Chapter 1

Analytic Free Maps

We show analytic fee maps analogues of classical analytic functions in several complex variables and are defined in terms of noncommuting variables amongst which there are on relations.Analytic free maps include vector valued polynomials in free noncommuting variables and form a canonical class of mappings form one noncommutative domain in say as a natural extension.

Section(1.1): Free Maps and a Proper Free Map is Bianalytic

The notion of an analytic, free or non-commutative, map arises naturally in free probability, the study of non-commutative (free) rational functions and systems theory . Rigidity results for such functions paralleling those for their classical commutative counterparts are established. The free setting leads to substantially stronger results. Namely, if f is a proper analytic free map from a non-commutative domain in g variables to another in $\tilde{g}$ variables, then f is injective and $\tilde{g} \geq g$. If in addition $\tilde{g} = g$, then f is onto and has an inverse which is itself a (proper) analytic free map. This injectivity conclusion contrasts markedly to the classical case where a (commutative) proper analytic function f from one domain in $\mathbb{C}^g$ to another in $\mathbb{C}$, need not be injective, although it must be onto.

This section contains the background on non-commutative sets and on free maps. As we shall see, free maps which are continuous are also analytic in several senses. Fix a positive integer g. Given a positive integer n, let $(\mathbb{C})$ denote g-tuples of $n\times n$ matrices. Of course, $(\mathbb{C})$ is naturally identified with $(\mathbb{C})\otimes \mathbb{C}^g$.

A sequence $\mathcal{U} = ((n))_{\mathbb{N}}$, where $\mathcal{U}(n) \subseteq M_n(\mathbb{C})^g$ is a non-commutative set if it is closed with respect to simultaneous unitary similarity; i.e., if $X \in \mathcal{U}(n)$ and U is an $n\times n$ unitary matrix, then

$$U^*XU = (U^*X_1U, \ldots, U^*X_gU) \in \mathcal{U}(n);$$

and if it is closed with respect to direct sums; i.e., if $X \in \mathcal{U}(n)$ and $Y \in \mathcal{U}(m)$ implies

$$X \oplus Y = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \in \mathcal{U}(n+m).$$

Non-commutative sets differ from the fully matricial $\mathbb{C}^g$-sets in that the latter are closed with respect to simultaneous similarity, not just simultaneous *unitary* similarity. Remark (1.1.4) below briefly discusses the significance of this distinction for the results on proper analytic free maps.

The non-commutative set $\mathcal{U}$ is a non-commutative domain if each (n) is open and connected. Of course the sequence $(\mathbb{C})^g = (M_n(\mathbb{C})^g)$ is itself a non-commutative domain. Given $\varepsilon > 0$, the set $\mathcal{N}_\varepsilon = ((n))$ given by

$$\mathcal{N}_\varepsilon\,(n) = \left\{X \in (\mathbb{C}) : \sum X_j X_j^* < \varepsilon^2\right\} \qquad (1)$$

is a non-commutative domain which we call the non-commutative ε-neighborhood of 0 in $\mathbb{C}^g$.

The non-commutative set $\mathcal{U}$ is bounded if there is a $\mathbb{C} \in \mathbb{R}$ such that

$$\mathbb{C}^2 - \sum X_j X^*_j > 0 \qquad (2)$$

for every n and $X \in \mathcal{U}(n)$. Equivalently, for some $\lambda \in \mathbb{R}$, we have $\mathcal{U} \subseteq \mathcal{N}_\lambda$. Note that this condition is stronger than asking that each (n) is bounded.

Let $\mathbb{C}\langle x_1, \ldots, x_g\rangle$ denote the $\mathbb{C}$-algebra freely generated by g non-commuting letters $x = (x_1, \ldots, x_g)$. Its elements are linear combinations of words in x and are called polynomials. Given an $r \times r$ matrix-valued polynomial $p \in M_r(\mathbb{C}) \otimes \mathbb{C}\langle x_1, \ldots, {}_g\rangle$ with $P(0) = 0$, let $\mathcal{D}(n)$ denote the connected component of

$$\{x \in (\mathbb{C})^g : 1 + P(X) + P(X)^* > 0\}$$

containing the origin. The sequence $\mathcal{D} = ((n))$ is a non-commutative domain which is semi algebraic in nature. Note that $\mathcal{D}$ contains an $\varepsilon > 0$ neighborhood of 0, and that the choice gives $\mathcal{D} = \mathcal{N}_\varepsilon$. Further examples of natural non-commutative domains can be generated by considering

$$P = \frac{1}{\varepsilon}\begin{pmatrix} 0_{g\times g} & \begin{matrix} x_1 \\ . \\ . \\ x_g \end{matrix} \\ 0_{1\times g} & 0_{1\times 1} \end{pmatrix}$$

non-commutative polynomials in both the variables $x = (x_1,\ldots,x_g)$ and their formal adjoints, $x^* = (x_1^*,\ldots, x_g^*)$.

The case of domains determined by linear matrix inequalities appears.

Let $\mathcal{U}$ denote a non-commutative subset of $(\mathbb{C})^g$ and let $\tilde{g}$ be a positive integer. A free map f from $\mathcal{U}$ into $M(\mathbb{C})^{\tilde{g}}$ is a sequence of function $f[n] : \mathcal{U}(n) \to M_n(\mathbb{C})^{\tilde{g}}$ which respects intertwining maps; i.e., if $X \in \mathcal{U}(n)$, $Y \in \mathcal{U}(m)$, $\Gamma: \mathbb{C}^m \to \mathbb{C}^n$, and

$$X\Gamma = (X_1\Gamma, \ldots, {}_g\Gamma) = (\Gamma Y_1,\ldots, \Gamma Y_g) = \Gamma Y,$$

then $f[n](X)(\Gamma) = (\Gamma)f[m](Y)$. Note for (n) it is natural to write simply (X) instead of the more cumbersome $[n](X)$ and likewise $f\colon \mathcal{U} \to M(\mathbb{C})^{\tilde{g}}$

In a similar fashion, we will often write $(X) = \Gamma f(Y)$.

Remark (1.1.1)[1]: Each $[n]$ can be represented as

$$f[n] = \begin{pmatrix} f[n]^1 \\ \vdots \\ f[n]\tilde{g} \end{pmatrix}$$

where $f[n]_j : \mathcal{U}(n) \to M_n(\mathbb{C})$. (Of course, for each j, the sequence $([n])$ is a free map $: \mathcal{U} \to (\mathbb{C})$ with $f_j[n] = f[n]_j$. In particular, if $: \mathcal{U} \to M(\mathbb{C})^{\tilde{g}}$, and $v = \sum e_j \otimes v_j$, Then

$$f(X)^* v = \sum f_j(X)^* v_j.$$

Let $\mathcal{U}$ be a given non-commutative subset of $(\mathbb{C})^g$ and suppose $f = (f[n])$ is a sequence of functions $f[n] : \mathcal{U}(n) \to M_n(\mathbb{C})^{\tilde{g}}$. The sequence f respects direct sums if, for each n, m and (n) and (m),

$$f(X \otimes Y) = f(X) \otimes f(Y).$$

similarly, f respects similarity if for each n and $X, Y \in \mathcal{U}(n)$ and invertible $n\times n$ matrix S such that $XS = SX$,

$$f(X)S = Sf(X).$$

The following proposition gives an alternate characterization of free maps.

Proposition (1.1.2)[1]: Suppose $\mathcal{U}$ is a non-commutative subset of $(\mathbb{C})^{g}$. A sequence $f = (f[n])$ functions $f[n] : (n) \to M(\mathbb{C})^{\tilde{g}}$ is a free map if and only if it respects direct sums and similarity.

Proof: Observe $(X)\Gamma = \Gamma f(Y)$ if and only if

$$\begin{pmatrix} f(X) & 0 \\ 0 & f(Y) \end{pmatrix} \begin{pmatrix} I & \Gamma \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & \Gamma \\ 0 & I \end{pmatrix} \begin{pmatrix} f(X) & 0 \\ 0 & f(Y) \end{pmatrix}.$$

Thus if f respects direct sums and similarity, then f respects intertwining. the other hand, if f respects intertwining then, by choosing Γ to be an appropriate projection, it is easily seen that $f$respects direct sums too.

Remark (1.1.3)[1]: Let $\mathcal{U}$ be a non-commutative domain in $(\mathbb{C})^{g}$ and suppose $f : \mathcal{U} \to M(\mathbb{C})^{\tilde{g}}$ is a free map. If $X \in \mathcal{U}$ is similar to Y with $Y = S^{-1}XS$, then we can define $(Y) = S^{-1}XS$. In this way f naturally extends to a free map on $\mathcal{H}(\mathcal{U}) \subseteq M(\mathbb{C})^{g}$ defined by

$$\mathcal{H}(\mathcal{U})(n) = \{Y \in M_n(\mathbb{C})^{g} : \text{there is an } X \in \mathcal{U}(n) \text{ such that } Y \text{ is similar to } X\}.$$

Thus if $\mathcal{U}$ is a domain of holomorphy, then $\mathcal{H}(\mathcal{U}) = \mathcal{U}$.

On the other hand, because our results on proper analytic free maps to come depend strongly upon the non-commutative set $\mathcal{U}$ itself, the distinction between non-commutative sets and fully matricial sets are important.

Proposition (1.1.4)[1]: If $\mathcal{U}$ is a non-commutative subset of $(\mathbb{C})^{g}$ and $f : \mathcal{U} \to M(\mathbb{C})^{\tilde{g}}$ is a free map,then the range of f , equal to the sequence $f(\mathcal{U}) = (f(\mathcal{U})(n)))$, is itself a non-commutative subset of $M(\mathbb{C})^{\tilde{g}}$.

Let $\mathcal{U} \subseteq (\mathbb{C})^{g}$ be a non-commutative set. A free map $: \mathcal{U} \to M(\mathbb{C})^{\tilde{g}}$ is continuous if each $f : \mathcal{U}(n) \to M_n(\mathbb{C})^{\tilde{g}}$ is continuous. Likewise, if $\mathcal{U}$ is a non-commutative domain, then f is called analytic if each $f\ [n]$ is analytic. This implies the existence of directional derivatives for all directions at each point in the domain**.**

Lemma (1.1.5)[1]: Suppose $\mathcal{U} \subseteq (\mathbb{C})^g$ is a non-commutative set and f : $\mathcal{U} \to M(\mathbb{C})^{\tilde{g}}$ is a free map.

Suppose $X \in (n)$, $Y \in (m)$, and Γ is an $n \times$ matrix. Let

$$C_j = X_j\Gamma - \Gamma Y_j, \quad Z_j = \begin{pmatrix} Xj & Cj \\ 0 & Yj \end{pmatrix}. \quad (3)$$

$If\ Z = (Z_{1,\dots,g}) \in \mathcal{U}(n+m)$, then

$$f_j(Z) = \begin{pmatrix} fj(X) & fj(X)\Gamma - \Gamma fj(Y) \\ 0 & fj(Y) \end{pmatrix}. \quad (4)$$

This formula generalizes to larger block matrices.

Proof: With

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

we have

$$\tilde{Z}_j = \begin{pmatrix} Xj & 0 \\ 0 & Yj \end{pmatrix} = SZ_jS^{-1}.$$

Thus, writing $f = (f_{1,\dots,\tilde{g}})^T$ and using the fact that f respects intertwining maps, for each j,

$$(Z) = (\tilde{Z})\, S^{-1} = \begin{pmatrix} fj(X) & fj(X)\Gamma - \Gamma fj(Y) \\ 0 & fj(Y) \end{pmatrix}. \quad (5)$$

Proposition (1.1.6)[1]: Suppose $\mathcal{U}$ is a non-commutative domain in $(\mathbb{C})^g$.

(i) A continuous free map $: \mathcal{U} \to M\mathbb{C})^{\tilde{g}}$ is analytic.

(ii) $If\ X \in (n)$, and $H \in (\mathbb{C})^g$ has sufficiently small norm, then

$$f\begin{pmatrix} X & H \\ 0 & X \end{pmatrix} = \begin{pmatrix})f(X) & f(X[H] \\ 0 & f(X) \end{pmatrix}.$$

Proof : Fix n and $\in (n)$. Because $(2n)$ is open and $X \oplus (2n)$, for every $H \in (\mathbb{C})^g$ of sufficiently small norm the tuple with j-th entry

$$\begin{pmatrix} Xj & Hj \\ 0 & Xj \end{pmatrix}. \quad (6)$$

is in $\mathcal{U}(2n)$. Hence, for $z \in \mathbb{C}$ of small modulus, the tuple (z) with j-th entry

$$Z(z) = \begin{pmatrix} Xj + zHj & Hj \\ 0 & Xj \end{pmatrix} \quad (7)$$

is in $\mathcal{U}(2n)$. Note that the choice (when $z \neq 0$) of $(z) = 1/z$, $X = X +$ and

$= X$ in Lemma(1.1.7) gives this $Z(z)$.Hence, by Lemma (1.1.7),

$$f(Z(z)) = \begin{pmatrix} f(X+zH) & \frac{f(X+zH)-f(X)}{z} \\ 0 & f(X) \end{pmatrix}.$$

Since (z) converges as z tends to 0 and $[2n]$ is assumed continuous, the limit

$$\lim_{z\to 0} \frac{f(X+zH)-f(X)}{z}$$

exists. This proves that f is analytic at X. It also establishes the moreover portion of the proposition.

For perspective we mention power series. an analytic free map f has a formal power series expansion in the non-commuting variables, which indeed is a powerful way to think of analytic free maps.

Voiculescu also gives elegant formulas for the coefficients of the power series expansion of f in terms of clever evaluations of f. Convergence properties for bounded analytic free maps are studied for a bad unbounded example.

Given non-commutative domains $\mathcal{U}$ and $\mathcal{V}$ in $(\mathbb{C})^g$ and $M(\mathbb{C})^{\tilde{g}}$ respectively, a free map $: \mathcal{U} \to \mathcal{V}$ is *proper* if each $[n] : \mathcal{U}(n) \to \mathcal{V}(n)$ is proper in the sense that if $K \subseteq \mathcal{V}(n)$ is compact, then $f^{-1}(K)$ is compact. In particular, for all n, if (z_j) is a sequence in (n) and $z_j \to (n)$, then $(z_j) \to \partial(n)$. In the case $g = \tilde{g}$ and both f and f^{-1} are (proper) analytic free maps we say f is a bi analytic free map.

Corollary (1.1.7)[1]: Suppose $\mathcal{U}$ and $\mathcal{V}$ are non-commutative domains in $(\mathbb{C})^g$. If f: $\mathcal{U}\to\mathcal{V}$ is a free map and if each $[n]$ is bianalytic , then f is a bianalytic free map. Before proving Theorem (1.1.5) we establish the following preliminary result which is of independent interest and whose proof uses the full strength of Lemma (1.1.5).

Proposition (1.1.8)[1]: Let $\mathcal{U} \subseteq (\mathbb{C})$ be a non-commutative domain and suppose $f : \mathcal{U} \to M(\mathbb{C})^{\tilde{g}}$ is a free map. Suppose further that $X \in (n)$, $\in \mathcal{U}(m)$, Γ is an $n\times m$ matrix, and

$$f(X)\Gamma = \Gamma f(Y).$$

if $f^{-1}(\{f(X)\oplus f(Y)\})$ has compact closure in $\mathcal{U}$, then $X\Gamma = \Gamma Y$.

Proof: As in Lemma (1.1.5), let $C_j = X_j\Gamma$- . For $0 < t$ sufficiently small, $(t) \in (n+m)$, Where

$$Z_j(t) = \begin{pmatrix} Xj & tCj \\ 0 & Yj \end{pmatrix}. \qquad (8)$$

if $f(X)\Gamma = \Gamma f(Y)$, then, by Lemma (1.1.5),

$$f(Z(t)) = \begin{pmatrix} fj(X) & t(fj(X)\Gamma - \Gamma fj(Y)) \\ 0 & fj(Y) \end{pmatrix} \qquad (9)$$

$$= \begin{pmatrix} fj(X) & 0 \\ 0 & fj(Y) \end{pmatrix}$$

Thus, $((t)) = (Z(0))$. In particular,

$$f^{-1}(\{((0))\}) \supseteq \{(t) : t \in \mathbb{C}\} \cap \mathcal{U}.$$

Since this set has, by assumption, compact closure in $\mathcal{U}$ it follows that $C = 0$; i.e., $X\Gamma = \Gamma Y$.

We are now ready to prove that a proper free map is one-to-one and even a bi analytic free map if continuous and mapping between domains of the same dimension.

Theorem (1.1.9)[1]: Let $\mathcal{U}$ and $\mathcal{V}$ be non-commutative domains containing 0 in $(\mathbb{C})^g$ and $M(\mathbb{C})^{\tilde{g}}$, respectively and suppose $f : \mathcal{U} \to \mathcal{V}$ is a free map.

(i) If f is proper, then it is one-to-one, and $f^{-1} : f(\mathcal{U}) \to \mathcal{U}$ is a free map.

(ii) If, for each n and $Z \in (\mathbb{C})^{\tilde{g}}$, the set $f[n]^{-1}(\{Z\})$ has compact closure in $\mathcal{U}$ then f is one-to-one and moreover, $f^{-1} : f(\mathcal{U}) \to \mathcal{U}$ is a free map.

(iii) If= $\tilde{g}$ and $f : \mathcal{U} \to \mathcal{V}$ is proper and continuous, then f is bianalytic.

Proof : If f is proper, then $f^{-1}(\{Z\})$ has compact closure in $\mathcal{U}$ for every $Z \in (\mathbb{C})^g$.

Hence (i)is a consequence of (ii). For (ii), invoke Proposition (1.1.9) with $\Gamma = \gamma I$ to conclude that f is injective. Thus $: \mathcal{U} \to (\mathcal{U})$ is abijection from one non-commutative set to another. Given $W,(\mathcal{U})$ there exists $X,Y \in \mathcal{U}$ such that $f(X) = W$ and $f(Y) = Z$ If moreover, $W\Gamma = \Gamma Z$, then $f(X)\Gamma = \Gamma f(Y)$

and Proposition (1.1.9) implies $X\Gamma = \Gamma\gamma$ i.e., $f^{-1}(W)\Gamma = \Gamma f^{-1}(Z)$. Hence f is itself a free map.

Let us now consider (iii). Using the continuity hypothesis and Proposition (1.1.6), for each n the map $[n] : \mathcal{U}(n) \to \mathcal{V}(n)$ is analytic. By hypothesis each $[n]$ is also proper and hence its range is (n) .

Now $[n] : (n) \to \mathcal{V}(n)$ is one-to-one, onto and analytic, so its inverse is analytic. Further, by the already proved part of the theorem f^{-1} is an analytic free map. For both completeness and later use we record the following companion to Lemma (1.1.5).

Proposition(1.1.10)[1]: Let $\mathcal{U} \subseteq$ M($\mathbb{C}$) and $\mathcal{V} \subseteq$ ($\mathbb{C}$)$\tilde{}$ be non-commutative domains. If f: $\mathcal{U} \to \mathcal{V}$ is a proper analytic free map and if $X \epsilon \mathcal{U}(n)$, then $f'(X)$: $M_n(\mathbb{C})^{g} \to M_n(\mathbb{C})^{\tilde{g}}$ is one-to-one in particular, if g = $\tilde{g}$, then $f'(X)$ is a vector space isomorphism.

Proof: Suppose $f'(X)[H] = 0$. We scale H so that $\begin{pmatrix} X & H \\ 0 & X \end{pmatrix} \epsilon\ \mathcal{U}$. From Proposition (1.1.8)

$$f\begin{pmatrix} X & H \\ 0 & X \end{pmatrix} = \begin{pmatrix} f(X) & f'(X)[H] \\ 0 & f(X) \end{pmatrix} = \begin{pmatrix} f(X) & 0 \\ 0 & f(X) \end{pmatrix} = f\begin{pmatrix} X & H \\ 0 & X \end{pmatrix}. \quad (10)$$

By the injectivity of f established in Theorem (1.1.11), $H=0$.

Key to the proof of Theorem (1.1.9) is testing f on the special class of matrices of the form (1.1.9). One naturally asks if the hypotheses of the theorem in fact yield stronger conclusions, say by plugging in richer classes of test matrices. The answer to this question is no: suppose f is any analytic free map from g to g variables defined on a neighborhood (n) of 0 with $f(0)=0$ and $[1]'(0)$ invertible. Under mild additional assumptions (e.g. the lowest eigenvalue of $'(X)$ or the norm $\|f'(X)\|$ is bounded away from 0 for $X\epsilon\mathcal{N}_{\varepsilon}(n)$ independently of the size n) then there are non-commutative domains $\mathcal{U}$ and $\mathcal{V}$ with f:$\mathcal{U}\to\mathcal{V}$ meeting the hypotheses of the theorem.

Indeed, consider (for fixed n) the analytic function $[n]$ on (n). Its derivative at 0 is invertible; in fact, $[n]'(0)$ is unitarily equivalent to $I_n \otimes$

$f[1]'(0)$, cf. Lemma (1.2.2) below. By the implicit function theorem, there is a small δ-neighborhood of 0 on which $[n]^{-1}$ is defined and analytic.

By our assumptions and the bounds on the size of this neighborhood given in [ ], $\delta>0$ may be chosen to be independent of n.This gives rise to a non-commutative domain $\mathcal{V}$ and the analytic free map f^{-1}: $\mathcal{U} \to \mathcal{V}$, where $\mathcal{U} = f^{-1}(\mathcal{V})$. Note $\mathcal{U}$ is open (and hence a non commutative domain) since $f^{-1}(n)$ is analytic and one-to-one. It is now clear that : $\mathcal{U} \to \mathcal{V}$ satisfies the hypotheses of Theorem (1.1.10).

We just saw that absent more conditions on the non-commutative domains $\mathcal{D}$ and $\mathcal{D}$, nothing beyond bianalytic free can be concluded about

Section (1.2): Maps in One Variable and Examples

In this section analytic free map analogs of classical several complex variable theorems are obtained by combining the corresponding classical theorem and Theorem (1.1.10).Indeed, hypotheses for these analytic free map results are weaker than their classical analogs would suggest.

The commutative Caratheodory-Cartan-Kaup-Wu (CCKW) Theorem (1.1.10) say that if f is an analytic self-map of a bounded domain in C^g which fixes a point P, then the eigenvalues of $f'(P)$ have modulus at most one. Conversely, if the eigenvalues all have modulus one, then f is in fact an automorphism; and further if $'(P) = 1$, then f is the identity.

The CCKW theorem together with Corollary (1.1.8) yields Corollary (1.2.1) below . We note that Theorem can also be thought of as a non-commutative CCKW theorem in that it concludes, like the CCKW theorem does, that a map f is bianalytic, but under the (rather different) assumption that f is proper. Then one works with the formal power series representation for a free analytic function.

Lemma (1.2.1)[1]: keep the notation and hypothesis of Corollary (1.2.2) If n is a positive integer and ϕ denotes the mapping $[n] : \mathcal{D}(n)$

$\to \mathcal{D}(n)$, then $\Phi'(0)$ is unitarily equivalent to $I_n \otimes \phi'(0)$.

Proof: Let $E_{i,}$ denote the matrix units for $(\mathbb{C})$. Fix $h \epsilon \mathbb{C}^g$. Arguing as in the proof of Proposition $(1.1.10)$ gives. for $k \neq \ell$ and $z \epsilon \mathbb{C}$ of small modulus,

$$\Phi((E_{k,k} + E_{k,\ell}) \otimes zh) = (E_{k,k} + E_{k,\ell}) \otimes (zh).$$

It follows that

$$\Phi'(0)[(E_{k,k} + E_{k,\ell}) \otimes h] = (E_{k,k} + E_{k,\ell}) \phi'(0)[h].$$

On the other hand ,

$$\Phi'(0)[E_{k,k} \otimes h] = E_{k,k} \otimes \phi'(0)[h].$$

By linearity of $\Phi'(0)$, it follows that

$$\Phi'(0)[Ek,\ell \otimes h] = Ek,\ell \otimes \phi'(0)[h].$$

Thus,$'(0)$ is unitarily equivalent to $In \otimes \phi(0)$.

Corollary (1.2.2)[1]:

Let $\mathcal{D}$ be a given bounded non-commutative domain which contains 0. Suppose $: \mathcal{D} \to \mathcal{D}$ is an analytic free map. Let ϕ denote the mapping [1] $: \mathcal{D}(1) \to \mathcal{D}(1)$ and assume $(0) = 0$.

(i) If all the eigenvalues of $'(0)$ have modulus one, then f is a bi analytic free map

(ii) If $\phi'(0) = 1$, then f is the identity.

The proof uses the following lemma(1.2.1), whose proof is trivial if it is assumed that f is continuous (and hence analytic) and then one works with the formal power series representation for a free analytic function.

Proof : The hypothesis that $\phi'(0)$ has eigenvalues of modulus one, implies, by Lemma $(1.2.1)$,that, for each n, the eigenvalues of $f[n]'(0)$ all have modulus one. Thus, by the CCKW theorem, each $[n]$ is an automorphism. Now Corollary $(1.1.8)$ implies f is a bianalytic free map.

Similarly, if $\phi'(0) = I_g$, then $f[n]'(0) = I_{ng}$ for each n. Hence, by the CCKW theorem, $[n]$ is the identity for every n and therefore f is itself the identity.

Note a classical bianalytic function f is completely determined by its value and differential at a point. Much the same is true for analytic free maps and for the same reason.

Proposition (1.2.3)[1]: Suppose $\mathcal{U}$, $\mathcal{V} \subseteq$ $(\mathbb{C})$ are non-commutative domains, $\mathcal{U}$ is bounded, both contain 0, and f ,g $: \mathcal{U} \to \mathcal{V}$ are proper analytic free maps. If (0) = (0) and $f'(0) = g'(0)$, then f = g.

Proof: By Theorem (1.1.10) both $f$and g are bianalytic free maps. Thus $h = f o g^{-1}: \mathcal{U} \to \mathcal{U}$ is a bianalytic free map fixing 0 with $h[1]'(0) = I$. Thus, by Corollary (1.2.2), h is the identity consequently f = g.

A subset S of a complex vector space is circular if exp(it)$s \epsilon S$ whenever $s \epsilon S$ and $t \epsilon \mathbb{R}$. A non-commutative domain $\mathcal{U}$ is circular if each (n) is circular. Compare the following theorem to its commutative counterpart where the domains $\mathcal{U}$ and $\mathcal{V}$ are the same.

Corollary (1.2.4)[1]: Let $\mathcal{U}$ and $\mathcal{V}$ be bounded non-commutative domains in $(\mathbb{C})^{g}$ both of which contain 0.

Suppose : $\mathcal{U} \to \mathcal{V}$ is a proper analytic free map If both $\mathcal{U}$ and $\mathcal{V}$ are circular and if one is convex, then so is the other.

This corollary is an immediate consequence of Theorem (1.2.5) and the fact (see Theorem ((1.1.9) (iii)) that f is onto $\mathcal{V}$.

We admit the hypothesis that the range $\mathcal{R} = f(\mathcal{U})$ of f in Theorem (1.2.5) is circular seems pretty contrived when the domains $\mathcal{U}$ and $\mathcal{V}$ have a different number of variables. On the other hand if they have the same number of variables it is the same as $\mathcal{V}$ being circular since by Theorem (1.1.9), f is onto.

Theorem (1.2.5)[1]: Let $\mathcal{U}$ and $\mathcal{V}$ be bounded non-commutative domains in $(\mathbb{C})$and $M(\mathbb{C})^{\tilde{g}}$, respectively, both of which contain 0. Suppose : $\mathcal{U} \to \mathcal{V}$ is a proper analytic free map with $f(0) = 0$.

If $\mathcal{U}$ and the range $\mathcal{R} := (\mathcal{U})$ of f are circular, then f is linear. The domain $\mathcal{U} = ((n))$ is convex if each $\mathcal{U}(n)$ is a convex set.

Proof : Because f is a proper free map it is injective and its inverse (defined on $\mathcal{R}$) is a free map by Theorem (1.1.10) Moreover, using the analyticity of f ,its derivative is point-wise injective by Proposition (1.1.11) It follows that each $f[n] : \mathcal{U}(n) \to M_n(\mathbb{C})^{\tilde{g}}$ is an embedding. Thus, each $[n]$ is a homeomorphism onto its range and its inverse $[n]^{-1}$=$f^{-1}[n]$ is continuous. Define $: \mathcal{U} \to \mathcal{U}$ by

$$(x) := f^{-1}(e^{i\theta}(e^{i\theta} x)). \qquad (11)$$

This function respects direct sums and similarities, since it is the composition of maps which do. Moreover, it is continuous by the discussion above. Thus F is an analytic free map. Using the relation $e^{i\theta}$ $((x)) = (e^{i\theta})$ we find $e^{i\theta} f'(F(0))F'(0) = f'(0)$. Since $f'(0)$ is injective, $e^{i\theta}\ F'(0)\ = 1$ It follows from Corollary (1.2.2)(ii) that $F(x) = e^{i\theta}x$ and thus, by (1.2.2), $f(e^{i\theta}x) = e^{i\theta}f(x)$. Since this holds for every θ, it follows that f is linear. if f is not assumed to map 0 to 0 (but instead fixes some other point), then a proper self-map need not be linear.

This section contains two examples. The first shows that the circled hypothesis is needed in Theorem (1.2.5) Our second example concerns $\mathcal{D}$, a non-commutative domain in one variable containing the origin, and $: \mathcal{D} \to \mathcal{D}$ a proper analytic free map with $b(0) = 0$. It follows that b is bianalytic and hence $[1]'(0)$ has modulus one. Our second example shows that this setting can force further restrictions on $[1]'(0)$. The non-commutative domains of both examples are LMI domains; i.e., they are the non-commutative solution set of a linear matrix inequality (LMI). Such domains are convex, and play a major role in the important area of semidefinite programming .

A special case of the non-commutative domains are those described by a linear matrix inequality. Given a positive integer d and $A_1,\dots, A_g \in M_d(\mathbb{C})$, the linear matrix-valued polynomial

$$(x) = \sum A_j\, x_j \in M_d(\mathbb{C}) \otimes \mathbb{C}\langle x_1,\dots,x_g\rangle$$

is a truly linear pencil. Its adjoint is, by definition, $(x)^* = \sum A^*_j x^*_j$. Let

$$\mathcal{L}(x) = I_d + L(x) + L(x)^* .$$

If $(\mathbb{C})^g$,then $\mathcal{L}(X)$ is defined by the canonical substitution,

$$\mathcal{L}(X) = I_d \otimes I_n + \sum A_j \otimes X_j + \sum A^*_j \otimes X^*_j,$$

and yields a symmetric $d_n \times d_n$ matrix. The inequality $\mathcal{L}(X) > 0$ for tuples $X \epsilon M(\mathbb{C})^g$ is a linear matrix inequality (LMI). The sequence of solution sets $\mathcal{D}_{\mathcal{L}}$ defined by

$$\mathcal{D}_{\mathcal{L}}(n) = \{X \epsilon M_n(\mathbb{C})^g : \mathcal{L}(X) > 0\}$$

is a non-commutative domain which contains a neighborhood of 0. It is called a non-commutative (NC) LMI domain.

It is surprisingly difficulty to find proper self-maps on LMI domains which are not linear.

This section contains the only such example, up to trivial modification, of which we are aware.By Theorem (1.2.5) the underlying domain cannot be circular.

In this example the domain is a one-variable LMI domain. Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

and let $\mathcal{L}$ denote the univariate 2×2 linear pencil,

$$\mathcal{L}(x) := I + Ax + A^* x^* = \begin{pmatrix} 1+x+x^* & x \\ x^* & 1 \end{pmatrix}. \qquad (12)$$

Then

$$\mathcal{D}_{\mathcal{L}} = \{X \mid \|X-1\| < \sqrt{2}\}.$$

To see this note $\mathcal{L}(X) > 0$ if and only if $1+X+X^*-XX^*>0$, which is in turn equivalent to $(1-X)(1-X)^* < 2$.

Proposition (1.2.6)[1]: For real θ, consider

$$f_\theta(x) := \frac{e^{i\theta}x}{1+x-e^{i\theta}x}.$$

(i) : $\mathcal{D}_{\mathcal{L}} \to \mathcal{D}_{\mathcal{L}}$ is a proper analytic free map, $f_\theta(0) = 0$, and $f'_\theta(0) = \exp(i\theta)$.

(ii) Every proper analytic free map $f: \mathcal{D}_{\mathcal{L}} \to \mathcal{D}_{\mathcal{L}}$ fixing the origin equals one of the f_θ.

Proof: Item (i) follows from a straightforward computation:

(1-(X))(1-(X))* < 2

$$\Leftrightarrow \left(1-\frac{e^{i\theta}X}{1+X-e^{i\theta}X}\right)\left(1-\frac{e^{i\theta}X}{1+X-e^{i\theta}X}\right)^* < 2$$

$$\Leftrightarrow \left(\frac{1+X-2e^{i\theta}X}{1+X-e^{i\theta}X}\right)\left(\frac{1+X-2e^{i\theta}X}{1+X-e^{i\theta}X}\right)^* < 2$$

$$\Leftrightarrow (1+X-2e^{i\theta}X)(1+X-2e^{i\theta}X)^* < 2(1+X-e^{i\theta}X)(1+X-e^{i\theta}X)^*$$

$$\Leftrightarrow 1+X+X^*-XX^* > 0$$

$$\Leftrightarrow (1-X)(1-X)^* < 2.$$

Statement (ii) follows from the uniqueness of a bianalytic map carrying 0 to 0 with a prescribed derivative.

Recall that a bianalytic f with $(0)=0$ is completely determined by its differential at a point. Clearly, when $f'(0) = 1$, then $f(x) = x$. Does a proper analytic free self-map exist for each $f'(0)$ of modulus one? In the previous example this was the case. For the domain in the example in this subsection, again in one variable, there is no proper analytic free self-map whose derivative at the origin is i.

The domain will be a "non-commutative ellipse" described as $\mathcal{D}_\mathcal{L}$ with $\mathcal{L}(x) \coloneqq I+Ax+A^*x^*$ for A of the form

$$A \coloneqq \begin{pmatrix} C_1 & C_2 \\ 0 & -C_1 \end{pmatrix}$$

where $C_1, C_2 \epsilon \mathbb{R}$. There is a choice of parameters in $\mathcal{L}$ such that there is no proper analytic free self-map b on $\mathcal{D}_\mathcal{L}$ with $(0) = 0$, and $b'(0) = i$.

Suppose $: \mathcal{D}_\mathcal{L} \to \mathcal{D}_\mathcal{L}$ is a proper analytic free self-map with $b(0) = 0$, and $b'(0) = i$. By Theorem (1.1.10), b is bianalytic. In particular, $b[1] : \mathcal{D}_\mathcal{L}(1) \to \mathcal{D}_\mathcal{L}(1)$ is bianalytic. By the Riemann mapping theorem there is a conformal map f of the unit disk onto $\mathcal{D}_\mathcal{L}(1)$ satisfying $f(0) = 0$. Then

$$b[1](z) = f(if^{-1}(z)). \qquad (13)$$

(Note that $b[1] \, o \, b[1] o \, b[1] o \, b[1]$ is the identity.)

To give an explicit example, we recall some special functions involving elliptic integrals. Let $(z,\ t)$ and (t) be the normal and complete elliptic integrals of the first kind, respectively, that is,

$$(z, t) = \int_0^z \frac{dx}{\sqrt{(1-x^2)(1-t^2x^2)}}, \qquad (t) = (1, t).$$

Furthermore, let

$$\mu(t) = \frac{\pi}{2} \frac{K(\sqrt{(1-t^2)}}{K(t)}.$$

Choose the axis for the non-commutative ellipse as follows:

$$a = \cosh\left(\tfrac{1}{2}\mu\left(\tfrac{2}{3}\right)\right), \quad b = \sinh\left(\tfrac{1}{2}\mu\left(\tfrac{2}{3}\right)\right).$$

Then

$$C_1 = \frac{1}{2}\sqrt{\frac{1}{a^2} - \frac{1}{b^2}}, \quad C_2 = \frac{1}{b}.$$

The desired conformal mapping is

$$f(z) = \sin\left(\frac{\pi}{2K\left(\frac{2}{3}\right)} K\left(\frac{z}{\sqrt{\frac{2}{3}}}, \frac{2}{3}\right)\right).$$

Hence [1] in (1.2.6) can be explicitly computed has a power series expansion

$$\begin{aligned} b[1](z) &= iz - \frac{1}{27} i\left(9 - \frac{52K\left(\frac{4}{9}\right)^2}{\pi^2}\right) z^3 + i\frac{\left(9\pi^2 - 52K\left(\frac{4}{9}\right)^2\right)}{486\pi^4} z^5 + O(z^7) \\ &\approx i(1 + 0.30572 z^3 + 0.140197 z^5). \end{aligned}$$

This power series expansion has a radius of convergence $\geq \varepsilon > 0$ and thus induces an analytic free mapping $\mathcal{N}_\varepsilon \to (\mathbb{C})$. By analytic continuation, this function coincides with b.This enables us to evaluate (zN) for a nilpotent N.

Let N be an order 3 nilpotent,

$$N = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $r \in \mathbb{R}$ satisfies $rN \in \mathcal{D}_{\mathcal{L}}$ if and only if $-1.00033 \leq r \leq 1.00033 =: r_0$.

This has been computed symbolically in the exact arithmetic using Mathematica, and the bounds given here are just approximations. However, $(r_0N)_{\mathcal{L}} \backslash \partial\mathcal{D}_{\mathcal{L}}$ contradicting the properness. This was established by computing the 8×8 matrix $\mathcal{L}\ ((r_0N)\ )$ symbolically thus *ensuring it is exact. Then* we apply a numerical eigenvalue solver to see that it is positive definite with smallest eigenvalue 0.0114903. . . . We conclude that the proper analytic free self-map *b* does not exist.