Sudan University of Science and Technology

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Subdiagonal Subalgebras and Commutative Banach Algebras by Toeplitz Operators

الجبريات الجزئية القطرية الجزئية و جبريات باناخ التبديلية بواسطة مؤثرات تبوليتز

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By

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Dedication

To my Family

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Abstract

We determine the eigenvalues inequalities, sums of hermitian and normal matrices, Schubert calculus, Wielant's theorem with spectral sets and Banach algebra. The principal submatrices with noncommutative function theory and unique extensions was shown. We give applications of the Fuglede-Kadison determinant, Riesz and Szegö type factorizations theorem for noncommutative Hardy spaces and for a Helson-Szegö theorem noncommutative Hardy-Lorentz spaces. We also give a Helson-Szegö subdiagonal subalgebras with applications to Toeplitz operators. The algebraic structure of non-commutative analytic with quasi-radial quasi-homogeneous symbols and commutative Banach algebra of Toeplitz algebra and operators are presented, the structure of a commutative Banach algebra on the unit ball and quasi-nilpotent group action, generated by Toeplitz operators with quasi-radial quasi-homogeneous symbols are discussed.

الخلاصة

حددنا متباينات القيم الذاتية والمجاميع الهيرميتية والمصفوفات الناظمة وحسبان شيبورت ومبرهنة ويلانت مع الفئات الطيفية وجبر باناخ. تم ايضاح المصفوفات الجزئية الاساسية مع نظرية الدالة غير التبديلية والتمديدات الوحيدة. اعطينا تطبيقات لمحددة فيقليد—كادسون ومبرهنة التحليل الى عوامل نوع ريس—سيزيقو لأجل فضاءات هاردي—لورنتز. أيضا اعطينا الجبريات الجزئية القطرية جزئية هيلسون—سيزيقو مع التطبيقات الى مؤثرات تبوليتز. تم احضار التشييد الجبري للتحليل غير التبديلي مع الرموز شبه— المتجانسة شبه— نصف القطرية وجبر باناخ التبديلي لجبر مؤثرات تبوليتز. تم مناقشة تشييد جبرباناخ التبديلي على كرة الوحدة وفعل زمرة شبه—متلاشية القوى والمولدة بواسطة مؤثرات تبوليتز مع الرموز شبه—المتجانسة شبه—نصف القطرية.

Introduction

We examine, simultaneously, all of the k-square principal submatrices of an n-square matrix A. Usually A has been symmetric or Hermitian, and much of our effort has centered around the well-known fact asserting that the eigenvalues of an $(n - I)$-square principal submatrix of Hermitian A always interlace the eigenvalues of A.

We generalize to the setting of Arveson's maximal subdiagonal subalgebras of finite von Neumann algebras, the Szegö L^p-distance estimate, and classical theorems of F. and M. Riesz, Gleason and Whitney, and Kolmogorov. We first use properties of the Fuglede-Kadison determinant on $L^p(M)$, for a finite von Neumann algebra M, to give several useful variants of the noncommutative Szegö theorem of $L^p(M)$, including the one usually attributed to Kolmogorov and Krein.

The non-commutative analytic Toeplitz algebra is the WOT-closed algebra generated by the left regular representation of the free semigroup on n generators. We develop a detailed picture of the algebraic structure of this algebra. In particalur, we show that there is a canonical homomorphism of group of the automorphism group onto the of conformal automorphisms of the complex n-ball. We present here a quite unexpected result: Apart from already known commutative C^*- algebras generated by Toeplitz operators on the unit ball, there are many other Banach algebras generated by Toeplitz operators which are commutative on each weighted Bergaman space.

We extend eigenvalue inequalities due of Freede-Thompson and Horn for sums of eignevalues of two Hermitian matrices.Let A be a complex unital Banach algebra and let $a, b \in A$. We give regions of the complex plane which contain the spectrum of $a + b$ or ab using von Neumann spectral set theory.

Let A be a finite subdiagonal algebra in Arveson's sense. Let $\mathrm{H}^{\mathrm{p}}(\mathrm{A})$ be the associated noncommutative Hardy spaces, $0 < p \leq \infty$. We extend to the case of all positive indices most recent results about these spaces, which include notably the Riesz, Szegö and inner-outer type factorizations. We formulate and establish a noncommutative version of the well-known Helson- Szegö theorem about the angle between past and future for subdiagonal subalgebras.

Studying commutative C^*-algebras generated by Toeplitz operators on the unit ball it was proved that, given a maximal commutative subgroup of biholomorphisms of the unit ball, the C^*-algebra generated by Toeplitz operators, whose symbols are invariant under the action of this subgroup, is commutative on each standard weighted Bergman space. There are five different pairwise non-conjugate model classes of such subgroups: quasi-elliptic, quasi-parabolic, quasi-hyperbolic, nilpotent, and quasi-nilpotent. It was observed in Vasilevski that there are many other, not geometrically defined, classes of symbols which generate commutative Toeplitz operator algebras on each weighted Bergman space. These classes of symbols were subordinated to the quasi-elliptic group. The corresponding commutative operator algebras were Banach, and being extended to C^*-algebras they became non-commutative. These result were extended then to the classes of symbols, subordinated to the quasi-hyperbolic and quasi-parabolic groups. Let $\mathcal{A}_\lambda^2(\mathbb{B}^n)$ denote the standard weighted Bergman space over the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$. New classes of commutative Banach algebras $\mathcal{T}(\lambda)$ which are generated by Toeplitz operators on $\mathcal{A}_\lambda^2(\mathbb{B}^n)$ have been recently discovered in Vasilevski.). These algebras are induced by the action of the quasi-elliptic group of biholomorphisms of $\mathbb{B}^n$. we analyze in detail the internal structure of such an algebra in the lowest dimensional case $n = 2$. Extending recent results to the higher dimensional setting $n \geq 3$ we provide a futher step in the structural analysis of a class of commutative Banach algebras generated by Toeplitz operators on the standard weighted Bergman space over the n-dimensional complex unit ball. The algebras $\mathcal{B}_k(h)$ under study are subordinated to the qausi-elliptic group of automorphisms of $\mathbb{B}^n$ and in term of their generators they were described.

The contents

Subject		Page on
Dedication		I
Acknowledgements		II
Abstract		III
Abstract (Arabic)		IV
Introduction		V
The contents		VII
Chapter 1 Eigenvalues of Sums and Principal Submatrices		
Section (1.1)	Hermitian Matrices	1
Section (1.2)	Interlacing Inequalities for Singular Values Of Submatrices	13
Chapter 2 Noncommutative Function Theory and Applications of The Fuglede-Kadison Determinant		
Section (2.1)	Unique Extensions	22
Section (2.2)	Szegö's Theorem and Outers for Noncommutative H^p	34
Chapter 3 The Algebraic Structure and Quasi-Radial Quasi-Homogeneous Symbols		
Section (3.1)	Non-commutative Analytic Toeplitz Algebras	47
Section (3.2)	Commutative Banach Algebras of Teoplitz Operators	70

Chapter 4 Eigenvalue Inequalities and on the Eigenvalues of Normal Matrices		
Section (4.1)	Schubert Calculus	79
Section (4.2)	Wielant's Theorem with Spectral sets and Banach Algebra	93
Chapter 5 Riesz and Szegö Type Szegö Factorization Theorem for a Helson-Szegö Theorem		
Section (5.1)	Factorizations for Noncommutative Hardy Spaces	100
Section (5.2)	Noncommutative Hardy-Lorentz Spaces	111
Section (5.3)	Subdiagonal Subalgebras with Applications to Toeplitz Operators	126
Chapter 6 Structure of a Commutative Banach Algebra		
Section (6.1)	Quasi-Nilpotent Group Action	138
Section (6.2)	Toeplitz Operators with Quasi-Radial Quasi-Homogeneous Symbols	163
Section (6.3)	Toeplitz Operators on the Unit Baal	189
list of symbols		218
References		220

Chapter 1
Eigenvalues of Sums and Principal Submatrices

We study the singular values of the submatrices (not necessarily principal submatrices) of an arbitrary matrix A. Although we study not necessarily principal submatrices, we Principal Submatrices series because the singular values of an arbitrary submatrix of matrix A may be approached through an examination of the principal submatrices of AA^*.

Section (1.1) Hermitian Matrices

Let $a = \alpha(\alpha_1, \dots \alpha_n)$ and $\beta = (\beta_1 \dots , \beta_n)$ be arbitrary nonincreasing sequences of real numbers. We consider the question: for which nonincreasing sequences$\gamma = (\gamma_1, \dots \gamma_n)$do there exist Hermitian matrices A and B such that A, B and $A + B$ have α, β and γ respectively as their sequences of eigenvalues. Necessary conditions have been obtained by Weyl [108], Lidskii [292], Wielandt [312, 263, 278, 289], and Amir-Moez [18], Besides the obvious condition

$$\gamma_1 + \cdots + \gamma_n = \alpha_1 + \cdots + \alpha_n + \beta_1 + \cdots + \beta_n, \tag{1}$$

these conditions are linear inequalities of the form

$$\gamma_{k_i} + \cdots + \gamma_{k_r} \leq \alpha_{i_i} + \cdots + \alpha_{i_r} + \beta_{j_i} + \cdots + \beta_{j_r}, \tag{2}$$

where i, j and k are increasing sequences of integers. As far as we know all other known necessary conditions are consequences of these inequalities. It is therefore natural to conjecture that the set E of all possible γ forms a convex subset of the hyperplane (1). The set E has hitherto not been determined except in the simple cases $n = 1, 2$, and will not be determined in general here.

We give a method of finding conditions of the form (2) which will yield many new ones. We shall find all possible inequalities (2) for $r = 1, 2$, and arbitrary n, and establish a large class of such inequalities for r=3. We use Lidskii's method to find a necessary condition on the boundary points of a subset E' of E. These results are used to determine the set E for n = 3, 4. In addition a conjecture is given for E in general.

If x is a sequence, x_p denotes the p^{th}component of x. If A is a matrix, A^* and AA' denote the conjugate transpose and transpose of A. If i is a sequence of integers such that $1 \leqq i_1 < \cdots < i_r \leqq n$, by the complement of i with respect to n we mean the sequence obtained by deleting the terms of i from the sequence $1,2, \dots, n$. If a is a sequence of numbers, *diag* $(\alpha_1, \dots \alpha_n)$ denotes the diagonal matrix with diagonal a. If M and N are matrices, diag (M, N) denotes the direct sum matrix

$$\begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}$$

The inner product of the vectors x and y is written $(x. y)$.I_r is the unit matrix of order r. Finally expB denotes the $\sum_{b=0}^{\infty} B^n/n!$.

We are going to use methods introduced by Lidskii [292,31.1]. Lidskii gave sketchy proofs of his results and it is not obvious how to reconstruct his argument, see [312]. we will therefore derive the results of Lidskii which are needed.

The set E referred to in the introduction is the set of points γ such that $\gamma_1 \geqq \cdots \geqq \gamma_r$ and γ is the sequence of eigenvalues of $diag\ (\alpha_1, \dots, \alpha_n) + U^* diag\ (\beta_1, \dots, \beta_n)U$, where U ranges over all unitary matrices. Fix α, β, with $\alpha_1 > \cdots > \alpha_n$, and $\beta_1 > \cdots > \beta_n$. let E' be the subset of E obtained by letting U range over real orthogonal matrices. To indicate the dependence of E' on α and β we write $E'(\alpha_1, \dots \alpha_n; \beta_1, \dots \beta_n)$. Boundary points and interior points of E' are always taken with respect to the relative topology of the hyperplane (1).

Theorem (1.1.1)[10]: If γ is a boundary point of E' with distinct coordinates then there exist a positive integer $r < n$ and increasing sequences $i, j,$ and k of order r such that

$$(\gamma_{k_i}. \dots . \gamma_{k_r}) \in E'(\alpha_{i_1}, \dots \alpha_{k_r}; \beta_{i_1}, \dots \beta_{j_r})$$

And

$$(\gamma_{k'_i}. \dots . \gamma_{k'_{n-r}}) \in E'(\alpha_{i'_1}, \dots \alpha_{i'_{n-r}}; \beta_{j'_1}, \dots \beta_{j'_{r-n}})$$

where i', j' and k' are the complements of i, j and k with respect to n.

Proof: Let U_0 be a real orthogonal matrix such that $diag\ (\alpha_1, \dots \alpha_n) + U_0' diag\ (\beta_1, \dots, \beta_n) U_0$ has eigenvalues γ. If $T = (t_{pq})$ is a real anti-symmetric matrix, $expT$ is orthogonal. For sufficiently small values of t_{pq}, the eigenvalues $\lambda_1 > \cdots > \lambda_n$ of

$diag\ (\alpha_1, \dots \alpha_n) + U_0' exp(-T) B exp\ (T) U_0$

where $B = diag(\beta_1, \dots \beta_n)$, are distinct and determine a point of E'. Let $A = U_o diag\ (\alpha_1, \dots \alpha_n) U_0'$, and let x_ι be a unit eigenvector of $A + exp(-T) B expT$ corresponding to the eigenvector λ_ι which varies continuously with T. We have

$$Ax_i + exp(-T) B\ exp(T) x_i = \lambda_i x_i. \tag{3}$$

Using superscripts to denote derivatives with respect to $t_{pq} p < q$, it follows that

$$Ax_\iota^{pq} + exp(-T) B\ exp(T) x_\iota^{pq} + (exp(-T)\ B\ exp(T))^{pq} x_\iota$$
$$= \lambda_\iota^{pq} x_\iota + \lambda_\iota x_\iota^{pq}. \tag{4}$$

It is easily seen that $(expT)^{pq}$ reduces to T^{pq} when $T = 0$. Hence when $T = 0, (exp(-T) B expT)^{pq} = (\beta_p - \beta_q) Z^{pq}$ where Z^{pq} is the matrix whose (p, q) and (q, p) entries are 1 and whose other entries are 0. .Since x_J is a unit vector, $(x_J, x_J^{Iq}) = 0$. Therefore by (3),

$$(Ax_\iota, x_\iota^{Iq}) + (exp(-T) B\ exp(T) x_\iota, x_\iota^{Iq} = 0 \tag{5}$$

Taking the inner product of (4) with x_ι we find by (5) and the symmetry of A and B,

$$\lambda_\iota^{pq} = ((exp(-T)\ B\ exp\ T)^{pq}\ x_\iota, x_\iota)$$

Setting $T = 0$,

$$\gamma_J^{pq} = 2(\beta - \beta) \omega_{\iota p} \omega_{\iota q}, \tag{6}$$

where w_J and γ_J^{pq} denote the values of x_J and $x_{\iota\iota}^{Iq}$ when $T = 0$. If γ is not an interior point of E' the rank of the n by $n(n - l)/2$ matrix $G = (\gamma_\iota^{pq})$ must be less than $n - 1$. Now let $D = (\omega_{\iota p} \omega_{\iota q})$ be the n by $n(n - 1)$ matrix whose rows are indexed by J, where $1 \leqq p \leqq n$, and whose columns are indexed by (p, q), where p and q vary over the range $1 \leqq p \leqq n$, and $p \neq q$ rather than $p < q$. Clearly D, and hence DD' has the same rank as G. If F is the square matrix $(\omega_{\iota m}^2)$ of order n, then $DD' = I - FF'$. Thus if $rank\ D < n - 1, FF'$ has 1 as a multiple eigenvalue. Since FF' is stochastic, it follows that FF' is decomposable [91,158,310,122]. That is to say, $FF' = P\ diag\ (M, N) P'$, where M and N are square matrices and P is a permutation matrix. Let

$$F = P \begin{pmatrix} G & H \\ J & K \end{pmatrix} P'$$

be the decomposition of F corresponding to that of FF'. Then $GJ' + HK' = 0$. Since the entries of F are nonnegative, we have $GJ' = HK' = 0$. It follows that if a column of G contains a nonzero term then all terms of the corresponding column of J vanish, and similarly for H and K. Moving all nonzero columns of G and H to the left, we find

$$\mathrm{F} = \mathrm{P}\begin{pmatrix} \mathrm{S}_1 & 0 \\ 0 & \mathrm{S}_2 \end{pmatrix}\mathrm{P}'$$

where R is another permutation matrix. Since F is doubly stochastic, $S_1$and S_2 must be square matrices. If $\mathrm{W} = (\omega_{\iota\mathrm{m}})$ then $W = P\begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix} R$, where $W_1$and W_2 are square. Setting I' =$diag(\gamma_1, \ldots, \gamma_n)$, we have $A + B = W'I'W$. Therefore $RAR' + RBR' = G$, where$C = diag\ (W_1', W_2')P'I'P\ diag(W_1, W_2)$ Let j and k be such that $RBR' = diag\ (\beta_{j_1}, \ldots, \beta_{J_n})$ and $P'\ I'P = diag\ (\gamma_{k_1}, \ldots, \gamma_{k_n}\ )$. If W_1 is of order r, then C =$diag\ \ (C_1, C_2)$ where C_1 has eigenvalues $\gamma_{k_1}, \ldots, \gamma_{k_n}$and C_2 has eigenvalues $\gamma_{k_{r+1}}, \ldots, \gamma_{k_n}$. Therefore $RAR' = diag\ (A_1, A_2)$, where $A_1 + diag\ (\beta_{j_1}, \ldots, \beta_{J_r}) = C_1$ and A_2 +$diag\ (\beta_{j_{r+1}}, \ldots, \beta_{J_n}) = C_2$. This completes the proof.

If $M = (m_{i,j})\ 1 \leqq i \leqq r, 1 \leqq j \leqq r$ a matrix of order r and $N = (n_{k\iota}), r + 1 \leqq k \leqq n, r + 1 \leqq \iota \leqq n$ is a matrix of order $n - r$, we define $\mathrm{M} \times \mathrm{N}$ to be the matrix (m_{iJ}, n_{kJ}) of order $r(n - r)$ whose rows are indexed by pairs $(i, k)$and whose columns are indexed by pairs (j, l). This product is left and right distributive and $(M \times N)' = M' \times N'$. Also $(M_1 \times N_1)(M_2 \times N_2) = (M_1M_2 \times N_1N_2)$. We set $M \ominus N = (M \times I_{n-r}) - (I_r - N)$. It follows from these remarks that if W_1 and W_2 are orthogonal then so is $W_1 \times W_2$ and

$$(W_1'MW) \ominus (W_1'NW_2) = (W_1 \times W_2)(M \ominus N)(W_1 \times W_2) \qquad (7)$$

The index of a real symmetric matrix is the number of its positive eigenvalues.

Lemma (1.1.2)[10]: If M, N, and $M + N$ are nonsingular real symmetric matrices then *index M* + *index N* = *index (M* + *N)* + *index* $(M^{-1} + N^{-1})$.

Proof: We have $M^{-1} + N^{-1} = N^{-1}(N + M)M^{-1}$so that $M^{-1} + N^{-1}$ is nonsingular. Also

$$\begin{pmatrix} 1 & 1 \\ M^{-1} & -N^{-1} \end{pmatrix}\begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}\begin{pmatrix} 1 & M^{-1} \\ 1 & -N^{-1} \end{pmatrix}\begin{pmatrix} M + N & 0 \\ 0 & M^{-1} + N^{-1} \end{pmatrix}$$

The result now follows by the Law of Inertia.

Theorem (1.1.3)[10]: Let γ be a boundary point of E' with distinct coordinates. Then there exist sequences i, j and k satisfying the conclusion of Theorem (1.1.1) and such that

$$i_1 + \cdots + i_r + j_1 + \cdots + j_r = k_1 + \cdots + k_r + r(r + 1)/2.$$

Proof: Using a slight change of notation, we have seen that there exist permutations i, j and k of (1,…,n)and real symmetric matrices $A_1, A_2, B_1, B_2, C_1\ , C_2$ such that A_1 has eigenvalues $\alpha_{i_1}, \ldots \alpha_{i_r}$,$A_2$ has. eigenvalues $\alpha_{i_{r+1}}, \ldots \alpha_{i_n}, B_1 = diag(\beta_{i_1}, \ldots \beta_{i_r}), B = diag(\beta_{i_{r+1}}, \ldots \beta_{i_n}), C_1$ has eigenvalues $\gamma_{k_1}, \ldots, \gamma_{k_r}, C_2$ has eigenvalues $\gamma_{k_{r+1}}, \ldots, \gamma_{k_n}$, and$A + B = C$, where $A = diag(A_1, A_2), B = diag(B_1, B_2)\ C = diag(C_1, C_2\ )$. We also assume $i_1 < \cdots < i_r$, and $i_{r+1} < \cdots < i_n$ and similarly for the $j's$ and $k's$. We set $\bar{\alpha}_\iota = \alpha_{i\iota}, \bar{\beta}_\iota = \beta_{i\iota}$ and $\bar{\gamma}_\iota = \gamma_{k\iota}, 1 \leqq \iota \leqq n$. Let $T = (t_{pq})$ be a real anti-symmetric matrix and let $\lambda_1 > \cdots > \lambda_n$be the eigenvalues of $A + exp(-T)BexpT$. If $\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}$ is a real orthonormal system of corresponding eigenvectors, we let w_l and ω_l^{pq} be the values of x_k and x_{kl}^{pq} when $T = 0$, where x_l^{pq} denotes the derivative of x_l with respect to $t_{pq}, p < q$. If W is the matrix whose rows are $\omega_1, \dots, \omega_n$, then $W = diag(W_1, W_2)$and $C_1 = W_1' \Gamma_1 W_1, C_2 = W_2' \Gamma_2 W_2$where$\Gamma_1 = diag(\gamma_{k_1}, \dots, \gamma_{k_r})\Gamma_2 = diag\ (\gamma_{k_{r+1}}, \dots, \gamma_{k_n})$. Clearly λ_{k_l} reduces to $\bar{\gamma}_l$ when *T = 0*, and we let $\bar{\gamma}_l^{pq}$ be the value of $\lambda_k^{pq} = \partial \lambda_{kl} / \partial t_{pq}$ when *T = 0.*

Starting from the equation

$$A\lambda_{pq} + (exp(-t)\, B expT) x_{kl} = \lambda_{kl} x_{kl} \tag{8}$$

We find

$$\mathrm{A}\lambda_{kl}^{pq} + (exp(-t)\, B expT) x_{kl}^{pq} + (exp(-T)\, B\ expT)^{pq} x_{kl} = \lambda_{kl}^{pq} x_{kl} + \lambda_{kl} x_{kl}^{pq} \tag{9}$$

As in Theorem (1.1.1) is follows that

$$\lambda_l^{pq} = ((exp(-T)\, B\ expT)^{pq} x_{kl}, x_{kl}) \tag{10}$$

and therefore

$$\bar{\gamma}_l^{pq} = 2(\bar{\beta}_p - \bar{\beta}_q)\omega_{lp}\omega_{lq} \tag{11}$$

We are going to test $\sigma = \lambda_{l_l} + \dots + \lambda_{k_r}$ for a local extreme at *T = 0*. If p and q are ≦ r, then *expT* has the form $diag\ (exp\ T_{1,}, 0)$ when $T_{uv} = 0$for $(u, v) \neq (p, q)$, and hence σ remains constant for T_{pq} in a neighborhood of 0. Therefore all partial derivatives of σ with respect to t_{pq} vanish at the origin when $p < q \leqq r,$ and similarly when $r < p < q$. by (11), $\sigma^{pq} = 0$ at *T=0* when $p \leqq r < q$, since that last $n - r$ components or ω_l are 0 when $1 \leqq I \leqq r$. we now calculate $\lambda_{k_l}^{pq.uv}$ at T=0 when

$$1 \leqq p \leqq r < q \leqq n, \qquad 1 \leqq u \leqq r < v \leqq n, \qquad 1 \leqq I \leqq r. \tag{12}$$

Differentiation of (10) yields

$$\lambda_{k_l}^{pq.uv} = ((exp(-T)\, B\ expT)^{pq} x_{k_l}^{uv}, x_{kl} + 2((exp(-T)\, B\ expT)^{pq} x_{kl}^{uv}, x_{kl}) \tag{13}$$

It is easily seen that when $T = 0$

$$(exp(-T) B exp\ T)^{pq.uv}$$
$$= -(T^{pq}\ BT^{uv} + T^{uv} BT^{pq}) + \frac{1}{2} B(T^{pq}T^{uv} + T^{uv}T^{pq})$$
$$+ \frac{1}{2}(T^{pq}T^{uv} + T^{uv}T^{pq})B$$

Considering only the cases (12), a straightforward calculation shows that when *T = 0,*

$$((exp(-T)\, B\ expT)^{pq} x_{k_l}^{uv}, x_{kl} = 0\ for\ p \neq u, q \neq v$$
$$= (2\bar{\beta}_q - \bar{\beta}_p - \bar{\beta}_u)\omega_{lp}\omega_{lu} \quad for\ p \neq u, q = v$$
$$= (2\bar{\beta}_p - \bar{\beta}_q - \bar{\beta}_v)\omega_{lp}\omega_{lv}\ for\ p = u, q \neq v$$
$$= -2(\bar{\beta}_p - \bar{\beta}_p)(\omega_{lp}^2 - \omega_{lp}^2) for\ p = u, q = v$$

Recalling that $\omega_{tq} = 0$for $I\ \leqq r < q$, we find that when $\Gamma = 0$,

$$\sum_{J=1}^{r} ((exp(-T)\, B\ exp\ T)^{pq.uv} x_{kl}, x_{kl}) = -2\left(\bar{\beta}_p - \bar{\beta}_p\right) \text{ for } p = u, q = v$$
$$= 0 \text{ otherwise.} \tag{14}$$

The second term on the right of (13) reduces when $T = 0$ to

$$2(\bar{\beta}_p - \bar{\beta}_p)\omega_{lp}^{uv}\omega_{lp} \qquad (15)$$

To compute ω_{lp}^{uv}, rewrite (9) in the form

$$(A + exp(-T)B\, exp\, T - \lambda_{l_l}\, I_n)x_{kl}^{uv}$$
$$= -(exp(-T)B\, exp\, T)^{uv} x_{k_l} + \lambda_{k_l}^{uv}\, x_{k_l}$$

Setting T = 0 and using (11), we find, since ω = 0,

$$(c - \bar{\gamma}_l\, I_n)x_l^{uv} = -(\bar{\beta}_u - \bar{\beta}_v)y,$$

where y is the vector such that $y_u = \omega_{lv} = 0, y_v = \omega_{lv}$ and $y_v = 0$ for $m \neq u, m \neq v$. Therefore

$$\omega_{lp}^{uv} = (\bar{\beta}_u - \bar{\beta}_v)((\gamma_{l_l}\, I_n - C)^{-1}y)_q$$

Since $q > r$, and $C = diag\,(C_1, C_2)$, we may replace C by C_2 and I_n by $I_{n-r,}$Thus

$$\omega_{lp}^{uv} = (\bar{\beta}_u - \bar{\beta}_v)d_{qu}\omega_{lu} \qquad (16)$$

where d_{qu} is the (q, v) entry of $((\bar{\gamma}_l\, I_{n-r} - C_2)^{-1}$. Now

$$(\bar{\gamma}_l\, I_{n-r} - C_2)^{-1} = (W_2'((\bar{\gamma}_l\, I_{n-r} - \Gamma_2)^{-1}W_2)^{-1}$$

Therefore

$$d_{qu} = \sum_{m=r+1}^{n} \frac{\omega_{mq}\omega_{mv}}{\bar{\gamma}_l - \bar{\gamma}_m} \qquad (17)$$

Combining (13), (14), (15), (16), and (17), we find at $T = 0$

$$\sigma^{pq.uv} = 2(\bar{\beta}_p - \bar{\beta}_q)(\bar{\beta}_u - \bar{\beta}_v)\sum_{I=1}^{r}\sum_{m=r+1}^{n} \frac{\omega_{lp}\omega_{lu}\omega_{mq}\omega_{mv}}{\bar{\gamma}_l - \bar{\gamma}_m} - 2\delta_{uv}^{pq}(\bar{\beta}_p - \bar{\beta}_q), \qquad (18)$$

where $\delta_{uv}^{pq} = 1$ when $(p, q) = (u, v)$, and = 0 otherwise.

We must now determine the index of the matrix $G = (\sigma^{pq.uv})_{r=0}$ of order $r(n - r)$ whose rows and columns are indexed by pairs (p, q) and (u, v) satisfying (12).

The double sum on the right of (18) is the (pq, uv) entry of

$$(W_1 \times W_2)'(\Gamma_1 \ominus \Gamma_2)^{-1}(W_1 \times W_2) = ((W_1 \times W_2)'(\Gamma_1 \ominus \Gamma_2)(W_1 \times W_2))^{-1}$$

By (7) this reduces to

$$(C_1 \ominus C_2)^{-1} = (A_1 + B_2) \ominus (A_1 + B_2) = ((A_1 \ominus A_2) + (B_1 \ominus B_2)^{-1}$$

Therefore by (18)

$$\frac{1}{2}G = (B_1 \ominus B_2)((A_1 \ominus A_2) + (B_1 \ominus B_2))^{-1}\,(B_1 - B_2) \ominus (B_1 \ominus B_2)$$
$$= (B_1 \ominus B_2(((A_1 \ominus A_2) + (B_1 \ominus B_2))^{-1} - (B_1 \ominus B_2)^{-1})(B_1 \ominus B_2)$$

Thus G has the same index as$((A_1 \ominus A_2) + (B_1 \ominus B_2))^{-1} - (B_1 \ominus B_2)^{-1}$.

Applying Lemma (1.1.2) with

$$M = ((A_1 \ominus A_2) + (B_1 \ominus B_2)), N = -(B_1 \ominus B_2),$$

$$index\; G = index\;((C_1 \ominus C_2)^{-1} - (B_1 \ominus B_2)^{-1})$$
$$= index(C_1 \ominus C_2) + index - (B_1 \ominus B_2) = index\;(A_1 \ominus A_2)$$
$$= r(n - r) + index(C_1 \ominus C_2) - index\;(B_1 \ominus B_2)$$
$$- index\;(A_1 \ominus A_2).$$

Thus G is positive definite if and only if *index* $(C_1 \ominus C_2)$ = *index* $(A_1 \ominus A_2)$ *index* $(B_1 \ominus B_2)$, and G is negative definite if and only *if neg* $(C_1 \ominus C_2)$= *neg* $(A_1 \ominus A_2)$+ *neg* $(B_1 \ominus B_2)$, where neg H is the number of negative eigenvalues of H. Next we determine

$$i_1 + \cdots + i_r + j_1 + \cdots + j_r = k_1 + \cdots + k_r + r(r+1)/2, \tag{19}$$

and G is positive definite if and only if

$$i_{r+1} + \cdots + i_n + j_{r+1} + \cdots + j_n = k_{r+1} + \cdots + k_n + (n-r)(n-r+1)/2. \tag{20}$$

By Theorem (1.1.1) the boundary points of E'' lie on a finite number of hyperplanes of the form

$$\gamma_{k_1} + \cdots + \gamma_{k_r} = \alpha_{i_1} + \cdots + \alpha_{i_r} + \beta_{j_1} + \cdots + \beta_{j_r} \tag{21}$$

The hyperplane

$$\gamma_{k'_1} + \cdots + \gamma_{k'_n} = \alpha_{k'_1} + \cdots + \alpha_{i'_{n-r}} + \beta_{j'_1} + \cdots + \beta_{j'_{n-r}}$$

intersects the hyperplane (1) in the same set. If γ lies on only one of these hyperplanes (21) and does not satisfy (19) or (20), then in every small sphere about γ there exist points of E' on both sides of the hyperplane (21). Therefore E' must fill the sphere, for otherwise there would be boundary points of E' inside the sphere and off the hyperplane (21). This being impossible,λ must satisfy (19) or (20). Now suppose γ lies on several hyperplanes (21), and (19) and (20) both fail for each of these hyperplanes. By continuity the quadratic form G is not definite for all points near γ which satisfy the conclusion of Theorem (1.1.1). Therefore in a neighborhood of γ all points of E' lying on only one hyperplane (21) are interior points of E'. Therefore γ cannot be a boundary point of E'. since E' is the closure of its interior, and a finite union of linear varieties of deficienc$y \geqq 2$ cannot separate the interior of a sphere. The proof is complete.

If i, j and k are increasing sequences of integers of order r and (2) holds for the eigenvalues of $A + B$ for any Hermitian A, B with arbitrary eigenvalues $\alpha_1 \geqq \cdots \geqq \alpha_n$and $\beta_1 \geqq \cdots \geqq \beta_n$we write $(i;\ j\ ;\ k) \in\ S_r^n$. If

$$\gamma_{k_1} + \cdots + \gamma_{k_n} \geq \alpha_{i_1} + \cdots + \alpha_{i_n} + \beta_{j_1} + \cdots + \beta_{j_n}$$

for any such A, B we write$(i;\ j\ ;\ k) \in\ \tilde{S}_r^n$

Theorem (1.1.4)[10]: The following conditions are equivalent:

(i) $(i;\ j\ ;\ k) \in\ \tilde{S}_r^n$

(ii) $(n - i, +1, \ldots, n - i_1 + 1; n - j_r + 1, \ldots n - j_1 + 1; n - k_r + 1, \ldots n, -k_1 + 1) \in \tilde{S}_r^n$

(iii) $(k_1, \ldots, k_r; n - j_r + 1, \ldots, n - j_1 + 1; i_1, \ldots i_r)) \in\ \tilde{S}_r^n$

(iv) $(i', j', k')) \in\ \tilde{S}_{n-r}^n$, where i', j', k' are the compelements of i, j, k with respect to n.

Proof: The equation $A + B = C$ may be written $-A - B = -C$ or $A = C - B$. This proves the equivalence of (i) with (ii) and (iii). The equivalence of (i) and (iv) is immediate by the trace Condition (1).

If A is a Hermitian matrix with eigenvalues $\alpha_1 \geqq \cdots \geqq \alpha_n$ and M is a linear subspace of dimension $n - 1$, let A_M be the transformation PA with domain restricted to M, where P is the orthogonal projection on M. A_M is a Hermitian transformation on M to $M$and $(A + B)_M = A_M + B_M$. It is well known that the eigenvalues α'_p of A_M separate those of A, that is $\alpha_{p+1} \leqq \alpha'_p \leqq \alpha_p$. For $1 \leqq p \leqq n - 1$. If (x_{p}) is an orthonormal sequence of eigenvectors corresponding to (α_p) and if M contains $x_1, \dots, x_m$ then $\alpha'_p = \alpha_p$ for $1 \leqq p \leqq m$. This is an immediate consequence of the minimax principle, since $(A_M x, x) = (Ax, x)$for $x \in M$. Dually if M contains $x_{m+1}, \dots, x_n$, then $\alpha'_p = \alpha_{p+1}$ for $m \leqq p \leqq n-1$. The next theorem shows that S is essentially independent of n.

Theorem (1.1.5)[10]: If $(i; j; k) \in S_r^n$ for some n then $i_p \leqq k_p$and $j_p \leqq k_p$ for all p, and $(i; j; k) \in S_r^n$? For $all\ \ n \geq k_r$

Proof: Suppose $(i; j; k) \in S_r^n$ for some n. Considering the case $\beta = 0$, it is clear that $i_p \leqq k_p$and $j_p \leqq k_p$ for all p. If A and B are of order k_r, the identity $diag\ (A - \lambda I) + diag\ (B - \lambda I)\ diag\ (A\ +\ B -\ 2\lambda l)$ for large λ shows that $(i; j; k) \in S_r^{k_r}$. It remains to prove $(i; j; k) \in S_r^{n+1}$. Let A and B be of order $n+1$ with eigenvalues $(\alpha_p), (\beta_p)$, and let (z_p) be an orthonormal sequence of eigenvectors of $A + B$ corresponding to the eigenvalues (γ_p). Let M be the subspace spanned by $z_1, \dots, z_n$. Letting $(\alpha'_p), (\beta'_p)$ and (γ'_p) be the eigenvalues of A_M, B_M and $(A+B)_M$, we have by hypothesis

$$\gamma'_{j_1} + .. + \gamma'_{j_1} \leqq \alpha'_{i_1} + \cdots + \alpha'_{i_r} + \cdots + \beta'_{j_1} + \cdots + \beta'_{j_r}.$$

But $\gamma'_{k_p} = \gamma_{k_p}, \alpha'_{i_p} \leq \alpha_{i_p}$ and $\beta'_{j_p} \leq \beta_{j_p}$ for $1 \leqq p \leqq r$. Therefore $(i; j; k) \in\ S_r^{n+1}$

Theorem (1.1.6)[10]: If $(i; j; k) \in S_r^n$ and u, v and w are integers such that $r\ +\ 1 \geqq u \geqq 1, r+1 \geqq v \geqq 1$ and $r \geqq w \geqq 1,$ and if $i_u +\ j_v \geqq k_{w-1} + k_r + 2$ then$\big(i_1, \dots, i_{u-1,}, i_u + 1, \dots, i_r + 1; j_1 \dots, j_{v-1}, j_v + 1, \dots, j_r + 1; k_1, \dots, k_{\omega-1}, k_\omega + 1, \dots, k_r + 1\big) \in S_r^{n+1}$. Here $k_0 = 0$ and $i_{r+1} = j_{r+1} = k_r + 1$ by definition. In particular, $(i_1 + 1, \dots, i_r + 1; j_1 \dots, j_r; k_1 + 1) \in S_r^{n+1}$.

Proof: By Theorem (1.1.5) we may assume $n = k_r$. Let $(x_p), (y_p)$ and $\big(z_p\big), 1 \leqq P \leqq n+1$, be orthonormal sequences of eigenvectors corresponding to the eigenvalues $(\alpha_p), (\beta_p)$ and (γ_P) of A, B and $A + B$. Since $x_p, i_v \geqq k_{w-1} +\ n\ +\ 2$, there exists an n dimensional subspace M containing the vectors $x_p, i_u \geqq k_{\omega-1} + n + 1$, the vectors $y_p, i_v + 1 \leqq p \leqq\ n+1$, and the vectors $z_p, 1 \leqq p \leqq\ k_{\omega-1} + 1$. Let $\big(\alpha'_p\big), \big(\beta'_p\big)$ and (γ'_p) be the eigenvalues of A_M, B_M, and$(A\ +\ B)$. By hypothesis

$$\gamma'_{k_1} + \dots + \gamma'_{k_r} \leqq \alpha'_{i_1} + \cdots + \alpha'_{i_r} + \cdots \beta'_{j_1} + \cdots + \beta'_{j_r}.$$

The theorem now follows because $\gamma'_p =\ \gamma_p$ for $1\ \leqq p \leqq k_{w-1}, \gamma_{p+1} \leqq \gamma'_p$ for $k_w \leqq p \leqq n, \alpha'_p \leqq \alpha_p$ for $1\ \leqq p \leqq i_{u-1}, \alpha'_p = \alpha_{p+1}$for $i_u\ \leqq p \leqq n, \beta'_p \leqq \beta_p\ for\ 1\ \leqq p \leqq j_{u-1}$ and $\beta'_p = \beta_{p+1}\ for\ j_v \leqq p \leqq n.$

Theorem (1.1.6) yields a simple proof of the following theorem due to Lidskii.

Theorem (1.1.7)[10]:[292]. If $1\ \leqq p < \cdots < p_r \leqq n$, then $(p_1, \dots, p_r; 1, \dots, r;) \in\ S_r^n$.

Proof: Obviously $(1_1, \dots, r; 1, \dots, r; 1, \dots, r) \in\ S_r^n$. Using Theorem (1.1.6) $p_1 - 1$ times with $u = \omega = 1, v = r+1$, we find $(p_1, p_1 + 1, \dots, p_1 + r - 1, ; 1, \dots, r: p_1, p_1 + 1, \dots, p_1 + r - 1) \in\ S_r^n$. Such use of Theorem (1.1.6) is justified since $i_1 + j_{r+1} = i_1 + k_r + 1 \geqq k_r + 2 = k_0 + k_r + 2$ at each stage. We may now apply Theorem (1.1.6) $p_2 - (p_1 + 1)$ times with $u = \omega = 2. v = r + 1$since at each stage $i_1 + j_{r+1} = i_1 + k_r + 1 \geqq i_1 + k_r + 2 = k_1 + k_r + 2$. The result is

$$(p_1, p_2, p_2 + 1, \dots, p_2 + r - 2: 1, \dots, r; p_1, p_2, p_2$$

$$+1,\dots,p_2+r-2) \in S_r^{p_2+r-2}$$

Continuing in this way we find

$$(p_1,\dots,p_r;1,\dots,r;p_1,\dots,p_r) \in S_r^{p_r}$$

By Theorem (1.1.5) the proof is complete.

Theorem (1.1.8)[10]: $(i_1;j_1;k_1) \in S_1^n$ for $n \geqq k_1$ and only if $1 \leqq i_1 \leqq k_1, 1 \leqq j_1 \leqq k_1$, and $i_1+j_1=k_1+1$.

Proof: The sufficiency of the conditions, due to Weyl, is usually proved by the minimax principle. It can also be proved using Theorem (1.1.6) . We have already seen the necessity of $i_1 \leqq k_1$ and $j_1 \leqq k_1$ in the proof of Theorem (1.1.5). Now suppose $i_1+j_1 \leqq k_1+2$. Let $A=\operatorname{diag}(1,\dots,1,0,\dots,0)$ with i_1-1 ones, and $B=diag(0,\dots,0,1,\dots,1)$ with j_1-1 ones, where the orders of A and B are k_1. Since $k_1-j_1+1 \leqq i_1-1$, all the eigenvalues of $A+B$ are $\geqq 1$. Therefore $\gamma_{k_1} \geqq 1$, while $\alpha_{i_1}=\beta_{j_1}=0$, contradicting $(i_1,j_1,k_1) \in S_1^k$

Theorem (1.1.9)[10]: If i, j and k are ordered pairs of integers satisfying

$$1 \leqq i_1 \leqq i_2 \leqq n, 1 \leqq j_1 \leqq j_2 \leqq n, 1<k_1<k_2 \leqq n, \tag{22}$$

$$i_1+j_1 \leqq k_1+1 \tag{23}$$

$$\left.\begin{matrix} i_1+j_2 \\ i_2+j_1 \end{matrix}\right\} \leqq k_2+1 \tag{24}$$

$$i_1+i_2+j_1+j_2=k_1+k_2+3 \tag{25}$$

then $(i;j;k) \in S_2^n$

Proof: By Theorem (1.1.5) we may assume $n=k_2$. We proceed by induction on n. If $n=2$ the theorem follows from (1). Suppose the theorem holds for all $n<N$, where $N>2$. By (22), (23) and (24), $i_p \leqq k_p$ and $j_p \leqq k_p, p=1,2$. Suppose $i_1>1$. Then the pairs (i_1-1,i_2-1),$(j_1,j_2)$and(k_1-1,k_2-1)satisfy (22)-(25). Therefore by the induction hypothesis $(i_1-1,i_2-1;j_1,j_2;k_1-1,k_2-1) \in S_2^{N-1}$If we apply Theorem (1.1.6) with $u=w=1, v=3$, we find $(i;j;k) \in S_2^N$. A similar method takes care of the case $j_1>1$. Therefore we may assume

$$i_1=j_1=1 \tag{26}$$

If

$$(i_1,i_2-1;,j_1,j_2-1;\ k_1,k_2-1,k_2-1) \in S_2^{N-1} \tag{27}$$

and if

$$i_2+j_2 \geqq +3+k_2 \tag{28}$$

then Theorem (1.1.6) with $u=v=2, \omega=1$ allows us to conclude $(i;j;k) \in S_2^N$. But the Condition (28) which is needed for the application of Theorem (1.1.6) will also guarantee (27). To see this, first note that (27) can fail only when

(i) $i_2=i_1+1=2$

or

(ii) $j_2=j_1+1=2$

or

(iii) $k_1=1$

or

(iv) $i_1+j_1=k_1+1$.

If (i) holds then $i_1+j_2=2+j_2 \leqq 2+k_2$, contradicting (28). Similarly

(ii) cannot hold. If (iii) holds, then by (26), $i_1 + i_2 + j_1 + j_2 = 2 + i_1 + j_2 = k_1 + k_2 + 3 = k_2 + 4, or\ i_1 + j_2 - k_2 + 2$, contradicting (28). Condition
(iv) implies (iii) by (26). Therefore we may assume

$$i_1 + j_2 \leqq 2 + k_2 \tag{29}$$

If $i_2 \leqq k_1 + 2$, it is easy to show by the induction hypothesis that $(i_1, i_2 - 1; j_1, j_2; k_1, k_2 - 1) \in S_2^{N-1}$ and Theorem (1.1.6)with $u = \omega = 2, v = 3$implies $(i; j; k) \in S_2^N$. Hence we assume

$$i_2 \leqq k_1 + 1 \text{ and } j_2 \leqq k_1 + 1. \tag{30}$$

Now (25) and (26) imply $i_2 + j_2 = k_1 + k_2 + 1$, which with (29) implies $k_2 = 1$. Therefore by (30) and (22), $i_2 + j_2 = 2$ and hence $i_1 + j_1 = 1$. Using (25) we find $k_2 = 2$, contradicting $N > 2$.

The proof is complete.

If in (25) we replace the equality sign by $\leqq$, Theorem (1.1.9) remains true. For if i,j and k satisfy (22)-(24) and the modified (25), there exists a pair $k' = \{k_1', k_2')$ such that $k_1' \leqq k_1, k_2' \leq k_2$ and i, j , k′ satisfy (22)-(25). However Theorem (1.1.3) suggests that we consider only cases where (19) holds. Conditions (23) and (24) combined may be expressed as follows: $i_u + j_v \leqq k_\omega + 1$ whenever$1 \leqq i \leqq 2, 1 \leqq j \leqq 2, 1 \leqq \omega \leqq 2$, and $u + v = \omega + 1$. This suggests the following conjecture. Let us define inductively the following sequence of sets of triples of sequences of integers:Let $(i_1, j_1, K_1) \in T_1^n$ if $1 \leqq i_1 \leqq n, 1 \leqq j_1 \leqq n, 1 \leqq k_1 \leqq n$, and $\mathrm{i}_1 + \mathrm{j}_1 = \mathrm{K}_1 + 1$ and let $(i_1, \dots, i_r;\ j_1, \dots, j_r;\ K_1, \dots, k_r) \in T_r^n$, if $1 \leqq i_1 < \cdots < i_r \leqq n, 1 \leqq j_1 \leqq n, 1 \leqq k_1 < \cdots < k_r \leqq n$, and

$$i_1 + \cdots + i_r + j_1 + \cdots + j_r \leqq k_1 + \cdots + k_r + \frac{r(r+1)}{2} \tag{31}$$

And

$$i_{u_1} + \cdots + i_{u_s} + j_{v_1} + \cdots + j_{v_s} \leqq k_{\omega_1} + \cdots + k_{\omega_s} + \frac{s(s+1)}{2} \tag{32}$$

Whenever

$$(u; v; \omega) \in T_s^r, 1 \leq s \leq r - 1.$$

Theorem (1.1.8) and (1.1.9) show that $T_r^n \subset S_r^n$ for $r = 1, 2$. It seems reasonable to conjecture $T_r^n \subset S_r^n$ for all r. The case $r = 3$is the following.

Theorem (1.1.10)[10]: If i, j and k are ordered triples of integers such that

$$1 \leqq i_1 < i_2 < i_2 \leqq n, 1 \leqq j_1 < j_2 < j_2 \leqq n, 1 \leqq k_1 < k_2 < k_3 \leqq n \tag{33}$$

$$i_1 + j_1 \leqq k_1 + 1 \tag{34}$$

$$\left.\begin{matrix} i_1 + j_2 \\ i_2 + j_1 \end{matrix}\right\} \leqq k_2 + 1 \tag{35}$$

$$\left.\begin{matrix} i_1 + j_3 \\ i_2 + j_2 \\ i_3 + j_1 \end{matrix}\right\} \leqq k_3 + 1 \tag{36}$$

$$i_1 + i_2 + j_1 + j_2 \leqq k_1 + k_2 + 3 \tag{37}$$

$$\left.\begin{matrix} i_1 + i_2 + j_1 + j_3 \\ i_1 + i_3 + j_1 + j_2 \end{matrix}\right\} \leqq k_1 + k_3 + 3 \tag{38}$$

$$\left.\begin{array}{l} i_1+i_2+j_2+j_3 \\ i_2+i_3+j_1+j_2 \\ i_1+i_3+j_1+j_3 \end{array}\right\} \leqq k_2+k_3+3 \quad (39)$$

$$i_1+i_2+i_3+j_1+j_2+j_3=k_1+k_2+k_3+6, \quad (40)$$

Then $(i,j,k)\subset S_3^n$

Proof: The proof begins along the same lines as the proof of Theorem (1.1.9) and will only be sketched. We may assume $\mathrm{n}=\mathrm{k}_3$, and proceed by induction on n. When $n=3, i_1=j_1=k_1=1, i_2=j_2=k_2\ 2, i_3=j_3=k_3=3$, and the result follows from (1). Assume the theorem for all $n<N$, where $N>3$. As in Theorem (1.1.9), we may assume

$$i_1=j_1=1 \quad (41)$$

If

$$(i_1, i_2-1.i_3-1),(j_1, j_2, j_3-1),(k_1-1, k_2-2, k_3-1) \quad (42)$$

Satisfies (33)- (40) and if

$$i_1+j_3 \geqq k_3+3 \quad (43)$$

then the induction hypothesis and Theorem (1.1.6)with $u=2, v=3, w=1$ yield the theorem. Again the condition (43) which is needed for the application of Theorem (1.1.6) will guarantee (42). For example $k_1 - 1 \geqq 1$, because if $k_1=1$, then by (38) and (41), $i_1+j_3 \leqq k_3+2$, contradicting (43). The second inequality of (36) together with (43) and $j_3 \leqq k_3$ (which follows from (36)) ensure $j_3-1>j_2$. We may therefore assume

$$\left.\begin{array}{l} \mathrm{i}_2+\mathrm{j}_3 \\ \mathrm{i}_3+\mathrm{j}_2 \end{array}\right\} \leqq \mathrm{k}_3+2. \quad (44)$$

Next we show that we may assume

$$i_2 \leqq k_1+1 \text{ and } j_2 \leqq k_1+1 \quad (45)$$

by showing that if $i_2 \leqq k_1+2$, then$(i_1, i_2-1, i_3-1; j_1, j_2, j_3, k_1, k_2-1, k_3-1) \in S_3^N$ and Theorem (1.1.6) with $u=2, v=3, w=2$gives $(i;\ j\ ;\ k) \in S_3^N$ In a similar manner we may assume

$$i_3+j_3 \leqq k_1+k_3+2 \quad (46)$$

$$i_3 \leqq k_2+1, j_3 \leqq k_3+1 \quad (47)$$

Now (33)-(41) together with (44)-(47) are easily seen to imply $k_1+k_2-k_3, i_2=i_2=k_3+1, i_3=i_3=k_2+1$ and $k_1+1 \leqq k_2 \leqq 2k_1$. Therefor the theorem will be proved if we can show that

$$(1, p+1, p+q+1; 1, p+1, p+q+1; p, p+q, 2p+q) \in S_3^n$$

whenever$1 \leqq q \leqq p$and $2p+q=n.$

Let A, B and $A+B$ be of order n with eigenvalues $(\alpha_p),(\beta_p)$ and (γ_p). We have $q\,\gamma_p \leqq \gamma_p+\gamma_{p+1}+\cdots+{}'\Upsilon_{P-r+i,}\ q'\Upsilon_{P+q},+\cdots+{}'\Upsilon_{P+q} \leq \gamma_{p+q}+\cdots+\gamma_{p+1}$and $q'\Upsilon_{P+q} \leqq \gamma_{2p+q}+\cdots+\gamma_{2p+1}$. Hence

$$q\left(\gamma_p+{}'\Upsilon_{P+q}+{}'\Upsilon_{2P+q}\right) \leqq \operatorname{trace}(A+B) - (\gamma_1+\cdots+\gamma_{p+q}+\gamma_{p+q+1}+\cdots+\gamma_{2p}).$$

Similarly

$$q\left(\alpha_1+\alpha_{p+1}+\alpha_{p+q+1}\right) \geqq \operatorname{trace} A-(\alpha_{q+1}+\cdots+\alpha_p+\alpha_{p+2q+1}+\cdots+\alpha_{2p+q})$$

and we have a similar statement for the β's. Therefore we need only prove

$$(q + 1, \dots, p, p + 2p + 1, \dots, 2p + q;\ q + 1, \dots, p, p + 2q + 1, \dots ,2p + q; 1, \dots , p - q, p + 1, \dots, 2p) \in \bar{S}^{n}_{2p-2q}$$

This will follows from Theorem (1.1. 4) (ii) if we can show

$$(1, \dots , p - q, p + q + 1, \dots , 2p;\ 1,, p - q, p + q + 1, \dots, 2\,p\,;\ q + l, \dots, p\,, p + 2q + l,, 2p + q) \in S^{n}_{2p-2q}\ . \quad (48)$$

By Theorem (1.1.7) we have

$$(1, \dots, 2p - 2q\,;\ 1, \dots , p - q, p + 1, \dots , 2p) \in S^{p2}_{2p-2q}$$

$$p + 1, \dots, P - q, P + 1, \dots , 2p - q) \in S^{2p}_{2p-2q}$$

We may apply Theorem (1.1.6) q times with $u = w = p - q + 1, v = 2p - 2q + 1$ to obtain

$$(1,, \mathrm{p} - \mathrm{q}, \acute{\imath} > + 1,, 2p - q;\ 1,, p - q,$$

$$p + 1,, 2p - q;\ 1,, p - q, p + q + 1,, 2p) \in S^{p2}_{2p-2q}\,.$$

Theorem (1.1.6) applied q times with $u = v = p - q + 1, \omega = 1$yields (48).The proof is now complete.

A proof of $T^n_4 \subset S^n_4$ along the same lines runs into the following difficulty. The first half of the proof, that is, the application of Theorem (1.1.6) in all possible ways, carries through. However the cases left untouched turn out to be too numerous to handle by the methods of the second half of the proof of Theorem (1.1.10). We have verified $T^n_4 \subset S^n_4$ for $n \leq 8$.

As for the statement $S^n_r \subset T^n_r$, it is possible to show by a consideration of diagonal matrices that if $(i; j; k) \in S^n_r$ then (32) holds for $s = 1,2$. This together with the remark following Theorem (1.1,9) determines S^n_r. But the general statement $S^n_r \subset T^n_r$ is false even if we weaken the definition of T^n_r by replacing the equality sign in (31) by$\leqq$. For example a consideration of the trace condition shows that $(1, 5, 9, 12;\ 1, 5, 9, 12;\ 4, 8, 12, 16) \in S^{16}_4$

Guided by Theorem (1.1.4) (ii), the dual set $\tilde{T}^n_r$ may be defined inductively as follows: $(i_1, j_1, k_1) \in \tilde{T}^n_r$ if$i_1 + j_1 + k_1 + n$, and $(i, j, k) \in \tilde{T}^n_r$ if $i_1 + \cdots + i_r + j_1 + \cdots + j_r = k_1 + \cdots . + k_r + nr - r(r - 1)/2$ *and*

$$i_{u_1} + \cdots + i_{u_s} + j_{v_1} + \cdots + j_{v_s} \leqq k_{w_1} + \cdots + k_{w_s} + s(s + 1)/2$$

Wherever $(u; v; w) \in \tilde{T}^n_r$. It is easily seen that $(i, j, k) \in \tilde{T}^n_r$ if and only if $(n - i_r + 1 \dots ., n - i_1 + 1; n - j_r + 1 \dots ., n - j_{1-} + 1; n - k_r + 1, \dots, n - k_1 + 1) \in \tilde{T}^n_r$ Hence by Theorem (1.1,4), $\mathrm{T}^n_r \subset \mathrm{S}^n_r$ is equivalent to $T^n_r \subset \tilde{S}^n_r$. we have been unable to prove the analogue of the last transformation rule of Theorem (1.1.4). However we can prove that if $(i; j; k) \in \tilde{T}^n_r$ then $(i'; j'; k') \in T^n_{n-1,}$ where i$'$, j $'$, and k' are the complements with respect to n.

We return to the problem of determining the set E that being defined. Let F be the set of points γ defined by $\gamma_1 \geqq \cdots \geqq \gamma_n$,

$$\gamma_1 + \cdots + \gamma_n = \alpha_1 + \cdots + \alpha_n + \beta_1 + \cdots + \beta_n$$

And

$$\gamma_{k_1} + \cdots + \gamma_{k_r} \leqq \alpha_{i_1} + \cdots + \alpha_{k_r} + \beta_i + \cdots + \beta_{i_r}$$

Wherever

$$(i;j;k) \in \tilde{T}_r^n, 1 \leqq r \leqq n-1.$$

We have shown that $E \subset F$ for $n \leqq 4$. We will prove that $E = F$ for $n \leqq 4$.

We assuming $\alpha_1 > \cdots > \alpha_n$ and $\beta_1 > \cdots > \beta_n$. The set E' defined is a closed subset of E. Since F is closed and convex, it will follow that $E' = F$, and therefore $E = F$, if the boundary of E' is contained in the boundary of F. To see this, let γ' be an interior point of E' and supposeγ' is any point of F. If γ' is not in E' there must be a boundary point of E' in the open segment joining γ and γ'. But all points of this open segment are interior points of F.

A boundary points of E' with at least two equal coordinates is obviously a boundary point of F. If γ is a boundary point of E' with distinct coordinates, there is associated with γ a triple $(i;j;k)$satisfying the conditions of Theorem (1.1.3). All that remains to prove is that $(i;j;k) \in T_r^n$ To this end we first prove the following theorem.

Theorem (1.1.11)[10]: Ifγ is a boundary point of E' with associated sequences $(i;j;k)$ of order r, then for any $(x;\ y;\ z) \in \tilde{S}_{m,}^r$ there cannot exist a triple $(u;v;\omega) \in T_m^{n-r}$ such that $i_{x_p} \leqq x_p + u_p - 1, j_{v_p} \leqq y_p + v_p - 1$, and $k_{z_p} \geqq z_p + \omega_p$, for $1 \leqq p \leqq m$.

Proof: For convenience, we write $\alpha(p)$ instead of α_p**.** By hypothesis there exist Hermitian matrices A_1, B_1 and $A_1 + B_1$ with eigenvalues $\left(\alpha\left(i_p\right)\right), \left(\beta\left(j_p\right)\right)$, and$\left(\gamma\left(k_p\right)\right), p = 1, r$,and Hermitian matrices A_2, B_2 and $A_2 + B_2$ with eigenvalues $(\alpha\left(i'_p\right), (\beta(j'_P))$, and $\left(\gamma\left(k'_p\right)\right), p = 1, \dots, n - r$, where i' is complement of i with respect to n. If there exists a triple $(u;\ v, w) \in S_m^{n-r}$such that $i_{X_p} < i'_{u_p}, j_{v_p} < j\,'_{v_p}$, and $k'_{z_p} > k'_{w_p}, 1 \leqq p \leqq m$, ,then we have

$$\sum_{p=1}^{m} \alpha\left(i_{z_p}\right) + \sum_{p=1}^{m} \beta\left(j_{v_p}\right) \leqq \sum_{p=1}^{m} \gamma\left(k_{z_p}\right) < \sum_{p=1}^{m} \gamma\left(k'_{w_p}\right) \leqq \sum_{p=1}^{m} \alpha(i'_u) + \sum_{p=1}^{m} \beta(j'_v)$$

This is impossible since $\alpha\left(i_{z_p}\right) > \alpha\left(i'_{u_p}\right)$ and $\beta\left(j_{v_p}\right) < \beta\left(j'_{v_p}\right)$. Therefore it remains only to show that $i_p < i'_q$ is implied by $i_p \leqq p + q — 1$. If $i_p \leqq p + q - 1$, then at least p terms of the sequence i are $\leqq p + q - 1$. Therefore at most $q - 1$ positive integers $\leqq p + q - 1$ are not in i. Hence $i'_p > p + q - 1 \geqq i_p$

Theorem (1.1.12)[10]: If $\gamma$is a boundary point of E' with associated sequences i, j, k of order r, then $i_x + j_y \geqq k_z + r$ whenever $(x, y, z) \in \tilde{T}_1^r$. More generally,if $x + y \geqq z + r$, the $i_x - x + j_y - y \geqq k_z - z$.

Proof: We have $n \geqq r + 1 \geqq 2$. Since $(x;\ y;\ x + y - r) \in \tilde{T}_1^r \subset \tilde{S}_1^r$, it follows that $(x;\ y;\ z) \in \tilde{S}_1^r$. Let $u = i_x - x + 1, v = j_y - y + 1$, and $\omega = k_z - z$. Clearly, $u \geqq 1, v \geqq 1$, and $w \leqq n - r$ since $k_1 - 1 \leqq k_2 - 2 \leqq \cdots \leqq k_r - r \leqq n - r$. We must prove $u + v \leqq \omega + 2$. It $u + v \leqq \omega + 1$, then $\omega \geqq l, u \leqq \omega$, and $v \leqq \omega$. Therefore $(u;\ v;\ \omega) \in \tilde{T}_1^{n-r}$ This contradicts Theorem (1.1.11)

Theorem (1.1.13)[10]: Under the same hypothesis as Theorem (1.1.12),if n $\geqq r + 2$, then $i_{z_1} + i_{x_2} + j_{v_1} + i_{y_2} \geqq k_{z_1} + k_{z_2} + 2r - 1$ whenever $(x, y, z) \in \tilde{T}_z^r$.

Proof: We are given $x_1 + y_2 \geqq z_1 + r, x_2 + y_1 \geqq z_1 + r, x_2 + y_2 \geqq z_2 + r.$ and$x_1 + x_2 + y_1 + y_2 + z_1 + z_2 + 2r - 1$.Let $\alpha_p = i_{x_p} - x_p + 1, b_p = i_{v_p} - y_p + 1.$ and $C_p = \omega_{z_p} - z_p, p = 1.2$. By Theorem (1.1.12), $\alpha_1 + b_2 \geqq c_1 + 2, \alpha_2 + \geqq c_1 + c_2 + 3$. Therefore

$$a_1 + b_2 \geqq c_1 + 1 \quad (49)$$

$$a_1 + b_2 \geqq c_2 + 1 \quad (50)$$

$$a_2 + b_1 \geqq c_2 + 1. \quad (51)$$

Also $1 \geqq a_1 \geqq a_2, 1 \geqq b_1 \geqq b_2$ and $c_2 \geqq n - r$. By (49), $c_1 \geqq 1$. Moreover $c_1 + 2 \leqq a_1 + b_2 \leqq c_2 + 1$, so that $c_1 + 1 \leqq c_2$. Now let $u_1 = \alpha_1, u_2 = max\ (\alpha_2, \alpha_1 + 1), v_1 = b_1, v_2 = max\ (b_2, b_1 + 1), \omega_1 = c_1$, and $w_2 = c_2$. It is easy to see that, and $u_1 + v_1 \leqq \omega_1 + 1, u_1 + v_2 \leqq \omega_2 + 1, u_2 + v_1 \leqq \omega_2 + 1$ and $u_1 + v_2 + v_1 + v_2 \leqq \omega_1 + \omega_2 + 3$. As previously remarked there exists a pair (ω_1', ω_2') such that$\omega_1' \leqq \omega_1 \omega_2' \leqq \omega_2$, and $(u;\ v;\ \omega)) \in \tilde{T}_z^{n-r}$. This contradicts Theorem (1.1.11).

Using a generalized version of Theorem (1.1.13), it is possible to show that

$$i_{x_1} + i_{x_2} + j_{v_1} + j_{v_2} \leqq k_{z_1} + k_{z_2} + k_{z_3} + r + r - 1 + r - 2$$

Whenever$(x; y; z) \in \tilde{T}_z^{n-r}, n \geqq r + 2$

Theorem (1.1.14)[10]: If γ is a boundary point of E′ with associated sequences i, j , k of order $r = 1, 2, 3$ or $n - 1$, then $(i; j; k) \in \tilde{T}_z^n$.

Proof: For$r = 1$this is obvious. For $r = n - 1$, the complementary sequences with respect to n are of order 1 and satisfy $i_1' + j_1' = k_1' + n$. Therefore $(i'; j';\ k') \in \tilde{T}_1^n$. By the last sentence, it follows that$(i'j;\ k) \in \tilde{T}_1^n$. For the cases $n = 3,4$ this can be easily verified by listing cases. Now suppose $r = 2$. We must prove that (23) and (24) hold. In view of (25), this means we must show that $i_x + j_v \geqq k_z + 2$ whenever $(x; y; z) \in \tilde{T}_1^2$. But this follows from Theorem (1.1.12). Suppose $r = 3$. We may assume $n \geqq 5$. By (40) and Theorems(1.1.12) and(1.1.13) we have (34)-(39), since if $(x;\ y;\ z) \in \tilde{T}_p^2$ then$(x'; y'; z') \in T_{3-q}^3, p = 1, 2$.

Theorem (1.1.14) completes the proof that $E = F$ for $n \leqq 5$. It is possible to extend the proof to $n \leqq 8$.

Section (1.2) Interlacing Inequalities for Singular Values of Submatrices

We give a brief summary of certain particular cases of our results that merit special attention. Let A bean $n \times n$ real or complex matrix, and let $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_n$ be the singular values of A. (They are defined to be the eigenvalues of the positive semidefinite matrix $(AA^*)^{1/2}$.) Let $B = A_{ij}$ be the $(n-1)$-square submatrix of A obtained by deleting row i and column j, and let $\beta_1 \geqq \beta_2 \geqq \cdots \geqq \beta_{n-1}$ be the singular values of B. Our first theorem yields, as a special case, these interlacing inequalities:

$$\begin{gathered} \alpha_1 \geqq \beta_1 \geqq \alpha_3 \\ \alpha_2 \geqq \beta_2 \geqq \alpha_4 \\ \cdots \\ \alpha_t \geqq \beta_t \geqq \alpha_{t+2,} 1 \leqq t \leqq n-2, \\ \cdots \\ \alpha_{n-2} \geqq \beta_{n-2} \geqq \alpha_{n,} \\ \alpha_{n-2} \geqq \beta_{n-1.} \end{gathered} \quad (52)$$

That inequalities (52) are the best that can be asserted is shown by (a special case of) Theorem (1.2.3). It follows from Theorem (1.2.3) that, if arbitrary nonnegative numbers $\beta_1 \geqq \cdots \geqq \beta_{n-1}$ are given satisfying (52), there will always exist unitary matrices U and Vsuch that the singular values of $(UA\,V)_{ij}$ are $\beta_1, \dots, \beta_{n-1}$. (Of course, A and $UA\,V$ always have the same singular values $\alpha_1, \dots, \alpha_{n.}$)Thus nothing more than (52) can hold in general, when looking at a fixed submatrix. Further results can be obtained, however, by examining all the submatrices of A of fixed degree. Now let $\beta_{ij,1} \geqq \cdots \beta_{ij,n-1}$denote the singular values of A_{ij}.We obtain the following estimates on the mean square of the tth singular value of all the$(n-1)$-square submatrices $A_{ij}$of A:

$$\left(\frac{1}{2}\right)^2 \alpha_t^2 + \frac{2(n-1)}{n^2}\alpha_t^2 + \left(\frac{n-1}{2}\right)^2 \alpha_{t+2}^2 \leqq \frac{1}{n^2}\sum_{i,j=1}^{n} \beta_{ij,t}^2$$

$$\leqq \left(\frac{n-1}{2}\right)^2 \alpha_t^2 + \frac{2(n-1)}{n^2}\alpha_{t+1}^2 + \left(\frac{1}{2}\right)^2 \alpha_{t+2}^2,$$

$$1 \leqq t \leqq n-2, \tag{53}$$

$$\left(\frac{1}{2}\right)^2 \alpha_{n-1}^2 + \frac{n-1}{n^2}\frac{1}{n}\alpha_n^2 \leqq \sum_{i,j=1}^{n} \beta_{ij,n-1}^2 \leqq \left(\frac{n-1}{2}\right)^2 \alpha_{n-1}^2 + \frac{n-1}{n^2}\frac{1}{n}\alpha_{n.}^2 \tag{54}$$

In (53) we have displayed convex combinations of $\alpha_t^2, \alpha_{t+1}^2, \alpha_{t+2}^2$ which serve as upper and lower bounds for the mean square of the tth singular value $(t \leqq n-2)$ of the different $(n-1)$-square submatrices A_{ij} of A. (By (52), this mean lies between α_t^2 and $\alpha_{t+2.}^2$) In (54), we have similar, though not precisely the same, convex combinations of $\alpha_{n-1,}^2, \alpha_n^2$, and 0 yielding bounds for the mean square of the $\beta_{ij,n-l}$. These results, (53) and (54), will appear as special cases of Theorem (1.2.5).

Let

$$f_{ij}(\lambda) = \left(\lambda - \beta_{ij,}\right) \dots \left(\lambda - \beta_{ij,n-l}\right) \tag{55}$$

be the singular value polynomial ofA_{ij}. This is the polynomial whose roots are the squares of the singular values of A_{ij}. Let

$$f(\lambda) = (\lambda - \alpha_1^2) \dots (\lambda - \alpha_n^2) \tag{56}$$

be the corresponding polynomial for A. As a particular instance of Theorem (1.2.6), we obtain

$$\sum_{i=1}^{n}\sum_{j=1}^{n} f_{ij}(\lambda) = \frac{d}{d\lambda}\lambda\frac{\lambda d}{d\lambda}f(\lambda) \tag{57}$$

It is interesting to contrast formula (57) with the well-known result asserting that the sum of the characteristic polynomials of all the principal $(n-1)-$ quare submatrices of A is just the derivative of the characteristic polynomial of A.

We give first the definition of the singular values of a rectangular matrix.

Definition (1.2.1)[238]: Let A be an m x n matrix. The singular values

$$\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_{min(m,n)} \tag{58}$$

of A are the common eigenvalues of the positive semidefinite matrices $(AA^*)^{1/2}$ and $(A^*A)^{1/2}$.

Since AA^* is m-square and A^*A is n-square, the eigenvalues of $(AA^*)^{1/2}$and $((AA^*)^{1/2}$do not coincide in full. However, it is well known that the nonzero eigenvalues (including multiplicities) of these two matrices always coincide. It is frequently convenient to define x_t to be zero for $min(m,n) < t \leqq max(m,n)$. Then $\alpha_1^2 \geqq \cdots \geqq \alpha_{max(m,n)}^2$ and the roots of AA^* (respectively A^*A) are the first m (respectively n) of these numbers.

We are the following

Theorem (1.2.2)[238]: Let A be an $m \times n$ matrix with singular values (58). Let B be a $p \times q$ submatrix of A, with singular values

$\beta_1 \geqq \beta_2 \geqq \cdots \geqq \beta_{min(p,q)}$(59)

Then

$$\alpha_i \geqq \beta_{i,} \quad for\ i = 1,2,\dots,min(p,q). \tag{60}$$

$$\beta_{i,} \geqq \alpha_{i+(m-p)+(n-q),}\ for\ i \leqq min(p+q-m, p+q-n). \tag{61}$$

Proof: For an arbitrary matrix M, let $M[i_1,\dots,i_p : j_1,\dots,j_p]$denote the submatrix of M lying at the intersection of rows $i_1,..,i_p$, and columns $j_1,\dots,j_q$.

Suppose that $B = A\ [i_1,\dots,i_p : j_1,\dots,j_q]$. To simplify notation let $\omega = \{i_1,\dots,i_p\}$and $\tau = \{j_1,\dots,j_q\}$ denote the sets of integers giving the rows and columns of A used to form B, and denote B by $B = A\ [\omega,\tau]$.

Let us view B as a submatrix of UAV, where U is an m-square unitary matrix and V is an n-square unitary matrix. In this proof we may take $U = I_m$, and $V = I_n$,. (In the next theorem, U and V will become variable.) Then

$$B = U[i_1,\dots,i_p : 1,\dots,m]\ AV[1,\dots,n : j_1,\dots,j_q]. \tag{62}$$

Thus

$$BB^* = U[i_1,\dots,i_p : 1,\dots,m]\ AV[1,\dots,n : j_1,\dots,j_q]. \tag{63}$$

Where

$$X = AV[1,\dots,n : j_1,\dots,j_q]\ . \tag{64}$$

is $m \times q$. Thus BB^* is a principal $p-$square submatrix of the m-square Hermitian matrix UXX^*U^*. Let

$$x_1^2 \geqq x_2^2 \geqq \cdots \geqq x_{min(m,q)}^2 \geqq x_{min(m,q)+1}^2 = \cdots x_m^2 = 0, \tag{65}$$

denote the eigenvalues of XX^*. Thus $x_1,\dots,x_{\min(m,q)}$ are the singular values of X. From the well-known formulas linking the eigenvalues of a Hermitian matrix with the eigenvalues of a principal submatrix, we obtain

$$x_i^2 \geqq \beta_i^2 \geqq x_{i+m-p}^2, for\ i = 1,2,\dots,p. \tag{66}$$

Now $x_{\mathrm{i}}^2,\dots,x_{\min(m,q)}^2$, $0(\mathrm{q} - \min(\mathrm{m,q})$ times) are the eigenvalues of

$$X^*X = V^*[j_1,\dots,j_q : 1,\dots,n]AA^*V[1,\dots,n : j_1,\dots,j_q] \tag{67}$$

Thus X^*X is a principal q-square submatrix of the $n-$square Hermitian matrix $V^*A^*A\ V$. Hence

$$\alpha_i^2 \geqq x_i^2 \geqq \alpha\ _{i+n-q}^2 \quad \text{for } i = 1,2,\dots,p. \tag{68}$$

Thus for $i \leqq min(p,q)$ we have $\alpha_i^2 \geqq x_i^2 \geqq \beta_i^2$, yielding (60). And for $i \leqq min\ (p+q-m, q+q-n)$ we have $\beta_i^2 \geqq \beta_{i+m-p}^2 \geqq \alpha_{i+(n-q)+(m-p)}^2$ yielding (61).

The proof of Theorem (1.2.1) is now complete. We shall present a second proof of Theorem (1.2.1) at the later.

Theorem (1.2.3)[238]: Let A be an $m \times n$ matrix with singular values (58). Let arbitrary nonnegative numbers (59) be given, satisfying both (60) and (61).

Then m-square unitary matrix U and n-square unitary matrix V exist such that the singular values of the $p \times q$ submatrix

$$(UAC)[i_1, \ldots, i_p]\,AV[j_1, \ldots, j_q]$$

of UAV are the numbers (59).

Proof: Define β_i to be zero if $i > min(p, q)$, and define α_i to be zero if i $> min(m, n)$. Now define inductively nonnegative numbers $x_1, \ldots, X_{nin(m,q)}$ by

$$x_1 = min \begin{cases} \alpha_1 \\ \beta_{1-m+p} \end{cases} \quad \text{if } m - p < 1 \qquad (69)$$

And

$$x_i = min \begin{cases} \alpha_1 \\ \beta_{i-m+p} \\ x_{i-1} \end{cases} \quad \text{if } m - p < i$$

For $2 \le i \le min(m, q)$. (70)

(We include β_{i-m+p} in (69) and (70) only if i satisfies the indicated condition.) For all $t > min(m, q)$, define x_t by $x_t = 0$.

It is plain that $x_1 \geqq \cdots \geqq x_{min(m,q)}$. We claim that inequalities (68) are satisfied. Plainly, $x_i \geqq \alpha_i$ for $i \geqq min(m, q)$, and this also holds for $min(m, q) < i \geqq q$ since then $x_i = 0$. We show by induction on i that the lower bounds in (68) are satisfied. To show that $x_1 \geqq \alpha_{1+n-q,}$, we must show that both of the quantities entering into the minimum in (68) exceed $\alpha_{1+n-q,}$. Plainly, by (58), $\alpha_{1,} \geqq \alpha_{1+n-q,}$. If $m - p < i$ we obtain from (thus $m = p$), **(**61) tells us that $\beta_1 \geqq \alpha_{1+n-q,}$, provided $i \geqq min(q, m + q - n)$.However, *if* $m + q - n \geqq 0$, we have $m + 1 \geqq 1 + n - q$ and thus automatically $0 = \alpha_{1+n-q,} \geqq \beta_1$.Hence $m + 1 \geqq 1 + n - q$. Suppose (induction) $x_{i-1} \geqq \alpha_{i-1+n-q,}$. Let $i \geqq min(m, q)$. If we show that each of the three quantities entering into the minimum in (70) exceeds $\alpha_{i+n-q,}$, it will follow that $x_i \geqq \alpha_{i-1+n-q,}$ Plainly, by (58), $\alpha_i \geqq \alpha_{i-1+n-q,}$. If $m - p < i$, we obtain from (61) that $\beta_{i-m+p} \geqq \alpha_{i-1+n-q,}$ provided $i \geqq min(q, m + q - n)$.By induction, $x_i \geqq \alpha_{i-1+n-q,} \geqq \alpha_{i+n-q,}$(by (58)).Thus $x_i \geqq \alpha_{i-1+n-q,}$, except perhaps if $i > min(q, m + q - n)$. However, if $i > min(q, m + q - n)$, then $i + n - q > min(n, m)$, so that $\alpha_{i+n-q,} = 0$ and hence automatically $x_i \geqq \alpha_{i-1+n-q,}$.Therefore $x_i \geqq \alpha_{i+n-q,}$is established if $i \geqq min(m, q)$.If $i > min(m, q)$, then $i + n - q > min(n - q + m, n) \geqq min(m, n)$, so that automatically $0 = \alpha_{i+n-q,} \geqq x_i$. Therefore inequalities (68) are established.

We now claim that inequalities (66) are satisfied. By (70), $x_{i+m-p} \geqq \beta_{i,}$, for $i + m - p \le min(m, q)$. Thus the lower inequality in (66) is satisfied, provided $i \geqq min(p, p + q - m)$. If $i > min(p, p + q - m)$, then $i + m - p > min(m, q)$and hence $x_{i+m-p} = 0$, so that automatically $\beta_{i,} \geqq x_{i+m-p}$. Thus the lower inequalities in (66) are satisfied. We show by induction on i that $x_i \geqq \beta_{i,}$. For $i = 1$ this follows immediately from (69), since $\alpha_1 \geqq \beta_{i,}$.

Suppose $x_{i-1} \geqq \beta_{i-1}$. If we show that each of the three quantities entering into the minimum in (70) exceeds β_i, we may conclude that $x_i \geqq \beta_i$. We may assume also that $i \leqq min(p,q)$, since $\beta_i = 0$ ($\leqq x_1$) for $i > min(p,q)$.Thus (by (60)) $\alpha_i \geqq \beta_i$ if $m-p < i, \beta_{i-m+p,} \geqq \beta_{i,}$by (59). By induction $x_{i-1} \geqq \beta_{i-1} \geqq \beta_i$. Hence the inequality $x_i \geqq \beta_i$ for all $i \leqq p$ is established.

It is a known fact (see [231]), because $x_1^2 \geqq \cdots \geqq x_q^2$ satisfy (68), there exists an n-square unitary matrix V such that the eigenvalues of

$$X^*X = V\left[j_1, \dots, j_q | 1, \dots, n\right] A^*AV\left[1, \dots, n \middle| j_1, \dots, j_q\right] \quad (71)$$

are

$$x_1^2, \dots, x_{min(m,q)}^2, \dots, x_q^2 \quad (72)$$

Here

$$X = AV\left[1, \dots, n \middle| j_1, \dots, j_q\right]$$

Thus XX^* has

$$x_1^2, \dots, x_{min(m,q)}^2, \dots, x_m^2 \quad (73)$$

as eigenvalues. Because the inequalities (66) are satisfied, there exists an m-square unitary matrix U such that

$$UXX^*U^*\left[i_1, \dots, i_p \middle| i_1, \dots, i_q\right]$$

has eigenvalues $\beta_1^2 \geqq \beta_{min(p,q)}^2 \geqq \cdots = \beta_p^2$. It is now immediate that the submatrix

$$U\left[i_1, \dots, i_p | 1, \dots, m\right] AV\left[1, \dots, n \middle| j_1, \dots, j_q\right]$$

of UAV has (59) as its singular values. The proof of Theorem (1.2.3) is now finished.
We remark that the nonincreasing condition (59) is actually superfluous. We have Theorem (1.2.4).

Theorem (1.2.4)[238]: Let arbitrary numbers $\beta_1 \dots . \beta_{min(p,q)}$be given, such that (60) and (61) hold. Then the conclusions of Theorem (1.2.3) aye valid.

Proof: The proof amounts to showing that, if (60) and (61) are valid for not necessarily decreasing numbers $\beta_1 \dots . \beta_{min(p,q)}$then (60) and (61) remain valid if $\beta_1 \dots . \beta_{min(p,q)}$are rearranged into decreasing order. More precisely, let σ be a permutation of 1, 2,..., $s = min(p,q)$ such that $\beta_{\sigma(1)} \geqq \beta_{\sigma(2)} \geqq \cdots \geqq \beta_{\sigma(s)}$. If $\sigma(i) \geqq i$we then have $\beta_{\sigma(i)} \leqq \alpha_{\sigma(i)} \leqq \alpha_i$ if $\sigma(i) < i$ then for some j $< i$ we have $\sigma(j) \geqq i$ and hance $\beta_{\sigma(i)} \leqq \beta_{\sigma(j)} \leqq \alpha_{\sigma(i)}$. thus $\beta_{\sigma(i)} \leqq \alpha_i$holds for all i. similarly, for $i \leqq min(p+q-m, p+q-n)$,if $\sigma(i) \leqq i$then $\beta_{\sigma(i)} \geqq \beta_i \geqq \alpha_{i+m-p+n-q.}$ if $\sigma(i) > i$, then for some $j > i$ we have $\sigma(j) \leqq i$ But then $\beta_{\sigma(i)} \geqq \beta_{\sigma(j)} \geqq \alpha_{\sigma(j)+m-p+n-q} \geqq \alpha_{i+m-p+n-q}$ thus $\beta_{\sigma(j)} \geqq \alpha_{i+m-p+n-q}$ For all $i \leqq min(p+q-m, p+q-n)$.

For the next theorems we let Q_{mp} denote the totality of $\binom{m}{p}$sequences $\omega = \{i_1, \dots, i_p\}$ of integers for which $1 \leqq i_1 < \cdots < i_p \leqq m$and we let Q_{na} denote the totality of sequences$r = \{j_1, \dots, j_p\}$ of integers for which $1 \leqq j_1 < \cdots < j_p \leqq n$. We let

$$A\ [\omega|\tau] = A\left[i_1, \dots, i_p \middle| j_1, \dots, j_p\right] \quad (74)$$

be *the* $p \times q$ submatrix of A at the intersection of the rows w and the olumns τ, and we let

$$\beta_{\omega\tau.1} \geqq \beta_{\omega\tau.2} \geqq \cdots \geqq \beta_{\omega\tau.nim(p,q)}$$

be the singular values of (74). As before, we let $\beta_{\omega\tau.t} = 0$for $t > min(p,q)$.
Theorem (1.2.5)[238]: Define rational numbers $\varphi_0, \dots, \varphi_{m-p}$, and $\psi_0, \dots, \psi_{n-q}$by the polynomial identities

$$\prod_{i=p}^{m-1}\left(\frac{\lambda+i}{1+i}\right) = \sum_{t=0}^{m-p} \varphi_t \lambda^{m-p-t} \qquad (75)$$

$$\prod_{i=q}^{n-1}\left(\frac{\lambda+i}{1+i}\right) = \sum_{t=0}^{n-p} \psi_t \lambda^{n-p-t} \qquad (76)$$

For $i \leqq min(p,q)$, define rational numbers $d_0, \dots, d_{min(m+n-p-q,n-i)}$, and $d'_0, \dots, d'_{min(m+n-p-q,n-i)}$ (depending on i) by the polynomial identities

$$\left(\sum_{r=0}^{\min(m-p,q-i)} \varphi_t \lambda^{n-p-r}\right)\left(\sum_{s=0}^{n-q} \psi_s \lambda^{n-q-s}\right) = \sum_{\rho=0}^{min(m+n-p-q,n-i)} d_\rho \lambda^{m+n-p-q-\rho} \qquad (77)$$

and

$$\left(\sum_{r=0}^{min(m-p,q-i)} \varphi_{m-p-r} \lambda^{n-p-r}\right)\left(\sum_{s=0}^{n-q} \psi_{n-q-s} \lambda^{n-q-s}\right) = \sum_{\rho=0}^{min(m+n-p-q,n-i)} d'_\rho \lambda^{m+n-p-q-\rho} \qquad (78)$$

then

$$\sum_{\rho=0}^{min(m+n-p-q,n-i)} d_\rho \alpha_{i+\rho}^2 \leqq \frac{1}{\binom{m}{p}}\frac{1}{\binom{n}{q}} \sum_{\omega\in Q_{mp}\tau\in Q_{nq}} \beta_{\omega\tau,i}^2 \leqq \sum_{\rho=0}^{min(m+n-p-q,n-i)} d'_\rho \alpha_{i+\rho}^2 \qquad (79)$$

Proof: Let $\mathrm{X}_\tau = \mathrm{AV}[1, \dots, \mathrm{n}: \mathrm{j}_1, \dots, \mathrm{j}_\mathrm{q}]$ and let

$$x_{\tau,1}^2 \geqq x_{\tau,2}^2 \geqq \cdots \geqq x_{\tau,\min(p,q)+1}^2 = \cdots = x_{\tau,m}^2 (= 0)$$

be the roots of $X_\tau X_\tau^*$,. Then by (66) we have

$$x_{\tau,1}^2 \geqq \beta_{\omega\tau,i}^2 \geqq x_{\tau,i+m-p}^2 \qquad 1 \leqq i \leqq p.$$

Using [231], we see that, for $i \leqq p$ (and so for $i \leqq min(p,q)$),

$$\sum_{\tau=0}^{m-p} \varphi_\tau x_{r,i+r}^2 \leqq \frac{1}{\binom{m}{p}} \sum_{\omega\in Q\tau\in Q} \beta_{\omega\tau,i}^2 \leqq \sum_{r=0}^{m-p} \varphi_{m-p-r}\, x_{r,i+r.}^2$$

Since $x_{\tau,i+r}$ whenever $i + r > min(m,q)$, we get

$$\sum_{s=0}^{n-p} \psi_s \alpha_{i+s}^2 \leqq \frac{1}{\binom{n}{p}} \sum_{\omega\in Q_{nq}} x_{\tau,i}^2 \leqq \sum_{s=0}^{n-q} \psi_{n-q-s}\, x_{\tau,i+r.}^2 \qquad (80)$$

By (68), $\alpha_i \geqq x_{\tau,i}^2 \geqq \alpha_{i+n-q}^2$ for $1 \leqq i \leqq q$, and hence, by [231,304],

$$\sum_{s=0}^{n-q} \psi_s \alpha_{i+s}^2 \leqq \frac{1}{\binom{n}{q}} \sum_{\omega \in Q_{nq}} x_{r,i}^2 \leqq \sum_{s=0}^{n-q} \psi_{n-q-s}\, x_{i+s,}^2 \tag{81}$$

For $i \leqq q$.

Summing (80) over -c and dividing by $\binom{n}{q}$, , upon using (81) we obtain

$$\sum_{\tau=0}^{min(m-p,q-i)} \varphi_\tau \sum_{s=0}^{n-q} \psi_s \alpha_{i+r+s}^2 \leqq \frac{1}{\binom{m}{p}} \frac{1}{\binom{n}{q}} \sum_{\omega \in Q_{mp} \tau \in Q_{nq}} \beta_{\omega\tau,i}^2$$

$$\leqq \sum_{\tau=0}^{\min(m-p,q-i)} \varphi_{m-p-\tau} \sum_{s=0}^{n-q} \psi_{sn-p-s} \alpha_{i+\tau+s}^2 \tag{82}$$

For $i \leqq min(p,q)$.

On the left side of (82), the coefficient of α_{i+p}^2, is

$$\sum_{r=0,r+s=\rho}^{min(m-p,q-i)} \sum_{s=0,r+s=\rho}^{n-q} \varphi_r \psi_s$$

For $0 \leqq \rho \leqq min(m+n-p-q, n-i)$.

However,

$$d_\rho = \sum_{r=0,r+s=\rho}^{min(m-p,q-i)} \sum_{s=0,r+s=\rho}^{n-q} \varphi_r \psi_s$$

for $0 \leqq \rho \leqq min(m+n-p-q, n-i)$.

Thus the lower bound in (79) is established.

On the right side of (82), the coefficient of α_{i+p}^2, is

$$\sum_{\tau=0,\tau+s=\rho}^{\min(m-p,q-i)} \sum_{s=0,\tau+s=\rho}^{n-q} \varphi_{m-p-\tau} \psi_{n-q-s}$$

for0 $\leqq \rho \leqq min(m+n-p-q, n-i)$.

However,

$$d'_\rho = \sum_{\tau=0,\tau+s=\rho}^{min(m-p,q-i)} \sum_{s=0,r+s=\rho}^{n-q} \varphi_{m-p-\tau} \psi_{n-q-s}$$

for0 $\leqq \rho \leqq min(m+n-p-q, n-i)$.

The result is now at hand.

If p and q are large and i is small, so that $\min(m+n-p-q.n-i) = m + n-p-q$, formulas (79) provide convex combinations of $\alpha_i^2, \ldots, \alpha_{i+m-p-n-q.}^2$ which serve as upper and lower bounds for the mean of the $\beta_{\omega\tau,i}^2$ Thus Theorem (1.2.5) provides a result sharper than can be established by applying Theorem (1.2.2), since Theorem (1.2.2) only asserts that the

$\beta^2_{\omega\tau,i}$between $\alpha_i^2$and $\alpha^2_{i+m-p-n-q.}$. To see that in fact we have convex combinations, notice that (for these values $of\ p,q,i$),

$$\sum_{\rho=0}^{m+n-p-q} d_\rho = \sum_{\rho=0}^{m+n-p-q} \sum_{s=0,\tau+s=\rho}^{m-q} \sum_{s=0,r+s=\rho}^{n-q} \varphi_\tau \psi_s = \sum_{s=0}^{m-p} \sum_{s=0}^{n-q} \psi_s = 1,$$

since

$$\sum_{\tau=0}^{m-p} \varphi_\tau = 1 = \sum_{s=0}^{n-q} \psi_s.$$

Similarly

$$\sum_{\rho=0}^{m-n-p-q} d'_\rho = \sum_{\rho=0}^{m-n-p-q} \sum_{r=0,r+s=\rho}^{m-p} \sum_{s=0,r+s=\rho}^{n-q} \varphi_{m-p-r}\psi_{n-q-s}$$
$$= \sum_{\tau=0,\tau+s=\rho}^{m-p} \sum_{s=0,\tau+s=\rho}^{n-q} \varphi_{m-p-\tau}\psi_{n-q-s} = 1$$

When p and q are small and i large, so that $min(m-n-p-q, n-i) = n-i$, formula (79) may be regarded as providing subconvex combinations of $\alpha_i^2, \dots, \alpha_{n.}^2$ (convex combinations of $\alpha_i^2, \dots, \alpha_{n.}^2, 0$) which serve as bounds for the mean of the $\beta^2_{\omega\tau,i}$.

Theroem(1.2.6)[238]:. Let

$$f_{\omega,\tau}(\lambda) = \left(\lambda - \beta_{\omega,\tau,i}\right) \dots \left(\lambda - \beta_{\omega,\tau,min(p,q)}\right),$$
$$f_{\omega,\tau}(\lambda) = (\lambda - \alpha_{1.}^2) \dots \left(\lambda - \alpha^2_{min(m,n).}\right)$$

Then

$$\sum_{\omega\in Q_{mp}\tau\in Q_{nq}} \lambda^{p-min(p,q)} f_{\omega,\tau}(\lambda)$$
$$= \frac{1}{(m-q)!}\frac{1}{(n-q)!}\frac{d^{m-p}}{d\lambda^{m-q}}\lambda^{m-q}\frac{d^{n-q}}{d\lambda^{n-q}}\lambda^{n-min(p,q)}f(\lambda) \qquad (83)$$

Proof: Since the matrices $\beta_{\omega,\tau}\beta^*_{\omega,\tau}$ are $p\times p$ principal submatrices of the $m\times m$ matrix $X_\tau X_\tau^*$, we find (see [231,33]) that

$$\sum_{\omega\in Q_{mp}\tau\in Q_{nq}} (\lambda - \beta^*_{\omega\tau.1}) \dots \left(\lambda - \beta^2_{\omega\tau,min(p,q)}\right)\lambda^{m-min(m,q).}$$
$$= \frac{1}{(n-q)!}\frac{d^{n-p}}{d\lambda^{n-q}}(\lambda - \alpha_1^2) \dots \left(\lambda - \alpha^2_{min(m,q)}\right)\lambda^{m-min(m,n).}$$

Since $X_\tau^* X_\tau$, is a principal $q\times q$ submatrix of the $n\ x\ n$ matrix A^*A, we have

$$\sum_{\tau\in Q_{nq}} (\lambda - x^2_{\tau.1}) \dots \left(\lambda - x^2_{\tau,min(p,q)}\right)\lambda^{\rho-min(m,q).}$$
$$= \frac{1}{(n-q)!}\frac{d^{n-p}}{d\lambda^{n-q}}(\lambda - \alpha_1^2) \dots \left(\lambda - \alpha^2_{min(m,n)}\right)\lambda^{n-min(m,n).}$$

Thus

$$\sum_{\tau\in Q_{nq}} f_{\omega,\tau}(\lambda)\lambda^{\rho-min(p,q).}$$

$$= \frac{1}{(n-q)!}\frac{d^{m-p}}{d\lambda^{m-q}}\lambda^{m-\rho}\sum_{\tau\in Q_{nq}}(\lambda - x_{\tau.1}^2)\ldots\left(\lambda - x_{\tau,min(m,q)}^2\right)\lambda^{\rho-min(m,q).}$$

$$= \frac{1}{(m-q)!}\frac{1}{(n-q)!}\frac{d^{m-p}}{d\lambda^{n-q}}\lambda^{m-\rho}\frac{d^{n-p}}{d\lambda^{n-q}}\lambda^{m-min(m,n)f(\lambda).}$$

The proof is complete.

We now give the promised second proof of Theorem (1.2.2). For any $m \times n$ matrix A with singular values $\alpha_1 \geqq \cdots \geqq \alpha_{min(m,n)}$the roots of the $(m + n)$ −square Hermitian matrix

$$M = \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}$$

Are $\pm\alpha_1, \ldots, \pm\alpha_{\min(m,n)}$,0 (with multiplicity $m + n - 2\ min\ (m,n)$). to see this, observe that

$$\det(\lambda 1_{m+n} - M) = det\begin{bmatrix} 1_m & \lambda^{-1}A \\ 0 & I_n \end{bmatrix} de\begin{bmatrix} \lambda I_n & -A \\ -A^* & \lambda I_n \end{bmatrix}$$

$$= det\begin{bmatrix} \lambda I_m - \lambda^{-1}AA^* & 0 \\ -A^* & \lambda I_n \end{bmatrix}$$

$$= \lambda^n det(\lambda 1_m - \lambda^{-1}AA^*) = \lambda^{n-m}det(\lambda^2 1_m - AA^*)$$

The principle $(p+q)$square submatrix of M, obtained by deleting all rows and columns except rows and columns $i_1, \ldots, i_p, m + j_1, \ldots, m{+}j_q$, is

$$\begin{bmatrix} 0 & \mathrm{A}\left[\mathrm{i}_1, \ldots, \mathrm{i}_p : \mathrm{j}_1, \ldots, \mathrm{j}_\mathrm{q}\right] \\ \mathrm{A}\left[\mathrm{i}_1, \ldots, \mathrm{i}_p : \mathrm{j}_1, \ldots, \mathrm{j}_\mathrm{q}\right]^* & 0 \end{bmatrix}.$$

Using the inequalities connecting the eigenvalues of a $(p + q)$ −principal submatrix of Hermitian matrix M with the eigenvalues of M, we obtain the inequalities (60) and (61).

Chapter 2
Interlacing Inequalities for Singular Values of Submatrices

We provide a complete noncommutative analog of the famous cycle of theorems characterizing the function theoretic generalizations of H^{∞}. A sample of our other results: we prove a Kaplansky density result for a large class of these algebras, and give a necessary condition for when every completely contractive homomorphism on a unital subalgebra of a C*-algebra possesses a unique completely positive extension. As an application, we solve the longstanding open problem concerning the noncommutative generalization, to Arveson's noncommutative H^p spaces, of the famous 'outer factorization' of functions f with log |f| integrable. Using the Fuglede-Kadison determinant, we also generalize many other classical results concerning outer functions.

Section (2.1) Unique Extensions

Function algebras are subalgebras of $C(K)$-spaces, or equivalently, subalgebras of commutative C^* −algebras. Thus function algebras are examples of operator algebras (subalgebras of general C^*-algebras). Much work has been done to transfer results or perspectives from function theory to operator algebraic settings. One such setting, is the theory of noncommutative H^p spaces associated with Arveson's maximal subdiagonal subalgebras of finite von Neumann algebras. Many of the central results from abstract analytic function theory, and in particular much of the classical generalized H^{p} function theory, may be generalized almost verbatim to subdiagonal algebras. The proofs in the noncommutative case however, while often modeled loosely on the 'commutative' arguments of Helson and Lowdenslager [100,13,79,32] and others, usually require substantial input from the theory of von Neumann algebras and noncommutative L^p-spaces, see [300,177, 153, 176, 279, 163, 58, 56]. In fact in many cases like Szegö's theorem – completely new proofs have had to be invented. We tackle what appears to us to be the main 'classical' results which have resisted generalization to date, namely those referred to in the generalized function theory literature from the 1960's as, respectively, the F. and M. Riesz, Gleason and Whitney, Szegö L^{p}, and Kolmogorov, theorems. We the following statement: essentially all of the generalized H^p function theory as summarized in [282] for example, extends further to the setting of subdiagonal algebras.

In Arveson's setting, we have a weak*-closed unital subalgebra A of a von Neumann algebra M possessing a faithful normal tracial state τ, such that if Φ is the unique conditional expectation from M onto $\mathcal{D} = A \cap A^*$ satisfying $\tau = \tau \circ \Phi$, then Φ is a homomorphism on A. Take note that here A^* denotes the set $\{a: a^* \in A\}$ and not the Banach dual of A. For the sake of clarity we will write X^* for the Banach dual of a normed space X. We say that a subalgebra A of the type described above is a tracial subalgebra of M. If in addition $A + A^*$ is weak* dense in M then we say that A is maximal subdiagonal (see [300, 240]). A large number of very interesting examples of these objects were given by Arveson [300], and others (see e.g. [161, 178]). If $\mathcal{D}$ is one dimensional we say that A is antisymmetric; if further M is commutative then A is called a weak* Dirichlet algebra [282]. For antisymmetric maximal subdiagonal algebras, many of the 'commutative' proofs from [282] require almost no change at all. It is worth saying that classical notions of 'analyticity' correspond in some very vague sense to the case that $\mathcal{D}$ is 'small'. Indeed if $A = M$ then $\mathcal{D} = M$ and Φ is the identity map, so that the theory essentially collapses to the theory of finite von Neumann algebras, which clearly is far removed from classical concepts of 'analyticity. Indeed for our $F.$ and $M.$ Riesz theorem to hold, we show that it is necessary and sufficient for $\mathcal{D}$ to be finite dimensional. Because of this, in our several applications of this theorem we assume $dim(\mathcal{D}) < \infty$.

A subsidiary theme is 'unique extensions' of maps on A. We begin with some results on this topic. Recall from [58] that a subalgebra A of M has the unique normal state extension property if there is a unique normal state on M extending τ|A. If, on the other hand, for every state ω of M with $\omega \circ \Phi = \omega$ on A, we always have that $\omega \circ \Phi = \omega$ on M, then we say that A has the Φ -state property. The major unresolved question in [58] was whether a tracial subalgebra with the unique normal state extension property is maximal subdiagonal. We make what we feel is substantial progress on this question. In particular, we show that the question is equivalent to the question of whether every tracial subalgebra with the Φ-state property is maximal subdiagonal, and equivalent to whether every tracial subalgebra satisfying a certain variant of the well known 'factorization' property actually has 'factorization'.We also give an interesting necessary condition for when completely contractive homomorphisms possess a unique completely positive extension. Our unique extension results play a role in the proof of our $F.$ and $M.$ Riesz theorem, and are the primary thrust of the Gleason-Whitney theorem. We prove our SzegöL^p formula, and generalized Kolmogorov theorem.

The first noncommutative $F.$ and $M.$ Riesz theorem for subdiagonal algebras was the pretty theorem of Exel in [241]. This result assumes norm density of $A + A^*$, and antisymmetry. (We are aware of the $F.$ and $M.$ Riesz theorem of Arveson [301,9,54] and Zsido's extension there of [161,259,4,8,85,160,17,284], but this result is quite distinct from the ones discussed above.) Although some of the steps of our proof parallel those of [241,204,99,19,12], the arguments are for the most part quite different. Indeed generally the proofs will be modeled on the classical ones, but do however require some rather delicate additional machinery.

We remark that there are other, more recent, noncommutative variants of H^∞ besides the subdiagonal algebras see e.g. [90,5]. Here too one finds noncommutative generalizations of classical H^p-theoretic results, such as the Szegö infimum theorem, these variants are in general quite unrelated, with only a formal correspondence.

For a functional $\omega \in M^\star$, we will need to compare the property $\omega = \omega \circ \Phi$ on A, with the property $\omega = \omega \circ \Phi$ on M. On this topic we begin with the following remarks. It is easy to see, since Φ is idempotent, that $\omega = \omega \circ \Phi$ on A iff $A_0 \subset Ker(\omega)$. Here, $A_0 = A \cap Ker(\omega)$, a closed two-sided ideal in A. For normal functionals one can say more, although this will not play an important role for us. If $f \in L_1(M)$ let $\omega_f = \tau(f\ \cdot)$. From the last paragraph, $\omega_f = \omega_f \circ \Phi$ on A ff $\tau(fA_0) = (0)$. On the other hand, $\omega_f = \omega_f \circ \Phi$on M iff $\tau(fa) = \tau(f\Phi(a)) = \tau(\Phi(f)a)$ for all $a \in M$ iff $f = \Phi(f)$ iff $f \in L^1(\mathcal{D})$.

Proposition (2.1.1)[61]: If A is a tracial subalgebra of M then the unique normal state extension property is equivalent to the following property: whenever ω is a normal state of M satisfying $\omega = \omega \circ \Phi$ on A, then $\omega = \omega \circ \Phi$ on M.

Proof: Suppose that A has the unique normal state extension property, and suppose that ω is a normal state of M satisfying $\omega = \omega \circ \Phi$ on A. If $\omega = \tau(f\ \cdot)$, where $f \in L^1(M)_+$, then by the remarks preceding Proposition (2.1.1) we have that $\tau(fA_0) = (0)$. Hence $f \in L^1(D)$ by [58,307,11]. Hence $\omega = \omega \circ \Phi$ on M.

For the converse, note that if $g \in L^1(M)_+$with$\tau = \tau(g\ \cdot\ )$ on A, then since $\tau = \tau \circ \Phi$, we have that $\tau(g.) = \tau(g\ \cdot\ ) \circ \Phi$ on A, and hence that $\tau(g.) = \tau(g.) \circ \Phi$ on M. By the remarks above, $g \in L^1(\mathcal{D})_+$. But then the fact that $\tau = \tau(g\ \cdot)$ on D is enough to force $g = \mathbb{I}$. So A has the unique normal state extension property.

We say that a subalgebra A of M has factorization if given $b \in M^+ \cap M^{-1}$ we can find $a \in A^{-1}$ with $b = a^*a$(or equivalently $b = aa^*$). It is shown in [300,310,18] that any maximal subdiagonal algebra has factorization. Thus it is logmodular, namely any such b is a limit of terms of the form a^*a with $a \in A^{-1}$. In fact, in the category of tracial algebras factorization or logmodularity are equivalent to maximal subdiagonality [58]. By the next result such algebras satisfy a formally much stronger property than that of the last proposition:

Theorem (2.1.2)[61]: Let A be a logmodular subalgebra of a C^*-algebra M, and let ψ be a positive contractive projection from M onto a subalgebra of A containing $\mathbb{I}_M$, which is a homomorphism on A. Then for any state ω of M, we have that $\omega = \omega \circ \psi$on M, whenever $\omega = \omega \circ \psi$ on A.

Proof: If $a \in A^{-1}$ then by hypothesis we have

$$\omega(\psi(a)a^{-1}) = \omega\big(\psi(\psi(a)a^{-1})\big) = \omega\big(\psi(a)\psi(a^{-1})\big) = \omega(\mathbb{I}) = 1$$

By the Cauchy-Schwarz and Kadison-Schwarz inequality we deduce:

$$1 \leq \omega(\psi(a)\psi(a)^*)\,\omega((a^{-1})^*a^{-1}) \leq \omega(\psi(aa^*))\,\omega((a^{-1})^*a^{-1}) = \omega((aa^*))\,\omega((aa^*)^{-1}).$$

We can now follow the proof of [56] or [61,227]. Since A is logmodular, for any $b \in M^{-1} \cap M^+$ we have that $1 \leq \omega\big(\psi(b)\big)\omega(b^{-1})$. This leads to the equation $1 \leq \omega(\psi(e^{tu}))\,\omega(e^{-tu}) = f(t)$, for$u \in M_{sa}$. Differentiating and noting that $f'(0) = 0$, yields $\omega(u) = \omega(\psi(u))$ as required.

When applied to tracial algebras and their associated canonical conditional expectations, the preceding result still holds under a formally weaker hypothesis. Specifically we say that a tracial subalgebra A of M with canonical conditional expectation Φ has conditional factorization if given any $b \in M^+ \cap M^{-1}$, we have $b = |a|$for some element $a \in A \cap M^{-1}$ with $\Phi(a)\Phi(a^{-1}) = 1$.

Corollary (2.1.3)[61]:. A tracial subalgebra of M with conditional factorization has the Φ −state property.

We say that A has the unique state extension property if if there is a unique state on M extending $\tau|A$. This is a formally weaker property than the Φ-state property:

Proposition (2.1.4)[61]: Let A be a weak* closed unital subalgebra of M. If A has the Φ-state property then it has the unique state extension property. The converse is true if A is antisymmetric.

Proof: Suppose that ω is a state of M extending τ|A. Then $\omega \circ \Phi = \tau \circ \Phi = \tau = \omega$ on A. By the Φ-state property, on M we have $\omega = \omega \circ \Phi = \tau \circ \Phi = \tau$. For the converse we need only note that if A is antisymmetric, then $\omega \circ \Phi = \omega$ on A forces $\tau = \omega$ on A.

Corollary (2.1.5)[61]: Suppose that A is a tracial subalgebra of M with the unique normal state extension property. Then $A_\infty = M \cap [A]_2$is a tracial subalgebra with the Φ-state property.
Proof: First note that by [58], $A_\infty$is a tracial subalgebra of M with respect to the same Φ and τ. By [58], A_∞has conditional factorization. Corollary (2.1.3) now gives the conclusion.
Corollary (2.1.6)[61]: The open question from [58] as to whether every tracial subalgebra with the unique normal state extension property is maximal subdiagonal, is equivalent to the question of whether every tracial subalgebra with the Φ-state property is maximal subdiagonal. It is also equivalent to whether every tracial subalgebra with the unique state extension property is maximal subdiagonal. It is also equivalent to whether every tracial subalgebra with conditional factorization has factorization.
Proof: Suppose that every tracial subalgebra with the Φ-state property is maximal subdiagonal, and suppose that A has the unique normal state extension property. By Corollary (2.1.5), A_∞ has the Φ -state property.Henceit is maximal subdiagonal, and therefore satisfies L^2-density. Consequently A satisfies L^2-density, and so A is maximal subdiagonal by [58].

Similarly, suppose that every tracial subalgebra with conditional factorization has factorization, and suppose that A has the Φ -state property. By results above, A has the unique normal state extension property, and so by [58], A_∞ has conditional factorization. By hypothesis, A_∞ has factorization. Thus it is maximal subdiagonal by[4], and thus as in the last paragraph A is maximal subdiagonal.
In [86], Lumer considered the property of 'uniqueness of representing measure', namely the property that every multiplicative functional on $A \subset C(K)$ has a unique extension to a state on $C(K)$, He showed how this condition could be used as another possible axiom from which all the generalized H^p theory may be derived. The natural noncommutative generalization of Lumer's property, is that every completely contractive representation of A has a unique completely positive extension to$M$. It is known that maximal subdiagonal algebras have this property [60, 55]. Although we have not settled the converse yet, we can say that every unital subalgebra of M which has this property must in some sense be a large subalgebra of M. The following result represents some sort of converse to many of the preceding results which established various unique extension properties as a consequence of maximal subdiagonality.

In the following result we use the C^*-envelope C_e^* *(A)* of an operator algebra A. See e.g. [56,16,85,131,160] for the definition of this, and for its universal property.
Theorem (2.1.7)[61]: Suppose that A is a subalgebra of a unital C^*-algebra B such that $\mathbb{I}_B \in A$, and suppose that A has the property that for every Hilbert space H, every completely contractive unital homomorphism π: A → B(H) has a unique completely contractive (or equiv. completely positive) extension $B \to$*B(H*). Then $B = C_e^*\ (A)$, the C∗-envelope of A.
Proof: (i) (The case that A is a $C*$-subalgebra of B.) In this case, since contractive homomorphisms on $C*$-algebras are $*$-homomorphisms (see e.g. [56]), we must prove that if every unital $*$-homomorphism $\pi: A \to B(H)$ has a unique completely contractive extension $B \to B(H)$, then $A = B$. To see this, let $\rho: B \to B(H)$ be the universal representation of B. Thenρ is unital, and hence so is $\pi = \rho|A$. Let U be a unitary in $(A)'$. Then since $U^*\rho(.)U = \rho$ on A, we have by hypothesis that $U^*\rho(\cdot)U = \rho$ on B, and thus $U \in \rho(B)'$. Thus $\pi(A)' = \rho(B)'$, and it follows that $\pi(A)'' = \rho(B)''$. If $\tilde{\rho}$ is the unique normal extension of ρ

to B^{**}, then $\tilde{\rho}$ is faithful on B^{**} and it has range $\rho(B)''$. The restriction of $\tilde{\rho}$ to the copy $A^{\perp\perp}$of A^{**}inside B^{**} has range $\pi(A)'' = \overline{\pi(A)}^{w^*}$, and is therefore surjective. This forces the copy of A^{**} inside B^{**} to be all of B^{**}. Thus $A = B \cap A^{\perp\perp} = B$.

(ii) (The general case.) Let $C = C^*(A)$, the C^*-algebra generated by A in B. Since $A \subset C$, it follows from the hypothesis that every unital $*$-homomorphism $\pi: C \to B(H)$ has a unique completely contractive extension $B \to B(H)$. By (i),$C = B$.

By virtue of this fact, we need only prove that $C^*(A) = C_e^*(A)$ under the assumptions of the theorem. By the universal property of $C_e^*(A)$, there is a $*$-epimorphism $\theta: B = C^*(A) \to C_e^*(A)$ restricting to the 'identity map' on A. If $B \subset B(H)$ then the canonical map from the copy of A in $C_e^*(A)$, to $A \subset B(H)$, has a completely positive extension $\Phi: C_e^*(A) \to B(H)$. On A, the map $\Phi \circ \theta$ is the identity map, so that by hypothesis $\Phi \circ \theta = i_B$. Thus θ is one-to-one, and hence $C^*(A)$is a C^*-envelope of A.

Corollary (2.1.8)[61]: Suppose that A is a tracial subalgebra of $M$with the property that for every Hilbert space H, every completely contractive unital homomorphism $\pi: A \to B(H)$ has a unique completely contractive (or equiv. completely positive) extension $B \to B(H)$. Then A generates M as a C^*-algebra. Indeed, M is a C^*-envelope of A.

The classical form of the F. and M. Riesz theorem (see e.g. [147]) is known to fail for weak* Dirichlet algebras; and hence it will fail for subdiagonal algebras too. However there is an equivalent version of the theorem which is true for weak* Dirichlet algebras[146, 282], and we will focus on this variant here. We shall say that a tracial subalgebra A of M has the F &M Riesz property if for every bounded function ρ on M which annihilates A_0, the normal and singular partsρ_n and ρ_s annihilate A_0 and A respectively. During our investigation we shall have occasion to make use of the polar decomposition of normal functionals on a von Neumann algebra. We take the opportunity to point out that for our purposes we shall assume such a polar decomposition to be of the form $\omega(a) = |\omega|(ua)$ for some partial isometry, rather than$\omega(a) = |\omega|(au)$ which seems to be more common among the proponents of noncommutative L^p-spaces.

The following result shows that to study the F & M Riesz property, we may restrict our attention to algebras for which the diagonal $\mathcal{D}$ is finite dimensional:

Proposition (2.1.9)[61]: If a tracial subalgebra A of M satisfies the F & M Riesz property then the diagonal $D$is finite dimensional.

Proof: Let $\psi \in D^\star$. Then $\psi \in M^\star$ annihilates A_0. By the F&M Riesz property, $\psi \circ \Phi$ agrees with $(\psi \circ \Phi)$ on A, and so $\psi = \psi \circ \Phi|_D$ is weak* continuous on $\mathcal{D}$. Thus $\mathcal{D}$ is reflexive, and therefore finite dimensional.

Lemma 2.1.10)[61]: Let A be a maximal subdiagonal subalgebra of M. Let ω be a state of M, and let $(\pi_\omega, \mathfrak{h}_\omega, \Omega_\omega)$ be the GNS representation of M. Further, let ω be the orthogonal projection of ω onto the closed subspace $\overline{\pi_\omega(A_0)\Omega_\omega}$.

(a) The following holds:

(i) There exists a central projection p_0 in $\pi_\omega(M)''$such that for any $\xi, \psi \in \mathfrak{h}_\omega$the functionals $a \to \pi_\omega(a)p_0\xi, \psi$ and $a \to \pi_\omega(\mathbb{I} - p_0)\xi, \psi$ on M are respectively the normal and singular parts of the functional $a \to \pi_\omega(a)p_0\xi, \psi$. In particular, the triples

$(p_0\pi_\omega, p_0\mathfrak{h}_\omega, p_0\Omega_\omega)$ and $((\mathbb{1}-p_0)\,\pi_\omega, (\mathbb{1}-p_0)h\omega, (\mathbb{1}-p_0)\Omega_\omega)$ are copies of the GNS representations of ω_n and ω_s respectively.

(ii) $\omega_0: a \to \langle \pi_\omega(a)(\Omega_\omega - \Omega_0), \Omega_\omega - \Omega_0\rangle$ defines a positive functional of M satisfying $\omega_0 = \omega_0 \circ \Phi$.

(b) Suppose that in addition $dim(\mathcal{D}) < \infty$.

(i) Then ω_0 is a normal functional of the form $\omega_0 = \tau(g^{1/2} \cdot g^{1/2})$ for some $g \in D_+$. Moreover $p_0(\Omega_\omega - \Omega_0) = \Omega_\omega - \Omega_0$, and $p_0\Omega_\omega$ is the orthogonal projection of $p_0\Omega_\omega$ onto $\overline{p_0(\pi_\omega(A_0)\Omega_\omega)}$.

(ii) If ω is singular, then for any $f \in D$ we have that $\pi_\omega(f)\Omega_\omega \in \overline{\pi_\omega(A_0)\Omega_\omega}$.

(c) Suppose that $dim(D) < \infty$ and $\Omega_\omega \notin \pi_\omega(A_0)\Omega_\omega$. If ω_0 is faithful on D, then there exists a sequence $\{a_n\} \subset A$ such that $\pi_\omega(a_n)(\Omega_\omega - \Omega_0) \to p_0\Omega_\omega$.

Proof: (a)(i): This is essentially the content of [180].

(a)(ii): Let $(\pi_\omega, \mathfrak{h}_\omega, \Omega_\omega)$ and Ω_0 be as in the hypothesis, and define a positive functional ω_0 on M by

$$\omega_0: a \to \langle \pi_\omega(a)(\Omega_\omega - \Omega_0), \Omega_\omega - \Omega_0\rangle.$$

Let $f \in A_0$ be given. By construction

$$\pi_\omega(f)\Omega_\omega \perp (\Omega_\omega - \Omega_0).$$

Since A_0 is an ideal, $\pi_\omega(fa)\Omega_\omega \in \pi_\omega(A_0)\Omega_\omega$ for each $a \in A_0$. Since A_0 belongs to $\pi_\omega(A_0)\Omega_\omega$, we may of course select a sequence $\{b_n\} \subset A_0$ for which $\pi_\omega(b_n)\Omega_\omega$convergeto Ω_0. Hence $\pi_\omega(fb_0)\Omega_\omega$ converges to $\pi_\omega(f)\Omega_0$. Thus $\pi_\omega(fb_0)\Omega_\omega \in \pi_\omega(f)\Omega_0$, which forces

$$\pi_\omega(f)\Omega_0 \perp (\Omega_\omega - \Omega_0).$$

From the previous two centered equations it is now clear that $A_0 \subset Ker(\omega_0)$. Thus $\omega_0 = \omega_0 \circ \Phi$ on A by the remarks preceding Proposition (2.1.1). Hence $\omega_0 = \omega_0 \circ \Phi$on M by Corollary (2.1.3).

(b) (i): Since D is finite dimensional, we can find $g \in D_+$ so that

$\omega_0(a) = \tau(ga)$ for all$a \in \mathcal{D}$.

Since $\omega_0 \circ \Phi = \omega_0$, we conclude that for any $a \in M$,

$$\omega_0(a) = \omega_0(\Phi(a)) = \tau(g\Phi(a)) = \tau(\Phi(ga)) = \tau(ga),$$

There by establishing the first part of the claim.

For the second part, note that since ω_0 is clearly normal, we have by part (a)(i) that

$$0 = \pi_\omega(a)(\mathbb{1} - p_0)(\Omega_\omega - \Omega_0), \Omega_\omega - \Omega_0 \text{ for all } a \in M.$$

For $a = \mathbb{1}$ this yields $0 = \|(\mathbb{1} - p_0)(\Omega_\omega - \Omega_0)\|$, or equivalently

$$p_0(\Omega_\omega - \Omega_0) = \Omega_\omega - \Omega_0.$$

From this fact, we may now conclude that

$$\langle p_0\pi_\omega(a)\Omega_\omega, p_0(\Omega_\omega - \Omega_0)\rangle = \pi_\omega(a)\Omega_\omega, \Omega_\omega - \Omega_0 = 0 \text{ for all } a \in A_0.$$

Thus $p_0(\Omega_\omega - \Omega_0) \perp p_0\pi_\omega(A_0)\Omega_\omega$. Now select a sequence $\{b_n\} \subset A_0$ so that $\pi_\omega(b_n)\Omega_\omega \to 0$. By continuity,

$$p_0\Omega_0 = lim_n\, p_0\pi_\omega(b_n)\Omega_\omega \in p_0\pi_\omega(A_0)\Omega_\omega.$$

From these considerations it is clear that $p_0\Omega_0$ is the orthogonal projection of $p_0\Omega_\omega$ onto $p_0\pi_\omega(A_0)\Omega_\omega$.

(b) (ii): If ω is singular, then

$$0 = \omega_n(ab) = \langle \pi_\omega(ab)p_0\Omega_\omega, \Omega_\omega\rangle = \langle\, p_0\pi_\omega(b_n)\Omega_\omega,\ \pi_\omega\,(a^*)\,\rangle$$

for all$a, b \in M$ Since Ω_ω is cyclic, this is sufficient to force $p_0 = 0$. But then $\Omega_\omega = \Omega_0 = p_0(\Omega_\omega - \Omega_0) = 0$by part (b)(i). As before select $\{b_n\} \subset A_0$ so that $\pi_\omega(b_n)\Omega_\omega \to \Omega_0 = \Omega_\omega$. For any $f \in D$ the ideal property of A_0 then ensures that $\pi_\omega(f)\Omega_\omega = lim_n \pi_\omega(fb_n)\Omega_\omega \in \pi_\omega(A_0)\Omega_\omega$.

(c): Suppose that ω_n, the normal part of ω, is of the form $\omega_n = \tau(h\ \cdot)$ for some $h \in L^1(M)_+$. As noted earlier, $(p_0\pi_\omega, p_0\mathfrak{h}_\omega, p_0\Omega_\omega)$ is a copy of the GNS representation engendered by ω_n. If now we compute the representation of ω_n from first principles, it is clear that $p_0\mathfrak{h}_\omega$corresponds to the weighted Hilbert space $L^2(M, h)$ obtained by equipping M with the inner product

$$\langle a, b\rangle = \tau(h^{1/2\sigma}b^*ah^{1/2}), \quad a, b \in M,$$

and taking the completion. Note that $L^2(M, h)$ can be identified unitarily, and as M modules, with the closure of $Mh^{1/2}$ in $L^2(M)$. For any a $\in M$ considered as an element of $L^2(M, h)$ we will write a instead of a. The canonical $*$-homomorphism representing M as an algebra of bounded operators on $L^2(M, h)$ is of course given by defining

$$\pi_n(b)\psi_a = \psi_{ab}\,, \qquad a, b \in M,$$

and then extending this action to all of $L^2(M, h)$. Since ω_n is normal, π_n (corresponding to $p_0\pi_\omega$) is σ-weakly continuous and satisfies $\pi_n(M) = \pi_n(M)''$. Thus $Ker(\pi_n)$ is σ -weakly closed two-sided ideal, and hence we can find a central projection $e \in M$ so that $(\mathbb{1} - e)M = ker(\pi_n)$. Restrict $\pi_n$to $a*$-isomorphism from eM onto $\pi_n(M)$. Then for any $a, b, c \in M$ we have

$$\langle \pi_n(c)\psi_a, \psi_b\rangle h = \tau(h^{1/2}b^*(ece)ah^{1/2})$$

Let $\psi^{(0)}$ denote the orthogonal projection of $\psi_{\mathbb{1}}$ onto the closure of $\{\psi_a: a \in A_0\}$.
(Note that $\psi_{\mathbb{1}}$ and $\psi^{(0)}$ of course correspond to $p_0\Omega_\omega$ and $p_0\Omega_0$ in parts (a) and (b) of the proof.) Since $L^2(M, h)$ may be viewed as a subspace of $L^2(M)$, let $F \in L^2(M)$be the element corresponding to $\psi^{(0)}$. It is easy to see that $eF = F$. From parts (a) and (b) we now have that

$$\omega_0 = \langle \pi_n(\cdot)(\psi_{\mathbb{1}} - \psi^{(0)}), \psi_{\mathbb{1}} - \psi^{(0)}\rangle_h = \tau\left(\left(h^{\frac{1}{2}}e - F^*\right).(h^{1/2}e - F)\right).$$

This in turn ensures that

$$|h^{\frac{1}{2}}e - F^*|^2 = g$$

where g is as in part (b). Thus $h^{1/2}e - F \in M$. Since by assumption ω_0 is faithful on $\mathcal{D}$, it follows that $Supp(g) = \mathbb{1}$. Since $\mathcal{D}$ is finite dimensional, g must be invertible. But then $h^{1/2}e - F$ must also be invertible, by the previous centered equation. (Recall that if ab is invertible in a finite von Neumann algebra then both a and b are invertible.) The polar decomposition of $h^{1/2}e - F^*$ is of the form $h^{1/2}e - F^* = ug^{1/2}$ for some unitary $u \in M$. From this it is clear that

$$(h^{\frac{1}{2}}e - F)^{-1} = ug^{-1/2}.$$

Clearly $h^{1/2}ug^{-1/2} \in L^2(M)$. Hence we may select $\{a_n\} \subset M$ converging in $L^2(M)$ to $h^{1/2}ug^{-1/2} = h^{1/2}(h^{1/}2e - F)^{-1}$.By the previously established correspondences we then have

$$\left\|\psi_{\mathbb{1}} - \pi_n(a_n)(\psi_{\mathbb{1}} - \psi^{(0)})\right\|_h = \tau(|h^{\frac{1}{2}}e - (a_ne)(h^{1/2}e - F)|^2)^{1/2}$$

$$\to \tau(|h^{1/2}e - h^{1/2}e|^2)^{1/2} = 0.$$

This implies, in the notation of parts (a) and (b), that $\pi_\omega(a_n)(\Omega_\omega - \Omega_0) \to p_0\Omega_\omega$. It remains to show that we may select $\{a_n\} \subset A$, or equivalently, that $h^{1/2}ug^{-1/2} \in [A]_2$. For this, it suffices by the L_2 density of $A + A^*$ to show that $h^{1/2}ug^{-1/2} \perp [A_0^*]_2$.So let $a \in A_0$ be given, and observe that

$$\tau\left(ah^{\frac{1}{2}}ug^{-\frac{1}{2}}\right) = \tau\left(g^{-1}ah^{\frac{1}{2}}ug^{-\frac{1}{2}}g\right) = \tau\left(g^{-1}ah^{\frac{1}{2}}ug^{-\frac{1}{2}}\right) = \tau\left(g^{-1}ah^{\frac{1}{2}}(h^{\frac{1}{2}}e - F^*\right)$$
$$= (h^{1/2}e - F^*)(\, g^{-1}ah^{1/2}) = \langle \psi_{g^{-1}a}, \psi_{\mathbb{1}} - \psi^{(0)} \rangle_h = 0$$

(The last equality follows from the ideal property of A_0 and the fact that $\psi_{\mathbb{1}} - \psi^{(0)}$ is orthogonal to $\{\psi_a : a \in A_0\}$.) The claim therefore follows.

Corollary (2.1.11)[61]: Let A be a maximal subdiagonal algebra with $dim(\mathcal{D}) < \infty$. The following are equivalent:

(i) A satisfies the $F\&M$ Riesz property.

(ii) Whenever ω annihilates A_0, the normal and singular parts ω_n and ω_s, will separately annihilate A_0.

(iii) Whenever ω annihilates A, the normal and singular parts, ω_n and ω_s, will separately annihilateA_0.

(iv) Whenever ω annihilates A, the normal and singular parts, ω_n and ω_s will separately annihilate A.

Proof: The implications (i) ⇒ (ii) ⇒ (iii) are clear. If (iii) holds, let ω be a bounded linear functional which annihilates A_0. Since Φ is a normal map onto $\mathcal{D}$, and $\mathcal{D}$ is finite dimensional, the functional defined by

$$\omega_{\mathcal{D}} = \omega|_{\mathcal{D}} \circ \Phi$$

is normal. Then $\rho = \omega - \omega_{\mathcal{D}}$ defines a functional which annihilates A. From (iii) we then have that ρ_n and ρ_s separately annihilate A_0. The normality of ω_D ensures that

$$\rho_n = \omega_n - \omega_{\mathcal{D}}\,, \rho_s = \omega_s.$$

Since by construction $\rho = \omega - \omega_{\mathcal{D}}$ annihilates A_0, we conclude that $\omega_n$and $\omega_s$separately annihilate A_0. This proves (ii). To prove the validity of (i), it remains to show that any singular functional ω which annihilates A_0, also annihilates $\mathcal{D}$. For such , the 'modulus' $|\omega|$ is still singular (see e.g. [174, 241], or the argument in the first part of the proof of the next theorem). Let $(\pi_\omega, \mathfrak{h}_\omega, \Omega_\omega)$ be the GNS representation of $|\omega|$. For each $a \in M$ we have $|\omega(a)|^2 \leq \|\omega\| |\omega|(a^*a)$. By a standard argument this implies that there exists a vector $\eta \in \mathfrak{y}_\omega$ such that

$$\omega(\cdot) = \langle \pi_\omega(\cdot)\Omega_\omega, \eta \rangle.$$

Let $d \in \mathcal{D}$ be given. By part (b)(ii) of Lemma (2.1.10) we may select a sequence $\{f_n\} \subset A_0$ so that $\pi_\omega(d)\Omega_\omega = \lim_n \pi_\omega(f_n)\Omega_\omega$. But then

$$\omega(d) = \langle \pi_\omega(d)\Omega_\omega, \eta \rangle = \lim_n \langle \pi_\omega(d)\Omega_\omega, \eta \rangle = \lim_n \omega(f_n) = 0$$

as required.

The equivalence with (iv) is now obvious.

Theorem (2.1.12)[61]: Let A be a maximal subdiagonal algebra. Then A satisfies the $F\&M$ Riesz property if and only if $dim(\mathcal{D}) < \infty$.

Proof: We proved the one direction in Proposition (2.1.9). For the other, let ω be a bounded linear functional on M which annihilates A_0, and let ω_n and ω_s be the normal and singular parts of ω. Write $\omega_n = \tau(h\,\cdot)$, for some $h \in L^1(M)$. We extend ω, ω_n, and ω_s, uniquely to normal functionals on the enveloping von Neumann algebra (the double commutant in the universal representation) and define $|\omega|, |\omega_n|$ and $|\omega_s|$, to be the absolute values of these extensions restricted to M. Then from for example ([173], cf. [240]) applied to ω and τ, we have that as functionals on M, $|\omega_n|$ and $|\omega_s|$ are respectively the normal and singular parts of $|\omega|$, and that $|\omega| = |\omega_n| + |\omega_s|$. We note from [134] that there is no danger of confusion as regards the absolute value of ω_n since the absolute value of ω_n as a functional on M and as a functional on the enveloping von Neumann algebra coincide on M. Now consider the positive functional ρ given by

$$\rho = \tau + |\omega|.$$

Let $(\pi_\rho, \mathfrak{y}_\rho, \Omega_\rho)$ be the GNS representation constructed from ρ, and define ρ_0 by $\rho_0(a) = \langle \pi_\rho(a)(\Omega_\rho - \Omega_0), \Omega_\rho - \Omega_0 \rangle$, where ρ_0 is the orthogonal projection of ρ onto the closure of $\{\pi_\rho(a)\Omega_\rho : a \in A_0\}$. For any $f \in A_0$ and any $d \in \mathcal{D}^+$, we have by construction.That

$$\left\|\pi_\rho(d^{1/2})(\Omega_\rho - \pi_\rho(f)\Omega_\rho)\right\|^2 = \rho(\left|d^{\frac{1}{2}}(\mathbb{I} - f)\right|^2) \geq \tau(\left|d^{\frac{1}{2}}(\mathbb{I} - f)\right|^2)$$
$$= \tau(d - df - f^*d + |d^{1/2}f|^2) = \tau(d + |d^{1/2}f|^2) \geq \tau(d).$$

On selecting a sequence $\{f_n\} \subset A_0$ so that $\pi_\rho(f)\Omega_\rho \to 0$, it follows that $\rho_0(d) = \left\|\pi_\rho(d^{1/2})(\Omega_\rho - \Omega_0)\right\|^2 \geq \tau(d)$. Hence ρ_0 is faithful on $\mathcal{D}$, and $\Omega_\rho \neq \Omega_0$. Thus we may apply all of Lemm(2.1.10) to $(\pi_0, \mathfrak{h}_0, \Omega_0)$.

Next notice that for each a in the enveloping von Neumann algebra we have

$$|\omega(a)|^2 \leq \|\omega\| |\omega|(a^*a) \leq \|\omega\| \rho(a^*a).$$

Thus on restricting to elements of M, and employing a standard argument, this implies that there exists a vector $\eta \in \pi_\rho$ such that

$$\omega(\cdot) = \langle \pi_\rho(\cdot)\Omega_\rho, \eta \rangle.$$

Now consider the related functional

$$\widetilde{\omega}(\cdot) = \langle \pi_\rho(\cdot)(\Omega_\rho - \Omega_0), \eta \rangle.$$

Select a sequence $\{f_n \subset A_0$ so that $\pi_\rho(f_n)\Omega_0 \to \Omega_0$. Let $a \in A_0$ be given. Since A_0 is an ideal, and since ω annihilates A_0, we conclude that

$$\widetilde{\omega}(a) = \langle \pi_\rho(a)(\Omega_\rho - \Omega_0), \eta \rangle$$
$$= \lim_n \langle \pi_\rho(a(\mathbb{I} - f_n))(\Omega_\rho, \eta \rangle = \lim_n \omega(a(\mathbb{I} - f_n)) = 0.$$

Thus $\widetilde{\omega}$ also annihilates A_0.

By part (c) of the Lemma we can find a sequence $\{a_n\} \subset A$ such that $\pi_\rho(a_n)((\Omega_\rho - (\Omega_0) \to p_0(\Omega_\rho$. Let $a \in A_0$ be given. Since A_0 is an ideal, and since $\widetilde{\omega}$ annihilates A_0, we may now conclude that

$$\omega_n(a) = \langle \pi_\rho(a)\Omega_\rho, \eta \rangle = \lim_n \langle \pi_0(a(\mathbb{I} - f_n))(\Omega_\rho, \eta \rangle = \lim_n \widetilde{\omega}(a(\mathbb{I} - f_n)] = 0$$

Thus ω_n annihilates A_0. But then so does $\omega_s = \omega - \omega_n$. It now follows from Corollary (2.1.11) that A satisfies the F & M Riesz property.

Corollary (2.1.13)[61]: If A is a maximal subdiagonal algebra with $\mathcal{D}$ finite dimensional, and if $\omega \in M^*$annihilates $A + A^*$, then ω is singular.

Proof: Since A satisfies the $F\&M$ Riesz property, ω_n. annihilates A. Similarly, since A^*. satisfies the $F\&M$ Riesz property, $\omega_n$annihilates A*. Since A is subdiagonal, $\omega_n = 0$.

Corollary (2.1.14)[61]: If A has the F & M Riesz property, then any positive functional on M which annihilates A_0 is normal.

Proof: If ω is a state on M which annihilates A_0, and if A has the F & M Riesz property, then the (positive) singular part of ω is 0 since it must annihilate $\mathbb{1}$.

We say that an extension in $M^\star$ of a functional in $A^\star$ is a Hahn-Banach extension if it has the same norm. If A is a weak* closed subalgebra of M then we say that A has property (GW_1) if every Hahn-Banach extension to M of any normal functional on A, is normal on M. We say that A has property (GW_2) if there is at most one normal Hahn-Banach extension to M of any normal functional on A. We say that A has the Gleason-Whitney property (GW) if it possesses (GW_1) and (GW_2). This is simply saying that there is a unique Hahn-Banach extension to M of any normal functional on A, and this extension is normal. Of course normal functionals on A or on M have to be of the form $\tau(g\ \cdot)$ for some $g \in L^1(M)$.

Theorem (2.1.15)[61]: If A is a tracial subalgebra of M then A is maximal subdiagonal if and only if it possesses property (GW_2). If $\mathcal{D}$ is finite dimensional, then A is maximal subdiagonal if and only if it possesses property (GW).

Proof: Suppose that A possesses property (GW_2). To show that A is maximal subdiagonal, it suffices to show that if $g \in L^1(M)$,with $\tau(g(A + A^*)) = 0$, then $g = 0$.

By considering real and imaginary parts we may assume that $g = g^*$. Then $\tau(|g|\ \cdot)$ and $\tau((|g| + g)\ \cdot)$ are positive normal functionals on M which agree on A. They are also Hahn-Banach extensions, since the norm of a positive functional is achieved at 1. Thus by (GW_2), these functionals agree on M, and so $|g| + g = |g|$. That is, $g = 0$.

In the remainder of the proof suppose that A is maximal subdiagonal. Suppose that $f, g \in L^1(M)$ correspond to two normal Hahn-Banach extensions to M of a given functional on A. Then $\|f\|_1 = \|g\|_1$, and this quantity equals the norm of the restriction to A. We have $\tau((f - g)A) = 0$; since A is subdiagonal it follows from [153] that $\mathrm{h} = g - f \in [A_0]_1$. In order to establish (GW_2), we need to show that $h = 0$. Since Ball (A) is weak* compact, and since $\|f\|_1$ equals the norm of the above-mentioned restriction to A, there exists a $\in$ A of norm 1 with $\tau(fa) = \|f\|_1$. It is evident that

$$|af|^2 = f^*a^*af \leq f^*f = |f|^2 .$$

Now $0 \leq T \leq S$ in $L^p(M)$ implies that $T^{\frac{1}{2}} \leq S^{\frac{1}{2}}$(see e.g. [163], and we thank David Sherman for this reference). It follows that $|af| \leq |f|$. On the other hand, $\tau(|f|) = \tau(fa) = \tau(af) \leq \tau(|af|)$.Thus$\|\,|f| - |af|\,\|_1 = \tau(|f| - |af|) = 0$, and so $|f| = |af|$.The functional $\psi = \tau(af.)$ on M must be positive since $\psi(\mathbb{1}) = \tau(af) = \tau(|f|) = \tau(|af|) = \|\psi\|$. Thus $af \geq 0$, and $af = |af| = |f|$. Since $h \in [A_0]_1$ we have

$$\tau\big((f + h)a\big) = \tau(fa)\|f\|_1 = \|g\|_1 = \|f + h\|_1$$

An argument similar to that of the last paragraph shows that $a(f + h) = |f + h| \geq 0$.Thus ah is self-adjoint. Since $h \in [A_0]_1$ it is easy to see that $\tau(ahA) = 0$. Therefore from the selfadjointness of ah one may deduce that $\tau(ah(A + A^*)) = 0$. Because A is subdiagonal, it follows that $ah = 0$. Thus

$$|f| = af = a(f + h) = |f + h|$$

Let e be the left support projection of a. Then $e^\perp$ is the projection onto $Ker(a^*)$. We have $|f|e^\perp = f^*a^*e^\perp = 0$. It follows that $fe^\perp = 0$. Thus $0 = e^\perp f^* f e^\perp = e^\perp |f + h|^2 e^\perp = e^\perp (f + h)^*(f + h)e^\perp = e^\perp h^* h e^\perp$.

Hence $he^\perp = 0$. To show that $he = 0$, we reproduce the ideas in the argument in the second paragraph of the proof. Namely, note that $|(fa)^*|^2 \le |f^*|^2$, so that $|(fa)^*| \le |f^*|$. But $(|f^*|) = \|f\|_1 = \tau(fa) \le \tau(|(fa)^*|)$, and as before this shows that $|(fa)^*| = |f^*|$. Then also $\tau(fa) = \tau(|(fa)^*|)$, and as before this shows that $fa \ge 0$. Similarly, $(f + h)a \ge 0$. So ha is again selfadjoint, and this implies as before that $ha = 0$. Thus $he^\perp = 0$, and so $h = he + he^\perp = 0$ as required.

Now suppose that, in addition, $\mathcal{D}$ is finite dimensional, and that ρ is a Hahn-Banach extension of a normal functional ω on A. By basic functional analysis, $\omega$is the restriction of a normal functional ω on M. We may write $\rho = \rho_n + \rho_s$, where ω_n and ω_s are respectively the normal and singular parts, and $\|\rho\| = \|\rho_n\| + \|\rho_s\|$.Then $\rho - \widetilde{\omega}$ annihilates A, and hence by our F. and M. Riesz theorem both the normal and singular parts, $\rho_n - \widetilde{\omega}$ and ρ_s respectively, annihilate A_0. Hence they annihilate A, and in particular $\rho_n = \omega$ on A. But this implies that

$$\|\rho_n\| + \|\rho_s\| = \|\rho\| = \|\omega\| \le \|\rho_n\|$$

We conclude that $\rho_s = 0$. Thus A also satisfies (GW_1), and hence (GW).There is another (simpler) variant of the Gleason-Whitney theorem [149], which transfers more easily to our setting:

Theorem (2.1.16)[61]: Let A be a maximal subdiagonal subalgebra of M with $\mathcal{D}$finite dimensional. If $\omega$is a normal functional on M then ω is the unique Hahn-Banach extension of its restriction to $A + A^*$. In particular, $\|\omega\| = \|\omega_{A+A^*}\|$for any$\omega \in M_*$.

Proof: Let ρ be a Hahn-Banach extension of the restriction of ω to $A + A^*$. We may write $\rho = \rho_n + \rho_s$, where ρ_n and ρ_s, are respectively the normal and singular parts, and $\|\rho\| = \|\rho_n\| + \|\rho_s\|$. Then $\rho - \omega$ annihilates $A + A^*$. By Corollary (2.1.14), $\rho_n - \omega = (\rho - \omega)_n = 0$. As in the last part of the previous proof, this implies that $\rho_s = 0$. So $\rho_s = \rho_n = \omega$.

Corollary (2.1.17)[61]:(Kaplan sky density theorem for subdiagonal algebras) Let A be a maximal subdiagonal subalgebra of M with D finite dimensional. Then the unit ball of $A + A^*$is weak* dense in Ball(M).

Proof: If C is the unit ball of $A + A^*$, it follows from the last remark that thepre-polar of C is Ball$(M_\star)$. By the bipolar theorem, C is weak* dense in Ball (M).

Arveson formulated the Szegö theorem for $L^2(M)$ in terms of the Kadison-Fuglede determinant $\Delta(\cdot)$. The long-outstanding open question of whether general maximal subdiagonal algebras satisfy the Szego theorem for $L^2(M)$, was eventually settled in the affirmative in [163]. We will now extend this result to $L^p(M)$. We refer the reader to [300, 58] for the properties of the Kadison-Fuglede determinant which we shall need.

Lemma (2.1.18)[61]:$\Delta(b^p) = \Delta(b)^p$ for $p \ge 1$ and $b \in M_+$.

Proof: By the multiplicativity property of Δ, the relation clearly holds for dyadic rationals. We may assume that $0 \leq b \leq 1$. In this case, by the functional calculus it is clear that $b^q \leq b^p$ if $0 < p \leq q$. If q is any dyadic rational bigger than p then

$$\Delta(b)^p = \Delta(b^p) \leq \Delta(b^p)$$

It follows that $\Delta(b^p) \leq \Delta(b^p)$. Replacing p by $1/p$, we have $\Delta(\mathrm{b}^\mathrm{p})^{\frac{1}{\mathrm{p}}} \leq \Delta\left((\mathrm{b}^\mathrm{p})^{\frac{1}{\mathrm{p}}}\right) = \Delta(\mathrm{b})$, which gives the other direction.

Theorem (2.1.19)[61]:.(Szegö theorem for $L^p(M)$) Suppose that A is maximal subdiagonal, and $1 \leq p < \infty$. If $h \in L^1(M)_+$ then
$\Delta(h) = inf\ \{\tau(h|a + d|^p) : a \in A_0, d \in \mathcal{D}, \Delta(d) \geq 1\}$.

Proof: We set

$$S_p = \{|a|^p : a \in A, (\phi(a)) \geq 1\},$$
$$S = \{a^*a : a \in A^{-1}, \Delta(a) \geq 1\}.$$

By the modification in [58] of a trick of Aversion's from [300], it suffices to show that the closure of S_p equals the closure of S. First we show that $S \subset S_p$. Indeed, if $b \in S$ then b is invertible, and therefore so is $\frac{1}{p}$. Since A has factorization, there is an $a \in A^{-1}$ with $|\mathrm{a}| = \mathrm{b}^{\frac{1}{\mathrm{p}}}$. By Lemma (2.1.19) and Jensen's formula [300, 163] we have

$$\Delta(\Phi(a)) = \Delta(a) = \Delta(|a|) = \Delta\left(b^{\frac{1}{p}}\right) = \Delta(b)^{\frac{1}{p}} \geq 1.$$

Hence $b = |a|^p \in S_p$.

Suppose that $b \in S_p$ .If $b = |a|^p$ where $\Delta(\Phi(a)) \geq 1$ then by Jensen's inequality [300, 241] we have $\Delta(a) = \Delta(|a|) \geq 1$. Hence by Lemma (2.1.18) we have $\Delta(b) \geq 1$. If $n \in \mathbb{N}$ then since A has factorization, there exists a $\mathrm{c} \in \mathrm{A}^{-1}$with $b + \frac{1}{n}1 = c^*c$. Thus

$$\Delta(c)^2 = \Delta(b + \frac{1}{n}1) \geq \Delta(b) \geq 1.$$

Thus $\mathrm{b} + \frac{1}{\mathrm{n}}1 = \mathrm{c}^*\mathrm{c} \in \mathrm{S}$, and we deduce that $b \in \bar{S}$. Hence $\overline{\mathrm{S_p}} \subset \bar{\mathrm{S}}$.

Note that the following generalized Kolmogorov theorem is not true for all maximal subdiagonal algebras. For example, take $A = M = L^\infty[0,1]$.

Theorem (2.1.20)[61]:.Suppose that A is an antisymmetric maximal subdiagonal algebra. If$h \in L^1(M)_+$ then $\inf\{\tau(h|\mathbb{1} + f|^2) : f \in A_0 + A_0^*\}$ is either $\tau(h^{-1})^{-\frac{1}{2}}$, if h^{-1} exists in the sense of unbounded operators and is in $L^1(M)$; or the infimum is 0 if $\mathrm{h}^{-1} \notin \mathrm{L}^1(\mathrm{M})$. More generally, if $1 \leq p < \infty$ then $inf\{\tau(|(\mathbb{1} + f)h^{\frac{1}{p}}|^p) : f \in A_0 + A_0^*\}$ is either 0 if $h^{-1} \notin L^{1/(p-1)}(M), or\ \tau(h^{-1/p-1})\frac{1}{p} - 1$ if$h^{-1} \in L^{1/(p-1)}(M)$.

Proof: We formally follow the proof of Forelli as adapted in [282]. Let $h \in L^1(M)_+$, and $\frac{1}{p} + \frac{1}{q} = 1$. Define $L^p(M,h)$ to be the completion in $L^p(M)$ of $Mh^{\frac{1}{p}}$. Note that if e is the support projection of a positive $x \in L^p(M)$ then it is well known (see e.g. [175]) that $L^p(M)e$ equals the closure in $L^p(M)$ of M_x. Hence $L^p(M,h) = L^p(M)e$, where e is the support projection of

h. Now for any projection $e \in M$ it is an easy exercise to prove that the dual of $L^p(M)$ is $eL^q(M)$ (see e.g. [175]). It follows that the dual of $L^p(M,h)$ is $L^q(M,h)$.

If $k \in L^p(M,h)$ then $kh^{\frac{1}{q}} \in L^p(M)L^q(M) \subset L^1(M)$. We view $A_0 + A_0^*$ in $L^p(M,h)$ as its image $(A_0 + A_0^*)h^{\frac{1}{p}}$, and let N be the annihilator of this in $L^q(M,h)$. That is, $g \in N$ iff $g \in L^q(M,h)$ and

$$0 = \tau(h^{\frac{1}{p}}(A_0 + A_0^*)g) = \tau((A_0 + A_0^*)gh^{\frac{1}{p}}).$$

Since $gh^{\frac{1}{p}} \in L^1(M)$ the last equation holds iff $gh^{\frac{1}{p}} = c\mathbb{1}$, where c is a constant. Since h is selfadjoint, if $c \neq 0$ then it follows that $h^{-\frac{1}{p}}$ exists in the sense of unbounded operators, and its closure is the constant multiple $dg \in L^q(M)$, where $d = c^{-1}$. (Since we are in the finite case, there is no difficulty with τ-measurability here, this is automatic [183]). If $c = 0$ then $gh^{\frac{1}{p}} = 0$ which implies that $g = 0$. To see the last statement note that if $h^{\frac{1}{p}}$ is viewed as a selfadjoint unbounded operator on a Hilbert space H, and if e is its support projection, which equals the support projection of $h^{\frac{1}{q}}$, then $eh^{\frac{1}{p}} = h^{\frac{1}{p}}$, and so $eh^{\frac{1}{p}}e = eh^{\frac{1}{p}}$. Since $g \in \overline{Mh^{\frac{1}{q}}}$, we have $ge = g$. However $ge = 0$ since $gh^{\frac{1}{p}} = 0$. Thus if g has norm 1 then $c \neq 0, h^{\frac{1}{p}} \in L^q(M)$ and $|d| = \left\|h^{\frac{1}{p}}\right\|_{L^q(M)} = \tau(h^{-\frac{q}{p}})^{\frac{1}{q}}$.

The infimum in the theorem is the pth power of the norm of $\mathbb{1}$ in the quotient space of $L^p(M,h)$ modulo the closure of $A_0 + A_0^*$. Since the dual of this quotient is $(A_0 + A_0^*)^{\perp} = N$, this infimum equals the pth power of $sup\{\left|\tau\left(gh^{\frac{1}{p}}\right)\right| : g \in N, \|g\|_{L^q}(M) \leq 1\}$. This equals 0 if no $g \in N$ has norm 1; otherwise it equals $\tau(h^{-\frac{q}{p}})^{-\frac{1}{q}} = \tau(h^{-\frac{1}{p-1}})^{-\frac{1}{q}}$ by the above. Indeed, the infimum is 0 iff $\tau(gh^{\frac{1}{p}}) = 0$ for all $g \in N$. Since $gh^{\frac{1}{p}}$ is constant, this occurs iff $gh^{\frac{1}{p}} = 0$, which as we saw above happens iff $g = 0$. Thus the infimum is 0 iff $N = (0)$ iff $(A_0 + A_0^*)h^{\frac{1}{p}}$ is dense in $L^p(M,h)$. Since $h^{\frac{1}{p}} \in L^p(M,h)$, the latter condition implies that there is a sequence (g_n) in $A_0 + A_0^*$ with $g_n h^{\frac{1}{p}} \to h^{\frac{1}{p}}$ in p −norm. If $h^{-1/p} \in L^q(M)$ then by Hölder's inequality we have $\tau(|g_n - \mathbb{1}|) \to 0$, which is impossible since $1 = |\tau(g_n - \mathbb{1})| \leq \tau(|g_n - \mathbb{1}|)$.

Section (2.2) Szegö's Theorem and Outers for Noncommutative H^p

It has long been of great importance to operator theorists and operator algebraists to find noncommutative analogues of the classical 'inner-outer factorization' of analytic functions. We recall some classical results: If $f \in L^1$ with $f \geq 0$, then $\int log\,|f| > -\infty$ iff $f = |h|$ for an outer $h \in H^1$ (iff $f = |h|^p$ for an outer $h \in H^p$). We will call this the Riesz-Szegö theorem. If $f \in L^1$ with $\int log\,|f| > -\infty$, then $f = uh$, where u is unimodular and h is outer. Outer functions may be defined in terms of a simple equation involving $\int log\,|f|$Such results are usually treated as consequences of the classical Szegö theorem, which is really a distance formula in terms of the entropy $exp(\int\ log\,|f|)$, and which in turn is intimately related to the

Jensen inequality (see e.g. [147]). In the noncommutative situation one wishes, for example, to find conditions on a positive operator T which imply that $T = |S|$ for an operator S which is in a 'noncommutative Hardy class', or, even better, which is 'outer' in some sense. There are too many such results to attempt a listing of them (see e.g. [88]). Central parts of this topic still seem to be poorly understood. As an example of this, we cite the main and now classical result of [6], concerning a Riesz-Szegö like factorization of a class of $B(H)$-valued functions on the unit interval, which has resisted generalization in some important directions. We generalize the classical results above to the noncommutative H^p spaces associated with Arveson's remarkable subdiagonal algebras [300]. Our generalization solves an old open problem (see the discussion in [88], and [178]). The approach which we take has been unavailable until now (since it relies ultimately on the recent solution in [163] of a 40 year old open problem from [300]). Moreover, the approach is very faithful to the original classical function theoretic route (see e.g. [147]), proceeding via noncommutative Szegö theorems.

We have attempted to demonstrate that all the results in ([282]) the 'generalized H^p-theory' for abstract function algebras from the 1960s, extend in an extremely complete and literal fashion, to the noncommutative setting of Arveson's subdiagonal subalgebras of von Neumann algebras [300]. This may be viewed as a very natural merging of generalized Hardy space, von Neumann algebra, and noncommutative L^p space, techniques. See [62]. It completes the noncommutative extension of the basic Hardy space theory. As posited by Arveson, one should use the Fuglede-Kadison determinant $\Delta(a) = exp(\tau(log\,|a|))$ where τ is a trace, as a natural replacement in the noncommutative case for the quantity $\int log\, f$ above. We use properties of the Fuglede-Kadison determinant to give several useful variants of the noncommutative Szegö theorem for $L^p(M)$, including the one usually attributed to Kolmogorov and Krein. As applications, we generalize the noncommutative Jensen inequality, and generalize many of the classical resuts concerning outer functions, to the noncommutative H^p context.

For a set S, we write $S_+$for the set $\{x \in S : x \geq 0\}$ see [62]. . We assume throughout that M is a von Neumann algebra possessing a faithful normal tracial state τ. The existence of such τ implies that M is a so-called finite von Neumann algebra, and that if $x^*x = 1$ in M, then $xx^* = 1$ too. Indeed, for any $a, b \in M$, ab will be invertible precisely when a and b are separately invertible. We will also need to use a well known fact about inverses of an unbounded operator , and in our case T will be positive, selfadjoint, closed, and densely defined. We recall that T is bounded below if for some $\lambda > 0$ one has $kT(\eta)k \geq \lambda\|\eta\|$for all $\eta \in dom(T)$. This is equivalent to demanding that $|T| \geq \varepsilon_1$ for some $\varepsilon > 0$, and of course in this case, $|T|$ has a bounded positive inverse.

A (finite maximal) subdiagonal subalgebra of M is a weak* closed unital subalgebra A of M such that if Φ is the unique conditional expectation guaranteed by [181] from M onto $A \cap A^* \stackrel{\text{def}}{=} D$ which is trace preserving (that is, $\tau \circ \Phi = \tau$), then:

$$\Phi(a_1a_2) = \Phi(a_1)\Phi(a_2), a_1, a_2 \in A. \qquad (1)$$

One also must impose one further condition on A. There is a choice of at least eight equivalent, but quite different looking, conditions [62]; Arveson's original one (see also [240]) is that $A + A^*$ is weak* dense in M. In the classical function algebra setting [288], one assumes that $\mathcal{D} = A \cap A^*$ is one dimensional, which forces $\Phi = \tau(\cdot)1$. If in our setting this is the case, then we say that A is antisymmetric.

The simplest example of a maximal subdiagonal algebra is the upper triangular matrices A in M_n. Here Φ is the expectation onto the main diagonal. There are much more interesting examples from free group von Neumann algebras, Tomita-Takesaki theory, etc (see e.g. [300,161,178]). On the other end of the spectrum, M itself is a maximal subdiagonal algebra (take $\Phi = Id$). It is therefore remarkable that so much of the classical H^p theory does extend to all maximal subdiagonal algebras.

By analogy with the classical case, we set $A_0 = A \cap Ker(\Phi)$ and set H^p or $H^p(A)$ to be $[A]_p$, the closure of A in the noncommutative L^pspace $L^p(M)$, for $p \geq 1$. More generally we write $[S]_p$ for this closure of any subset S. We will often view $L^p(M)$ inside $\widetilde{M}$, the set of unbounded, but closed and densely defined, operators on H which are affiliated to M. This is a $*$-algebra with respect to the 'strong' sum and product (see [183]). We order $\widetilde{M}$ by its cone of positive (selfadjoint) elements. The trace τ extends naturally to the positive operators in $\widetilde{M}$,. If $1 \leq p < \infty$, then $L^p(M,\tau) = \{a \in \widetilde{M} : \tau(|a|p) < \infty\}$, equipped with the norm $\|\cdot\|_p = \tau(|\cdot|^p)^{1/p}$ (see e.g.[77]). We abbreviate $L^p(M,\tau)$ to $L^p(M)$. Arveson's Szegö formula is:

$$\Delta(h) = inf\{\tau(h|a + d|^2) : a \in A_0, d \in D, \Delta(d) \geq 1\}$$

for all $h \in L^1(M)_+$. Here Δ is the Fuglede-Kadison determinant, originally defined on M by $\Delta(a) = exp\,\tau(log\,|a|)\ if\ |a| > 0$, and otherwise, $\Delta(a) = inf\,\Delta(|a| + \mathcal{E}1)$, the infimum taken over all scalars $\mathcal{E} > 0$ (see [26, 300]). We will discuss this determinant in more detail. Unfortunately, the just-stated noncommutative Szegö formula, and the (no doubt more important) associated Jensen's inequality

$$\Delta(\Phi(a)) \leq \Delta(a), a \in A,$$

resisted proof for nearly 40 years, although Arveson did prove them in his extraordinary original [300] for the examples that he was most interested in. The second proved in [163] that all maximal subdiagonal algebras satisfy these formulae. Settling this old open problem opened up the theory.

An element $h \in H^p$ is said to be outer if $1 \in [hA]_p$. This definition is in line with e.g. Helson's definition of outers in the matrix valued case he considers in [102]. We now state a sample of our results about outers. For example, we are able to improve on the factorization theorems from e.g. [58] in several ways: namely we show that if $f \in L^p(M)$ with $\Delta(f) > 0$ then f may be essentially uniquely factored $f = uh$ with u unitary and h outer. There is a much more obvious converse to this, too. We now have an explicit formula for the u and h. We refer to a factorization $f = uh$ of this form as a Beurling-Nevanlinna factorization. It follows that in this case if $f \geq 0$ then $f = |h|$ with h outer. This gives a solution to the problem posed in [88], and in [178]. If $h \in H^p$, and h is outer then $\Delta(h) = \Delta(\Phi(h))$. A converse is true: if $\Delta(h) = \Delta(\Phi(h)) > 0$ then h is outer. It follows that under some restrictions on $\mathcal{D} = A \cap A^*$, h is outer iff $\Delta(h) = \Delta(\Phi(h)) > 0$.

There are many factorization theorems for subdiagonal algebras (see e.g. [300, 178, 149, 176, 88]), but as far as we know there are no noncommutative factorization results involving outers or the Fuglede-Kadison determinant. We mention for example Arveson's original factorization result from [300], or Marsalli and West's Riesz factorization of any $f \in H^1$ as a product$f = gh$ with $g \in H^p, h \in H^q, \frac{1}{p} + \frac{1}{q} = 1$. Some also require rather stronger hypotheses, such as $f^{-1} \in L^2(M)$ (see e.g. [178])

The commutative case of most of the topics was settled in [281]. While certainly gave us motivation to persevere in our endeavor, we follow completely different lines, and indeed the results work out rather differently too. In particular, the quantity $\tau(exp(\Phi(log\,|f|)))$, which plays a central role in most of the results in [281], seems to us to be unrelated to outers or factorization in the noncommutative setting. We remark that numerical experiments do seem to confirm the existence of a Jensen inequality $\tau(exp(\Phi(log|a|))) \geq \tau(|\Phi(a)|)$ for subdiagonal algebras.

We remark that there are many other, more recent, generalizations of H^∞, based around multivariable analogues of the S_z-Nagy-Foias model theory for contractions. The unilateral shift is replaced by left creation operators on some variant of Fock space. Many are currently intensively pursuing these topics, they are very important and are evolving in many directions. Although these theories also contain variants of Hardy space theory, they are quite far removed from subdiagonal algebras. For example, if one compares Popescu's theorem of Szegö type from [90] with the Szegö theorem for subdiagonal algebras discussed here, one sees that they are only related in a very formal sense.

The Fuglede-Kadison determinant Δ, and its amazing properties, is perhaps the main tool in the noncommutative H^p theory. In [277], Fuglede and Kadison study the determinant as a function on M. We will define it for elements of $L^q(M)$ for any $q > 0$. In fact, as was pointed out to us by Quanhua Xu, L. G. Brown investigated the determinant and its properties in the early 1980s, on a much larger class than $L^q(M)$ (see [163]); indeed recently Haagerup and Schultz have thoroughly explicated the basic theory of this determinant for a very general class of τ −measurable operators (see [286]) as part of Haagerup's amazing attack onthe invariant subspace problem relative to a finite von Neumann algebra.

We will define the Fuglede-Kadison determinant for an elementh $h \in L^q(M)$, for any $q > 0$, as follows. We set $\Delta(h) = exp\,\tau(log\,|h|)$ if $|h| > \varepsilon 1$ for some $\varepsilon > 0$, and otherwise, $\Delta(h) = inf\,\Delta(|h| + \epsilon 1)$, the infimum taken over all scalars $\epsilon > 0$. To see that this is well-defined, we adapt the argument in [57], making use of the Borel functional calculus for unbounded operators applied to the inequality

$$0 \leq log\,t \leq \frac{1}{q} t^q, \qquad t \in [1, \infty).$$

Notice that for any $0 < \varepsilon < 1$, the function $log\,t$ is bounded on $[\varepsilon, 1]$. So given $h \in L^1(M)_+$ with $h \geq \varepsilon\mathbb{1}$, it follows that $(log\,h)e_{[0,1]}$ is similarly bounded. The previous centered equation ensures that $0 \leq (log\,h)_{e[1,\infty)} \leq \frac{1}{q} h^q e[1,\infty) \leq \frac{1}{q} h^q$. Here$e_{[0,\lambda]}$ denotes the spectral resolution of h. Thus if $h \in L^q(M)$ and $h \geq \epsilon$ then $log\,h \in L^1(M)$.

The following are the basic properties of this extended determinant which we shall need. Full proofs may be found in [286], which are valid for a very general class of unbounded operators (see also [62] for another (later) proof for the $L^p(M)$ class).

Theorem (2.2.1)[66]: If $p > 0$ and $h \in L^p(M)$ then

(i) $\Delta(h) = \Delta(h^*) = \Delta(|h|)$.
(ii) If $h \geq g$ in $L^p(M)_+$ then $\Delta(h) \geq \Delta(g)$.
(iii) If $h \geq 0$ then $\Delta(h^q) = \Delta(h)^q$ for any $q > 0$.
(iv) $\Delta(hb) = \Delta(h)\Delta(b) = \Delta(bh)$ for any $b \in L^q(M)$ and any $q > 0$.

Throughout, A is a maximal subdiagonal algebra in M. We consider versions of Szegö's formula valid in $L^p(M)$ rather than $L^2(M)$. We will also prove a generalized Jensen inequality, and show that the classical Verblunsky-Kolmogorov- Krein strengthening of Szegö's formula extends even to the noncommutative context.

It is proved in [61] that for $h \in L^1(M)_+$ and $1 \leq p < \infty$, we have

$$\Delta(h) = inf\{\tau(h|a + d|^p) : a \in A_0, d \in \mathcal{D}, \Delta(d) \geq 1\}.$$

We now prove some perhaps more useful variants of this formula

Lemma (2.2.2)[66]: If $h \in L^q(M)_+$ and $0 < p, q < \infty$, we have $\Delta(\mathrm{h}) = inf\{\tau(|h^{\frac{q}{p}} b|^p)^{\frac{1}{q}} : b \in M_+, \Delta(b) \geq 1\} = inf\{\tau(|bh^{\frac{q}{p}}|^p)^{\frac{1}{q}} : b \in M_+, \Delta(b) \geq \}$. The infimums are realized on the commutative von Neumann subalgebra M_0 generated by h, and are unchanged if in addition we also require b to be invertible in B.

Proof: That the two infimums in the displayed equation are equal follows from the fact that $\|x\|_p = \|x^*\|_p$ for $x \in L^p(M)$ (see [275]). Thus we just prove the first equality in that line.

For $b \in M_+, \Delta(b) \geq 1$, we have by Theorem (2.2.1) (iii) that

$$\Delta(|h^{q/p}b|^p) = \Delta(|h^{q/p}b|^p) = \Delta(|h^{q/p}b|^p).$$

Consequently, using facts from Theorem (2.2.1) again, we have

$$\tau(|h^{q/p}b|^p) \geq \Delta(|h^{q/p}b|^p) = [\Delta(h^{q/p})\Delta(b)]^p \geq \Delta(h^{q/p})^p = \Delta(h)^q.$$

To complete the proof, it suffices to find, given $\varepsilon > 0$, an invertible b in $(\mathrm{M}_0)_+$, the von Neumann algebra generated by h, with $\Delta(b) \geq 1$ and $\tau(|h^{\frac{q}{p}} b|^p)^{\frac{1}{q}} < \Delta(h) + \varepsilon$. But for any $b \in (M_0)_+$ we have $|h^{\frac{q}{p}}b|^p = h^q b^p$ by commutativity, and then the result follows from an analysis of Arveson's original definition of $\Delta(\mathrm{h})$ (see [59]). In particular since $\Delta(h^q) = inf\{\tau(h^q b^p) : b \in (M_0)+, \Delta(b) \geq 1\}$ by [59], an application of Theorem (2.2.1) (3) ensuresthat $\Delta(h) = inf\{\tau(h^q b^p)^{\frac{1}{q}} : b \in (M_0)_+, \Delta(b) \geq 1\}$.

Corollary (2.2.3)[66]: If $\mathrm{h} \in \mathrm{L}^{\mathrm{q}}(\mathrm{M})_+$ and 0 $p, q < \infty$, we have

$$\Delta(h) = inf\{\tau(|h^{\frac{q}{p}} a|^p)^{\frac{1}{q}} : a \in A, \Delta(\Phi(a)) \geq 1\}$$
$$= inf\{\tau(|h^{\frac{q}{p}}|^p)^{\frac{1}{q}} : a \in A, \Delta(\Phi(a)) \geq 1\}.$$

The infimums are unchanged if we also require a to be invertible in A, or if we require $\Phi(a)$ to be invertible in $\mathcal{D}$.

Proof: That the two infimums in the displayed equation are equal follows from the fact that $\|x\|_p = \|x^*\|$ for $x \in L^p(M)$ (see [277]), and by replacing A with A^*, which is also subdiagonal. Thus we just prove the first equality in that line.

For $a \in A, \Delta(\Phi(a)) \geq 1$ we have

$$\tau(|h^{\frac{q}{p}} a|^p)^{\frac{1}{q}} = \tau(|a^* h^{\frac{q}{p}}|^p)^{\frac{1}{q}} = \tau(||a^*|h^{\frac{q}{p}}|^p)^{\frac{1}{q}} \geq \Delta(h),$$

by Lemma (2.2.2), since $\Delta(|a^*|) = \Delta(a^*) = \Delta(a) \geq \Delta(\Phi(a)) \geq 1$ (using Jensen's inequality). Thus Δ(h) is dominated by the first infimum. On the other hand, by the previous result there is an invertible b ∈ M_+ with $\Delta(b) \geq 1$ and $\tau(|h^{\frac{q}{p}} a|^p)^{\frac{1}{q}} < \Delta(h) + \varepsilon$. By factorization, we can write b = $|a^*|$ for an invertible a in A, and by Jensen's formula [300, 163] we have $\Delta(\Phi(a)) = \Delta(a) = \Delta(a^*) = \Delta(b \geq 1$. Then
$\tau(|h^{\frac{q}{p}} a|^p)^{\frac{1}{q}} = \tau(|a^* h^{\frac{q}{p}}|^p)^{\frac{1}{q}} = \tau(|b\, a|^p)^{\frac{1}{q}} < \Phi(h) + \varepsilon.$

Corollary (2.2.4)[66]: (Generalized Jensen inequality) Let A be a maximal subdiagonal algebra. For any $h \in H^1$ we have $\Delta(h) \geq \Delta(\Phi(h))$.

Proof: Using theL^1-contractivity of Φ we get

$$\tau(\big||h|a\big|) = \tau(|ha|) = \tau(\big||\Phi(h)|\Phi(a)\big|), \quad a \in A.$$

Taking the infimum over such a with $\Delta(\Phi(a)) \geq 1$, we obtain from Corollary (2.2.3), and Theorem (2.2.1) applied to $\mathcal{D}$, that $\Delta(h) = \Delta(|h|) \geq \Delta(|\Phi(h)|) = \Delta(\Phi(h))$.

We recall that although $L^p(M)$ is not a normed space if $1 > p > 0$, it is a socalled vlinear metric space with metric given by $\|x - y\|_p^p$
for any $x, y \in L^p$(see [277]). Thus although the unit ball may not be convex, continuity still respects all elementary linear operations.

Corollary (2.2.5)[66]:.Let $h \in L^q(M)_+$ and $0 < p, q < \infty$. If $h^{\frac{q}{p}} \in [h^{\frac{q}{p}} A_0]p$, then$\Delta(h) = 0$. Conversely, if A is antisymmetric and $\Delta(h) = 0$, then $h^{\frac{q}{p}} \in [h^{\frac{q}{p}} A_0]_p$. Indeed if A is antisymmetric, then

$$\Delta(h) = inf\{\tau(|h^{\frac{q}{p}}(1 - a_0)|^p)^{\frac{1}{q}} : a_0 \in A_0\}.$$

Proof: The first assertion follows by taking a in the infimum in Corollary (2.2.3) to be of the form $1 - a_0$ for $a_0 \in A_0$.

If A is antisymmetric, and if $t \geq 1$ with $\tau(|h^{\frac{q}{p}}(t1 - a_0)|^p)^{\frac{1}{q}} < \Delta(h) + \varepsilon$, then $\tau(|h^{\frac{q}{p}}(1 - a_0)|^p)^{\frac{1}{q}} < \Delta(h) + \varepsilon$. From this the last assertion follows that the infimum's in Corollary (2.2.3) can be taken over terms of the form $1 + a_0$ where $a_0 \in A_0$. If this infimum was 0 we could then find a sequence $a_n \in A_0$ with $h^{\frac{q}{p}}(1 + a_n) \to 0$ with respect to $\|\cdot\|_p$. Thus $h^{\frac{q}{p}} \in [h^{\frac{q}{p}} A_0]_p$.

We close with the following version of the Szegö formula valid for general positive linear functionals. Although the classical version of this theorem is usually attributed to Kolmogorov and Krein, we have been informed by Barry Simon that Verblunsky proved it first, in the mid 1930's (see e.g. [267]):

Theorem (2.2.6)[66]: Noncommutative Szegö -Verblunsky- Kolmogorov- Krein theorem) Let ωbe a positive linear functional on M, and let ω_n and ω_s be its normal and singular parts respectively, with $\omega_n = \tau(h)\, for h \in L^1(M)_+$. If $dim(D) < \infty$, then

$$\Delta(h) = inf\{\omega(|a|^2) : a \in A, \Delta(\Phi(a)) \geq 1\}.$$

The infimum remains unchanged if we also require $\Phi(a)$ to be invertible in $\mathcal{D}$.

Proof: Suppose that $dim(\mathcal{D}) < \infty$. All terminology and notation will be as [61], the preamble to the proof of the noncommutative $F\&M$. Riesz [61]. For the sake of clarity we pause to highlight the most important of these. If $(\pi_\omega, H_\omega, \Omega_\omega)$ is the GNS representation engendered by ω, there exists a central projection p_0 in $\pi_\omega(M)''$ such that for any $\xi, \psi \in H_\omega$ the functionals $a \to \langle \pi_\omega(a)p_0\xi, \psi\rangle$ *and* $a \to \langle \pi_\omega(a)(1 - p_0)\xi, \psi\rangle$ on M are respectively the normal and singular parts of the functional $a \to \langle \pi_\omega(a)\xi, \psi\rangle$ [180]. In this representation Ω_0 will denote the orthogonal projection of Ω_ω onto the closed subspace $\pi_\omega(A_0)\Omega_\omega$.

Let $d \in \mathcal{D}$ be given. We may of course select a sequence $(f_n) \subset A_0$ so that $lim_{n\to\infty}\pi_\omega(f_n)\Omega_\omega = 0$. By the ideal property of A_0 and continuity, it then follows that

$$\pi_\omega(d)\Omega_0 = \lim_{n\to\infty} \pi_\omega(df_n)\Omega_\omega \in \overline{\pi_\omega(A_0)\Omega_\omega}.$$

Once again using the ideal property of A_0, the fact that $\Omega_\omega - \Omega_0 \perp \pi_\omega(A_0)\Omega_\omega$ now forces $\pi_\omega(d)\Omega_0, \pi_\omega(a)\Omega_\omega = \langle \Omega_\omega - \Omega_0, \pi_\omega(d^*a)\Omega_\omega\rangle = 0$ for every $a \in A_0$. Therefore

$$\pi_\omega(d)\Omega_0(\Omega_\omega - \Omega_0) \perp \overline{\pi_\omega(A_0)\Omega_\omega}.$$

From the facts in the previous two centered equations, it now follows that $\pi_\omega(d)\Omega_0$is the orthogonal projection of $\pi_\omega(d)\Omega_\omega$ onto $\overline{\pi_\omega(A_0)\Omega_\omega}$. Using this fact we may now repeat the arguments of [61] for the functional

$$\omega_d(\cdot) = \langle \pi_\omega(\cdot)\pi_\omega d)(\Omega_\omega - \Omega_0), \pi_\omega(d)(\Omega_\omega - \Omega_0)\rangle$$

to conclude that ω_d is normal, with $p_0(\pi_\omega(d)(\Omega_\omega - \Omega_0)) = \pi_\omega(d)(\Omega_\omega - \Omega_0)$, where p_0 is the central projection in $\pi_\omega(M)''$ mentioned above, and also: $p_0\pi_\omega(d)(\Omega_\omega - \Omega_0) \perp p_0\pi_\omega(A_0)\Omega_\omega$ and $p_0(\pi_\omega(d)\Omega_0)$ is the orthogonal projection of $p_0(\pi_\omega(d)\Omega_\omega)$ onto $p_0(\pi_\omega(A_0)\Omega_\omega)$. Thus we arrive at the fact that

$$\begin{aligned}\inf_{a\in A_0} \omega(|d + a|^2) &= \inf_{a\in A_0} \langle \pi_\omega(d)\Omega_\omega + \pi_\omega(a)\Omega_\omega, \pi_\omega(d)\Omega_\omega + \pi_\omega(a)\rangle \\ &= \inf_{a\in A_0} \|\pi_\omega(d)\Omega_\omega - \pi_\omega(a)\|^2 = \langle \pi_\omega(d)(\Omega_\omega - \Omega_0), \pi_\omega(\Omega_\omega - \Omega_0)\rangle \\ &= \langle p_0\pi_\omega(d)(\Omega_\omega - \Omega_0), p_0\pi_\omega(d)(\Omega_\omega - \Omega_0)\rangle \\ &= \inf_{a\in A_0} \langle p_0\pi_\omega(d)\Omega_\omega + p_0\pi_\omega(a)\Omega_\omega, p_0\pi_\omega(d)\Omega_\omega + p_0\pi_\omega(a)\rangle \\ &= \inf_{a\in A_0} \omega_n(|d + a|^2) = \inf_{a\in A_0} \tau(h|d + a|^2.\end{aligned}$$

On taking the infimum over all $d \in \mathcal{D}$ with $\Delta(d) \geq 1$, the result follows from Corollary (2.2.3).

Throughout A is a maximal subdiagonal algebra. We recall that if $h \in H^1$then h is outer if $[hA]_1 = H^1$. An inner element is a unitary which happens to be in A.

Lemma (2.2.7)[66]: Let $1 \leq p \leq \infty$. Then $h \in L^p(M)$ and h is outer in H^1, iff $[hA]_p = H^p$. (Note that $[\cdot]_\infty$ is the weak* closure.)

If these hold, then $h \notin [hA_0]_p$.

Proof: It is obvious that if $[hA]_p = H^p$. then $[hA]_1 = H^1$. Conversely, if $[hA]_1 = H^1$and $h \in L^p(M)$, then the first part of the proof of [58] applied to $[hA]_p$actually shows that $[hA]_p = [hA]_1 \cap L^p(M)$for all $1 \leq p \leq \infty$. Hence by [148], we have

$$[hA]_p = [hA]_1 \cap L^p(M) = H^1 \cap L^p(M) = H^p.$$

If $h \in [hA_0]_p$ then $1 \in [hA]_p \subset [[hA_0]_p A]_p \subset [hA_0]_p$. Now continuously extends to a map which contractively maps $L^p(M)$ onto $L^p(\mathcal{D})$ (see e.g. [176]). If $ha_n \to 1$ in L^p, with $a_n \in A_0$, then

$$\Phi(ha_n) = \Phi(h)\Phi(a_n) = 0 \to \Phi(1) = 1,$$

This forces $\Phi(\mathbb{1}) = 1$, a contradiction.

Lemma (2.2.8)[66]: If $h \in H^1$ is outer then as an unbounded operator h has dense range and trivial kernel. Thus $h = u|h|$ for a unitary $u \in M$. Also, $\Phi(h)$ has dense rangeand trivial kernel.

Proof: If h is considered as an unbounded operator, and if p is the range projection of h, then since there exists a sequence (a_n)in A with $ha_n \to 1$in L^1-norm, we have that $p^\perp = 0$. Thus the partial isometry u in the polar decomposition of h is an isometry, and hence is a unitary, in M. It follows that |h| has dense range, and hence it, and h also, have trivial kernel.

For the last part note that

$$L^1(\mathcal{D}) = \Phi(H^1) = \Phi([hA]_1) = [\Phi(h)\mathcal{D}]_1.$$

Thus we can apply the above arguments to $\Phi(h)$too.

There is a natural equivalence relation on outers:

Proposition (2.2.9)[66]: If $h \in H^p$ is outer, and if u is a unitary in $\mathcal{D}$, then $h' = uh$ is outer in H^p too. If $h, k \in H^p$ are outer, then $|h| = |k|$ iff there is unitary $u \in \mathcal{D}$ with $h = uk$. Such u is unique.

Proof: The first part is just as in the classical case. If $h, k \in H^1$ and $|h| = |k|$, then it follows, as in [281], that $h = uk$ for a unitary $u \in M$ with $u, u^* \in H^1$. Thus $u \in H^1 \cap M = A$ (see [148]), and similarly $u^* \in A$, and so $u \in \mathcal{D}$. The uniqueness of u follows since the left support projection of an outer is 1 (see proof of Lemma (2.2.8)).

As in the classical case, if $h \in H^2$ is outer, then h_2 is outer in H^1. Indeed one may follow the proof on [281], and the same proof shows that a product of any two outers is outer (see also the last lines of the proof of Theorem (2.2.13) below). We do not know whether every outer in H^1 is the square of an outer in H^2.

The first theorem is a generalization of the classical characterization of outers in H^p:

Theorem (2.2.10)[66]: Let A be a subdiagonal algebra, *let* $1 \leq p \leq \infty$ and $h \in H^p$. If h is outer then $\Delta(h) = \Delta(\Phi(h))$. If $\Delta(h) > 0$, this condition is also sufficient for h to be outer.

Note that if $dim(D) < \infty$, then $\Phi(h)$ will be invertible for any outer h by Lemma (2.2.8). Thus in this case it is automatic that $\Delta(\Phi(h)) > 0$.

Proof: The case for general p follows from the $p = 1$ case and Lemma (2.2.7). Hence we may suppose that $p = 1$.

First suppose that $h$is outer. Given any $d \in L^1(D)$ and any $a_0 \in [A_0]_1$, we clearly have that $\tau(|d - a_0|) \geq \tau(|d| - u^* a_0) = \tau(|d|)$, where $u$is the partial isometry in the polar decomposition of d. In other words, for any $a \in [A]_1$ we have

$$\|\Phi(a)\|_1 = \inf_{a_0 \in A_0} \|a - a_0\|_1 = \inf_{a_0 \in |A_0|_1} \|a - a_0\|_1 .$$

Therefore

$$\tau(|\Phi(h)\, \tilde{d}|) = \inf_{a_0 \in A_0} \tau(|h\, \tilde{d} - a_0|), \qquad \tilde{d} \in \mathcal{D}.$$

Notice that $[hA_0]_1 = [[hA]_1 A_0]_1 = [[A]_1 A_0]_1 = [A_0]_1$. Thus the above equality may alternatively be written as

$$\tau(|\Phi(h)\,\tilde{d}|) = \inf_{a_0 \in A_0} \tau(|h\,\tilde{d} - a_0|), \qquad \tilde{d} \in \mathcal{D}.$$

Finally notice that $|\Phi(h)\tilde{d}| = ||\Phi(h)|\tilde{d}|$ and $|h(\tilde{d} - a_0)| = ||h|(\tilde{d} - a_0)|$.Therefore if now we take the infimum over all $\tilde{d} \in \mathcal{D}$with$\Delta(\tilde{d}) \geq 1$, Szegö's theorem will force

$$\Delta(\Phi(h)) = \Delta(|\Phi(h)|) = \Delta(|h|) = \Delta(h).$$

Next suppose that $\Delta(h) = \Delta(\Phi(h)) > 0$. We will first consider the case $h \in [A]_2$.By Lemma (2.2.7) we then need only show that h is outer with respect to$[A]_2$. Replace h by $\tilde{h} = u^*h$ where u is the partial isometry in the polar decomposition of $\Phi(h)$. If we can show that eh is outer, it (and hence also u^*) will have dense range, which would force u^* to be a unitary. Thus $h = u\tilde{h}$ would then also be outer. Now notice that by construction $|h| \geq |\tilde{h}|$ and $\Phi(\tilde{h}) = |\Phi(h)|$. From this and the generalized Jensen inequality we have

$$\Delta(h) = \Delta(|h|) \geq \Delta(|\tilde{h}|) = \Delta(\tilde{h}) \geq \Delta(\Phi(\tilde{h})) = \Delta(X(h)) = \Delta(h).$$

Thus $\Delta(\tilde{h}) = \Delta(\Phi(\tilde{h})) > 0$. We may therefore safely pass to the case where $\Phi(h) \geq 0$. By multiplying with a scaling constant, we may also clearly assume that $\Delta(h) = 1$.

For any $d \in \mathcal{D}$ and $a \in A_0$ we have

$$\tau(|1 - h(d+a)|^2) = \tau(1 - \Phi(h)d - d^*\Phi(h)) + \tau(|h(d+a)|^2). \qquad (2)$$

To see this, simply combine the fact that $\tau \circ \Phi = \tau$ with the observation that $\Phi(h(d+a)) = \Phi(h)\Phi(d+a) = \Phi(h)d$. With d, a as above, notice that $\tau(|h(d+a)|^2) = \tau(||h|(d+a)|^2)$. By Szegö's theorem in the form of Corollary (2.2.3), we may select sequences $(d_n) \subset \{d \in \mathcal{D}^{-1} : \Delta(d) \geq 1\}, (a_n) \subset A_0$, such that

$$\lim_{n\to\infty\tau}(|h(d_n + a_n)|^2) = \Delta(|h|^2) = \Delta(h)^2 = 1.$$

Claim: we may assume the d_n's to be positive. To see this, notice that the invertibility of the d_n's means that for each n we can find a unitary $u_n \in \mathcal{D}$ so that $d_n u_n = |d^*{}_n|$. Since for each n we have

$$\tau(|h(d_n + a_n)|^2) = \tau(|h(d_n + a_n)u_n|^2) = \tau(|h(|d^*{}_n| + a_n u_n)|^2),$$

the claim follows. Notice that then $\tau(\Phi(h)d_n) = \tau(d_n^{1/2}\,\Phi(h)d_n^{1/2}\,) \geq 0$. Using in turn the L^2-contractivity of Φ, the fact that $\Phi(h(d_n + a_n)) = \Phi(h)d_n$, and Hölder's inequality, we conclude that

$$\tau(|h(d_n + a_n)|^2) \geq \tau(|\Phi(h)d_n|^2) \geq \tau(|\Phi(h)d_n|)^2$$
$$\geq \tau(\Phi(h)d_n)^2 \geq \Delta(\Phi(h))^2 = 1.$$

Since $\lim_{n\to\infty} \tau(|h(d_n + a_n)|^2) = 1$, we must therefore also have $\lim_{n\to\infty} \tau(\Phi(h)d_n) = 1$.But if this is the case then equation (2) assures us that $h(d_n + a_n) \to 1$ in L^2-norm as$n \to \infty$. That is, $1 \in [hA]_2$. Clearly h must then be outer.Now let $h \in [A]_1$. By noncommutative Riesz factorization (see [181]) we may select $h_1, h_2 \in [A]_2$ so that $h = h_1 h_2$. Since$\Delta(h_1)\Delta(h_2) = \Delta(h) = \Delta\big(\Phi(h)\big) = \Delta(\Phi(h_1))\Delta(\Phi(h_2)) > 0$ and $\Delta(h_i) \geq \Delta(\Phi(h_i))$ for each $i = 1, 2$ (by the generalized Jensen inequality), we must have $\Delta(h_i) = \Delta(\Phi(h_i))$ for each $i = 1, 2$. Thus both h_1 and h_2 must be outer elements of $[A]_2$. Consequently

$[hA]_1 = [h_1 h_2 A]_1 = [h_1[h_2 A]_2]_1 = [h_1[A]_2]_1 = [[h_1[A]_2]_2]_1 = [[A]_2]_1 = [A]_1$, so that h is outer as required.

Note that: In the general non-antisymmetric case, one can have outers with

$\Delta(h) = 0$. Indeed in the case that $A = M = L\infty[0,1]$, then outer functions in L^2 are exactly the ones which are a.e. nonzero. One can easily find an increasing function $h:[0,1] \to (0,1]$ satisfying $\Delta(h) = 0$, or equivalently $\int_0^1 \log h = -\infty$. See also [280]. This prompts the following:

Definition (2.2.11)[66]: We say that h is strongly outer if it is outer and $\Delta(h) > 0$.Note that if $dim(D) < \infty$, then every outer h is strongly outer.

Corollary (2.2.12)[66]: Let $1 \leq p,q,r \leq \infty$ with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ and let $h = h_1 h_2$ with $h_1 \in H^q$and $h_2 \in H^r$. If $\Delta(h) > 0$ then h is outer in H^r iff both $h_1$and h_2 are outer.

Corollary (2.2.13)[66]: If$f \in L^p(D)$ with $\Delta(f) > 0$ then f is outer. Indeed there exist $d_n \in \mathcal{D}$ with $\Delta(fd_n) \geq 1$, and $fd_n \to 1$ in 2-norm. Also, any $f \in L^p(M)$ with $\Delta(f) > 0$ has left and right support projections equal to 1. That is, as an unbounded operator it is one-to-one and has dense range.

Proof: For the first assertion note that $\Phi(f) = f$ and so $\Delta(f) = \Delta(\Phi(f)) > 0$. An inspection of the proof of the theorem gives the d_n with the asserted properties. Thus f clearly has left support projection 1, and by symmetry the right projection is 1 too. Finally note that for the last assertion we may assume that $M = \mathcal{D}$.

Corollary (2.2.14)[66]: If $1 \leq p \leq \infty$and $\Delta(h) > 0$ then h is outer in H^p iff $[Ah]_p = H^p$.

Proof: Replacing A by A^*, it is trivial to see that $\Delta(h) = \Delta(\Phi(h)) > 0$, is equivalent to $\Delta(h^*) = \Delta(\Phi(h^*)) > 0$. The latter is equivalent to h∗ being outer in $H^2(A^*) = (H^2)^*$; or equivalently, to $(H^2)^* = [h^*A^*]_2$. Taking adjoints again gives the result.

Proposition (2.2.15)[66]: If $h \in H^2$, then h is outer iff the wandering subspace of $[hA]_2$(see [280, 62]) has a separating cyclic vector for the $\mathcal{D}$ action, and

$$\|\Phi(h)\|^2 = inf\{\tau(|h(1-a_0)|^2): a_0 \in A_0\}.$$

Proof: (Following [281].) For $x \in L^1(M)$ set $\delta(x) = inf\{\tau(||x|^{\frac{1}{2}}(1-a_0)|^2): a_0 \in A_0\}$.

First suppose that $h \in H^2$ is outer. Then $[hA]_2 \ominus [hA_0]_2 = H^2 \ominus [A_0]_2 = L^2(\mathcal{D})$, which has a separating cyclic vector. We next prove that if $h \in H^2$ is outer, then $\|\Phi(h)\|^2 = \delta(|h|^2)$. To do this we view Φ as the orthogonal projection from $L^2(M)$ onto $L^2(\mathcal{D})$, which restricts to the orthogonal projection P from $[A]_2$ onto $L^2(\mathcal{D})$. For any orthogonal projection P from a Hilbert space onto a subspace K, we have $\|P(\zeta)\| = inf\{\|\zeta - \eta\|: \eta \in K^\perp\}$. Thus$\|\Phi(h)\|^2 = inf\{\tau(|h-a_0|^2): a_0 \in [A_0]_2\}$.Since h is outer, we have $[[hA]_2 A_0]_2 = [H^2 A_0]_2, or\ [hA_0]_2 = [A_0]_2$. Thus

$$\|\Phi(h)\|^2 = inf\{\tau(|h-ha_0|^2: a_0 \in A_0\} = \delta(|h|^2).$$

Conversely, suppose that the wandering subspace of $[hA]_2$ has a separating cyclic vector. By [57], we have $[hA]_2 = uH^2$ for a unitary $u \in [hA]_2 \subset H^2$. We have $h = uk$, with $k \in H^2$, and $[A]_2 = u^*[hA]_2 = H^2$. So k is outer. If $\|\Phi(h)\|^2 = \delta(h)$, then using the notation in the last paragraph,

$$\|\Phi(u)\Phi(k)\|^2 = \delta(|uk|^2) = \delta(|k|^2) = \|\Phi(k)\|^2.$$

That is, $\tau(\Phi(k)^*(1-\Phi(u)^*\Phi(u))\Phi(k)) = 0$. Since by Lemma (2.2.8) the left support projection of $\Phi(k)$ is 1, the functional $a \to \tau(\Phi(k)^* a\Phi(k))$ is faithful on M_+ (indeed, $\tau(\Phi(k)^* a\Phi(k) \neq 0$ for any non-zero $a \in M_+$, which forces $\Phi(u)^*\Phi(u) = 1$. A simple computation shows that $\Phi(|u-\Phi(u)|^2)=1-\Phi(u)^*\Phi(u)=0$, so that $u=\Phi(u)\in\mathcal{D}$.Thus h = uk is outer.A classical theorem of Riesz-Szegö states that if $f\in L^1$ with $f\geq 0$, then R log $f>-\infty$ iff $f=|h|$ for an outer $h\in H^1$ iff $f=|h|^2$ for an outer $h\in H^2$. We now turn to this issue in the noncommutative case.

We are adapting ideas of Helson-Lowdenslager and Hoffman:

Lemma (2.2.16)[66]: Suppose that A is a maximal subdiagonal algebra, and that $k\in L^2(M)$with $k\notin[kA_0]_2$. Let v be the orthogonal projection of k onto $[kA_0]_2$. Then $|k-v|^2=\Phi(|k-v|^2)\in L^1(\mathcal{D})$. Also, $\Delta(|k-v|)\geq\Delta(k)$.

Proof: Suppose that $ka_n\to v$,with $a_n\in A_0$.Clearly $k-v\perp k(1-a_n)a_0\in kA_0$for all $a_0\in A_0$. In the limit, $k-v\perp(k-v)a_0$.That is,$\tau(|k-v|^2a_0)=0$, which by [59] implies that $|k-v|^2=\Phi(|k-v|^2)\in L^1(\mathcal{D})$.For the last assertion, note that by Lemma (2.2.2) we have $\Delta(|k-v|^2)=inf\{\tau(|(k-v)d|^2):d\in\mathcal{D}_+\text{with }\Delta(d)\geq 1\}$.But since $vd\in[kA_0]_2$ for every $d\in\mathcal{D}$, we may apply Szegö's theorem to conclude that this infimum majorises $\inf\{\tau(|kd-ka_0|^2):d\in D_+\text{with } \Delta(d)\geq 1,a_0\in A_0\}=\Delta(|k|^2)=\Delta(k)^2$,using the fact that $|kd-ka_0|=||k|(d-a_0)|$.

Theorem (2.2.17)[66]: Suppose that A is a maximal subdiagonal algebra, and that $k\in L^2(M)$. Let $v$be the orthogonal projection of k onto $[kA_0]_2$. If $\Delta(k)>0$, then k has an (essentially unique) Beurling-Nevanlinna factorization $k=uh$, where u is a unitary in M, and equals the partial isometry in the polar decomposition of $k-v$, and h is strongly outer and equals u^*k. We also have $\Delta(k)=\Delta(k-v)$.If$|k-v|$is bounded below then $(k-v)d=u$ for some $d\in\mathcal{D}$.

Proof: By Corollary (2.2.5), $k\notin[kA_0]_2$. By the Lemma, $|k-v|_2\in L^1(\mathcal{D})$. Let u be the partial isometry in the polar decomposition of $k-v$. Since $\Delta(k-v)\geq\Delta(k)>0$ by the Lemma, we deduce from Corollary (2.2.13) that u is surjective, and hence is a unitary. In the case that $|k-v|$ is bounded below let $d=|k-v|-1\in\mathcal{D}_+$, and then $u=(k-v)d$. Let $h=u^*k\in L^2(M)$. We claim that $\tau(u^*ka_0)=0$ for all $a_0\in A_0$, so that $h=u^*k\in L^2(M)\ominus[A_0^*]_2=H^2$. To see this, let e_n be the spectral projection of $|k-v|$ corresponding to the interval $[0,1/n]$. Then by elementary spectral theory, and since $k-v=u|k-v|$, we have $1-e_n=|k-v|r$ for some $r\in\mathcal{D}$. (Take $r=g(|k-v|)$ where g is $\frac{1}{t}\chi(\frac{1}{n},\infty)$). Thus

$$\tau(a_0^*k^*u(1-e_n))=\tau(a_0^*k^*(k-v)r)=0,$$

since $ka_0r^*\in[kA_0]_2$ and $k-v\perp[kA_0]_2$. On the other hand, by the Borel functional calculus it is clear that $e_n\to e$ strongly, where e is the spectral projection of $|k-v|$ corresponding to $\{0\}$. Since $\Delta(|k-v|)\geq\Delta(k)>0$ by the Lemma, it is easy to see by spectral theory that $e=0$ (this is essentially corresponds to the fact that a positive function f which is 0 on a nonnull set has $\int log\, f=-\infty$). We conclude that $(a_0^*k^*ue_n)\to 0$, and it follows that

$$\tau(a_0^*k^*u)=\tau(a_0^*k^*ue_n)+\tau(a_0^*k^*u(1-e_n))=0.$$

To see that u^*k is outer, we will use the criterion in Theorem (2.2.10). We claim that $\Phi(u^*k)=|k-v|$. To see this, note that by the last paragraph we have $\tau(u^*x)=0$ for any $x\in[kA_0]_2$ and in particular for $x=vc$ for any $c\in\mathcal{D}$. We have

$$\tau(\Phi(u^*k)c)=\tau(u^*kc)=\tau(u^*(k-v)c)=\tau(|k-v|c).$$

Since this holds for any $c \in \mathcal{D}$ we have $\Phi(u^*k) = |k - v|$. Thus we have by the generalized Jensen inequality (2.2.4) that

$$\Delta(k) = \Delta(u^*k) \geq \Delta(\Phi(u^*k)) = \Delta(|k - v|) \geq \Delta(k).$$

Hence $h = u^*k$ is outer by Theorem (2.2.10).

The uniqueness now follows.

Corollary (2.2.18)[66]: Suppose that A is a maximal subdiagonal algebra with $\mathcal{D}$finite dimensional, and that $k \in L^2(M)$ with $\Delta(k) > 0$. Let v be the orthogonal projection of k onto $[kA_0]_2$. Then $|k - v|$ is invertible, and all the conclusions of the previous theorem hold.

Proof: By the above, $|k - v| \in L^1(\mathcal{D}) = \mathcal{D}$, and $\Delta(|k - v|) \geq \Delta(k) > 0$. Thus $|k - v|$is invertible since $\mathcal{D}$ is finite dimensional. The rest follows from the previous theorem.

We next give a refinement of the 'Riesz factorization' into a product of two H^2 functions:

Corollary (2.2.19)[66]:. If A is a maximal subdiagonal algebra with $\mathcal{D}$finite dimensional, and if $f \in L^1(M)$ with $\Delta(f) > 0$, then there exists an outer $h^2 \in H^2$, an invertible $d \in \mathcal{D}$ with $\Delta(d) = \frac{1}{\sqrt{\Delta(f)}}$, and an $h_1 \in [fA_0]_1$ such that $f - h_1 \in L^2(M)$, and $f = (f - h_1)dh_2$. If also $f \in H^1$, then this can be arranged with $h_1 \in H^1, \Phi(h_1) = 0$, and $f - h_1 \in H^2$.

Proof: Let $k = |f|^{\frac{1}{2}}$. By Corollary (2.2.5) we have $k \notin [kA_0]_2$. If u, v are as in Theorem (2.2.16), and if $f = w|f| = wk^2$ is the polar decomposition of f, then

$$f = (wku)(u^*k) = (wk(k - v))dh_2 = (f - h_1)dh_2$$

where $h_2 = u^*k$ and $h_1 = wkv$.

If $ka_n \to v$ in L^2 norm, with $a_n \in A_0$, then $fa_n = wk^2a_n \to wkv$ in L^1 norm. Thus $h_1 \in [fA_0]_1$. Also, $f - h_1 = wkud^{-1} \in L^2(M)$ (recall that since $\mathcal{D}$ is finite dimensional, $d - 1 = |k - v| \in \mathcal{D}$). If $f \in H^1$, then $h_1 \in [fA_0]_1 \subset H^1$, and $\Phi(h_1) = 0$.So $f - h_1 \in H^1 \cap L^2(M) \subset L^2(M) \ominus [A^*]_2 = H^2$.

Corollary (2.2.20)[66]:.If A is a maximal subdiagonal algebra, and if $f \in L^1(M)$ with $\Delta(f) > 0$, then there exists a strongly outer $h \in H^1$, and a unitary $u \in M$ with $f = uh$.

Proof: By the proof of Corollary (2.2.19), and in that notation, we have $f = wkuh_2$ for an outer h_2. Note that w is a unitary, since f has dense range (Corollary (2.2.13)). Since $\Delta(wku) = \Delta(k) > 0$, we have by the last theorem that $wku = Uh_1$ for a unitary U and strongly outer $h_1 \in H^2$. Let $h = h_1h_2$.

Corollary (2.2.21)[66]: If A is a maximal subdiagonal algebra, and $f \in L^p(M)$ then $\Delta(f) > 0$ iff $f = uh$ for a unitary u and a strongly outer $h \in H^p$. Moreover, this factorization is unique up to a unitary in $\mathcal{D}$.

Proof: (⇒) By Corollary (2.2.20) we obtain the factorization with outer $h \in H^1$. Since $|f| = |h|$ we have $h \in L^p(M) \cap H^1 = H^p$ (using [153]), and $\Delta(h) > 0$. (⇐) We have $\Delta(f) = \Delta(u)\Delta(h) > 0$.

The uniqueness of the factorization was discussed after Proposition (2.2.9).

Note that. The u in the last result is necessarily in $[fA]_p$indeed if $ha_n \to 1$ with $a_n \in A$, then $fa_n = uha_n \to u$.

Corollary (2.2.22)[66]: If A is a maximal subdiagonal algebra, then $f \in H^p$ with $\Delta(f) > 0$ iff $f = uh$ for an inner u and a strongly outer $h \in H^p$. Moreover, this factorization is unique up to a unitary in $\mathcal{D}$.

Proof: Clearly f is also in H^1. Then u is necessarily in $[fA]_p \subset H^1$. So $u \in H^1 \cap M = A$(see [149]). Thus $u$is 'inner' (i.e.is a unitary in $H^\infty = A$).

An obvious question is whether there are larger classes of subalgebras of M besides subdiagonal algebras for which such classical factorization theorems hold. The following shows that, with a qualification, the answer to this is in the negative:

Proposition (2.2.23)[66]: Suppose that $A$is a tracial subalgebra of M in the sense, such that every $f \in L^2(M)$ with $\Delta(f) > 0$ is a product $f = uh$ for a unitary u and an outer h $\in [A]_2$. Then A is a finite maximal subdiagonal algebra.

Proof: Suppose that A is a tracial subalgebra of M with this factorization property. We will show that A satisfies the 'L^2-density' and the 'unique normal state extension' properties which together were shown in [59] to imply that $A$is subdiagonal. As in [59, 153,149], A_∞ is the tracial algebra $A_\infty = M \cap [A]_2$ extending A. If $x \in M$ is strictly positive, then $\Delta(x) > 0$ by e.g. Theorem (2.2.1) (ii). So $x = uh$ for a unitary u and $h \in H^2$. Clearly h is bounded, so that $h \in A_\infty$, and $x = (x^*x)^{\frac{1}{2}} = |h|$. Also, $h^{-1} \in A_\infty$, since if $ha_n \to 1$ then $a_n \to h^{-1}$. Thus A_∞, has the 'factorization' property and so is maximal subdiagonal [59]. Hence $A_\infty + A_\infty^*$, and therefore also $A + A^*$, is dense in $L^2(M)$. Next, suppose that $g \in L^1(M)_+$ satisfies $\tau(gA_0) = 0$. We need to show that $g \in L^1(\mathcal{D})_+$. Since $\tau((g+1)A_0) = 0$, we can replace g with $g + 1$ if necessary, to ensure that $\Delta(g) > 0$. Let $f = g^{\frac{1}{2}} \in L^2(M)$. Then $\Delta(f) > 0, f \perp [fA_0]_2$, and by hypothesis $f = uh$for an outer $h \in [A]_2$ and some unitary u in M. Since $h = u^*f \perp u^* \in [fA]_2 = [hA]_2 = [A_0]_2$, and $h \in [A]_2$, it follows that $h \in [D]_2$. Thus $g \in [\mathcal{D}]_1 = L^1(\mathcal{D})$. This verifies the 'unique normal state extension' property of [59]. The following generalizes [145]:

Corollary (2.2.24)[66]: If $f \in L^1(M)_+$, then the following are equivalent:

(i) $\Delta(f) > 0$,

(ii) $f = |h|^p$for a strongly outer $h \in H^p$,

(iii) $f = |k|^p$ for $k \in H^p$ with $\Delta(\Phi(k)) > 0$.

Proof: (i) ⇒(ii) By a previous result, $\Delta(f^{\frac{1}{p}}) > 0$, and so by the last result we have$f^{\frac{1}{p}} = uh$, where h is outer in H^p, and u is unitary. Thus $f = (f^{\frac{1}{p}} f^{\frac{1}{p}})^{\frac{p}{2}} = (h^*h)^{\frac{p}{2}} = |h|^p$.

(ii) ⇒(iii) This follows from Theorem (2.2.10) .

(iii) ⇒(i) If $f = |k|^p$ for $k \in H^p$ with $\Delta(\Phi(k)) > 0$, then $\Delta(f) = \Delta(k)p \geq \Delta(\Phi(k))p > 0$ by Theorem (2.2.1) and the generalized Jensen inequality.

Of course in the case that $\mathcal{D}$ is finite dimensional one can drop the word 'strongly' in the last several results. In particular, in the case that the algebra A is antisymmetric, these results and their proofs are much simpler and are spelled out in our survey [62].

Chapter 3
The Algebraic Structure and Quasi-Radial Quasi-Homogegeneous Symbols

The k-dimensional representations form a generalized maximal ideal space with a canonical surjection onto the ball of $k \times k_n$ matrices which is a homeomorphism over the open ball analogous to the fibration of the maximal ideal space of H^∞ over the unit disk. The algebras are non conjugated via biholomorphisms of the unit ball, non of them is a C^*-algebra, and for $n = 1$ all of them collapse to the algebra generated by Toeplitz operators with radial symbols.

Section (3.1)The Non-Commutative Analytic Toeplitz Algebras

In [68, 208, 211, 212], a good case is made that the appropriate analogue for the analytic Toeplitz algebra in n non-commuting variables is the WOT-closed algebra generated by the left regular representation of the free semigroupon n generators. It obtain a compelling analogue of Beurling's theorem and inner-outer factorization. We add further evidence. The result is a short exact sequence determined bya canonical homomorphism of the automorphism group onto this algebra onto the group of conformal automorphisms of the unit ball of $\mathbb{C}^N$. The kernelis the subgroup of quasi-inner automorphisms, which are trivial modulothe WOT-closed commutator ideal. Additional evidence of analytic properties comes from the structure of k-dimensional (completely contractive) representations, which have a structure very similar to the fibration of the maximal ideal space of H^∞ over the unit disk. An important tool in our analysis is a detailed structure theory for WOT-closed right ideals. Curiously,left ideals remain more obscure.

The non-commutative analytic Toeplitz algebra $\mathfrak{L}_n$ is determined by the left regular representation of the free semigroup $\mathcal{F}_n$ on n generators $z_1, \dots, z_n$which acts on $\ell_2(\mathcal{F}_n)$ by $\lambda(\omega)\, \xi_v = \xi_{\omega v}$ for v, ω in $\mathcal{F}_n$. In particular, the algebra $\mathfrak{L}_n$ is the unital, WOT-closed algebra generated by the isometries $L_I = \lambda(z_i)$ for $1 \le i \le n$. This algebra and its norm-closed version (the non-commutativedisk algebra) were introduced by Popescu [214] in an abstract sense in connection with a non-commutative von Neumann inequality and further studied in [208, 214, 209, 212, 213]. For $n = 1$, weobtain the algebra generated by the unilateral shift, the analytic Toeplitz algebra. The corresponding algebra for the right regular representation is denoted $\mathfrak{R}_n$. This algebra is unitarily equivalent to $\mathfrak{L}_n$ and is also equal to the commutantof $\mathfrak{L}_n$. (see [68,98]).

It contains the classification of the WOT-closed right and two-sided ideals of $\mathfrak{L}_n$. These ideals are determined by their range, which is always a subspace in $Lat\mathfrak{R}_n$; and this pairing is a complete lattice isomorphism.The ideal is two-sided when the range is also in $Lat\ \mathfrak{L}_n$. This is the keytool needed to establish classify the weak-$*$ continuous multiplicative linear functionals on $\mathfrak{L}_n$. We obtain some factorization results for right ideals that allow us to show that a WOT-closed right ideal is finitely generated algebraically precisely when the wandering subspace of the range space is finite dimensional; and otherwise, they require a countably infinite set of generators even as a WOT-closed right ideal.

We examine the representation space of $\mathfrak{L}_n$. The multiplicative linear functionals have a structure that parallels the maximal ideal space of H^∞. This provides a natural homomorphism of $\mathfrak{L}_n$ into the space $H^\infty(\mathbb{B}_n)$ of bounded analytic functions on the ball. Strikingly, the dilation theory for non-commuting n-tuples allows us to obtain an analogous structurefor k-

dimensional representations for every $k < \infty$. In particular, the open ball $\mathbb{B}_{n,k}$ of strict contractions in $\mathcal{M}_{k,nk}$ sits homeomorphically in a canonical way in this space.

Automorphisms of $\mathfrak{L}_n$ are shown to be automatically norm and WOT continuous. We show that there is a natural homomorphism from $Aut\mathfrak{L}_n$ onto $Aut(\mathbb{B}_n)$, the group of conformal automorphisms of the ball of $\mathbb{C}^n$, determined by their action on the WOT $-$continuous linear functional φ_λ for$\lambda \in \mathbb{B}_n$. The kernel of this map is the ideal of automorphisms which are trivial modulo the WOT-closed commutator ideal. In order to show that this homomorphismis surjective, we determine all automorphisms of $\mathfrak{L}_n$ of the form AdW for unitary W. Using certain automorphisms of the Cuntz-Toeplitz algebra found by Voiculescu [221,299], we are able to obtain an isomorphism of this subgroup $Aut_u(\mathfrak{L}_n)$ with $Aut(\mathbb{B}_n)$. Thus the automorphism group of $\mathfrak{L}_n$ is a semidirect product.

We will write $L = [L_1 \dots L_2]$ both for then-tuple of isometries and the isometric operator from $\mathcal{H}_n^{(n)}$into $\mathcal{H}_n$. By L_v or $v(L)$we will denote the corresponding word $\lambda(v)$ in the n tuple. We allow $n = \infty$. In this case, $\mathbb{C}^n$ is replaced by a separable Hilbert space $\mathcal{H}$, and the unit ball $\mathbb{B}_n$becomes the unit ball of $\mathcal{H}$ endowed with the weak topology.

This occasionally causes additional difficulties which will be pointed out as necessary.

The full Fock space of a Hilbert space $\mathcal{H}$ is the Hilbert space

$$F(\mathcal{H}) = \sum_{k\geq 0} \oplus \mathcal{H}^{\oplus k}$$

where $\mathcal{H}^{\oplus 0} = \mathbb{C}$ and $\mathcal{H}^{\oplus k}$ is the tensor product of k copies of $\mathcal{H}$. When $\mathcal{H} = \mathbb{C}^n$ with orthonormal basis $\zeta_i$for $1 < i < n$, the Fock space has anorthonormal basis $\zeta_\omega = \zeta_{i1} \otimes \dots \otimes \zeta_{ik}$ for all choices of $\omega = (i_1 \dots i_k)$ in$\{1, \dots, n\}^k$ and $k \geq 0$ (with the convention that $\zeta_\emptyset$ spans $\mathcal{H}^{\oplus 0}$. For each vector ζin $\mathcal{H}$, there is a left creation operator $\ell(\zeta)\xi = \zeta \otimes \xi$. Clearly, there is a natural isomorphism of Fock space onto $\mathcal{H}_n$, where $n = dim\,\mathcal{H}$given by sending ζ_ω to ξ_ω. This unitary equivalence sends $\ell(\zeta_i)$ to L_i.

The following heuristic is useful when working with operators in $\mathfrak{L}_n$. If $A = \sum_\omega a_\omega L_\omega$is a finite linear combination of the set $\{L_\omega : \omega \in \mathcal{F}_n\}$, then $A\xi_1 = \sum_\omega a_\omega \xi_\omega$; conversely, given a finite linear combination of basis vectors $\zeta = \sum_\omega a_\omega \xi_\omega$, the operator $A = \sum_\omega a_\omega L_\omega$ belongs to $\mathfrak{L}_n$ and satisfies $A\xi_1 = \zeta$. Sometimes this operator will be denoted by L_ζ . This correspondence of course cannot be extended to infinite combinations. However, notice that for an arbitrary element A of $\mathfrak{L}_n$, A is completely determined by its action on ξ_1: indeed,$A\xi_v = AR_v\xi_1 = R_vA\xi_1$. So if $A\xi_1 = \sum_\omega a_\omega \xi_\omega$, we have

$$A\xi_v = \sum a_\omega \xi_{\omega v} = \sum_\omega a_\omega (L_\omega \xi_v).$$

It is useful to view the formal sum $\sum_w a_w L_w$ as the Fourier expansion of A.

In particular [68], the Cesaro sums

$$\sum_n (A) = \sum_{|\omega|<n} (1 - \frac{|\omega|}{n})\, a_\omega L_\omega$$

converge in the strong-* topology to A.

The algebra $\mathfrak{L}_n$ contains no non-scalar normal elements. Every non-scalar element of $\mathfrak{L}_n$ is injective and has connected spectrum containing more than one point. So $\mathfrak{L}_n$ contains no non-zero compact operators,quasinilpotent elements or non-scalar idempotents. In particular, $\mathfrak{L}_n$ is semisimple.(See [114,68].)

If $\mathcal{M}$ is an invariant subspace for $\mathfrak{L}_n$, the wandering subspace is $\mathcal{W} = \mathcal{M} \ominus \sum_{i=1}^{n} L_i\mathcal{M}$. By the analogue of the $\mathcal{W}$old decomposition [210], it follows that $\mathcal{M} = \sum_{\omega\in\mathcal{F}_n} \oplus L_\omega \mathcal{W}$. The invariant subspaces of the analytic Toeplitz algebra are determined by Beurling's [40] as the subspaces ωH^2 where ω is an inner function in H^∞. These subspaces are always cyclic with wandering subspace $\omega H^2 \ominus z\omega H^2 = \mathbb{C}\omega$. The subspace ωH^2 is the range of T_ω, which is an isometry in $H^\infty = \mathfrak{L}_1 = \mathfrak{R}_1$. The analogue of Beurling'stheorem is:

Theorem (3.1.1)[156]:([208, 68]). Every invariant subspace of $\mathfrak{L}_n$ is generated by a wandering subspace. Thus it is the direct sum of cyclic subspaces. The cyclic invariant subspaces of $\mathfrak{L}_n$ are precisely the ranges of isometries in $\mathfrak{R}_n$; and the choice of isometry is unique up to a scalar. If $\mathcal{M}$ is a cyclic invariant subspace for $\mathfrak{L}_n$, then its wandering subspace is 1-dimensional. If ξ is a wandering vector for $\mathcal{M}$, then we denote the corresponding isometry in $\mathfrak{R}_n$ by $\mathfrak{R}_\xi$. Explicitly, we have the formula, $\mathfrak{R}_\xi \xi_\omega = L_\omega \xi$. Conversely, any isometry in $\mathfrak{R}_n$is an $\mathfrak{R}_\xi$ for some $\mathfrak{L}_n$-wandering vectorξ: Similarly, we see that any isometry in $\mathfrak{L}_n$ has the form L_ξ for some$\mathfrak{R}_n$-wandering vector ξ.

By analogy, the isometries of $\mathfrak{L}_n$ are called inner; and the elements with dense range are called outer. An element A in $\mathfrak{L}_n$ is inner if and only if $\|A\| = \|A\xi_1\| = 1$. As a corollary, one obtains the following analogue of inner-outer factorization:

Corollary (3.1.2)[156]: Every A in $\mathfrak{L}_n$ factors as $A = L_\xi B$ where L_ξ is an isometry in $\mathfrak{L}_n$ and B belongs to $\mathfrak{L}_n$ and has dense range. This factorization is uniqueup to a scalar. The operator $B$is invertible if and only if A has closed range.

We also need to understand the structure of the eigenvectors for the adjoint analogous to the point evaluations in the unit disk associated to eigen-values of the backward shift.

Theorem (3.1.3)[156]: (cf. [21] and[68,69,196,197]). The eigenvectors for $\mathfrak{L}_n^*$ are the vectors

$$v_\lambda = (1 - \|\lambda\|^2)^{1/2} \sum_{\omega\in\mathcal{F}_n} \overline{\omega(\lambda)}\xi_\omega = (1 - \|\lambda\|^2)^{1/2} (I - \sum_{i=1}^{n} \overline{\lambda_i} L_i)^{-1}\xi_1$$

For λ in the unit ball $\mathbb{B}_n$. They satisfy

$$L_i^* v_\lambda = \overline{\lambda_i} v_\lambda$$

And $(p(L)v_\lambda, v_\lambda) = p(\lambda)$ for every polynomial $p = \sum_\omega a_\omega \omega$ in the semigroup algebra $\mathbb{C}\mathcal{F}_n$. This extends to the map $\varphi_\lambda(A) = (Av_\lambda, v_\lambda)$, which is a WOT-continuous multiplicative linear functional on $\mathfrak{L}_n$. The vector v_λ is cyclic for$\mathfrak{L}_n$. The subspace $\mathcal{M}_\lambda = \{v_\lambda\}^\perp$ is $\mathfrak{L}_n$invariant, and its wandering subspace$\mathcal{W}_\lambda$is n-dimensional, spanned by

$\xi_{\lambda,i} = \lambda_i \xi_1 - (1 - \|\lambda\|^2)^{1/2} L_i v_\lambda$for$1 \le i \le n$

These results are used in [68] to show that $\mathfrak{L}_n$ is hyper-reexive. Moreover,for every weak-* continuous linear functional f on $\mathfrak{L}_n$with $\|f\| < 1$, there are vectors ξ and ζ such that $f(A) = (A\xi, \zeta)$for all A in $\mathfrak{L}_n$ and$\|\xi\|\|\zeta\| < 1$.

This yields the immediate consequence which will be important on several occasions.

Corollary (3.1.4)[156]: ([68,199]). The weak-* and WOT topologies on $\mathfrak{L}_n$ coincide.

We identify the WOT-closed right and two-sided ideals of $\mathfrak{L}_n$. Let $Id_r(\mathfrak{L}_n)$,$Id_\ell(\mathfrak{L}_n)$ and $Id(\mathfrak{L}_n)$ denote the sets of all WOT-closed right, left and two-sided ideals respectively. The important observation is that these ideals can be identified by their ranges. If $\mathfrak{J}$ belongs to $Id_r(\mathfrak{L}_n)$, then the subspace $\overline{\mathfrak{J}\xi_1}$belongs to $Lat\mathfrak{R}_n$. To see this, note that

$$\mathfrak{R}_n\overline{\mathfrak{J}\xi_1} = \overline{\mathfrak{J}\mathcal{H}_n\xi_1} = \overline{\mathfrak{J}\mathcal{H}_n} = \overline{\mathfrak{J}\mathfrak{L}_n\xi_1} = \overline{\mathfrak{J}\xi_1}$$

Thus $\overline{\mathfrak{J}\xi_1} = \overline{\mathfrak{J}\mathcal{H}_n}$ is the range of $\mathfrak{J}$ and is $\mathfrak{R}_n$ invariant.

When $\mathfrak{J}$ belongs to $\mathrm{Id}_\ell(\mathfrak{L}_n)$; we have $\mathfrak{L}_n\mathfrak{J}\xi_1 = \mathfrak{J}\xi_1$; so $\overline{\mathfrak{J}\xi_1}$ is $\mathfrak{L}_n$ invariant.

Hence when $\mathfrak{J}$ is a two-sided ideal, $\overline{\mathfrak{J}\xi_1}$ belongs to $Lat(\mathfrak{L}_n)\cap Lat(\mathfrak{R}_n)$:

Conversely, if $\mathcal{M}$ belongs to $Lat(\mathfrak{R}_n)$, we shall see during the proof of Theorem (3.1.5) that the set $\{A \in \mathfrak{L}_n : A\xi_1 \in \mathcal{M}\}$ belongs to $Id_r(\mathfrak{L}_n)$. It will follow that when $\mathfrak{J}$ is a right ideal, the subspace $\overline{\mathfrak{J}\xi_1}$determines $\mathfrak{J}$ and moreover $\mathfrak{J}$ is two-sided precisely when $\overline{\mathfrak{J}\xi_1}$is also $\mathfrak{L}_n$ invariant.

We do not make the same claims for left ideals. One should note that when $\mathfrak{J}$ is a left ideal, $\overline{\mathfrak{J}\xi_1}$is not equal to $\overline{\mathfrak{J}\mathcal{H}_n}$. The full range of the ideal is not a complete invariant. There are technical difficulties for left ideals that we were not able to resolve; but analogous results are plausible.

We remark that $Id_r(\mathfrak{L}_n)$ and $Id(\mathfrak{L}_n)$ form complete lattices with the operations of intersection and WOT-closed sum.

Theorem (3.1.5)[156]: Let $\mu: Id_r(\mathfrak{L}_n) \to Lat(\mathfrak{R}_n)$ be given by $\mu(\mathfrak{J}) = \overline{\mathfrak{J}\xi_1}$. Then μ a complete lattice isomorphism. The restriction of μ to the set $\mathrm{Id}(\mathfrak{L}_n)$ is a complete lattice isomorphism onto $Lat\,\mathfrak{L}_n \cap Lat\mathfrak{R}_n$. The inverse map ι sends a subspace $\mathcal{M}$ to

$$\iota(\mathcal{M}) = \{J \in \mathfrak{L}_n : J\xi_1 \in \xi\}$$

Proof: We have seen above that $\mathcal{M} = \mu(\mathfrak{J})$ is a subspace of the appropriate type for right and two-sided (and even left) ideals.

Conversely, we now check that ι sends invariant subspaces to ideals of the appropriate type. So fix a subspace $\mathcal{M}$ in $Lat(\mathfrak{R}_n)$ and consider $\iota(\mathcal{M})$. It is clear that $\iota(\mathcal{M})$ is a WOT −closed subspace of $\mathfrak{L}_n$. Suppose that J is in $\iota(\mathcal{M})$and A belongs to $\mathfrak{L}_n$. Then

$$JA\xi_1 \in \overline{J\mathcal{H}_n} = \overline{J\mathfrak{R}_n\xi_1} = \overline{\mathfrak{R}_n J \xi_1} \subset \mathcal{M}$$

Whence $\iota(\mathcal{M})$ is a right ideal. And if $\mathcal{M}$ is in $Lat\,\mathfrak{L}_n$, then for J in $\iota(\mathcal{M})$ and A in $\mathfrak{L}_n$, :

$$A J \xi_1 \in A\mathcal{M} \subset \mathcal{M}$$

So $\iota(\mathcal{M})$ is a left ideal. Thus $\iota(Lat\mathfrak{L}_n \cap Lat\mathfrak{R}_n)$ is contained in $Id(\mathfrak{L}_n)$.

Next we show that μι is the identity map. Fix $\mathcal{M}$ in $Lat(\mathfrak{R}_n)$. By the definitions of the maps involved, we have $\mu\iota(\mathcal{M})$ is contained in $\mathcal{M}$. To see the opposite inclusion, let $\{\zeta_j\}$ be an orthonormal basis for the $\mathfrak{R}_n$ wandering subspace $\mathcal{W} = \mathcal{M} \ominus \sum_{i=1}^{n} \oplus R_i\mathcal{M}$. Then

$$\mathcal{M} = \sum_j \oplus \mathfrak{R}_n[\zeta_j] = \sum_j \oplus Ra_n L_{\zeta_j}$$

Since $L_{\zeta_j}\xi_1 = \zeta_j$ belongs to $\boldsymbol{\mathcal{M}}$, it follows that L_{ζ_j} lies in ι($\mathcal{M}$). So

$$\sum_j \oplus Ra_n L_{\zeta_j} \subset \overline{\iota(\mathcal{M})\mathcal{H}_n} = \overline{\iota(\mathcal{M})\xi_1} = \mu(\iota(\mathcal{M}))$$

Therefore $\mu\,\iota(\mathcal{M}) = \mathcal{M}$.

Now fix $\mathfrak{J}$ in $Id_r(\mathfrak{L}_n)$. As before, the defnitions involved show that $\mathfrak{J}$ is contained in $\iota\,\mu\,(\mathfrak{J})$. To see that this is an equality, we first show that for every $\mathfrak{J}$ in $\mathcal{H}_n$,

$$\overline{\mathfrak{J}\xi} = \overline{\iota\,\mu\,(\mathfrak{J})\xi} \qquad (1)$$

Since $\mathfrak{L}_n|\xi|$ is a cyclic invariant subspace for $\mathfrak{L}_n$, it may be written as $\mathfrak{L}_n[\xi] = RanR_\eta$ where η is a wandering vector for $\mathfrak{L}_n[\xi]$. Thus

$$\overline{\mathfrak{J}\xi} = \overline{\mathfrak{J}\mathfrak{L}_n\xi} = \overline{\mathfrak{J}\,R_\eta\mathcal{H}_n} = \overline{R_\eta\mathfrak{J}\,\mathcal{H}_n} = R_\eta\mathcal{M}$$

evidently, the same computation for ι μ ($\mathfrak{J}$) yields the same result; hence (1) holds.

Suppose that f is a WOT-continuous linear functional on $\mathfrak{L}_n$ which annihilates the ideal $\mathfrak{J}$. By [68,152], there are vectors ξ and η such that $f(A) = (A\xi, \eta)$ for all A in $\mathfrak{L}_n$. Since $f(\mathfrak{J}) = 0$, it follows that η isorthogonal to $\mathfrak{J}\xi$. Then by the previous paragraph, η is also orthogonal to$\iota\,\mu\,(\mathfrak{J})\xi$ and thus f also annihilates $\iota\,\mu\,(\mathfrak{J})$. By the Hahn-Banach Theorem,we therefore have $\mathfrak{J} = \iota\,\mu\,(\mathfrak{J})$.

Thus we have established that μ is a bijective pairing between $Id_r(\mathfrak{L}_n)$and $Lat\mathfrak{R}_n$ which carries $Id(\mathfrak{L}_n)$ onto $Lat\mathfrak{L}_n \cap Lat\mathfrak{R}_n$and $\iota = \mu^{-1}$. If $\mathfrak{J}_1$and $\mathfrak{J}_2$ are WOT-closed right ideals, then

$$\mu(\mathfrak{J}_1 + \mathfrak{J}_2) = \overline{(\mathfrak{J}_1 + \mathfrak{J}_2)\mathcal{H}_n} = \overline{\mathfrak{J}_1\mathcal{H}_n + \mathfrak{J}_2\mathcal{H}_n} = \mu(\mathfrak{J}_1) \vee \mu(\mathfrak{J}_2)$$

and hence sums are sent to spans. Similarly, if$\mathcal{M}_1$ and $\mathcal{M}_2$ are subspaces in $Lat(\mathfrak{L}_n) \cap Lat\mathfrak{R}_n$, then

$$\iota(\mathcal{M}_1 \cap \mathcal{M}_2) = \{J \in \mathfrak{L}_n : J\xi_1 \in \mathcal{M}_1 \cap \mathcal{M}_2\} = \{J \in \mathfrak{L}_n : J\xi_1 \in \mathcal{M}_1\} \cap \{J \in \mathfrak{L}_n : J\xi_1 \in \mathcal{M}_2\} = \iota(\mathcal{M}_1) \cap \iota(\mathcal{M}_2)$$

It follows that μ preserves intersections. Finally, to see that μ is complete, note that if $\mathfrak{J}_k$ is an increasing union (or decreasing intersection) of ideals,we have

$$\mu\left(\overline{\bigcup_k \mathfrak{J}_k}\right) = \overline{\bigcup_k \mathfrak{J}_k\,\mathcal{H}_n} = \bigvee_k \mu(\mathfrak{J}_k)$$

and similarly for intersections. Therefore μ is a complete lattice isomorphism.

Corollary (3.1.6)[156]: If J belongs to $\mathfrak{L}_n$, then the WOT-closed (two-sided) ideal$\langle J\rangle$generated by J consists of all elements A in $\mathfrak{L}_n$ such that $A\xi_1$ lies in$\mathfrak{L}_n J\mathcal{H}_n$.

Proof: The ideal $\langle J\rangle$ is determined by its range, and this must be the least element $\mathcal{M}$ of $Lat(\mathfrak{L}_n) \cap Lat\mathfrak{R}_n$containing J$\xi_1$. Thus

$$\mathcal{M} = \overline{\mathfrak{L}_n\mathfrak{R}_n J\xi_1} = \overline{\mathfrak{L}_n\,J\,\mathfrak{R}_n\xi_1} = \overline{\mathfrak{L}_n\,J\mathcal{H}_n}$$

By Theorem (3.1.5), it follows tha$\langle J\rangle = \iota(\mathcal{M})$t.

Theorem (3.1.5) enables us to characterize the WOT-continuous multiplicative linear functionals on $\mathfrak{L}_n$.

Theorem (3.1.7)[156]: Suppose φ is a (non-zero) WOT-continuous multiplicative linear functional on $\mathfrak{L}_n$. Then there exists λ in $\mathbb{B}_n$ such that $\varphi = \varphi_\lambda$

Proof: Let $\mathfrak{J} = ker\,\varphi$. Then $\mathfrak{J}$is a WOT-closed two sided maximal ideal of codimension one. Set $\mathcal{M} = \mu(\mathfrak{J})$ and note that $\xi_1 \notin \mathcal{M}$. (If not, Theorem (3.1.5) implies I belongs to $\iota(\mathcal{M}) = \mathfrak{J}$, which is impossible.) In fact,$\mathcal{M}$ has codimension one. To see this, let $\mathcal{N} = \mathcal{M} + \mathbb{C}\xi_1$. If $\mathcal{N} \neq \mathcal{H}_n$, let ζ be a unit vector in $\mathcal{N}^\perp$. Choose A in $\mathfrak{L}_n$so that $\|\zeta - A\xi_1\| < 1$ and set $\alpha = \varphi(A)$. Since $A - \alpha I$ is in $\mathfrak{J}, A\xi_1 - \alpha\xi_1$belongs to $\mathcal{M}$. Hence

$$1 = |(\zeta - \alpha\xi_1, \zeta)| = |(\zeta - A\xi_1, \zeta) + (A\xi_1 - \alpha\xi_1, \zeta)| = |(\zeta - A\xi_1, \zeta)| < 1,$$

which is absurd. So $\mathcal{N} = \mathcal{H}_n$ and hence $\mathcal{M}$ belongs to $Lat(\mathfrak{L}_n) \cap Lat\,\mathfrak{R}_n$and has codimension one. Thus $\mathcal{M}^\perp$ is a 1-dimensional invariant subspace for$\mathfrak{L}_n^*$. By Theorem (3.1.3), there is a point λ in $\mathbb{B}_n$ such that $\mathcal{M} = \{v_\lambda\}^\perp$. ByTheorem (3.1,5), $ker\,\varphi = \iota(\mathcal{M}) = ker\,\varphi_\lambda$. Therefore $\varphi = \varphi_\lambda$.

We present, as an example, an ideal which will be important later. Let $\overline{e}$ denote the WOT-closure of the commutator ideal of $\mathfrak{L}_n$. The space $\mathcal{H}_n^s$ is the symmetric Focz space spanned by the vectors $\frac{1}{k!}\sum_{\sigma\in S_k}\xi_{\sigma(\omega)}$, where ω is in$\mathcal{F}_n, k = |\omega|, S_k$ is the symmetric group on k elements, and $\sigma(\omega)$ is the word with the terms in ω permuted by σ. Also recall that for λ in $\mathbb{B}_n$, $\varphi_\lambda$is the multiplicative linear functional on $\mathfrak{L}_n$ given by $\varphi_\lambda(A) = (Av_\lambda, v_\lambda)$ as in Theorem (3.1.3).

Proposition (3.1.8)[156]: The WOT-closure of the commutator ideal is

$$\overline{e} = \langle L_iL_j - L_jL_i : i \neq j\rangle = \bigcap_{\lambda\in\mathbb{B}_n} ker\,\varphi_\lambda$$

The corresponding subspace in $Lat(\mathfrak{L}_n) \cap Lat\,\mathfrak{R}_n$is

$$\mu(e) = span\left\{\xi_{uz_iz_jv} - \xi_{uz_jz_iv} \;:\; i \neq j,\; u, v \in \mathcal{F}_n\right\}$$
$$= \mathcal{H}_n^{s\perp} = span\,\{v_\lambda : \lambda \in \mathbb{B}_n\}^\perp$$

Proof: Let $\mathfrak{J}$ be the WOT-closed ideal generated by the set of commutators $\{L_iL_j - L_jL_i : i \neq j\}$. Clearly $\bar{e} \supset \mathfrak{J}$. On the other hand, consider the set of operators of the form $A(BC - CB)D$for A, B, C, D in $\mathcal{L}_n$. These elements span a WOT-dense subset of $\bar{e}$. Moreover, since the polynomials in the $L_i$are WOT-dense in $\mathcal{L}_n$, we may further suppose that each of A, B, C, D is such a polynomial. Thus by expanding, it suffices to show that operators of the form$L_u(L_vL_\omega - L_\omega L_v)L_x$ belong to $\mathfrak{J}$ for all words u, v, ω, x in $\mathcal{F}_n$. Now, every permutation of $\mathcal{K}$ objects is the product of interchanges $(i, i+1)$ for some$1 \leq i < \mathcal{K}$. Using this, it follows that $L_\omega - L_{\sigma(\omega)}$belongs to $\mathfrak{J}$ for everyω in $\mathcal{F}_n$ and every σin$S_{|\omega|}$. Therefore it follows that $L_u(L_vL_\omega - L_\omega L_v)L_x$belongs to $\mathfrak{J}$. Thus $\mathfrak{J} = \bar{e}$.

The subspace $\mu(e) = \mu(\mathfrak{J})$ is the smallest $\mathcal{L}_n\mathcal{R}_n$invariant subspace containing $\{\xi_{z_iz_j} - \xi_{z_jz_i} : i \neq j\}$ which is the subspace spanned by the vectors of the form $\xi_{uz_iz_jv} - \xi_{uz_jz_iv}$.

It is now clear that $\mathcal{H}_n^s$ is orthogonal to $\mu(e)$.On the other hand, a vector $\zeta = \sum_\omega a_\omega\,\xi_\omega$is orthogonal to $\mu(e)$ if and only if it is orthogonal to every $\xi_\omega - \xi_{\sigma(\omega)}$ for $\omega \in \mathcal{F}_n$and $\sigma \in S_{|\omega|}$. Hence $a_{\sigma(\omega)} = a_\omega$; whence it follows that ζ belongs to $\mathcal{H}_n^s$.

Next we show $span\{v_\lambda : \lambda \in \mathbb{B}_n\} = \mathcal{H}_n^s$. Evidently, each v_λ belongs to $\mathcal{H}_n^s$. Let $Q_k$denote the projection onto $span\{\xi_\omega : |\omega| = k\}$. For each λ in$\mathbb{B}_n$and $z$in $\mathbb{T}$,

$$v_{\bar{z}\lambda} = \sum_{m\geq 0} z^m Q_m v_\lambda$$

Thus by considering the $\mathcal{H}_n$-valued integrals$\int_{\pi} \bar{z}^k v_{\bar{z}\lambda}\, dz$.
For $k \geq 0$, it follows that $Q_k v_\lambda$ lies in $span\{v_\lambda : \lambda \in \mathbb{B}_n\}$. Now it is an easy exercise to show that the set of all $Q_k v_\lambda{}'$s contains each $\frac{1}{k!}\sum_{\sigma \in S_k} \xi_{\sigma(\omega)}$ for $|\omega| = k$. Hence $span\{v_\lambda : \lambda \in \mathbb{B}_n\} = \mathcal{H}_n^s$.
It is clear that the multiplicative linear functionals $\varphi_\lambda$vanish on the commutator,and hence on the γ closed ideal that it generates. Conversely, suppose that A in $\mathfrak{L}_n$is not in é. Then A in ξ_1 is not contained in$\mu(é)$. Therefore, since $\mu(é)$ is the orthogonal complement of the set$\{v_\lambda : \lambda \in \mathbb{B}_n\}$, thereis a λ in $\mathbb{B}_n$ such that

$$0 \neq (A\xi_1, v_\lambda) = (\xi_1, A^* v_\lambda) = \varphi_\lambda(A)(\xi_1, v_\lambda) = (1 - \|\lambda\|^2)^{1/2}\varphi_\lambda(A).$$

Thus $\varphi_\lambda(A) \neq 0$; whence A is not in $ker\ \varphi_\lambda$.
Next we develop some useful lemmas about factorization in right ideals. In particular, they will allow us to determine when a right ideal is finitely generated. Recall from Theorem (3.1.1) that each isometry in $\mathfrak{L}_n$has the form $\mathfrak{L}_\zeta$for some $\mathcal{R}_n$-wandering vector ζ.

Lemma (3.1.9)[156]: Let L_{ζ_j}, for$1 \leq j \leq k$, be a finite set of isometries in $\mathfrak{L}_n$with pairwise orthogonal ranges $\mathcal{M}_j$. *Let* $\mathcal{M} = \sum_j^k \mathcal{M}_j$ and $\mathfrak{J} = \iota(\mathcal{M})$. Then $\mathfrak{J}$ equals $\{A \in \mathfrak{L}_n : Ran(A) \subset \mathcal{M}\}$and every element of $\mathfrak{J}$ factors uniquely as

$$A = \sum_{j=1}^{k} L_{\xi_j} A_j \qquad \text{for} \qquad A_j \in \mathfrak{L}_n$$

Thus the (algebraic) right ideal generated by $\left\{L_{\zeta_j} : 1 \leq j \leq k\right\}$ equals $\mathcal{J}$.

Proof: Clearly each $\mathcal{M}_j$ is $\mathcal{R}_n$invariant, and thus so is $\mathcal{M}$. Hence if A in $\mathfrak{L}_n$satisfies $A\xi_1 \in \mathcal{M}$, then $A\mathcal{H}_n$is contained in $\mathcal{M}$. Thus

$$\mathfrak{J} = \{A \in \mathcal{L}_n : Ran(A) \subset \mathcal{M}\}$$

So $\mathfrak{J}$ is a WOT-closed right ideal containing L_{ζ_j} for $1 \leq j \leq k$.
Conversely, suppose that A belongs to$\mathfrak{J}$. Then since $L_{\zeta_j} L_{\zeta_j}^*$is the orthogonal projection onto $\mathcal{M}_j$, we obtain the factorization.

$$A = \left(\sum_{j=1}^{k} L_{\zeta_j} L_{\zeta_j}^*\right) A = \sum_{j=1}^{k} L_{\zeta_j} A_j,$$

Where $A_j = L_{\zeta_j}^* A$. This decomposition is unique because the$L_{\zeta j}$,s are isometries with orthogonal ranges. We will show that each $A_j$belongs to$\mathcal{L}_n$.As $\zeta_j$is a $\mathcal{R}_n$-wandering vector for$\mathcal{M}$, it is orthogonal to $\sum_{i=1}^n \mathcal{R}_i \mathcal{M}$.
Now$N = Ran(A)$is contained in $\mathcal{M}$, whence ζ_j is also orthogonal to $\sum_{i=1}^n \mathcal{R}_i \mathcal{N}$. Therefore, for any word win $\mathcal{F}_n$,

$$\left(R_i^* A^* \zeta_j \xi_\omega\right) = \left(\zeta_j, A R_i \xi_\omega\right) = \left(\zeta_j, R_i A \xi_\omega\right) = 0,$$

and so $R_i^* A^* \zeta_j = 0$. Now compute using [68]

$$A_j R_i - R_i A_j = L_{\zeta_j}^* A R_i - R_i L_{\zeta_j}^* A = \left(L_{\zeta_j}^* R_i - R_i L_{\zeta_j}^*\right) A$$
$$\xi_1 (R_i^* L_{\zeta_j} \xi_1)^* A = \xi_1 (A^* R_i^* \zeta_j)^* = \xi_1 (R_i^* A^* \zeta_j)^* = 0$$

Therefore $A_j$belongs to $\mathcal{R}'_n = \mathfrak{L}_n$. It is now evident that A belongs to the Algebraic right ideal generated by $\left\{L_{\zeta_j}: 1 \leq j \leq k\right\}$.

An important special case concerns the (two-sided) ideals $\mathfrak{L}_n^{0,k}$ generated by $\{L_\omega: |\omega| = k\}$, which yields a useful decomposition of an arbitrary element of $\mathfrak{L}_n$. In particular, the ideal $\mathfrak{L}_n^0 := \mathcal{L}_n^{0,1}$leads us to a unique decomposition of $\mathcal{L}_n$ as $\mathfrak{L}_n = \mathbb{C}I + \sum_{i=1}^n L_i \mathfrak{L}_n$.

This provides a handle on the algebraic rigidity of $\mathfrak{L}_n$that will prove useful for analyzing the automorphism group.

Corollary (3.1.10)[156]: For $1 \leq n < \infty$ and $k \geq 1$, every A in $\mathfrak{L}_n$ can be written uniquely as a sum

$$A = \sum_{|\omega|<k} a_\omega L_\omega + \sum_{|\omega|=k} L_\omega A_\omega$$

Where $a_\omega \in \mathbb{C}$ and $A_\omega \in \mathfrak{L}_n$.

Proof: The isometries $\{L_\omega: |\omega| = k\}$ have pairwise orthogonal ranges summing to $\mathcal{M} = span\{\xi_v: v \geq k\}$. This subspace is$\mathfrak{L}_n \mathcal{R}_n$ invariant, and thus by Theorem (3.1.5), the right ideal i$(\mathcal{M})$is in fact two-sided. Lemma (3.1.9) shows that ı$(\mathcal{M})$ coincides with $\mathfrak{L}_n^{0,k}$.

Given Ain $\mathfrak{L}_n$, write$A\xi_1 = \sum a_\omega \xi_\omega$. The coefficients a_ω for $|\omega| < k$arethe unique constants such that $\left(A - \sum_{|\omega|<k} a_\omega L_\omega\right)\xi_1$lies in $\mathcal{M}$. Therefore by Lemma (3.1.9), this difference can be written uniquely as $\sum_{|\omega|=k} L_\omega A_\omega$.

Example (3.1.11)[156]: Lemma (3.1.9) is not valid for countably many generators even with norm closure. Indeed, consider the isometries $L_1^k L_2$in $\mathfrak{L}_2$for $k \geq 0$. Their ranges are orthogonal, summing to the $\mathfrak{L}_n \mathcal{R}_n$-invariant subspace generated byξ_{z_2}. So the WOT-closed right ideal $\mathfrak{J}$ that they generate is the two-sided WOT-closed ideal generated by L_2. Consider a sum of the form

$$A = \sum_{k\geq 0} L_1^k L_2 h_k(L_1)$$

Where h_k will be functions in H^∞. This will lie in $\mathfrak{J}$ provided that A is a bounded operator. However, it is a norm limit of finite sums of this type only if the series converges in norm.

An easy computation shows that

$$A^* A = \sum_{k\geq 0} h_k (L_1)^* h_k(L_1).$$

As $L_1$is a unilateral shift of infinite multiplicity, this sums to an operator unitarily equivalent to the infinite implication of a Toeplitz operator with symbol $\sum_{k\geq 0} |h_k|^2$. Thus A is bounded precisely when this sum is bounded.

The sum of operators is norm convergent exactly when this sum of functionsis norm convergent. Constructing a sequence which is bounded but not norm convergent is easy.

The algebra H^∞ is logmodular [118,251,200], and so if f is a non-negative real function in L^∞such that $logf$ is integrable, then there is a function h in H^∞ such that $|h| = f$. Choose a sequence of disjoint closed intervals J_k of the unit circle, each of positive length. Let $f_k = 2^{-k} + \chi_{j_k}$for $k \geq 0$, and let h_k be analytic functions with $|h_k|^2 = f_k$.

Then

$$\sum_{k\geq 0}|h_k|^2 = 2+\chi_J \text{ where } J=\bigcup_{k\geq 0}J_k$$

This sum is bounded. However, $\|h_k\| > 1$ for all k, and thus this sum is not norm convergent. Moreover, this ideal is not finitely generated as a right ideal because $\mathcal{M}$ has an infinite dimensional $\mathcal{R}_\mathrm{n}$ wandering space Ω. Any element J in $\mathfrak{J}$ has $Ran(J)$ contained in $\mathcal{M}$, and its projection onto Ω. is a subspace of at most one dimension. The ranges of a set of generators must necessarily $spanM$; and thus countably many are required.

We need a variant of Lemma (3.1.9) which is valid for countably generated ideals. Let $\mathrm{X}_\mathrm{k}(\mathfrak{L}_\mathrm{n})$ denote the order k column space of $\mathfrak{L}_\mathrm{n}$, which is the set of all k-tuples of the form

$$A=\begin{bmatrix}A_1\\A_2\\\vdots\\A_k\end{bmatrix}, A_i\in\mathfrak{L}_n,\qquad 1\leq i\leq k$$

such that A is bounded with respect to the norm obtained by considering A as an element of $\mathcal{B}(\mathcal{H}_n,\mathcal{H}_n^{(k)})$. Similarly, let $\mathcal{R}_\mathrm{k}(\mathfrak{L}_\mathrm{n})$ denote the order k row space of $\mathfrak{L}_\mathrm{n}$consisting of operators

$$A=[A_1,A_2,\dots A_K], A_i\in\ \mathcal{L}_n,\qquad 1\leq i\leq k$$

such that A is bounded with respect to the norm obtained by considering A as an element of $\mathcal{B}(\mathcal{H}_n^{(k)},\mathcal{H}_n)$.. For $k\ <\ \infty$, this is all k-tuples, but the bounded-ness condition is non-trivial for $k=\ \infty$.

The following lemma shows, in particular, that the infinite row matrix, $L=[L_1,L_2,L_3,\dots]$, maps $\mathcal{C}_\infty(\mathfrak{L}_\infty)$ bijectively onto $\mathfrak{L}_\infty^0$This result also applies to the two-sided WOT-closed ideal $\mathcal{J}$ generated by the set $\{L_2,\cdots,L_n\}$.

The range of this ideal is the sum of the pairwise orthogonal ranges of $\{L_1^kL_j\colon k\geq 0, 2\leq j\leq n\}$.

Lemma (3.1.12)[156]: Let, forL_{ζ_j}, for $j\geq 1$, be a countably infinite set of isometries in $\mathfrak{L}_\mathrm{n}$with pairwise orthogonal ranges $\mathcal{M}_\mathrm{j}$. Let $\mathcal{M}=\sum_{\mathrm{j}=1}^{\infty}\mathcal{M}_{\mathrm{j}}$, and let J be the WOT-closed right ideal $\iota(\mathcal{M})$. Then every element of J factors uniquely as A = ZX, where Z is the fixed isometry in $\mathcal{R}_\infty(\mathfrak{L}_\mathrm{n})$ given by$Z=\left[L_{\zeta_2},L_{\zeta_1},\dots\right]$ and X is a bounded operator in $\mathcal{C}_\infty(\mathfrak{L}_\mathrm{n})$. Hence A can be written uniquely as the WOT limit

$$A=\mathrm{WOT}\underset{k-\infty}{-}Lim\sum_{j=1}^{k}L_{\zeta_j}A_j$$

Proof: The proof begins as in Lemma (3.1.9). There is a unique decompositionof A as a WOT-convergent sum,

$A=\mathrm{WOT}-\sum_{j\geq 1}L_{\zeta_j}X_j$,where$X_j=L_{\zeta_j}^*A$.

The$X_j$are elements of $\mathfrak{L}_\mathrm{n}$ by the same computation. Thus defining X to be the column operator with entries X_j, we obtain a formal factorization $A\ =\ ZX$. To see that X is bounded, it suffices to compute that $X^*X=A^*A$.

Corollary (3.1.13)[156]: Every element A in $\mathfrak{L}_\infty$ decomposes uniquely as

$$\mathrm{A} = \sum_{|\omega|<k} a_\omega L_\omega + \omega_k X_k$$

Where $(a_\omega)_{|\omega|<k}$belongs to $\ell^2, W_k = [L_{\omega_{k,2}}, L_{\omega_{k,1}}, \dots]$, and $\{\omega_{k,i}\}$ is a listing of all words of length k, and X_k belongs to $C_\infty(\mathfrak{L}_\infty)$.

Proof: The identity $A\xi_1 = \sum_{\omega\in\mathcal{F}_n} a_\omega \xi_\omega$determines the coefficients $a_\omega$uniquely, and shows that they belong to ℓ^2. For each j, the isometries $L_{\omega_{j,i}}$have pairwise orthogonal ranges, and hence the sun $\sum_{|\omega|=j} a_\omega L_\omega$is norm convergent. Summing this over $j \ < \ k$ yields the unique operator of this formin the same coset of $A + \mathfrak{L}_\infty^{0,k}$The remainder is factored by Lemma (3.1.12).These lemmas allow us to determine when a right ideal is finitely generated.

Theorem (3.1.14)[156]: Let J be a WOT-closed right ideal. If $\mathcal{M} = \mu(\mathfrak{J})$in $Lat\mathcal{R}_n$has a finite dimensional wandering space of dimension k, then $\mathfrak{J}$ is generated by k isometries as an algebraic right ideal. When this wandering subspace is infinite dimensional, $\mathfrak{J}$ is not finitely generated even as a WOT-closed rightideal. However, it is generated by countably many isometries as a WOT-closed right ideal.

Proof: When the wandering space Ω is finite dimensional, choose an orthonormal basis $\{\zeta_j: 1 \le j \le k\}$. Then $\mathcal{M} = \sum_{j=1}^k \oplus Ran\, L_{\zeta_j}$. Thus by Lemma (3.1.9), the isometries $\{L_{\zeta_j}: 1 \le j \le k\}$ generate $\mathfrak{J}$ as an algebraic right ideal. Similarly, when Ω is infinite dimensional, Lemma (3.1.12) yields a countable set of isometries which generate $\mathfrak{J}$ as a WOT-closed right ideal.

Finally, suppose that $\mathfrak{J}$ is finitely generated as a WOT-closed right ideal, say by $\{A_j: 1 \le j \le k\}$. Then the operators of the form $\sum_{j=1}^k A_j\, B_j$ for B_j in $\mathfrak{L}_n$ are WOT-dense in $\mathfrak{J}$. Therefore

$$\mu(\mathfrak{J}) = \overline{\sum_{j=1}^{k} A_j \mathcal{H}_n} = \mathcal{R}_n[\{A_j\xi_1: 1 \le j \le k\}]$$

This subspace is finitely generated, and therefore has finite dimensional wandering space.

In the category of unital operator algebras, we take the view point that the natural representations are the completely contractive unital representations. Given an operator algebra $\mathfrak{A}$, for each $1 \le k \le \mathcal{N}_0$ we let $Rep_k(\mathcal{Y})$ denote the set of completely contractive representations of $\mathfrak{A}$ into $\mathcal{B}(\mathcal{H})$, where$\mathcal{H}$ is a fixed Hilbert space of dimension k. Put the topology of pointwise weak-* convergence on this space. When $k < \infty$, this is the topology of point wise (norm) convergence. Since the unit ball of $\mathcal{B}(\mathcal{H})$ is weak-* compact(and norm compact when $k < \infty$), Tychonoff's Theorem shows that the set of contractive maps from $\mathfrak{A}$ into $\mathcal{B}(\mathcal{H})$ is pointwise weak-* compact. When $k < 1$, the collection of representations is closed in this topology, and thus is also compact. Unfortunately, the collection of representations is not closed when $k = \infty$. For an example, consider the direct sum $id^{(n)}$ of n copies of the identity representation of $\mathcal{B}(\mathcal{H})$for $n \ \ge \ 1$. Since the direct sumof n copies of the unilateral shift S is unitarily equivalent to S^n, we may find representations σ_n of $\mathcal{B}(\mathcal{H})$ on $\mathcal{H}$such that $\sigma_n(S) = S^n$ for every n. Note that no point wise weak-* limit point of this sequence

of representations is multiplicative, and hence the space of representations is not closed when $k = \infty$.
The natural equivalence relation on representations is unitary equivalence. When k $< \infty$, the unitary group $\mathcal{U}_k$ is compact and acts on $Rep_k(\mathcal{Y})$. Thus the quotient space is also compact and Hausdorff. This need not be the case for $k = \mathcal{N}_0$ since unitary orbits of representations need not be closed in general.
For these reasons, our standing assumption is that all representations of $\mathcal{L}_\mathrm{n}$ are on finite dimensional spaces.
The familiar case of $k = 1$ yields the set of multiplicative linear functionals. It is well known that multiplicative linear functionals are automatically completely contractive. In this case, unitary equivalence is the identity relation. Moreover, there is an objective pairing between the multiplicative linear functional and its kernel, a maximal ideal of co dimension 1. So $Rep_1(\mathcal{Y})$ is the direct analogue of the maximal space of a commutative Banach algebra. In a non-abelian algebra, there may be many maximal ideals of other co-dimensions.
For $k > 1$, it is clear that two similar representations will have the same kernel. In the case of $\mathfrak{L}_\mathrm{n}$, similar representations which are both completely contractive need not be unitarily equivalent. (Indeed, when $n = 1$, simply consider two similar, but non-unitarily equivalent, contractions.) When $k < \infty$ and a representation Φ in $Rep_k(\mathcal{Y})$ is irreducible (no invariant subspaces),the range must be all of $\mathcal{M}_\mathrm{k} = \mathcal{B}(\mathcal{H})$. This is shows that every proper subalgebra of $\mathcal{M}_\mathrm{k}$ has a proper invariant subspace. Thus the kernel will be a maximal ideal of codimension k^2.Conversely, if M is a maximal ideal of $\mathcal{Y}$ of finite codimension, then there is a finite dimensional representation of $\mathcal{Y}$ on $\mathcal{Y}/M$ with kernel M. This quotient is simple, and thus by Wedderburn's Theorem, $\mathcal{Y}/M$ is isomorphicto $\mathcal{M}_k$ for some positive integer k. In particular, M has codimension k^2.Restrict this representation to a minimal invariant subspace $\mathcal{M}$ to obtaina representation π and note that $\mathcal{M}$ must have dimension k. Now π doesnot act on a Hilbert space. However, it is clearly a completely contractive representation. Any Hilbert space norm on M is equivalent to the quotient norm, and thus will yield a completely bounded Hilbert space representation. Then by Paulsen's [206], this is similar to a completely contractive representation. This shows that the map from irreducible representations in $Rep_k(\mathcal{Y})$ to the set of maximal ideals of codimension k^2 is surjective.The algebra $\mathfrak{L}_\mathrm{n}$ has many representations of every dimension. This will follow from Popescu's work on dilation theory for non-commuting n-tuples of operators. The case of $k = 1$ is special and has some extra structure. So we will handle these special features separately.
Recall the situation for $n = 1$ in which $\mathfrak{L}_1$ is isomorphic to H^∞. There is a natural continuous projection π_1 of the maximal ideal space M_{H^∞} of H^∞ onto the closed disk $\overline{\mathbb{D}}$ given by evaluation at the coordinate function z. For each point λin $\mathbb{D}$, there is a unique multiplicative linear functional $\varphi_\lambda(h) = h(\lambda)$ extending evaluation of z at λ. But for$|\lambda| = 1$, there is a very large space M_λ of multiplicative linear functionals taking the value λ at z. (See Hoffman [118] or Garnett [95].) The famous corona theorem of Carles on [47] shows that the point evaluations in the open unit disk are dense in M_{H^∞}.
Even though $\mathfrak{L}_\mathrm{n}$ is not commutative, the space $Rep_1(\mathfrak{L}_n)$ of multiplicative linear functionals is very large. For representations of dimension greater than one, there are interesting parallels

with the case of multiplicative linear functional. The analysis is based on the extensive knowledge of dilation theory for non-commuting n-tuples. We reprise the results that we will need.

Recall that if Φ is a linear map of an operator algebra $\mathcal{Y}$ into $\mathcal{B}(\mathcal{H})$, then $\Phi^{(k,\ell)}$ is the map from $\mathcal{M}_{k,\ell}(\mathcal{Y})$ into $\mathcal{M}_{k,\ell}(\mathcal{B}(\mathcal{H}))$, each endowed withthe usual operator norms, given by $\Phi^{(k,\ell)}([A_{ij}]) = [\Phi(A_{ij})]$. When $\ell = k$, we write $\Phi^{(k)}$ instead. The complete bound norm of Φ is defined to be $\|\Phi\|_{cb} = sup_{k,\ell}\|\Phi^{(k,\ell)}\|$. The map Φ is completely contractive if $\|\Phi\|_{cb} \le 1$.See Paulsen's book [206] for details.

Let $\overline{\mathbb{B}_{n,k}}$denote the collection of all contractions in $\mathcal{R}_n(\mathcal{B}(\mathcal{H}))$ where $dim\,\mathcal{H} = k$; namely all n-tuples$T = [T_1 \dots T_n]$ in $\mathcal{B}(\mathcal{H}^{(\mathrm{n})}, \mathcal{H})$, such that $dim\,\mathcal{H} = k$ and $\|T\| = \|\sum_{i=1}^{n} T_i T_i^*\|^{1/2} \le 1$. This is the higher dimension alanalogue of the n-ball. It is endowed with the product norm topology when $k < \infty$ and the product weak-* topology when $k = \mathcal{N}_0$.

If Φ is a (completely contractive) representation of $\mathcal{L}_\mathrm{n}$ on a Hilbert space $\mathcal{H}$, then the n-tuple$T = \Phi^{(1,n)}(L) = (\Phi(L_1), \dots, \Phi(L_n))$ is a contraction.Bunce [43], generalizing Frahzo [82], showed that every contraction T has a dilation to an n-tuple of isometries $S = (S_1, \dots, S_n)$ with orthogonal ranges. Popescu [210] extended this to $n = \infty$ and showed that there is a unique minimal isometric dilation of T. This yields a representation of the norm-closed algebra generated by L because the map taking each L_i to S_i is a completely isometric isomorphism. Following this with the compression to the original space yields a homomorphism taking $L_i$to T_i. However, this map usually does not extend naturally to a continuous map from $\mathfrak{L}_\mathrm{n}$ into $Alg(S)$. Popescu [209] determines when this has a wot-continuous extension to a representation of $\mathfrak{L}_\mathrm{n}$. Nevertheless, when $k = dim\,\mathcal{H} < \infty$, we shall see that norm-continuous extensions always exist.The following is a technical lemma used in the proof of Theorem (3.1.16) below. Recall that $\mathfrak{L}_n^{0,j}$ is the WOT-closed ideal of $\mathfrak{L}_\mathrm{n}$ generated by the set $\{L_\omega : |\omega| = j\}$.

Lemma (3.1.15)[156]: Let Φ belong to $Rep_k(\mathcal{L}_n)$. If $T := (\Phi(L_1), \dots, \Phi(L_n))$ satisfies $\|T\| = r < 1,$ then $\|\Phi(A)\| \le r^j \|A\|$ for every A in $\mathcal{L}_n^{0,j}$.

Proof. Let ψ be the $1 \times n^j$ row matrix with entries $L_\omega$for $|\omega| = j$. And let $\psi(T)$ denote the row matrix with entries $\psi(T)$ for $|\omega| = j$. By Corollary (3.1.10)for $n < \infty$ and Corollary (3.1.13) for $n = \infty$, we may factor $A = \omega X$ for some X in $C_{n^j}(\mathfrak{L}_n)$. Notice that ω is an isometry, and therefore $\|A\| = \|X\|$. By the Frahzo-Bunce dilation result [82,43] for $n < \infty$ and Popescu [210] for$n = \infty$, the n-tuple$r^{-1}T$ dilates to an n-tuple of isometries S, and therefore $\omega_j(r^{-1}T)$dilates to the isometry $\omega_j(S)$. Hence

$$\|\omega_j(T)\| = r^j \|\omega_j(r^{-1}T)\| \le r^j \|\omega_j(S)\| = r^j$$

Then since Φ is completely contractive,

$$\|\Phi(A)\| = \left\|\Phi^{(1,n^j)}(\omega)\Phi^{(1,n^j)}(X)\right\| \le \|\omega_j(T)\|\|X\| \le r^j \|A\|.$$

The first result generalizes the fact that there is a natural map of $\mathcal{M}_{H^\infty}$. onto the closed unit disk. The uniqueness result appears to be new even for $n = 1$ when $k > 1$. Recall that for $n = \infty$, $\mathbb{B}_\infty$ denotes theunit ball of Hilbert space with the weak topology.

Theorem (3.1.16)[156]: For $k < \infty$, there is a natural continuous projection $\pi_{n,k}$of $Rep_k(\mathfrak{L}_n)$ onto the closed unit ball $\overline{\mathbb{B}_{n,k}}$ given by

$$\pi_{n,k}(\Phi) = (\Phi(L_1), \dots, \Phi(L_n))$$

For each Tin $\mathbb{B}_{n,k}$, the open unit ball, there is a unique representation in$\pi_{n,k}^{-1}(T)$. It is WOT-continuous and is given by Popescu's functional calculus.The restriction of $\pi_{n,k}^{-1}$ to $\mathbb{B}_{n,k}$ is a homeomorphism.

Proof: Since Φ in $\mathrm{Rep}_k(\mathfrak{L}_n)$ is completely contractive, it follows that

$$T = \Phi^{(1,n)}(L) = [\Phi(L_1) \ \dots \ \Phi(L_n)]$$

is a contraction. Hence $\pi_{n,k}$ is a well defined map of $Rep_k(\mathfrak{L}_n)$into $\overline{\mathbb{B}_{n,k}}$.

Since it is determined by evaluation at the points L_i, this is a continuous map from $Rep_k(\mathfrak{L}_n)$with the topology of pointwise convergence into the ball with the product topology. By Popescu's functional calculus, there is a representation Φ_T for every T in the interior $\mathbb{B}_{n,k}$ (and in fact, for every completely non-coisometric contraction). Since $Rep_k(\mathfrak{L}_n)$ is compact, the image is compact and therefore maps onto $\overline{\mathbb{B}_{n,k}}$.

When $\|T\| = r < 1$, the WOT-continuous representation $\Phi_T$is defined as follows. Each A in $\mathfrak{L}_n$ is determined by $A\xi_1 = \sum_\omega a_\omega \xi_\omega$as a formal sum $A = \sum_\omega a_\omega L_\omega$. The image $\Phi_T(A)$ is determined as a norm convergent sum

$$\Phi_T(A) = \sum_\omega a_\omega \omega(T)$$

To see this, apply Lemma (3.1.15) for each $j \geq 0$ to obtain

$$\left\| \sum_{|\omega|=j} a_\omega \omega(T) \right\| \leq r^j \left\| \sum_{|\omega|=j} a_\omega \omega(L) \right\| = r^j \left(\sum_{|\omega|=j} |a_\omega|^2 \right)^{1/2}$$

Thus two partial sums of $\sum_\omega a_\omega \omega(T)$which both contain all words of lengthless than j will differ in norm by at most

$\sum_{k\geq j} r^k \left(\sum_{|\omega|=k} |a_\omega|^2 \right)^{1/2} \leq \sum_{k \geq j} r^k \|A\| = r^j (1-r)^{-1} \|A\|$,(2)

which tends to zero as j tends to infinity.Therefore this series is norm convergent.The fact that it is WOT-continuous was shown by Popescu in [209].

The proof of uniqueness follows similar lines. Let Φ in $Rep_k(\mathfrak{L}_n)$ be a completely contractive representation of $\mathfrak{L}_n$ such that $\pi_{n,k}(\Phi) = T$, where $\|T\| = r < 1$. We shall show that $\Phi = \Phi_T$. So let A be an element of $\mathfrak{L}_n$.Then by Corollary (3.1.10)for $n < \infty$ and Corollary (3.1.13) for $n = \infty$, A can be written uniquely as

$$A = \sum_{|\omega|<j} a_\omega L_\omega + \sum_{|\omega|=j} L_\omega A_\omega \text{ with } A_\omega \in \mathfrak{L}_n$$

Therefore

$$\Phi(A) = \sum_{|\omega|<j} a_\omega \omega(T) + \sum_{|\omega|=j} \omega(T) \Phi(A_\omega)$$

Let $\sum_k(A) = \sum_{|\omega|<k} \left(1 - \frac{|\omega|}{k}\right) a_\omega L_\omega$ denote the Cesaro sums. Recall that $\|\sum_k(A)\| \leq \|A\|$, and that they converge to A in the strong-* operator topology. For each integer j, there is an integer k sufficiently large that

$$\left\|\sum_{|\omega|<j}\frac{|\omega|}{k}a_\omega L_\omega\right\| < r^j\|A\|$$

Then $A-\sum_k(A)=A_1+\sum_{\|\omega\|<j}\frac{|\omega|}{k}a_\omega L_\omega$ where A_1 belong to $\mathcal{L}_n^{0,j}$. Clearly, $\|A_1\|<(2+r^j)\|A\|$. Hence, using the fact that Φ is contractive and Lemma (3.1.15),

$$\left\|\Phi(A)-\Phi\left(\sum\nolimits_k(A)\right)\right\| \leq \left\|\Phi(A)-\Phi\left(\sum_{|\omega|<j}\frac{|\omega|}{k}a_\omega L_\omega\right)\right\|$$

$$+\|\Phi(A_1)\| \leq r^j\|A\|+r^j(2+r^j)\|A\| < 4r^j\|A\|$$

Since Φ and Φ_T agree on polynomials in L, it follows that

$$\Phi(A)=\lim_{k\to\infty}\Phi\left(\sum\nolimits_k(A)\right)=\lim_{k\to\infty}\Phi_T\left(\sum\nolimits_k(A)\right)=\Phi_T(A).$$

Finally, we verify that the map sending T to Φ_T maps $\mathbb{B}_{n,k}$ homeomorphically onto the open set $\pi_{n,k}^{-1}(\mathbb{B}_{n,k})$. It is evident from the series representationof Φ_T and estimate (2) above, that if$\|T\|\leq r<1, \|T'\|\leq r$, and A is in $\mathfrak{L}_n$,

$$\|\Phi_T(A)-\Phi_{T'}(A)\| \leq \sum_{|\omega|\leq j}|a_\omega|\,\|\omega(T)-\omega(T')\|+2r^j(1-r)^{-1}\|A\|$$

Thus as T' converges to T, it follows that $\Phi_{T'}(A)$ converges to $\Phi_T(A)$. So this mapping of $\mathbb{B}_{n,k}$into $Rep_k(\mathfrak{L}_n)$ is continuous.

Now we specialize to 1-dimensional representations. In this case, each fibre over a point on the boundary of the ball is homeomorphic to every other because the gauge automorphisms act on the ball by the unitary group, and thus is transitive on the boundary. Moreover, this fibre is always very large based on the fact that it is known to be very large for $n=1$.

Theorem (3.1.17)[156]: There is a natural continuous projection $\pi_{n,1}$of the space $Rep_1(\mathfrak{L}_n)$ onto the closed unit ball $\overline{\mathbb{B}_n}$ in $\mathbb{C}^n$ given by evaluation at the n-tuple$(L_1,\ldots,L_n)$.

For each point λ in $\mathbb{B}_n$, there is a unique multiplicative linear functionalin $\pi_{n,1}^{-1}(\lambda)$; and it is given by $\varphi_\lambda(A)=(Av_\lambda,v_\lambda$. The set $\pi_{n,1}^{-1}(\mathbb{B}_n)$is homeomorphic to $\mathbb{B}_n$ and the restriction of the Gelfand transform to this ball is a contractive homomorphism of $\mathcal{L}_n$ into $H^\infty(\mathbb{B}_n)$.The ball $\mathbb{B}_n$ forms a Gleason part of $Rep_1(\mathcal{L}_n)$. These are the only weak-* continuous functionals on $\mathcal{L}_n$.

For each point λ in $\partial\mathbb{B}_n$, $\pi_{n,1}^{-1}(\lambda)$ is homeomorphic to $\pi_{n,1}^{-1}(1,0,\ldots 0)$.

There is a canonical surjection of $\pi_{n,1}^{-1}(\lambda)$ onto the fibre M_1 of M_{H^∞} given by restricting φ in $\pi_{n,1}^{-1}$ to $Alg\left(\sum_{i=1}^n\bar{\lambda}_iL_i\right)\simeq H^\infty$. This map has a continuous section.

Proof: ByTheorem (3.1.15), the map $\pi_{n,1}$ maps $Rep_1(\mathfrak{L}_n)$ onto $\overline{\mathbb{B}_n}$ by evaluation at L. For each point λ in the open ball, there is a unique preimage $\pi_{n,1}^{-1}(\lambda)$which is evidently φ_λ. Also, the preimage of $\mathbb{B}_n$ is homeomorphic to the open ball. By Theorem (3.1.7), these are the only WOT-continuous multiplicative linear functional son $\mathfrak{L}_n$. By Corollary (3.1.4), these coincide with the weak-* continuous ones.

For each polynomial $p(z) = \sum a_\omega \omega$ in $\mathbb{C}\mathcal{F}_n$, the Gelfand transform $\widehat{p(L)}(\lambda) = p(\lambda)$is evidently a contractive homomorphism of $\mathbb{C}\mathcal{F}_n$ into $\mathbb{C}[z]$ normed as a subset of $H^\infty(\mathbb{B}_n)$. Suppose that $p_n(L)$ is a WOT-Cauchy sequence in $\mathfrak{L}_n$. Since the set $\{\varphi_\lambda: \|\lambda\| \le r\}$ is a compact set of WOT-continuous linear functionals for each $0 < r < 1$, the restriction of $\widehat{p_n(L)}$ to this set converges uniformly. Thus the limit lies in $H^\infty(\mathbb{B}_n)$. This shows that the Gelfand map yields a contractive homomorphism into $\mathcal{H}^\infty(\mathbb{B}_n)$. which carries WOT-convergent sequences to sequences converging uniformly on compact subsets of the ball.

Now recall that the Gleason part containing φ_0 is the equivalence class

$$\{\varphi \in Rep_1(\mathfrak{L}_n): \|\varphi - \varphi_0\| < 2\}.$$

Consider the positive linear functional $\delta_\xi(T) = (T\xi, \xi)$on $\mathcal{B}(\mathcal{H})$ for a unit vector ξ. Let ζ be another unit vector with $|(\xi, \zeta)| = \cos\theta$ for $0 \le \theta \le \frac{\pi}{2}$ it is a well known fact that

$$\left\|\delta_\xi - \delta_\zeta\right\| = \sup_{\|T\| \le 1} |(T\xi, \xi) - (T\zeta, \zeta)| = 2(1 - \sin\theta)$$

Since$(v_0, v_\lambda) = (1 - \|\lambda\|^2)^{1/2} \ne 0$, it follows that $\|\varphi_0 - \varphi_\lambda\| < 2$ for λ in $\mathbb{B}_n$. On the other hand, if $\|\lambda\| = 1$, then $S = \sum_{i=1}^n \overline{\lambda_i}\, L_i$is a proper isometry in$\mathcal{L}_n$such that $\varphi_0(S) = 0$ and $\varphi_\lambda(S) = 1$. So the Möbius map $b_r(z) = \frac{z-r}{1-r^2}$ for $0 < r < 1$ can be used to obtain

$\varphi_0\big(b_r(S)\big)$ =-r and $\varphi_\lambda\big(b_r(S)\big) = 1$.

Hence $\|\varphi_0 - \varphi_\lambda\| = 2$. So the Gleason part of φ_0 is precisely $\mathbb{B}_n$.

Next consider the point $\lambda = (1,0,\dots,0)$in $\partial\mathbb{B}_n$. The algebra *Alg*(L_1) is isomorphic to H^∞. For φ in $\pi_{n,1}^{-1}(1,0,\dots,0)$, let $\rho(\varphi)$ be the restriction of φ to *Alg*(L_1). Clearly, (φ) belongs to $\mathcal{M}_1$, the fibre of $\mathcal{M}_{H^\infty}$ over the point 1, and ρ is continuous. We now produce a right inverse for ρ. Let P be the projection onto the subspace $span\left\{\xi_{z_1^k}: k \ge 0\right\}$, and notice that $P^\perp\mathcal{H}_n$ is an $\mathfrak{L}_n$-invariant subspace. So the map $\psi(A) = PA|_{P\mathcal{H}_n}$ is a homomorphism of $\mathfrak{L}_n$. In fact, $P^\perp\mathcal{H}_n$ is also $\mathfrak{R}_n$ invariant. Thus the kernel of this homomorphism is $\mathfrak{J} = \{A \in \mathcal{L}_n: PA\xi_1 = 0\}$, which is the WOT-closed ideal generated by $\{L_2, \dots, L_n\}$.

The range of ψ is contained in the WOT-closed algebra generated by the operators PL_iP, which are all 0 except for PL_1P which is a unilateral shift.The map taking L_1 to PL_1P is isometric and WOT −continuous, and carries $Alg(L_1)$ onto $\mathcal{T}(H^\infty)$. By composing ψ with the isomorphism of *Alg*(L_1)onto H^∞, we may regard as a surjection of $\mathcal{L}_n$ onto H^∞.Let $\alpha \coloneqq \psi^*|_{\mathcal{M}_1}$ be the restriction of the Banach space adjoint of ψ to $\mathcal{M}_1$. Clearly α is a continuous map; and if $\varphi = \alpha(\psi)$, we have

$$\pi_{n,1}(\varphi) = (\pi_{1,1}((\psi), 0, \dots, 0)$$

So α maps M_1into $\pi_{n,1}^{-1}(1,0,\dots,0)$and $\rho \circ \alpha(\psi) = \rho(\psi\Psi)$ for in M_1.

Therefore this is a continuous section, and ρ is surjective. In particular, this yields a homeomorphism of M_1 into $\pi_{n,1}^{-1}(1,0,\dots,0)$.

For any other λ with $\|\lambda\| = 1$, choose a unitary $U = \left[u_{ij}\right]$ in $\mathcal{M}_n$ such that $u_{ij} = \lambda_j$. We will show that the gauge automorphism Θ_U maps $\pi_{n,1}^{-1}(1,0,\dots,0)$ onto $\pi_{n,1}^{-1}(\lambda)$.Indeed, for any φ in $\pi_{n,1}^{-1}(1,0,\dots,0)$.

$$\varphi\Theta_U(L_j) = \varphi\left(\sum_{i=1}^{n} u_{ij}L_i\right) = u_{1j} = \lambda_j$$

It is evident that this map is continuous with inverse obtained by sending φ to$\varphi\Theta_U^{-1}$. So it induces a homeomorphism between $\pi_{n,1}^{-1}(1,0,\dots,0)$.And$\pi_{n,1}^{-1}(\lambda)$. The role of L_1 is played by $\Theta_U^{-1}(L_1) = \sum_{i=1}^{n} \bar{\lambda}_i L_i$.

Example(3.1.18)[156]: This example is to illustrate some of the possibilities on the boundary when $k > 1$.

It is possible that the fibre over a boundary point is a singleton. Consider $Rep_3(\mathfrak{L}_2)$, the pair

$$T_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad T_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and a representation Φ such that $\Phi(L_i) = T_i$for $i = 1,2$. Then since $T_1^2 = T_2^2 = T_1T_2 = T_2T_1 = 0$, it follows from Lemma (3.1.9) that $ker\Phi$ contains the ideal $\mathcal{L}_2^{0,2}$. Every element A in $\mathcal{L}_2$can be represented uniquely as $A = a_0I + a_1L_1 + a_2L_2 + A'$where A' belongs to $\mathcal{L}_2^{0,2}$. Therefore $\Phi(A) = a_0I + a_1T_1 + a_2T_2$is uniquely determined.

On the other hand, the fibre over Tmay be very large indeed. Let

$$T_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad T_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

We consider a class of homomorphisms Φ of $\mathcal{L}_2$in $\pi_{2,2}^{-1}(T)$. Let ζ_i denote the standard basis for $\mathbb{C}^2$. Both T_i are lower triangular, so we will consider those representations Φ which map $\mathfrak{L}_2$into the algebra $\mathcal{T}_2$ of 2×2 lower triangular matrices.

The functionals$\varphi_i(A) = (\Phi(A)\zeta_i, \zeta_i)$ are multiplicative since compression to the diagonal is multiplicative on $\mathcal{T}_2$.Moreover, $\varphi_1(L_1) = 1$and $\varphi_1(L_2) = 0$, and hence $\varphi_1$lies in $\pi_{2,2}^{-1}(1,0)$. Likewise,$\varphi_2(L_1) = \varphi_2(L_2) = 0$. So $\varphi_2 = \varphi_0$ is evaluation at 0 by Theorem (3.1.17). Recall from Corollary (3.1.10) that every A in $\mathcal{L}_2$ can be uniquely written as $A = a_0I + L_1A_1 + L_2A_2$, where $a_0 = \varphi_0(A)$ and $A_i = L_i^*(A - a_0I)$. Define $\delta(A) = \varphi_1(A_2) = \varphi_1(L_2^*(A - \varphi_0(A)I)$. Then

$$\begin{aligned} \Phi(A) &= a_0I + \Phi(L_1)\Phi(A_1) + \Phi(L_2)\Phi(A_2) \\ &= \begin{bmatrix} a_0 & 0 \\ 0 & a_0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \varphi_1(A_1) & 0 \\ * & * \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \varphi_1(A_2) & 0 \\ * & * \end{bmatrix} \\ &= \begin{bmatrix} a_0 + \varphi_1(A_1) & 0 \\ \varphi_1(A_2) & a_0 \end{bmatrix} = \begin{bmatrix} \varphi_1(A) & 0 \\ \delta(A) & \varphi_0(A) \end{bmatrix} \end{aligned}$$

In order to have a representation, it remains to verify complete contractivity.

An explicit family of such representations may be obtained as follows.

Let $\mathcal{M}_1 = span\left\{\xi_{z_1^k} : k \geq 0\right\}$and $\mathcal{M}_2 = span\left\{\xi_{z_2z_1^k} : k \geq 0\right\}$, and set$\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$. Then $\mathcal{M}^\perp$is invariant for $\mathfrak{L}_2$ and $\mathfrak{R}_2$. Thus compression to $\mathcal{M}$is a WOT-continuous homomorphism. The compression to $\mathcal{M}_1$is a homomorphism onto $H^\infty(L_1)$, sending $L_1$to the unilateral shift as we have discussed before. The compressions of both $L_i$to $\mathcal{M}$vanish on $\mathcal{M}_2$, and L_2maps $\mathcal{M}_1$isometrically onto $\mathcal{M}_2$. Hence the compressions are

$$P_{\mathcal{M}}L_1|_{\mathcal{M}} \simeq \begin{bmatrix} T_z & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad P_{\mathcal{M}}L_2|_{\mathcal{M}} \simeq \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}$$

Thus Ψ maps $\mathfrak{L}_n$ onto the algebra of operators of the form

$$\begin{bmatrix} T_{h_0} & 0 \\ T_{h_1} & h_0(O) \end{bmatrix} \quad \text{for } h_i \in H^\infty$$

Indeed, this shows that every element of $\mathfrak{L}_n$ may be written uniquely as

$$A = h_0(L_1) + L_2 h_1(L_1) + A' \text{where } h_i \in H^\infty \text{and } P_{\mathcal{M}}A' = 0.$$

Now let ψ be any multiplicative linear functional on H^∞ in the fibre M_1.

Then $\Phi = \psi^{(2)}\Psi$ is a completely contractive homomorphism of $\mathfrak{L}_2$ onto $\mathcal{T}_2$ such that $\Phi(L_i) = T_i$. Indeed,

$$\Phi(A) = \begin{bmatrix} T_{h_0} & 0 \\ T_{h_1} & h_0(O) \end{bmatrix} = \begin{bmatrix} \psi(h_0) & 0 \\ \psi(h_1) & h_0(O) \end{bmatrix}$$

Hence we have shown that the fibre $\pi_{2,2}^{-1}(T)$ *i*s very large.

We analyze the automorphism group of $\mathfrak{L}_n$. The automorphismsof the algebra $L_1 = H^\infty$ are precisely the maps $\Theta_{\mathcal{T}}(\mathrm{h}) = \mathrm{h}(\mathcal{T})$ where $\mathcal{T}$ is a conformal automorphism of the unit disk. So *Aut*(L_1) is isomorphic to $Aut(\mathbb{B}_1)$, the group of conformal automorphisms of the unit disk. In particular, they are all norm and wot-continuous. See [118], where two proofs are given, both based on factorization of analytic functions. Our main result is Theorem (3.1.19), which is valid even for $n = \infty$. Our original proof of Theorem (3.1.19) failed when $n = \infty$.

An automorphism of $\mathfrak{L}_n$ will be called quasi-inner if it is trivial modulothe WOT-closed commutator ideal $\bar{\mathfrak{e}}$(see Proposition (3.1.8)). Denote the set of all quasi-inner automorphisms by q-Inn($\mathfrak{L}_n$). In particular, this contains the subgroup Inn(Ln$\mathfrak{L}_n$) of inner automorphisms.

Theorem (3.1.19)[156]: There is a natural short exact sequence

$$0 \longrightarrow q\text{-}Inn(\mathfrak{L}_n) \longrightarrow Aut(\mathfrak{L}_n) \xrightarrow{\mathcal{T}} Aut(\mathbb{B}_n) \longrightarrow (0)$$

The map $\mathcal{T}$ takes Θ to $\mathcal{T}_\Theta(\lambda) = (\varphi_\lambda \Theta^{-1})^{1,n}(L)$ for $\lambda \in \mathbb{B}_n$. Moreover, $\mathcal{T}$ has a continuous section kwhich carries *Aut*($\mathbb{B}_n$) onto the subgroup *Autu*($\mathfrak{L}_n$) of unitarily implemented automorphisms. Thus *Aut*($\mathfrak{L}_n$) is a semi direct product.

The proof will be carried out in stages. First we establish an automatic continuity result for automorphisms.

Lemma (3.1.20)[156]: Every automorphism Θ of $\mathfrak{L}_n$, for $n \geq 2$, is continuous.

Proof: The proof is a standard gliding bump argument. We define $B_i = \Theta^{-1}(L_i)$ and set $\Lambda = max\{1, \|B_1\|, \|B_2\|\}$. Suppose that Θ is not continuous . Then there is a sequence A_k in $\mathfrak{L}_n$ such that

$$\|A_k\| \leq (2\Lambda)^{-k} \text{ and } \|\Theta(A_k)\| > k.$$

Let A be defined by the norm convergent sum

$$A = \sum_{k\geq 1} B_2^k B_1 A_k = \sum_{k=1}^{m} B_2^k B_1 A_k + B_2^{m+1} \sum_{k\geq 1} B_2^k B_1 A_{m+1+k}:$$

Set $X_m = \sum_{k\geq 0} B_2^k B_1 A_{m+1+k}$. Then for all $k > 0$,

$$\|\Theta(A)k\| \geq \|L_1^* L_2^{*m} \Theta(A)\| = \left\| \sum_{k=1}^{m} L_1^* L_2^{*m} L_2^k L_1(A_k) + L_1^* L_2^{*m} L_2^{m+1} \Theta(X_m) \right\| = \|\Theta(A_k)\| > k.$$

This is absurd, and consequently Θ must be continuous.
We show that every automorphism determines a special point in the ball.
Proposition (3.1.21)[156]: Let Θ be an automorphism of $\mathfrak{L}_n$. Then there is a unique point λ in $\mathbb{B}_n$ such that $\Theta(\mathfrak{L}_n^0) = ker\ \varphi_\lambda$. Indeed, $\varphi_\lambda = \varphi_0\Theta^{-1}$.
Proof: Let

$$S = \Theta^{(1;n)}(L) := [S_1 \dots S_n]:$$

By Corollaries (3.1.10) and (3.1.13), $\mathfrak{L}_n = \mathbb{C}I + L\mathcal{C}_n(\mathfrak{L}_n)$, and this decomposition is unique. Applying Θ yields $\mathfrak{L}_n = \mathbb{C}I + S\mathcal{C}_n(\mathfrak{L}_n)$, and every A in $\mathfrak{L}_n$ has a unique decomposition as $A = \alpha I + SB$ for some B in $\mathcal{C}_n(\mathfrak{L}_n)$. Hence the continuous linear map T from $\mathbb{C} \oplus \mathcal{C}_n(\mathfrak{L}_n) to \mathfrak{L}_n$ given by

$$T(\alpha; B) = \alpha I + SB$$

is a bijection. By Banach's isomorphism Theorem, T is invertible. So there is a constant $c > 0$ so that

$$c^{-1}\| |\alpha|^2 1 + B^*B \|^{1/2} \le \|\alpha 1 + SB\| \le c \| |\alpha|^2 1 + B^*B \|^{1/2}:$$

Let $\mathfrak{J} = S\mathcal{C}_n(\mathfrak{L}_n) = \Theta(\mathfrak{L}_n^0)$. Since T maps a subspace of codimension one onto $\mathfrak{J}$, it also has codimension one. We claim that this ideal is WOT-closed. Suppose that $J_\beta = SB_\beta$ is a bounded net in $\mathfrak{J}$ which converges weak-$*$ to an operator X in $\mathfrak{L}_n$. Then the net B_β is bounded in $\mathcal{C}_n(\mathfrak{L}_n)$ by the previous paragraph. Hence there is a cofinal subnet $B_{\beta'}$, which converges weak-* to an operator B in $\mathcal{C}_n(\mathfrak{L}_n)$. Consequently, it follows that $X = SB$ belongs to J. This shows that the intersection of $\mathfrak{L}$ with each closed ball is weak-* closed. by the Krein-Smulian Theorem (c.f. [73]), $\mathfrak{L}$ is weak-* closed. By Corollary (3.1.4), the weak-* and WOT topologies coincide on $\mathfrak{L}_n$. Hence J is WOT − closed.
Consider the multiplicative linear functional $\varphi = \varphi_0\Theta^{-}1$, which yields the formula $\varphi(\alpha I + SB) = \alpha$. Since $\mathfrak{L}$ is WOT-closed, this functional is WOT-continuous. Therefore by Theorem (3.1.7), there is a point λ in $\mathbb{B}_n$ such that $\varphi = \varphi_\lambda$.
We will show that automorphisms of $\mathfrak{L}_n$ are automatically WOT-continuous. First we establish a criterion for WOT-convergence in $\mathfrak{L}_n$. Recall that $C_n, \mathfrak{L}_n$ is the ideal generated by $\{L_\omega : |\omega| = k\}$.
Lemma (3.1.22)[156]: For a bounded net A_α in $\mathfrak{L}_n$, $n < 1$, the following are equivalent:
(i) $WOT\text{-}\lim_\alpha A_\alpha = 0$.
(ii) $w - \lim_\alpha A_\alpha\ \xi_1 = 0$.
(iii) $\lim_\alpha dist(A_\alpha ; \mathfrak{L}_n^{0,k}) = 0$ for all $k \ge 1$.
Proof: It is evident that (i) implies (ii). If (ii) holds, then write

$$A_\alpha \xi_1 = \sum_\omega a_\omega^\alpha \xi_\omega :$$

Then $A_{\alpha,k} := A_\alpha \sum_{|\omega|<k} a_\omega^\alpha L_\omega$ belongs to $\mathfrak{L}_n^{0,k}$ by Lemma (3.1.9) Condition (ii) Clearly implies that $\lim_\alpha a_\omega^\alpha = 0$ for every ω. Hence

$$\limsup_\alpha\ \mathrm{dist}(A_\alpha , \mathfrak{L}_n^{0,k}) \ge \limsup_\alpha \|A_\alpha - A_{\alpha,k}\| \le \limsup_\alpha \sum_{|\omega|<k} |a_\omega^\alpha|^2 = 0$$

for every k ≥ 1. Now if (iii) holds, then

$$0 = \lim_{\alpha} \operatorname{dist}\left(A_\alpha , \mathfrak{L}_n^{0,k}\right) \lim_{\alpha} \sup \operatorname{dist}(A_\alpha\, \xi_1; \mathfrak{L}_n^{0,k}\xi_1) = \left(\sum_{|\omega|<k} |a_\omega^\alpha|^2\right)^{1/2}$$

A fortiori, $\lim_{\alpha} a_w^\alpha \ = \ 0$ for every w in$\mathcal{F}_n$. Therefore

$$\lim_{\alpha}(A_\alpha\, \xi_u;\ \xi_v) = \lim_{\alpha}(A_\alpha\, \xi_1; R_u^*\xi_v) \ = \ 0$$

for every pair of words $u; v$ in $\mathcal{F}_n$. These vectors span a dense subset of $\mathcal{H}_n$.
As the net A_α is bounded, it converges WOT to 0.

Theorem (3.1.23)[156]: Every automorphism Θ of $\mathfrak{L}_n$ is WOT-continuous.

Proof: By Lemma (3.1.22), there is a point λ in $\mathbb{B}_n$ such that $\varphi_0\Theta^{-1} = \varphi_\lambda$.
Thus

$$\mathfrak{J} = \Theta(\mathfrak{L}_n^0) = ker\varphi_\lambda:$$

Hence

$$\left(\mathfrak{L}_n^{0,k}\right) = \Theta(\mathfrak{L}_n^0)^k = \mathfrak{J}^k \text{for all } k \geq 1:$$

Clearly $\cap_{k\leq 1} \mathfrak{J}^k = \{0\}$ since

$$\Theta^{-1}(\cap_{k\leq 1} \mathfrak{J}^k) \subset \Theta^{-1}(\mathfrak{J}^k) = \mathfrak{L}_n^{0,k} \text{ for all } k \geq 1:$$

Thus by Theorem (3.1.5), we have

$$\cap_{k\leq 1} \overline{\mathfrak{J}^k\mathcal{H}_n} = \|0\| \quad (3)$$

Set $\zeta_\omega = \ \Theta(L_\omega)\xi_1$ for $\omega \in \mathcal{F}_n$. Fix an integer j and let $\mathcal{M}_j$ and $\mathcal{N}_j$ be the closed linear spans of $\{\xi_\omega : |\omega| = j\}$ and $\{\xi_\omega : |\omega| = j\}$ respectively. If $\beta = \sum_{|\omega|=j} b_\omega\xi_\omega$ is a finite linear combination of the ξ_ω, put B $= \sum_{|\omega|=j} b_\omega L_\omega$. Then since Θ is bounded,

$$\left\|\sum_{|\omega|=j} b_\omega\zeta_\omega\right\| = \|\Theta(B)\xi_1\| \ \leq \ \|\Theta\|\|B\| \ = \|\Theta\|\|\beta\|.$$

Thus the map $\sum_{|\omega|=j} b_\omega\xi_\omega \longmapsto \sum_{|\omega|=j} b_\omega\zeta_\omega$ extends to a bounded linear operator $T_j : \mathcal{M}_j \ \longmapsto \mathcal{N}_j$.

Now consider a bounded net $B_\alpha = \sum_{|\omega|=j} b_\omega^\alpha\xi_\omega$ such that $\lim_{\omega\to 0} b_\omega^\alpha\xi_\omega = 0$ for all ω. Let $\beta_\alpha = \sum_{|\omega|=j} b_\omega^\alpha\xi_\omega$. It follows that

$$\omega - \lim_{\alpha} \Theta(\,B_\alpha)\xi_1 = \omega - \lim_{\alpha} T_j\beta_\alpha = T_j\omega - \lim_{\alpha}\beta_\alpha = 0:$$

Hence $\Theta(\beta_\alpha)$ converges WOT to 0. Again let A_α be a bounded net converging WOT to 0 in $\mathfrak{L}_n$ and let $\Lambda \ = sup\|A_\alpha\|$, it suffices to show that

$$\lim_{\alpha}(\Theta(\,A_\alpha)\xi_1\zeta) = \ 0$$

for ζ in a dense subset of $\mathcal{H}_n$. A convenient choice is $\cup_{k\leq 1} (\mathfrak{J}^k\mathcal{H}_n)^\perp$, which is dense by the equality (3). Choose ζ in $(\mathfrak{J}^k\mathcal{H}_n)^\perp$, and set $p \ = \ k^2$. Decompose $A_\alpha = \ B_\alpha \ + \ C_\alpha$ where

$$B_\alpha \ = \sum_p (A_\alpha) + \sum_{|\omega|=j} \frac{|\omega|}{p} a_\omega^\alpha L_\omega \text{ and } C_\alpha \ = A_\alpha - \ B_\alpha \in \ \mathfrak{L}_n^k\,.$$

Since the Cesaro mean $\sum_p(A_\alpha)$ is contractive, it follows that $\sum_p(A_\alpha) \leq \ \Lambda$.
Also the terms $A_{\alpha,j} \ = \sum_{|\omega|=j} a_\omega^\alpha L_\omega$ are bounded by s, and thus

$$\|B_\alpha\| \le \Lambda + p^{-1}\sum_{j=1}^{k-1}\|A_{\alpha,j}\| \le 2\Lambda:$$

Hence $\|C_\alpha\| \le 3\Lambda$. Moreover each net B_α and C_α converge wot to 0.
Now since B_α is supported on words of length less than p, we have seen that $\Theta(B_\alpha)$ converges WOT to 0. Finally by construction,

$$(\Theta(C_\alpha)\xi_1;\ \zeta) = 0$$

since $\Theta(D_\alpha)\xi_1$ lies in $\mathfrak{J}^k\mathcal{H}_n$, which is orthogonal to ζ.
The weak-$*$ topology on the unit ball of $B(\mathcal{H}_n)$ is metrizable, and the ball is compact. Thus it follows readily that a linear mapΘ is weak-$*$ continuous on a bounded convex set if and only if it takes every weak-$*$ null sequence to a weak-$*$ null sequence. Hence we see that Θ is weak-$*$ continuous on every closed ball of $\mathfrak{L}_n$. Therefore by the Krein-Smulian (c.f [73]), it follows that Θ is weak-$*$ continuous. By Corollary (3.1.4), the weak-$\mathfrak{L}_n$ and WOT topologies coincide on $\mathfrak{L}_n$. Thus $\Theta$is WOT-continuous.
The tools are now available to define the map τ given in Theorem (3.1.19). Using Theorem (3.1.17), we identify $\mathbb{B}_n$ with $Rep_1(\mathfrak{L}_n)$ by associating λ in $\mathbb{B}_n$ with the multiplicative linear functional φ_λ in $Rep_1(\mathfrak{L}_n)$.
Theorem (3.1.24)[156]: For each Θ in $Aut(\mathfrak{L}_n)$, the dual map τ_Θ on $Rep_1(\mathfrak{L}_n)$ given by $\tau_\Theta(\varphi) := \varphi \circ \Theta^{-1}$ maps the open ball $\mathbb{B}_n$ conformally onto itself. This determines a homomorphism of $Aut(\mathfrak{L}_n)$ into the group $\mathrm{Aut}(\mathbb{B}_n)$ of conformal automorphisms. If $\tau_\Theta(\varphi_0) = \varphi_0$, then there is a unitary matrix U in $\mathcal{U}_n$ such that $\tau_\Theta(\varphi_\lambda) = \varphi_{U\lambda}$.
Proof: Since Θ is WOT-continuous by Theorem (3.1.24), it follows that $\tau_\Theta(\varphi_\lambda) = \varphi_\lambda \circ \Theta^{-1}$ is a WOT-continuous multiplicative linear functional. Hence by Theorem (3.1.7), this is a functional φ_μ. Thus τ_Θ maps $\mathbb{B}_n$ into itself. We obtain an explicit formula for this map using the fact that $\varphi_\lambda^{(1;n)}(L) = \lambda$, whence

$$\tau_\Theta(\lambda) = (\varphi\lambda\Theta^{-1})^{(1;n)}(L) = \varphi_\lambda^{(1;n)}(T) = \hat{T}(\lambda);$$

where

$$T = (\Theta^{-1})^{(1;n)}(L) = [\Theta^{-1}(L_1) \dots \Theta^{-1}(L_n)].$$

By Theorem (3.1.17), $\widehat{T}$ is analytic and thus so is τ_Θ.
Next notice that the map τ taking Θ to τ_Θ is a homomorphism. It is evident that $\tau_{Id} = id$;that is, the identity automorphism induces the identity map on $\mathbb{B}_n$. Suppose that $\Theta_j$belong to$Aut(L_n)$, and $\tau_j = \tau(\Theta_j)$ for j = 1; 2.Then

$$\tau(\Theta_1\Theta_2)(\lambda) = (\varphi_\lambda(\Theta_1\Theta_2)^{-1})^{(1;n)}(L) = (\varphi_\lambda\Theta_2^{-1}\Theta_1^{-1})^{(1;n)}(L) = (\varphi\tau_2(\lambda)(\Theta_1^{-1})^{(1;n)}(L$$
$$= (\varphi_{\tau_1}(\tau_2(\lambda)))^{(1;n)}(L) = \tau_1(\tau_2(\lambda)):$$

Hence $\tau(\Theta_1\Theta_2) = \tau(\Theta_1) \circ \tau(\Theta_2)$.
Consequently

$$\tau_\Theta\tau_{\Theta^{-1}} = id = \tau_{\Theta^{-1}}\tau_\Theta;$$

from which we deduce that τ_Θ is a bijection. Therefore $\tau_\Theta$is a biholomorphic bijection (i.e. a conformal automorphism) of the ball.
If $\tau$is a conformal automorphism of $\mathbb{B}_n$. such that $\tau(0) = 0$, then by Schwarz's Lemma, there is a unitary operator U in U_n such that $\tau(\lambda) = U_\lambda$[256] for $n < \infty$and [113] for $n = \infty$.

The following corollary characterizes the quasi-inner automorphisms.

Corollary (3.1.25)[156]: For Θ in $Aut(\mathfrak{L}_n)$, the following are equivalent:

(i) Θ belongs to $ker\ \tau$.

(ii) $\Theta(L_i) - L_i$ belongs to $\bar{\mathfrak{e}}$for $1 \le i \le n$.

(iii) $\Theta(A) - A$ belongs to $\bar{\mathfrak{e}}$ for every A in $\mathfrak{L}_n$.

Proof: If Θ belongs to *ker* τ , then so does Θ^{-1}; whence

$$\varphi_\lambda(\Theta(L_i) - L_i \ = \ \tau_{\Theta^{-1}}(\lambda) - \lambda$$

is zero for everyλin $\mathbb{B}_n$ if and only if $\Theta(L_i) - L_i$ belongs to $\bigcap_{\lambda \in \mathbb{B}_n} \ \ ker\ \varphi_\lambda$ for $1 \le i \le n$. But this set equals $\mathfrak{e}$ by Proposition (3.1.8). So (i) and (ii) are equivalent.

Suppose that (ii) holds. As $\mathfrak{e}$ is an ideal, it readily follows that $\Theta\big(p(L)\big) - p(L)$ belongs to $\mathfrak{e}$ for every polynomial in L. Then because Θ is wotcontinuous and $\mathfrak{e}$ is -closed, this extends to the WOT-closure of these polynomials, which is all of $\mathfrak{L}_n$. This establishes (iii). Clearly (iii) implies (ii).

To complete the picture, we need to construct explicit automorphisms to demonstrate that the map τ is surjective. In fact, much more will be established. An explicit section of τ will be found that maps $Aut(\mathbb{B}_n)$ onto the subgroup $Aut_u(\mathfrak{L}_n)$ of unitarily implemented automorphisms. This will establish that $Aut(\mathfrak{L}_n)$ actually has the structure of a semidirect product.

A certain class of unitarily implemented automorphisms of $\mathfrak{L}_n$ are well known from quantum mechanics, and are called gauge automorphisms. Think of $\mathcal{H}_n$ as being identified with the Fock space $F(H)$ with $dimH \ = \ n$. For any unitary U on H, form the unitary operator

$$\widetilde{U} = \sum_{k \ge 0} \otimes\, U^{\otimes k}$$

which acts on Fock space by acting as the k-fold tensor product of U on the k-fold tensor product of $\mathcal{H}$. It is evident that

$$\widetilde{U}\ell(\zeta)\ \widetilde{U}^* = \ell(U\zeta) \text{ for } \zeta \in \mathcal{H}:$$

therefore $\Theta_U = Ad\ \widetilde{U}$determines an automorphism of $\mathfrak{L}_n$. If $U = [u_{ij}\ ]$ is an $n \times n$ unitary matrix, this automorphism can also be seen to be given by

$$\Theta_U\big(L_j\big) = \sum_{i=1}^{n} u_{ij} L_i \text{ for } 1 \le j \le n.$$

An easy calculation shows that $\Theta_U \Theta_V = \Theta_{UV}$; so this is a homomorphism of the unitary group U_n into the automorphism group $Aut(\mathfrak{L}_n)$. It follows from Lemma (3.1.27) below that $\tau_\Theta U = \overline{U}$, the coordinatewise conjugate of U.So τ maps the group of gauge automorphisms onto the unitary group.

In [299], Voiculescu constructed a larger subgroup of automorphisms of the Cuntz-Toeplitz algebra $\mathcal{E}_n$ which turn out to be the one we want. He starts with the group $U(1, n)$consisting of those $(n + 1) \times (n + 1)$ matrices X such that $X^*JX = J$, where $J = \begin{bmatrix} 1 & 0 \\ 0 & I_n \end{bmatrix}$. One may compute that these matrices have the form $X = \begin{bmatrix} x_0 & \eta_1^* \\ \eta_2 & X_1 \end{bmatrix}$ where the coefficients satisfy the (redundant) relations:

(i) $\|\eta_1\|^2 \ = \|\eta_2\|^2 \ = \|x_0\|^2 - 1$

(ii) $X_1\eta_1 = \overline{x_0}\eta_2$ and $X_1^*\eta_2 = x_0\eta_1$
(iii) $X_1^*X_1 = I_n + \eta_1\eta_1^*$ and $X_1X_1^* = I_n + \eta_2\eta_2$.
Let us write $L_\zeta = \sum_{i=1}^{n} \zeta_i \ L_i$ for $\zeta \in \mathbb{C}^n$. Voiculescu shows that there is a (unique) automorphism Θ_X of $\mathcal{E}_n$ such that the restriction to the generators
is given by

$$\Theta_X(L_\zeta) = (x_0 I \ \ L_{\eta_2})^{-1}(LX_{1\zeta -} \langle \zeta, \eta_1 \rangle I):$$

It is easy to verify that the kernel of this map consists of the scalar matrices $x_0 I_{n+1}$ for x_0 in the circle $\mathbb{T}$. Moreover Voiculescu constructs a unitary operator U_X by

$$U_X(A\xi_1) = \Theta_X(A)(x_0 I \ \ L_{\eta_2})^{-1}\xi_1 \text{ for all } A \in \mathfrak{L}_n$$

so that $\Theta_X(A) = U_X A U_X^*$ for A in $\mathfrak{L}_n$.
It is apparent that the norm-closed (nonself-adjoint) algebra $\mathfrak{A}_n$ generated by $fL_i : 1 \le i \le n$ is mapped into itself by this map. Since this is a group homomorphism, it maps $\mathfrak{A}_n$ onto itself. Then because Θ_X is unitarily implemented, it is WOT-continuous and thus determines an automorphism of $\mathfrak{L}_n$. We will provide discussion below to indicate another method of obtaining these automorphisms that fits into our framework some what better.

There is also a natural map from $U(1,n)$ onto $Aut(\mathbb{B}_n)$ by fractional linear maps. This result must be well known. We do not have a reference, but the results of Phillips [211] on simplectic automorphisms of the ball of $\mathcal{B}(H)$ may be modified to apply to the ball of $\mathcal{B}(\mathcal{H};\mathcal{K})$ for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. Taking $\mathcal{H} = \mathbb{C}^n$ and $\mathcal{K} = \mathbb{C}$ yields our map.
Lemma (3.1.27)[156]: For X in $U(1,n)$, define a map $\theta_X : \mathbb{B}_n \to \mathbb{C}^n$ by

$$\theta_X(\lambda) = \frac{X_1\lambda + \eta_2}{x_0 + \langle \lambda, \eta_1 \rangle} \text{ for } \lambda \in \mathbb{B}_n.$$

Then θ_X belongs to $Aut(\mathbb{B}_n)$ and the associated map $\theta : U(1,n) \to Aut(\mathbb{B}_n)$ is a surjective homomorphism with kernel equal to the scalars.
Proof: First one computes using (i) and (ii) above:

$$\begin{aligned} &|x_0 + \langle \lambda, \eta_1 \rangle|^2 - \|\langle X_1 - \lambda, \eta_2 \rangle\|^2 \\ &= |x_0|^2 + |\langle \lambda, \eta_1 \rangle|^2 - \|X_1\lambda\|^2 - \|\eta_2\|^2 + 2\,Re(\langle \lambda, x_0\eta_1 \rangle - \langle X_1\lambda, \eta_2 \rangle) \\ &= (|x_0|^2 - \|\eta_2\|^2) - (\|X_1\lambda\|^2 - |\langle \lambda, x_0\eta_1 \rangle|^2) + 2\,Re\langle \lambda, x_0\eta_1 \rangle - \langle X_1^*\eta_2 \rangle \\ &= 1 - |\lambda|^2. \end{aligned}$$

Thus this map carries $\mathbb{B}_n$ onto itself, and so belongs to $Aut(\mathbb{B}_n)$.
A straightforward calculation shows that this map is a group homomorphism. Again the kernel of this map is the circle of scalar matrices in $U(1,n)$. The unitary operator $X = \begin{bmatrix} 1 & 0 \\ 0 & U \end{bmatrix}$ is sent to U. Now $\theta_X(0) = x_0^{-1}\eta_2$ is an arbitrary point in the ball. Hence the range of θ is a transitive subgroup of $Aut(\mathbb{B}_n)$ containing the unitary group. By Schwarz's lemma [256, 113], the range is the whole group of conformal automorphisms.
To see the relationship between Θ and θ, we make the following computation.

Lemma (3.1.27)[156]: $\tau(\Theta\bar{X}) = \theta(\bar{X})$ for all X in $U(1,n)$.
Proof: Compute for $X = \begin{bmatrix} x_0 & \eta_1^* \\ \eta_2 & X_1 \end{bmatrix}$ that

$$X^{-1} = JX^*J = \begin{bmatrix} \bar{x}_0 & -\eta_1^* \\ -\eta_2 & X_1^* \end{bmatrix}.$$

Therefore if e_i form the standard basis for $\mathbb{C}^n$, then

$$\begin{aligned} \tau(\Theta_X)(\lambda) &= [\varphi_\lambda \Theta_{X^{-1}}(L_i]) = \varphi_\lambda \left(\left(\bar{x}_0 I \ + \ L_{\eta_1} \right)^{-1} (L_{X_1^*} e_i \ + \ \langle e_i, \eta_2 \rangle) \right) \\ &= (\bar{x}_0 \ + \ \langle \lambda, \bar{\eta}_1 \rangle)^{-1} \sum_{i=1}^{n} \left(\langle \lambda, L_{X_1^*} e_i \rangle + \langle e_i, \eta_2 \rangle \right) e_i \\ &= (\bar{x}_0 \ + \ \langle \lambda, \bar{\eta}_1 \rangle)^{-1} \sum_{i=1}^{n} (\langle \bar{X}_1 \lambda, e_i \rangle + \langle \eta_2, e_i, \rangle) e_i = (\bar{x}_0 \ + \ \langle \lambda, \bar{\eta}_1 \rangle)^{-1} (\bar{X}_1 \lambda + \bar{\eta}_2) \\ &= \Theta_{\bar{X}}(\lambda): \end{aligned}$$

Theorem (3.1.28)[156]: The restriction of τ to the subgroup $\mathrm{Aut_u}(\mathfrak{L}_n)$ of unitarily implemented automorphisms is an isomorphism onto $\mathrm{Aut}(\mathbb{B}_n)$.

Proof: Define a map $\mathcal{K}: Aut(\mathbb{B}_n) \to Aut_u(\mathfrak{L}_n)$ as follows: given αα in $Aut(\mathbb{B}_n)$,pick $X \in U(1,n)$ belonging to $\theta^{-1}(\alpha)$and set $\mathcal{K}(\alpha) = \Theta_{\bar{X}}$. This is a well defined monomorphism because Θ and θ have the same kernel and complex conjugation is an automorphism of $U(1,n)$.By the previous lemma, it follows that $\tau\mathcal{K}$ is the identity on $Aut(\mathbb{B}_n)$. In particular, τ restricted to $Aut_u(\mathfrak{L}_n)$ is a surjective homomorphism.

To prove that this map is injective, suppose that Θ is a unitarily implemented automorphism such that $\tau(\Theta) = id$. A fortiori, Θ is contractive.

But $\Theta(L_i) = L_i + C_i$ where $C_i \in \bar{e}$whence

$$1 \geq \|\Theta\|_2 \geq (\|L_i \ + C_i)\xi_1\|^2 = 1 \ + \ \|C_i \xi_1\|^2.$$

Consequently,$C_i \xi_1 = 0$ which implies that $C_i = 0$. Therefore $\Theta = Id$ andour map is an isomorphism.

We record an immediate consequence of the proof.

Corollary (3.1.29)[156]: Every contractive automorphism of $\mathfrak{L}_n$ is unitarily implemented. In particular, it is completely isometric.

It would be interesting to know if automorphisms of $\mathfrak{L}_n$ are automatically completely bounded.

All the necessary parts for Theorem (3.1.19) have now been accumulated. The homomorphism τ is now known to be surjective, with kernel $q - Inn(\mathfrak{L}_n)$ and a continuous section k onto $Aut(\mathfrak{L}_n)$ as required.

Notice that if Θ is unitarily implemented, then $S_i = \Theta^{-1}(L_i)$ will be isometries with pairwise orthogonal ranges. They generate the ideal

$$\sum_{i=1}^{n} S_i \, \mathfrak{L}_n = \Theta^{-1}(\mathfrak{L}_n^0).$$

This is a WOT-closed two-sided ideal of codimension one, and thus by Theorem (3.1.5) its range is a $\mathfrak{L}_n\mathfrak{R}_n$ invariant subspace of codimension one. The complement is a one-dimensional invariant subspace for $\mathfrak{L}_n^0$, and thus by Theorem (3.1.3) is spanned by ν_λ for some λ in $\mathbb{B}_n$. It is easy to check that $\tau_\Theta(0) = \lambda$.

Conversely, given λ, we can construct such isometries. By Theorem (3.1.3), the subspace $\{\nu\lambda\}^\perp$ is $\mathfrak{R}_n$ invariant and has an n-dimensional wandering space Ω_λ. Let ξ_i for $1 \leq i \leq n$ be an orthonormal basis for W_λ. Then by [68], the operators $S_i = L_{\xi_i}$ are isometries in $\mathfrak{L}_n$ with ranges summing to $\{\nu_\lambda\}^\perp$. We will sketch how to construct the automorphism _ which takes L_i to Si for $1 \le i \le n$.

The first step is to show that ν_λ is cyclic for the WOT-closed subalgebra $\mathfrak{A}$ generated by $\{S_1, \dots, S_n\}$. This is established by showing that $\zeta_\omega = (S)\nu_\lambda, \omega \in \mathcal{F}_n$, is an orthonormal basis for $\mathcal{H}_n$. This immediately yields a unitary operator W such that $WL_iW^* = S_i$ such that AdW is an endomorphism of $\mathfrak{L}_n$.

The second step is to show that $\mathcal{Y} = \mathfrak{L}_n$. Since it is contained in $\mathfrak{L}_n$, we see that $\nu_0 = \xi_1$ is an eigenvalue for $\mathfrak{A}^*$. Since $\mathfrak{A}$ is unitarily equivalent to $\mathfrak{L}_n$, there is a non-zero μ such that $W_{\nu_\mu} = \xi_1$. Apply the argument again to obtain a second unitary W' so that $AdW'W(L_i) = S_i' = L_{\zeta_I'}$ where ζ_I' form an orthonormal basis for the wandering space of $\{\xi_1\}^\perp$. But then (when $n < 1$) there is a unitary U in U_n such that $\zeta_I' = U_{e_i} = U\xi_{z_i}$.
Unfortunately, this argument fails for $n = \infty$. Consequently, it follows that $AdW'W = \Theta_U$. Thus the two endomorphisms AdW and AdW' must have been automorphisms.

Section (3.2) Cmmutative Banach Algebras of Teoplitz Operators

Recall first that the C^*-algebras generated by Toeplitz operators which are commutative on each weighted Bergman space over the unit disk were completely classified in [262]. Under some technical assumption on “richness” of a class of generating symbols the result was as follows. A C^*-algebra generated by Toeplitz operators is commutative on each weighted Bergman space if and only if the corresponding symbols of Toeplitz operators are constant on cycles of a pencil hyperbolic geodesics on the unit disk, or if and only if the corresponding symbols of Toeplitz operators are invariant under the action a maximal commutative subgroup of the Möbius transformations of the unit disk. We note that the commutativity on each weighted Bergman space was crucial in the part “only if” of the above result.

Generalizing this result to Toeplitz operators on the unit ball, it was proved in [245, 251] that, given a maximal commutative subgroup of biholomorphisms of the unit ball, the C^*-algebra generated by Toeplitz operators, whose symbols are invariant under the action of this subgroup, is commutative on each weighted Bergman space. The geometric description of corresponding symbols in terms of so-called Lagrangian foliations (which generalize the notion of a pencil of hyperbolic geodesics to multidimensional case) was also given. It turned out that for the unit ball of dimension n there are $n + 2$ essentially different “model” commutative C^*-algebras, all others are conjugated with one of them via biholomorphisms of the unit ball. It was firmly expected that the above algebras exhaust all possible algebras of Toeplitz operators on the unit ball which are commutative on each weighted Bergman space.

We present here a quite unexpected result. There exist other Banach algebras generated by Toeplitz operators which are commutative on each weighted Bergman space. These algebras are non conjugated via biholomorphisms of the unit ball, non of them is a C^*-algebra, and for $n = 1$ all of them collapse to the C^*-algebra, which is generated by Toeplitz operators with radial symbols.
Let $\mathbb{B}^n$ be the unit ball in $\mathbb{C}^n$, that is,

$$\mathbb{B}^n = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z|^2 = |z_1|^2 + \dots + |z_n|^2 < 1\},$$

and let $\mathbb{S}^n$ be the corresponding unit sphere, the boundary of the unit ball $\mathbb{B}^n$. In what follows we will use the notation $\tau(\mathbb{B}^m)$ for the base of the unit ball $\mathbb{B}^m$, considered as a Reinhard domain, i.e.,

$$\tau\,(\mathbb{B}^m) = \{(r_1, \ldots, r_m) = (|z_1|, \ldots, |z_m|) : r^2 = r_1^2 + \ldots + r_m^2 \in [0,1)\}.$$

Given a multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ we will use the standard notation,

$$|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n,$$
$$\alpha! = \alpha_1!\ \alpha_2! \cdots \alpha_n!,$$
$$z^\alpha = z_1^{\alpha 1} z_2^{\alpha 2} \cdots z_n^{\alpha n}.$$

Two multi-indices α and β are called orthogonal, $\alpha \perp \beta$, if

$$\alpha.\beta = \alpha_1\beta_1 + \alpha_2\beta_2 + \ldots + \alpha_n\beta_n = 0. \quad (4)$$

Denote *by* $dV = dx_1dy_1 \ldots dx_ndy_n$, where $z_l = x_l + iy_l, l = 1, 2, \ldots, n$, the standard Lebesgue measure in $\mathbb{C}^n$; and let dS be the corresponding surface measure on $\mathbb{S}_n$. We introduce the one-parameter family of weighted measures,

$$dv_\lambda(z) = \frac{\Gamma(n + \lambda + 1)}{\pi^n \Gamma(\lambda + 1)} (1 - |z|^2)^\lambda\, dV\,(z), \;\; \lambda > -1,$$

which are probability ones in $\mathbb{B}^n$; and recall two known equalities (see, for example, [150])

$$\int_{\mathbb{S}^n} \xi^\alpha \xi^{-\beta}\, dS(\zeta) = \delta_{\alpha,\beta}, \frac{2\pi^n \alpha!}{(n - 1 + |\alpha|)!,} \quad (5)$$

$$\int_{\mathbb{B}^n} z^\alpha z^{-\beta}\, dv_\lambda(z) = \delta_{\alpha,\beta} \frac{\alpha!\ (n + \lambda + 1)}{\Gamma(n + |\alpha| + \lambda + 1).} \quad (6)$$

We introduce the weighted space $L_2(\mathbb{B}^n, dv_\lambda)$ and its subspace, the weighted Bergman space $A_\lambda^2 = A_\lambda^2(\mathbb{B}^n)$, which consists of all functions analytic in $\mathbb{B}^n$. The (orthogonal) Bergman projection $\mathbb{B}^n$of $L_2(\mathbb{B}^n, dv_\lambda)$onto $A_\lambda^2(\mathbb{B}^n)$ is known to have the following integral form

$$(B_\lambda \varphi)(z) = \int_{\mathbb{B}^n} \frac{\varphi(\zeta)\, dv_\lambda(\zeta)}{(1 - z \cdot \bar{\zeta})^{n+\zeta+1}}.$$

Finally, given a function $a(z) \in L_\infty(\mathbb{B}^n)$, the Toeplitz operator T_a with symbol a acts on $A_\lambda^2(\mathbb{B}^n)$as follows

$$T_a\colon \varphi \in A_\lambda^2(\mathbb{B}^n) \to B_\lambda(a\varphi) \in A_\lambda^2(\mathbb{B}^n).$$

Let$k = (k_1, \ldots, k_m)$ be a tuple of positive integers whose sum is equal to n: $k_1 + \ldots + k_m = n$. The length of such a tuple may obviously vary from 1, for $k = (n)$,to n, for$k = (1, \ldots, 1)$. Given a tuple $k = (k_1, \ldots, k_m)$, we rearrange the n coordinates of $z \in \mathbb{B}^n$ in$m$ groups, each one of which has k_j , $j = 1, \ldots, m$, entries and introduce the notation

$$z_{(1)} = (z_{1,1}, \ldots, z_{1,}k_1), z_{(2)} = (z_{2,1}, \ldots, z_{2,}k_2), \ldots, z_{(m)} = (z_{m,1}, \ldots, z_m, k_m).$$

We represent then each $z_{(j)} = (z_{j,1}, \ldots, z_j, k_j\,) \in \mathbb{B}^{k_j}$ in the form

$$z_{(j)} = r_j \xi_{(j)}, \text{where} r_j = \sqrt{|z_{j,1}|^2 + \ldots |z_j, k_j\,|^2} \text{ and } \xi_{(j)} \in \mathbb{S}^{k_j}.$$

Given a tuple $k = (k_1, \ldots, k_m)$, a bounded measurable function $a = a(z), z \in \mathbb{B}^n$, will be called k-quasi-radial if it depends only on $r_1, \ldots, r_m$.

Varying k we have a collection of the partially ordered by inclusion sets $\mathcal{R}_{\mathrm{k}}$ of k-quasiradial functions. The minimal among these sets is the set $\mathcal{R}_{(n)}$ of radial functions and the maximal one is the set $\mathcal{R}_{(1,\dots,1)}$ of separately radial functions.

There is some ambiguity in the above definition. Indeed given a tuple k there are many corresponding sets $\mathcal{R}_{\mathrm{k}}$ which differ by perturbation of coordinates. At the same time each perturbation of coordinates of z is a biholomorphism, say k, of the unit ball $\mathbb{B}^{\mathrm{n}}$, which generates the unitary equivalence of the Toeplitz operators $\mathrm{T_a}$ and $\mathrm{T_{a\circ k}}$ Thus it is sufficient, in fact, to consider only one of these perturbation different sets.

To avoid all possible repetitions and ambiguities in what follows we will always assume first, that $k_1 \leq k_2 \leq \ldots \leq k_m$, and second, that

$$z_{1,1} = z_1, z_{1,2} = z_2, \dots, z_1, k_1 = z_{k_1}, z_{2,1} = z_{k_{1+1}}, \dots, z_2, k_2 = z_{k_{1+k_2}}, \dots, z_m, k_m = z_n. \tag{7}$$

Given $k = (k_1, \dots, k_m)$and any n-tuple $\alpha = (\alpha_1, \dots, \alpha_n)$, we define

$$\alpha_{(1)} = (\alpha_1, \dots, \alpha_{k1}), \alpha_{(2)} = (\alpha_{k1+1}, \dots, \alpha_{k1+k2}), \dots, \alpha_{(m)} = (\alpha_{n-km+1}, \dots, \alpha_n).$$

as each set $\mathcal{R}_{\mathrm{k}}$is a subset of the set $\mathcal{R}_{(1,\dots,1)}$ of separately radial functions, the Toeplitz operator T_a with symbol $a \in \mathcal{R}_k$, by [245], is diagonal with respect to the standard monomial basis in $\mathcal{A}^2_\lambda(\mathbb{B}^n)$. The exact form of the corresponding spectral sequence gives the next lemma.

Lemma (3.2.1)[193]: Given a k −quasi-radial function $a = a(r_1, \dots, r_m)$, we have

$$T_a z^\alpha = \gamma_{a,k,\lambda}(\alpha)\, z^\alpha, \alpha \in \mathbb{Z}^n_+,$$

Where

$$\begin{aligned}\gamma_{a,k,\lambda}(\alpha) = {} & \gamma_{a,k,\lambda}\left(\left|\alpha_{(1)}\right|, \dots, \alpha_{(m)}\right) \\ = {} & \frac{2^m\,(n + |\_| + \_ + 1)}{\Gamma(\lambda + 1)\prod_{j=1}^m\left(k_j - 1 + \left|\alpha_{(j)}\right|\right)!} \\ & \times \int\limits_{\tau(\mathbb{B}^m)} a(r_1, \dots, r_m)\,(1 - |r|^2)\lambda \prod_{j=1}^m r_j^{2|\alpha_{(j)}|+2k_j-1j}\, dr_j\end{aligned}$$

Proof: We calculate

$$\langle T_a z^\alpha, z^\alpha\rangle = \langle z^\alpha, z^\alpha\rangle = \frac{\Gamma(n + \lambda + 1)}{\pi^n\,\Gamma(\lambda + 1)} \times \int_{\mathbb{B}^m} a(r_1, \dots, r_m)\, z^\alpha, z^{-\alpha}(1 - |z|^2)^\lambda dV(z).$$

Changing the variables $z_{(j)} = r_j \xi_{(j)}$, where $r_j \in [0, 1]$ and $\xi_{(j)} \in \mathbb{S}^{k_j}$, $j = 1, \dots, m$, we have

$$\begin{aligned}\langle z^\alpha, z^\alpha\rangle = {} & \frac{\Gamma(n + \lambda + 1)}{\pi^n\,\Gamma(\lambda + 1)} \int_{\mathbb{B}^m} a(r_1, \dots, r_m)\,(1 - |z|^2)^\lambda \prod_{j=1}^m r_j^{2|\alpha_{(j)}|+2k_j-1j} dr_j \\ & \times \prod_{j=1}^m \int_{\mathbb{S}^{k_j}} \xi_{(j)}^{\alpha_{(j)}} \xi_{(j)}^{-\alpha_{(j)}} dS\left(\xi_{(j)}\right)\end{aligned}$$

$$= \frac{2^m(n + |\alpha| + \alpha + 1)}{\Gamma(\lambda + 1)\prod_{j=1}^m\left(k_j - 1 + \left|\alpha_{(j)}\right|\right)!} \int\limits_{\tau(\mathbb{B}^m)} a(r_1, \dots, r_m)\,(1 - |r|^2)\lambda \prod_{j=1}^m r_j^{2|\alpha_{(j)}|+2k_j-1j} dr_j\ .$$

Then the result follows by (11).

Given $k = (k_1, \ldots, k_m)$ we use the representations $z_{(j)} = r_j \xi_{(j)}, j = 1, \ldots, m$,to define the vector

$$\xi = \left(\xi_{(1)}, \xi_{(2)}, \ldots, \xi_{(m)}\right) \in \mathbb{S}^{k_1} \times \mathbb{S}^{k_2} \times \ldots \times \mathbb{S}^{k_m}.$$

we introduce now an extension of k −quasi-radial functions, which may be called following [125, 129, 317] the quasi-homogeneous functions. A function $\varphi(z)$ is called quasi-homogeneous (or k −quasi-homogeneous) function if it has the form

$$\varphi(z) = \varphi(z_{(1)}, z_{(2)}, \ldots, z_{(m)}) = a(r_1, r_2, \ldots, r_m)\xi^s = a(r_1, r_2, \ldots, r_m)\xi_{(2)}^{s_{(1)}} \xi_{(2)}^{s_{(2)}} \ldots \xi_{(m)}^{s_{(m)}},$$

where $a(r_1, r_2, \ldots, r_m) \in \mathcal{R}_k$and $s \in \mathbb{Z}^n$.

After separating positive and negative entries in s, it admits the unique representation $s = p - q$, where $p, q \in \mathbb{Z}_+^n$and $p \perp q$. Then ξ^s, for $s \in \mathbb{Z}^n$, is always understood as

$$\xi^s = \xi^p \xi^{-q},$$

where $s = p - q$, with $p, q \in \mathbb{Z}_+^n$ and $p \perp q$. We will call the pair (p, q) the quasi-homogeneous degree of the k-quasi-homogeneous function $a(r_1, r_2, \ldots, r_m)\xi^p \xi^{-q}$.

Lemma (3.2.2)[193]: The Toeplitz operator $T_a \xi^p \xi^{-q}$with k-quasi-homogeneous symbol $a\xi^p \xi^{-q}$acts on monomials $z^\alpha, \alpha \in \mathbb{Z}_+^n$ as follows

$$T_{a\xi^p \xi^{-q}} z^\alpha = \begin{cases} 0 & if\ \exists\ l\ such\ that\ \alpha_l < q_l - p_l \\ \tilde{\gamma}_{a,k,p,q,\lambda}(\alpha) z^{\alpha + p - q}, & if\ \forall\ l\ \alpha_l \geq q_l - p_l, \end{cases}$$

where

$$\tilde{\gamma}_{a,k,p,q,\lambda}(\alpha) = \frac{2^m\,(n + |\alpha| + \alpha + 1)}{\Gamma(\lambda + 1) \prod_{j=1}^m (k_j - 1 + |\alpha_{(j)}|)(\alpha + p - q)\,!}$$
$$\times \int_{\tau(\mathbb{B}^m)} a(r_1, \ldots, r_m)\,(1 - |r|^2)\lambda \prod_{j=1}^m r_j^{2|\alpha_{(j)} + p_{(j)} - q_{(j)}| + 2k_j - 1} dr_j \quad (8)$$

Proof: For each two multi-indices $\alpha, \beta \in \mathbb{Z}_+^n$, we calculate

$$\langle T_{a\xi^p \bar{\xi}^q} z^\alpha, z^\beta \rangle = \langle \alpha \xi^p \bar{\xi}^q z^\alpha, z^\beta \rangle$$
$$= \frac{\Gamma(n + \lambda + 1)}{\pi^n\, \Gamma(\lambda + 1)} \int_{\mathbb{B}^n} a(r_1, \ldots, r_m)\, \xi^p \bar{\xi}^q z^\alpha, \bar{z}^\beta (1 - |z|^2)^\lambda dV(z).$$

Changing the variables $z_{(j)} = r_j \xi_{(j)}$, where $r_j \in [0, 1)$ and $\xi_{(j)} \in \mathbb{S}^{k_j}, j = 1, \ldots, m$ we have

$$\langle \xi^p \bar{\xi}^q z^\alpha, z^\beta \rangle$$
$$= \frac{\Gamma(n + \lambda + 1)}{\pi^n\, \Gamma(\lambda + 1)} \times \int_{\mathbb{B}^n} a(r_1, \ldots, r_m)\,(1 - |r|^2)^\lambda \prod_{j=1}^m r_j^{2|\alpha_{(j)} + p_{(j)} - q_{(j)}| + 2k_j - 1} dr_j$$
$$\times \prod_{j=1}^m \int_{\mathbb{S}^{k_j}} \xi_{(j)}^{\alpha_{(j)} + p_{(j)}} \bar{\xi}_{(j)}^{\beta_{(j)} + p_{(j)}} dS(\xi_{(j)})$$

$$= \delta_{\alpha+p,\beta+q} \frac{2^m\,(n + |\alpha| + \alpha + p)}{\Gamma(\lambda + 1)\prod_{j=1}^{m}\left(k_j - 1 + |\alpha_{(j)}+p_{(j)}|\right)!}$$
$$\times \int_{\tau(\mathbb{B}^m)} a(r_1, \dots, r_m)\,(1 - |r|^2)^{\lambda} \prod_{j=1}^{m} r_j^{|\lambda_{(j)}+\beta_{(j)}|+2k_j-1} dr_j$$

The last integral is non zero if and only if $\alpha + p = \beta + q$ and $\alpha_l + p_l - q_l \geq 0,$ for each $l = 1, 2, \dots, n$. Now for $\beta = \alpha + p - q$, with $\alpha_l + p_l - q_l \geq 0$, for each $l = 1, 2, \dots, n$, we have by (11),

$$\langle z^{\beta}, z^{\beta}\rangle = \langle z^{\alpha+p-g}, z^{\alpha+p-g}\rangle = \frac{(\alpha + p - q)!\,\Gamma(n + \lambda + 1)}{\Gamma(n + |\alpha + p - q| + \lambda + 1)},$$

and the result follows.

A particular case of the next theorem when $k = (n)$ and $\lambda = 0$ was proved in [317].

Theorem (3.2.3)[193]: Let $k = (k_1, k_2, \dots, k_m)$ and p, q be a pair of orthogonal multi-indices. Then for each pair of non identically zero k-quasi-radial functions a_1 and a_2, the Toeplitz operators T_{a_1} and $T_{a_2\xi^p\xi^{-q}}$ commute on each weighted Bergman space $\mathcal{A}_\lambda^2(\mathbb{B}^n)$ if and only if $|p_{(j)}| = |q_{(j)}|$ for each $j = 1, 2, \dots, m$.

Proof: For those multi-indices α with $\alpha_l + p_l - q_l \geq 0$, for each $l = 1, 2, \dots, n$, by Lemmas(3.2.1) and (3.2.2) we have

$$T_{a_2\xi^p\bar{\xi}^q} T_{\alpha_l} z^{\alpha}$$
$$= \frac{2^m(n + |\alpha + p + q| + \lambda + 1)(\alpha + p)!}{\Gamma(\lambda + 1)\prod_{j=1}^{m}\left(k_j - 1 + |\alpha_{(j)}| + p_{(j)}\right)!\,(\alpha + p - q)!}$$
$$\times \int_{\tau(\mathbb{B}^m)} a_2(r_1, \dots, r_m)\,(1 - |r|^2)^{\lambda} \prod_{j=1}^{m} r_j^{|2\alpha_{(j)}+p_{(j)}+q_{(j)}|+2k_j-1} dr_j$$
$$\times \frac{2^m\,(n + |\alpha| + \lambda + 1)}{\Gamma(\lambda + 1)\prod_{j=1}^{m}(k_j - 1 + |\alpha_{(j)}|)!}$$
$$\times \int_{\tau(\mathbb{B}^m)} a_2(r_1, \dots, r_m)\,(1 - |r|^2)^{\lambda} \prod_{j=1}^{m} r_j^{2|\alpha_{(j)}|+2k_j-1} dr_j\, z^{\alpha+p-q}$$

and

$T_{a_1}T_{a_2}\xi^p\bar{\xi}^q\, z^\alpha$

$$= \frac{2^m\,(n + |\alpha| + \lambda + 1)}{\Gamma(\lambda + 1)\prod_{j=1}^{m}(k_j - 1 + |\alpha_{(j)}|)\,!}$$

$$\times \int_{\tau(\mathbb{B}^m)} a_1(r_1, \dots, r_m)\,(1 - |r|^2)^\lambda \prod_{j=1}^{m} r_j^{2|\alpha_{(j)}p_{(j)}+q_{(j)}|+2k_j-1} dr_j$$

$$\times \frac{2^m\,\Gamma(n + |\alpha + p - q| + \lambda + 1)(\alpha + p)!}{\Gamma(\lambda + 1)\prod_{j=1}^{m}(k_j - 1 + |\alpha_{(j)}p_{(j)} + q_{(j)}|)\,(\alpha + p - q)!}$$

$$\times \int_{\tau(\mathbb{B}^m)} a_1(r_1, \dots, r_m)\,(1 - |r|^2)^\lambda \prod_{j=1}^{m} r_j^{2|\alpha_{(j)}p_{(j)}+q_{(j)}|+2k_j-1} dr_j\, z^{\alpha+p-q}$$

That is $T_{a_2}T_{a_1\xi^p\bar{\xi}^q}z^\alpha = T_{a_1}T_{a_2\xi^p\bar{\xi}^q}z^\alpha$if and only if $|p_{(j)}| = |q_{(j)}|$ for each $j = 1, 2, \dots, m$

We note that under the condition $p_{(j)}| = |q_{(j)}|$, for each $j = 1, 2, \dots, m$, formula (13) reads as

$$\tilde{\gamma}_{a,k,p,q,\lambda}(\alpha) = \frac{2^m\,(n + |\alpha| + \alpha + 1)}{\Gamma(\lambda + 1)\prod_{j=1}^{m}\left(k_j - 1 + \left|\alpha_{(j)+}p_{(j)}\right|\right)!\,(\alpha + p - q)\,!}$$

$$\times \int_{\tau(\mathbb{B}^m)} a(r_1, \dots, r_m)\,(1 - |r|^2)^\lambda \prod_{j=1}^{m} r_j^{2|\alpha_{(j)}|+2k_j-1} dr_j\,.$$

$$= \frac{\prod_{j=1}^{m}\left(k_j - 1 + \left|\alpha_{(j)}\right|\right)!\,(\alpha + p)!}{\prod_{j=1}^{m}\left(k_j - 1 + \left|\alpha_{(j)+}p_{(j)}\right|\right)!\,(\alpha + p - q)}\gamma_{\alpha,k\lambda}(\alpha)$$

$$= \prod_{j=1}^{m}\left[\frac{\left(k_j - 1 + \left|\alpha_{(j)}\right|\right)!}{\left(k_j - 1 + \left|\alpha_{(j)+}p_{(j)}\right|\right)!}\frac{\left(\alpha_{(j)+}p_{(j)}\right)!}{\left(\alpha_{(j)+}p_{(j)} - q_{(j)}\right)!}\right]\gamma_{\alpha,k\lambda}(\alpha) \quad (9)$$

As surprising corollaries we have:

Corollary (3.2.4)[193]: Given $k = (k_1, k_2, \dots, k_m)$, for each pair of orthogonal multi-indices p and q with $|p_{(j)}| = |q_{(j)}|$, for all $j = 1, 2, \dots, m$, and each $a(r_1, r_2, \dots, r_m) \in \mathcal{R}_k$, we have

$$T_aT_{\xi^p\bar{\xi}^q} = T_{\xi^p\bar{\xi}^q}T_a = T_{a\xi^p\bar{\xi}^q}.$$

Given$k = (k_1, k_2, \dots, k_m)$, and a pair of orthogonal multi-indices p and q with $|p_{(j)}| = |q_{(j)}|$,, for all $j = 1, 2, \dots, m$, let

$\tilde{p}(j) = (0, \dots, 0, p_{(j)}, 0, \dots, 0)$ and $\tilde{q}_{(j)} = (0, \dots, 0, q_{(j)}, 0, \dots, 0)$.

Then, of course, $p = \tilde{p}_{(1)} + \tilde{p}_{(2)} + \dots + \tilde{p}_{(m)}$ and $q = \tilde{p}_{(1)} + \tilde{q}_{(2)} + \dots + \tilde{q}_{(m)}$.

For each $j = 1, 2, \dots, m$, we introduce the Toeplitz operator $T_j = T_{\xi^{\tilde{p}_{(j)}}\bar{\xi}^{\tilde{p}_{(j)}}}$.

Corollary (3.2.5)[193]: The operators $T_j, j = 1, 2, \dots, m$, mutually commute.Given an h-tuple of indices $(j_1, j_2, \dots, j_h)$, where $2 \le h \le m$, let

$\tilde{p}_h = \tilde{p}_{(j_1)} + \tilde{p}_{(j_2)} + \dots + \tilde{p}_{(j_h)}$ and$\tilde{q}_h = \tilde{q}_{(j_1)} + \tilde{q}_{(j_2)} + \dots + \tilde{q}_{(j_h)}$.

Then

$$\prod_{g=1}^{h} T_{i_g} = T_{\xi^{\bar{p}_h}\bar{\xi}^{\bar{g}h}}$$

In particular,

$$\prod_{j=1}^{m} T_i = T_{\xi^p \bar{\xi}^q}.$$

Given $k = (k_1, k_2, \ldots, k_m)$, we consider any two bounded measurable k −quasi-homogeneous symbols $a(r_1, r_2, \ldots, r_m)\xi^p\bar{\xi}^q$ and $b(r_1, r_2, \ldots, r_m)\xi^u\bar{\xi}^v$, which satisfy the conditions of Theorem (3.2.4) , i.e., $a(r_1, r_2, \ldots, r_m)$ and $b(r_1, r_2, \ldots, r_m)$are arbitrary k-quasi-radial functions, $p \perp q, u \perp v$, and$|p_{(j)}| = |q_{(j)}|$ and $|u_{(j)}| = |v_{(j)}|$, for all $j = 1, 2, \ldots, m$.

Theorem (3.2.6)[193]: Let $(r_1, r_2, \ldots, r_m)\xi^p\bar{\xi}^q$and $b(r_1, r_2, \ldots, r_m)\xi^u\bar{\xi}^v$, be as above. Then the Toeplitz operators $T_{a\xi^p\bar{\xi}^q}$ and $T_{b\xi^u\bar{\xi}^v,}$ commute on each weighted Bergman space $\mathcal{A}_\lambda^2(\mathbb{B}^n)$ if and only if for each $l = 1, 2, \ldots, n$ one of the next conditions is fulfilled

(i) $p_l = q_l = 0$;
(ii) $u_l = v_l = 0$;
(iii) $p_l = u_l = 0$;
(iv) $q_l = v_l = 0$.

Proof: We calculate and compare first $T_{a\xi^p\bar{\xi}^q}T_{b\xi^u\bar{\xi}^v}z^\alpha$ and $T_{b\xi^u\bar{\xi}^v}T_{a\xi^p\bar{\xi}^q}z^\alpha$ for those multindices α when both these expressions are non zero.By (8) we have

$$T_{a\xi^p\bar{\xi}^q}T_{b\xi^u\bar{\xi}^v,}$$

$$= \frac{2^m\Gamma(n + |\alpha| + \lambda + 1)(\alpha + u - v + p)!}{\Gamma(\lambda + 1)\prod_{j=1}^m\left(k_j - 1 + \left|\alpha_{(j)+}p_{(j)}\right|\right)!\,(\alpha + u - v + p - q)\,!}$$

$$\times \int_{\tau(\mathbb{B}^m)} a(r_1, \ldots, r_m)\,(1 - |r|^2)^\lambda \prod_{j=1}^m r_j^{2|\alpha_{(j)}|+2k_j-1} dr_j$$

$$\times \frac{2^m\Gamma(n + |\alpha| + \lambda + 1)(\alpha + u)!}{\Gamma(\lambda + 1)\prod_{j=1}^m\left(k_j - 1 + \left|\alpha_{(j)+}u_{(j)}\right|\right)!\,(\alpha + u - v)\,!}$$

$$\times \int_{\tau(\mathbb{B}^m)} b(r_1, \ldots, r_m)\,(1 - |r|^2)^\lambda \prod_{j=1}^m r_j^{2|\alpha_{(j)}|+2k_j-1} dr_j\, z^{\alpha+u-v+p-q}$$

and

$T_{b\xi^u\bar{\xi}^v,}T_{a\xi^p\bar{\xi}^q}z^\alpha$

$$= \frac{2^m\Gamma(n + |\alpha| + \lambda + 1)(\alpha + u - v + p)!}{\Gamma(\lambda + 1)\prod_{j=1}^{m}\left(k_j - 1 + \left|\alpha_{(j)+}u_{(j)}\right|\right)!\,(\alpha + p - q + u - v)\,!}$$

$$\times \int_{\tau(\mathbb{B}^m)} b(r_1, \dots, r_m)\,(1 - |r|^2)^\lambda \prod_{j=1}^{m} r_j^{2|\alpha_{(j)}|+2k_j-1} dr_j$$

$$\times \frac{2^m\Gamma(n + |\alpha| + \lambda + 1)(\alpha + p)!}{\Gamma(\lambda + 1)\prod_{j=1}^{m}\left(k_j - 1 + \left|\alpha_{(j)+}p_{(j)}\right|\right)!\,(\alpha + p - q)\,!}$$

$$\times \int_{\tau(\mathbb{B}^m)} b(r_1, \dots, r_m)\,(1 - |r|^2)^\lambda \prod_{j=1}^{m} r_j^{2|\alpha_{(j)}|+2k_j-1} dr_j\, z^{\alpha+u-v+p-q}$$

That is $T_{a\xi^p\bar{\xi}^q}T_{b\xi^u\bar{\xi}^v,} = T_{b\xi^u\bar{\xi}^v,}T_{a\xi^p\bar{\xi}^q}z^\alpha$ if and only if

$$\frac{(\alpha + u - v + p)!\,(\alpha + u)!}{(\alpha + u - v)!} = \frac{(\alpha + p - q + u)!\,(\alpha + p)!}{(\alpha + p - q)!}$$

Varying α it is easy to see that the last equality holds if and only if for each $l = 1, 2, \dots, n$ one of the next conditions is fulfilled

(i) $p_l = q_l = 0;$

(ii) $u_l = v_l = 0;$

(iii) $p_l = u_l = 0;$

(iv) $q_l = v_l = 0.$

To finish the proof we mention that under either of the above conditions both quantities $T_{a\xi^p\bar{\xi}^q}T_{b\xi^u\bar{\xi}^v}z^\alpha$and $T_{b\xi^u\bar{\xi}^v}T_{a\xi^p\bar{\xi}^q}z^\alpha$ are zero or non zero simultaneously only.

Example (3.2.7)[193]: Let $n = 7$ and k $= (2, 5)$. Then by Theorem (3.2.4) the Toeplitz operators with symbols $a(r_1, r_2) \in \mathcal{R}_k$ and $b\,\xi^p\bar{\xi}^q$, where $b(r_1, r_2) \in \mathcal{R}_k, p = (1, 0, 0, 3, 0, 1, 0), q = (0, 1, 1, 0, 1, 0, 2)$, commute. We mention that here

$p(1) = (1, 0), p(2) = (0, 3, 0, 1, 0)$ and $q(1) = (0, 1), q(2) = (1, 0, 1, 0, 2)$.

As easy to see, all pairs (u, v) of orthogonal multi-indices such that (by Theorem (3.2.8)) the Toeplitz operators with k-quasi-homogeneous symbols having that quasi-homogeneous degrees mutually commute, and commute with both T_a and $T_{b\xi^p\bar{\xi}^q}$ are of the form

$$u = (u_1, 0, 0, u_4, 0, u_6, 0), v = (0, v_2, v_3, 0, v_5, 0, v_7), \qquad (10)$$

where $u_1, u_4, u_6 \in \mathbb{Z}_+, v_2, v_3, v_5, v_7 \in \mathbb{Z}_+$, and

$$u_1 = v_2, u_4 + u_6 = v_3 + v_5 + v_7. \qquad (11)$$

that is, the Banach algebra generated by all Toeplitz operators$T_{a\xi^u\xi^{-v}}$, where $a(r_1, r_2) \in \mathcal{R}_k$, and the orhogonal multi-indices u and v of the form (10) satisfy the condition (11), is commutative.

We formalize the above example as follows. First, to avoid the repetition of the unitary equivalent algebras and to simplify the classification of the (non unitary equivalent) algebras, in addition to (7), we can rearrange the variables z_l and correspondingly the components of multi-indices in p and q so that

(i) for each j with $k_j > 1$, we have

$$p_{(j)} = \left(p_{j,1}, \dots, p_{j,hj}, 0, \dots, 0\right) \text{ and } q_{(j)} = \left(0, \dots, 0, q_{j,hj+1}, \dots, q_{j,k_j}\right); \quad (12)$$

(ii) if $k_j{}' = k_j{}''$ with$j' < j''$, then $h_j{}' \le h_j{}''$.

Now, given $k = (k_1, \dots, k_m)$, we start with m-tuple $h = (h_1, \dots, h_m)$, where $h_j = 0$ if $\mathrm{k_j} = 1$ and $1 \le h_j \le k_j - 1$ if $\mathrm{k_j} \ge 1$; in the last case, if $k_j{}' = k_j{}''$ with $j' < j''$, then $h_j{}' \le h_j{}''$. .

We denote by $\mathcal{R}_k(h)$ the linear space generated by all k-quasi-homogeneous functions

$$a(r_1, r_2, \dots, r_m)\ \xi^p \xi^{-q},$$

where $a(r_1, r_2, \dots, r_m) \in \mathcal{R}_k$, and the components $p_{(j)}$ and $q_{(j)}, j = 1, 2, \dots, m$, of multi-indices p and q are of the form (12) with

$$p_{j,1} + \dots + p_{j,h_j} = q_{j,h_{j+1}} + \dots + q_{j,k_j}, p_{j,1}, \dots, p_{j,h_j}, q_{j,h_{j+1}} + 1, \dots, q_{j,k_j} \in \mathbb{Z}_+.$$

We note that $\mathcal{R}_\mathrm{k} \subset \mathcal{R}_\mathrm{k}(\mathrm{h})$ and that the identity function $\mathrm{e(z)} \equiv 1$ belongsto$\mathcal{R}_k(h)$.

We have the following result.

Corollary (3.2.8)[193]: The Banach algebra generated by Toeplitz operators with symbols from $\mathcal{R}_k(h)$is commutative.

We would like to emphasize the following features of such algebras:

(i) For different k and h these algebras are not conjugated via biholomorphisms of the unit ball;

(ii) These algebras are just Banach and not C^*-algebras; extending them to C^*-algebras they become non commutative;

(iii) Given $k \neq (1, 1, \dots, 1)$, there is a finite number of different m-tuples h and thus a finite number of different corresponding commutative algebras;

(iv) These algebras remain commutative for each weighted Bergman space $\mathcal{A}_\lambda^2(\mathbb{B}^n)$, with $\lambda > -1$,

(v) For $n = 1$ all of them collapse to the single C^*-algebra generated by Toeplitz operators with radial symbols.

We finish presenting another application of Theorems (3.2.4) and(3.2.8),Studying commutativity properties of Toeplitz operators on the Bergman space on the unit disk I. Louhichi and N. V. Rao [124] conjectured that if two Topelitz operators commute with a third one, none of them being the identity, then they commute with each other.

As next example shows, this conjecture is wrong when formulated for Toeplitz operatorson the unit ball $(\mathbb{B}^\mathrm{n})$, with $\mathrm{n} > 1$.

Example (3.2.9)[193]: Given $n > 1$, let $k = (2, 1, \dots, 1)$. Consider the following three symbols

$$a_0 = a(r_1, r_2, \dots, r_{n-1}), a_1 = b(r_1, r_2, \dots, r_{n-1})\xi_{(1)}^{(1,0)}\bar{\xi}_{(1)}^{(0,1)},$$

$$a_2 = c(r_1, r_2, \dots, r_{n-1})\xi_{(1)}^{(1,0)}\bar{\xi}_{(1)}^{(0,1)}$$

where $\mathrm{a, b, c} \in \mathcal{R}_\mathrm{k}$.

Then by Theorem (3.2.3) T_{a_0} commutes with both T_{a_1} and T_{a_2}, while by Theorem (3.2.6) the operators T_{a_1} and T_{a_2} do not commute.

Chapter 4
Eigenvalue Inequalities and on the Eigenvalues of Normal Matrices

Using techniques from algebraic topology we derive linear inequalities which relate the spectrum of a set of Hermitian matrices $A_1, \ldots, A_r \in \mathbb{C}^{n\times n}$ with the spectrum of the sum $A_1 + \cdots + A_r$. There results are a direct generalization of a theorem of Wielandt on the eigenvalues of the sum of two normal matrices. Characterizations of eigenvalues of normal matrices using the lexicographical order in $\mathbb{C}$ are presented, with some applications.

Section (4.1) Schubert Calculus

Consider real $n \times n$ diagonal matrices $D_1, \ldots, D_r$with diagonal elements $\lambda_1(D_1) \geq \lambda_2(D_1) \geq \ldots \geq \lambda_n(D_1), l = 1, \ldots, r$. In this section we are concerned with geometric properties of the set of possible spectrums of the matrices

$$\left\{\sum_{l=1}^{r} U_l^* \, U_l : U_l \quad \text{are unitary}\right\}. \tag{1}$$

Equivalently we are interested in the following question:
Given Hermitian matrices $A_1, \ldots, A_r\, 2 \in \mathbb{C}^{n\times n}$ each with a fixed spectrum $\lambda_1(A_l) \geq \cdots \lambda_n(A_l), j = 1, \ldots, r$ and arbitrary else. Is it possible to find then linear inequalities which describe the possible spectrum of the matrix $A_1 \ + \cdots \ + A_r$?
For $r = 1$ this question is of course trivial. For $r = 2$ the question is classical and very well studied (compare with [139, 10, 263, 34, 239, 237, 225, 108]).
An early example of an eigenvalue inequality for a sum of two Hermitian matrices is that of Weyl [108,112,158,33,78]. A generalization of the Weyl inequalities to k −fold partial sums of eigenvalues of Hermitian matrices A, B and $A + B$ is due to Freede and Thompson [225]. Still more general is the class of eigenvalue inequalities described by Horn [10,32] for sums of two eigenvalues.

We will present a systematic geometric approach to obtain such eigenvalue inequalities. Although our main results are in the case of two matrices, where $r = 2$, the approach works equally well in the case of r-fold sums $A_1 + \cdots + A_r$ of Hermitian matrices $A_1, \ldots, A_r$. Our interest in this problem originates in the observation by Thompson [239, 237] who indicates that most of the known inequalities for the case $r = 2$ can be derived using methods from algebraic topology, i.e. by the Schubert calculus of complex Grassmann manifolds. As this topological approach is described only in a rudimentary form in [239, 237,35,132] we first present a rigorous development of the Schubert calculus technique towards eigenvalue inequalities. We then show that it is also possible to derive with the same method a large set of inequalities for the case $r > 2$ as well.

The algebraic topology approach to solving inverse eigenvalue problems is by no means limited to the task of finding eigenvalue inequalities for sums of Hermitian matrices. In fact, the technique has been already successfully applied to solve an outstanding inverse eigenvalue problem arising in control theory, i.e. the pole placement problem for multivariable linear systems by static output feedback. See. [248, 138].

The minmax principles of Wielandt and Hersch-Zwahlen are reviewed, which characterize in geometric terms partial sums of eigenvalues of a Hermitian matrix. We review the relevant results from the Schubert calculus of Grassmann manifolds. We apply the technique and state the main results. We show how the inequalities of Weyl [108], [34] and Freede-Thompson [225] follow from the main theorem. In the last section we describe a large set of nonzero products in the cohomology ring $H^*(G_k\mathbb{C}^n),\mathcal{Z})$ of the Grassmann manifold, leading to a new class of inequalities for sums of eigenvalues of Hermitian matrices $A_1,\cdots,A_r$. Let $A \in \mathbb{C}^{n\times n}$ be a complex Hermitian matrix with eigenvalues

$$\lambda_1(A) \geq \lambda_2(A) \geq \ldots \geq \lambda_n(A). \quad (2)$$

The classical Courant-Fischer minmax principle then asserts that (compare e.g. [222]):

Theorem (4.1.1)[289]: For $1 \leq i \leq n$:

$$\lambda_1(A) = \max_{dimV=i} \min_{x\in V} tr(Axx^*) \quad (3)$$

$$= \max_{dimW=n-i+1} \min_{\substack{x\in W \\ \|x\|=1}} tr(Axx^*) \quad (4)$$

Amore general version of the minmax principle is due to Wielandt [109] and Hersch- Zwahlen [138] and characterizes partial sums of eigenvalues via flags of subspaces of $\mathbb{C}^n$. To state their result we first recall some basic notions and definitions from geometry:

The complex projective space $\mathbb{CP}^n$ is defined as the set of all one-dimensional complex subspaces of $\mathbb{C}^{n+1}$, i.e. as the set of all complex lines passing through the origin $0 \epsilon \mathbb{C}^{n+1}$. More generally, the complex Grassmann manifold $G_k(\mathbb{C}^n)$ is defined as the set of all k-dimensional complex linear subspaces of $\mathbb{C}^n$. In particular for $k = 1$ one has the complex projective space $G_1(\mathbb{C}^n) = \mathbb{CP}^{n-1}$. The Grassmannian is a smooth, compact manifold of real dimension $2k(n - k)$.

Equivalently, the Grassmannian $G_k(\mathbb{C}^n)$ may be defined as the set of all Hermitian projection operators $P: \mathbb{C}^n \to \mathbb{C}^n$ of rank k. A Hermitian projection operator of $\mathbb{C}^n$ is a Hermitian matrix $P \in \mathbb{C}^{n\times n}$satisfying

$$P^* = P, P^2 = P, \text{and rank } P = k. \quad (5)$$

For any k-dimensional complex linear subspace $L \subset \mathbb{C}^n$ let P_L: $\mathbb{C}^n \to \mathbb{C}^n$ be the uniquely determined projection operator satisfying

$$im(P_L) = L, ker(P_L) = L^\perp, \quad (6)$$

where $L^\perp$ denotes the orthogonal complement of L in $\mathbb{C}^n$ with respect of the standard Hermitian inner product. Thus P_L is the orthogonal projection of $\mathbb{C}^n$ onto L along $L^\perp$. If $X \in \mathbb{C}^{n\times k}$ is any full rank matrix whose columns form a basis of L, then one has

$$P_L = X(X^*X)^{-1}X^* \quad (7)$$

Conversely, for any full rank matrix $X \in \mathbb{C}^{n\times k}$, the operator defined by (7) is a rank k Hermitian projection operator on $\mathbb{C}^n$. Thus the map $L^\perp \to P_L$ is a bijection of $G_k(\mathbb{C}^n)$ onto the set

$$\{P \in \mathbb{C}^{n\times n}: P^* = P, P^2 = P, and\ rankP = k\}.$$

Given any k-dimensional linear subspace $L \subset \mathbb{C}^n$ let $P_L: \mathbb{C}^n \to \mathbb{C}^n$ denote the associated Hermitian projection operator. We then define

$$tr(A|_L):= tr(P_LAP_L) = tr(AP_L) = tr(AX(X^*X)^{-1}X^*), \quad (8)$$

where $X \in C^{n\times k}$ is any full rank matrix whose columns form a basis of L. Note that $tr(A|_L))$ is the trace of a Hermitian operator and therefore a real number.

Definition (4.1.2)[289]: The smooth map

$$\mathrm{R_A}: \mathrm{G_k}(\mathbb{C}^n) \to \mathbb{R}$$
$$\mathrm{L} \to tr(A|_L) \quad (9)$$

is called the Rayleigh quotient of A on $G_k(\mathbb{C}^n)$.

If $k = 1$ the map R_A coincides with the classical Rayleigh quotient

$$R_A(x) = \frac{< Ax, x >}{< x, x >} \quad (10)$$

The extremal principles for the partial sums of eigenvalues of a Hermitian matrix A of Wielandt, Hersch-Zwahlen and Riddel are now stated as follows:

Theorem (4.1.3)[289]: (Wielandt [19]) For $1 < i_1 < \ldots < i_k < n$:

$$\lambda_{i1}(A) + \cdots + \lambda_{ik}(A) = \max_{\substack{V_1\subset\cdots\subset V_k \\ dimV_j = i_j}} \min_{\substack{V_1 \in G_k(\mathbb{C}^n) \\ dim(L\cap V_j)\geq i_j}} tr(A|_L) \quad (11)$$

$$= \min_{\substack{\mathrm{dimW_j = n - i_{j+1}} \\ \mathrm{dimW_j = n - i_j + 1}}} \max_{\substack{\mathrm{L \in G_k(\mathbb{C}^n)} \\ \mathrm{dim(L\cap V_j) \geq j}}} tr(A|_L) \quad (12)$$

In particular, for $k = 1$, Theorem (4.1.3) specializes to the Courant-Fischer minmax principle as formulated in Theorem (4.1.1).

Note that ,it can be shown (see [222]) that the maximal value of (11) is assumed at a “partial flag of eigenspaces”, i.e. at a flag $(V_1, \ldots, V_k)$ having the property that

$dim(V_j) = i_j$ and $V_j \subset ker\,(\lambda_1 I - A) \oplus \ldots \oplus ker\,\left(\lambda_{i_j} I - A\right)$,for $j = 1, \ldots, k$

We conclude with the following result from Hersch-Zwahlen [138]:

Theorem (4.1.4)[289]: Let A be a Hermitian matrix with eigenvalues $\lambda_1(A) \geq \ldots \geq \lambda_1(A)$ and a corresponding orthogonal set of eigenvectors $v_1, \ldots, v_n$. Denote with

$$V_m \coloneqq span\,(v_1, \ldots, v_n), m = 1, \ldots, n. \quad (13)$$

Let $1 \leq i_1 < \cdots < i_k \leq n$ then one has:

$$\lambda_1(A) + \cdots + \lambda_{ik}(A) = \min_{L\in G_K(\mathbb{C}^n)} \{tr(A|_L): dim(L \cap V) \geq j, j = 1, \ldots, k\}. \quad (14)$$

Thus the result of Hersch-Zwahlen just says that the sum of eigenvalues $\lambda_1(A) + \cdots + \lambda_{ik}(A)$ is characterized as the minimal value of the trace function $tr(A|_L)$when evaluated on a Schubert subvariety of $G_k(\mathbb{C}^n)$.

Consider again the Grassmann manifold $G_k(\mathbb{C}^n)$consisting of k-dimensional linear subspaces of the vector space $\mathrm{C^n}$. Using the Plücker embedding $G_k(\mathbb{C}^n)$can be embedded into the projective space $\mathbb{C}^n\mathbb{P}^N$ of dimension $N = \frac{n!}{k!(n-k)!} - 1$. Under this embedding $G_k(\mathbb{C}^n)$ is a projective variety described by a famous set of quadratic relations (see e.g. [203]).

Definition (4.1.5)[289]: A flag $\mathcal{F}$ is a sequence of nested subspaces

$$\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_3 = \mathbb{C}^n \quad (15)$$

where we assume that $dim\, V_i = i$ for $i = 1, \ldots, n$.

Let $i = (i_1, \ldots, i_k)$ denote a sequence of numbers having the property that

$$i \leq i_1 \leq \ldots \leq i_k \leq n. \quad (16)$$

Definition (4.1.6)[289]: For each flag $\mathcal{F}$ and each multiindex i define

$$C(\underline{i};\mathcal{F}) \coloneqq \{W \in G_k(\mathbb{C}^n): dim(w \cap V_{i_s}) = s\}$$
is called a Schubert cell and
$$S(\underline{i};\mathcal{F}) \coloneqq \{W \in G_k(\mathbb{C}^n): dim(w \cap V_{i_s}) = s\}$$
is called a Schubert variety.
We emphasize that the Schubert cell $C(\underline{i};\mathcal{F})$is indeed a cell, i.e. isomorphic to theaffine space $\mathbb{C}^N$ where $N: = \sum_{j=1}^{k} i_j - j$ is the dimension of the cell $C(\underline{i};\mathcal{F})$. (Compare with [203].) Moreover the Zariski closure of the cell $C(\underline{i};\mathcal{F})$is the variety $S(\underline{i};\mathcal{F})$which is a projective algebraic subvariety of $G_k(\mathbb{C}^n)$.

The following results are well known and we refer e.g. to [309, 203].

Theorem (4.1.7)[289]: For every fixed flag $\mathcal{F}$ the Schubert cells$C(\underline{i};\mathcal{F})$ decompose the Grassmann variety $G_k(\mathbb{C}^n)$ into a finite cellular CW–complex. The integral homology $H_{2m}(G_k(\mathbb{C}^n), Z)$ has no torsion and is freely generated by the fundamental classes of the Schubert varieties $S(\underline{i};\mathcal{F})$of real dimension $2m$.

Consider a fixed Schubert variety $S(\underline{i};\mathcal{F})$Its homology class is independent of the choice of the flag $\mathcal{F}$ and therefore depends only on the numbers $i_1, \dots, i_k$. we will use the symbol $(i_1, \dots, i_k)$ to denote this homology class. The Poincar´e-dual of the class $(i_1, \dots, i_k)$ will be denoted by
$$\{\mu_1, \dots, \mu_k\} \coloneqq \{n - k - i_1 + 1, n - k - i_2 + 2, \dots, n - i_k\} \in H^*(G_k(\mathbb{C}^n), Z) \quad (17)$$
At this point we want to mention that our notation was already used by Schubert (compare with the book of Fulton [309,306,258]) and is slightly different to the one used in [203, 103,46,173,30]. The cohomology ring
$$H^*(G_k(\mathbb{C}^n), Z) \coloneqq \bigoplus_{m=0}^{k(n-k)} H^{2m}(G_k(\mathbb{C}^n), Z) \quad (18)$$
has in a natural way the structure of a graded ring. From Poincar´e-duality and Theorem (4.1.8) it follows in particular that each graded component $H^{2m}(G_k(\mathbb{C}^n), Z)$ is a free Z-module with basis the set of Schubert cocycles $\{\mu_1, \dots, \mu_k\}$ where $n \geq k \geq \mu_1 \geq \dots \geq \mu_k \geq 0$and $\sum_{j=1}^{k} i_j = j = m$.

Before we describe the multiplicative structure of this ring we formulate the following proposition which establishes the crucial link between geometric intersection properties of Schubert varieties and algebraic properties of the ring $H^*(C_k(C^n), Z)$. A proof of this as well as more general theorems can be found e.g. in [308, 202].

Proposition (4.1.8)[289]: Consider r Schubert varieties $S(\underline{i}_l;\mathcal{F}_l), l = 1, \dots, r. I$
$$\prod_{l=1}^{r+1} \{n - k - i_{jl} + 1, \dots, n - i_{kl}\} \neq 0, \quad (19)$$
then the intersection
$$\bigcap_{l=1}^{r} S(\underline{i}_l;\mathcal{F}_l) \neq 0. \quad (20)$$

The multiplicative structure of $H^*(G_k(\mathbb{C}^n), Z)$ is described by the classical formulas of Pieri and Giambelli. For this denote with

$$\sigma_j := \{k, 0, \dots, \dots, 0\}, \quad j = 1, \dots, n-k. \tag{21}$$

In fact σ_j is the $j-th$ Chern class of the universal (classifying) bundle over $G_k(\mathbb{C}^n)$.

In the following we describe the formulas of Pieri and Giambelli. Giambelli's formula expresses a general Schubert cocycle $\{\mu_1, \dots, \mu_k\}$ as a polynomial in the special Schubert cocycle σ_j and Pieri's formula expresses the product of a general Schubert cocycle with a special Schubert cocycle. Pieri's formula:

$$\{\mu_1, \dots, \mu_k\}.\sigma_j = \sum_{\substack{\mu_{i-1} \geq V_i \geq \mu_i \\ \sum_{i=1}^k V_i = (\sum_{i=1}^k \mu_i) + j}} \{V_1, \dots, V_k\} \tag{22}$$

Giambelli's formula:

$$\{\mu_1, \dots, \mu_k\} = det\left(\sigma_{\mu_{i+j-i}}\right) = det\begin{pmatrix} \sigma_{\mu_1} & \sigma_{\mu_{1+1}} & \cdots & \sigma_{\mu_{1+k-1}} \\ \sigma_{\mu_{2-1}} & \sigma_{\mu_2} & & \vdots \\ \vdots & & \ddots & \vdots \\ \sigma_{\mu_{k-k+1}} & & \cdots & \sigma_{\mu_k} \end{pmatrix} \tag{23}$$

Note that Giambelli's formula implies that the Chern classes σ_j generate the ring $H^*(G_k(\mathbb{C}^n), Z)$.

There is a deep relationship between the ring $H^*(G_k(\mathbb{C}^n), Z)$and the ring of symmetric functions $Z[x_1, \dots, x_k]^{S_k}$, where S_k denotes the group of permutations, acting on k letters. To explain this relationship we consider a special set of symmetric functions called Schur functions. (See e.g. [123, 247,]). For this let $\mu := (\mu_1, \dots, \mu_k)$ and define

$$s_\mu := \frac{det\left[x^{\mu_j+k-j}\right]}{det\left[x^{k-j}\right]}; i, j = 1, \dots, k. \tag{24}$$

Note that s_μ is the quotient of two alternating functions and therefore a symmetric function, called a Schur function. As explained in detail in [123,109,29] the set of Schur functions

$$\left\{s_\mu : \mu_1 \geq \mu_2 \geq \cdots \geq \mu_k > 0 \; and \sum \mu_i = q\right\} \tag{25}$$

is an additive basis of the space of symmetric functions of degree q. As explained in [103, 136, 247] one has a ring epimorphism

$$\begin{aligned} \psi: Z[x_1, \dots, x_k]^{S_k} &\to H^*(G_k(\mathbb{C}^n), Z) \\ s_\mu &\to \{\mu_1, \dots, \mu_k\} \end{aligned} \tag{26}$$

The kernel of this map has as an additive basis the set of Schur functions s_μ with $s_\mu > n-k$. Using this epimorphism any calculation in the ring $H^*(G_k(\mathbb{C}^n), Z)$ can be formally done in the ring $Z[x_1, \dots, x_k]^{S_k}$ .We want to mention the rule of Littlewood and Richardson which explains how to additively expand a product of Schur functions in terms of Schur functions:

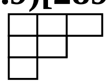
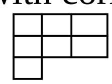
Consider two Schur functions s_μ and s_v. The product $s_\mu s_v$ is a symmetric functionof degree $\sum \mu_i + \sum v_i$ and has therefore an expansion in terms of Schur functions:

$$s_\mu s_v = \sum_\lambda C_{\mu,v}^\lambda s_\lambda \tag{27}$$

The appearing coordinates $C_{\mu,v}^{\lambda}$ are usually called the Littlewood Richardson coefficients [123,107,247]. In order to give a combinatorial characterization of those coefficients let $\mu = (\mu_1, \ldots, \mu_k)$ be a partition of n representing the Schur functionss_μ. In other words we assume that $n \geq k \geq \mu_1 \geq \mu_2 \geq \ldots \geq \mu_k \geq 0$and $\sum_{i=1}^{k} \mu_i = n$ If the integer μ_i is repeated r_i–times in the partition μ, the abbreviated notation $\mu = (\mu_1^{r1}, \ldots, \mu_t^{rt})$ will be used. The number $|\mu| := \sum_{i=1}^{k} \mu_i$ is sometimes called the weight of the partition μ and the numbers μ_i are called the parts of the partition.

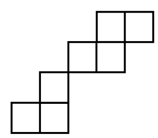
It is usual to present a partition by a left based array of boxes which has exactly μ_i boxes in the $i-th$ row. Such an array is sometimes called a tableau.

Example (4.1.9)[289]: Two partitions with corresponding diagrams are illustrated:

(3,2,1)⟷  $(3^2, 1)$ ⟷ 

Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a second partition. One writes$\lambda \geq \mu$ if $\lambda_i \geq \mu_i, i = 1, \ldots, k$. If $\lambda \geq \mu$ one defines the skew tableau λ/μ as the tableau obtained from the tableau λ by removing the first μ_i boxes in the row i of the tableau λ.

Example (4.1.10)[289]:$\lambda = (5, 4, 2, 2), \mu = (3, 2, 1)$ then λ/μ is given



We are now in a position to formulate the theorem of Littlewood and Richardson. The following formulation as well as the subsequent example can be found in the article of Stanley [247,228,236,249,87].

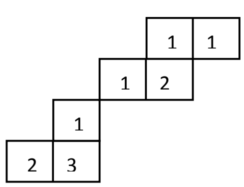
Theorem (4.1.11)[289]: Let s_μ and $s_v$be two Schur functions represented by two partitions μ, v. Then the Littlewood Richardson coefficient $C_{\mu,v}^{\lambda}$ of s_λ in the expansion of the product $s_\mu s_v$ is zero unless $\lambda \geq \mu$. In this case the coefficient is equal to the number of ways of inserting v_1 $1's, v_2$ $2's, v_3$ $3's, \ldots$ into the skew tableau λ/μ subject to the conditions:

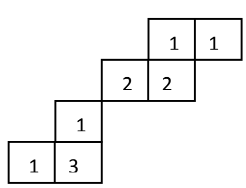
(i) The numbers are weakly increasing in each row and strictly increasing in each column.

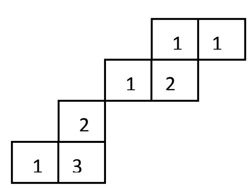
(ii) If $\alpha_1, \alpha_2, \ldots$ is the set of numbers obtained when reading of the numbers inserted in λ/μ from right to left then for any i, j the numbers of $i's$ among $\alpha_1, \alpha_2, \ldots, \alpha_j$ is not less than the number$(i + 1)'s$ among the numbers $\alpha_1, \alpha_2, \ldots, \alpha_j$.

The following example given in [247]:

Example(4.1.12)[289]: Let $\lambda = (5, 4, 2, 2)$, $\mu = (3, 2, 1)$ and $v = (4, 2, 1)$. Then the following skew diagrams λ/μ are the only ones which satisfy (i). and (ii). In particular the coefficient of s_λ in the expansion of the product $s_\mu s_v$is equal to







Using the Littlewood Richardson rule together with the description of the ring $H^*(G_k(\mathbb{C}^n), Z)$ as given in (26) we are in a position to multiply arbitrary cocycles in $H^*(G_k(\mathbb{C}^n), Z)$The following example illustrates the procedure:

Example(4.1.13)[289]: Consider the elements {3, 2, 0} and {2, 1, 0} in $H^*(G_3(\mathbb{C}^6), Z)$Then $\{3,2,0\}\{2,1,0\} = \{5,3,0\} + \{5,2,1\} + \{4,4,0\} + 2\{4,3,1\} + \{4,2,2\} + \{3,3,2\}$ (28)
We conclude with the Poincar´e duality theorem of cocycles. For this consider a cocycle $\{\mu_1, \ldots, \mu_k\}$. The dual cocycle in $H^*(G_k(\mathbb{C}^n), Z)$ is defined as the cocycle $\lambda := \{n - k - \mu_k, \ldots, n - k - \mu_1\}$.Using this notation one has:
Theorem (4.1.14)[289]:

$$\{\mu_1, \ldots, \mu_k\}\{v_1, \ldots, v_k\} = \{n-k, \ldots, n-k\}$$

Proof: Apply Theorem (1.3.12)of Littlewood and Richardson together with the description of $H^*(G_k(\mathbb{C}^n), Z)$ induced by the representation (26).
In order to derive result we will use the following simple lemma,.
Lemma (4.1.15)[289]: Suppose the eigenvalues of a Hermitian $n \times n$ matrix A are ordered as $\lambda_1(A) \geq \ldots \geq \lambda_n(A)$. Then for any $1 \leq i_1 < \ldots < i_k \leq n$ one h*as:*

$$\lambda_{i_1}(-A) + \cdots + \lambda_{i_k}(-A) = -\sum_{j=1}^{k} \lambda_{n-i_j+1}(A) \tag{29}$$

In the following we will consider Hermitian matrices $A_1, \ldots, A_{r+1} \in \mathbb{C}^{n\times n}$with corresponding eigenvalues

$$\lambda_{i_1}(A_l) \geq \ldots \geq \lambda_n(A_l), l = 1, \ldots, r+1 \tag{30}$$

and corresponding orthogonal sets of eigenvectors $v_{1l}, \ldots, v_{nl}$. Assume that

$$A_{r+1} = A_1 + \cdots + A_r. \tag{31}$$

For each Hermitian operator $A_l, l = 1, \ldots, r + 1$ construct a flag of eigenspaces

$$\mathcal{F}_l: \{0\} \subset V_{1l} \subset V_{2l} \subset \ldots \subset V_{nl} = \mathbb{C}^n \tag{32}$$

defined through the property:

$$V_{ml} := span(v_{1l}, \ldots, v_{ml}), m = 1, \ldots, n. \tag{33}$$

The following result, which has been first proved by Thompson [225] for the case $r = 2$, establishes the crucial relationship between matrix spectral inequalities and the Schubert calculus.
Lemma (4.1.16)[289]: Let $A_1, \ldots, A_r$be complex Hermitian n × n matrices and denote with $\mathcal{F}_l, \ldots, \mathcal{F}_{r+1}$ the corresponding flags of eigenspaces defined by (33). Assume $A_{r+1} = A_1 + \cdots + A_r$. and let $i_l = (i_{1l}, \ldots, i_{kl})$ be $r + 1$ sequences of integers satisfying

$$1 \leq i_{1l} < \ldots < i_{kl} \leq n, l = 1, \ldots, r+1. \tag{34}$$

Suppose the intersection of the $r + 1$ Schubert subvarieties of $G_k(\mathbb{C}^n)$ is nonempty, i.e.:

$$S\left(\underline{i_l}; \mathcal{F}_l\right) \cap \ldots \cap S(\underline{i}_{r+1}; \mathcal{F}_{r+1}) \neq 0 \tag{35}$$

Then the following matrix eigenvalue inequalities hold:

$$\sum_{j=1}^{k} \lambda_{n-i_{j,r+1}}(A_1 + \cdots + A_r) \leq \sum_{l=1}^{r}\sum_{j=1}^{k} \lambda_{n-i_j l+1}(A_l) \tag{36}$$

$$\sum_{j=1}^{k}\lambda_{n-i_{j,r+1+1}}(A_1+\cdots+A_r)\geq\sum_{l=1}^{r}\sum_{j=1}^{k}\lambda_{i_{jl}}(A_l) \tag{37}$$

Proof: Consider L $\in$ $G_k(C^n)$ with

$$L\in\bigcap_{l=1}^{r+1}S\left(\underline{i}_l;\mathcal{F}_l\right)\neq 0. \tag{38}$$

Then, by using the Hersch-Zwahlen extremal principle (Theorem (4.1.4)) one has:

$$0 \;=\; tr((A_1+\cdots+A_r-A_{r+1})|_L) \tag{39}$$

$$=\sum_{l=1}^{k}tr(A_l|_L)-\;tr(A_{r+l}|_L) \tag{40}$$

$$\geq\sum_{j=1}^{k}min\{tr(A_l|_L):L\in S(\underline{i}_{r+1};\mathcal{F}_{r+1})\}+min\{tr(-A_{r+1}|_L):L\in S(\underline{i}_{r+1};\mathcal{F}_{r+1})\} \tag{41}$$

$$=\sum_{l=1}^{r}\sum_{j=1}^{k}\lambda_{i_{jl}}(A_l)+\sum_{j=1}^{k}\lambda_{i_{j,r+1}}(-A_{r+1}). \tag{42}$$

Thus by Lemma (4.1.15) one has:

$$\sum_{j=1}^{k}\lambda_{n-i_{j,r+1+1}}(A_{r+})\geq\sum_{l=1}^{r}\sum_{j=1}^{k}\lambda_{i_{jl}}(A_l) \tag{43}$$

which proves (36). The inequality (37) follows from (36) by replacing the matrices A_l by $-A_l, l=1,\ldots,r+1$and using Lemma (4.1.15). This completes the proof.

In general it will be difficult to verify the intersection property (35) as it assumes the knowledge of the eigenspaces of $A_1,\ldots,A_r$ and of $A_{r+1}=A_1+\cdots+A_r$. By combining Lemma (4.1.16) with the intersection theoretic result of Proposition (4.1.8) we obtain a result with a more easily verifiable hypothesis.

Theorem (4.1.17)[289]: Let $i_l=(i_{1l},\ldots,i_{kl})$ be $r+1$ sequences of integers satisfying

$$1\leq i_{1l}<\ldots<i_{kl}\leq n, l=1,\ldots,r\;+1. \tag{44}$$

Let $\{n-k-i_{1l}+1,\ldots,n-i_{kl}\}\in H^*(G_k(\mathbb{C}^n),Z)$denote the Schubert cocycle that is the Poincar´e dual of the fundamental homology class of the Schubert variety $S(\underline{i}_1;\mathcal{F}_l)$ for $l=1,\ldots,r+1$. If the $(r+1)$ $-$fold product of the Schubert cocycles in $H^*(G_k(\mathbb{C}^n),Z)$)

$$\prod_{l=1}^{r+1}\{n-k-\;i_{1l}\;+1,\ldots,n-i_{kl}\}\;\neq\;0 \tag{45}$$

then the eigenvalue inequality (36) and (37) holds for any set of Hermitian matrices $A^1,\ldots,A_r\;\in\;\mathbb{C}^{n\times n}$.

Corollary (4.1.18)[289]: Let $i:=(i_1,\ldots,i_k), j:=(j_1,\ldots,j_k), p:=(p_1,\ldots,p_k)$,be sequences satisfying $1\le i_1<\ldots<i_k\le n, 1\le j_1<\ldots<j_k\le n$ and$1\le p_1<\ldots<p_k\le n$. If the triple product

$$\{n-k-i_1+1,\ldots,n-i_k\}\{n-k-j_1+1,\ldots,n-j_k\} \times\{n-k-p_1+1,\ldots,n-p_k\}\neq 0 \quad (46)$$

is nonzero then for any pair of complex Hermitian matrices $A,B\in\mathbb{C}^{n\times n}$ the following eigenvalue inequalities hold:

$$\sum_{v=1}^{k}\lambda_{n-p_v+1}(A+B)\ge\sum_{v=1}^{k}\lambda_{i_v}(A)+\sum_{v=1}^{k}\lambda_{jv}(B) \quad (47)$$

$$\sum_{v=1}^{k}\lambda_{p_v}(A+B)\le\sum_{v=1}^{k}\lambda_{n-p_v+1}(A)+\sum_{v=1}^{k}\lambda_{n-p_v+1}(B). \quad (48)$$

We conclude with a simple example.

Example (4.1.19)[289]: In $H^*(G_k(\mathbb{C}^n),Z)$) the following nonzero products exist:

$$\{1,0\}\{1,0\}\{2,0\} = \{2,2\} \quad (49)$$

$$\{1,0\}\{1,0\}\{1,1\} = \{2,2\} \quad (50)$$

$$\{1,0\}\{1,0\}\{1,0\}\{1,0\} = 2\{2,2\}. \quad (51)$$

By Theorem (4.1.16) and Corollary (4.1.17) the following eigenvalue inequalities hold for arbitrary 4×4 Hermitian matrices:

$$\lambda_1(A+B)+\lambda_4(A+B)\lambda_1(A)+\lambda_3(A)+\lambda_1(B)+\lambda_3(B) \quad (52)$$

$$\lambda_2(A+B)+\lambda_3(A+B)\le\lambda_1(A)+\lambda_3(A)+\lambda_1(B)+\lambda_3(B), \quad (53)$$

$$\lambda_2(A+B+C)+\lambda_4(A+B+C) \le\lambda_1(A)+\lambda_3(A)+\lambda_1(B)+\lambda_3(B)+\lambda_1(C)+\lambda_3(C). \quad (54)$$

We apply the preceding results to verify some classical eigenvalue inequalities. The first inequality is given in [108].

Weyl Inequality [108]: (4.1.20)[289]:

For any indices $1\le i,j\le n$ with$1\le i+j-1\le n$ and any Hermitian matrices $\mathrm{A},\mathrm{B}\in\mathbb{C}^{\mathrm{n}\times\mathrm{n}}$ one has:

$$\lambda_{i+j-1}(A+B)\le\lambda_i(A)+\lambda_j(B). \quad (55)$$

Proof: Here $k=1, G_1(\mathbb{C}^n)=\mathbb{CP}^{n-1}$and $H^*(\mathbb{CP}^{n-1}), Z)=Z[x]/(x^n)$ is a truncated polynomial ring. Using this classical description of the cohomology ring of the projective space, the Schubert cocycles are

$$\{i\}=x_i, i=0,\ldots,n-1. \quad (56)$$

Let i_1, j_1 and p_1 defined by:

$$i_1:=n-i+1, j_1:=n-j+1, p_1\coloneqq i+j-1. \quad (57)$$

Then (46) reduces to

$$\{i-1\}\{j-1\}\{n-i-j+1\}=\{n-1\}. \quad (58)$$

But since x^{n-1} generates $H^{2(n-1)}(\mathbb{CP}^{n-1}),Z)\equiv Z$ one has $\{n-1\}\neq 0$. Thus the Weyl inequality follows immediately from Corollary (4.1.18).

Lidskii Inequality: (4.1.21)[289]:

For $1 \leq \alpha_1 < \ldots < \alpha_k \leq n$and for any Hermitian matrices $A, B \in \mathbb{C}^{n\times n}$ one has the matrix eigenvalue inequality:

$$\sum_{j=1}^{k} \lambda_{\alpha_j}(A+B) \leq \sum_{j=1}^{k} \lambda_{\alpha_j}(A) + \sum_{j=1}^{k} \lambda_j(B) \tag{59}$$

Proof: Consider $i \coloneqq (n-\alpha_k+1, \ldots, n-\alpha_1+1), j := (n-k+1, \ldots, n), p \coloneqq (\alpha_1, \ldots, \alpha_k)$. Then the product in condition (46) of Corollary (4.1.18) is given by

$$\{\alpha_k - k, \ldots, \alpha_1 - 1\}\{0, \ldots, 0\}\{n-k-\alpha_1+1, \ldots, n-\alpha_k\}. \tag{60}$$

Since $\{0, \ldots, 0\} = 1 \in H^*(G_k(\mathbb{C}^n), Z))$ and $\{n-k-\alpha_1+1, \ldots, n-\alpha_k\}$ is Poincarè dual to $\{\alpha_k - k, \ldots, \alpha_1 - 1\}$ the above triple product is equal to $\{n-k, \ldots, n-k\}$ and hence nonzero. This completes the proof of the Lidskii Inequality.

Thus both the Weyl and the Lidskii inequality are direct consequences of the Poincar´e duality of the projective space $\mathbb{CP}^{n-1}$ and of the Grassmannian $G_k(\mathbb{C}^n)$respectively. A proof of the next inequality requires a more subtle topological argument.

Freede-Thompson Inequality [225]: (4.1.22)[289]:

For any $1 \leq \alpha_1 < \ldots < \alpha_k \leq n, 1 \leq b_1 < \ldots < b_k \leq n$ with $\alpha_k + b_k - k \leq n$ and Hermitian matrices $A, B \in \mathbb{C}^{n\times n}$ one has:

$$\sum_{v=1}^{k} \lambda_{\alpha_v + b_v - v}(A+B) \leq \sum_{v=1}^{k} \lambda_{\alpha_v}(A) + \sum_{j=1}^{k} \lambda_{b_v}(B). \tag{61}$$

Proof: Consider $\underline{i} \coloneqq (n-\alpha_k+1, \ldots, n-\alpha_1+1), j := (n-b_k+1, \ldots, n-b_1+1), p \coloneqq (\alpha_1+b_1-1, \ldots, \alpha_k+b_k-k)$. Then the product in condition (46) of Corollary (4.1.18) is given by

$$\{\alpha_k - k - \alpha_1 + 1\}\{b_k - k - b_1 - 1\}\{n-k-\alpha_1+2, \ldots, n+k-\alpha_k-b_k\}. \tag{62}$$

By assumption one has $\alpha_k + b_k - 2k \leq n - k$. From the Littlewood Richardson rule it follows that the product of the first two factors is of the form:

$$\{\alpha_k - k - \alpha_1 + 1\}\{b_k - k - b_1 - 1\}\{\alpha_k - b_k - 2k, \ldots, \alpha_1 + b_1 - 2\} + \sum_{\lambda} c_{\mu v}^{\lambda} \tag{63}$$

where $c_{\mu v}^{\lambda}$ are again the Littlewood Richardson coefficients and the sum is taken overall partitions $\lambda,\ \lambda \neq \alpha_k - b_k - 2k, \ldots, \alpha_1 + b_1 - 2$. Now the result follows from the observation that the cocycle $\alpha_k - b_k - 2k, \ldots, \alpha_1 + b_1 - 2$ is (compare with Theorem (4.1.14)) dual to the cocycle $\{n-k-\alpha_1-b_1+2, \ldots, n+k-\alpha_k-b_k\}$, i.e. the product (62) is nonzero and Theorem (4.1.18) applies.

It is a consequence of Theorem (4.1.17) that any nonzero product in $H^*(G_k(\mathbb{C}^n), Z)$implies an eigenvalue inequality of the form (36) and an inequality of the form (37). We describe a large class of nonzero products. In particular we will describe all maximal nonzero products in $H^*(G_k(\mathbb{C}^n), Z)$ and we will describe all maximal nonzero products in $H^*(G_k(\mathbb{C}^n), Z)$consisting of 3 factors. The following lemmas prepare for those results.

Lemma (4.1.23)[289]: Assume $\mu := \{\mu_1, \ldots, \mu_k\}$ and $v := \{v_1, \ldots, v_k\}$ are two cocycles in $H^*(G_k(\mathbb{C}^n), Z)$ which are complimentary in dimension, i.e. there weights satisfy $|\mu| + |v| = k(n-k)$. Then $\mu v \neq 0$ if, and only if μ and v are dual to each other, i.e. $v = \{n-k-\mu_k, \ldots, n-k-\mu_1\}$.

Proof: See also [203] for a different proof based on Poincar'e-duality. From the description of $H^*(G_k(\mathbb{C}^n), Z)$ in (25) it is clear that $\mu v \neq 0$ exactly when the coefficient of $\{(n-k)^k\} = \{n-k, \ldots, n-k\}$ in the expansion μv is nonzero. Applying the rule of Littlewood and Richardson to the skew tableau $(n - k)^k/\mu$ one verifies that there is only one possibility to fill this tableau with v_1 $1's, v_2$ $2's, \ldots, v_k$ $k's$, and in this case one necessarily has $v_1 = n - k - \mu_k, \ldots, v_k = n - k - \mu_1$.

Lemma (4.1.24)[289]: Assume $\mu_l = \{\mu_{1l}, \ldots, \mu_{kl}\}, l = 1, \ldots, r$, are cocycles with $\sum_{l=1}^{r} \mu_{1l} \leq n - k$. Then the following identity holds in $H^*(G_k(\mathbb{C}^n), Z)$:

$$\left\{n - k - \sum_{l=1}^{r} \mu_{kl}, \ldots, n - k - \sum_{l=1}^{r} \{\mu_{1l}\right\} \prod_{l=1}^{r} \{\mu_{1l}, \ldots, \mu_{kl}\} = \{n-k, \ldots, n-k\}. \tag{64}$$

Proof: Using inductively Littlewood Richardson's rule it follows that

$$\prod_{l=1}^{r} \{\mu_{1l}, \ldots, \mu_{kl}\} = \left\{\sum_{l=1}^{r} \mu_{kl}, \ldots, \sum_{l=1}^{r} \mu_{kl}\right\} + \sum_{\mu} C_\mu \{\mu_{1l}, \ldots, \mu_{kl}\}. \tag{65}$$

(Compare with (63)). Because $\{n - k - \sum_{l=1}^{r} \mu_{kl}, \ldots, n - k - \sum_{l=1}^{r} \mu_{1l}\}$ is the Poincarè dual of the first term after the equality sign the result follows from the previous Lemma.

In the next Lemma we will identify the Schubert symbol $\{x_1, x_2\} \in H^*(G_k(\mathbb{C}^n), Z)$ with zero for $x_1 > n - 2$.

Lemma (4.1.25)[289]: If $\{\alpha_1, \alpha_2\}, \{b_1, b_2\}$ are two cocycles in $H^*(G_k(\mathbb{C}^n), Z$ and

$$m := min\{(\alpha_1 - \alpha_2), (b_1 - b_2)\} \tag{66}$$

then one has

$$\{\alpha_1, \alpha_2\}\{b_1, b_2\} = \sum_{i=0}^{m} \{\alpha_1 + b_1 - i, \alpha_2 + b_2\ i\}. \tag{67}$$

Proof: Direct consequence of the Littlewood Richardson rule. (Compare with [107].)

For the following Lemma let $[x]$ denote the largest integer smaller or equal to x.

Lemma (4.1.26)[289]: If $\{\alpha_{1l}, \alpha_{2l}\} \in H^*(G_k(\mathbb{C}^n), Z), l = 1, \ldots, r, \}$ are r Schubert cocycles with

$$\alpha_{11} - \alpha_{21} \geq \cdots \geq \alpha_{1r} - \alpha_{2r} \tag{68}$$

and

$$m := min\left\{\left[\frac{1}{2}\sum_{l=1}^{m}(a_{1l} - a_{2l})\right], \sum_{l=2}^{m}(a_{1l} - a_{2l})\right\} \tag{69}$$

then there are positive nonzero integers ci such that

$$\prod_{l=1}^{r} \{a_{1l}, a_{2l}\} = \sum_{i=0}^{m} c_i \left\{\sum_{l=2}^{r} a_{1l} - i, \sum_{i=0}^{m} a_{2l} + i\right\}. \tag{70}$$

In particular if $\sum_{i=0}^{m} c_i\, a_{1l} \leq m + n - 2$ at least one summand is nonzero and therefore the whole product is nonzero.

Proof: Let $\alpha \in \{2, \ldots, r\}$ be the largest integer with the property that

$$(a_{11}-a_{21}) \geq \sum_{l=0}^{m}(a_{1l}-a_{2l}). \tag{71}$$

Denote with $\widetilde{m}:=\sum_{l=2}^{m}(a_{1l}\ -\ a_{2l})$. Using inductively Lemma (4.1.25) one sees that

$$\prod_{l=1}^{\sigma}\{a_{1l},a_{2l}\} = \sum_{i=0}^{\widetilde{m}} \tilde{c}_i \left\{\sum_{l=1}^{\alpha} a_{1l}-i, \sum_{l=0}^{\alpha} a_{2l}+i\right\}. \tag{72}$$

with positive, nonzero constants $\widetilde{c_\iota}$. In particular if $\alpha = r$ then $m = \widetilde{m}$ and the result is proven. If $\alpha < r$ then $(a_{1l}-a_{2l}) < \sum_{l=2}^{m}(a_{1l}-a_{2l})$ and therefore $m = [\frac{1}{2}\sum_{l=1}^{m}(a_{1l}-a_{2l})]$. Multiplying inductively expression (71) with the factors $\{a_{1l}, a_{2l}\}, l = \alpha+1,\ldots,r$ one deduces also in this case, using the fact that all Littlewood Richardson coefficients are positive, that $\prod_{l=1}^{\sigma}\{a_{1l}-a_{2l}\} = \sum_{i=0}^{m} C_i \{\sum_{l=1}^{m} C\{x_i, y_i\}$, where

$$\sum_{i=0}^{r} a_{1l}-m \leq x_i \leq \sum_{l=1}^{r} a_{1l} \ \ and \ \sum_{l=1}^{r} a_{2l} \leq y_i \sum_{l=1}^{r} a_{2l}+m. \tag{73}$$

In particular, if $\sum_{l=1}^{r} a_{1l}-m \leq n-2$, the product is nonzero, which completes the proof.

As a direct consequence of this Lemma we obtain a description of all maximal nonzero products in $H^*(G_k(\mathbb{C}^n), Z)$

Theorem (4.1.27)[289]: Assume $\{a_{1l}, a_{2l}\} \in H^*(G_k(\mathbb{C}^n), Z), l = 1,\ldots,r,$ are r cocycles with

$$\sum_{l=1}^{r}(a_{1l}, a_{2l}) = 2(n-2) \tag{74}$$

Then $\prod_{l=1}^{r}\{a_{1l}, a_{2l}\} \neq 0$ if, and only if

$$(a_{1l}, a_{2l}) \leq \sum_{l\in(1,\ldots,j-1,j+1,\ldots,r} (a_{1l}, a_{2l}), j = 1,\ldots,r. \tag{75}$$

Proof: After a possible reindexing we can assume that

$$a_{1l}-a_{2l} \geq \cdots \geq a_{1l}-a_{2l}. \tag{76}$$

Because of assumption (74), $m = [\frac{1}{2}\sum_{l=1}^{r}(a_{1l}-a_{2l})]$. Because of the description of $H^*(G_k(\mathbb{C}^n), Z)$ in (25) it is clear that the product is nonzero if, and only if the coefficient of $\{n-2, n-2\} \in H^{2(n-2)}(G_k(\mathbb{C}^n), Z)$ in the product expansion is nonzero. By the last Lemma this is the case iff $\sum_{l=1}^{r} a_{1l} \leq m+n-2$. Moreover because of (73) the number $\frac{1}{2}\sum_{l=1}^{r}(a_{1l}-a_{2l})]$ is an integer. But then $\sum_{l=1}^{r} a_{1l} \leq m+n-2$. is equivalent to $\sum_{l=1}^{r}(a_{1l}+a_{2l})] \leq 2(n-2)$which is true by assumption (74).

Combining Theorem (4.1.27) with Theorem (4.1.17) one finally has:

Theorem (4.1.28)[289]: Let (i_{1l}, i_{2l}) be $r+1$ pairs of integers with:

$$1 \leq i_{1l} < i_{2l} \leq n, l = 1,\ldots,r+1 \tag{77}$$

$$r(2n-1)+3 \leq \sum_{l=1}^{r+1}(i_{1l}+i_{2l}) \tag{78}$$

$$i_{2j} - i_{1j} \leq 1-r+\sum_{l\in\{1,\ldots,j-1,j+1,\ldots,r+1}(i_{1l}+i_{2l}) \ \ j = 1,\ldots,r+1. \tag{79}$$

Then for any set of Hermitian matrices $A_1, \dots, A_{r+1} \in \mathbb{C}^{n\times n}$ satisfying the relation $A_{r+1} = A_1 + \cdots + A_r$ the following eigenvalue inequalities hold:

$$\lambda_{n-i1,r+1+1}(A_{r+1}) + \lambda_{n-i2,r+1+1}(A_{r+1}) \geq \sum_{l=1}^{r}(A_l) + \lambda_{i_{2l}}(A_l)) \tag{80}$$

$$\lambda_{i1,r+1}(A_{r+1}) + \lambda_{i2,r+1}(A_{r+1}) \leq \sum_{l=1}^{r}(\lambda_{n-i_{1l}+1}(A_l) + \lambda_{n-i_{2l}+1}(A_l)) \tag{81}$$

Proof: Denote with $a_{1l} = n - i_{1l} - 1$and$a_{2l} = n - i_{2l}$. Then condition (78) is equivalent to the condition $\sum_{l=1}^{r+1}(a_{1l} + a_{2l}) \leq 2(n-2)$ and condition (79) is equivalent to inequality (75). The product $\prod_{l=1}^{r+1}\{n - i_{1l} - 1, n - i_{2l}\}$is nonzero and the result follows once again from Theorem (4.1,17) .

In order to illustrate the theorem in the case $r = 2$, let $A = A_1, B = A_2$ and let

$$(i_{1,1}, i_{2,1}) = (n - a_2 + 1, n - a_1 + 1), \tag{82}$$

$$(i_{1,2}, i_{2,2}) = (n - b_2 + 1, n - b_1 + 1), \tag{83}$$

$$(i_{1,3}, i_{2,3}) = (c_1, c_2). \tag{84}$$

Then we obtain

Corollary (4.1.29)[289]: Let $1 \leq a_1 < a_2 \leq n, 1 \leq b_1 < b_2 \leq n$ and $1 \leq c_1 < c_2 \leq n$ satisfy the system of linear inequalities

$$a_1 + a_2 + b_1 + b_2 \leq c_1 + c_2 + 3 \tag{85}$$

$$a_1 + a_2 + b_1 + b_2 \leq c_1 + c_2 - 1 \tag{86}$$

$$b_2 - b_1 \leq a_2 - a_1 + c_2 - c_1 - 1 \tag{87}$$

$$c_2 - c_1 \leq a_2 - a_1 + b_2 - b_1 - 1. \tag{88}$$

Then the eigenvalue inequality

$$\lambda_{c1}(A+B) + \lambda_{c2}(A+B) \leq \lambda_{a1}(A) + \lambda_{a2}(A) + \lambda_{b1}(B) + \lambda_{b2}(B) \tag{89}$$

holds for any pair of Hermitian $n \times n$ matrices A, B. We would like to remark that the assumptions in Corollary (4.1.29) imply the assumptions in Horn [10]. It is also possible to derive the inequality (89) by the methods developed in [10].

We describe all maximal nonzero products of $H^*(G_k(\mathbb{C}^n), Z$consisting of 3 factors. The results are based on a description of the Littlewood Richardson coefficients as given by Schlosser in [107].

In the following we explain his description and simultaneously adapt the notation for our purposes.

Let $\mu := (\mu_1, \dots, \mu_k), v := (v_1, \dots, v_k)and\lambda := (\lambda_1, \dots, \lambda_k)$be partitions. We are interested in conditions when the Littlewood Richardson coefficient $c_{\mu,v}^{\lambda}$ is nonzero.We will use the combinatorial description of $c_{\mu,v}^{\lambda}$ as given in Theorem (4.1.11) and the following parameterization by Schlosser [107].

Consider the tableau λ and denote with $p_{hi}$the number of boxes in the skew tableau λ/μ with label i in the $h-th$ row. This gives us the following description forthe tableau λ:

Row							(90)
1	μ_1	p_{11}				λ_1	
2	μ_2	p_{21}	p_{22}			λ_2	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$		$\vdots$	
K	μ_k	p_{k1}	p_{k2}	$\cdots$	p_{kk}	λ_k	
	$\|\mu\|$	v_1	v_{k2}	$\cdots$	v_k	*Total*	

Of course not all configurations of numbers p_{hi} will result in a filling compatible with the rule of Littlewood and Richardson. On the other hand, as shown in [107], one can iteratively fill the skew tableau λ / μ, starting with pk1 and proceeding inductively with

$$p_{hi}, h = k, \ldots, i+1, i = 1, \ldots, k-1,$$

subject to the following inequalities:

$$Max(h, i; (v) \leq p_{hi} \leq Min(h, i;\ (v), (\mu)) \tag{91}$$

Where

$$Max\big(h, i;\ (v)\big) = max\{0, v_i - v_i - 1 - \sum_{k=h+1}^{k} p_{ji} + \sum_{k=h-1}^{k} p_j, i+1\}$$

$$Min\big(h, i; (v), (\mu)\big) = min\{\mu_{h-1} - \mu_h + \sum_{j=1}^{i-1} \big(p_{h_{-1},j} - p_{h,j}\big), v_i - \sum_{j=h+1}^{k} p_{ji}\}$$

and

$$p_{ji} = v_h - \sum_{j=h+1}^{k} p_{hi}, \quad i = 1, \ldots, k \tag{92}$$

we assume that

$$\mathrm{v}_0 = 0, \mathrm{p}_{0,\mathrm{j}} = 0,\ \mathrm{p}_{\mathrm{h},0} = 0. \tag{93}$$

For our purposes, which is stated in similar form in [107], is:

Theorem (4.1.30)[289]: Let μ, v be partitions and let phi be iteratively described through (91) and (92). Denote with

$$\lambda_h \coloneqq \mu_h + \sum_{i=1}^{h} p_{hi}, \qquad h = 1, \ldots, k. \tag{94}$$

Then $\lambda := (\lambda_1, \ldots, \lambda_k)$describes a tableau and the Littlewood Richardson coefficient $c_{\mu,v}^{\lambda}$ is nonzero

Corollary (4.1.31)[289]: Let μ, v be partitions and let λ satisfy the inequalities induced by the iterative scheme (91) and (92). Then

$$\{\mu\}\{v\}\{n-k-\lambda_k,\ldots,n-k-\lambda_1\} \neq 0 \qquad (95)$$

Proof: The cocycle $\{n-k-\lambda_k,\ldots,n-k-1\}$is the Poincarè dual of the cocycle $\{\lambda\}$ and because the Littlewood Richardson coefficient $c^{\lambda}_{\mu,v}$is non- zero the results follows from Lemma (4.1.23).

Corollary (4.1.32)[289]: Let A, B be complex Hermitian $n \times n$ matrices. Let μ, v be partitions and let λ satisfy the inequalities induced by (90) and (91). Let

$$a_1 := \mu_k + 1,\ldots,a_k := \mu_1 + k \qquad (96)$$

$$b_1 := v_k + 1,\ldots,b_k := v_1 + k \qquad (97)$$

$$c_1 := \lambda_k + 1,\ldots,c_k := \lambda_1 + k \qquad (98)$$

Then

$$\sum_{v=1}^{k} \lambda_{cv}(A+B) \leq \sum_{v=1}^{k} \lambda_{av}(A) + \sum_{v=1}^{k} \lambda_{bv}(B). \qquad (99)$$

Section (4.2) Wielant's Theorem with Spectral sets and Banach Algebra

The classes of Hermitian and unitary matrices have a rich structure and much is known about the eigenvalues of these types of matrices. The more general class of normal (i.e. unitarily diagonalizable) complex matrices is less well understood. And not much is known about spectral problems involving normal matrices even with their eigenvalues being described in terms of those of their Hermitian and skew-Hermitian parts.

The dierence between Hermitian and general normal matrices is that the latter can have as eigenvalues arbitrary complex numbers. $\mathbb{C}$, of course $\mathbb{C}$ is not an ordered field. But it turns out that the simple fact that $\mathbb{C}$ can be totally ordered as a vector space over the reals is enough to obtain useful information on spectra of normal matrices using Hermitian matrices as an inspiration.

A total order in $\mathbb{C}$ compatible with addition of complex numbers and multiplication by positive reals is the lexicographic order. It is characterized by its positive cone $H = \{a+ib: a > 0\ or, if\ a = 0, b > 0\}$.

Compatibility with addition means $H + H \subseteq H$, and compatibility with multiplication by positive reals means$\lambda H \subseteq H$ for$\lambda > 0$. The order being total means $H \cup -H = \mathbb{C} \setminus \{0\}$.

The lexicographic order is not Archimedian and, apart from rotations of the positive cone, is the only total order in $\mathbb{C}$ compatible with the above mentioned operations. We shall use the notation $\leq^{\text{lex}}$ for it, and, for real θwe use $\leq^{\text{lex}}_{\theta}$ for the total order with positive cone $e^{i\theta}H$.

Let A be an $n \times n$ complex normal matrix. Let $\alpha_1,\ldots,\alpha_n$be its eigenvalues ordered so that $\alpha_1 \geq^{lex} \ldots \geq^{lex} \alpha_n$and let $v_1,\ldots,v_n$ be corresponding orthonormal eigenvectors of A. For $j = 1,\ldots,n$ denote by $Ej$and $E'_j$the subspaces of $\mathbb{C}_n$ spanned by $v_1,\ldots,v_j$ and $v_j,\ldots,v_n$respectively. Applying the argument used to obtain the corresponding result for Hermitian matrices, we get:

Theorem (4.2.1)[143]: For $j-1,\ldots,n$ we have

$$\alpha_j = \min_{x\in E_j,\|x\|=1} x^*Ax = \max_{x\in E'_j,\|x\|=1} x^*Ax$$

In addition, we have

$$\alpha_j = \max_{dimH=j} \min_{x\in E_j,\|x\|=1} x^*Ax = \min_{adimH=n-j+1} \max_{x\in E_j,\|x\|=1} x^*Ax$$

(Here max and min are used in the lexicographic sense).

Analogous characterizations hold for any order of the type $\leq_\theta^{lex}$ either using the same proof or applying the theorem to the normal matrix $e^{-i\theta}A$. Note how these results make immediately visible the fact that the numerical range $W(A) = \{x^*Ax : \|x\| = 1\}$ of a normal matrix A is the convex hull of its eigenvalues any straight line moving in the plane parallel to itself must touch $W(A)$ first at an eigenvalue of A.

From the above theorem we immediately obtain, again repeating the Hermitian argument, a result concerning principal normal submatrices of normal matrices:

Theorem (4.2.2)[143]: Let A be an $n \times n$ normal matrix with eigenvalues $\alpha_1 \geq^{lex} ... \geq^{lex} \alpha_n$ If B is a principal $k \times k$ normal submatrix of A with eigenvalues $\beta_1 \geq^{lex} ... \geq^{lex} \beta_{k,}$ we have

$$\alpha_j \geq^{lex} \beta_j \geq^{lex} \alpha_j + n - k, j = 1, ..., k.$$

An analogous result holds for any order of the type $\leq_\theta^{lex}$.

For other interlacing results in this setting see [154,160,126,135], [50].

The result in [154] shows that for a $n \times n$ normal matrix to have a principal $(n-1) \times (n-1)$ normal principal submatrix is a highly restrictive condition, essentially forcing the matrix apart from a rotation and a translation, to be Hermitian. It seems plausible that one can obtain this from Theorem (4.2.2) above.

In [50,139] an interlacing result is presented for the arguments of eigenvalues of a normal matrix and a normal principal submatrix a relation with Theorem (4.2.2) above is unclear.

and then there is the general interlacing theorem for singular values which for normal matrix and submatrix yields a statement whose relation with the above result is again unclear.

Note also that Theorem (4.2.2). does not follow directly from the interlacing theorem for Hermitian matrices applied to the Hermitian and skew-Hermitian parts of A and B.

A generalization of the first part of Theorem (4.2.1) can be obtained by mimicking the corresponding result for Hermitian matrices [139].

Take a sequence $V = (V_1, ..., V_n)$ of subspaces of $\mathbb{C}^n$ with $V_1 \subset \cdots \subset V_n$ and $dim(V_i) = i$ for $i = 1, ..., n$_Given a sequence $I = (i_1, ..., i_r)$, with $i \leq i_1 < \cdots < i_r \leq n$, the Schubert variety associated to V and I is $\Omega_I(V) = \{L$subspace of $\mathbb{C}^n$ $dim(L) = r, dim(L \cap V_{id} \geq d, d = 1, ..., r\}$.

Keep the notation and write $E = (E_1, ..., E_n)$,

$E' = (E'_n, ..., E'_1)$. Put also $I' = (n - i_r + 1, ..., n - i_1 + 1)$

If L is a subspace of dimension r and $x_1, ..., x_r$ is an orthonormal basis of L, the Rayleigh trace of Awith respect to L is

$$tr(A|_L) = \sum_{d=1}^{r} d_d^* A x_d.$$

(This does not depend on the basis)

Theorem (4.2.3)[143]: If the eigenvalues of a normal matrix A are $\alpha_1 \geq^{lex} ... \geq^{lex} \alpha_n$ one has

$$\alpha_{i_1} + \cdots + \alpha_{i_r} = \min_{L\in\Omega_1(E)} tr(A|_L) = \max_{L\in\Omega_1\prime(E\prime)} tr(A|_L)$$

where again max and min are used in the lexicographic sense.

This characterization of course also valid for any order of the type $\leq_{\theta}^{lex}$ can be applied to obtaining inequalities for the eigenvalues of a sum of two normal matrices if this sum is itself normal.

Let A and B be $n \times n$ normal matrices with eigenvalues $\alpha_1 \geq^{lex} \dots \geq^{lex} \alpha_n$and $\beta_1 \geq^{lex} \dots \geq^{lex} \beta_k$, respectively. Suppose that $A+B$ is normal, with eigenvalues $\gamma_1 \geq^{lex} \dots \geq^{lex} \gamma_n$. Let E, E', F, F' and G, G'' be sequences of subspaces built from the eigenvectors of A, B and $A+ B$,as before. Let $I, J$and K be sequences of r indices:

$$I = (i_1, \dots, i_r), 1 \leq i_1 < \cdots < i_r \leq n,$$
$$J = (j_1, \dots, j_r), 1 \leq j_1 < \cdots < j_r \leq n,$$
$$K = (k_1, \dots, k_r), 1 \leq k_1 < \cdots < k_r \leq n.$$

Then, using the characterizations of Theorem (4.2.3), it is easy to see that

Theorem (4.2.4) [143]: If

$$\Omega_K(G) \cap \Omega_{I'}(E') \cap \Omega_{J'}(F') \neq 0,$$

Then

$$\gamma_{k_1} + \cdots + \gamma_{k_r} \leq^{lex} \alpha_{i_1} + \cdots + \alpha_{i_r} + \beta_{j_1} + \cdots \beta_{j_{1r}}.$$

For the Hermitian case this appears in [263], [289]

So a geometric condition, nonempty intersection of the three Schubert Varieties, implies a linear inequality between the eigenvalues of the three normal matrices A, B and $A+ B$. We abreviate this inequality to

$$\sum \gamma_k \leq^{lex} \alpha_I + \sum \beta_{J.}$$

For the Hermitian case,by Klyachko, has shown that the inequalities arising from all such geometric conditions actually yield a complete list of restrictions for the eigenvalues of a sum of two Hermitian matrices in terms of the eigenvalues of the summands, For recent surveys on this see [306], [13].

Klyachko's results, coupled with the combinatorial work of Knutson and Tao [11] imply the classical Horn conjecture, on eigenvalues of Hermitian Matrices, which we now recall. For two real ordered spectra α and β denote by $E(\alpha, \beta)$ the set of all possible ordered spectra of sums of two Hermitian matrices with spectra α and β. For each r-tuple $I = (i_1 \dots i_r)$ with $1 \leq i_1 < \cdots < i_r \leq n$ define

$$\rho(I) = (i_r - r, \dots, i_2 - 2, i_1 - 1).$$

Then Horn's conjecture, now proved, can be presented as the following recursive description of the set E:

$$E(\alpha, \beta) = \{\gamma: \sum \gamma = \sum \alpha + \sum \beta \text{ and} \sum \gamma_k = \sum \alpha_I + \sum \beta_J$$

wherever $\rho(K) \in E[\rho(I), \rho(J)], 1 \leq r < n\}$.

By the Schubert calculus (see for example [287]), the geometric condition $\Omega_K(G) \cap \Omega_{I'}(E') \cap \Omega_{J'}(F') \neq 0$ is equivalent to $\rho(K) \in LR[\rho(I), \rho(J)]$, meaning that the r-tuple ρ(K) can be obtained from ρ(I) and ρ(J)using the combinatorial Littlewood-Richardson rule. From the results in [1] and [11] it turns out that it is also equivalent to $\rho(K) \in E[\rho(I), \rho(J)]$.

Return now to normal matrices A with spectrum α, β with spectrum β and $A + B$ with spectrum γ with notations as above. As we have seen, the condition $\Omega_K(G) \cap \Omega_{I'}(E') \cap \Omega_{J'}(F') \neq 0$ implies $\rho(K) \in LR[\rho(I), \rho(J)]$.
Therefore, bearing in mind the results quoted, we can now state:
Theorem (4.2.5)[143]: For $1 \leq r < n$, whenever one has $\rho(K) \in E[\rho(I), \rho(J)]$ the inequality

$$\sum \gamma_K \leq^{lex} \sum \alpha_I + \sum \beta_J$$

holds for the eigenvalues of the normal matrices A, B and $A + B$. And the same, of course, for any order of the type $\leq^{lex}$.

In [113], Helmut Wielandt proved an interesting result which gave regions in the complex plane which contain all the eigenvalues of the sum of two normal matrices A and B in terms of the spectra of A and B. We give a generalization of Wielandt's result to Banach algebras and we also give a multiplicative version of Wielandt's theorem. Before stating Wielandt's theorem, we need to review some geometric concepts in elementary complex function theory.
Definition (4.2.6)[244]: A generalized circle is either a circle ($\{z \in \mathbb{C} : |z - \kappa| = r\}$ where $\kappa \in \mathbb{C}$ and $r > 0$) or a straight line ($\{z \in \mathbb{C}: Re(\alpha z) = \beta\}$ where $\alpha \in \mathbb{C}\backslash\{0\}$ and $\beta \geq 0$) in the complex plane.
Definition (4.2.7) [244]: A Mobius transformation is a function of the form $f(z) = \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.
We note that a Mobius transformation maps generalized circles to generalized circles.
Definition (4.2.8) [244]: A circular region is a subset of the complex plane of the form $f(K)$ where f is a Möbius transform and K is either the open unit disk $\{z \in \mathbb{C}: |z| < 1\}$ or the closed unit disk $\{z \in \mathbb{C}: |z| \leq 1\}$.
A subset of a complex plane is a circular region if it is either an open or closed disk, a complement of an open or closed disk or a half-plane. The boundary of a circular region is always a generalized circle. We can now state Wielandt's theorem. We let $\sigma(M)$ denote the spectrum of the matrix M. If S and T are non-empty subsets of the complex plane, then $S + T = \{s + t: s \in S, t \in T\}$ and $S \cdot T = \{st: s \in S, t \in T\}$.
Theorem (4.2.9)[244]: [113]. Let A and B be two n by n normal matrices. Let K be a circular region which contains all of the eigenvalues of B, then $\sigma(A + B) \subseteq \sigma(A) + K$.
We will give a generalization of this result to Banach algebras. We will assume no knowledge of normed algebras and Banach algebras beyond their definitions. (See [311]. All of the normed algebras and Banach algebras in this section will be automatically assumed to be complex and unital. We denote the unit of a unital normed algebra as 1. If a is an element in a unital normed algebra $\mathcal{A}$, then the spectrum of a is the set $\{\lambda \in \mathbb{C}: (\lambda 1 - a) \text{ is not invertible}\}$. The spectrum of a Banach algebra is always non-empty and compact. We note that $B(\mathcal{H})$, the set of all bounded linear operators on a Hilbert space is an example of a Banach algebra; if further $\mathcal{H}$ is finite dimensional then $B(\mathcal{H})$ is the algebra of n by n complex matrices where $n = dim(\mathcal{H})$.
We have the following.

Proposition (4.2.10)[244]: Let $\mathcal{A}$ be a Banach algebra then $1 + x$ is invertible whenever $x \in \mathcal{A}$ with $\|x\| < 1$.

While this result may fail if$\mathcal{A}$ is an incomplete normed algebra, there are incomplete normed algebras for which the above result also holds as noted in [242]. All of our results below hold if we replace Banach algebra with complex unital normed algebra for which the conclusion of Proposition (4.2.10) holds. Since the proof that every element of a complex Banach algebra has compact spectrum uses Proposition (4.2.10) rather than using completeness directly, if A is a complex unital normed algebra for which the conclusion of Proposition (4.2.10) holds all its elements will have non-empty compact spectrum.

The term spectral set has several different meanings in mathematics. We will always use the term in the following sense:

Definition (4.2.11)[244]: Let $\mathcal{A}$ be a complex unital Banach algebra and let $a \in \mathcal{A}$. A closed subset S of the complex plane which contains the spectrum of a is called a spectral set of a if $\|r(a)\| \leq \ sup_{z\in S}|r(z)|$for all rational functions r which have no poles in S.

This concept is due to von Neumann [138] who gave the definition in the special case where the Banach algebra is $B(\mathcal{H})$,. Many different subsets of the complex plane may be the spectral set for the same element $a \in \mathcal{A}$. In general, the intersection of two spectral sets is not necessarily a spectral setand hence there is no minimal spectral set of a unless the spectrum of a is itself a spectral set of a. Any set which contains a spectral set must itself be a spectral set. We note that the rational functions form amonoid under composition; the invertible elements of thismonoid are the Mobius transformations. This observation leads to the following proposition:

Proposition (4.2.12)[244]: Let $\mathcal{A}$ be a Banach algebra and let $a \in \mathcal{A}$. Let $m(z)$ be a Mobius transformation whose pole lies outside the spectrum of a. If S is a spectral set for a, then $m(S)$ is a spectral set for $m(a)$.

Spectral sets play an especially important role in the special case where the Banach algebra $\mathcal{A} = B(\mathcal{H})$ where $\mathcal{H}$ is a Hilbert space. We note that in his original work [138], von Neumann gave necessary and sufficient conditions for a closed circular region in the complex plane to be a spectral set of a bounded linear operator.

Proposition (4.2.13)[244]: [138]. Let $\mathcal{H}$ be a complex Hilbert space and let $A \in B(\mathcal{H})$. Let $\kappa \in \mathbb{C}$ and $r > 0$, then the circular region $\{z \in \mathbb{C}: |z - \kappa| \leq r\}$ is a spectral set for A if and only if $\|A \ - \kappa I\| \leq r$.

Proposition (4.2.14)[244]:[138] Let $\mathcal{H}$be a complex Hilbert space and let $A \in B(\mathcal{H})$. Let $\kappa \in \mathbb{C}$and$r > 0$, then the circular region $\{z \in \mathbb{C}: |z - \kappa| \geq \ r\}$ is a spectral set for A if and only if $\|(A - \kappa I)^{-1}\| \leq r - 1$.

We note that the forward implications of the two previous propositions follow immediately from the definition of a spectral set and apply to general Banach algebras. The backwards implications of the two previous propositions are false for general Banach algebras.

Proposition (4.2.15)[244]: [138]. Let $\mathcal{H}$ be a complex Hilbert space and let $A \in B(\mathcal{H})$. Let $\alpha \in \mathbb{C}\backslash\{0\}$ and $\beta \geq 0$, then the circular region $\{z \ \in \mathbb{C} : Re(\alpha z) \geq \ \beta\}$ is a spectral set for A if and only if $\alpha(\frac{A+A^*}{2}) \geq \beta I$.

Since a spectral set of an operator must contain its spectrum; the smallest possible candidate to be a spectral set of an operator is the spectrum. The class of operators whose spectrum is itself a spectral set are important enough to be named (in honour of von Neumann fittingly enough).

Definition (4.2.16)[244]: Let $\mathcal{H}$ be a Hilbert space and let $A \in B(\mathcal{H})$.A is said to be a von Neumann operator if the spectrum of A is a spectral set of A.

The class of von Neumann operators are an important class of operators; important subsets of von Neumann operators include the normal operators and the subnormal operators [140]. If $\mathcal{H}$ is finite dimensional, then $A \in B(\mathcal{H})$ is a von Neumann operator if and only if it is normal. We now extend the definition of a von Neumann operator to Banach algebras in the obvious way.

Definition (4.2.17)[244]: Let $\mathcal{A}$ be a unital Banach algebra and let $a \in \mathcal{A}$. Then a is called a von Neumann element if the spectrum of a is itself a spectral set of a.

Theorem (4.2.18)[244]: Let $\mathcal{A}$ be a unital Banach algebra and let a, b $\in \mathcal{A}$. Let S_a and S_b be spectral sets of a and b respectively. If S_a and S_b are separated by a generalized circle then $a - b$ is an invertible element of A.

Proof: Let G be a generalized circle which separates S_a and S_b. If G is a circle, then $G = \{z \in \mathbb{C}: |z - \kappa| = r\}$ for some $\kappa \in \mathbb{C}$ and $r > 0$. One of Sa and S_b is contained in the set $\{z \in \mathbb{C}: |z - \kappa| < r\}$ and the other is contained in the set $\{z \in \mathbb{C}: |z - \kappa| > r\}$. WLOG let $S_a \subseteq \{z \in \mathbb{C}: |z - \kappa| > r\}$ and $S_b \subseteq \{z \in \mathbb{C}: |z - \kappa| < r\}$. From this it follows that $a - \kappa 1$ is invertible, $\|(a - \kappa 1)^{-1}\| < r^{-1}$ and $\|b - \kappa 1\| < r$.Therefore $\|(a - \kappa 1)^{-1}(b - \kappa 1)\| < 1$ and $a - b = (a - \kappa 1) - (b - \kappa 1) = (a - \kappa 1)[1 - (a - \kappa 1)^{-1}(b - \kappa 1)]$ is invertible.

Now suppose G is a line.Now choose $\tau \in \mathbb{C}$ such that $\tau \notin G \cup \sigma_a \cup \sigma_b$. Then $a - \tau 1$ and $b - \tau 1$ are two invertible elements of the Banach algebra having spectral sets $S_a - \tau$and $S_b - \tau$ respectively which are separated by a line $G - \tau$ which does not contain the origin. Hence $(a - \tau I)^{-1}$ and $(b - \tau I)^{-1}$ have spectral sets $(S_a - \tau)^{-1}$ and $(Sb - \tau) - 1$ which are separated by a circle$(G - \tau)^{-1}$. Hence $(a - \tau 1)^{-1} - (b - \tau 1)^{-1}$ is invertible which means $a - b = (a - \tau 1) - (b - \tau 1) = (a - \tau 1)[(b - \tau 1)^{-1} - (a - \tau 1)^{-1}](b - \tau 1)$ is invertible.

Corollary (4.2.19)[244]:. Let $\mathcal{A}$ be a unital Banach algebra and let $a, b \in \mathcal{A}$. Let S_a nd S_b be spectral sets of a and b respectively and let K any circular region which contains S_b. Then $\sigma(a + b) \subseteq S_a + K$.

Proof: Suppose $\lambda \in \sigma(a + b)$, then $b - (\lambda 1 - a) = a + b - \lambda 1$ is not invertible. The intersection of $\lambda - S_a$and K must be non-empty by Theorem (4.2.18), since the boundary of K is a generalized circle. Now let $\mu \in (\lambda - S_a) \cap K$. We note that since $\lambda - \mu \in S_a, \mu \in K$and $\lambda = (\lambda - \mu) + \mu$, our result follows.

Our generalization of Wielandt's theorem now follows immediately.

Theorem (4.2.20)[244]:. Let $\mathcal{A}$ be a unital Banach algebra and let a, b be von Neumann elements in $\mathcal{A}$. Let K be a circular region containing *σ(b).* Then $\sigma(a + b) \subseteq \sigma(a) + K$.

We also have a multiplicative version of Corollary (4.2.19).

Theorem (4.2.21)[244]: Let $\mathcal{A}$ be a unital Banach algebra and let $a, b \in \mathcal{A}$ with at least one of a or b being invertible. Let S_a and S_b be spectral sets of a and b respectively and let K be any circular region which contains S_b. Then $\sigma(ab) \subseteq S_a \cdot K$.

Proof: If a invertible then $\lambda 1 - ab = a(\lambda a^{-1} - b)$. Suppose λ cannot be expressed as a product of a number in S_a and a number in K. Then $\{\lambda/z : z \in S_a)\}$is a spectral set for λa^{-1} which lies entirelyoutside K. Then by Theorem (4.3.13), $\lambda a^{-1} - b$ and hence $\lambda 1 - ab$ is invertible. The proof where b is invertible is similar.

The special case where a and b are von Neumann elements of the Banach algebra is a multiplicative version of Wielandt's theorem.

Theorem (4.2.22)[244]: Let $\mathcal{A}$ be a unital Banach algebra and let a and b be von Neumann elements of $\mathcal{A}$with at least one of a or b being invertible. Let K be any circular region which contains the spectrum of b. Then $\sigma(ab) \subseteq \sigma(a) \cdot K$.

Chapter 5
Riesz and Szegö Type Factorizations with Helson -Szegö Theorem

We show the contractivity of the underlying conditional expectation on $H^p(A)$ for $p < 1$.We introduce noncommutative Hardy-Lorentz spaces and give the Szegöand inner – outer type factorizations of these spaces.We then proceed to use Helson Szegö theorem to haracterise the symbols of invertible Toeplitz operators on the noncommutative Hardy spaces associated to subdiagonal subalgebras.

Section (5.1) Factorizations for Noncommutative Hardy Spaces

We deal with the Riesz and Szegö type factorizations for noncommutative Hardy spaces associated with a finite subdiagonal algebra in Arveson's sense [300]. Let M be a finite von Neumann algebra equipped with a normal faithful tracial state τ. Let D be a von Neumann subalgebra of M, and let $\Phi: M \to D$ be the unique normal faithful conditional expectation such that $\tau \circ \Phi = \tau$. A finite subdiagonal algebra of M with respect to Φ (or D) is a ω^*-closed subalgebra A of M satisfying the following conditions

(i) $A + A^*$is ω^*-dense in M;

(ii) Φ is multiplicative on A, i.e., $\Phi(ab) = \Phi(a)\Phi(b)$ for all $a, b \in A;$

(iii) $A \cap A^* = D$.

We should call to fact that A^*denotes the family of the adjoints of the elements of A, i.e., $A^* = \{a^*: a \in A\}$. The algebra D is called the diagonal of A. It is proved by Exel [240] that a finite subdiagonal algebra A is automatically maximal in the sense that if B is another subdiagonal algebra with respect to Φ containing A, then $B = A$. This maximality yields the following useful characterization of A:

$$A = \{x \in M: \tau(xa) = 0, \forall\, a \in A_0\}, \quad (1)$$

where $A_0 = A \cap ker\Phi$ (see [300]).

Given $0 < p \leq \infty$ we denote by $L^p(M)$ the usual noncommutative L^p-space associated with (M, τ). Recall that $L^\infty(M) = M$, equipped with the operator norm. The norm of$L^p(M)$ will be denoted by $\| \cdot \|_p$. For $p < \infty$ we define $H^p(A)$ to be the closure of A in $L^p(M)$, and for $p = \infty$ we simply set $H^\infty(A) = A$ for convenience. These are the so-called Hardy spaces associated with A. They are noncommutative extensions of the classical Hardy spaces on the torus T. On the other hand, the theory of matrix-valued analytic functions provides an important noncommutative example. We see [300] and [89]for more examples. We will use the following standard notation in the theory: If S is a subset of $L^p(M)$, $[S]_p$ will denote the closure of S in $L^p(M)$ (with respect to the ω^*-topology in the case of $p = \infty$). Thus $H^p(A) = [A]_p$. Formula (1) admits the following $H^p(A)$ analogue proved by Saito [153]:

$$H^p(A) = \{x \in L^p(M): \tau(xa) = 0, \forall\, a \in A_0\}, 1 \leq p < \infty. \quad (2)$$

Moreover,

$$H^p(A) \cap L^p(M) = H^q(A), 1 \leq p < q \leq \infty. \quad (3)$$

These noncommutative Hardy spaces have received a lot of attention since Arveson's pioneer work. We refer Marsalli- West [176] and Blecher-Labuschagne [66, 58, 56], whereas more references on previous works can be found in the survey [89]. Most results on the classical Hardy spaces on the torus have been established in this noncommutative setting. Here we mention only two of them directly related with the objective. The first one is the Szegö factorization theorem. Already in the fundamental work [300], Arveson proved the following factorization theorem: For any invertible $x \in M$ there exist a unitary $u \in M$ and $a \in A$ such that $x = ua$ and $a^{-1} \in A$. This theorem is a base of all subsequent works on noncommutative Hardy spaces. It has been largely improved and extended. The most general form up to date was newly obtained by Blecher and Labuschagne [66]: Given $x \in L^p(M)$ with $1 \leq p \leq \infty$ such that $\Delta(x) > 0$ there exists $h \in H^p(M)$ such that $|x| = |h|$. Moreover, h is outer in the sense that $[hA]_p = H^p(M)$. Here $\Delta(x)$ denotes the Fuglede-Kadison determinant of x and $|x| = (x^*x)^{1/2}$ denotes the absolute value of x. We should emphasize that this result is the (almost) perfect analogue of the classical Szegö theorem which asserts that given a positive measurable function w on the torus there exists an outer function ' such that $w = |\varphi|$ iff logw is integrable. The second result we wish to mention concerns the Riesz factorization, which asserts that $H^p(A) = H^q(A).H^r(A)$ for any $1 \leq p, q, r \leq \infty$ such that $1/p = 1/q + 1/r$. More precisely, given $x \in H^p(A)$ and $\varepsilon > 0$there exist $y \in H^q(A)$ and $z \in H^r(A)$ such that

$$x = yz \text{ and } \|y\|_q \|z\|_r \leq \|x\|_p + \varepsilon.$$

This result is proved in [153] for $p = q$ =2, in [176] for $r = 1$and independently in [164] and in [89] for the general case as above.

Recall that in the case of the classical Hardy spaces the preceding theorems hold for all positive indices. The problem of extending these results to the case of indices less than one was left unsolved in these works. (We mentioned this problem for the Riesz factorization explicitly in [89]). The main purpose is to solve the problem above. As a byproduct, we also extend all results on outer operators in [66] to indices less than one.

A major obstacle to the solution of the previous problem is the use of duality, often in a crucial way, on noncommutative Hardy spaces. For instance, duality plays an important role in proving formulas (2) and (3), which are key ingredients for the Riesz factorization in [89]. In a similar fashion, we will see that their extensions to indices less than one will be essential for our proof of the Riesz factorization for all positive indices.

Our key new tool is the contractivity of the conditional expectation Φ on A with respect to $\|.\|_p$for $0 < p < 1$. Consequently, Φ extends to a contractive projection from $H^p(A)$ onto $L^p(D)$. This result is of independent interest and proved .

We devoted to the Szegö and Riesz type factorizations. In particular, we extend to all positive indices Marsalli-West's theorem quoted previously. It contains some results on outer operators, notably those in $H^p(A)$ for $p < 1$. see [66]. We devoted to a noncommutative Szegö formula, which wasobtained in [66] with the additional assumption that $dimD < \infty$.

In particular, A will always denote a finite subdiagonal algebra of (M, τ) with diagonal D.

It is well-known that extends to a contractive projection from $L^p(M)$ onto $L^p(D)$ for every $1 \leq p \leq \infty$. In general, Φ cannot be, of course, continuously extended to $H^p(A)$ for $p < 1$. Surprisingly, Φ denotes extend to a contractive projection on $H^p(A)$.

Theorem (5.1.1)[279]: Let $0 < p < 1$. Then

$$\forall\, a \in A\,, \|\Phi(a)\|_p \leq \|a\|_p\,. \tag{4}$$

Consequently, Φ extends to a contractive projection from $H^p(A)$. onto $L^p(D)$.The extension will be denoted still byΦ.

Inequality (4) is proved by Labuschagne [163] for $p = 2^{-n}$ and for operators a in A which are invertible with inverses in A too. Labuschagne's proof is a very elegant and simple argument by induction. It can be adapted to our general situation.

Proof: Since $\{k2^{-n}: k, n \in N, k \geq 1\}$ is dense in $(0, 1)$, it suffices to prove (4) for $p = k2^{-n}$. Thus we must show

$$\tau\left(|\Phi(a)|^{k2^{-n}}\right) \leq \tau\left(|a|^{k2^{-n}}\right), \forall\, a \in A\,. \tag{5}$$

This inequality holds for $n = 0$ because of the contractivity of Φ on $L^k(M)$. Now suppose its validity for some k and n. We will prove the same inequality with $n + 1$ instead of n. To this end fix $a \in A$ and $\varepsilon > 0$. Define, by induction, a sequence (x_m) by

$$x_1 = (|a| + \varepsilon)^{k2^{-n}} \text{and } x_{m+1} = \frac{1}{2}\left[x_m + (|a| + \varepsilon)^{k2^{-n}} x_m^{-1}\right].$$

Observe that all x_m belong to the commutative C*-subalgebra generated by $|a|$. Then it is an easy exercise to show that the sequence (x_m) is nonincreasing and converges to $(|a| + \varepsilon)^{k2^{-n-1}}$ uniformly (see [158]). We also have

$$\tau(x_{m+1}) = \frac{1}{2}\left[\tau(x_m) + \tau\left(x_m^{-1/2}\,(|a| + \varepsilon)^{k2^{-n}} x_m^{-1/2}\right)\right] \geq \frac{1}{2}\left[\tau(x_m) + \tau\left(x_m^{-1/2}\,|a|^{k2^{-n}} x_m^{-1/2}\right)\right]$$
$$= \frac{1}{2}\left[\tau(x_m) + \tau\left(\,|a|^{k2^{-n}} x_m^{-1}\right)\right].$$

Now applying Arveson's factorization theorem to each x_m, we find an invertible $b_m \in A$ with $b_m^{-1} \in A$ such that

$$|b_m| = x_m^{2n/k}\,.$$

Let $\mathrm{p} = k2^{-n}$. Then

$$\|ab_m^{-1}\|_p = \||a|\, b_m^{-1}\|_p = \||a|\,|(b_m^{-1}\,)^*|\|_p = \||a||b_m|^{-1}\|_p = \left(\tau(|a|^p|b_m|^{-p})\right)^{\frac{1}{p}}$$
$$= (\tau(|a|^p x_m^{-1}\,))^{1/p}$$

where we have used the commutation between $|a|$ and $|b_m|$ for the next to the last equality. Therefore, by the induction hypothesis and the multiplicativity of Φ on A

$$\tau(x_{m+1}) \geq \frac{1}{2}\left[\tau\left(|b_m|^{k2^{-n}}\right) + \tau\left(|a_m^{b-1}\,|^{k2^{-n}}\right)\right]$$
$$\geq \frac{1}{2}\left[\tau\left(\Phi|b_m|^{k2^{-n}}\right) + \tau\left(\Phi|(a)\Phi(b_m)^{-1}\,|^{k2^{-n}}\right)\right].$$

However, by the Hölder inequality

$$\left(\tau\left(\Phi|(a)\,|^{k2^{-n-1}}\right)\right)^2 \leq \tau(|\Phi((a)\Phi(b_m)^{-1}|^{k2-n})|\tau(b_m)|^{k2^{-n}}).$$

It thus follows that

$$\tau(x_{m+1}) \geq \frac{1}{2}\left[\tau\left(|\Phi(b_m)|^{k2^{-n}}\right) + (\tau\left(|\Phi(a)\Phi\,|^{k2^{-n-1}}\right))^2 \tau\left(\Phi|(b_m)|^{k2^{-n}}\right)^{-1}\right]$$
$$\geq \tau\left(|\Phi(a)|^{k2^{-n-1}}\right).$$

Recalling that $x_m \to (|a| + \varepsilon)^{k2^{-n-1}}$as $m \to \infty$, *we* deduce

$$\tau\left((|a| + \varepsilon)^{k2^{-n-1}} \geq \tau|\Phi(a)|^{k2^{-n-1}}\right).$$

Letting $\varepsilon \to 0$ we obtain inequality (5) at the $(n + 1) - th$ step.

Corollary (5.1.2)[279]:$\Phi$is multiplicative on Hardy spaces. More precisely, $\Phi(ab = \Phi(a)\Phi(b)$for$a \in H^p(A)$ and $b \in H^q(A)$with$0 < p, q \leq \infty$.

Proof: Note that $ab \in H^r(A)$ for any $a \in H^p(A)$ and $b \in H^q(A)$, where r is determined by $1/r = 1/p + 1/q$. Thus $\Phi(ab)$ is well defined. Then the corollary follows immediately from the multiplicativity of Φ on A and Theorem (5.1.1).

The following is the extension to the case $p < 1$ of Arveson-Labuschagne's Jensen inequality (cf. [300, 163]). Recall that the Fuglede-Kadison determinant $\Delta(x)$ of an operator $x \in L^p(M)$ $(0 < p \leq \infty)$can be defined by

$$\Delta(x) = exp(\tau(log\,|x|)) = exp\left(\int_0^\infty log\, t\, dv_{|x|}(t)\right),$$

where $dv_{|x|}$ denotes the probability measure on $\mathbb{R}_+$ which is obtained by composing the spectral measure of $|x|$ with the trace τ. It is easy to check that

$$\Delta(x) = \lim_{p\to 0}\|x\|_p .$$

As the usual determinant of matrices, Δ is also multiplicative: $\Delta(xy) = \Delta(x)\Delta(x)$. We refer for information on determinant to [26, 300] in the case of bounded operators, and to [157, 76] for unbounded operators.

Corollary (5.1.3)[278]: For any $0 < p \leq \infty$ and $x \in H^p(A)$ we have $\Delta(\Phi(x)) \leq \Delta(x)$.

Proof: Let$x \in H^p(A)$.Then $x \in H^q(A)$too for $q \leq p$. Thus by Theorem (5.1.1)

$$\|\Phi(x)\|_q \leq \| x\|_q .$$

Letting $q \to 0$ yields $\Delta(\Phi(x)) \leq \Delta(x)$.

The following result is a Szegö type factorization theorem. It is stated in [89]. We take this opportunity to provide a proof. It is an improvement of the previous factorization theorems of Arveson [300] and Saito [153]. As already quoted in the introduction, Blecher and Labuschagne newly obtained a Szegö factorization for any $\omega \in L^p(M)$ with $1 \leq p \leq \infty$ such that $\Delta(\omega) > 0$. Note that the property that $h^{-1} \in H^q(A)$ whenever $\omega^{-1} \in L^q(M)$ will be important for our proof of the Riesz factorization below. Let us also point out that although not in full generality, this result has hitherto been strong enough for applications. See Theorem (5.1.13) below for an improvement.

Theorem (5.1.4)[279]: Let $0 < p, q \leq \infty$. Let $\omega \in L^p(M)$ be an invertible operator such that $\omega^{-1} \in L^q(M)$. Then there exist a unitary $u \in M$ and $h \in H^p(A)$ such that $\omega = uh$ and $h^{-1} \in H^q(A)$.

Proof: We first consider the case $p = q = 2$. The proof of this special case is modelled on Arveson's original proof of his Szeg¨o factorization theorem (see also [153]). Let x be the orthogonal projection of w in $[wA_0]_2$; and set $y = w - x$. Thus $y \perp [wA_0]_2$; whence $y \perp [yA_0]_2$. It follows that

$$\forall\, a \in A_0, \quad \tau(y^*ya) = 0.$$

Hence by (2), $y^*y \in H^1(A) = [A]_1$, and $y^*y \in [A^*]_1$ too. On the other hand, it is easy to see that $[A]_1 \cap [A^*]_1 = L^1(D)$. Indeed, if $a \in [A]_1 \cap [A^*]_1$, then $\tau(ab) = 0$ for any $b \in A_0 + A_0^*$; so $\tau(ab) = \tau(\Phi(a)b)$ for any $b \in A + A^*$. It follows that $a = \Phi(a) \in L^1(D)$. Consequently, $y^*y \in L^1(D)$, so$|y| \in L^2(D)$.

Regarding M as a von Neumann algebra acting on $L^2(M)$ by left multiplication, we claim that y is cyclic for M. This is equivalent to showing that y is separating for the commutant of M. However, this commutant coincides with the algebra of all right multiplications on $L^2(M)$by the elements of M. Thus we are reduced to prove that if $z \in M$ is such that $yz = 0$, then $z = 0$. We have:

$$0 = \tau(z^*y^*yz) = \tau(|y|^2|z^*|^2) = \tau(|y|^2\,\Phi(|z^*|^2)) = \|yd\|_2^2,$$

where $d = \Phi(|z^*|^2)^{1/2} \in D$; whence $yd = 0$. Choose a sequence $(a_n) \subset A_0$ such that

$$x = \lim \omega a_n. \tag{6}$$

Then (recalling that $\omega^{-1} \in L^2(M)$)

$$0 = \tau(\omega^{-1}yd) = \lim_n \tau(\omega^{-1}(\omega - \omega a_n)d) = \tau(d) - \lim_n \tau(a_n d) = \tau(d)$$

It follows that $d = 0$, so by virtue of the faithfulness of $\Phi, z = 0$ too. This yields our claim. Therefore, $[My]_2 = L^2(M)$. It turns out that the right support of y is 1. Since M is finite, the left support of y is also equal to 1, so y is of full support. Consequently$[My]_2 = L^2(M)$too. Let $y = u|y|$be the polar decomposition of y. Then u is a unitary in M. Let $h = u^*w$. We are going to prove that $h \in H^2(A)$. To this end we first note the following orthogonal decomposition of $L^2(M)$:

$$L^2(M) = [yA_0]_2 \oplus [yD]_2 \oplus [yA_0^*]_2 \tag{7}$$

Indeed, for any $a \in A$ and $b \in A_0$ we have

$$\langle ya, yb^*\rangle = \tau(by^*ya) = \tau(|y|^2ab) = 0;$$

so $[yA_0]_2 \oplus [yD]_2 \oplus [yA_0^*]_2$ is really an orthogonal sum. On the other hand, by the previous paragraph, we see that

$$L^2(M) = [yM]_2 \subset [yA_0]_2 \oplus [yD]_2 \oplus [yA_0^*]_2\,.$$

therefore, decomposition (7) follows. Applying u^* to both sides of (7), we deduce

$$\begin{aligned} L^2(M) &= [u^*yA_0]_2 \oplus [u^*yD]_2 \oplus [u^*yA_0^*]_2 \\ &= [|y|A_0]_2 \oplus [|y|D]_2 \oplus [|y|A_0^*]_2\,. \end{aligned}$$

Since $|y| \in L^2(D)$, $[|y|A_0]_2 \subset [A_0]_2$, and similarly for the two other terms on the right. Therefore,

$$L^2(M) = [|y|A_0]_2 \oplus [|y|D]_2 \oplus [|y|A_0^*]_2 \subset [A_0]_2 \oplus [D]_2 \oplus [A_0^*]_2 = L^2(M)\,.$$

Hence

$$[|y|A_0]_2 = [A_0]_2, [|y|D]_2 = [D]_2, [|y|A_0^*]_2 = [A_0^*]_2 \tag{8}$$

Passing to adjoints, we also have

$$[A_0|y|]_2 = [A_0]_2, [D|y|]_2 = [D]_2, [A_0^*|y|]_2 = [A_0^*]_2\ .$$

Now it is easy to show that $h = u^*\omega \in H^2(A)$. Indeed, since $y \perp [\omega A_0], \tau(y^*\omega a) = 0$ for all $a \in A_0$; so $\tau(a|y|u^*\omega) = 0$. However, $[A_0|y|]_2 = [A_0]_2$. Thus

$$\forall\, a \in H_0^2(A),\ \ \tau(ah) = 0.$$

Hence by (1), $h \in H^2(A)$, as desired.

It remains to show that $h^{-1} \in H^2(A)$.To this end we first observe that $\Phi(h)\Phi(h^{-1}) = 1$.
Indeed, given $d \in D$ we have, by (6)

$$\begin{aligned} \tau((h)\Phi(h^{-1})|y|d) &= \tau(h^{-1}|y|d\Phi(h)) = \tau(\omega^{-1}u|y|d\Phi(h)) \\ &= \lim_n \tau(\omega^{-1}(\omega - \omega a_n)d\Phi(h)) = \tau d\Phi(h)) = \tau(hd) = \tau(u^*\omega d) = \tau(u^*yd) \\ &= \tau(|y|d), \end{aligned}$$

where we have used the fact that

$$\tau(u^* x d) = \lim_n \Phi\,(u^* \omega a_n d) = \lim_n \tau\,(h a_n d) = 0.$$

Since $[|y|D]_2 = L^2(D)$, we deduce our observation. Therefore, $\Phi(h)$ is invertible and its inverse is $\Phi(h-1)$. On the other hand, by (6)

$$\Phi(h) = \lim_n \Phi\,(u^*(y + \omega a_n)) = \Phi(|y|) + \lim_n \Phi\,(h a_n) = u^* y.$$

Hence,

$$u = y\Phi(h)^{-1} = y\Phi(h^{-1}).$$

Now let $a \in A_0$. Then

$$\tau(h^{-1}a) = \tau(\omega^{-1}ua) = \tau(\omega^{-1}y\Phi(h-1)a) = \lim_n \tau\,\omega^{-1}(\omega - \omega a_n)\Phi(h-1)a) = 0.$$

It follows that $h^{-1} \in H^2(A)$.Therefore, we are done in the case $p = q = 2$.

The general case can be easily reduced to this special one. Indeed, if $p \geq 2$and $q \geq 2$, then given $\omega \in L^p(M)$ with $\omega^{-1} \in L^q(M)$, we can apply the preceding part and then find a unitary $u \in M$ and $h \in H^2(A)$ such that $\omega = uh$ and$h^{-1} \in H^2(A)$. Then $h = u^*\omega \in L^p(M)$, so $\omega \in H^2(A) \cap L^p(M) = H^p(A)$ by (3). Similarly, $h^{-1} \in H^q(A)$.

Suppose $min(p,q) < 2$. Choose an integer n such that $min(np,nq) \geq 2$. Let $\omega = v|\omega|$ be the polar decomposition of ω. Note that $v \in M$ is a unitary. Write

$$\omega = v|\omega|^{1/n}\,|\omega|^{\frac{1}{n}} \ldots\, |\omega|^{1/n} = \omega_1\omega_2 \ldots \omega_n,$$

where $\omega_1 = v|\omega|^{1/n}$ and $\omega_k = |\omega|^{1/n}$ for$2 \leq k \leq n$. Since $\omega_k \in L^{np}(M)$ and $\omega_k^{-1} \in L^{nq}(M)$, by what is already proved we have a factorization

$$\omega_n = u_n h_n$$

with $u_n \in M$a unitary, $h_n \in H^{np}(A)$such that $h_n^{-1} \in H^{nq}(A)$. Repeating this argument, we again get a same factorization for $\omega_{n-1}u_n$:

$$\omega_{n-1}u_n = u_{n-1}h_{n-1}\,;$$

and then for $\omega_{n-2}u_{n-1}$, and so on. In this way, we obtain a factorization:

$$\omega = uh_1 \ldots h_n,$$

where $u \in M$ is a unitary, $h_k \in H^{np}(A)$ such that $h_k^{-1} \in H^{nq}(A)$. Setting $h = uh_1 \ldots h_n$, we then see that $\omega = uh$ is the desired factorization. Hence the proof of the theorem is complete.

Remark(5.1.5) [279]: Let $\omega \in L^2(M)$ be an invertible operator such that $\omega^{-1} \in L^2(M)$. Let $\omega = uh$ be the factorization in Theorem (5.1.4). The preceding proof shows that $[hA]_2 = H^2(A)$. Indeed, it is clear that$[yA]_2 \subset [\omega A]_2$. Using decomposition (7), we get

$$[\omega A]_2 \ominus [yA]_2 = [\omega A]_2 \cap [yA_0^*]_2\,.$$

Now for any $a \in A$ and $b \in A_0$,

$$\langle \omega a, yb^* \rangle = \tau(y^*\omega ab) = 0$$

since $y \perp [\omega A_0]$. It then follows that $[\omega A]_2 \ominus [yA]_2 = \{0\}$, so$[\omega A]_2 = [yA]_2$. Hence, by (8)

$$[yA]_2 = [u^*\omega A]_2 = [u^* yA]_2 = [|y|A]_2 = H^2(A).$$

We turn to the Riesz factorization. We first need to extend (3) to all indices.

Proposition (5.1.6)[279]: Let $0 < p < q \leq \infty$. Then

$$H^p(A) \cap L^q(M) = H^q(A) \text{ and } H^p(A) \cap L^q(M) = H_0^q\,(A),$$

where $H_0^p(A) = [A_0]_p$.
Proof: It is obvious that $H^q(A) \subset H^p(A) \cap L^q(M)$. To prove the converse inclusion, we first consider the case q $= \infty$. Thus let $x \in H^p(A) \cap M$. Then by Corollary (5.1.2),
$$\forall\, a \in A_0\,, \Phi(xa) = \Phi(x)\Phi(a) = 0.$$
Hence by (1), $x \in A$.

Now consider the general case. Fix an $x \in H^p(A) \cap L^q(M)$. Applying Theorem (5.1.4) to $\omega = (x^*x + 1)^{1/2}$, we get an invertible $h \in H^q(A)$such that
$$h^*h = x^*x + 1 \text{ and } h^{-1} \in A.$$
Since $h^*h \leq x^*x$, there exists a contraction $v \in M$ such that $x = vh$. Then $v = xh^{-1} \in H^p(A) \cap M$, so $v \in A$. Consequently, $x \in A.H^q(A) = H^q(A)$. Thus we proved the first equality. The second is then an easy consequence. For this it suffices to note that $H_0^p\,(A) = \{x \in H^p(A)\colon \Phi(x) = 0\}$. The later equality follows from the continuity of Φ on $H^p(A)$.
Theorem (5.1.7)[279]: Let $0 < p, q, r \leq \infty$ such that $1/p = 1/q + 1/r$. Then for $x \in H^p(A)$ and $\varepsilon > 0$ there exist $y \in H^q(A)$ and $z \in H^r(A)$ such that $x = yz$ and $\|y\|_q \|z\|_r \leq \|x\|_p + \varepsilon$. Consequently,
$$\|x\|_p = inf\{\|y\|_q \|z\|_r\colon x = yz, y \in H^q(A), z \in H^r(A)\}$$
Proof: The case where $max(q, r) = \infty$ is trivial. Thus we assume both q and r to be finite. Let $\omega = (x^*x + \varepsilon)^{1/2}$. Then $\omega \in L^p(M)$ and $\omega^{-1} \in M$. Let $v \in M$ be a contraction such that $x = v\omega$. Now applying Theorem (5.1.4) to $\omega^{p/r}$, we have: $\omega^{p/r} = uz$, where u is a unitary in M and $z \in H^r(A)$ such that $z^{-1} \in A$. Set $y = v\omega^{p/q}\, u$. Then $x = yz$, so $y = xz^{-1}$. Since $x \in H^p(A)$ and $z^{-1} \in A, y \in H^p(A)$. On the other hand, y belongs to $L^q(M)$ too. Therefore, $y \in H^q(A)$ by virtue of Proposition (5.1.6). The norm estimate is clear.
We consider outer operators. All results below on the left and right outers are due to Blecher and Labuschagne [66] in the case of indices not less than one. The notion of bilaterally outer is new. We start with the following result.
Proposition (5.1.8)[279]: Let $0 < p < q \leq \infty$ and let $h \in H^q(A)$. Then
(i) $[hA]_p = H^p(A)$ iff $[hA]_q = H^q(A)$;
(ii) $[Ah]_p = H^p(A)$ iff $[Ah]_q = H^q(A)$;
(iii) $[AhA]_p = H^p(A)$ iff $[AhA]_q = H^q(A)$.
Proof: We prove only the third equivalence. The proofs of the two others are similar (and even simpler). It is clear that $[AhA]_p = H^p(A) \Rightarrow [AhA]_q = H^q(A)$. To prove the converse implication we first consider the case $q \geq 1$.Let q' be the conjugate index of q. Let $x \in L^{q'}(M)$ be such that
$$\forall\, a, b \in A, \tau(xahb) = 0.$$

Then $xah \in H_0^1\,(A)$ for any $a \in A$by virtue of (2) (more rigorously, its H^p -analogue as in Proposition (5.1.6)). On the other hand, by the assumption that $[AhA]_p = H^p(A)$, there exist two sequences $(a_n), (b_n) \subset A$ such that
$$\lim_n a_n h b_n = 1 \; in \; H^p(A).$$

Consequently,

$$\lim_n xa_n hb_n = x \text{ in } L^r(M),$$

where $\frac{1}{r} = 1/q' + 1/p$.Since $xa_n hb_n = (xa_n h)b_n \in H_0^1(A) \subset H_0^r(A)$, we deduce that $x \in H_0^r(A)$. Therefore, $x \in H_0^r(A) \cap L^{q'}(M)$, so by Proposition (5.1.6), $x \in H_0^{q'}(A)$ Hence, $\tau(xy) = 0$ for all y $\in H^q(A)$. Thus $[AhA]_q = H^q(A)$.

Now assume $q < 1$. Choose an integer n such that $np \geq 2$. By the proof of Theorem (5.1.7) and Remark(5.1.5), we deduce a factorization:

$$h = h_1 h_2 \dots h_n ,$$

where $h_k \in H^{np}(A)$ for every $1 \leq k \leq n$ and $[h_k A]_2 = H^2(A)$for $2 \leq k \leq n$. By the left version (i.e;part i) of the previous case already proved, we also have $[h_k A]_{nq} = H^{nq}(A)$ and $[h_k A]_{nq} = H^{nq}(A)$for $2 \leq k \leq n$. Let us deal with the first factor h_1. Using $[AhA]_p = H^p(A)$ and $[h_k A]_{np} = H^{np}(A)$for $2 \leq k \leq n$, we see that $[Ah_1 A]_p = H^p(A)$;so again $[Ah_1 A]_p = H^p(A)$ by virtue of the first part. It is then clear that $[AhA]_q = H^q(A)$.

The previous result justifies the relative independence of the index p in the following definition.

Definition (5.1.9)[279]: Let $0 < p \leq \infty$. An operator $h \in H^p(A)$ is called left outer, right outer or bilaterally outer according to

$$[\mathrm{hA}]_\mathrm{p} = H^p(A), [hA]_p = H^p(A) \text{or} [AhA]_p = H^p(A).$$

Theorem (5.1.10)[278]: Let $0 < p \leq \infty$ and $h \in H^p(A)$.

(i) If h is left or right outer, then $\Delta(h) = \Delta\Phi(h))$. Conversely, if $\Delta(h) = \Delta(\Phi(h))$ and $\Delta(h) > 0$, then h is left and right outer (so bilaterally outer too).

(ii) If $A$is antisymmetric (i.e; $dimD = 1$) and h is bilaterally outer, then $\Delta(h) = \Delta(\Phi(h))$.

Proof: (i) This part is proved in [66] for $p \geq 1$. Assume h is left outer. Let $d \in D$. Using Theorem (5.1.1), we obtain

$$\|\Phi(h)d\|_p = \inf\{\|hd + x_0\|_p : x \in H_0^p(A)\}.$$

On the other hand,

$$[hA_0]_p = \left[[hA]_p A_0\right]_p = [[A]pA_0]_p = [A_0]_p = H_0^p\,(A).$$

Therefore,

$$\|\Phi(h)d\|_p = \inf\{\|h(d + a_0)\|_p : a \in A_0\}.$$

Recall the following characterization of$\Delta(x)$ from [66]:

$$\Delta(x) = \inf\{\|xa\|_p : a \in A, \Delta(\Phi(a)) \geq 1\}. \quad (9)$$

Now using this formula twice, we obtain

$$\tau\big(\Phi(h)\big) = \inf\{\|\Phi(h)d\|_p : d \in D, \Delta(d) \geq 1\}$$
$$= \inf\{\|h(d + a_0)\|_p : d \in D, \Delta(d) \geq 1, a_0 \in A_0\} = \Delta(h).$$

Let us show the converse under the additional assumption that $\Delta(h) > 0$. We will use the case $p \geq 1$ already proved in [66]. Thus assume $p < 1$. Choose an integer n such that $np \geq 1$. By Theorem (5.1.7), there exist $h_1, \dots, h_n \in H^{np}(A)$ such that $h = h_1 \dots h_n$.Then $\Delta(h) = \Delta(h_1) \dots \Delta(h_n)$; so $\Delta(h_k) > 0$ for all $1 \leq k \leq n$. On the other hand, by Arveson-Labuschagne's Jensen inequality [300,163] (or Corollary (5.1.3)), $\Delta(\Phi(h_k)) \leq \Delta(h_k)$. However,

$$\Delta(\Phi(h)) = \Delta(\Phi(h_1)) \dots \Delta(\Phi(h_n)) \leq \Delta(h_1) \dots \Delta(hn) = \Delta(h) = \Delta(\Phi(h)).$$

It then follows that $\Delta(\Phi(h_k)) \leq \Delta(h_k)$ for all k. Now $h_k \in H^{np}(A)$ with $np \geq 1$, so h_k is left and right outer. Consequently, h is left and right outer.

(ii) This proof is similar to that of the first part of i). We will use the following variant of (9)

$$\Delta(x) = inf\{\|(axb)\|_p : a, b \in A, \Delta(\Phi(a)) \geq 1, \Delta(\Phi(b)) \geq 1\} \quad (10)$$

for every $x \in L^p(M)$. This formula immediately follows from (9). Indeed, by (9) and the multiplicativity of Δ

$$inf\{\|(axb)\|_p : a, b \in A, \Delta(\Phi(a)) \geq 1, \Delta(\Phi(b)) \geq 1\} = inf\{\Delta(ax) : a \in A, \Delta(\Phi(a)) \geq 1\}$$
$$= inf\{\Delta(a)\Delta(x) : a \in A, \Delta(\Phi(a)) \geq 1\} = \Delta(x).$$

Now assume $h \in H^p(A)$ is bilaterally outer and A is antisymmetric. Then $\Phi(h)$ is a multiple of the unit of M. As in the proof of i), We have

$$\|\Phi(h)\|_p = inf\{\|h + x\|_p : x \in H_0^p(A)\} = inf\{\|h + ahb_0\|_p : a \in A, b_0 \in A_0\}. \quad (11)$$

Using $dimD = 1$, we easily check that

$$inf\{\|h + ahb_0\|_p : a \in A, b_0 \in A_0\} = inf\{\|(1 + a0)h(1 + b_0)\|_p : a_0, b_0 \in A_0\}. \quad (12)$$

Indeed, it suffices to show that both sets$\{h + ahb_0 : a \in A, b_0 \in A_0\}$and $\{(1 + a_0)h(1 + b_0) : a_0, b_0 \in A_0\}$ are dense in $\{x \in H^p(A) : \Phi(x) = \Phi(h)\}$. The first density immediately follows from the density of AhA_0 in $H_0^p\ (A)$. On the other hand, let$x \in H^p(A)$with $\Phi(x) = \Phi(h)$ and let $a_n, b_n \in A$ such that $\lim_n a_n h b_n = x$. By Theorem (5.1.1),

$$\lim_n \Phi(a_n)\Phi(h)\Phi(b_n)\ \Phi(x).$$

Since $\Phi(x) = \tau(x)1 = \tau(h)1 = \Phi(h) \neq 0$, we deduce that $\lim_n \tau(a_n)\tau(b_n) = 1$. Thus replacing a_n and b_n by $a_n/\tau(a_n)$ and $b_n/\tau(b_n)$, respectively, we can assume that $a_n = 1 + \tilde{a}_n$ and $b_n = 1 + \tilde{b}_n$with $\tilde{a}_n, \tilde{b}_n \in A_0$; whence the desired density of $\{(1 + a_0)h(1 + b_0) : a_0, b_0 \in A_0\}$ in $\{x \in H^p(A) : \Phi(x) = \Phi(h)\}$. Finally, combining (10), (11) and (12), we get $\Delta(\Phi(h)) = \Delta(h)$.

Note that, the assumption that A is antisymmetric in Theorem (5.1.12), ii) cannot be removed in general, as shown by the following example. Keep the notation introduced and consider the case where $M = L^\infty(T; \mathbb{M}_2)$ and $A = H^\infty(T; \mathbb{M}_2)$. Let φ_1 and φ_2 be two outer functions in $H^p(T)$, and let $h = \varphi_1 \otimes e_{11} + ze_{22} \otimes e_{22}$, where z denotes the identity function on $\mathbb{T}$. Then it is easy to check that h is bilaterally outer and

$$\Delta(h) = exp\left(\frac{1}{2}\int_{\mathbb{T}} log\ |\varphi_1| + \int_{\mathbb{T}} log\ |\varphi_2|\right) > 0.$$

However,$\Phi(h) = \varphi_1(0)e_{11}, so\ \Delta(\Phi(h)) = 0$.

The following is an immediate consequence of Theorem (5.1.12). We do not know, however, whether the condition $\Delta(h) > 0$ in i) can be removed or not.

Corollary (5.1.11)[279]: Let $h \in H^p(A), 0 < p \leq \infty$.

(i) If $\Delta(h) > 0$, then h is left outer iff h is right outer.

(ii) Assume that A is antisymmetric. Then the following properties are equivalent:

(1) h is left outer;
(2) h is right outer;
(3) h is bilaterally outer;
(4) $\Delta(\Phi(h)) = \Delta(h) > 0$.

We will say that h is outer if it is at the same time left and right outer. Thus if $h \in H^p(A)$with $\Delta(h) > 0$, then h is outer iff $\Delta(h) = \Delta(\Phi(h))$. Also in the case where A is antisymmetric, an h with $\Delta(h) > 0$is outer iff it is left, right or bilaterally outer.

Corollary (5.1.12)[279]: Let $h \in H^p(A)$ such that $h^{-1} \in H^q(A)$ with $0 < p, q \leq \infty$. Then h is outer.

Proof: By the multiplicativity of $\Delta, \Delta(h)\Delta(h^{-1}) = 1$ and $\Delta\big(\Phi(h)\big)\Delta\big(\Phi(h^{-1})\big) = 1$. Thus by Jensen's inequality (Corollary (5.1.3)),

$$\Delta(h) = \Delta(h^{-1})^{-1} \leq \Delta(\Phi(h^{-1}))^{-1} = \Delta(\Phi(h));$$

whence the assertion because of Theorem (5.1.10).

The following improves Theorem (5.1.4).

Theorem (5.1.13)[279]: Let $\omega \in L^p(M)$ with $0 < p \leq \infty$ such that $\Delta(\omega) > 0$. Then there exist a unitary $u \in M$ and an outer $h \in H^p(A)$ such that $\omega = uh$.

Proof: Based on the case$p \geq 1$ from [67], the proof below is similar to the end of the proof of Theorem (5.1.4). For simplicity we consider only the case where $p \geq 1/2$. Write the polar decomposition of ω: $\omega = v|\omega|$. Applying [66] to $|\omega|^{1/2}$ we get a factorization: $|\omega|^{1/2} = u_2 h_2$with u_2 unitary and $h_2 \in H^{2p}(A)$ left outer. Since $\Delta(h_2) > 0, h_2$ is also right outer; so h_2 is outer. Similarly, we have: $v|w|^{1/2}u_2 = u_1 h_1$. Then $u = u_1$and $h = h_1 h_2$ yield the desired factorization of ω.

The following is the inner-outer factorization for operators in $H^p(A)$, which is already in [66] for $p \geq 1$.

Corollary (5.1.14)[279]: Let $0 < p \leq \infty$ and $x \in H^p(A)$ with$\Delta(x) > 0$. Then there exist a unitary $u \in A$(inner) and an outer $h \in H^p(A)$ such that $x = uh$.

Proof: Applying the previous theorem, we get $x = uh$ with h outer and u a unitary in M. Let$a_n \in A$ such that $lim h a_n = 1$ in $H^p(A)$. Then $u = lim x a_n$ in $H^p(A)$ too; so $u \in H^p(A) \cap M$. By Proposition (5.1.5), $u \in A$.

Corollary (5.1.15)[279]: Let $0 < p \leq \infty$ and $h \in H^p(A)$with $\Delta(h) > 0$. Then h is outer iff for any $x \in H^p(A)$with $|x| = |h|$ we have $\Delta(\Phi(x)) \leq \Delta(\Phi(h))$.

Proof: Assume h outer. Then by Corollary (5.1.3) and Theorem (5.1.9),

$$\Delta(\Phi(x)) \leq \Delta(x) = \Delta(h) = \Delta(\Phi(h)).$$

Conversely, let $h = uk$ be the decomposition given by Theorem (5.1.13) with k outer. Then

$$\Delta(h) = \Delta(k) = \Delta(\Phi(k)) \leq \Delta(\Phi(h));$$

so h is outer by Theorem (5.1.12).

Corollary (5.1.16)[279]: Let $0 < p, q, r \leq \infty$ such that $1/p = 1/q + 1/r$. Let $x \in H^p(A)$ be such that $\Delta(x) > 0$. Then there exist $y \in H^q(A)$ and $z \in H^r(A)$ such that
$x = yz$ and$\|x\|_p = \|y\|_q \|z\|_r$.

Proof: This proof is similar to that of Theorem (5.1.7) Instead of Theorem (5.1.4), we now use Theorem (5.1.13). Indeed, by the later theorem, we can find a unitary $u_2 \in M$ and an outer $h_2 \in$

$H^{p/r}(A)$ such that $|x|^{p/r} = u_2h_2$. Once more applying this theorem to $v|x|^{p/q}u_2$, we have a similar factorization: $v|x|^{p/q}u_2 = u_1h_1$, where v is the unitary in the polar decomposition of x. Since h_1 andh_2 are outer, we deduce, as in the proof of Corollary (5.1.14), that $u_1 \in A$. Then $y = u_1h_1$ and $z = h_2$ give the desired factorization of x.

Let $\omega \in L^1(T)$be a positive function and let $d\mu = \omega dm$. Then we have the following well-known Szeg¨o formula [92]:

$$inf\{\int_{\mathbb{T}}|1-f|^2 d\mu: f \ mean \ zero \ analytic \ polynomial\} = exp\int_{\mathbb{T}} logwdm.$$

This formula was later proved for any positive measure μ on T independently by Kolmogorov/Krein [15] and Verblunsky [268]. Then the singular part of μ with respect to the Lebesgue measure dmdoes not contribute to the preceding infimum and w on the right hand side is the density of the absolute part of μ (also see [141]). This latter result was extended to the noncommutative setting in [66]. More precisely, let ω be a positive linear functional on M, and let $\omega = \omega_n + \omega_s$be the decomposition of ω into its normal and singular parts. Let $\omega \in L^1(M)$ be the density of ω_n with respect to τ, i.e., $\omega_n = \tau(\omega\ \cdot)$. Then Blecher and Labuschagne proved that if $dimD < \infty$,

$$\Delta(\omega) = inf\{\omega(|a|^2): a \in A, \Delta(\Phi(a)) \geq 1\}.$$

It is left open in [66] whether the condition $dimD < \infty$ can be removed or not. We will solve this problem in the affirmative. At the same time, we show that the square in the above formula can be replaced by any power p.

Theorem (5.1.17)[279]: Let $\omega = \omega_n + \omega_s$be as above and $0 < p < \infty$. Then

$$\Delta(\omega) = inf\{(\omega|a|^p): a \in A, \Delta(\Phi(a)) \geq 1\}.$$

Proof: Let

$$\delta(\omega) = inf\{\omega(|a|^p): a \in A, \Delta(\Phi(a)) \geq 1\}.$$

First we show that

$$\delta(\omega) = inf\{\omega(x): a \in A, \Delta(\Phi(a)) \geq 1\},$$

where M_+^{-1} denotes the family of invertible positive operators in M with bounded inverses. Given any $x \in M_+^{-1}$, by Arveson's factorization theorem there exists $a \in A$ such that $|a| = x^{1/p}$ and $a^{-1} \in A$. Then $x = |a|^p$, so $\Delta(x) = \Delta(|a|^p) = \Delta(a)^p$. Since a is invertible with $a^{-1} \ \in A$, by Jensen's formula in [300], $\Delta(a) = \Delta(\Phi(a))$. It then follows that

$$\delta(\omega) \leq \{\omega(x): x \in M_+^{-1}, \Delta(x) \geq 1\}.$$

The converse inequality is easier. Indeed, given $a \in A$ with $\Delta(\Phi(a)) \geq 1$and $\varepsilon > 0$, set $x = |a|^p + \varepsilon$. Then $x \in M_+^{-1}$and $\Delta(x) \geq \Delta(a)^p \geq \Delta(\Phi(a))^p$ by virtue of Jensen's inequality. Since $\lim_{\varepsilon\to 0} \omega\ (|a|^p + \varepsilon) = \omega(|a|^p)$, we deduce the desired converse inequality.

Next we show that $\delta(\omega) = \delta(\omega_n)$. The singularity of $\omega_s$implies that there exists an increasing net (e_i) of projections in M such that $e_i \to 1$ strongly and $\omega_s(e_i) = 0$ for every i (see [181]). Let ε >0. Set

$$x_i = \varepsilon^{\tau(e_i)^{-1}}(e_i + \varepsilon e_i^{\perp}), \text{where } e^{\perp} = 1 - e.$$

Clearly, $x_i \in M_+^{-1}$ and $(x_i) = 1$. Let $x \in M_+^{-1}$and $\Delta(x) \geq 1$. Then $\Delta(x_i x x_i) = \Delta(x) \geq 1$, and $x_i x x_i \to x$ in the ω^*-topology. On the other hand, note that

$$\omega_s(x_i x x_i) = \varepsilon^{2\tau(e_i)}\omega_s(e_i^{\perp} x e_i^{\perp}).$$

Therefore,

$$\delta(\omega) \leq limsup\omega(x_i x x_i) = \omega_n(x) + limsup\omega_s(x_i x x_i)$$
$$\leq \omega_n(x) + limsup\varepsilon^{2\tau(e_i)}\omega_s(e_i^{\perp} x e_i^{\perp}) \leq \omega_n(x) + \varepsilon^2\|\omega_s\|\|x\|.$$
It thus follows that $\delta(\omega) \leq \delta(\omega_n)$, so $\delta(\omega) = \delta(\omega_n)$. Now it is easy to conclude the validity of the result. Indeed, the preceding two parts imply
$$\delta(\omega) = inf\{\tau(wx): x \in M_+^{-1}, \Delta(x) \geq 1\}.$$
By a formula on determinants from [300], the last infimum is nothing but $\Delta(\omega)$. Therefore, the theorem is proved.

Section (5.2) Noncommutative Hardy-Lorentz Spaces

The classical Hardy spaces $H^p(D), 1 \leq p \leq \infty$, are Banach spaces of analytic functions on the unit disk satisfying that
$$\sup_{0<r<1}\left\{\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right\} < \infty .$$
by taking radial limits,$H^p(D)$ can be identified with $H^p(T)$, the space of functions on the unit circle which are in $L^p(T)$with respect to Lebesgue measure and whose negative Fourier coefficients vanish. These spaces have played an important role in modern analysis and prediction theory. One of the key results in the functional analytic approach to Hardy spaces is Szegö theorem (see [274]), which is formula for the weighted $L^2(T)$ distance from 1 to the analytic polynomials which vanish at the origin.

The theory of Hardy spaces was generalized in two directions. Masani and Wiener [186, 185] extended Szegö theorem to the theory of multivariate stochastic processes by studying matrix valued functions. Concurrently, Helson and Lowdenslager [115] adapting techniques from functional analysis to extend the theory to the setting of a compact group with ordered dual, thus laying the foundation for the theory of function algebras. This eventually led to the definition of a weak∗-Dirichlet algebra of functions by Srinivasan and Wang [272]. Srinivasan and Wang were able to prove Szegö's theorem and several other important results in the theory of function algebras.

Arveson [25] introduced the concept of maximal subdiagonal algebras, unifying analytic function spaces and nonselfadjoint operator algebras. Subsequently, Arveson's pioneer work was extended to different cases by several authors. In 1997, Marsalli and West [184] defined noncommutative Hardy spaces for finite von Neumann algebras and obtained a series of results including a Riesz factorization theorem, the dual relations between $H^p(\mathcal{A})$ and $H^q(\mathcal{A})$ and so on. Labuschagne [166] proved the universal validity of Szegö's theorem for finite subdiagonal algebras. Blecher and Labuschagne [41] gave several useful variants of the noncommutative Szegötheorem for $L^p(\mathcal{M})$. Tt was also in [42] that the longstanding open problem concerning the noncommutative generalization of the famous 'outer factorization' of functions f with $log|f|$ integrable was solved. Recently, Bekjan and Xu [39] presented the more general form of Szegö type factorization for the noncommutative Hardy spaces defined in [184].

We introduce the noncommutative Hardy-Lorentz spaces. By adapting the techniques in [39], we establish the Szegö factorization theorem of these spaces. Section contains some preliminaries and notations on the noncommutative $L^{p,q}$-spaces and noncommutative $H^{p,q}$-spaces. The proof of the Szegö factorization of noncommutative Hardy-Lorentz spaces are presented. Finally mainly devoted to the inner-outer type factorization of noncommutative Hardy- Lorentz spaces.

We denote by $\mathcal{M}$ a finite von Neumann algebra on the Hilbert space $\mathcal{H}$, equipped with a normal faithful tracial state τ. The identity in $\mathcal{M}$ is denoted by 1 and we denote by D a von Neumann subalgebra of $\mathcal{M}$, moreover, we let $\varepsilon: \mathcal{M} \to D$ be the unique normal faithful conditional expectation such that $\tau \circ \varepsilon = \tau$. $\mathcal{A}$ is a finite subdiagonal algebra of $\mathcal{M}$. A finite subdiagonal algebra of $\mathcal{M}$ with respect to ε (or D) is a w^* −closed subalgebra $\mathcal{A}$ of $\mathcal{M}$ satisfying the following conditions:

(i) $\mathcal{A} + \mathcal{A}^*$ is ω^* −dense in $\mathcal{M}$;

(ii) ε is multiplicative on A, i.e.,$\varepsilon(ab) = \varepsilon(a)\, \varepsilon(b)$ for all $a, b \in \mathcal{A}$;

(iii) $\mathcal{A} \cap \mathcal{A}^* = D$.

We denote by $\mathcal{M}_{proj}$ the lattice of (orthogonal) projections in $\mathcal{M}$. A linear operator $x: dom(x) \to \mathcal{H}$, with domain $dom(x) \subseteq \mathcal{H}$, is said to be affiliated with $\mathcal{M}$ if $ux = xu$ for all unitary u in the commutant $\mathcal{M}'$ of $\mathcal{M}$. The closed densely defined linear operators x affiliated with $\mathcal{M}$ is called τ −measurable if for every $\varepsilon > 0$ there exists an orthogonal projection $P \in \mathcal{M}_{\mathrm{proj}}$ such that $P(H) \subseteq dom(x)$ and $\tau(1 - P) < \varepsilon$. The collection of all τ-measurable operators is denoted by $\widetilde{\mathcal{M}}$. With the sum and product defined as the respective closures of the algebraic sum and product, $\widetilde{\mathcal{M}}$ is a $*$-algebra. For a positive self-adjoint operator x affiliated with M, we set

$$\tau(x) = \sup_n \tau\left(\int_0^n \lambda dE_\lambda\right) = \int_0^\infty \lambda d\tau(E_\lambda),$$

where $x = \int_0^\infty \lambda dE_\lambda$ is the spectral decomposition of x.

Let $0 < p < \infty$, $L^p(\mathcal{M}; \tau)$ is defined as the set of all τ-measurable operators x affiliated with $\mathcal{M}$ such that

$$\|x\|_p = \tau(|x|^p)^{\frac{1}{p}} < \infty.$$

In addition, we put $L^\infty(\mathcal{M};\, \tau) = \mathcal{M}$ and denote by $\|\cdot\|_\infty$ $(= \|\cdot\|)$ the usual operator norm. It is well known that $L^p(\mathcal{M};\, \tau)$ is a Banach space under $\|\cdot\|_p$ for $1 \leq p < \infty$. They have all the expected properties of classical L^p-spaces (see also [89]).

Let x be a τ-measurable operator and $t > 0$. The "t-th singular number (or generalized s-number) of x" is defined by

$$\mu_t(x) = inf\{\|xE\|: E \in \mathcal{M}_{proj}, \tau(1 - E) \leq t\}.$$

See [81] for basic properties and detailed information on the generalized s −numbers.

Let x be a τ-measurable operator in $L^p(\mathcal{M})$ with $0 < p \leq \infty$. The Fuglede-Kadison determinant $\Delta(x)$ is defined by

$$\Delta(x) = exp\big(\tau(log\,|x|)\big) = exp\int_0^\infty log\, t dv_{|x|}(t),$$

where $d\nu_{|x|}$ denotes the probability measure on $\mathcal{R}_+$ which is obtained by composing the spectral measure of $|x|$ with the trace τ. We refer to [25, 84] for more information on determinant in the case of bounded operators, and to [42, 111] for unbounded operators.

Definition (5.2.1)[142]: Let x be a τ-measurable operator affiliated with a finite von Neumann algebra $\mathcal{M}$ and $0 < p, q \leq \infty$, define

$$\|x\|_{L^{p,q}(\mathcal{M})} = \begin{cases} \left(\int_0^\infty (t^{\frac{1}{p}}\mu_t(x))^q \frac{dt}{t}\right)^{\frac{1}{q}}, if\ q < \infty, \\ \sup_{t>0} t^{\frac{1}{p}}\mu_t(x), if\ q = \infty\,. \end{cases} \tag{13}$$

The set of all $x \in \widetilde{\mathcal{M}}$ with $\|x\|_{L^{p,q}(\mathcal{M})} < \infty$ is denoted by $L^{p,q}(\mathcal{M})$ and is called the noncommutative Lorentz space with indices p and q.

Note that

(i) If $1 < p < \infty, 1 \leq q < \infty$, and $\frac{1}{p} + \frac{1}{p'} = 1, \frac{1}{q} + \frac{1}{q} = 1$, then by Xu [325], we obtain the following result

$$(L^{p,q}(\mathcal{M}))^* = L^{p',q'}(\mathcal{M}).$$

For more information on $L^{p,q}(\mathcal{M})$ we refer to [81, 315].

(ii) Since $\tau(1) = 1$, in (13) we can write

$$\|x\|_{L^{p,q}(\mathcal{M})} = \left(\int_0^\infty (t^{\frac{1}{p}}\mu_t(x))^q \frac{dt}{t}\right)^{\frac{1}{q}}, if\ q < \infty.$$

Definition (5.2.2)[142]: Let $\mathcal{A}$ be a finite subdiagonal algebra of $\mathcal{M}$. For $0 < p, q < \infty$, we define the noncommutative Hardy-Lorentz spaces to be the closure of $\mathcal{A}$ in $L^{p,q}(\mathcal{M})$, denoted by $H^{p,q}(\mathcal{M})$

Lemma (5.2.3)[142]: Let $0 < p_1 < p < \infty, 0 < q, s < \infty$, then

$$L^{p,q}(\mathcal{M}) \subset L^{p_1,s}(\mathcal{M}).$$

Consequently,

$$H^{p,q}(\mathcal{A}) \subset H^{p_1,s}(\mathcal{A}).$$

Proof: Similarly to the proof of [112] we can prove that $L^{p,q}(\mathcal{M}) \subset L^{p,\infty}(\mathcal{M})$ with $q < \infty$, and $L^{p_1,u}(\mathcal{M}) \subset L^{p_1,s}(\mathcal{M})$ with $u \leq s$. Now it suffices to prove that $\|x\|_{L^{p_1,u}(\mathcal{M})} \leq C\|x\|_{L^{p,\infty}(\mathcal{M})}, \forall\, x \in L^{p,\infty}(\mathcal{M})$ and $0 < u < \infty$. Indeed, $\forall\, x \in L^{p,\infty}(\mathcal{M})$, we have

$$\|x\|_{L^{p_1,u}(\mathcal{M})} = \left\{\int_0^1 (t^{\frac{1}{p1}}\mu_t(x))^u \frac{dt}{t}\right\}^{\frac{1}{u}} = \left\{\int_0^1 t^{\frac{u}{p1}-\frac{u}{p}-1}(t^{\frac{1}{p1}}\mu_t(x))^u\, dt\right\}^{\frac{1}{u}}$$

$$\leq \left\{\int_0^1 t^{\frac{u}{p1}-\frac{u}{p}-1}(\sup_{0<s<\tau(1)} s^{\frac{1}{p}}\mu_s(x))^u\, dt\right\}^{\frac{1}{u}} = \|x\|_{L^{p,\infty}(M)}\left\{\int_0^1 t^{\frac{u}{p1}-\frac{u}{p}-1}dt\right\}^{\frac{1}{u}}$$

$$= \frac{1}{\left(\frac{u}{p1}-\frac{u}{p}\right)^{\frac{1}{u}}}\|x\|_{L^{p,\infty}(M)}$$

which gives the first inclusion of the lemma. Consequently, we obtain

$$H^{p,q}(\mathcal{A}) \subset H^{p_1,u}(\mathcal{A}) \subset H^{p_1,s}(\mathcal{A}).$$

Definition (5.2.4)[142]: Let x be a τ-measurable operator affiliated with a finite von Neumann algebra $\mathcal{M}$ and $0 < p,q \leq \infty$, define

$$\|x\|^*_{L_r^{p,q}(\mathcal{M})} = \begin{cases} \left(\int_0^\infty \left[t^{\frac{1}{p}}x^{**}(x)\right]^q \frac{dt}{t}\right)^{\frac{1}{q}}, & if\ q < \infty, \\ \sup_{t>0} t^{\frac{1}{p}}x^{**}(x), & if\ q = \infty. \end{cases} \tag{14}$$

where $x^{**}(t) = (\frac{1}{t}\int_0^t(\mu_s(x))^r ds)^{\frac{1}{r}}$, $0 < r \leq min\,(1,q), r < p$. The set of all $x \in \widetilde{\mathcal{M}}$ with $\|x\|^*_{L_r^{p,q}(\mathcal{M})} < \infty$ is denoted by $L_r^{p,q}(\mathcal{M})$.

Lemma (5.2.5)[142]: Let $0 < p,q < \infty$, then

$$\|x\|_{L^{p,q}(\mathcal{M})} \leq \|x\|^*_{L_r^{p,q}(\mathcal{M})} \leq e^{\frac{1}{p}}\|x\|_{L^{p,q}(\mathcal{M})},$$

where r is as in Definition (5.2.4).

Proof: The first inequality is an immediate result from the following estimate

$$\mu_t(x)) \leq \left(\frac{1}{t}\int_0^t (\mu_s(x))^r ds\right)^{\frac{1}{r}}.$$

Now we turn to prove the second inequality. Hardy's first inequality of [119] tells us that

$$\|x\|^{*q}_{L_r^{p,q}(\mathcal{M})} = \int_0^\infty \left[t^{\frac{1}{p}-\frac{1}{r}} \left(\int_0^t (\mu_s(x))^r ds \right)^{\frac{1}{r}} \right] \frac{dt}{r} = \int_0^\infty t^{-\left(\frac{q}{r}-\frac{q}{p}\right)-1} \left(\int_0^t (\mu_s(x))^r ds \right)^{\frac{q}{r}} dt$$

$$\leq \left(\frac{\frac{q}{r}}{\frac{q}{r}-\frac{q}{p}} \right)^{\frac{q}{r}} \int_0^t [s(\mu_s(x))^r]^{\frac{s}{r}} s^{-\left(\frac{q}{r}-\frac{q}{p}\right)-1} ds = \left(\frac{q}{p-r}\right)^{\frac{q}{r}} \int_0^t s^{\frac{q}{p}} (\mu_s(x))^q \frac{ds}{s}$$

$$= \left(\frac{q}{p-r}\right)^{\frac{q}{r}} \|x\|^q_{L^{q,q}(\mathcal{M})} \leq e^{\frac{q}{p}} \|x\|^q_{L^{q,q}(\mathcal{M})}.$$

Lemma (5.2.6)[142]: Let $0 < p, q < \infty$, assume $\mathcal{M}$ has no minimal projection, then

$$\|\varepsilon(a)\|^*_{L^{p,q}_{r(\mathcal{M})}} \leq \|a\|^*_{L^{p,q}_{r(\mathcal{M})}};\ \|\varepsilon(a)\|_{L^{p,q}(\mathcal{M})} \leq e^{\frac{1}{p}} \|a\|_{Lp,q(\mathcal{M}),}$$

where r is as in Definition (5.2.4).

Proof : [81] gives that

$$\int_0^t (\mu_s(\varepsilon(a))^r ds = \int_0^t \mu_s(|\varepsilon(a)|^r) ds \ = \sup_t \{\tau(e|\varepsilon(a)|^r e) : e \in \mathcal{N}_{proj}, \tau(e) \leq t\},$$

where $\mathcal{N}$ is a von Neumann subalgebra generated by all spectral projections of $|\varepsilon(a)|$. It is clear that $\mathcal{N}_{proj} \subset D = \mathcal{A} \cap \mathcal{A}^*$, then we get

$$\int_0^t \big(\mu s(\varepsilon(a))\big)^r ds \ = \sup_t \{\tau(|\varepsilon(a)e|r) : e \in \mathcal{N}_{proj}, \tau(e) \leq t\}$$

$$\leq \sup_t \{\tau(|\varepsilon(a)e|^r) : e \in D, \tau(e) \leq t\}$$

$$= \ \sup_t \{\tau(|\varepsilon(a)e|^r) : e \in D, \tau(e) \leq t\}$$

$$\leq \ \sup_t \{\|\varepsilon(ae)\|_r^r : e \in D, \tau(e) \leq t\}.$$

$$\leq \int_0^t (\mu_s\,(a))^r ds.$$

It follows that

$$\|\varepsilon(a)\|^{*q}_{L^{p,qr}(\mathcal{M})} \ = \int_0^1 t^{\frac{q}{p}} \left(\frac{1}{t} \int_0^1 (\mu_s(\varepsilon(a)))^r ds \right)^{\frac{q}{r}} \frac{dt}{t} \leq \int_0^1 t^{\frac{q}{p}} \left(\frac{1}{t} \int_0^1 \big(\mu_s(a)\big)^r ds \right)^{\frac{q}{r}} \frac{dt}{t}$$

$$= \|(a)\|^{*q}_{L_r^{p,q}(\mathcal{M}),}$$

i.e.,

$$\|\varepsilon(a)\|_{L^{p,q}\,(\mathcal{M})} \leq \|\varepsilon(a)\|^*_{L_r^{p,q}\,(\mathcal{M})} \leq \ \|a\|^*_{L^{p,q}\,(\mathcal{M})} \leq e^{\frac{1}{p}} \|\varepsilon(a)\|_{L^{p,q}\,(\mathcal{M})}$$

Lemma (5.2.7)[142]: Let $0 < r_1 < r_2 \le 1, r_1 < r_2 < p, r_1 < r_2 \le q$, then $\|x\|^*_{L^{q,q}_{r_1}(\mathcal{M})}$ is equivalent to $\|x\|^*_{L^{q,q}_{r_2}(\mathcal{M})}$.

Proposition (5.2.8)[142]: Let $0 < p, p_0, p_1, q, q_0, q_1 < \infty$ such that $\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1}, \frac{1}{q} = \frac{1}{q_0} + \frac{1}{q_1}$, then

$$\|yz\|_{Lp,q(\mathcal{M})} \le e^{\frac{1}{p}} \|y\|_{L^{p_0,q_0}(\mathcal{M})} \|z\|_{L^{p_0,p_1}(\mathcal{M})},$$

where $y \in L^{p_0,q_0}(\mathcal{M}), z \in L^{p_1,q_1}(\mathcal{M})$.

Proof: Let $0 < 2r < min(1, p, q)$, we have

$$(yz)^{**}(t,r) = \left(\frac{1}{t}\int_0^t (\mu_s(yz))^r ds\right)^{\frac{1}{r}} \le \left(\frac{1}{t}\right)^{\frac{1}{r}} \left(\int_0^t (\mu_s(yz))^{2r} ds\right)^{\frac{1}{2r}} \left(\int_0^t (\mu_s(z))^{2r} ds\right)^{\frac{1}{2r}}$$
$$= y^{**}(t, 2r) z^{**}(t, 2r).$$

Combing the above estimate with Lemma (5. 2.5) we infer that

$$\|yz\|_{L^{p,q}(\mathcal{M})} \le \|yz\|^*_{L^{p,q}_r(\mathcal{M})} = \left(\int_0^\infty (t^{\frac{1}{p}} (yz)^{**}(t,r))^q \frac{dt}{t}\right)^{\frac{1}{q}}$$
$$\le \left(\int_0^\infty (t^{\frac{1}{p_0}} y^{**}(t,r))^{\frac{1}{p_1}} z^{**}(r,2r)^q \frac{dt}{t}\right)^{\frac{1}{q}}$$
$$\le \left(\int_0^\infty (t^{\frac{1}{p_0}} y^{**}(r,2r)^{q_0} \frac{dt}{t}\right)^{\frac{1}{q_0}} \left(\int_0^\infty (t^{\frac{1}{p_1}} z^{**}(r,2r)^{q_1} \frac{dt}{t}\right)^{\frac{1}{q_1}}$$
$$= \| y\|^*_{L^{p_0,q_0}_{2r}(\mathcal{M})} \|z\|^*_{L^{p_1,q_1}_{2r}(\mathcal{M})} \le (e^{\frac{1}{p_0}} \| y\|_{L^{p_0,q_0}(\mathcal{M})})(e^{\frac{1}{p_1}} \|z\|_{L^{p_1,q_1}(\mathcal{M})})$$
$$= e^{\frac{1}{p}} \|y\|_{L^{p_0,q_0}(\mathcal{M})} \|z\|_{L^{p_1,q_1}(\mathcal{M})},$$

which gives the result.

Lemma (5.2.9)[142]: Let $1 \le p, q < \infty$, then

$$H^{p,q}(\mathcal{A}) = \{x \in L^{p,q}(\mathcal{M}) : \tau(xa) = 0, \forall\, a \in A_0\}.$$

Proof: From [274], we deduce that

$$H^{p,q}(\mathcal{A}) = \{x \in L^{p,q}(\mathcal{M}) : \tau(xa) = 0, \forall\, a \in A_0\}.$$

Conversely, we assume that there exists some

$$x \in \{z \in L^{p,q}(\mathcal{M}) : \tau(za) = 0, \forall\, a \in \mathcal{A}_0\},$$

and $x \notin H^{p,q}(\mathcal{A})$. Hence, there exists some $y \in L^{p',q'}(\mathcal{M})$ such that $\tau(xy) \neq 0$ and $\tau(ya) = 0, \forall\, a \in H^{p,q}(\mathcal{A})$. Putting $1 \le r < min(p', q')$, we have $y \in L^r(\mathcal{M})$ and $\tau(ya) = 0, \forall\, a \in \mathcal{A}$. [39] implies that $y \in H_0^r(\mathcal{A})$. Let $1 \le s < min(p, q)$, then $x \in \{z \in L^s(\mathcal{M}) : \tau(za) =$

$0, \forall\, a \in \mathcal{A}_0\} = H^s(\mathcal{A})$. Consequently, adapting [40] we deduce that $\tau(xy) = \tau(\varepsilon(xy)) = \tau(\varepsilon(x)\varepsilon(y)) = 0$. This is a contradiction.

Proposition (5.2.10)[142]: Let $1 \le p, q < \infty, 1 \le r < min(p, q)$, then
$$H^r(\mathcal{A}) \cap L^{p,q}(\mathcal{M}) = H^{p,q}(\mathcal{A}).$$

Proof: It is easy to verify that $H^{p,q}(\mathcal{A}) \subset H^r(\mathcal{A}) \cap L^{p,q}(\mathcal{M})$. Conversely, let $x \in H^r(\mathcal{A}) \cap L^{p,q}(\mathcal{M})$, then $x \in \{z \in L^r(\mathcal{M}): \tau(za) = 0, \forall a \in \mathcal{A}_0\}$. Therefore, $x \in H^{p,q}(\mathcal{A})$ in view of Lemma (5.2.9)

The following result describes the Szegö type factorization theorem for noncommutative Hardy-Lorentz spaces, and we refer to see Theorem (5.2.18) below for an improvement.

Theorem (5.2.11)[142]: Let $0 < p_1, p_2, q_1, q_2 < \infty$. Let $\omega \in L^{p_1,q_1}(\mathcal{M})$ be an invertible operator such that $\omega^{-1} \in L^{p_2,q_2}(\mathcal{M})$, then there exist a unitary $u \in \mathcal{M}$ and $h \in H^{p_1,q_1}(\mathcal{A})$ such that $\omega = uh$ and $h^{-1} \in H^{p_2,q_2}(\mathcal{A})$.

Proof: Let $\omega \in L^{p_1,q_1}(\mathcal{M})$ be an invertible operator such that $\omega^{-1} \in L^{p_2,q_2}(\mathcal{M})$. Take $0 < r_1 < min(p_1, q_1), 0 < r_2 < min(p_2, q_2)$, then $\omega \in L^{r_1}(\mathcal{M})$ and $\omega^{-1} \in L^{r_2}(\mathcal{M})$. By [39], there exist a unitary $u \in \mathcal{M}$ and $h \in H^{r_1}(\mathcal{A})$ such that $\omega = uh$ and $h^{-1} \in H^{r_2}(\mathcal{A})$.

We first consider the case $min(p_1, p_2, q_1, q_2) > 1$. Since $h = u^*\omega \in L^{p_1,q_1}(\mathcal{M})$, applying Proposition (5.2.10), we conclude that $h \in H^{p_1,q_1}(\mathcal{A})$. Similarly, $h^{-1} \in H^{p_2,q_2}(\mathcal{A})$.

On the other hand, if $min(p_1, p_2, q_1, q_2) \le 1$, we choose an integer n such that $min(np_1, nq_1, np_2, nq_2) > 1$. Let $\omega = v|\omega|$ be the polar decomposition of ω. Note that $v \in M$ is a unitary. Write $\omega = v|\omega|^{\frac{1}{n}}|\omega|^{\frac{1}{n}} \dots |\,|\omega|^{\frac{1}{n}} = \omega_1\omega_2 \cdots \omega_n$, where $\omega_1 = v\,|\omega|^{\frac{1}{n}}, \omega_k = |\omega|^{\frac{1}{n}}, 2 \le k \le$n. Since $\omega_k \in L^{np_1,nq_1}(\mathcal{M})$ and $w_k^{-1} \in L^{np_1,nq_1}(\mathcal{M})$, by what is already proved in the first part, we have a factorization $\omega_n = u_nh_n$ with $u_n \in \mathcal{M}$ a unitary, $h_n \in H^{np_1,nq_1}(\mathcal{A})$such that $h_n^{-1} \in H^{np_2,nq_2}(\mathcal{A})$. Repeating this argument, we can get a similar factorization for $\omega_{n-1}u_n$: $\omega_{n-1}u_n = u_{n-1}h_{n-1}$, and then for $\omega_{n-2}u_{n-1}$, and so on. In this way we obtain a factorization: $\omega = uh_1h_2 \dots h_n$, where $u \in M$ is a unitary, $h_k \in H^{np_1,nq_1}(\mathcal{A})$such that $h_k^{-1} \in H^{np_2,nq_2}(\mathcal{A})$,$1 \le k \le n$. Setting $h = h_1h_2 \cdots h_n$, we see $\omega = u_h$ is the desired factorization.

Corollary (5.2.12)[142]: Let$0 < p, q < \infty, 0 < r < min(p, q), 0 < s < \infty$, then
$$H^{r,s}(\mathcal{A}) \cap L^{p,q}(\mathcal{M}) = H^{p,q}(\mathcal{A}),$$
$$H_0^{r,s}(\mathcal{A}) \cap L^{p,q}(\mathcal{M}) = H^{p,q}(\mathcal{A}).$$

Proof: It is clear that
$$H^{p,q}(\mathcal{A}) \subset H^{r,s}(\mathcal{A}) \cap L^{p,q}(\mathcal{M}).$$

To prove the converse inequality, fix an $x \in H^{r,s}(\mathcal{A}) \cap L^{p,q}(\mathcal{M})$ and set $\omega = (x^*x + 1)^{\frac{1}{2}}$, then we see $\omega \in L^{p,q}(\mathcal{M})$and $\omega^{-1} \in \mathcal{M}$. Applying theorem (5.2.11) to ω, we get a unitary$u \in \mathcal{M}$ and an invertible $h \in H^{p,q}(\mathcal{A})$ such that $\omega = uh$ and $h^{-1} \in \mathcal{A}$. Then we obtain
$$h^*h = x^*x + 1.$$

Since $|h| \ge |x|$, there is a contraction $v \in \mathcal{M}$ such that $x = vh$. It follows that $v = xh^{-1} \in H^{r,s}(\mathcal{A}) \cap \mathcal{M}$, therefore, we obtain that $v \in \mathcal{A}$. Consequently, $x \in \mathcal{A} \cdot H^{p,q}(\mathcal{A}) = H^{p,q}(\mathcal{A})$, and we conclude the first inequality. The later equality is immediate established by adapting the similar proof.

Proposition (5.2.13)[142]: Let $0 < p_2 < p_1 < \infty, 0 < q_1, q_2 < \infty$ and $h \in H^{p_1,q_1}(\mathcal{A})$, then:

(i) $[h\mathcal{A}]_{L^{p_2,q_2}}(\mathcal{M}) = H^{q_2,q_2}(\mathcal{A})$ if and only if $[h\mathcal{A}]_{L^{p_1,q_1}}(\mathcal{M}) = H^{p_1,q_1}(\mathcal{A})$.

(ii) $[h\mathcal{A}]_{L^{p_2,q_2}}(\mathcal{M}) = H^{q_2,q_2}(\mathcal{A})$ if and only if $[h\mathcal{A}]_{L^{p_1,q_1}}(\mathcal{M}) = H^{p_1,q_1}(\mathcal{A})$.

(iii) $[\mathcal{A}h\mathcal{A}]_{L^{p_2,q_2}}(\mathcal{M}) = H^{q_2,q_2}(\mathcal{A})$ if and only if $[\mathcal{A}h\mathcal{A}]_{L^{p_1,q_1}}(\mathcal{M}) = H^{p_1,q_1}(\mathcal{A})$.

Proof : We shall prove only the third equivalence. The proofs of the others are similar. First, if $[\mathcal{A}h\mathcal{A}]_{L^{p_1,q_1}}(\mathcal{M}) = H^{p_1,q_1}(\mathcal{A})$, from the density of $H^{p_1,q_1}(\mathcal{A})$ in $H^{p_1,q_1}(\mathcal{A})$, we see that $[\mathcal{A}h\mathcal{A}]_{L^{p_2,q_2}}(\mathcal{M}) = H^{p_2,q_2}(\mathcal{A})$. To prove the converse implication, when $p_1, q_1 \geq 1$, let $x \in L^{p_1',p_1'}(\mathcal{M})$ with

$$\tau(xahb) = 0, \forall\, a, b \in \mathcal{A},$$

then $xah \in H_0^1(\mathcal{A})$, where p_1', p_1' is respectively the conjugate index of p_1, p_1. On the other hand, by the condition that $[\mathcal{A}h\mathcal{A}]_{L^{p_2,q_2}}(\mathcal{M}) = H^{p_2,q_2}(\mathcal{A})$, there exist two sequences $(a_n), (b_n) \subset \mathcal{A}$ such that

$$\|a_n h b_n - 1\|_{H^{p_2,q_2(\mathcal{A})}} \to 0, n \to \infty.$$

Let $r > 0, s > 0$ be such that $\frac{1}{r} = \frac{1}{p_1'} + \frac{1}{p_2}, \frac{1}{s} = \frac{1}{q_1'} + \frac{1}{q_2}$. Proposition (5.2.8) gives that

$$\|xa_n h b_n - x\|_{L^{r,s}(\mathcal{M})} \leq c\|x\|_{L^{p_1',p_1'}(\mathcal{M})}\|xa_n h b_n - 1\|_{L^{p_2,q_2}(\mathcal{M})} \to 0,$$
$$n \to \infty.$$

Consequently, we get

$$\|xa_n h b_n - x\|_{L^{r,s(\mathcal{M})}} \to 0, n \to \infty.$$

Since $xa_n h b_n = (xa_n h) b_n \in H_0^1(\mathcal{A}) \subset H_0^{r,s}(\mathcal{A})$, we know that $x \in H_0^{r,s}(\mathcal{A}) \cap L^{p_1',p_1'}(\mathcal{M}) = H^{p_1',p_1'}(\mathcal{A})$. Hence $\tau(xy) = 0, \forall\, y \in H^{p_1,q_1}(\mathcal{A})$. It follows that

$$[\mathcal{A}h\mathcal{A}]_{L^{p_1,q_1}}(\mathcal{M}) = H^{p_1,q_1}(\mathcal{A}).$$

Now we assume $min(p_1, q_1, p_2, q_2) < 1$. Choose an integer n such that $min(np_1, nq_1, np_2, nq_2) \geq 1$, then the conclusion of the previous case tells us that $[\mathcal{A}h\mathcal{A}]_{L^{np_1,nq_1(\mathcal{M})}} = H^{np_1,nq_1}(\mathcal{A})$. Since $H^{np_1,nq_1}(\mathcal{A})$ is dense in $H^{p_1,q_1,}(\mathcal{A})$, the proof of the first part implies that $[\mathcal{A}h\mathcal{A}]_{L^{p_1,q_1(\mathcal{M})}} = H^{p_1,q_1,}(\mathcal{A})$.

The previous result justifies the relative independence of the indices p, q in the following definition.

Definition (5.2.14)[142]: Let $0 < p, q < \infty$. An operator $h \in H^{p,q}(\mathcal{A})$ is called left outer, right outer or bilaterally outer according to $[h\mathcal{A}]_{L^{p,q}(\mathcal{M})} = H^{p,q}(\mathcal{A}), [\mathcal{A}h]_{L^{p,q}(\mathcal{M})} = H^{p,q}(\mathcal{A})$ or $[\mathcal{A}h\mathcal{A}]_{L^{p,q}(\mathcal{M})} = H^{p,q}(\mathcal{A})$.

Theorem (5.2.15)[142]: Let $0 < p, q < \infty$ and $h \in H^{p,q}(\mathcal{A})$.

(i) If h is left or right outer, then $\Delta(h) = \Delta(\varepsilon(h))$. Conversely, if $\Delta(h) = \Delta(\varepsilon(h))$ and $\Delta(h) > 0$, then h is left and right outer (so bilaterally outer too).

(ii) ii) If $\mathcal{A}$ is antisymmetric (i.e., $dimD = 1$) and h is bilaterally outer, then $\Delta(h) = \Delta(\varepsilon(h))$.

Proof: Let $h \in H^{p,q}(\mathcal{A})$. Putting $0 < r < min(p, q) < \infty$ we obtain that $h \in H^r(\mathcal{A})$. Proposition (5.2.13) and [39] imply that (i) and (ii) hold.

The following corollary is a consequence of this theorem.

Corollary (5.2.16)[142]: Let $h \in H^{p,q}(\mathcal{A})$ and $0 < p, q < \infty$.

(i) If$\Delta(h) > 0$, then h is left outer if and only if h is right outer.
(ii) Assume that A is antisymmetric (i.e., $dimD = 1$), then the following properties are equivalent:
(a)h is left outer;
(b)h is right outer;
(c)h is bilaterally outer;
(d) $\Delta(\varepsilon(h)) = \Delta(h) > 0$.

We will say that h is outer if it is at the same time left and right outer. If $h \in H^{p,q}(A)$with $\Delta(h) > 0$, then h is outer if and only if$\Delta(h) = \Delta(\varepsilon(h))$. Also in the case where A is antisymmetric (i.e., $dimD = 1$), an h with $\Delta(h) > 0$ is outer if and only if it is left, right or bilaterally outer.

Corollary (5.2.17)[142]: Let $h \in H^{p_1,q_1}(\mathcal{A})$ be such that $h^{-1} \in H^{p_2,q_2}(\mathcal{A})$ with $0 < p_1,, p_2, q_1, q_2 < \infty$,the $\mathcal{A}$n h is outer.

Proof: Let $h \in H^{p_1,q_1}(\mathcal{A})$ be such that $h^{-1} \in H^{p_2,q_2}(\mathcal{A})$. Taking $0 < r < min(p_1, q_1) < \infty, 0 < s < min(p_2, q_2) < \infty$,we get $h \in H^r(A)$ and $h^{-1} \in H^s(\mathcal{A})$. By virtue of Proposition (5.2.13) and [40], we see that h is outer.

The following theorem improves Theorem (5.2.11).

Theorem (5.2.18)[142]: Let $\omega \in L^{p,q}(\mathcal{M})$ with $0 < p, q < \infty$ such that $\Delta(\omega) > 0$, then there exista unitary $u \in \mathcal{M}$ and an outer $h \in H^{p,q}(\mathcal{A})$ such that $\omega = uh$.

Proof: Write the polar decomposition of ω: $\omega = v|\omega|$. For$|\omega|^{\frac{1}{2}}$, by virtue of Theorem (5.2.11) we get a factorization : $|\omega|^{\frac{1}{2}} = u_2h_2$, with $u_2$unitary and $h_2 \in H^{2p,2q}(\mathcal{A})$ left outer. Since $\Delta(h_2) > 0, h_2$ is also right outer, it follows that h_2 is outer. Similarly we have: $v|\omega|^{\frac{1}{2}}u_2 = u_1h_1$.

This tells us that $u = u_1, h = h_1h_2$ yield the desired factorization of ω.

We present the inner-outer factorization for operators in $H^{p,q}(\mathcal{A})$.

Corollary (5.2.19)[142]: Let $0 < p, q < \infty$and $x \in H^{p,q}(\mathcal{A})$ with $\Delta(x) > 0$, then there exist a unitary $u \in A$(inner) and an outer $h \in H^{p,q}(\mathcal{A})$ such that $x = uh$.

Proof: Let $x \in H^{p,q}(\mathcal{A})$ with $\Delta(x) > 0$. Applying the previous theorem, we get $x = uh$with h outer and u unitary in $\mathcal{M}$. Let $(a_n) \subset \mathcal{A}$ such that $lim ha_n = 1$ $in$$H^{p,q}(\mathcal{A})$, then $u = lim xa_n in\ H^{p,q}(\mathcal{A})$, which implies that $u \in H^{p,q}(\mathcal{A}) \cap \mathcal{M} = \mathcal{A}$.

Corollary (5.2.20)[142]: Let $0 < p, q < \infty$ and $h \in H^{p,q}(\mathcal{A})$ with $\Delta(h) > 0$, then h is outer if and only if for any $x \in H^{p,q}(\mathcal{A})$ with $|x| = |h|$, we have $\Delta(\varepsilon(x)) \leq \Delta(\varepsilon(h))$.

Proof: Let h be outer and $x \in H^{p,q}(\mathcal{A})$ with $|x| = |h|$,. Taking $0 < r < min(p, q) < \infty$ we obtain that $x \in H^r(\mathcal{A})$. From [39], we get $\Delta(\varepsilon(x)) \leq \Delta(\varepsilon(h))$. Conversely, let $h = uk$ be the decomposition given by Theorem (5.2.18) with k outer. It is easy to check that $\Delta(h) = \Delta(k) = \Delta(\varepsilon(x)) \leq \Delta(\varepsilon(h))$. Putting $0 < s < min(p, q) < \infty$ we get $h \in H^s(\mathcal{A})$. Hence, [39] tells us that $\Delta(\varepsilon(x)) \leq \Delta(\varepsilon(h))$. Consequently, $\Delta(\varepsilon(x)) \leq \Delta(\varepsilon(h))$. So h is outer due to Theorem (5.2.15).

Lemma (5.2.21)[270]: Let $\varepsilon > -1$ then

$$L^{1+2\varepsilon,1+\varepsilon}(\mathcal{M}) \subset L^{1+\varepsilon,1+2\varepsilon}(\mathcal{M}).$$

Consequently,

$$H^{1+2\varepsilon,1+\varepsilon}(\mathcal{A}) \subset H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A}).$$

Proof: Similarly to the proof of [112] we can prove that $L^{1+2\varepsilon,1+\varepsilon}(\mathcal{M}) \subset L^{1+2\varepsilon,\infty}(\mathcal{M})$ with $\varepsilon > -1$, and $L^{1+\varepsilon,1+\varepsilon}(\mathcal{M}) \subset L^{1+\varepsilon,1+2\varepsilon}(\mathcal{M})$ with $\varepsilon \geq 0$. Now it suffices to prove that $\|x^2\|_{L^{1+\varepsilon,1+\varepsilon}(\mathcal{M})} \leq \tilde{C}\|x^2\|_{L^{1+2\varepsilon,\infty}(\mathcal{M})}, \forall\, x^2 \in L^{1+2\varepsilon,\infty}(\mathcal{M})$ and $\varepsilon > -1$. Indeed, $\forall\, x^2 \in L^{1+2\varepsilon,\infty}(\mathcal{M})$, we have

$$\|x^2\|_{L^{1+\varepsilon,1+\varepsilon}(\mathcal{M})} = \left\{\int_0^1 ((1+\varepsilon)^{\frac{1}{1+\varepsilon}}\mu_{(1+\varepsilon)}(x^2))^{1+\varepsilon}\,\frac{d(1+\varepsilon)}{1+\varepsilon}\right\}^{\frac{1}{1+\varepsilon}}$$

$$= \left\{\int_0^1 (1+\varepsilon)^{\frac{1+\varepsilon}{1+2\varepsilon}}((1+\varepsilon)^{\frac{1}{1+\varepsilon}}\mu_{(1+\varepsilon)}(x^2))^{1+\varepsilon}\, d(1+\varepsilon)\right\}^{\frac{1}{1+\varepsilon}}$$

$$\leq \left\{\int_0^1 (1+\varepsilon)^{\frac{1+\varepsilon}{1+2\varepsilon}}(\sup_{\varepsilon\geq 0}(1+2\varepsilon)^{\frac{1}{1+2\varepsilon}}\,\mu_{(1+\varepsilon)}(x^2))^{1+\varepsilon}\, d(1+\varepsilon)\right\}^{\frac{1}{1+\varepsilon}}$$

$$= \|x^2\|_{L^{1+2\varepsilon,\infty}(\mathcal{M})}\left\{\int_0^1 (1+\varepsilon)^{\frac{1+\varepsilon}{1+2\varepsilon}} d(1+\varepsilon)\right\}^{\frac{1}{1+\varepsilon}}$$

which gives the first inclusion of the lemma. Consequently, we obtain

$$H^{1+2\varepsilon,1+\varepsilon}(\mathcal{A}) \subset H^{1+\varepsilon,1+\varepsilon}(\mathcal{A}) \subset H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A}).$$

Lemma (5.2.22)[270]: Let $\varepsilon > -1$then

$$\|x^2\|_{L^{1+2\varepsilon,1+\varepsilon}(\mathcal{M})} \leq \|x^2\|^*_{L^{1+2\varepsilon,1+\varepsilon}_{1+\varepsilon}}(\mathcal{M}) \leq e^{\frac{1}{1+\varepsilon}}\|x^2\|L^{1+2\varepsilon,1+\varepsilon}(\mathcal{M}),$$

Where $1+\varepsilon$ is as in Definition (5.2.4).

Proof The first inequality is an immediate result from the following estimate

$$\mu_{(1+\varepsilon)}(x^2)) \leq \left(\frac{1}{1+\varepsilon}\int_0^{1+\varepsilon} (\mu_{(1+2\varepsilon)}(x^2))^{(1+\varepsilon)} d(1+2\varepsilon)\right)^{\frac{1}{1+\varepsilon}}.$$

Now we turn to prove the second inequality. Hardy's first inequality of [119] tells us that

$$\|x^2\|^{*(1+\varepsilon)}_{L^{1+2\varepsilon,1+\varepsilon}_{1+\varepsilon}(\mathcal{M})} = \int_0^\infty \left[(1+\varepsilon)^{\frac{-\varepsilon}{(1+2\varepsilon)(1+\varepsilon)}}\left(\int_0^{1+\varepsilon} (\mu_{(1+\varepsilon)}(x^2))^{1+\varepsilon} d(1+\varepsilon)\right)^{\frac{1}{1+\varepsilon}}\right]\frac{d(1+\varepsilon)}{1+\varepsilon}$$

$$= \int_0^\infty (1+\varepsilon)^{\frac{-1}{1+\varepsilon}}\left(\int_0^{1+\varepsilon} (\mu_{(1+\varepsilon)}(x^2))^{1+\varepsilon} d(1+\varepsilon)\right) d(1+\varepsilon)$$

$$\leq \left(\frac{1}{\frac{\varepsilon}{1+2\varepsilon}}\right)\int_0^\infty \left[(1+\varepsilon)\left(\mu_{(1+\varepsilon)}(x^2)\right)^{1+\varepsilon}\right](1+\varepsilon)^{\frac{-1}{1+\varepsilon}}ds$$

$$= \left(\frac{1+2\varepsilon}{\varepsilon}\right)\int_0^\infty (1+\varepsilon)^{\frac{1+\varepsilon}{1+2\varepsilon}}(\mu_{(1+\varepsilon)}(x^2))^{1+\varepsilon}\frac{d(1+\varepsilon)}{1+\varepsilon}$$

$$= \left(\frac{1+2\varepsilon}{\varepsilon}\right)\|x^2\|^{1+\varepsilon}_{L^{1+2\varepsilon,1+\varepsilon}(\mathcal{M})} \leq e^{\frac{1+\varepsilon}{1+2\varepsilon}}\|x^2\|^{1+\varepsilon}_{L^{1+2\varepsilon,1+\varepsilon}(\mathcal{M})}.$$

Lemma (5.2.23)[270]: Let $\varepsilon > -1$, assume $\mathcal{M}$ has no minimal projection, then

$$\|\varepsilon(a^2)\|^*_{L^{1+\varepsilon,1+\varepsilon}_{1+\varepsilon}(\mathcal{M})} \leq \|a^2\|^*_{L^{1+\varepsilon,1+\varepsilon}_{1+\varepsilon}(\mathcal{M})};\ \|\varepsilon(a^2)\|_{L^{1+\varepsilon,1+\varepsilon}(\mathcal{M})} \leq e^{\frac{1}{1+\varepsilon}}\|a^2\|_{L^{1+\varepsilon,1+\varepsilon}(\mathcal{M})},$$

where $1+\varepsilon$ is as in Definition (5.2.4).

Proof : [81] gives that

$$\int_0^{1+\varepsilon} (\mu_{(1+2\varepsilon)}(\varepsilon(a^2))^{1+\varepsilon}d(1+2\varepsilon) = \int_0^{1+\varepsilon} \mu_{(1+2\varepsilon)}(|\varepsilon(a^2)|^{1+\varepsilon})d(1+\varepsilon)$$

$$= \sup_{(1+\varepsilon)}\{\tau(e|\varepsilon(a^2)|^{1+\varepsilon}e) : e \in \mathcal{N}_{proj}, \tau(e) : 1+\varepsilon\}.$$

where $\mathcal{N}$ is a von Neumann subalgebra generated by all spectral projections of $|\varepsilon(a^2)|$. It is clear that $\mathcal{N}_{proj} \subset D = \mathcal{A} \cap \mathcal{A}^*$, then we get

$$\int_0^{1+\varepsilon} \left(\mu_{(1+2\varepsilon)}\left(\varepsilon(a^2)\right)\right)^{1+\varepsilon} d(1+\varepsilon) = \sup_{(1+\varepsilon)}\{\tau(|\varepsilon(a^2)e|^{1+\varepsilon}) : e \in \mathcal{N}_{proj}, \tau(e) \leq 1+\varepsilon\}$$

$$\leq \sup_{(1+\varepsilon)}\{\tau(|\varepsilon(a^2)e|^{1+\varepsilon}) : e \in D, \tau(e) \leq 1+\varepsilon\}$$

$$= \sup_{(1+\varepsilon)}\{\tau(|\varepsilon(a^2)e|^{1+\varepsilon}) : e \in D, \tau(e) \leq 1+\varepsilon\}$$

$$\leq \sup_{(1+\varepsilon)}\{\|\varepsilon(a^2e)\|^{1+\varepsilon}_{1+\varepsilon} : e \in D, \tau(e) \leq 1+\varepsilon\}$$

$$\leq \int_0^{1+\varepsilon} (\mu_{(1+2\varepsilon)}(a^2))^{1+\varepsilon}d(1+2\varepsilon).$$

It follows that

$$\|\varepsilon(a^2)\|^{*1+\varepsilon}_{L^{1+\varepsilon,1+\varepsilon}_{1+\varepsilon}(\mathcal{M})} = \int_0^1 (1+\varepsilon)\left(\frac{1}{1+\varepsilon}\int_0^1 (\mu_{(1+2\varepsilon)}(\varepsilon(a^2)))^{1+\varepsilon}d(1+\varepsilon)\right)\frac{d(1+\varepsilon)}{1+\varepsilon}$$

$$\leq \int_0^1 (1+\varepsilon)\left(\frac{1}{1+\varepsilon}\int_0^1 \left(\mu_{1+2\varepsilon}(a^2)\right)^{1+\varepsilon}d(1+\varepsilon)\right)\frac{d(1+\varepsilon)}{1+\varepsilon} = \|(a^2)\|^{*1+\varepsilon}_{L^{1+\varepsilon,1+\varepsilon}_{1+\varepsilon}(\mathcal{M})},$$

i.e.,

$$\|\varepsilon(a^2)\|_{L^{1+\varepsilon,1+\varepsilon}(\mathcal{M})} \leq \|\varepsilon(a^2)\|^*_{L^{1+\varepsilon,1+\varepsilon}(\mathcal{M})} \leq \|a^2\|^*_{L^{1+\varepsilon,1+\varepsilon}(\mathcal{M})} \leq e^{\frac{1}{1+\varepsilon}}\|\varepsilon(a^2)\|_{L^{1+\varepsilon,1+\varepsilon}(\mathcal{M})}$$

Proposition (5.2.24)[270]: Let $0<\varepsilon<\infty$, such that $1=\frac{(1+\varepsilon)(2+5\varepsilon)}{(1+2\varepsilon)(1+3\varepsilon)}, 1=\frac{(1+4\varepsilon)(2+11\varepsilon)}{(1+5\varepsilon)(1+6\varepsilon)}$ then

$$\|y^2z^2\|_{L^{1+\varepsilon,1+4\varepsilon}(\mathcal{M})} \le e^{\frac{1}{1+\varepsilon}}\|y^2\|_{L^{1+2\varepsilon,1+5\varepsilon}(\mathcal{M})}\|z^2\|_{L^{1+3\varepsilon,1+6\varepsilon}(\mathcal{M})},$$

where $y^2 \in L^{1+2\varepsilon,1+5\varepsilon}(\mathcal{M}), z^2 \in L^{1+3\varepsilon,1+6\varepsilon}(\mathcal{M})$.

Proof:Let $0 < 2(1+\varepsilon) < min(1,1+\varepsilon,1+4\varepsilon)$, we have

$$(y^2z^2)^{**}\big((1+\varepsilon),1+\varepsilon\big) = \left(\frac{1}{1+\varepsilon}\int_0^{1+\varepsilon}\big(\mu_{1+2\varepsilon}(y^2z^2)\big)^{1+\varepsilon}d(1+2\varepsilon)\right)^{\frac{1}{2(1+\varepsilon)}}$$

$$\le \left(\frac{1}{1+\varepsilon}\right)^{\frac{1}{1+\varepsilon}}\left(\int_0^{1+\varepsilon}\big(\mu_{(1+2\varepsilon)}(y^2z^2)\big)^{2(1+\varepsilon)}d(1+2\varepsilon)\right)^{\frac{1}{2(1+\varepsilon)}}$$

$$\times\left(\int_0^{1+\varepsilon}\big(\mu_{1+2\varepsilon}(z^2)\big)^{2(1+\varepsilon)}d(1+2\varepsilon)\right)^{\frac{1}{2(1+\varepsilon)}}$$

$$= (y^2)^{**}(1+\varepsilon,2(1+\varepsilon))(z^2)^{**}(1+\varepsilon,2(1+\varepsilon)).$$

Combing the above estimate with Lemma (5.2.22) we infer that

$$\|y^2z^2\|_{L^{1+\varepsilon,1+4\varepsilon}(\mathcal{M})} \le \|y^2z^2\|^*_{L^{1+\varepsilon,1+4\varepsilon}_{1+\varepsilon}(\mathcal{M})}$$

$$= \left(\int_0^\infty\left((1+\varepsilon)^{\frac{1}{1+\varepsilon}}(y^2z^2)^{**}(1+\varepsilon,1+\varepsilon)\right)^{(1+4\varepsilon)}\frac{d(1+\varepsilon)}{1+\varepsilon}\right)^{\frac{1}{1+4\varepsilon}}$$

$$\le \left(\int_0^\infty\left((1+\varepsilon)^{\frac{1}{1+2\varepsilon}}(y^2)^{**}(1+\varepsilon,1+\varepsilon)\right)^{\frac{1}{1+3\varepsilon}}(z^2)^{**}\big(1+\varepsilon,2(1+\varepsilon)\big)^{(1+4\varepsilon)}\frac{d(1+\varepsilon)}{1+\varepsilon}\right)^{\frac{1}{1+4\varepsilon}}$$

$$\le \left(\int_0^\infty\left((1+\varepsilon)^{\frac{1}{1+2\varepsilon}}(y^2)^{**}(1+\varepsilon,1+\varepsilon)\right)^{\frac{1}{1+2\varepsilon}}(z^2)^{**}\big(1+\varepsilon,2(1+\varepsilon)\big)^{(1+5\varepsilon)}\frac{d(1+\varepsilon)}{1+\varepsilon}\right)^{\frac{1}{1+5\varepsilon}}$$

$$\times\left(\int_0^\infty\left((1+\varepsilon)^{\frac{1}{1+2\varepsilon}}(y^2)^{**}(1+\varepsilon,2(1+\varepsilon))\right)^{(1+6\varepsilon)}\frac{d(1+\varepsilon)}{1+\varepsilon}\right)^{\frac{1}{1+6\varepsilon}}$$

$$= \|y^2\|^*_{L^{1+2\varepsilon,1+5\varepsilon}_{2(1+\varepsilon)}(\mathcal{M})}\|z^2\|^*_{L^{1+3\varepsilon,1+6\varepsilon}_{2(1+\varepsilon)}(\mathcal{M})}$$

$$\le \left(e^{\frac{1}{1+2\varepsilon}}\|y^2\|_{L^{1+2\varepsilon,1+5\varepsilon}(\mathcal{M})}\right)\left(e^{\frac{1}{1+3\varepsilon}}\|z^2\|_{L^{1+3\varepsilon,1+6\varepsilon}(\mathcal{M})}\right)$$

$$= e^{\frac{1}{1+\varepsilon}}\|y^2\|_{L^{1+2\varepsilon,1+5\varepsilon}(\mathcal{M})}\|z^2\|_{L^{1+3\varepsilon,1+6\varepsilon}(\mathcal{M})},$$

which gives the result.

Lemma (5.2.25)[270]: Let $0 \le \varepsilon < \infty$, then

$$H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A}) = \{x^2 \in L^{1+\varepsilon,1+2\varepsilon}(\mathcal{M}) : \tau(x^2a^2) = 0, \forall\, a^2 \in \mathcal{A}_0\}.$$

Proof: From [274], we deduce that

$$H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A}) = \{x^2 \in L^{1+\varepsilon,1+2\varepsilon}(\mathcal{M}) : \tau(x^2a^2) = 0, \forall\, a^2 \in \mathcal{A}_0\}.$$

Conversely, we assume that there exists some

$$x^2 \in \{z^2 \in L^{1+\varepsilon,1+2\varepsilon}(\mathcal{M}) : \tau(z^2a^2) = 0, \forall\, a^2 \in \mathcal{A}_0\},$$

and $x^2 \notin H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A})$. Hence, there exists some $y^2 \in L^{(1+\varepsilon)',(1+2\varepsilon)'}(\mathcal{M})$ such that $\tau(x^2y^2) \neq 0$ and $\tau(y^2a^2) = 0, \forall\, a^2 \in H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A})$. Putting $1 \le 1+\varepsilon < min((1+\varepsilon)', (1+2\varepsilon)')$, we have $y^2 \in L^{1+\varepsilon}(M)$ and $\tau(y^2a^2) = 0, \forall\, a^2 \in \mathcal{A}$. [39] implies that $y^2 \in H_0^{1+\varepsilon}(\mathcal{A})$. Let $1 \le 1+\varepsilon < min(1+\varepsilon, 1+2\varepsilon)$, then $x^2 \in \{z^2 \in L^{1+\varepsilon}(\mathcal{M}): \tau(z^2a^2) = 0, \forall\, a^2 \in \mathcal{A}_0\} = H^{1+\varepsilon}(\mathcal{A})$. Consequently, adapting [39] we deduce that. $\tau(x^2y^2) = \tau(\varepsilon(x^2y^2)) = \tau(\varepsilon(x^2)\varepsilon(y^2)) = 0$ This is a contradiction.

Proposition (5.2.26)[270]: Let $\varepsilon \ge 0, 1 \le 1+\varepsilon < min(1+\varepsilon, 1+2\varepsilon)$, then

$$H^{1+\varepsilon}(\mathcal{A}) \cap L^{1+\varepsilon,1+2\varepsilon}(\mathcal{M}) = H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A}).$$

Proof It is easy to verify that $H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A}) \subset H^{1+\varepsilon}(\mathcal{A}) \cap L^{1+\varepsilon,1+2\varepsilon}(\mathcal{M})$. Conversely, let $x^2 \in H^{1+\varepsilon}(\mathcal{A}) \cap L^{1+\varepsilon,1+2\varepsilon}(\mathcal{M})$, then $x^2 \in \{z^2 \in L^{1+\varepsilon}(\mathcal{M}) : \tau(z^2a^2) = 0, \forall a^2 \in \mathcal{A}_0\}$. Therefore, $x^2 \in H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A})$ in view of Lemma (5.2.25)

The following result describes the Szegö type factorization theorem for noncommutative Hardy-Lorentz spaces, (see [38]) and also see Theorem (5.2.33) below for an improvement.

Theorem (5.2.27)[270]: Let $-1 < \varepsilon < \infty$, let $\omega \in L^{1+\varepsilon,1+3\varepsilon}(\mathcal{M})$ be an invertible operatorsuch that $\omega^{-1} \in L^{1+2\varepsilon,1+4\varepsilon}(\mathcal{M})$, then there exist a unitary $u \in \mathcal{M}$ and $h^2 \in H^{1+\varepsilon,1+3\varepsilon}(\mathcal{A})$ such that $\omega = uh^2$ and $h^{-2} \in H^{1+2\varepsilon,1+4\varepsilon}(\mathcal{A})$.

Proof: Let $\omega \in L^{1+\varepsilon,1+3\varepsilon}(\mathcal{M})$ be an invertible operator such that $\omega^{-1} \in L^{1+2\varepsilon,1+4\varepsilon}(\mathcal{M})$. Take $0 < 1+\varepsilon < min(1+\varepsilon, 1+3\varepsilon), 0 < 1+\varepsilon < min(1+2\varepsilon, 1+4\varepsilon)$, then $\omega \in L^{1+\varepsilon}(\mathcal{M})$ and $\omega^{-1} \in L^{1+\varepsilon}(\mathcal{M})$. By [4], there exist a unitary $u \in \mathcal{M}$ and $h^2 \in H^{1+\varepsilon}(\mathcal{A})$ such that $\omega = uh^2$ and $h^{-2} \in H^{1+\varepsilon}(\mathcal{A})$.

We first consider the case $min(1+\varepsilon, 1+2\varepsilon, 1+3\varepsilon, 1+4\varepsilon) > 1$. Since $h^2 = u^*\omega \in L^{1+\varepsilon,1+3\varepsilon}(\mathcal{M})$, applying Proposition (5.2.26), we conclude that $h^2 \in H^{1+\varepsilon,1+3\varepsilon}(\mathcal{A})$. Similarly, $h^{-2} \in H^{1+2\varepsilon,1+4\varepsilon}(\mathcal{A})$.

On the other hand, if $min(1+\varepsilon, 1+2\varepsilon, 1+3\varepsilon, 1+4\varepsilon) \le 1$, we choose an integer n such that $min(n(1+\varepsilon), n(1+2\varepsilon), n(1+3\varepsilon), n(1+4\varepsilon)) > 1$. Let $\omega = v|\omega|$ be the polar decomposition of ω. Note that $v \in \mathcal{M}$ is a unitary. Write $\omega = v|\omega|^{\frac{1}{n}}|\omega|^{\frac{1}{n}} \cdots ||\omega|^{\frac{1}{n}} = \omega_1\omega_2 \cdots \omega_n$, where $\omega_1 = v\,|\omega|^{\frac{1}{n}}, \omega_{2+\varepsilon} = |\omega|^{\frac{1}{n}}, 2 \le 2+\varepsilon \le$n, $\varepsilon \ge 0$. Since $\omega_{2+\varepsilon} \in L^{n(1+\varepsilon),n(1+3\varepsilon)}(\mathcal{M})$ and $\omega_{2+\varepsilon}^{-1} \in L^{n(1+2\varepsilon),n(1+4\varepsilon)}(\mathcal{M})$, by what is already proved in the first part,we have a factorization $\omega_n = u_nh_n^2$ with $u_n \in \mathcal{M}$ a unitary, $h_n^2 \in H^{n(1+\varepsilon),n(1+3\varepsilon)}(\mathcal{A})$ such that $h_n^{-2} \in H^{n(1+2\varepsilon),n(1+4\varepsilon)}(\mathcal{A})$. Repeating this argument, we can get a similar factorization for $\omega_{n-1}u_n$: $\omega_{n-1}u_n = u_{n-1}h_{n-1}^2$, and then for $\omega_{n-2}u_{n-1}$, and so on. In this way we obtain a factorization: $\omega = uh_1^2h_2^2 \dots h_n^2$, where $u \in \mathcal{M}$ is a unitary, $h_{2+\varepsilon}^2 \in H^{n(1+\varepsilon),n(1+3\varepsilon)}(\mathcal{A})$ such that $h_{2+\varepsilon}^{-2} \in H^{n(1+2\varepsilon),n(1+4\varepsilon)}(\mathcal{A}), 1 \le 1+\varepsilon \le n, \varepsilon \ge 0$. Setting $h^2 = h_1^2h_2^2 \dots h_n^2$, we see $\omega = u_n$ is the desired factorization.

Corollary (5.2.28)[270]: Let, $0 < 1+\varepsilon < min(1+\varepsilon, 1+2\varepsilon), -1 < \varepsilon < \infty$, then

$$H^{1+\varepsilon,1+\varepsilon}(\mathcal{A}) \cap L^{1+\varepsilon,1+2\varepsilon}(\mathcal{M}) = H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A}),$$
$$H_0^{1+\varepsilon,1+\varepsilon}(\mathcal{A}) \cap L^{1+\varepsilon,1+2\varepsilon}(\mathcal{M}) = H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A}).$$

Proof: It is clear that
$$H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A}) \subset H^{1+\varepsilon,1+\varepsilon}(\mathcal{A}) \cap L^{1+\varepsilon,1+2\varepsilon}(\mathcal{M}).$$
To prove the converse inequality, fix an $x^2 \in H^{1+\varepsilon,1+\varepsilon}(\mathcal{A}) \cap L^{1+\varepsilon,1+2\varepsilon}(\mathcal{M})$ and set $= ((x^2)^*x^2 + 1)^{\frac{1}{2}}$, thenwe see $\omega \in L^{1+\varepsilon,1+2\varepsilon}(\mathcal{M})$and $\omega^{-1} \in \mathcal{M}$. Applying Theorem (5.2.27) to ω, we get a unitary $u \in \mathcal{M}$and an invertible $h^2 \in H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A})$ such that $\omega = uh^2$ and $h^{-2} \in \mathcal{A}$. Then we obtain
$$(h^2)^*h^2 = (x^2)^*(x^2) + 1.$$
Since $|h^2| \geq |x^2|$, there is a contraction $v \in \mathcal{M}$ such that $x^2 = vh^2$. It follows that $v = x^2h^{-2} \in H^{1+\varepsilon,1+\varepsilon}(\mathcal{A}) \cap \mathcal{M}$, therefore, we obtain that $v \in \mathcal{A}$. Consequently, $x^2 \in \mathcal{A} \cdot H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A}) = H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A})$, and we conclude the first inequality. The later equality is immediate established by adapting the similar proof

Proposition (5.2.29)[270]: Let$-1 < \varepsilon < \infty$, and $h^2 \in H^{1+2\varepsilon,1+\varepsilon}(\mathcal{A})$, then:

i) $[h^2\mathcal{A}]_{L^{1+\varepsilon,1+2\varepsilon}(\mathcal{M})} = H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A})$if and only if $[h^2\mathcal{A}]_{L^{1+2\varepsilon,1+\varepsilon,}(\mathcal{M})} = H^{1+2\varepsilon,1+\varepsilon}(\mathcal{A})$.

ii) $[h^2\mathcal{A}]_{L^{1+\varepsilon,1+2\varepsilon}(\mathcal{M})} = H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A})$if and only if $[h^2\mathcal{A}]_{L^{1+2\varepsilon,1+\varepsilon,}(\mathcal{M})} = H^{1+2\varepsilon,1+\varepsilon}(\mathcal{A})$.

iii)$[\mathcal{A}h^2\mathcal{A}]_{L^{1+\varepsilon,1+2\varepsilon}(\mathcal{M})} = H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A})$if and only if $[\mathcal{A}h^2\mathcal{A}]_{L^{1+2\varepsilon,1+\varepsilon,}(\mathcal{M})} = H^{1+2\varepsilon,1+\varepsilon}(\mathcal{A})$.

Proof : We shall prove only the third equivalence. The proofs of the others are similar.

First, if $[\mathcal{A}h^2\mathcal{A}]_{L^{1+2\varepsilon,1+\varepsilon}(\mathcal{M})} = H^{1+2\varepsilon,1+\varepsilon}(\mathcal{A})$, from the density of $H^{1+2\varepsilon,1+\varepsilon}(\mathcal{A})$ in $H^{1+2\varepsilon,1+\varepsilon}(\mathcal{A})$, we see that $[\mathcal{A}h^2\mathcal{A}]_{L^{1+\varepsilon,1+2\varepsilon}(\mathcal{M})} = H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A})$. To prove the converse implication, when $\varepsilon \geq 0$, let $x^2 \in L^{(1+2\varepsilon)\prime,(1+\varepsilon)\prime}(\mathcal{M})$ with
$$\tau\big(x^2a^2h^2(a^2+\varepsilon)\big) = 0, \forall\, a^2, a^2+\varepsilon \in \mathcal{A},$$
then $x^2a^2h^2 \in H_0^1(\mathcal{A})$, where $(1+2\varepsilon)', (1+\varepsilon)'$is respectively the conjugate index of $1+2\varepsilon, 1+\varepsilon$. On the other hand, by the condition that $[\mathcal{A}h^2\mathcal{A}]_{L^{1+\varepsilon,1+2\varepsilon}(M)} = H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A})$, there exist two sequences $({a_n}^2), ({a_n}^2+\varepsilon_n) \subset \mathcal{A}$ such that
$$\left\|{a_n}^2h^2({a_n}^2+\varepsilon_n) - 1\right\|_{H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A})} \to 0, n \to \infty.$$
Let $\varepsilon > -1$, be such that $1 = \frac{(2+3\varepsilon\prime)}{(1+2\varepsilon\prime)}$. Proposition (5.2.24) gives that
$$\left\|x^2{a_n}^2h^2({a_n}^2+\varepsilon_n) - x^2\right\|_{L^{1+\varepsilon,1+\varepsilon}(\mathcal{M})}$$
$$\leq \tilde{c}\|x^2\|_{L^{(1+2\varepsilon)\prime,(1+\varepsilon)\prime}(\mathcal{M})}\left\|x^2{a_n}^2h({a_n}^2+\varepsilon_n) - 1\right\|_{L^{1+\varepsilon,1+2\varepsilon}(\mathcal{M})} \to 0, n \to \infty.$$
Consequently, we get
$$\left\|x^2{a_n}^2h^2({a_n}^2+\varepsilon_n) - x^2\right\|_{L^{1+\varepsilon,1+\varepsilon}(\mathcal{M})} \to 0, n \to \infty.$$
Since $x^2a_nh^2({a_n}^2+\varepsilon_n) = (x^2{a_n}^2h)({a_n}^2+\varepsilon_n) \in H_0^1(\mathcal{A}) \subset H_0^{1+\varepsilon,1+\varepsilon}(\mathcal{A})$, we know that $x^2 \in H_0^{1+\varepsilon,1+\varepsilon}(\mathcal{A}) \cap L^{(1+2\varepsilon)\prime,(1+\varepsilon)\prime}(\mathcal{M}) = H^{(1+2\varepsilon)\prime,(1+\varepsilon)\prime}(\mathcal{A})$. Hence $\tau(x^2y^2) = 0, \forall\, y^2 \in H^{1+2\varepsilon,1+\varepsilon}(\mathcal{A})$. It follows that
$$[\mathcal{A}h^2\mathcal{A}]_{L^{1+2\varepsilon,1+\varepsilon}(\mathcal{M})} = H^{1+2\varepsilon,1+\varepsilon}(\mathcal{A}).$$

Now we assume $min(1+2\varepsilon, 1+\varepsilon, 1+\varepsilon, 1+2\varepsilon) < 1$. Choose an integer n such that $min(n(1+2\varepsilon), n(1+\varepsilon), n(1+\varepsilon), n(1+2\varepsilon)) \geq 1$, then the conclusion of the previous case tells us that $[\mathcal{A}h^2\mathcal{A}]_{L^{n(1+2\varepsilon),n(1+\varepsilon)}(\mathcal{M})} = H^{n(1+2\varepsilon),n(1+\varepsilon)}(\mathcal{A})$.Since $H^{n(1+2\varepsilon),n(1+\varepsilon)}(\mathcal{A})$ is dense in $H^{1+2\varepsilon,1+\varepsilon,}(\mathcal{A})$, the proof of the first part implies that $[\mathcal{A}h^2\mathcal{A}]_{L^{1+2\varepsilon,1+\varepsilon}(\mathcal{M})} = H^{1+2\varepsilon,1+\varepsilon,}(\mathcal{A})$.

The previous result justifies the relative independence of the indices $1+\varepsilon, 1+2\varepsilon$ in the following definition (see [38]).

Theorem (5.2.30)[270]: Let $-1 < \varepsilon < \infty$ and $h^2 \in H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A})$.

(i) If h^2 is left or right outer, then $\Delta(h^2) = \Delta(\varepsilon(h^2))$. Conversely, if $\Delta(h^2) = \Delta(\varepsilon(h^2))$ and $\Delta(h^2) > 0$, then h is left and right outer (so bilaterally outer too).

(ii) ii) If $\mathcal{A}$ is antisymmetric (i.e., $dimD = 1$) and h^2 is bilaterally outer, then $\Delta(h^2) = \Delta(\varepsilon(h^2))$.

Proof Let $h^2 \in H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A})$. Putting $0 < 1+\varepsilon < min(1+\varepsilon, 1+2\varepsilon) < \infty$ we obtain that $h^2 \in H^{1+\varepsilon}(\mathcal{A})$.

Proposition (5.2.29)and [39] imply that (i) and (ii) hold.

The following corollary is a consequence of this theorem.

Corollary (5.2.31)[270]: Let $h^2 \in H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A})$ and $-1 < \varepsilon < \infty$.

i) If $\Delta(h^2) > 0$, then$h^2$is left outer if and only if h^2 is right outer.

ii) Assume that $\mathcal{A}$ is antisymmetric (i.e.$, dimD = 1$), then the following properties areequivalent:

(a) $h^2$is left outer;

(b) h^2 is right outer;

(c) h^2 is bilaterally outer;

(d) $\Delta(\varepsilon(h^2)) = \Delta(h^2) > 0$.

We will say that h^2 is outer if it is at the same time left and right outer. If $h^2 \in H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A})$ with $\Delta(h^2) > 0$, then h^2 is outer if and only if $\Delta(h^2) = \Delta(\varepsilon(h^2))$. Also in the case where $\mathcal{A}$ is antisymmetric (i.e., $dimD = 1$), an h^2 with $\Delta(h^2) > 0$ is outer if and only if it is left, right orbilaterally outer.

Corollary (5.2.32)[270]: Let $h^2 \in H^{1+\varepsilon,1+3\varepsilon}(\mathcal{A})$ be such that $h^{-2} \in H^{1+2\varepsilon,1+4\varepsilon}(\mathcal{A})$ with $0 < \varepsilon < \infty$,then h^2 is outer.

Proof: Let $h^2 \in H^{1+\varepsilon,1+3\varepsilon}(\mathcal{A})$ be such that $h^{-2} \in H^{1+2\varepsilon,1+4\varepsilon}(\mathcal{A})$. Taking $0 < 1+\varepsilon < min(1+\varepsilon, 1+3\varepsilon) < \infty, 0 < 1+\varepsilon < min(1+2\varepsilon, 1+4\varepsilon) < \infty$, we get $h^2 \in H^{1+\varepsilon}(\mathcal{A})$ and $h^{-2} \in H^{1+\varepsilon}(\mathcal{A})$. By virtue of Proposition (5.2.29) and [39], we see that h^2 is outer.

The following theorem improves Theorem (5.2.27).

Theorem (5.2.33)[270]: Let $\omega \in L^{1+\varepsilon,1+2\varepsilon}(\mathcal{M})$ with $-1 < \varepsilon < \infty$ such that $\Delta(\omega) > 0$, then there exist a unitary $u \in \mathcal{M}$and an outer $h^2 \in H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A})$ such that $\omega = uh^2$.

Proof Write the polar decomposition of ω: $\omega = v|\omega|$. For $|\omega|^{\frac{1}{2}}$, by virtue of Theorem (5.2.27) we get a factorization: $|\omega|^{\frac{1}{2}} = u_2 h_2^2$, with $u_2$unitary and $h_2^2 \in H^{2(1+\varepsilon),2(1+2\varepsilon)}(\mathcal{A})$ left outer. Since $\Delta({h_2}^2) > 0, {h_2}^2$is also right outer, itfollows that $h_2^2$is outer. Similarly we have: $v|\omega|^{\frac{1}{2}}u_2 = u_1 h_1^2$.

This tells us that $u = u_1, h^2 = h_1^2 h_2^2$ yield the desired factorization of ω.

Here we present the inner-outer factorization for operators in $H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A})$.

Corollary (5.2.34)[270]: Let $-1 < \varepsilon < \infty$ and $x^2 \in H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A})$ with $\Delta(x^2) > 0$, then there exist aunitary $u \in \mathcal{A}$(inner) and an outer $h^2 \in H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A})$ such that $x^2 = uh^2$.

Proof Let $x^2 \in H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A})$ with $\Delta(x^2) > 0$. Applying the previous theorem, we get $x^2 = uh^2$ with h^2outer and u unitary in $\mathcal{M}$. Let $({a_n}^2) \subset \mathcal{A}$ such that $lim h^2 a_n = 1$ in $H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A})$, then $u = lim x^2 {a_n}^2$ in $H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A})$, which implies that $u \in H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A}) \cap \mathcal{A} = \mathcal{A}$.

Corollary (5.2.35)[270]: Let $-1 < \varepsilon < \infty$ and $h^2 \in H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A})$ with $\Delta(h^2) > 0$, then h^2 is outer if and only if for any $x^2 \in H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A})$ with $|x^2| = |h^2|$, we have $\Delta(\varepsilon(x^2)) \leq \Delta(\varepsilon(h^2))$.

Proof Let h^2 be outer and $x^2 \in H^{1+\varepsilon,1+2\varepsilon}(\mathcal{A})$ with $|x^2| = |h^2|$,. Taking $0 < 1+\varepsilon < min(1+\varepsilon, 1+2\varepsilon) < \infty$ we obtain that $x^2 \in H^{1+\varepsilon}(\mathcal{A})$. From [39], we get $\Delta(\varepsilon(x^2)) \leq \Delta(\varepsilon(h^2))$. Conversely, let $h^2 = uk^2$ be the decomposition given by Theorem (5.2.33) with k^2 outer. It is easy to check that $\Delta(h^2) = \Delta(k^2) = \Delta(\varepsilon(x^2)) \leq \Delta(\varepsilon(h^2))$. Putting $0 < 1+\varepsilon < min(1+\varepsilon, 1+2\varepsilon) < \infty$ we get $h^2 \in H^{1+\varepsilon}(\mathcal{A})$. Hence, [39] tells us that $\Delta(\varepsilon(x^2)) \leq \Delta(\varepsilon(h^2))$. Consequently, $\Delta(\varepsilon(x^2)) \leq \Delta(\varepsilon(h^2))$. So h^2 is outer due to Theorem (5.2.30).

Section (5.3) Subdiagonal Subalgebras with Applications to Toeplitz Operators

Let $\mathbb{T}$ be the unit circle of the complex plane equipped with normalised Lebesgue measure dm. We denote by $H^p(\mathbb{T})$ the usual Hardy spaces on $\mathbb{T}$. Let P_+ be the orthogonal projection from $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. The classical Helson-SzegÖ theorem [101] (see also [141]), characterises those positive measures μ on $\mathbb{T}$ such that P_+ is bounded on $L^2(\mathbb{T}, \mu)$. The condition is that μ is absolutely continuous with respect to dm and the corresponding Radon-Nikod´ym derivative w satisfies

$$\omega = e^{u+\tilde{v}} \text{ for two functions } u, v \in L^\infty(\mathbb{T}) \text{ with } \|\tilde{v}\|_\infty < \pi/2, \qquad (15)$$

where $\tilde{v}$ denotes the conjugate function of v.

The motivation of this theorem comes from univariate prediction theory. Let $\mathcal{P}_+$ denote the space of all polynomials in z, and $\mathcal{P}_-$ the space of all polynomials in $\bar{z}$ without constant term. $\mathcal{P} = \mathcal{P}_+ + \mathcal{P}_-$ is the space of all trigonometric polynomials. Then P_+ is bounded on $L^2(\mathbb{T}, \mu)$ if and only if $\mathcal{P}_+$ and $\mathcal{P}_-$ are at positive angle in $L^2(\mathbb{T}, \mu)$. Recall that the angle between $\mathcal{P}_+$ and $\mathcal{P}_-$ is defined as arccos of the following quantity

$$\rho = sup\left\{\left|\int_{\mathbb{T}} f\bar{g}d\mu\right| : f \in \mathcal{P}_+, g \in \mathcal{P}_-, \|f\|_{L^2(\mathbb{T},\mu)} = \|g\|_{L^2(\mathbb{T},\mu)} = 1\right\}.$$

Thus P_+ is bounded on $L^2(\mathbb{T}, \mu)$ if and only if $\rho < 1$.

In multivariate prediction theory one needs to consider the matrix-valued extension of the Helson-Szegö theorem. Let $\mathbb{M}_n$ denote the full algebra of complex $n \times n$-matrices, equipped with the normalised trace tr. Let $\mathcal{P}_+(\mathbb{M}_n)$ denote the space of all polynomials in z with coefficients in $\mathbb{M}_n$. $\mathcal{P}_-(\mathbb{M}_n)$ and $\mathcal{P}(\mathbb{M}_n)$ have similar meanings. Let ω be an $\mathbb{M}_n$-valued weight on $\mathbb{T}$, i.e. ω is an integrable function on $\mathbb{T}$ with values in the family of semidefinite nonnegative matrices. For any trigonometric polynomials f and g in $\mathcal{P}(\mathbb{M}_n)$ define

$$\langle f, g\rangle_\omega = \int_{\mathbb{T}} tr(g^* f\omega)dm \quad and \ \|f\|_\omega = \|f, f\|_\omega^{1/2},$$

where a^* denotes the adjoint of a matrix a. Like in the scalar case, define

$$\rho = sup\{|\int_{\mathbb{T}} tr(g^* f\omega)dm| : f \in \mathcal{P}_+(\mathbb{M}_n), g \in \mathcal{P}_-(\mathbb{M}_n), \|f\|_\omega = \|g\|_\omega = 1\}.$$

Again, $\rho < 1$ if and only if $P_+ \otimes Id\mathbb{M}_n$ is bounded on $\mathcal{P}(\mathbb{M}_n)$ with respect to $\|.\|_w$. The problem here is, of course, to characterise w such that $\rho < 1$ in a way similar to the scalar case. This time the task is much harder, and it is impossible to find a characterisation as nice as (15). Numerous works have been devoted to this subject, see, for instance [223, 172, 171, 180, 106, 266]. In particular, Pousson's characterisation in [106] is the matrix-valued analogue of a key intermediate step to (15). It is strong enough for applications to the invertibility of Toeplitz operators.

The preceding two cases can be put into the more general setting of subdiagonal algebras in the sense of [300]. We will provide an extension of the Helson-Szegö theorem in this general setting.

We study the invertibility of Toeplitz operators. It is well known that the Helson-Szegö theorem is closely related to the invertibility of Toeplitz operators. This relationship was remarkably exploited by Devinatz [7]. Pousson [105, 106] then subsequently extended Devinatz's work to the matrix-valued case. Using our extension of the Helson- Szegö theorem, we will characterize the symbols of invertible Toeplitz operators in the very general setting of subdiagonal algebras.

We end this introduction by mentioning the link between the Helson-Szegö theorem and Muckenhoupt's A_2 weights. Let ω be a weight on $\mathbb{T}$. Hunt, Muckenhoupt and Wheeden [243] proved that the Riesz projection P_+ is bounded on $L^2(\mathbb{T}, w)$ if and only if

$$sup \frac{1}{|I|}\int_1 \omega \frac{1}{|I|}\int_1 \omega^{-1} < \infty, \tag{16}$$

where the supremum runs over all arcs of $\mathbb{T}$. Such a ω is called an A_2-weight. Thus for a weight ω the two conditions (15) and (16) are equivalent via the boundedness of the Riesz projection. It seems that it is still an open problem to find a direct proof of this equivalence.

Hunt, Muckenhoupt and Wheeden's theorem was extended to the matrix-valued case by Treil and Volberg [266]. Namely, let w now be an $\mathbb{M}_n$-valued weight on $\mathbb{T}$. Then $P_+ \otimes Id\mathbb{M}_n$ is bounded on $\mathcal{P}(\mathbb{M}_n)$ with respect to $\|.\|_\omega$ if and only if

$$\sup_1 \left\| \left(\frac{1}{|I|}\int_1 \omega\right)^{1/2} \left(\frac{1}{|I|}\int_1 \omega^{-1}\right) \left(\frac{1}{|I|}\int_1 \omega\right)^{1/2} \right\|_{\mathbb{M}_n} < \infty.$$

It is not clear for us how to extend Treil and Volberg's theorem to the case of subdiagonal algebras. On the other hand, Hunt, Muckenhoupt and Wheeden also characterised the boundedness of P_+ on $L^p(\mathbb{T}, \omega)$for any $1 < p < \infty$ by the so-called A_p weights. A well known open problem in matrix-valued harmonic analysis is to extend this result to the matrix-valued case; even to the very general one of subdiagonal algebras.

M will be a von Neumann algebra possessing a faithful normal tracial state τ. The associated noncommutative L^p-spaces are denoted by $L^p(M)$. We refer to [89] for noncommutative integration. For a subset S of $L^p(M)$, we will write $[S]_p$for the closure of S in the L^p $-$topology. On the other hand, S^* will denote the set of all Hilbert-adjoints of elements of S. When an actual Banach dual of some Banach space is in view, we will for the sake of avoiding confusion prefer the superscript $\star$. For example the dual of M will be denoted by $M^\star$. Because M is finite, there will for any von Neumann subalgebra N of M, always exist a normal contractive projection $\psi: M \to N$ satisfying $\tau \circ \psi = \tau$.This is the so-called normal faithful conditional expectation onto N with respect to τ.

A finite subdiagonal algebra of M is a weak* closed unital subalgebra A of M satisfying the following conditions

(i) $A + A^*$ is weak* dense in M;

(ii) the trace preserving conditional expectation $\Phi: M \to A \cap A^* = D$ is multiplicative on A:

$$\Phi(ab) = \Phi(a)\Phi(b), a, b \in A.$$

In this case, D is called the diagonal of A. We also set $A_0 = A \cap Ker(\Phi)$. In the sequel, A will always denote a finite subdiagonal algebra of M.

Subdiagonal algebras are our noncommutative H^∞'s. The most important example is, of course, the classical $H^\infty(\mathbb{T})$ on the unit circle. Another example important for multivariate prediction theory is the matrix-valued $H^\infty(\mathbb{T})$. More precisely, let $M = L^\infty(\mathbb{T}) \otimes \mathbb{M}_n = L^\infty(\mathbb{T}; \mathbb{M}_n)$ equipped with the product trace, and let $A = H^\infty(\mathbb{T}; \mathbb{M}_n)$ the subalgebra of M consisting of $n \times n$-matrices with entries in $H^\infty(\mathbb{T})$. Many classical results about Hardy spaces on $\mathbb{T}$ have been transferred to the matrix-valued case. A third example is the upper triangle subalgebra $\mathbb{T}_n$of $\mathbb{M}_n$. This example is closely related to the second one, and is a finite dimensional nest algebra. We refer to [89] for more information and historical references on subdiagonal algebras, in particular, on matrix-valued analytic functions.

For $p < \infty$ the Hardy space $H^p(A)$ associated with a finite subdiagonal algebra A is defined to be $[A]_p$. The closure of A_0 in $L^p(M)$will be denoted by $H_0^p(M)$. By convention, we put $H^\infty(A) = A$ and $H_0^\infty(A) = A_0$. These spaces exhibit many of the properties of classical H^p spaces (see [278, 64, 65, 175, 190, 152]). In particular for $1 < p < \infty, L^p(M)$ appears as the Banach space direct sum of $H^p(M)$ and $H_0^p(M)^*$, with $H^p(M)$ appearing as the Banach space direct sum of $H_0^p(M)$ and $L^p(D)$. In the case $p = 2$, these direct sums are even orthogonal direct sums.

Recall that if a weight ω on $\mathbb{T}$ satisfies (15), then necessarily $log\omega \in L^1(\mathbb{T})$, or equivalently,

$$exp\left(\int_{\mathbb{T}} log\omega\right) > 0. \tag{17}$$

The integrability of logw is also equivalent to the existence of an outer function $h \in H^1(T)$ such that $\omega = |h|$. To state the outer-inner factorisation and prove the Helson-Szegö analogue for subdiagonal algebras, we need an appropriate substitute of the latter condition. This is achieved by the Fuglede-Kadison determinant. Recall that the Fuglede-Kadison determinant $\Delta(a)$ of an operator $a \in L^p(M)$ $(p > 0)$ can be defined by

$$\Delta(a) = exp(\tau(log\,|a|)) = exp\left(\int_0^\infty log\ t\ dv_{|a|}(t)\right),$$

where $dv_{|a|}$denotes the probability measure on $\mathbb{R}_+$ which is obtained by composing the spectral measure of |a| with the trace τ. It is easy to check that

$$\Delta(a) = \lim_{p\to 0} \|a\|_p \ and\ \Delta(a) = \inf_{\varepsilon>0} exp\,\tau(log(|a| + \varepsilon\mathbb{1}))\,.$$

As the usual determinant of matrices, Δ is also multiplicative: $\Delta(ab) = \Delta(a)\Delta(b)$. We refer for information on determinant to [26, 300] in the case of bounded operators, and to [157, 288] for unbounded operators.

Return to our Hardy spaces. An element h of $H^p(M)$ with $p < \infty$ is said to be an outer element if hA is dense in $H^p(M)$. If in addition $\Delta(h) > 0$, we call such an h strongly outer. See [63] for $p \geq 1$ and [284] for $p < 1$. We will however pause to summarise the essential points of the theory. For any outer element h of $H^p(M)$, both h and $\Phi(h)$ necessarily have dense range and trivial kernel. Hence their inverses exist as affiliated operators. For such an outer element, we also necessarily have that $\Delta(h) = \Delta(\Phi(h))$. If indeed $\Delta(h) > 0$, the equality $\Delta(h) = \Delta(\Phi(h))$is sufficient for h to be outer. Using this fact it is now an easy exercise to see that if $\Delta(h) > 0$, then h is an outer element of $H^p(M)$ if and only if h^* is an outer element of $H^p(M)^*$ if and only if h is right outer in the sense that Ahwill also be dense in $H^p(M)$. In this theory one also has a type of noncommutative Riesz-Szegö theorem, in that any $f \in L^p(M)$ for which $\Delta(f) > 0$, may be written in the form $f = uh$ where $u \in M$ is unitary and $h \in H^p(M)$an outer element of $H^p(M)$.

Given a state ω on M, we write $(\pi_\omega, L^2(\omega)\Omega_\omega)$for the cyclic representation associated to ω. The subspaces A^* and A_0 embed canonically into $L^2(\omega)$ by means of the operation $a \to \pi_\omega(a)\Omega_\omega$. The angle between A^* and A_0 in $L^2(\omega)$is defined to be that between the closed subspaces $\overline{\pi_\omega(A^*)\Omega_\omega}$and $\overline{\pi_\omega(A_0)\Omega_\omega}$. The latter is equal to $\cos^{-1}\rho$ with ρ given by

$$\rho = sup\{|\langle\pi_\omega(a)\Omega_\omega, \pi_\omega(b)\Omega_\omega\rangle|: a \in A_0, b \in A^*, \|\pi_\omega(a)\Omega_\omega\| \leq 1, \|\pi_\omega(b)\Omega_\omega\| \leq 1\}.$$

In view of the fact that $\pi_\omega(a)\Omega_\omega, \pi_\omega(b)\Omega_\omega = \omega(b^*a)$, this may be rewritten as

$$\rho = sup\{|\omega(b^*a)|: a \in A_0, b \in A^*, \omega(|a|^2) \leq 1, \omega(|b|^2) \leq 1\}.$$

In general $0 \leq \rho \leq 1$. A^* and A_0 are said to be at positive angle in $L^2(\omega)$if $\rho < 1$. Let P_+ be the orthogonal projection from $L^2(M)$ onto $H^2(M)$. It is then clear that P_+ defines a bounded operator on $L^2(\omega)$ if *and* only if $\rho < 1$.

We present our noncommutative Helson-Szegö theorem. This theorem will prove to be an important ingredient in our onslaught on Toeplitz operators . The classical Helson-Szegö theorem contains the information that any finite Borel measure for which the angle between A and A_0^* is positive must necessarily be absolutely continuou with respect to Lebesgue measure, and moreover that the Radon-Nikod´ym derivative of this measure must have a strictly positive geometric mean (17). Before presenting our noncommutative Helson-Szegö theorem, we first show that under some mild restrictions the same claims are true in the noncommutative case. $L_+^p(M)$ will denote the positive cone of $L^p(M)$.

Proposition (5.3.1)[170]: Let $D = A \cap A^*$be finite dimensional, and let ω be a state on M for which$\rho < 1$. Then ω is of the form $\omega = \tau(g\,\cdot)$ for some $g \in L_+^1(M)$.

Proof: We keep the notation introduced at the end of the previous section. Let ω_n and ω_s respectively be the normal and singular parts of ω. Firstly note that by [183], there exists a central projection $e_0$in $\pi_\omega(M)''$ such that for any $\xi, \psi \in L^2(\omega)$ the functionals $a \to \langle\pi_\omega(a)e_0\xi, \psi\rangle$and $a \to \langle\pi_\omega(a)e_0^\perp\xi, \psi\rangle$ on M are respectively the normal and singular parts of the functional $a \to \langle\pi_\omega(a)\xi, \psi\rangle$, where $e_0^\perp = \mathbb{1} - e_0$. In particular, the triples $(e_0\pi_\omega, e_0L^2(\omega), e_0\Omega_\omega)$ and $(e_0^\perp\pi_\omega, e_0^\perp L^2(\omega), e_0^\perp\Omega_\omega)$ are copies of the GNS representations of ω_n and $\omega_s$respectively. Since $\rho < 1$, we must have that

$$\overline{\pi_\omega(A_0)\omega} \cap \overline{\pi_\omega(A^*)\Omega_\omega} = \{0\}.$$

Now suppose that the singular part ω_s of ω is nonzero. By Ueda's noncommutative peak-set theorem [315] there exist an orthogonal projection e in the second dual $M^{\star\star}$ of M and a contractive element a of A so that

(i) a^n converges to e in the weak*-topology on $M^{\star\star}$;

(ii) $\omega_s(e) = \omega_s(1)$ (here ω_s is identified with its canonical extension to $M^{\star\star}$);

(iii) a^n converges to 0 in the weak*-topology on M.

Since the expectation Φ is weak*-continuous on $M, \Phi(a^n)$is weak* convergent to 0. But then the finite dimensionality of D ensures that $\Phi(a^n)$converges to 0 in norm.

Recall that the bidual $M^{\star\star}$ of M may be represented as the double commutant of M in its universal representation. So when this realisation of $M^{\star\star}$is compressed to the specific representation engendered by ω, it follows that e yields a projection $\tilde{e}$ in $\pi_\omega(M)''$to which $\pi_\omega(a^n)$ converges in the weak*-topology on $\pi_\omega(M)''$. This weak* convergence in $\pi_\omega(M)''$ together with the second bullet above, then yield the facts that

(i) $\pi\omega(a^n)\Omega_\omega$ converges to $\tilde{e}\Omega_\omega$ in the weak-topology on $L^2(\omega)$;

(ii)$\langle \tilde{e}\Omega_\omega, e_0^\perp\Omega_\omega\rangle = \omega_s(\mathbb{1})$.

From the first bullet and the fact that $\{\Phi(a^n)\}$ is a norm-null sequence, it follows that $\pi_\omega(a^n - \Phi(a^n))\Omega_\omega$ is weakly convergent to $\tilde{e}\omega$, and hence that $\tilde{e}\Omega_\omega \in \overline{\pi_\omega(A_0)\Omega_\omega}$. But if an converges to e in the weak*-topology on $M^{\star\star}$, then surely so does (a^{*n}). In terms of the GNS representation for ω, this means that $\pi_\omega((a^*)^n)\Omega_\omega$ also converges to$\tilde{e}\omega$in the weak-topology on $L^2(\omega)$. But then $\tilde{e}\omega \in \overline{\pi_\omega(A^*)\Omega_\omega}$. Then $\tilde{e}\omega = 0$ since $\tilde{e}\omega \in \overline{\pi_\omega(A_0)\omega} \cap \overline{\pi_\omega(A^*)\Omega_\omega}$. But this cannot be, since by the second bullet this would mean that $\omega_s(\mathbb{1}) = \langle \tilde{e}\omega, e_0^\perp\Omega_\omega\rangle = 0$. Thus our supposition that ω_s is nonzero, must be false. The condition that $\rho < 1$, is therefore sufficient to force ω to be normal. That isω is of the form $\omega = \tau(g.)$ for some $g \in L^1_+(M)$.

The following lemmata present two known elementary facts.

Lemma (5.3.2)[170]: For any $g \in L^1_+(M)$ we have that

$$s(\Phi(g)) \geq s(g),$$

where $s(g)$ denotes the support projection of g.

Proof: For simplicity of notation we respectively write s and $s_\Phi$for s(g) and $s(\Phi(g))$. Since $s_\Phi \in D$, we have that

$$\tau(s_\Phi^\perp g s_\Phi^\perp) = \tau \circ \Phi(s_\Phi^\perp g s_\Phi^\perp) = \tau(s_\Phi^\perp \Phi(g) s_\Phi^\perp) = 0.$$

Therefore $g^{1/2}s_\Phi^\perp = s_\Phi^\perp g^{1/2} = 0$. This is sufficient to force $s_\Phi^\perp \perp s$, which in turn suffices to show that $s_\Phi \geq s$.

Lemma (5.3.3)[170]: Let e be a nonzero projection in D. Then eAe is a finite maximal subdiagonal subalgebra of eMe (equipped with the trace $\tau e(.) = \frac{1}{\tau(e)}\tau(.)$) with diagonal $eAe \cap (eAe)^* = eDe$.

Proof: The expectation Φ is trivially still multiplicative on the compression eAe. Using the fact that $e \in D$, it is an exercise to see thatΦ mapseAe onto eDe. It is also straightforward to see that the weak*-density of $A + A^*$ in M forces the weak*-density of $eAe + (eAe)^*$ in eMe, and that $(eAe)_0 = eA_0e$.

Definition (5.3.4)[170]: Adopting the notation of the previous two lemmata, given a nonzero element $g \in L^1_+(M)$, we define Δ_Φ(g) to be the determinant of $s_\Phi g s_\Phi$ regarded as an element of $(s_\Phi M s_\Phi, \tau s_\Phi)$

Proposition (5.3.5)[170]: Let $D = A \cap A^*$ be finite dimensional, and let $g \in L^1_+(M)$ be a norm-one element for which the state $\omega = \tau(g\ \cdot)$ satisfies $\rho < 1$. Then $\Delta_\Phi(g) > 0$.

Proof: It is clear from the previous lemmata that we may reduce matters to the case where$s(\Phi(g)) = \mathbb{I}$, and hence we will assume this to be the case. Suppose by way of contradiction that $\Delta(g) = 0$. By the Szegö formula for subdiagonal algebras [159], we then have that

$$0 = \Delta(g) = inf\{\tau(g|a - d|^2): a \in A_0, d \in D, \Delta(d) \geq 1\}.$$

Thus there exist sequences $\{a_n\} \subset A_0 and \{a_n\} \subset D$ with$\Delta(d_n) \geq 1$ for all n, so that

$$\tau(g|a_n - d_n|^2) \to 0\ as \quad n \to \infty.$$

By [60] we may assume all the a_n's to be invertible. Now let $u_n \in D$be the unitary in the polar decomposition $d_n = u_n|d_n|$. It is an exercise to see that then $\{u_n^* a_n\} \subset A_0$with $|a_n - d_n|^2 = |u_n^* a_n - |d_n||^2$. Making the required replacements, we may therefore also assume that $\{d_n\} \subset D^+$.

Since $1 \leq (d_n) \leq \|d_n\|_\infty$ for all n, we will for the sequences $\widetilde{d_n} = \frac{1}{\|d_n\|_\infty} d_nand\widetilde{a_n} = \frac{1}{\|d_n\|_\infty} a_n$ $(n \in \mathbb{N})$, still have that $\tau(g|\widetilde{a_n} - \widetilde{d_n}|^2) \to 0$ as $n \to \infty$. Now recall that D is finite dimensional. So by passing to a subsequence if necessary, we may assume that $\{\widetilde{d_n}\}$ converges uniformly to some norm one element $d_0$of D^+. But then by what we showed above,

$$\begin{aligned} \left\|\pi_g(\widetilde{a_n}) - \pi(d_0)\right\|_2 &= \tau(g|\widetilde{a_n} - d_0|^2)^{1/2} \\ &\leq \tau(g|\widetilde{a_n} - \widetilde{d_n}|^2)^{1/2} + \tau(g|\widetilde{d_n} - d_0|^2)^{1/2} \\ &\leq \tau(g|\widetilde{a_n} - \widetilde{d_n}|^2)^{1/2} + \left\|\widetilde{d_n} - d_0\right\|_\infty \tau(g)^{1/2} \\ &\to 0. \end{aligned}$$

Thus $\pi_g(d_0) \in \pi_g(A_0) \cap \pi_g(A_*)$. Since $\Phi(g)$ is of full support, we have that $\Phi(g)^{1/2} d_0 \Phi(g)^{1/2} \neq 0$. So

$$0 < \tau(\Phi(g)^{1/2} d_0 \Phi(g)^{1/2} = \tau(\Phi(g) d_0) = \tau(\Phi(g d_0)) = \tau(g d_0).$$

Therefore $\pi_g(d_0) \neq 0$. But this proves that the subspaces $\pi_g(A_0)$ and $\pi_g(A^*)$ have a nonzero intersection, and hence that $\rho = 1$.

The following technical lemma is a crucial step in the proof of the classical Helson-Szegö theorem. The challenge one faces in the noncommutative world is that the functional calculus at our disposal in that context is simply not strong enough to reproduce so detailed a statement in that framework. However in the lemma following this one, we present what we believe to be a reasonable noncommutative substitute of this interesting lemma.

Lemma (5.3.6)[170]: Let $u = e^{-i\psi}$ withψ a real measurable function on $\mathbb{T}$. Then $inf_{g \in H^\infty(T)} \left\|e^{-i\psi} - g\right\|_\infty < 1$ if and only if there exist an $\varepsilon > 0$ and a $k_0 \in H^\infty(\mathbb{T})$so that $|k_0| \geq \varepsilon$ and $|\psi| + arg(k_0)| \leq \frac{\pi}{2} - \varepsilon$ almost everywhere.

Lemma (5.3.7)[170]: Let u be a unitary element of M. Then there exists some $f \in A$ so that $\|u-f\|_\infty < 1$ if and only if there exists $h \in A$so that$\Re(u^*h)$ is strictly positive.

Proof: Suppose first that there exists $f \in A$ with $\|u-f\|_\infty < 1$. We then equivalently have that $\|1-u^*f\| = \|1-f^*u\| < 1$. On setting
$\alpha = \|1-u^*f\|$, it follows that $\|\mathbb{I} - \Re(u * f)\| \le \alpha < 1$, and hence that

$$-\alpha\mathbb{I} \le \Re(u^*f) - 1 \le \alpha\mathbb{I}.$$

This in turn ensures that $0 < (1-\alpha)\mathbb{I} \le \Re(u^*f)$.

Conversely suppose that there exists $h \in A \cap M^{-1}$ so that $\Re(u^*h) \ge \alpha\mathbb{I}$ for some $0 < \alpha \le \|\Re(u^*h)\| \le \|h\|$, where M^{-1} denotes the subset of invertible elements of M. Given $\varepsilon > 0$, set$\lambda = \frac{\varepsilon}{\|h\|}$. It then follows that

$$-2\lambda\Re(u^*h) + \lambda^2|h|^2 \le -\left(\frac{2\alpha\epsilon}{\|h\|} - \varepsilon^2\right)\mathbb{I}.$$

(Observe that $\frac{\alpha}{\|h\|} \le 1$ in the above inequality.) It is clear that if ε is small enough, we would have that $1 > \left(\frac{2\alpha\varepsilon}{\|h\|} - \varepsilon^2\right) > 0$. Thus we may assume this to be the case. For simplicity of notation we now set $\delta = \left(\frac{2\alpha\varepsilon}{\|h\|} - \varepsilon^2\right)$. It therefore follows from the previous centered inequality that

$$0 \le |1-u^*(\lambda h)|^2 = \mathbb{I} - 2\Re(u^*(\lambda h) + |\lambda h|^2 \le (1-\delta)\mathbb{I}.$$

Hence as required, $\|\mathbb{I} - u^*(\lambda h)\|^2 \le (1-\delta) < 1$.

We are now finally ready to present our noncommutative Helson-Szegötheorem. In view of Propositions (5.3.1) and(5. 3.5), it is not unreasonable to restrict attention to normal states $\tau(g.)$ in this theorem for which $\Delta_\Phi(g > 0$. The following result is a sharpening of the result of Pousson [108], in that here the conditions imposed on the unitary u are less restrictive. This sharpening is achieved by means of the preceding Lemma.

Theorem (5.3.8)[170]: Let $g \in L^1_+(M)$ be given with $\|g\|_1 = 1$, and denote $s(\Phi(g))$ by s_Φ. Consider the state $\omega = \tau(g.)$. Then $\rho < 1$ and $\Delta_\Phi(g) > 0$ if and only if $g$is of the form $g = f_R u f_L$where

(i) $u \in M$ is a partial isometry with initial and final projections s_Φ for which there exists some $k \in s_\Phi A s_\Phi$ so that $\Re(u^*k) \ge \alpha s_\Phi$ for some $\alpha > 0$,•

(ii) and f_L and f_R are strongly outer elements of $H^2(M)$ commuting with s_Φ for which$g + (\mathbb{I} - s_\Phi) = |f_L|^2 = |f_R^*|^2$.

If in addition $dimD < \infty$, we may dispense with the restrictions that ω is normal, and that $\Delta_\Phi(g) > 0$.

Proof: Set $s = s_\Phi$for simplicity. Suppose that g satisfies the condition $\Delta_\Phi(g) > 0$. Using the fact that then $\Delta_\Phi(g^{1/2}) = \Delta_\Phi(g)^{1/2} > 0$, it follows from the noncommutative Riesz-Szegötheorem (see [63]) that there exist strongly outer elements $h_L, h_R \in H^2(sMs)$ and unitaries $v_L, v_R \in sMs$ for which $g^{1/2} = v_L h_L = h_R v_R$. (Then also $g^{1/2} = |h_L| = |h_R^*|$.) We set

$$u = v_R v_L, \quad f_L = h_L + s^\perp, f_R = h_R + s^\perp.$$

It is then clear that

$$g = f_R u f_L \text{ and } g + s^\perp = |f_L|^2 = |f_R^*|^2.$$

We proceed to show that f_L and $f_R$are strongly outer. The proofs of the two cases are identical, and hence we do this for f_L only. Notice that

$$log(|f_L|) = log(|h_L| + s^{\perp}) = log(|h_L|)s.$$

Since $\Phi(f_L) = \Phi(h_L) + s^{\perp}$, we similarly have that

$$log(|\Phi(f_L)|) = log(|\Phi(h_L)|)s.$$

It then follows that

$\tau(log\,|f_L|) = \tau(s)\tau_s(log\,|h_L|)$and $\tau(log\,|\Phi(f_L)|) = \tau(s)\tau_s(log\,|\Phi(h_L)|)$.

Thus the outerness of h_L yields that

$$\tau(log\,|f_L|) = \tau(log\,|\Phi(f_L)|) > -\infty, \text{so } \Delta(f_L) = \Delta(\Phi(f_L)) > 0.$$

Then an application of [63] now shows that f_L is strongly outer. On the other hand, we have

$$\langle \pi_g(a)\Omega_g, \pi(b)\Omega_g \rangle = \tau(gb^*a) = \tau(uf_Lb^*af_R), a \in A_0, b \in A^*.$$

So

$$\begin{aligned}\rho &= sup\{|\tau(gb^*a)| : a \in A_0, b \in A^*, \tau(g|a|^2) \le 1, \tau(g|b|^2) \le 1\} \\ &= sup\{|\tau((u(sf_Lb^*)(af_Rs))| : a \in A_0, b \in A^*, \tau(|af_Rs|^2) \le 1, \tau(|bf_L^*s|^2) \le 1\} \\ &= sup\{|\tau(uF_1F_2)| : F_1 \in sH^2(M), F_2 \in H_0^2(M)s, \|F_1\|_2 \le 1, \|F_2\|_2 \le 1\}.\end{aligned}$$

In the above computation one has used the fact that$f_L$and f_R are strongly outer to approximate F_1 and F_2 with elements of the formssf_Lb^* and $af_Rs$where $a \in A_0$ and $b \in A^*$. However, it is easy to check that for $F_1 \in sH^2(M), F_2 \in H_0^2(M)s$

$$F_1F_2 \in H_0^1\,(sMs) \text{ } and \text{ } \|F_1F_2\|_1 \le \|F_1\|_2\|F_2\|_2.$$

Conversely, by the Noncommutative Riesz Factorisation theorem [176, 153], for any $\varepsilon > 0$ and any $F \in H_0^1\,(sMs)$ there exist $F_1 \in H^2(sMs) \subset sH^2(M)$ and $F_2 \in H_0^2(sMs) \subset H^2(M)s$ such that

$$F = F_1F_2 \text{ } and \text{ } \|F_1\|_2\|F_2\|_2 \le \|F_1\|_1 + \varepsilon.$$

From these discussions we conclude that

$$\rho = sup\{|\tau(uF)| : F \in H_0^1(sMs), \|F\|_1 \le 1\} = sup\{|\tau s(uF)| : F \in H_0^1\,(sMs), \tau_s(|F|) \le 1\}.$$

The norm of the restriction of the functional $L^1(sMs) \to \mathbb{C} : a \to \tau_s(ua)$ to $H_0^1\,(sMs)$ is by duality precisely the norm of the equivalence class $[u]$ in the quotient space $sMs/(H_0^1(sMs))$. However, it is well known that

$$sAs = \{a \in sMs : \tau_s(ab) = 0, b \in sA_0s\}$$

(cf. e.g., [153]). From this fact it is now an easy exercise to see that the polar $(H_0^1\,(sMs))$ is nothing but sAs. It therefore follows that

$$\rho = inf\{\|u - k\|_\infty : k \in sAs\}.$$

The result now follows from an application of the preceding Lemma.

We start by recalling the definition of Toeplitz operators. Given$a \in M$, the Toeplitz operator T_a with symbol a is defined to be the map

$$T_a : H^2(M) \to H^2(M) : b \to P_+(ab),$$

where P_+denotes the orthogonal projection from $L^2(M)$ onto $H^2(M)$. see [177] (see also [30]). We will characterise the symbols of invertible Toeplitz operators. We point out that these results are new even for the matrix-valued case. In achieving this characterisation, we will follow the same basic strategy as Devinatz [7] in his remarkable solution of this problem in the classic setting. Our first result essentially reduces the problem to that of characterising invertible Toeplitz operators with unitary symbols.

Theorem (5.3.9)[170]: Let $a \in M$ be given. A necessary and sufficient condition for T_a to be invertible is that it can be written in the form $a = uk$ where $k \in A^{-1}$, and $u \in M$ is a unitary for which T_u is invertible.

Suppose that $a \in M$ is indeed of the form $a = uk$ where $k \in A^{-1}$, and $u \in M$ is a unitary. It is a simple exercise to see that then T_k is invertible with inverse T_{k-1} . Since $T_a T_{k-1} = T_u$ and $T_u T_k = T_a$, it is now clear that T_a will then be invertible if and only if T_u is invertible.

Proof: The sufficiency of the stated condition was noted in the above discussion. To see the necessity, assume T_a to be invertible. There must therefore exist some $g \in H^2(M)$ so that $T_a g = \mathbb{I}$. This in turn can only be true if there exists some $h \in H_0^2\,(M)$ so that $ag = \mathbb{I} + h^*$. By the generalised Jensen inequality [63] we have that

$$\Delta(a)\Delta(g) = \Delta(ag) = \Delta(\mathbb{I} + h^*) \geq \Delta\big(\Phi(\mathbb{I} + h^*)\big) = \Delta(\mathbb{I}) = 1.$$

Clearly we then have that $\Delta(|a|^{1/2}) = \Delta(a)^{1/2} > 0$. So by the noncommutative Riesz-Szegö theorem [64], there must exist an outer element $f \in H^2(M)$ and a unitary v so that $|a|^{1/2} = vf$. (Note then that $f \in M$, so f must belong to A too.) Let ω be the unitary in the polar decomposition $a = \omega|a|$, and consider $b = \omega|a|^{1/2}v$. Notice that by construction $bf = a$. Thus $T_b T_f = T_a$. We will use this formula to show that T_f is invertible, from which the result will then follow.

Firstly note that the injectivity of T_a combined with the above equality, ensures that T_f is injective. Next notice that the equality $T_b T_f = T_a$. ensures that $(T_a)^{-1} T_b$ is a left inverse for T_f. So T_f must have a closed range. However since f is outer, we also have that $[fA]_2 = H_2(M)$. Since $fA \subset T_f\,(H_2(M))$, these two facts ensure that the range of T_f is all of $H_2(M)$. Hence T_f must be invertible.

But if T_f is invertible, then so is $T_f^* = T_{f^*}$. Since $T_{f^*} T_f = T_{|f|^2} = T_{|a|}$, the operator $T_{|a|}$ must be invertible. Since $\sigma(|a|) \subset \sigma(T_{|a|})$ by [177], we must have that $0 \notin \sigma(|a|)$. In other words $|a|$ must be strictly positive. But if $|a|$ is strictly positive, then by Arveson's factorization theorem there exists some $k \in A^{-1}$ with $|a| = |k|$. Finally let ω_0 be the unitary in the polar form $k = \omega_0|k|$. Then $a = \omega\omega_0^* k$, which proves the theorem with $u = \omega\omega_0^*$.

Our next step in achieving the desired characterisation, is to present some necessary structural information regarding unitaries u for which T_u is invertible. We then subsequently use this structural information to obtain a characterisation of invertibility in terms of positive angle.

Lemma (5.3.10)[170]: Let $u \in M$ be a unitary. A necessary condition for T_u to be invertible is that it is of the form $u = (g_1^*)^{-1} d {g_0}^{-1}$ where g_0, g_1 are strongly outer elements of $H^2(M)$ and d a strongly outer
element of $L^2(D)$ related by the conditions that

$$d = \Phi(g_0) = \Phi(g_1^*), d{g_0}^{-1}, d^* g_1^{-1} \in H^2(M) \text{ and } g_0^* g_0 = d^*(g_1^* g_1)^{-1} d.$$

Proof: Let $u \in M$ be a unitary for which T_u is invertible. Since $T_u^* = T_{u^*}$ is then also invertible, itfollows that there must exist $g_0, g_1 \in H^2(M)$ so that $T_u g_0 = \mathbb{I} = T_u g_1$. This in turn means that there exist $h_0, h_1 \in H_0^2\,(M)$ with

$$ug_0 = \mathbb{I} + h_0^*, u^* g_1 = \mathbb{I} + h_1^*.$$

Notice that we may then apply the generalised Jensen inequality [64] to conclude that

$$\Delta(g_0) = \Delta(u)\Delta(g_0) = \Delta(ug_0) \geq \Delta(\mathbb{I}) = 1.$$

Similarly $\Delta(g_1) \geq 1$. By [63] this means that both g_0 and g_1 are injective with dense range, and hence that $g_0^{-1}$and g_1^{-1} exist as affiliated operators. On the other hand, we have that

$$g_1^* u g_0 = g_1^*(\mathbb{I} + h_0^*) \in H^1(M)^* \text{ and } g_1^* u g_1 = g_1^*(\mathbb{I} + h_1^*) \in H^1(M)^*.$$

Hence

$$g_1^* u g_0 \in H^1(M) \cap H^1(M)^* = L^1(D).$$

If we denote this element by d, it follows that u is of the form $u = (g_1^*)^{-1} d\, g_0^{-1}$. It is then clearthat $d^*(g_0^* g_1)^{-1} d = g_1^* g_0$.

It remains to show that g_0 and g_1 are outer and that $d = \Delta(g_0) = \Delta(g_1^*)$. To see this notice that since $g_1^* \in H^2(M)^*$and$u g_0 = \mathbb{I} + h_0^* \in H^2(M)^*$, we have that

$$d = \Phi(d) = \Phi(g_1^* u g_0) = \Phi(g_1^*(\mathbb{I} + h_0^*))\, \Phi(g_1^*)\Phi(\mathbb{I} + h_0^*) = \Phi(g_1^*).$$

Similarly, $d = \Phi(g_0)$. (Since Φ maps $H^2(M)$ onto $L^2(D)$, this equality also shows that d is in fact in $L^2(D)$,, and not just $L^1(D)$,.) It now follows from the equality $g_0^* g_0 = d^*(g_1^* g_1)^{-1} d$, that

$$\Delta(g_0)^2 = \Delta(g_0^* g_0) = \Delta(d^*(g_1^* g_1)^{-1} d) = \Delta(d^*)^2 \Delta(g_1)^{-2} = \Delta(\Phi(g_1))^2 \Delta(g_1)^{-2}.$$

Since as was shown earlier we have that $\Delta(g_0) \geq 1$, it therefore follows that $0 < \Delta(g_1) \leq \Delta(\Phi(g_1))$.If we combine this with the generalised Jensen inequality [63],we obtain $0 < \Delta(g_1) = \Delta(\Phi(g_1)$.Similarly, $0 < \Delta(g_0) = \Delta(\Phi(g_0))$. Thus by [64] both g_0 and g_1 are strongly outer.

When combined with Theorem (5.3.9), the following lemma characterises the invertibility of Toeplitz operators in terms of positive angle. If we further combine this lemma with the noncommutative Helson-Szegö theorem obtained, we end up with the promised structural characterisation of invertible Toeplitz operators with unitary symbols.

Lemma (5.3.11)[170]: Let $u \in M$ be a unitary of the form described in the previous lemma. Then $T_u$is invertible if and only ifA^* and A_0 are at positive angle with respect to the functional $\tau(\omega\, \cdot)$, where

$\omega = g_0^* g_0 = d^*(g_1^* g_1)^{-1} d.$

Proof: First suppose that T_u is invertible. For any $a \in A$ the element $g_0 a$ will belong to $H^2(M)$. So the invertibility of T_u ensures that we can find a constant $K > 0$ so that

$$\|g_0 a\|_2 \leq \mathrm{K}\|T_u(g_0 a)\|_2, a \in A.$$

Recall that by Lemma (5.3.10) u is of the form $u = (g_1^*)^{-1} d g_0^{-1}$. Thus the former inequality translates to

$$\|g_0 a\|_2 \leq \mathrm{K}\|P_+ (g_1^*)^{-1} d a\|_2, a \in A.$$

Now observe that for any$b \in A_0$, the element $(g_1^*)^{-1} d b^*$ will belong to $H^2(M)^* A_0^* \subset H_0^2(M)^*$. Hence

$$P_+((g_1^*)^{-1} d a) = P_+((g_1^*)^{-1} d a + (g_1^*)^{-1} d b^*).$$

If we now write $\|f\|_\omega$for$\tau(\omega f^* f)^{1/2}$, then for any $a \in A$and $b \in A_0$ we have that

$$\begin{aligned} \|a^*\|_\omega = \tau(a^* w a)^{1/2} &= \|g_0 a\|_2 \\ &\leq \mathrm{K}\|P_+((g_1^*)^{-1} d a + (g_1^*)^{-1} d b^*)\|_2 \\ &\leq K\|(g_1^*)^{-1} d(a + b^*)\|_2 \\ &= K\tau((a^* + b)\omega(a + b^*)) = K\|a^* + b\|_\omega \end{aligned}$$

Thus A^* and $A_0$are at positive angle with respect to the functional $\tau(w.)$.

Conversely, suppose that A^* and $A_0$are at positive angle with respect to the functional $\tau(\omega\cdot)$.We first show that T_u has dense range, and hence that it will be invertible whenever it is bounded below. Let $a_0 \in H^2(M)$ be orthogonal to $T_u(H^2(M))$. We will show that a_0 must then be the zero vector. Given $a \in A$, the orthogonality of a_0 to $T_u(H^2(M))$ together with the fact that $u = (g_1^*)^{-1} d g_0^{-1}$, ensures that

$$0 = \langle T_u(g_0 a), a_0\rangle = \tau(a_0^* T_u(g_0 a)) = \tau(a_0^* P_+((g_1^*)^{-1} da) = \tau(a_0^* (g_1^*)^{-1} da).$$

However, as was noted in the first part of the proof, for any $b \in A_0$we have that

$$a_0^*(g_1^*)^{-1} db^* \in H_0^2(M)^*,$$

which implies that

$$\tau(a_0^*(g_1^*)^{-1} db^*) = \tau\big(\Phi(a_0^*(g_1^*)^{-1} db^*)\big) = 0.$$

Thus

$\tau(a_0^*(g_1^*)^{-1} d(a + b^*)) = 0$for all$a \in A, b \in A_0$.

Hence $d^* g_1^{-1} a_0 = 0,$ so $a_0 = 0$.

It remains to show that T_u is bounded below whenever A^* and A_0 are at positive angle with respect to the functional $\tau(w\,\cdot)$. Hence assume that there exists a constant $B > 0$ so that

$$\|a^*\|_\omega \le B\|a^* + b\|_\omega \text{ for all } a \in A, b \in A_0.$$

Since by assumption we have that d $= \Phi(g_1^*)$, and since both g_1^* and $((g_1^*)^{-1} d)$belong to $H^2(M)^*$,it follows that

$$d = \Phi(d) = \Phi(g_1^*[(g_1^*)^{-1} d]) = \Phi(g_1^*)\Phi((g_1^*)^{-1} d) = d\Phi((g_1^*)^{-1} d).$$

This yields that $\Phi((g_1^*)^{-1} d)$ =1. Now since g_1^* is by assumption strongly outer, we have that $\Delta(g_1) = \Delta(\Phi(g_1)) > 0$ by [63]. Consequently

$$\Delta(d) = \Delta(g_1^*)\Delta((g_1^*)^{-1} d) = \Delta(\Phi(g_1^*))\Delta((g_1^*)^{-1} d) = \Delta(d)\Delta((g_1^*)^{-1} d).$$

Thus since $\Delta(\mathrm{d}) > 0$ by the strong outerness of d, we must have that

$$\Delta((\mathrm{g}_1^*)^{-1}\mathrm{d}) = 1 = \Delta(\mathbb{I}) = \Delta(\Phi(\mathrm{g}_1^*)^{-1}\mathrm{d})).$$

Hence by [63] $((\mathrm{g}_1^*)^{-1}\mathrm{d})$ is a strongly outer element of $H^2(M)^*$. But this ensures that $[((g_1^*)^{-1} d)A_0^*] = H_0^2(M)^*$. Hence for any fixed $a \in A$, we may select a sequence $\{b_n\} \subset A_0$ so that

$$(g_1^*)^{-1} d b_n^* \to (P_+ - Id)[(g_1^*)^{-1} da] \in H_0^2(M)^* \; in \; L^2(M).$$

Finally recall that by assumption $|g_0| = |(g_1^*)^{-1} d|$. So given any $a \in A$, with $\{b_n\} \subset A_0$ the sequence as constructed above, we have that

$$\|g_0 a\|_2 = \|a^*\|_\omega \le B\|a^* + b_n\|_\omega = B\|g_0(a + b_n^*)\|_2 = B\||g_0|(a + b_n^*)\|_2$$
$$= B\||(g_1^*)^{-1}|(a + b_n^*)\|_2 = B\||(g_1^*)^{-1}|(a + b_n^*)\|_2.$$

Letting $n \to \infty$ now yields

$\|g_0 a\|_2 \le B\|P_+[(g_1^*)^{-1} da]\| = B\|T_u(g_0 a)\|_2$ for any$a \in A$.

Finally note that by assumption $g_0$is an outer element of $H^2(M)$. With $g_0 A$ therefore being dense in $H^2(M$, the above inequality extends by continuity to the claim that

$\|a\|_2 \le B\|T_u(a)\|_2$for any$a \in H^2(M)$.

Thus $T_u$is invertible.

Definition (5.3.12)[170]: Given $f \in M$ we define the Hankel operator with symbol f by means of the prescription

$$\mathcal{H}_f : H^2(M) \to H^2(M)^* : x \to P_-(fx),$$

where P_- is the orthogonal projection from $L^2(M)$ onto$H^2(M)^*$.

The following lemma is entirely elementary.

Lemma (5.3.13)[170]: Let $f \in M$ be given. Then

$$\|\mathcal{H}_f|_{H_0^2}\| = sup\{|\tau(fF)|: F \in H_0^1\,(M), \tau(|F|) \le 1\}.$$

Proof: Since for every $x \in H^2(M)$ we have that$(Id - P_-)(x) \in H_0^2\,(M)$, it is clear that such an $(Id - P_-)(x)$will be orthogonal to any $y \in H^2(M)^*$. Thus $\langle P_-(fa), b\rangle = \langle fa, b\rangle$for any $a \in H_0^2\,(M)$ and $b \in H^2(M)^*$.Thus

$$\|\mathcal{H}_f|_{H_0^2}\| = sup\{\|P - (fa)\|: a \in H_0^2\,(M), \|a\|_2 \le 1\} = sup\{|\langle P_-(fa), b\rangle|: a \in H_0^2\,(M), b \in H^2(M)^*\|a\|_2 \le 1, \|b\|_2 \le 1\} = sup\{|\langle fa, b\rangle|: a \in H_0^2\,(M), b \in H^2(M)^*\|a\|_2 \le 1, \|b\|_2 \le 1\} = sup\{|\tau(fa, b^*)|: a \in H_0^2\,(M), b \in H^2(M)^*\|a\|_2 \le 1, \|b\|_2 \le 1\} = sup\{|\tau(fF)|: F \in H_0^1(M), \tau(|F|) \le 1\}.$$

Here the last equality follows from the Noncommutative Riesz Factorisation theorem from [153] and[176].

When taken alongside Theorem (5.3.9), this result fully characterises invertible Toeplitz operators.

Theorem (5.3.14)[170]: Let $u \in M$ be a unitary of the form described in Lemma (5.3.10). Then the following are equivalent:

(i) $T_u$is invertible;

(ii)there exists $k \in A$ such that $\Re(u^*k)$ is strictly positive;

(iii) The Hankel operator H_u restricted to $H_0^2\,(M)$ has norm less than 1.

Proof: Our aim is to apply Theorem (5.3.8). In this regard we point out that although this theorem is formulated for norm one elements of $L^1(M)^+$, that assumption is one of convenience and not necessity. Hence the value of $\|\omega\|_1$ is no essential obstruction to applying this theorem. Next observe that the fact that $\omega = g_0^*g_0$, not only ensures that $\Delta(\omega) = \Delta(g_0)^2 > 0$, but also that w is injective. Thus by Lemma (5.3.2), $s\big(\Phi(\omega)\big) = \mathbb{1}$. We showed in the proof of the preceding Lemmathat $\Delta((g_1^*)^{-1}d) = 1 = \Delta(\Phi((g_1^*)^{-1}d))$. Applying this fact to $d^*g_1^{-1}$ enables us to conclude from [63] that $d^*g_1^{-1}$ is a strongly outer element of $H^2(M)$. On setting $h_R = d^*g_1^{-1}$ and $h_L = g_0$, it follows that ω is of the form

$$\omega = d^*g_1^{-1}(g_1^*)^{-1}d = d^*g_1^{-1}\,[(g_1^*)^{-1}dg_0^{-1}\,]g_0 = h_R u h_L$$

with h_R and h_L strongly outer elements of $H^2(M)$ for which we have that $|h_L| = |g_0| = \omega^{1/2}and|h_R^*| = |(g_1^*)^{-1}d| = |\omega|^{1/2}$.

With all the other conditions of this theorem being satisfied, we may now conclude from Theorem (5.3.9) that A and A_0^* are at positive angle with respect to the functional $\tau(\omega)$ if and only if there exists a $k \in A$ such that $\Re(u^*k)$ is strictly positive. From the proof of Theorem (5.3.8) we also have that A and A_0^* are at positive angle if and only if $sup\{|\tau(fF)|: F \in H_0^1(M), \tau(|F|) \le 1\} < 1$. The result now follows from an application of the preceding two lemmata.

Chapter 6
Structure of Commutative Toeplitz Banach Algebras

.We prove the analogous commutability result for Toeplitz operators whose symbols are subordinated to the quasi-nilpotent group. At the same time we conjecture that apart from the known C^*-algebra cases there are no more new Banach algebras generated by Toeplitz operators whose symbols are subordinated to the nilpotent group and which are commutative on each weighted Bergman space.We explicitly describe the maximal ideal space and the Gelfand map of $\mathcal{T}(\lambda)$. Since $\mathcal{T}(\lambda)$ is not invariant under the $*$-operation of $\mathcal{L}(\mathcal{A}_\lambda^2(\mathbb{B}^n))$ its inverse closedness is not obvious and is shown. We remark that the algebra $\mathcal{T}(\lambda)$ is not semi-simple and we derive its radical. Several applications of our results are given and, in particular, we conclude that the essential spectrum of elements in $\mathcal{T}(\lambda)$ is always connected. We show that $\mathcal{B}_k(h)$is generated in fact by an essentially smaller set of operators,i.e.,theToeplitz operators with k-quasi-radial symbols and a finite set of Toeplitz operators with "elementary"k-quasi-homogeneous symbols.Then we analyze the structure of the commutative subalgebras corresponding to these two types of generating symbols . In particular,we describe spectra, joint spectra, maximal ideal spaces and the Gelfand transform.

Section (6.1) Quasi-Nilpotent Group Action

We finish the classification of the Banach and C^*-algebras generated by Toeplitz operators that are commutativeon each (commonly considered) weighted Bergman space over the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ .The short history of this problem is as follows.

The C^*-algebras generated by Toeplitz operators which are commutative on each weighted Bergman space over the unit disk were completely classified in [98] .Under some technical assumption on "richness"of a class of generating symbols the result was as follows. A C^*-algebra generated by Toeplitz operators is commutative on each weighted Bergman space if and only if the corresponding symbols of Toeplitz operators are constant on cycles of a pencil of hyperbolic geodesicson the unit disk, or if and only if the corresponding symbols of Toeplitz operators areinvariant under the action of a maximal commutative subgroupof the Mobius transformations of the unit disk. The commutativity on each weighted Bergman spacewas crucial in the part"only if" of the above result.

Generalizing this result to Toeplitz operators on the unit ball, it was proved in[219,218], that, given a maximal commutative subgroup of biholomorphisms of the unit ball, theC^*-algebra generated by Toeplitz operators, whose symbols are invariant under the action of this subgroup, is commutative on each weighted Bergman space. There are five different pair wise non-conjugate model classes of such subgroups :quasi-elliptic, quasi-parabolic، quasi-hyperbolic, nilpotent, andquasi-nilpotent) the last one depends on a parameter, giving in total $n+2$ model classes for then-dimensional unit ball). As a consequence, for the unit ball of dimension n, there are $n+2$ essentially different "model" commutative -C^*-algebras, all others are conjugated with one of them via biholomorphisms of the unit ball.

It was firmly expected that the above algebras exhaust all possible algebras of Toeplitz operators which are commutative on each weighted Bergman space. That is, the invariance under the action of a maximal commutative subgroup of biholomorphisms for generating symbols is the only reason for the appearance of Toeplitz operator algebras which are commutative on each weighted Bergman space.

Recently and quite unexpectedly it was observed in[306] that for $n - 1$ there are many other, not geometrically defined, classes of symbols which generate commutative Toeplitz operator algebras on each weighted Bergman space. These classes of symbols were in a sense originated from, or subordinated to the quasi-elliptic group, the corresponding commutative operator algebras were Banach ‹and being extended toC^*-algebras they became non-commutative. Moreover, for $n = 1$ all of them collapsed to the commutativeC^*-algebra generated by Toeplitz operators with radial symbols (one-dimensional quasi-elliptic case). These results were extended in [36,297] then to the classes of symbols, subordinated to the quasi-hyperbolic and quasi parabolic groups, which as well generate via corresponding Toeplitz operators classes of Banach algebras being commutative on each weighted Bergman space. That is, together with[193] ‹cover the multi-dimensional extensions of the (only) three model cases on the unit disk. The study of the last two model cases of maximal commutative subgroup of biholomorphisms of the unit ball, the nilpotent, and quasi-nilpotent groups (which appear only for $n > 1$and $n < 2$ respectively), was left as an important and interesting open question.
After many unsuccessful attempts to find commutative algebras generated by Toeplitz operators and subordinated to the nilpotent group we conjecture that a part from the known cases there are no more new Banach algebras generated by Toeplitz operators with symbols subordinated to the nilpotent group of biholomorphisms of the unit ball$\mathbb{B}^n$and commutative on each weighted Bergman space.
At the same time such commutative algebras subordinated to the quasi- nilpotent group do exist, is devoted to their description. According to our current understanding the only additional source for the appearance of (Banach) Toeplitz operator algebras which are commutative on each weighted Bergman space comes from a torus action on $\mathbb{B}^n$. The maximal commutative group of biholomorphisms, to which the symbols are subordinated, must contain the torus $\mathbb{T}^k$with $k \geq 2$, as a subgroup. In the case of the one-dimensional torus$\mathbb{T}$ the above commutative Toeplitz operator algebras collapse to known commutative C^*-algebras generated by Toeplitz operators whose symbols are invariant under the action of the maximal commutative group of biholomorphisms in question.

We recall some notation from [218] that are used throughout Let

$$\mathbb{B}^n := \{z = (z_1, \dots, z_2) \in \mathbb{C}^n : |z|^2 = |z_1|^2 + \cdots + |z_n|^2 < 1\}$$

be the unit ball in $\mathbb{C}^n$. The Siegel domain $D_n$in $\mathbb{C}^n$, which is an unbounded realization of the unit ball $\mathbb{B}^n$,has the form

$$D_n = \{z = (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : Im z_n - |z'|^2 > 0\}$$

Recall that the Cayley transform $\omega\colon \mathbb{B}^n \to D_n$ maps biholomorphically the unit ball$\mathbb{B}^n$ontoD_n . Let$v$be the usual Lebesgue measure on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and fix $\lambda > -1$. Then the standard weighted measure$\mu_\lambda$on$\mathbb{B}^n$with weight parameter$\lambda$is given by:

$$d\mu_\lambda := c_\lambda (1 - |z|^2)^\lambda \, dv \quad \text{and} \quad c_\lambda := \frac{\Gamma(n+\lambda+1)}{\pi^n \Gamma(\lambda+1)}$$

Here$c_\lambda$is a normalizing constant such that $\mu_\lambda(\mathbb{B}^n) = 1$ On$D_n$we can consider the corresponding weighted measure $\tilde{\mu}_\lambda$defined by. 𝕀

$$d\tilde{\mu}_\lambda(\xi', \xi_n) = \frac{c_\lambda}{4} (Im \xi_n - |\xi'|^2)^\lambda dv(\xi', \xi_n).$$

Let f be a function on $\mathbb{B}^n$, then we put $(u_\lambda f)(\xi) := 2^{n+\lambda+1}(1-i\xi_n)^{-n-\lambda-1} f \circ \omega^{-1}(\xi)$ where $\xi \in D_n$.A straightforward calculation shows, cf [36,218]

Lemma (6.1.1)[38]: Let $\lambda > -1$, then u_λ defines a unitary transformation of $L_2(\mathbb{B}^n, \mu_\lambda)$ onto $L_2(D_n, \tilde{\mu}_\lambda)$.

In the following we write $\mathcal{A}_\lambda^2(\mathbb{B}^n)$ and $\mathcal{A}_\lambda^2(D_n)$ for the weighted Bergman spaces of all complex analytic functions in $L_2(\mathbb{B}^n, \mu_\lambda)$ and $L_2(D_n, \tilde{\mu}_\lambda)$, respectively. It is known that by restriction $\mathcal{U}_\lambda$ defines a unitary transformation of $\mathcal{A}_\lambda^2(\mathbb{B}^n)$ and $\mathcal{A}_\lambda^2(D_n)$.

Let $B_{D_n,\lambda}$ be the Bergman projection of $L_2(D_n, \tilde{\mu}_\lambda)$ onto $\mathcal{A}_\lambda^2(D_n)$. Given a bounded measurable function $f \in L^\infty(D_n)$ we define the Toeplitz operator T_f acting on the weighted Bergman space $\mathcal{A}_\lambda^2(D_n)$ in the usual way by

$$T_f := B_{D_{n,\lambda}} M_f,$$

Where M_f denotes the multiplication by f. We study a class of commutative Banach algebras generated by Toeplitz operators on $\mathcal{A}_\lambda^2(D_n)$.

To simplify the notation we will not indicate the dependence of T_f on the weight parameter λ . Note that via the unitary transformation $\mathcal{U}_\lambda$ the results on Toeplitz operators acting on weighted Bergman spaces over D_n can be directly translated to the corresponding setting of Toeplitz operators on $\mathcal{A}_\lambda^2(\mathbb{B}^n)$.

Put $\mathcal{D} := \mathbb{C}^{n-1} \times \mathbb{R} \times \mathbb{R}_+$. Then the map:

$$\kappa: \mathcal{D} \longrightarrow \mathcal{D}_n: (z', u, v) \longmapsto (z', u + iv + i|z'|^2)$$

defines a diffeomorphism with inverse

$$\kappa^{-1}(z', z_n) = (z', Re\, z_n, Im z_n - |z'|^2).$$

Given a function f on D_n, we define $U_0 f := f \circ \kappa$ to obtain a function $U_0 f$ $on \mathcal{D}$. On the domain $\mathcal{D}$ we consider the measure

$$d\eta_\lambda(z', u, v) := \frac{c_\lambda}{4} v^\lambda dv(z', u, v).$$

We have the following, of. [36,219]

Lemma (6.1.2)[38]: The operator U_0 is unitary from $L_2(\mathcal{D}_n, \tilde{\mu}_\lambda)$ to $L_2(\mathcal{D}, \mu_\lambda)$ with inverse $U_0^{-1} = U_0^*$ given by $U_0^* f = f \circ \kappa^{-1}$.

We occasionally omit the dependence of the weight $\lambda > -1$ and put $\mathcal{A}_0(\mathcal{D}) := U_0 \mathcal{A}_\lambda^2(\mathcal{D}_n)$ which clearly forms a closed subspace of $L_2(\mathcal{D}, \eta_\lambda)$.

As was explained in [219,218] the classification of maximal commutative subgroups G of biholomorphisms of $\mathcal{D}_n$ or $\mathbb{B}^n$ yields five essentially different types. Corresponding to each type there are commutative Banach or C^*-algebras of Toeplitz operators acting on weighted Bergman spaces. The aim is to define such algebras in case of the quasi-nilpotent group G of biholomorphisms. We recall the definition.

Let $1 \leq k \leq n-2$. We rather use the notation $z = (z', \omega', z_n)$ for $z \in D_n$ where $z' \in \mathbb{C}^k$ and $\omega' \in \mathbb{C}^{n-k-1}$. The quasi-nilpotent group $\mathbb{T}^k \times \mathbb{R}^{n-k-1} \times \mathbb{R}$ acts on $\mathcal{D}_n$, cf. [218], as follows: given $(t, b, h) \in \mathbb{T}^k \times \mathbb{R}^{n-k-1} \times \mathbb{R}$, we have:

$$\mathcal{T}(t, b, h): (z', \omega', z_n) \to (tz', \omega' + b, z_n + h + 2i\omega'.b + i|b|^2).$$

Note that in the case $k = n-1$ we obtain the quasi-parabolic group, while

for $k=0$ the group action is called nilpotent.
On the domain $\mathcal{D}=\mathbb{C}^k\times\mathbb{C}^{n-k-1}\times\mathbb{R}\times\mathbb{R}_+$ we use the variables (z',ω',u,v) and we represent $L_2(\mathcal{D},\eta_\lambda)$in the form:

$$L_2(\mathcal{D},\eta_\lambda)=L_2(\mathbb{C}^k)\otimes L_2(\mathbb{C}^{n-k-1})\otimes L_2(\mathbb{R})\otimes L_2(\mathbb{R}_+,\eta_\lambda) \tag{1}$$

Let F be the Fourier transform on $L_2(\mathbb{R})$, and with respect to the decomposition (1) consider the unitary operators $U_1:=I\otimes I\otimes F\otimes I$ acting on $L_2(\mathcal{D},\eta_\lambda)$. With this notation we put $\mathcal{A}_1(D):=U_1(\mathcal{A}_0(\mathcal{D}))$.

Next, we introduce polar coordinates on $\mathbb{C}^{\mathrm{k}}$ and put $r=(r_1,\dots,r_k)=(|z_1'|,\dots,|z_k'|)$. Moreover, in the following we write $x':=Re\ \omega'$ and $y':=Im\ \omega$. Then one can check that r,y' and $Im_{z_n}-|\omega'|^2$ are invariant under the action of the quasi-nilpotent group. Following the ideas in [218] and with $rd_r=r_1d_{r_1}\dots r_kd_{rk}$ we represent $L_2(\mathcal{D},\eta_\lambda)$ in the form

$$L_2(\mathbb{R}_+^k,rdr)\otimes L_2(\mathbb{T}^k)L_2(\mathbb{R}^{n-k-1})\otimes L_2(\mathbb{R}^{n-k-1})\otimes L_2(\mathbb{R})\otimes L_2(\mathbb{R}_+,\eta_\lambda) \tag{2}$$

We define the unitary operator U_2 on $L_2(\mathcal{D},\eta_\lambda)$ by $U_2=I\otimes F_{(k)}\otimes F_{(n-k-1)}\otimes I\otimes I\otimes I$. Here $\mathcal{F}_{(k)}=\mathcal{F}\otimes\dots\otimes\mathcal{F}$ is the k-dimensional discrete Fourier transform and $F_{(n-k-1)}=F\otimes\dots\otimes F$ denotes the $(n-k-1)$-dimensional Fourier transform on$L_2(\mathbb{R}^{n-k-1})$. Note that $L_2(\mathcal{D},\eta_\lambda)$ is isometrically mapped by U_2 onto

$$\ell_2((\mathbb{Z}^k,L_2(\mathbb{R}_+^k)\otimes L_2(\mathbb{R}^{n-k-1})\otimes)L_2(\mathbb{R}^{n-k-1})\otimes L_2(\mathbb{R})\otimes L_2(\mathbb{R}_+,\eta_\lambda)) \tag{3}$$

We put $\mathcal{A}_2(D)=U_2(\mathcal{A}_1(D))$ and we write elements in (3) as $\{f_\beta(r,x',y',\xi,v)\}_{\beta\in\mathbb{Z}^k}$, where

$$(r,x',y',\xi,v)\in\mathbb{R}_+^k\times\mathbb{R}^{n-k-1}\times\mathbb{R}^{n-k-1}\times\mathbb{R}\times\mathbb{R}_+.$$

Next we recall the definition of the unitary operator U_3 which acts on (3) by:

$$U_3:\{f_\beta(r,x',y',\xi,v)\}_{\beta\in\mathbb{Z}}\mapsto\left\{f_\beta\left(r,\sqrt{\xi}(x',y'),\frac{1}{2\sqrt{\xi}}(-x',y'),\xi,v\right)\right\}_{\beta\in\mathbb{Z}^k}$$

One immediately checks that the inverse U_3^{-1} has the form

$$U_3^{-1}:\{f_\beta(r,x',y',\xi,v)\}_{\beta\in\mathbb{Z}}\mapsto\left\{f_\beta\left(r,\frac{x'}{2\sqrt{\xi}}-\sqrt{\xi}\mathcal{Y}',\frac{x'}{2\sqrt{\xi}}+\sqrt{\xi}\mathcal{Y}',\xi,v\right)\right\}_{\beta\in\mathbb{Z}^k}$$

In the following we write $\mathbb{Z}_+=\mathbb{N}\cup\{0\}=\{0,1,2,\dots\}$ for the nonnegative integers. In order to state the main result of Section in [218] we need to introduce the operator R_0, which defines an isometric embedding of $\ell_2\left(\mathbb{Z}_+^k,L_2(\mathbb{R}^{n-k-1}\times\mathbb{R}_+)\right)$into (3). It is explicitly given by

$$R_0:\{c_\partial(x',\xi)\}_{\beta\in\mathbb{Z}_+^k}\mapsto\left\{\chi_{\mathbb{Z}_+^k\times\mathbb{R}_+}(\beta,\xi)A_\beta(\xi)r^\beta e^{-\xi(|r|^2+v)-\frac{|v'|^2}{2}}c_\beta(x',\xi)\right\}_{\beta\in\mathbb{Z}^k}$$
$$=\{g_\beta(r,x',y',\xi,v)\}_{\beta\in\mathbb{Z}^k}$$

Here $\chi_{\mathbb{Z}_+^k\times\mathbb{R}_+}(\beta,\xi)$denotes the characteristic function of $\mathbb{Z}_+^k\times\mathbb{R}_+$ and $c_\beta(x',\xi)$ is extended by zero for $\xi\in(-\infty,0)$ and all $x'\in\mathbb{R}^{n-k-1}$. Moreover, we have used the abbreviation

$$A_\beta(\xi):=\pi^{-\frac{n-k-1}{4}}\sqrt{\frac{2^{k+2}}{C_\lambda}\frac{(2\xi)^{|\beta|+\lambda+k+1}}{\beta!\,\Gamma(\lambda+1)}} \tag{4}$$

The ad joint operator R_0^* is given by:

$$R_0^*:\{f_\beta(r,x',y',\xi,v)\}_{\beta\in\mathbb{Z}^k}\longmapsto\{A_\beta(\xi)\times\int_{\mathbb{R}_+^k\times\mathbb{R}^{n-k-1}\times\mathbb{R}_+}r^\beta e^{-\xi(|r|^2+v)-\frac{|v'|^2}{2}}f_\beta(r,x',y',\xi,v)rdrdy'\frac{C_\lambda v^\lambda}{4}dv\}_{\beta\in\mathbb{Z}_+^k} \tag{5}$$

We set $U:=U_3U_2U_1U_0$, which gives a unitary operator from $R_\lambda^2(\mathcal{D}_n)$ onto $\mathcal{A}_3(\mathcal{D}):=U_3(\mathcal{A}_2(\mathcal{D}))$. The following result has been proved in [218], and it provides a decomposition of the Bergman projection $B_{\mathcal{D}_n,\lambda}$ in form of a certain operator product.

Theorem (6.1.3)[38]: [218] The operator $R:=R_0^*U$ maps $L_2(\mathcal{D}_n,\tilde{\mu}_\lambda)$ onto the space $\ell_2\left(\mathbb{Z}_+^k,L_2(\mathbb{R}^{n-k-1}\times\mathbb{R}_+)\right)$, and the restriction

$$R_{|\mathcal{A}_\lambda^2(\mathcal{D})}\mathcal{A}_\lambda^2(\mathcal{D})\to\ell_2\left(\mathbb{Z}_+^k,L_2(\mathbb{R}^{n-k-1}\times\mathbb{R}_+)\right)$$

is an isometric isomorphism. The ad joint operator

$$R^*=U^*R_0:L_2\left(\mathbb{Z}_+^k,L_2(\mathbb{R}^{n-k-1}\times\mathbb{R}_+)\right)\to\mathcal{A}_\lambda^2(\mathcal{D})\subset L_2(\mathcal{D}_n,\tilde{\mu}_\lambda)$$

is an isometric isomorphism of $L_2\left(\mathbb{Z}_+^k,L_2(\mathbb{R}^{n-k-1}\times\mathbb{R}_+)\right)$onto the subspace $\mathcal{A}_\lambda^2(\mathcal{D})$of $L_2(\mathcal{D}_n,\tilde{\mu}_\lambda)$. Furthermore one has:

$$RR^*=I:L_2\left(\mathbb{Z}_+^k,L_2(\mathbb{R}^{n-k-1}\times\mathbb{R}_+)\right)\to L_2\left(\mathbb{Z}_+^k,L_2(\mathbb{R}^{n-k-1}\times\mathbb{R}_+)\right),$$
$$R^*R=B_{\mathcal{D}_{n,\lambda}}:L_2(\mathcal{D}_n,\tilde{\mu}_\lambda)\to\mathcal{A}_\lambda^2(\mathcal{D})$$

Now, we restrict our attention to bounded measurable symbols on $\mathcal{D}_n$ that are invariant or have a certain homogeneity with respect to the quasi-nilpotent group action on $\mathcal{D}_n$.

Definition (6.1.4)[38]: A bounded measurable function $\alpha:\mathcal{D}_n\to\mathbb{C}$ is called quasi-nilpotent if it has the form $\alpha(z)=\alpha(r,y',Imz_n-|\omega'|^2)$. In particular, such a is invariant under the action of the quasi-nilpotent group.

The following theorem was proved in [218].

Theorem (6.1.5)[38]: [218] Let $\alpha(z)=\alpha(r,y',Imz_n-|\omega'|^2)$ be a bounded measurable quasi-nilpotent function on $\mathcal{D}_n$. Then the Toeplitz operator T_α acting on $\mathcal{A}_\lambda^2(\mathcal{D})$*)* is unitary equivalent to the multiplication operator $\gamma_\alpha I=RT_\alpha R^*$ acting on the space $L_2\left(\mathbb{Z}_+^k,L_2(\mathbb{R}^{n-k-1}\times\mathbb{R}_+)\right)$. The sequence

$$\gamma_\alpha=\{\gamma_\alpha(\beta,x',\xi)\}_{\beta\in\mathbb{Z}_+^k}\in L_2\left(\mathbb{Z}_+^k,L_2(\mathbb{R}^{n-k-1}\times\mathbb{R}_+)\right)$$

With $(x',\xi)\in\mathbb{R}^{n-k-1}\times\mathbb{R}_+$ is given by

$$\gamma_\alpha(\beta,x',\xi)=2^k\pi^{-\frac{n-k-1}{2}}\frac{(2\xi)^{|\beta|+\lambda+k+1}}{\beta!\,\Gamma(\lambda+1)}\int_{\mathbb{R}_+^k\times\mathbb{R}^{n-k-1}\times\mathbb{R}_+}\alpha\left(r,\frac{1}{2\sqrt{\xi}}(-x'+y'),v+(|r|^2\right)$$
$$\times r^\beta e^{-2\xi(v+(|r|^2)-|y'|^2}v^\lambda r dr dy' dv.$$

We need to prove a similar result for a class of more general symbols.

Recall that we use the notation $x' \coloneqq Re\omega' \in \mathbb{R}^{n-k-1}$where $\omega' \in \mathbb{C}^{n-k-1}$and let $\alpha = (\alpha_1,\dots,\alpha_m)$be a tuple in $\mathbb{Z}_+^m$such that $|\alpha| = (\alpha_1,\dots,\alpha_m) = k$ Similar to [38,193] we divide the coordinates of $z' \in \mathbb{C}^k$ into m groups as follows:

$z'_{(1)} = \left(z'_{1,1},\dots,z'_{1,\alpha_1}\right), z'_{(2)} = \left(z'_{2,1},\dots,z'_{2,\alpha_2}\right),\dots, z'_{(m)} = \left(z'_{m,1},\dots,z'_{m,\alpha_m}\right)$and such that $z' = \left(z'_{(1)},z'_{(2)},\dots,z'_{(m)}\right)$. In the following we will use the same notation also in case of multi-indices $\beta \in \mathbb{Z}^k$instead of vectors $z' \in \mathbb{C}^k$. By passing to polar coordinates, we write each tuple $z'_{(j)} = \left(z'_{j,1},\dots,z'_{j,\alpha j}\right)$where $j = 1,\dots,m,$ in the form

$$z'_{(j)} \text{ or } r_j\xi_{(j)} \text{ with } r_j = \sqrt{\left|Z'_{j,1}\right|^2+\cdots+\left|Z'_{j,\alpha j}\right|^2} \text{ and} \xi_{(j)} \in \mathbb{S}^{2\alpha_j-1} \subset \mathbb{C}^{\alpha_j}.$$

Here $\mathbb{S}^{2\alpha_j-1}$denotes the real $(2n-1)$ -dimensional boundary of $\mathbb{B}^n$.

Definition (6.1.6))[38]: Let $\alpha(r,y'\mathrm{Im}\mathbb{Z}_n-|\omega'|^2)$ be a quasi-nilpotent function and $\alpha \in \mathbb{Z}_+^m$as above.

(i) Then α is called “α-quasi-nilpotent quasi-radial” if its radial dependence on r can be expressed as a function of $r_1,\dots,r_m$.

(ii) The function $b = (z',\omega',\mathbb{Z}_n)$ is called “α-quasi-nilpotent quasi-homogeneous”if it is α-quasi-nilpotent quasi-homogeneous with respect to the variable z', i.e.

$$b = (z',\omega',z_n) = b_0(r_1,\dots,r_m,y'Im\mathbb{Z}_n-|\omega'|^2)\xi^p\bar{\xi}^q \tag{6}$$

Where $\xi = (\xi_1,\xi_2,\dots,\xi_m) \in \mathbb{S}^{2\alpha_1-1}\times\mathbb{S}^{2\alpha_2-1}\times\dots\times\mathbb{S}^{2\alpha_m-1}$ and $p,q \in \mathbb{Z}_+^m$ are orthogonal. The pair (p,q) is then called the “degree”of b.

Note that there is a one-to-one correspondence between the set of tuples $\{(p,q) \in \mathbb{Z}_+^m \times \mathbb{Z}_+^m : p \perp q\}$ and $\mathbb{Z}^k$ via $(p,q) \longmapsto p-q$.

Consider an α-quasi-nilpotent quasi-homogeneous symbol $b = (z',\omega',\mathbb{Z}_n)$ as in (6) and of degree $(p,q) \in \mathbb{Z}_+^k \times \mathbb{Z}_+^k$ with $p \perp q$. Our next aim is to calculate the operator RT_bR^*. On the domain $\mathcal{D} = \mathbb{C}^k \times \mathbb{C}^{n-k-1} \times \mathbb{R} \times \mathbb{R}_+$we use the variables (z',ω',u,v).Moreover,we express z'in polar coordinates $z' = t_{1r1},\dots,t_{krk}$ where $r_s \geq 0$ and $t_s \in \mathbb{s} = S^1$for $s = 1,\dots,k$. Then we have the relations

$$z_{j,\ell} = r_j\xi_{j,\ell} = t_{j,\ell}r_{j,\ell}$$

for $\ell = \{1,\dots,\alpha_j\}$ and $j = 1,\dots,m$. It follows that $\xi_{j,\ell} = t_{j,\ell}r_{j,\ell}{r_j}^{-1}$ in the case of$r_j \neq 0$ and therefore:

$$\xi^p\bar{\xi}^q = t^p\bar{t}^q r^{p+q}\prod_{j=1}^m {r_j}^{-|p_j|-|q_j|} \tag{7}$$

Note that the assignment $z' \longmapsto \xi^p\bar{\xi}^q$ depends on the initial choice of $\alpha \in \mathbb{Z}_+^m$.

Using Theorem (6.1.1) we can write:

$$RT_bR^* = RB_{\mathcal{D}_{n,\lambda}}bB_{\mathcal{D}_{n,\lambda}}R^* = R(R^*R)b(R^*R)R^* = (RR^*)RbR^*(RR^*) = RbR^*$$
$$= R_0^*U_3U_2U_1U_0bU_0^{-1}U_1^{-1}U_2^{-1}U_3^{-1}R_0$$
$$= R_0^*U_3U_2U_1b_0(r_1,\dots,r_m,y'Imz_n - |w'|^2)\xi^p\bar{\xi}^qU_0^{-1}U_1^{-1}U_2^{-1}U_3^{-1}R_0$$

First we calculate the operator $U_0bU_0^{-1}$. Let $\{f_\beta(r,x',y',\xi,v)\}_{\beta\in Z_+^k}$be an element in the space (3) and write $r := (r_1,\dots,r_m)$. Since the ymbol $b_0(r_1,y',v+|r|^2)\xi^p\bar{\xi}^q$ is independent of x' we obtain from (7) that:

$$U_2b_0(r,y',v+|r|^2)\xi^p\bar{\xi}^qU_2^{-1}\{f_\beta(r,x',y',\xi,v)\}_{\beta\in\mathbb{Z}_+^k} \quad (8)$$
$$\left\{b_0(r,y',v+|r|^2)r^{p-q}\left(\prod_{j=1}^m r_j^{-|p_j|-|q_j|}\right)f_{\beta-p+q}(r,x',y',\xi,v)\right\}_{\beta\in\mathbb{Z}^k}$$

Combining (8) and (5) gives:

$$RT_bR^*:\{C_\beta(x',\xi)\}_{\beta\in\mathbb{Z}_+^k} = R_0^*U_3U_2bU_2^{-1}U_3^{-1}\Big\{\chi_{\mathbb{Z}_{+\times\mathbb{R}_+}^k}(\beta,\xi)$$
$$A_\beta(\xi)r^\beta e^{-\xi(|r|^2+v)-\frac{|v'|^2}{2}}c_\beta(x',\xi)\Big\}_{\beta\in\mathbb{Z}^k}$$
$$= R_0^*U_3U_2bU_2^{-1}\Big\{\chi_{\mathbb{Z}_+^k\times\mathbb{R}_+}(\beta,\xi)A_\beta(\xi)r^\beta$$
$$\times e^{-\xi(|r|^2+v)-\frac{1}{2}-\left|\frac{1}{2\sqrt{\xi}}x'+\sqrt{\xi}y'\right|^2}c_\beta\left(\frac{1}{2\sqrt{\xi}}x'-\sqrt{\xi}y',\xi\right)\Big\}_{\beta\in\mathbb{Z}^k}$$
$$= R_0^*U_3\Big\{\chi_{\mathbb{Z}_+^k\times\mathbb{R}_+}(\beta-p+q,\xi)A_{\beta-p+q}(\xi)r^{\beta+2q}b_0\times(r,y',v+|r|^2)$$
$$\times\left(\prod_{j=1}^m r_j^{-|p_j|-|q_j|}\right)e^{-\xi(|r|^2+v)-\frac{1}{2}-\left|\frac{1}{2\sqrt{\xi}}x'+\sqrt{\xi}y'\right|^2}$$
$$\times C_{\beta-p+q}\left(\frac{1}{2\sqrt{\xi}}x'-\sqrt{\xi}y',\xi\right)\Big\}_{\beta\in\mathbb{Z}^k}$$
$$= R_0^*\Big\{\chi_{\mathbb{Z}_+^k\times\mathbb{R}_+}(\beta-p+q,\xi)A_{\beta-p+q}(\xi)b_0\left(r,\frac{-x'+y'}{2\sqrt{\xi}},v+|r|^2\right)$$
$$\times\left(\prod_{j=1}^m r_j^{-|p_j|-|q_j|}\right)r^{\beta+2q}e^{-\xi(|r|^2+v)-\frac{1}{2}-|y'|^2}C_{\beta-p+q}(x',\xi)\Big\}_{\beta\in\mathbb{Z}^k}$$

$$= \left\{A_\beta(\xi)A_{\beta-p+q}(\xi)\chi_{\mathbb{Z}_+^k\times\mathbb{R}_+}(\beta-p+q,\xi)C_{\beta-p+q}(x',\xi)\right\}_{\beta\in Z^k}$$
$$\times\int_{\mathbb{R}_+^k\times\mathbb{R}^{n-k-1}\times\mathbb{R}_+} r^{2(\beta+q)}\left(\prod_{j=1}^m r_j^{-|p_j|-|q_j|}\right)e^{-\xi(|r|^2+v)-|y|^2}$$
$$\left.\times\, b_0\left(r,\frac{-x'+y'}{2\sqrt{\xi}},v+|r|^2\right)rdrdy'\,\frac{C_\lambda v^\lambda}{4}\,dv\right\}_{\beta\in\mathbb{Z}_+^k}$$

Now put:

$\gamma_{b,p,q}(\beta,x',\xi)$:

$$= A_\beta(\xi)A_{\beta-p+q}(\xi)\chi_{\mathbb{Z}_+^k\times\mathbb{R}_+}(\beta-p+q,\xi)\int_{\mathbb{R}_+^k\times\mathbb{R}^{n-k-1}\times\mathbb{R}_+}\prod_{j=1}^m r_j^{-|p_j|-|q_j|}$$
$$\times\, r^{2(\beta+q)}e^{-2\xi(|r|^2+v)-|y|^2}\times b_0\left(r,\frac{-x'+y'}{2\sqrt{\xi}},v+|r|^2\right)rdrdy'$$
$$\times\frac{C_\lambda v^\lambda}{4}dv \qquad (9)$$

Hence, we have proved:

Theorem (6.1.7))[38]: Let b be defined as in (6). The operator RT_bR^* acts on the Hilbert space $\ell_2\left(\mathbb{Z}_+^k,L_2(\mathbb{R}^{n-k-1}\times\mathbb{R}_+)\right)$by the rule:

$RT_bR^*\colon\left\{C_\beta(x',\xi)\right\}_{\beta\in\mathbb{Z}_+^k} = \left\{\gamma_{b,p,q}(\beta,x',\xi).\,C_{\beta-p+q}(x',\xi)\right\}_{\beta\in Z^k}$

Note that, in the case $p=q=0$,Theorem (6.1.8) reduces to Theorem (6.1.5).

Example(6.1.8)[38]: We calculate RT_bR^*more explicitly in the special case where$b_0\equiv 1$ and we choose $k=m$, i.e. $\alpha=(1,\dots,1)\in\mathbb{Z}_+^k$. Let $(p,q)\in\mathbb{Z}_+^k$such that $p\perp q$ and put

$$b(z',\omega',z_n)=\xi^p\bar{\xi}^q=t^p\bar{t}^q$$

According to Theorem (6.1.7) it is sufficient to calculate the functions:

$\gamma_{b,p,q}(\beta,x',\xi)$:

$$= A_\beta(\xi)A_{\beta-p+q}(\xi)\chi_{\mathbb{R}_+}(\xi)$$
$$\times\int_{\mathbb{R}_+^k\times\mathbb{R}^{n-k-1}\times\mathbb{R}_+} r^{2\beta+q-p}\,e^{-2\xi(|r|^2+v)-|y|^2}rdrdy'\,\frac{C_\lambda v^\lambda}{4}dv$$

for all $\beta\in\mathbb{Z}_+^k$ with $\beta-p+q\in\mathbb{Z}_+^k$. We use the identity:

$$\int_{\mathbb{R}_+^k\times\mathbb{R}^{n-k-1}\times\mathbb{R}_+} e^{-2\xi v-|y|^2}\,dy'v^\lambda dv=\pi^{\frac{n-k-1}{2}}\Gamma(\lambda+1)(2\xi)^{-(\lambda+1)}$$

(cf [97]) where $\xi>0$, which together with (4) shows that

$$\gamma_{b,p,q}(\beta,x',\xi)=2^k(2\xi)^{|\beta|+k+\frac{|q|-|p|}{2}}\frac{1}{\sqrt{\beta!\,(\beta-p+q)!}}\int_{\mathbb{R}_+^k} r^{2\beta+q-p}\,e^{-2\xi-|y|^2}dr$$
$$=\frac{\prod_{j=1}^k\Gamma\left(\beta_j+\dfrac{q_j-p_j}{2}+1\right)}{\sqrt{\beta!\,(\beta-p+q)!}}.$$

In particular, in this case $\gamma_{b,p,q}(\beta,x',\xi)$is independent of x' and ξ.

The goal is to study the commutatively of Toeplitz operators with symbols having certain invariance properties. We will use the above notation. Fix $\alpha \in \mathbb{Z}_+^m$ with $|\alpha| = k$ as before and let $\alpha = \alpha_0(r_1, \dots, r_m, y' Imz_n - |\omega'|^2)$ be a bounded measurable α-quasi-nilpotent quasi-radial function on D_n. Consider the symbol:

$$b(z', \omega', z_n) = b_0(r_1, \dots, r_m, y' Imz_n - |\omega'|^2). \xi^p \bar{\xi}^q \qquad (10)$$

We calculate the operator products $RT_bT_aR^*$ and $RT_aT_bR^*$. According to Theorem (6.1.7) and Theorem (6.1.1) we have

$$\begin{aligned} RT_bT_aR^*\{c_\beta\}_{\beta\in\mathbb{Z}_+^k} &= (RT_bR^*)(RT_aR^*)\{c_\beta\}_{\beta\in\mathbb{Z}_+^k} \\ &= (RT_bR^*)\{\bar{\gamma}_{b,p,q}(\beta, x', \xi). c_\beta(x', \xi)\}_{\beta\in\mathbb{Z}_+^k} \\ &= \{\bar{\gamma}_{b,p,q}(\beta, x', \xi)\gamma_{a,0,0}(\beta - p + q, x', \xi)c_{\beta-p+q}(x', \xi)\}_{\beta\in\mathbb{Z}_+^k} \qquad (11) \end{aligned}$$

On the other hand it follows:

$$\begin{aligned} RT_aT_bR^*\{c_\beta\}_{\beta\in\mathbb{Z}_+^k} &= (RT_aR^*)(RT_bR^*)\{c_\beta\}_{\beta\in\mathbb{Z}_+^k} = (RT_aR^*)\{\bar{\gamma}_{b,p,q}(\beta, x', \xi). c_\beta(x', \xi)\}_{\beta\in\mathbb{Z}_+^k} \\ &= \{\bar{\gamma}_{a,0,0}(\beta, x', \xi)\bar{\gamma}_{b,p,q}(\beta, x', \xi)c_{\beta-p+q}(x', \xi)\}_{\beta\in\mathbb{Z}_+^k} \qquad (12) \end{aligned}$$

Hence, we conclude from (11) and (12) that both operators T_a and T_b commute if and only if

$$\bar{\gamma}_{a,0,0}(\beta, x', \xi) = \bar{\gamma}_{a,0,0}(\beta - p + q, x', \xi)$$

for all $\beta \in \mathbb{Z}_+^k$. According to (9) this is equivalent to:

$$\begin{aligned} &\frac{1}{\beta!}\int_{\mathbb{R}_+^k\times\mathbb{R}^{n-k-1}\times\mathbb{R}_+} a_0\left(r, \frac{-x'+y'}{2\sqrt{\xi}}, v + |r|^2\right) r^{2\beta} e^{-2\xi(v-|r|^2)-|y'|^2} v^\lambda r dr dy' dv \\ &= \frac{(2\xi)^{-|p|+|q|}}{(\beta - p + q)!}\int_{\mathbb{R}_+^k\times\mathbb{R}^{n-k-1}\times\mathbb{R}_+} a_0\left(r, \frac{-x'+y'}{2\sqrt{\xi}}, v + |r|^2\right) \\ &\qquad \times r^{2\beta+q-p} e^{-2\xi(v+|r|^2)-|y'|^2} v^\lambda r dr dy' dv \qquad (13) \end{aligned}$$

Since $a_0(r, y', Imz_n - |\omega'|^2)$only depends on $r = (r_1, \dots, r_m)$we can assume that the above integral has the form:

$$\int_{\mathbb{R}_+^k\times\mathbb{R}^{n-k-1}\times\mathbb{R}_+} a_0\left(r, \frac{-x'+y'}{2\sqrt{\xi}}, v + |r|^2\right) r^{2\beta} e^{-2\xi(v+|r|^2)-|y'|^2} v^\lambda r dr dy' dv =: (*),$$

Where $\beta \in \mathbb{Z}_+^k$. With $e = (1,1, \dots, 1) \in \mathbb{Z}_+^k$ we obtain

$$\begin{aligned} (*) &= \frac{1}{2^k}\int_{\mathbb{R}^k\times\mathbb{R}^{n-k-1}\times\mathbb{R}_+} a_0\left(r, \frac{-x'+y'}{2\sqrt{\xi}}, v + |r|^2\right) |r^{2\beta}| e^{-2\xi(v+|r|^2)-|y'|^2} v^\lambda r dr dy' dv \\ &= \frac{1}{2^k}\int_{\mathbb{R}^{n-k-1}\times\mathbb{R}_+}\int_{\mathbb{R}_+^m\times\mathbb{S}^{a_1-1}\times\mathbb{S}^{a_m-1}} a_0\left(r, \frac{-x'+y'}{2\sqrt{\xi}}, v + |r|^2\right) \\ &\times |\rho^{2\beta+e}|. \left(\prod_{j=1}^m r_j^{2|\beta_{(j)}|+2a_j-1}\right) e^{-2\xi(v+|r|^2)-|y'|^2} v^\lambda d\sigma(\rho_{(1)}) \dots d\sigma(\rho_{(m)}) r dr dy' dv. \end{aligned}$$

In the last integral we wrote $d\sigma(\rho_{(j)})$ for the standard area measure on the sphere $\mathbb{S}^{a_j-1}$. The integral over the m-fold product $\mathbb{S}^{a_1-1} \times \dots \times \mathbb{S}^{a_m-1}$can be calculated explicitly by using the following well-known formula:

Lemma (6.1.9))[38]: Let $d\sigma$ denote the usual surface measure on the $(n-1)$ −dimensional sphere $\mathbb{S}^{n-1}$ and let $\theta \in \mathbb{Z}_+^k$. Then

$$\int_{\mathbb{S}^{n-1}} |y^\theta| d\sigma(y) = \frac{2\Gamma\left(\frac{\theta_1 - 1}{2}\right) \dots \Gamma\left(\frac{\theta_n - 1}{2}\right)}{\Gamma\left(\frac{n-|\theta|}{2}\right)}$$

Using the formula in Lemma (6.1.9) we define:

$$\Theta_\beta := \int_{\mathbb{S}^{a_1-1}\times\dots\times\mathbb{S}^{a_m-1}} |\rho^{2\beta+e}| d\sigma(\rho_{(1)}) \dots d\sigma(\rho_{(m)})$$
$$= 2^m \beta! \prod_{j=1}^{m} r\left(\frac{a_j+1}{2} + |\beta_{(j)}|\right)^{-1} \qquad (14)$$

This finally gives:

$$(*) = \frac{\ominus_\beta}{2^k} := \int_{\mathbb{R}_+^m \times \mathbb{R}^{n-k-1} \times \mathbb{R}_+} a_0\left(r, \frac{1}{2\sqrt{\xi}}(-x'+y'), v + |r|^2\right)$$
$$\times r_1^{2|\beta_{(1)}|+2a_1-1} \dots r_m^{2|\beta_{(m)}|+2a_1-m} e^{-2\xi(v+|r|^2)-|y'|^2} v^\lambda dr dy' dv.$$

Note that the last integral does not depend on the full multi-index β but rather on the values $|\beta_{(1)}|$ for $j = 1, \dots, m$. We denote this integral by $G_a(|\beta_{(1)}|, \dots, |\beta_{(m)}|)$. Then the commutativity condition (13) can be written in the form:

$$\frac{\Theta_\beta}{\beta!} G_a(|\beta_{(1)}|, \dots, |\beta_{(m)}|)$$
$$= (2\xi)^{-|p|+|q|} \frac{\Theta_{\beta-p+q}}{(\beta - P - q)!}$$
$$\times G_a(|\beta_{(1)}| - |p_{(1)}| + |q_{(1)}|, \dots, |\beta_{(m)}| - |p_{(m)}| + |q_{(m)}|).$$

According to the definition (14) this is equivalent to:

$$G_a(|\beta_{(1)}|, \dots, |\beta_{(m)}|) \prod_{j=1}^{m} \Gamma\left(\frac{a_j+1}{2} + |\beta_{(j)}|\right)^{-1}$$
$$= (2\xi)^{-|p|+|q|} G_a(|\beta_{(1)}| - |p_{(1)}| + |q_{(1)}|, \dots, |\beta_{(m)}| - |p_{(m)}| + |q_{(m)}|)$$
$$\times \prod_{j=1}^{m} \Gamma\left(\frac{a_j+1}{2} + |\beta_{(j)}| - |p_{(j)}| + |q_{(j)}|\right)^{-1}$$

This equality can be only true simultaneously for all α-quasi-nilpotent quasi-radial functions a and all $\beta \in \mathbb{Z}_+^k$ if $|p_{(j)}| = |q_{(j)}|$ for $j = 1, \dots, m$. Hence, we obtain:

Theorem (6.1.10))[38]: Let $a \in \mathbb{Z}_+^k$ be given. Then the statements (i), (ii) and (iii)below are equivalent:

(i) For each α-quasi-nilpotent quasi-radial function $a = a_0(r_1, y' Imz_n - |\omega'|^2) \in L^\infty(D_n)$ and each α-quasi-nilpotent quasi-homogeneous function

$$b = b_0(r_1, \dots, r_m, y' Imz_n - |\omega'|^2)\xi^p \bar{\xi}^q \in L^\infty(D_n) \qquad (15)$$

of degree $(p,q) \in \mathbb{Z}_+^k \times \mathbb{Z}_+^k$ the Toeplitz operators T_a and T_a commute on each weighted Bergman space $\mathcal{A}_\lambda^2(\mathrm{D_n})$.

(ii) The equality $\bar{\gamma}_{a,0,0}(\beta,x',\xi) = \bar{\gamma}_{a,0,0}(\beta-p+q,x',\xi)$holds for all $\beta \in \mathbb{Z}_+^k$ and for each α-quasi-nilpotent quasi-radial functions a.

(iii) The equality $|p_{(j)}| = |q_{(j)}|$ holds for each $j = 1,\dots,m$.

Now, let us assume that $b \in L^\infty(D_n)$ is of the form (15). Under the assumption $|p_{(j)}| = |q_{(j)}|$, for each $j = 1,\dots,m$, we calculate $\bar{\gamma}_{b,p,q}(\beta,x',\xi)$ in (9) more explicitly by reducing the order of integration. Assume that $\beta - p + q \in \mathbb{z}_+^k$. Then:

$$\begin{aligned}
&\bar{\gamma}_{b,p,q}(\beta,x',\xi)\\
&= A_\beta(\xi)A_{\beta-p+q}(\xi)\chi_{\mathbb{R}_+}(\xi)\int_{\mathbb{R}_+^k\times\mathbb{R}^{n-k-1}\times\mathbb{R}_+} r^{2(\beta+q)}\\
&\times\prod_{j=1}^{m} r_j^{-|p(j)|-|q(j)|}e^{-2\xi(|r|^2+v)-|y'|^2}b_0\\
&\times\left(r,\frac{-x'+y'}{2\sqrt{\xi}},v+|r|^2\right)rdrdy'\frac{c\lambda v^\lambda}{4}dv\\
&= \Theta_{\beta+q}A_\beta(\xi)A_{\beta-p+q}(\xi)\chi_{\mathbb{R}_+}(\xi)2^{-k}\\
&\times\int_{\mathbb{R}_+^m\times\mathbb{R}^{n-k-1}\times\mathbb{R}_+}\prod_{j=1}^{m} r_j^{2|\beta(j)|+|q(j)|-|p(j)|+2\alpha_j-1}\\
&\times e^{-2\xi(|r|^2+v)-|y'|^2}b_0\left(r,\frac{-x'+y'}{2\sqrt{\xi}},v+|r|^2\right)rdrdy'\frac{c\lambda v^\lambda}{4}dv\\
&= \frac{\Theta_{\beta+q}}{\Theta_\beta}\frac{A_{\beta-p+q}(\xi)}{A_\beta(\xi)}.D_b(\beta,x',\xi)\\
&= \frac{(\beta+q)!}{\sqrt{\beta!\,(\beta-p+q)!}}\prod_{j=1}^{m}\frac{\Gamma\left(\frac{\alpha_j+1}{2}+|\beta_{(j)}|\right)}{\Gamma\left(\frac{\alpha_j+1}{2}+|\beta_{(j)}|+|q_{(j)}|\right)}.D_b(\beta,x',\xi),
\end{aligned}$$

where $D_b(\beta,x',\xi)$= $\bar{\gamma}_{b,p,q}(\beta,x',\xi)$, which can be seen by choosing $p = q = 0$in the above equalities. Hence we have proved:

Proposition (6.1.11))[38]: Let $\alpha \in \mathbb{z}_+^m$ be given. Assume that $b \in L^\infty(D_n)$ is of theform (15) and let $|p_{(j)}| = |q_{(j)}|$, for each $j = 1,\dots,m$.Then in the case of$\beta - p + q \in \mathbb{z}_+^m$ we have

$$\tilde{\gamma}_{b,p,q}(\beta,x',\xi) = \frac{(\beta+q)!}{\sqrt{\beta!\,(\beta-p+q)!}}\prod_{j=1}^{m}\frac{\Gamma\left(\frac{\alpha_j+1}{2}+|\beta_{(j)}|\right)}{\Gamma\left(\frac{\alpha_j+1}{2}+|\beta_{(j)}|+|q_{(j)}|\right)}.\tilde{\gamma}_{b,0,0}(\beta,x',\xi).$$

In the case *of* $\beta - p + q \notin \mathbb{z}_+^k$we have $\tilde{\gamma}_{b,p,q}(\beta,x',\xi) = 0$. The factor $\tilde{\gamma}_{b,p,q}(\beta,x',\xi)$can be expressed int he form

$$(\beta,x',\xi)=\Theta_\beta A_\beta^2(\xi)\chi_{\mathbb{R}_+}(\xi)2^{-k}\int_{\mathbb{R}_+^m\times\mathbb{R}^{n-k-1}\times\mathbb{R}_+}\prod_{j=1}^{m} r_j^{2|\beta(j)|+|q(j)|-|p(j)|+2\alpha_j-1}$$
$$\times e^{-2\xi(|r|^2+v)-|y'|^2} b_0\left(r,\frac{-x'+y'}{2\sqrt{\xi}},\nu+|r|^2\right) rdrdy' \frac{c\lambda v^\lambda}{4} d \qquad (16)$$

Let $\alpha\in\mathbb{Z}_+^m$ be given and $(p+q)\in\mathbb{Z}_+^k\times\mathbb{Z}_+^k$. From Proposition (6.1.11) we conclude:

Corollary (6.1.12))[38]: Let $\alpha=\alpha_0(r,y',Imz_n-|\omega'|^2)\in L^\infty(D_n)$ be an $\alpha-$quasinilpotent quasiradial function. Under the assumption $|p(j)|$ for all
$j=1,2,\dots,m$we have

$$T_\alpha T_{\xi^p\bar{\xi}^q}=T_{\xi^p\bar{\xi}^q}T_\alpha=T_{\alpha\xi^p\bar{\xi}^q} \qquad (17)$$

on each weighted Bergman space.

Proof: The first equality in (17) is a direct consequence of Theorem (6.1.10). If $e(z)\equiv$ 1 $t$hen $T_e\ =\ Id$, and thus $\tilde{\gamma}_{b,0,0}(\beta,x',\xi)\equiv 1$. Hence, Proposition (6.1.11) implies that in the case of a symbol $b_{\xi^p\bar{\xi}^q}$ with $|p_{(j)}|=|q_{(j)}|$, for all $j=1,2,\dots,m$, one has

$$\tilde{\gamma}_{b,p,q}(\beta,x',\xi)=\frac{(\beta+q)!}{\sqrt{\beta!\,(\beta-p+q)!}}\prod_{j=1}^{m}\frac{\Gamma\left(\frac{\alpha_j+1}{2}+|\beta_{(j)}|\right)}{\Gamma\left(\frac{\alpha_j+1}{2}+|\beta_{(j)}|+|q_{(j)}|\right)}, \qquad (18)$$

Whenever$\beta-p+q\in\mathbb{Z}_+^k$(cf. Example(6.1.8) for the choice of $\alpha=(1,\dots,1)\in\mathbb{Z}_+^k$ and the case $p_j=q_j, j=1,\dots,k$). Moreover, if $\beta-p+q\in\mathbb{Z}_+^k$, then it holds $\tilde{\gamma}_{b,p,q}(\beta,x',\xi)$. Theorem (6.1.10), Proposition (6.1.11) and the assumption that $|p_{(j)}|\ =\ |q_{(j)}|$, for all$j\ =$ $1,2,\cdots,m$, imply now that

$$\tilde{\gamma}_{ab,p,q}(\beta,x',\xi)=\tilde{\gamma}_{b,p,q}(\beta,x',\xi)\,.\,\tilde{\gamma}_{a,0,0}(\beta,x',\xi)$$
$$=\tilde{\gamma}_{b,p,q}(\beta,x',\xi)\,.\,\tilde{\gamma}_{a,0,0}(\beta-p+q,x',\xi)$$

This together with (11) and Theorem (6.1.7) yields the second equality in (17).

We define commutative Banach algebras of Toeplitz operators which are induced by the quasi-nilpotent group action. Given a pair of multi-indices $(p,q)\in\mathbb{Z}_+^m\times\mathbb{Z}_+^m$, we put
$\tilde{p}_{(j)}:=\left(0,\dots,p_{(j)},0,\dots,\right)$ and $\tilde{q}_{(j)}:=\left(0,\dots,q_{(j)},0,\dots,\right)$
so that $p=\tilde{p}_{(1)}+\tilde{p}_{(2)}+\cdots+\tilde{p}_{(m)}$ and $q=\tilde{q}_{(1)}+\tilde{q}_{(2)}+\cdots+\tilde{q}_{(m)}$
Consider the Toeplitz operators:

$$T_j:=T_{\xi^{\tilde{p}(j)}\bar{\xi}^{\tilde{q}(j)}}$$

(cf. Definition (6.1.6)). Now, we can prove that certain products of Toeplitz operators are Toeplitz operators again with the product symbol.

Proposition (6.1.13))[38]: Let us assume that $|p_{(j)}|=|q_{(j)}|$ for all $j=\ 1,2,\dots,m$. Then the Toeplitz operators T_j commute mutually. Moreover,

$$\prod_{j=1}^{m} T_j:=T_{\xi^p\bar{\xi}^q} \qquad (19)$$

on each weighted Bergman space.

Proof: Let $b_j := \xi^{\tilde{p}(j)}\bar{\xi}^{\tilde{q}(j)}$, for $j = 1, \dots, m$, We only prove the following product rule:

$$T_j T_i := T_{\xi^{\tilde{p}(i)}\bar{\xi}^{\tilde{q}(i)+\tilde{q}(j)}} \tag{20}$$

for $i, j \in \{1, \dots, m\}$and $i \neq j$. According to Theorem (6.1.7) the operator $RT_jT_iR^*$ acts on the sequence space $\ell_2\left(\mathbb{Z}_+^k, L_2(\mathbb{R}^{n-k-1} \times \mathbb{R}_+)\right)$ by the rule:

$$\begin{aligned} &RT_jT_iR^*\{c_\beta(x',\xi)\}_{\beta\in\mathbb{Z}_+^k} \\ &\quad = RT_jR^*\left\{\tilde{\gamma}_{b_j,\tilde{p}_{(j)},\tilde{q}_{(j)}}(\beta,x',\xi).\tilde{\gamma}_{b_i,\tilde{p}_{(i)},\tilde{q}_{(i)}}\left(\beta-\tilde{p}_{(j)},\tilde{q}_{(j)},x',\xi\right)\right. \\ &\quad \left.\times\, \tilde{\gamma}_{c_{\beta-\tilde{p}_{(i)}-\tilde{p}_{(j)}+\tilde{q}_{(i)}+\tilde{q}_{(j)}}}(x',\xi)\right\}_{\beta\in\mathbb{Z}_+^k} \end{aligned}$$

Hence it is clear that (20) is equivalent to:

$$\tilde{\gamma}_{b_j,\tilde{p}_{(j)},\tilde{q}_{(j)}}(\beta,x',\xi).\tilde{\gamma}_{b_i,\tilde{p}_{(i)},\tilde{q}_{(i)}}\left(\beta-\tilde{p}_{(j)},\tilde{q}_{(j)},x',\xi\right) = \tilde{\gamma}_{b_i,b_j,\tilde{p}_{(i)},\tilde{p}_{(j)},\tilde{q}_{(i)},\tilde{q}_{(j)}}(\beta,x',\xi) \tag{21}$$

By (18) we have

$$\tilde{\gamma}_{b_j,\tilde{p}_{(j)},\tilde{q}_{(j)}}(\beta,x',\xi) = \frac{\left(\beta_{(j)}+\tilde{q}_{(j)}\right)!}{\sqrt{\beta_{(j)}!\left(\beta_{(j)}-\tilde{p}_{(j)}+\tilde{q}_{(j)}\right)!}}\frac{\Gamma\left(\frac{\alpha_j+1}{2}+\left|\beta_{(j)}\right|\right)}{\Gamma\left(\frac{\alpha_j+1}{2}+\left|\beta_{(j)}\right|+\left|\tilde{q}_{(j)}\right|\right)},$$

and similar for i replaced by j. Moreover, the function on the right hand side of (21) has the explicit form:

$$\begin{aligned} &\tilde{\gamma}_{b_i,b_j,\tilde{p}_{(i)},\tilde{p}_{(j)},\tilde{q}_{(i)},\tilde{q}_{(j)}}(\beta,x',\xi) \\ &\quad = \frac{\left(\beta+\tilde{q}_{(i)}+\tilde{q}_{(j)}\right)!}{\sqrt{\beta!\left(\beta-\tilde{p}_{(i)}-\tilde{p}_{(j)}+\tilde{q}_{(i)}+\tilde{q}_{(j)}\right)!}}\prod_{\ell\in\{i,j\}}\frac{\Gamma\left(\frac{\alpha_\ell+1}{2}+\left|\beta_{(\ell)}\right|\right)}{\Gamma\left(\frac{\alpha_\ell+1}{2}+\left|\beta_{(\ell)}\right|+\left|\bar{q}_{(\ell)}\right|\right)} \end{aligned}$$

Now, (21) can be easily checked from these identities.

Let $\alpha \in \mathbb{Z}_+^m$ with $|\alpha| = k$ as before and consider two α-quasi-nilpotent quasi-homogeneous functions $\varphi_j \in L^\infty(D_n)$ where j = 1, 2. We express φ_j , for $j = 1{,}2$ in the form

$$\varphi_1(z',\omega',z_n) = \alpha_1(r_1,\dots,r_m,y'Imz_n-|\omega'|^2)\zeta^p\bar{\zeta}^q,$$
$$\varphi_2(z',\omega',z_n) = \alpha_2(r_1,\dots,r_m,y'Imz_n-|\omega'|^2)\zeta^p\bar{\zeta}^q,$$

where $(p,q),(u,v)\epsilon\mathbb{Z}_+^k\times\mathbb{Z}_+^k$ with $p\perp q$ and $u\perp v$ are the degrees of φ_1 and φ_2, respectively. Moreover, assume that $|p_{(j)}| = |q_{(j)}|$and $\left|u_{(j)}\right| = |v_{(j)}|$, for $j = 1, 2, \dots, m$.

Theorem (6.1.14))[38]: The Toeplitz operators $T\varphi_1$ and $T\varphi_2$ commute on each weighted Bergman space $\mathcal{A}_\lambda^2(D_n)$ if and only if for each ℓ=1,2,…, k one of the conditions (i) − (iv) is fulfilled:

(i) $p_\ell = q_\ell = 0$
(ii) $u_\ell = v_\ell = 0$
(iii) $p_\ell = u_\ell = 0$
(iv) $q_\ell = v_\ell = 0$

Proof: Similar to the argument in the proof of Proposition (6.1.13) it follows that the operators $T\varphi_1$ and $T\varphi_2$ commute on $\mathcal{A}_\lambda^2(D_n)$ if and only if for all $\beta \in \mathbb{Z}_+^k$:

$$\tilde{\gamma}_{\varphi_1,p,q}(\beta, x', \xi).\tilde{\gamma}_{\varphi_2 u,v}(\beta - p + q, x', \xi) = \tilde{\gamma}_{\varphi_2 u,v}(\beta, x', \xi).\tilde{\gamma}_{\varphi_1,p,q}(\beta - p + q, x', \xi)$$

Since $|p_{(j)}| = |q_{(j)}|$and $|u_{(j)}| = |v_{(j)}|$, for $j = 1, 2, \dots, m$we can use the factorization of$\tilde{\gamma}_{\varphi_1,p,q}(\beta, x', \xi)$ and $\tilde{\gamma}_{\varphi_2 u,v}(\beta, x', \xi)$ in Proposition (6.1.11):

$$\tilde{\gamma}_{\varphi_1,p,q}(\beta, x', \xi) = \Phi_{p,q}(\beta).\tilde{\gamma}_{\varphi_1 0,0}(\beta, x', \xi),$$
$$\tilde{\gamma}_{\varphi_2 u,v}(\beta, x', \xi) = \Phi_{u,v}(\beta).\tilde{\gamma}_{\varphi_2 0,0}(\beta, x', \xi),$$

where we use the notation:

$$\Phi_{p,q}(\beta) = \frac{(\beta + q)!}{\sqrt{\beta!\,(\beta - p + q)!}} \prod_{j=1}^{m} \frac{\Gamma\left(\frac{\alpha_j + 1}{2} + |\beta_{(j)}|\right)}{\Gamma\left(\frac{\alpha_j + 1}{2} + |\beta_{(j)}| + |q_{(j)}|\right)} \qquad (22)$$

Moreover, it follows from Theorem (6.1.10) and again by the conditions on (p, q) and (u, v) that

$$\tilde{\gamma}_{\varphi_1 0,0}(\beta, x', \xi) = \tilde{\gamma}_{\varphi_1 0,0}(\beta - u + v, x', \xi)$$
$$\tilde{\gamma}_{\varphi_2 0,0}(\beta, x', \xi) = \tilde{\gamma}_{\varphi_2 0,0}(\beta - p + q, x', \xi)$$

Therefore we only need to verify that

$$\Phi_{p,q}(\beta).\Phi_{u,v}(\beta - p + q) = \Phi_{u,v}(\beta).\Phi_{p,q}(\beta - u + v).$$

By a straightforward calculation this is equivalent to:

$(\beta + q)!\frac{(\beta-p+q+v)!}{(\beta-p+q)!} = (\beta + v)!\frac{(\beta-u+v+q)!}{(\beta-u+v)!}$.

Varyingβ it can be seen that this equality holds if and only if for each $\ell = 1, 2, \dots, k$ one of the conditions (i) − (iv) is fulfilled.

Let $(p, q) \in \mathbb{Z}_+^k \times \mathbb{Z}_+^k$ and $\alpha \in \mathbb{Z}_+^m$such that $|\alpha| = k$. Let $h \in \mathbb{Z}_+^m$be given with the properties:

(i) $h_j = 0$, if $\alpha_j = 1$,

(ii) $1 \le h_j \le \alpha_j - 1$, if$\alpha_j > 1$. In the case of $\alpha_{j_1} = \alpha_{j_2}$ with $j_1 < j_2$ we assume that $h_{j_1} \le h_{j_2}$.

In the following we assume that $p_{(j)}$ and $q_{(j)}$ for $j = 1, \dots, m$ are of the particular form

$$p_{(j)} = (p_{j,1}, \dots, p_{j,h_j}, 0, \dots, 0) \text{ and } q_{(j)} = (0, \dots, 0, q_{j,h_{j+1}}, \dots, q_{j,\alpha_j}). \qquad (23)$$

below we will use the data α and h to define commutative Banach algebras of Toeplitz operators.The second assumption in (ii) serves to avoid repetition of the unitary equivalent algebras.

Define $\mathcal{R}_\alpha(h)$ to be the linear space generated by all bounded measurableα-quasi-nilpotent quasi-homogeneous functions

$$b(z', \omega', z_n) = b_0(r_1, \dots, r_m, y', Imz_n - |\omega'|^2).\zeta^p\bar{\zeta}^q \qquad (24)$$

Moreover, in (24) we assume that $p_{(j)}$ and $q_{(j)}$ are of the form (23) with:

$$p_{j,1} + \cdots + p_{j,h_j} = q_{j,h_{j+1}} + \cdots + q_{j,\alpha_j}.$$

As a corollary to Theorem (6.1.14) we obtain:

Theorem (6.1.15)[38]: The Banach algebra generated by Toeplitz operators with symbols from $\mathcal{R}_\alpha(h)$ is commutative.

Finally, we remark:

(i) For $k > 2$ and$\alpha \neq (1,1,\dots,1)$ the commutative algebras $\mathcal{R}_\alpha(h)$ are just Banach algebras, while the C*-algebras generated by them are noncommutative.

(ii) These algebras are commutative for each weighted Bergman space $\mathcal{A}_\lambda^2(D_n)$ with $\lambda > -1$.

(iii) For $k = 0$ (nilpotent case) or $k = 1, 2$ these algebras collapse to the single C*-algebras which are generated by Toeplitz operators with quasinilpotent symbols $b(r, y', Imz_n - |z'|)$.

Let $0 \leq \varepsilon \leq n-3, \varepsilon \geq 0$. We rather use the notation $(x+2\varepsilon) = ((x+2\varepsilon)', (u+2\varepsilon)', (x+2\varepsilon)_n)$ for $(x+2\varepsilon) \in D_{n+s-1}$where $(x+2\varepsilon)' \in \mathbb{C}^{n-\varepsilon-2}$ and $(u+2\varepsilon)' \in \mathbb{C}^{n-\varepsilon}$. The quasi-nilpotent group $\mathbb{T}^{1+\varepsilon} \times \mathbb{R}^{n-\varepsilon-2} \times \mathbb{R}$ acts on D_{n+s-1}, cf. [218], as follows: given $(t, b_{s-2}, h) \in \mathbb{T}^{2+\varepsilon} \times \mathbb{R}^{n-\varepsilon-2} \times \mathbb{R}$, we have:

$$\mathcal{T}(t, b_{s-2}, h)\colon \big((x+2\varepsilon), (u+2\varepsilon)'(x+2\varepsilon)_n\big)$$
$$\to (t(x+2\varepsilon)', (u+2\varepsilon)' + b_{s-2}, (x+2\varepsilon)_n + h + 2i\omega.b_{s-2} + i|b_{s-2}|^2).$$

Note that in the case $\varepsilon = n-2$, $\varepsilon \geq 0$we obtain the quasi-parabolic group, while for $\varepsilon = -2$ the group action is called nilpotent.

On the domain $\mathcal{D}_{s-2} = \mathbb{C}^{1+\varepsilon} \times \mathbb{C}^{n-\varepsilon-2} \times \mathbb{R} \times \mathbb{R}_+$we use the variables $((x+2\varepsilon)', (u+2\varepsilon)', u, (u+\varepsilon))$ and we represent $L_2(\mathcal{D}_{s-2}, \eta_{\varepsilon-1})$ in the form:

$$L_2\big(\mathcal{D}_{s-2}, \eta_{(\varepsilon-1)}\big) = L_2(\mathbb{C}^{1+\varepsilon}) \otimes L_2(\mathbb{C}^{n-\varepsilon-2}) \otimes L_2(\mathbb{R}) \otimes L_2\big(\mathbb{R}_+, \eta_{(\varepsilon-1)}\big). \tag{25}$$

Let F be the Fourier transform on $L_2(\mathbb{R})$, and with respect to the decomposition (25) consider the unitary operators $U_s := I \otimes I \otimes F \otimes I$ acting on $L_2\big(\mathcal{D}_{s-2}, \eta_{(\varepsilon-1)}\big)$. With this notation we put $\mathcal{A}_s(\mathcal{D}_{s-2}) := U_s(\mathcal{A}_{s-1}(\mathcal{D}_{s-2}))$.

We introduce polar coordinates on $\mathbb{C}^{\varepsilon+1}$ and put $r = (r_1, \dots, r_{\varepsilon+1}) = (|(x+2\varepsilon)_1'|, \dots, |(x+2\varepsilon)_{\varepsilon+1}'|)$. In the following we write $x' :=$ Re $(u+2\varepsilon)'$ and $(x+\varepsilon)' := Im(u+2\varepsilon)'$. Then one can check that $r, (x+\varepsilon)'$ and $Im(x+2\varepsilon)_n - |(u+2\varepsilon)'|^2$ are invariant under the action of the quasi-nilpotent group. Following the ideas in [218] and with $rd = r_1 d_{r_1} \cdots r_{(1+\varepsilon)} d_{r_{(1+\varepsilon)}}$ we represent $L_2(\mathcal{D}_{s-2}, \eta_{\varepsilon-1})$ in the for

$$L_2\left(\mathbb{R}_+^{(1+\varepsilon)}, r_1 dr_1\right) \otimes L_2(\mathbb{T}^{1+\varepsilon}) \otimes L_2(\mathbb{R}^{n-\varepsilon-2})$$
$$\otimes L_2(\mathbb{R}^{n-\varepsilon-2}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_{\varepsilon-1}) \tag{26}$$

We define the unitary operator U_{s+1} on $L_2(\mathcal{D}_{s-2}, \eta_{\varepsilon-1})$ by $U_{s+1} = I \otimes \mathcal{F}_{(1+\varepsilon)} \otimes F_{(n-\varepsilon-2)} \otimes I \otimes I \otimes I$. Here $\mathcal{F}_{(1+\varepsilon)} = \mathcal{F} \otimes \dots \otimes \mathcal{F}$ is the $(1+\varepsilon)$-dimensional discrete Fourier transform and $F_{(n-\varepsilon-2)} = F \otimes \dots \otimes F$ denotes the $(n-\varepsilon-2)$-dimensional Fourier transform on $L_2(\mathbb{R}^{n-\varepsilon-2})$. Note that $L_2\big(\mathcal{D}_{s-2}, \eta_{(1+\varepsilon)}\big)$ is isometrically mapped by U_{s+1} onto

$$\ell_2\Big(\big(\mathbb{Z}^{(\varepsilon+1)}, L_2(\mathbb{R}_+^{1+\varepsilon}) \otimes L_2(\mathbb{R}^{n-\varepsilon-2})\big) \otimes L_2(\mathbb{R}^{n-\varepsilon-2}) \otimes L_2(\mathbb{R})$$
$$\otimes L_2\big(\mathbb{R}_+, \eta_{(\varepsilon-1)}\big).\ \Big) \tag{27}$$

We put $\mathcal{A}_{s+1}(\mathcal{D}_{s-2}) := U_{s+1}(\mathcal{A}_s(\mathcal{D}_{s-2}))$ and we write elements in (27) as $\{f_\beta(r,x',(x+\varepsilon)',\xi,u+\varepsilon)\}_{\beta\in\mathbb{Z}^k}$, where $(r,x',(x+\varepsilon)',\xi,u+\varepsilon)\in\mathbb{R}_+^{\varepsilon+1}\times\mathbb{R}^{n-\varepsilon-2}\times\mathbb{R}^{n-\varepsilon-2}\times\mathbb{R}\times\mathbb{R}_+$.

Next we recall the definition of the unitary operator U_{s+2} which acts on (27) by:

$$U_{s+2}:\{f_\beta(r,x',(x+\varepsilon)',\zeta,u+\varepsilon)\}_{\beta\in\mathbb{Z}} \mapsto \left\{f_\beta\left(r,\sqrt{\zeta}(x',(x+\varepsilon)'),\frac{1}{2\sqrt{\xi}}(-x',(x+\varepsilon)'),\zeta,u+\varepsilon\right)\right\}_{\beta\in\mathbb{Z}^{(1+\varepsilon)}}$$

One immediately checks that the inverse U_{s+2}^{-1} has the form

$$U_{s+2}^{-1}:\{f_\beta(r,x',(x+\varepsilon)',\zeta,u+\varepsilon)\}_{\beta\in\mathbb{Z}} \mapsto \left\{f_\beta\left(r,\frac{x'}{2\sqrt{\zeta}}-\sqrt{\zeta}(x+\varepsilon)',\frac{x'}{2\sqrt{\zeta}}+\sqrt{\zeta}(x+\varepsilon)',\zeta,u+\varepsilon\right)\right\}_{\beta\in\mathbb{Z}^{(1+\varepsilon)}}$$

In the following we write $\mathbb{Z}_+=\mathbb{N}\cup\{0\}=\{0,1,2,\dots\}$ for the nonnegative integers. In order to state the main result of Section 8 in [218] we need to introduce the operator R_{s-2}, which defines an isometric embedding of $\ell_2\left(\mathbb{Z}_+^{(1+\varepsilon)},L_2(\mathbb{R}^{n-\varepsilon-2}\times\mathbb{R}_+)\right)$ into (27). It is explicitly given by

$$\begin{aligned} R_{s-2}:\{c_\beta(x',\zeta)\}_{\beta\in\mathbb{Z}_+^{(1+\varepsilon)}} &\mapsto \left\{\chi_{\mathbb{Z}_+^{(\varepsilon+1)}\times\mathbb{R}_+}(\beta,\xi)(R_{s-2})_\beta(\zeta)r^\beta e^{-\zeta(|r|^2+(u+\varepsilon))-\frac{|(u+\varepsilon)\prime|^2}{2}}c_\beta(x',\zeta)\right\}_{\beta\in\mathbb{Z}^{(1+\varepsilon)}} \\ &= \{g_\beta(r,x',(x+\varepsilon)',\zeta,u+\varepsilon)\}_{\beta\in\mathbb{Z}^{(1+\varepsilon)}} \end{aligned}$$

Here $\chi_{\mathbb{Z}_+^{(\varepsilon+1)}\times\mathbb{R}_+}(\beta,\zeta)$denotes the characteristic function of $\mathbb{Z}_+^{(\varepsilon+1)}\times\mathbb{R}_+$ and $c_\beta(x',\zeta)$ is extended by zero for $\zeta\in(-\infty,0)$ and all $x'\in\mathbb{R}^{n-\varepsilon-2}$. Moreover, we have used the abbreviation

$$A_\beta(\zeta):=\pi^{-\frac{n-\varepsilon-2}{4}}\sqrt{\frac{2^{\varepsilon+3}}{C_{\varepsilon-1}}\frac{(2\zeta)^{|\beta|+2\varepsilon+1}}{\beta!\Gamma(\varepsilon)}} \tag{28}$$

The ad joint operator R_{s-1}^* is given by:

$$\begin{aligned} R_{s-1}^*:\{f_\beta(r,x',(x+\varepsilon)',\zeta,u+\varepsilon)\}_{\beta\in\mathbb{Z}^{\varepsilon+1}} &\mapsto \{(A_{s-2})_\beta(\zeta) \\ \times\int_{\mathbb{R}_+^{\varepsilon+1}\times\mathbb{R}^{n-\varepsilon-2}\times\mathbb{R}_+} r^\beta e^{-\zeta(|r|^2+(u+\varepsilon))-\frac{|(u+\varepsilon)'|^2}{2}} & f_\beta(r,x',(x+\varepsilon)',\zeta,u+\varepsilon)rdrd(x+\varepsilon)' \\ \frac{C_\lambda(u+\varepsilon)^{(\varepsilon-1)}}{4}d(u+\varepsilon)\}_{\beta\in\mathbb{Z}_+^{(1+\varepsilon)}} \end{aligned} \tag{29}$$

We set $U_{s-2}:=U_{s+2}U_{s+1}U_sU_{s-1}$, which gives a unitary operator from $R_{\varepsilon-1}^2(\mathcal{D}_{n+s-1})$ onto $A_{s+2}(\mathcal{D}_{s-2}) \coloneqq U_{s+3}A_{s+1}(\mathcal{D}_{s-2})$. The following result has been proved in [218], and it provides a decomposition of the Bergman projection $B_{\mathcal{D}_{n+s-1,\varepsilon-1}}$ in form of a certain operator product.

Theorem (6.1.16)[270]: [218] The operator $R_{s-2} := R^*_{s-1}U_{s-2}$ maps $L_2\big(\mathcal{D}_{n+s-1}, \bar{\mu}_{(\varepsilon-1)}\big)$ onto the space $\ell_2\big(\mathbb{Z}_+^{\varepsilon+1}, L_2(\mathbb{R}^{n-\varepsilon-2}\times\mathbb{R}_+)\big)$, and the restriction

$$R_{s-2|(\mathcal{A}^2_{s-2})_{\varepsilon-1}(\mathcal{D}_{s-2})}(\mathcal{A}^2_{s-2})_{\varepsilon-1}(\mathcal{D}_{s-2}) \to \ell_2\big(\mathbb{Z}_+^{\varepsilon+1}, L_2(\mathbb{R}^{n-\varepsilon-2}\times\mathbb{R}_+)\big)$$

is an isometric isomorphism. The ad joint operator

$$R^*_{s-2} = U^*_{s-2}\; R_{s-1}: L_2\big(\mathbb{Z}_+^{\varepsilon+1}, L_2(\mathbb{R}^{n-\varepsilon-2}\times\mathbb{R}_+)\big) \to (\mathcal{A}^2_{s-2})_{\varepsilon-1}(\mathcal{D}_{s-2}) \subset L_2(\mathcal{D}_{n+s-1}, \bar{\mu}_{\varepsilon-1})$$

is an isometric isomorphism of $L_2\big(\mathbb{Z}_+^{\varepsilon+1}, L_2(\mathbb{R}^{n-\varepsilon-2}\times\mathbb{R}_+)\big)$ onto the subspace $(\mathcal{A}^2_{s-2})_{\varepsilon-1}(\mathcal{D}_{s-2})$ of $L_2(\mathcal{D}_{n+s-1}, \bar{\mu}_{\varepsilon-1})$. Furthermore one has:

$$R_{s-2}\, R^*_{s-2} = I: L_2\big(\mathbb{Z}_+^{\varepsilon+1}, L_2(\mathbb{R}^{n-\varepsilon-2}\times\mathbb{R}_+)\big) \to L_2\big(\mathbb{Z}_+^{\varepsilon+1}, L_2(\mathbb{R}^{n-\varepsilon-2}\times\mathbb{R}_+)\big),$$

$$R^*_{s-2}R_{s-2} = B_{\mathcal{D}_{n+s-1,\varepsilon-1}}: L_2(\mathcal{D}_{n+s-1}, \bar{\mu}_{\varepsilon-1}) \to (\mathcal{A}^2_{s-2})_{\varepsilon-1}(\mathcal{D}_{s-2})$$

Consider an $(\beta+\varepsilon)$-quasi-nilpotent quasi-homogeneous symbol

$$b_{s-2}((x+2\varepsilon)', (u+2\varepsilon)', (x+2\varepsilon)_n)$$

as in[38] and of degree $(p, p+\varepsilon) \in \mathbb{Z}_+^{(1+\varepsilon)} \times \mathbb{Z}_+^{(1+\varepsilon)}$ with $p \perp (p+\varepsilon)$. Our next aim is to calculate the operator $R_{s-2}\, T_{b_{s-2}} R^*_{s-2}$. On the domain $\mathcal{D}_{s-2} = \mathbb{C}^{(1+\varepsilon)} \times \mathbb{C}^{n-\varepsilon-2} \times \mathbb{R} \times \mathbb{R}_+$ we use the variables $((x+2\varepsilon)', (u+2\varepsilon)', u, u+\varepsilon)$. Moreover, we express $(u+2\varepsilon)'$in polar coordinates $(u+2\varepsilon)' = t_1 r_1, \dots, t_{(1+\varepsilon)} r_{(1+\varepsilon)}$ where $r_s \ge 0$ and $t_s \in \mathbb{S} = \mathbb{S}$ for $s = 1, \dots, (1+\varepsilon)$. Then we have the relations

$$(u+2\varepsilon)_{j,\ell} = \mathrm{r}_j \zeta_{j,\ell} = t_{j,\ell} r_{j,\ell}$$

for $\ell = \{s, \dots, (\beta_j+\varepsilon)\}$and $j = 1, \dots, m$. It follows that $\xi_{j,\ell} = t_{j,\ell} r_{j,\ell} \mathrm{r}_j^{-1}$ in the case of $\mathrm{r}_j \neq 0$ and therefore:

$$\overline{(\zeta+\varepsilon)}^{\mathrm{p}+\varepsilon} = t^p \bar{t}^{\mathrm{p}+\varepsilon} r^{p+\mathrm{p}+\varepsilon} \prod_{j=1}^{m} \mathrm{r}_j^{-|p_j|-|(\mathrm{p}+\varepsilon)_j|} \tag{30}$$

Note that the assignment $(x+2\varepsilon)' \longmapsto (\zeta+\varepsilon)^p \overline{(\zeta+\varepsilon)}^{\mathrm{p}+\varepsilon}$ depends on the initial choice of $(\beta+\varepsilon) \in \mathbb{Z}_+^m$.Using Theorem (6.1.16) we can write:

$$\begin{aligned}
R_{s-2}\, T_{b_{s-2}} R^*_{s-2} &= R_{s-2}\; B_{\mathcal{D}_{n+s-1,\varepsilon-1}} b_{s-2} B_{\mathcal{D}_{n,\varepsilon-1}} R^*_{s-2} \\
&= R_{s-2}\; \Big(R^*_{s-2}\; R_{s-2}\Big) b_{s-2} \Big(R^*_{s-2}\; R_{s-2}\Big) R^*_{s-2} \\
&= \Big(R_{s-2}\; R^*_{s-2}\Big) R_{s-2}\; b_{s-2} R^*_{s-2}\; \Big(R_{s-2}\; R^*_{s-2}\Big) = R_{s-2} b_{s-2} R^*_{s-2} \\
&= R^*_{s-1} U_{s+2} U_{s+1} U_s U_{s-1} b_{s-2} U^{-1}_{s-1} U_s^{-1} U^{-1}_{s+1} U^{-1}_{s+2} R_{s-1} \\
&= R^*_{s-2}\; U_{s+2} U_{s+1} U_s b_{s-1}(r_1, \dots, r_m, (x+\varepsilon)', \mathrm{Im}(x+2\varepsilon)_n \\
&\quad - |(u+2\varepsilon)'|^2)(\zeta+\varepsilon)^p \overline{(\zeta+\varepsilon)}^{\mathrm{p}+\varepsilon} U^{-1}_{s-1} U_s^{-1} U^{-1}_{s+1} U^{-1}_{s+2} R_{s-1}
\end{aligned}$$

First we calculate the operator $U_{s-1} b U^{-1}_{s-1}$. Let $\{f_\beta(r, x', (x+\varepsilon)', \zeta, u+\varepsilon)\}_{\beta \in Z_+^{(1+\varepsilon)}}$be an element in the space (27) and write $r := \underline{(r_1, \dots, r_m)}$. Since the ymbol $b_{s-1}(r, (x+\varepsilon)', (u+\varepsilon) + |r|^2)(\zeta+\varepsilon)^p \overline{(\zeta+\varepsilon)}^{\mathrm{p}+\varepsilon}$ is independent of x' we obtain from (4.2) that:

$$U_{s+1}b_{s-1}(r,(x+\varepsilon)',(u+\varepsilon)$$
$$+|r|^2)(\zeta+\varepsilon)^p\overline{(\zeta+\varepsilon)}^{\mathrm{p}+\varepsilon}U_{s+1}^{-1}\{f_\beta\big(r,x',(x+\varepsilon)',\zeta,(u+\varepsilon)\big)\}_{\beta\in\mathbb{Z}_+^{1+\varepsilon}}$$
$$=\Bigg\{b_{s-1}(r,(x+\varepsilon)',(u+\varepsilon)+|r|^2)r^{2\mathrm{p}+\varepsilon}$$
$$\times\left(\prod_{j=1}^{m}\mathrm{r}_j^{-|p_j|-|\mathrm{p}+\varepsilon_j|}\right)f_{\beta+\varepsilon}\big(r,x',(x+\varepsilon)',\zeta,(u+\varepsilon)\big)\Bigg\}_{\beta\in\mathbb{Z}^{1+\varepsilon}} \tag{31}$$

Combining (31) and (29) gives:

$$R_{s-2}\,T_{b_{s-2}}R_{s-2}^*\ :\{C_\beta(x',\zeta)\}_{\beta\in\mathbb{Z}_+^{1+\varepsilon}}=R_{s-2}^*U_{s+2}U_{s+1}b_{s-2}U_{s+1}^{-1}U_{s+1}^{-1}\Big\{\chi_{\mathbb{Z}_{+\times\mathbb{R}_+}^{1+\varepsilon}}(\beta,\zeta)$$
$$(A_{s-2})_\beta(\zeta)r^\beta\,e^{-\zeta(|r|^2+(u+\varepsilon))-\frac{|(u+\varepsilon)'|^2}{2}}c_\beta(x',\zeta)\Big\}_{\beta\in\mathbb{Z}^{1+\varepsilon}}$$
$$=R_{s-2}^{*\,*}U_{s+2}U_{s+1}b_{s-2}U_{s+1}^{-1}\{\chi_{\mathbb{Z}_+^{1+\varepsilon}\times\mathbb{R}_+}(\beta,\zeta)(A_{s-2})_\beta(\zeta)r^\beta$$
$$\times e^{-\zeta(|r|^2+(u+\varepsilon))-\frac{1}{2}\left|\frac{1}{2\sqrt{\zeta}}x'+\sqrt{\zeta}\,(x+\varepsilon)'\right|^2}c_\beta\left(\frac{1}{2\sqrt{\zeta}}x'-\sqrt{\zeta}\,(x+\varepsilon)',\zeta\right)\Bigg\}_{\beta\in\mathbb{Z}^{1+\varepsilon}}$$
$$=R_{s-2}^*U_{s+2}\{\chi_{\mathbb{Z}_+^{1+\varepsilon}\times\mathbb{R}_+}(\beta+\varepsilon,\zeta)(A_{s-2})_{\beta+\varepsilon}(\zeta)r^{\beta+2(\mathrm{p}+\varepsilon)}b_{s-1}$$
$$\times(r_{s-2},(x+\varepsilon)',(u+\varepsilon)+|r|^2)$$
$$\times\left(\prod_{j=1}^{m+s-1}\mathrm{r}_j^{-|p_j|-|(\mathrm{p}+\varepsilon)_j|}\right)e^{-\zeta(|r|^2+(u+\varepsilon))-\frac{1}{2}-\left|\frac{1}{2\sqrt{\zeta}}x'+\sqrt{\zeta}\,(x+\varepsilon)'\right|^2}$$
$$\times C_{\beta+\varepsilon}\left(\frac{1}{2\sqrt{\zeta}}x'-\sqrt{\zeta}\,(x+\varepsilon)',\zeta\right)\Bigg\}_{\beta\in\mathbb{Z}^{1+\varepsilon}}$$
$$=R_{s-2}^*\Bigg\{\chi_{\mathbb{Z}_+^{1+\varepsilon}\times\mathbb{R}_+}(\beta+\varepsilon,\zeta)(A_{s-2})_{\beta+\varepsilon}(\zeta)b_{s-1}\Bigg(r,\frac{-x'+(x+\varepsilon)'}{2\sqrt{\zeta}},(u+\varepsilon)$$
$$+|r|^2\Bigg)\times\left(\prod_{j=1}^{m}\mathrm{r}_j^{-|p_j|-|(\mathrm{p}+\varepsilon)_j|}\right)r^{\beta+2(p+\varepsilon)}e^{-\zeta(|r|^2+(u+\varepsilon))-\frac{1}{2}|\,(x+\varepsilon)'|^2}C_{\beta+\varepsilon}$$
$$(x',\zeta)\}_{\beta\in\mathbb{Z}^{1+\varepsilon}}$$
$$=\{(A_{s-2})_\beta(\zeta)(A_{s-2})_{\beta+\varepsilon}(\zeta)\chi_{\mathbb{Z}_+^{1+\varepsilon}\times\mathbb{R}_+}(\beta+\varepsilon,\zeta)\,C_{\beta+\varepsilon}(x',\zeta)\}_{\beta\in\mathbb{Z}^{1+\varepsilon}}$$
$$\times\int_{\mathbb{R}_+^{1+\varepsilon}\times\mathbb{R}^{n-\varepsilon-2}\times\mathbb{R}_+}r^{2(\beta+\mathrm{p}+\varepsilon)}\left(\prod_{j=1}^{m}\mathrm{r}_j^{-|p_j|-|(\mathrm{p}+\varepsilon)_j|}\right)e^{-\zeta(|r|^2+(u+\varepsilon))-|x+\varepsilon|^2}$$
$$\times b_{s-1}\left(r,\frac{-x'+(x+\varepsilon)'}{2\sqrt{\zeta}},(u+\varepsilon)+|r|^2\right)rdrd(x+\varepsilon)'\frac{\mathrm{C}_{\varepsilon-1}(u+\varepsilon)^{\varepsilon-1}}{4}d(u+\varepsilon)\Bigg\}_{\beta\in\mathbb{Z}_+^{1+\varepsilon}}$$

Now put:

$$\gamma_{b,p,p+\varepsilon}(\beta,x',\zeta):$$
$$= (A_{s-2})_\beta(\xi)(A_{s-2})_{\beta+\varepsilon}(\xi)\chi_{\mathbb{Z}_+^{1+\varepsilon}\times\mathbb{R}_+}(\beta+\varepsilon,\zeta)$$
$$\times\int_{\mathbb{R}_+^{1+\varepsilon}\times\mathbb{R}^{n-\varepsilon-2}\times\mathbb{R}_+}\prod_{j=1}^{m}\mathrm{r}_j^{-|p_j|-|(\mathrm{p}+\varepsilon)_j|}r^{2(\beta+\mathrm{p}+\varepsilon)}e^{-2\zeta(|r|^2+u+\varepsilon)-|x+\varepsilon|^2}$$
$$\times b_{s-1}\left(r,\frac{-x'+(x+\varepsilon)'}{2\sqrt{\zeta}},(u+\varepsilon)+|r|^2\right)rdrd(x+\varepsilon)'\frac{\mathrm{C}_{\varepsilon-1}(u+\varepsilon)^{\varepsilon-1}}{4}d(u+\varepsilon) \qquad (32)$$

Hence, we have proved:

Theorem (6.1.17)[270]: Let b_{s-2} be defined as in [38]. The operator $R_{s-2}\,T_{b_{s-2}}R_{s-2}^*$ acts on the Hilbert space $\ell_2\big(\mathbb{Z}_+^{1+\varepsilon},L_2(\mathbb{R}^{n-\varepsilon-2}\times\mathbb{R}_+)\big)$ by the rule:

$R_{s-2}\,T_{b_{s-2}}R_{s-2}^*\ :\{C_\beta(x',\zeta)\}_{\beta\in\mathbb{Z}_+^{1+\varepsilon}}=\ \{\gamma_{b_{s-2},p,p+\varepsilon}(\beta,x',\zeta).\,C_{\beta+\varepsilon}(x',\zeta)\}_{\beta\in\mathbb{Z}^{1+\varepsilon}}$

Note that, in the case $\varepsilon\ =\ 0$, Theorem 4.5 reduces to Theorem 4.2.

Example (6.1.18)[270]: We calculate $R_{s-2}\,T_{b_{s-2}}R_{s-2}^*$ more explicitly in the special case where

$b_{s-1}\ \equiv\ 1$ and we choose $\varepsilon\ =\ m-1$, i.e. $\alpha=(1,\dots,1)\in\mathbb{Z}_+^{1+\varepsilon}$. Let $(p,p+\varepsilon)\in\mathbb{Z}_+^{1+\varepsilon}$ such that $p\ \perp(\ p+\varepsilon)$ and put

$$b_{s-2}((x+2\varepsilon)',(u+2\varepsilon)',(x+2\varepsilon)_n)=(\zeta+\varepsilon)^p\overline{(\zeta+\varepsilon)}^{\mathrm{p}+\varepsilon}=t^p\bar{t}^{\mathrm{p}+\varepsilon}$$

According to Theorem (6.1.17) it is sufficient to calculate the functions:

$$\gamma_{b_{s-2},p,p+\varepsilon}(\beta,x',\zeta):$$
$$=(A_{s-2})_\beta(\zeta)(A_{s-2})_{\beta+\varepsilon}(\zeta)\chi_{\mathbb{R}_+}(\zeta)$$
$$\times\int_{\mathbb{R}_+^{1+\varepsilon}\times\mathbb{R}^{n-\varepsilon-2}\times\mathbb{R}_+}r^{2\beta+\varepsilon}{}_{s-2}\,e^{-2\zeta(|r|^2+(u+\varepsilon))-|x+\varepsilon|^2}rdrd(x+\varepsilon)'$$
$$\times\frac{\mathrm{C}_{(\varepsilon-1)}(u+\varepsilon)^{(\varepsilon-1)}}{4}d(u+\varepsilon)$$

for all $\beta\in\mathbb{Z}_+^{1+\varepsilon}$ with $(\beta+\varepsilon)\in\mathbb{Z}_+^{1+\varepsilon}$. We use the identity:

$$\int_{\mathbb{R}_+^{1+\varepsilon}\times\mathbb{R}^{n-\varepsilon-2}\times\mathbb{R}_+}e^{-2\zeta(u+\varepsilon)-|x+\varepsilon|^2}\,d(x+\varepsilon)'(u+\varepsilon)^{\varepsilon-1}d(u+\varepsilon)=\pi^{\frac{n-\varepsilon-2}{2}}\Gamma(\varepsilon)(2\zeta)^{-(\lambda+1)}$$

(see [97]) where $\zeta>0$, which together with (28) shows that

$$\gamma_{b_{s-2},p,\mathrm{p}+\varepsilon}(\beta,x',\zeta)=2^{(1+\varepsilon)}(2\zeta)^{|\beta|+\varepsilon+1+\frac{|\mathrm{p}+\varepsilon|-|p|}{2}}\frac{1}{\sqrt{\beta!\,(\beta+\varepsilon)!}}\int_{\mathbb{R}_+^k}r^{2\beta+\varepsilon}\,e^{-2\zeta-|(x+\varepsilon)|^2}dr$$
$$=\frac{\prod_{j=1}^{1+\varepsilon}\Gamma\left(\beta_{\mathrm{j}}+\frac{(\mathrm{p}+\varepsilon)_{\mathrm{j}}-\mathrm{p}_{\mathrm{j}}}{2}+1\right)}{\sqrt{\beta!\,(\beta+\varepsilon)!}}.$$

In particular, in this case $\gamma_{b_{s-2},p,p+\varepsilon}(\beta,x',\zeta)$ is independent of x' and ζ

Fix $(\beta+\varepsilon)\in\mathbb{Z}_+^m$ with $|\beta+\varepsilon|=\varepsilon+1$ as before and let $a_{s-2}=a_{s-1}(r_1,\dots,r_m,(x+\varepsilon)'\mathrm{Im}(x+2\varepsilon)_n-|(u+2\varepsilon)'|^2)$ be a bounded measurable $(\beta+\varepsilon)$-quasi-nilpotent quasi-radial function on D_{n+s-1}. Consider the symbol:

$$b_{s-2}((x+2\varepsilon)',(u+2\varepsilon)',(x+2\varepsilon)_n) = b_{s-1}(r_1,\dots,r_m,(x+\varepsilon)',\mathrm{Im}(x+2\varepsilon)_n - |(u+2\varepsilon)'|^2).(\zeta+\varepsilon)^p\overline{(\zeta+\varepsilon)}^{\mathrm{p+\varepsilon}}. \quad (33)$$

We calculate the operator products $R_{s-2}\,T_{b_{s-2}}T_{a_{s-2}}R^*_{s-2}$ and $R_{s-2}\,T_{a_{s-2}}T_{b_{s-2}}R^{\,*}_{\,s-2}$. According to Theorem (6.1.7) and Theorem (6.1.1) we have

$$\begin{aligned} R_{s-2}\,T_{b_{s-2}}T_{a_{s-2}}R^*_{s-2}\ \{c_\beta\}_{\beta\in\mathbb{Z}_+^k} &= \left(R_{s-2}\,T_{b_{s-2}}R^*_{s-2}\ \right)\left(R_{s-2}\,T_{a_{s-2}}R^*_{s-2}\ \right)\{c_\beta\}_{\beta\in\mathbb{Z}_+^{\varepsilon+1}} \\ &= \left(R_{s-2}\,T_{b_{s-2}}R^*_{s-2}\ \right)\left\{\bar{\gamma}_{b_{s-2},p,p+\varepsilon}(\beta,x',\xi).\,c_\beta(x',\xi)\right\}_{\beta\in\mathbb{Z}_+^{\varepsilon+1}} \\ &= \left\{\bar{\gamma}_{b_{s-2},p,p+\varepsilon}(\beta,x',\xi)\gamma_{a_{s-2},0,0}(\beta+\varepsilon,x',\xi)c_{\beta+\varepsilon}(x',\xi)\right\}_{\beta\in\mathbb{Z}_+^{\varepsilon+1}} \end{aligned} \quad (34)$$

On the other hand it follo

$$\begin{aligned} R_{s-2}\,T_{a_{s-2}}T_{b_{s-2}}R^{\,*}_{\,s-2}\{c_\beta\}_{\beta\in\mathbb{Z}_+^{\varepsilon+1}} &= \left(R_{s-2}\,T_{a_{s-2}}R^*_{s-2}\ \right)\left(R_{s-2}\,T_{b_{s-2}}R^*_{s-2}\ \right) \\ &= \left(R_{s-2}\,T_{a_{s-2}}R^*_{s-2}\ \right)\left\{\bar{\gamma}_{b_{s-2},p,P+\varepsilon}(\beta,x',\zeta).\,c_\beta(x',\zeta)\right\}_{\beta\in\mathbb{Z}_+^{\varepsilon+1}} \\ &= \left\{\bar{\gamma}_{a_{s-2},0,0}(\beta,x',\zeta)\bar{\gamma}_{b_{s-2},p,p+\varepsilon}(\beta,x',\zeta)c_{\beta+\varepsilon}(x',\zeta)\right\}_{\beta\in\mathbb{Z}_+^{\varepsilon+1}} \end{aligned} \quad (35)$$

Hence, we conclude from (34) and (35) that both operators $T_{a_{s-2}}$ and $T_{b_{s-2}}$commute if and only if

$$\bar{\gamma}_{a_{s-2},0,0}(\beta,x',\zeta) = \bar{\gamma}_{a_{s-2},0,0}(\beta+\varepsilon,x',\zeta)$$

for all $\beta\in\mathbb{Z}_+^{1+\varepsilon}$. According to (9) this is equivalent to:

$$\begin{aligned} &\frac{1}{\beta!}\int_{\mathbb{R}_+^{\varepsilon+1}\times\mathbb{R}^{n-\varepsilon-2}\times\mathbb{R}_+} a_{s-1}\left(r,\frac{-x'+(x+\varepsilon)'}{2\sqrt{\xi}},(u+\varepsilon)+|r|^2\right) \\ &\qquad\times r^{2\beta}e^{-2\zeta\left((u+\varepsilon)-|r|^2\right)-\left|(x+\varepsilon)'\right|^2}(u+\varepsilon)^{\varepsilon-1}rdrd \\ &\qquad= \frac{(2\zeta)^{-|p|+|p+\varepsilon|}}{(\beta+\varepsilon)!}\int_{\mathbb{R}_+^{\varepsilon+1}\times\mathbb{R}^{n-\varepsilon-2}\times\mathbb{R}_+} a_{s-1}\left(r,\frac{-x'+(x+\varepsilon)'}{2\sqrt{\zeta}},(u+\varepsilon)+|r|^2\right) \\ &\qquad\times r^{2\beta+\varepsilon}e^{-2\zeta\left((u+\varepsilon)+|r|^2\right)-\left|(x+\varepsilon)'\right|^2}(u+\varepsilon)^{\varepsilon-1}rdrd(x+\varepsilon)'d(u+\varepsilon) \end{aligned}$$

Since $a_{s-1}(r,(x+\varepsilon)',\mathrm{Im}(x+2\varepsilon)_n - |(u+2\varepsilon)|^2)$ only depends on $r = (r_1,\dots,r_m)$ we can assume that the above integral has the form

$$\begin{aligned} &\int_{\mathbb{R}_+^{\varepsilon+1}\times\mathbb{R}^{n-\varepsilon-2}\times\mathbb{R}_+} a_{s-1}\left(r,\frac{-x'+(x+\varepsilon)'}{2\sqrt{\zeta}},(u+\varepsilon)+|r|^2\right)r^{2\beta}\,e^{-2\zeta\left((u+\varepsilon)+|r|^2\right)-\left|(x+\varepsilon)'\right|^2} \\ &\qquad\times(u+\varepsilon)^{\varepsilon-1}rdrd(x+\varepsilon)'d(u+\varepsilon) =: (*), \end{aligned}$$

Where $\beta\in\mathbb{Z}_+^{\varepsilon+1}$. With $e = (1,1,\dots,1)\in\mathbb{Z}_+^{\varepsilon+1}$ we obtain

$$(*) = \frac{1}{2^{\varepsilon+1}} \int_{\mathbb{R}^{\varepsilon+1}\times\mathbb{R}^{n-\varepsilon-2}\times\mathbb{R}_+} a_{s-1}\left(r, \frac{-x' + (x+\varepsilon)'}{2\sqrt{\zeta}}, (u+\varepsilon) + |r|^2\right)$$
$$\times \left|r^{2\beta}\right| e^{-2\zeta\left((u+\varepsilon)+|r|^2\right)-\left|(x+\varepsilon)'\right|^2} (u+\varepsilon)^{\varepsilon-1} r dr d(x+\varepsilon)' d(u+\varepsilon)$$
$$= \frac{1}{2^{\varepsilon+1}} \int_{\mathbb{R}^{n-\varepsilon-2}\times\mathbb{R}_+} \int_{\mathbb{R}_+^m \times \mathbb{S}^{\beta_1+\varepsilon-1}\times\ldots\times\mathbb{S}^{\beta_m+\varepsilon-1}} a_{s-1}\left(r, \frac{-x' + (x+\varepsilon)'}{2\sqrt{\zeta}}, (u+\varepsilon)\right.$$
$$\left. + |r|^2\right) \left|\rho^{2\beta+e}\right| . \left(\prod_{j=1}^{m} \mathrm{r}_j^{2\left|\beta_{(j)}\right| + 2(\beta+\varepsilon)_j - 1}\right) e^{-2\zeta\left((u+\varepsilon)+|r|^2\right)-\left|(x+\varepsilon)'\right|^2}$$
$$\times (u+\varepsilon)^{\varepsilon-1} d\sigma\left(\rho_{(m)}\right) r dr d(x+\varepsilon)' d(u+\varepsilon)$$

In the last integral we wrote $d\sigma\left(\rho_{(j)}\right)$ for the standard area measure on the sphere $\mathbb{S}^{(\beta+\varepsilon)_j-1}$. The integral over the *m*-fold product $\mathbb{S}^{(\beta+\varepsilon)_1-1} \times \ldots \times \mathbb{S}^{(\beta+\varepsilon)_m-1}$ can be calculated explicitly by using the following well-known formula:

Lemma (6.1.19)[270]: Let $d\sigma$ denote the usual surface measure on the $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$ and let $\theta \in \mathbb{Z}_+^{1+\varepsilon}$. Then

$$\int_{\mathbb{S}^{n-1}} \left|y^\theta\right| d\sigma(1+\varepsilon) = \frac{2\Gamma\left(\frac{\theta_1 - 1}{2}\right) \ldots \Gamma\left(\frac{\theta_\mathrm{n} - 1}{2}\right)}{\Gamma\left(\frac{\mathrm{n} - |\theta|}{2}\right)}$$

Using the formula in Lemma (6.1.19)we define:

$$\ominus_\beta := \int_{\mathbb{S}^{\beta_1+\varepsilon-1}\times\ldots\times\mathbb{S}^{\beta_m+\varepsilon-1}} \left|\rho^{2\beta+e}\right| d\sigma\left(\rho_{(1)}\right) \ldots d\sigma\left(\rho_{(m)}\right)$$
$$= 2^m \beta! \prod_{j=1}^{m} r\left(\frac{\beta_j + \varepsilon + 1}{2} + \left|\beta_{(j)}\right|\right)^{-1} \tag{36}$$

This finally gives:

$$(*) = \frac{\ominus_\beta}{2^{1+\varepsilon}} \int_{\mathbb{R}_+^m\times\mathbb{R}^{n-\varepsilon-2}\times\mathbb{R}_+} a_{s-1}\left(r, \frac{1}{2\sqrt{\zeta}}\left(-x' + (x+\varepsilon)'\right), (u+\varepsilon) + |r|^2\right)$$
$$\times \mathrm{r}_1^{2\left|\beta_{(s)}\right| + 2(\beta+\varepsilon)_1 - 1} \ldots \mathrm{r}_m^{2\left|\beta_{(m)}\right| + 2(\beta+\varepsilon)_m - 1} e^{-2\zeta\left((u+\varepsilon)+|r|^2\right)-\left|(x+\varepsilon)'\right|^2} (u+\varepsilon)^{\varepsilon-1} dr d(x+\varepsilon)' d(u$$
$$+ \varepsilon).$$

Note that the last integral does not depend on the full multi-index β but rather on the values $\left|\beta_{(1)}\right|$ for $j = 1, \ldots, m$. We denote this integral by $G_{a_{s-2}}\left(\left|\beta_{(1)}\right|, \ldots, \left|\beta_{(m)}\right|\right)$. Then the commutativity condition (35) can be written in the form:

$$\frac{\ominus_\beta}{\beta!} G_{a_{s-2}}\left(\left|\beta_{(1)}\right|, \ldots, \left|\beta_{(m)}\right|\right) = (2\zeta)^{-|p|+|p+\varepsilon|} \frac{\ominus_{\beta+\varepsilon}}{(\beta+\varepsilon)!} G_{a_{s-2}}\left(\left|\beta_{(1)}\right| - \left|p_{(1)}\right| + \left|(p+\varepsilon)_{(1)}\right|, \ldots, \left|\beta_{(m)}\right| - \left|p_{(m)}\right| + \left|(p+\varepsilon)_{(m)}\right|\right).$$

According to the definition (36) this is equivalent to

$$G_{a_{s-2}}(|\beta_{(1)}|,\dots,|\beta_{(m)}|)=\prod_{j=1}^{m}\Gamma\left(\frac{(\beta+\varepsilon)_j+1}{2}+|\beta_{(j)}|\right)^{-1}$$
$$=(2\zeta)^{-|p|+|p+\varepsilon|}G_{a_{s-2}}(|\beta_{(1)}|-|p_{(1)}|+|(p+\varepsilon)_{(j)}|,\dots,|\beta_{(m)}|-|p_{(m)}|$$
$$+|(p+\varepsilon)_{(m)}|)\prod_{j=1}^{m}\Gamma\left(\frac{(\beta+\varepsilon)_j+1}{2}+|\beta_{(j)}|-|p_{(j)}|+|(p+\varepsilon)_{(j)}|\right)^{-1}$$

This equality can be only true simultaneously for all $\beta+\varepsilon$ -quasi-nilpotent quasi-radial functions a_{s-2} and all $\beta\in\mathbb{Z}_+^{1+\varepsilon}$ if $|p_{(j)}|=|(p+\varepsilon)_{(j)}|$ for $j=1,\dots,m$. Hence, we obtain see[38]:

Theorem (6.1.20)[270]: Let $(\beta+\varepsilon)\in\mathbb{Z}_+^{1+\varepsilon}$ be given. Then the statements (a), (b) and (c) below are equivalent:

(a) For each $(\beta+\varepsilon)$-quasi-nilpotent quasi-radial function $a_{s-2}=a_{s-1}(r_1,\ (x+\varepsilon)',\mathrm{Im}(x+2\varepsilon)_n-|(u+2\varepsilon)'|^2)\in L^\infty(D_{n+s-1})$ and each $(\beta+\varepsilon)$-quasi-nilpotent quasi-homogeneous function

$$b_{s-2}=b_{s-1}(r_1,\dots,r_m,(x+\varepsilon)',\mathrm{Im}(x+2\varepsilon)_n-|(x+2\varepsilon)'|^2).(\zeta+\varepsilon)^p\overline{(\zeta+\varepsilon)}^{p+\varepsilon}\in L^\infty(D_{n+s-1})\qquad(37)$$

of degree $(p,p+\varepsilon)\in\mathbb{Z}_+^{1+\varepsilon}\times\mathbb{Z}_+^{1+\varepsilon}$ the Toeplitz operators $T_{a_{s-2}}$ and $T_{b_{s-2}}$ commute on each weighted Bergman space $\mathcal{A}_{\varepsilon-1}^2(D_{n+s-1})$.

(b) The equality $\bar{\gamma}_{a_{s-2},0,0}(\beta,x',\xi)=\bar{\gamma}_{a_{s-2},0,0}(\beta+\varepsilon,x',\xi)$ holds for all $\beta\in\mathbb{Z}_+^k$ and for each $(\beta+\varepsilon)$-quasi-nilpotent quasi-radial functions a_{s-2}.

(c) The equality $|p_{(j)}|=|(p+\varepsilon)_{(j)}|$ holds for each $j=1,\dots,m$.

Now, let us assume that $b_{s-2}\in L^\infty(D_{n+s-1})$ is of the form (37). Under the assumption $|p_{(j)}|=|(p+\varepsilon)_{(j)}|$, for each $j=1,\dots,m$, we calculate $\bar{\gamma}_{b_{s-2},p,p+\varepsilon}(\beta,x',\zeta)$ in (32) more explicitly by reducing the order of integration. Assume that $\beta+\varepsilon\in\mathbb{Z}_+^{1+\varepsilon}$. Then:

$$\bar{\gamma}_{b_{s-2},p,p+\varepsilon}(\beta,x',\zeta)$$
$$=(A_{s-2})_\beta(\zeta)(A_{s-2})_{\beta+\varepsilon}(\zeta)\chi_{\mathbb{R}_+}(\zeta)\int_{\mathbb{R}_+^{\varepsilon+1}\times\mathbb{R}^{n-\varepsilon-2}\times\mathbb{R}_+}r^{2(\beta+p+\varepsilon)}$$
$$\times\prod_{j=1}^{m}\mathrm{r}_j^{-|p_{(j)}|-|(p+\varepsilon)_{(j)}|}e^{-2\zeta(|r|^2+(u+\varepsilon))-|\widetilde{(x+\varepsilon)}'|^2}b_{s-1}$$
$$\times\left(r,\frac{-x'+\widetilde{(x+\varepsilon)}'}{2\sqrt{\zeta}},(u+\varepsilon)+|r|^2\right)rdrd\widetilde{(x+\varepsilon)}'\frac{c_{\varepsilon-1}(u+\varepsilon)^{\varepsilon-1}}{4}d(u+\varepsilon)$$
$$=\ominus_{\beta+p+\varepsilon}(A_{s-2})_\beta(\zeta)(A_{s-2})_{\beta+\varepsilon}(\xi)\chi_{\mathbb{R}_+}(\zeta)2^{-(1+\varepsilon)}$$
$$\times\int_{\mathbb{R}_+^{m}\times\mathbb{R}^{n-\varepsilon-2}\times\mathbb{R}_+}\prod_{j=1}^{m}\mathrm{r}_j^{2|\beta_{(j)}|+|(p+\varepsilon)_{(j)}|-|p_{(j)}|+2\alpha_j-1}e^{-2\zeta(|r|^2+(u+\varepsilon))-|\widetilde{(x+\varepsilon)}'|^2}b_{s-1}$$
$$\times\left(r,\frac{-x'+\widetilde{(x+\varepsilon)}'}{2\sqrt{\zeta}},(u+\varepsilon)+|r|^2\right)rdrd\widetilde{(x+\varepsilon)}'\frac{c_{\varepsilon-1}(u+\varepsilon)^{\varepsilon-1}}{4}d(u+\varepsilon)$$

$$= \frac{\ominus_{\beta+p+\varepsilon}}{\ominus_\beta}\frac{A_{\beta+\varepsilon}(\zeta)}{A_\beta(\zeta)}.D_{b_{s-2}}(\beta,x',\zeta)$$

$$= \frac{(\beta+p+\varepsilon)!}{\sqrt{\beta!\,(\beta+\varepsilon)!}}\prod_{j=1}^{m}\frac{\Gamma\left(\frac{(\beta+\varepsilon)_j+1}{2}+|\beta_{(j)}|\right)}{\Gamma\left(\frac{\beta_j+\varepsilon+1}{2}+|\beta_{(j)}|+|(p+\varepsilon)_{(j)}|\right)}.D_{b_{s-2}}(\beta,x',\zeta),$$

where $D_{b_{s-2}}(\beta,x',\zeta)=\bar{\gamma}_{b_{s-2},p,p+\varepsilon}$, which can be seen by choosing $\varepsilon=0$ in the above equalities. Hence we have proved:

Proposition (6.1.21)[270]:. Let $(\beta+\varepsilon)\in\mathbb{Z}_+^m$ be given. Assume that $b_{s-2}\in L^\infty(D_{n+s-1})$ is of the
form (37) and let $|p_{(j)}|=|(p+\varepsilon)_{(j)}|$, for each $j=1,\ldots,m$. Then in the case of $\beta+\varepsilon\in\mathbb{Z}_+^m$ we have

$$\bar{\gamma}_{b_{s-2},p,p+\varepsilon}(\beta,x',\zeta)$$
$$= \frac{(\beta+p+\varepsilon)!}{\sqrt{\beta!\,(\beta+\varepsilon)!}}\prod_{j=1}^{m}\frac{\Gamma\left(\frac{(\beta+\varepsilon)_j+1}{2}+|\beta_{(j)}|\right)}{\Gamma\left(\frac{(\beta+\varepsilon)_j+1}{2}+|\beta_{(j)}|+|p+\varepsilon_{(j)}|\right)}.\bar{\gamma}_{b_{s-2},0,0}(\beta,x',\xi).$$

In the case of $(\beta+\varepsilon)\notin\mathbb{Z}_+^{1+\varepsilon}$ we have $\bar{\gamma}_{b_{s-2},p,p+\varepsilon}(\beta,x',\zeta)=0$. the factor $\bar{\gamma}_{b_{s-2},p,p+\varepsilon}(\beta,x',\zeta)$ can be expressed in the form

$$\bar{\gamma}_{b_{s-2},p,p+\varepsilon}(\beta,x',\zeta)$$
$$=\ominus_\beta A_\beta^2(\zeta)\chi_{\mathbb{R}_+}(\zeta)2^{-(1+\varepsilon)}\int_{\mathbb{R}_+^m\times\mathbb{R}^{n-\varepsilon-2}\times\mathbb{R}_+}\prod_{j=1}^{m}\mathrm{r}_j^{2|\beta(j)|+|q(j)|-|p(j)|+2(\beta+\varepsilon)_j-1}$$
$$\times e^{-2\zeta(|r|^2+(u+\varepsilon))-|(x+\varepsilon)'|^2}b_{s-1}\left(r,\frac{-x'+(x+\varepsilon)'}{2\sqrt{\zeta}},(u+\varepsilon)+|r|^2\right)$$
$$\times rdrd(x+\varepsilon)'\frac{c_{\varepsilon-1}(u+\varepsilon)^{\varepsilon-1}}{4}d(u+\varepsilon) \qquad (38)$$

Let $\beta+\varepsilon\in\mathbb{Z}_+^m$ be given and $(p,p+\varepsilon)\in\mathbb{Z}_+^{1+\varepsilon}\times\mathbb{Z}_+^{1+\varepsilon}$. From Proposition (6.1.21) we conclude:

Corollary (6.1.22)[270]: Let $a_{s-2}=a_{s-1}(r,(x+\varepsilon)',Im(x+2\varepsilon)_n-|(u+2\varepsilon)'|^2)\in L^\infty(D_{n+s-1})$ be an $(\beta+\varepsilon)-$ quasi-nilpotent quasi-radial function. Under the assumption $|p(j)|$ for all $j=1,2,\ldots,m$ we have

$$T_{a_{s-2}}T_{(\zeta+\varepsilon)^p\overline{(\zeta+\varepsilon)}^{p+\varepsilon}}=T_{(\zeta+\varepsilon)^p\overline{(\zeta+\varepsilon)}^{p+\varepsilon}}T_{a_{s-2}}=T_{a_{s-2}(\zeta+\varepsilon)^p\overline{(\zeta+\varepsilon)}^{p+\varepsilon}} \qquad (39)$$

on each weighted Bergman space.

Proof: The first equality in (39) is a direct consequence of Theorem (6.1.20). If $e(x+2\varepsilon)\equiv 1$ then $T_e=Id$, and thus $\bar{\gamma}_{e,0,0}(\beta,x',\xi)\equiv 1$. Hence, Proposition (6.1.21)implies that in the case of a symbol $b_{s-2}=(\zeta+\varepsilon)^p\overline{(\zeta+\varepsilon)}^{p+\varepsilon}$ with $|p_{(j)}|=|(p+\varepsilon)_{(j)}|$, for all $j=1,2,\ldots,m$, one has

$$\bar{\gamma}_{b_{s-2},p,p+\varepsilon}(\beta,x',\zeta)$$
$$=\frac{(\beta+p+\varepsilon)!}{\sqrt{\beta!\,(\beta+\varepsilon)!}}\prod_{j=1}^{m}\frac{\Gamma\left(\frac{(\beta+\varepsilon)_j+1}{2}+|\beta_{(j)}|\right)}{\Gamma\left(\frac{(\beta+\varepsilon)_j+1}{2}+|\beta_{(j)}+(p+\varepsilon)_{(j)}|\right)}, \tag{40}$$

Whenever $(\beta+\varepsilon)\in\mathbb{Z}_+^{1+\varepsilon}$ (cf. Example (6.1.18) for the choice of $(\beta+\varepsilon)=(1,\dots,1)\in\mathbb{Z}_+^{1+\varepsilon}$ and the case $p_j = (p+\varepsilon)_j$, for $1,\dots,1+\varepsilon$) Moreover, if $\beta+\varepsilon\in\mathbb{Z}_+^{1+\varepsilon}$, then it holds $\bar{\gamma}_{b_{s-2},p,p+\varepsilon}(\beta,x',\zeta)$. Theorem (6.1.20), Proposition (6.1.21) and the assumption that $|p_{(j)}| = |(p+\varepsilon)_{(j)}|$, for all $j=1,2,\dots,m$, imply now that

$$\bar{\gamma}_{a_{s-2}b_{s-2},p,p+\varepsilon}(\beta,x',\zeta)=\bar{\gamma}_{b_{s-2},p,p+\varepsilon}(\beta,x',\zeta)\,.\,\bar{\gamma}_{a_{s-2},0,0}(\beta,x',\zeta)$$
$$=\bar{\gamma}_{b_{s-2},p,p+\varepsilon}(\beta,x',\zeta)\,.\,\bar{\gamma}_{a_{s-2},0,0}(\beta+\varepsilon,x',\zeta)$$

This together with (33) and Theorem 4.5 yields the second equality in (39).

Proposition (6.1.23)[270]: Let us assume that $|p_{(j)}| = |(p+\varepsilon)_{(j)}|$ for all $j=1,2,\dots,m,$. Then the Toeplitz operators T_j commute mutually. Moreover,

$$\prod_{j=1}^{m}T_j := T_{(\zeta+\varepsilon)^p\overline{(\zeta+\varepsilon)}^{p+\varepsilon}} \tag{41}$$

on each weighted Bergman space.

Proof: *Let* $b_j := (\zeta+\varepsilon)^{\bar{p}_{(j)}}\overline{(\zeta+\varepsilon)}^{(p+\varepsilon)_{(j)}}$, *for* $j=1,2,\dots,m$, We only prove the following product rule:

$$T_jT_i := T_{(\zeta+\varepsilon)^{\bar{p}_{(i)}+\bar{p}_{(j)}}\left(\overline{\zeta+\varepsilon}\right)^{\overline{(p+\varepsilon)}_{(i)}+\overline{(p+\varepsilon)}_{(j)}}} \tag{42}$$

for $i,j\in\{j=1,2,\dots,m,\}$ and $i\neq j$. According to Theorem (6.1.17)the operator $R_{s-2}\,T_jT_iR_{s-2}^*$ acts on the sequence space $\ell_2\left(\mathbb{Z}_+^{\varepsilon+1},L_2(\mathbb{R}^{n-\varepsilon-2}\times\mathbb{R}_+)\right)$ by the rule:

$$R_{s-2}\,T_jT_iR_{s-2}^*\,\{c_\beta(x',\zeta)\}_{\beta\in\mathbb{Z}_+^{\varepsilon+1}}$$
$$=R_{s-2}T_jR_{s-2}^*\,\Big\{\tilde{\gamma}_{b_j,\bar{p}_{(j)},\overline{(p+\varepsilon)}_{(j)}}(\beta,x',\zeta).\tilde{\gamma}_{b_i,\bar{p}_{(i)},\overline{(p+\varepsilon)}_{(i)}}\left(\beta-\bar{p}_{(j)}\overline{(p+\varepsilon)}_{(j)},x',\zeta\right)$$
$$\times c_{\beta-\bar{p}_{(i)}-\bar{p}_{(j)}+\overline{(p+\varepsilon)}_{(i)}+\overline{(p+\varepsilon)}_{(j)}}(x',\zeta)\Big\}_{\beta\in\mathbb{Z}_+^{\varepsilon+1}}$$

Hence it is clear that (42) is equivalent to:

$$\tilde{\gamma}_{b_j,\bar{p}_{(j)},\overline{(p+\varepsilon)}_{(j)}}(\beta,x',\zeta).\tilde{\gamma}_{b_i,\bar{p}_{(i)},\overline{(p+\varepsilon)}_{(i)}}\left(\beta-\bar{p}_{(j)},\overline{(p+\varepsilon)}_{(j)},x',\zeta\right)$$
$$=\tilde{\gamma}_{b_i,b_j,\bar{p}_{(i)},\bar{p}_{(j)},\overline{(p+\varepsilon)}_{(i)},\overline{(p+\varepsilon)}_{(j)}}(\beta,x',\zeta). \tag{43}$$

By (40) we have

$$\tilde{\gamma}_{b_j,\bar{p}_{(j)},(p+\varepsilon)_{(j)}}(\beta,x',\zeta)$$
$$=\frac{\left(\beta_{(j)}+\overline{(p+\varepsilon)}_{(j)}\right)!}{\sqrt{\beta_{(j)}!\left(\beta_{(j)}-\bar{p}_{(j)}+\overline{(p+\varepsilon)}_{(j)}\right)!}}\frac{\Gamma\left(\frac{(\beta+\varepsilon)_j+1}{2}+|\beta_{(j)}|\right)}{\Gamma\left(\frac{(\beta+\varepsilon)_j+1}{2}+|\beta_{(j)}+\overline{(p+\varepsilon)}_{(j)}|\right)},$$

and similar for i replaced by j. Moreover, the function on the right hand side of (43) has the explicit form:

$$\tilde{\gamma}_{b_i,b_j,\tilde{p}_{(i)},\tilde{p}_{(j)},\widetilde{(p+\varepsilon)}_{(i)},\widetilde{(p+\varepsilon)}_{(j)}}(\beta,x',\zeta)$$
$$=\frac{\left(\beta+\widetilde{(p+\varepsilon)}_{(i)}+\widetilde{(p+\varepsilon)}_{(j)}\right)!}{\sqrt{\beta!\left(\beta-\tilde{p}_{(i)}-\tilde{p}_{(j)}+\widetilde{(p+\varepsilon)}_{(i)}+\widetilde{(p+\varepsilon)}_{(j)}\right)!}}$$
$$\times\prod_{\ell\in\{i,j\}}\frac{\Gamma\left(\frac{(\beta+\varepsilon)_\ell+1}{2}+|\beta_{(\ell)}|\right)}{\Gamma\left(\frac{(\beta+\varepsilon)_\ell+1}{2}+\left|\beta_{(\ell)}+\widetilde{(p+\varepsilon)}_{(\ell)}\right|\right)}$$

Now, (43) can be easily checked from these identities.

Theorem (6.1.24)[270]: *The Toeplitz operators $T\varphi_s$ and $T\varphi_{s+1}$ commute on each weighted Bergman space $(\mathcal{A}^2_{s-2})_{\varepsilon-1}(D_{n+s-1})$ if and only if for each $\ell=1,2,\dots,\varepsilon+1$ one of the conditions (a)* − (d) is fulfilled*:*

(a) $p_\ell=(p+\varepsilon)_\ell=0$

(b) $u_\ell=(u+\varepsilon)_\ell=0$

(c) $p_\ell=u_\ell=0$

(d) $(p+\varepsilon)_\ell=(u+\varepsilon)_\ell=0$

Proof: Similar hat the operators $T\varphi_s$ and $T\varphi_{s+1}$ commute on $(\mathcal{A}^2_{s-2})_{\varepsilon-1}(D_{n+s-1})$ if and only if for all $\in\mathbb{Z}_+^{\varepsilon+1}$:

$$\tilde{\gamma}_{\varphi_s,p,p+\varepsilon}(\beta,x',\zeta).\tilde{\gamma}_{\varphi_{s+1}u,u+\varepsilon}(\beta+\varepsilon,x',\zeta)=\tilde{\gamma}_{\varphi_{s+1}u,u+\varepsilon}(\beta,x',\zeta).\tilde{\gamma}_{\varphi_s,p,p+\varepsilon}(\beta+\varepsilon,x',\zeta)$$

Since $|p_{(j)}|=|(p+\varepsilon)_{(j)}|$ *and* $|u_{(j)}|=|\,(u+\varepsilon)_{(j)}|$ for $j=1,2,\dots,m$we can use the factorization of $\tilde{\gamma}_{\varphi_s,p,p+\varepsilon}(\beta,x',\zeta)$ and $\tilde{\gamma}_{\varphi_{s+1}u,u+\varepsilon}(\beta,x',\zeta)$ in Proposition (6.1.21):

$$\tilde{\gamma}_{\varphi_s,p,p+\varepsilon}(\beta,x',\zeta)=\Phi_{p,p+\varepsilon}(\beta).\tilde{\gamma}_{\varphi_s0,0}(\beta,x',\zeta),$$
$$\tilde{\gamma}_{\varphi_{s+1}u,u+\varepsilon}(\beta,x',\zeta)=\Phi_{u,u+\varepsilon}(\beta).\tilde{\gamma}_{\varphi_{s+1}0,0}(\beta,x',\zeta),$$

where we use the notation:

$$(\beta)=\frac{(\beta+p+\varepsilon)!}{\sqrt{\beta!\,(\beta+\varepsilon)!}}\prod_{j=1}^{m}\frac{\Gamma\left(\frac{(\beta+\varepsilon)_j+1}{2}+|\beta_{(j)}|\right)}{\Gamma\left(\frac{(\beta+\varepsilon)_j+1}{2}+\left|\beta_{(j)}+(p+\varepsilon)_{(j)}\right|\right)}\qquad(44)$$

Moreover, it follows from Theorem (6.1.20) and again by the conditions on $(p,p+\varepsilon)$ and $(u,u+\varepsilon)$ that

$$\tilde{\gamma}_{\varphi_s0,0}(\beta,x',\zeta)=\tilde{\gamma}_{\varphi_s0,0}(\beta+\varepsilon,x',\zeta)$$
$$\tilde{\gamma}_{\varphi_{s+1}0,0}(\beta,x',\zeta)=\tilde{\gamma}_{\varphi_{s+1}0,0}(\beta+\varepsilon,x',\zeta)$$

Therefore we only need to verify that

$$\Phi_{p,p+\varepsilon}(\beta).\Phi_{u,u+\varepsilon}(\beta+\varepsilon)=\Phi_{u,u+\varepsilon}(\beta).\Phi_{p,p+\varepsilon}(\beta+\varepsilon).$$

By a straightforward calculation this is equivalent to:

$$(\beta+p+\varepsilon)!\,\tfrac{(\beta+u+2\varepsilon)!}{(\beta+\varepsilon)!}=(\beta+u+\varepsilon)!\,\tfrac{(\beta+p+2\varepsilon)!}{(\beta+\varepsilon)!}.$$

Varying β it can be seen that this equality holds if and only if for each $\ell=1,2,\dots,\varepsilon+1$ one of the conditions (a)-(d) is fulfilled.

Section (6.2) Toeplitz Operators with Quasi-Radial Quasi-Homogeneous Symbols

We study of commutative algebras generated by Toeplitz operators acting on the Bergman spaces over the unit ball. The fact of just an existence of such algebras was quite unexpected and its exploration for the unit disk case was started in [294, 296, 295]. The final result on classification and description of the C*-algebras generated by Toeplitz operators being commutative on all weighted Bergman spaces $\mathcal{A}_{\lambda}^{2}(\mathbb{D}^{n})$ on the unit disk was obtained in [98]. In an equivalent reformulation it states that, under some technical assumption on the "richness" of a class of generating symbols, a C*-algebra generated by Toeplitz operators is commutative on each weighted Bergman space if and only if the corresponding symbols of Toeplitz operators are constant on the orbits of a maximal commutative subgroup of the M̈obius transformations of the unit disk.

This result was extended then to the case of the unit ball. As proved in [218, 219], given a maximal commutative subgroup of biholomorphisms of the unit ball, the C*-algebra generated by Toeplitz operators, whose symbols are constant on the orbits of this subgroup, is commutative on each weighted Bergman space.

There are five different pairwise non-conjugate model classes of such subgroups: quasi-elliptic, quasi-parabolic, quasi-hyperbolic, nilpotent, and quasi-nilpotent (the last one depends on a parameter, giving in total $n+2$ model classes for the n-dimensional unit ball). As a consequence, for the unit ball of dimension n, there are $n+2$ essentially different "model" commutative C*-algebras, all others are conjugated with one of them via biholomorphisms of the unit ball.The next surprise came first in [193] and was developed then in [37,38,195].

As it turned out, for n > 1 there exist many other, not geometrically defined, classes of symbols which generate commutative Toeplitz operator algebras on each weighted Bergman space. These classes of symbols were always subordinated to one of the above model classes of the maximal commutative subgroup (with the exception of the nilpotent subgroup). The corresponding commutative operator algebras were Banach, and being extended to C*-algebras they became non-commutative.

We note that in all above cases of the commutative C*-algebras generated by Toeplitz operators these algebras always come with an unitary operator (specific for each algebra) that reduces each operator from the algebra to a multiplication operator, giving thus, among other results, a complete spectral picture of the operators under study.

As for the commutative Banach algebras generated by Toeplitz operators, the results obtained so far give just the description of these algebras in terms of their generators. The next challenging task is to develop their Gelfand theory, obtaining thus more detailed information on the operators forming the algebra.

We study the case of a commutative Banach algebra generated by Toeplitz operators with quasi-radial quasi-homogeneous symbols (i.e. an algebra subordinated to the quasi-elliptic group). To simplify the considerations we restrict our attention to the lowest dimensional case $n=2$.The corresponding (unique) commutative Toeplitz operator algebra $\mathcal{T}(\lambda)$ is Banach (not C*), and can be described as follows:

Let $H:=\mathcal{A}_{\lambda}^{2}(\mathbb{B}^{n})$ be the weighted Bergman space over $\mathbb{B}^{2}$ with parameter $\lambda>-1$, and write $\mathcal{T}_{rad}(\lambda)$ for the commutative C*-subalgebra of $\mathcal{L}(H)$ generated by all Toeplitz operators T_a

with radial bounded measurable symbols a on $\mathbb{B}^2$ (i.e.$\alpha(z) = \alpha(|z|)$). Further, we denote by T_φ the unital Banach Structure of A Commutative Banach Algebra algebra with a single generator T_φ, where φ is the "simplest" quasi-homogeneous symbol on $\mathbb{B}^2$ .It is easy to see that operators in $\mathcal{T}_{\text{rad}}(\lambda)$ and $\mathcal{T}_\varphi$ commute and, as an important observation, we remark that $\mathcal{T}(\lambda)$is generated by these two algebras (cf. Corollary (6.2.5)).

Our results on the structure of$\mathcal{T}(\lambda)$already reveal some important features which we expect to be useful under a further study of the higher dimensional case $n > 2$, and in a situation where the quasi-elliptic group of automorphisms of $\mathbb{B}^n$ is replaced by another group among the above model classes.

The main theorem explicitly expresses the maximal ideals of $\mathcal{T}(\lambda)$ and the Gelfand map.

Theorem (Theorem (6.2.28)) The compact set $M\ (\mathcal{T}(\lambda))$ of maximal ideals of the algebra $\mathcal{T}(\lambda)$ has the form

$$M(\mathcal{T}(\lambda)\ ) = \mathbb{Z}_+ \times \{0\} \cup M_\infty(\lambda) \times \overline{D}\left(0, \frac{1}{2}\right)$$

where $M_\infty(\lambda)$ can be identified with the subset of all multiplicative functional of $\mathcal{T}_{\text{rad}}(\lambda)$ that map compact operators to zero. The Gelfand transform is generated by the following mapping of the elements of a dense (non-closed) subalgebra of $\mathcal{T}(\lambda)$:

$$\sum_{j=0}^{n} D_{\gamma j} T_\Phi^j \longmapsto \begin{cases} \gamma_0(k) & (k, 0) \in \mathbb{Z}_+ \times \{0\} \\ \sum_{j=0}^{n} \mu(D_{\gamma j}) \xi j & (\mu, \xi) \in M_\infty(\lambda) \times \overline{D}\left(0, \frac{1}{2}\right) \end{cases}$$

Here $D_{\gamma_j} \in \mathcal{T}_{\text{rad}}(\lambda)$ is a diagonal operator with respect to the standard orthonormal basis $[e_\alpha : \alpha \in \mathbb{Z}_+^2]$ of H with the sequence $\gamma_j = \{\gamma_j(|\alpha|)\}_\alpha$of the corresponding eigenvalues.

As an important ingredient of the proof we carefully analyze the structure of the algebras $\mathcal{T}_{rad}(\lambda)$, $\mathcal{T}_\varphi$, and of C^*-algebras that are generated by just a finite number of Toeplitz operators with radial symbols. In order to identify the multiplicative functionals of the previous algebras we essentially employ the concept of the "joint spectrum" and the "joint approximate spectrum"of finite tuples of operators together with the Berezin transform on functions with respect to suitable subspaces of H. It is important to note that the arguments are not purely algebraic but heavily rely on the analytic structure of the generating Toeplitz operators and the underlying Bergman space.

Some important properties of $\mathcal{T}(\lambda)$ can be deduced by the help of the previous theorem. Since $\mathcal{T}(\lambda)$is not invariant under the $*-$operation of $\mathcal{L}(H)$ the inverse closedness of this algebra is not obvious, and usually such a feature is hard to show. Here we can prove the inverse closedness of $\mathcal{T}(\lambda)$ from the explicit description of the maximal ideal space and by extending multiplicative functionals (in a multiplicative way) from commutative Banach algebras to an enveloping (non-commutative) $C^* -$algebra.

We show that $\mathcal{T}(\lambda)$ is not semi-simple, and in Lemma (6.2.7) we describe some of the elements in its radical $Rad\mathcal{T}(\lambda)$. However, to calculate the radical precisely we again need Theorem (6.2.28) together with additional arguments (see Lemma (6.2.38) and Theorem (6.2.43)).

Finally we wish to point out that the derivation of the maximal ideal space has important consequences for the operator theory of the elements in $\mathcal{T}(\lambda)$We give some remarks on the essential spectrum and the Fredholm property of (certain) operators $A \in \mathcal{T}(\lambda)$ (cf. Theorem (6.2.32) and Corollary (6.2.36)) and solve a"zero-product-problem"for Toeplitz operators (cf. Corollary (6.2.10)), which holds true despite of a certain ambiguity in the representation of operators as a finite sum of products of elements in$\mathcal{T}_{rad}(\lambda)$ and $\mathcal{T}_{\varphi}$(see Lemma (6.2.8)).

We recall the construction of commutative Banach Toeplitz algebras that are subordinate to the quasielliptic group.We obtain a set of generators for this algebra $\mathcal{T}(\lambda)$in the lowest dimensional case and we study some of its subalgebras. contains preliminary results on the maximal ideals for certain finitely generated subalgebras of $\mathcal{T}(\lambda)$. Our main result (Theorem (6.2.28)) on the Gelfand theory of $\mathcal{T}(\lambda)$is proved and we give some applications.
Among them we prove the inverse closedness of $\mathcal{T}(\lambda)$and calculate its radical.

We write $\mathbb{B}^n$, $(n \in \mathbb{N})$ for the open Euclidean unit ball in $\mathbb{C}^n$, i.e.

$$\mathbb{B}^n := \{z = (z_1, \dots, z_2) \in \mathbb{C}^n : |z|^2 = |z_1|^2 + \cdots + |z_n|^2 < 1\}$$

Let dv denote the standard volume form on $\mathbb{B}^n$. With $\lambda > -1$ we consider the one-parameter family of the standard weighted measures

$$d\mu_\lambda(z) = c_\lambda (1 - |z|^2)^\lambda \, dv$$

where $c_\lambda > 0$ is a normalizing constant such that $v_\lambda(\mathbb{B}^n)$. More precisely, c_λ is explicitly given by the formula:

$$c_\lambda := \frac{\Gamma(n + \lambda + 1)}{\pi^n \Gamma(\lambda + 1)}$$

The weighted Bergman space $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ is the closed subspace in $L_2(\mathbb{B}^n, dv_\lambda)$ consisting of all functions analytic in $\mathbb{B}^n$. We write $\mathbb{B}_\lambda$ for the orthogonal Bergman projection from $L_2(\mathbb{B}^n, dv_\lambda)$ onto $\mathcal{A}^2_\lambda(\mathbb{B}^n)$. It is well-known that B_λ can be expressed as the following integral operator:

$$[B_\lambda \varphi](z) = \int_{\mathbb{B}^n} \frac{\varphi(\xi)}{(1 - \langle z, \xi \rangle^{n+\lambda+1})}, dv_\lambda(\xi).$$

where $\varphi \in L_2(\mathbb{B}^n, dv_\lambda)$ and $\langle z, \xi \rangle := z_1 \bar{\xi}_1 + \cdots + z_n \bar{\xi}_n$ .Given a function $g \in L^\infty(\mathbb{B}^n)$ the Toeplitz operator T_g with symbol g acting on $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ is defined by:

$$T_g \varphi := B_\lambda(g\varphi) \qquad \varphi \in \mathcal{A}^2_\lambda(\mathbb{B}^n)$$

For $n > 1$, new classes of commutative Banach algebras generated by Toeplitz operators with specific bounded symbols on $\mathbb{B}^n$ have been constructed in [37,38,193,195]. As we have remarked already these algebras remain commutative on each weighted Bergman spaces $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ with $\lambda > -1$, and are induced by the maximal commutative subgroups of the biholomorphisms of the unit ball: quasi-elliptic, quasi-parabolic, quasi-hyperbolic and quasi-nilpotent.

These Toeplitz Banach algebras are not invariant under the $*$-operation of $\mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{B}^n))$, and being extended to C^*-algebras they become non-commutative.

We shortly recall now the definition of the commutative algebras that are subordinated to the quasi-elliptic group of biholomorphisms (for further details see [217, 192]). Let $k = (k_1, k_2, \cdots, k_m)$ be a tuple of positive integers with $|k| = k_1 + k_2 + \cdots + k_m = n$. We divide the coordinates of $z \in \mathbb{B}^n$ into m groups of k_j entries, respectively by using the notation $z = z_{(1)}, \dots, z_{(m)} \in \mathbb{C}^n$ with

$$z_{(j)} = \left(z_{j,1}, \dots, z_{j,k_j}\right) \in \mathbb{C}^{k_j}$$

where $j = 1, \dots, m$. Let $\mathbb{S}^{2k_j-1} \subset \mathbb{C}^{k_j}$ denote the (real) $(2k_j - 1)$ –dimensionalunit sphere in $\mathbb{C}^{k_j}$. We express $z_{(j)} = 0$ in polar-coordinates $z_{(j)} = r_j z_{(j)} \xi_{(j)}$ with

$$\xi_{(j)} = \frac{z_{(j)}}{\|z_{(j)}\|} \in \mathbb{S}^{2k_j-1} \quad \text{and} \quad r_j = \|z_{(j)}\| \in \mathbb{R}_+ \qquad (45)$$

A bounded function $\varphi(z)$ on $\mathbb{B}^n$a 3 is called k-quasi-homogeneous if it has the form:

$$\varphi(z) = a(r_1, \dots, r_m) \xi_{(1)}^{p(1)} \xi_{(2)}^{p(2)} \dots \xi_{(m)}^{p(m)} \xi_{(1)}^{-q(1)} \xi_{(2)}^{-q(2)} \dots \xi_{(m)}^{-q(m)} \qquad (46)$$

and $a$is a function of the m non-negative real variables $r_1, \dots, r_m$.The tuple $(p, q) \in \mathbb{Z}_+^n \times \mathbb{Z}_+^n$ with $p \perp q$is called the quasi-homogeneous degree of $\varphi(z)$.

Fix a tuple $h = (h_1, \dots, h_m) \in \mathbb{Z}_+^m$,with $h_j = 0$ if $k_j = 1$ and $1 \le h_j \le k_j - 1$ if $k_j > 1$.

We denote by $\mathcal{R}_k(h)$ the linear space generated by all k-quasi-homogeneous functions of the form (46) such that

(i) For j with $k_j > 1$: $p_{(j)} = (p_{j,1}, \dots, p_{j,h_j}, 0, \dots, 0)$ and $q_{(j)} = (0, \dots, 0, q_{j,h_{j+1}}, \dots, q_{j,k_j})$, with $p_{j,1}, \dots, p_{j,h_j}, q_{j,h_{j+1}}, \dots, q_{j,k_j} \in \mathbb{Z}_+$, and$p_{j,1} + \dots + p_{j,h_j} = q_{j,h_{j+1}}, \dots, q_{j,k_j}$.

(ii) If $k_{j'} = k_{j''}$ with $j' < j''$ then$h_{j'} \le h_{j''}$.

Recall that the quasi-elliptic group of biholomorphisms of $\mathbb{B}^n$ is isomorphic with the n-torus ($\mathbb{T}^n$ (here $\mathbb{T} = \mathbb{S}^1$) and acts on $\mathbb{B}^n$ as follows

$$\mathbb{T}^n \ni t = (t_1, \dots, t_n) : \; z = (z_1, \dots, z_n) \longmapsto tz = (t_1 z_1, \dots, t_n z_n).$$

Note that the functions from $\mathcal{R}_k(h)$ are invariant under the subgroup $\mathbb{T}^m$ of the quasi-elliptic group $\mathbb{T}^n$, which acts on$\mathbb{B}^n$ as follows

$$\mathbb{T}^m \ni t = (t_1, \dots, t_m) : \; z = (z_1, \dots, z_m) \longmapsto \left(t_1 z_{(1)}, \dots, t_m z_{(m)}\right).$$

The main result in [193] states the following:

Theorem (6.2.1)[303]: The Banach algebra $\mathcal{B}_k(h)$ generated by Toeplitz operators with symbols from $\mathcal{R}_k(h)$ is commutative.

In the case of $n > 1$ the algebras $\mathcal{B}_k(h)$ do not extend to commutative C^*-algebras. This effect arise from the multidimensional setting and has no counterpart in the case of $n = 1$.

Our next global plan is to study the internal structure of $\mathcal{B}_k(h)$ more precisely and, in particular, we wish to determine their maximal ideal spaces.

We consider the simplest model case where

$$n = 2, \text{and } k = 2$$

that is, we fix the dimension $n = 2$ and we choose $m = 1$. As a consequence we need to put $h = (1) = 1$, and our main object to study, the commutative Banach algebra $\mathcal{T}(\lambda) := \mathcal{B}_2(1)$, is generated by the operators of the form $T_{a(r)\xi(p,0)\xi(0,p)}$, where $a \in L_\infty[0,1)$ and $p \in \mathbb{Z}_+$.

By [192], for any bounded measurable function a(r) we have

$$T_\alpha z^\alpha = \gamma_{\alpha,\lambda}(|\alpha|) z^\alpha, \qquad \alpha \in \mathbb{Z}_+^2$$

Where

$$\gamma_{\alpha,\lambda}(|\alpha|) = \frac{\Gamma(|\alpha|+\lambda+3)}{\Gamma(\lambda+1)\Gamma(|\alpha|+2)} \int_0^1 \alpha(\sqrt{r})(1-r)^{\lambda} r^{|\alpha|+1} dr. \quad (47)$$

According to [192], for $|p| = |q|$ we have

$$T_{\xi^{(p,0)}\bar{\xi}^{(o,p)}} z^{\alpha} = \bar{\gamma}_{p,\lambda}(\alpha) z_1^{\alpha_1+p} z_2^{\alpha_2-p}, \alpha \in \mathbb{Z}_+^2,$$

Where

$$\bar{\gamma}_{p,\lambda}(\alpha) = \frac{\alpha_2(\alpha_2-1)\ldots\alpha_2(\alpha_2-p+1)}{(p+1+|\alpha|)(p+|\alpha|)\ldots(2+|\alpha|)} \quad (48)$$

We mention that $\bar{\gamma}_{p,\lambda}$ does not depend on the weight parameter λ. By [193], for any bounded measurable $\alpha = \alpha(r)$ and $p \in \mathbb{Z}^+$ we have

$$T_{\alpha} T_{\xi^{(p,0)}\bar{\xi}^{(o,p)}} = T_{\xi^{(p,0)}\bar{\xi}^{(o,p)}} T_{\alpha} = T_{\alpha\xi^{(p,0)}\bar{\xi}^{(o,p)}} \quad (49)$$

As a consequence the algebra $\mathcal{T}(\lambda)$ is generated by the operators T_{α}, with $a \in L_{\infty}[0,1)$, and $T_{\xi^{(p,0)}\bar{\xi}^{(o,p)}}$, where $p \in \mathbb{Z}^+$ (see Corollary (6.2.3) for an even"smaller"set of generators).

We start our analysis by studying separately the different types of Toeplitz operators that, according to generate $\mathcal{T}(\lambda)$. First we consider operators with radial and then with quasi-homogeneous symbols.

Let $\gamma = \{\gamma(|\alpha|)\}\ |\alpha| \in \mathbb{Z}^+$ be a bounded sequence. Denote by D_{γ} the (bounded linear) diagonal operator which acts on the weighted Bergman space $\mathcal{A}_{\lambda}^2(\mathbb{B}^2)$, by the rule

$$\mathcal{D}_{\alpha} z^{\alpha} = \gamma(|\alpha|)\, z^{\alpha}\ , \quad \alpha \in \mathbb{Z}_+^2$$

Of course each Toeplitz operator with bounded measurable radial symbol α(r) is diagonal, and $T_{\alpha} = D_{\gamma\alpha,\lambda}$. However, as the next lemma states, not all bounded diagonal operators D_{γ} can be represented in such a form since the eigenvalue sequence $\gamma_{\alpha,\lambda}$ of T_{α} is always slowly oscillating.

Lemma (6.2.2)[303]: Let $\alpha(r) \in L_{\infty}[0,1)$and $k = |\alpha|$.Then $\lim_{k\to\infty}(\gamma_{\alpha,\lambda}(k) - \gamma_{\alpha,\lambda}(k+1)) = 0.$

Proof: Let $M = ess - sup\,|\alpha(r)|$. By (27), we have

$$\begin{aligned} &|\gamma_{\alpha,\lambda}(k) - \gamma_{\alpha,\lambda}(k+1)| \\ &= \left| \frac{\Gamma(k+\lambda+3)}{\Gamma(\lambda+1)\Gamma(k+2)} \int_0^1 \alpha(\sqrt{r})(1-r)^{\lambda} r^{k+1} dr \right. \\ &\left. - \frac{\Gamma(k+\lambda+4)}{\Gamma(\lambda+1)\Gamma(k+3)} \int_0^1 \alpha(\sqrt{r})(1-r)^{\lambda} r^{k+2} dr \right| \\ &= \left| \frac{\Gamma(k+\lambda+3)}{\Gamma(\lambda+1)\Gamma(k+2)} \int_0^1 \alpha(\sqrt{r})(1-r)^{\lambda+1} r^{k+1} dr \right. \\ &\left. - \frac{\lambda+1}{k+\lambda+3} \frac{\Gamma(k+\lambda+4)}{\Gamma(\lambda+1)\Gamma(k+3)} \int_0^1 \alpha(\sqrt{r})(1-r)^{\lambda} r^{k+2} dr \right| \end{aligned}$$

$$= \frac{\lambda+1}{k+\lambda+3}\gamma_{\alpha,\lambda+1}(k) - \gamma_{\alpha,\lambda}(k+1)\Big|$$
$$\leq 2M\frac{\lambda+1}{k+\lambda+3}$$
The last expression tends to 0 when $k \to \infty$.
It follows from the lemma that the set of (partial) limit points of the sequence $\gamma_{\alpha,\lambda}$ is connected and compact.
According to the results in [194] and for the unweighted case ($\lambda = 0$) the algebra $\mathcal{T}_{rad}(\lambda)$is isomorphic and isometric to the C*-algebra $SO_{(0)}$ which consists of all slowly oscillating sequences satisfying the condition
$$\lim_{\frac{m}{n}\to 1} |\gamma(m) - \gamma(n)| = 0.$$
We denote by $M(\mathcal{T}_{rad}(\lambda))$the compact set of maximal ideals of the algebra $\mathcal{T}_{rad}(\lambda)$ (or, which is the same, of the algebra $SO(\lambda)$). Let $M_\infty(\lambda)$ be the fiber of$M(\mathcal{T}_{rad}(\lambda))$ consisting of all multiplicative functionals ψ such that $\psi(D_\gamma) = 0$whenever D_γ is compact (or whenever $\gamma \in c_0$, where c_0 denotes set of all sequences converging to zero).
Each point $k \in \mathbb{Z}_+$ defines a multiplicative functional $\psi(k)$ on $\mathcal{T}_{rad}(\lambda)$:
$$\psi(k): D_\gamma \longmapsto \gamma(k),$$
and thus the set $\mathbb{Z}_+$ can be considered as a part of $M(\mathcal{T}_{rad}(\lambda))$. Moreover,
$$M(\mathcal{T}_{rad}(\lambda)) = \mathbb{Z}_+ \cup M_\infty(\lambda) \tag{50}$$
and by [95], the set $\mathbb{Z}_+$ is densely and homeomorphically embedded into $M(\mathcal{T}_{rad}(\lambda))$ with respect to the Gelfand topology on $M(\mathcal{T}_{rad}(\lambda))$ Furthermore, by [250], the set $M_\infty(\lambda)$ is connected.

We mention for completeness that none of the points of $M_\infty(\lambda)$ can be reached by subsequences of $\mathbb{Z}_+$; its topological nature requires to use nets (subnets of $\mathbb{Z}_+$). That is, for each point $\mu \in M_\infty(\lambda)$ there is a net $\{n_\beta\}_{\beta\in\beta}$, valued in $\mathbb{Z}_+$, which tends to μ in the Gelfand topology of $M(\mathcal{T}_{rad}(\lambda))$. Or, in other words, for each $\gamma = \{\gamma(n)\}_{n\in\mathbb{Z}_+} \in SO(\lambda)$, we have that
$$\lim_{\beta\in\beta} \gamma(n_\beta) = \gamma(\mu) \tag{51}$$
where we identify $\gamma(\mu)$ with $\mu(\gamma)$, the value of the functional $\mu \in M_\infty(\lambda)$ on the element$\gamma \in SO(\lambda)$.

Consider now the special case of a radial symbol: $(r) = r^2$. By (47) we have
$$\gamma_{r^2},\lambda(|\alpha|) = \frac{\Gamma(|\alpha|+\lambda+3)}{\Gamma(\lambda+1)\Gamma(|\alpha|+2)}\int_0^1 \alpha(\sqrt{r})(1-r)^\lambda r^{|\alpha|+1}dr$$
$$= \frac{\Gamma(|\alpha|+\lambda+3)}{\Gamma(\lambda+1)\Gamma(|\alpha|+2)}\int_0^1 \alpha(1-r)^{\lambda+1} r^{|\alpha|+2}dr$$
$$= \frac{\Gamma(|\alpha|+\lambda+3)}{\Gamma(\lambda+1)\Gamma(|\alpha|+2)} B(\lambda+1, |\alpha|+3) = \frac{|\alpha|+2}{|\alpha|+\lambda+3}$$

The sequence $\gamma_{r^2,\lambda}(n),\ n \in \mathbb{Z}_+$, is real valued, strictly monotone (thus separatingpoints of $\mathbb{Z}_+$), and convergent when $n \to \infty$. Hence, by the Stone-Weierstrass theorem, the unital C*-algebra generated by a single Toeplitz operator T_{r^2} coincides with the algebra of all diagonal operators D_γ with $\gamma \in c$, where c denotes the set of all convergent sequences.

Corollary (6.2.3)[303]: Let $\gamma \in c$, then $D_\gamma \in \mathcal{T}(\lambda)$. In particular, for all $n \in \mathbb{Z}_+$ the orthogonal projection P_n of $\mathcal{A}^2_\lambda(\mathbb{B}^2)$ onto span$\{z^\alpha : |\alpha| = n\}$ belongs to the algebra $\mathcal{T}(\lambda)$.

In order to simplify formulas, and with the coordinates in (45) we will use the notation $\phi_p = \phi_p(\xi) = \xi^{(p,0)}\bar{\xi}^{(o,p)}$, where $\xi = (\xi_1, \xi_2) \in \mathbb{S}^3 \subset \mathbb{C}^2$and $p \in \mathbb{N}$; for $p = 1$ we simply write $\phi = \phi_1$.

We start with some calculations based on (48):

$$T_\phi z_1^{\alpha_1} z_2^{\alpha_2} = \frac{\alpha_2}{2 + |\alpha|} z_1^{\alpha_1+1} z_2^{\alpha_2-1},\ \ \alpha_2 \geq 1$$

$$T_\phi^2 z_1^{\alpha_1} z_2^{\alpha_2} = \frac{\alpha_2(\alpha_2 - 1)}{(2 + |\alpha|)^2} z_1^{\alpha_1+2} z_2^{\alpha_2-2},\ \ \alpha_2 \geq 2$$

$$T_{\phi 2} z_1^{\alpha_1} z_2^{\alpha_2} = \frac{\alpha_2(\alpha_2 - 1)}{(3 + |\alpha|)(2 + |\alpha|)} z_1^{\alpha_1+2} z_2^{\alpha_2-2},\ \ \alpha_2 \geq 2$$

Thus

$$\left(T_\phi^2 - T_{\phi 2}\right) z_1^{\alpha_1} z_2^{\alpha_2} = \left(\frac{1}{2 + |\alpha|} - \frac{1}{3 + |\alpha|}\right) \frac{\alpha_2(\alpha_2 - 1)}{2 + |\alpha|} z_1^{\alpha_1+2} z_2^{\alpha_2-2} = \frac{1}{3 + |\alpha|} T_\phi^2 z_1^{\alpha_1} z_2^{\alpha_2},$$

or

$$T_{\phi 2} = D_{d2} T_\phi^2,$$

where

$$d_2(|\alpha|) = \frac{2 + |\alpha|}{3 + |\alpha|}, \qquad \alpha \in \mathbb{Z}_+^2.$$

Due to the remark before Corollary (6.2.3) we conclude that the Toeplitz operator T_{ϕ_2} belongs to the unital algebra generated by T_{r^2} and T_ϕ. Similarly, for any $p \in \mathbb{N}$, and $\alpha_2 > p$ we have

$$T_{\phi_p} z_1^{\alpha_1} z_2^{\alpha_2} = \frac{\alpha_2(\alpha_2 - 1) \dots (\alpha_2 - p + 1)}{(p + 1 + |\alpha|) \dots (2 + |\alpha|)} z_1^{\alpha_1+p} z_2^{\alpha_2-p}$$

$$\left(T_\phi\, T_{\phi_p}\right) z_1^{\alpha_1} z_2^{\alpha_2} = \frac{\alpha_2(\alpha_2 - 1) \dots\ (\alpha_2 - p + 1)(\alpha_2 - p)}{(p + 1 + |\alpha|)\ \ \dots (2 + |\alpha|)(2 + |\alpha|)} z_1^{\alpha_1+p+1} z_2^{\alpha_2-p-1}$$

$$T_{\phi_{p+1}} z_1^{\alpha_1} z_2^{\alpha_2} = \frac{\alpha_2(\alpha_2 - 1) \dots\ (\alpha_2 - p)}{(p + 1 + |\alpha|)\ \dots (2 + |\alpha|)} z_1^{\alpha_1+p+1} z_2^{\alpha_2-p-1}$$

By comparing these relations we obtain:

$T_\phi\, T_{\phi_p} - T_{\phi_{p+1}} z_1^{\alpha_1} z_2^{\alpha_2} = \frac{p}{p+2+|\alpha|} T_\phi\, T_{\phi_p} z_1^{\alpha_1} z_2^{\alpha_2}$.

Note that the last equality is also valid in the case of $0 \leq \alpha_2 \leq p$ and hence

$$T_{\phi_{p+1}} = D_{d_{p+1}} T_\phi\, T_{\phi_p}$$

where

$$d_{p+1}(|\alpha|) = \frac{2 + |\alpha|}{p + 1 + |\alpha|} = \frac{2 + |\alpha|}{(p + 1) + 1 + |\alpha|},\ \ \ \alpha \in \mathbb{Z}_+^2$$

By induction we finally have

$$T_{\phi_p} = \left(\prod_{k=1}^{p} D_{d_k} \right) T_{\phi}^{p}, \tag{52}$$

where the eigenvalue sequence $d_k = \{d_k(|\alpha|)\}_{|\alpha| \in \mathbb{Z}_+}$ is given by

$$d_k(|\alpha|) = \frac{2 + |\alpha|}{k + 1 + |\alpha|}$$

Note that $d_1(|\alpha|) \equiv 1$, as it should be.
An alternative form of (52) is

$$T_{\phi_p} = D_{\bar{d}_p} T_{\phi}^{p} \tag{53}$$

where the eigenvalue sequence $\tilde{d}_p = \left\{\tilde{d}_p|\alpha|\right\}_{|\alpha| \in \mathbb{Z}_+}$ is given by

$$\tilde{d}_p|\alpha| = \frac{(2 + |\alpha|)^p}{(p + 1 + |\alpha|)_{(p)}}, \tag{54}$$

and $(x)_{(p)} = x(x-1) \cdots (x - p + 1)$ is a kind of Pochhammer symbol.
We note that, for each $p \in \mathbb{N}$, both sequences d_p and $\tilde{d}_p$ tend to 1 when $|\alpha| \to \infty$. Thus we have according to the remark before Corollary (6.2.3):
Theorem (6.2.4)[303]: For each $p \in \mathbb{N}$, the Toeplitz operator T_{ϕ_p} belongs to the unital algebra generated by T_{r^2} and T_{ϕ}.
Corollary (6.2.5)[303]: The Banach algebra $\mathcal{T}(\lambda)$ is generated, in fact, just by Toeplitz operators T_a with bounded measurable radial symbols $a(r)$ and the single Toeplitz operator T_{ϕ} (with the simplest quasi-homogeneous symbol $\phi(\xi)$).
We normalize the monomials z^{α} to the standard orthonormal basis $[e_{\alpha}: \alpha \in \mathbb{Z}_+^2]$ of the Bergman space $\mathcal{A}_{\lambda}^2(\mathbb{B}^2)$, i.e.

$$e_{\alpha} = \sqrt{\frac{\Gamma(|\alpha| + \lambda + 3)}{\alpha!\, \Gamma(\lambda + 3)}}\, z^{\alpha}, \quad \alpha \in \mathbb{Z}_+^2. \tag{55}$$

Then we have in the case of $\alpha \in \mathbb{Z}_+^2$ with $\alpha_2 \geq p$:

$$T_{\phi} e_{\alpha} = \frac{\sqrt{(\alpha_1 + 1)\alpha_2}}{2 + |\alpha|} e_{(\alpha_1+1, \alpha_2-1)}, \tag{56}$$

$$T_{\phi}^{p} e_{\alpha} = \frac{\sqrt{(\alpha_1 + p) \dots (\alpha_1 + 1)\alpha_2 \dots (\alpha_2 - p + 1)}}{(2 + |\alpha|)^p} e_{(\alpha_1+p, \alpha_2-p)}, \tag{57}$$

which implies that $\left\| T_{\phi}^{p} \right\| = 2^{-p}$ for all $p \in \mathbb{N}$, and thus the spectral radius of T_{ϕ} is equal to $\frac{1}{2}$.
Note that (48) and (56) imply that the action of T_{ϕ} does not depend on the weight parameter λ. Thus the structure of the unital Banach algebra $\mathcal{T}_{\phi}$ generated by T_{ϕ} does not depend on the weight parameter λ as well, and thus the spectrum of T_{ϕ} is independent of λ.
Consider now the case $\lambda = 0$. Since φ extends continuously to the boundary $\partial\mathbb{B}^2$ of $\mathbb{B}^2$ it follows from the results in [298] that the essential spectrum of the operator T_{ϕ} is given by

$$ess - spT_{\phi} = ImT_{\phi}(\xi)|_{\partial\mathbb{B}^2} = Im\left(\frac{r_1 r_2}{r_1^2 + r_2^2} t_1 \bar{t}_2\right)\Bigg|_{r_1^2+r_2^2=1, t_1, t_2 \in \mathbb{S}^1} = \bar{D}\left(0, \frac{1}{2}\right)$$

where $\bar{D}\left(0, \frac{1}{2}\right)$ is the closed disk centered at origin and with the radius $\frac{1}{2}$.
Finally,

$$\bar{D}\left(0, \frac{1}{2}\right) = ess - sp\, T_{\phi} \subset sp\ \ T_{\phi} \subset \bar{D}\left(0, \frac{1}{2}\right)$$

implies that $sp\, T_{\phi} = \bar{D}\left(0, \frac{1}{2}\right)$.
By [75], the maximal ideal space $M(\,T_{\phi})$ of the commutative Banach algebra T_{ϕ} coincides with the spectrum of the operator T_{ϕ},i.e. $M(\,\mathcal{T}_{\phi}) = \bar{D}\left(0, \frac{1}{2}\right)$.

Theorem (6.2.6)[303]: The Banach algebra $\mathcal{T}_{\phi}$ is isomorphic via the Gelfand transform to the algebra $C_a\bar{D}\left(0, \frac{1}{2}\right)$, which consists of all functions analytic in $D\left(0, \frac{1}{2}\right)$ and continuous on $\bar{D}\left(0, \frac{1}{2}\right)$.

Proof: Consider two unital algebras: the Banach algebra $\mathcal{T}_{\phi}$ and the C*-algebra $\mathcal{T}_{\phi}^*$, both are generated by T_{ϕ}. The operator T_{ϕ} commutes with its adjoint $T_{\phi}^* = T_{\bar{\phi}}$ modulo a compact operator, thus the quotient algebra $\widehat{\mathcal{T}_{\phi}^*} = \mathcal{T}_{\phi}^*/(\mathcal{T}_{\phi}^* \cap \mathcal{K})$. where $\mathcal{K}$ denotes the ideal of all compact operators, is a commutativeC*-algebra which is isomorphic and isometric to $C(ess - sp\, T_{\phi}) = C(D\left(0, \frac{1}{2}\right))$. As the spectrum of any compact operator is at most countable and having at most one limit point 0, which is not the case for any non-zero operator from $\mathcal{T}_{\phi}$, we have $\mathcal{T}_{\phi} \cap \mathcal{K} = \{0\}$. Thus

$$\mathcal{T}_{\phi} = \mathcal{T}_{\phi}/(\,\mathcal{T}_{\phi} \cap \mathcal{K}) \cong \left(\mathcal{T}_{\phi} + \mathcal{T}_{\phi}^* \cap \mathcal{K}\right)/(\mathcal{T}_{\phi}^* \cap \mathcal{K}) \subset \mathcal{T}_{\phi}^*/(\mathcal{T}_{\phi}^* \cap \mathcal{K}) \cong C(\bar{D}\left(0, \frac{1}{2}\right)$$

That is, the algebra $\mathcal{T}_{\phi}$, being isomorphic to the uniform closure of all polynomials of ζ defined on $(\bar{D}\left(0, \frac{1}{2}\right)$, is isomorphic to $Ca\ (\bar{D}\left(0, \frac{1}{2}\right))$.

For each $k \in \mathbb{Z}_+^2$, we denote by H_k the following subspace of $\mathcal{A}_{\lambda}^2(\mathbb{B}^2)$:
$H_k = span\,\{e_a : \alpha \in \mathbb{Z}_+^2,\ \ |\alpha| = k\}$, (38)
and, of course, we have an orthogonal decomposition of $\mathcal{A}_{\lambda}^2(\mathbb{B}^2)$ into finite dimensional Hilbert spaces

$$\mathcal{A}_{\lambda}^2(\mathbb{B}^2) = \bigoplus_{k=0}^{\infty} H_k$$

each space H_k is obviously invariant for all operators from $\mathcal{T}(\lambda)$. The diagonal operator D_{λ}, restricted to H_k, is just the scalar operator $\gamma(k)I$, while the operator T_{ϕ} acts on H_k as a weighted shift operator. Moreover, the operator T_{ϕ}, restricted to H_k, is nilpotent,

$$\left(\,T_{\phi|H_k}\right)^{k+1} = 0.$$

In particular, this implies that, for all $p \in \mathbb{N}$,

$$\bigoplus_{k=0}^{p-1} H_k \subset ker\ T_\phi \qquad (59)$$

Recall as well that each orthogonal projection P_k of $\mathcal{A}_\lambda^2(\mathbb{B}^2)$ onto H_k is a diagonal operator, which belongs to the unital subalgebra of $\mathcal{T}(\lambda)$generated by T_{r^2}.
An important information on the structure of $\mathcal{T}(\lambda)$ gives the next lemma.
A much stronger result can be found at the end of the section (cf. Lemma (6.2.38) and Theorem (6.2.43)):

Lemma (6.2.7)[303]: The algebra $\mathcal{T}(\lambda)$is not semi-simple. Its radical $Rad\mathcal{T}(\lambda)$contains, in particular, all operators of the form $T_{\phi_p} = D_{\boldsymbol{\gamma}}$ where $\gamma \in c_0$ and p $\in \mathbb{N}$.

Proof: In virtue of (53) it is sufficient to prove that $D_\gamma\ T_\phi \in Rad\mathcal{T}(\lambda)$, or (see, for example, [75]) that the operator $A = D_\gamma\ T_\phi$ is topologically nilpotent, i.e.,

$$\lim_{k\longrightarrow\infty} \|A^k\|^{\frac{1}{k}} = 0\,.$$

we have

$$A^k = D_\gamma^k T_\phi^k = D_\gamma^k T_\phi^k\big(I-(P_0+\cdots+P_{k-1})\big) = \big[D_\gamma\big(I-(P_0+\cdots+P_{k-1})\big)\big]^k T_\phi^k$$

Thus

$$\|A^k\|^{\frac{1}{\mathrm{k}}} \le \big\|D_\gamma\big(I-(P_0+\cdots+P_{k-1})\big)\big\|.\,\|T_\phi^k\| = \|T_\phi^k\|.\sup_{1>k}|\gamma(l)|.$$

As $\gamma \in c_0$, the last expression tends to 0 when $k \longrightarrow \infty$.
We denote by $D(\lambda)$ the dense (non-closed) subalgebra of $\mathcal{T}(\lambda)$ formed by finite sums of finite products of its generators: T_a with bounded measurable radial symbols $a(r)$ and T_{ϕ_p} with$p \in \mathbb{N}$.
An operator A from $D(\lambda)$ has the form

$$A = \sum_{p=0}^{m} D_{\gamma_p} T_\phi^p$$

We mention that for arbitrary diagonal operators D_{γ_p} the above representation is not unique. To describe this ambiguity we will use the notation $K_\gamma(p)$, with $p \in \mathbb{Z}_+$, for a finite dimensional diagonal operator, whose eigenvalue sequence has the form

$$\gamma = \{\gamma(0), \gamma(1), \dots, \gamma(p-1), 0,0, \dots\},$$

of course, for $p = 0$, it is just the zero operator.

Lemma (6.2.8)[303]: We have
$\sum_{p=0}^m D_{\gamma_p} T_\phi^p = 0(60)$
if and only if $D_{\gamma_p} = k_{\gamma_p}(p)$, for each $p = 0,1, \dots, m$.

Proof: The part “if” follows from (59).
To prove the “only if” part, consider any n ≥ m and note that each $D_{\gamma_p}$is diagonal with respect to the basis (55) with

$$D_{\gamma_p} e_\alpha = \gamma_p(|a|) e_\alpha.$$

Moreover, by (57), the operator T_ϕ^p acts on (55) as $T_\phi^p e_{(\alpha_1, a_2)} = \tau_p(\alpha)$.

$e_{(\alpha_1+p,\alpha_2-p)}$, where $0 \neq \tau_p(\alpha) \in \mathbb{R}if\alpha_2 \geq p$. Then, by (60), we have

$$0 = \sum_{p=0}^{m} D_{\gamma_p} T_\phi^p e_{(0,n)} = \sum_{p=0}^{m} \gamma_p(n)\, \tau_p(0,n) e_{(p,n-p)}.$$

Since$\{e_{(p,n-p)}: p = 0, 1, \ldots, m\}$forms a system of orthogonal vectors and $\tau_p(0,n)$=0 we conclude that $\gamma_p(n) = 0$ for $p = 0, 1, \ldots, m$ and $n \geq m$.

Therefore D_{γ_p} is finite dimensional for $p = 0, 1, \ldots, m$, and, in particular, $D_{\gamma_m} = K_{\gamma_m}(m)$.

It follows that $D_{\gamma_m} T_\phi^m = K_{\gamma_m}(m) T_\phi^m = 0$, and thus we have

$$\sum_{p=0}^{m-1} D_{\gamma_p} T_\phi^p = 0$$

Repeating the above arguments m times (each time lowering the sum upperlimit), we have consequently

$$D_{\gamma_{m-1}} = K_{\gamma_{m-1}}(m-1), D_{\gamma_{m-2}} = K_{\gamma_{m-2}}(m-2), \ldots, D_{\gamma_0} = K_{\gamma_0}(0) = 0$$

At the same time the situation is quite different for special finite sums of finite products of generators from $D(\lambda)$.

To proceed with the result we define the"grade"for some operators by:

$grade\left(D_\gamma\right) := 0$, and $grade\left(T_{\phi_p}\right) := p$.

Moreover, if $\prod_{k=1}^{m} A_k$ is the product of the above operators, then we put

$$grade\left(\prod_{k=1}^{m} A_k\right) := \sum_{k=1}^{m} grade(A_k)$$

Theorem (6.2.9)[303]: Let us assume that all summands of the operator

$$A = \sum_{k=1}^{m} \left(\prod_{q=1}^{m_k} T_{a_{k,q}} \prod_{s=1}^{n_k} T_{\phi_{p_{k,s}}} \right) = 0$$

have different grades. Then it follows that for each k at least one radial symbol $a_{k,q}$ is identically zero.

Proof: By (53) and (54), we have

$$A = \sum_{k=1}^{m} \left(\prod_{q=1}^{m_k} T_{a_{k,q}}\, D_{\gamma_k} T_\phi^{pk,1+\cdots+pk,n_k} \right) = 0,$$

where each D_{γ_k} is invertible and its eigenvalue sequence tends to 1. Thus, by Lemma (6.2.8), we obtain that each diagonal operator $\prod_{q=1}^{m_k} T_{a_{k,q}}$ is finite dimensional.

Then the result follows by [319], Theorem (6.2.1), and Theorem (6.2.6).

As a corollary to the previous theorem we give a result on the so-called zero-product problem (see, for example, [168,169,319]).

Corollary (6.2.10)[303]: For the operator

$$A = \prod_{q=1}^{m} T_{a_q} \prod_{s=1}^{n} T_{\phi_{p_s}}$$

the following statements are equivalent:

(i) $A = 0$,

(ii) A is finite dimensional,

(iii) At least one radial symbol a_q is identically zero.

We start from recalling some known facts and definitions: Let $\mathcal{A} = \mathcal{A}(x_1, \dots x_n)$ be a unital commutative Banach algebra generated by the elements $x_1, \dots x_n$, and let $M(\mathcal{A})$ denote the compact set of its maximal ideals. Then (cf. [167]) the joint spectrum $\sigma(x_1, \dots x_n)$ of $x_1, \dots, x_n$ is the subset of $\mathbb{C}^n$ defined by

$$\sigma(x_1, \dots x_n) = \{m(x_1), m(x_2), \dots . m(x_n): m \in M(\mathcal{A})\}. \quad (61)$$

In (61)we identify maximal ideals in $\mathcal{A}$ and multiplicative functional on $\mathcal{A}$in the usual way. As is well known, the mapping

$m \in M(\mathcal{A}) \to \big(m(x_1), m(x_2), \dots . m(x_n)\big) \in \sigma(x_1, \dots x_n)$

defines a homeomorphism between $M(\mathcal{A})$ and $\sigma(x_1, \dots x_n)$

If$e \in \mathcal{A}$ denotes the unit element then we can also write (cf.[318])

$$\sigma(x_1, \dots, x_n) = \{(\mu_1, \dots \mu_n) \in \mathbb{C}^n: J(x_1 - \mu_1 e, \dots, x_n - \mu_n e) \neq \mathcal{A}\} \quad (62)$$

where $J(x_1 - \mu_1 e, \dots, x_n - \mu_n e)$ denotes the smallest ideal in the algebra $\mathcal{A}$ which contains the elements $x_j - \mu_j e$, with $j = 1, \dots, n$.

Let H be a complex Hilbert space and (A_1, A_2) be a tuple of (bounded) commuting operators on H.

We say (cf. [67]) that $(\mu_1, \mu_2) \in \mathbb{C}^2$ is in the joint approximate point spectrum $\sigma\pi(A_1, A_2)$ of (A_1, A_2) if and only if, for all $B_1, B_2 \in \mathcal{L}(H)$,

$$B_1(A_1 - \mu_1 I) + B_2(A_2 - \mu_2 I) \neq I$$

Now, let $\mathcal{A} = \mathcal{A}(A_1, A_2)$ be the Banach algebra in $\mathcal{L}(H)$ generated by the commuting operators A_1, A_2 and the identity element. The next statement is well-known and quite standard.

Lemma (6.2.11)[303]: The following inclusions hold:

$$\sigma\pi(A_1, A_2) \subset \sigma(A_1, A_2) \cong M(\mathcal{A}) \subset M(\mathcal{A}_1) \times M(\mathcal{A}_2)$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ denote the unital Banach algebras generated by $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively.

Proof: Note that, by restriction, each element $\varphi \in M(\mathcal{A})$ defines a functional in $M(\mathcal{A}_j)$, $j = 1,2$ which proves the second inclusion. The first inclusion directly follows from the characterization (62).

Below we need a more concrete characterization of the joint approximate point spectrum which has been given by A.T. Dash:

Proposition (6.2.12)[303]: [67] A tuple $(\mu_1, \mu_2) \in \mathbb{C}^2$ is in $\sigma\pi(A_1, A_2)$ if and only if there is a sequence $\{f_n\}_n \subset H$ of unit vectors such that

$\|(A_1 - \mu_1 I) f_n\| \to 0$ and $\|(A_2 - \mu_2 I) f_n\| \to 0$

as n tends to infinity.

It is instructive to consider first the unital algebra with just two generators:

a diagonal operator D_γ and T_ϕ. The result will already give a good approximation to what one can expect for the algebra $\mathcal{T}(\lambda)$.
We fix $\lambda \in (-1, \infty)$ and a diagonal operator $A_1 = D_\gamma \in \mathcal{T}_{rad}(\lambda)$, whose eigenvalue sequence $\gamma = \{\gamma(k)\} \in \mathbb{Z}_+$ satisfies the conditions:
(i)$\gamma(k_1) \neq \gamma(k_2)$, for all $k_1 \neq k_2$,
(ii) none of the eigenvalues $\gamma(k)$ belongs to the set of limit points of γ.
We note that the aim of these conditions is to separate"as much as possible" the points of the compact set of maximal ideals of the unital algebra generated by D_γ. Furthermore they will be important in the proof of Lemma (6.2.14).
Let $A_2 = T_\phi$. We consider the unital Banach algebra $\mathcal{A} = \mathcal{A}(A_1, A_2)$ generated by two commuting elements A_1 and A_2.
If we denote by$Lim(\gamma)$ the set of all limit points of the sequenceγ, then the spectrum of the operator D_γ is given by

$$\sigma(D_\gamma) = clos\, Im(\gamma) = \{\gamma(k): k \in \mathbb{Z}_+\} \cup Lim\,(\gamma)$$

Recall that the operators A_1 and A_2 act on the basis elements (55) as follows

$$A_1 e_\alpha = D_\gamma e_\alpha = \gamma|\alpha| e_\alpha\,,$$

$$A_2 e_\alpha = T_\phi e_\alpha = \frac{\sqrt{(\alpha_1+1)\alpha_2}}{2+|\alpha|} e_{(\alpha_1+1, \alpha_2-1)}.$$

Lemma (6.2.13)[303]: We have $(D_\gamma) \times \{0\} \subset \sigma\pi(A_1, A_2)$.
Proof: Fix first$(k) \in \gamma$, and observe that $e_{(k,0)} \in ker\, T_{\xi^{(1,0)}\bar{\xi}^{(0,1)}}$. If we put $f_n \equiv e_{(k,0)}$, for $n \in \mathbb{N}$, then we have

$$\|(D_\gamma - \gamma(k)I)f_n\| = 0 \quad \text{and} \quad \|(T_{\xi^{(1,0)}\bar{\xi}^{(0,1)}} - 0I)f_n\| = 0.$$

That is, $Lim(\gamma) \times \{0\} \subset \sigma\pi(A_1, A_2)$.
Lemma (6.2.14)[303]: None of the points $(\gamma(k), \zeta)$,where k $\in \mathbb{Z}_+$ and $k \in \mathbb{Z}_+ \Big(\zeta \in D(0, \frac{1}{2}\Big)\backslash\{0\}$belong to the joint spectrum $\sigma(A_1, A_2)$.
Proof: We fix a pair$(\gamma(k), \zeta)$, with $\zeta = 0$, and show that the ideal in $\mathcal{A}(A_1, A_2)$, which is generated by the operators $A_1 - \gamma(k_0)I = D_\gamma - \gamma(k_0)I$, and $A_2 - \zeta I = T_\phi - \zeta I$, coincides with the whole algebra $\mathcal{A}(A_1, A_2)$.

Consider the finite dimensional space H_{k_0} in (58). Then it can be easily seen that H_{k_0} and its orthogonal complement $H_{k_0}^\perp = \mathcal{A}_\lambda^2(\mathbb{B}^2) \ominus H_{k_0}$are invariant under the operators D_γ and T_ϕ. Moreover, the restriction of $\mathcal{A}_2 = T_\phi$ to H_{k_0} is nilpotent,

$$\left((\mathcal{A}_2|H_{k_0}\right)^{k_0+1} = 0.$$

The operator $\mathcal{A}_1 - \gamma(k_0)I$ is diagonal, and its eigenvalue sequence is of the form $\bar{\gamma} = \{\gamma(k) - \gamma(k_0)\}_{k \in \mathbb{Z}_+}$.The above conditions (i) and (ii) guarantee that

$$\inf_{k \neq k_0} |\gamma(k) - \gamma(k_0)| > 0$$

for each $k_0 \in \mathbb{Z}_+$. Thus the diagonal operator $D_{\overline{\gamma}(-1)}$ with the eigenvalue sequence

$$\bar{\gamma}^{-1} = \left\{ \left(1 - \delta_{k,k_0}\right) \frac{1}{(k) - \gamma(k_0)} \right\}_{k \in \mathbb{Z}_+}$$

is well defined, it obviously belongs to $\mathcal{A}(A_1) \subset \mathcal{A}(A_1, A_2)$, and

$$D_{\overline{\gamma}(-1)}(A_1 - \gamma(k_0)I\ ) = I - P_{k_0},$$

where P_{k_0} is the orthogonal projection of $\mathcal{A}_\lambda^2(\mathbb{B}^2)$ onto H_{k_0}. From this relation we also conclude that $\gamma(k_0)$ defines an element in $\mathcal{A}(A_1, A_2)$. The operator

$$D_{\overline{\gamma}(-1)}(A_1 - \gamma(k_0)I\ ) + (A_2 - \zeta I)P_{k_0}$$

belongs to $\mathcal{A}(A_1, A_2)$ and is invertible. Its inverse belongs to $\mathcal{A}(A_1, A_2)$ as well and has the form

$$D_{\overline{\gamma}(-1)}(A_1 - \gamma(k_0)I\ ) - \zeta^{-1}\left(\xi^{-1}A_2 + \zeta^{-2}A_2^2 + \cdots + \zeta^{-k_0}A_2^{k_0}\right)P_{k_0}$$

which implies that the ideal generated by the operators $A_1 - \gamma(k_0)I = D_\gamma - \gamma(k_0)I$ and $A_2 - \zeta I = T_\phi - \zeta I$ coincides with the whole algebra $\mathcal{A}(A_1, A_2)$.

To finish the description of the joint spectrum $\sigma(A_1, A_2)$ we need first some preliminary facts on the Berezin transform corresponding to certain subspaces of $\mathcal{A}_\lambda^2(\mathbb{B}^2)$.

Let $S \subset \mathbb{Z}_+$ be infinite. We introduce the Hilbert space

$$H_S := \overline{span}\{e_\alpha : |\alpha| \in S\} \subset \mathcal{A}_\lambda^2(\mathbb{B}^2).$$

Its reproducing kernel function $K_S(z, \omega)$ has the form of a power series converging uniformly on compacts of $\mathbb{B}^2 \times \mathbb{B}^2$:

$$K_S(z, \omega) = \sum_{|\alpha| \in S} e_\alpha(z)\, \overline{e_\alpha(\omega)}.$$

Lemma (6.2.15)[303]: Let $\{\omega_n\}_n \subset \mathbb{B}^2$ be such that the sequence $(\|\omega_n\|)_n$ of real numbers is increasing and $\lim\limits_{n\to\infty} \omega_n = v\, k \in \partial\mathbb{B}^2$. Then

$$\|K_S(.\,, \omega_n)\|^2 = \sum_{|\alpha| \in S} |e_\alpha(\omega_n)|^2 \to \infty, \text{ as } n \to \infty.$$

Moreover, for all $z, \omega \in \mathbb{B}^2$ the following estimate holds:

$$|K_S(z, \omega)| \leq \frac{1}{(1 - |\langle z, w\rangle|)^{3+\lambda}}. \tag{63}$$

Proof: According to the multinomial theorem we have for $j \in \mathbb{Z}_+$:

$$\sum_{|\alpha| \in S} |e_\alpha(\omega)|^2 = \frac{\Gamma(j + \lambda + 3)}{\Gamma(\lambda + 3)j!} \sum_{|\alpha| \in S} \frac{j!}{\alpha!} |(\omega_1)|^{2\alpha_1} |(\omega_2)|^{2\alpha_2} = \frac{\Gamma(j + \lambda + 3)}{\Gamma(\lambda + 3)j!} \|\omega\|^{2j}.$$

Since $\lambda > -1$ it follows with $|\alpha| = j \to \infty$ that

$$\frac{\Gamma(j + \lambda + 3)}{j!} > \frac{\Gamma(j + 2)}{j!} = j + 1 \to \infty.$$

As $\{\omega_n\}_n$ is increasing and S is infinite, the first assertion follows from

$$\|K_S(.\,, \omega_n)\|^2 = \frac{1}{\Gamma(\lambda + 3)} \sum_{j \in S} \frac{\Gamma(j + \lambda + 3)}{j!} \|\omega_n\|^{2j}$$

and the monotone convergence theorem. The inequality(63) is a consequence of:

$$K_S(z,\omega) \geq \frac{1}{\Gamma(\lambda+3)} \sum_{j\in S} \frac{\Gamma(j+\lambda+3)}{j!} |\langle z,\omega\rangle|^j \geq \frac{1}{\Gamma(\lambda+3)} \sum_{j\in \mathbb{Z}_+} \frac{\Gamma(j+\lambda+3)}{j!} |\langle z,\omega\rangle|^j$$
$$= \frac{1}{(1-|\langle z,\omega\rangle|)^{3+\lambda}}$$

and the lemma is proved.

Let $\overline{\mathbb{B}^2}$ be the closed unit ball in $\mathbb{C}^2$. Given a function $\psi \in C\overline{\mathbb{B}^2}$, we define its Berezin transform with respect to H_S as

$$B_S[\psi](z) := \frac{1}{\|K_S(.,\omega_n)\|^2} \langle \psi K_S(.,z), K_S(.,z)\rangle, \quad z \in \mathbb{B}^2.$$

Now, we can prove:

Proposition (6.2.16)[303]: Let $\psi \in C\overline{\mathbb{B}^2}$ be invariant under the componentwise $\mathbb{S}^1$-action on $\overline{\mathbb{B}^2}$, i.e. for all $(\lambda, z) \in \mathbb{S}^1 \times \overline{\mathbb{B}^2}$ we have:

$$\psi(\lambda z) = \psi(z) \tag{64}$$

Let $v \in \partial\mathbb{B}^2$ and put $\omega_n = \frac{n-1}{n} v$ such that $\lim_{n\to\infty} \omega_n = v$. Then, we have:

$$\lim_{n\to\infty} B_S[\psi](\omega_n) = \psi(v).$$

Proof: Let $\varepsilon > 0$ and consider the orbit $O_v := \{\lambda v: \lambda \in \mathbb{S}^1\} \subset \partial\overline{\mathbb{B}^2}$.Let$\delta > 0$ and define a δ-neighborhood of O_v by

$$O_v^\delta := \left\{z \in \overline{\mathbb{B}^2} : dis(z, O_v) < \delta\right\}.$$

Let $z \in O_v^\delta$, then there is $\lambda_0 \in \mathbb{S}^1$such that $|z - \lambda_0 v| < \delta$.Since ψ is continuous up to the boundary of $\overline{\mathbb{B}^2}$ and due to the invariance (64) it follows that one can choose $\delta > 0$ sufficiently small with

$$\varepsilon > |\psi(z) - \psi(\lambda_0 v)| = |\psi(z) - \psi(v)|. \tag{65}$$

Fix $\delta > 0$ with (65). Then there is $\gamma \in (0,1)$ such that for all $z \in \overline{\mathbb{B}^2} \backslash O_v^\delta$ and $n \in \mathbb{N}$ one has

$$|\langle z, \omega_n\rangle| \leq \gamma < 1.$$

Hence it follows that:

$$|K_S(z,\omega_n)|^2 \leq \frac{1}{\Gamma(\lambda+3)} \sum_{j\in S} \frac{\Gamma(j+\lambda+3)}{j!} |\langle z,\omega\rangle|^j \leq \frac{1}{\Gamma(\lambda+3)} \sum_{j\in \mathbb{Z}_+} \frac{\Gamma(j+\lambda+3)}{j!} |\langle z,\omega\rangle|^j \gamma^j$$
$$= c_\delta.$$

Now we calculate

$$|\psi(v) - B_S[\psi](\omega_n)|$$
$$\leq \frac{1}{\|K_S(.,\omega_n)\|^2} \int_{O_v^\delta} |\psi(v)$$
$$- \psi(z)| \, |K_S(z,\omega_n)|^2 \, dv\lambda(z). + \frac{1}{\|K_S(.,\omega_n)\|^2} \int_{\mathbb{B}^2 \backslash O_v^\delta} |\psi(v)$$
$$- \psi(z)| \, |K_S(z,\omega_n)|^2 \, dv\lambda(z) \leq \varepsilon + 2c_\delta \frac{||\psi||_{\mathcal{L}^\infty \overline{\mathbb{B}^2}}}{\|K_S(.,\omega_n)\|^2}.$$

The last term on the right tends to zero as $n \to \infty$ (by Lemma (6.2.15)), and hence the proposition is proved.

Lemma (6.2.17)[303]: We have $Lim(\gamma) \times \overline{D}\left(0, \frac{1}{2}\right) \subset \sigma\pi(A_1, A_2)$.

Proof: Let $(\mu, \zeta) \in Lim(\gamma) \times \overline{D}\left(0, \frac{1}{2}\right)$. Recall that $\overline{D}\left(0, \frac{1}{2}\right) = Im\phi|_{\partial\mathbb{B}^2}$. Then we choose $v \in \partial\mathbb{B}^2$ such that $\xi = \phi(v)$ and fix a sequence $\{m_e\}_e \subset \mathbb{Z}_+$ such that

$$\lim_{e\to\infty} \gamma(m_e) = \mu.$$

Consider the set $S_\mu := \{m_\ell : \ell \in N\} \subset \mathbb{Z}_+$ and define the Hilbert space H_{S_μ}.

$$H_{S_\mu}\overline{span}\{e_\alpha : |\alpha| = m_\ell, \ \ \ell \in \mathbb{N}\}.$$

Let $\{\omega_n\}_n \subset \mathbb{B}^2$ be a sequence as in Lemma (6.2.15) with $\lim_{n\to\infty} \omega_n = v$. We define a corresponding sequence $\{f_n\}_n$ of unit vectors by

$$f_n = \frac{K_S(.,\omega_n)}{\|K_S(.,\omega_n)\|} \tag{66}$$

Then, we have

$$\left\|\left(D_\gamma - \mu I\right) f_n\right\|^2 = \frac{1}{\|K_S(.,\omega_n)\|^2} \sum_{\ell\in\mathbb{N}} \sum_{|\alpha|=m_\ell} |\gamma(m_\ell - \mu)|^2 |e_\alpha(\omega_n)|^2 .$$

Given $\varepsilon > 0$ we choose $\ell_0 \in \mathbb{N}$ such that $|\gamma(m_\ell - \mu)|^2 < \varepsilon$ for $\ell \geq \ell_0$. Then,

$$\left\|\left(D_\gamma - \mu I\right) f_n\right\|^2 \leq \frac{1}{\|K_S(.,\omega_n)\|^2} \sum_{\ell=1}^{\ell_0} \sum_{|\alpha|=m_\ell} |\gamma(m_\ell - \mu)|^2 |e_\alpha(\omega_n)|^2 + \varepsilon.$$

by Lemma (6.2.15) the first term on the right hand side tends to zero as $n \to \infty$, and therefore

$$\underset{e\to\infty}{Lim} \left\|\left(D_\gamma - \mu I\right) f_n\right\| = 0 \tag{67}$$

Using $0 \leq T_{\overline{\phi}} T_\phi \leq T_{|\phi|^2}$ together with Proposition (6.2.16) we have

$$\begin{aligned} \left\|\left(T_\phi - \xi I\right) f_n\right\|^2 &= \langle T_{\overline{\phi}} T_\phi f_n, f_n\rangle - \bar{\xi} B_{S_\mu}[\phi](\omega_n) - \xi B_{S_\mu}\overline{[\phi]}(\omega_n) + |\xi|^2 \\ &\leq B_{S_\mu}[|\phi - \xi|^2](\omega_n) \\ &\to |\phi - \xi|^2(v) = 0, \qquad \text{as} \quad n \to \infty \end{aligned} \tag{68}$$

Finally, (67), (68) and Proposition (6.2.12) imply that $(\mu, \xi) \in \sigma\pi\left(D_\gamma, T_\phi\right)$.

Theorem (6.2.18)[303]: We have

$$\begin{aligned} M(\mathcal{A}(A_1, A_2)) &= \sigma\left(D_\gamma\right) \times \{0\} \cup Lim(\gamma) \times \overline{D}\left(0, \frac{1}{2}\right). \\ &= \{\gamma(k) : k \in \mathbb{Z}_+\} \times \{0\} \cup Lim(\gamma) \times \overline{D}\left(0, \frac{1}{2}\right) \end{aligned}$$

To uniform the result and make it independent of a concrete choice of the diagonal operator $A_1 = D_\gamma$ we proceed as follows. Given the operator D_γ, the compact set of maximal ideals of the C∗-algebra $\mathcal{A}(A_1)$ was identified with

$$\sigma\left(D_\gamma\right) = \{\gamma(k) : k \in \mathbb{Z}_+\} \times \cup \ Lim(\gamma).$$

The last set is homeomorphic to a certain compactification of $\mathbb{Z}_+$, and this homeomorphism is given by

$$\gamma(k) \in \gamma \longmapsto k,$$

$\gamma^* = \lim_{e\to\infty} \gamma(k_n) \in Lim(\gamma) \longmapsto$ class of equivalence containing $\{k_n\}_{n\in k}$.

We say that two subsequences $\{k_n\}$ and $\{k_m\}$ are equivalent if and only if the following limits exist and coincide:

$$\lim_{e\to\infty} \gamma(k_n) = \lim_{e\to\infty} \gamma(k_m)$$

We denote by $M_\infty(\gamma)$ the part of the maximal ideals of $\mathcal{A}(A_1)$ which is homeomorphic to$Lim(\gamma)$. Now Theorem (6.2.18) reads as follows.

Theorem (6.2.19)[303]: We have

$$M(\mathcal{A}(A_1, A_2)) = \mathbb{Z}_+ \times \{0\} \cup M_\infty(\gamma) \times \overline{D}\left(0, \frac{1}{2}\right).$$

The difference among different choices of the generating operator D_γ is reflected now in the different corresponding compactifications of $\mathbb{Z}_+$, i.e. in different sets $M_\infty(\gamma)$.

We describe first multiplicative functionals of the C^*-algebra generated by a finite number of diagonal operators: Let $D_{\gamma_1}, \ldots, D_{\gamma_n}$ be bounded diagonal operators on $H = \mathcal{A}_\lambda^2(\mathbb{B}^2)$ acting on elements of the basis (55) as

$$D_{\gamma_j} e_\alpha = \gamma_j(|\alpha|) e_\alpha, \qquad \text{for} \qquad \alpha \in \mathbb{Z}_+^2$$

and whose eigenvalue sequence γ_j belongs to $SO(\lambda)$. Consider then the unital C^* −algebra

$$\mathcal{A}_D^* := \mathcal{A}\left(D_{\gamma_1}, \ldots, D_{\gamma_n}\right) \subset \mathcal{L}(H) \tag{69}$$

which is generated by elements of $\boldsymbol{D} = \left(D_{\gamma_1}, \ldots, D_{\gamma_n}\right)$.

Recall that the joint spectrum $\sigma\left(D_{\gamma_1}, \ldots, D_{\gamma_n}\right)$ of the operators $D_{\gamma_1}, \ldots, D_{\gamma_n}$, which is identified with the maximal ideal space $M(\mathcal{A}_D^*)$ of $\mathcal{A}_\mathrm{D}^*$, has the form:

$$\sigma\left(D_{\gamma_1}, \ldots, D_{\gamma_n}\right) = \left\{(\mu_1, \ldots, \mu_n) \in \mathbb{C}^n : J\left(D_{\gamma_1} - \mu_1 I, \ldots, D_{\gamma_n} - \mu_n I\right) \neq \mathcal{A}_D^*\right\}$$

where$J\left(D_{\gamma_1} - \mu_1 I, \ldots, D_{\gamma_n} - \mu_n I\right)$denotes the smallest ideal in the algebra $\mathcal{A}_\mathrm{D}^*$ containing the elements $D_{\gamma_j} - \mu_j I$, for $j = 1, \ldots, n. (\mu_1, \ldots, \mu_n) \in \sigma\left(D_{\gamma_1}, \ldots, D_{\gamma_n}\right)$. Then

$$D = \left(D_{\gamma_1}^* - \overline{\mu_1} I\right)\left(D_{\gamma_1 1} - \mu_1 I\right) + \cdots + \left(D_{\gamma_n}^* - \overline{\mu_n} I\right)\left(D_{\gamma_n} - \mu_n I\right)$$

is an element of this ideal, and hence D is not invertible in $\mathcal{A}_\mathrm{D}^*$. Since $\mathcal{A}_\mathrm{D}^*$ (as a C^*-algebra) is inverse closed, the operator D is not invertible in $\mathcal{L}$(H) either. Note that D is diagonal with the eigenvalues

$$\gamma(|\alpha|) = |\gamma_1(|\alpha|) - \mu_1|^2 + \cdots + |\gamma_n(|\alpha|) - \mu_n|^n \geq 0.$$

Corollary (6.2.20)[303]: Either there is $k \in \mathbb{Z}_+$ such that $\gamma_j(k) = \mu_j$ for all $j = 1, \ldots, n$, or there is a sequence $\{m_\ell\} \subset \mathbb{Z}_+$ such that for all$j = 1, \ldots, n$:

$$\lim_{e\to\infty} \gamma_j(m_\ell) = \mu_j. \tag{70}$$

Let $(\mu_1, \ldots, \mu_n) \in \sigma\left(D_{\gamma_1}, \ldots, D_{\gamma_n}\right)$. According to Corollary (6.2.20), we assume first that there is $k \in \mathbb{Z}_+$ such that for all$j = 1, \ldots, n$:

$$\gamma_j(k) = \mu_j$$

Then we define a multiplicative functional $\varphi_{(k)}$on$\mathcal{A}_D^*$ by:

$$\varphi_{(k)}(D) := \langle D e_{(k,0)}, e_{(k,0)} \rangle \tag{71}$$

for all $D \in \mathcal{A}_D^*$. Note that $\psi_{(k)}(D_{\gamma j}) = \mu_j$ for all j = 1, ..., n.
If such k $\in \mathbb{Z}_+$ does not exists, then, by the second option of Corollary (6.2.21), there is a sequence $\{m_\ell : \ell \in N\} \subset \mathbb{Z}_+$ having the property (70). Define the functional on $\mathcal{A}_D^*$ by:

$$\psi_{\{m_\ell\}}(D) := \langle De_{(k,0)}, e_{(k,0)}\rangle \tag{72}$$

Lemma (6.2.21)[303]: The limit (72) exists for all $D \in \mathcal{A}_D^*$, and the functional $\psi_{\{m_\ell\}}$ is multiplicative with $\psi_{\{m_\ell\}}(D_{\gamma j}) = \mu_j$, for all $j = 1, \ldots, n$.

Proof: Similar to the proof of Lemma (6.2.22) below.

Now we modify the definition (72) so that the right hand side extends to a larger algebra (see Lemma (6.2.23) below). Consider the infinite set $S = \{m_\ell : \ell \in \mathbb{N}\} \subset \mathbb{Z}_+$ and define the Hilbert space

$$H_S = \overline{span}\{e_\alpha : \alpha \in \mathbb{Z}_+, |\alpha| \in S\}$$

Let K_S be the reproducing kernel of H_S, i.e. for all $z, \omega \in \mathbb{B}^2$ we have

$$K_S = (z, \omega) := \sum_{|\alpha| \in S} e_\alpha(z)\overline{e_\alpha(\omega)}. \tag{73}$$

Let $\xi \in \overline{D}\left(0, \frac{1}{2}\right)$ and $v \in \partial\mathbb{B}^2$ such that $\phi(v) = \xi$. Let $\{\omega_k\}_k \subset \mathbb{B}^2$ be a sequence with $\omega_k \longrightarrow v \in \partial\mathbb{B}^2$ as $k \longrightarrow \infty$, and assume that $\{\omega_k\}_k$ is increasing. Define a sequence $\{f_k\}_k$ of unit vectors in H by

$$f_k = \frac{K_S(., \omega_k)}{\|K_S(., \omega_k)\|} \in H. \tag{74}$$

Lemma (6.2.22)[303]:. The multiplicative functional $\psi_{\{m_\ell\}}$ in (72) can be also defined as

$$\psi_{\{m_\ell\}}(D_\gamma) = \underset{e\to\infty}{Lim}\langle D_\gamma f_k, f_k\rangle, \quad \text{where } D_\gamma \in \mathcal{A}_D^* \tag{75}$$

Proof: Since the functional $\psi_{\{m_\ell\}}$ is continuous and due to Lemma (6.2.21) it is sufficient to show that for all $(i_1, \ldots, i_n) \in \mathbb{Z}_+^n$:

$$\underset{e\to\infty}{Lim}\langle D_{\gamma_1}^{i_1} D_{\gamma_2}^{i_2} \ldots D_{\gamma_n}^{i_n} f_k, f_k\rangle = \mu_{\gamma_1}^{i_1}\mu_{\gamma_2}^{i_2} \ldots \mu_{\gamma_n}^{i_n}.$$

A simple argument using the Cauchy-Schwarz and triangle inequality together with $\|f_k\| = 1$ shows that it is sufficient to prove for $j = 1, \ldots, n$ that

$$\underset{k\to\infty}{Lim}\left\|\left(D_{\gamma_j}^{i_j} - \mu_{\gamma_j}^{i_j} I\right) f_k\right\| = 0. \tag{76}$$

But this has been already shown for the above choice of $\{f_k\}_k$ in the proof of Lemma (6.2.17) by using the convergence (70).

Further, for each operator $D_\gamma \in \mathcal{A}_D^*$, the limit along the subsequence $\{m_\ell\}$ of its eigenvalue sequence γ exists and is equal to the value of the functional $\psi_{\{m_\ell\}}$ on the diagonal operator D_γ:

$$\underset{e\to\infty}{lim}\, \gamma_{(m_\ell)} = \psi_{\{m_\ell\}}(D_\gamma).$$

Let $\xi \in \overline{D}\left(0, \frac{1}{2}\right)$ be as above, with corresponding sequences $w_k \longrightarrow v \in \partial\mathbb{B}^2$ such that $\phi(v) = \xi$ and $\{f_k\}$ is of the form (74).

Lemma (6.2.23)[303]: The functional $\psi_{\{m_\ell\}}$ extends to the functional $\psi = (\psi_{\{m_\ell\}}, \xi)$ on the algebra generated by elements of $\mathcal{A}_D^*$ and T_ϕ via

$$\psi\left(DT_\phi^j\right) = \lim_{k\to\infty} \langle DT_\phi^j f_k, f_k\rangle$$

with $j \in \mathbb{Z}_+$ and $D \in \mathcal{A}_D^*$. Moreover, for elements of the form $\sum_{j=0}^{m} D_j T_\phi^j$we have

$$\psi\left(\sum_{j=0}^{m} D_j T_\phi^j\right) = \sum_{j=0}^{m} \psi_{\{m_\ell\}},(D_j)\xi^j.$$

Proof: Similar to the proof of Lemma (6.2.22) and using the convergence $\lim_{e\to\infty}\left\|\left(T_\phi - \xi I\right)f_k\right\| = 0$ (see (68)).

Let $\mathcal{A}$ be a unital commutative Banach algebra which is generated by two of its unital subalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ sharing the same identity, and let $M(\mathcal{A})$, $M(\mathcal{A}_1)$, and $M(\mathcal{A}_2)$ be their respective sets of maximal ideals (≡multiplicative functionals). Recall that, since the restrictions of a multiplicative functional $\psi \in M(\mathcal{A})$ onto subalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are multiplicative functional $\psi_1 \in M(\mathcal{A}_1)$ and $\psi_2 \in M(\mathcal{A}_2)$, correspondingly, we have a natural continuous mapping

$$\kappa: \psi \in M(\mathcal{A}) \longmapsto (\psi_1, \psi_2) \in M(\mathcal{A}_1) \times M(\mathcal{A}_2)$$

As $\mathcal{A}$ is generated by $\mathcal{A}_1$ and $\mathcal{A}_2$, the mapping κ is obviously injective, and thus its range can be identified with $M(\mathcal{A})$.

The unital Banach algebra $\mathcal{T}(\lambda)$, we are interested in, is generated by two algebras sharing the same identity: the C*-algebra $\mathcal{T}_{rad}(\lambda)$, which is generated by all Toeplitz operators T_a with radial symbols $a \in L^\infty[0,1)$, and the Banach algebra $\mathcal{T}_\phi$, which is generated by a single Toeplitz operator T_ϕ, where $\phi(\xi) = \xi(1,0)\xi(0,1)$. Thus, by (50) and the last paragraph of the section, the mapping κ identifies $M(\mathcal{T}(\lambda))$ with a subset of $\left(\mathbb{Z}_+ \times \cup M_\infty(\lambda)\right) \times \bar{D}\left(0,\frac{1}{2}\right)$.

Lemma (6.2.24)[303]: None of the points of the set $\mathbb{Z}_+ \times \left(\bar{D}\left(0,\frac{1}{2}\right)\setminus\{0\}\right)$ belongs to $M(\mathcal{T}(\lambda))$.

Proof: Let us assume that a point $(k,\xi) \in \mathbb{Z}_+ \times \left(\bar{D}\left(0,\frac{1}{2}\right)\setminus\{0\}\right)$ belongs to $M(\mathcal{T}(\lambda))$. Then, for the operator $A = P_k T_\phi \in \mathcal{T}(\lambda)$, where $P_k$is the orthogonal projection onto Hk (see (58)), we have$\psi(A) = 1, \quad \xi = 0$. At the same time, by Lemma (6.2.7), the operator A belongs to the radical of the algebra $\mathcal{T}(\lambda)$, and thus $\psi(A) = 0$. Contradiction.

Lemma (6.2.25)[303]: The set $\mathbb{Z}_+ \times \{0\}$ belongs to $M(\mathcal{T}(\lambda))$.

Proof: Let $\psi = (k,0) \in \mathbb{Z}_+ \times \{0\}$. Denote by $\psi_{(k)}$ the multiplicative functional on $\mathcal{T}_{rad}(\lambda)$ (see (71)) given by:

$$\psi_{(k)}(D_\gamma) := \langle D_\gamma e_{k,o}, e_{k,o}\rangle = \gamma_{(k)}, \quad \text{where } D_\gamma \in \mathcal{T}_{rad}(\lambda).$$

Then the functional $\psi = (k,0) = \left(\psi_{(k)}, 0\right)$ is defined on a dense subalgebra D_γ of $\mathcal{T}(\lambda)$ as follows: for any $A = \sum_{j=0}^{m} D_p T_\phi^p \in D(\gamma)$ where $D_p \in \mathcal{T}_{rad}(\lambda)$ we put:

$$\psi(A) := \langle Ae_{(k,0)}, e_{(k,0)}\rangle = \psi_{(k)}(D_0) = \gamma_{D_0}(k). \tag{77}$$

Note that the functional ψ is well-defined, since $\sum_{j=0}^{m} D_p T_\phi^p = 0$ implies $D_0 = 0$, according to Lemma (6.2.8). Moreover, we have

$$\left|\psi\left(\sum_{p=0}^{m} D_p T_\phi^p\right)\right| = \left|\langle\sum_{p=0}^{m} D_p T_\phi^p e_{(k,0)}, e_{(k,0)}\rangle\right| \le \left\|\sum_{p=0}^{m} D_p T_\phi^p\right\|$$

Hence ψ is continuous and extents to a multiplicative functional on $\mathcal{T}(\lambda)$.

Recall that $M_\infty(\lambda)$ denotes the multiplicative functionals on $\mathcal{T}_{rad}(\lambda)$ that map compact operators to zero.

Lemma (6.2.26)[303]: The set $M_\infty(\lambda) \times \overline{D}\left(0,\frac{1}{2}\right)$ belongs to $\mathrm{M}(\mathcal{T}(\lambda))$.

Proof: We define the functional $\psi = (\mu, \xi) \in M_\infty(\lambda) \times \overline{D}\left(0,\frac{1}{2}\right)$ on a dense subalgebra $\mathcal{D}_\gamma$ of $\mathcal{T}(\lambda)$ as follows: for any $A = \sum_{p=0}^{m} D_p T_\phi^p \in \mathcal{D}(\lambda)$, where $D_p(\lambda) \in \mathcal{T}_{rad}(\lambda)$, we put:

$$\psi\left(\sum_{p=0}^{m} D_p T_\phi^p\right) := \sum_{p=0}^{m} \mu(D_p)\xi^p = \sum_{p=0}^{m} \gamma D_p(\mu)\xi^p$$

First we need to show that ψ is well-defined. Indeed, according to Lemma (6.2.8), the equality $A = \sum_{p=0}^{m} D_p T_\phi^p = 0$ implies that $\mathrm{D_p}$ is compact for all $p = 0, \dots, m$. But $\mu \in M_\infty(\lambda)$, and thus $\mu(D_p) = 0$ for all $p = 0, \dots, m$,which implies that $\psi(A) = 0$. The functional $\psi$is obviously multiplicative on $D(\gamma)$, and thus it remains to show that it is continuous and therefore extents to a multiplicative functional on $\mathcal{T}(\lambda)$.

Fix now any $A = \sum_{p=0}^{m} D_p T_\phi^p \in D(\lambda)$ and consider the unital C*-algebra $\mathcal{A}_{\mathrm{D}}^*$generated by $D_0, \dots, D_m$. Clearly, the restriction $\hat{\psi}$ of ψ to $\mathcal{A}_D^*$ defines a multiplicative functional on $\mathcal{A}_{\mathrm{D}}^*$. Note that

$$(\mu_0, \dots, \mu_m) = \left(\psi(D_0), \dots, \psi(D_m)\right) \in \sigma(D_0, \dots, D_m).$$

Since $\widehat{\psi}$ maps compact operators in $\mathcal{A}_{\mathrm{D}}^*$to zero and because of Lemma (6.2.23), we have $\hat{\psi}$has the form (75):

$$\hat{\psi}(D_\gamma) = \lim_{k\to\infty} \langle D_\gamma f_k, f_k\rangle \,, D_\gamma \in \mathcal{A}_D^*$$

where $\{m_\ell : \ell \in N\} \subset \mathbb{Z}_+$ is a suitable sequence which is induced by $(\mu_0, \dots, \mu_m)$ as was explained, and f_k, with $k \in \mathbb{N}$, are given by (74) with $\xi = \phi(v)$.

Now from Lemma (6.2.23) it follows that

$$\left|\psi\left(\sum_{p=0}^{m} D_p T_\phi^p\right)\right| = \left|\psi\left\langle\sum_{p=0}^{m} D_p T_\phi^p f_k, f_k\right\rangle\right| \le \left\|\sum_{p=0}^{m} D_p T_\phi^p\right\|,$$

and thus ϕ is continuous on D_γ and extends to a multiplicative functional on $D_{(\lambda)}$.

Theorem (6.2.27)[303]: The compact set $M(\mathcal{T}(\lambda))$ of maximal ideals of the algebra $\mathcal{T}(\lambda)$ has the form

$$M\left(\mathcal{T}(\lambda)\right) = \mathbb{Z}_+ \times \{0\} \cup M_\infty(\lambda) \times \overline{D}\left(0,\frac{1}{2}\right),$$

(i) The Gelfand image of the algebra $\mathcal{T}(\lambda)$ is isomorphic to $\mathcal{T}(\lambda)/Rad\,\mathcal{T}(\lambda)$ and coincides with the algebra

$$SO(\lambda) \cup \left[C\left(M_\infty(\lambda)\right) \widehat{\otimes}_\varepsilon C_a\left(\overline{D}\left(0,\frac{1}{2}\right)\right)\right],$$

which is identified with the set of all pairs

$$(\gamma, f) \in SO(\lambda) \times \left[C\left(M_\infty(\lambda)\right) \widehat{\otimes}_\varepsilon C_a\left(\overline{D}\left(0,\frac{1}{2}\right)\right)\right]$$

satisfying the following compatibility condition $\gamma(\mu) = f(\mu, 0)$, for all $\mu \in M_\infty(\lambda)$.

Here $\widehat{\otimes}_\varepsilon$ denotes the injective tensor product, and we identify $\gamma(\mu)$ with the value of the functional $\mu \in M_\infty(\lambda)$ on the element $\gamma \in SO(\lambda)$.

(ii) The Gelfand transform is generated by the following mapping of elements of $SO(\lambda)$:

$$\sum_{j=0}^{n} D_{\gamma_j} T_\phi^p \mapsto \begin{cases} \gamma_0(k) & if\ (k,0) \in \mathbb{Z}_+ \times \{0\} \\ \sum_{j=0}^{n} \gamma_j(\mu)\xi^j, & if\ (\mu,\xi) \in M_\infty(\lambda) \times \overline{D}\left(0,\frac{1}{2}\right) \end{cases}.$$

Proof: Follows directly from Lemmas (6.2.24), (6.2.25), and(6.2.26), Theorem (6.2.6) and the injective tensor product description (see [256]).

Our next aim is to show that the algebra $\mathcal{T}(\lambda)$ is inverse closed, that is: each operator $A \in \mathcal{T}(\lambda)$ which is invertible in $\mathcal{L}(\mathcal{A}_\lambda^2(\mathbb{B}^2))$ is invertible in $\mathcal{T}(\lambda)$, i.e., $A^{-1} \in \mathcal{T}(\lambda)$. The proof of this fact essentially relies on Theorem (6.2.27).

Lemma (6.2.28)[303]: Let $\varphi = (k,0) \in \mathbb{Z}_+ \times \{0\} \subset M(\mathcal{T}(\lambda))$, and assume that $A \in \mathcal{T}(\lambda)$ is invertible in $\mathcal{L}(\mathcal{A}_\lambda^2(\mathbb{B}^2))$. Then $\psi(A) = 0$.

Proof: Recall that the functional$\psi = (k,0)$ on $\mathcal{T}(\lambda)$ is defined on the dense subalgebra $D(\lambda)$ by (77). Clearly, it extents by the same formula (77) to a continuous (not necessarily multiplicative) functional on $\mathcal{A}_\lambda^2(\mathbb{B}^2)$.

Let $\sum_{p=0}^{m} D_{\gamma j} T_\phi^j \in D(\lambda)$, and let $B \in \mathcal{L}(\mathcal{A}_\lambda^2(\mathbb{B}^2))$be arbitrary.Then:

$$\psi(BA) = \langle BAe_{(k,0)}, e_{(k,0)} \rangle = \langle BD_{\gamma 0} e_{(k,0)}, e_{(k,0)} \rangle = \gamma_0(k) \langle Be_{(k,0)}, e_{(k,0)} \rangle = \psi(A)\psi(B) \\ = \psi(B)\psi(A)$$

By continuity it follows that $\psi(BA) = \psi(B)\psi(A)$ for all $A \in \mathcal{T}(\lambda)$. In particular, if A is invertible in $\mathcal{A}_\lambda^2(\mathbb{B}^2)$, then

$$1 = \psi(I) = \psi(A^{-1}A) = \psi(A^{-1})\psi(A),$$

and we conclude that $\psi(A) \neq 0$.

Given $D = (D_{\gamma_1}, \dots, D_{\gamma_n}) \subset \mathcal{T}_{rad}(\lambda)$, we consider the unital C*-algebra $\mathcal{A}_D^*$ generated by $D_{\gamma_1}, \dots, D_{\gamma_n}$ (see (69)). Let $\tilde{\mathcal{A}}_D$ and $\tilde{\mathcal{A}}_D^*$ be the Banach algebra and the C*-algebra which are generated by elements of $\mathcal{A}_D \cup T_\phi$ and of $\mathcal{A}_D^* \cup T_\phi$, respectively. Clearly we have $\tilde{\mathcal{A}}_D \subset \tilde{\mathcal{A}}_D^*$.

Consider now the functional $\psi = (\mu,\xi) \in M_\infty(\lambda) \times \overline{D}\left(0,\frac{1}{2}\right) \subset M(\mathcal{T}(\lambda))$. Its restriction to the algebra $\mathcal{A}_D^*$ maps compact operators to zero. Thus, according to Lemmas(6.2.23) and(6.2.24), we can construct the sequence of unit vectors $\{f_k\}_k$ by (54) with $\xi = \phi(v)$ such that

$$\psi\left(D_\gamma T_\phi^j\right) = \lim_{k\to\infty} \langle D_\gamma T_\phi^j f_k, f_k \rangle, D_\gamma \in \mathcal{A}_D^* \quad (78)$$

Moreover, by continuous extension the right hand side of (78) defines a multiplicative continuous functional on $\tilde{\mathcal{A}}_D$ (which coincides with the restriction of $\psi \in M_\infty(\lambda) \times \overline{D}\left(0,\frac{1}{2}\right)$to the algebra $\tilde{\mathcal{A}}_D$.

We show now that the restriction of $\psi$to $\tilde{\mathcal{A}}_D$ extends further to a multiplicative and continuous functional on the (non-commutative) C*-algebra $\tilde{\mathcal{A}}_D^*$. The nature of such an extension is very simple. Let $\mathcal{K}$ be the ideal of all compact operators on $\mathcal{A}_\lambda^2(\mathbb{B}^2)$. Two quotient algebras

$$\hat{\mathcal{A}}_D = \tilde{\mathcal{A}}_D(\tilde{\mathcal{A}}_D \cap \mathcal{K}) \quad and \quad \hat{\mathcal{A}}_D^* = \tilde{\mathcal{A}}_D^*/(\tilde{\mathcal{A}}_D^* \cap \mathcal{K})$$

will be involved. For an algebra $\mathcal{A} \subset \mathcal{L}(\mathcal{A}_\lambda^2(\mathbb{B}^n))$, we denote by pr the natural projection

$$pr: \mathcal{A} \to \hat{\mathcal{A}} = \tilde{\mathcal{A}}/(\tilde{\mathcal{A}} \cap \mathcal{K})$$

As $[T_\phi, T_{\overline{\phi}}] \in \mathcal{K}$ and both operators $T_\phi$and $T_{\overline{\phi}}$ commute with diagonal operators D_γ, the C*-algebra$\hat{\mathcal{A}}_D^*$ is commutative, and furthermore $\hat{\mathcal{A}}_D \subset \tilde{\mathcal{A}}_D^*$.

The functional ψ, restricted to $\hat{\mathcal{A}}_D$, maps compact operators from $\hat{\mathcal{A}}_D$ to zero, and thus it admits the natural decomposition.

$$\psi: \tilde{\mathcal{A}}_D \overset{pr}{\to} \hat{\mathcal{A}}_D \overset{\hat{\psi}}{\to} \mathbb{C}$$

for a suitable multiplicative functional $\hat{\psi}$ on $\hat{\mathcal{A}}_D$.

We extend now the functional $\hat{\psi}$ from $\hat{\mathcal{A}}_D$ to the functional (one-dimensional representation) $\widehat{\psi^*}$ on $\hat{\mathcal{A}}_D^*$ on the C*-algebra $\hat{\mathcal{A}}_D^*$ defining it on the extra generator $[T_{\overline{\phi}}] = T_{\overline{\phi}} + \hat{\mathcal{A}}_D^* \cap \mathcal{K}$ as it should be:

$$\hat{\psi}^*([T_{\overline{\phi}}]) = \overline{\psi([T_{\overline{\phi}}])}$$

The extension ψ^* of the functional ψ from $\hat{\mathcal{A}}_D$ onto $\hat{\mathcal{A}}_D^*$ is thus given by

$$\psi^*: \tilde{\mathcal{A}}_D \overset{pr}{\to} \hat{\mathcal{A}}_D \overset{\widehat{\psi^*}}{\to} \mathbb{C}$$

As the next lemma shows the functional ψ^*has the same form as in (78):

$$\psi^*\left(D_\gamma T_\phi^{j_1} T_\phi^{j_2}\right) = \lim_{k\to\infty} \langle D_\gamma T_\phi^{j_1} T_\phi^{j_2} f_k, f_k\rangle \quad (79)$$

Lemma (6.2.29)[303]: The limit on the right hand side of (79) exists for all $D_\gamma \in \mathcal{A}_D^*$ and $j_1, j_2 \in \mathbb{Z}_+$. Moreover, it has the value:

$$\lim_{k\to\infty} \langle D_\gamma T_\phi^{j_1} T_\phi^{j_2} f_k, f_k\rangle = \psi(D_\gamma)\psi(T_\phi)^{j_1}\overline{\psi([T_{\overline{\phi}}])}^{j_2}$$

In particular, by linearity and continuity it induces a multiplicative functional ψ^* on $\tilde{\mathcal{A}}_D^*$ which extends ψ.

Proof: Similar to the proof of Lemma (6.2.22) or Lemma (6.2.23) and by using the convergence $\lim_{k\to\infty}\|(T_{\overline{\phi}} - \bar{\xi}I)f_k\| = 0$, where $\xi = \psi(T_\phi)$.

Corollary (6.2.30)[303]:let$\psi = (\mu, \xi) \in M_\infty(\lambda) \times \overline{D}\left(0, \frac{1}{2}\right) \subset M(\mathcal{T}(\lambda))$, and let $\mathcal{A} \in \tilde{\mathcal{A}}_D$ be invertible as an element in $\mathcal{L}\left(\mathcal{A}_\lambda^2(\mathbb{B}^n)\right)$. Then $\psi(\mathcal{A}) \neq 0$.

Proof. As each C*-subalgebra of $\mathcal{L}(\mathcal{A}_\lambda^2(\mathbb{B}^n))$, the algebra $\tilde{\mathcal{A}}_D$ is inverse closed and thus we have $\mathcal{A}^{-1} + \tilde{\mathcal{A}}_D^*$. According to the previous lemma the functional ψextends to a multiplicative functional ψ^* on $\tilde{\mathcal{A}}_D^*$, and therefore

$$1 = \psi^*(AA^{-1}) = \psi(A)\psi(A^{-1}),$$

which shows that $\psi(A) \neq 0$.

Now, we deal with the general case: Let $A \in \mathcal{T}(\lambda)$ be invertible as an element in $\mathcal{L}(\mathcal{A}_\lambda^2(\mathbb{B}^n))$, then we wish to show that $A^* \in \mathcal{T}(\lambda)$. Choose a sequence $\{A_k\}_k \subset D(\lambda)$ such that

$$\lim_{k\to\infty} A_k = A$$

Since the group of invertible elements is open, we can assume that A_k is invertible for all $k \in \mathbb{N}$. Moreover, by the continuity of inversion we have

$$A^{-1} = \lim_{k\to\infty} A_k^{-1}$$

and hence it is sufficient to show that $A_k^{-1} \in \mathcal{T}(\lambda)$ for each k. Fix $D = \left(D_{\gamma_1}, \dots, D_{\gamma_n}\right) \subset \mathcal{T}_{rad}(\lambda)$, such that (with our notation above)

$$A_k \in \tilde{\mathcal{A}}_D$$

From Theorem (6.2.27), Lemma (6.2.28) and Corollary (6.2.30), together with the fact that A_k is invertible, we have

$$\psi(A_k) \neq 0$$

For all multiplicative functionals on $\tilde{\mathcal{A}}_D$, and thus $A_k^{-1} \in \tilde{\mathcal{A}}_D \subset \mathcal{T}(\lambda)$.
Hence we have shown:

Theorem (6.2.31)[303]: The commutative Banach algebra $\mathcal{T}(\lambda)$ is inverse closed, and, in particular, for each $A \in \mathcal{T}(\lambda)$,

$$sp_{\mathcal{T}(\lambda)}A = sp_{\mathcal{L}(\mathcal{A}_\lambda^2(\mathbb{B}^n)),}A$$

The next assertions give, in particular, some information on the spectra of elements of the algebra $\mathcal{T}(\lambda)$.

Lemma (6.2.32)[303]: The difference $T_{\phi_p} - T_\phi^p$, where p $\in \mathbb{N}$, belongs to the radical of the algebra $\mathcal{T}(\lambda)$.

Proof: By (53) we have $T_{\phi_p} - T_\phi^p = D_{\tilde{d}_p - 1} T_\phi^p$, and the assertion follows from Lemma (6.2.7) and the convergence $\lim_{k\to\infty} d_p\,(k) = 1$ (see (54)).

Corollary (6.2.33)[303]: The operators

$$\sum_{k=1}^{m}\left(\prod_{q=1}^{m_k} T_{a_{k,p}} \prod_{s=1}^{n_k} T_{\phi_{p_{k,s}}}\right) \text{and} \sum_{k=1}^{m}\left(\prod_{q=1}^{m_k} T_{a_{k,p}}\, T_\phi^{p_{k,1}+\cdots+p_{k,n_k}}\right)$$

differ by an element in the radical and thus have the same Gelfand images and the same spectra.

Theorem (6.2.35)[303]: With our previous notation we have:

(i) The Calkin algebra $\hat{\mathcal{T}}(\lambda) = \mathcal{T}(\lambda) \,/\, (\mathcal{T}(\lambda) \cap \mathcal{K})$ is semi-simple and isomorphic to the injective tensor product $\left(M_\infty(\lambda)\right) \times \widehat{\otimes}_\varepsilon\, C_a \overline{D}\left(0, \frac{1}{2}\right)$.

(ii) The Calkin algebra $\widehat{\mathcal{T}^*}(\lambda) = \mathcal{T}^*(\lambda)/(\mathcal{T}^*(\lambda) \cap \mathcal{K})$ of the C*-extension $\mathcal{T}^*(\lambda)$ of the Banach algebra $\mathcal{T}(\lambda)$ is isomorphic and isometric to $C\left(M_\infty(\lambda)\right) \times \overline{D}\left(0, \frac{1}{2}\right)$.

(iii) Both natural homomorphisms

$$\mathcal{T}(\lambda) \to \hat{\mathcal{T}}(\lambda) \cong C\left(M_\infty(\lambda)\right) \times \widehat{\otimes}_\varepsilon\, C_a \left(\overline{D}\left(0, \frac{1}{2}\right)\right),$$

$$\mathcal{T}^*(\lambda) \to \hat{\mathcal{T}}(\lambda) \cong C\left(M_\infty(\lambda)\right) \times \left(\overline{D}\left(0, \frac{1}{2}\right)\right).$$

are generated by the following mapping:

$$\sum_{j=0}^{n} D_{\gamma_j} T_\phi^j \mapsto \sum\nolimits_{j=0}^{n} \gamma_j(\mu)\xi^j \,, (\mu, \xi) \in M_\infty(\lambda) \times \overline{D}\left(0, \frac{1}{2}\right)$$

(iv) The essential spectrum of all elements of both algebras $\mathcal{T}(\lambda)$ and $\mathcal{T}^*(\lambda)$ is connected.

Proof: Follows from Theorem (6.2.27) and by arguments similar to the ones used in the proof of Theorem (6.2.31) The last assertion is a consequence of the connectedness of both sets $M_\infty(\lambda)$ and $\overline{D}\left(0, \frac{1}{2}\right)$.

Corollary (6.2.35)[303]: For all elements of the form $I+\sum_{j=1}^{n} D_{\gamma_j} T_{\phi}^{j}$, where $\eta \in \mathbb{C}$, and, in particular, for finite sums of finite product of generators of the algebra $\mathcal{T}(\lambda)$.

$$\eta I+\sum_{k=1}^{n}\left(\prod_{q=1}^{m_k} T_{a_{k,p}} \prod_{s=1}^{n_k} T_{\phi_{p_{k,s}}}\right),$$

where $\eta \in \mathbb{C}$ and none of n_k is equal to zero, their spectrum and essential spectrum coincide. In particular, for such operators the spectrum is connected, and being Fredholm this operator is invertible.

Corollary (6.2.36)[303]: An element $\sum_{j=1}^{n} D_{\gamma_j} T_{\phi}^{j}$ from the dense subalgebra $D(\lambda)$ is compact if and only if each diagonal operator D_{γ_j} is compact, or if and only if each eigenvalue sequence γ_j belongs to c_0.

Our last aim is to describe explicitely the radical of the algebra $\mathcal{T}(\lambda)$.

Consider the multiplicative functional $\psi(k,0) \in \mathbb{Z}_+ \times \{0\} \subset M(\mathcal{T}(\lambda))$.

First we would like to analyze the structure of operators $A \in \mathcal{T}(\lambda)$ satisfying the following property

$$\psi_{(k,0)}(A)=0 \text{ for all } k \in \mathbb{Z}_+. \tag{80}$$

For an element $A=\sum_{j=0}^{p} D_{\gamma_j} T_{\phi}^{j}$ from the dense subalgebra $D(\lambda)$ of $\mathcal{T}(\lambda)$, we have

$$\psi_{(k,0)}(A)=\psi_{(k,0)}\left(\sum_{j=0}^{p} D_{\gamma_j} T_{\phi}^{j}\right)=\gamma_0(k)$$

That is, the operator $A=\sum_{j=0}^{p} D_{\gamma_j} T_{\phi}^{j}$ satisfies property (80) if and only if $\gamma_0 \equiv 0$, i.e. A has to be of the form

$$A=\sum_{j=0}^{p} D_{\gamma_j} T_{\phi}^{j}=T_{\phi} C, \tag{81}$$

where $C=\sum_{j=0}^{p} D_{\gamma j} T_{\phi}^{j-1} \in \mathcal{T}(\lambda)$. The description of $Rad\, \mathcal{T}(\lambda) \cap D(\lambda)$ is straightforward:

Lemma (6.2.37)[303]: We have

$$Rad\, \mathcal{T}(\lambda) \cap D(\lambda)=\{T_{\phi} C: C \in D(\lambda) \cap K\}$$

Proof: Observe first that $A \in Rad\, T(\lambda) \cap D(\lambda)$ if and only if A satisfies property (80) and A is compact. That is A is of the form $A=T_{\phi} C$ in (81) with $C \in D(\lambda)$; and by Corollary (6.2.36) compactness of A is equivalent to compactness of C.

The description of $Rad\, \mathcal{T}(\lambda) \cap (\mathcal{T}(\lambda) \backslash D(\lambda))$ is more elaborated. We start with an easy lemma.

Lemma (6.2.38)[303]: Each operator $A \in \mathcal{T}(\lambda) \backslash D(\lambda)$ satisfying the property (80) can be approximated in norm by the operators of the form (81).

Proof: Given $A \in \mathcal{T}(\lambda) \backslash D(\lambda)$, there is a sequence of operators

$$\tilde{A}_n=\sum_{j=0}^{pn} D_{\gamma_j}(n) T_{\phi}^{j} \in D(\lambda) \tag{82}$$

such that $\|A-\tilde{A}_n\|<\frac{1}{n}$. Then, for each k $\in \mathbb{Z}_+$, we have

$$\left|\gamma_0^{(n)}(k)\right| = \left|\psi_{(k,0)}(\tilde{A}_n)\right| = \left|\psi_{(k,0)}(\tilde{A}_n - A)\right| \leq \|A-\tilde{A}_n\| < \frac{1}{n}$$

That is, $\|D_{\gamma 0}(n)\| \leq \frac{1}{n}$ for all$n \in \mathbb{N}$. Then for the operators

$$\tilde{A}_n = \sum_{j=1}^{pn} D_{\gamma_j}(n) T_\phi^j = T_\phi C_n$$

we have $\|\tilde{A}_n - A_n\| = \|D_{\gamma 0}(n)\| \leq \frac{1}{n}$.

Finally, the inequality $\|A-A_n\| \leq \|A-\tilde{A}_n\| + \|\tilde{A}_n - A_n\| \leq \frac{2}{n}$ implies that the sequence $\{A_n\}$ converges to the operator A.

Let $H := \mathcal{A}_\lambda^2(\mathbb{B}^2)$. We consider two of its subspaces V and W defined by:

$V = \overline{span}\{e_{(k,0)}: k \in \mathbb{Z}_+\}$ and $W = \overline{span}\{e_{(0,k)}: k \in \mathbb{Z}_+\}$

It is easy to see that $V = ker\, T_\phi$ and $W = coker\, T_\phi = (ImT_\phi)^\perp$. We introduce as well the orthogonal projections $P : H \to V$ and $Q : H \to W$.

Corollary (6.2.39)[303]: For each operator $A \in \mathcal{T}(\lambda)$ satisfying the property (80), $ImA \perp W$.

Proof: For operators $A \in \mathcal{T}(\lambda)$ it follows from (81). Then, by Lemma (6.2.38), each operator $A \in \mathcal{T}(\lambda) \backslash \mathcal{D}(\lambda)$ with (80) can be uniformly approximated by operators whose range is orthogonal to W. Thus the limit operator A obeys the same property.

The following result is known (see, for example, [48,253]).

Lemma (6.2.40)[303]: Let H be a Hilbert space and $\mathcal{A}$ α C^*-algebra in $\mathcal{L}(H)$. Let $A \in \mathcal{A}$ have a closed range. Then:

(1)The orthogonal projection P onto $kerA = kerA^*A$ belongs to the algebra $\mathcal{A}$.

(2) There exists $B \in \mathcal{A}$, namely $B = (P + A^*A)^{-1}, A^* \in \mathcal{A}$, such that

(i)$P = I - BA$ is the orthogonal projection onto ker $\mathcal{A}$,

(ii) $Q = I - AB$ is the orthogonal projection onto $(Im\, \mathcal{A})^\perp$,

(iii) $ABA = A$ and $BAB = B$, i.e. B is a relative inverse of A.

With $n \in \mathbb{Z}_+$ we recall the definition $H_n = span\{e_\alpha : \alpha \in \mathbb{Z}_+^2, |\alpha| = n\}$, and let:

$$\tilde{P}_n : H \to \bigoplus_{j=0}^{n} H_j = \tilde{H}_n$$

denote the orthogonal projection. Recall that the finite dimensional spaces $\tilde{H}_n$ and hence $\tilde{H}_n$ are invariant for all elements of $\mathcal{T}^*(\lambda)$.

Consider the sequence of C^*-algebras

$$\mathcal{T}_n^*(\lambda) = \{A\tilde{P}_n : A \in \mathcal{T}^*(\lambda)\} \subset \mathcal{T}^*(\lambda),$$

where we have used that $\tilde{P}_n \in \mathcal{T}_{rad}(\lambda) \subset \mathcal{T}^*(\lambda)$. Then the restriction of T_ϕ to $\tilde{H}_n$ defines an element in $\mathcal{T}_n^*(\lambda)$ for all $n \in \mathbb{Z}_+$ (we keep denoting the restriction by T_ϕ and do not indicate the n-dependence). Since $dim\tilde{H}_n < \infty$, the range of T_ϕ in $\tilde{H}_n$ is closed.

Let $\prod_n \tilde{H}_n \to ker\, T_\phi = V \cap \tilde{H}_n \subset \tilde{H}_n$ denote the orthogonal projection.

Letting $\prod_n \equiv 0$ on $H \ominus \widetilde{H}_n$, we can consider $\prod_n as$ a (finite dimensional) orthogonal projection on H.

By Lemma (6.2.40)applied to $T_\phi \in \mathcal{T}_n^*(\lambda) \subset \mathcal{L}(\widetilde{H}_n)$, we have $\prod_n \in \mathcal{T}_n^*(\lambda)$.

As $\widetilde{H}_n$ and $H \ominus \widetilde{H}_n$ are invariant under P we remark that the projections P and $\tilde{P}_n$ commute, and, in particular, $\prod_n = \tilde{P}_n P \tilde{P}_n$.

Now we give a generalization of (a slightly weaker version of) Lemma (6.2.37).

Lemma (6.2.41)[303]: Let $A \in Rad\,\mathcal{T}(\lambda)$, then there is $C \in \mathcal{T}^*(\lambda) \cap \mathcal{K}$ such that $A = T_\phi C$.

Proof: Recall that if $A \in Rad\,\mathcal{T}(\lambda)$ then A is compact and satisfies property (80). According to Corollary (6.2.39), we have $(Im\,A \cap \widetilde{H}_n) \perp (W \cap \widetilde{H}_n)$, for all $n \in \mathbb{N}$, According to Lemma (6.2.40) (1) the operator $\mathrm{B_n} = (P + T_{\bar{\phi}} T_\phi)^{-1} T_{\bar{\phi}} \tilde{P}_n$ belongs to $\mathcal{T}_n^*(\lambda)$, and Lemma (6.2.40), (ii) gives

$$A\tilde{P}_n = I_n \tilde{P}_n = (Q_n + T_\phi B_n) A\tilde{P}_n = T_\phi B_n A \tilde{P}_n \tag{83}$$

where In is the identity element in $\mathcal{L}(\widetilde{H}_n)$ and $Q_n = I_n - T_\phi B_n \in \mathcal{T}_n^*(\lambda)$is the orthogonal projection onto $W \cap \widetilde{H}_n$.

Since $\tilde{P}_n \to I$ as $n \to \infty$ in the strong topology, it immediately follows that

$$B_n \to B = (P + T_{\bar{\phi}} T_\phi)^{-1} T_{\bar{\phi}}$$

in the strong sense as $n \to \infty$. Since A is compact it follows that $B_n A \to BA$ and $A\tilde{P}_n \to A$ as $n \to \infty$ with respect to the norm topology. The inequality

$$0 \leq \|B_n A \tilde{P}_n - BA\| \leq \|B_n A\tilde{P}_n - BA\tilde{P}_n\| + \|BA\tilde{P}_n - BA\| \leq \|B_n A - BA\| + \|B\|\|A\tilde{P}_n - A\|$$

implies that $\|B_n A\tilde{P}_n - BA\| \to 0$ as $n \to \infty$. If n on both sides of (83) tends to infinity then we conclude from $B_n A\tilde{P}_n \subset \mathcal{T}_n^*(\lambda) \subset \mathcal{T}^*(\lambda)$ for all $n \in \mathbb{N}$ that $BA \in T_*(\lambda)$, and, in addition, $A = T_\phi BA = T_\phi C$ with $C = BA \in \mathcal{T}^*(\lambda)$.

Finally the simple arguments based on results of Theorem (6.2.34) show that the operator $A = T_\phi C \in \mathcal{T}^*(\lambda)$ is compact if and only if the operator $C \in \mathcal{T}^*(\lambda)$ is compact.

Summarizing our previous results we obtain the following description of $Rad\,\mathcal{T}(\lambda)$:

Theorem (6.2.42)[303]: We have

$$\mathrm{Rad}\,\mathcal{T}(\lambda) = \{\mathrm{T}_\phi \mathrm{C} : \mathrm{C} \in \mathrm{T}_*(\lambda) \cap \mathrm{K}\} \cap \mathcal{T}(\lambda) \tag{84}$$

If $A \in Rad\,\mathcal{T}(\lambda) \cap \mathcal{D}(\lambda)$, then A can be even expressed in the form $A = T_\phi C$, where C is chosen in $\mathcal{D}(\lambda) \cap \mathcal{K}$.

Proof: The first assertion follows directly from Lemma (6.2.41) and the simple observation that the elements on the right hand side of (84) do belong to the radical. The last statement has been shown in Lemma (6.2.37).

Remark(6.2.43)[303]: Lemma (6.2.41) says that $A \in Rad\,\mathcal{T}(\lambda)$ has the form $A = T_\phi C$, where $C \in \mathcal{T}^*(\lambda) \cap \mathcal{K}$, while Lemma (6.2.37) gives a more precise information in case of $A \in \mathcal{D}(\lambda)$: C can be taken from $\mathcal{D}(\lambda) \cap \mathcal{K}$. Let us comment this ambiguity for $A \in Rad\,\mathcal{T}(\lambda) \cap \mathcal{D}(\lambda)$.

We start from the representation $A = T_\phi C_1$, with $C_1 \in \mathcal{D}(\lambda) \cap \mathcal{K}$given by Lemma (6.2.37). In turn, Lemma (6.2.41) gives a different representation $A = T_\phi C_2$ of A, where $C_2 \in \mathcal{T}^*(\lambda) \cap \mathcal{K}$. In particular, from $0 = T_\phi(C_1 - C_2)$ it follows that $Im(C_1 - C_2) \subset KerT_\phi$. More precisely, the operators C_1 and C_2 are related in the following way:

$$C_2 = BA = \left(P + T_{\overline{\phi}}T_{\phi}\right)^{-1}T_{\overline{\phi}}A = \left(P + T_{\overline{\phi}}T_{\phi}\right)^{-1}T_{\overline{\phi}}T_{\phi}C_1 = \left(1 - \left(P + T_{\overline{\phi}}T_{\phi}\right)^{-1}P\right)C_1.$$

The operator P obviously commutes with $\left(P + T_{\overline{\phi}}T_{\phi}\right)^{-1}$, and (we recall that) P is the orthogonal projection onto $KerT_{\phi}$. Thus $C_2 = C_1 - P\left(P + T_{\overline{\phi}}T_{\phi}\right)^{-1}C_1$, and we arrive again to the previous two different representations of the same operator A:

$$A = T_{\phi}C_2 = T_{\phi}C_1$$

We mention as well that the projection P does not belong to the algebra $\mathcal{T}^*(\lambda)$, while the operator$P\left(P + T_{\overline{\phi}}T_{\phi}\right)^{-1}C_1$ does.

Section (6.3) Toeplitz Operators on the Unit Ball

Let $\mathbb{B}^n$ denote the complex n-dimensional open unit ball in $\mathbb{C}^n$. We introduce the standard weighted L^2 −spaces $L^2(\mathbb{B}^n, dv_\lambda)$, where the family of probability measures

$$dv_\lambda(z) = c_\lambda(1 - |z|^2\lambda dv(z)$$

is parameterized by $\lambda \in (-1, \infty)$. The normalizing constant λ is given in (85) below and by dv we denote the usual volume form on $\mathbb{B}^n$. We write $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ for the classical weighted Bergman space, being the closed subspace of $L^2(\mathbb{B}^n, dv_\lambda)$ that consists of all complex-valued analytic functions. The Toeplitz operator T_a with symbol $a \in L_\infty(\mathbb{B}^n)$ acting on $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ is defined as the compression of a multiplication operator on $L^2(\mathbb{B}^n, dv_\lambda)$ on to the Bergman space, i.e., $T_a f = B_\lambda(af)$, where B_λ is the Bergman (orthogonal) projection of $L^2(\mathbb{B}^n, dv_\lambda)$ on to $\mathcal{A}^2_\lambda(\mathbb{B}^n)$.

For a generic subclass $S \subset L_\infty(\mathbb{B}^n)$ of symbols the algebra $\mathcal{T}(S)$ generated by Toeplitz operators T_a with $a \in S$ is non-commutative and practically nothing can be said on its properties .However, if $S \subset L_\infty(\mathbb{B}^n)$ has a more specific structure (e.g. induced by the geometry of $\mathbb{B}^n$, or with a certain smoothness properties) the study of operator algebras $\mathcal{T}(S)$ is quite important and has attracted lots of interest during the last decades. The reason of such an interest is caused, in particular, by the fact that such algebras provide rather simple but tractable examples of non-commutative algebras and play an important role in the applications. At the turn of this century it was observed (see [192] that, unexpectedly and contrary to the case of Toeplitz operators acting on the Hardy space over the circle, there exist many non-trivial algebras $\mathcal{T}(S)$ that are commutative on each standard weighted Bergman space. The detailed structural analysis of such commutative algebras became then an important task sit provides an explicit information on many essential properties of Toeplitz operators such as compactness, boundedness, invariant subspaces, spectral properties, etc.

We continue a project on the classification and structural analysis of commutative Banach algebras that are generated by Toeplitz operators with a specific class of the so-called quasi-homogeneous symbols acting on the weighted Bergman space $\mathcal{A}^2_\lambda(\mathbb{B}^n)$. The classification of these algebras has been given earlier in [193]. Subsequently in [303], and as a model case, we have analyzed the examples example $\mathcal{T}(S)$ of such type. More precisely, $\mathcal{T}(S)$ is the unique commutative Banach algebra in the above classification geverated by Toeplitz operators over the two- dimensional unit ball $\mathbb{B}^2$. As it turned out $\mathcal{T}(\lambda)$ is generated, in fact, by all Toeplitz operator with bounded measurable complex-valued radial symbols a on$\mathbb{B}^2$ (i.e. $a(z) = a(|(z)|)$togather with a single Toeplitz operator T_ϕ having a certain quasi-homogeneous

symbol ϕ. Among other results we explicitly described the space of maximal ideals of $\mathcal{T}(\lambda)$, results we were able to prove the inverse closeness of $\mathcal{T}(\lambda)$ and state some spectral properties of the elements in $\mathcal{T}(\lambda)$.
Our next plan is to extend the results in [303] to the case of all commutative Toeplitz Banach algebras in [193] that are induced by the quasi-elliptic group of biholomorphisms of the unit ball $\mathbb{B}^n$. In [193] these algebras have been described in terms of their generators. Thus developing the Gelf and theory for these algebras will permit us to obtain more detailed (and, in particular, spectral) information on their elements. As it turned out, the algebra $\mathcal{T}(\lambda)$. studied in [303] indeed was the simplest in many respects.
Let us single out some of the principal difficulties that bring the general multi-dimensional case studied here compared to the analysis of $\mathcal{T}(\lambda)$. We mention first that in all cases the algebra under study is generated by two mutually commuting commutative subalgebras: the C^*-algebra generated by Toeplitz operators with bounded (quasi-) radial symbols and the Banach algebra generated by Toeplitz operators with quasi-homogeneous symbols. In the case of [303] this last Banach algebra was generated by a single operator T_ϕ with the simplest quasi-homogeneous symbol ϕ.
In the general case we can as well essentially reduce the number of generators having quasi-homogeneous symbols. However, still a finite number $N \geq 1$ of them remains. Due to this reduction some (bounded) Toeplitz operators with unbounded quasi-radial symbols come into play, and therefore immediately: realize whether these Toeplitz operators with unbounded symbols belong or do not belong to the C^*-algebra generated by Toeplitz operators with bounded quasi-radial symbols. Fortunately we manage to prove that the answer is positive, and so we do not need any further extension of the generator set.

In order to describe the compact set of maximal ideals of the Banach algebra generated by Toeplitz operators with quasi-homogeneous symbols we need to calculate the joint spectrum of its generators (not just the spectrum of the unique generator, as in [303]). In addition this task be-comes more difficult as, contrary to the case of [303], the quasi-homogeneous functions in general do not extend continuously to the boundary $\partial\mathbb{B}^n$ of the unit ball. As a consequence our previous approach in [303] does not apply anymore in its full power. Finally we mention that C^*-algebra generated by Toeplitz operators with bounded quasi-radial symbols has a more complicated structure and involves a new class of slowly oscillating sequences which is defined.

We recall the construction in [193] of a class of commutative Banach algebras $\mathcal{B}_k(h)$ that are generated by Toeplitz operators on the weighted Bergman space $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ and have k $-$quasi-radialquasi-homogeneous symbols. These algebras are indexed by pair (k, h) of multi-indexes that fulfill certain relations. We point out that this class of Banach algebras is subordinated to the group of quasi-elliptic automorphisms of the unit ball since the k-radial part of the symbols of the generating operator is invariant under the action of this quasi-elliptic group, defined by the n-torus action on the ball $\mathbb{B}^n$ (see [245,192]).

We devoted to the analysis of the subalgebra $\mathcal{T}_\lambda(L^\infty_{k-qr})$ of $\mathcal{B}_k(h)$, which is generated byToeplitz operators withbounded k-quasi-radial symbols .All elements $T \in \mathcal{T}_\lambda(L^\infty_{k-qr})$ are operators that are diagonal with respect to the standard monomial basis in $\mathcal{A}^2_\lambda(\mathbb{B}^n)$). Therefore

we can identify T with its eigenvalue sequence and interpret $\mathcal{T}_\lambda(L^\infty_{k-qr})$ as a C*-subalgebra of all bounded complex-valued sequences on $\mathbb{Z}_+^m$ (here $m \in \mathbb{N}$ is the length of the multi-indexk). As for bounded Toeplitz operators with k-quasi-radial symbols we give an explicit integral form for their eigenvalues, and show that the eigenvalue sequences for all operators from $\mathcal{T}_\lambda(L^\infty_{k-qr})$ slowly oscillate in a specific sense.Quite a delicate task in this section is the proof that some Toeplitz operators with unbounded quasi-radial symbols belong to the algebra $\mathcal{T}_\lambda(L^\infty_{k-qr})$. We prove as well that $\mathcal{T}_\lambda(L^\infty_{k-qr})$ contains the orthogonal projections onto certain types of closed subspaces of $\mathcal{A}^2_\lambda(\mathbb{B}^n)$. These two assertions are crucial for reducing the set of generators of $\mathcal{B}_k(h)$in Theorem (6.3.20). With an analysis of the maximal ideal space $M(\mathcal{T}_\lambda(L^\infty_{k-qr}))$ of the algebra $\mathcal{T}_\lambda(L^\infty_{k-qr})$. In particular, we give a useful stratification of $M(\mathcal{T}_\lambda(L^\infty_{k-qr}))$in Lemma (6.3.15).This stratification will be important in the description of the maximal ideals of the full algebra $\mathcal{B}_k(h)$.
We define a second subalgebra $\mathcal{T}_\lambda(\varepsilon_k(h))$ of $\mathcal{B}_k(h)$, which is generated by a finite set of Toeplitz operators with so-called elementary k-quasi-homogeneous symbols. Contrary to the case, the generators of $\mathcal{T}_\lambda(\varepsilon_k(h))$ are not anymore diagonal with respect to the standard monomial basis and their action on the elements of this basis is independent of the weight parameter $\lambda > -1$. We show that the union of generators of both above sub algebras gives in fact a reduced set of generators for $\mathcal{B}_k(h)$.
As was already mentioned, a special care has to be taken since the elementary k-quasi-homogeneous functions do not admit continuous extensions to the boundary $\partial\mathbb{B}^n$, unless $m = 1$. The main result is description of the maximal ideal space of $\mathcal{T}_\lambda(\varepsilon_k(h))$ and the corresponding Gelf and transform. First we treat the case where $m = 1$ and in the final part we generalize the result to the case $m > 1$ by using an appropriate tensor product description.
We list several open problems closely related to the results.Finally we wish to remark that in the case of dimension $n \geq 3$ adscription of the maximal ideal space of the full commutative algebra $\mathcal{B}_k(h)$, the Gelf and map, a characterization of the radical, and a spectral analysis of its elements can be achieved.
Consider the open complex unit ball $\mathbb{B}^n := \{z \in \mathbb{C}^n : |z|^2 = |z_1|^2 + \cdots + |z_n|^2 < 1\}$ in $\mathbb{C}^n$ equipped with the standard weighted measure.

$$dv_\lambda(z) = c_\lambda(1 - |z|^2)^\lambda \, dv(z),$$

Where $\lambda > -1$ is fixed. Here we write dv for the usual volume form on Bn. Recall that due to the assumption $\lambda > -1$ the measure $v_\lambda(\mathbb{B}^n)$ of the unit ball is finite and we chose $c_\lambda > 0$ such that $v_\lambda(\mathbb{B}^n) = 1$. In fact this is realized by defining

$$c_\lambda := \frac{\Gamma\,(n + \lambda + 1)}{\pi^n \Gamma\,(\lambda + 1)} \quad . \tag{85}$$

We write $L^2(\mathbb{B}^n, dv_\lambda)$ for the Hilbert space of all squareintegrable functions with respect to v_λ. The corresponding norm and inner product are denoted by$\|\cdot\|_\lambda$ and $\langle\cdot,\cdot\rangle_\lambda$, respectively.
The weighted Bergman space $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ over $\mathbb{B}^n$ consists of all complex-valued analytic functions that are squareintegrable with respect to the measured v_λ. It is a standard fact that $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ is a closed sub space of $L^2(\mathbb{B}^n, dv_\lambda)$ and that the orthogonal projection (Bergman projection) B_λ from $L^2(\mathbb{B}^n, dv_\lambda)$ on to $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ has the following explicit form

$$[B_\lambda\varphi](z) = \int_{\mathbb{B}^n} \frac{\varphi(\omega)}{(1 - \langle z, w\rangle)^{n+\lambda+1}} dv\lambda(\omega), \text{where } \varphi \in L^2(\mathbb{B}^n, dv_\lambda).$$

In the following we write $\langle z, \omega\rangle := z_1\overline{\omega}_1 + \cdots + z_n\overline{\omega}_n$ for the usual Euclidean inner product. Let $\mathbb{Z}_+$ be the set of all non-negative integers. Recall as well that the standard orthonormal basis $[e_\alpha : \alpha \in \mathbb{Z}_+^n]$ of $\mathcal{A}_\lambda^2(\mathbb{B}^n)$is given by the monomials

$$e_\alpha(z) := \sqrt{\frac{\Gamma\ (n\ +\ |\alpha| +\ \lambda\ + 1)}{\alpha!\, \Gamma\ (n\ + \lambda +\ 1)}}\, z^\alpha. \tag{86}$$

Although the functions e_α depend on the particular choice of the weight λ we do not indicate this dependence in our notation and we assume that $\lambda \in (-1, \infty)$ is arbitrary but fixed.
Given $g \in L_\infty(\mathbb{B}^n)$, the Toeplitz operator$T_g$ with symbol g on $\mathcal{A}_\lambda^2(\mathbb{B}^n)$ is defined by:

$$[T_g f\ ](z) := [B_\lambda(gf)](z) = \int_{\mathbb{B}^n} \frac{g(w)f(\omega)}{(1 - \langle\ z, \omega\rangle)^{n+\lambda+1}} dv_\lambda(\omega).$$

Recall that the algebra generated by the set of all Toeplitz operators with bounded measurable symbols is non-commutative. However, after a restriction of the symbol class to a certain type of functions it turns out that the induced Toeplitz algebra becomes commutative. In sense, commutative Banach algebras generated by Toeplitz operators of such type are subordinate to (some of) the maximal abeliansub groups of the automorphism group $Aut\ (\mathbb{B}^n)$ of $\mathbb{B}^n$. The classification of commutative algebras interms of the generating symbols has been given in [37,38,193,195]. Here we are interested in a class of commutative Toeplitz Banach algebras that is induced by the quasi-elliptic group of biholomorphisms of $\mathbb{B}^n$, cf.[193].For completeness were call some notation and results from [193]:
Let $m \in \{1, \ldots, n\}$, and fix a tuple $k = (k_1, \ldots, k_m) \in \mathbb{Z}_+^m$ with $|k| = k_1 + \cdots + k_m = n$. Then we can interpret $\mathbb{C}^n$ as a product space $\mathbb{C}^n = \mathbb{C}^{k_1} \times \mathbb{C}^{k_2} \times \cdots \times \mathbb{C}^{k_m}$, and we use the notation

$$z = \left(z_{(1)}, \ldots, z_{(m)}\right) \in \mathbb{C}^n, \qquad \text{where } z_{(j\,)} = (z_{j,1}, \ldots, z_j, k_j\,) \in \mathbb{C}^{k_j}\,.$$

We will frequently employ polar coordinates. Let us write $S^{2k_j-1}1 \subset \mathbb{C}^{k_j}$ for the (real) $(2k_j - 1)$-dimensional units herein $\mathbb{C}^{k_j}$. We express non-zero vectors $z_{(j)} \in \mathbb{C}^{k_j}$ in the form$z_{(j)} = r_j\xi_{(j)}$,where

$$\xi_{(j)} = \frac{z_{(j)}}{|z_{(j)}|} \in S^{2k_{j-1}} \text{ and } r_j = |z_{(j)}| \in \mathbb{R}_+.$$

Let us recall now the notion of k-quasi-homogeneous functions on the unit ball $\mathbb{B}^n$ in [303,193]:

Definition (6.3.1)[302]: Fix $(p, q) \in \mathbb{Z}_+^n \times \mathbb{Z}_+^n$ with $p \perp q$(i.e. $\langle p, q\rangle = 0$). Abounded function $\varphi(z)$on $\mathbb{B}^n$ is called “k-quasi-radialquasi-homogeneous” with the quasi-homogeneous degree (p, q) if it has the form

$$\varphi(z) = a(r_1, \ldots, r_m)\xi_{(1)}^{p_{(1)}} \xi_{(2)}^{p_{(2)}} \ldots \xi_{(m)}^{p_{(m)}}\, \xi_{(1)}^{q_{(1)}}\, \xi_{(2)}^{q_{(2)}} \ldots \xi_{\ (m)}^{\ q_{(m)}} \tag{87}$$

and $a = a(r_1, \ldots, r_m)$ is a function of them non-negative real variables $r_1, \ldots, r_m$. A function that can be expressed in the form $a = a(r_1, \ldots, r_m)$ is called “k-quasi-radial”.

In what follows we denote by L^{∞}_{k-qr} the Banach space of all bounded measurable k-quasi-radial functions on $\mathbb{B}^{n}$.
In order to define a class of c commutative Toepliz Banach algebras we choose a second tuple $h \in \mathbb{Z}^{n}_{+}$ which is subordinate of k in the sense that it fulfills the following conditions: $h_j = 0$ if$k_j = 1$ and $1 \leq h_j \leq k_j - 1$ if $k_j > 1$.
We denote by $\mathcal{R}_k(h)$ the linear space generated by all k-quasi-radialquasi-homogeneous functions (87) which satisfy (i) and (ii):
(i) For j with $k_j > 1$ the tuples $p_{(j)}, q_{(j)} \in \mathbb{Z}^{k_j}_{+}$ have the form:

$$p_{(j)} = \left(p_j, 1, \dots, p_{j,h_j}, 0, \dots, 0\right) \text{ and } q_{(j)} = (0, \dots, 0, q_{j,h_{j+1}}, \dots, q_{j,k_j}) \quad (88)$$

and, in addition, $|p_{(j)}| = |q_{(j)}|$.
(ii)If $k_{j'} = k_{j''}$ with $j' < j''$, then$h_{j'} \leq h_{j''}$.
For a given set $\mathcal{F}$ of bounded measurable complex-valued functions on $\mathbb{B}^{n}$ we denote by $\mathcal{T}_\lambda(F)$ the unital Banach algebra in $\mathcal{L}(A^2_\lambda(\mathbb{B}^n))$ which is generated by all Toeplitz operators with symbols in $\mathcal{F}$. Note that $\mathcal{T}_\lambda(F)$ has the structure of a C^*-algebra in the case when $\mathcal{F} = \bar{\mathcal{F}}$, i.e. $\mathcal{F}$ is invariant under complex conjugation.[22] states:
Theorem (6.3.2)[302]: The Banach subalgebra $\mathcal{B}_k(h) := \mathcal{T}_\lambda(\mathcal{R}_k(h))$ of $\mathcal{L}(A^2_\lambda(\mathbb{B}^n))$ which is generated by Toeplitz operators with symbols from $\mathcal{R}_k(h)$ is commutative for all $\lambda > -1$.
We study separately two commutative subalgebras of $\mathcal{B}_k(h)$which together generate $\mathcal{B}_k(h)$The first one has the structure of a commutativeC^* $-$algebra and is generated by Toeplitz operators with k $-$quasi-radial symbols. The second one is a commutative Banach algebra generated by a finite number of Toeplitz operators with certain quasi-homogeneous symbols. The analysis of these subalgebras serves as an important tool for studying the complete Banach algebra structure of $\mathcal{B}_k(h)$ in an upcoming work.
As is shown [192], a Toeplitz operator with a bounded measurable k-quasi-radial symbol $a = a(r_1, \dots, r_m)$ has the monomialsz^α (or the normalizedmonomials $e_\alpha(z)$ (66)), where $\alpha \in \mathbb{Z}^n_+$, as eigenfunctions. In what follows, for each multi-index$\alpha = (\alpha_1, \dots, \alpha_n) = (\alpha_{(1)}, \dots, \alpha_{(m)}) \in \mathbb{Z}^n_+$,
we denote by $\kappa = \kappa(\alpha) = (\kappa_1, \dots, \kappa_m) \in \mathbb{Z}^m_+$ the multi-index with the entries $\kappa_j = |\alpha_{(j)}|$, for all $j = 1, \dots, m$. In particular, $|\alpha| = |\kappa|$ and the eigenvalue $\gamma_{a,k,\lambda}(\kappa)$ of T_a with respect to z^α depends only on $\kappa = \kappa(\alpha)$, i.e.,
$T_a z^\alpha = \gamma_{a,k,\lambda}(\kappa) z^\alpha$. (89)
Moreover, we have the following explicit expression of $\gamma_{a,k,\lambda}(\kappa)$

$$\gamma_{a,k,\lambda}(\kappa) = \frac{2^m \Gamma(n + |\kappa| + \lambda + 1)}{\Gamma(\lambda + 1) \prod_{j=1}^{m}\left(k_j - 1 + k_j\right)!} \int_{\tau(\mathbb{B}^m)} a(r)\,(1 - |r|^2)\lambda \prod_{j=1}^{m} r_j^{2k_j - 1} dr \quad (90)$$

The integration is taken over the base $\tau(\mathbb{B}^m) = \{r = (r_1, \dots, r_m) \in \mathbb{R}^m_+ : 0 \leq |r| < 1\}$of the unit ball $\mathbb{B}^m$ considered as a Reinhardt domain.

Corollary (6.3.3)[302]: Let $\alpha \in \mathbb{Z}_+^n$, and $\kappa = (|\alpha_{(1)}|, \dots, |\alpha_{(m)}|) \in \mathbb{Z}_+^m$as above. Then we have

$$\int_{\tau(\mathbb{B}^m)} a(r)\,(1-|r|^2)\lambda \prod_{j=1}^{m} r_j^{2k_j+2k_j-1} dr_j = \frac{\Gamma(\lambda+1)\prod_{j=1}^{m}\left(k_j - 1 + k_j\right)!}{2^m \Gamma(n+|\kappa|+\lambda+1)} \tag{91}$$

Proof: Put $a(r) \equiv 1$, then T_a is the identity operator. Thus$\gamma_{a,k,\lambda}(\kappa) = 1$ for all $\kappa \in \mathbb{Z}_+^m$, and the assertion follows from the expression (90) of $\gamma_{a,k,\lambda}(\kappa)$.

Let $\gamma = \{\gamma(\kappa)\}_{\kappa \in \mathbb{Z}_+^m}$bean arbitrary bounded sequence of complex numbers. Denoteby D_γ the diagonal operator which acts on the weighted Bergman space $\mathcal{A}_\lambda^2(\mathbb{B}^n)$ by the rule

$$D_\gamma e_\alpha(z) = \gamma\,(\kappa) e_\alpha(z), \alpha \in \mathbb{Z}_+^n. \tag{92}$$

According to the above remarks each Toeplitz operator with bounded measurable k-quasi-radial symbol $a(r_1, \dots, r_m)$ is diagonal, and $T_a = D\gamma_{a,k,\lambda}$. However, as we will show later on, not all bounded diagonal operators D_γ can be represented in such a form since the eigenvalue sequence $\gamma_{a,k,\lambda}$ of T_a possesses certain specific features (cf.Lemma (6.3.6)). Now consider a particular case of k-quasi-radial symbols:

$$\begin{aligned}\gamma_{r_1^2,k,\lambda}(\kappa) &= \frac{2^m\Gamma(n+|\kappa|+\lambda+1)}{\Gamma(\lambda+1)\prod_{j=1}^{m}\left(k_j - 1 + k_j\right)!} \int_{\tau(\mathbb{B}^m)} a(r)\,(1-|r|^2)\lambda \prod_{j=1}^{m} r_j^{2\left(k_j+\delta_{j,1}\right)+2k_j-1} dr \\ &= \frac{2^m\Gamma(n+|\kappa|+\lambda+1)}{\Gamma(\lambda+1)\prod_{j=1}^{m}\left(k_j - 1 + k_j\right)!} \cdot \frac{\Gamma(\lambda+1)\prod_{j=1}^{m}\left(k_j - 1 + k_j + \delta_{j,1}\right)!}{2^m\Gamma(n+|\kappa|+\lambda+1)} \\ &= \frac{k_l + \kappa_l}{n + |\kappa| + \lambda + 1},\end{aligned}$$

Where $\delta_{j,1}$ is the standard Kronecker delta.

We mention that neither of the sequences $\{\gamma_{r_1^2,k,\lambda}(k)\}_\kappa$, where $l = 1, \dots, m$, has a limit when$\kappa \to \infty$ (or, which is the same, $|k| \to \infty$), though all of them possess many partial limit values. In particular, given any m-tuple$s = (s_1, \dots, s_m) \in \mathbb{Z}_+^m$, we introduce the subsequence $\{k_s(n)\}_{n\in\mathbb{Z}_+}of\mathbb{Z}_+^m$, where $k_s(n) = (s_1 n, s_2 n, \dots, s_m n)$. Then, for each $l = 1, \dots, m$,

$$\lim_{n\to\infty} \gamma_{r_1^2,k,\lambda}\,(\kappa_s(n)) = \frac{s_l}{|s|}.$$

In general, let $\{k(n)\}_{n\in\mathbb{Z}_+,}$ where $k(n) = (k_1(n), k_2(n), \dots, k_m(n))$be subsequence of $\mathbb{Z}_+^m$. Then the limit of the sequence $\gamma_{r_1^2,k,\lambda}$ along the subsequence $\{k(n)\}_{n\in\mathbb{Z}_+}$ exists if and only if the sequence $\{\frac{k_l(n)}{|k(n)|}\}_{n\in\mathbb{Z}_+}$has a limit, and in the last case both limits coincide.

At the same time, regardless of the way how κ tend so infinity, we always have

$$\lim_{|\kappa|\to\infty} \gamma_{r_1^2,k,\lambda}(k) + \cdots + \gamma_{r_m^2,k,\lambda}(k) = 1.$$

We denote by Δ^{m-1} the standard $(m-1)$- dimensional simplex with the vertices $(1,0,\dots,0), (0,1,0,\dots,0), \dots, (0,\dots,0,1) \in \mathbb{R}_+^m$. Summarizing the above results to:

Lemma (6.3.4)[302]: For any subsequence $\{k(n)\}_{n\in\mathbb{Z}_+}$ of $\mathbb{Z}_+^m$, all limits

$$\lim_{|\kappa(n)|\to\infty} \gamma_{r_l^2,k,\lambda}(k(n)) = \beta_l, l = 1,\ldots,m,$$

Exist if and only if $(\beta_1,\ldots,\beta_m) \in \Delta^{m-1}$ and

$$\lim_{|\kappa(n)|\to\infty} \frac{k_l(n)}{|k(n)|} = \beta_l, l = 1,\ldots,m.$$

Moreover, for each point $(\beta_1,\ldots,\beta_m) \in \Delta^{m-1}$, there is a subsequence $\{k(n)\}_{n\in\mathbb{Z}_+}$ with the above properties.

In order to make our considerations geometrically more transparent we proceed as follows. Denote by $\mathbb{R}_1^m = (\mathbb{R}^m, \|\cdot\|_1)$ the Banach space $\mathbb{R}^m$ equipped with the norm $\|x\|_1 = \sum_{j=1}^m |x_j|$, where $x = (x_1,\ldots,x_m) \in \mathbb{R}^m$. Let $S_1^{m-1}(0,R)$ be the sphere in $\mathbb{R}_1^m$ of radius R centered at the origin. We interpret $\mathbb{Z}_+^m = \mathbb{Z}_+^m \cap \mathbb{Z}_1^m$ as a metric space with the metric $\rho_1(\kappa',\kappa'') = \|k'-k''\|_1$ in herited from $\mathbb{R}_1^m$. Then $|k| = \rho_1(0,\kappa)$, and $\Delta^{m-1} = S_1^{m-1}(0,1) \cap \mathbb{R}_+^m$ is nothing but the corresponding part of the unit sphere $S_1^{m-1}(0,1)$.

We denote by $\overline{\mathbb{R}}_1^m$ the compactification of $\mathbb{R}_1^m$ by the "infinitely far" sphere $S_1^{m-1}(0,\infty)$ consisting of rays through the origin. Each such ray and its intersection with the unit sphere $S_1^{m-1}(0,1)$ are identified. This yields a parametrization of the points of the "infinitely far" sphere $S_1^{m-1}(0,\infty)$ by elements of the unit sphere $S_1^{m-1}(0,1)$, identifying these two objects.

Let now $\overline{\mathbb{Z}}_+^m$ be the closure of $\mathbb{Z}_+^m$ in $\overline{\mathbb{R}}_1^m$, being the compactification of $\mathbb{Z}_+^m$ by the corresponding part $\Delta^{m-1}(\infty)$ of the "infinitely far" sphere $S_1^{m-1}(0,\infty)$. That is $\overline{\mathbb{Z}}_+^m = \mathbb{Z}_+^m \cup \Delta^{m-1}(\infty)$, where we also parameterize the points of $\Delta^{m-1}(\infty)$ by the points of Δ^{m-1}, identifying them.

Lemma (6.3.4) states that each sequence $\gamma_{r_l^2,k,\lambda}, l = 1,\ldots,m$, admits continuous extension to $\Delta^{m-1}(\infty)$.

We denote by $c(\overline{\mathbb{Z}}_+^m)$ the unital C*-algebra which consists of all bounded sequences $\{\gamma(k)\}_{\kappa\in\mathbb{Z}_+^m}$ that admit a continuous extension to $\Delta^{m-1}(\infty)$.

The elements of them- tuple $(\gamma_{r_1^2,k,\lambda},\ldots,\gamma_{r_m^2,k,\lambda})$ are real-valued sequences and separate the points of $\mathbb{Z}_+^m$. Then, by the above discussion and the Stone–Weierstrass theorem, the unital C*-algebra $\mathcal{T}_\lambda(\{r_1^2,r_2^2,\ldots,r_m^2\})$ coincides with the algebra of all diagonal operators D_γ with $\gamma \in c(\overline{\mathbb{Z}}_+^m)$. Furthermore, its maximal ideal space coincides with $\mathbb{Z}_+^m$.

For each $k = (k_1,\ldots,k_m) \in \mathbb{Z}_+^m$ introduce the finite dimensional space H_k defined by $H_k \coloneqq span\{e_\alpha : \alpha \in \mathbb{Z}_+^n, |\alpha_{(j)}| = k_j, j = 1,\ldots,m\}$(93)

And let

$$P_k : A_\lambda^2(\mathbb{B}^n) \to H_k \tag{94}$$

Be the orthogonal projection onto H_k. Then we have:

Corollary (6.3.5)[302]: Let $\gamma \in c(\overline{\mathbb{Z}}_+^m)$, then $D\gamma \in \mathcal{T}_\lambda(L_{k-qr}^\infty)$. In particular, for all $\kappa \in \mathbb{Z}_+^m$:

$$P_k \in \mathcal{T}_\lambda(\{r_1^2,r_2^2,\ldots,r_m^2\}) \subset \mathcal{T}_\lambda(L_{k-qr}^\infty \subset B_k(h).$$

For each $j = 1,2,\ldots,m$, introducethestandardunit vector $\boldsymbol{e}_j := (0,\ldots,0,j\downarrow 1,0,\ldots,0) \in \mathbb{Z}_+^m$, being also the *j-th* vertex of Δ^{m-1}. We denote by $d_1(m)$ these to fall bounded sequences $\gamma = \{\gamma(\kappa)\}_{\kappa\in\mathbb{Z}_+^m}$ that satisfy the following condition

$$\sup_{\kappa\neq 0}|k|\left|\gamma\,(k)-\sum_{s=1}^{m}\frac{\kappa s}{|\kappa|}\;\gamma\,(k+\boldsymbol{e}_s\,)\right|<+\infty. \tag{95}$$

Note that the point $(\frac{\kappa_1}{|k|},\dots,\frac{\kappa_m}{|k|})\in\Delta^{m-1}$corresponds to the point of $\Delta^{m-1}(\infty)$defined by the ray passing through $\kappa\in\mathbb{Z}_+^m$.

Lemma (6.3.6)[302]: For each $a=a(r_1,\dots,r_m)\in L_\infty(\tau(\mathbb{B}^m))$,the eigenvalues equence $\gamma_{a,k,\lambda}$of the Toeplitz operator T_a belongs to $d_1(m)$.

Proof: For all$k\neq 0$ we have by (90)

$$\gamma_{a,k,\lambda}=\frac{2^m\Gamma(n+|\kappa|+\lambda+1)}{\Gamma(\lambda+1)\prod_{j=1}^{m}\left(k_j-1+k_j\right)!}\int_{\tau(\mathbb{B}^m)}a(r)\,(1-|r|^2)^{\lambda+1}\prod_{j=1}^{m}r_j^{2k_j-2k_j-1}dr$$

$$=\frac{2^m\Gamma(`|\kappa|+\lambda+2)}{\Gamma(\lambda+1)\prod_{j=1}^{m}\left(k_j-1+k_j\right)!}\left[\int_{\tau(\mathbb{B}^m)}a(r)\,(1-|r|^2)^{\lambda}\prod_{j=1}^{m}r_j^{2k_j-2k_j-1}dr\right.$$

$$\left.-\sum_{s=1}^{m}\int_{\tau(\mathbb{B}^m)}a(r)\,(1-|r|^2)^{\lambda}\prod_{j=1}^{m}r_j^{2k_j-2k_j-1+2\delta_{j,s}}dr\right]$$

$$=\frac{n+|\kappa|+\lambda+1}{\lambda+1}\gamma_{a,k,\lambda}(k)-\sum_{s=1}^{m}\frac{k_s+\kappa_s}{\lambda+1}\gamma_{a,k,\lambda}(k+\boldsymbol{e}_s)$$

Or,

$$N(a,\lambda,k):=(\lambda+1)\left[\gamma_{a,k,\lambda+1}(k)-\gamma_{a,k,\lambda}(k)\right]+\sum_{s=1}^{m}k_s\gamma_{a,k,\lambda}(k+\boldsymbol{e}_s)-n\gamma_{a,k,\lambda}(k)$$

$$=|k|\left[\gamma a,k,\lambda(\kappa)-\sum_{s=1}^{m}\frac{ks}{|k|}\gamma_{a,k,\lambda}(k+\boldsymbol{e}_s)\right].$$

But

$$|N(a,\lambda,k)|\leq 2(n+\lambda+1)\|a\|_\infty,$$

And the result follows.

Some important comments to the definition of $d_1(m)$ and Lemma (6.3.6) have to be added. First of all we mention that the condition (95) pacifies the form in which the sequences $\gamma\in d_1(m)$ may oscillate at infinity. In particular, (95) implies that

$$\lim_{k\to\infty}\left[\gamma_{a,k,\lambda}(k)-\sum_{s=1}^{m}\frac{ks}{|k|}\gamma_{a,k,\lambda}(k+\boldsymbol{e}_s)\right]=0.$$

Then, for $m=1$, the class $d_1(1)$ coincides with the class which in the work of Suárez [290] was denoted by d_1 and is commonly used in Tauberian theory. For each $m\neq 1, d_1(m)$ Is just a (non-closed) linear space, and only for $m=1$ it is an algebra

Further, for $m=1$,i.e., in the case of radial symbols, and the un weighted Bergman spaces, it is known [271] (see as well [76,290] for the one-dimensional case $n=1$) that the set of Toeplitz operators with bounded radial symbolsis dense in the C*-algebra generated by these operators, and that the l_∞-closure of the set of corresponding eigenvalue sequences γ_a coincides with the l_∞- closure of d_1. Moreover, by [271] this closure coincides with the C*-algebra $SO_1(\mathbb{Z}_+)$ of all slowly oscillating sequences introduced by Schmidt [246], i.e., of all bounded sequences $\gamma = \gamma(p)_{p=0}^{\infty}$such that

$$\lim_{\frac{p+1}{q+1}\to 1} |\gamma\,(p) - \gamma\,(q)| = 0.$$

This gives the exact characterization of the algebra

According to [194] the set $\mathrm{SO}_1(\mathbb{Z}_+)$ is apropersubset of the standard $\mathrm{SO}(\mathbb{Z}_+)$ Recall in this connection (see, for example, [293]) that the algebra $\mathrm{SO}(\mathbb{Z}_+^{\mathrm{m}})$ consists of all sequence $\gamma = \{\gamma(k)\}_{\kappa\in\mathbb{Z}_+^m}$ such that

$$\lim_{\kappa\to\infty}(\gamma\,(k)\ -\gamma\,(k+\rho))\ =\ 0\ for\ all\ \rho\ \in\ \mathbb{Z}_+^m.$$

For $m = 1$and anarbitrary weight parameter $\lambda \in (-1,\infty)$ Lemma (6.3.6) just ensures that the algebra $\mathcal{T}_\lambda(\mathrm{L}_{\mathrm{rad}}^\infty)$. is isomorphic to a subalgebra of $SO_1(\mathbb{Z}_+)$.

It is clear that for $m > 1$ the C*-algebra $\mathcal{T}_\lambda(L_{k-qr}^\infty)$ is isomorphic to a certain subalgebra of the C*-algebra generated by sequences in $d_1(m)$.But the exact description of this subalgebra is unknown. As partial information we mention that the algebra $\mathcal{T}_\lambda(L_{k-qr}^\infty)$ intersects $SO(\mathbb{Z}_+^m)$, for example, by diagonal operators whose eigenvalue sequences have a limit as $k = (k_1,\dots,k_m) \to \infty$. At the sametime, contrary to the case $m = 1, \mathcal{T}_\lambda(L_{k-qr}^\infty)$is not a subalgebra of $SO(\mathbb{Z}_+^m)$,, as will be shown in Lemma (6.3.13).

It is unclear whether the set of all Toeplitz operators with bounded k-quasi-radial symbols is dense in $\mathcal{T}_\lambda(L_{k-qr}^\infty)$, as it is the case for $m = 1$ and $\lambda = 0$. The fact that $d_1(m)$(contrarytod1) is not an algebra suggests that the answer might be negative.

We list the above open questions among others.Then extcorollaries to Lemma (6.3.6) give a further characterization of the eigenvalue sequences $\gamma_{a,k,\lambda}$ of Toeplitz operators $T_a \in \mathcal{T}_\lambda(L_{k-qr}^\infty)$.

Corollary (6.3.7)[302]: We have that

$$\gamma_{a,k,\lambda}(\kappa)\ - \sum_{s=1}^{m}\frac{\kappa s}{|\kappa|}\gamma_{a,k,\lambda}(\kappa\ + \boldsymbol{e}_s) =\ \mathcal{O}(1/|\kappa|)\ \ as\ \kappa\to\infty.$$

For each $j = 1,\dots,m$ let us fix the values of κ_S for all $s \neq j$arbitrarily, and define the "$j-th$ coordinate" sequence $\hat{\kappa}_j = \{\hat{\kappa}_j(n)\}_{n\in\mathbb{Z}_+}$, where $\hat{\kappa}_j(n) = (\kappa_1(n),\dots,\kappa_m(n))$ has the entries $\hat{\kappa}_j(n) := n$ and $\kappa_s(n) := \kappa_s$. By considering $\boldsymbol{e}_j$ as the j-th vertex of $\Delta^{m-1}(\infty)$ it is clear that

$$\lim_{n\to\infty} \hat{\kappa}j(n) = e_j \in \Delta^{m-1}(\infty).$$

Corollary (6.3.8)[302]: For each$j = 1,\dots,m$ the sequence $\{\gamma_{a,k,\lambda}(\hat{\kappa}_j(n))\}_{n\in\mathbb{Z}_+}$ be longs to d_1, and thus to $SO_1(\mathbb{Z}_+)$.

Proof: Follows from Lemma (6.3.6) together with the observation that

$$\lim_{n\to\infty}\frac{\hat{\kappa}_j\,(n)}{|\hat{\kappa}_j\,(n)|} = 1,$$

And that the sum $\sum_{s\neq j}|\kappa_s(n)|$ does not depend on n and is bounded.

The nextimportant particular case of k-quasi-radial symbols is as follows: for each $j = 1,\dots,m$, we introduce the unbounded k-quasi-radial function

$$a^{(j)}(z) := \frac{1 - |z|^2}{|z(j)|^2} = \frac{1 - |r|^2}{r_j^2} \in R_+ \cup \{+\infty\}.$$

Although this symbol is unbounded, it generates a bounded Toeplitz operator. Moreover, Corollary (6.3.3) yields the explicit formula for its eigenvalue sequence (see the notation in (89)):

$$\gamma a^{(j)}, k, \lambda\big(\kappa(\alpha)\big) = \frac{\lambda + 1}{\kappa_j + k_j - 1} \ , \alpha \in \mathbb{Z}_+^n.$$

In particular, since$k_j > 1$ we have that $a^{(j)} \in L_1(\mathbb{B}^n, dv_\lambda)$ for $j = 1,\dots,m$ with norm

$$\left\|a^{(j)}\right\|_{L_1(\mathbb{B}^n dv_\lambda)} = \langle a^{(j)}, 1\rangle_\lambda = \langle T_{a^{(j)}} 1, 1\rangle_\lambda = \gamma_{a^{(j)},k,\lambda}(0) = \frac{\lambda + 1}{k_j - 1}.$$

The last formula implies an interesting characterization of the first (i.e., for $\kappa = 0$) eigenvalue of a Toeplitz operator with quasi-radial symbol.

Lemma (6.3.9)[302]: Let $a(r) \in L_1(\mathbb{B}^n, dv_\lambda)$ be aquasi-radial symbol. Then the first eigenvalue (ground state) of a bounded, or densely defined unbounded Toeplitz operator T_a is given by

$$\gamma_{a,k,\lambda}(0) = \int_{\mathbb{B}^n} a(r)\, dv_\lambda.$$

In the case of a non-negative symbol a it coincides with $\|a(r)\|_{L_1(\mathbb{B}^n, dv_\lambda)}$.

Recall that a finite positive measure ν on $\mathbb{B}^n$ is called a Carleson measure with respect to $\mathcal{A}_\lambda^2(\mathbb{B}^n)$ (shortly: Carleson measure) if the Toeplitz operator T_ν with measure symbol ν defined by

$$T_\nu f := \int_{\mathbb{B}^n} \frac{f(\omega)}{(1 - \langle z, \omega\rangle)^{n+1+\lambda}} dv(\omega), \ \ where\ f \in H^\infty(\mathbb{B}^n),$$

has a bounded extension from the space $H^\infty(\mathbb{B}^n)$ of all bounded analytic functions on $\mathbb{B}^n$ to the Bergman space $\mathcal{A}_\lambda^2(\mathbb{B}^n)$ (e.g. see [179], [65]). From the above discussion it follows that $\nu_j := a^{(j)} dv_\lambda$ for $j = 1, \dots, m$ is a Carleson measure.

Let us recall the notion of the (ℓ, λ)-Berezin-transform where$\ell \geq \lambda$ (for the unweighted case $\lambda = 0$ see [148, 65], and for an arbitrary weight $\lambda > -1$ see [179]). If μ is a complex-valued, Bore regular measure on $\mathbb{B}^n$ and $z \in \mathbb{B}^n$ then we set

$$B_\ell(\mu)(z) := \frac{c_\ell}{c_\lambda} \int_{\mathbb{B}^n} \frac{(1 - |\phi_z(\omega)|2)^{n+1+\ell}}{(1 - |\omega|^2)^{n+1+\ell}} d\mu(\omega) \quad . \tag{96}$$

Here the constants c_λ and c_ℓ were defined in (85) and φ_z denotes the (unique up to unitary multiples) automorphism of $\mathbb{B}^n$ with $\varphi_z \circ \varphi_z = id$ and $\varphi_z(0) = z$. When $\mu = a dv_\lambda$ with $a \in L_1(\mathbb{B}^n, dv_\lambda)$ then a change of variables shows that (76) takes the form

$$B_\ell(a\,dv_\lambda)(z) \;=\; \int_{\mathbb{B}^n} a\circ\varphi_z(\omega)dv_\ell(\omega).$$

Note that the right hand side of this equation is independent of $\lambda > -1$. On the other hand, by inserting the well-known relation

$$1\;-|\phi z(\omega)| = \frac{(1\;-\;|z|^2)(1\;-\;|\omega|^2)}{|1\;-\langle\, z,\omega\rangle|^2}$$

Into (96) we obtain the following expression for the $(\ell,\lambda)-$ Berezin-transform, which will be most suitable for the considerations below,

$$B_\ell(a\,dv_\lambda)(z) = (1\;-\;|z|^2)^{n+1+\ell}\int_{\mathbb{B}^n}\frac{a(\omega)}{(1-\langle z.\omega\rangle)^{2(n+1+\ell)}}dv_\lambda(\omega). \qquad (97)$$

The following result has been proved in the unweighted situation $\lambda = 0$ in [66], and in the general weighted case $\lambda > -1$ in [179]:

Proposition (6.3.10)[302]: (See [65,179].) Let the positive measure ν be Carleson with respect to the weighted measured v_λ on $\mathbb{B}^n$ where $\lambda > -1$. Then:

(i)The functions $B_\ell(\nu)$ are bounded and continuous on $\mathbb{B}^n$ for all$\ell \geq \lambda$.

(ii)The convergence $T_{B_\ell}(\nu) \to T_\nu$ holds, as $\ell \to \infty$, in the uniform topology of $\mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{B}^n))$.

Corollary (6.3.11)[302]: The Toeplitz operators $T_{a^{(j)}}$ with L_1-symbol $a^{(j)}$, where $j \in \{1,\ldots,m\}$, belong to the algebra $\mathcal{T}_\lambda(L^\infty_{k-qr})$.

Proof: Since the measures $\nu_j := a^{(j)}dv_\lambda$, for $j= 1,\ldots,m$, are Carleson we conclude from Proposition (6.3.10) that the functions $\nu_j,\ell := B_\ell(\nu_j)$ are bounded and as $\ell \to \infty$ the norm convergence $T_{\nu_{j,\ell}} \to T_{a^{(j)}}$ holds. Hence it remains to show that $\nu_{j,\ell}(z)$ for all j andℓ are k-quasi-radial.

For any given $r \in \mathbb{N}$ let us denote by $U(r)$ the group of unitary $r\times r$-matrices. Then we have the natural embedding $G := U(k_1)\times\cdots\times U(k_m) \subset U(n)$ of groups. It can be easily seen from the expression (97) of the $(\ell,\lambda)-$Berezin transform and the transformation rule of the integral that in the case of a k-quasi-radial symbols $a(w)$ the integral transform $B_\ell(a)(z)$ is invariant under the action of G. Hence $B_\ell(a)(z)$ defines a k-quasi-radial function, as well for all $\ell \geq \lambda$. Since $a^{(j)}$ is k-quasi-radial this observation finishes the proof.

The eigenvalues of the Toeplitz operator $T_{a^{(j)}}$ are real-valued and monotonically decreasing when $\kappa_j = |a^{(j)}|$ tends to infinity. By c we denoe the set of all converging sequences. The Stone– Weierstrass the ore directly leads to the following lemma.

Lemma (6.3.12)[302]: The unital C^*-subalgebra in $\mathcal{T}_\lambda(L^\infty_{k-qr})$ which is generated by the operators $T_{a^{(j)}}$, where $j = 1,\ldots,m$, coincides with the set of all diagonal operators $D_{\gamma^{(j)}}$ whose eigenvalue sequences $\gamma^{(j)} = \{\gamma^{(j)}(\kappa)\}_{\kappa\in\mathbb{Z}^m_+}$ depend on κ_j only; more over being considered with respect to their dependence on κ_j, i.e., $\gamma^{(j)} = \{\gamma^{(j)}(\kappa_j)\}_{\kappa_j\in\mathbb{Z}_+}$, they belong to c.

We mention that the operators $T_{a^{(j)}} \in \mathcal{T}_\lambda(L^\infty_{k-qr})$ where $j = 1, \ldots, m$, commute among themselves. Moreover, they commute with each Toeplitz operator T_φ having a k-quasi-radialquasi-homogeneous symbol $\varphi \in R_k(h)$.
The next lemma shows that $\mathcal{T}_\lambda(L^\infty_{k-qr})$ is not a subalgebra of $\mathrm{SO}(\mathbb{Z}^{\mathrm{m}}_+)$.
Lemma (6.3.13)[302]: For $j = 1, \ldots, m$, the eigenvalue sequence $\gamma_{a^{(j)},k,\lambda}$of the Toeplitz operator $T_{a^{(j)}}$ is not slowly oscillating when $\kappa = (\kappa_1, \ldots, \kappa_m) \to \infty$.
Proof: The assumption that the sequence $\gamma_{a^{(j)},k,\lambda}$ belongs to $SO(\mathbb{Z}^m_+)$. Implies that the set of its partial limits is connected (see, [293]). At the same time this set has the form

$$\left\{\frac{\lambda + 1}{\kappa_j + k_j - 1}\right\}_{\kappa_j \in \mathbb{Z}_+} \cup \{0\}$$

and hence is discrete, which leads to a contradiction.
Let $j \in \{1, \ldots, m\}$ be fixed and for each $d \in \mathbb{Z}_+$ consider the closed subspace:
$H^{(j)}_d := span\{e_\alpha : |\alpha^{(j)}| = \kappa_j = d\} \subset \mathcal{A}^2_\lambda(\mathbb{B}^n)$. (98)
By $Q^{(j)}_d : \mathcal{A}^2_\lambda(\mathbb{B}^n). \to H^{(j)}_d$ we denote the orthogonal projection of $\mathcal{A}^2_\lambda(\mathbb{B}^{\mathrm{n}})$ on to $H^{(j)}_d$. Moreover, given $(a, b) \in \mathbb{R}^2_+$, we define family of diagonal operators

$$D^{(j)}_{a,b} e_\alpha := \frac{a + \kappa_j}{a + b + \kappa_j} e_\alpha, \tag{99}$$

where $\alpha \in \mathbb{Z}^n_+, \kappa_j = |\alpha^{(j)}|$. These operators will appear in (107) below. Then Lemma (6.3.12) yields
Corollary (6.3.14)[302]: The operators $Q^{(j)}_d$ and $D^{(j)}_{a,b}$ belong to the algebra $\mathcal{T}_\lambda(L^\infty_{k-qr})$.
Next we describe a fibration of the compact set $M(\mathcal{T}_\lambda(L^\infty_{k-qr}))$ of maximal ideals of the commutative C^*-algebra $\mathcal{T}_\lambda(L^\infty_{k-qr})$. We identify a maximal ideal with a corresponding multiplicative functional in the standard way. First we introduce some notation:
Let $\theta = (\theta_1, \ldots, \theta_m) \in \{0,1\}^m =: \Theta$, and with $\mathbf{1} = (1,1, \ldots, 1)$ we write $\theta^c = 1 - \theta$ for the "complementary" m-tuple. In particular, we have $\mathbf{1}^c = \mathbf{0} = (0,0, \ldots, 0)$. Using the notation$J_\theta = \{j : \theta_j = 1\}$, we introduce

$$\mathbb{Z}^\theta_+ = \bigoplus_{j \in J_\theta} \mathbb{Z}_+(j) \quad and\ \kappa_\theta = \left\{(\kappa_{j_1}, \ldots, \kappa_{j_{|\theta|}}) : j_p \in J_\theta\right\}.$$

Given $\theta \in \Theta$, let

$$M_\theta = \left\{\mu \in M(\mathcal{T}_\lambda(L^\infty_{k-qr})) : \mu\left(Q^{(j)}_d\right) = \begin{cases} 0 \text{ for all } d \in \mathbb{Z}_+, & if\ \theta_j = 0 \\ 1 \text{ for some } d \in \mathbb{Z}_+, & if\ \theta_j = 1 \end{cases}\right\}.$$

Lemma (6.3.15)[302]: The following decomposition onto mutually disjoint sets holds

$$M(\mathcal{T}_\lambda(L^\infty_{k-qr})) = \bigcup_{\theta \in \Theta} M_\theta .$$

Proof: From the definition of the setsM_θ it follows that the union is disjoint and varying $\theta \in \Theta$ we cover all possible cases.
We note that the set M_1 admits a simple alternative description. For each $\mu \in M_1$ there is a $\kappa \in \mathbb{Z}^{\mathrm{m}}_+$ such that

$$\mu(P_\kappa) = \mu\left(\prod_{j=1}^{m} Q_{\kappa_j}^{(j)}\right) = \prod_{j=1}^{m} \mu\left(Q_{\kappa_j}^{(j)}\right) = 1,$$

Where P_κ is the orthogonal projection defined in (94). Thus, given any operator $D_\gamma = \sum_{\rho\in\mathbb{Z}_+^m}\gamma(\rho)P_\rho \in \mathcal{T}_\lambda(L_{k-qr}^\infty)$, we have

$$\mu(D_\gamma) = \mu(P_\kappa)\mu(D_\gamma) = \mu(P_\kappa D_\gamma) = \gamma\,(\kappa)\mu(P_\kappa) = \gamma\,(\kappa).$$

Identifying the functional μ with $\kappa \in \mathbb{Z}_+^m$, we have that $M_1 = \mathbb{Z}_+^m$. Moreover, each functional $\mu \in M_1$ can be defined by the formula

$$\mu(D) := \langle De_{\alpha_\kappa}, e_{\alpha_\kappa}\rangle_\lambda,$$

Where $\alpha_\kappa := ((\kappa_1, 0, \ldots, 0), \ldots, (\kappa_m, 0, \ldots, 0)) \in \mathbb{Z}_+^{k_1} \times\cdots\times \mathbb{Z}_+^{k_m} = \mathbb{Z}_+^n$.

We note as well that all functional from $M(\mathcal{T}_\lambda(L_{k-qr}^\infty))\backslash M_1$ map compact operators of the algebra $\mathcal{T}_\lambda(L_{k-qr}^\infty)$ to zero.

In order to analyze the sets M_θ, with $\theta \neq 1$, we mention first that for each $\mu \in M_\theta$ there is auniquetuple $\kappa_\theta \in \mathbb{Z}_+^\theta$ such that $\mu(Q_{\kappa_j}^{(j)}) = 1$ for all $j \in J_\theta$. Therefore, we have the following decomposition of M_θ in to disjoint sets

$$M_\theta = \bigcup_{\kappa_\theta\in\mathbb{Z}_+^\theta} M_\theta\,(\kappa_\theta\,),$$

Where $M_\theta(\kappa_\theta) = \{\mu \in M_\theta : \mu(Q_{\kappa_j}^{(j)}) = 1 \text{forall} \quad j \in J_\theta\}$.

Note that,we mention for the completeness that none of the points of $M_\theta(\kappa_\theta)$ can be reached by subsequences; its topological nature requires to use nets (subnets of $\mathbb{Z}_+^{|\theta^c|}$). That is, for each point $\mu \in M_\theta(\kappa_\theta)$ there is a net $\{\kappa_{\theta^c}(\beta)\}_{\beta\in B}$, valued in $\mathbb{Z}_+^{|\theta^c|}$, such that $\kappa = \sigma(\kappa_\theta, \kappa_{\theta^c}(\beta))$ tends to μ in the Gelfand topology of $M(\mathcal{T}_\lambda(L_{k-qr}^\infty))$. Here σ is the permutation of m- tuples such that $(\kappa_\theta, \kappa_{\theta^c}(\beta)) = \sigma - 1(\kappa_1, \ldots, \kappa_m)$. In other words, for each $\gamma = \{\gamma(\kappa)\}_{\kappa\in\mathbb{Z}_+^m} \in \mathcal{T}_\lambda(L_{k-qr}^\infty)$, we have that

$$\lim_{\beta\in B} \gamma\,(\kappa) = \gamma\,(\mu), \tag{100}$$

Where $\kappa = \sigma(\kappa_\theta, \kappa_{\theta^c}(\beta))$ with $(\kappa_\theta, \kappa_{\theta^c}(\beta)) \in \mathbb{Z}_+^\theta \times Z\theta c+ = \mathbb{Z}_+^m$, and where we identify $\gamma(\mu)$ with $\mu(\gamma)$, the value of the functional μ on the element $\gamma \in \mathcal{T}_\lambda(L_{k-qr}^\infty)$.

We consider a second intrinsic commutative subalgebra $\mathcal{T}_\lambda(\varepsilon_k(h))$ of $B_k(h)$, which is generated by a finite set of elementary k-quasi-homogeneous symbols (see the definition in (105)). The structure of this algebra is independent of the weight parameter λ and different from $\mathcal{T}_\lambda(L_{k-qr}^\infty)$. its elements are not diagonal operators with respect to the standard orthonormal basis of$\mathcal{A}_\lambda^2(\mathbb{B}^n)$.

We start by recalling some notation and results from [193]:

Let $(p,q) \in \mathbb{Z}_+^n \times \mathbb{Z}_+^n$ be a pair of orthogonal multi-indices with $|p_{(j)}| = |q_{(j)}|$ for all $j = 1,2,\ldots,m$. We use the notation in [293] and write

$$\tilde{p}_{(j\,)} := (0, \ldots, 0, p_{(j\,)}, 0, \ldots, 0) \text{ and } \tilde{q}_{(j\,)} = (0, \ldots, 0, q_{(j\,)}, 0, \ldots, 0),$$

Where the entries $p_{(j)}$ and $q_{(j)}$ are at the j-th position, respectively. For each $j = 1, \dots, m$ with $k_j > 1$consider the Toeplitz operator $T\xi^{\bar{p}_{(j)}}\xi^{\bar{p}_{(j)}}$.Aswas shown in [293],we have

$$T_{\xi_p \bar{\xi}_q} = \prod_{j=1}^{m} T_{\xi^{\tilde{p}_{(j)}}\xi^{\tilde{q}_{(j)}}} \tag{101}$$

Let us recall the action of $T_{\xi^{\tilde{p}_{(j)}}\xi^{\tilde{q}_{(j)}}}$ on the monomials $z^\alpha, \alpha \in \mathbb{Z}_+^n$. According to [293] we have

$$T_{\xi^{\tilde{p}_{(j)}}\xi^{\tilde{q}_{(j)}}} z^\alpha = \begin{cases} 0, & \text{if } \exists \ell \in \{1, \dots, k_j\} \text{ such that} \alpha_{j,\ell} < q_{j,\ell} - p_{j,\ell,} \\ \tilde{\gamma}_{k,p_{(j)},q_{(j)},\lambda}(\alpha) z^{\alpha + \tilde{p}_{(j)} - \tilde{q}_{(j)}}, & \text{otherwise} \end{cases}, \tag{102}$$

Where the numbers $\tilde{\gamma}_{k,p_{(j)},q_{(j)},\lambda}(\alpha) \in \mathbb{Z}_+^n$ explicitly are given by

$$\tilde{\gamma}_{k,p_{(j)},q_{(j)},\lambda}(\alpha) = \frac{2^m \Gamma(n + |\alpha| + \lambda + 1)\left(\alpha + u - v + \tilde{p}_{(j)}\right)!}{\Gamma(\lambda + 1)\left(k_j - 1 + |\alpha_{(j)} + p_{(j)}|\right)! \prod_{\ell \neq j}\left(k_\ell - 1 + |\alpha_{(j)}|\right)! \left(\alpha_{(j)} + \tilde{p}_{(j)} - \tilde{q}_{(j)}\right)!}$$
$$\times \int_{\tau(\mathbb{B}^m)} (1 - |r|^2)\lambda \prod_{\ell=1}^{m} r_\ell^{2|\alpha_{(j)}| + 2k_j - 1} dr.$$

The integral on the right hand side can be calculated by using Corollary (6.3.3)

$$\tilde{\gamma}_{k,p_{(j)},q_{(j)},\lambda}(\alpha) = \frac{\left(\alpha + \tilde{p}_{(j)}\right)! \left(k_j - 1 + |\alpha_{(j)}|\right)!}{\left(k_j - 1 + |\alpha_{(j)} + p_{(j)}|\right)! \left(\alpha_{(j)} + \tilde{p}_{(j)} - \tilde{q}_{(j)}\right)!}$$
$$= \frac{\left(\alpha_{(j)} + p_{(j)}\right)! \left(k_j - 1 + |\alpha_{(j)} + p_{(j)}|\right)!}{\left(k_j - 1 + |\alpha_{(j)} + p_{(j)}|\right)! \left(\alpha_{(j)} + p_{(j)} - q_{(j)}\right)}.$$

Note that this expression only depends on the portion $\alpha_{(j)}$of α. For simplicity

$$\Phi_{p,q}(z) := \xi_{(1)}^{p_{(1)}} \dots \xi_{(m)}^{p_{(m)}} \xi_{(1)}^{q_{(1)}} \dots \xi_{(m)}^{q_{(m)}} \tag{103}$$

For a k-quasi-homogeneous function of degree (p, q). Replacing z^α by the normalized monomial $e_\alpha(z)$ in (86) gives:

Lemma (6.3.16)[302]: For $j = 1, \dots, m$ with $k_j > 1$the Toeplitz operator $T_{\Phi_{\tilde{p}_{(j)},\tilde{q}_{(j)}}}$ acts on the or honor-mal basis $[e_\alpha : \alpha \in \mathbb{Z}_+^n]$ by the rule

$$T_{\Phi_{\tilde{p}_{(j)},\tilde{q}_{(j)}}} e_\alpha = \begin{cases} 0, & \text{if } \exists \ell \in \{1, \dots, k_j\} \text{ such that } \alpha_{j,\ell} < q_{j,\ell} - p_{j,\ell,} \\ \rho_{k,p_{(j)},q_{(j)},\lambda}(\alpha) z^{\alpha + \tilde{p}(j) - \tilde{q}(j)}, & \text{otherwise} \end{cases}$$

The factor $\rho_{k,p_{(j)},q_{(j)},\lambda}(\alpha)$ is independent of the weight parameter λ and given by

$$\rho_{k,p_{(j)},q_{(j)}(\alpha)} = \frac{\left(\alpha_{(j)} + p_{(j)}\right)! \left(k_j - 1 + |\alpha_{(j)}|\right)!}{\sqrt{(\alpha_{(j)}! \, (\alpha_{(j)} + p_{(j)} - q_{(j)})! \, (k_j - 1 + |\alpha_{(j)} + p_{(j)}|)!}}. \tag{104}$$

In particular, (104) only depends on $\alpha_{(j)}$.

Now assume in addition that the tuples $p_{(j)}$ and $q_{(j)}$ are of the form (88), for $j = 1, \dots, m$, and such that (by definition) $\Phi_{\tilde{p}_{(j)},\tilde{q}_{(j)}} \in R_k(h)$. Then we have

$$\Phi_{\tilde{p}_{(j)},\tilde{q}_{(j)}}(z) := \xi_{1,j}^{p_{j,1}} \dots \xi_{j,h_j}^{p_{j,h_j}} \xi_{j,h_j+1}^{q_{j,h_j+1}} \dots \xi_{q_{j,k_j}}^{-q_{j,k_j}}.$$

Let $k_j > 1$ and $(\ell_j, r_j) \in \{1, \dots, h_j\} \times \{h_j + 1, \dots, k_j\} =: J_{h,j}$ be arbitrary. We define set of so-called elementary k-quasi-homogeneous functions by

$$\Psi_{\ell_j, r_j}(z) := \xi_{j,\ell_j} \bar{\xi}_{j,r_j} \tag{105}$$

Since $|p_{(j)}| = |q_{(j)}|$ we can express $p_{(j)} + q_{(j)}$ (non-uniquely) in the form

$$\mathrm{P}_{(j)} + \mathrm{q}_{(j)} = \sum_{(\ell_j, r_j) \in J_{h,j}} \beta_{\ell_j, r_j} \mathbf{e}_{\ell_j, r_j},$$

where $\mathbf{e}_{\ell_j, r_j} = (0, \dots, \underset{\underset{\ell_j}{\uparrow}}{1}, 0, \dots, \underset{\underset{\ell_j}{\uparrow}}{1}, 0, \dots, 0) \in \mathbb{Z}_+^{k_j}$,and $\beta_{\ell_j, r_j} \in \mathbb{Z}_+$. We also write et for the t-th standard unit vector in $\mathbb{R}^{k_j}$. For each $j = 1, \dots, m$we now can decompose $\Phi_{\tilde{p}_{(j)}, \tilde{q}_{(j)}}(z)$ (non-uniquely) in the form of a product

$$\Phi_{\tilde{p}_{(j)}, \tilde{q}_{(j)}}(z) = \prod_{(\ell_j, r_j) \in J_{h,j}} \psi_{\ell_j, r_j}(z)^{\beta_{\ell_j, r_j}} \tag{106}$$

After a straight forward computation using Lemma (6.3.17) we have

$$T_{\Psi_{\ell_j, r_j}} T_{\tilde{p}_{(j)}, \tilde{q}_{(j)}} e_\alpha$$

$$= \begin{cases} 0, & if\ \exists t \in 1, \dots, k_j \ with \ \alpha_{j,t} < \left(-e_{\ell_j} - p_{(j)} + e_{r_j} + q_{(j)}\right)_t, \\ \dfrac{\left(\alpha + \tilde{p}_{(j)} + \tilde{\boldsymbol{e}}_{\ell_j}\right)! \left[\left(k_j - 1 + \left|\alpha_{(j)}\right|!\right)\right]^2}{\sqrt{\alpha! \left(\alpha + \tilde{\boldsymbol{e}}_{\ell_j} + \tilde{p}_{(j)} - \tilde{\boldsymbol{e}}_{r_j} - , \tilde{q}_{(j)}\right)\left(k_j - 1 + \left|\alpha_{(j)} + p_{(j)}\right|\right)! \left(k_j - 1 + \left|\alpha_{(j)}\right|\right)!}} e_{\alpha + \bar{\boldsymbol{e}}_{\ell_j} + \tilde{p}_{(j)} - \tilde{\boldsymbol{e}}_{r_j} - \tilde{q}_{(j)}}, & \text{otherwise.} \end{cases}$$

On the other hand it is easy to verify that

$$T_{\Psi_{\ell_j, r_j}} T_{\tilde{p}_{(j)}, \tilde{q}_{(j)}} e_\alpha = T_{\Phi_{\tilde{p}_{(j)}, + \tilde{e}_{\ell_j} \tilde{q}_{(j)} + \tilde{e}_{r_j}}} e_\alpha$$

$$= \begin{cases} 0., & \text{if} \exists t \in 1, \dots, k_j \ \text{with} \alpha_{j,t} < \left(-e_{\ell_j} - p_{(j)} + e_{r_j} + q_{(j)}\right)_t, \\ \dfrac{\left(\alpha + \tilde{p}_{(j)} + \tilde{\boldsymbol{e}}_{\ell_j}\right)! \left(k_j - 1 + \left|\alpha_{(j)}\right|\right)!}{\sqrt{\alpha! \left(\alpha + \tilde{p}_{(j)} - \tilde{\boldsymbol{e}}_{\ell_j} - , \tilde{q}_{(j)} - e_{r_j}\right) \left(k_j + \left|\alpha_{(j)} + p_{(j)}\right|\right)!}} e_{\alpha + \tilde{\boldsymbol{e}}_{\ell_j} + \tilde{p}_{(j)} - \tilde{\boldsymbol{e}}_{r_j} - \bar{q}_{(j)}} & \text{otherwise.} \end{cases}$$

by comparing the action of the above operators on $[e_\alpha : \alpha \in \mathbb{Z}_+^n]$ we conclude that

$$T_{\psi_{\ell_j, r_j}} T_{\Phi_{\tilde{p}_{(j)}, \tilde{q}_{(j)}}} - T_{\psi_{\ell_j, r_j}} T_{\Phi_{\tilde{p}_{(j)}, \tilde{q}_{(j)}}} = \frac{\left|p_{(j)}\right|}{k_j + \left|\alpha_{(j)} + p_{(j)}\right|} T_{\psi_{\ell_j, r_j}} T_{\Phi_{\tilde{p}_{(j)}, \tilde{q}_{(j)}}}.$$

Or equivalently

$$T_{\psi_{\ell_j,r_j}\Phi_{\tilde{p}_{(j)},\tilde{q}_{(j)}}} = \frac{k_j+|\alpha_{(j)}|}{k_j+|\alpha_{(j)}+p_{(j)}|} T_{\psi_{\ell_j,r_j}} T_{\Phi_{\tilde{p}_{(j)},\tilde{q}_{(j)}}} e_\alpha \tag{107}$$

using the notation in (99) we can rewrite (107) in the form

$$T_{\psi_{\ell_j,r_j}} T_{\Phi_{\tilde{p}_{(j)},\tilde{q}_{(j)}}} = D^{(j)}_{k_j,|p_{(j)}|} T_{\psi_{\ell_j,r_j}} T_{\Phi_{\tilde{p}_{(j)},\tilde{q}_{(j)}}}$$

Note that according to Corollary (6.3.14) one has $D^{(j)}_{k_j,|p_{(j)}|} \in \mathcal{T}_\lambda(L^\infty_{k-qr})$.

Now, we return to symbols $\Phi_{\tilde{p}_{(j)},\tilde{q}_{(j)}}$ of the form (106). With $j \in \{1, \dots, m\}$ consider the sequences $\{\tilde{\gamma}_j(r)\}_{r\in\mathbb{Z}_+}$ defined by

$$\tilde{\gamma}_j(r) := \prod_{\ell=1}^{|p_{(j)}|-1} \frac{k_j + r}{k_j + r + \ell} = \frac{(k_j + r)|p_{(j)}|(k_j - 1 + r)!}{(k_j - 1 + r + |p_{(j)}|)!}, \tag{108}$$

and note that $lim_{r\to\infty}\tilde{\gamma}_j(r) = 1$. Let D_{γ_j} be the diagonal operator acting one α, for all $\alpha \in \mathbb{Z}^n_+$, by

$$D_{\gamma_j} e_\alpha = \tilde{\gamma}_j(\alpha_{(j)}) e_\alpha.$$

then it follows from Lemma (6.3.12) that $D_{\gamma_j} \in \mathcal{T}_\lambda(L^\infty_{k-qr})$. Moreover, by induction it is clear from (87) that

$$T_{\xi^{\tilde{p}_{(j)}}\xi^{\tilde{q}_{(j)}}} = T_{\Phi_{\tilde{p}_{(j)},\tilde{q}_{(j)}}} = D_{\gamma_j} \prod_{\ell=1}^{|p_{(j)}|-1} T^{\beta_{\ell_j,r_j}}_{T_{\Psi_{\ell_j,r_j}}}. \tag{109}$$

We also need the exact action of a product of Toeplitz operators with elementary k-quasi-homogeneous symbols.

Lemma (6.3.17)[302]: Let $\Phi_{\tilde{p}_{(j)},\tilde{q}_{(j)}}$ be given by (106) and define the operator A appearing in (89) by

$$A := \prod_{(\ell_j,r_j)\in J_{h,j}} T^{\beta_{\ell_j,r_j}}_{T_{\Psi_{\ell_j,r_j}}}. \tag{110}$$

Then A acts on the basis $[e_\alpha : \alpha \in \mathbb{Z}^n_+]$ by the rule

$$A\, e_\alpha = D^{-1}_{\gamma_j}\Phi_{\tilde{p}_{(j)},\tilde{q}_{(j)}} e_\alpha = \begin{cases} 0, & \text{if} \exists \ell \in \{h_j + 1, \dots, k_j\} : \alpha_{j,\ell} < q_{j,\ell}, \\ \dfrac{1}{(k_j + |\alpha_{(j)}|)^{|p_{(j)}|}} \sqrt{\dfrac{(\alpha_{(j)} + p_{(j)})!}{(\alpha_{(j)} + q_{(j)})!}} \cdot e_{\alpha+\tilde{p}_{(j)}-\tilde{q}_{(j)}}. & \text{otherwise.} \end{cases}$$

In particular, the operator A acts on $[e_\alpha : \alpha \in \mathbb{Z}^n_+]$.in the form

$$A_{e_\alpha} = m(\alpha_{(j)}) e_{\alpha+\tilde{p}_{(j)}-\tilde{q}_{(j)}}, \tag{111}$$

Where the scalar factors $m(\alpha_{(j)})$ only depend on the *j-th* portion $\alpha_{(j)}$ of $[e_\alpha : \alpha \in \mathbb{Z}^n_+]$.

Proof: The first assertion follows from Lemma (6.3.16), (109), and the relation

$$\frac{(\alpha + \tilde{p}_{(j)})!}{\sqrt{\alpha!\,(\alpha + \tilde{p}_{(j)} - \tilde{q}_{(j)})!}} = \frac{(\alpha_{(j)} + p_{(j)})}{\sqrt{\alpha_{(j)}!\,(\alpha_{(j)} + p_{(j)} - q_{(j)})!}}$$

The second statement is an immediate consequence of the first.

Remark(6.3.18)[302]: Recall that the decomposition (106) of $T_{\Phi_{\tilde{p}_{(j)},\tilde{q}_{(j)}}}$ is not unique. However, Lemma (6.3.18) shows that the map

$$\Phi_{\tilde{p}_{(j)},\tilde{q}_{(j)}} \to \prod_{(\ell_j,r_j)\in J_{h,j}} T_{T_{\Psi_{\ell_j,r_j}}}^{\beta_{\ell_j,r_j}} .= A$$

Is well-defined, i.e. the operator A is independent of there presentation (106) of $T_{\Phi_{\tilde{p}_{(j)},\tilde{q}_{(j)}}}$.

We associate to the space $R_k(h)$ the following set $\mathcal{E}_k(h)$ of elementary k-quasi-homogeneous functions of quasi-homogeneous degree (1,1)

$$\mathcal{E}_k(h) := \bigcup_{j=1}^{m} \mathcal{E}_{k,j}(h). \tag{112}$$

Where for each $j \in \{1, \dots, m\}$ with $k_j = 1$ we put$\mathcal{E}_{k,j}(h) = \emptyset$, and in the case where $k_j > 1$wedefine

$$\mathcal{E}_{k,j}(h) := \left\{\psi_{\ell_j,r_j}(z) = \xi_{j_{\ell_j}}, \bar{\xi}_{j_{r_j}}, : \left(\ell_j, r_j\right) \in \{1, \dots, h_j\} \times \{h_j + 1, \dots, k_j\} = J_{h,j}\right\}. \tag{113}$$

Clearly, $\mathcal{E}_{k,j}(h)$contains $\sum_{j=1}^{m} h_j(k_j - h_j)$ elements. These symbols ets define the corresponding commutative Toeplitz Banach algebras $\mathcal{T}_\lambda(\mathcal{E}_{k,j}(h))$and $\mathcal{T}_\lambda(\mathcal{E}_k(h))$. The following result has been proved in [193].

Proposition (6.3.19)[302]: Let $\mathrm{k} = (\mathrm{k}_1, \dots, \mathrm{k}_m) \in \mathbb{Z}_+^m$. For each pair of orthogonal multi-indices p and q with $|p_{(j)}| = |q_{(j)}|$ for all $j = 1, \dots, m$ and each $a = a(r_1, \dots, r_m) \in L_{k-qr}^\infty$ we have

$$T_a T_{\Phi_{p,q}} = T_{\Phi_{p,q}} T_a = T_{a\Phi_{p,q}}.$$

Now formula (101), Proposition (6.3.19), the decomposition (109) with $D_{\gamma_j} \in \mathcal{T}_\lambda(L_{k-qr}^\infty)$, and the notation in Theorem (6.3.2) permit us to essentially reduce the set of generators of the algebra $\mathcal{B}_k(h)$.Namely, we have:

Theorem (6.3.20)[302]: The following commutative Banach algebras coincide

$$\mathcal{T}_\lambda\left(L_{k-qr}^\infty \cup \mathcal{E}_k(h)\right) = B_k(h) \tag{114}$$

The algebra on the left hand side of (114) is clearly generated by its two commutative subalgebras $\mathcal{T}_\lambda(L_{k-qr}^\infty)$ and $\mathcal{T}_\lambda(\mathcal{E}_k(h))$. Whereas the first one is a commutative C^*-algebra, the second algebra is just a commutative Banach algebra and is not in variant under the $*$-operationof $\mathcal{L}(\mathcal{A}_\lambda^2(\mathbb{B}^n))$. Recall that $\mathcal{T}_\lambda(L_{k-qr}^\infty)$was analyzed. Our next aim is to analyze the structure of the finitely generated algebra $\mathcal{T}_\lambda(\mathcal{E}_k(h))$.

As was already mentioned, the structure of the algebra$\mathcal{T}_\lambda(\mathcal{E}_k(h))$does not depend on the weight parameter λ. Thus in what follows we will always assume that $\lambda = 0$, i.e., the operators will act on the unweighted Bergman space $\mathcal{A}^2(\mathbb{B}^n) := \mathcal{A}_0^2(\mathbb{B}^n)$ and clarify the structure of the algebra $\mathcal{T}(\mathcal{E}_k(h) := \mathcal{T}_0(\mathcal{E}_k(h)$.

By Lemma (6.3.16)the Toeplitz operator with symbol $\psi_{\ell_j,r_j}(z) = \xi_{j,\ell_j}\xi_{j,r_j}$, defined in (105), acts on the orthonormal basis $[e_\alpha : \alpha \in \mathbb{Z}_+^n]$as follows

$$T_{\psi_{\ell_j,r_j}} e_\alpha = \begin{cases} 0, & \text{if } \alpha_{j_{r_j}} = 0, \\ \dfrac{\sqrt{(a_{j_{\ell_j}}+1)a_{j_{r_j}}}}{k_j + |\alpha_{(j)}|} . e_{\alpha + \tilde{e}_{\ell_j} - \tilde{e}_{r_j}} & \text{otherwise.} \end{cases} \tag{115}$$

Let H bean abstract Hilbert space whose orthonormal basis is enumerated by $\mathbb{Z}_+^n$. If we consider set of operators T_{ℓ_j,r_j}on H which act on the basis elements according to (115), then the unital Banach algebra generated by such operators would be isomorphic and isometric to the algebra $T(\mathcal{E}_k(h)$Now, we will choose a particular realization of H which is different from our present setting of weighted Bergman spaces over the unit ball together with a set of operators T_{ℓ_j,r_j}. We introduce the Segal–Bargmann space and certain Toeplitz operators acting on it.

The main simplification we achieve in this way lies in the additional tensor product structure of the multi-dimensional Segal–Bargmann space. This feature will allow us tore present the corresponding Toeplitz operator algebra(and hence the algebra $\mathcal{T}\ (\mathcal{E}_k(h))$)in the form of a tensor product, cf. (130).

Westar with $\mathbb{C}^n$ equipped with the standard Gaussian measure

$$d\mu_n(z) := \pi^{-n} e^{-|z|^2}\, dv(z),$$

where dv denotes the Lebesgue measure on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. Denote by $H(\mathbb{C}^n)$ the space of entire functions on $\mathbb{C}^n$. The Segal–Bargmann space (or Fock space) $\mathcal{F}^2(\mathbb{C}^n)$ is defined as

$$\mathcal{F}^2(\mathbb{C}^n) := H(\mathbb{C}^n) \cap L_2(\mathbb{C}^n, d\mu_n).$$

Denote by $\boldsymbol{P}$ the orthogonal projection from $L_2(\mathbb{C}^n, d\mu_n)$onto$\mathcal{F}^2(\mathbb{C}^n)$. with $g \in L_\infty(\mathbb{C}^n)$ the Toeplitz operator $\boldsymbol{T}_g$ with symbol g acts on $\mathcal{F}^2(\mathbb{C}^n)$ in standard way

$$\boldsymbol{T}_g : f \in \mathcal{F}^2(\mathbb{C}^n) \to \boldsymbol{P}(gf) \in \mathcal{F}^2(\mathbb{C}^n).$$

Given $k = (k_1, \dots, k_m) \in \mathbb{Z}_+^m$ with $|k| = n$, we interpret $\mathbb{C}^n$ as a product space

$$\mathbb{C}^n = \mathbb{C}^{k_1} \times \dots \times \mathbb{C}^{k_m}$$

and we write $z = (z_{(1)}, \dots, z_{(m)}) \in \mathbb{C}^n$, where $z_{(j)} := (z_j, 1, \dots, z_j, k_j) \in \mathbb{C}^{k_j}$. With respect to polar coordinates we express $z_{(j)} \neq 0$ in the form $z_{(j)} = r_j \xi_{(j)}$, where

$$\xi_{(j)} = \frac{z_{(j)}}{|z_{(j)}|} \in S^{2k_j - 1} \qquad \text{and } r_i := |z_{(j)}| \in \mathbb{R}_+$$

Let $(\ell_j, r_j) \in \{1, \dots, h_j\} \times \{h_{j+1}, \dots, k_j\}$. We interpret the elementary k-quasi-homogeneous functions $\Psi_{\ell_j,r_j} := \xi_{\ell_j,r_j} \bar{\xi}_{\ell_j,r_j}$ as elements in $L\infty(\mathbb{C}^n)$. It can be checked by an easy calculation (see [104]) that

$$T_{\psi_{\ell_j,r_j}} f_\alpha == \begin{cases} 0, & \text{if} \alpha_{j_{r_j}} = 0, \\ \dfrac{\sqrt{(a_{j_{\ell_j}}+1)a_{j_{r_j}}}}{kj + |\alpha_{(j)}|} . f_{\alpha + \tilde{e}_{\ell_j} + \tilde{e}_{r_j}} & \text{otherwise.} \end{cases}$$

Here the monomials

$$f_\alpha = (z) \frac{1}{\sqrt{\alpha!}} z^\alpha \text{with } \alpha \in \mathbb{Z}_+^n$$

form the standard orthonormal basis in $\mathcal{F}^2(\mathbb{C}^n)$.

That is the set of Toeplitz operators $T_{\psi_{\ell_j,r_j}}$, where $T_{\psi_{\ell_j,r_j}} \in \mathcal{E}_k(h)$, obeys the relations (115). Denote by $\mathfrak{T}(\mathcal{E}_k(h))$ the unital Banach algebra generated by $T_{\psi_{\ell_j,r_j}}$ with $T_{\psi_{\ell_j,r_j}} \in \mathcal{E}_k(h)$. According to the above remarks we have

Lemma (6.3.21)[302]: The assignment $T_{\psi_{\ell_j,r_j}} \to T_{\psi_{\ell_j,r_j}}$ extends to an isometric isomorphism between the Banach algebras $\mathcal{T}\ (\mathcal{E}_k\ (h))$ and $\mathfrak{T}(\mathcal{E}_k(h))$.

We start with a classical result on Toeplitz operators with continuous symbols, see [162,291]. Denote by $C(\overline{\mathbb{B}}^n)$ the algebra of all functions continuous on the closed unit ball $\overline{\mathbb{B}}^n$, and let $\mathcal{T}(C(\overline{\mathbb{B}}^n))$be the C*-algebra generated by all Toeplitz operators Ta act in g on the Bergman space $\mathcal{A}^2(\overline{\mathbb{B}}^n)$and having symbols$a \in C(\overline{\mathbb{B}}^n)$.

Theorem (6.3.22)[302]: (See [162,291].) The algebra $\mathcal{T}(C(\overline{\mathbb{B}}^n))$ is irreducible and contains the ideal K of all compact operators on $\mathcal{A}^2(\mathbb{B}^n)$. Each operator $\mathcal{T} \in \mathcal{T}(C(\overline{\mathbb{B}}^n))$ has the form

$$T = T_a + K,\ \text{where } a \in C(\mathbb{B}^n) \text{ and } K \in \mathcal{K}. \tag{116}$$

The quotient algebra $\hat{\mathcal{T}}(C(\overline{\mathbb{B}}^n)) = \mathcal{T}(C(\overline{\mathbb{B}}^n))/\mathcal{K}$ is isomorphic and isometric to $C(S^{2n-1})$, and under their identification the homomorphism

$$\pi: \mathcal{T}\big(C(\overline{\mathbb{B}}^n)\big) \to \hat{\mathcal{T}}\big(C(\overline{\mathbb{B}}^n)\big) \cong C(S^{2n-1}) \tag{117}$$

is given by

$$\pi: T = T_a + K \to a|_{S^{2n-1}}\ .$$

we note that the representation (116) is not unique. An ambiguity comes from the fact that for any two functions $a, a_1 \in C(\mathbb{B}^n)$ with $(a - a_1)|_{S^{2n-1}} \equiv 0$ the difference $T_a - T_{a_1}$ is compact, and thus $T_a + \mathrm{K} = T_{a_1} + K_1$, for $K_1 = K + (T_a - T_{a_1}) \in \mathcal{K}$.

In order to make the representation (116) unique (and in a sense canonical) we proceed as follows. We introduce the C*-algebra $H(C(S^{2n-1}))$ consisting of all functions that are homogeneous of order zero on $\mathbb{B}^n$ and continuous on $S^{2n-1} \cong \partial\mathbb{B}^n$.Let $\mathcal{T}(H(C(S^{2n-1})))$ be the C*-algebra generated by all Toeplitz operators acting on the Bergman space $\mathcal{A}^2(\mathbb{B}^n)$ having symbols in $H(C(S^{2n-1}))$. With any pair of functions $a \in C(\overline{\mathbb{B}}^n)$ and $\hat{a} \in H(C(S^{2n-1}))$ such that $(a - \hat{a})|_{S^{2n-1}} \equiv 0$ we have $T_a - T_{\hat{a}} \in \mathcal{K}$. Moreover, the algebras $\mathcal{T}(C(\mathbb{B}^n))$ and $\mathcal{T}(H(C(S^{2n-1})))$consist of the same operators, in spite of the fact that they have different systems of generators.

Each operator $T \in \mathcal{T}(C(\mathbb{B}^n)) = \mathcal{T}(H(C(S^{2n-1})))$ admits the (unique) canonical representation

$$T = T_{\hat{a}}\ + K, \text{where } \hat{a} \in H\big(C(S^{2n-1})\big) \text{ and } K \in \mathcal{K}. \tag{118}$$

we note that none of the above operator $T_{\hat{a}}$is compact (unless $T_{\hat{a}} \equiv 0$ and thus $T_{\hat{a}} = 0$), and the essential spectrum of $T_{\hat{a}}$ is given by

$$ess - sp\ T_{\hat{a}} = \hat{a}(S^{2n-1}) = \hat{a}(\mathbb{B}^n).$$

This implies the estimate $r(T_{\hat{a}}) \geq \|\hat{a}\|_{L_\infty}$ *f*or the spectral radius $r(T_{\hat{a}})$ of $T_{\hat{a}}$ which, together with $\|T_{\hat{a}}\| \leq \|\hat{a}\|_{L_\infty}$, shows that

$$\|T_{\hat{a}}\| = r(T_{\hat{a}}) = \|\hat{a}\|_{L_\infty} \quad \text{for all } \hat{a} \in (H\big(C(S^{2n-1})\big). \tag{119}$$

the above observations permit us to give another equivalent description of the quotient algebra $\hat{\mathcal{T}}(C(\overline{\mathbb{B}}^n)) = \hat{\mathcal{T}}(H(C(S^{2n-1}))) \cong C(S^{2n-1})$. Indeed, the assignment

$$\hat{T} = \widehat{T_{\hat{a}} + k} \mapsto T_{\hat{a}}$$

gives a Banach space isometric isomorphism

$$\hat{\mathcal{T}}(C(\overline{\mathbb{B}}^n)) \to T_H(C(S^{2n-1})) := \{T_{\hat{a}} : \hat{a} \in (H(C(S^{2n-1}))\}. \tag{120}$$

This isomorphism becomes algebraic after introducing the multiplication law in $T_H(C(S^{2n-1}))$ as $T_{\hat{a}_1} \odot T_{\hat{a}_2} = T_{\hat{a}_1\hat{a}_2}$.

With our previous notation let $\lambda = 0, n \in \mathbb{N}$ and $m = 1$, so that $k = (n)$, and $h = (h)$, with $1 < h < n-1$. Let the tuple

$$\psi = (\psi_1, \ldots, \psi_\gamma) \tag{121}$$

be the (somehow) ordered set of $\gamma := h(n-h)$ elementary quasi-homogeneous symbols in

$$\mathcal{E}_k(h) = \left\{\psi_{j,l(z)} := \frac{z_j \bar{z}_l}{|z|^2} : j = 1, \ldots, h, l = h+1, \ldots, n\right\},$$

And let $T(\psi) = (T_{\psi_1}, \ldots, T_{\psi_\gamma})$ be the ordered set of the corresponding Toeplitz operators. We introduce the Banach algebra $\mathcal{B}(\mathcal{E}_k(h))$, asubalgebra of $H(C(S^{2n-1}))$, being the unital algebra enerated by all elementary quasi-homogeneous functions from $\mathcal{E}_k(h)$, as well as the unital Banach algebra $\mathcal{T}(\mathcal{E}_k(h))$, a subalgebra of $\mathcal{T}(H(C(S^{2n-1})))$, which is generated by the Toeplitz operators in $T(\psi)$.

Note that in our particular case ($m = 1$) the functions $\psi_j \in \psi$ continuously extend to the sphere $S^{2n-1} \cong \partial\mathbb{B}^n$, and thus we are in the framework of the previous subsection. The general case of $m > 1$ where such a continuity on the boundary is not fulfilled will be treated in the section.

We define the following tuples of multi-indexes

$$P := \{(p,q) \in \mathbb{Z}_+^n \times \mathbb{Z}_+^n : p = (p_1, \ldots, p_h, 0, \ldots, 0), q = (0, \ldots, 0, q_{h+1}, \ldots, q_n), |p| = |q|\}.$$

If $(p,q) \in \boldsymbol{P}$, then there is $\alpha = \alpha(p,q) \in \mathbb{Z}_+^\gamma$ such that

$$\frac{z^p \bar{z}^q}{|z|^{2|p|}} = \psi_1^{\alpha_1} \ldots \psi_\gamma^{\alpha_\gamma}. \tag{122}$$

Consider the space of polynomials $\mathcal{F}_{\boldsymbol{P}} := \{\varphi_{(p,q)}(z) := z^p \bar{z}^q : (p,q) \in \boldsymbol{P}\}$. It is easy to see that all the elements of $\mathcal{F}_{\boldsymbol{P}}$ are harmonic polynomials on $\mathbb{R}^{2n} \cong \mathbb{C}^n$. Moreover, we have:

Lemma (6.3.23)[302]: The functions in $\mathcal{F}_{\boldsymbol{P}}$ are orthogonal in $L_2(\mathbb{B}^n)$. If were strict them to a sphere rS^{2n-1} of radius $r \in (0,1)$, then they define orthogonal functions in $L_2(rS^{2n-1}, \sigma)$ where σ denotes the usual surface measure on rS^{2n-1}.

Proof: With $(p,q), (r,s) \in P$and suitable numbers $C(p,q) > 0$ it holds

$$\int_{\mathbb{B}^n} z^p \tilde{z}^q \, \overline{z^r \tilde{z}^s} \, dv(z) = \int_{\mathbb{B}^n} z^{p+s} \tilde{z}^{q+r} \, dv(z)$$

$$= \int_{\mathbb{B}^n} z_1^{p_1} \ldots z_h^{p_h} z_{h+1}^{s_{h+1}} \ldots z_n^{s_n} \tilde{z}_1^{r_1} \ldots \tilde{z}_h^{r_h} z_{h+1}^{q_{h+1}} \ldots \tilde{z}_n^{q_n} dv(z) = \mathrm{C}(p,q)\delta_{p,r}\delta_{s,q}.$$

The second assertion follows by the same argument.

Let $A \in \mathcal{L}(A^2(\mathbb{B}^n))$, then we write $B[A](z) \in L_\infty(\mathbb{B}^n)$ for the Berezin transform of A. More precisely, $B[A](z)$ is defined by

$$B[A](z) \coloneqq \frac{1}{\|K(\cdot,z)\|_0^2} A\langle K(\cdot,z), K(\cdot,z)\rangle_0, \tag{123}$$

Where $\mathrm{K}: \mathbb{B}^n \times \mathbb{B}^n \to \mathrm{C}$ is the reproducing kernel of the unweighted Bergman space $A^2(\mathbb{B}^n)$

$$K(u,z) = \frac{1}{(1 - \langle u,z\rangle)^{n+1}} .$$

Recall that $B: \mathcal{L}(A^2(\mathbb{B}^n)) \to C_b^{\omega}(\mathbb{B}^n)$ is linear and injective. Here we write $C_b^{\omega}(\mathbb{B}^n)$ for the space of bounded real analytic functions on $\mathbb{B}^n$.

Lemma (6.3.24)[302]: Let $\alpha \in \mathbb{Z}_+^{\gamma}$ and $(p,q) \in \boldsymbol{P}$ be related to α via (122), then

$$B\left[T_{\psi_1}^{\alpha_1} \dots T_{\psi_\gamma}^{\alpha_\gamma}\right](z) = z^p \tilde{z}^q H_{|p|}(|z|), \tag{124}$$

Where the function $H_{|p|}$ fulfills $\lim_{\rho \uparrow 1} H_{|p|}(\rho) = 1$ and it has the explicit form

$$H_{|p|}(|z|) = \frac{2}{(|p|-1)!}(1 - |z|^2)^{n+1} \frac{(n+|p|)!}{n!}.$$

$$\times \int_0^{\infty} \frac{e - (n+|p|)s^2}{\left(1 - |z|^2 e^{-s^2}\right)^{n+|p|+1}} s^{2|p|-1}\, ds \tag{125}$$

Proof: According to Lemma (6.3.17) the operator product $T_{\psi_1}^{\alpha_1} \dots T_{\psi_\gamma}^{\alpha_\gamma}$ acts on the orthonormal basis $[e_\beta : \beta \in \mathbb{Z}_+^n]$ of $A^2(\mathbb{B}^n)$ in the form

$$T_{\psi_1}^{\alpha_1} \dots T_{\psi_\gamma}^{\alpha_\gamma} e_\beta = m(\beta) e_{\beta+p-q},$$

Where $m(\beta)$ is defined by

$$m(\beta) := \begin{cases} \dfrac{1}{(n+|\beta|)^{|p|}!}\sqrt{\dfrac{(\beta+p)!}{(\beta-q)!}}, & if\ \beta - q \geq 0 \text{ (componentwise)}, \\ 0, & \text{otherwise} \end{cases}$$

Hence it follows

$$\langle T_{\psi_1}^{\alpha_1} \dots T_{\psi_\gamma}^{\alpha_\gamma} e_\beta\, K(\cdot,z), K(\cdot,z)\rangle_0 = \sum_{\beta,\eta \in \mathbb{Z}_+^n} \langle T_{\psi_1}^{\alpha_1} \dots T_{\psi_\gamma}^{\alpha_\gamma} e_\beta e_\beta, e_\eta\rangle_0\, \overline{e_\beta(z)} e_\eta(z)$$

$$= \sum_{\beta_j \geq q_j} m(\beta)\overline{e_\beta(z)} e_{\beta+p-q}(z) = \sum_{\beta \in \mathbb{Z}_+^n} m(\beta + q)\overline{e_\beta(z)} e_{\beta+p}(z) = (*).$$

Using the explicit form of $m(\beta+q)$ above and the expression for $e_\beta(z)$ in (86) together with $|p| = |q|$ we obtain

$$(*) = \frac{z^p \tilde{z}^q}{n!} \sum_{\beta \in \mathbb{Z}_+^n} \frac{\Gamma(n+|\beta|+|p|+1)}{(n+|\beta|+|p|)^{|p|}} \frac{|z^\beta|^2}{\beta!} = \frac{z^p \tilde{z}^q}{n!} \sum_{\ell=0}^{\infty} \frac{\Gamma(n+\ell+|p|+1)}{(n+\ell+|p|)^{|p|}} \frac{|z|^{2\ell}}{\ell!}.$$

In the last equation we have applied the multinomial theorem. In order to obtain an integral representation of $(*)$ we use the well-known relations

$$\frac{1}{(n+\ell+|p|)^{|p|}} = \frac{1}{\pi|p|} \int_{\mathbb{B}^{2|p|}} e^{-(n+\ell+|p|)|x|^2}\, dx,$$

$$\sum_{\ell=0}^{\infty} \Gamma(t+\ell)\frac{u^\ell}{\ell!} = \frac{\Gamma(t)}{(1-u)^t},$$

Which hold true for $u \in [0.1]$and $t \geq 0$. Interchanging summation and integration implies

$$(*) = \frac{z^p\tilde{z}^q}{n!}\frac{1}{\pi^{|p|}}\int_{\mathbb{R}^{2|p|}} e^{-(n+|p|)|x|^2}\sum_{\ell=0}^{\infty}\Gamma(n+\ell+|p|+1)\frac{\left(e^{-|x|^2}|z|^2\right)^{\ell}}{\ell!}dx$$
$$= \frac{z^p\tilde{z}^q}{n!}\frac{(n+|p|)!}{\pi^{|p|}}\int_{\mathbb{R}^{2|p|}}\frac{e^{-(n+|p|)|x|^2}}{\left(1-e^{-|x|^2}|z|^2\right)^{n+|p|+1}}dx$$

Finally, by changing to polar coordinates in the last integral, we obtain the expression (125). The limit behavior $\lim_{\rho\uparrow 1} H_{|p|}(\rho) = 1$ can be directly checked from (125). However, we can also give a more abstract argument. By (118) we can write the product of Toeplitz operators in the form

$$T^{\alpha_1}_{\psi_1} \dots T^{\alpha_\gamma}_{\psi_\gamma} = T_{\frac{z^p\tilde{z}^q}{|z|^{2|p|}}} + K, \tag{126}$$

Where K is a compact operator. If f is a continuous function in a neighborhood of S^{2n-1} and bounded on $\mathbb{B}^n$ then it is well-known that $\lim_{\rho\uparrow 1} B[T_f](\rho z) = f(z)$ for all $z \in S^{2n-1}$ (cf.[21]). Moreover, it holds $\lim_{|z|\uparrow 1} B[K](z) = 0$. From (126) we obtain

$$\lim_{\rho\uparrow 1} B\left[T^{\alpha_1}_{\psi_1} \dots T^{\alpha_\gamma}_{\psi_\gamma}\right](\rho z) = z^p\tilde{z}^q$$

For all$z \in S^{2n-1}$.Together with (124) it follows that $\lim_{\rho\uparrow 1} H_{|p|}(\rho) = 1$.

Recall (see (117)) that the Banach algebra homomorphism $\pi: \mathrm{T}(\mathrm{C}(\mathbb{B}^n)) \to C(S^{2n-1})$ is given by

$$\pi(T_a + K) = a|_{S^{2n-1}}.$$

Here K is compact and $a \in \mathrm{C}(\overline{\mathbb{B}}^n)$ is continuous up to the boundary.

Proposition (6.3.25)[302]: The restriction of π to the algebra $\mathcal{T}(\mathcal{E}_k(h))$ is injective.

Proof: Let $T \in \mathcal{T}(\mathcal{E}_k(h))$ with $\pi(T) = 0$, then we want to show that $T = 0$. Choose sequence

$$T_\ell = \sum_{\alpha\in\mathbb{Z}_+^n} a_\alpha(\ell) T^{\alpha_1}_{\psi_1} \dots T^{\alpha_\gamma}_{\psi_\gamma} \in \mathcal{T}(\mathcal{E}_k(h))$$

Such that $\lim_{\ell\to\infty} T_\ell = T$ in the norm topology. Here for each $\ell \in \mathbb{Z}_+$only finitely many coefficients$a_\alpha(\ell)$are non-zero. As was already mentioned the operator product $T^{\alpha_1}_{\psi_1} \dots T^{\alpha_\gamma}_{\psi_\gamma}$ admits the decomposition

$$T^{\alpha_1}_{\psi_1} \dots T^{\alpha_\gamma}_{\psi_\gamma} = T^{\alpha_1}_{\psi_1} \dots T^{\alpha_\gamma}_{\psi_\gamma} + K_\alpha, K_\alpha \in \mathcal{K},$$

And we have $\pi(T^{\alpha_1}_{\psi_1} \dots T^{\alpha_\gamma}_{\psi_\gamma}) = z^p\tilde{z}^q|z|^{-2|p|} = z^p\tilde{z}^q$, where (p,q) and α are related sin (122). Since π is continuous we have

$$0 = \pi(T) = \lim_{\ell\to\infty}\pi(T_\ell) = \lim_{\ell\to\infty}\sum_{p,q} a_\alpha(p,q)(\ell) z^p\tilde{z}^q \quad ,$$

Where the convergence on the right hand side is in $C(S^{2n-1})$. In particular, the convergence takes place in $L_2(S^{2n-1},\sigma)$. Due to the orthogonality result in Lemma (6.3.23) it follows

$$\left\|\sum_{P,q} a_\alpha(p,q)(\ell) z^p \tilde{z}^q\right\|^2_{L_2(S^{2n-1},\sigma)} = \sum_{p,q} |a_\alpha(p,q)(\ell)|^2 \, \|z^p \tilde{z}^q\|^2_{L_2(S^{2n-1},\sigma)} \to 0(\ell \to \infty).$$

as a consequence we have that $\lim_{\ell\to\infty} a_\alpha(p,q)(\ell) = 0.$

Now we consider the sequence of Berezin transforms $B[T_\ell]$. According to Lemma (6.3.24) we have for all $\ell \in \mathbb{Z}_+$

$$B[T_\ell](z) = \sum_{p,q} a_\alpha(p,q)(\ell) z^p \tilde{z}^q \, H_{|p|}(|z|) = \sum_{l=1}^{\infty} H_l(|z|) \sum_{\substack{(p,q)\\ |p|=|q|=l}} a_\alpha(p,q)\,(\ell) z^p \tilde{z}^q\,.$$

by continuity of the Berezin transform, it follows that $\lim_{\ell\to\infty} B[T_\ell]\,(z) = B[T](z)$uniformly for $z \in \mathbb{B}^n$. In particular, if we fix $r \in (0,1)$ and restrict $B[T_\ell]\, to\ rS^{2n-1}$, then we obtain a convergent sequence in $L_2(rS^{2n-1})$ Since $z^p\tilde{z}^q$ are orthogonal in $L_2(rS^{2n-1})$ and by applying $\lim_{\ell\to\infty} a_\alpha(p,q)(\ell) = 0$, we obtain

$$\lim_{\ell\to\infty} H_{|p|}(|z|)\, a_{\alpha(p,q)}(\ell) \ = \ 0$$

for all (p,q) and therefore $B[T_\ell](z)$ converges to zero in $L_2(rS^{2n-1})$. Since r was arbitrary it follows that $\lim_{\ell\to\infty} B\,[T_\ell](z) = B[T](z) = 0$ a.e. on Bnand from the continuity of $B[T]$ we see that $B[T]$ identically vanishes on $\mathbb{B}^n$. Since the Berezin transform B is injective on bounded operators we have $T = 0$.

Corollary (6.3.26)[302]: The algebra $\mathcal{T}(\mathcal{E}_k(h))$ does not contain any non-zero compact operator. Each element $T \in \mathcal{T}(\mathcal{T}(\mathcal{E}_k(h)))$ admits a unique representation

$$T \ = \ T_\psi \ + K_\psi, \tag{127}$$

Where $\psi \in B(\mathcal{E}_k(h))$ and $K_\psi$is the compact operator from $K \cap \mathcal{T}_0(L^\infty_{k-qr} \cup \mathcal{E}_k(h))$ uniquely determined byψ;i.e.,if both operators $T_1 = T_\psi + K_1$ and $T_2 = T_\psi + K_2$ belong to $T(\mathcal{E}_k(h))$, then $K_1 = K_{2,}$and thus $T_1 = T_2$.

Proof: According to Proposition (6.3.24)we know that the homomorphism π is injective on $\mathcal{T}(\mathcal{E}_k(h))$; as it vanishes on compact operators, we have that $K \cap \mathcal{T}(\mathcal{E}_k(h)) = \{0\}$.

Representation (127) follows from(118). Assuming that both $T_1 = T_\psi + K_1$ and$T_2 = T_\psi + K_2$ belong to $\mathcal{T}(\mathcal{E}_k(h))$, we have $T_1 - T_2 = K_1 - K_2 \in K \cap \mathcal{T}(\mathcal{E}_k(h)) = \{0\}$. Thus $K_1 = K_2$ is uniquely determined by ψ.

We mention that the latter result remains true for the algebras $\mathcal{T}_\lambda(\mathcal{E}_k(h))$with the only difference that $K_\psi \in K \cap \mathcal{T}_\lambda(L^\infty_{k-qr} \cup \mathcal{E}_k(h))$.

We mention that the exact form of the compact operator in (127) can be easily figured out. Indeed, let $\mathcal{F}$ be a dense subset of $\mathcal{T}(\mathcal{E}_k(h))$consisting of finite sums of finite products of its generators. For elements of $\mathcal{F}$ the concrete form of the compact operator in (127) can be obtained using (108) and (109). Let now $\{T_p\}_{p\in\mathbb{N}}$, where $T_p = T_{\psi_p} + K_{\psi_p} \in \mathcal{F}$be a sequence that uniformly converges to $T = T_\psi + K \in \mathcal{T}(\mathcal{E}_k(h))\backslash\mathcal{F}$. Then the sequence $\{T_p\}_{p\in\mathbb{N}}$ converges to $\hat{\mathcal{T}}$ in the quotient algebra $\hat{\mathcal{T}}(H(C(S^{2n-1})))$. The isomorphism (120) implies that $T_{\psi_p} \to T_\psi$ uniformly, and thus $K = \lim_{p\to\infty} K_{\psi_p}$.

We have then

$$\mathcal{T}(\mathcal{E}_k(h)) = \mathcal{T}(\mathcal{E}_k(h))/(\mathcal{T}(\mathcal{E}_k(h)) \cap \mathcal{K}) \cong (\mathcal{T}(\mathcal{E}_k(h)) + \mathcal{K})/\mathcal{K} \subset \mathcal{T}(H(C(S^{2n-1})))/\mathcal{K} \cong C(S^{2n-1}). \quad (128)$$

Lemma (6.3.27)[302]: For each $\psi_{j,l} \in \mathcal{E}_k(h)$, the spectrum of $T_{\psi_{j,l}}$, is given by $spT_{\psi_{j,l}} = \overline{\mathbb{D}}\left(0, \frac{1}{2}\right)$.

Proof: Follows from two facts: $ess - sp\, T_{\psi_{j,l}} = Range\ \psi_{j,l}|_{S^{2n-1}} = \overline{\mathbb{D}}\left(0, \frac{1}{2}\right)$, and the spectral radius of $T_{\psi_{j,l}}$ is equal to $\frac{1}{2}$ which follows from (119).

We recall the notion of the joint spectrum (see [285] or [27]). Let A be commutative Banach algebra with identity and let $x_1, \dots, x_n \in A$. The joint spectrum of $x_1, \dots, x_n$ is the subset $\sigma_A(x_1, \dots, x_n)$ of $\mathbb{C}^n$ defined by

$$\sigma_A(x_1, \dots, x_n) = \{(\varphi(x_1), \dots, \varphi(x_n)) : \varphi \in M(A)\},$$

Where $M(A)$ is the compact set of maximal ideals (≡multiplicative functionals) of A.

By (128), the algebra $\mathcal{T}(\mathcal{E}_k(h))$ is isomorphic to the unital subalgebra of $C(S^{2n-1})$ generated by the elements of ψ in (101), with the following assignment: $T_{\psi_j} \longmapsto \psi_j|_{S^{2n-1}}$. Identifying them we calculate the joint spectrum of elements of $T(\psi)$, relative to $C(S^{2n-1})$, as the joint spectrum of ψ in the algebra $C(S^{2n-1})$.

Lemma (6.3.28)[302]: The joint spectrum of the Toeplitz operators with symbols in ψ is given by

$$\sigma(T(\psi)) = \psi(S^{2n-1}) \subset \mathbb{C}^{h(n-h)}.$$

Proof: As S^{2n-1} is the compact set of maximal ideals of $C(S^{2n-1})$, we have

$$\sigma(T(\psi)) = \sigma_{C(S^{2n-1})}(\psi) = \psi(S^{2n-1}).$$

At the same time the unital Banach algebra $\mathcal{T}(\mathcal{E}_k(h))$ it self, considered as a finitely generated algebra by elements of $T(\psi)$ is isomorphic to the "polynomial" algebra $P(\sigma(T(\psi)))$, and, by [285], its compact set of maximal ideals coincides with the poly nominally convex hull $\hat{\sigma}(T(\psi))$ of $\sigma(T(\psi))$, i. e.

$$M(\mathcal{T}(\mathcal{E}_k(h)) = \hat{\sigma}(T(\psi)).$$

From [285] yields

Theorem (6.3.29)[302]: The Banach algebra $\mathcal{T}(\mathcal{E}_k(h))$ is isomorphic to the algebra $P(\hat{\sigma}(T(\psi)))$.

We proceed now with the description of the set $\sigma(T(\psi))$. With $r \in \mathbb{N}$ and $s > 0$ let $\overline{\mathbb{B}}^r(0, s)$ be the closed ball in $\mathbb{R}^r$ of radius s centered at the origin.

Example(6.3.30)[302]: Given $n > 1$, consider $m = 1$, so that $k = (n)$, and $h = (1)$. In this case we have with our former notation

$$\mathcal{E}_k(h) = \left\{\psi_{1,2}(z) = \frac{z_1\bar{z}_2}{|z|^2}, \psi_{1,3}(z) = \frac{z_1\bar{z}_3}{|z|^2}, \dots, \psi_{1,n}(z) = \frac{z_1\bar{z}_n}{|z|^2}\right\}.$$

By changing to polar coordinates $z_j = r_j e^{i\theta_j}$, with $r_j \in \mathbb{R}_+$ for $j = 1, \dots, n$ and writing $r = (r_1, r_2, \dots, r_n)$ we obtain:

$$\psi(S^{2n-1}) = \{(r_1 r_2 e^{i(\theta_1-\theta_2)}, \dots, r_1 r_n e^{i(\theta_1-\theta_n)}) : |r| = 1, \theta_j \in [0, 2\pi), j = 1, \dots, n\}$$
$$\subset \overline{\mathbb{D}}\left(0, \frac{1}{2}\right)^{n-1}.$$
Setting $\omega_j = |\omega_j| t_j = r_1 r_{j+1} e^{i(\theta_1-\theta_{j+1})}forj = 1,2, \dots, n-1$gives
$$|\omega|^2 = \sum_{j=1}^{n-1} |\omega_j|^2 = r_1^2 (1 - r_1^2) \le \frac{1}{4},$$
Which implies that
$$\psi(S^{2n-1}) = \left\{(\omega_1, \dots, \omega_{n-1}) \in \mathbb{C}^{n-1} : |w| \le \frac{1}{2}\right\} = \overline{\mathbb{B}}^{n-1}\left(0, \frac{1}{2}\right).$$
Notethatthe*casen* $> 1, m = 1, k = (n)$,and$h = (n-1)$givesthesame result:for
$$\psi = \left(\psi_{1,n}(z) = \frac{z_1 \bar{z}_n}{|z|^{2,}} \psi_{2,n}(z) = \frac{z_2 \bar{z}_n}{|z|^{2,}} \dots, \psi_{n-1,n}(z) = \frac{z_{n-1} \bar{z}_n}{|z|^2}\right),$$
We have that
$$\psi(S^{2n-1}) = \left\{(\omega_1, \dots, \omega_{n-1}) \in \mathbb{C}^{n-1} : |\omega| \le \frac{1}{2}\right\} = \overline{\mathbb{B}}^{n-1}\left(0, \frac{1}{2}\right).$$
Example(6.3.31)[302]: Consider now the case: $n > 3, m = 1, k = (n)$, and $h = (h)$, with $1 < h < n-1$. In this case we have $h(n-h)$ elementary quasi-homogeneous symbols,
$$\mathcal{E}_k(h) = \left\{\psi_{j,l}(z) = \frac{z_j \bar{z}_l}{|z|^{2,}} : j = 1, \dots, h, l = h+1, \dots, n\right\}.$$
Passing to the polar coordinates $z_j = r_j e^{i\theta_j}$, for $j = 1, \dots, n$, and with $|z| = 1$ we have
$$\psi_{j,l} = r_j r_l e^{i(\theta_j - \theta_l)}, \qquad \text{where } j = 1, \dots, h, l = h+1, \dots, n.$$
The range of $\psi = (\psi_{1,h+1}, \dots, \psi_{1,n}, \dots, \psi_{h,h+1}, \dots, \psi_{h,n})$ on S^{2n-1} is calculated as
$$\psi(S^{2n-1}) = \{(r_1 r_{h+1} e^{i(\theta_j-\theta_l)}, \dots, r_j r_l e^{i(\theta_j-\theta_l)}, \dots, r_h r_n e^{i(\theta_h-\theta_n)})$$
$$|r| = 1, \theta_j - \theta_l \in [0, 2\pi), j = 1, \dots, h,$$
$$l = h+1, \dots, n \subset \overline{\mathbb{D}}\left(0. \frac{1}{2}\right)^{(n-h)}.$$
Let
$$r_1 r_{h+1} e^{i(\theta_j-\theta_l)} = \omega_{1,h+1} = a_{1,h+1} t_{1,h+1}, \dots,$$
$$r_j r_l e^{i(\theta_j-\theta_l)} = \omega_{j,l} = a_{j,l} t_{j,l}, \dots,$$
$$r_h r_n e^{i(\theta_h-\theta_n)} = \omega_{h,n} = a_{h,n} t_{h,n},$$
here $a_{j,l} \in \mathbb{R}_+, t_{j,l} \in S^1, j = 1, \dots, h$and $l = h+1, \dots, n$. Moreover, the above components $t_{j,l}$ obey the relations
$T = \{t_{j_1,l1} t_{j_1,l2} t_{j_2,l_1} t_{j_2,l_2} = 1$: forall $j_1, j_2 \in 1, \dots, h$ and $l_1, l_2 \in h+1, \dots, n\}$.
note that not all relations in T are independent. An equivalent reformulation of the relations Tisa follows. The equation $t_{j_1,l1} \overline{t_{j_1,l2} t_{j_2,l_1}} t_{j_2,l_2} = 1$ is equivalent to $t_{j_2,l_2} = \overline{t_{j_1,l1}} t_{j_1,l2} t_{j_2,l_1}$showing that only $n-1$ of the $h(n-h)$ variables $t_{j.1}$ are actually independent (e.g. take$t_{1,h+1}, \dots, t_{1,n,t2,h+1}, \dots, t_{h,h+1}$) which yields
$$T = \{t_{j,l} = \overline{t_{1,h+1}} t_{1,l} t_{j,h+1} : j = 2, \dots, h, l = h+2, \dots, n\}.$$
Then we have

$$|\omega|^2 = \sum_{j=1}^{h} \sum_{l=h+1}^{n} |\omega_{j,l}|^2 = (r_1^2 + \cdots + r_h^2)[1 - (r_1^2 + \cdots + r_h^2)] \leq \frac{1}{4},$$

Which implies that

$$\psi(S^{2n-1}) = \left\{(\omega_{1,h+1}, \dots, \omega_{h,n}) \in \mathbb{C}^{h(n-h)} : |\omega| \leq \frac{1}{2} \text{ subject to T}\right\} \subset \overline{\mathbb{B}}^{h(n-h)}\left(0, \frac{1}{2}\right),$$

Where $\omega_{j,l} = a_{j,l} t_{j,l}$, with $j = 1, \dots, h$and $l = h+1, \dots, n$, as above.

We unify the results of the above examples in the next lemma.

Lemma (6.3.32)[302]: Let $n > 1, k = (n)$, and $h = (h)$, with$1 \leq \mathrm{h} \leq \mathrm{n} - 1$.Then the vector ψ in (4.21) has $\mathrm{h(n-h)}$ components, and

$$\psi(S^{2n-1}) = \Lambda(n,h) \subseteq \overline{\mathbb{B}}^{h(n-h)}\left(0, \frac{1}{2}\right),$$

Where

$$\Lambda(n,h) = \begin{cases} \overline{\mathbb{B}}^{n-1}(0, \frac{1}{2}), & \text{if } h = 1, or\ h = n-1, \\ \{\omega = (\omega_{1,h+1}, \dots, \omega_{h,n}) \in \overline{\mathbb{B}}^{h(n-h)}\left(0, \frac{1}{2}\right): \\ \omega \text{ subject to } T\}, & \text{otherwise.} \end{cases} \quad (129)$$

Corollary (6.3.33)[302]: The Banach algebra $\mathcal{T}(\mathcal{E}_k(h))$ is isomorphic to the polynomial algebra $P(\hat{\Lambda}(n,h))$, where $\hat{\Lambda}(n,h)$is the polynomially convex hull of $\Lambda(n,h)$.

Proof: Follows from the above lemma and Theorem (6.3.29)

Note that, if $h = 1$ or $h = n-1$, then the set $\Lambda(n,h) = \overline{\mathbb{B}}^{h(n-h)}\left(0, \frac{1}{2}\right)$ is convex and thus coincides with its polynomially convex hull. At the same time, in the case $n > 3$ and $1 < h < n-1$, an explicit description of the polynomially convex hull $\hat{\Lambda}(n,h)$: of $\Lambda(n,h)$ seems to be quite anon-trivial task. We do not know the answer, and leave it as a problem (v) in the section. The following discussion provides a (rough) upper bound for $\hat{\Lambda}(n,h)$:

It is easy to see that $\hat{\Lambda}(n,h)$: is a subset of $\overline{\mathbb{B}}^{\mathrm{h(n-h)}}\left(0, \frac{1}{2}\right)$. We will show here that it is even proper subset. With$r \in \left(0, \frac{1}{2}\right)$ consider the sets

$$\Lambda_r\ (n,h) := \{\omega \in \Lambda(n,h) : |\omega| = r\}$$

And for each fixed $z \in \mathbb{C}^{h(n-h)}$ define the holomorphic polynomial $P_z(\omega) := r^{-2}\langle \omega, z\rangle$. Let $\Lambda_r^c(n,h)$ be the (open) complement of $\Lambda_r(n,h)$ in the r-sphere $S_r^{2h(n-h)-1}$. For any $z \in \Lambda_r^c(n,h)$ there is $0 < \gamma < 1$ such that

$|\langle \omega, z\rangle| \leq \gamma\, r^2$, for all $\omega \in \Lambda_r(n,h)$.

Hence it follows for $\omega \in \Lambda_r(n,h)$ and by using the maximum principle in the last equality:

$$1 = |P_z(z)| \geq \gamma^{-1} \sup_{\omega \in \Lambda_r\ (n,h)} |P_z(\omega)| \geq \gamma^{-1} \sup_{\omega \in \Lambda_{1/2}\ (n,h)} |P_z(2r\omega)| = \frac{2r}{\gamma} \sup_{\omega \in \Lambda(n,h)} P_z(\omega)\ .$$

Thus, no point in $\Lambda_r^c(n,h)$ with $\gamma \leq 2r$ be longs to the polynomial convex hull of $\Lambda_r^c(n,h)$. In particular this holds for all points in $\Lambda_{1/2}^c(n,h) = \partial\overline{\mathbb{B}}^{h(n-h)}\left(0, \frac{1}{2}\right) \backslash \Lambda(n,h) \neq \emptyset$.

Now we list several properties of $\Lambda(n,h)$ f or the case $n \geq 3$and $1 < h < n-1$.

(i) $\hat{\Lambda}(n,h)$ is a compact subset of $\overline{\mathbb{B}}^{h(n-h)}\left(0,\frac{1}{2}\right)$.
(ii) None of the points of $\Lambda(n,h)$ is interior, i.e., this set has an empty interior.
(iii)The set $\Lambda(n,h)$ is a contractible star-like set, i.e., together with each of its point the set $\Lambda(n,h)$ contains the radius of $\overline{\mathbb{B}}^{h(n-h)}\left(0,\frac{1}{2}\right)$ passing through this point.
(iv).Both $\Lambda(n,h)$ and its complement $\mathbb{C}^{h(n-h)}\backslash\Lambda(n,h)$ are connected sets.
(v)The set $\Lambda(n,h)$ is invariant under the following action of the $(n-1)$-dimensional torus $\mathbb{T}^{n-1}$. For $\tau=(\tau_{1,h+1},\ldots,\tau_{1,n},\tau_{2,h+1},\ldots,\tau_{h,h+1})\in\mathbb{T}^{n-1}$ and each point $\omega=(\omega_{1,h+1},\ldots,\omega_{h,n})\in\Lambda(n,h)$ the coordinates of $u=\tau\cdot\omega\in\Lambda(n,h)$ are of the form

$$u_{j,l}=\begin{cases}\tau_{j,l}\omega_{j,l}\,, & \text{if } j=1 \text{ and } l=h+1,\ldots,n,.or\ j=2,\ldots,h\ and\ l=h+1,\\ \overline{\tau_{1,h+1}}\tau_{1,l}\tau_{j,h+1}\omega_{j,l}\,, & \text{otherwise.}\end{cases}$$

Consider now the general case of $n>1$, with $m>1$, that islet $k=(k_1,\ldots,k_m)$ and $h=(h_1,\ldots,h_m)$. In this case, by (112),

$$\mathcal{E}_k(h):=\bigcup_{j=1}^{m}\mathcal{E}_{k,j}(h),$$

Where for each j ∈ {1, … , m} we have used the notation in (113).
We consider as well the ordered sets ψ and $\psi[j]$ formed by elements of $\mathcal{E}_{\mathrm{k}}(\mathrm{h})$ and $\mathcal{E}_{k,j}(h), j=1,\ldots,m$, respectively, together with the corresponding unital Banach Toeplitz operator algebras: $\mathcal{T}(\mathcal{E}_k(h))$, generated by operators acting on $\mathcal{A}_2(\mathbb{B}^{\mathrm{n}})$, and $\mathcal{T}(\mathcal{E}_{\mathrm{k,j}}(\mathrm{h}))$, generated by operators acting on $\mathcal{A}_2(\mathbb{B}^{k_j})$, where $j=1,\ldots,m$.
With the above multi- index $k=(k_1,\ldots,k_m)\in\mathbb{Z}_+^m$, we interpret $\mathbb{C}^n$as a product space

$$\mathbb{C}^n=\mathbb{C}^{k_1}\times\cdots\times\mathbb{C}^{k_m}.$$

As is well known, in this situation the standard Hilbert space tensor product decomposition holds

$$\mathcal{F}^2(\mathbb{C}^n)=\mathcal{F}^2(\mathbb{C}^{k_1})\otimes\cdots\otimes\mathcal{F}^2(\mathbb{C}^{k_m}).$$

Similarly we have

$$\mathcal{A}_k:=\mathcal{A}_2(\mathbb{B}^{k_1})\times\cdots\times(\mathbb{B}^{k_n})=\mathcal{A}_2(\mathbb{B}^{k_1})\otimes\cdots\otimes\mathcal{A}_2(\mathbb{B}^{k_m}).$$

We introduce as well the unital Banach algebras: $\mathfrak{T}(\mathcal{E}_{\mathrm{k}}(\mathrm{h}))$, generated by Toeplitz operators on $\mathcal{F}^2(\mathbb{C}^n)$ with symbols in $\mathcal{E}_k(h)$, and $\mathcal{T}(\mathcal{E}_{k,j}(h))$, generated by Toeplitz operators on$\mathcal{F}^2(\mathbb{C}^{\mathrm{k_j}})$ with symbols in $\mathcal{E}_{\mathrm{k,j}}(\mathrm{h})$, where $j=1,\ldots,m$. By Lemma (6.3.21) we have

$$\mathcal{T}(\mathcal{E}_{k,j}(h)\cong\mathfrak{T}(\mathcal{E}_k(h))=\bigotimes_{j=1}^{m}\mathfrak{T}\mathcal{E}_{k,j}(h)\cong\bigotimes_{j=1}^{m}\mathcal{T}(\mathcal{E}_{k,j}(h)).\tag{130}$$

since the choice of the tensor product norm is somehow tricky, some comments to the above formula have to be added. In the setting of C*-algebras the task is simpler and therefore we first extend the algebras $\mathcal{T}(\mathcal{E}_{k,j}(h))$and $\mathcal{T}(\mathcal{E}_{k,j}(h))$ to the corresponding C*-algebras $\mathcal{T}^*(\mathcal{E}_{k,j}(h))$ and $\mathcal{T}^*(\mathcal{E}_{k,j}(h))$ where $j=1,\ldots,m$. Note that all these C*-algebras are of type I. A simple method of proving that $\mathcal{T}^*(\mathcal{E}_{k,j}(h))$ is of type I uses the observation that it is asubalgebra of $\mathcal{T}(C(\overline{\mathbb{B}}^{k_{mj}}k))$,which by its description in Theorem (6.3.23) is a GCR-algebra. Thus these C*-

algebras are nuclear (see, for example, [44,182]). This in turn implies that the C*-cross-norms on the tensor products

$$\bigotimes_{j=1}^{m} \mathfrak{T}^*(\mathcal{E}_{k,j}(h)) \text{ and } \bigotimes_{j=1}^{m} \mathcal{T}^*(\mathcal{E}_{k,j}(h))$$

are uniquely defined and coincide, in particular, with the spatial cross-norm [80,44], being the standard norm of operators acting on the Hilbert spaces $\mathcal{F}^2(\mathbb{C}^n)$) and A_k, respectively. Finally the tensor products of Banach algebras

$$\bigotimes_{j=1}^{m} \mathfrak{T}(\mathcal{E}_{k,j}(h)) \text{ and } \bigotimes_{j=1}^{m} \mathcal{T}(\mathcal{E}_{k,j}(h))$$

In her it the operator norm from their C*-algebra extensions.

Theorem (6.3.34)[302]: The compact set $M(\mathcal{T}_\lambda(\mathcal{E}_k(h)))$ of maximal ideals of the algebra $\mathcal{T}_\lambda(\mathcal{E}_k(h))$is given by

$M\mathcal{T}_\lambda\big(\mathcal{E}_k(h)\big) = \hat{\Lambda}(k_1, h_1) \times\cdots\times \hat{\Lambda}(k_m, h_m)$. (131)

Proof: As was stated before, the description of the algebra $\mathcal{T}_\lambda(\mathcal{E}_k(h))$ does not depend on the weight parameter λ. That is all algebras $\mathcal{T}_\lambda(\mathcal{E}_k(h))$, where $\lambda \in (-1, \infty)$, are isomorphic and thus have the same compact set of maximal ideals. By (110) the compact set of maximal ideals of the algebra $\mathcal{T}(\mathcal{E}_k(h))$ (the unweighted case$\lambda = 0$) coincides with the one of the algebra' $\otimes_{j=1}^{m} \mathcal{T}(\mathcal{E}_{k,j}(h))$. Then by the terminology of [136], the norm on A_k is uniform, and the corresponding operator (spatial) norm is ordinary. By [136],

$$M\left(\bigotimes_{j=1}^{m} \mathcal{T}(\mathcal{E}_{k,j}(h))\right) = M\ (\mathcal{T}(\mathcal{E}_k(h)) \times\cdots\times M\big(\mathcal{T}(\mathcal{E}_{k,m}(h)\big),$$

which, together with Corollary (6.3.34) and there marks after Lemma (6.3.29) finishes the proof.

We now describe the space of maximal ideals in (131) in a different form. As before let $k = (k_1, \dots, k_m)$and consider the following compact subset of the boundary $\partial\mathbb{B}^n$

$$D_{comp} := \left\{(z_{(1)}, \dots, z_{(m)}) \in \mathbb{C}^n : |z_{(j)}| = \frac{1}{\sqrt{m}} \quad for\ j = 1, \dots, m\right\}$$
$$\cong S^{2k_1-1}_{\frac{1}{\sqrt{m}}} \times\cdots\times S^{2k_m-1}_{\frac{1}{\sqrt{m}}}.$$

We interpret the tuple $\psi = (\psi[1], \dots, \psi[m])$ of ordered elementary k-quasi-homogeneous symbols on $\mathbb{C}^n$ as a vector valued function

$$\psi : D_{comp} \to \mathbb{C}^\gamma,$$

Here $\gamma = \sum_{j=1}^{m} h_j(k_j - h_j)$. As a consequence of Theorem (6.3.35) we have:

Corollary (6.3.35)[302]: The compact set $M(T\lambda(Ek(h)))$in(131) coincides with the polynomial convex hull $\hat{\psi}\ (D_{comp})$ of the range $\psi(D_{comp})$.

Proof: It is easy to check that $\psi(D_{comp}) = \Lambda(k_1, h_1) \times\cdots\times \Lambda(k_m, h_m)$. Note that the relation $\hat{X} \times \hat{Y} = \widehat{X \times Y}$ holds for compact subsets $X \subset \mathbb{C}^{n_1}$ and $Y \subset \mathbb{C}^{n_2}$. Hence we have

$$\hat{\psi}\ (D_{comp}) = [\Lambda(k_1, h_1) \times\cdots\times \Lambda(k_m, h_m)] = \hat{\Lambda}(k_1, h_1) \times\cdots\times (k_m, h_m).$$

Now, the assertion follows from Theorem (6.3.35).

We recall some standard notation (see [285]). Given acompact (polynomially convex) set $M \subset \mathbb{C}^q$, we denote by $P(M)$ the closed subalgebra of $C(M)$ consisting of all functions that uniformly on M can be approximated by analytic polynomials. The algebra $A(M)$ is the subalgebra of C(M) consisting of all functions that are analytic on the interior int(M) of M. Note that $A(M) = C(M)$ in the case where $int(M) = \emptyset$.
Recall as well that the inclusion $P(M) \subset A(M)$ holds. Although many partial results (both positive andcounterexamples)areknown,thequestionwhetherthe algebras $P(M)$and $A(M)$ coincide still remains open for general subsets $M \subset \mathbb{C}^q$.
Theorem (6.3.36)[302]: The Gelf and trans form is generated by the following mapping of generators of the algebra $\mathcal{T}(\mathcal{E}_k(h))$

$$T_{\psi_{\ell_j,r_j}} \longmapsto \omega_{j,\ell_j,r_j} \tag{132}$$

Where ψ_{ℓ_j,r_j} is given in (93) and $\omega = (\omega_{1,1,h_1+1}, \ldots, \omega_{j,\ell_j,r_j}, \ldots, \omega_m, h_m, k_m) \in M(\mathcal{T}_\lambda(\mathcal{E}_k(h)))$.
(i)The Gelf and image of the algebra $\mathcal{T}_\lambda(\mathcal{E}_k(h))$ coincides with P(M),where

$$M = M(\mathcal{T}_\lambda(\mathcal{E}_k(h))) = \hat{\Lambda}(k_1, h_1) \times \cdots \times \hat{\Lambda}(k_m, h_m).$$

(ii)The isomorphism $\mathcal{T}_\lambda(\mathcal{E}_k(h))) \to P(M)$ is generated by the mapping (132) of generators of the algebra $(\mathcal{T}_\lambda(\mathcal{E}_k(h)))$.
(iii)In the case where either $h_j = 1$or$h_j = k_j - 1$ for all $j = 1, \ldots, m$ we have that

$$M = \overline{\mathbb{B}}^{k_1-1}\left(0, \frac{1}{2}\right) \times \cdots \times \overline{\mathbb{B}}^{k_m-1}\left(0, \frac{1}{2}\right) \text{ and } P(M) = A(M).$$

List of Symbols

Symbol		Page
$diag$	diagonal	1
$\ominus$	Direct difference	3
min	minimum	14
max	maximum	15
det	Determinant	21
H^{∞}	essential Hardy space	22
H^{p}	Hardy space	22
dim	dimension	23
Ker	kernel	23
L^{1}	Lebesgue space of the real line	24
L^{2}	Hibert space	25
$supp$	support	28
GW	Gleason – Whitny	31
inf	infimum	33
L^{q}	Dual Lebesgue space	34
$\sup$	supremum	34
H^{1}	Hardy space	34
dom	domain	35
H^{2}	Hardy space	41
L^{∞}	essential Lebesgue space	43`
WOT	weak operator topology	47
Aut	Automorphism	48
$\oplus$	orthogonal sum	48

$\otimes$	tensor product	48
lat	lattice	50
Ran	range	53
ℓ^2	Hilbert space of sequences	56
Rep	representation	56
Alg	Algebra	60
$q - Inn$	quasi-inner	63
$dist$	distance	65
im	imaginary	80
tr	trace	86
lex	lexicographic	93
Re	real	96
$proj$	projection	112
arg	argument	131
H_u	Hankel operator	137
$\mathcal{A}_\lambda^2$	Bergman spaces	163
rad	radial	163
$clos$	closure	175
$\widehat{\otimes}_\varepsilon$	Injective tensor product	182
sp	spectrum	185
ess	essential	207
$\odot$	multiplication law	208
$comp$	compact	216

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