ x^{1\acute{0}} & x^{1\acute{1}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{t+x} & \mathrm{y+iz} \\ \mathrm{y-iz} & \mathrm{t-x} \end{pmatrix} \qquad (3.47)$$

we get

$$\varphi_{AB} = \frac{1}{2\pi i}\oint d\zeta\, \frac{(\delta_A^1\pi_{\acute{1}} - \delta_A^0\,\zeta)(\delta_B^1\pi_{\acute{1}} - \delta_B^0\zeta)}{\left(1/\sqrt{2}(y-iz) + \sqrt{2}x\zeta - 1/\sqrt{2}(y+iz)\zeta^2\right)^2}\pi_{\acute{E}}d\pi^{\acute{E}} \qquad (3.48)$$

This has double poles at

$$\zeta = \frac{-\sqrt{2}\,x \pm \sqrt{2x^2+2y^2+2z^2}}{-\sqrt{2}\,(y+iz)} \tag{3.49}$$

Denote these ζ_1 and ζ_2. The residue at ζ_1 is

$$r_1 = \rho\rho_{\zeta_1} \frac{d}{d\zeta} \frac{2(\delta_A^1 \pi_{\hat{1}} - \delta_A^0 \zeta)(\delta_B^1 \pi_{\hat{1}} - \delta_B^0 \zeta)}{(y+iz)^2(\zeta-\zeta_2)^2}$$

$$= \frac{1}{2r^2}\left(-\delta_A^0(\delta_B^1 - \delta_B^0\zeta_1) - \delta_B^0(\delta_A^1 - \delta_A^0\zeta_1)\right)$$

$$+(\delta_B^0(\delta_A^1 - \delta_A^0\zeta_1)(\delta_B^1 - \delta_B^0\zeta_1)\,(y+iz)/r) \tag{3.50}$$

Now we calculate explicitly

$$\varphi_{01} = \frac{1}{2r^2}(-1-\zeta\_1\,(y+iz) \quad /\, r) = \frac{x}{2r^3}$$

$$\varphi_{00} = \frac{1}{2r^2}\left(2\zeta_1 + {\zeta_1}^2(y+iz)\,/\,r\right)/r = \frac{(y-iz)}{2r^3}$$

$$\varphi_{00} = \frac{(y-iz)}{2r^3} \tag{3.51}$$

Whence we fiend

$$E_x = \frac{x}{2r^3}, \qquad E_y = \frac{y}{2r^3} \qquad E_z = \frac{z}{2r^3} \tag{3.52}$$

as required.

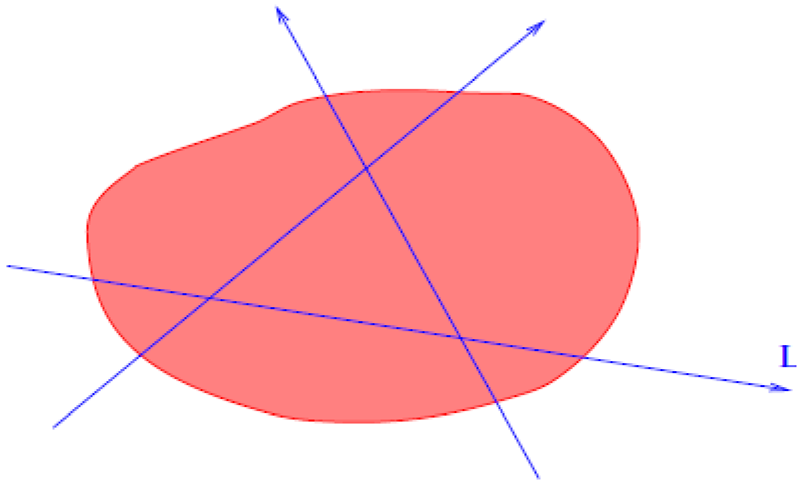
3.3.9 Remark

It is natural to ask whether we can formulate an inverse twistor transform. Given a ZRM field φ on CM^+, what is the set of twistor functions which yield φ under the Penrose integral? This is not immediately obvious. Suppose we are given f producing φ via the integral formula with contour Γ at x. Let h and $\tilde{h}$h be holomorphic on opposite sides of Γ. Then certainly $f + h - \tilde{h}$ will also generate φ. Indeed we now proceed to reformulate the ideas of this section in the language of sheaves, thus obtaining a bijective transform.

3.4 Penrose Transform

3.4.1. Radon Transform

Integral geometry goes back to Radon who considered the following problem: let $f: R^2 \rightarrow R$ be a smooth function with suitable decay conditions at ∞ (for example a function of compact support as shown below)



and let $L \rightarrow R^2$ be an oriented line. Define a function on the space of oriented lines in R^2 by

$$\phi(L) = \int_L f \qquad (3.53)$$

Radon has demonstrated that there exists an inversion formula $\phi \rightarrow f$. Radon's construction can be generalized in many ways and it will become clear that Penrose's twistor theory is its far reaching generalization. Before moving on, it is however worth remarking that an extension of Radon's work has led to Nobel Prize awarded (in medicine) for pure mathematical research! It was given in 1979 to Cormack, who unaware of Radon's results had rediscovered the inversion formula for (3.53), and had explored the set-up allowing the function f to be defined on a non-simply connected region in R^2 with a convex boundary. If one only allows the lines which do not pass through the black region

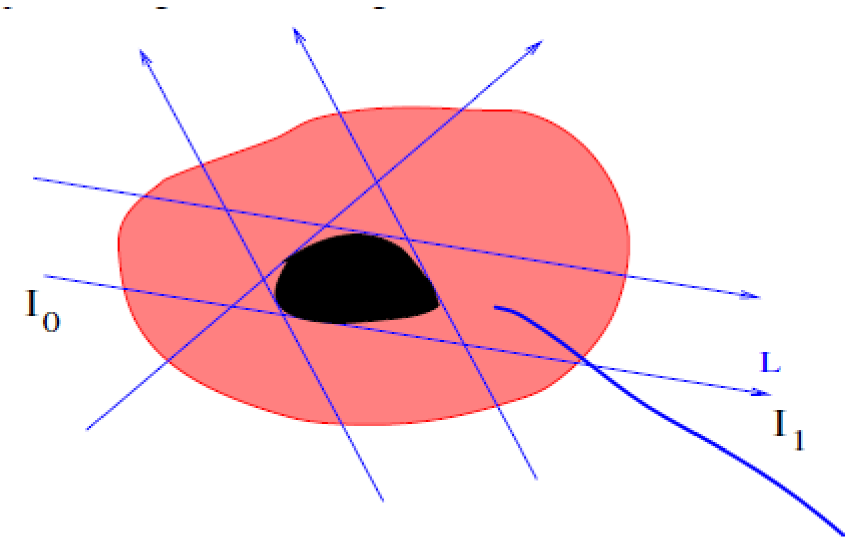


Fig (17)

or are tangent to the boundary of this region, the original function f may still be reconstructed from its integrals along such lines. In the application to computer tomography, one takes a number of $2D$ planar sections of 3D objects and relates the function f to the (unknown) density of these objects. The input data given to a radiologist consist of the intensity of the incoming and outgoing x-rays passing through the object with intensities I_0 and I_1 respectively

$$\phi(L) = \int_L \frac{dI}{I} = log I_1 - log I_0 = -\int_L f \qquad (3.54)$$

where d$I/I = -f(s)$ ds$\frac{dI}{I} = -f(s)ds$ is the relative infinitesimal intensity loss inside the body on aninterval of length ds.

The Radon transform then allows to recover f from this data, and the generalization provided by the support theorem becomes important if not all regions in the object (for example patient's heart) can be x-rayed.

3.4.2John Transform

The inversion formula for the Radon transform (3.53) can exist because both R^2 and the space of oriented lines in R^2 are two dimensional. Thus, at least naively, one function of two variables can be constructed from another such function (albeit defined on a different space). This symmetry does not hold in higher dimensions, and this underlines the following important result of John. Let $f: \mathbb{R}^3 \to \mathbb{R}$ be a function (again, subject to some decay conditions which makes the integrals well defined) and let $L \subset\ L \subset R^3$be an oriented line. Define

$$\phi(L) = \int_L f \qquad (3.55)$$

, or

$$\phi(\alpha_1, \alpha_2, \beta_1, \beta_2) = \int_{-\infty}^{\infty} f(\alpha_1 s + \beta_1, \alpha_2 s + \beta_2, s) ds \qquad (3.56)$$

where *(α,β)* parametrize the four-dimensional space T of oriented lines in R^3. (Note that this parametrization misses out the lines parallel to the plane x_3 = const. The whole construction can be done invariantly without choosing any parametrization, but here we choose the explicit approach for clarity.) The space of oriented lines is four dimensional, and 4 > 3 so expect one condition on *φ*. Differentiating under the integral sign yields the ultrahyperbolic wave equation

$$\frac{\partial^2 \emptyset}{\partial \alpha_1 \partial \beta_2} - \frac{\partial^2 \emptyset}{\partial \alpha_2 \partial \beta_1} = 0 \qquad (3.57)$$

And John has shown that all smooth solutions to this equation arise from some function on R^3. This is a feature of twistor theory an unconstrained function on twistor space (which in this case is identified with R^3) yields a solution to a differential equation on spacetime. After the change of coordinates

$$\alpha_1 = x + y\ ,\ \alpha_2 = t + z, \beta_1 = t - z = t - z,\ \beta_2 = x - y \qquad (3.58)$$

the equation becomes which may be relevant to physics two times! The integral formula given in the following section corrects the 'wrong' signature to that of the Minkowski space and is a starting point of twistor theory.

3.4.3 Penrose Transform

In 1969, Penrose gave a formula for solutions to the wave equation in the Minkowski space

$$\emptyset(x, y, z, t) = \oint_{\Gamma \subset \mathbb{CP}^1} \mathrm{f}\left((z + t) + (x + iy)\lambda, (x - iy) - (z - t)\lambda, \lambda\right) d\lambda \qquad (3.59)$$

Here $\Gamma \subset \mathbb{CP}^1$ is a closed contour and the function f is holomorphic on $\mathbb{CP}^1$ except some number of poles. Differentiating the RHS verifies that

$$\frac{\partial^2 \emptyset}{\partial t^2} - \frac{\partial^2 \emptyset}{\partial x^2} - \frac{\partial^2 \emptyset}{\partial y^2} - \frac{\partial^2 \emptyset}{\partial z^2} = 0 \tag{3.60}$$

Despite the superficial similarities, the Penrose formula is mathematically much more sophisticated than John's formula (3.56). One could modify a contour and add a holomorphic function inside the contour to f without changing the solution φ.

The question we now discuss is how fields in M are represented in twistor space. We shall find that the general zero-rest-mass free fields can be remarkably concisely represented by holomorphic (complex analytic) functions g(Z^α) and $f(W_\alpha)$on the twistor space and its dual, C*. But in order to make the correspondence we must take suitable contour integrals. Thus only the residues at the poles of f will be physically meaningful; consequently formalism will be based on contour integrals in C.

3.4.4 Lemma

A function $f\left(x^{A\acute{A}}, \pi^{\acute{A}}\right)$ on F pushes down to a function on P iff $\pi^{\acute{A}} \nabla_{A\acute{A}} f = 0$ in every coordinate chart.

Proof

We demonstrate that this is equivalent to the stated condition in our preferred patch (P^I, M^I, F^I). Then the general result follows by a combinatorial argument. Clearly $f\left(x^{A\acute{A}}, \pi_{\acute{A}}\right)$ yields a function on P^I iff is constant each α-plane defined by $x^{A\acute{A}}$ and $\pi_{\acute{A}}$. We observe

$$\begin{aligned} \pi^{\acute{A}} \nabla_{A\acute{A}} f = 0 &\Leftrightarrow \nabla_{A\acute{A}} f = \xi_A \pi_{\acute{A}} \text{ for some } \xi_A(\pi) \\ &\Leftrightarrow f = \xi_A \pi_{\acute{A}} x^{A\acute{A}} = \xi_A w_A \end{aligned} \tag{3.61}$$

and the result follows.

3.4.5 Remark

In particular a function $f\left(x^{A\acute{A}}, \pi_{\acute{A}}\right)$ on F pushes down to a twistor function iff the given condition holds in the non-projective sense. We shall make frequent use of this observation

3.4.6 Theorem

$$H^1\big(\mathbb{P}T^+, O(-n-\ 2)\big) \cong \{\text{ZRM fields } \varphi_{\acute{A}.......\acute{B}}\ of\ helicity\ n/2\, on\ CM^+\} \qquad (3.62)$$

where we may view the set of ZRM fields as a group under addition since the ZRM equations are linear.

Proof

The avour of the proof is as follows. We construct a short exact sequence of sheaves culminating in the sheaf of germs of the desired ZRM fields. Recalling the long exact sequence in cohomology, we obtain the require disomorphism by identifying certain sheaves as zero.

Define the sheaves $\mathcal{Z}_n(m)$ on F^+ by stipulating that $\varphi_{\acute{A}.....\acute{B}}(x,\mu)$ must satisfy the following conditions

(i) $\varphi_{\acute{A}.....\acute{B}}$ is a symmetric holomorphic valence n primed spinor field on F^+

(ii) $\varphi_{\acute{A}.....\acute{B}}$ is homogeneous of degree m in π

(iii) $\varphi_{\acute{A}.....\acute{B}}$ satisfies the ZRM equation $\nabla^{A\acute{A}}\ \varphi_{\acute{A}.....\acute{B}} = 0$ throughout F^+

3.4.7 Note

Immediately that $\mathcal{Z}_n(0)$ consists of symmetric n index primed spinor fields which are independent of π, so there is a canonical sheaf isomorphism

$$\mathcal{Z}_n(0) \cong \{ZRM\ fields\ \ \varphi_{\acute{A}.....\acute{B}}\ \ of\ helicity\ n/2\ on\ \mathbb{C}M^+\} \qquad (3.63)$$

Define a sheaf morphism

$$P:\ \mathcal{Z}_{n+1}(m-1) \rightarrow\ \mathcal{Z}_n(m) \qquad (3.64)$$

$$\varphi_{\acute{A}\acute{B}.....\acute{C}} \longmapsto\ \pi^{\acute{A}}\ \varphi_{\acute{A}\acute{B}.....\acute{C}} \qquad (3.65)$$

We claim that this morphism is surjective, and it suffices to check this locally by Theorem. Let $\varphi_{\acute{A}\acute{B}.....\acute{C}} \in \mathcal{Z}_n(m)$ be arbitrary. Define pointwise for each $\left(x^{A\acute{A}}, \pi_{\acute{A}}\right) \in F^+$

$$(3.70)\varphi_{0\acute{B}.....\acute{C}} \ \frac{1}{2\pi^0}\varphi_{\acute{B}.....\acute{C}}$$

$$(3.71)\varphi_{1\acute{B}.....\acute{C}} \ \frac{1}{2\pi^1}\varphi_{\acute{B}.....\acute{C}}$$

which we can do since $\pi_{\acute{A}} \neq 0 \ \in F$ by definition. When $\pi_{\acute{A}} = 0$ or $\pi_{\acute{A}} \neq 0$ individually an obvious modiffcation can be made. Then clearly $\varphi_{\acute{A}\acute{B}.....\acute{C}} \in \mathcal{Z}_{n+1}(m-1)$ and around every point of F^+ there exists a neighbourhood in which $P(\varphi_{\acute{A}.....\acute{C}}) = \varphi_{\acute{B}.....\acute{C}}$.

Consider the special case $m = 0$. Let K denote the sheaf kernel of P: $\mathcal{Z}_{n+1}(-1)$. Define on F^+ the sheaves

$$T\ (n) = \begin{Bmatrix} scalarfields\ f(\ x, \pi) homogeneous\ of\ degree\ n \\ in\ \pi\ which\ push\ down\ to\ twistor\ functions \end{Bmatrix}$$

We claim that K is isomorphic to $T\ (-n - 2)$. Indeed let $\mathcal{X}_{\acute{A}.....\acute{B}} \in K$ be an $(n + 1)$ index spinor field on F^+, homogeneous of degree -1 in π. Then since $\mathcal{X}_{\acute{A}.....\acute{B}}$ symm`etric we may write

$$\mathcal{X}_{\acute{A}.....\acute{B}} = \alpha_{(\acute{A}} \ldots\ldots \beta_{\acute{B})} \tag{3.72}$$

We then deduce

$$\pi^{\acute{A}}\ \alpha_{(\acute{A}} \ldots\ldots \beta_{\acute{B})} = 0 \Longrightarrow \pi^{\acute{A}} \ldots . \pi^{\acute{B}} \alpha_{(\acute{A}} \ldots\ldots \beta_{\acute{B}} = 0$$
$$\Longrightarrow \pi^{\acute{A}}\alpha_{\acute{A}} = 0$$
$$\Longrightarrow \pi^{\acute{A}}\alpha_{\acute{A}} = 0, \ldots . \pi^{\acute{B}}\beta_{\acute{B}} = 0$$
$$(3.73) \Longrightarrow \mathcal{X}_{\acute{A}.....\acute{B}} = \pi_{\acute{A}} \ldots . \pi_{\acute{B}} f(x, \pi) = 0$$

Now since $\pi \neq 0$ the ZRM equations imply

$$\pi_{\acute{A}}\, \nabla^{A\acute{A}} f = 0 \tag{3.74}$$

which is precisely the condition that f pushes down to a twistor function. Observe also that f is homogeneous of degree $(-n-2)$ in π. The converse is obvious.

We thus have a short exact sequence of sheaves

$$0 \to T\,(-n-2) \xrightarrow{\pi_{\acute{A}} \cdots \pi_{\acute{B}}} \mathcal{Z}_{n+1}(-1) \xrightarrow{\pi_{\acute{A}}} \mathcal{Z}_n(0) \longrightarrow 0 \qquad (3.75)$$

Whence we obtain a long exact sequence of cohomology

$$\ldots \to H^0\big(F^+, Z_{n+1}(-1)\big) \to H^0\big(F^+, Z_n(0)\big) \overset{\delta^*}{\to}$$

$$H^1\big(F^+, Z_{n+1}(-n-2)\big) \to H^1\big(F^+, Z_{n+1}(-1)\big) \qquad (3.76)$$

We now identify these groups.

(i) Suppose $s(x,\pi) \in H^0\big(F^+, Z_{n+1}(-1)\big)$. Then s is a global section of $Z_{n+1}(-1)$ over F^+. For fixed x, s defines a global section of $O(-1)$ over $\mathbb{P}^1$, so $s = 0$. Thus $H^0\big(F^+, Z_{n+1}(-1)\big) = 0$.

(ii) $H^0\big(F^+, Z_n(0)\big)$ is clearly the desired group of ZRM fields on F^+.

(iii) Observe that we may canonically identify $T\,(-n-2)$ with the sheaf of twistor functions homogeneous of degree $(-n-2)$ on T^+ which itself is naturally intepreted as the sheaf $O(-n-2)$ on $\mathbb{P}T^+$, . We may therefore write $H^1\big(F^+, T(-n-2)\big) \cong H^1(\mathbb{P}T^+, O(-n-2),)$

(iv) We note without proof that $\mathbb{C}M^+$ is Stein. Since $Z_{n+1}(-1)$ is a sheaf of holomorphic sections of a vector bundle. Thus the pullback $\mathcal{G}$ of $Z_{n+1}(-1)$ to $\mathbb{C}M^+$ has $H^1(\mathbb{C}M^+, \mathcal{G}) = 0$. Recall that $H^1(\mathbb{P}^1, O(-1)) = 0$. Hence the pullback $\mathcal{H}$ of $Z_{n+1}(-1)$ to $\mathbb{P}^1$ has $H^1(\mathbb{P}^1, \mathcal{H}) = 0$. Applying a suitable K$\ddot{u}$nneth formula, we get $H^1\big(F^+, Z_{n+1}(-1)\big) = 0$.

Therefore we may conclude that δ^* provides the required isomorphism in the statement of the theorem, and our proof is complete.

3.4.8 Remark

We may regain the contour integral formulation of the Penrose transform by explicitly analysing the map$(\delta^*)^{-1}$. Recall that to define δ^* we F^+ consider the cochain complex of sheaves on F^+

$$\begin{array}{ccccccccc} 0 \to & C^0\big(T(-n-2)\big) & \to & C^0\big(Z_{n+1}(-1)\big) & \to & C^0\big(Z_n(-1)\big) & \to 0 \\ & \downarrow d & & \downarrow d & & \downarrow d & \\ 0 \to & C^1\big(T(-n-2)\big) & \to & C^1\big(Z_{n+1}(-1)\big) & \to & C^1\big(Z_n(-1)\big) & \to 0 \\ & \downarrow d & & \downarrow d & & \downarrow d & \end{array}$$

Choose a cover which is Leray for all the given sheaves on F^+ and work with Cech cohomology.

Let $f_{ij} \in H^1(\mathbb{P}T^+, O(-n-2))$. Then by commutativity of the above diagram

$$\pi_{\acute{A}} \ldots . \pi_{\acute{B}}\, f_{ij} \in H^1(\mathbb{P}T^+, O(-n-2)) \tag{3.78}$$

Therefore we may write

$$\pi_{\acute{A}} \ldots . \pi_{\acute{B}}\, f_{ij} = \rho_{[i}\psi_{j]\acute{A}.....\acute{C}} \tag{3.79}$$

for $\psi_{j\acute{A}.....\acute{C}} \in C^0\big(Z_{n+1}(-1)\big)$. Now define

$$\psi_{j\acute{A}.....\acute{B}} = \psi_{j\acute{A}.....\acute{B}}\, \pi^{\acute{C}} \in C^0\big(Z_n(0)\big) \tag{3.80}$$

and note that $\psi_{j\acute{A}.....\acute{B}} \in H^0\big(Z_n(0)\big)$ by the isomorphism $H^1\big(T(-n-2)\big) \cong H^0\big(Z_n(0)\big)$ proved above. Thus there is a ZRM field $\psi_{\acute{A}.....\acute{B}}$ with

$$\rho_j\psi_{\acute{A}.....\acute{B}} = \psi_{j\acute{A}.....\acute{B}} = \psi_{j\acute{A}.....\acute{B}}\ \pi^{\acute{C}} \tag{3.81}$$

Now for fixed x we know that $\rho_x f_{ij}$ defines an element of $O(-n-2)$ over $\mathbb{P}^1$.Therefore $\pi_{\acute{A}} \ldots . \pi_{\acute{B}}\, \rho_x\, f_{ij}$ is an element of $O(-1)$ over $\mathbb{P}^1$. Employing Sparling's formula we may therefore write

$$\psi_{j\acute{A}.....\acute{B}} = \pi^{\acute{C}} \frac{1}{2\pi i} \oint \left(\zeta^{\acute{F}} \pi_{\acute{F}}\right)^{-1} \zeta_{\acute{A}} \ldots \zeta_{\acute{C}}\, \rho_x f_{01}(w^A, \zeta_{\acute{A}}) \zeta_{\acute{G}} d\zeta^{\acute{C}}$$
$$= \frac{1}{2\pi i} \oint \zeta_{\acute{A}} \ldots \zeta_{\acute{B}}\, \rho_x f_{01}(w^A, \zeta_{\acute{A}}) \zeta_{\acute{C}} d\zeta^{\acute{C}} \quad (3.82)$$

3.4.9 Remark

We lacked some rigour in our proof above, failing to mention the subtleties involved in comparing sheaves on different spaces. More complete reasoning requires the use of spectral sequences, which we have not discussed.

3.4.10 Theorem

$$H^1(\mathbb{P}T^+, O(n-2)) \cong \{ZRM\ fields\ \psi_{A.....B}\ of\ helicity\ -n/2\ on\ \mathbb{C}M^+\} \quad (3.83)$$

Proof

This proof has a similar flavor to the previous argument. Define on F^+ the following sheaves

$$K(n) = \{holomorphic\ functions\ f(x,\pi)\ homogeneous\ of\ degree\ n\ in\ \pi\}$$
$$Q_A(n+1) = f\ spinor\ fields\ \psi_A(x,\pi) homogeneous\ of\ degree$$
$$(n+1) in\ \pi_{\acute{A}}\ and\ satisfying\ \pi_{\acute{A}} \nabla^{A\acute{A}} \psi_A = 0$$

Define a sheaf morphism $D_A : \kappa(n) \to Q_A(n+1)$ by

$$D_A f = \pi^{\acute{A}}\, \nabla_{A\acute{A}} f \quad (3.84)$$

Let $T\,(n)$ denote the kernel of D_A and identify as before

$$T\,(n) = \{scalar\ fields\ f(x,\pi)\ homogeneous\ of\ degree\ n$$
$$in\ \pi\ which\ push\ down\ to\ twistor\ functions\}$$

Now we have a short exact sequence of sheaves

$$0 \to T\,(n) \hookrightarrow \kappa(n) \xrightarrow{D_A} Q_A(\mathrm{n}+1)$$

whence we obtain a long exact sequence of cohomology

$$0 \to H^0\big(F^+, T(n)\big) \to H^0\big(F^+, \kappa(n)\big) \to H^0(F^+, Q_A(\mathrm{n}\ +\ 1)) \xrightarrow{\delta^*} H^1\big(F^+, T(n)\big) H^1\big(F^+, \kappa(n)\big)$$

We investigate each of these groups in turn.

(i) Let $f \in \big(F^+, T(n)\big)$. Then we may write

$$f(x, \pi) = \mu_{\acute{A}\ldots\acute{B}}(x)\, \pi^{\acute{A}} \ldots \pi^{\acute{B}} \tag{3.85}$$

where $\mu_{\acute{A}\ldots\acute{B}}$ is a symmetric holomorphic spinor field on '$\mathbb{C}M^+$. The push down condition is

$$\pi^{\acute{C}}\pi^{\acute{A}} \ldots \pi^{\acute{B}} \nabla_{C\acute{C}} \mu_{\acute{A}\ldots\acute{B}} = 0 \tag{3.86}$$

$$\nabla_{C(\acute{C}}\, \mu_{\acute{A}\ldots\acute{B})} = 0 \tag{3.87}$$

Hence we may identify $H^0(F^+, T\ (n))$ with the group $T\ (n)$ of $\mu_{\acute{A}\ldots\acute{B}}$ on $\mathbb{C}M^+$ satisfying this equation.

(ii) Let $\lambda \in H^0(F^+,\ \kappa(n))$. Then we may write

$$\lambda = \lambda_{\acute{A}\ldots\acute{B}}(x)\, \pi^{\acute{A}} \ldots \pi^{\acute{B}} \tag{3.88}$$

where $\lambda_{\acute{A}\ldots\acute{B}}$ is a symmetric holomorphic spinor field on $\mathbb{C}M^+$. There are no additional constraints on $\lambda_{\acute{A}\ \ldots\ \acute{B}}$ so we identify $H^0(F^+,\ \kappa(n))$ with the group Λ_n of such $\lambda_{\acute{A}\ \ldots\ \acute{B}}$.

(iii) Let $\psi_A \in H^0(F^+,\ \mathrm{Q}_A(n+1))$ and write

$$\psi_A = \psi_{A\acute{A}\ldots.\acute{C}}(x) \pi^{\acute{A}} \ldots \pi^{\acute{B}} \tag{3.89}$$

where $\psi_{A\acute{A}\ldots.\acute{C}}$ is a holomorphic spinor field on $\mathbb{C}M^+$ symmetric in its $(n\ +\ 1)$ primed indices. The defining condition for $\mathrm{Q}_A(n+1)$ gives

$$\pi^{\acute{D}}\pi^{\acute{A}} \dots \pi^{\acute{C}}\nabla^{A}_{\acute{D}}\psi_{\acute{A}.....\acute{C}A} = 0$$

$$\Leftrightarrow \nabla^{A}_{(\acute{D}}\psi_{\acute{A}.....\acute{C})A} = 0 \qquad (3.90)$$

We identify $H^0(F^+,\ Q_A(n+1))$with the group ψ^1_{n+1} of $\psi^A_{\acute{A}....\acute{C}}$ on $\mathbb{C}M^+$ satisfying this equation.

(iv) As in the previous proof, we somewhat unrigorously write $H^1(F^+, T\ (n)) = H^1(\mathbb{P}T^+, O(n))$.

(v) Recall that $H^1(\mathbb{P}^1, O(n))$. Also $\kappa(n)$ is coherent analytic as a sheaf of sections of the trivial $\mathbb{C}$-bundle over F^+. Using again that $\mathbb{C}M^+$ is Stein, and an appropriate Kunneth formula we obtain $H^1(F^+,\ \kappa(n)) = 0$.

Rewriting the long exact sequence in our new notation we have the section

$$0 \to T(n) \hookrightarrow \Lambda_n \xrightarrow{\sigma} \psi^1_{n+1} \xrightarrow{\delta^*} H^1(\mathbb{P}T^+, O(n)) \to 0$$

where the reader may easily check that σ is given by

$$\sigma(\lambda_{\acute{B}...\acute{C}}) = \nabla^{A}_{(\acute{A}}\lambda_{\acute{B}...\acute{C})} \qquad (3.91)$$

We now relate this sequence to ZRM fields using Hertz potentials. Let Φ_{n+2} denote the group consisting of $(n+2)$ unprimed index ZRM fields $\varphi_{A...D}$ on $\mathbb{C}M^+$. Define a group homomorphism $P:\ \psi^1_{n+1} \to \Phi_{n+2}$ by

$$P(\psi_{A\acute{B}.....\acute{D}}) = \nabla^{\acute{B}}_{(B...}\nabla^{D}_{\acute{D}}\psi_{A)\acute{B}....\acute{D}} \qquad (3.92)$$

We check that this is well-defined by computing

$$\nabla^{A}_{\acute{A}}\nabla^{\acute{B}}_{(B...}\nabla^{D}_{\acute{D}}\psi_{A)\acute{B}....\acute{D}} = \nabla^{B}_{\acute{B}} \dots . \nabla^{D}_{\acute{D}}\nabla^{A}_{(\acute{A}}\psi_{\acute{B}....D)A} = 0 \qquad (3.93)$$

which may be verified by expanding out the symmetrisers on each side. Moreover observe that P is surjective. We know that given $\varphi_{A\ldots D} \in \Phi_{n+2}$ there exists $\psi_{A\acute{B}\ldots\acute{D}}$ defined on $\mathbb{C}M^{+}$ such that

$$\varphi_{A\ldots D} = \nabla^{B}_{\acute{B}} \ldots \nabla^{D}_{\acute{D}} \psi_{A\acute{B}\ldots\acute{D}}$$

$$\nabla^{A}_{\acute{A}} \psi_{A\acute{B}\ldots\acute{D}} = 0 \tag{3.94}$$

since $\mathbb{C}M^{+}$ is simply connected and has vanishing second homotopy group. In particular we immediately have $\psi_{A\acute{B}\ldots\acute{D}} \in \psi^{1}_{n+1}$ as required.

Finally we claim that $ker(P) = im(\sigma)$. For the reverse inclusion we compute

$$\nabla^{\acute{B}}_{(B\ldots} \nabla^{D}_{\acute{D}} \nabla_{A)\acute{B}} \lambda_{\acute{C}\ldots\acute{D}\ldots\acute{D}} = \nabla^{B}_{\acute{B}} \ldots \nabla^{D}_{\acute{D}} \nabla^{A}_{(\acute{A}} \psi_{\acute{B}\ldots D\acute{)}A} = 0 \tag{3.95}$$

We therefore have an exact sequence

$$0 \to T(n) \hookrightarrow \Lambda_{n} \xrightarrow{\sigma} \psi^{1}_{n+1} \xrightarrow{P} \Phi_{n+2} \tag{3.96}$$

Comparing with (3.1) we obtain $\Phi_{n+2} \cong H^{1}(\mathbb{P}T^{+}, O(n))$ as required.

3.4.11 Remark

We observe that an explicit inverse twistor transform exists in this case. Given a ZRM field $\psi_{A\ldots D}$
let $\psi_{A\acute{B}\ldots\acute{D}}$ be a Hertz potential. We must construct a cover $\{U_{j}\}$ of $\mathbb{P}T^{+}$ and twistor functions f_{jk} on U_{jk}. Choose $\{U_{j}\}$ with the property that.

There exists $Y^{\alpha}_{j} \in \mathbb{P}T^{+}$ such that for all $z^{\alpha} \in U_{j}$ the line joining Y^{α}_{j} and z^{α} lies entirely in $\mathbb{P}T^{+}$.

Now suppose $z^{\alpha} \in U_{j} \cap U_{k}$. Denote by Y_{j}, Y_{k} and Z the $\propto$-planes in $\mathbb{C}M^{+}$corresponding to Y^{α}_{j}, Y^{α}_{k} and z^{α}. Observe that Y_{j} intersects Z in a point $p_j \in \mathbb{C}M^+$ defined by the line joining Y_j^{α} and z^{α} in $\mathbb{P}T^+$. Similarly we define $p_k = Y_k \cap Z \in \mathbb{C}M^+$.

We now hypothesise an integral formula for f_{jk}. Let $Z^{\alpha} = (w^A, \pi_{\acute{A}})$.

Choose an arbitrary contour Γ_{jk} from p_j to p_k lying in Z and define

$$f_{jk}(Z^{\alpha}) = \int_{\Gamma_{jk}} \psi_{A\acute{B}\acute{C}\ldots\acute{D}}\, \pi^{\acute{C}} \ldots \pi^{\acute{D}} dx^{A\acute{B}} \qquad (3.97)$$

We must check that f_{jk} is indendendent of Γ_{jk}, defines a 1-cocycle and reproduces the potential $\psi_{A\acute{B}\acute{C}\ldots\acute{D}}$ under $\delta^{*^{-1}}$.

3.5 The Solutions of Zero Rest Mass Equation

The question we now discuss is how fields in M are represented in twistor space. We shall find that the general zero-rest-mass free fields can be remarkably concisely represented by holomorphic (complex analytic) functions g(Z^{α}) and $f(W_{\alpha})$ on the twistor space and its dual C*. But in order to make the correspondence we must take suitable contour integrals. Thus only the residues at the poles of f will be physically meaningful; consequently the subsequent formalism will be based on contour integration in C.

The solutions of the equations (3.13) can be represented by a set of quantities $\phi_r(\boldsymbol{P},\ O^A, \iota^B)$ where $r = 0,1, \ldots n$; O^A, ι^B are a pair of basis spinors at the point **P** and

$$\phi_r = \phi_{AB\ldots L}\ \underbrace{\iota^A \ldots \iota^D}_{r}\ \underbrace{O^E \ldots O^L}_{n-r} \qquad (3.98)$$

Now O_A and ι_B define null twistor through **P**, namely U_{α}, V_{β} say, i.e.

$$U_{\alpha} \leftrightarrow \left(O_A, -ip^{AA'} O_A\right), \quad V_{\beta} \leftrightarrow \left(\iota_B, -ip^{BB'} \iota_B\right). \qquad (3.99)$$

Thus we have the quantities:

$$\Phi_r\left(U_{\alpha}, V_{\beta}\right) = \phi_r(P, O^A, \iota^B), \qquad r = 0, \ldots\ldots n \qquad (3.100)$$

If U_α and $,V_\beta$ are restricted to be null twistors with real intersection, Φ_r represent a zero-rest-mass field in M. Such a field may be regarded as defined on some three-parameter initial set (Cauchy hypersurface) and thence extended over the rest of space by the field equations. In twistor terms it would be economic if we could describe the field on M by some field on the (complex) 3-space C, or C*. So far it appears that we must define the field on pairs of points $U,\ V$ in C*.

Let us take the point **P** and define a standard tensor and spinor reference frame such that:

$$u = p^{00'} = \frac{p^0 +\ p^1}{\sqrt{2}}\ ; \qquad \xi =\ p^{01'} = \frac{p^2 + i\,p^3}{\sqrt{2}}$$

$$\tilde{\xi} = p^{01'} = \frac{p^2 - i\,p^3}{\sqrt{2}} \qquad v =\ p^{11'} = \frac{p^0 - p^1}{\sqrt{2}} \tag{3.101}$$

$\bar{\xi}\ =\ \tilde{\xi}\,,\ u =\ \bar{u}\ ,\ v = \bar{v}$ if and only if p^a is real. The field equations (3.13) become:

$$\frac{\partial\phi_r}{\partial\tilde{\xi}} = \frac{\partial\phi_{r+1}}{\partial u}; \quad \partial\phi_r/\partial v =\ \partial\phi_{r+1}/\partial\xi\ r = o, \dots, n-1 \tag{3.102}$$

These equation are automatically satisfied if

$$\phi_r =\ \frac{1}{2\pi i}\oint_K \lambda^r F\big(\lambda, u + \lambda\tilde{\xi}\,, \xi +\ \lambda v\big)d\lambda \tag{3.103}$$

Where F is a holomorphic (i.e. analytic or regular in the complex sense) function of three complex variables, the contour K being taken to surround the poles of F in a suitable way. The resulting field will always be analytic in the real sense with respect to $u, v, \xi, \tilde{\xi}$, but we may represent non- analytic fields as limits of analytic fields as limits of analytic ones.

A real null factor at $p^a = (u, v, \xi, \tilde{\xi}\,)$ has direction given by $du\!:dv\!:d\xi\!:d\tilde{\xi}$ where:

$$du + \lambda d\tilde{\xi}\ = 0 = d\xi + \lambda dv \tag{3.104}$$

for some complex λ (possibly infinite). For the Minkowski metric is $2\left(du\,dv - d\xi\,d\tilde{\xi}\right)$ so that $du\,dv = d\xi\,d\tilde{\xi}$ for all null direction. Thus $du : dv : d\xi : d\tilde{\xi} = \lambda\bar{\lambda} : 1 : -\lambda : -\bar{\lambda}$. The corresponding null twistor is $U_\alpha + \lambda V_\alpha = W_\alpha \leftrightarrow (\bar{\pi}_A, \bar{\omega}^{A'})$ where

$$\bar{\pi}_{\mathfrak{A}}\pi_{\mathfrak{A}}, \propto \begin{pmatrix} dv & -d\tilde{\xi} \\ -d\xi & du \end{pmatrix} \tag{3.105}$$

And

$$\lambda = \bar{\pi}_1/\bar{\pi}_0 = W_1/W_0 .$$

Thence, as

$$\bar{\omega}^{A'} = -ip^{AA'}\pi_{A'},$$

$$(W_2, W_3) = \left(\bar{\omega}^{0'}, \bar{\omega}^{1'}\right) = -i(\bar{\pi}_0, \bar{\pi}_1)\begin{pmatrix} u & \xi \\ \tilde{\xi} & v \end{pmatrix} = -iW_0(u + \lambda\tilde{\xi}, \xi + \lambda v). \tag{3.106}$$

Thus

$$(W_0, W_1, W_2, W_3) = W_0(1, \lambda, -\left(u + \lambda\tilde{\xi}\right), -i(\xi + \lambda v)).$$

If we therefore set:

$$f(W_\alpha) = (W_0)^{-n-2}F(W_1/W_0, iW_2/W_0, iW_3/W_0) \tag{3.107}$$

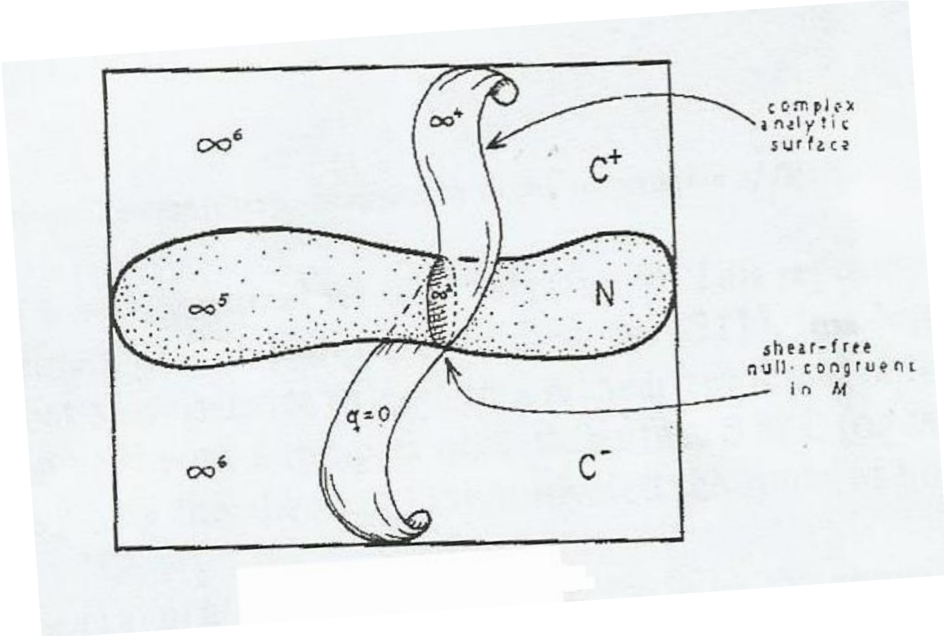


Figure17. The Kerr theorem

Then $f(W_\alpha)$ is homogeneous of degree $-n-2$ in W_α. (We can now check that this has the correct transformation properties under rotation for spin $\frac{1}{2}$ n). The final formula is:

$$\Phi_r(U_\alpha, V_\beta) = \frac{1}{2\pi i} \oint_K \lambda^r f(U_\alpha + \lambda V_\alpha) d\lambda \qquad (3.108)$$

We may now generalize by taking any U_α, V_β (no longer necessary null) thus defining complex fields on complex points $U_{[\alpha} V_{\beta]}$. It seems (although there is as yet no completely satisfactory theorem) that the set of such fields is extremely general. For a particular field it is clear that f is not unique since all the contour integrals remain the same under $f \to f + h$ where h is regular inside the contour. We may regard this as a sort of gauge invariance. This non-uniqueness of f would clearly lead to difficulties for any proposed explicit formula giving f in terms of $\phi_{A....L}$.

It is however easy to construct special type of solution for f. For example $\phi_{A....L}$ is called null if:

$$\phi_{AB....L} = \alpha_A \alpha_B \ldots . \alpha_L \qquad (3.109)$$

And such a field arises when the contour surrounds only a single simple pole [24]. (Note that a general symmetric spinor may be written as symmetrized product of one-spinors). More generally, the algebraically special fields

$$\phi_{AB....L} = \alpha_{(A}\alpha_B\beta_c \ldots \lambda_{L)} \tag{3.110}$$

If ϕ is algebraically special (e.g.null) there is associated with it a shearfree null congruence.

If

$$f(W_\alpha) = p(W_\alpha)/q(W_\alpha) \tag{3.111}$$

Then $q(W_\alpha) = 0$ is a four (real) dimensional surface in a six dimensional space (C), and intersects. The 5-dimensional surface N in a 3-dimensional set of points (fig.17). This represents 3-parameter null congruence in M. By a theorem of R. P. Kerr this congruence must be shearfree. The theorem is that acongruence of null lines is shearfree if and only if it is representable in C as the intersection of N with a complex analytic surface S in C (or as a limiting case of such an intersection). It was partly this theorem that motivated the study of holomorphic functions in twistor space.

If we suppose $q = 0$ is a plane (i.g $q(W_\alpha) = A^\alpha W_\alpha$) then we obtain the above method a “linear” system of null lines in M (a Robinson congruence), which we may consider to be a geometrical picture of the (complex) twistor A^α (which previously had no intuitively obvious picture associated with it). These “Robinson” congruences are largely what led to the name twistor, for they are shearfree, and twist with a handedness dependent on the sign of $A^\alpha \bar{A}_\alpha$.

If we consider the source free spin $\frac{1}{2}$ n massless field in M (compactified Minkowski space), which has the correct peeling-off behavior toward infinity, then the field will not match at infinity unless we take a fourfold covering for odd n (two fold for $n \equiv 0$ mod 4). (This reflected in the behavior of the integrals introduced above since the homogeneity degree of $f(Z)$ is $-n-2$ and twistors are 4-valued). Rather than work with awkward covering spaces, however, we shall make the convention that a source-free field with the correct peeling-off properties is to be regarded as continuous across infinity if it has the right “Grgin discontinuity” at infinity (i.e. a general free wave of spin $\frac{1}{2}$ n should jump by a factor of i^{n+2}).

Consider then the fields with correct peeling-off and Grgin behavior (which momentum Eigen state, for example, do not have). These may be uniquely) split into positive and negative energy fields. A process equivalent to Grgin's harmonic analysis technique applied to the positive energy fields is the following. Instead of $\bar{Z}_0 = \bar{Z}^2$ etc., let us take twistor coordinates so that we get the more natural-looking $\bar{Z}_{\aleph} = (\bar{Z}^0, \bar{Z}^1, -\bar{Z}^2, -\bar{Z}^3)$, the Hermitiam form $Z^\alpha \bar{Z}_\alpha$, of signature $(+ + --)$, being now diagonalised. The orthonormal basis $\{E_{i_\alpha}\}$ then has two vectors of positive and two of negative length. These points give us four planes (fig.18) and the simplest possible function of positive frequency has as its singular region just the planes shaded in fig.18. A general function for spin $\frac{1}{2}$ n fields of positive frequency is:

$$f(\bar{Z}_\alpha) = \sum_{a_0 a_1 a_2 a_3} \frac{(\bar{Z}_0)^{a_0} (\bar{Z}_\alpha)^{a_1}}{(\bar{Z}_2)^{a_2+1} (\bar{Z}_3)^{a_3+1}} f_{a_0 a_1 a_2 a_3} \qquad (3.112)$$

Where $f_{a_0 a_1 a_2 a_3}$ is a constant and $a_0 a_1 a_2 a_3$ are non negative integers satisfying $a_0 + a_1 + n = a_2 + a_3$. If S is the set of singularities of this function then assuming suitable convergence $S \cap C^{-*}$ is disconnected in two pieces, and so will yield a positive frequency field.

3.6 Quantization

We start out by considering how to connect the spin s of relativistic dynamics, which appeared in the classical twistor picture of regular momentum discussed above with spin s of the zero-rest-mass fields just considered.

The momentum of a particle with zero-spin was described by $\pi_A, (\bar{\pi}_A \pi_{A'} = p_a)$ while the position of the centre of mass is then determined by $\omega^A = iX^{AA'}\pi_{A'}$, As

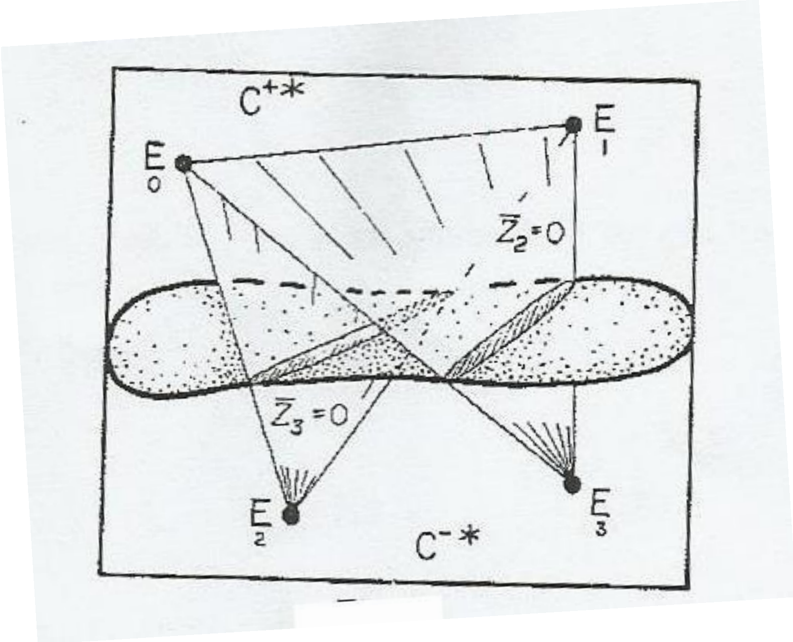


Figure.18

$$Z^\alpha \leftrightarrow (\omega^A, \pi_{A'}) \qquad\qquad \bar{Z}_\alpha \leftrightarrow (\bar{\pi}_{A'}, \bar{\omega}^{A'}$$

We find that

$$iZ^\alpha d\,\bar{Z}_\alpha \;\leftrightarrow i\omega^A d\bar{\pi}_A + \pi_{A'}\, d\bar{\omega}^{A'}$$

$$= -X^{AA'}\pi_{A'}d\bar{\pi}_A + \pi_{A'}d\big(X^{AA'}\bar{\pi}_A\big)$$

$$= X^{AA'}\pi_{A'}d\bar{\pi}_A + \pi_{A'}d\big(X^{AA'}\big)\bar{\pi}_A + \pi_{A'}X^{AA'}d\bar{\pi}_A$$

$$= \pi_{A'}\,\bar{\pi}_A dX^{AA'} = P_a dX^a \qquad (3.113)$$

If $X^{AA'}$ is real. Thus, taking the exterior derivative,

$$idZ^\alpha \wedge d\,\bar{Z}_\alpha = dP_a \wedge dX^a \qquad (3.114)$$

And the right hand side is just the two-form preserved under canonical transformations, i.e. by Hamiltonian equations. (For a fuller account of this correspondence). This suggests that we should regard iZ^α, $\bar{Z}_\alpha$ as canonically conjugate variables. Thus in the passage to a quantum theory we should expect iZ^α, $\bar{Z}_\alpha$ to become canonically conjugate operators (with $\bar{Z}_\alpha \propto \partial/\partial Z^\alpha$, etc.).

In the operator form

$$P_a = i\partial/\partial x^a \qquad \text{(and } X^a = -i\partial/\partial P_a\text{)}$$

$$P_a X^b - X^b P_a = i\delta_a^b, \tag{3.115}$$

Units being chosen so that $h = 1$. Thus we shall want

$$Z^\alpha = \partial/\partial\, \bar{Z}_\alpha \qquad (\,\bar{Z}_\alpha = -\partial/\partial Z^\alpha)$$

And

$$Z^\alpha\, \bar{Z}_\beta - \bar{Z}_\beta Z^\alpha = \delta_\beta^\alpha\ , \tag{3.116}$$

Where these operators are taken to act on functions $f(\,\bar{Z}_\alpha)$. Now ϕ is essentially given by $f(\,\bar{Z}_\alpha)$, and it is clear from taking complex conjugates that solutions of $\nabla^{\acute{A}P}\theta_{\acute{A}\,\acute{B}\ldots\ldots\acute{L}} = 0$ are similarly described by a function $g(Z^\alpha)$. Now

$$Z^\alpha f(\bar{Z}) = \frac{\partial}{\partial\, \bar{Z}_\alpha} f(\bar{Z}); \qquad \bar{Z}_\alpha f(\bar{Z}) = \bar{Z}_\alpha f(\bar{Z})$$

$$Z^\alpha g = Z^\alpha g(Z); \qquad \bar{Z}_\alpha g(Z) = -\frac{\partial}{\partial Z^\alpha} g(Z) \tag{3.117}$$

Previously we had $Z^\alpha\, \bar{Z}_\alpha = 2s$, where $S^a = sP^a, s$ being the spin parallel to the direction of motion. So consider the operator S defined by

$$4S \coloneqq Z^\alpha\, \bar{Z}_\alpha + \bar{Z}_\alpha Z^\alpha = 2(\,\bar{Z}_\alpha Z^\alpha + 2) = 2(Z^\alpha\, \bar{Z}_\alpha - 2) \tag{3.118}$$

Then

$$sg(Z^\alpha) = \tfrac{1}{2}\,\big((n+2) - 2\big)g(Z^\alpha) = sg(Z^\alpha) \tag{3.119}$$

For g is homogeneous of degree $(-n-2)$ and $2s = n$ whereas $Z^\alpha \partial g(Z)/\partial Z^\alpha$ gives $kg(Z^\alpha)$ where k is the homogeneity degree. (One may, incidentally, say that the fact that $\delta_\beta^\alpha = 4$ in twistor space, i.e. its 4-dimensionality, is related to the need for the degree $(-n-2)$ in the definition of f). We also find $Sf(\,\bar{Z}_\alpha) = sf(\,\bar{Z}_\alpha)$ if $n = -2s$, so that the twistor field corresponding to spinors with primed indices are of opposite helicity, as we expect. The fact the spin is half-integral is a consequence of the one valuedness of f.

We may inquire what is the effect of Z^{α}, $\bar{Z}_{\alpha}$ when acting on the fields ϕ .... Consider

$$f(W_{\alpha}) \to (Q^{\alpha} W_{\alpha}) f(W_{\alpha}), \tag{3.120}$$

Which is the result of $Q^{\alpha}\,\bar{Z}_{\alpha}$. If $Q^{\alpha} \leftrightarrow (Q^{A}, Q_{A'})$, eq. (3.120) corresponds to:

$$\phi_{AB\,....L} \to \tilde{Q}^{A} \phi_{AB\,....L} = \psi_{B\,....L} \tag{3.121}$$

Where $\tilde{Q}^{A} = Q^{A} - iX^{AA'} Q_{A'}$, and $\psi_{B\,....L}$ satisfies the zero-rest-mass field equation for spin $(n-1)$.

Similarly, if $R_{\alpha} \leftrightarrow (R_{A}, R^{A'})$, the operator $R_{\alpha} Z^{\alpha}$ acts so that:

$$f(W_{\alpha}) \to R_{\beta} \frac{\partial}{\partial W_{\beta}} f(W_{\alpha}); \tag{3.122}$$

$$\phi_{AB\,....L} \to \frac{1}{2} i(n+1) \phi_{(AB\,....L} \nabla_{M)M'} \tilde{R}^{M'} + i\tilde{R}^{M'} \nabla_{M'M} \phi_{AB\,....L} = X_{AB\,....M}, \tag{3.123}$$

where $X_{AB\,....M}$ is a solution of the zero-rest-mass field equation for spin$(n+1)$. Thus $\bar{Z}_{\alpha}$ raises, and Z^{α} lowers, the helicity by one half.

Chapter Four

Applications of Twistor Space in 3D

4.1 3D Twistors and The Biharmonic Equation

The use of complex variable techniques in applied mathematics, and especially fluid dynamics, is dominated by two-dimensional applications through the prescription

$$w = x + iy \tag{4.1}$$

On the other hand, in the context of relativistic physics in four or more dimensions, the use of twistor methods due to R. Penrose and co-workers is becoming an ever more present tool in the hands of theoretical physicists. The focus of much of the published work has been on time-independent problems within the general context of theoretical relativistic physics. In this section the idea is to present such methods as being a routinely useful tool in traditional applied mathematics. To this end, an example of the application of twistor theory to viscous fluid flow is presented. In particular, the solution of various biharmonic problems will be presented using contour integral techniques. The ultimate goal of this work is a better understanding of the Navier-Stokes equations through the geometry of holomorphic complex variable techniques at first sight, even our most basic goal might seem to be an unreasonable proposal. For example, the biharmonic equation in two dimensions, with the $w = x + iy$ prescription, amounts to

$$\partial_w^2 \partial_{\bar{w}}^2 = 0 \tag{4.2}$$

with the general real solution

$$\Psi = Re\{\bar{w}f(w) + g(w)\} \tag{4.3}$$

where f and g are both locally holomorphic. This is generally regarded as going outside the holomorphic context as it involves $\bar{w}$ in an essential way. We shall show that equation (4.3) is in fact the two-dimensional projection of an essentially holomorphic three-dimensional result.

4.2.1 The Navier-Stokes Equations:

The Navier-Stokes equations are the set of nonlinear partial differential equations that describe the flow of fluids

4.2.2 Steady Viscous Incompressible Flow

A large class of fluids can be characterized by their density ρ a scalar field not presumed to be constant, and their dynamic viscosity μ. The flow is characterized by a velocity vector field $\underline{v}$, and an associated scalar pressure field p .

Conservation of mass is expressed by the continuity equation

$$\frac{\partial \rho}{\partial t} + \underline{\nabla}.\left(\rho \underline{v}\right) = 0 \tag{4.4}$$

and the conservation of momentum is expressed by the Navier-Stokes equations

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}.\underline{\nabla}\,\underline{v}\right) = -\underline{\nabla}\,p + \mu\nabla^2\underline{v} \tag{4.5}$$

4.2.3 Remedial Fluids

If the fluid is incompressible in the sense that is a constant in both time and space, we have the condition:

$$\underline{\nabla}.\underline{v} = 0 \tag{4.6}$$

To analyze matters further, we introduce the vorticity vector

$$\underline{w} = \underline{\nabla} \times \underline{v} \tag{4.7}$$

We demand incompressibility but allow for non-zero vorticity. We let

$$H = \left(p + \frac{1}{2}p\underline{v}^2\right) \tag{4.8}$$

4.2.4 Recasting of Navier-Stokes

If $\underline{\nabla}.\underline{v} = 0$ then $\nabla^2\underline{v} = -\underline{\nabla} \times \underline{v}$ using simple identities from vector calculus the Navier-Stokes equations may then be recast in the form

$$\rho\left(\frac{\partial \underline{v}}{\partial t} - \underline{v} \times \underline{w}\right) + \underline{\nabla}\left(p + \frac{1}{2}\rho\underline{v}^2\right) = -\mu\underline{\nabla} \times w \qquad (4.9)$$

Taking the curl of this, we get vorticity equation

$$\frac{\partial \underline{w}}{\partial t} + \underline{v}.\underline{\nabla}\,\underline{w} - \underline{w}.\underline{\nabla}\,\underline{v} = v\nabla^2\underline{w} \qquad (4.10)$$

where the kinematic viscosity $v = \mu/\rho$

4.2.5 The 'Stream Vector Potential

Since the velocity field is divergence-free, we may introduce a vector potential $\underline{\Psi}$ such that

$$\underline{v} = \underline{\nabla} \times \underline{\Psi} \qquad (4.11)$$

and furthermore we may choose it so that it is divergence free

$$\underline{\nabla}.\underline{\Psi} = 0 \qquad (4.12)$$

In theoretical physics, notably electromagnetic theory, this is known as setting a gauge condition. The tradition in fluid dynamics is to mainly use the vector potential only when it can be reduced to a single function using some type of symmetry. The resulting object is a stream function. For example, planar $2D$ flow is obtained by setting (and note that this automatically satisfies the gaugecondition)

$$\underline{\Psi} = -\psi(x, y)\underline{e}_z \qquad (4.13)$$

4.2.6 Good Idea in 3D Too

We will work with the full vector form. First of all we note that under the assumption that $\underline{\Psi}$ satisfies $\underline{\nabla}.\underline{\Psi} = 0$.

$$w = -\underline{\nabla}^2\underline{\Psi} \qquad (4.14)$$

and the vorticity equation becomes, denoting $\frac{\partial}{\partial t}$ by:

$$\nabla^4\underline{\Psi} = \frac{1}{v}\left\{\left(\left(\nabla^2\underline{\Psi}\right).\underline{\nabla}\right)\underline{\nabla}\times\underline{\Psi} - \left(\left(\underline{\nabla}\times\underline{\Psi}\right).\underline{\nabla}\right)\left(\underline{\nabla}^2\underline{\Psi}\right) + \underline{\nabla}^2\underline{\dot{\Psi}}\right\} \qquad (4.15)$$

or indeed as

$$\nabla^4\underline{\Psi} = \frac{1}{v}\left\{\underline{\nabla}\times\underline{\left(\left(\underline{\nabla}\times\underline{\Psi}\right)\times\underline{\nabla}^2\underline{\Psi}\right)\underline{\Psi}}\right\} + \underline{\nabla}^2\underline{\dot{\Psi}} \qquad (4.16)$$

This latter representation of the Navier-Stokes equations is well-known in the $2D$ planar casewhere it reduces to the equation

$$\nabla^4\underline{\Psi} = \frac{1}{v}\frac{\partial(\psi.\nabla^2\psi)}{\partial(x,y)} \qquad (4.17)$$

4.1.7 Potential Flow

Here $\underline{w}$ is zero and $\nabla^4\underline{\Psi} = \underline{0}$ vorticity equations satisfied as identity. The potential ϕ is

$$\phi(\underline{r}) = \int\left(\underline{\nabla}\times\underline{\Psi}\right).d\underline{r}' \qquad (4.18)$$

and is harmonic conjugate of ψ in 2D case

$$\nabla^4\phi = 0 \qquad , \qquad \nabla^2\underline{\Psi} = \underline{0} \qquad (4.19)$$

4.2.8 The Biharmonic Limit

When viscosity $v \to \infty$, ignore non-linearities time-independent Navier-Stokes equations reduce to

$$\nabla^4\underline{\Psi} = 0 \qquad (4.20)$$

which is the biharmonic limit, also known as Stokes flow. We want to understand the biharmonic structure for the 3D vector version of

$$\frac{\partial^4\psi}{\partial w^2\partial\overline{w}^2} = 0 \qquad (4.21)$$

We shall focus on the solution of this equation by complex variable methods. It is now well known (see for example lack of any solution for asymptotically uniform two-dimensional flow past a cylinder. However, in attempting to construct a twistor description of fluid flow we must be able to at least solve the biharmonic equation. It is to this that we now turn.

4.2.9 Twistor Solutions of The Laplace and Biharmonic Equation

We need anew picture to proceed. It is very well known that the Laplace equation can be solved in terms of holomorphic functions in two dimensions. Among devotees of twistor methods, and student of Bateman, Whittaker, it is known that this can be carried out in three dimensions. Want to extend to the biharmonic case .This can be done. This is at least an opportunity to explain how to use complex methods in 3 dimensions. In 2 D we let $z = x + iy$. What also have a z ? (Never put $z = x + iy$) but what complex structure do we use ? the key is twistor space for $3D$

The twistor space associated with R^3 is first, as a real space, the set of oriented straight lines in R^3. Relative to some origin O. Let r denote the position vector of the point on a given line nearest to O. Then r is orthogonal to the direction of the line, which we denote by $\underline{u}$ with $u.u = 1$. So the set of oriented straight lines is the set.

$$TS^2 = \{(\underline{r}, \underline{u}) \epsilon R^3 \times S^2 | \underline{r}.\underline{u} = 0\} \tag{4.22}$$

This set is a naturally the tangent bundle to a complex manifold, where S^2as the Riemann sphere CP^1. This TCP^1 (complex tangent bundle) is the twistor space.

4.2.11 Defining A Points I

How do we define a point in ordinary space in terms of some structure on TCP^1? A point may be regarded as the intersection of all straight lines through it. This means that a point is necessarily some vector field in TCP^1 that is defined globally.

4.2.12 Defining A Points II

To see the implications of this we introduce two open sets that cover CP^1. We can take coordinates for the sphere as ξ on one patch (covering everything except infinity), and $\bar{\xi} = \frac{1}{\xi}$ on another patch, covering everything except $\bar{\xi} = 0$. Over each of these respective patches we can define coordinates for the tangent bundle as (η, ξ) and $(\bar{\eta}, \bar{\xi})$, where the relevant vector fields are, respectively

$$\eta \frac{\partial}{\partial \xi} \quad , \quad \bar{\eta} \frac{\partial}{\partial \bar{\xi}} \tag{4.23}$$

4.2.13 Defining A Points III

Consider now a holomorphic vector field. On the ζ patch it can be written as

$$f_0(\xi) \frac{\partial}{\partial \xi} \tag{4.24}$$

for some f_0, and on the $\tilde{\xi}$ patch, it can be written as

$$f_1(\tilde{\xi}) \frac{\partial}{\partial \tilde{\xi}} \tag{4.25}$$

for some f_1.

4.1.14 Defining A Points IV

On the intersection of the two patches equality of the two representations gives us

$$f_1(\xi^{-1})(-\xi^2) \frac{\partial}{\partial \xi} = f_0(\xi) \frac{\partial}{\partial \xi} \tag{4.26}$$

If we make a Taylor series expansion of both functions,

$$f_i(\xi) = \sum_n^0 a_n'^i \, \xi^n \tag{4.27}$$

Deduce that the coefficients a_n^i vanish if $n > 2$. , the global vector field smust be of the form, for example on the η patch:

$$\eta(\xi) = a + b\xi + c\xi^2 \tag{4.28}$$

so that such quadratics are the only holomorphic vector fields, and these correspond to points of C^3, parametrized in some way by (a, b, c).

4.1.15 Summary of Reality and Metric

Further analysis of this system allows the identification of real points in R^3, and the construction of a natural metric. The points are real if and only if

$$c = -\bar{a} \qquad \text{and} \qquad b = -\bar{b} \tag{4.29}$$

The induced metric is proportional to the discriminant of the quadratic, and we shall normalize matters such that

$$ds^2 = dx^2 + dy^2 + dz^2 = \frac{1}{4}db^2 - dadc \tag{4.30}$$

The metric for real points:

$$ds^2 = dx^2 + dy^2 + dz^2 = \frac{1}{4}db^2 + dad\bar{a} \tag{4.31}$$

4.1.16 The Final Point Correspondence

If we pick our coordinate system such that the real part of a is , we see that we can take the imaginary part of a to be ±y and set b = ±2z.
The convention is to set:

$$\eta_{\underline{r}}(\xi) = (x + iy) + 2z\xi - (x - iy)\xi^2 \tag{4.32}$$

This gives us the correspondence between real points in $3D$ and global holomorphic vector fields

4.1.17 Solving the Scalar Laplace Equation

We consider a function $f(\eta, \xi)$ defined on twistor space. This can then be thought of as restricted to the special global sections of twistor space represented by $\eta_{\underline{r}}(\xi)$, and the ξ dependence integrated out by integration over a contour C. We set:

$$\phi(\underline{r}) = \int_c f(\eta_r(\xi), \xi) d\xi \tag{4.33}$$

4.1.18 Laplace II

It is easy to check ϕ satisfies the scalar Laplace's equation. To see this observe that

$$\frac{\partial^k f(\eta_{\underline{r}}(\xi), \xi)}{\partial x^k} = (1 - \xi^2)^k \frac{\partial^k f}{\partial \eta^k} | \eta = \eta_{\underline{r}} \tag{4.34}$$

$$\frac{\partial^k f(\eta_{\underline{r}}(\xi), \xi)}{\partial y^k} = i^k (1 + \xi^2)^k \frac{\partial^k f}{\partial \eta^k} | \eta = \eta_{\underline{r}} \tag{4.35}$$

$$\frac{\partial^k f(\eta_{\underline{r}}(\xi), \xi)}{\partial z^k} = (2\xi)^k \frac{\partial^k f}{\partial \eta^k} | \eta = \eta_{\underline{r}} \tag{4.36}$$

and that adding these three expressions with $k = 2$ gives zero identically for any choice of f.

4.1.19 Note

many different choices of f will give rise to the same. Such choices differ by the additions of functions that are holomorphic inside or outside of C, so that one must pursue a cohomological approach in order to state a formal isomorphism between structures on twistor space and solutions of the Laplace equation.

4.1.20 Solving the Scalar Biharmonic Equation

How do we modify the integrand $f(\eta_{\underline{r}}(\xi), \xi)$, say to some holomorphic function g, to arrange that $\nabla^4 g = 0$ but $h(\underline{r}, \xi)$? We try to build g from f by multiplying by some prefactor, so that:

$$g = h(\underline{r}, \xi) f(\eta_{\underline{r}}, \xi) \tag{4.37}$$

Now

$$\nabla^2 g = \nabla^2 hf = f\nabla^2 h + h\nabla^2 f + 2\underline{\nabla} h \underline{\nabla} f = f\nabla^2 h + 2\underline{\nabla} h \underline{\nabla} f \tag{4.38}$$

That is, as f satisfies the Laplace equation.

$$\nabla^2 g = f\nabla^2 h = 2\underline{\nabla} h \underline{\nabla} f \tag{4.39}$$

4.1.21 Biharmonics II

If we furthermore choose h to be linear in $\underline{r}$ matters simplify further and we have

$$\nabla^2 g = 2\underline{\nabla} h \underline{\nabla} f \tag{4.40}$$

Let us set, w.l.o.g, $h = \underline{u}(\xi).\underline{r}$, so that. $\underline{\nabla} h = \underline{u}(\xi)$ We also note that

$$\underline{\nabla} f = \frac{\partial f}{\partial \eta} \underline{\nabla} \eta = \frac{\partial f}{\partial \eta}(1 - \xi^2, i(1 + \xi^2), 2\xi) \tag{4.41}$$

4.1.22 Biharmonics III

Putting this all together, we arrive at

$$\nabla^2 g = \nabla^2 hf = 2\underline{u}(\xi).(1 - \xi^2, i(1 + \xi^2), 2\xi)\frac{\partial f}{\partial \eta} = 2\eta_{\underline{u}(\xi)}(\xi)\frac{\partial f}{\partial \eta} \quad (4.42)$$

We can now see that $\nabla^4 g = 0$ of this last expression vanishes identically, while this expression does not itself vanish unless

$$\eta_{\underline{u}(\xi)}(\xi) = 0 \quad (4.42)$$

4.1.23 Biharmonics IV

To see what is happening, we can now make matters more explicit. We let

$$\underline{u}(\xi) = u_1(\xi), u_2(\xi), u_3(\xi) \tag{4.43}$$

Then

$$\underline{u}.\underline{r} = u_1(\xi)x + u_2(\xi)y + u_3(\xi)z \tag{4.44}$$

And

$$\eta_{\underline{u}(\xi)}(\xi) = \big(u_1(\xi) + iu_2(\xi)\big) + 2u_3(\xi)\xi - \big(u_1(\xi) - iu_2(\xi)\big)\xi^2 \tag{4.45}$$

4.2.24 Biharmonics

In terms of these variables the proposed integral representation for solutions of the $3D$ scalar biharmonic equation is just

$$\Psi = \int_c d\xi[xu_1(\xi) + yu_2(\xi) + zu_3(\xi)]f\left(\eta_{\underline{r}(\xi)}, \xi\right) \tag{4.46}$$

or indeed , with $w = x + iy$

$$\Psi = \frac{1}{2}\int_c d\xi[wg - (\xi) + \bar{w}g + (\xi) + 2zu_3(\xi)]f\left(\eta_{\underline{r}(\xi)}, \xi\right) \tag{4.47}$$

where

$$g_{\pm}(\xi) = u_1\left((\xi) \pm iu_2(\xi)\right) \tag{4.48}$$

4.2.25 The Scalar Biharmonic Problem In 2D

Suppose we want no z-dependence. We set $u_3 = 0$ and $w = x + iy$, so that

$$\Psi = \int_c d\xi[wg - (\xi) + \bar{w}g + (\xi)]f\left(\eta_{\underline{r}(\xi)}, \xi\right) \tag{4.49}$$

We can write this in the equivalent form

$$\Psi = w\int_c d\xi f_1\left(\eta_{\underline{r}(\xi)}, \xi\right) + \bar{w}\int_c d\xi f_2\left(\eta_{r(\xi)}, \xi\right) \tag{4.50}$$

Now consider the second term. This is $\bar{w}\emptyset(x, y, z)$ where $\emptyset$ is a solution of Laplace's equation and is just

$$\phi(x, y, z) = \int_c d\xi f_2\left(\eta_{r(\xi)}, \xi\right) \tag{4.51}$$

We want this not to depend on z either. But this looks awkward given that

$$\eta_{\underline{r}(\xi)} = (x + iy) + 2z\xi - (x - iy)\xi^2 \tag{4.52}$$

Actually it is not

$$\phi(x,y,z+h/2) = \int_C d\xi f_2\big(\eta_{\underline{r}}(\xi) + h\xi, \xi\big) \tag{4.53}$$

The equation we need is

$$\phi(x,y,z+h/2) = \phi(x,y,z) \tag{4.54}$$

This does not require that

$$f_2\big(\eta_{\underline{r}}(\xi) + h\xi, \xi\big) = f_2\big(\eta_{\underline{r}}(\xi), \xi\big) \tag{4.55}$$

Instead we need

$$f_2\big(\eta_{\underline{r}}(\xi) + h\xi, \xi\big) = f_2{}'\big(\eta_{\underline{r}}(\xi), \xi\big) + g_0(\eta, \xi, h) - g_1(\eta, \xi, h) \tag{4.56}$$

where g_0 is holomorphic on and inside C and g_1 is likewise outside. Cauchy's theorem. Let's take C to be unit circle, or to be deformable to the unit circle. Now differentiate w.r.t h then set $h = 0$. We obtain, for some G_i,

$$\xi \frac{\partial f_2}{\partial \eta} = G_0(\eta, \xi) - G_1(\eta, \xi) \tag{4.57}$$

We integrate this w.r.t. η and divide by ξ. We obtain, for some H_i,

$$f_2 = \frac{H_0(\eta, \xi)}{\xi} - \frac{H_1(\eta, \xi)}{\xi} \tag{4.58}$$

and recall that H_0 must be holomorphic inside C and H_1 holomorphic outside. Now we evaluate the integral of Eq. (4.51) using calculus of residues. The first term in Eq. (4.55) is easy, and we get

$$2\pi i H_0\big(\eta_{\underline{r}}(0), 0\big) = K(w) \tag{4.59}$$

for some function $K(w)$, giving a contribution to ϕ of $\bar{w}k(w)$ When we calculate the contribution of the second term of Eq. (4.55) to the integral of Eq. (4.51), we make the transformation $\xi \to \tilde{\xi}$ and obtain an integrand that is a function of $\tilde{\eta} = (x - iy) - 2z\tilde{\xi} - (x + iy)\xi^2$. Taking the residue at $\hat{\xi} = 0$ gives a function of $\bar{w} = x - iy$, also to be multiplied by $\bar{w}$.

The other two terms in Eq. (4.50) may be treated similarly. We end up with four terms contributing to:

$$\Psi = \bar{w}K_2(w) + \bar{w}\tilde{k}_2(\bar{w}) + wK_1(\bar{w}) + w\tilde{k}_1(w) \qquad (4.60)$$

When Ψ is real we must have Eq. (4.24). So the fully holomorphic picture in three dimensions projects, via the calculus of residues, to a two-dimensional picture and generates the familiar yet superficially non-holomorphic two-dimensional representation of solutions to biharmonic (and Laplace) equations. In three dimensions our functions are contour integrals.

4.1.26 The Axis-Symmetric Scalar Problem

We go back to the representation

$$\Psi = \frac{1}{2}\int_c d\xi[wg - (\xi) + \bar{w}g + (\xi) + 2zu_3(\xi)]\, f\big(\eta_{\underline{r}}(\xi), \xi\big) \qquad (4.61)$$

with $w = x + iy$. We can regard this as three pieces, where we discard the factors of a half:

$$\Psi_- = w\int_c d\xi g - (\xi)f\big(\eta_{\underline{r}}(\xi), \xi\big) = w\psi_- \qquad (4.62)$$

$$\Psi_+ = \bar{w}\int_c d\xi g + (\xi)f\big(\eta_{\underline{r}}(\xi), \xi\big) = \bar{w}\psi_+ \qquad (4.62)$$

$$\Psi_3 = z\int_c d\xi u_3(\xi)f\big(\eta_{\underline{r}}(\xi), \xi\big) = z\psi_0 \qquad (4.63)$$

In order to develop axis-symmetric solutions, we need to understand the action of the group of rotations about the z-axis. We need to bear in mind the formula

$$\eta = w + 2z\xi - \bar{w}\xi^2 \qquad (4.64)$$

with w = x + iy. Under a rotation about the z-axis, $z \to z$ and $w \to exp(i\emptyset)w$ this is compatible with the action $(\eta, \xi) \to exp(i\emptyset)(\eta, \xi)$. In seeking axis-symmetric solutions for $\psi_{\pm,0}$ we need to arrange that

$$d\xi g - f \to exp(-i\emptyset)d\xi g - f \qquad (4.65)$$
$$d\xi g + f \to exp(i\emptyset)d\xi g + f \qquad (4.66)$$
$$d\xi u_3 f \to d\xi u_3 f \qquad (4.66)$$

To treat all of these situations together, we consider the case where. $d\xi h(\eta, \xi) \to exp(im\emptyset)d\xi h(\eta, \xi)$. To this end we consider the contour of integration to be the unit circle and consider a basic set

$$\psi_{n,m} = \frac{1}{2\pi i}\int d\xi \frac{\eta^n}{\xi^{n+1-m}} \qquad (4.68)$$

where for ψ_0 , $m = 0$, and for $\phi_\pm$, $m = \pm 1$. So our task now is to calculate

$$\psi_{n,m} = \frac{1}{2\pi i}\int d\xi \frac{\eta^n}{\xi^{n+1-m}}(w + 2z\xi - \bar{w}\xi^2)^n \qquad (4.69)$$

By multiplying these by the relevant factors of $w, \bar{w}, z$ for $m = -1, 1, 0$ we get an interesting set of axis-symmetric biharmonic functions. The functions n, m themselves are now contour integral solutions of Laplace's equation. This is of course of interest in itself.

To evaluate this set we let y = 0 since $\psi_{n,m}(r, \theta, \phi) = e^{im\phi}(r, \theta, 0)$. Then, we have, in spherical polar coordinates

$$\psi_{n,m} = \frac{1}{2\pi i}\int d\xi \xi^{m-1}(2\cos(\theta) + \sin(\theta))^n \qquad (4.70)$$

Parametrizing the integral as $\xi = e^{it}$, we obtain

$$\psi_{n,m} = \frac{(2r)^n}{2\pi}\int dt e^{imt}\left(\cos(\theta) + \left(\frac{1}{\xi} - \xi\right)\sin(\theta)\sin(t)\right)^n \qquad (4.71)$$

Performing some manipulations, we see that, discarding normalizations, if $n \neq 1$.

$$\psi_{n,m}\alpha\begin{cases} r^n P_n^m(\cos(\theta)) & n = 0,1,2,3,\dots\dots\dots\dots\dots\dots \\ \frac{1}{r^{k+1}}P_k^m(\cos(\theta)) & n = -k-1, k = 1,2,3,\dots \end{cases} \quad (4.72)$$

When $n = -1$ matters are quite subtle as the integral branches depending on the sign of z! A full treatment of this is rather beyond the scope of this paper but we note that in this case,

$$\psi_{n,m} = \frac{1}{2\pi i}\int d\xi \frac{\xi^m}{w + 2z\xi - \bar{w}\xi^2} \quad (4.73)$$

The quadratic in the denominator has two roots ± given by

$$\xi_\pm = \frac{-z\pm r}{-\bar{w}}, \quad r = \sqrt{x^2+y^2+z^2} \quad , \bar{w} = x - iy \quad (4.73)$$

These roots are located, using standard spherical polar coordinates, at

$$\xi_+ = e^{i\theta}\tan\left(\frac{\theta}{2}\right) \ , \ \xi_- = e^{i\theta}\cot\left(\frac{\theta}{2}\right) \quad (4.74)$$

and we can write

$$\psi_{-1,m} = \frac{-1}{2\pi i\bar{w}}\int d\xi \frac{\eta^m}{\left((\xi-\xi_+)(\xi-\xi_-)\right)} \quad (4.75)$$

The details of the global evaluation of this are lengthy. We note here that when $z > 0$, $|\xi_+| < 1$ and when $m \geq 0$, the single residue inside the unit circle

$$\psi_{-1,m} = \frac{-\xi_+^m}{\bar{w}(\xi_+ - \xi_-)} = \frac{\xi_+^m}{2r} \quad (4.76)$$

so in particular we obtain the Coulomb field in the region $z > 0$ when $m = 0$. The reader is invited to explore the other cases.

3.2.27 Axis-Symmetric Stokes Flow

This is traditionally modelled in terms of the Stokes stream function $\Psi_S(r,\theta)$. The components of the velocity field are given by

$$\mathrm{u}_r = \frac{1}{r^2 \sin\theta}\frac{\partial \Psi_s}{\partial \theta} \quad , \qquad \mathrm{u}_\theta = \frac{1}{r \sin\theta}\frac{\partial \Psi_s}{\partial r} \qquad (4.77)$$

What this representation is really telling us, as is made clear in modern fluid theory, is that the vector potential $\underline{\Psi}$ for the flow is given by

$$\underline{\Psi} = -\frac{\Psi_s}{r\sin(\theta)}\underline{e}_\phi \qquad (4.78)$$

as is revealed, together with the fact that $\underline{\Psi}$ it is divergence-free, by elementary calculations with the curl and div operator expressed in a spherical basis. A further elementary calculation in vector calculus shows that symmetric for an axis-symmetric function $f(r,\theta)$

$$\underline{\nabla} \times \left(\underline{\nabla} \times \frac{f}{r\sin(\theta)}\underline{e}_\phi\right) = -\frac{-1}{r\sin(\theta)}(E^2 f)\underline{e}_\phi \qquad (4.79)$$

where the operator D^2 is given by

$$D^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial f}{\partial \theta}\right) \qquad (4.80)$$

The biharmonic condition may be expressed as the scalar PDE

$$D^4 \Psi_s = 0 \qquad (4.81)$$

4.1.29 Cartesian Basis

However, this representation in some ways obscures the underlying simplicity of the problem. To see why, we need to work with the problem in the full vector form, and, perhaps surprisingly, recast it in a Cartesian basis. In this way we can use the contour integral technology already developed for the scalar biharmonic problem. We write the basis vector $\underline{e}_\phi$ in the form

$$\underline{e}_\phi = \frac{-y\underline{e}_x + x\underline{e}_y}{\sqrt{x^2+y^2}} = \frac{-y\underline{e}_x + x\underline{e}_y}{r\sin(\theta)} = \frac{1}{r\sin(\theta)} Re\{w[i\underline{e}_x + \underline{e}_y]\} \quad (4.82)$$

where $w = x + iy$ as before. We could equally well write this down in terms of $\overline{w}$. Now recall that the full vector potential is given in terms of the Stokes streamfunction by the relation)

$$\underline{\Psi} = -\frac{\Psi_s}{r\sin(\theta)}\underline{e}_\phi \tag{4.83}$$

and if Ψ_s is real we can write the vector potential as

$$\underline{\Psi} = Re\left\{\Psi_s \frac{\Psi_s w}{r^2\sin^2(\theta)}\left[i\underline{e}_x + \underline{e}_y\right]\right\} \tag{4.84}$$

The components of this with respect to a Cartesian basis must satisfy the scalar biharmonic equation, or indeed, as a special case, the Laplace equation. We now appeal to equation (4.62), where we note that Ψ_- is just a harmonic function. It follows that we can write the parts of Ψ_s that are biharmonic

4.1.30 Cartesian Analysis

Cartesian components must satisfy the scalar biharmonic equation

$$\Psi_s = r^2\sin^2(\theta)\, g(r,\theta) = w\overline{w}\frac{1}{2\pi i}\int d\xi\,\frac{1}{\xi}f\left(\frac{\eta}{\xi}\right)$$

for some complex function f. f can be expanded as a Laurent series: Here g iharmonic and axis-symmetric and can therefore be written in terms of the $\psi_{n,0}$ functions given in equation (4.69) or indeed in terms of normal Legendre functions and powers of r via a Laurent expansion of f in the form

$$\Psi_s = w\overline{w}\frac{1}{2\pi i}\int d\xi \sum_{n=-\infty}^{n=\infty}\frac{a_n}{\xi^{n+1}}(w + 2z\xi - \overline{w}\xi^2)^{\,n} \tag{4.85}$$

The terms in the series can be evaluated in terms of Legendre functions.We argue that these relations are the natural axis-symmetric versions of (4.60). Of course, in general, we need to add in harmonic components, just as in the $2D$ planar case where we can add to Ψ any pair $k_1(w)$ and $k_2(\overline{w})$of holomorphic and anti-holomorphic functions. To treat this we look again at the representation in a Cartesian basis, this time in the form:

$$\underline{\Psi} = \frac{\Psi_s}{r\sin(\theta)}\left(-\sin(\phi)\underline{e}_x - \cos(\phi)\underline{e}_y\right) \tag{4.86}$$

We deduce that the function

$$\underline{\Psi} = -\frac{\Psi_s}{r\sin(\theta)}e^{\pm i\emptyset} \tag{4.87}$$

must be harmonic and therefore a solution of Laplace's equation with $m = \pm 1$ as described above. By packaging this up as before, we can write these harmonic contributions to Ψ_s, say Ψ_{SH} in the elegant form

$$\Psi_{SH} = \frac{\overline{\mathrm{w}}}{2\pi i}\int d\xi\beta\left(\frac{\eta}{\xi}\right) + \frac{\mathrm{w}}{2\pi i}\int d\xi\frac{1}{\xi^2}\gamma\left(\frac{\eta}{\xi}\right) \tag{4.88}$$

for some choice of complex functions β and γ .

4.1.31 The Stokes Stream Function

The contour integral solution for the Stokes stream function for axis-symmetric biharmonic flow

$$\Psi_S = w\overline{w}\frac{1}{2\pi i}\int d\xi\frac{1}{\xi}f\left(\frac{\eta}{\xi}\right) + \frac{\overline{\mathrm{w}}}{2\pi i}\int d\xi\beta\left(\frac{\eta}{\xi}\right) + \frac{\mathrm{w}}{2\pi i}\int d\xi\frac{1}{\xi^2}\gamma\left(\frac{\eta}{\xi}\right) \tag{4.89}$$

where η is written in terms of x, y, z and where $f, \beta, \eta,$ have Laurent series expansions that generate expansions in terms of powers of γ and regular η and modified (β, γ) functions.

4.1.34 Example: (Simple Twistor Function)

f constant gives a contribution to Ψ_S proportional to

$$w\overline{w} = r^2\sin^2(\theta) \tag{4.90}$$

We also know (at least locally) that the choice $f(z) = 1/z$ gives a Coulomb field and a contribution to Ψ_s proportional to

$$\frac{w\overline{w}}{r} = r\sin^2(\theta) \tag{4.91}$$

Another interesting contribution can be generated by the choice $\beta(z) = \frac{1}{z^2}$, where an elementary exercise in the calculus of residues leads to a contribution to Ψ_s of the form

$$\frac{w\overline{w}}{\mathrm{r}^3} = \frac{1}{\mathrm{r}}\sin^2(\theta) \tag{4.92}$$

If we take a general linear combination of these three in the form

$$\sin^2(\theta)\left\{A\,\mathrm{r}^2 + Br + \frac{C}{\mathrm{r}}\right\} \tag{4.93}$$

we obtain a valid stream function. The particular choice

$$\Psi_s = \frac{U}{2}\sin^2(\theta)\left\{\mathrm{r}^2 + \frac{3ar}{2} + \frac{a^3}{2\mathrm{r}}\right\} \tag{4.94}$$

gives the well-known stream function for very viscous flow around a sphere of radius a and uniform flow at rate U at infinity. Having non-dimensionalized we would, for example, scale so that $a = 1$. In general we have a contour integral technique for solving the PDE given by equation (4.81).

4.1.35 Small but Non-Vanishing Reynolds Number

A natural question to ask is to wonder how much of the above is dependent on the purely linear structure that arises in the biharmonic limit. We cannot yet answer this question for a general Reynolds number in three dimensions, but we can observe that something very interesting happens when we (i) go back to two dimensions and (ii) consider the case of a small but non-zero Reynolds number.

Let us go back to the non-dimensional form of (4.17). This is

$$D^4\psi = \mathrm{R}\frac{\partial(\psi, \nabla^2\psi)}{\partial(x,y)} \tag{4.95}$$

In terms of the complex variable $w = x + iy$, we can write this in the form

$$i\frac{\partial^4\psi}{\partial w^2 \partial \overline{w}^2} = \frac{R}{2}\left(\frac{\partial\psi}{\partial w}\frac{\partial^3\psi}{\partial w \partial \overline{w}^2} - \frac{\partial\psi}{\partial \overline{w}}\frac{\partial^3\psi}{\partial \overline{w} \partial w^2}\right) \tag{4.96}$$

Rather than pursuing the approach of Legendre (1949) and Ranger (1991, 1994) we can consider instead the interesting physical case of small but non-vanishing Reynolds number. Let us assume that the solution for may be written as

$$\psi = \psi_0 + R\psi_1 + O(R^2) \tag{4.97}$$

and that

$$\psi_0 = Re\{\overline{w} f_0(w) + g_0(w)\} \tag{4.98}$$

This is a very strong assumption, and it is well known that this assumption of a power series dependence on the Reynolds number may fail. There may not indeed be a sensible form for ψ_0 over a simple domain of interest. We illustrate that the low Reynolds number perturbation equation may indeed be integrated using holomorphic methods. The result may be of use in refining the results for a certain sub-class of problems where there is both a meaningful ψ_0 and the inertia terms in the Navier-Stokes equations (i.e. the non-linear terms) arising from ψ_0 remain small over the entire domain of interest. Under these strong assumptions we can proceed. The equation for ψ_1 is, under these assumptions,

$$i\frac{\partial^4\psi_1}{\partial w^2 \partial \overline{w}^2} = \frac{1}{8}\left(\overline{f_0''(w)}\left[\overline{w}\, f_0'(w) + g_0'(w) + \overline{f_0(w)}\right] - f_0''(w)\left[w\overline{f_0'(w)} + \overline{g_0'(w)} + f_0(w)\right]\right) \tag{4.99}$$

This may be solved almost explicitly as follows. We let $F(w), G(w), H(w)$ be holomorphic functions with the properties

$$F'(w) = f_0(w)\,, \quad G'(w) = g_0(w)\,, \quad H''(w) = f_0(w) f_0''(w) \qquad (4.100)$$

Then a particular solution to Eq. (4.99) is given by

$$\psi_{1P} = \frac{1}{4}\,\mathrm{Im}\left(\left(\overline{wF'(w) - 2F(w)}\right) F(w) + \overline{F'(w)}\,G(w) + \frac{w^2}{2}\overline{H(w)} \right)$$

and a complementary function exists in the obvious form

$$\psi_{1CF} = \mathrm{Re}\{\overline{w} f_1(w) + g_1(w)\} \qquad (4.101)$$

Where f_1 and g_1 are arbitrary holomorphic functions. So we can see that apart from the practical issue on constructing the integrals in Eq. (4.100), the first perturbation can be constructed by separate integration of the w and $\overline{w}$ components. In fact, we have shown that the perturbative non-linear problem may be solved in terms of free holomorphic functions F, G, f_1, g_1 and the solution, apart from the construction of H, is given explicitly in terms of this holomorphic information. This observation gives some hope that a corresponding three-dimensional structure might exist

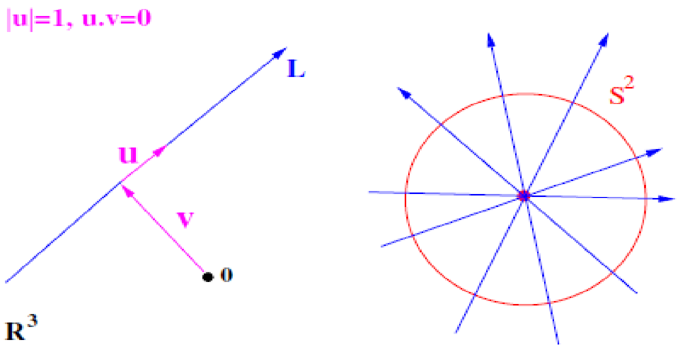
4.2 Non-Abelian Monopoles and Euclidean Mini-Twistors

4.2.1 Complex structure on $\mathbf{R}^3$

It is well known that the problem of finding harmonic functions in R^2 can be solved 'in one line' by introducing complex numbers: any solution of a two-dimensional Laplace equation $\emptyset_{xx} + \emptyset_{yy} = 0$ is a real part of a function holomorphic in $x + iy$. This technique fails when applied to the Laplace equation in three dimensions as R^3 cannot be identified with C^n for any n.

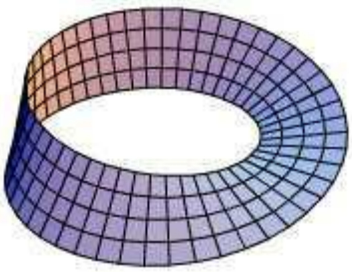
We shall associate a two-dimensional complex manifold with the three-dimensional Euclidean space. Define the twistor space T to be the space of oriented lines in R^3. Any oriented line is of form $\boldsymbol{v} + s\boldsymbol{u}, s \in R$ where $\boldsymbol{u}$ is a unit vector giving the direction of the line and $\boldsymbol{v}$ is orthogonal to $\boldsymbol{u}$ and joins the line with some chosen point (say the origin) in R^3.

$$T = \{(u, v) \in s^2 \times R^3, u \cdot v = 0\}$$



and the dimension of T is a four. For each fixed $\boldsymbol{u} \in S^2$ this space restricts to a tangent plane to S^2. The twistor space is the union of all tangent planes—the tangent bundle $T\,S^2$**.**

This is a topologically non-trivial manifold: locally it is diffeomorphic to $S^2 \times R^2$ but globally it is twisted in a way analogous to the M¨obius strip.



Reversing the orientation of lines induces a map $\tau : T \rightarrow T$ given by $\tau(u, v) = (-u, v)$. The points $p = (x, y, z)$ in R^3 correspond to two spheres in T given by τ -invariant maps

$$u \rightarrow (u, v(u) = p - (p \cdot u)u) \in T \qquad (102)$$

which are sections of the projection $T \rightarrow S^2$.

4.2.2 Differential Equations and Twistor Functions

Introduce the local holomorphic coordinates on an open set $U \subset T$ where $u = (0,0,1)$ by

$$\lambda = \frac{u_1 + iu_2}{1 - u_3} \in CP_1 = S^2, \quad \eta = \frac{v_1 + iv_2}{1 - v_3} + \frac{u_1 + iu_2}{(1 - u_3)^2} v_3 \tag{103}$$

and analogous complex coordinates $(\tilde{\lambda}, \tilde{\eta})$ in an open set $\tilde{U}$ containing $u = (0,0,1)$. On the overlap

$$\tilde{\lambda} = 1/\lambda \quad , \quad \tilde{\eta} = -\eta/\lambda^2. \tag{104}$$

In the holomorphic coordinates, the line orientation reversing involution τ is given by

$$\tau(\lambda, \eta) = (-\frac{1}{\lambda}, -\frac{\bar{\eta}}{\bar{\lambda}^2}). \tag{4.105}$$

From equation (102) we get the τ -invariant holomorphic map

$$\lambda \to (\lambda, \eta = (x + iy) + 2\lambda z - \lambda^2(x - iy)). \tag{4.106}$$

4.2.3 Harmonic Functions

To find a harmonic function at $P = (x, y, z)$

(i) Restrict a twistor function $f(\lambda, \eta)$ defined on $U \cap \tilde{U}$ to a line (4.106) $\breve{P} = CP^1 = S^2$

(ii) Integrate along a closed contour integral

$$\emptyset(x, y, z) = \oint_{\Gamma \subset \breve{P}} f(\lambda, (x + iy) + 2\lambda z - \lambda^2(x - iy))d\lambda \tag{4.107}$$

(iii) Differentiate under the integral to verify

$$\frac{\partial^2 \emptyset}{\partial x^2} + \frac{\partial^2 \emptyset}{\partial y^2} + \frac{\partial^2 \emptyset}{\partial z^2} = 0 \tag{4.108}$$

This formula was already known to Whittaker

4.2.4 Abelian Monopole Equation

Small modification of the formula can be used to solve a first-order linear equation for a function ϕ and a magnetic Potential $A = (A_1, A_2, A_3)$ of the form

$$\nabla\emptyset = \nabla \wedge A \tag{4.109}$$

This is the abelian monopole equation.

Geometrically, the one-form $A = A_j dx^j$ is a connection on a $U(1)$ principal bundle over R^3 , and $\emptyset$ is a section of the adjoint bundle.

Taking the **curl** of both sides of this equation implies that $\emptyset$ is harmonic, and conversely given a harmonic function $\emptyset$ locally one can always find a one-form A such that the abelian monopole equation holds.

4.2.5 Non-Abelian Monopoles Equation

Replacing $U(1)$ by a non-Abelian Lie group generalizes this picture to some equations on R^3 in the following way:

Let $(A_j, \emptyset)$ be anti- Hermitian traceless n by n matrices on R^3. Define the non-abelian magnetic field

$$F_{kl} = \frac{\partial A_l}{\partial x^k} - \frac{\partial A_k}{\partial x^l} + [A_k, A_l], \quad k, l = 1,2,3. \tag{4.110}$$

The non-Abelian monopole equation is a system of nonlinear PDEs

$$\frac{\partial\emptyset}{\partial x^j} + [A_j, \emptyset] = \frac{1}{2}\varepsilon_{jkl}F_{kl} \quad . \tag{4.111}$$

These are three equations for three unknowns as $(A, \emptyset)$ are defined up to gauge transformations

$$A \to gAg^{-1} - dgg^{-1}, \emptyset \to g\emptyset g^{-1} \text{for } g = g(x, y, t) \in SU(n) \tag{4.112}$$

and one component of A (say A_1) can always be set to zero.

4.2.6 Twistor Solution to the Monopole Equation

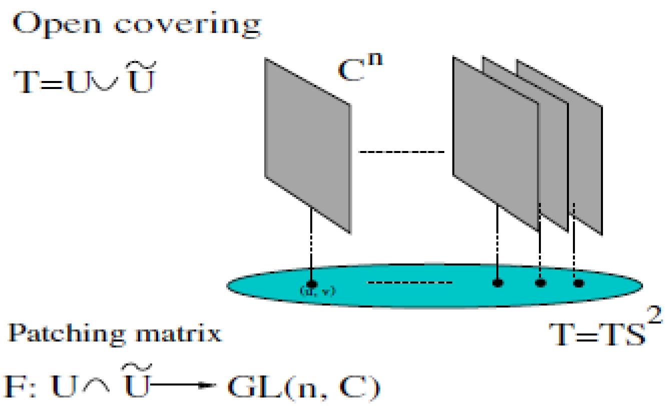
The twistor solution to the monopole equation consists of the following steps:

• Given $\left(A_j(X), \emptyset(X)\right)$ solve a matrix ODE along each oriented line $x(s) = v + su$

$$\frac{dV}{ds} + \left(u^j A_j + i\emptyset\right)V = 0 \qquad (4.113)$$

Space of solutions at $p \in R^3$ is a complex vector space C^n.

• This assigns a complex vector space C^n to each point of T, thus giving rise to a complex vector bundle over T with patching matrix$\left(\lambda, \bar{\lambda}, \eta, \bar{\eta}\right) \in GL(n, C)$.



• The monopole equation (4.111) on R^3 holds if and only if this vector bundle is holomorphic, i.e. the Cauchy–Riemann equations

$$\frac{\partial F}{\partial \bar{\lambda},} = 0\,, \qquad \frac{\partial F}{\partial \bar{\eta}} = 0 \qquad (4.114)$$

hold.

• Holomorphic vector bundles over $TCP^1$are well understood. Take one and work backwards to construct a monopole. We shall work through the details of this reconstruction in the proof of theorem 4.2.8

4.2.7 The Ward Model and Lorentzian Mini-Twistors

In this section, we shall demonstrate how mini-twistor theory can be used to solve nonlinear equations in 2+1 dimensions.

Let $A = d\mu dx^{\mu}$ and $\emptyset$ be a one-form and a function respectively on the Minkowski space,1 with values in a Lie algebra of the general linear group. They are defined up to gauge transformations (4.112) where g takes values in $GL(n, R)$.

Let $D_\mu = \partial_\mu + A_\mu$ be a covariant derivative, and define $D_\emptyset = d\emptyset + [A, \emptyset]$. The Ward model is a system of PDEs (4.111) where now the indices are raised using the metric on $R^{2,1}$. If the metric and the volume form are chosen to be

$$h = dx^2 - 4dudv\,, \qquad vol = du\, \wedge dx\, \wedge dv \qquad (4.115)$$

where the coordinates (x, u, v) are real the equations become

$$D_x\emptyset = \frac{1}{2} F_{uv} \qquad D_u\emptyset = F_{ux}\,, \qquad D_v\emptyset = F_{xv} \qquad (4.116)$$

where $F_{\mu v} = [D_\mu\,, D_v]$. These equations arise as the integrability conditions for an over determined system of linear Lax equations

$$L_0\Psi = 0\ \ , L_1\Psi = 0 \qquad (4.117)$$

Where

$$L_0 = D_u - \lambda(D_x + \emptyset), \qquad L_1 = D_x - \emptyset - \lambda D_v \qquad (4.118)$$

and $\Psi = \Psi(x\,, u\,, v\,, \lambda)$ takes values in $GL(n, C)$. We shall follow and 'solve' the system by establishing a one-to-one correspondence between its solutions and certain holomorphic vector bundles over the twistor space T. This construction is of interest in soliton theory as many known integrable models arise as symmetry reduction and/or choosing a gauge in (4.117). To this end, we note a few examples of such reductions.

- Choose the unitary gauge group $G = U(n)$. The integrability conditions for (4.118) imply the existence of a gauge $A_v = 0$, and $A_x = -\phi$, and a matrix $J : R^{2,1} \to U(n)$ such that

$$A_u = J^{-1}\partial_u J\ \ , \quad A_x = -\emptyset = \frac{1}{2} J^{-1}\partial_x J. \qquad (4.119)$$

With this gauge choice equations (4.116) become the integrable chiral model

$$\partial_v(J^{-1}\partial_u J) - \partial_x(J^{-1}\partial_x J) = 0 \qquad (4.120)$$

This formulation breaks the Lorentz invariance of (4.116) but it allows the introduction of a positive definite energy functional. Where more details can be found.

• Solutions to equation (4.116) with the gauge group $SL(2,R)$ which are invariant under a null translation given by a Killing vector K such that the matrix $K \lrcorner A$ is nilpotent are characterized by the KdV equation.

• The direct calculation shows that the Ward equations with the gauge group $SL(3,R)$ are solved by the ansatz

$$\emptyset = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -e^{\psi} & 0 & 0 \end{pmatrix}$$

$$A = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ e^{\psi} & 0 & 0 \end{pmatrix} dx + \begin{pmatrix} \psi_u & 0 & 0 \\ 1 & -\psi_u & 0 \\ 0 & 0A & 0 \end{pmatrix} du + \begin{pmatrix} e^{-2\psi} & 0 & 0 \\ 1 & 0 & e^{\psi} \\ 0 & 0 & 0 \end{pmatrix} dv$$

iff $\psi(u,v)$ satisfies the Tzitz´eica equation

$$\frac{\partial^2 \psi}{\partial u \partial v} = e^{\psi} - e^{-2\psi} \tag{4.121}$$

This reduction can also be characterized in a gauge invariant manner using the Jordan normal forms for the Higgs fields for details.

4.2.8 Null Planes and Ward Correspondence

The geometric interpretation of the Lax representation (4.117) is the following. For any fixed pair of real numbers (η, λ) the plane

$$\eta = v + x\lambda + u\lambda^2 \tag{4.122}$$

is null with respect to the Minkowski metric on $R^{2,1}$, and conversely all null planes can be put in this form if one allows $\lambda = \infty$. The two vector fields

$$\delta_0 = \partial_u - \lambda\partial_x , \qquad \delta_1 = \partial_x - \lambda\partial_v \tag{4.123}$$

span this null plane. Thus the Lax equations (4.117) imply that the generalized connection $(A, \emptyset)$ is flat on null planes. This underlies the twistor approach, where one works in a complexified Minkowski space$M = C^3$, and interprets $(\eta\,,\ \lambda)$ as coordinates in a patch of the twistor space $T = TCP^1$, with $\eta\ \in\ C$ being a coordinate on the fibres and $\lambda\ \in\ CP^1$ being an affine coordinate on the base. We shall adopt this complexified point of view from now on.

It is convenient to make use of the spinor formalism based on the isomorphism

$$T = S \otimes S, \tag{4.124}$$

where S is the rank two complex vector bundle (spin bundle) over M and $\odot$ is the symmetrized tensor product. The fibre coordinates of this bundle are denoted by (π^0, π^1) and the sections $M \rightarrow S$ are called spinors. We shall regard S as a symplectic bundle with an anti-symmetric product

$$k \cdot \rho = k^0 \rho^1 - k^1 \rho^0 = \varepsilon(k, \rho) \tag{4.125}$$

on its sections. The constant symplectic form ε is represented by a matrix

$$\varepsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.126}$$

This gives an isomorphism between S and its dual bundle, and thus can be used to 'rise and lower the indices' according to $k_A = k^B \varepsilon_{AB}, k^A = k^B \varepsilon^{AB}$, where $\varepsilon_{AB}\varepsilon^{AB}$ is an identity endomorphism.

Rearrange the spacetime coordinates (u, x, v) of a displacement vector as a symmetric two-spinor

$$\varepsilon^{AB} = \begin{pmatrix} u & x/2 \\ x/2 & v \end{pmatrix} \tag{4.127}$$

such that the spacetime metric is

$$h = -2dx_{AB}dx^{AB} \tag{4.128}$$

The twistor space of M is the two-dimensional complex manifold $T = TCP^1$. Points of T correspond to null 2-planes in M via the incidence relation

$$x^{AB}\pi_A\pi_B = \omega \tag{4.129}$$

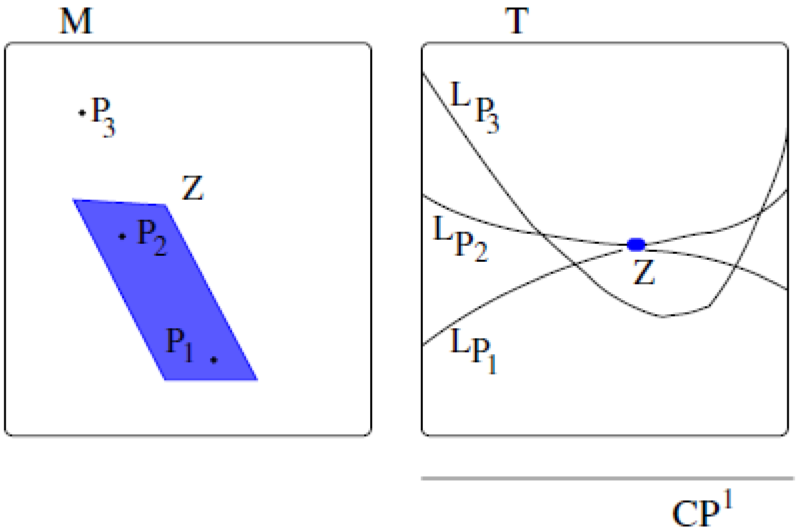
Here $(\omega\,,\pi_{0\,,}\pi_1)$ are homogeneous coordinates on T as $(\omega,\pi_A) \sim (c^2w, c\pi_A)$, where $c \in C^*$. In the affine coordinates $\lambda := \frac{\pi 0}{\pi 1}$, $\eta := \omega/(\pi)^2$ equation (4.129) gives (4.128).

The projective spin space $P(S)$ is the complex projective line CP^1. The homogeneous coordinates are denoted by $\pi_A = (\pi_0\,,\pi_1\,)$, and the two-set covering of CP^1 lifts to a covering of the twistor space T:

$$U = \{(w,\pi_A),\pi_1 \neq 0\}\,, \qquad \widetilde{U} = \{(w,\pi_A),\pi_0 \neq 0\}. \tag{4.130}$$

The functions $\lambda = \pi_0/\pi_1$, $\tilde{\lambda} = 1/\lambda$ are the inhomogeneous coordinates in U and $\widetilde{U}$, respectively. It then follows that $\lambda = -\pi^1/\pi^0$.

Fixing (w,π_A)gives a null plane in M. An alternative interpretation of (4.128) is to fix x^{AB}.This determines $w$as a function of π_A, i.e. a section of $\lambda = \pi_0/\pi_1$ when factored out by the relation $(w,\pi_A) = (c^2w, c\pi_A)$. These are embedded rational curves with self-intersection number 2, as infinitesimally perturbed curve $\eta + \delta\eta \;\; with \; \delta\eta = \delta v - \lambda\delta x + \lambda^2 \, \delta u$ generically intersects (4.126) at two points. Two curves intersect at one point if the corresponding points in M are null separated. This defines a conformal structure on M.



The space of holomorphic sections of $TCP^1 \to CP^1$ is $M = C^3$. The real spacetime R^{2+1} arises as the moduli space of those sections that are invariant under the conjugation

$$\tau\left(w, \pi_A = (\bar{w}, \bar{\pi}_A)\right) \tag{4.131}$$

which corresponds to real x^{AB}. The points in T fixed by τ correspond to real null planes in $R^{2,1}$. The following result makes the mini-twistors worthwhile.

4.2.8 Theorem

There is a one-to-one correspondence between:

(i) The gauge equivalence classes of complex solutions to (4.112) in the complexified Minkowski space M with the gauge group $GL(n, C)$.

(ii) Holomorphic rank n vector bundles E over the twistor space T which are trivial on the holomorphic sections of $TCP^1 \to CP^1$.

Proof.

Let $(A, \emptyset)$ be a solution to (4.116). Therefore we can integrate a pair of linear PDEs $L_0 V = L_1 V = 0$, where L_0, L_1 are given by (4.118). This assigns an n-dimensional vector space to each null plane Z in a complexified Minkowski space, and so to each point $Z \in T$. It is a fibre of a holomorphic vector bundle $\mu: E \to$ T. *The* bundle E is trivial on each section, since we can identify fibres of $E|L_P$ at Z_0, Z_1 because covariantly constant vector fields at null planes Z_1, Z_2 coincide at a common point $p \in M$.

Conversely, assume that we are given a holomorphic vector bundle E over T which is trivial on each section. Since $E|L_P$ is trivial and $L_P \cong CP^1$, the Birkhoff–Grothendieck theorem gives

$$E \,\big|\, L_p = O \oplus O \ldots\ldots\ldots CX \ldots\ldots \oplus O \qquad (4.132)$$

and the space of sections of E restricted to L_P is C^n. This gives us a holomorphic rank n vector bundle $\tilde{E}$ over the complexified three-dimensional Minkowski space. We shall give a concrete method of constructing a pair (A, ϕ) on this bundle which satisfies (4.131). Let us cover the twistor space with two open sets U and $\hat{U}$ as in (4.130).
Let

$$\aleph : \mu^{-1}(U) \to U \times C^n, \qquad \tilde{\aleph} : \mu^{-1}(\tilde{U}) \to \tilde{U} \times C^n \qquad (4.133)$$

be local trivializations of E, and let $F = \tilde{\aleph} \circ \aleph : C^n \to C^n$ be a holomorphic patching matrix for a vector bundle E over TCP^1 defined on $U \cap \hat{U}$. Restrict F to a section (4.129) where the bundle is trivial, and therefore F can be spli:

$$F = \tilde{H}\, H^{-1}, \qquad (4.134)$$

where the matrices H and $\tilde{H}$ are defined on $M \times CP^1$ and are holomorphic in π^A around $\pi^A = 0^A = (1,0)$ and $\pi^A = l^A = (0,1)$ respectively. As a consequence of $\delta_A F = 0$ the splitting matrices satisfy

$$H^{-1}\delta_A H = \tilde{H}^{-1}\delta_A \tilde{H} = \pi^B \Phi_{AB} \qquad (4.135)$$

for some $\Phi_{AB}(x^\mu)$ which does not depend on λ. This is because the RHS and LHS are homogeneous of degree 1 in π^A and holomorphic around $\lambda = 0$ and $\lambda = \infty$, respectively.
Decomposing

$$\Phi_{AB} = \Phi_{(AB)} + \varepsilon_{AB}\emptyset \tag{4.136}$$

gives a one-form $\Phi_{AB}dx^{AB}$ and a scalar field $\phi = (^1/_2)\varepsilon^{AB}\Phi_{AB}$ on the complexified Minkowski space, i.e.

$$\Phi_{AB} = \begin{pmatrix} A_u & A_x + \emptyset \\ A_x - \emptyset & A_v \end{pmatrix} \tag{4.137}$$

The Lax pair (4.116) becomes

$$L_A = \delta_A + H^{-1}\delta_A H \tag{4.138}$$

where $\delta_A = \pi^B \partial_{AB}$, so that

$$L_A(H^{-1}) = -H^{-1}(\delta_A H)H^{-1} + H^{-1}(\delta_A H)H^{-1} = 0 \tag{4.139}$$

and $\Psi = H^{-1}$ is a solution to the Lax equations regular around $\lambda = 0$. Let us show explicitlythat (4.117) holds. Differentiating (4.135) with respect to δ_A yields

$$\delta^A(\mathrm{H}^{-1}\delta_A \mathrm{H}) = -(\mathrm{H}^{-1}\delta^A H)(\mathrm{H}^{-1}\delta_A \mathrm{H}) \tag{4.140}$$

which holds for all π^A if

$$D_A(C\Phi^A_B) = 0, \tag{4.141}$$

where $D_{AC} = \partial_{AC} + \Phi_{AC}$. This is the spinor form of the Yang–Mills–Higgs system (3.112).

• To single out the Euclidean reality conditions leading to non-abelian monopoles (4.111) on R^3 with the gauge group *SU(n)*, the vector bundle E must be compatible with the involution. This comes down to $detF = 1$ and

$$F^*(Z) = F(\tau(Z)) \tag{4.142}$$

where $Z \in T$ and * denotes the Hermitian conjugation.

• To single out the Lorentzian reality conditions, the bundle must be invariant under the involution (4.131). Below we shall demonstrate how the gauge choices leading to the integrable chiral model (4.120) can be made at the twistor level.

Let

$$h = H(x^{\mu}, \pi^A = O^A), \qquad \tilde{h} = \tilde{H}(x^{\mu}, \pi^A = l^A) \tag{4.143}$$

so that

$$\Phi_{A0} = h^{-1}\,\partial_{A0}\,h, \qquad \Phi_{A1} = \tilde{h}^{-1}\,\partial_{A1}\,\tilde{h} \tag{4.144}$$

The splitting matrices are defined up to a multiple by an inverse of a non-singular matrix $g = g(x^{\mu})$ independent of π^A

$$H = Hg^{-1}, \qquad \widetilde{H} = \widetilde{H}g^{-1} \tag{4.145}$$

We choose g such that $\widetilde{h} = 1$ so

$$\Phi_{A1} = l^A\,\Phi_{AB} = 0 \tag{4.146}$$

and

$$\Phi_{AB} = -l_B O^C h^{-1}\,\partial_{AC}\,h \tag{4.147}$$

i.e.

$$A_x + \emptyset = A_v = 0$$

$$\Phi_{AB} = -A_x + \phi = A_V = 0 \tag{4.148}$$

This is the Ward gauge with $J(x^{\mu}) = h$. In this gauge, the system (3.141) reduces to

$$\partial_1^A \Phi_{A0} = 0 \tag{4.149}$$

which is (4.120). The solution is given by

$$J\,(x^{\mu}) = \Psi^{-1}\,(x^{\mu}, \lambda = 0) \tag{4.150}$$

where $\Psi = H^{-1}$ is a solution to the Lax pair.

• In the abelian case $n = 1$ the patching matrix becomes a function defined on the intersection of two open sets, and we can set $F = exp(f)$ for some f. The nonlinear splitting (4.145) reduces to the additive splitting of f which can be carried out explicitly using the Cauchy integral formula. The Higgs field is now a function that satisfies thewave equation and is given by the formula

$$\phi = \int_{\Gamma} \frac{\partial f}{\partial w} \rho \, . d\rho \tag{4.151}$$

where Γ is a real contour in a rational curve $w = x^{AB}\pi_A \pi_B$. If the Euclidean reality conditions are chosen, we recover the Whittaker formula (4.107).

CHAPTER FIVE

Applications of Twistor Space in Six Dimensions

5.1 Spinors in Six Dimensions

In the following, we shall be working with the complexification of flat six-dimensional space-time $M^6 \coloneqq C^6$. Notice that reality conditions leading to real slices of M^6 with Minkowski or split signature can be imposed if desired.

5.1.1 The Spin Bundle

The spin bundle on M^6 is of rank eight and decomposes into the direct sum $S \oplus \tilde{S}$ of the two rank-4 sub bundles of anti-chiral spinors, S, and chiral spinors, $\tilde{S}$. There is a natural isomorphism identifying S and $\tilde{S}$ with the duals $\tilde{S}^\vee$ and $S^\vee$ (for details; this identification basically works via an automorphism of the Clifford algebra corresponding to charge conjugation). Therefore, we may exclusively work with, say, S and $S^\vee$.

We shall label the corresponding spinors by upper and lower capital Latin letters from the beginning of the alphabet, e.g. ψ^A for a section of S and ψ_A for a section of $S^\vee$, with $A, B, \ldots. = 1, \ldots, 4$.

5.1.2 The Tangent Bundle

We may identify the tangent bundle T_{M^6} with the anti-symmetric tensor product of the chiral spinor bundle with itself via

$$T_{M^6} \cong S \wedge S \tag{5.1}$$

$$\partial_M \coloneqq \frac{\partial}{\partial x^M} \overset{\tilde{\sigma}_*}{\leftrightarrow} \partial_{AB} \coloneqq \frac{\partial}{\partial x^{AB}} \tag{5.2}$$

Here, we coordinatised M^6 by x^M, for $M, N, \ldots = 1, \ldots, 6$ and used the identification

$$\tilde{\sigma} : x = x^M \to \tilde{\sigma}(x) = x^{AB} \tag{5.3}$$

with

$$x^{AB} = \tilde{\sigma}^{AB}_{M} x^{M} \Leftrightarrow x^{M} = \frac{1}{4}\sigma^{M}_{AB} x^{AB} \tag{5.4}$$

where $\tilde{\sigma}^{AB}_{M}$, σ^{M}_{AB} are the six-dimensional sigma-matrices.

The induced linear mapping $\tilde{\sigma}^{*}$ is explicitly given as

$$\partial_{AB} = \frac{1}{4}\sigma^{M}_{AB}\,\partial_{M} \tag{5.5}$$

and the (flat) metric η_{MN} on M^6 can be identified with the Levi-Civita symbol $\frac{1}{2}\varepsilon_{ABCD}$ in spinor notation. Hence,

$$\sigma^{M}_{AB} = \frac{1}{2}\varepsilon_{ABCD}\tilde{\sigma}^{MCD} \tag{5.6}$$

And we can raise and lower indices according to:

$$\partial_{AB} = \frac{1}{2}\varepsilon_{ABCD}\,\partial^{CD} \quad \Leftrightarrow \quad \partial^{AB} = \frac{1}{2}\varepsilon^{ABCD}\,\partial_{CD} \tag{5.7}$$

For any two six-vectors $p = (p^M)$ and $q = (q^M)$, we shall write:

$$p \cdot q := p_M q^M = \frac{1}{4} p_{AB} q^{AB} = \frac{1}{8}\varepsilon_{ABCD} p^{AB} q^{CD}, \tag{5.8}$$

and we have

$$p^2 := p \cdot p = \sqrt{\det p^{AB}}. \tag{5.9}$$

5.2. Zero Rest Mass Fields in Six Dimensions:

Next we wish to discuss zero-rest-mass fields in the six-dimensional spinor-helicity formalism.

Let us start by considering a momentum six-vector $p = (p_M)$. If we impose the null-condition $p^2 = 0$, then we have

$$\det p_{AB} = 0 = \det p^{AB}. \tag{5.10}$$

These equations are solved most generally by:

$$P_{AB} = k_{Aa}k_{Bb}\varepsilon^{ab} \quad \text{and} \qquad p^{AB} = \tilde{k}^{A\dot{a}}\tilde{k}^{B\dot{b}}\varepsilon_{\dot{a}\dot{b}} \qquad (5.11)$$

with $a, b, \ldots, \dot{a}, \dot{b}, \ldots = 1, 2$ and $\varepsilon^{ab} = -\varepsilon^{ba}$ and $\varepsilon_{\dot{a}\dot{b}} = -\varepsilon_{\dot{b}\dot{a}}$.

We shall refer to such a momentum as null-momentum. Moreover, transformations of the form $k_{Aa} \to M_a^b k_{Ab}$ and $\tilde{k}^{A\dot{a}} \to \tilde{M}_{\dot{b}}^{\dot{a}}\tilde{k}^{A\dot{b}}$ with $\det M = 1 = \det \tilde{M}$ will leave p invariant, which shows that the indices $a, \dot{a}, \ldots$ are little group indices. The little group of (complex) null-vectors in six dimensions is therefore $SL(2,\mathbb{C}) \times \widetilde{SL(2,\mathbb{C})}$.

Notice that $k_{Aa}\tilde{k}^{A\dot{b}} = 0$ since $p_{AB} = \frac{1}{2}\varepsilon_{ABCD}p^{CD}$, which, in turn, shows that k_{Aa} and $\tilde{k}^{A\dot{b}}$ are not independent. Notice also that k_{Aa} has $4 \times 2 = 8$ components, but three of them can be fixed by little group transformations.

Thus, k_{Aa} has indeed exactly the five independent components needed to describe the (five-dimensional) null-cone in six dimensions.

Fields form irreducible representations of the Lorentz group which are induced from representations of the little group. In six dimensions, the spin label of fields therefore has to be generalized to a pair of integers, labelling the irreducible representations of the little group $SL(2,\mathbb{C}) \times \widetilde{SL(2,\mathbb{C})}$.

As an example of zero-rest-mass fields, let us consider the fields in the $\mathcal{N} = (2,0)$ tensor multiplet. This multiplet is a chiral multiplet and hence the fields transform trivially under the $\widetilde{SL(2,\mathbb{C})}$ subgroup.

Amongst these fields, there is a self-dual three-form $H = dB$, which transforms as the (**5.29**) of the little group.

In spinor notation, H has components $H_{AB} = \partial_{C(A}B_{B)}{}^C$, where $B_B{}^C$ is trace-less and denotes the components of a two-form potential B in spinor notation. In addition, we have four Weyl spinors ψ^I_A in the $(\mathbf{5.1})$ and five scalars ϕ^{IJ} in the trivial representation $(\mathbf{1,1})$ of the little group.

Notice that the a priori six components of $\phi^{IJ} = -\phi^{JI}$ are reduced to five by the condition $\phi^{IJ}\Omega_{IJ} = 0$, where $I, J, \ldots = 1, \ldots, 4$ and Ω_{IJ} is an invariant form of the underlying R-symmetry.

In the following, we shall work with complex fields. The zero-rest-mass field equations (i.e. the free equations of motion) for the fields in the tensor multiplet read as:

$$H^{AB} = 0 \text{ with } \partial^{AC}H_{CB} = 0, \quad \partial^{AB}\psi_B = 0, \quad \Box_\phi = 0 \qquad (5.12)$$

where we suppressed the R-symmetry indices.

Notice that the second equation is the Bianchi identity (which, of course, is equivalent to the field equation for self-dual three forms). The corresponding plane waves are given by the expressions ($i := \sqrt{-1}$)

$$H_{AB\,ab} = k_{A(a}k_{Bb)}e^{ix.p}, \quad \psi_{Aa} = k_{Aa}e^{ix.p} \text{ and } \phi = e^{ix.p} \qquad (5.13)$$

This follows from straightforward differentiation. Here, the representations of the little group formed by the fields become explicit.

Furthermore, since

$$H_{AB} = \partial_{C(A}B_{B)}{}^C \qquad (5.14)$$

we can express the plane waves of H_{AB} in terms of the plane waves of the potential two-form $B_B{}^A$. To this end, we note that in spinor notation, gauge transformations of $B_B{}^A$ are mediated by gauge parameters

$$\Lambda_{AB} = \Lambda_{[AB]} \text{ via } B_B{}^A \to B_B{}^A + \partial^{AC}\Lambda_{CB} - \partial_{BC}\Lambda^{CA}. \qquad (5.15)$$

We shall choose Lorenz gauge, which in spinor notation reads as

$$\partial_{[A}B_{B]}{}^C = 0 = \partial^{C[A}B_C{}^{B]}. \qquad (5.16)$$

The residual gauge transformations are given by gauge parameters that obey $\partial \cdot A = 0$. Let us now choose reference spinors μ_{Aa} and define the null-momentum

$$q_{AB} \coloneqq \mu_{Aa}\mu_{Bb}\varepsilon^{ab} \text{ so that } p \cdot q \neq 0. \tag{5.17}$$

Then the plane waves of the potential two-form ${B_B}^A$ in Lorenz gauge are given by:

$$ {B_B}^A{}_{ab} = k^A_{(a}k_{Bb)}e^{ix\cdot p} \quad \text{with} \quad k^A_a \coloneqq -2i\frac{q^{AB}k_{Ba}}{p\cdot q} \tag{5.18}$$

Clearly, ${B_B}^A$ is trace-less and one can check that $\partial_{C(A}{B_{B)}}^C$ yields the components for $H^{AB}H_{AB}$, given in (5.13).Since

$$\partial^{CA}{B_C}^B = 0 \tag{5.19}$$

we also have

$$H^{AB} = \partial^{C(A}{B_C}^{B)} = 0 \tag{5.20}$$

Which implies that ${B_B}^A$ does indeed yield a self-dual field strength. Furthermore, the choice of μ_{Aa} is irrelevant since changes in μ_{Aa} merely correspond to (residual) gauge transformations of ${B_B}^A$, a fact that is already familiar from four dimensions. One may analyze other spin fields in a very similar way and we shall present a few more comments in Remark 5.1 below.

We shall mostly be interested in chiral zero-rest-mass fields, i.e. fields forming representations $(2h+1,1), h \in \frac{1}{2}\mathbb{N}_0$, of the little group $SL(2,\mathbb{C}) \times \widetilde{SL(2,\mathbb{C})}$. These fields will carry 2h symmetrized spinor indices. Specifically, using the conventions:

$$[k] \coloneqq \otimes^k \det S^\vee \ , \quad [-k] \coloneqq [k]^\vee \quad \text{and} \quad [0] \coloneqq [k] \otimes [-k] \quad \text{for } k \in \mathbb{N}$$
$$, S[\pm k] \coloneqq S \otimes_{O_{M^6}} [\pm k] \text{ for some Abelian sheaf } S \text{ on } M^6. \tag{5.21}$$

We shall denote the sheaf of chiral zero-rest-mass fields on M^6 by $\mathcal{Z}_h$,

$$\mathcal{Z}_h = \begin{cases} \ker\{\partial^{AB}: (\odot^{2h} S^\vee)[1] \to (\odot^{2h-1} S^\vee \otimes_{O_{M^6}} S)_0[2]\} & \text{for } h \geq \frac{1}{2}, \\ \ker\{\Box := \frac{1}{4}\partial^{AB}\partial_{AB} : [1] \to [2]\} & \text{for } h = 0, \end{cases} \tag{5.22}$$

Here, the subscript zero refers to the totally trace-less part. The factors $[\pm k]$ are referred to as conformal weights, as they render the zero-rest-mass field equations conformably invariant.

For the discussion of conformal weights in the four-dimensional setting.

Remark

Recall that there is a potential formulation of zero-rest-mass fields in four dimensions. This formulation generalizes to six dimensions, as we shall demonstrate now. Consider an $h \in \frac{1}{2}\mathbb{N}^*$ from the potential fields

$$B_A{}^{A_1 \ldots A_{2h-1}} = B_A{}^{(A_1 \ldots A_{2h-1})} \in H^0(M^6, (\odot^{2h-1} S \otimes_{O_{M^6}} S^\vee)_0[1] \tag{5.23}$$

We derive a field strength $H_{A_1 \ldots A_{2s}} \in H^0(U, \odot^{2h} S^\vee))$ according to

$$H_{A_1 \ldots A_{2h}} := \partial_{(A_1 B_1} \ldots \partial_{A_{2h} \ 1 B_{2h} \ \ 1} B_{A_{2h})}{}^{B_1 \ldots B_{2h-1}} \tag{5.24}$$

The equations

$$H^{A_1 \ldots A_{2h}} := \partial^{A(A_1} B_A{}^{A_2 \ldots A_{2h})} = 0 \tag{5.25}$$

Then imply that

$$\partial^{AA_1} H_{A_1 \ldots A_{2h}} = 0 \tag{5.26}$$

Furthermore, the pair of spinors $\left(H_{A_1 \ldots A_{2h}}, H^{A_1 \ldots A_{2h}}\right)$ is invariant under gauge transformations of the form

$$B_B{}^{AA_1....A_{2h-2}} \rightarrow B_B{}^{AA_1....A_{2h-2}} + \left[\partial_{CB}\Lambda^{C(AA_1....A_{2h-2})} - \partial^{C(A}\Lambda_{CB}{}^{A_1....A_{2h-2}}\right]_0 \qquad (5.27)$$

Where the subscript zero refers again to the totally trace-less part and $\Lambda_{AB}{}^{A_1...A_{2h-2}} = \Lambda_{[AB]}{}^{(A_1...A_{2h-2})}$ is totally trace-less itself.

Note

the traces of $\left[\partial_{CB}\Lambda^{C(AA_1.....A_{2h-2})} - \partial^{C(A}\Lambda_{CB}{}^{A_1...A_{2h-2)}}\right]$ always drop out of the above definition of $\left(H_{A_1....A_{2h}}, H^{A_1....A_{2h}}\right)$. Altogether, the spinor field $H_{A_1....A_{2h}}$ can therefore be regarded as a section of the sheaf $\mathcal{Z}_h$.

5.3 Twistor Space of Six-Dimensional Space-Time

In this section, we shall review a particular twistor space associated with M^6 that is a very natural generalization of known twistor spaces and suitable for the description of chiral theories in six dimensions. Here we shall present a detailed discussion of its constructions from an alternative point of view.

5.3.1 Remark.

We shall always be working with locally free sheaves and therefore we shall not make a notational distinction between vector bundles and their corresponding sheaves of sections. We shall switch between the two notions freely depending on context.

5.3.2 Twistor Space From Space-Time

Let us consider the projectivisation $\mathbb{P}(S^\vee)$ of the dual anti-chiral spin bundle $S^\vee$.

Since $S^\vee$ is of rank four, $\mathbb{P}(S^\vee) \rightarrow M^6$ is a $\mathbb{P}^3$-bundle over M^6. Hence, the projectivisation $\mathbb{P}(S^\vee)$ is a nine-dimensional complex manifold $F^9 \cong \mathbb{C}^6 \times \mathbb{P}^3$, the correspondence space.

We take

$$(x, \lambda) = (x^{AB}, \lambda_A) \qquad (5.28)$$

as coordinates on F^9, where λ_A are homogeneous coordinates on $\mathbb{P}^3$.

Consider now the following vector fields on F^9:

$$V^A = \lambda_B \frac{\partial}{\partial x_{AB}} \tag{5.29}$$

5.3.3 Note

$\lambda_A V^A = 0$ because of the anti-symmetry of the spinor indices in the partial derivative. These vector fields define an integrable rank-3 distribution on F^9, which we call twistor distribution. Therefore, we have a foliation of F^9 by three-dimensional complex manifolds.

The resulting quotient will be twistor space, a six-dimensional manifold denoted by $\mathbb{P}^6$. We have thus established the following double fibration:

$$\begin{array}{ccc} & F^9 & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ P^6 & & M^6 \end{array} \tag{5.30}$$

Let $(z, \lambda) = (z^A, \lambda_A)$ be homogeneous coordinates on P^7 and assume that $\lambda_A \neq 0$. This effectively means that we are working on the open subset

$$\mathbb{P}^7_0 \coloneqq \mathbb{P}^7 \backslash \mathbb{P}^3 \tag{5.31}$$

of $\mathbb{P}^7$, where the removed $\mathbb{P}^3$ is given by $z^A \neq 0$ and $\lambda_A = 0$.

In the double fibration (5.30), the projection π_2 is the trivial projection and

$$\pi_1 : (x^{AB}, \lambda_A) \to (z^A, \lambda_A) = (x^{AB}, \lambda_B) = (x^{AB}\lambda_B, \lambda_A). \tag{5.32}$$

Thus, P^6 forms a quadric hypersurface inside P^7_0 , which is given by the equation

$$z^A \lambda_A = 0 \tag{5.33}$$

We shall refer to the relation

$$z^A = x^{AB}\,\lambda_B \qquad (5.34)$$

as incidence relation, because it is a direct generalisation of Penrose's incidence relation in four dimensions.

5.4 Geometric Twistor Correspondence.

The double fibration (5.30) shows that points in either of the base spaces M^6 and P^6 correspond to subspaces of the other base space:

For any point $x \in M^6$, the corresponding manifold $\hat{x} := \pi_1(\pi_2^{-1}(x)) \hookrightarrow P^6$ is a three dimensional complex manifold bi-holomorphic to $\mathbb{P}^3$ as follows from (5.34).

Conversely, for any fixed $p = (z, \lambda) \in P^6$, the most general solution to the incidence relation (5.34) is given by

$$x^{AB} = x_0^{AB} + \varepsilon^{ABCD}\mu_C\lambda_D, \qquad (5.35)$$

where x_0^{AB} is a particular solution and μ_A is arbitrary. This defines a totally null-plane $\pi_2(\pi_1^{-1}(p))$ in M^6.

This plane is three-dimensional because of the freedom in the choice of μ_A given by the shifts $\mu_A \to \mu_A + \varrho\lambda_A$ for $\varrho \in \mathbb{C}$ which do not alter the solution (5.35).

Altogether, points in space-time correspond to complex projective three-spaces in twistor space while points in twistor space correspond to totally null three-planes in space-time.

Thus, twistor space parametrises all totally null three-planes of space-time.

5.5 Twistor Space as A normal Bundle.

The above considerations imply that P^6 can be viewed as a holomorphic vector bundle over $\mathbb{P}^3$, where the global holomorphic sections are given by the incidence relation (5.34). In fact, (5.34) shows that P^6 is a rank-3 subbundle of the bundle $\mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathbb{C}^4 \to \mathbb{P}^3$, whose total space is $\mathbb{P}^7_0$. Here and in the following, $\mathcal{O}_{\mathbb{P}^3}(1)$ denotes the dual tautological bundle over $\mathbb{P}^3$.

To identify the sub bundle P^6, let us denote by $N_{Y|X}$ the normal bundle of some complex sub manifold Y of a complex manifold $X, i : Y \hookrightarrow X$. This bundle is defined by the following short exact sequence:

$$0 \to T_Y \to i^* T_X \to N_{Y|X} \to 0 \qquad (5.36)$$

Let us now specialise to $Y = \mathbb{P}^3$ and $X = \mathbb{P}^7$ with coordinates (z^A, λ_A) on $\mathbb{P}^7$ as before.

If $\mathbb{P}^3 \hookrightarrow \mathbb{P}^7$ is given by $z^A = 0$ and $\lambda_A \neq 0$, then

$$T_{\mathbb{P}^3} = \langle \frac{\partial}{\partial \lambda_A} \rangle \qquad (5.37)$$

And

$$T_{\mathbb{P}^7} = \langle \frac{\partial}{\partial z^A} , \frac{\partial}{\partial \lambda_A} \rangle. \qquad (5.38)$$

The normal bundle of $N_{\mathbb{P}^3|\mathbb{P}^7}$ of $\mathbb{P}^3$ inside $\mathbb{P}^7$ is given by

$$0 \to T_{\mathbb{P}^3} \to i^* T_{\mathbb{P}^7} \to N_{\mathbb{P}^3|\mathbb{P}^7} \to 0 \qquad (5.39a)$$

which implies that

$$N_{\mathbb{P}^3|\mathbb{P}^7} \cong \mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathbb{C}^4, \qquad (5.39b)$$

Since the coefficient functions of the basis vector fields $\frac{\partial}{\partial z^A}$ and $\frac{\partial}{\partial \lambda_A}$ are linear in the coordinates. Hence, the z^A can be regarded as fibre coordinates of $N_{\mathbb{P}^3|\mathbb{P}^7}$, while the λ_A are base coordinates. Using these results, we find that our twistor space P^6 fits into the short exact sequence

$$0 \to P^6 \to N_{\mathbb{P}^3|\mathbb{P}^7} \xrightarrow{k} \mathcal{O}_{\mathbb{P}^3}(2) \to 0\,, \qquad (5.40a)$$

Where

$$k : (z^A, \lambda_A) \to z^A \lambda_A. \tag{5.40b}$$

Note

The sequence (5.40a) can be regarded as an alternative definition of twistor space.

Again, we see that P^6 is a rank-3 subbundle of $\mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathbb{C}^4 \to \mathbb{P}^3$ as stated earlier.

It also shows that P^6 is the normal bundle of $\mathbb{P}^3$ inside the quadric hypersurface $\mathbb{Q}^6 \hookrightarrow \mathbb{P}^7$ given by the zero locus

$$z^A \lambda_A = 0. \tag{5.41}$$

Moreover, notice that the open subset $\mathbb{Q}^6 \cap \mathbb{P}^7_0$ can be identified with P^6.

5.6 Space-Time From Twistor Space

Next we wish to address the problem of how to obtain space-time M^6, and in particular the factorisation (5.2) of the tangent bundle, from twistor space using (5.40a). To this end, consider the long exact sequence of cohomology groups induced by the short exact sequence (5.40a),

$$\begin{aligned} &0 \to H^0(\mathbb{P}^3, P^6) \to H^0(\mathbb{P}^3, N_{\mathbb{P}^3|\mathbb{P}^7}) \xrightarrow{k} H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) \to \\ &\to H^1(\mathbb{P}^3, P^6) \to H^1(\mathbb{P}^3, N_{\mathbb{P}^3|\mathbb{P}^7}) \to H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) \to \cdots. \end{aligned} \tag{5.42}$$

where we have slightly abused notation by again using the letter k. To compute these cohomology groups, we recall a special case of the Borel–Weil–Bott theorem:

5.6.1 Lemma: (Bott's Rule)

Let V be an n-dimensional complex vector space. Consider its projectivisation $\mathbb{P}(V)$ together with the hyperplane bundle $\mathcal{O}_{\mathbb{P}(V)}(1)$. Furthermore, set

$$\mathcal{O}_{\mathbb{P}(V)}(k) := \otimes^k \mathcal{O}_{\mathbb{P}(V)}(1), \mathcal{O}_{\mathbb{P}(V)}(-k) := \mathcal{O}^\vee_{\mathbb{P}(V)}(k) \tag{5.44}$$

And

$$\mathcal{O}_{\mathbb{P}(V)}(0) \coloneqq \mathcal{O}_{\mathbb{P}(V)} \text{ for } k \in \mathbb{N}\,. \tag{5.45}$$

Then

$$H^q\left(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(k)\right) \cong \begin{cases} \bigodot^k V^\vee & \text{for } q = 0 \ \mathfrak{E}\ k \geq 0 \\ \bigodot^{-k-n} V \otimes \det V & \text{for } q = n-1 \quad \mathfrak{E} k \leq -n \\ 0 & \text{otherwise}\,, \end{cases} \tag{5.46}$$

where $\det\, V \equiv \Lambda^n V$. From Bott's rule for $V = \mathbb{C}^4$, we find that

$$H^1\left(\mathbb{P}^3, N_{\mathbb{P}^3|\mathbb{P}^7}\right) = 0 = H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) \tag{5.47}$$

and furthermore

$$H^1(\mathbb{P}^3, P^6) = 0, \tag{5.48}$$

Since k is surjective. Therefore, the long exact sequence of cohomology groups (5.42) reduces to

$$0 \to H^0(\mathbb{P}^3, P^6) \to H^0(\mathbb{P}^3, N_{\mathbb{P}^3|\mathbb{P}^7}) \xrightarrow{k} H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) \to 0 \tag{5.49}$$

By applying Bott's rule again, we deduce from the latter sequence that

$$\dim_{\mathbb{C}} H^0\,(\mathbb{P}^3, P^6) \;=\; 6 \tag{5.50}$$

Because of (5.43) and (5.45), we may now apply Kodaira's theorem of relative deformation theory to conclude that there is a six-dimensional family of deformations of $\mathbb{P}^3$ inside the quadric hypersurface $\mathbb{Q}^6 \hookrightarrow \mathbb{P}^7$. We shall denote this family by M^6 and the individual deformation of $\mathbb{P}^3$ labelled by $x \in M^6$ as $\hat{x}$.

Next we define the correspondence space F^9 according to

$$F^9 \coloneqq \{(p, x) \in P^6 \times M^6 | p \in \hat{x}\}, \tag{5.51}$$

5.6.2 Note

F^9 is fibred over both P^6 and M^6. The typical fibres of $\pi_2 : F^9 \to P^6$ are complex projective three-spaces $\mathbb{P}^3$. Hence, we have again established a double fibration of the form (5.30), where the fibres of $F^9 \to P^6$ are three-dimensional complex submanifolds of M^6.

On F^9, we may consider the relative tangent bundle, denoted by T_{π_1}, along the fibration $\pi_1 : F^9 \to P^6$. It is of rank three and defined by

$$0 \to T_{\pi_1} \to T_{F^9} \to \pi_1{}^* T_{\mathbb{P}^7} \to 0. \tag{5.52}$$

By construction, the vector fields V^A given in (5.29) annihilate $z^A = x^{AB}\lambda_B$ and therefore, T_{π_1} can be identified with the twistor distribution generated by V^A, cf. (5.29). Hence, sections μ_A of T_{π_1} are defined up to shifts by terms proportional to λ_A (recall that $\lambda_A V^A = 0$). Then we define a bundle N on F^9 by

$$\begin{gathered} 0 \to T_{\pi_1} \to \pi_2^* T_{M^6} \to N \to 0 \\ \mu_A \to \varepsilon^{ABCD}\mu_C\lambda_D \\ \xi^{AB} \to \xi^{AB}\lambda_B \end{gathered} \tag{5.53}$$

Clearly, the rank of N is three and the restriction of N to the fibre $\pi_2^{-1}(x)$ of $F^9 \to P^6$ for $x \in M^6$ is isomorphic to the pull-back $\pi_1^* N_{\hat{X}|P^6}$ of the normal bundle $N_{\hat{x}|P^6}$ of $\hat{x} \hookrightarrow P^6$. Thus, N can be identified with $\pi_1^* N_{\hat{x}|P^6}$.

These considerations allow us to reconstruct the tangent bundle T_{M^6} from twistor space. In fact, we may apply the direct image functor (with regard to π_2) to the short exact sequence (5.53). Since both direct images $\pi_{2*}T_{\pi_1}$ and $\pi^1_{2*}T_{\pi_1}$ vanish, we obtain

$$T_{M^6} \cong \pi_{2*}\pi_1^* N_{\hat{x}|P^6} \quad \Leftrightarrow \quad (T_{M^6})_x \cong H^0(\hat{x}, N_{\hat{x}|P^6}). \tag{5.54}$$

Elements of $H^0(\hat{x}, N_{\hat{x}|P^6})$ are given in terms of elements of $H^0(\mathbb{P}^3, P^6)$ by allowing the latter to depend on x. One can check that this dependence is holomorphic in an open neighbourhood of x.

What remains to be understood is how the explicit factorisation (5.29) of the tangent bundle emerges from the above construction and in particular from $H^0(\mathbb{P}^3, P^6)$. To show this, we consider the Euler sequence for $\mathbb{P}^3$,

$$0 \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathbb{C}^4 \to T_{\mathbb{P}^3} \to 0. \qquad (5.55)$$

Upon dualising this sequence and twisting by $\mathcal{O}_{\mathbb{P}^3}(2)$, we

$$0 \to \Omega^1_{\mathbb{P}^3}\mathcal{O}_{\mathbb{P}^3}(2) \otimes\to \mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathbb{C}^4 \to \mathcal{O}_{\mathbb{P}^3}(2) \to 0 \qquad (5.56)$$

By comparing with (5.40a), we may conclude that

$$P^6 \cong \Omega^1(2) \ \text{ with } \ \Omega^p(k) : \ \Omega^p_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(k), \qquad (5.57)$$

Thus, elements of $H^0(\mathbb{P}^3, P^6)$ can also be viewed as elements of $H^0(\mathbb{P}^3, \Omega^1(2))$. The latter are of the form $\omega = \omega^{AB}\lambda_A d\lambda_B$ with $\omega^{AB} = -\omega^{BA}$. Since

$$S_x \cong H^0(\hat{x}, \mathcal{O}_{\hat{x}}(1)) \qquad (5.58)$$

via $s^A \to s^A\lambda_A$ for $s^A \in S_x$, we indeed find the factorization $(T_{M^6})_x \cong S_x \wedge S_x$. This concludes our construction of space-time from twistor space.

5.6.2 Remark

Notice that an identification of the form (5.29) amounts to choosing a (holomorphic) conformal structure. This can be seen as follows:

Let X be a six-dimensional complex spin manifold. The first definition of a conformal structure on X (and perhaps the standard one) assumes an equivalence class [g], the conformal class, of holomorphic metrics g on X.

Two given metrics g and g' are called equivalent if

$$g' = \gamma^2 g \qquad (5.59)$$

For some nowhere vanishing holomorphic function γ. Thus, a conformal structure is a line subbundle L in $T_X^\vee \odot T_X^\vee$.

An alternative definition of a conformal structure assumes a factorisation of the form $T_X \cong S \wedge S$, where S is the rank-4 chiral spin bundle.

This isomorphism in turn gives (canonically) the line subbundle $\det S^\vee \equiv \Lambda^4 S^\vee$ in $T_X^\vee \odot T_X^\vee$ since upon using splitting principle arguments, one finds the identification

$$K_X \coloneqq \det T_X^\vee \cong \otimes^3 \det S^\vee \tag{5.60}$$

for the canonical bundle K_X. Hence, det $S^\vee$ can be identified with the line bundle L from above, and the metric g is then of the form $\gamma^2 \varepsilon_{ABCD}$.

5.7 Penrose Transform in Six Dimensions

Having defined twistor space, we would like to understand differentially constrained data on space-time in terms of differentially unconstrained data on twistor space. Specifically, we are interested in the chiral fields introduced in Section 5.2 and prove the following theorem:

5.7.1 Theorem

Consider the double fibration (5.30). Let $U \subset M^6$ be open and convex and set $U' \coloneqq \pi_2^{-1}(U) \subset F^9$ and $\hat{U} \coloneqq \pi_1(\pi_2^{-1}(U)) \subset P^6$, respectively. For $h \in \frac{1}{2}\mathbb{N}_0$, there is a canonical isomorphism

$$\mathcal{P}: H^3\left(\hat{U}, \mathcal{O}_{\hat{U}}(-2h-4)\right) \to H^0(U, z_h), \tag{4.1}$$

Where z_h is the sheaf of chiral zero-rest-mass fields defined in (5.22). This transformation is called the Penrose transform.

5.8 Cohomological Considerations

5.8.1Relative de Rham Complex

The starting point of our considerations is the double fibration (5.30). As a first tool in proving the Penrose transform, we introduce the relative differential forms $\Omega^p_{\pi_1}$, i.e. the differential p-forms along the fibres of the fibration $\pi_1: F^9 \to P^6$.

We have already introduced the corresponding relative tangent bundle in (5.52). Simply dualising this sequence, we obtain the definition of the sheaf of relative one-forms from

$$0 \to \pi_1^*\Omega^1_{P^6} \to \Omega^1_{F^9} \to \Omega^1_{\pi_1} \to 0. \tag{5.61}$$

Recall from our previous discussion that in our parametrisation, sections μ_A of the relative tangent bundle T_{π_1} are defined up to shifts by terms proportional to λ_A.

This, in turn, induces the condition $\omega^A\lambda_A = 0$ on sections ω^A of $\Omega^1_{\pi_1}$. We shall come back to this point when discussing the direct images of $\Omega^1_{\pi_1}$.

In general, we introduce the relative p-forms $\Omega^p_{\pi_1}$ on F^9 with respect to the fibration $\pi_1\colon F^9 \to P^6$ according to

$$0 \to \pi_1^*\Omega^1_{P^6} \wedge \Omega^{p-1}_{F^9} \to \Omega^p_{F^9} \to \Omega^p_{\pi_1} \to 0 \tag{5.62}$$

Thus, relative p-forms have components only along the fibres of $\pi_1\colon F^9 \to P^6$ (i.e. any contraction with a vector field which is a section of $\pi_1^*TP^6$ vanishes). The coefficient functions in local coordinates, however, depend on both the base and the fibre coordinates. Note that the maximum value of p here is three.

If we let $Pr_{\pi_1}\colon \Omega^p_{F^9} \to \Omega^p_{\pi_1}$ be the quotient mapping, we can define the relative exterior derivative d_{π_1} by setting

$$d_{\pi_1} \coloneqq Pr_{\pi_1} \circ d : \Omega^p_{\pi_1} \to \Omega^{p+1}_{\pi_1} \tag{5.63}$$

Where d is the usual exterior derivative on F^9.

In local coordinates (x^{AB}, λ_A) on F^9, the relative exterior derivative can be presented in terms of the vector fields (5.29).

Next, observe that the relative differential d_{π_1} induces the relative de Rham complex.

This complex is given in terms of an injective resolution of the topological inverse $\pi_1^{-1}\mathcal{O}_{P^6}$ of $\mathcal{O}_{P^6}$ on the correspondence space F^9:

$$0 \to \pi_1^{-1}\mathcal{O}_{P^6} \to \mathcal{O}_{F^9} \xrightarrow{d_{\pi_1}} \Omega^1_{\pi_1} \xrightarrow{d_{\pi_1}} \Omega^2_{\pi_1} \xrightarrow{d_{\pi_1}} \Omega^3_{\pi_1} \to 0. \quad (5.64)$$

A natural question is now if the sheaves $\Omega^p_{\pi_1}$ have an interpretation in terms of certain pull-back sheaves from space-time and twistor space.

5.8.2 **Note**

The vectors fields (5.29) are given by

$$V^A = \frac{1}{2}\varepsilon^{ABCD}\lambda_B\,\partial_{CD} \quad (5.65)$$

where ∂_{AB} are the vector fields spanning T_{M^6}.

In terms of the V^A, the map d_{π_1}: $\mathcal{O}_{F^9} \to \Omega^1_{\pi_1}$ reads explicitly as

$$V^A: f \to \omega^A = V^A f = \frac{1}{2}\varepsilon^{ABCD}\lambda_B\,\partial_{CD}f, \qquad f \in \mathcal{O}_{F^9}. \quad (5.66)$$

This shows that $\omega^A = V^A f$ is a section of $\pi_2^*(\det S^\vee \otimes_{\mathcal{O}_{M^6}} S)\otimes_{\mathcal{O}_{F^9}}\pi_1^*\,\mathcal{O}_{P^6}(1)$.

Clearly, it is not the most general section of this sheaf, since we have

$$\lambda_A\omega^A = \lambda_A V^A f = 0 \quad (5.67)$$

see also our comments given below (5.62). For a general section s^A of $\pi_2^*(\det S^\vee \otimes_{\mathcal{O}_{M^6}} S)\otimes_{\mathcal{O}_{F^9}}\pi_1^*\,\mathcal{O}_{P^6}(1)$, the map $\lambda_A : s^A \to s^A\lambda_A$ gives a section of $\pi_2^*\det S^\vee \otimes_{\mathcal{O}_{F^9}}\pi_1^*\,\mathcal{O}_{P^6}(2)$ and its kernel gives $\Omega^1_{\pi_1}$. Altogether, we conclude that $\Omega^1_{\pi_1}$ fits into the following short exact sequence:

$$0 \rightarrow \Omega^1_{\pi_1} \rightarrow \pi_2^*(\det S^\vee \otimes_{\mathcal{O}_{M^6}} S) \otimes_{\mathcal{O}_{F^9}} \pi_1^* \mathcal{O}_{P^6}(1) \rightarrow$$
$$\rightarrow \pi_2^* \det S^\vee \otimes_{\mathcal{O}_{F^9}} \pi_1^* \mathcal{O}_{P^6}(2) \rightarrow 0 \qquad (5.68)$$

Using the notation (5.21), we then obtain the following proposition:

5.8.3 Lemma

The sheaves appearing in the relative de Rham sequence (4.5) can be canonically identified as follows. With $\Omega^p_{\pi_1}(k) \coloneqq \Omega^p_{\pi_1} \otimes_{\mathcal{O}_{F^9}} \pi_1^* \mathcal{O}_{P^6}(k)$, we have

$$0 \rightarrow \Omega^p_{\pi_1} \rightarrow \pi_2^*(\Lambda^p S)[p] \otimes_{\mathcal{O}_{F^9}} \pi_1^* \mathcal{O}_{P^6}(p) \rightarrow \pi_2^*[1] \otimes_{\mathcal{O}_{F^9}} \Omega^{p-1}_{\pi_1}(2) \rightarrow 0 \qquad (5.69)$$

Proof:

Using the fact that short exact sequences of the form $0 \rightarrow \varepsilon \rightarrow \mathcal{F} \rightarrow \mathcal{L} \rightarrow 0$, where $\mathcal{L}$ is the sheaf of sections of some line bundle, always induce $0 \rightarrow \Lambda^p \mathcal{E} \rightarrow \Lambda^p \mathcal{F} \rightarrow \Lambda^{p-1} \mathcal{E} \otimes \mathcal{L} \rightarrow 0$, the sequence (5.68) immediately leads to (5.69).

Finally, we point out that the relative de Rham sequence (5.64) has a natural extension via twisting by a holomorphic vector bundle. Specifically, let $E \rightarrow P^6$ be a holomorphic vector bundle over P^6 and consider the pull-back bundle $\pi_1^* E$ over the correspondence space F^9. We may tensor (5.64) by $\pi_1^{-1} \mathcal{O}_{P^6}(E)$, which is the sheaf of sections of $\pi_1^* E$ that are constant along $\pi_1 \colon F^9 \rightarrow P^6$. Because $\mathcal{O}_{F^9}(\pi_1^* E) \cong \pi_1^* \mathcal{O}_{P^6}(E)$ and $\mathcal{O}_{F^9} \otimes_{\pi_1^{-1} \mathcal{O}_{P^6}} \pi_1^{-1} \mathcal{O}_{P^6}(E)$ are canonically isomorphic, we find

$$0 \rightarrow \pi_1^{-1} \mathcal{O}_{P^6}(E) \rightarrow \Omega^0_{\pi_1}(E) \xrightarrow{d_{\pi_1}} \ldots . \xrightarrow{d_{\pi_1}} \Omega^3_{\pi_1}(E) \rightarrow 0, \qquad (5.70a)$$

where we have defined

$$\Omega^0_{\pi_1}(E) \coloneqq \mathcal{O}_{F^9}(\pi_1^* E) \text{ and } \Omega^p_{\pi_1}(E) \coloneqq \Omega^p_{\pi_1} \otimes_{\mathcal{O}_{F^9}} \mathcal{O}_{F^9}(\pi_1^* E), \qquad (5.70b)$$

5.8.4 Direct Image Sheaves.

The next important ingredient for our subsequent discussion isthe direct images of $\Omega^p_{\pi_1}(E)$ with respect to the fibration $\pi_2: F^9 \to P^6$ for the special case $E = \mathcal{O}_{P^6}(k),\ k \in \mathbb{Z}$. To compute those, we shall make use of the following lemma:

5.8.5 Lemma

Let V be a four-dimensional complex vector space together with its projectivisation $\mathbb{P}(V)$. Using the shorthand notations $\Omega^p(k) : \Omega^p_{\mathbb{P}(V)} \otimes \mathcal{O}_{\mathbb{P}(V)}(k)$ and $\Omega^0(k) \coloneqq \mathcal{O}_{\mathbb{P}(V)}(k)$, we have the following list of sheaf cohomology groups:

$$H^q(\mathbb{P}(V), \Omega^0(k)) \cong \begin{cases} \odot^k V^\vee & \text{for } q = 0 \ \mathfrak{E} \ k \geq 0 \\ \odot^{-k-4} V \otimes \det V & \text{for } q = 3 \ \mathfrak{E} \ k \leq -4 \\ 0 & \text{otherwise} \end{cases} \tag{5.71a}$$

$$H^q(\mathbb{P}(V), \Omega^1(k)) \cong \begin{cases} \left[\dfrac{\odot^{k-1} V \otimes V}{\odot^k V}\right]^\vee & \text{for } q = 0 \ \mathfrak{E} \ k \geq 2 \\ \mathbb{C} & \text{for } q = 1 \ \mathfrak{E} \ k = 0 \\ V^\vee \otimes \det V & \text{for } q = 3 \ \mathfrak{E} \ k = -3 \\ \left[\dfrac{\odot^{-k-3} V^\vee \otimes V}{\odot^{-k-4} V^\vee}\right]^\vee \otimes \det V & \text{for } q = 3 \ \mathfrak{E} \ k < -3 \\ 0 & \text{otherwise,} \end{cases} \tag{5.71b}$$

$$H^q(\mathbb{P}(V), \Omega^2(k)) \cong \begin{cases} V \otimes \det V^\vee & \text{for } q = 0 \ \mathfrak{E} \ k = 3, \\ \dfrac{\odot^{k-3} V^\vee \otimes V}{\odot^{k-4} V^\vee} \otimes \det V^\vee & \text{for } q = 0 \ \mathfrak{E} \ k > 3, \\ \mathbb{C} & \text{for } q = 2 \ \mathfrak{E} \ k = 0, \\ \dfrac{\odot^{-k-1} V \otimes V}{\odot^{-k} V} & \text{for } q = 3 \ \mathfrak{E} \ k \leq -2, \\ 0 & \text{otherwise,} \end{cases} \tag{5.71c}$$

$$H^q(\mathbb{P}(V), \Omega^3(k)) \cong \begin{cases} \odot^{-k-4} V^\vee \otimes \det V^\vee & \text{for } q = 0 \ \mathfrak{E} \ k \geq 4 \\ \odot^{-k} V & \text{for } q = 3 \ \mathfrak{E} \ k \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

(5.71d)

Notice that here, we are essentially computing the Dolbeault cohomology groups $H^{p,q}_{\bar\partial}(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(k))$ of the complex projective three-space $\mathbb{P}^3$ with values in $\mathcal{O}_{\mathbb{P}^3}(k)$ via the Dolbeault isomorphism.

Proof:

We already know the cohomology groups (5.71a) from Bott's rule given in Lemma 5.6.1 Moreover, after computing (5.71b), all remaining cases follow directly from (5.715.71a) and (5.71b) via Serre duality.17 In fact, we find the cohomology groups (5.71c) and (5.71d) from

$$H^q\big(\mathbb{P}(V), \Omega^2(k)\big) \cong \big[H^{3-q}\big(\mathbb{P}(V), \Omega^1(-k)\big)\big]^\vee$$
$$H^q(\mathbb{P}(V), \Omega^1(k)) \cong [H^{3-q}(\mathbb{P}(V), \Omega^0(-k))]^\vee \tag{5.72}$$

To compute (5.71b), let us consider the Euler sequence (5.55). We can dualise this sequence and twist by $\mathcal{O}_{\mathbb{P}(V)}(k)$ to obtain

$$0 \to \Omega^1(k) \to \Omega^0(k-1)\otimes V^\vee \to \Omega^0(k) \to 0. \tag{5.73}$$

From this sequence and Bott's rule, we derive the long exact sequences of cohomology groups

$$0 \to H^0\big(\mathbb{P}(V), \Omega^1(k)\big) \to H^0\big(\mathbb{P}(V), \Omega^0(k-1)\otimes V^\vee\big) \overset{k}{\to}$$
$$\overset{k}{\to} H^0\big(\mathbb{P}(V), \Omega^0(k)\big) \to H^1\big(\mathbb{P}(V), \Omega^1(k)\big) \to 0\,, \tag{5.74a}$$

And

$$0 \to H^3\big(\mathbb{P}(V), \Omega^1(k)\big) \to H^3\big(\mathbb{P}(V), \Omega^0(k-1)\otimes V^\vee\big) \to$$
$$\to H^3\big(\mathbb{P}(V), \Omega^0(k)\big) \to 0, \tag{5.74b}$$

where we used $H^2\big(\mathbb{P}(V), \Omega^1(k)\big) = 0$.

Let us start with $H^q(\mathbb{P}(V), \Omega^1(k))$ for $q = 0, 1$. For $k < 0$, the sequence (5.74a) together with Bott's rule yield that

$$H^0\big(\mathbb{P}(V), \Omega^1(k)\big) = 0 = H^1(\mathbb{P}(V), \Omega^1(k)) \tag{5.75}$$

while for $k = 0$ we find

$$H^0\big(\mathbb{P}(V), \Omega^1(0)\big) = 0 \tag{5.76}$$

And

$$H^1(\mathbb{P}(V), \Omega^1(0)) \cong H^0\big(\mathbb{P}(V), \Omega^0(0)\big) \cong \mathbb{C}. \tag{5.77}$$

For $k = 1$, (5.74a) also shows that

$$H^0\big(\mathbb{P}(V), \Omega^1(1)\big) = 0 = H^1\big(\mathbb{P}(V), \Omega^1(1)\big) \tag{5.78}$$

while for $k \geq 2$ we find $H^1\big(\mathbb{P}(V), \Omega^1(k)\big) = 0$ since k is surjective.

The rest of $H^0\big(\mathbb{P}(V), \Omega^1(k)\big)$ then follows from the short exact sequence

$$0 \to H^0\big(\mathbb{P}(V), \Omega^1(k)\big) \to \odot^{k-1} V^\vee \otimes V^\vee \to \odot^k V^\vee \to 0. \tag{5.79}$$

It remains to find $H^3\big(\mathbb{P}(V), \Omega^1(k)\big)$. The sequence (4.13b) and Bott's rule show that for $k \geq -2$, $H^3\big(\mathbb{P}(V), \Omega^1(k)\big) = 0$. while for $k = -3$, we get $H^3\big(\mathbb{P}(V), \Omega^1(-3)\big) \cong V^\vee \otimes \det V$. For $k < -3$, (4.13b) reads as

$$0 \to H^3\big(\mathbb{P}(V), \Omega^1(k)\big) \to \odot^{-k-3} V \otimes \det V \otimes V^\vee \to \odot^{-k-4} V \otimes \det V \to 0 \tag{5.80}$$

which gives the remaining cases for $H^3\big(\mathbb{P}(V), \Omega^1(k)\big)$. This completes the proof.

Next, we compute the direct image sheaves $\pi^q_{2*}\Omega^p_{\pi_1}\big(\mathcal{O}_{P^6}(k)\big)$. Using the short-hand notation $\Omega^p_{\pi_1}(k) : \Omega^p_{\pi_1}\big(\mathcal{O}_{P^6}(k)\big)$, we have the following proposition:

5.8.6 Proposition

Let $k_p := 2p + k$. The direct image sheaves $\pi^q_{2*}\Omega^p_{\pi_1}(k)$ are given by

$$\pi^q_{2*}\Omega^0_{\pi_1}(k) \cong \begin{cases} \odot^{k_0} S & \text{for } q = 0 \ \mathfrak{E} \ k_0 \geq 0 \\ (\odot^{-k_0-4} S^\vee)[1] & \text{for } q = 3 \ \mathfrak{E} \ k_0 \leq -4 \\ 0 & \text{otherwise} \end{cases} \tag{5.81a}$$

$$\pi^q_{2*}\Omega^1_{\pi_1}(k) \cong \begin{cases} \left(\dfrac{\odot^{k_1-1} S^\vee \otimes_{\mathcal{O}_{M^6}} S^\vee}{\odot^{k_1} S^\vee}\right)^\vee [1] & \text{for } q = 0 \ \mathfrak{E} \ k_1 \geq 2 \\ [1] & \text{for } q = 1 \ \mathfrak{E} \ k_1 = 0 \\ \left(\odot^{-k_1-3} S^\vee \otimes_{\mathcal{O}_{M^6}} S\right)_0 [2] & \text{for } q = 3 \ \mathfrak{E} \ k_1 \leq -3 \\ 0 & \text{otherwise} \end{cases} \tag{5.81b}$$

$$\pi^q_{2*}\Omega^2_{\pi_1}(k) \cong \begin{cases} \left(\odot^{k_2-3} S \otimes_{\mathcal{O}_{M^6}} S\right)_0 [1] & \text{for } q = 0 \ \mathfrak{E} \ k_2 \geq 3 \\ [2] & \text{for } q = 2 \ \mathfrak{E} \ k_2 = 0 \\ \left(\dfrac{\odot^{-k_2-1} S^\vee \otimes_{\mathcal{O}_{M^6}} S^\vee}{\odot^{-k_2} S^\vee}\right)[2] & \text{for } q = 3 \ \mathfrak{E} \ k_2 \leq -2 \\ 0 & \text{otherwise} \end{cases} \tag{5.81c}$$

And

$$\pi^q_{2*}\Omega^3_{\pi_1}(k) \cong \begin{cases} \left(\odot^{k_3-4} S\right)[2] & \text{for } q = 0 \ \mathfrak{E} \ k_3 \geq 4 \\ \left(\odot^{-k_3} S^\vee\right)[3] & \text{for } q = 3 \ \mathfrak{E} \ k_3 \leq 0 \\ 0 & \text{otherwise} \end{cases} \tag{5.81d}$$

where $\left(\odot^\iota S^\vee \otimes_{\mathcal{O}_{M^6}} S\right)_0$ is the totally trace-less part of $\odot^\iota S^\vee \otimes_{\mathcal{O}_{M^6}} S$ which is

$$\left(\bigodot^{\iota} S^{\vee} \otimes_{\mathcal{O}_{M^6}} S\right)_0 \cong \begin{cases} S & \text{for } \iota = 0, \\ \dfrac{\bigodot^{\iota} S^{\vee} \otimes_{\mathcal{O}_{M^6}} S}{\bigodot^{\iota-1} S^{\vee}} & \text{for } \iota \geq 0 \end{cases} \tag{5.82}$$

Proof

By definition of direct image sheaves, our task is to compute the cohomology groups $H^q(\pi_2^{-1}(U), \Omega^p_{\pi_1}(k))$ for open sets $U \subset M^6$.

Notice that it suffices to work with Stein open sets U so that $U' := \pi_2^{-1}(U) \cong U \times \mathbb{P}^3 \subset F^9$ since there are arbitrarily small Stein open sets on M^6. We could now apply the direct image functor to the short exact sequences of Proposition 5.8.3 to obtain the direct images. There is, however, a quicker way of computing these.

Consider the case when $p = 0$. It is rather straightforward to see that in this case, we have the identification

$$H^q(U', \Omega^0_{\pi_1}(k)) \cong \{\text{holomorphic functions} : U \to H^q(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(k))\} \tag{5.83}$$

and we can directly apply the results of Lemma 5.8.3 The other cohomology groups can be characterised analogously.

We first recall our discussion of the relative one-forms, $\Omega^1_{\pi_1}(0) = \Omega^1_{\pi_1}$ that led to the sequence (5.69).

Let

$$(x, \lambda) = (x^{AB}, \lambda_A) \tag{5.84}$$

be local coordinates on F^9, as before.

Then the components ω^A of a relative one-form ω are of weight one in λ and obey $\omega^A \lambda_A$.

This essentially implies that

$$\omega^A = \frac{1}{2} \varepsilon^{ABCD} \omega_{BC} \lambda_D, \tag{5.85}$$

where $\omega_{AB} = -\omega_{BA}$ depends (holomorphically) on x. Together with our results for the twistor space P^6 presented at the end of Section 5.3.2, we may conclude that:

$$H^q(U', \Omega^1_{\pi_1}(0)) \cong \{\text{holomorphic functions} : U \to H^q(\mathbb{P}^3, \Omega^1(2)\} \quad (5.86)$$

This argument generalizes to the remaining cohomology groups $H^q(U', \Omega^p_{\pi_1})$ for $p = 2, 3$, and we have

$$H^q(U', \Omega^p_{\pi_1}(0)) \cong \{\text{holomorphic functions} : U \to H^q\big(\mathbb{P}^3, \Omega^p(2p)\big)[p]\} \quad (5.87)$$

Therefore, if we let $k_p := 2p + k$, we obtain

$$H^q(U', \Omega^p_{\pi_1}(k)) \cong \{\text{holomorphic functions} : U \to H^q\left(\mathbb{P}^3, \Omega^p\left(k_p\right)\right)[p]\} \quad (5.88)$$

In summary, all the cohomology groups $H^q(\pi_2^{-1}(U), \Omega^p_{\pi_1}(k))$ are characterised in terms of the cohomology groups appearing in Lemma 5.8.5 for $V = S^\vee$, which yields (5.81).

So far, we have computed the direct images of the sheaves $\Omega^p_{\pi_1}(k)$.The resolutions (5.64) and (5.70a) also contain the topological inverse sheaves $\pi_1^{-1}\mathcal{O}_{P^6}$ and $\pi_1^{-1}\mathcal{O}_{P^6}(\mathcal{O}_{P^6}(k))$, respectively. The direct images of these sheaves are computed using spectral sequences.

In the following, we shall merely recall a few facts about spectral sequences and we refer to for a more detailed account.

For us, a spectral sequence is basically a sequence of two-dimensional arrays of Abelian groups

$$E_r = (E_r^{p,q}) \text{ for } r = 1, 2, \ldots \quad (5.89)$$

which are labelled by $p, q = 0, 1, 2, \ldots$ together with differential operators $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ that obey

$$d_r \circ d_r = 0. \tag{5.90}$$

In addition, the arrays are linked cohomologically from one order to the next. Specifically, we have

$$E_{r+1}^{p,q} \cong H^{p,q}(E_r) := \frac{\ker d_r \colon E_r^{p,q} \to E_r^{p+r,q-r+1}}{\operatorname{im} d_r \colon E_r^{p-r,q+r-1} \to E_r^{p,q}} \tag{5.91}$$

There also is a well-defined limit of the spectral sequence in terms of the inductive limit

$$E_\infty^{p,q} = \lim_{r\to\infty} ind\, E_r^{p,q} \tag{5.92}$$

If $U \subset M^6$ is open and $U' := \pi_2^{-1}(U)$, the resolution (5.70a) yields a spectral sequence with initial terms

$$E_1^{p,q} \cong H^q(U', \Omega_{\pi_1}^p(E)) \tag{5.93}$$

and differential operators $d_1 \colon E_1^{p,q} \to E_1^{p+1,q}$ induced by $d_{\pi_1} \colon \Omega_{\pi_1}^p(E) \to \Omega_{\pi_1}^{p+1}(E)$.

This spectral sequence converges to the cohomology group

$$E_\infty^{p,q} \cong H^{p+q}(U', \pi_1^{-1}\mathcal{O}_{P^6}(E)) \tag{5.94}$$

which is mnemonically written as

$$H^q\left(U', \Omega_{\pi_1}^p(E)\right) \Longrightarrow H^{p+q}(U', \pi_1^{-1}\mathcal{O}_{P^6}(E)). \tag{5.95}$$

``

Altogether, we have the following proposition:

5.8.6 Proposition

Let U be an open set in M^6 and let $U' := \pi_2^{-1}(U) \subset F^9$. Then there is a spectral sequence

$$E_1^{p,q} \cong H^q\left(U', \Omega_{\pi_1}^p(k)\right) \Longrightarrow H^{p+q}(U', \pi_1^{-1}\mathcal{O}_{P^6}(k)) \tag{5.96}$$

where the differential operators $d_1 : E_1^{p,q} \rightarrow E_1^{p+1,q}$ are induced by the relative exterior derivative $d_{\pi_1} : \Omega^p_{\pi_1}(k) \rightarrow \Omega^{p+1}_{\pi_1}(k)$.

Hence, we have an explicit way of computing $H^q(U', \pi_1^{-1}\mathcal{O}_{P^6}(k))$ in terms of the cohomology groups $H^q(U', \Omega^p_{\pi_1}(k))$.

5.9 Cohomology Groups of Topological Inverse Sheaves.

The final ingredient we need is a result due to Buchdahl. Above we have computed the direct images of sheaves on the correspondence space F^9 along the fibration $\pi_2 : F^9 \rightarrow M^6$ to obtain certain sheaves on space-time M^6.

In the Penrose transform, these sheaves on F^9 originate from sheaves on twistor space. To connect the cohomology groups of both kinds of sheaves, we can use the following proposition:

5.8.7 Lemma

Let X and Y be complex manifolds and $\pi : X \rightarrow Y$ a surjective holomorphic mapping of maximal rank with connected fibres.

Furthermore, let S be an Abelian sheaf on Y . If there is an $n_0 > 0$ such that $H^q(\pi^{-1}(p), \mathbb{C}) = 0$ for $q = 1, \ldots, n_0$ and for all $p \in Y$, then

$$\pi^* : H^q(Y, S) \rightarrow H^q(X, \pi^{-1}S) \qquad (5.97)$$

is an isomorphism for $q = 0, \ldots, n_0$ and a monomorphism for $q = n_0 + 1$. The requirements of this proposition for the projection $\pi_1 : F^9 \rightarrow M^6$ are always satisfied in our setting. Because we always work with convex subsets $U \subset M^6$, we always have the isomorphism $H^q(U', S) \cong H^q(\widehat{U}, \pi_1^{-1} S)$, where $U' := \pi_2^{-1}(U) \subset F^9$ and $\widehat{U} \coloneqq \pi_1(\pi_2^{-1}(U)) \subset P^6$.

In a compactified version of the twistor correspondence, one has to supplement Theorem 4.1 by the above requirements.

Proof

We are now ready to prove Theorem 5.8.3 We shall first proof the case $h > 0$, that is $-2h - 4 < -4$, and then come to the case $h = 0$, which is slightly more complicated.

Case $\mathbf{h > 0}$. Recall that sections ψ of the sheaf $\mathcal{Z}_h$ defined in (2.9) obey the free field equation

$$\partial^{AB}\psi_{BA_1.....A_{2h-1}} = 0 \tag{5.98}$$

We thus have to prove that,

$$\mathcal{P}: \ H^3(\widehat{U}, \mathcal{O}_{\widehat{U}}(-2h-4)) \to \ H^0(U, \mathcal{Z}_h) \tag{5.99}$$

is an isomorphism. We already know from Proposition 4.4 that

$$H^3(\widehat{U}, \mathcal{O}_{\widehat{U}}(-2h-4)) \cong H^3(U', \pi_1^{-1}\, \mathcal{O}_{\widehat{U}}(-2h-4) \tag{5.100}$$

which reduces (5.99) to

$$H^3(U', \pi_1^{-1}\, \mathcal{O}_{\widehat{U}}(-2h-4) \cong H^0(U, \mathcal{Z}_h) \tag{5.101}$$

Firstly, we notice that there is a particular spectral sequence, the Leray spectral sequence $L_r = \left(L_r^{p,q}\right)$, which gives

$$L_2^{p,q} \cong H^p\left(U, \pi_{2*}^q\, \Omega_{\pi_1}^l(-2h-4)\right) \Longrightarrow H^{p+q}\left(U', \Omega_{\pi_1}^l(-2h-4)\right) \tag{5.102}$$

For fixed l, Proposition 4.2 for $h > 0$ tells us that $\pi_{2*}^q\, \Omega_{\pi_1}^l(-2h-4) = 0$ if $q \neq 3$. Thus, the Leray spectral sequence $L_r^{p,q}$ is degenerate at the second level. Therefore, we have

$$L_\infty^{p,q} \cong L_2^{p,q} \ \text{ for } \ p, q \ \geq \ 0\,, \tag{5.103}$$

cf. (5.92). Recall that if a spectral sequence $(E_r^{p,q})$ has the property that for some $r_0, E_{r_0}^{p,q} = 0$ for $q \neq q_0$, then

$$E_{r_0}^{p,q_0} \cong H^{p+q_0}. \tag{5.104}$$

This property together with (5.102) then imply

$$H^p\left(U', \Omega^l_{\pi_1}(-2h-4)\right) \cong \begin{cases} H^{p-3}\left(U, \pi^3_{2*}\Omega^l_{\pi_1}(-2h-4)\right) & \text{for } p \geq 3 \\ 0 & \text{for } p < 3 \end{cases} \tag{5.106}$$

Secondly, Proposition 4.3 yields another spectral sequence $E_r = (E_r^{p,q})$ with

$$E_1^{p,q} \cong H^q\left(U', \Omega^p_{\pi_1}(-2h-4)\right) \Longrightarrow H^{p+q}\left(U', \pi_1^{-1}\mathcal{O}_{P^6}(-2h-4)\right) \tag{5.107}$$

Explicitly, the $r = 1$ array in this sequence reads as ($k = -2h-4$):

$$\begin{array}{ccccccc}
H^0\left(U', \Omega^0_{\pi_1}(k)\right) & \xrightarrow{d_{\pi_1}} & H^0\left(U', \Omega^1_{\pi_1}(k)\right) & \xrightarrow{d_{\pi_1}} & H^0\left(U', \Omega^2_{\pi_1}(k)\right) & \xrightarrow{d_{\pi_1}} & H^0\left(U', \Omega^3_{\pi_1}(k)\right) \\
H^1\left(U', \Omega^0_{\pi_1}(k)\right) & \xrightarrow{d_{\pi_1}} & H^1\left(U', \Omega^1_{\pi_1}(k)\right) & \xrightarrow{d_{\pi_1}} & H^1\left(U', \Omega^2_{\pi_1}(k)\right) & \xrightarrow{d_{\pi_1}} & H^1\left(U', \Omega^3_{\pi_1}(k)\right) \\
H^2\left(U', \Omega^0_{\pi_1}(k)\right) & \xrightarrow{d_{\pi_1}} & H^2\left(U', \Omega^1_{\pi_1}(k)\right) & \xrightarrow{d_{\pi_1}} & H^2\left(U', \Omega^2_{\pi_1}(k)\right) & \xrightarrow{d_{\pi_1}} & H^2\left(U', \Omega^3_{\pi_1}(k)\right) \\
H^3\left(U', \Omega^0_{\pi_1}(k)\right) & \xrightarrow{d_{\pi_1}} & H^3\left(U', \Omega^1_{\pi_1}(k)\right) & \xrightarrow{d_{\pi_1}} & H^3\left(U', \Omega^2_{\pi_1}(k)\right) & \xrightarrow{d_{\pi_1}} & H^3\left(U', \Omega^3_{\pi_1}(k)\right) \\
H^4\left(U', \Omega^0_{\pi_1}(k)\right) & \xrightarrow{d_{\pi_1}} & H^4\left(U', \Omega^1_{\pi_1}(k)\right) & \xrightarrow{d_{\pi_1}} & H^4\left(U', \Omega^2_{\pi_1}(k)\right) & \xrightarrow{d_{\pi_1}} & H^4\left(U', \Omega^3_{\pi_1}(k)\right) \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array} \tag{5.108}$$

We may now replace these cohomology groups by $H^q\left(U', \Omega^p_{\pi_1}(k)\right)$ using (5.106) to obtain

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
0 & & 0 & & 0 \\
0 & & 0 & & 0 \\
H^0\left(U, \pi^3_{2*}\,\Omega^0_{\pi_1}(k)\right) & \to & H^0\left(U', \pi^3_{2*}\Omega^1_{\pi_1}(k)\right) & \to \cdots \to & H^0\left(U', \pi^3_{2*}\Omega^3_{\pi_1}(k)\right) \\
H^1\left(U, \pi^3_{2*}\,\Omega^0_{\pi_1}(k)\right) & \to & H^1\left(U', \pi^3_{2*}\Omega^1_{\pi_1}(k)\right) & \to \cdots \to & H^1\left(U', \pi^3_{2*}\Omega^3_{\pi_1}(k)\right) \\
H^2\left(U, \pi^3_{2*}\,\Omega^0_{\pi_1}(k)\right) & \to & H^2\left(U', \pi^3_{2*}\Omega^1_{\pi_1}(k)\right) & \to \cdots \to & H^2\left(U', \pi^3_{2*}\Omega^3_{\pi_1}(k)\right) \\
H^3\left(U, \pi^3_{2*}\,\Omega^0_{\pi_1}(k)\right) & \to & H^3\left(U', \pi^3_{2*}\Omega^1_{\pi_1}(k)\right) & \to \cdots \to & H^3\left(U', \pi^3_{2*}\Omega^3_{\pi_1}(k)\right)
\end{array} \tag{5.109}$$

This diagram together with (5.91) then yield the following identification:

$$E_2^{0,3} \cong \ker\left\{H^0\left(U, \pi_{2*}^3\,\Omega_{\pi_1}^0(-2h-4)\right) \to H^0\left(U, \pi_{2*}^3\,\Omega_{\pi_1}^1(-2h-4)\right)\right\} \tag{5.110}$$

Furthermore, all $E_r^{p,q} = 0$ for $p + q = 3$ with $q \neq 3$, and $E_2^{0,3} \cong E_3^{0,3} \cong \cdots E_\infty^{0,3}$. From Proposition 4.2, it follows that $\pi_{2*}^3\,\Omega_{\pi_1}^0(-2h-4) \cong (\odot^{2h}S^\vee)[1]$ and $\pi_{2*}^3\,\Omega_{\pi_1}^1(-2h-4) \cong (\odot^{2h-1}S^\vee \otimes_{\mathcal{O}_U} S)_0[2]$.

In addition, the relative exterior derivative $d_{\pi_1}\colon H^3\left(U', \Omega_{\pi_1}^0(k)\right) \to H^3\left(U', \Omega_{\pi_1}^1(k)\right)$ induces the differential operator

$$\partial^{AB} :\ H^0\left(U, \pi_{2*}^3\Omega_{\pi_1}^0(-2h-4)\right) \to\ H^0\left(U, \pi_{2*}^3\Omega_{\pi_1}^1(-2h-4)\right) \tag{5.111}$$

In summary, from (4.28) and (5.107) we may therefore conclude that

$$H^3\left(\widehat{U}, \mathcal{O}_{\widehat{U}}(-2h-4)\right) \cong H^3\left(U', \pi_1^{-1}\mathcal{O}_{\widehat{U}}(-2h-4)\right) \cong E_2^{0,3} \cong H^0(U, \mathcal{Z}_h) \tag{5.112}$$

Case $h = 0$. The proof for $h = 0$ is similar to the one presented above albeit somewhat more difficult. Firstly, we shall be dealing with a second-order partial differential operator and secondly, on a more technical level, the appropriate spectral sequence will degenerate differently.

Recall that $\mathcal{Z}_0$ is the sheaf of solutions to the Klein–Gordon equation. That is, its sections describe scalar fields on space-time forming the trivial representation under the little group. We wish to prove that

$$\mathcal{P} :\ H^3\left(\widehat{U}, \mathcal{O}_{\widehat{U}}(-4)\right) \to H^0(U, \mathcal{Z}_0) \tag{5.113}$$

is an isomorphism. Again, by virtue of Proposition 4.4, we only need to show that

$$H^3\left(U', \pi_1^{-1}\mathcal{O}_{\widehat{U}}(-4)\right) \cong H^0(U, \mathcal{Z}_0) \tag{5.114}$$

From lemma 5.8.3, we see that

$$\pi_{2*}^{q}\,\Omega_{\pi_1}^{l}(-4) \cong \begin{cases} [1] & \text{for } (q,l) = (3,0) \\ [2] & \text{for } (q,l) = (2,2) \\ 0 & \text{otherwise} \end{cases} \tag{5.115}$$

When $(q,l) = (3,0)$, the corresponding Leray spectral sequence (5.102) yields

$$H^p\left(U', \Omega_{\pi_1}^0(-4)\right) \cong \begin{cases} H^{p-3}\left(U, \pi_{2*}^3\Omega_{\pi_1}^0(-4)\right) H^{p-3}(U,[1]) & \text{for} \quad p \geq 3 \\ 0 & \text{for} \quad p \geq 3 \end{cases} \tag{5.116}$$

Moreover, with (5.115) the Leray spectral sequence (5.102) also gives

$$H^p\left(U', \pi_{2*}^{p}\,\Omega_{\pi_1}^{l}(-4)\right) = 0 \text{ for } p,q \geq 0 \text{ and } l = 1,3 \tag{5.117}$$

When $(q,l) = 2,2)$, we derive

$$H^p\left(U', \Omega_{\pi_1}^2(-4)\right) \cong \begin{cases} H^{p-2}\left(U, \pi_{2*}^2\Omega_{\pi_1}^2(-4)\right) \cong H^{p-2}(U,[2]) & \text{for} \quad p \geq 2 \\ 0 & \text{for} \quad p > 2 \end{cases} \tag{5.118}$$

Next, the $r = 1$ part of the spectral sequence (5.107) for $h = 0$ is given by

$$\begin{array}{ccccccc}
H^0\left(U', \Omega_{\pi_1}^0(-4)\right) & \xrightarrow{d_{\pi_1}} & H^0\left(U', \Omega_{\pi_1}^1(-4)\right) & \xrightarrow{d_{\pi_1}} & H^0\left(U', \Omega_{\pi_1}^2(-4)\right) & \xrightarrow{d_{\pi_1}} & H^0\left(U', \Omega_{\pi_1}^3(-4)\right) \\
H^1\left(U', \Omega_{\pi_1}^0(-4)\right) & \xrightarrow{d_{\pi_1}} & H^1\left(U', \Omega_{\pi_1}^1(-4)\right) & \xrightarrow{d_{\pi_1}} & H^1\left(U', \Omega_{\pi_1}^2(-4)\right) & \xrightarrow{d_{\pi_1}} & H^1\left(U', \Omega_{\pi_1}^3(-4)\right) \\
H^2\left(U', \Omega_{\pi_1}^0(-4)\right) & \xrightarrow{d_{\pi_1}} & H^2\left(U', \Omega_{\pi_1}^1(-4)\right) & \xrightarrow{d_{\pi_1}} & H^2\left(U', \Omega_{\pi_1}^2(-4)\right) & \xrightarrow{d_{\pi_1}} & H^2\left(U', \Omega_{\pi_1}^3(-4)\right) \\
H^3\left(U', \Omega_{\pi_1}^0(-4)\right) & \xrightarrow{d_{\pi_1}} & H^3\left(U', \Omega_{\pi_1}^1(-4)\right) & \xrightarrow{d_{\pi_1}} & H^3\left(U', \Omega_{\pi_1}^2(-4)\right) & \xrightarrow{d_{\pi_1}} & H^3\left(U', \Omega_{\pi_1}^3(-4)\right) \\
H^4\left(U', \Omega_{\pi_1}^0(-4)\right) & \xrightarrow{d_{\pi_1}} & H^4\left(U', \Omega_{\pi_1}^1(-4)\right) & \xrightarrow{d_{\pi_1}} & H^4\left(U', \Omega_{\pi_1}^2(-4)\right) & \xrightarrow{d_{\pi_1}} & H^4\left(U', \Omega_{\pi_1}^3(-4)\right) \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array} \tag{5.119}$$

Our above calculations show that the second and fourth columns of this diagram are zero, while the first and third ones are non-zero in general. Hence, the differential operator d_1 on $E_1^{p,q}$ vanishes identically and therefore, we have the identification $E_1^{p,q} \cong E_2^{p,q}$. Substituting (5.116) – (5.118) into this diagram, we eventually find

$$\begin{array}{llll} 0 & \to 0 \to & 0 & \to 0 \\ 0 & \to 0 \to & 0 & \to 0 \\ 0 & \to 0 \to & H^0(U,[2]) & \to 0 \\ H^0(U,[1]) & \to 0 \to & H^1(U,[2]) & \to 0 \\ H^1(U,[1]) & \to 0 \to & H^2(U,[2]) & \to 0 \\ H^2(U,[1]) & \to 0 \to & H^3(U,[2]) & \to 0 \\ H^3(U,[1]) & \to 0 \to & H^4(U,[2]) & \to 0 \end{array} \tag{5.120}$$

Furthermore, the differential operator d_2 on $E_2^{0,3}$ maps $E_2^{0,3}$ to $E_2^{2,2}$ and since $E_1^{p,q} \cong E_2^{p,q}$ and thus, $E_2^{0,3} \cong H^0(U,[1])$ and $E_2^{2,2} \cong H^0(U,[2])$, respectively, we have a map $\square : H^0(U,[1]) \to H^0(U,[2]$ which is induced by d_2. One can see that this map is a composition of first-order differential operators and it is indeed the one we defined in (5.22).
Finally, we note that

$$E_2^{0,3} \cong \ker\{\square : H^0(U,[1]) \to H^0(U,[2]\} \tag{5.121}$$

together with $E_2^{0,3} \cong \cdots \cong E_\infty^{0,3}$. Altogether,

$$H^3\left(\hat{U}, \mathcal{O}_{\hat{U}}(-4)\right) \cong H^3\left(U', \pi_1^{-1}\mathcal{O}_{\hat{U}}(-4)\right) \cong E_3^{0,3} \cong H^0(U, \mathcal{Z}_0) \tag{5.122}$$

which completes the proof for $h = 0$.

5.10. Integral Formulæ

Similarly to four dimensions, we can write down certain contour integral formulæ yielding solutions to the zero-rest-mass field equations in six dimensions.

5.10.1 Integral Formulæ on Twistor Space

Let us choose a sufficiently fine open Stein covering

$$\hat{\mathfrak{U}} = \{\hat{\mathrm{U}}_a\} \qquad (5.123)$$

of $\hat{\mathrm{U}}$. We shall make use of the abbreviations $\hat{\mathrm{U}}_{ab} := \hat{\mathrm{U}}_a \cap \hat{\mathrm{U}}_b$, $\hat{\mathrm{U}}_{abc} := \hat{\mathrm{U}}_a \cap \hat{\mathrm{U}}_b \cap \hat{\mathrm{U}}_c$, etc.

The simplest choice for $\hat{\mathrm{U}}$ is a lift of the standard cover of $\mathbb{P}^3$ to $\hat{\mathrm{U}}$ requiring four patches $\hat{\mathrm{U}}_a, a = 1, \dots, 4,$. In this case, there is only one quadruple overlap of four patches, and a holomorphic function

$$\hat{f}_{-2h-4} = \hat{f}_{-2h-4}(\mathcal{z}, \lambda) \qquad (5.124)$$

on $\hat{\mathrm{U}}_{1234} \subset \hat{\mathrm{U}}$ of homogeneity $-2h-4$ represents an element of $H^3(\hat{\mathrm{U}}, \mathcal{O}_{\hat{\mathrm{U}}}(-2h-4))$. For simplicity, we shall assume a ˇCech cocycle $\hat{f}_{-2h-4}$ of this form in the following. Note that this is not the most general way of representing elements of $H^3(\hat{\mathrm{U}}, \mathcal{O}_{\hat{\mathrm{U}}})$. This, however, requires merely a technical extension of our discussion below using branched contour integrals.

Let us now restrict to $h \geq 0$ and construct zero-rest-mass fields $\psi \in H^0(\mathrm{U}, \mathcal{Z}_h)$. That is, ψ forms the representation $(2h+1, 1)$ of the little group $\mathsf{SL}(2, \mathbb{C}) \times \mathsf{SL}(2, \mathbb{C})$ cf. (5.22).

We start from a ˇCech cocycle $\hat{f}_{-2h-4}$, which we restrict to $\hat{x} \cong \mathbb{P}^3$ to obtain $\hat{f}_{-2h-4} = \hat{f}_{-2h-4}(x \cdot \lambda, \lambda)$ on the intersection $\hat{\mathrm{U}}_{1234} \cap \hat{x}$. Using the holomorphic $\mathsf{SL}(4, \mathbb{C})$-invariant measure on $\mathbb{P}^3$ given by

$$\Omega^{(3,0)} := \frac{1}{4!} \varepsilon^{ABCD} \lambda_A d\lambda_B \wedge d\lambda_C \wedge d\lambda_D \qquad (5.125)$$

we can write down the contour integral

$$\psi_{A \dots A_{2h}}(x) = \oint_{\mathcal{C}} \Omega^{(3,0)} \lambda_A \dots \lambda_{A_{2h}} \hat{f}_{-2h-4}(x \cdot \lambda, \lambda) \qquad (5.126)$$

where the contour $\mathcal{C}$ is topologically a three-torus contained in $\hat{\mathrm{U}}_{1234}$. Clearly

$$\partial^{AB} \psi_{BA_1 \dots A_{2h-1}} = 0 \text{ for } h > 0 \text{ and } \Box\psi = 0 \text{ for } h = 0, \qquad (5.127)$$

as follows from straightforward differentiation under the integral.

5.10.2 Integral Formulæ on Thickened Twistor Space.

More recently, similar integral formulæ were discussed by the cohomology groups $H^3(\hat{U}, \mathcal{O}_{\hat{U}}(2h-4))$. with $h > 0$. However, these cohomology groups yield trivial space-time fields. Therefore, their integral formulæ make only sense if one thickens (via infinitesimal neighbourhoods) P^6 into its ambient space $\mathbb{P}^7_\circ \cong \mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathbb{C}^4$. Thickenings of manifolds occur in various twistor geometric contexts. The most prominent examples appear in the twistor descriptions of Yang–Mills theory and Einstein gravity in four space-time dimensions.

To thicken our twistor space P^6, consider $\mathcal{O}_{\mathbb{P}^7}$, the sheaf of holomorphic functionson $\mathbb{P}^7_\circ$, and $\mathcal{J}$, the ideal subsheaf of $\mathcal{O}_{\mathbb{P}^7_\circ}$ consisting of those functions that vanish on $P^6 \to \mathbb{P}^7_\circ$. ℓ-th order thickening (or ℓ-th infinitesimal neighbourhood) of P^6 inside $\mathbb{P}^7_\circ$ is the scheme $P^6_{[\ell]}$ defined by

$$P^6_{[\ell]} := (P^6, \mathcal{O}_{\mathbb{P}^7}/\mathcal{J}^{\ell+1} \tag{5.128}$$

5.10.3 Note

We recover the twistor space as the zeroth order thickening, i.e. $P^6_{[0]} = P^6$ Moreover, a cover of P^6 will also form a cover of $P^6_{[\ell]}$. The spaces $P^6_{[\ell]}$ can be thought of as the jets of the embedding of P^6 into the larger manifold $\mathbb{P}^7_\circ$.

In local coordinates (z^A, λ_A) on $\mathbb{P}^7_\circ$, we have

$$(z^A, \lambda_A)^{i+1} = 0 \text{ for } i \geq \ell \tag{5.129}$$

But

$$(z^A, \lambda_A)^i \neq 0 \text{ for } 0 < i \leq \ell \text{ on } P^6_{[\ell]}. \tag{5.130}$$

This implies that on the first order thickening $P^6_{[1]}$, the four vector fields $\frac{\partial}{\partial z^A}$ are linearly independent and act freely on functions on $P^6_{[1]}$. Differential operators of order ℓ constructed out of these four vector fields act freely on functions on $P^6_{[\ell]}$. As we shall see momentarily, this fact is the essential ingredient for writing down a contour integral leading to zero-rest-mass fields.

Proceeding analogously to four dimensions, we shall now construct a second contour integral by replacing λ_A in (5.126) by the derivatives $\frac{\partial}{\partial z^A}$ and adjusting the homogeneity of $\hat{f}_{-2h-4}$ for $h > 0$ accordingly. The resulting $2h$ derivatives in the contour integral should act freely, and therefore we have to consider a thickening of $\widehat{U} \subset P^6$ to $2h$-th order, that is, $\widehat{U}_{[2h]} \subset P^6{}_{[2h]}$.

Let

$$\hat{f}^{[2h]}_{2h-4} = \hat{f}^{[2h]}_{2h-4}(z, \lambda) \tag{5.131}$$

be a representative of the cohomology group

$$H^3(\widehat{U}_{[2h]}, \mathcal{O}_{\widehat{U}_{[2h]}}(2h-4)) \text{ for } h > 0. \tag{5.132}$$

It is expanded as

$$\hat{f}^{[2h]}_{2h-4}(z, \lambda) = \hat{g}(\lambda) + \sum_{l \geq 1} \frac{1}{l!} z^{A_1} \dots z^{A_{2h}} \hat{g}_{A_1 \dots A_l}(\lambda) \tag{5.133}$$

where the coefficients $\hat{g}_{A_1 \dots A_l}$ for $l \leq 2h$ are uniquely defined for $0 < l \leq 2h$. We may rewrite the above expansion as

$$\hat{f}^{[2h]}_{2h-4}(z, \lambda) = \frac{1}{(2h)!} z^{A_1} \dots z^{A_{2h}} \hat{f}_{A_1 \dots A_{2h}}(z, \lambda) + \cdots, \tag{5.134}$$

where the ellipsis denotes terms that contain at most $2h - 1$ factors of z^A. As the coefficients $\hat{f}_{A_1 \dots A_{2h}}$ are uniquely fixed, they can be extracted from $\hat{f}^{[2h]}_{2h-4}$. Upon restriction to $\hat{x} \cong \mathbb{P}^3$ we may write

$$\hat{f}_{A_1 \dots A_{2h}}(x \cdot \lambda, \lambda) = \frac{\partial}{\partial z^{A_1}} \dots \frac{\partial}{\partial z^{A_{2h}}} \hat{f}^{[2h]}_{2h-4}(z, \lambda)\Big|_{z = x \cdot \lambda} \tag{5.135}$$

The latter relation can then be used to construct the contour integral formula

$$\psi_{A_1 \dots A_{2h}}(x) = \oint_{\mathcal{C}} \Omega^{(3,0)} \hat{f}_{A_1 \dots A_{2h}}(x \cdot \lambda, \lambda) \tag{5.136}$$

$$= \oint_{\mathcal{C}} \Omega^{(3,0)} \hat{\mathrm{f}}_{\mathrm{A}_1 \dots \mathrm{A}_{2h}} \frac{\partial}{\partial z^{\mathrm{A}_1}} \cdots \frac{\partial}{\partial z^{\mathrm{A}_{2s}}} \hat{\mathrm{f}}^{[2h]}_{2h-4}(z, \lambda)\Big|_{z=\mathrm{x}\cdot\lambda} \tag{5.137}$$

where the contour is again a three-torus. By differentiation under the integral, one may check that this is indeed a zero-rest-mass field, i.e.

$$\partial^{\mathrm{AB}} \psi_{\mathrm{BA}_1 \dots \mathrm{A}_{2h-1}} = 0, \tag{5.138}$$

Since

$$\frac{\partial}{\partial \mathrm{x}^{\mathrm{AB}}} = \lambda_{[\mathrm{A}} \frac{\partial}{\partial z^{\mathrm{B}]}} \tag{5.139}$$

under the integral.

More generally, we can write down the following contour integral, which interpolates between the above two formulæ (5.126) and (5.137):

$$\psi_{\mathrm{A}_1 \dots \mathrm{A}_{2h}}(\mathrm{x}) = \oint_{\mathcal{C}} \Omega^{(3,0)} \lambda_{(\mathrm{A}_1} \dots \lambda_{\mathrm{A}_{j+h}} \frac{\partial}{\partial z^{\mathrm{A}_{j+h+1}}} \cdots \frac{\partial}{\partial z^{\mathrm{A}_{2h})}} \hat{\mathrm{f}}^{[h-j]}_{-2j-4}(z, \lambda)\Big|_{z=\mathrm{x}\cdot\lambda} \tag{5.140}$$

Here, $\mathrm{j} = -\mathrm{h}, \dots, \mathrm{h}$ and the indices $\mathrm{A}_1 \dots \mathrm{A}_{2h}$ are symmetrised in the integrand. Again, it is straightforward to check that these fields satisfy the field equation $\partial^{\mathrm{AB}} \psi_{\mathrm{BA}_1 \dots \mathrm{A}_{2h-1}} = 0$.

5.11 Minitwistors and Monopoles

5.11.1Minitwistor Space

The twistor space used to describe monopoles on three dimensional space-time $M^3 := \mathbb{C}^3$ is Hitchin's minitwistor space P^2. It can be regarded as the tangent space of $\mathbb{P}^1$ or, equivalently, the total space of the holomorphic line bundle $O_{\mathbb{P}^1}(2) \to \mathbb{P}^1$.

In the twistor picture, the restriction of the moduli space of sections from M^4 to M^3 amounts to restricting the line bundle P^3 to the diagonal $\mathbb{P}^1$ with $\mu_\alpha = \lambda_\alpha$ in the base $\mathbb{P}^1 \times \mathbb{P}^1$ of P^3. We can achieve this by quotenting by the distribution

$$D_{P^3} = \langle \mu^\beta \lambda_\beta \left(\lambda_\alpha \frac{\partial}{\partial \mu_\alpha} - \mu_\alpha \frac{\partial}{\partial \lambda_\alpha} \right) \rangle \tag{5.141}$$

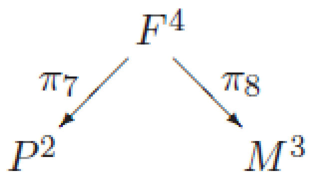
That is, $P^2 := P^2/D_{P^3}$, and the holomorphic line bundle $O_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1) \to \mathbb{P}^1 \times \mathbb{P}^1$ reduces to the line bundle $O_{\mathbb{P}^1}(2) \to \mathbb{P}^1$. The correspondence space is obtained by taking the quotient of F^6 by the distribution

$$D_{F^6} = \left\langle \frac{\partial}{\partial x^{[12]}}, \mu^\beta \lambda_\beta \left(\lambda_\alpha \frac{\partial}{\partial \mu_\alpha} - \mu_\alpha \frac{\partial}{\partial \lambda_\alpha} \right) \right\rangle \tag{5.142}$$

so that

$$P^4 := \frac{P^6}{D_{P^6}} \cong \mathbb{C}^3 \times \mathbb{P}^1 . \tag{5.143}$$

Here, we have the double fibration



with $\pi_7 : (x^{\alpha\beta}, \mu_\alpha)\, 7 \to (z, \mu\alpha) = (x^{\alpha\beta}\mu_\alpha\mu_\beta, \mu_\alpha)$ and π8 being the trivial projection. In the case of P^2, we have a geometric twistor correspondence between points in M^2 and holomorphic embeddings $\mathbb{P}^1 \hookrightarrow P^2$, as well as between points in P^2 and two-planes in M^3.

5.11.2Note

that the twistor distribution here is of rank two and it is generated by the vector fields $\mu_\alpha \partial^{\alpha\beta}$, i.e.

$$P^2 \cong F^4/\langle \mu_\alpha \partial^{\alpha\beta} \rangle \quad \text{with } \partial^{\alpha\beta} = \varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta} \frac{\partial}{\partial x^{\gamma\delta}} . \tag{5.144}$$

5.11.3 Remark

There is an alternative way of obtaining the minitwistor space from the ambitwistor space in the non-Abelian setting. Firstly, one reduces to the miniambitwistor space underlying a Penrose–Ward transform for solutions to the three-dimensional Yang–Mills–Higgs theory. Restricting to BPS solutions then amounts to restricting the miniambitwistor space to the minitwistor space.

Penrose–Ward transforms. The construction of the Abelian monopole equations in the twistor context has been discussed extensively in the literature comments in the following.
The Penrose–Ward transform works here in the familiar way. A holomorphic vector bundle over P^2 which becomes holomorphically trivial upon restriction to the submanifolds
$\hat{x} \cong \mathbb{P}^1 \hookrightarrow P^2$ can be pulled back to F^4.
Specifically, we have

$$\hat{f} = \{\hat{f}_{ab}\} \in H^1(\widehat{U}, O_{\widehat{U}}) \text{ for } \widehat{U} \subset P^2. \quad (5.145)$$

The pull-back of $\hat{f}$ can be split holomorphically,

$$f'_{ab} = \pi_7^* f'_{ab} = h'_a - h'_b. \quad (5.146)$$

Using the Liouville theorem, this allows us to introduce a global relative one-form A' with components

$$A'^{\alpha} := \mu_\beta\, \partial^{\alpha\beta} h'_a =: \mu_\beta (A^{\alpha\beta} - \varepsilon^{\alpha\beta} \emptyset), \quad (5.147)$$

where the fields on the right-hand-side depend only on space-time. From the flatness condition on the corresponding curvature, we obtain

$$f_{\alpha\beta} = \partial_{\alpha\beta} \emptyset, \quad (5.148)$$

where $f_{\alpha\beta}$ is the curvature of $A_{\alpha\beta}$. This is the spinorial form of the Bogomolny monopole equation

$$F := dA = \star_3\, d\emptyset \quad (5.149)$$

in three dimensions

List of Symbols

No	Symbols	Meaning
1	M	Topological n-manifold
2	T_pM	Tangent vector at p
3	T_pM^*	Cotangent space of M at p
4	TM	Tangent bundle of M
5	TM^*	Cotangent bundle of M
6	Λ	The wedge (or exterior) product
7	$\nabla^{\acute{A}A}$	Spinor covariant derivative
8	Z^{α}	Twistor
9	T	Twistor space
10	$\bar{Z}_{\alpha}$	Conjugate twsitor
11	T^*	Dual twistor space
12	ZRM	Zero Rest Mass
13	SD	Self-dual
14	ASD	Anti-self-dual
15	ρ	Density
16	Λ_b^a	Lorentz transformation

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