Chapter 3
Singular Integral Operators

We will see that some of our proofs immediately extend to analogues of Radon transforms for such singular integral operators.

Section (3.1): Introduction and Theorems in the Multiplicity One Case:

Let \( n \geq 2 \) and let \( b(x) \) be a real-analytic function on a neighborhood of the origin in \( \mathbb{R}^n \) with \( b(0) = 0 \). By resolution of singularities, there is a number \( \delta_0 > 0 \) such that on any sufficiently small neighborhood \( U \) of the origin, \( \int_U |f|^{-\delta} = \infty \) for \( \delta \geq \delta_0 \), and \( \int_U |f|^{-\delta} < \infty \) for \( \delta < \delta_0 \). The number \( \delta_0 \) is sometimes referred to as the "critical integrability exponent" of \( f \) at the origin. We consider operators of the form

\[
Tf(x) = \int_{\mathbb{R}^n} f(x - y) \alpha(x,y)m(y)|b(y)|^{-\delta_0}dy
\]

Here \( \alpha(x,y) \) is a Schwartz function, and \( m(y) \) is a bounded real-valued function on a neighborhood of the origin such that \( m(y)|b(y)|^{-\delta_0} \) satisfies natural derivative and cancellation conditions deriving from \( b(y) \) that allows \( T \) to be considered as a type of singular integral operator. The focus will be to determine the boundedness properties of such \( T \) on \( L^p \) spaces for \( 1 < p < \infty \). Most of our results will concern the \( L^2 \) situation. As we will see, the operators we will consider will generalize local singular integral operators such as local versions of Riesz transforms, and also classes of local multiparameter singular integrals.

We will see that some of the proofs immediately extend to analogues of singular Radon transforms for such singular integral operators. Namely,
results will cover some operators of the following form, where \( x \in \mathbb{R}^m \) and \( h \) is a real-analytic map from a neighborhood of the origin in \( \mathbb{R}^n \) into \( \mathbb{R}^m \) with \( h(0) = 0 \).

\[
T'f(x) = \int_{\mathbb{R}^n} f(x - h(y))a(x,y)m(y)|b(y)|^{-\delta}dy
\]

To help define what types of kernels we allow, we now delve into the resolution of singularities near the origin of a real-analytic function \( b(x) \) with \( b(0) = 0 \). For this we use the resolution of singularities theorem, but other resolution of singularities theorems including Hironaka’s famous work can be used in similar ways.

By [G1], there is a neighborhood \( U \) of the origin such that there exist finitely many coordinate change maps \( \{\beta_i(x)\}_{i=1}^M \) and finitely many vectors \( \{(m_{i1}, \ldots, m_{in})\}_{i=1}^M \) of nonnegative integers such that if \( \rho(x) \) is a nonnegative smooth bump function supported in \( U \) with \( \rho(0) \neq 0 \), then \( \rho(x) \) can be written in the form \( \rho(x) = \sum_{i=1}^M \rho_i(x) \) in such a way that each \( \rho_i \circ \beta_i(x) \), after an adjustment on a set of measure zero, is a smooth nonnegative bump function on a neighborhood of the origin with \( \rho_i \circ \beta_i(0) \neq 0 \). The components of each \( \beta_i(x) \) are real-analytic. In addition, \( \beta_i \) is a bijection from \( \{x : \rho_i \circ \beta_i(x) \neq 0, x_i \neq 0 \text{ for all } i\} \) to \( \{x : \rho_i(x) \neq 0\} - Z_i \) where \( Z_i \) has measure zero, and on a connected neighborhood \( U_i \) of the support of \( \rho_i \circ \beta_i(x) \) the function \( b \circ \beta_i(x) \) is well-defined and “comparable” to the monomial \( x_1^{m_{i1}} \cdots x_n^{m_{in}} \), meaning that there is a real-analytic function \( c_i(x) \) with \( |c_i(x)| > \epsilon > 0 \) on \( U_i \) such that \( b \circ \beta_i(x) = c_i(x)x_1^{m_{i1}} \cdots x_n^{m_{in}} \) on \( U_i \). This decomposition is such that the Jacobian determinant of \( \beta_i(x) \) can be
written in an analogous form \( d_i(x) x_1^{e_{i1}} \ldots x_n^{e_{in}} \) on \( U_i \); again the \( e_{ij} \) are integers and \( |d_i(x)| > \epsilon > 0 \) on \( U_i \).

In view of the above, one has

\[
\int_{\mathbb{R}^n} |b(x)|^{-\delta} \rho(x) \, dx = \sum_{i=1}^{M} \int_{\mathbb{R}^n} |b(x)|^{-\delta} \rho_i(x) \, dx
\]

\[
= \sum_{i=1}^{M} \int_{\mathbb{R}^n} |b \circ \beta_i(x)|^{-\delta} (\rho_i \circ \beta_i(x)) |d_i(x) x_1^{e_{i1}} \ldots x_n^{e_{in}}| \, dx
\]

(41)

\[
= \sum_{i=1}^{M} \int_{\mathbb{R}^n} |c_i(x)|^{-\delta} |d_i(x) x_1^{-\delta m_i + e_{i1}} \ldots x_n^{-\delta m_i + e_{in}}| (\rho_i \circ \beta_i(x)) \, dx
\]

(42)

Since \( \rho_i \circ \beta_i(0) \neq 0 \), the \( i \)th term of the sum (4) is finite if each \(-\delta m_{ij} + e_{ij} > -1\); that is, if \( \delta < \frac{e_{ij}+1}{m_{ij}} \). Thus the number \( \delta_0 \) is given in terms of the resolution of singularities of \( b(x) \) by \( \inf_{i,j} \frac{e_{ij}+1}{m_{ij}} \).

**Definition (3.1.1)[3]:** The multiplicity of the critical integrability exponent \( \delta_0 \) of \( b(x) \) at the origin is the maximum over all \( i \) of the cardinality of \( \left\{ j : \frac{e_{ij}+1}{m_{ij}} = \delta_0 \right\} \).

One example of the significance of the multiplicity is as follows. Let \( B_r \) denote \( \{ x \in \mathbb{R}^n : |x| < r \} \) and let \( m \) denote the multiplicity of the exponent \( \delta_0 \) for \( b(x) \) at the origin. It can be shown that if \( r > 0 \) is sufficiently small then as \( \epsilon \to 0 \) one has asymptotics of the form

\[
|\{ x \in B_r : |b(x)| < \epsilon \}| = c_r \epsilon^{\delta_0 (\ln \epsilon)^{m-1}} + o(\epsilon^{\delta_0 (\ln \epsilon)^{m-1}})
\]

(43)

Here \( c_r > 0 \). One obtains analogous asymptotics for various oscillatory integrals associated with \( b(x) \). Note that (5) shows that the multiplicity is independent of the which resolution of singularities process is being used.
Let $b(x)$ be a real-analytic function on a neighborhood of the origin, not identically zero, with $b(0) = 0$. Let $\rho(x), \{\beta_i(x)\}_{i=1}^M$, and $\{(m_{i1}, \ldots, m_{i\eta_i})\}_{i=1}^M$ be as above. We will define singular integrals associated to $b(x)$ as follows. For a given $i$, we move into the "blown-up" coordinates determined by $\beta_i(x)$ and define a type of singular integral that is of magnitude bounded by $C|x_1^{m_{i1}} \cdots x_n^{m_{i\eta_i}}|^{-\delta_0}$, with corresponding bounds on first derivatives, which is supported on the support of $\rho_i \circ \beta_i(x)$. An appropriate cancellation condition will be assumed that will ensure that the kernels are distributions. A singular integral associated to $b(x)$ will then be defined to be a sum from $i = 1$ to $i = M$ of the blow-downs of such singular integrals into the original coordinates.

Specifically, we consider $k_i(x) = \sum_{(j_1 \ldots j_n) \in \mathbb{Z}^n} k_{i,j_1 \ldots j_n}(x)$, where for some fixed $C_0$ the function $k_{i,j_1 \ldots j_n}(x)$ is $C_1$, supported on $\{x : |x_l| \in [2^{-jl}, C_0 2^{-jl}] \}$ for all $l$, and satisfies

$$|k_{i,j_1 \ldots j_n}(x_1, \ldots, x_n)| < C_0 |x_1^{m_{i1}} \cdots x_n^{m_{i\eta_i}}|^{-\delta_0}$$

(44)

We also assume that for each $l = 1, \ldots, n$ we have

$$|\partial_{x_l} k_{i,j_1 \ldots j_n}(x_1, \ldots, x_n)| < C_0 \frac{1}{|x_l|} C_0 |x_1^{m_{i1}} \cdots x_n^{m_{i\eta_i}}|^{-\delta_0}$$

(45)

The cancellation condition we assume for the multiplicity one case is that for some $\varepsilon_0 > 0$, whenever $i$ and $l$ are such that $\frac{\varepsilon_0 + 1}{m_{il}} = \delta_0$ (the minimum possible value), then where $Jac\beta_i(x)$ denotes the Jacobian determinant of $\beta_i$ we have

$$\left| \int_R k_{i,j_1 \ldots j_n}(x_1, \ldots, x_n) Jac\beta_i(x_1, \ldots, x_n) dx_l \right| < C_0 2^{-\varepsilon_0 jl}$$

(46)
To ensure that our singular integrals are well-defined, we also assume that the support of \( k_{i,j_1,\ldots,j_n}(x) \) is contained in that of \( \rho_i \circ \beta_i(x) \). We next make the following definition.

**Definition (3.1.2):** If \( b(x) \) has multiplicity one at the origin, we define a singular integral kernel associated to \( b(x) \) to be a function \( K(x) \) of the form

\[
K(x) = \sum_{i=1}^{M} \rho_i(x) k_i \left( \beta_i^{-1}(x) \right),
\]

where \( k_i \) satisfies (6)–(8) and the support condition stated afterwards.

One can simply explicitly construct \( K(x) \) satisfying Definition (3.1.2) for any given \( b(x) \) with multiplicity one at the origin, but a familiar example can be derived from local Riesz transforms:

**Example (3.1.3):** Let \( L(x) \) be the local Riesz transform kernel given by

\[
\phi(x) \frac{x_i}{|x|^{n+1}}
\]

for a cutoff function \( \phi(x) \) supported near the origin. Then \( L(x) \) satisfies Definition (3.1.2) for \( b(x) = x_1^2 + \cdots + x_n^2 \). Here \( \delta_0 = \frac{n}{2} \). For one can write

\[
L(x) = \sum_{i=1}^{n} L_i(x)
\]

where \( L_i(x) \) is supported on a cone centered at the \( x_i \)-axis. Then if \( \beta_i(x) = (x_i x_1, \ldots, x_i x_{i-1}, x_i, x_i x_{i+1}, \ldots, x_i x_n) \), the functions \( L_i \circ \beta_i(x) \) will satisfy (6)–(8) with \( x_1^{m_i} \ldots x_n^{m_n} = x_i^{-n} \). Nonisotropic versions of \( L(x) \) will satisfy Definition (3.1.2) for \( b(x) \) of the form \( x_1^{2k_1} + \cdots + x_n^{2k_n} \) for positive integers \( k_1, \ldots, k_n \).

Each \( K(x) \) satisfying Definition (3.1.2) can be viewed in a natural way as a distribution as follows. Let \( k_{iL}(x) \) denote the truncated version of \( k_i(x) \) given by

\[
k_{iL}(x) = \sum_{(j_1,\ldots,j_n) \in \mathbb{Z}^n: j_l < L \text{ for all } l} k_{i,j_1,\ldots,j_n}(x) \quad (47)
\]
Define the corresponding truncated $K_L(x)$ by $K_L(x) = \sum_{i=1}^M K_{iL}(x)$, where $K_{iL}(x) = \rho_i(x) k_{iL} \left( \beta_i^{-1}(x) \right)$. Note that $K_L(x)$ is a smooth compactly supported function. If $\phi(x)$ is a Schwartz function, then one has

$$\int_{\mathbb{R}^n} K_L(x) \phi(x) dx = \sum_{i=1}^M \int_{\mathbb{R}^n} \rho_i(x) k_{iL} \left( \beta_i^{-1}(x) \right) \phi(x) dx$$

$$= \sum_{i=1}^M \int_{\mathbb{R}^n} \left( \rho_i \circ \beta_i(x) \right) k_{iL}(x) J a c \beta_i(x) \left( \phi \circ \beta_i(x) \right) dx \quad (48)$$

$$= \sum_{i=1}^M \int_{\mathbb{R}^n} \sum_{(j_1, \ldots, j_n): j_l \leq L \text{ for all } l} \left( \rho_i \circ \beta_i(x) \right) k_{i,j_1,\ldots,j_n}(x) J a c \beta_i(x) \left( \phi \circ \beta_i(x) \right) dx \quad (49)$$

Let $\psi_i(x) = \left( \rho_i \circ \beta_i(x) \right) \left( \phi \circ \beta_i(x) \right)$. Then $\psi_i(x)$ is a smooth compactly supported function. Note that we do use the fact from that the function $\beta_i(x)$ is defined and smooth on a neighborhood of the support of $\rho_i \circ \beta_i(x)$ so that there are no issues concerning the smoothness of $\phi \circ \beta_i(x)$ on the boundary of the support of $\rho_i \circ \beta_i(x)$. Thus we can rewrite the expression (11) for $\int_{\mathbb{R}^n} K_L(x) \phi(x) dx$ as

$$= \sum_{i=1}^M \int_{\mathbb{R}^n} \sum_{(j_1, \ldots, j_n): j_l \leq L \text{ for all } l} k_{i,j_1,\ldots,j_n}(x) J a c \beta_i(x) \psi_i(x) dx \quad (50)$$

If $i$ is such that $|x_1^{m_{i1}} \ldots x_n^{m_{in}}|^{-\delta_0} |J a c \beta_i(x)| \sim |x_1^{e_{i1}} \ldots x_n^{e_{in}}|^{-\delta_0} |x_1^{e_{i1}} \ldots x_n^{e_{in}}|$ is integrable on a neighborhood of the origin, then by (6), the form of the $i$th term of (12) ensures that the kernel $K_{iL}(x)$ is a distribution that converges as $L \to \infty$ to a finite measure which we denote by $K_i(x)$.

Next, we show that for the $i$ for which $|x_1^{m_{i1}} \ldots x_n^{m_{in}}|^{-\delta_0} |J a c \beta_i(x)|$ is not integrable, the cancellation condition (8) ensures that such an $K_{iL}$ too converges as $L$ goes to infinity in the distribution sense to some $K_i(x)$. We
will then define $K(x) = \sum_{i=1}^{M} K_i(x)$. To see why this is the case, note that since $b(x)$ has multiplicity one, for each such $i$ there is exactly one value $l_0$ for which $\frac{e_{il_0} + 1}{m_{il_0}} = \delta_0$, and $\frac{e_{il} + 1}{m_{il}} > \delta_0$ for all other values of $l$. Write $\psi_i(x) = \psi_i(x_1, \ldots, x_{l_0-1}, 0, x_{l_0+1}, \ldots, x_n) + x_{l_0} \xi_i(x_1, \ldots, x_n)$, with $\xi_i$ smooth.

The $i$th term of (12) can be written as the sum of two terms. In the first, $\psi_i(x)$ is replaced by $\psi_i(x_1, \ldots, x_{l_0-1}, 0, x_{l_0+1}, \ldots, x_n)$ and in the second $\psi_i(x)$ is replaced by $\xi_i(x)$ and $k_{i,j_1,\ldots,j_n}(x)$ by $x_{l_0} k_{i,j_1,\ldots,j_n}(x)$. The second term is handled exactly as we handled the terms for which $|x_1^{m_1} \cdots x_n^{m_n}|^{-\delta_0} \mathcal{F} \psi_i(x)$ is integrable since the additional $x_{l_0}$ factor causes us to once again have absolute integrability of the limiting kernel. As for the first term, we perform the $x_{l_0}$ integration first in the $i$th term of (12). The cancellation condition (8) implies that the limiting kernel of the result of this integration is similarly absolutely integrable in the remaining $n-1$ variables, and thus the limit again defines a distribution.

Thus we see that $K_i(x)$ is a well defined distribution for all $i$ and therefore $K(x) = \sum_{i=1}^{M} K_i(x)$ gives a well-defined distribution. Hence if $\alpha(x,y)$ is a Schwartz function on $R^{n+m}$ and $f(x)$ is a Schwartz function on $R^n$ then $Tf(x) = \int_{R^n} f(x-y) \alpha(x,y) K(y) \, dy$ is well-defined. If for some $1 < p < \infty$ and some constant $C$ the operators $T_Lf(x) = \int_{R^n} f(x-y) \alpha(x,y) K_L(y) \, dy$ are such that $\|T_L\|_{L^p \to L^p} \leq C$ for all Schwartz functions $f$ and all $L$, then an application of the dominated convergence theorem gives that one also has $\|Tf\|_{L^p(R^n)} \leq C \|f\|_{L^p(R^n)}$ for all Schwartz functions, for the same constant $C$.

The first theorem is simply that $T$ is bounded on $L^2(R^n)$. 

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Theorem (3.1.4)[3]: Whenever the critical integrability exponent of \( b(x) \) at the origin has multiplicity one, then there is a neighborhood \( U \) of the origin such that if \( K(x) \) is supported in \( U \), there is a constant \( C \) such that for all Schwartz functions \( f(x) \) one has \( \|Tf\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)} \).

It turns out that it is no harder prove \( L^2 \) boundedness for singular Radon transform generalizations of \( T \). Namely, let \( K(x) \) be as above, and let \( h_1(x), \ldots, h_m(x) \) be real-analytic functions on a neighborhood of the origin in \( \mathbb{R}^n \) with \( h_i(x) = 0 \) for all \( i \). Let \( \alpha(x,y) \) be a Schwartz function on \( \mathbb{R}^m \times \mathbb{R}^n \). Then for a Schwartz function \( f(x) \) in \( m \) variables, we define \( T'f(x) \) by

\[
T'f(x) = \int_{\mathbb{R}^n} f(x_1 - h_1(y), \ldots, x_m - h_m(y)) \alpha(x,y)K(y)dy \quad (51)
\]

The operator \( T \) above corresponds to \( m = n \) and \( h_i(y) = y_i \) for all \( i \). We have the following theorem.

Section (3.2): Theorems when the Multiplicity is Greater than One

When the critical integrability exponent \( \delta_0 \) has multiplicity greater than one at the origin, the coordinate changes \( \beta_i(x) \) used in the multiplicity one case will lead to trying to prove \( L^2 \) boundedness of an operator that resembles a multiparameter singular Radon transform, rather than a (one-parameter) singular Radon transform. Unfortunately since the \( \beta_i(x) \) here involve blowups, one often ends out with a multiparameter singular Radon transform that is not bounded on \( L^2 \). As a result, instead of trying to find a general correct notion of singular integral and prove a general result, when the multiplicity is greater than one we will focus on theorems that can be proven in the original coordinates.
One can often determine the critical integrability exponent of a function at the origin and its multiplicity through the use of Newton polyhedron of the function. We turn to the relevant definitions.

**Definition (3.2.1)[3]:** Let \( b(x) \) be a real-analytic function with Taylor series \( \sum_{\alpha} b_\alpha x^\alpha \) on a neighborhood of the origin. For each \( \alpha \) for which \( b_\alpha \neq 0 \), let \( Q_\alpha \) be the octant \( \{ t \in \mathbb{R}^n : t_i \geq \alpha_i \text{ for all } i \} \). The Newton polyhedron \( N(b) \) of \( b(x) \) is defined to be the convex hull of all \( Q_\alpha \).

A Newton polyhedron can contain faces of various dimensions in various configurations. The faces can be either compact or unbounded. We consider each vertex of \( N(b) \) to be a compact face of dimension zero.

**Definition (3.2.2)[3]:** Let \( F \) be a compact face of \( N(b) \). Then if \( b(x) = \sum_{\alpha} b_\alpha x^\alpha \) denotes the Taylor expansion of \( b \) like above, we define \( b_F(x) = \sum_{\alpha \in F} b_\alpha x^\alpha \).

We will also use the following terminology.

**Definition (3.2.3)[3]:** Assume \( N(b) \) is nonempty. Then the Newton distance \( d(b) \) of \( b(x) \) is defined to be \( \inf \{ t : (t,t,\ldots,t,t) \in N(b) \} \).

**Definition (3.2.4)[3]:** The central face of \( N(b) \) is the face of \( N(b) \) of minimal dimension intersecting the line \( t_1 = t_2 = \cdots = t_n \).

In Definition (3.2.4), the central face of \( N(b) \) is well-defined since it is the intersection of all faces of \( N(b) \) intersecting the line \( t_1 = t_2 = \cdots = t_n \). An equivalent definition that can be used is that the central face of \( N(b) \) is the unique face of \( N(b) \) that intersects the line \( t_1 = t_2 = \cdots = t_n \) in its interior.

We showed that if the zeros of each \( b_F(x) \) on \( (\mathbb{R} - \{0\})^n \) are of order less than \( d(b) \), then the critical integrability index \( \delta_0 \) is equal to \( \frac{1}{d(b)} \) and the multiplicity is equal to \( n \) minus the dimension of the central face of \( N(b) \).
This can be used to compute $\delta_0$ and its multiplicity for specific examples of interest, such as in the following two examples. Suppose $b(x) = x_1^{k_1} + \cdots + x_n^{k_n}$ with each $k_i$ even. Then $\delta_0 = \left(\frac{1}{k_1} + \cdots + \frac{1}{k_n}\right)^{-1}$ and $m = 1$. On the other hand, if $b(x) = x_1^{l_1} + \cdots + x_n^{l_n}$ then $\delta_0 = \frac{1}{\max l_i}$ and $m$ is equal to the number of times $k$ that $\max l_i$ appears in $\{l_1, \ldots, l_n\}$. For in the former case the line $t_1 = \cdots = t_n$ intersects $N(b)$ in the interior of the $n - 1$ dimensional face with equation $\frac{t_1}{k_1} + \cdots + \frac{t_n}{k_n} = 1$, while in the latter case the line $t_1 = \cdots = t_n$ intersects $N(b)$ in the $n - k$ dimensional plane determined by the equations $t_l = \max l_i$ for all $l$ such that $l = \max l_i$.

In order to understand the behavior of functions satisfying the finite-type condition, it is often helpful to consider the function $b^*(x)$ defined by

$$b^*(x) = \sum_{(v_1, \ldots, v_n) \text{ a vertex of } N(b)} |x_1^{v_1} \cdots x_n^{v_n}| \quad (52)$$

There is a constant $C$ such that for all $x$ one has $|b(x)| \leq C b^*(x)$. In Lemma (4.2.1) we will see that given any $\delta > 0$ there is a $\delta' > 0$ such that $|b(x)| > \delta' b^*(x)$ on a portion of any dyadic rectangle with measure at least $1 - \delta$ times that of the rectangle. Hence $|b(x)| \sim b^*(x)$ except near the zeroes of $|b(x)|$.

Next, observe that the Newton polygon of any first partial $\partial_{x_l} b(x)$ is a subset of the shift of $N(b)$ by $-1$ units in the $x_l$ direction. Hence the above considerations tell us that

$$|\partial_{x_l} b(x)| \leq C \frac{1}{|x_l|} b^*(x) \quad (53)$$

If $b(x) \neq 0$, we also have
Singular integrals when the multiplicity is greater than one.

When the multiplicity is greater than one, the class of \( \mathcal{A}(\mathbf{x}) \) where we will prove \( L^2 \) boundedness of associated singular integrals are the \( \mathcal{B}(\mathbf{x}) \) analyzed that were discussed above (52). Namely, using the terminology of Definitions (3.2.1)-(3.2.4), we will assume that for each compact face \( F \) of \( N(b) \), each zero of each \( b_F(x) \) in \( (R - \{0\})^n \) has order less than \( d(b) \). As mentioned above, in this situation we have \( \delta_0 = \frac{1}{d(b)} \). Note that our theorems do not require the multiplicity to be greater than one, and in fact the theorems here will include some multiplicity one operators not covered by Theorem (3.1.4).

In the situation at hand, we define a singular integral associated to \( b(x) \) as follows. Let \( \alpha(x, y) \) be a Schwartz function on \( R^{n+n} \). We will consider kernels of the form \( \alpha(x, y)K(y) \), where \( K(y) \) is as follows. We assume that \( K(y) \) can be written as \( \sum_{(j_1, \ldots, j_n) \in Z^n} K_{j_1, \ldots, j_n}(y) \) such that for some fixed \( C_1 \) the function \( K_{j_1, \ldots, j_n}(y) \) is supported on \( \{x : |x| \in [2^{-j_l}, C_1 2^{-j_l}] \text{ for all } l\} \), is \( C_1 \) on \( \{x : b(x) \neq 0\} \), and satisfies the estimates

\[
|K_{j_1, \ldots, j_n}(y_1, \ldots, y_n)| < C_1 |b(y_1, \ldots, y_n)|^{-\delta_0} \quad (55)
\]

Motivated by (17), we also assume that if \( b(y_1, \ldots, y_n) \neq 0 \) then for each \( l \) we have

\[
|\partial_{y_l} K_{j_1, \ldots, j_n}(y_1, \ldots, y_n)| < C_1 \frac{1}{|y_l|} b^*(y_1, \ldots, y_n)|b(y_1, \ldots, y_n)|^{-1-\delta_0} \quad (56)
\]

We further assume the cancellation conditions that for each \( l \) we have

\[
\int_{R} K_{j_1, \ldots, j_n}(y_1, \ldots, y_n) dy_l = 0 \quad (57)
\]
In Lemma (3.2.7), we will see that in the settings of our theorems (Definitions (3.2.1) and (3.2.2)) each \( K_{j_1, \ldots, j_n}(y_1, \ldots, y_n) \) is integrable, so (57) makes sense if we assume it holds whenever \( K_{j_1, \ldots, j_n}(y_1, \ldots, y_n) \) is integrable in the \( y_i \) variable for fixed values of the other \( y \) variables.

We will also assume that \( K_{j_1, \ldots, j_n}(y_1, \ldots, y_n) \) is identically zero when 
\[
[2^{-j_1}, C_1 2^{-j_1}] \times \cdots \times [2^{-j_n}, C_1 2^{-j_n}]
\] is not contained in a certain neighborhood of the origin to be determined by our arguments.

Some motivation for our definition of a singular integral associated to \( b(x) \) is the fact that for traditional multiparameter singular integrals, often a sufficient and necessary condition for \( L^p \) boundedness is that the kernel be expressible as a dyadic sum of terms satisfying standardized estimates as well as a cancellation condition.

**Example (3.2.5)[3]:** Let \( b(x) = x_1^{a_1} \cdots x_n^{a_n} \) for nonnegative integers \( a_1, \ldots, a_n \) with at least one \( a_i \) being nonzero. Then \( \delta_0 = \frac{1}{\max a_i} \) here. If \( \phi(x) \) is a cutoff function supported on a sufficiently small neighborhood of the origin, 
\[ K(x) = (-1)^{\text{sgn}(x_1) + \cdots + \text{sgn}(x_n)} \phi(x_1^2, \ldots, x_n^2) |b(x)|^{-\delta_0} \] will satisfy (55)–(57).

In particular, \( K(x) = (-1)^{\text{sgn}(x_1) + \cdots + \text{sgn}(x_n)} \phi(x_1^2, \ldots, x_n^2) \frac{1}{|x_1, x_n|} \) qualifies.

**Example (3.2.6)[3]:** Let \( f(x_1, \ldots, x_n) \) be any real-analytic function with \( f(0, \ldots, 0) = 0 \), and let \( b(x) = f(x_1^2, \ldots, x_n^2) \). Then if \( \phi(x) \) is a cutoff function supported on a sufficiently small neighborhood of the origin, 
\[ K(x) = (-1)^{\text{sgn}(x_1) + \cdots + \text{sgn}(x_n)} \phi(x_1^2, \ldots, x_n^2) |b(x)|^{-\delta_0} \] will satisfy (55)–(57).

For a fixed value of \( x \), the function \( \alpha(x, y)K(y) \) can be viewed in a natural way as a distribution in the \( y \) variable as follows. This will resemble the discussion following (47). Let \( K_L(y) = \sum_{j_i < L} \text{for all } i K_{j_1, \ldots, j_n}(y) \) and let \( \phi(y) \) be a Schwartz function. Then
\[
\int_{\mathbb{R}^n} \alpha(x, y) K_L(y) \phi(y) dy \\
= \int_{\mathbb{R}^n} \sum_{j_l < L \text{ for all } l} K_{j_1, \ldots, j_n}(y) \alpha(x, y) \phi(y) dy_1 \ldots dy_n \quad (58)
\]
Let \( \sigma_x(y) = \alpha(x, y) \phi(y) \). Then we may write
\[
\sigma_x(y) = \sigma_x(0, y_2, \ldots, y_n) + y_1 \xi_x(y_1, \ldots, y_n)
\]
for some smooth \( \xi_x(y_1, \ldots, y_n) \). Then the right-hand side of (58) can be rewritten as
\[
\int_{\mathbb{R}^n} \sum_{j_l < L \text{ for all } l} K_{j_1, \ldots, j_n}(y_1, \ldots, y_n) \sigma_x(0, y_2, \ldots, y_n) dy_1 \ldots dy_n
\]
\[
+ \int_{\mathbb{R}^n} \sum_{j_l < L \text{ for all } l} K_{j_1, \ldots, j_n}(y_1, \ldots, y_n) y_1 \xi_x(y_1, \ldots, y_n) dy_1 \ldots dy_n \quad (59)
\]
Because of the cancellation condition (20) in the \( y_1 \) variable, the first integral of (59) is zero. We next similarly write
\[
\xi_x(y_1, \ldots, y_n) = \xi_x(y_1, 0, y_3, \ldots, y_n) + y_2 \tilde{\xi}_x(y_1, \ldots, y_n)
\]
and insert it into (59), obtaining
\[
\int_{\mathbb{R}^n} \sum_{j_l < L \text{ for all } l} K_{j_1, \ldots, j_n}(y_1, \ldots, y_n) y_1 y_2 \tilde{\xi}_x(y_1, \ldots, y_n) dy_1 \ldots dy_n \quad (60)
\]
Going through all the \( y_l \) variables in this way, we see that
\[
\int_{\mathbb{R}^n} \alpha(x, y) K_L(y) \phi(y) dy
\]
is equal to an expression
\[
\int_{\mathbb{R}^n} \sum_{j_l < L \text{ for all } l} K_{j_1, \ldots, j_n}(y_1, \ldots, y_n) y_1 y_2 \eta_x(y_1, \ldots, y_n) dy_1 \ldots dy_n \quad (61)
\]
Here \( \eta_x(y_1, \ldots, y_n) \) is smooth in both the \( x \) and \( y \) variables. We will see in Lemma (3.2.7) that the condition on the order of the zeroes of the functions \( b_F(x) \) on \( (R - \{0\})^n \) implies that the integral of \( |b(x)|^{-\delta_0} \) over any dyadic rectangle in \( U \) is uniformly bounded. Thus (55) implies that...
$K_{j_1,\ldots,j_n}(y)y_1y_2\ldots y_n$ is absolutely integrable over $U$. Hence $\alpha(x, y)K(y)$ is naturally a distribution in $y$ when $K(y)$ is supported in $U$ if its action on $\phi(y)$ is given by

$$\langle \alpha(x, y)K(y), \phi(y) \rangle$$

$$= \int_{\mathbb{R}^n} \sum_{(j_1, \ldots, j_n) \in \mathbb{Z}^n} K_{j_1,\ldots,j_n}(y_1,\ldots,y_n)y_1y_2\ldots y_n\eta_x(y_1,\ldots,y_n)dy_1\ldots dy_n \quad (62)$$

One can then use (61) to define $Tf(x) = \int_{\mathbb{R}^n} f(x-y)\alpha(x, y)K(y)dy$ for Schwartz functions $f$, and then examine boundedness of such integral operators on $L^p$ spaces. We have the following theorem in this regard for $p = 2$. 
Chapter 4

Kernels Associated with Negative Power of Real-Analytic Functions

We determine the bounded properties $L^p$ spaces for $1 < p < \alpha$ and we will generalize local singular integral operators such as local revisions of Pierz transforms.

Section (4.1): Proofs of Theorems When the Multiplicity is Equal to One:

Since Theorem (3.1.4) is a special case of Theorem (3.1.5), we prove Theorem (3.1.5).

Theorem (4.1.1)\[4]\: Whenever the critical integrability exponent of $b(x)$ at the origin has multiplicity one, then there is a neighborhood $U$ of the origin such that if $K(x)$ is supported in $U$, there is a constant $C$ such that for all Schwartz functions $f(x)$, one has $\|T'f\|_{L^2(R^m)} \leq C\|f\|_{L^2(R^m)}$.

To give a rough idea of how our proofs will work, note that (50) can be written as

$$\sum_{i=1}^{M} \int_{\mathbb{R}^n} f(x_1 - h_1(y), \ldots, x_m - h_m(y))\alpha(x, y)\rho_i(y)k_i\left(\beta_i^{-1}(y)\right)dy \quad (63)$$

Let $T_i$ be the operator corresponding to the $i$th term of (51). Doing a change of variables from $y$ to $\beta_i(y)$ in the integral of (51) leads to

$$T_if(x) = \int_{\mathbb{R}^n} f(x_1 - h_1 \circ \beta_i(y), \ldots, x_m - h_m \circ \beta_i(y))\alpha(x, \beta_i(y))$$

$$\left(\rho_i \circ \beta_i(y)\right)k_i(y)\text{Jac}_{\beta_i}(y)dy \quad (64)$$

$T_i$ is a sort of singular Radon transform with kernel $\alpha(x, \beta_i(y))(\rho_i \circ \beta_i(y))k_i(y)\text{Jac}_{\beta_i}(y)$, which we will be able to analyze by reducing to singular Radon transform estimates.
Proof: Theorem (3.1.5) will follow if we can prove each that for each \( i \) there is a constant \( C \) such that \( \| T_i f \|_{L^2(R^m)} \leq C \| f \|_{L^2(R^m)} \) for all Schwartz functions \( f(x) \). Here \( T_i \) is as in (53). Let \( m_{ij} \) and \( e_{ij} \) be the monomial exponents of \( b \circ \beta_i(x) \) and \( Jac \beta_i(x) \) as before. If \( i \) is such that \( \delta_0 < \frac{1 + e_{il}}{m_{ij}} \) for each \( j \), then the kernel of \( T_i \) is absolutely integrable and \( L^2 \) boundedness is immediate. Thus it suffices to consider only the \( i \) for which there is some \( l \) for which \( \delta_0 = \frac{1 + e_{il}}{m_{il}} \). Since we are assuming \( b(x) \) has multiplicity one, there will only be one such \( l \) for each such \( i \). Writing \( h \circ \beta_i(y) = (h_1 \circ \beta_i(y), \ldots, h_m \circ \beta_i(y)) \), can be rewritten as

\[
T_i f(x) = \int_{R^n} f(x - h \circ \beta_i(y)) \alpha(x, \beta_i(y))(\rho_i \circ \beta_i(y))k_i(y) Jac \beta_i(y) dy \quad (65)
\]

It is more convenient for our proofs that there be no nonzero \( \lambda \) such that \( \lambda \cdot (h \circ \beta_i(y)) \) is a linear function of \( y \). This can be accomplished as follows. If there is a nonzero \( \lambda \) such that \( \lambda \cdot (h \circ \beta_i(y)) \) is the zero function, then on each hyperplane orthogonal to \( \lambda \), the operator \( T_i \) restricts to an operator of the same type as \( T_i \) here, except the ambient space is of one lower dimension. Repeating as necessary, we may assume that \( \lambda \cdot (h \circ \beta_i(y)) \) is never the zero function for any nonzero \( \lambda \). It is worth mentioning that we are using the fact that \( h \circ \beta_i(y) \) extends to a connected neighborhood of the support of \( \rho_i \circ \beta_i(y) \) to ensure we don’t have different functions on different connected components to worry about.

One can further ensure that \( \lambda \cdot (h \circ \beta_i(y)) \) is never linear by letting \((y_1, \ldots, y_n) = (z_1^3, \ldots, z_n^3) \) with the corresponding change from \( \beta_i(y_1, \ldots, y_n) \) to \( \tilde{\beta}_i(z) = \beta_i(z_1^3, \ldots, z_n^3) \). The exponents \( m_{ij} \) and \( e_{ij} \) can change, but
(44)--(46) and the other properties from resolution of singularities that we are using will still hold. Thus in the following, without loss of generality we will always assume that we are working in a situation where $\lambda \cdot (h \circ \beta_i(y))$ is not linear for any nonzero $\lambda$.

Let $k_{iL}(x) = \sum_{(j_1,\ldots,j_n) : j_i < L} k_{i,j_1,\ldots,j_n}(x)$ as in (9), and let $T_{iL}$ be defined by

$$T_{iL}f(x) = \int_{\mathbb{R}^n} f(x - h \circ \beta_i(y))\alpha(x,\beta_i(y))(\rho_i \circ \beta_i(y))k_{iL}(y)jac\beta_i(y)dy$$ (66)

In order to prove Theorem (3.1.5), it suffices to show that $T_{iL}$ is bounded on $L^2$ with bounds uniform in $L$. We can reduce to the case where $(\rho_i \circ \beta_i(y))\alpha(x,\beta_i(y))$ is replaced by a function of $y$ (i.e. the operator is translation-invariant) through the following lemma.

**Lemma (4.1.2)[4]:** Define $U_{iL}$ by

$$U_{iL}f(x) = \int_{\mathbb{R}^n} f(x - h \circ \beta_i(y))k_{iL}(y)jac\beta_i(y)dy$$ (67)

If there is a constant $C$ depending on $b(x)$ (and the resolution of singularities procedure we are using on it), $h_1(x),\ldots,h_m(x)$, and the constant $C_0$ of (44)--(46), such that $\|U_{iL}f\|_p \leq C\|f\|_p$ for all Schwartz $f$ and all $L$, then there is a constant $C'$ such that $\|U_{iL}f\|_p \leq C'\|f\|_p$ for all Schwartz $f$ and all $L$.

**Proof.** Let $\gamma(x,y)$ be the Schwartz function $(\rho_i \circ \beta_i(y))\alpha(x,\beta_i(y))$. We use the Fourier inversion formula in the $x$ variable and write

$$\gamma(x,y) = \int_{\mathbb{R}^m} \tilde{\gamma}(t,y)e^{it_1x_1+\ldots+it_mx_m}dt$$ (68)

Here $\tilde{\gamma}(t,y)$ refers to the Fourier transform in the $x$ variable only. If $f$ and $g$ are Schwartz functions then $\int_{\mathbb{R}^n} T_{iL}f(x)g(x)dx$ is equal to
\[
\int \left[ \int \left[ \int f(x - h \circ \beta_i(y)) \tilde{y}(t, y) k_{iL}(y) \Jac_{\beta_i}(y) \, dy \right] \left( e^{it_1 x_1 + \cdots + it_m x_m} g(x) \right) \, dx \right] \, dt \tag{69}
\]

Stated another way, let \( U_{iLt} \) denote the operator

\[
U_{iLt} f(x) = \int f(x - h \circ \beta_i(y)) \tilde{y}(t, y) k_{iL}(y) \Jac_{\beta_i}(y) \, dy \tag{70}
\]

Then \( \int_{\mathbb{R}^n} T_{iL} f(x) \, g(x) \, dx \) is equal to

\[
\int \int \int U_{iLt} f(x) \left( e^{it_1 x_1 + \cdots + it_m x_m} g(x) \right) \, dx \, dt \tag{71}
\]

Since \( \tilde{y}(t, y) \) is Schwartz, under the assumptions of this lemma there is a constant \( K \) such that

\[
\| U_{iLt} \|_{L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)} \leq K \frac{1}{1 + |t|^{m+1}} \tag{72}
\]

Thus by (72) and Hölder's inequality we have

\[
\int_{\mathbb{R}^n} T_{iL} f(x) \, g(x) \, dx \leq K \| f \|_p \| g \|_{p'} \int_{\mathbb{R}^m} \frac{1}{1 + |t|^{m+1}} \, dt \tag{73}
\]

Thus the \( T_{iL} \) are bounded on \( L^p \) uniformly in \( L \) and we are done with the proof of Lemma (4.1.1).

We now proceed to proving uniform bounds on the \( U_{iL} \). Taking Fourier transforms, we get

\[
\hat{U}_{iL}(\lambda) = \hat{f}(\lambda) \int_{\mathbb{R}^n} e^{i\lambda \cdot (h \circ \beta_i(y))} k_{iL}(y) \Jac_{\beta_i}(y) \, dy \quad \tag{74}
\]
Hence in order to prove Theorem (3.1.5), it suffices to show that there is a constant $C$ such that $|B_{iL}(\lambda)| \leq C$ for each $i, L$ and $\lambda$, where

$$B_{iL}(\lambda) = \int_{\mathbb{R}^n} e^{i\lambda \cdot (\mathbf{h} \circ \beta_i(y))} k_{iL}(y) \text{Jac}_i(y) \, dy$$

(75)

Without loss of generality, to simplify notation in the following we will assume that the $l$ for which $\delta_0 = \frac{1+e_{il}}{m_{il}}$ is $l = 1$. Next, we write the factor $k_{iL}(y) \text{Jac}_i(y)$ in (75) as $\sum_{m<\lambda} p_{imL}(y)$, where we add over the dyadic pieces in the $y_2, \ldots, y_n$ variables to form $p_{imL}(y)$:

$$p_{imL}(y) = \sum_{(j_2, \ldots, j_n) : j_l < L \text{ for all } l > 1} k_{i,m,j_2,\ldots,j_n}(y) \text{Jac}_i(y)$$

(76)

Then (44) implies the estimates

$$|p_{imL}(y)| \leq C |y_1|^{-\delta_0 m_{i1} + e_{i1}} \ldots |y_n|^{-\delta_0 m_{in} + e_{in}}$$

(77)

Similarly, (44)–(45) implies that for each $l$ we have

$$|\partial_{y_l} p_{imL}(y)| \leq C \frac{1}{|y_l|} |y_1|^{-\delta_0 m_{i1} + e_{i1}} \ldots |y_n|^{-\delta_0 m_{in} + e_{in}}$$

(78)

The cancellation condition (46) gives

$$\left| \int_{\mathbb{R}} p_{imL}(y_1, \ldots, y_n) \, dy_1 \right| < C 2^{-e_0 m}$$

(79)

Note that $-\delta_0 m_{i1} + e_{i1} = -1$ here, while the other exponents are all greater than $-1$. If one changes variables $y_l = z_l^N$, in (75) for some $l > 1$, instead of having a factor $|y_l|^{-\delta_0 m_{il} + e_{il}}$ in (77)–(78) one has a factor of $|z_l|^{-\delta_0 m_{il} + e_{il}N}$. One also gains an additional factor of $N |z_l|^{N-1}$ from the Jacobian of the coordinate change. Thus overall one has a factor of $|z_l|^{(-\delta_0 m_{il} + e_{il}+1)N-1}$. Since $-\delta_0 m_{il} + e_{il} > -1$, if $N$ is large enough this factor will be bounded by just $|z_l|$ and thus one can remove the $y_l$ variable from (77)–(78). Stated
another way, if one changes $y_l = z_i^N$ for all $l > 1$ (making $N$ odd to ensure it’s a one-to-one map), (75) becomes

$$B_{iL}(\lambda) = \int_{R^n} e^{i\lambda \cdot (h \circ \beta_i)(z_1 z_2^N, \ldots, z_N^N)} q_{iL}(z) d\lambda$$  \hspace{1cm} (80)$$

Here $q_{iL}(z) = \sum_{m < L} q_{imL}(z)$, where $q_{imL}(z)$ satisfies

$$|q_{imL}(z)| \leq C |z_1|^{-1}$$ \hspace{1cm} (81)$$

$$|\partial_{z_1} q_{imL}(z)| \leq C |z_1|^{-2}, \quad \forall l > 1 |\partial_{z_l} q_{imL}(z)| \leq C |z_1|^{-1}$$ \hspace{1cm} (82)$$

$$\left| \int_{R} q_{imL}(z_1, \ldots, z_n) d\lambda \right| \leq C 2^{-E_0 m}$$ \hspace{1cm} (83)$$

Letting $f_i(z) = h \circ \beta_i(z_1, z_2^N, \ldots, z_N^N)$, (80) becomes

$$B_{iL}(\lambda) = \int_{R^n} e^{i\lambda \cdot f_i(z)} q_{iL}(z) d\lambda$$ \hspace{1cm} (84)$$

In view of the discussion above (84), we may assume that $\lambda \cdot f_i(z)$ is not linear for any nonzero $\lambda$. Therefore, writing $\lambda = |\lambda| \omega$ for $\omega \in S^{m-1}$, by a compactness argument on $S^{m-1} \times \text{supp} (q_{iL})$, we may restrict consideration to $\omega$ to a small neighborhood $N$ in $S^{m-1}$ and replace $q_{iL}(x)$ by $\sigma(x) q_{iL}(x)$ for a function $\sigma(x)$ supported on a ball $B(x_0, r_0)$ on which there is an $\epsilon > 0$ and a single directional derivative $\partial_{\nu}$ such that $|\partial_{\nu}^\alpha (\omega \cdot f_i)(z)| > \epsilon$ on $B(x_0, r_0)$ for some $\alpha \geq 2$. We can do this in such a way that $\nu$ has a positive $x_1$ component. Thus if $\bar{x}$ denotes the first component of $x_0$, for $\omega \in N$ we are attempting to bound $\int_{R^{n-1}} D_{iL}(\lambda, z_2, \ldots, z_n) dz_2 \ldots dz_n$, where

$$D_{iL}(\lambda, z_2, \ldots, z_n) = \int_{R} e^{i |(\omega \cdot f_i)(\bar{x}, z_2, \ldots, z_n) + tv)\sigma((\bar{x}, z_2, \ldots, z_n) + tv)} q_{iL}((\bar{x}, z_2, \ldots, z_n) + tv) dt$$ \hspace{1cm} (85)$$
Thus to prove Theorem (3.1.5), it suffices for our purposes to bound $D_{iL}(\lambda, z_2, \ldots, z_n)$ uniformly in $L, \lambda, z_2, \ldots, z_n$ for $\omega \in \mathbb{N}$. For this, it suffices to bound $\tilde{D}_{iL}(\lambda, \tilde{L}, z_2, \ldots, z_n)$ uniformly in $L, \lambda, \tilde{L}, z_2, \ldots, z_n$ for $\omega \in \mathbb{N}$, where

$$
\tilde{D}_{iL}(\lambda, \tilde{L}, z_2, \ldots, z_n) = \int_{\mathbb{R}} e^{i|\lambda|(\omega \cdot f_i)((\bar{x}, z_2, \ldots, z_n) + tv) + i\tilde{L}t} \sigma((\bar{x}, z_2, \ldots, z_n) + tv) \times q_{iL}((\bar{x}, z_2, \ldots, z_n) + tv)dt \tag{86}
$$

But similarly to (74)–(75), such uniform bounds for $\tilde{D}_{iL}(\lambda, \tilde{L}, z_2, \ldots, z_n)$ follows from uniform boundedness on $L^2$ of the singular Radon transforms along curves in $R^2$ of the form

$$
U_{iL\omega z_2 \ldots z_n}f(x_1, x_2) = \int_{\mathbb{R}} f(x_1 - t, x_2 - (\omega \cdot f_i)((\bar{x}, z_2, \ldots, z_n) + tv)) \times \sigma((\bar{x}, z_2, \ldots, z_n) + tv)q_{iL}((\bar{x}, z_2, \ldots, z_n) + tv)dt \tag{87}
$$

Note that

$$
|q_{imL}((\bar{x}, z_2, \ldots, z_n) + tv) - q_{imL}((\bar{x}, z_2, \ldots, z_n) + t(v_1, 0, \ldots, 0))| \leq C|t| \sup_{l>1} \max_{z} \partial_{z_l} q_{imL}(z) \tag{88}
$$

By (82), since $|t| < C^{2-m}$ and $|z_1| \sim 2^{-m}$ we have

$$
|q_{imL}((\bar{x}, z_2, \ldots, z_n) + tv) - q_{imL}((\bar{x}, z_2, \ldots, z_n) + t(v_1, 0, \ldots, 0))| \leq C' \tag{89}
$$

Hence by the cancellation condition (47) one has

$$
\int_{\mathbb{R}} q_{imL}((\bar{x}, z_2, \ldots, z_n) + tv)dt < C''2^{-\epsilon_0m} \tag{90}
$$

Since $\sigma((\bar{x}, z_2, \ldots, z_n) + tv) = \sigma(\bar{x}, z_2, \ldots, z_n) + O(|t|)$, using (81) one also

$$
\int_{\mathbb{R}} \sigma((\bar{x}, z_2, \ldots, z_n) + tv)q_{imL}((\bar{x}, z_2, \ldots, z_n) + tv)dt < C'''2^{-\epsilon_0m} \tag{91}
$$
In other words, we have a cancellation condition in (87) derived from (83). The constant $C'''$ in (91) depends on $b(x), h_1(x), \ldots, h_m(x)$ and the constant $C$ of (81)–(83), which in turn depends on $b(x)$ and the constant $C_0$ of (44)–(46).

The arguments of [G3] provide $L^2$ bounds for the operators $U_i L \omega z_2 \ldots z_n$ under the assumptions (81)–(83) and a lower bound on $|\partial_x \omega \cdot f_i(z)|$ that are uniform in $L, \omega, z_2, \ldots, z_n$ for $\omega \in N$. (A slightly stronger cancellation condition is assumed but (83) suffices). This is because the bounds obtained in [G3] are at least as strong as the bounds obtained when the convolution is over the curve $(t, t^\alpha)$, in which case the bounds can be expressed in terms of the constant $C$ of (81)–(83), the constant $C''$ of (90), and the function $h \circ \beta_i(x)$. For the ball $B(x_0, r_0)$ on which $\sigma(x)$ is supported, how small $r_0$ needs to be for the uniform bounds to hold will also be uniform in the various parameters but may be smaller than the $r_0$ we originally selected. However, this can be corrected by writing $\sigma(x)$ as a finite sum of bump functions with smaller support if needed. This completes the proof of Theorem (3.1.5).

**Section (4.2): Proof of Theorems when the Multiplicity is Greater than One:**

We start with some facts from [G1] - [G2] which will help us understand the distribution function of $b(x)$ and related properties of integrals of $|b(x)|^{-\delta_0}$. The constructions in [G1] are slightly better for our purposes so we bring our attention to them. If $U$ is a sufficiently small neighborhood of the origin, up to a set of measure zero one may write $U = \bigcup_{i=1}^N U_i$ as a finite union of open sets such that the following hold. Each $U_i$ is contained in one of the $2n$ octants determined by coordinate hyperplanes. For each $i$, there is some integer $1 \leq k_i \leq n$ and a function $\gamma_i : R^n \to R^n$ such that each
component of \( \gamma_i(x) \) is plus or minus a monomial and \( \gamma_i^{-1}(U_i) \) satisfies the following. If \( k_i < n \), then there are cubes \((0, \eta_i)^{k_i}\) and \((0, \eta_i')^{k_i}\) with \( \eta_i' > \eta_i \) and bounded open sets \( O_i \subset O_i' \) whose closures are a subset of \( \{ x \in \mathbb{R}^{n-k_i} : x_i > 0 \text{ for all } l \} \), such that
\[
(0, \eta_i)^{k_i} \times O_i \subset \gamma_i^{-1}(U_i) \subset (0, \eta_i')^{k_i} \times O_i' \quad (92)
\]
If \( k_i = n \), then there are cubes \((0, \eta_i)^{k_i}\) and \((0, \eta_i')^{k_i}\) with \( \eta_i' > \eta_i \) such that
\[
(0, \eta_i)^{k_i} \subset \gamma_i^{-1}(U_i) \subset (0, \eta_i')^{k_i} \quad (93)
\]
In either case, there is a monomial \( m_i(x_1, \ldots, x_{k_i}) \) and constants \( C_i, C'_i \) such that on \( \gamma_i^{-1}(U_i) \) one has
\[
C_i m_i(x_1, \ldots, x_{k_i}) < |b^* \gamma_i(x)| < C'_i m_i(x_1, \ldots, x_{k_i}) \quad (94)
\]
If \( k_i = n \), then (94) holds with \( b^* \circ \gamma_i(x) \) replaced by \( b \circ \gamma_i(x) \) if \( k_i < n \), then on \( \gamma_i^{-1}(U) \) the function \( b \circ \gamma_i(x) \) can be expressed as \( m_i(x_1, \ldots, x_{k_i}) \)
\( g_i(x_1, \ldots, x_n) \) where \( m_i(x_1, \ldots, x_{k_i}) \) is a monomial and where \( g_i(x_1, \ldots, x_n) \) satisfies the following. One may write \( \gamma_i^{-1}(U_i) = \bigcup_{j=1}^{M_i} V_{ij} \) such that for each \( i \) and \( j \) there is an \( \epsilon > 0 \), a compact face \( F_i \) of \( N(b) \), and a directional derivative \( \partial_{v_{ij}} \) in the last \( n-k_i \) variables, such that \| \partial_{v_{ij}} g_i(x_1, \ldots, x_n) \| > \epsilon \) on \( V_{ij} \) for some \( a_{ij} = 0 \), which is at most the maximum order of any zero of \( b_{F_i}(x_1, \ldots, x_n) \) on \( \mathbb{R} - \{0\}^n \). When \( a_{ij} = 0 \), we interpret \( \partial_{v_{ij}} g_i(x) \) to just mean \( g_i(x) \).

For a given \( \epsilon > 0 \), the following lemma explicitly bounds the measure of the portion of a dyadic rectangle where \( |b(x)/b^*(x)| < \epsilon \) in terms of the maximum order of the zeroes of the \( b_F(x) \) on \( \mathbb{R} - \{0\}^n \).

**Lemma (4.2.1)[4]:** Suppose \( p > 0 \) is an integer such that the zeroes of each \( b_F(x) \) on \( \mathbb{R} - \{0\}^n \) are all of order at most \( p \). Then there is a neighborhood
$U$ of the origin and a constant $C > 0$ such that if $R \subset \mathbb{R}^n$ is a set of the form 
\[ \{ x \in \mathbb{R}^n : 2^{-j_l} < |x_i| < 2^{-j_{l+1}} \} \] 
for integers $j_l$, then
\[ |\{ x \in R : |b(x)/b^*(x)| < \epsilon \}| < C \epsilon^{\frac{1}{p}} |R| \]  
(95)

*Proof.* It suffices to show for each $i$ an estimate of the form 
\[ |b(x)/b^*(x)| < \epsilon < C \epsilon^{\frac{1}{p}} |R|. \] 
Since the components of $\gamma_i(x)$ are all monomials, the absolute value of the Jacobian of $\gamma_i(x)$ is of the form 
\[ c_i x_1^{e_{i1}} \cdots x_n^{e_{in}} \] 
for some integers $e_{i1}, \ldots, e_{in}$ and some $c_i > 0$. Viewing 
\[ |\{ x \in R \cap U_i : |b(x)/b^*(x)| < \epsilon \}| \] 
as the integral of 1 over 
\[ |\{ x \in R \cap U_i : |b(x)/b^*(x)| < \epsilon \}| \] 
and changing coordinates via $\gamma_i(x)$, one obtains
\[ \int_{\{ x \in \gamma_i^{-1}(R) \cap \gamma_i^{-1}(U_i) : |b^* \gamma_i(x)/b^* \gamma_i(x)| < \epsilon \}} c_i x_1^{e_{i1}} \cdots x_n^{e_{in}} \, dx \]  
(96)
Note that by (94) and the following paragraph, one has $|b^* \gamma_i(x)/b^* \gamma_i(x)| > C' \gamma_i(x)$ for some constant $C'$. (We can include the $k_i = n$ situation here by defining $g_i(x) = 1$.) Thus in order to bound (96) by an expression of the form $C \epsilon^{\frac{1}{p}} |R|$, it suffices to show the following estimate of the following form for each $i$ and $j$.
\[ \int_{\{ x \in \gamma_i^{-1}(R) \cap V_{ij} : g_i(x)| < \epsilon \}} c_i x_1^{e_{i1}} \cdots x_n^{e_{in}} \, dx < C \epsilon^{\frac{1}{p}} |R| \]  
(97)
If the multi-index $a_{ij}$ in the paragraph after (94) is zero, then $g_i(x)$ is bounded below, and thus (97) reduces to showing that 
\[ \int_{\gamma_i^{-1}(R) \cap V_{ij}} x_1^{e_{i1}} \cdots x_n^{e_{in}} \, dx \] 
is bounded by a constant times $|R|$, which follows immediately from changing back into the original coordinates using $\gamma_i$. Thus it suffices to assume $a_{ij} \geq 1$.
Note that this only occurs if $k_i < n$. Since the final $n - k_i$ variables are bounded below on $V_{ij}$, it suffices to prove a bound
We now integrate the left-hand side of (98) starting with the \( v_{ij} \) direction. Since \( a_{ij} \) \( p \), by the measure version of the Van der Corput lemma (see [C] for details), the integral in the \( v_{ij} \) direction is at most \( Cx_1^{e_{i1}} \cdots x_n^{e_{nk_i}} \frac{1}{\epsilon^p} \). If we next perform the integration in the remaining \( n - k_i - 1 \) directions of last \( n - k_i \) variables (if any exist), then if \( \pi_i \) denotes the projection on \( R^n \) onto the first \( k_i \) variables, we obtain

\[
\int_{\{x \in \gamma_i^{-1}(R) \cap V_{ij} : \ g_i(x) < \epsilon\}} x_1^{e_{i1}} \cdots x_n^{e_{nk_i}} \, dx 
\leq C \epsilon^p \int_{\pi_i(\gamma_i^{-1}(R) \cap V_{ij})} x_1^{e_{i1}} \cdots x_n^{e_{nk_i}} \, dx_1 \cdots dx_k_i
\]

\[
= C \epsilon^p \int_{\pi_i(\gamma_i^{-1}(R) \cap V_{ij}) \times [1,2]^{n-k_i}} x_1^{e_{i1}} \cdots x_n^{e_{nk_i}} \, dx_1 \cdots dx
\]

\[
= C \epsilon^p \int_{\pi_i(\gamma_i^{-1}(R) \cap V_{ij}) \times [1,2]^{n-k_i}} x_1^{e_{i1}} \cdots x_n^{e_{nk_i}} \, dx_1 \cdots dx
\]

Because the last \( n - k_i \) coordinates of the points in \( U_i \) are bounded above and below away from zero, there is a constant \( C_0 > 1 \) such that if \( (x_1, \ldots, x_n) \in \pi_i(\gamma_i^{-1}(R) \cap V_{ij}) \times [1,2]^{n-k_i} \) then there is a point \( (y_1, \ldots, y_n) \in \gamma_i^{-1}(R) \cap V_{ij} \) such that \( \frac{1}{C_0} < \frac{y_l}{x_l} < C_0 \) for each \( l \). This property is preserved under monomial maps (perhaps with a different constant \( C_1 \)), so the image of \( \pi_i(\gamma_i^{-1}(R) \cap V_{ij}) \times [1,2]^{n-k_i} \) under \( \gamma_i \) is a subset of a corresponding dilate of \( \gamma_i(\gamma_i^{-1}(R) \cap V_{ij}) \), which in turn is a subset of the dilate of \( R \). Denote this
dilate by $R^*$. Changing coordinates in (101) back into the original coordinates via $\gamma_i$, we see that

$$C \epsilon \frac{1}{p} \int_{\pi_i(\gamma_i^{-1}(R) \cap V_{ij}) \times [1,2]^{n-k_i} x_1^{e_i_1} \cdots x_n^{e_i_n} \, dx_1 \cdots dx_n \leq C'' \epsilon \frac{1}{p} \int_{R^*} 1 \, dx$$

$$= C''' \epsilon \frac{1}{p} \, |R|$$

(102)

This is the desired estimate (98) and we are done.

We also will make use of the following result.

**Lemma (4.2.2)[4]:** Suppose the zeroes of each $b_F(x)$ on $(R - \{0\})^n$ are all of order less than $d(b)$. Then there is a neighborhood $U$ of the origin and constants $C, \eta > 0$ such that if $\epsilon > 0$ and $R \subset U$ is a set of the form

$$\{x \in R^n: 2^{-j} < |x| < 2^{-j+1}\},$$

then

$$\int_{\{x \in R: |b(x)| < |b^*(x)|\}} |b(x)|^{-\delta_0} < C \epsilon^n.$$  

In particular, since there is a constant $C'$ such that $|b(x)| \leq C' b^*(x)$ on any such $R$, there is a constant $C''$ such that

$$\int_R |b(x)|^{-\delta_0} < C''$$

for such $R \subset U$.

**Proof.** Since the terms of $b^*(x)$ are absolute values of monomials, there is a constant $c > 1$ and an $x_0 \in R$ such that

$$c b^*(x_0) \geq b^*(x) \geq \frac{1}{c} b^*(x_0)$$

on $R$. Hence it suffices to prove an estimate of the form

$$\int_{\{x \in R: |b(x)| < |b^*(x)|\}} |b(x)|^{-\delta_0} < C \epsilon^n.$$  

By the relation between $L^p$ norms and distribution functions, applied to $\frac{1}{|b(x)|}$, one has

$$\int_{\{x \in R: |b(x)| < |b^*(x)|\}} |b(x)|^{-\delta_0}$$

$$= \delta_0 \int_0^\infty t^{\delta_0-1} \left| \{x \in R: |b(x)| < \min \left(\epsilon |b^*(x_0)|, \frac{1}{t}\right)\} \right| \, dt$$

(103)

It is natural to break up (103) into two pieces, the first where $t < \frac{1}{\epsilon b^*(x_0)}$ and the second where $\geq \frac{1}{\epsilon b^*(x_0)}$. Then the right-hand side of (103) becomes
\[ = \delta_0 \int_0^1 \frac{1}{e^{b^*(x_0)}} t^{\delta_0 - 1} \left| \{x \in R : |b(x)| < \epsilon |b^*(x_0)| \} \right| dt \]
\[ + \delta_0 \int_{\frac{1}{e^{b^*(x_0)}}}^{\infty} t^{\delta_0 - 1} \left| \left\{ x \in R : |b(x)| < \frac{1}{t} \right\} \right| dt \]  
(104)

Performing the first integral in the first term of (67) results in
\[ \epsilon^{-\delta_0} |b^*(x_0)|^{-\delta_0} \left| \{x \in R : |b(x)| < \epsilon |b^*(x_0)| \} \right| \]  
(105)

By Lemma (4.2.1), (105) is bounded by \( C |b^*(x_0)|^{-\delta_0} e^{\frac{1}{p} - \delta_0} |R| \) for some \( p < d(b) = \frac{1}{\delta_0} \).

Hence we have
\[ \int_{\{x \in R : |b(x)| < \epsilon |b^*(x_0)| \}} |b(x)|^{-\delta_0} \leq C |b^*(x_0)|^{-\delta_0} e^{\frac{1}{p} - \delta_0} |R| \]
\[ + \delta_0 \int_{\frac{1}{e^{b^*(x_0)}}}^{\infty} t^{\delta_0 - 1} \left| \left\{ x \in R : |b(x)| < \frac{1}{t} \right\} \right| dt \]  
(106)

Note that \( \left\{ x \in R : |b(x)| < \frac{1}{t} \right\} \subseteq \left\{ x \in R : |b(x)| < \frac{c_0}{tb^*(x_0)} b^*(x) \right\} \), so by Lemma (4.2.1) for some constant \( C_0 \) we have
\[ \int_{\{x \in R : |b(x)| < \epsilon |b^*(x_0)| \}} |b(x)|^{-\delta_0} \leq C |b^*(x_0)|^{-\delta_0} e^{\frac{1}{p} - \delta_0} |R| \]
\[ + C_0 \int_{\frac{1}{e^{b^*(x_0)}}}^{\infty} t^{\delta_0 - 1} \left( \frac{c_0}{tb^*(x_0)} \right)^{\frac{1}{p}} |R| dt \]  
(107)

Note that the exponent \( \delta_0 - 1 - \frac{1}{p} \) is less than \(-1\) since \( p < d(b) = \frac{1}{\delta_0} \).

Hence integrating the second term on the right of (107) leads to the following for some constant \( C_1 \).
\[ \int_{\{x \in R : |b(x)| < \epsilon |b^*(x_0)| \}} |b(x)|^{-\delta_0} \leq C |b^*(x_0)|^{-\delta_0} e^{\frac{1}{p} - \delta_0} |R| \]
\[ + C_1 |b^*(x_0)|^{-\delta_0} e^{\frac{1}{p} - \delta_0} |R| \]  
(108)
Since $|R| \sim |x_1 \ldots x_n|$ for any $(x_1, \ldots, x_n) \in R$, in order to prove Lemma (4.2.2) with $\eta = \frac{1}{p} - \delta_0$, it suffices to show that there is a constant $C_2$ such that for any $x$ we have

$$|x_1 \ldots x_n|(b^*(x))^{-\delta_0} \leq C_2$$

(109)

Since $\delta_0 = \frac{1}{d(b)}$ in the case at hand, (72) is equivalent to the statement that

$$b^*(x) \geq C_3 |x_1 \ldots x_n|^{d(b)}$$

(110)

Since $(d(b), \ldots, d(b))$ is on the Newton polyhedron $N(b)$, there are nonnegative $\alpha_i$ with $\alpha_1 + \ldots + \alpha_k = 1$ such that each component of $(d(b), \ldots, d(b))$ is greater than or equal to that of $\alpha_1 v_1 + \ldots + \alpha_k v_k$ for some vertices $v_1, \ldots, v_k$ of $N(b)$. Hence $|x_1 \ldots x_n|^{d(b)} \leq |x^{v_1}|^{\alpha_1} \ldots |x^{v_k}|^{\alpha_k}$. So by the generalized AM-GM inequality one has $|x_1 \ldots x_n|^{d(b)} \leq \sum_{i=1}^{k} \alpha_i |x^{v_i}| \leq b^*(x)$ as needed. This completes the proof of Lemma (4.2.2).

Similar to the multiplicity one case, in order to show $\|Tf\|_p \leq C\|f\|_p$ for all Schwartz $f$ for a given $1 < p < \infty$, it suffices to show the that if $K(y)$ is supported on a sufficiently small neighborhood of the origin there is a constant $C$ such that $\|T_Lf\|_p \leq C\|f\|_p$ for all Schwartz $f$ and each $L$, where $T_Lf(x) = \int_{R^n} f(x - y)\alpha(x, y)K_L(y) \, dy$. Here $K_L(y) = \sum_{j_1 < L} \forall \, l \, K_{j_1 \ldots j_L}(y)$ as in (60). As in Lemma (4.1.1) for the multiplicity one case, we may also replace $\alpha(x, y)$ by just 1. Thus we focus our attention on $U_L$ given by

$$U_Lf(x) = \int_{R^n} f(x - y)K_L(y) \, dy$$

(111)

Our goal will be to prove $U_L$ is bounded on $L^p$ with a norm independent of $L$ under the hypotheses of Theorem (3.2.7) or (3.2.10). The next two lemmas provide bounds on the $|\widehat{K_{j_1 \ldots j_n}}(\xi)|$ that allow us to prove such uniform bounds.

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Lemma (4.2.3)[4]: Under the assumptions of Theorem (3.2.7), there is a constant $C > 0$ such that if $l$ is such that $2^{-l} |\xi| \leq 1$, then
\[
|K_{j_1,\ldots,j_n}(\xi)| \leq C 2^{-l} |\xi|
\]

Proof. $|K_{j_1,\ldots,j_n}(\xi)|$ is given by
\[
|K_{j_1,\ldots,j_n}(\xi)| = \int_{\mathbb{R}^n} K_{j_1,\ldots,j_n}(x) e^{-i\xi_1 x_1 - \cdots - i\xi_n x_n} \, dx \tag{112}
\]
Since the integral of $K_{j_1,\ldots,j_n}(x)$ in the $x_l$ variable is equal to zero by (59), one can subtract $K_{j_1,\ldots,j_n}(x) e^{\sum_{k=l}^n i\xi_k x_k}$ from the integrand in (112) without changing the integral, so we have
\[
|K_{j_1,\ldots,j_n}(\xi)| = \int_{\mathbb{R}^n} K_{j_1,\ldots,j_n}(x) (e^{-i\xi_l x_l} - 1) e^{\sum_{k=l}^n i\xi_k x_k} \, dx \tag{113}
\]
Since $|\xi_l x_l| \sim 2^{-l} |\xi| \leq C$ when $K_{j_1,\ldots,j_n}(x) \neq 0$, in (113) one has that $(e^{-i\xi_l x_l} - 1) \leq C |\xi_l x_l| < C' 2^{-l} |\xi|$ and we get
\[
|K_{j_1,\ldots,j_n}(\xi)| \leq C' 2^{-l} |\xi| \int_{\mathbb{R}^n} |K_{j_1,\ldots,j_n}(x)| \, dx \tag{114}
\]
Using Lemma (4.2.2) we obtain the desired estimate
\[
|K_{j_1,\ldots,j_n}(\xi)| \leq C'' 2^{-l} |\xi| \tag{115}
\]

Lemma (4.2.4)[4]: Under the assumptions of Theorem (3.2.7), there are constants $\rho, C > 0$ such that if $l$ is such that $2^{-l} |\xi| \geq 1$, then
\[
|K_{j_1,\ldots,j_n}(\xi)| \leq C \frac{1}{2^{-l} |\xi|^\rho} \tag{116}
\]

Proof. Let $\sigma_1(x)$ be a smooth increasing nonnegative function on $\mathbb{R}^+$ with $\sigma_1(x) = 1$ for $|x| < 1$ and $\sigma_1(x) = 0$ for $|x| > 2$. Let $\sigma_2(x) = 1 - \sigma_1(x).$ For a constant $\rho_0 > 0$ to be determined by our arguments, for any fixed $x_0$ in the dyadic rectangle corresponding to $(j_1,\ldots,j_n)$ we write
\[ |K_{j_1,...,j_n}(\xi)| = \int_{\mathbb{R}^n} \sigma_1 \left( (2^{-j_1} |\xi_l|)^{\rho_0} \frac{|b(x)|}{b^*(x_0)} \right) K_{j_1,...,j_n}(x) e^{-i\xi_1 x_1 - ... - i\xi_n x_n} \, dx \]

\[ + \int_{\mathbb{R}^n} \sigma_2 \left( (2^{-j_1} |\xi_l|)^{\rho_0} \frac{|b(x)|}{b^*(x_0)} \right) K_{j_1,...,j_n}(x) e^{-i\xi_1 x_1 - ... - i\xi_n x_n} \, dx \quad (117) \]

The first term of (117) is bounded by

\[ \int_{\{x: |b(x)| \leq 2(2^{-j_1} |\xi_l|)^{\rho_0} b^*(x_0)\}} |K_{j_1,...,j_n}(x)| \quad (118) \]

Using (57) and Lemma (4.1.1), we see that this term is at most \( C (2^{-j_1} |\xi_l|)^{-\frac{\rho_0}{d}} \) for some \( d \), which gives the bound of the right-hand side of (116).

Proceeding to the second term of (117), we integrate by parts, integrating the \( e^{-i\xi_1 x_1 - ... - i\xi_n x_n} \) factor in the \( x_l \) variable and differentiating the remaining factors. The resulting term is given by

\[ \frac{1}{i\xi_l} = \int_{\mathbb{R}^n} \partial_{x_l} \left[ \sigma_2 \left( (2^{-j_1} |\xi_l|)^{\rho_0} \frac{|b(x)|}{b^*(x_0)} \right) K_{j_1,...,j_n}(x) \right] e^{-i\xi_1 x_1 - ... - i\xi_n x_n} \, dx \quad (119) \]

If the \( x_l \) derivative in (119) lands on the \( K_{j_1,...,j_n}(x) \) factor, one obtains a term which by (58) is bounded by

\[ C_1 \left| \frac{1}{i\xi_l} \right| \int_{\mathbb{R}^n} \sigma_2 \left( (2^{-j_1} |\xi_l|)^{\rho_0} \frac{|b(x)|}{b^*(x_0)} \right) \frac{1}{|x_l|} b^*(x) b(x)^{-1-\delta_0} \, dx \quad (120) \]

Due to the \( \sigma_2 \) factor in (120), on the support of the integrand of (120) we have \( |b(x)| \geq (2^{-j_1} |\xi_l|) b^*(x_0) \). Thus (120) is bounded by

\[ C_1 \left| \frac{1}{i\xi_l} \right| \int_{\mathcal{R}_{j_1,...,j_n}} \sigma_2 \left( (2^{-j_l} |\xi_l|)^{\rho_0} \frac{|b(x)|}{b^*(x_0)} \right) \frac{1}{|x_l|} b^*(x) (2^{-j_l} |\xi_l|)^{\rho_0(1+\delta_0)} (b^*(x))^{-1-\delta_0} \, dx \quad (121) \]
Here $R_{j_1,\ldots,j_n}$ denotes the (expanded) dyadic rectangle-like set on which $K_{j_1,\ldots,j_n}$ is supported. Since $\sigma_2(t) \leq 1$ for all $t, |x_t| \sim 2^{-j_t}$ on $R_{j_1,\ldots,j_n}$, and $b^*(x)$ is within a constant factor of $b^*(x_0)$ on $R_{j_1,\ldots,j_n}$, (121) is bounded by

$$C_2 \left( \frac{1}{2^{-j|\xi_l|}} \right) (2^{-j_t|\xi_l|})^{\rho_0(1+\delta_0)} \int_{R_{j_1,\ldots,j_n}} (b^*(x_0))^{-\delta_0} dx$$

(122)

By Lemma (4.2.2) which applies to negative powers of the smaller function $|b(x)||$, we see that the above is bounded by

$$C_3 \left( \frac{1}{2^{-j|\xi_l|}} \right) (2^{-j_t|\xi_l|})^{\rho_0(1+\delta_0)}$$

(123)

Thus so long as $\rho_0$ is chosen so that $\rho_0(1 + \delta_0) < \frac{1}{2}$ for example, this term of (119) satisfies the bounds needed in this lemma.

We now bound the term where the derivative in (119) lands on the $\sigma_2 \left( (2^{-j_t|\xi_l|})^{\rho_0} \frac{|b(x)|}{b^*(x_0)} \right)$ factor. Observe that

$$\partial_{x_l} \left( \sigma_2 \left( (2^{-j_t|\xi_l|})^{\rho_0} \frac{|\partial_{x_l} b(x)|}{b^*(x_0)} \right) \right)$$

$$= \pm \left( (2^{-j_t|\xi_l|})^{\rho_0} \frac{|b(x)|}{b^*(x_0)} \right) \sigma_2' \left( (2^{-j_t|\xi_l|})^{\rho_0} \frac{|b(x)|}{b^*(x_0)} \right)$$

(124)

Since $|b(x)| \geq (2^{-j_t|\xi_l|})^{-\rho_0} b^*(x_0)$ in the support of the $\sigma'$ factor, by (57), on the support of the integrand of this term of (119) we have

$$|K_{j_1,\ldots,j_n}(x)| \leq C_4 (2^{-j_t|\xi_l|})^{\rho_0\delta_0} |b^*(x_0)|^{-\delta_0}$$

(125)

As a result, the absolute value of the term of (119) in question is bounded by

$$C_4 |\xi_l|^{-1} (2^{-j_t|\xi_l|})^{\rho_0\delta_0} b^*(x_0)^{-\delta_0} \int_{R_{j_1,\ldots,j_n}} \left| \partial_{x_l} \left( \sigma_2 \left( (2^{-j_t|\xi_l|})^{\rho_0} \frac{|b(x)|}{b^*(x_0)} \right) \right) \right| dx$$

(126)
We first integrate in the $x_l$ variable in (126). By the hypotheses of Theorem (3.2.7) concerning zeroes of $\partial_{x_l} b(x)$, for any fixed value of the remaining $n - 1$ variables (outside a set of measure zero) the domain of integration in the $x_l$ variable can be written as the union of boundedly many intervals on which
\[ \partial_{x_l} \left( \sigma_2 \left( (2^{-j_l} |\xi_l|) \rho_0 |b(x)| b^*(x_0) \right) \right) \] does not change sign. Thus on each of these intervals this derivative integrates back to the function. Since $\sigma_2$ is bounded this means the $x_l$ integrals in (126) are uniformly bounded in the remaining variables. Thus doing the $x_l$ integral first and then integrating over the remaining variables shows that (126) is bounded by
\[ C_5 |\xi_l|^{-1} (2^{-j_l} |\xi_l|)^{\rho_0 \delta_0} |b^*(x_0)|^{-\delta_0} 2^{j_l} \sum_{i \neq l} - j_i \] (127)
Since $b^*(x) \sim b^*(x_0)$ on $R_{j_1 \cdots j_n}$, (127) is bounded by
\[ C_6 |\xi_l|^{-1} (2^{-j_l} |\xi_l|)^{\rho_0 \delta_0} 2^{j_l} \int_{R_{j_1 \cdots j_n}} |b^*(x)|^{-\delta_0} dx \] (128)
As in (123), the integral in (128) is uniformly bounded and we obtain the bound
\[ C_7 (2^{-j_l} |\xi_l|)^{\rho_0 \delta_0} \frac{1}{2^{-j_l} |\xi_l|} \] (129)
So as long as $\rho_0 \delta_0 < 1$, we see from (129) that the term of (119) under consideration is also is bounded by the right-hand side of (116). We have now shown that the first term of (117) and each term of (119) all are bounded by the right-hand side of (116) and thus we are done with the proof of Lemma 4.4.

**Theorem (4.2.5)[4]:** Suppose each polynomial $b_F(x)$ only has zeroes of order less than $d(b)$ on $(R - \{0\})^n$. Suppose also that there is a $C_0 > 0$ and a neighborhood $U$ of the origin such that for each $l$, there is a set $Z \subset R^{n-1}$ of
measure zero such that the function $\partial_{x_l} b(x)$ has at most $C_0$ zeroes in $U$ on any line parallel to the $x_l$ coordinate axis whose projection onto the plane $x_l = 0$ is not in $Z$. Then there is an $R > 0$ such that if each $K_{j_1 \cdots j_n}(y)$ satisfies (18)–(20) and is supported on $|y| < R$, then there is a constant $C$ such that $\|Tf\|_{L^2(R^n)} \leq C\|f\|_{L^2(R^n)}$ for all Schwartz functions $f(x)$.

The condition concerning zeroes on lines parallel to the coordinates axes is needed for technical reasons in the proof. Note that this condition holds whenever $b(x)$ is a polynomial, and it is not hard to see that it always holds in two variables, using the Weierstrass Preparation Theorem for example. We does not know if it holds for all real-analytic functions, so it is included as an assumption in Theorem (3.2.7) (and in Theorem (3.2.8) below).

For $p \neq 2$, we have a weaker statement. To motivate the statement of the theorem, in Lemma (3.2.7) and the line afterwards we will see that if each polynomial $b_F(x)$ is nonvanishing on $(R - \{0\})^n$, then there are constants $C_1$ and $C_2$ such that

$$C_1 b^*(x) < |b(x)| < C_2 b^*(x)$$

Hence in this situation (94) becomes

$$\left| \partial_{y_1} K_{j_1 \cdots j_n}(y_1, \ldots, y_n) \right| < C_1' \frac{1}{|y_1|} \left( b^*(y_1, \ldots, y_n) \right)^{\delta_0}$$

For the $L^p$ theorem, we need bounds on derivatives of higher order in order to apply the Marcinkiewicz multiplier theorem. Hence we assume that each $K_{j_1 \cdots j_n}(y_1, \ldots, y_n)$ is a $C^{n+1}$ function and there is a constant $C$ such that for any multiindex $\alpha$ with $0 \leq |\alpha| \leq n + 1$ we have

$$\left| \partial^\alpha K_{j_1 \cdots j_n}(y_1, \ldots, y_n) \right| \leq C \frac{1}{|y_1|^{\alpha_1} \cdots |y_n|^{\alpha_n}} \left( b^*(y_1, \ldots, y_n) \right)^{-\delta_0}$$
The condition (104) is motivated by the fact that by iterating (55), the bounds (104) hold for $|b(x)|^{-\delta_0}$ whenever each $b_F(x)$ is nonvanishing on $(R - \{0\})^n$.

The $L^p$ theorem is as follows.

**Proof**: We will prove $L^2$ boundedness of $U_L$ uniformly in $L$ by bounding the Fourier transform $\hat{K}_L(\xi)$ uniformly in $L$ and $\xi$. Since $|\hat{K}_L(\xi)| = \sum_{(j_1, \ldots, j_n) \in \mathbb{Z}^n: j_l < L} \hat{K}_{j_1, \ldots, j_n}(\xi)$, we have the bound

$$|\hat{K}_L(\xi)| \leq \sum_{(j_1, \ldots, j_n) \in \mathbb{Z}^n: j_l < L} |\hat{K}_{j_1, \ldots, j_n}(\xi)| \quad (133)$$

We use the better of the two estimates from Lemmas (4.2.2) and (4.2.3) in each term of (133) then add the result. Let $(k_1, \ldots, k_n)$ be the vector of integers such that for each $l$, $2^{j_l}$ is the nearest power of 2 to $|\lambda_l|$. For any $M$ the number of $(j_1, \ldots, j_n)$ such that $\max_l |j_l - k_l| = M$ is bounded by $CM^{n-1}$, and for each such $(j_1, \ldots, j_n)$ Lemma (4.2.3) or (4.2.4) gives a bound $|\hat{K}_{j_1, \ldots, j_n}(\xi)| \leq C'2^{-\rho_1}M$ for some $\rho_1 > 0$. Hence in (133) the sum over all terms with $\max_l |j_l - k_l| = M$ is bounded by $C''M^{n-1}2^{-\rho_1}M$. Adding over all $M$ gives a uniform bound and we are done.

**Theorem (4.2.6)[4]**: Suppose each polynomial $b_F(x)$ is nonvanishing on $(R - \{0\})^n$. Suppose also that there is a $C_0 > 0$ and a neighborhood $U$ of the origin such that for each $l$, there is a set $Z \subset R^{n-1}$ of measure zero such that the function $\partial_{x_l} b(x)$ has at most $C_0$ zeroes in $U$ on any line parallel to the $x_l$ coordinate axis whose projection onto the plane $x_l = 0$ is not in $Z$. Then there is an $R > 0$ such that if each $K_{j_1, \ldots, j_n}(y)$ satisfies (57), (67), (59) and is supported on $|y| < R$, then if $1 < p < \infty$ there is a constant $C$ such that $\|Tf\|_{L^p(R^n)} \leq C\|f\|_{L^p(R^n)}$ for all Schwartz functions $f(x)$. 

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Going back to the examples preceding (59), in the first example where \( b(x) = x_1^{a_1} \ldots x_n^{a_n} \) each kernel
\[
K(x) = (-1)^{\text{sgn}(x_1) + \ldots + \text{sgn}(x_n)} \phi(x_1^2, \ldots, x_n^2) |b(x)|^{-\delta_0}
\]
will be covered by Theorems (3.2.1) and (3.2.2). As for the second example where \( b(x) = f(x_1^2, \ldots, x_n^2) \), the maximum order of any zero of any \( b_F(x) \) on \( (R - \{0\})^n \) is the same as the maximum order of any \( f_F(x) \) on \( (R^+)^n \). So when this quantity is less than \( d(b) = 2d(f) \), \( K(x) \) will fall under the conditions of Theorem (3.2.7). When each \( f_F(x) \) is nonvanishing on \( (R^+)^n \), then \( K(x) \) will fall under the conditions of Theorem (3.2.8) as well.

**Proof:** We will make use of the Marcinkiewicz multiplier theorem (see Theorem 6' on p.109 of [S]), which implies that \( L^p \) bounds on \( U_L \) that are uniform in \( L \) will follow if we can show that there is a constant \( C \) such that for each multiindex \( \alpha \) with \( |\alpha| \leq n \) and each \( L \) we have the estimate
\[
\left| \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n} \partial^\alpha K_L(\xi_1, \ldots, \xi_n) \right| \leq C \tag{134}
\]
Returning to the \( x \) variables, this will follow as in the proof of Theorem (3.2.7) if we can show that for each multiindex \( \alpha \) with \( 0 \leq |\alpha| \leq n \) the kernel \( \partial^\alpha \left( x^\alpha K_{j_1, \ldots, j_n}(x) \right) \) satisfies the conditions of Lemmas (4.2.3) and (4.2.4). But the fact that (67) holds for \( K_{j_1, \ldots, j_n}(x) \) immediately implies that (67) also holds for \( \partial^\alpha \left( x^\alpha K_{j_1, \ldots, j_n}(x) \right) \). The cancellation condition (59) also holds for \( \partial^\alpha \left( x^\alpha K_{j_1, \ldots, j_n}(x) \right) \); if the \( x_l \) variable is not represented in \( \alpha \) it can be shown by multiplying (59) through by \( x^\alpha \) and then applying \( \partial^\alpha \) under the integral sign, while if the \( x_l \) variable is represented in \( \alpha \), then the integral (59) is zero simply because one is integrating the derivative of a compactly supported \( C_1 \) function. Hence each kernel \( \partial^\alpha \left( x^\alpha K_{j_1, \ldots, j_n}(x) \right) \) satisfies Lemmas (4.2.3) and (4.2.4) and Theorem (3.2.8) follows.
List of symbol
References


