Sudan University of Science and Technology

On The Geometrical Approach to The Problem of Integrability of Hamiltonian Systems

حول الإقتراب الهندسي لمسألة قابلية التكامل لأنظمة هاملتون

A thesis submitted for degree of Ph.D. in Mathematics

By

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Dedication

To my husband Yahia Altaib and My big family

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Abstract

In this research we treat the problem of integrability of Hamiltonian systems . There have been several methods for treating this problem depending on different situations. These methods include the first integral method obtained via the Poission bracket and generalized in the context of Lie bracket . The latter generalization is based on Hamiltonian mechanics and symplectic structure. The method that we emphasized in this research is the Cartan method of moving frame. We have utilized this method of moving frame where the killing tensor is major entity that is involved in the treatment .In particular, we have used the intrinsic geometry provided by the Guass and main curvature to extract the separable system of coordinates by employing the method of moving frame. Then the corresponding Killing tensor , the potential function and the first integrals are recovered .We have applied this procedure of separation of variables to surfaces of rotation and surface of constants curvature .

المستخلص

فى هذا البحث عالجنا مشكلة التكامل للنظم الهاملتونيه، حيث تطرقنا لعدة طرق مختلفة. ووجدنا من خلال هذه الرسالة أن المعالجة الهندسية أفضت إلى حلول واضحة إعتماداً على المشاكل المختلفة. وتعتبر الصياغة المستخدمة صياغه هندسيه حيث أستعملنا طريقة كارتان للاطار المتحرك حيث إن ممتدة كلينق هو المعامل الرئيسى المستخدم فى معالجة المسألة .

ان الطرق المختلفة في هذه الرسالة تشمل طريقة التكاملي الاول من خلال قوس بوسون والذي تم تعميمه في قوس لي . هذا التعميم الاخير مؤسس على الميكانيكا الهاملتونية والتركيب السمبلكتيكي . ان الطريقة التي اكدنا عليها في هذه الرسالة هي طريقة كارتان للاطار المتحرك.

على وجه الخصوص استخدمنا الهندسة الذاتية المعطاة بانحناء جاوس والانحناء المتوسط من اجل استخلاص فصل المتغيرات وذلك بالاستعانة بطريقة كارتان للاطار المتحرك . وبذا امكن استرجاع ممتد كلينق ودوال الطاقة الكامنة وكذلك التكامليات الاولى . وطبقنا هذه الاجراءات لفصل المتغيرات على سطوح الدوران وسطوح ذات الانحناء الثابت .

Introduction

The problem of the itegrability of Hamiltonian system is a long standing problem. Several trials has been achieved to approach a complete solution. In fact there has been roughly three main approaches. The first approach is the classical approach where one seeks first integrals of the Hamiltonian system and then the solution is written via these integrals. A first integral F satisfies:

$\frac{\partial F}{\partial t} + \{F, H\} = 0$, where $\{,\}$ is the Poisson bracket and H is the Hamiltonian function . In this approach one uses Calculus as an analytical tool. However one can also utilize Lie bracket instead of Poisson bracket. The Lie bracket is considered as a geometrical approach, where we involve vector fields, called Hamiltonian vector fields, corresponding to Hamiltonian functions. The next stage in the development of integrability of Hamiltonian system is due to Eisenhart and Cartan approach. Eiserhart used the frame field and Cartan used the coframe field and thus exterior Calculus is to be the geometrical tool for a free coordinates description of Hamiltonian system and Hamilton's equation. In this late approach prove existance and uniqueness of solutions of the system , which is problem of integrability . Of particular interest to us as a technique to solve Hamiltonian system is the method of separation of variables . The key idea behind this method is to seek a k-set of special coordinates : $q = (q^1, \dots, q^k)$ in which corresponding Hamilton- Jacobi partial differential equation admits a complete integral of the form

$$w(q, C) = w_1(q^1, C) + \cdots + w_n(q^n, C).$$

This method of separability has been considered by several mathematician such as Dall' Acqua, Eisenhart, Levi-Civita , Riai , Stackel and others. In this research we develop the method of separability and use it in the same cases.

Lastly we want to mention that these is another third approach to the problem of integrability of Hamiltonian system. This approach is purely geometrical .

Indeed here mathematicians construct the solutions as submanifolds of some ambient manifold . The first integrals of the system or their corresponding vector fields are interpreted as generators of the flows and provide the symmetries of the system .The reduction of order of the Hamiltonian equations is achieved in such a way that for each symmetry the order is reduced by two .

So the geometrical properties of the symplectic form and the corresponding Lie symmetries that come from first integrals are utilized to construct the submanifolds of solutions.

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Differentiable Manifolds and Lie Groups

1.1Manifold

1. the definition of differentiable manifolds:

(1.1.1)Definition:

If $U \subset \mathfrak{R}^m$ is open and $V \subset \mathfrak{R}^m$ is open, $\psi: U \to V$ is said to be differomorphism if ψ is infinitely differentiable map with infinitely differentiable inverse, and the objects defined on U will counter parts on V.

(1.1.2)Definition:

An m-dimensional manifold is a set M, together with a countable collection of subsets for $U_\alpha \subset M$, called coordinate charts, and one-to-one functions φ: $U_\alpha \to V_\alpha$ onto connected open subset $V_\alpha$of $\mathfrak{R}^m$, where $V_\alpha \subset \mathfrak{R}^m$called local coordinate maps which satisfy:-

1. The coordinate charts cover M.
2. On the overlap of any pair of coordinate charts $U_\alpha \cap U_\beta$composite map

$$\varphi_\beta{}^\circ\varphi_\alpha^{-1}: \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is smooth (or infinitely differentiable) function.

3. W of $\varphi_\alpha(x)$ inV_α and $\breve{W}$ of $\varphi_\beta(\bar{x})$ in V_β such that

$$\varphi_\alpha^{-1}(w) \cap \varphi_\beta(\overline{w}) = \emptyset$$

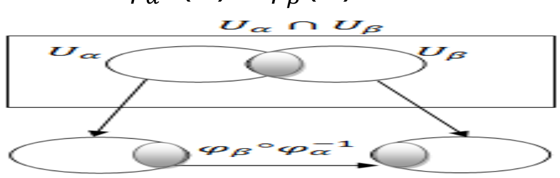


Figure 1.1

Coordinate Charts on manifold

Example(1):

The sample n-dimensional manifold is just Euclid's space $\mathfrak{R}^n$itself. There is a single coordinate chart $U = \mathfrak{R}^m$ with local coordinate map given by$X = I: \mathfrak{R}^n \to \mathfrak{R}^n$, more generally any open subset $U \subset \mathfrak{R}^n$ is an n-dimensional manifold with a single coordinate chart given Uitself, with local coordinate map the identity again. Conversely, if M is any manifold with a single global coordinate chart

$$X: M \to V \subset \Re^n$$

We can identify M with image V, an open subset of $\Re^n$.

Example(2):

The unit sphere

$$S^2 = \{(x, y, z): x^2 + y^2 + z^2 = 1\}$$

Is an example of non-trivial two-dimensional manifold realized as a surface in $\Re^3$ let $U_1 = S^2 \setminus \{(0,0)\}$, $U_2 = S^2 \setminus \{(0,0,1)\}$be the subset obtain by deleting the north and south poles respectively let

$$\varphi_\alpha: U_\alpha \to \Re^2 \approx \{(x, y, 0)\}, \alpha = 1,2$$

be stereographic projection from respective poles,

$$\varphi_1(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

$$\varphi_2(x, y, z) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right)$$

It can be easily checked that on the overlap $U_1 \cap U_2$.

The Hausdorff separation property follows easily from that of $\Re^3$, so S^2 is a smooth, indeed two-dimensional manifold. The unit sphere is particular case of the general concept of surface in $\Re^3$, which historically provided the principle motivating of the general theory of manifolds.

(1.1.3) Definition

The set $U \subset M$ is open if and only if $\forall x \in U$ there is a neighborhood of x contained in U so $x \in \varphi^{-1}(\omega)$, $x \in \varphi_\alpha^{-1}(\omega) \subset U_\alpha$where $\varphi_\alpha: U_\alpha \to V_\alpha$is coordinate chart containing x, and ω is open subset of V_α .

- The degree of differentiability of the overlap functions $\varphi_\beta \circ \varphi_\alpha^{-1}$determines the degree of smoothness of the manifold M.

2. Map between manifolds:

(1.1.4)Definition:

If M and N are smooth manifolds, a map $f: M \to N$ *is* said to be smooth if its local coordinate expression is a smooth map in every coordinate chart.

In other words, for every coordinate chart $\varphi_\alpha: U_\alpha \to V_\alpha \subset \Re^m$ on M and every chart $\tilde{\varphi}_\beta: \tilde{U}_\beta \to \tilde{V}_\beta \subset \Re^n$ *on* N.

The composite map

$$\tilde{\varphi}_\beta \circ f \circ \varphi_\alpha^{-1} : \mathfrak{R}^m \to \mathfrak{R}^n \quad (1.1)$$

is a smooth map. In other words, a smooth map is of the form y= $f(x)$, where f is a smooth function on the open subsets given local coordinates x on M and y on N.

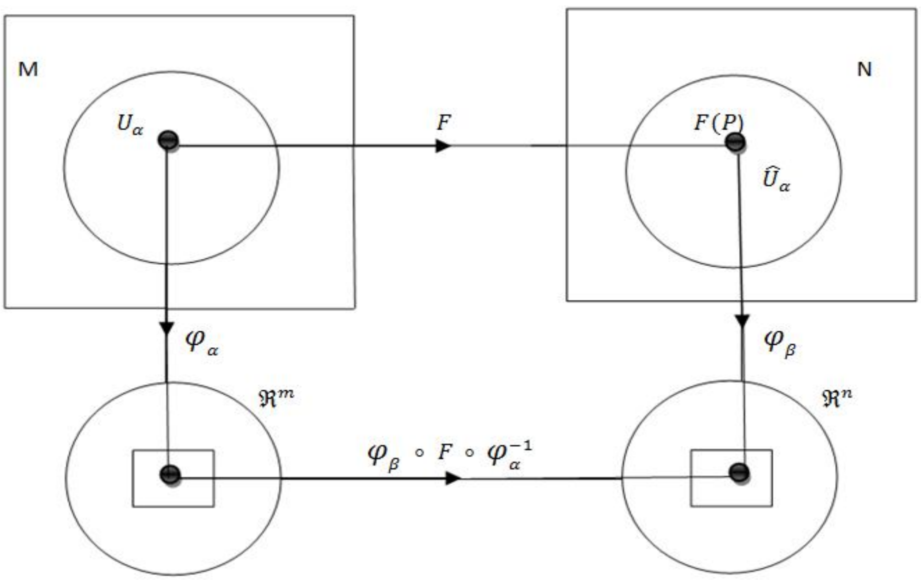


Figure 1.2

Map between manifolds

3. The maximal rank condition:

(1.1.5) Definition:

Let f: $M \to$N be a smooth maping from an m-dimensional manifold M to an n-dimensional manifold N. the rank of f at a point $x \in M$ is the rank of the $n \times m$ Jacobin matrix $(\partial f^i / \partial x^i)$ at x, where y= $f(x)$ is expressed in any convenient local coordinates near x. The mapping f is of maximal rank on a subset $S \subset$M if for each $x \in S$ the rank of f is large as possible (i.e. minimum of m and n).

4.Submainfold:

(1.1.6) Definition:

Let M be a smooth manifold. A submanifold of M is a manifold $N \subset M$, together with a smooth, one-to-one map $\emptyset$: $\widetilde{N} \to N \subset M$ satisfying the maximal rank condition everywhere,

where the parameter $\widetilde{N}$ is some other manifold and $N = \emptyset(\widetilde{N})$ is the image of $\emptyset$. In particular, the dimension of N is the same as that of $\widetilde{N}$, and does not exceed the dimension of M.

Example (3):

In all these examples of submanifolds, the parameter space $\widetilde{N} = \Re$ is the real line, with $\emptyset: \Re \to M$ parametrizinga one-dimensional submanifold $N = \emptyset(\Re)$ of some manifold M

a) Let $M = \Re^3$. Then

$$\emptyset(t) = (\cos t, \sin t, t)$$

defines a circular helix spiralling up the z-axis.

Here $\emptyset$ is clearly one to one and $\widetilde{\emptyset} = (-sint, cost, 1)$ never vanishes, so the maximal rank condition holds.

b) Let $M = \Re^2$ and
$\emptyset$(t) =($(1 + e^{-t})$ cost, $(1+e^{-t})\sin t$) then as t $\to\infty$, N spirals into the unit circle $x^2 + y^2 = 1$. Similarly, $\tilde{\phi}$(t) = $(e^{-t}\ cost, e^{-t}\ sint)$ defines a logarithmic spiral at origin.

(1.1.7)Definition:

A subset N of a C^∞- manifold M is said to have n-submanifold property if each $P \epsilon N$ has a coordinate neighborhood (U, ϕ) on$M$ with local coordinates $x^1, \dots, x^m$ such that

(i) $\phi(p) = (0,0,\dots,0)$

(ii) $\phi(u) = C^m(0)$

(iii) $\phi\ (u \cap N) = \{x \epsilon C^m(0) | \ x^{n+1} = \cdots = x^m = 0\ \}$

If N has this property, coordinate neighborhood of this type are called preferred coordinater (relative to N)

(1.1.8) Definition:

A regular submanifold N of a manifold M is a submanifold parametrized by $\phi: \widetilde{N} \to M$ with the property that for each x in N there exist arbitrarily small open neighbourhoods U of x in M such that) $\phi^{-1}[U \cap N]$ is a connected open subset of $\widetilde{N}$.

(1.1.9)Definition:

A differentiable mapping $F: M \to N$ is called an immersion if rank $F = dim\ M$ at all point of M. Thus every regular mapping from one manifold to another define an immersion provided $dimM \leq dim\ N$.

(1.1.10)Definition:

A subset $N \subset M$ (with a differentiable structure is called an immersed submanifold of M if the inclusion map $I: N \to M$ is an immersion.

(1.1.11)Definition:

Let $F: M \to N$ be one-to-one immersion such that M is homeomorphic to its image $\widetilde{M} = F(M)$with respect to topology which $\widetilde{M}$ receives as subspace of N, then F is called an imbedding and $\widetilde{M}$ is called an imbedded submanifold.

(1.1.12)Definition:

A subspace M of a C^∞- manifold N having the submanifold property is called a regular submanifold of N if the differentiable structure induced is one which is determined by preferred coordinate neighborhood of N (relative to M).

1.2 Lie Groups

1. Lie Groups:

A Lie group appears to be a somewhat unnatural marriage between on the one hand the algebraic concept of a group, and on the other hand the differential - geometric notion of a manifold.

(1.2.1) Definition:

A group G which is also a manifold is Lie group provided that mapping of $G \times G \to G$ defined by $(x, y) \to xy$ and the mapping of $G \to G$ defined $x \to x^{-1}$ are both C^∞mapping.

(1.2.2) Definition:

An r-parameter Lie group G, is a Lie group which also carries the structure of an r-dimensional manifold in such a way that both the group operation.

$$m: G \times G \to G$$

$$m(g, h) = g.h, \qquad g, h \epsilon G$$

and the inversion

$$i: G \to G, \qquad i(g) = g^{-1}, g \epsilon G$$

are Smooth maps between manifolds.

Example (4):

GL(n, $\Re$) the set of non singular n $\times n$ matrices, is as we have seen, an open submanifold of $\mu_n(\Re)$, the set of $n \times n$ real matrices identified with $\Re^{n^2}$ moreover GL(n, $\Re$) is a group with respect to matrix multiplication.

In fact an $n \times n$ matrix A is nonsingular if and only if $\det(A) \neq 0$, but $det\ (AB) = detA.det\ B$, so if A and B are non singular, that $det\ A \neq 0$ if an only if A has a multiplicative inverse, thus GL (n, $\Re$) is a group. Both the map $(A, B) \to AB\ and\ A \to A^{-1}$ are C^∞ - the product has entries which are polynomials in the entries of A and B and these entries are exactly the expression in local coordinates of the product map which is thus C^∞ here C^∞ the inverse of $A = (a_{ij})$ may written as $A^{-1} = (1/detA)(\bar{a}_{ij})$ where the $\bar{a}_{ij}$ are cofactor of A (thus polynomials in theses entries which does not vanish on GL(n, $\Re$). It follows that the entries of $A^{-1}$are rational function on GL (n, $\Re$) with non-vanishing denominators, hence C^∞, there for GL (n, $\Re$) is Lie group.

2. Local Lie groups

(1.2.3) Definition:

An r-parameter local Lie group consists of connected open subsets $V_o \subset V \subset \Re^r$ containing the origin 0, and smooth maps.

$$m: V \times V \to \Re^r \tag{1.2}$$

defining the group operation, and

$$i : V_o \to V,$$

defining the group inversion, with the following properties

a) Associativity if $x, y, z \in V_o$ and also $m(x, y)$ and $m(y, z)$ are in V, then
$$m(x, m(y, z)) = m(m(x, y), z).$$
b) Identity element, for all $x\ in\ V_o, m(o, x) = x = m(x, 0)$.
c) Inverses, for each $x\ in\ V_o, m(x, i(x)) = 0 = m(i(x), x)$.

Example (5):

Here we present a nontrivial example of a local about not global, one-parameter Lie group. Let $V = \{x: |x| < 1\} \subset \Re$ with group multiplication.

$$m(x, y) = \frac{2xy - x - y}{xy - 1}, \qquad x, y \in V$$

A straight forward computation verifies the associativity and identify laws for m. The inverse map is $i(x) = x/(2x-1)$, defined for $x \in V_O = \left\{x: |x| < \frac{1}{2}\right\}$. Thus m defines a local one-parameter Lie group.

3. Local transformation groups:

(1.2.4) Definition:

Let M be a smooth manifold. A local group of transformations acting on M is given by a local Lie group G, an open subset η, with

$$\{e\} \times M \subset \eta \subset G \times M$$

which is the domain of definition of the group action, and smooth map $\psi = \eta \to M$ with the following properties.

a) If $(h,x) \in \eta, \big(g, \psi(h,x)\big) \in \eta$ and also $(g.h,x) \in \eta$ then
$$\psi\big(g, \psi(h,x)\big) = \psi(g.h,x)$$

b) For all $x \in M$
$$\psi(e,x) = x$$

c) If $(g,x) \in \eta$,then $\left(g^{-1}, \psi(g,x)\right) \in \eta$ and
$$\psi(g^{-1}, \psi(g,x)) = x$$

4. Orbits

(1.2.5) Definition:

An orbit of a local transformation group is a minimal nonempty group invariant subset of the manifold M. In other words, $O \subset M$ is an orbit provided it satisfies the conditions.

a) If $x \in O, g \in G$and $g.x$ is defined, then $g.x \in O$
b) If$\check{O} \subset O$, and $\check{O}$ satisfied part (a) then either $\check{O} = O$ or is empty.

In the case of a global transformation group, for each $x \in M$ the orbit through x has the explicit definition $O_x = \{g.x: g \in G\}$.For local transformation group, we must look at products of group elements acting on x

$$O_x = \{g_1 \cdot g_2 \cdot \ldots g_k \cdot x: k \geq 1, g_i \in G\ , g_1 \cdot g_2 \cdot \ldots \cdot g_k \cdot x \ is\ defined\ \}.$$

As we will see, the orbits of a Lie group of transformations are in fact submanifolds of M, but they may be of varying dimensions, or may not be regular.

(1.2.6) Definition:

Let G be a local group of transformations acting on M.

(a) The group G acts semi-regularly if all the orbits O are of the same dimension as submanifolds of M.

(b) The group G acts regularly if the action is semi-regular, and in addition, for each point $x \in M$ there exist arbitrarily small neighbourhoods U of x with the property that each orbit of G intersects U in a pathwise connected subset.

Example (6):

Examples of transformation groups-

(a) The group of translations in $\Re^m$: $let\ a \neq 0$ be a fixed vector in $\Re^m$, and let $G = \Re$. Define

$$\Psi(\varepsilon, x) = x + \varepsilon a, \quad x \in \Re^m, \ \varepsilon \in \Re. \tag{1.3}$$

This is readily seen to give a global group action. The orbit are straight lines parallel to a, so the action is regular with one-dimensional orbits.

(b) Groups of scale transformations : Let $G = \Re^+$be the multiplication group. Fix real numbers $\alpha_1, \alpha_2, \dots, \alpha_m$not all zero. Then $\Re^+$ acts on $\Re^m$ by the scaling transformations

$$\Psi(\lambda, x) = (\lambda^{\alpha_1} x^1, \dots, \lambda^{\alpha_m} x^m), \quad \lambda \in \Re^+, \ x = (x^1, \dots, x^m) \in \Re^m.$$

The orbit of this action are all one- dimensional regular submanifold of $\Re^m$, except for the singular orbit consisting of just the origin $\{0\}$ for instance, in the special case of $\Re^2$ with $\Psi(\lambda, (x, y)) = (\lambda x, \lambda^2 y)$ the orbits are halves of the parabolas $y = kx^2$ (corresponding to either $x > 0$ or $x < 0$) the positive and negative y-axis, and the origin. In general, this scaling group action is regular on the open subset $\Re^m \setminus \{0\}$.

5. The Action of a Lie Group on manifold

(1.2.7) Definition:

Let G be a group and X a set, then G is said to act on X (on the left) if there is a mapping

$$\theta = G \times \mathrm{X} \to \mathrm{X} \tag{1.4}$$

Satisfying two conditions:

1) If e the identity element of G, then
$$\theta(e, x) = x \ \ for\ all\ x \in X$$

2) If $g_1, g_2 \in G$ then
$$\theta(g_1, \theta(g_2, x)) = \theta(g_1 g_2, x) \ \ for\ all\ x \in X \tag{1.5}$$

- When G is a topological group X is a topological space and θ is continuous, then the action is called continuous.
- When G is Lie group, X is a C^∞- manifold and θ is C^∞, then the action is called C^∞ action.
- We define right action

(1) $\theta(x, e) = x$

(2) $\theta(\theta(x, g_1). g_2) = \theta(x, g_1 g_2)$

(1.2.8) Definition:

If G acts on set X, then the map $g \to \theta_g$ is a homomorphism of G into $S(x)$. Conversely, any such homomorphism determines an action $\theta(g, x) = \theta_g(x)$.

(1.2.9) Definition:

Let a group G act on a set M and suppose that $A \subset M$ is subset, then GA denotes the set $\{ga: g \in G \; and \; a \in A\}$. The orbit of $x \in M$ is the set G_x, if G_x=x then x is fixed point of G, and if $G_x = M$ for some x, then G is said to be transitive on M. In this case $G_x = M$ for all x.

(1.2.10) Definition:

Let G be a group acting on set X and, let $x \in X$, the stability isotropy group of x denoted by G_x is the subgroup of all element of G leaving X a fixed G_x={g∈G | g. $x = x$}.

(1.2.11) Definition:

If G, X be as in previous definition then G is said to act freely on X if $gx = x$ implies $g = e$ the identity is the only element of G having a fixed point.

1.3 Vector Fields

1. The tangent space at a point of a manifold:

(1.3.1) Definition:

We define a differentiable curve (α) on manifold M as follows:

$\alpha: I \subset \Re \to M$ is differentiable function (C^∞differentiable means differentiable infinity many times) where I is an interval in $\Re$ i.e. $\alpha(t) = \left(\alpha_1(t), \alpha_2(t), \dots, \alpha_m(t)\right) = P(t)$ a point in M.

(1.3.2) Definition:

Suppose C is a smooth curve on a manifold M, parameterized by $\phi: I \to M$, where I is a subinterval of $\Re$. In local coordinates $x = (x^1, \dots, x^m)$, C is given by m smooth functions $\phi(\varepsilon) = (\phi^1(\varepsilon), \dots, \phi^m(\varepsilon))$ of the real variable ε. At each point $x = \phi(\varepsilon)$ of C the curve has a tangent vector, namely the derivative $\phi^{\cdot}(\varepsilon) = \partial\phi/\partial\varepsilon = (\phi^{\cdot 1}(\varepsilon), \dots, \phi^{\cdot m}(\varepsilon))$. In order to distinguish between tangent vectors and local coordinate expressions for points on the manifold, we adopt the notation.

$$V|_x = \phi^{\cdot}(\varepsilon) = \emptyset^{\cdot 1}(\varepsilon)\frac{\partial}{\partial x^1} + \emptyset^{\cdot 2}\frac{\partial}{\partial x^2} + \cdots + \emptyset^{\cdot m}(\varepsilon)\frac{\partial}{\partial x^m} \qquad (1.6)$$

for the tangent vector to C at $x = \emptyset(\varepsilon)$, then $V|_x$ is called the tangent vector.

Example (7):

The helix

$$\emptyset(\varepsilon) = (cos\varepsilon, sin\varepsilon, \varepsilon)$$

In $\mathfrak{R}^3$, with coordinates (x, y, z) has tangent vector

$$\phi^{\cdot}(\varepsilon) = -sin\varepsilon\frac{\partial}{\partial x} + cos\varepsilon\frac{\partial}{\partial y} + \frac{\partial}{\partial z} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

At the point $(x, y, z) = \emptyset(\varepsilon) = (cos\varepsilon, sin\varepsilon, \varepsilon)$.

(1.3.3)Remark:

Two curves $C = \{\emptyset(\varepsilon)\}$ *and* $\tilde{C} = \{\widetilde{\emptyset}(\theta)\}$ passing through the same point.

$$x = \emptyset(\varepsilon^*) = \widetilde{\emptyset}(\theta^*) \qquad (1.7)$$

For some ε^*, θ^* have the same tangent vector if and only if their derivatives agree at the point.

$$\frac{d\emptyset}{d\varepsilon}(\varepsilon^*) = \frac{d\emptyset^*}{d\theta}(\theta^*) \qquad (1.8)$$

This concept is independent of local coordinate system used near x. If $x = \emptyset(\varepsilon) = (\emptyset^1(\varepsilon), \ldots, \emptyset^m(\varepsilon))$ is the local coordinate expression in terms of $x_1 = (x^1, \ldots, x^m)$ and $y = \psi(x)$ is any diffeomorphism, then $y = \psi(\emptyset(\varepsilon))$ in the local coordinate formula for the curve in terms of the y-coordinates. The tangent vector $V^{\cdot}|_x = \emptyset^{\cdot}(\varepsilon)$, which has the formula (1.6) is the x-coordinates, takes the form.

$$V^{\cdot}|_y = \psi(x) = \sum_{j=1}^{m}\frac{d}{d\varepsilon}\psi^j(\emptyset(\varepsilon))\frac{\partial}{\partial y^j} = \sum_{j=1}^{m}\sum_{k=1}^{m}\frac{\partial\psi^j}{\partial x^k}(\emptyset(\varepsilon))\frac{\partial\emptyset^k}{\partial\varepsilon}\frac{\partial}{\partial y^j} \qquad (1.9)$$

In the y-coordinates the Jacobin matrix $\frac{\partial\psi^j}{\partial x^k}$ is invertible at each point (1.8) holds if and only if

$$\frac{d}{d\varepsilon}\psi(\emptyset(\varepsilon^*)) = \frac{d}{d\theta}\psi\left(\widetilde{\emptyset}(\theta^*)\right) \qquad (1.10)$$

The (1.9) tells how a tangent (1.6) behaves under the given change of coordinates $y = \psi(x)$.

(1.3.4) Definition:

The collection of all tangent vectors to all possible curves passing through a given point x in M is called the tangent space to M at x, and is denoted by $TM|_x$. If M is an m-dimensional manifold, then $TM|_x$ is an m-dimensional vector space, with $\{\frac{\partial}{\partial x^i}, \dots, \frac{\partial}{\partial x^m}\}$ providing a basis for $TM|_x$in the given local coordinate.

(1.3.5) Definition:

The tangent bundle of M is the collection of all tangent spaces corresponding to all points x in M denoted by

$$TM = \cup_{x \in M} TM|_x \qquad (1.11)$$

2. A vector field:

(1.3.6) Definition:

A vector field V on M assigns a tangent vector $V|_x \in TM|_x$to each point $x \in M$, with $V|_x$ varying smoothly from point. In local coordinates $(x^1, \dots, x^m)$, a vector field has form

$$V|_x = \xi^1(x)\frac{\partial}{\partial x^1} + \xi^2(x)\frac{\partial}{\partial x^2} + \cdots \dots \dots \dots + \xi^m(x)\frac{\partial}{\partial x^m} \qquad (1.12)$$

Where each $\xi^i(x)$ is a smooth function of x.

1.4 Flows

(1.4.1) Definition:

If V is a vector field, we denote the parameterized maximal integral curve passing through x in M by $\psi(\varepsilon, x)$ and call ψ the flow generated by V. The flow of a vector field has the basic properties:

1) $\psi(\delta, \psi(\varepsilon, x)) = \psi(\delta+\varepsilon, x), x \in M$ (1.13)

for all $\delta, \varepsilon \in \Re$ such that both sides of the equation are defined.

2) $\psi(0, x) = x.$ (1.14)

3) $\frac{d}{d\varepsilon}\psi(\varepsilon, x) = V|_{\psi(\varepsilon, x)}$ (1.15)

This mean that V is tangent to the curve $\psi(\varepsilon, x)$ for fixed point x.

The flow generated by a vector field is same as local a group action of Lie group $\Re$ on manifold M, often called one parameter group of transformation. The vector field is called the infinitesimal generator of the action since by Taylaor's theorem, in local coordinate

$$\psi(\varepsilon, x) = x + \varepsilon\, \xi\, (x) + O(\varepsilon^2)$$

where $\xi = (\xi^1, \ldots, \xi^m)$ are the coefficients of V . the orbits of the one-parameter group action are the maximal integral curves of the vector field V.

Conversely if ψ (ε, x) is any one-parameter group of transformations acting on M, then it is infinitesimal generator is obtained by specializing (1.15) at ε=0

$$V|_x = \frac{d}{d\varepsilon}|_{\varepsilon=0}\psi(\varepsilon, x) \qquad (1.16)$$

Uniqueness of solutions to $\frac{dx^i}{d\varepsilon} = \xi^i(x),\ i = 1,2,\ldots,m$ guarantees that the flow generated by V coincides with the given local action of $\mathfrak{R}$ on M the common domain of definition.

Thus, there is one -to-one correspondence between local one - parameter groups of transformation and their infinitesimal generators.

The computation of the flow or one - parameter group generated by a given vector field V (in other words, solving the system of ordinary differential equations) is often referred to as exponentiation of the vector field. The suggestive notation

$$exp(\varepsilon v)\, x \equiv \psi\, (\varepsilon, x)$$

In terms of this exponential notation, the above three properties can be restated as

$$exp[(\delta + \varepsilon)v]x = exp(\delta v)\, exp(\varepsilon v)x \qquad (1.17)$$

Whenever defined

$$exp(0v)\, x = x \qquad (1.18)$$

And

$$\frac{d}{d\varepsilon}[exp(\varepsilon v)x] = V|_{exp(\varepsilon v)x} \text{for all } x \in M \qquad (1.19)$$

Example(8):

Examples of vector field and flows.

a) Let $M = \mathfrak{R}$ with coordinate x, and consider the vector field $v = \frac{\partial}{\partial x} \equiv \partial_x$

(In the squad, we will often use the notation ∂_x for $\frac{\partial}{\partial x}$ to save space)

Then

$$exp(\varepsilon v)\, x = exp\, (\varepsilon \partial_x)\, x = x + \varepsilon$$

which is globally defined for the vector field $x\partial_x$ we recover the usual exponential

$$exp(\varepsilon x\partial_x)x = e^{\varepsilon}x,$$

Since it must be the solution to the ordinary differential equation$\dot{x} = x$ with initial value x at $\varepsilon =$ 0.

b) In the case of $\Re^m$, a constant vector field

$$V_a = \sum a^i \frac{\partial}{\partial x^i}$$

$a = (a^1, \dots, a^m)$ exponentiates to the group of translations

$$exp(\varepsilon V_a)x = x + \varepsilon a \ , \qquad x \in \Re^m \ ,$$

in direction a. similarly, a linear vector field

$$V_A = \sum_{i=1}^{m}\left(\sum_{j=1}^{m} a_{ij}x^j\right)\frac{\partial}{\partial x^i}$$

where $A = (a_{ij})$ is an $m \times m$ matrix of constants, has flow

$$exp(\varepsilon V_A)\, x = e^{\varepsilon A}x,$$

where $e^{\varepsilon A} = 1 + \varepsilon A + \frac{1}{2}\varepsilon^2 A^2 + \cdots$ is the usual matrix exponential

c) Consider the group of rotations in the plane

$$\Psi(\varepsilon, (x, y)) = (x cos\varepsilon - y sin\varepsilon, x\ sin\varepsilon + y\ cos\varepsilon)$$

Its infinitesimal generator is a vector field.

$$V = \xi(x, y)\partial_x + \eta(x, y)\partial_y \ ,$$

Where according to (1.16)

$$\xi(x, y) = \frac{d}{d\varepsilon}|_{\varepsilon=0}(x\ cos\varepsilon - y\ sin\varepsilon) = -y$$

$$\eta(x, y) = \frac{d}{d\varepsilon}|_{\varepsilon=0}(x\ sin\varepsilon + y\ cos\varepsilon) = x$$

Thus

$$V = -y\,\partial_x + x\,\partial_y$$

is the infinitesimal generator , and indeed, the above group transformations agree with the solution to the system of ordinary differential equations

$$\frac{dx}{d\varepsilon} = -y, \qquad \frac{dy}{d\varepsilon} = x$$

d)finally, consider the local group action

$$\psi(\varepsilon, (x, y)) = \left(\frac{x}{1-\varepsilon x}, \frac{y}{1-\varepsilon x}\right)$$

Differentiating as before, we find the infinitesimal generator to be

$$V = x^2\partial_x + xy\partial_y$$

This demonstrates that a smooth vector field may still generate only a local group action.

1. Action of functions:

(1.4.2) Definition:

Let V be a vector field on M and $f: M \to \mathfrak{R}$ a smooth function we are interested in seeing how f changes under the flow generated by V, meaning we look at $f(exp(\varepsilon v)x)$ as ε varies in local coordinates, if $V = \sum_{i=1}^{m} \xi^i(x)\frac{\partial}{\partial x^i}$ then using the chain rule and (1.19) we find

$$\frac{d}{d\varepsilon} f(exp(\varepsilon v)\,x) = \sum_{i=1}^{m} \xi^i(exp(\varepsilon v)\,x)\frac{\partial f}{\partial x^i}(exp(\varepsilon v)\,x) = V(f)[exp(\varepsilon v)\,x]$$

In particular at

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f(exp(\varepsilon v)\,x) = \sum_{i=1}^{m} \xi^i(x)\frac{\partial f}{\partial x^i}(x) = V(f)(x) \qquad (1.20)$$

Now the reason underlying our notation for vector fields becomes apparent, the vector field V acts as first order partial differential operator on real valued functions $f(x)\ on\ M$. Furthermore, by Taylor's theorem.

$$f(exp(\varepsilon v)\,x) = f(x) + \varepsilon v(f)(x) + o(\varepsilon^2),$$

So, $v(f)$ gives the infinitesimal change in the function f under the flow generated by v. We can continue the process of differentiation and substitution into the Taylor series, obtaining

$$f(exp(\varepsilon v)\,x) = f(x) + \varepsilon v(f)(x) + \frac{\varepsilon^2}{2!}v^2(f)(x) + \cdots + \frac{\varepsilon^k}{k!}v^k(f)(x) + O(\varepsilon^{k+1})$$

Where $v^2(f) = v(vf), v^3(f) = v(v^2(f))$, etc .If we assume convergence of the entire Taylor series in ε, then we obtain the Lie Series,

$$f(exp(\varepsilon v)\, x) = \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} v^k\, (f)(x) \tag{1.21}$$

for the action of the flow on f .The same result holds for vector - valued functions $F: M \to \Re^n, F(x) = (F^1(x) \dots, F^n(x))$ where we let v act component-wise on $F = v(F) = (v(F^1) \dots, v(F^n))$ in particular , If we let F be the coordinate functions X, we obtain (again under assumptions of convergence) a Lie series for the flow itself , given by :

$$exp(\varepsilon v)\, x = x + \varepsilon \xi(x) + \frac{\varepsilon^2}{2!} v(\xi)(x) + \cdots = \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} v^k(x), \tag{1.22}$$

Where $\xi = (\xi^1, \dots, \xi^m), v(\xi) = \left(v(\xi^1), \dots, v(\xi^m)\right), etc.$ Providing even further

justification for our exponential notation .According to our new interpretation of the symbols $\frac{\partial}{\partial x^i}$, each tangent vector $v|_x$ at a point x defines a derivation on the space of smooth real valued functions f defined near x in M.

2. Differentials:

(1.4.3) Definition:

Let M and N be smooth manifolds and $f: M \to N$ a smooth map between them. Each parameterized curve $c = \{\emptyset(\varepsilon): \varepsilon \in I\}$ on M is mapped by F to a parametrized curve $\check{c} = F(c) = \left\{\breve{\emptyset}(\varepsilon) = F\left(\emptyset(\varepsilon)\right): \varepsilon \in I\right\}$ on N.

Thus F induces a map from the tangent vector $d\emptyset/d\varepsilon$ to C at $x = \emptyset(\varepsilon)$ to the corresponding tangent vector $d\breve{\emptyset}/d\varepsilon$ to $\check{C}$ at the image point $F(x) = F\left(\emptyset(\varepsilon)\right) = \breve{\emptyset}(\varepsilon)$. This induced map is called the differential of , and denoted by :

$$dF\left(\dot{\emptyset}(\varepsilon)\right) = \frac{d}{d\varepsilon}\left\{F\left(\emptyset(\varepsilon)\right)\right\} \tag{1.23}$$

As every tangent vector $v|_x \in TM$ is tangent to some curve passing through x, the differential maps the tangent space to M at x to the tangent space to N at $F(x)$

$$dF: TM|_x \to TN|_{F(x)}$$

The local coordinate formula for the differential is found using the chain rule in same manner as the change of variables formula (1.9). If

$$v|_x = \sum_{i=1}^{m} \xi^i \frac{\partial}{\partial x^i}$$

is a tangent vector at $x \in M$ then

$$dF(v|_x) = \sum_{j=1}^{n}\left(\sum_{i=1}^{m}\xi^i \frac{\partial F^j}{\partial x^i}(x)\right)\frac{\partial}{\partial y^j} = \sum_{j=1}^{n} v\left(F^j(x)\right)\frac{\partial}{\partial y^j} \qquad (1.24)$$

Note that the differential $df|_x$is linear map from $TM|_x\ to\ TN|_{F(x)}$, whose matrix expression in local coordinates is just the jacobian matrix of F at x.

If we prefer to think of tangent vectors as derivations on the space of smooth function defined near a point x, then the differential dF has the alternative definition

$$dF(v|_x)f(y) = v(f \circ F)(x),\ y = F(x) \qquad (1.25)$$

For all $v|_x \in TM_x$ and all smooth $f: N \to \Re$, the equivalence of (1.23) and(1.25) is easily verified using local coordinates .

Example (9):

Let $M = \Re^2$, with coordinates (x, y) and $N = \Re$ with coordinates s, and let $f = \Re^2 \to \Re$ be any map $s = F(x, y)$. Given

$$V|_{(x,y)} = a\frac{d}{dx} + b\frac{d}{dy}$$

Then , by (1.24)

$$dF\ (V|_{(x,y)}) = \left\{a\frac{\partial F}{\partial x}(x,y) + b\frac{\partial F}{\partial y}(x,y)\right\}\frac{d}{ds}|_{F(x,y)}$$

For example, if $F(x, y)=\alpha x + \beta y$ is linear projection ,then

$$dF\left(V|_{(x,y)}\right) = (a\alpha + b\beta)\frac{\partial}{\partial s}|_{s=\alpha x+\beta y}$$

3.Lie Brackets

The most important operation on vector field is their Lie bracket or commutator.

(1.4.4)Definition:

If V and W are vector fields on M, then their Lie bracket $[V, W]$ is the unique vector field satisfying

$$[V, W](f) = V\left(W(f)\right) - W\left(V(f)\right) \qquad (1.26)$$

For all smooth functions $f: M \to \Re$. It is easy to verify that $[V, W]$ is indeed a vector field .in local coordinates if

$$V = \sum_{i=1}^{n} \xi^i (x) \frac{\partial}{\partial x^i} \quad , W = \sum_{i=1}^{m} \eta^i (x) \frac{\partial}{\partial x^i}$$

then:

$$[v, w] = \sum_{i=1}^{m} \xi \, v(\eta^i) \frac{\partial}{\partial x} - w(\xi^i) \frac{\partial}{\partial x^i} = \sum_{i=1}^{m} \sum_{j=1}^{m} \left\{ \xi^j \frac{\partial \eta^i}{\partial x^j} - \eta^j \frac{\partial}{\partial x^i} \right\} \frac{\partial \xi^i}{\partial x^i} \qquad (1.27)$$

Note that in (1.26) the terms involving second order derivatives of f cancel .

Example (10):

$$v = y \frac{\partial}{\partial x}, \quad W = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}$$

then

$$[V, W] = V(x^2) \frac{\partial}{\partial x} + V(xy) \frac{\partial}{\partial y} - W(y) \frac{\partial}{\partial x} = xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}$$

(1.4.5)Definition :

Let $v_1, \dots, v_r$ be vector field on a smooth manifold M. An integral submanifold of $\{v_1, \dots, v_r\}$ in a submanifold $N \subset M$ whose tangent space $TN|_y$ is spanned by the vector $\{v_1|_y, \dots, v_r|_y\}$ for each $y \in N$. The system of vector fields $\{v_1, \dots, v_r\}$ is integrable if through every point $x_\circ \in M$ there passes an integral submanifold.

(1.4.6)Definition:

A system of vector field $\{v_1, \dots, v_r\}$ on M is in involution if there exist smooth real - valued functions $C_{ij}^k(x), x \in M, i, j, k = 1, \dots, r$, such that for each i, j= 1,…,r,

$$[v_i, v_j] = \sum_{k=1}^{r} C_{ij}^k \cdot v_k \qquad (1.28)$$

4. Lie algebras

If G is a Lie group, then there are certain distinguished vector field on G characterized by their invariance (in sense to be defined shortly) under the group multiplication .As we see, these invariant vector fields from a finite dimensional vector space, called the Lie algebra of G, which is in a precise sense the “infinitesimal generator” of G.

(1.4.7) *Definition :*

The Lie algebra of a Lie group G, traditionally denoted by the corresponding lower case German letter g is the vector space of all right – invariant vector field on G.

Note that any right-invariant vector field is uniquely determined by its value at the identity because

$$v|_g = dR_g(v|_e), \tag{1.29}$$

Since $R_g(e) = g$. Conversely, any tangent vector to G at e uniquely determines a right - invariant vector field on G by formula (1.29). Indeed,

$$dR_g(v|_h) = dR_g\big(dR_h(v|_e)\big) = d\big(R_g \circ R_h\big)(v|_e) = dR_{hg}(v|_e) = v|_{hg}$$

Proving the right-invariance of v. Therefore we can identify the Lie algebra g of G with the tangent space to G at the indentify element.

$$g \simeq TG|_e \tag{1.30}$$

(1.4.8) Definition :

A Lie algebra is a vector space g together with bilinear operation

$$[.,.] = g \times g \to g.$$

called the Lie bracket for g, satisfying the axioms.

(a) Bilinearity

$$[cv + c'v', w] = c[v, w] + c'[v', w],$$

$$[v, cw + c'w'] = c[v, w] + c'[v, w'], \text{ for the constants } c, c' \in \Re$$

(b) skew-symmetry

$$[v, w] = -[w, v]$$

(c) Jacoi Identity

$$\big[u, [v, w]\big] + \big[w, [u, v]\big] + \big[v, [w, u]\big] = 0$$

for all u, v, v', w, w' in g

Example (11):

If $G = \Re$, then there is, up to constant multiple a single right- invariant vector field , namely

$$\partial_x = \partial / \partial x. \text{ In fact given } x , y \in \Re$$

$$\Re_y(x) = x + y$$

hence

$$d\Re_y(\partial_x) = \partial_x$$

Similarly , if $G = \Re^+$,then, the single independent right - invariant vector field is $x\partial_x$, finally ,for So(2) the vector field ∂_θ is again the unique independent right -invariant one .Note that the Lie algebras of $\Re . \Re^+$ and So(2) Are all the same, being one- dimensional vector space with trivial Lie brackets $([v, w] = 0 \; for \; all \; v, w)$.

Example (12):

Here we compute of the Lie algebra of the general linear group $GL(n)$. Note that since $GL(n)$ is n^2 - dimensional we can indentify the Lie algebra $g|_{(n)} \simeq \Re^{n^2}$ with the space of all $n \times n$ matrices .Indeed, coordinates on $GL(n)$ are provided by the matrix entries x_{ij}, $i, j = 1, \dots , n$, so the tangent space to $GL(n)$ at the identity is the set of all vector fields

$$v_A|_l = \sum_{i,j} a_{ij} \frac{\partial}{\partial x_{ij}} |_l$$

Where $A = (a_{ij})$ is an arbitrary $n \times n$ matrix. Now given $(y_{ij}) \in GL(n)$, the matrix $\Re_y \, (X) = X_y$ has entries

$$\sum_{k=1}^{n} x_{ik} \, y_{ki}$$

Therefore , according to (1.29) we find

$$v_A|_Y = d\Re_Y(v_A|_I)$$

$$= \sum_{i,m} \sum_{i,j} a_{ij} \tfrac{\partial}{\partial x_{ij}} (\textstyle\sum_k x_{lk} \, y_{km}) \tfrac{\partial}{\partial x_{lm}}$$

$$= \sum_{i,j,m} a_{ij} y_{jm} \tfrac{\partial}{\partial x_{im}} \quad ,$$

or, in terms of $X \in GL(n)$,

$$v_A|_x = \textstyle\sum_{i,j} (\sum_k a_{ik} \, x_{kj}) \tfrac{\partial}{\partial x_{ij}} \qquad (1.31)$$

To compute Lie bracket:

$$[v_A, v_B] = \sum_{\substack{i,j,k,\\ l,m,p}} \left[a_{ip}\, x_{pm} \frac{\partial}{\partial x_{im}}\left(b_{lk}x_{kj}\right) - b_{ip}\, x_{pm}\frac{\partial}{\partial x_{lm}}(a_{lk}\, x_{kj})\right] \frac{\partial}{\partial x_{ij}}$$

$$=\sum_{i,j,k} [\sum_l (b_{il}a_{lk} - a_{il}\, b_{lk})]\, x_{kj} \frac{\partial}{\partial x_{ij}}$$

where $[A, B] = BA - AB$ is the matrix commulator. Therefore, the Lie algebra $g|(n)$ of the general linear group $GL(n)$ is the space of all $n \times n$ matrices with the Lie bracket being the matrix commulator.

5. One - parameter subgroup of a Lie group

One parameter subgroups of Lie group G are one - to - one correspondence with element of $T_e(G)$.

We shall use this to help determine all one parameter subgroups of various matrix groups we first consider $G = GL(n, \Re)$, the matrix entries x_{ij}, $1 \le i, j \le n$ for any $x = (x_{ij}) \in GL(n, \Re)$ are coordinates on a single neighborhood covering the group which is an open subset of $\mu_n(\Re), the\ n \times n$ matrices over $\Re$. Therefore $\frac{\partial}{\partial x_{ij}}$, $1 \le i, j \le n$ is a field of frames on G and relative to these frames as a basis at e there is an isomorphism of $\mu_n(\Re)$, as a vector space on to $T_e(G)$ given by

$A = (a_{ij}) \to \sum_{i,j} a_{ij} (\frac{\partial}{\partial x_{ij}})_e$ [when $G = GL(n, \Re), e$ is the $n \times n$ identity matrix I].

6. Sub algebra

(1.4.9) Definition :

In general a sub algebra η of Lie algebra L is a vector space which is closed under the Lie bracket so $[v, w] \in \eta$ whenever $v, w \in \eta$ .if H is a Lie subgroup of a Lie group G, any right invariant vector field v on H can be extended to a right- invariant vector field on G .(Just set $v|_g = d\, \Re_g\, (v|_e), g \in G$.

In this way the Lie algebra η of H is realized as sub algebra of the Lie algebra L of G. Correspondence between one - parameter sub group of a Lie group G and one - dimensional sub spaces η (sub algebras) of its Lie algebra L generalized to provide a complete one - to - one correspondence between Lie sub groups or G and sub algebra of L.

7. Lie algebra of local lie groups

Turning to local version we consider a local Lie group $V \subset \Re^r$with multiplication $m(x,y)$. the corresponding right multiplication map $\Re_y: V \to \Re^r \, is \, \Re_Y(x) = m(x,y)$.A vector field $v \, on \, V$ is right invariant if and only if

$$d\Re_Y = (v|_x) = v|_{\Re(x)} = V|_{m(x,y)}$$

whenever $x, y \, and \, m(x.y)$ are in V . As in the case of global Lie groups, any right invariant vector field is determined uniquely by its value at the origin (identity element), $v|_x = d\Re_x(v|_0)$ and hence the Lie algebra g for the local Lie group V,determined as the space of right invariant vector field on V, is on r-dimensional vector space .

8. Infinitesimal group actions

Suppose G is a local group of transformations acting on manifold M, $g.x = \psi(g,x) \, for \, (g,x) \in \eta \subset G \times M$there is a corresponding "infinitesimal action" of the Lie algebra$L$ of G on M. Namely, if $v \in L$ we define $\psi(v)$,to be the vector field on M whose flow coincides with the action of the one parameter sub group $exp(\varepsilon v) \, of \, G \, on \, M$this means that for $x \in M$

$$\psi(v)|_x \equiv \frac{d}{d\varepsilon}|_{\varepsilon=0} \, \psi(exp(\varepsilon v), x) = d\psi_x(v|_e),$$

Where $\psi_x(g) = \psi(g,x). Note \, further \, that \, since$

$$\psi_x \circ \Re_g(h) = \psi(h.g,x) = \psi(h,g.x) = \psi_{gx}(h)$$

Where ever defined, we have

$$d\psi_n(v|_g) = d\psi_{gx}(v|_e) = \psi(v)|_{gx}$$

for any $g\epsilon G_x$.It follows from the property $df([v,w]) = [df(v), df(w)]$ of the

Lie bracket that ψ is a Lie algebra homomorphism from L to the Lie algebra of vector fields on M

$$[\psi\,(v), \psi(w)] = \psi([v,w]), \qquad v, w \,\epsilon\, L$$

Therefore the set of all vector field $\psi(v)$ corrosponding to $v\epsilon g$ forms a Lie algebra of vector field on M.

Conversely, given a finite- dimensional Lie algebra of vector field on M, there is always a local group of transformation whose infinitesimal action is generated by the given algebra .

(1.4.10) Definition):

Let M be a manifold of dimension $m = n + k$ and assume that to each $p \in M$is assigned an n-dimensional subspace Δ_p of $T_P(M)$. Suppose moreover that in a neighborhood $U\ of\ each\, p \in M$. There are n-linearly independent $C^\infty - vector$ fields $X_1, \dots, X_m$which form a basis of Δq for every $q \in U$ then we shall say that $\Delta\ is\ c^\infty$n-plane distribution of dimension$n$ on M and $x_1, \dots, x_n$ is local basis of Δ , we shall say that the distribution"Δ" is involutive if there exists a local basis $x_1, \dots, x_n$ in a neighborhood of each point such that

$$[x_i, x_j] = \sum_{k=1}^{n} c_{ij}^k\, x_k\, ;\ 1 \le i, j \le n$$

(the c_{ij}^kwill not in general be constant ,but will be c^∞function on aneighborhood)

Finally, if $\Delta\ is\ a\ C^\infty$ distribution of M and N is a connected C^∞ submanifold of M such that for each $q \in N$ we have $T_q(n) \subset \Delta_g$, then we shall say that Nisan integral manifold of Δ. Note that an integral manifold may be of lower dimension than Δ, and need not be a regular submanifold .

Let $\Delta\ be\ a\ C^\infty$ distribution on M of dimension n, the dimension of M being $m = n + k$.We shall say that is Δ completely integral if each point $p \in M$ has a cubical coordinate neighborhood $u\,.\,\phi$ such that if $x^1, \dots, x^n$denotethe local coordinate, then the n vectors $E_i = \phi_*^{-1}\left(\frac{\partial}{\partial x^i}\right), i = 1, \dots, n$ are local basis on U for Δ . Note that in this case there is an n-dimensional integral manifold N through each point q of U such that $T_P(N) = \Delta q\ that\ is$ dim $N = n$ in fact ,if $(a^1, \dots, a^m)$ denote the coordinates of q, then an integral manifold through q is the n-slice defined by

$$X^{n+1} = a^{n+1}, \dots, X^m = a^m$$

That is

$$N = \phi^{-1}[n \in \phi(u) X^j = a^i,\ j = n + 1, \dots, m]$$

a slice of U of course in this case the distribution is in volutive for

$$[E_i, E_j] = \phi_*^{-1}\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0\,, \qquad 1 \le i, j \le n$$

We shall call U, ϕ flat with respect to Δ. Thus complete integrability is equivalent to every point having a flat coordinate neighborhood .

Thus any completely integrable distribution is invlutive . However most distributions are involutive .

(1.4.11) Proposition:

Suppose v is a vector field not vanishing at a point $x \in M$: $v|_{x_0} \neq 0$. Then there is a local coordinate chart $y = (y^1, \dots, y^m)$ at x_0 such that in terms of these coordinates, $v = \partial / \partial y^I$.

Proof:

First linearly change coordinates so that $x_0 = 0$ and $v|_{x_0} = \partial / \partial x^I$. By the continuity the coefficient $\xi^I(x)$ of $\partial / \partial x^I$ is positive in a neighbourhood of x_0. Since $\xi^I(x) > 0$, the integral curves of v cross the hyper plane $\{(0, x^2, \dots, x^m)\}$ transversally, and hence in a neighbourhood of $x_0 = 0$, each point $x = (x^1, \dots, x^m)$ can be defined uniquely as the flow of some point $(0, y^2, \dots, y^m)$ on this hyper plane. Consequently

$$x = \exp(y^1 v)\,(0, y^2, \dots, y^m),$$

for y^1 near 0, gives a diffeomorphism form $(x^1, \dots, x^m)$ to $(y^1, \dots, y^m)$ which defines the y-coordinates. (Geometrically, we have "straightened out" the integral curves passing through the hyper plane perpendicular to the x^1-axis.) In terms of the y-coordinates, we have by (1.17), for small ε,

$$\exp(\varepsilon v)\,(y^1, \dots, y^m) = (y^1 + \varepsilon, y^2, \dots, y^m),$$

so the flow is just translation in the y^1-direction. Thus every nonvanishing vector field is locally equivalent to the infinitesimal generator of a group of translations. (course, the global picture can be very complicated, as the irrational flow on the torus makes clear.)

1.5 Riemannian Manifolds

1- Riemannian Metrics

(1.5.1) Definition:

Given differentiable manifold, define a Riemannian metric g on M, to be a mapping that associates with each $p \in M$ an inner product $g: M_p \times M_p \to \mathfrak{R}$ satisfying the following differentiability property : If U is any open set in M and X, Y are differentiable vector fields on U then the function $g(X, Y): U \to \mathfrak{R}$ given by

$$g(X, Y)(p) = g_p(X_p, Y_p)$$

g is differentiable on U.

- By a Riemannian manifold we mean a differentiable manifold with given Riemannian metric .

(1.5.2) Definition:

Let M, N be differentiable manifolds, h a Riemannian metric on $N, \varphi: M \to N$ differentiable, and $M_o = \{p \in M : \varphi_{*p} \quad is \ one \ to \ one\}$. Of course M_o is possibly empty, open submanifold of M. The pull back $\varphi^* h$ of h is defined to be Riemannian metric on M_o given by

$$(\varphi^* h)(\xi, \eta) = h(\varphi_* \xi, \varphi_* \eta) \tag{1.32}$$

$where \ \xi, \eta \in M_p, p \in M_{o.}$

If $\in M \backslash M_o$, then (1.32) defines a symmetric bilinear form on M_o , but the form is only nonnegative.

(1.5.3) Definition:

Let M be a Riemannian manifold with Riemannian metric g, then we say that φ is a local isometry of M_o into N if $g = \varphi^* h$ on M_o. If M is connected, then $g = \varphi^* h$ also implies that M_o= M, that, φ is a Riemannian immersion if φ is an imbedding satisfying $g = \varphi^* h$ then we call an isometry of M into N.

- An isometry of M is diffeomorphism of M onto itself that is an isometry.

2. *The metric space structure*

Let M, N be differentiable manifold and A an arbitrary set in M. Recall that a map $\varphi: A \to N$ is C^k on $A, k \geq 1$, if there exist an open set U such that $A \subseteq U \subseteq M$ and a map $\hat{\varphi}: U \to N \in C^k$ satisfying $\hat{\varphi}|A = \varphi$.

(1.5.4) Definition:

For a given differentiable manifold $M, k = 1, \dots, \infty,$ we let D^k denote the collection of all maps, ω form closed intervals of $\Re$ into M that are continuous and piecewise C^k, that is, ω is given by $\omega: [\alpha, \beta] \to M \subset C^\circ$ and there exist $\alpha = t_\circ < t_1 < \cdots < t_l = \beta$ such $\omega|[t_{j-1}, t_j] \in C^k$ for $j = 1, \dots, l$.

Let M be a Riemannian manifold. For any $\xi \epsilon TM$, define the length of ξ by

$$|\xi| = \langle \xi, \xi \rangle^{\frac{1}{2}}$$

For any path $\omega: [\alpha, \beta] \to \mathrm{M} \in \mathrm{D}^1$ define the length of $\omega, l(\omega)$ by

$$l(\omega) = \int_\alpha^\beta |\omega(t)| dt$$

For M connected (our usual assumption), $p, q \epsilon M$ define the distance between p and q, $d(p,q)$ by

$$d(p,q) = inf l(\omega)$$

Where ω range over all $\omega: [\alpha, \beta] \rightarrow M \in D^I$ satisfying $\omega(\alpha) = p$, $\omega(\beta) = q$.

Example (13):

We want to show $l_d(\omega) = \int_a^b |\omega'|\, dt$

For any D^I path $\omega: [a,b] \rightarrow M$. one easily form the definition of l_d that

$$l_d(\omega) \leq \int_a^b |\omega'|\, dt$$

so the real issue is the opposite inequality . the argument is as follow:

One proves that given any compact k in M and any real $\lambda > 1$, there exists a finite cover of k, $\{u_1, \dots, u_k\}$ with charts $x_j: U_j \rightarrow \mathfrak{R}^n$ such that

$$\lambda^{-1} \leq \frac{|\xi|}{|\xi|_{\mathfrak{R}^n}} < \lambda$$

(where $|\xi|_{\mathfrak{R}^n}$ denotes the standard norm on $\mathfrak{R}^n$) for all $\xi \in TU_j$, $j = 1, \dots, k$ and

$$\lambda^{-1} \leq \frac{d(p,q)}{|x_j(p) - x_j(q)|} < \lambda$$

For all $p, q \in U_j$, $j = 1, \dots, k$ form this

$$l_d(\omega) \geq \lambda^{-2} \int_a^b |\omega'|\, dt \text{ for all } \lambda > 1$$

Chapter Two

Exterior Differential Forms and Geometric Calculus

Chapter Two

Exterior Differential Forms and Geometric Calculus

Differential forms play a fundamental role in the topological aspect of differential geometry.

2.1 Differential Forms

(2.1.1)Definition:

A smooth 1-form ϕon $\mathfrak{R}^n$ is a real- valued function on the set of all tangent vectors to $\mathfrak{R}^n$, i.e…

$$\phi = T\mathfrak{R}^n \to \mathfrak{R} \quad (2.1)$$

with the properties that

1. $\phi$is linear on the tangent space $T_x\mathfrak{R}^n$ for each $x\epsilon\mathfrak{R}^n$
2. For any smooth vector field $V = v(x)$ the function $\phi(v): \mathfrak{R}^n \to \mathfrak{R}$ is smooth.

Given a 1-form ϕ, for each $x\epsilon\mathfrak{R}^n$ the map

$$\phi_x = T_x\mathfrak{R}^n \to \mathfrak{R} \qquad (2.2)$$

is an element of the dual space $(T_x\mathfrak{R}^n)$, when we extend this notion to all of $\mathfrak{R}^n$, we see that the space of 1-forms on $\mathfrak{R}^n$ is dual to the space of vector fields on $\mathfrak{R}^n$

In particular, the 1-form $dx^i, \dots, dx^n$ are defined by the property that for any vector $v = (v^1, \dots, v^n) \in T_x\mathfrak{R}^n$,

$$dx^i(v) = v^i \qquad (2.3)$$

The $dx^{i'}s$ form a basis for the 1-forms on $\mathfrak{R}^n$, so any other 1-form ϕ may be expressed in the form

$$\phi = \sum_{i=1}^{n} f_i\,(x)dx^i \qquad (2.4)$$

If a vector field v on $\mathfrak{R}^n$, has the form

$$v(x) = \left(v^1(x), \dots, v^n(x)\right),$$

then at any point $x\epsilon\mathfrak{R}^n$

$$\phi_x(v) = \sum_{i=1}^{n} f_i\,(x)v^i x \qquad (2.5)$$

(2.1.2)Definition:

Let M be a smooth manifold and $TM|_x$ its tangent space at x. The space $\wedge_k T^*M|_x$of differential k-form at x is set of all k-linear alternating functions

$$w: TM|_x \times, \dots, \times\ TM|_x \to \mathfrak{R}^n \qquad (2.6)$$

Specifically, if we denote the evaluation of w on the tangent vectors $v_1, \dots, v_k., \in TM|_x$ by$\langle w; v_1, \dots, v_k\rangle$, then the basic requirements are that for all tangent vectors at x

$$(w; cv_1, \dots, cv_l + c'v_i, \dots, c'v_k) = c(w; v_1, \dots, v_i, \dots, v_k) + c'(w; v_1, \dots, v_i', \dots, v_k)$$

for $c, c' \in \mathfrak{R}, 1 \le i \le k$,

$$(w;\ v_{\pi i}, \dots, v_{\pi k}) = (-1)^{\pi}(w; v_1, \dots, v_k) \qquad (2.7)$$

for every permutation π of the integers $\{1, \dots, k\}$ with $(-1)^{\pi}$ denoting the sign of π. The space $\Lambda_k T^*M|_x$is, in fact, a vector space under the obvious operations of addition and scalar multiplication, A 0-form at x is, convention just a real number, while the space $T^*M|_x = \wedge_i T^*M|_x$ of one forms, called the cotangent space to M at x, the space of linear functions on $TM|_x$ i.e. the dual vector space to the tangent at x. A smooth differential k- form w on M (or k-form for short) is a collection.

Smoothly varying alternating k-linear maps $w|_x \in \wedge_k T^*M|_x$ for each $x \in M$, where we require that for all smooth vector fields $v_1, \dots, v_k$

$$\langle w;\ v_1, \dots, v_k\rangle(x) = \langle w|_x;\ v_1|_x, \dots, v_k|_x\rangle \qquad (2.8)$$

is a smooth, real-valued function of x.In particular, a 0-form is just a smooth real-valued function $f: M \to \Re$

(2.1.3)Definition:

Let $(x^1, \dots, x^m)$ are local coordinates, then $TM|_x$ has basis $\{\partial/\partial x^1, \dots, \partial/\partial x^m\}$.The dual cotangent space has a dual basis, which is traditionally denoted $\{dx^1, \dots, dx^m\}$; Thus $\langle dx^i; \partial/\partial x^j\rangle = \delta_j^i$ for all i, j where δ_j^i is I for $i = j$ and 0 otherwise.

(2.1.4)Definition:

A differential one-form w thereby has the local coordinate expression

$$w = h_1(x)dx^1 + \cdots + h_m(x)dx^m, \qquad (2.9)$$

where each coefficient function $h_j(x)$ is smooth. Note that for any vector field $v = \sum \xi^i(x)\partial/\partial x^i$,

$$\langle w; v\rangle = \sum_{i=1}^{m} h_i(x)\xi^i(x)$$

is a smooth function. Of particular importance are the one- forms given by the differentials of real-valued functions.

$$df = \sum_{i=1}^{m} \frac{\partial f}{\partial x^i} dx^i, \quad with\ \langle df; v\rangle = v(f) \qquad (2.10)$$

(2.1.5)Definition:

To proceed to higher differential forms, we note that given a collection of differential one-forms $w_1, \dots, w_k$, we can form a differential k-form, $w_1 \wedge, \dots, \wedge w_k$, called the wedge product, using the determinately formula

$$\langle w_1 \wedge, \dots, \wedge w_k; v_1, \dots, v_k\rangle = \det(\langle w_i; v_j\rangle) \qquad (2.11)$$

the right-hand side being the determinant of a $k \times k$ matrix with indicated (i, j) entry. Note that the wedge product itself is both multi-linear and alternating

$$w_1 \wedge \ldots \wedge (cw_i + c'w_i') \wedge \ldots \wedge w_k = c(w_1 \wedge \ldots \wedge w_i \wedge \ldots \wedge w_k) + c'(w_1 \wedge \ldots \wedge w_i' \wedge \ldots w_k),$$

$$w_{\pi 1} \wedge \ldots \wedge w_{\pi k} = (-1)^{\pi} w_1 \wedge \ldots \wedge w_k \qquad (2.12)$$

In local coordinates, $\wedge_k T^*M|_x$ is spanned by the dasis k-forms:

$$dx^I \equiv dx^{i_1} \wedge \ldots \wedge dx^{i_k} \qquad (2.13)$$

where I ranges over all strictly increasing multi-indices $1 \le i_1 < i_2 < \cdots < i_k \le m$. Thus$\wedge_k T^*M|_x$ has dimension $\binom{m}{k}$; in particular, $\wedge_k T^*M|_x \simeq \{0\}$ If $k > m$.

Any smooth differential k-form on M has the local coordinate expression.

$$w = \sum_I \alpha_I(x)\, dx^I \qquad (2.14)$$

Where, for each strictly increasing multi-index I, the coefficient a_I is a smooth real-valued function.

Example (1):

A two-form in $\Re^3$ takes the form

$$w = \alpha(x, y, z) dy \wedge dz + \beta(x, y, z) dz \wedge dx \ \gamma (x, y, z) dx \wedge dy \ (1.15)$$

using the basis $dy \wedge dz$, $dz \wedge dx = -dx \wedge dz$, and $dx \wedge dy$, attuned to the notation for surface integrals we have

$$\langle \omega; \xi\partial x + \zeta\partial y + \eta\partial z, \hat{\xi}\partial x + \hat{\zeta}\partial y + \hat{\eta}\partial z \rangle = \alpha(\zeta\hat{\eta} - \hat{\zeta}\eta) + \beta(\eta\hat{\xi} - \hat{\eta}\xi) + \gamma(\xi\hat{\zeta} - \hat{\xi}\zeta).$$

If

$$\omega = \omega_1 \wedge \ldots \wedge \omega_k \quad , \theta = \theta_1 \wedge \ldots \wedge \theta_l$$

are decomposable forms, their wedge product is the form

$$\omega \wedge \theta = \omega_1 \wedge \ldots \wedge \omega_k \wedge \theta_1 \wedge \ldots \wedge \theta_l \ ,$$

with the definition extending bilinearly to more general types of forms:

$$(c\omega + c'\omega') \wedge \theta = c(\omega + \theta) + c'(\omega' + \theta),$$

$$\omega \wedge (c\theta + c'\theta') = c(\omega \wedge \theta) + c'(\omega \wedge \theta');$$

for c ,$c' \epsilon \Re$ this wedge product is associative:

$$\omega \wedge (\theta \wedge \xi) = (\omega \wedge \theta) \wedge \xi , \qquad (2.16)$$

And anti-commutative,

$$\omega \wedge \theta = (-1)^{k1} \theta \wedge \omega \qquad (2.17)$$

for ω a k-form and θ an I-form. For example the wedge product of (1.15) with a one-form $\theta = \lambda dx + \mu dy + \gamma dz$ is the three - form.

$$\omega \wedge \theta = (\alpha\lambda + \beta\mu + \sigma\gamma) dx \wedge dy \wedge dz \qquad (2.18)$$

2.2 Pull Back and change of coordinates

(2.2.1)Definition:

If $F: M \to N$ is a smooth map between manifolds its differential dF maps tangent vectors M to tangent vectors on N. There is thus an induced linear map F^*, called the pull-back or co-differential of F, which takes differential k-forms on N back to differential k-forms on M,

$$F^*: \wedge_k T^*N|_{F(x)} \to \wedge_k T^*M|_x \qquad (2.19)$$

It is defined so that if $\omega \epsilon \wedge_k T^*N|_{F(x)}$,

$$\langle F^*(\omega); v_1, \dots, v_k \rangle = \langle \omega; dF(v_1), \dots, dF(v_k) \rangle \qquad (2.20)$$

For any set of tangent vectors $v_1, \dots, v_k \epsilon TM|_x$. In contrast to the differential, the pull-back does take smooth differential forms on N back to smooth differential forms on M. if $x = (x^1, \dots, x^m)$ are local coordinates on M and $y = (y^1, \dots \dots, y^n)$ coordinates on N, then

$$F^*(dy^i) = \sum_{j=1}^{m} \frac{\partial y^i}{\partial x^j} \ . \partial x^j \qquad (2.21)$$

Where $y = F(x)$; gives the action of F^* on the basis one-forms. We conclude that in general

$$F^*(\sum_I \alpha_I(y) dy^I) = \sum_{I,J} \alpha_I(F(x)) \frac{\partial y^I}{\partial x^J} dx^J , \qquad (2.22)$$

Where $\frac{\partial y^I}{\partial x^J}$ stands for the Jacobian determinant $\det(\frac{\partial y^{i_k}}{\partial x^{j_v}})$ corresponding to the increasing multi-indices $I = (i_1, \dots, i_k)$, $J = (j_1, \dots \dots, j_K)$. In particular, if $y = F(x)$ determines a change of coordinates on M, then (2.22) provides the corresponding change of coordinates for differential k-form on M. Note also the pull-back preserves the algebraic operation of wedge product.

$$F^*(\omega \wedge \theta) = F^*(\omega) \wedge F^*(\theta) \qquad (2.23)$$

1- Closed and Exact form

(2.2.2)Definition:

A k-form ω is called closed if $d\omega = 0$,closed forms are the kernel of d.

(2.2.3) Definition:

$\omega$is called exact if $\omega = d\alpha$ for (k-1)-form α exact forms are the image of d because $d^2 = 0$ every exact form is closed.

(2.2.4)Definition:

A differential 1- form ω defined on a domain Ω is a map that to each point $p \in \Omega$ assigns $\omega(p) \epsilon (\Re^n)^*$ given by

$$\omega(p) = a_1(p)dx_1 + \cdots + a_n(p)dx_n \tag{2.24}$$

Such that each $a_i : \Omega \subset \Re^n \to \Re$ is smooth function.

Example (2):

The 1-form

$$\omega = -\frac{y}{x^2 + y^2}dx + \frac{x}{x^2 + y^2}dy$$

Defined on $\Omega = \Re^2 - (0,0)$

(2.2.5)Definition:

A differential 1- form ω defined on a domain Ω is said to be closed if

$$\frac{\partial a_i}{\partial x_j}(p) = \frac{\partial a_j}{\partial x_i}(p), \forall\, i, j \text{ and } x \in \Omega$$

we say that a differential 1- form ω is exact if there exists a smooth function $f : \Omega \to \Re$ such that

$$\omega = df \tag{2.25}$$

Example (3):

Let us consider the differential 1-form

$$\omega = \frac{y}{x^2 + y^2}dx + \frac{x}{x^2 + y^2}dy$$

We claim that ω is closed but not exact 1-form .In fact let γ be closed curve such that $\gamma : [0,2\pi] \to \Re^2$, $\theta \to (\cos\theta , \sin\theta)$

Computing the line integral

$$\int_\gamma \omega = \int_\gamma \frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

$$= \int_0^{2\pi} \frac{\sin\theta}{\sin^2\theta+\cos^2\theta}(-\sin\theta)dt \quad \frac{\cos\theta}{\sin^2\theta+\cos^2\theta}(\cos\theta)dt$$

$$= \int_0^{2\pi} dt$$

Since $\int_\gamma \omega \neq 0$, ω is not exact .

On odor hand it we compute $d\omega$ we have

$$d\omega = dA \wedge dx + dB \wedge dy$$

$$\text{A} = \frac{-y}{x^2+y^2} , B = \frac{x}{x^2+y^2}$$

Where

$$d\omega = \left(\frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy\right) \wedge dx + \left(\frac{\partial B}{\partial x}dx + \frac{\partial B}{\partial y}dy\right) \wedge dy$$

$$= \frac{\partial A}{\partial y}dy \wedge dx + \frac{\partial B}{\partial x}dx \wedge dy$$

$$= -\frac{\partial A}{\partial y}dx \wedge dy + \frac{\partial B}{\partial x}dx \wedge dy$$

$$= \left(-\frac{\partial A}{\partial y} + \frac{\partial B}{\partial x}\right) \wedge dxdy$$

$$= 0$$

Thus ω is a closed 1-fom but not exact.

2-Interior Products

(2.2.6)Definition:

If ω is a differential k-form and v a smooth vector field, then we can form a (k-1)- form $v\lrcorner\ \omega$ called the interior product of v with ω, defined so that

$$\langle v\lrcorner\ \omega; v_1, \dots, v_{k-1}\rangle = \langle \omega; v, v_1, \dots, v_{k-1}\rangle \qquad (2.26)$$

for every set of vector fields $v_1, \dots, v_{k-1}$. The is bilinear in both its arguments, so it suffices to determine it for basis elements:

$$\frac{\partial}{\partial x^i}\lrcorner\left(\partial x^{j_1}\wedge,\dots,\wedge\partial x^{j_k}\right)=\begin{cases}(-1)^{k-1} & dx^{j_1}\wedge,\dots,\wedge dx^{j_{k-1}}\wedge dx^{j_{k+1}}\wedge,\dots dx^{j_k}, i=j_k\\ 0 & , i\neq j_k\end{cases}$$

Example (4):

$\partial_x\lrcorner\, dx\wedge dy=dx$, $\partial_x\lrcorner\, dz\wedge dx=dx$, $\partial_x\lrcorner\, dx\wedge dz=0$ so that if ω is as in (1.15)

$$\left(\xi\partial_x+\zeta\partial_y+\eta\partial_z\right)\lrcorner\,\omega=(\eta\beta-\zeta\gamma)dx+(\xi\gamma-\eta\alpha)\partial y+(\zeta\alpha+\xi\beta)\partial z$$

Note that the interior product acts as an anti-derivation on forms, meaning that

$$v\lrcorner\,(\omega\wedge\theta)=\left(v\lrcorner\,\omega\right)\wedge\theta+(-1)^k\omega\wedge(v\lrcorner\,\theta) \qquad (2.27)$$

whenever ω is ak-form, θ an 1-form

3-The Differential Exterior derivative

The exterior derivative of differential form of degree k is a differential form of degree $k+1$.

IF f is a smooth function (a 0-form) then the exterior derivative of f is the deferential of that df is the unique 1-form such that for every smooth vector field X, $df(x)=d_x f$ where is the direction of X

(2.2.7)Definition:

In local coordinate , if $\omega=\sum\alpha_i(x)dx^i$ is a smooth differential k-form on a manifold M its differential or exterior derivative is the (k+1)-form

$$d\omega=\sum d\alpha_i\wedge dx^i=\sum\frac{\partial\alpha_i}{\partial x^j}dx^j\wedge dx^i \qquad (2.28)$$

The differential or the exterior derivative d , taking k- form to (k+1)-form has the following properties :

1- Linearity
$d(c\omega\wedge c'\omega')=cd\omega+c'd\omega'$ for c , c' constants.

2- Anti-derivation
$d(\omega\wedge\theta)=d(\omega)\wedge\theta+(-1)^k\omega d\theta$, for ω a k-form , θ an 1-from

3- Closure

$$d(d\,\omega)=0$$

4- Commuttion with Pull-Back
$f^*(d\omega)=d(f^*\omega)$for $f\colon M\to N$ smooth , ω a k-form on N.

Example (5):

If $M = \Re^3$, then the differential of one –form ,

$d(\lambda dx + \mu dy + \gamma dz) = (\gamma_y - \mu_z)dy \wedge dz + (\lambda_z - \gamma_x)dz\Lambda dx + (\mu_x - \lambda_y)dx \wedge$dy,

can be indentified with curl of its coefficients :

$\nabla \times \lambda \equiv \nabla \times (\lambda, \mu, \gamma)$. Similarly the differential of a two-form

d($\alpha dy \wedge dz + \beta dz \wedge dx + ydx \wedge dy$)= $(\alpha_x + \beta_y + \gamma_z)dx \wedge dy \wedge dz\beta$

can be identified $\nabla.\alpha \equiv \nabla.(\alpha, \beta, \gamma)$. The closure property therefore translates into the familiar vector calculus identities

$$\nabla \times (\nabla f) = 0 \quad , \qquad \nabla(\nabla \times \lambda) = 0$$

2.3 Lie Derivatives

Let σ be a differential form or vector field defined over . Given a point $x \epsilon M$, after "time" ε it has moved to $exp(\varepsilon v)x$ and the goal is to compare the value of σ at $exp(\varepsilon v)\, x$ with original value at x. However, $\sigma|_{exp(\varepsilon v)x}$ and $\sigma|_x$ as they stand are strictly speaking incomparable as they belong to different vector space e.g. $TM|_{exp(\varepsilon v)x}$ and $TM|_x$ in the case of vector field. To effect any comparison, we need to "transport"$\sigma|_{exp(\varepsilon v)x}$ back to x in some natural way, and then make our comparison. For vector field, this natural transport is the inverse differential.

$$\phi_\varepsilon^* \equiv d\ exp(-\varepsilon v): TM|_{exp(\varepsilon v)x} \to TM|_x, \qquad (2.29)$$

whereas for differential forms we use pull back map

$$\phi_\varepsilon^* = exp(\varepsilon v)^* : \wedge_k T^*M|_{exp(\varepsilon v)x} \to T^*M|_x, \qquad (2.30)$$

This allows to make the general definition of a Lie derivative

(2.3.1)Definition:

Let V be a vector field on M and σ a vector field or differential form defined on M. The Lie derivative of σ with respect to V is the object whose value at $x \epsilon M$is :

$$V(\sigma)|_x = \lim_{\varepsilon \to 0} \frac{\emptyset_\varepsilon^*(\sigma|_{exp(\varepsilon v)x}) - \sigma|_x}{\varepsilon} = \frac{\partial}{\partial \varepsilon_0}|_{\varepsilon=0} \emptyset_\varepsilon^*(\sigma|_{\exp(\varepsilon v)x} \qquad (2.31)$$

(Note that $V(\sigma)$ is an object of the same type as σ.)

(2.3.2)Proposition:

Let V and w be smooth vector fields on M. The Lie derivative of w with respect to$V$ coincides with the Lie bracket of $V$and w.

$$V(w) = [V, w] \tag{2.32}$$

Proof:

Let $(x^1, \dots, x^m)$ be local coordinates, with

$$V \equiv \sum \xi^i(x)\partial/\partial x^i,\ \omega = \omega|_{\exp(\varepsilon v)x}$$

$$= \sum_{i=1}^{m} \left[\eta^i(x) + \varepsilon V(\eta^i) + O(\varepsilon^2)\right] \frac{\partial}{\partial x^i},$$

Hence, using (1. 24) and (1. 22)

$$d\exp(-\varepsilon v)\left[\omega|_{\exp(\varepsilon v)x}\right] = \sum_{i=1}^{m} \{\eta^i(x) + \varepsilon\left[V(\eta^i) - \omega(\xi^i)\right] + O(\varepsilon^2)\} \frac{\partial}{\partial x^i}$$

Substituting into the definition (2.31). We deduce (2.32) from (1. 28).

Turning to differential forms, we find that the derivative can be most easily reconstructed form its basic properties.

a) Linearly

$$V(c\omega + c'\omega') = cV(\omega) + c'V(\omega'), \qquad c, c' \ constant \tag{2.33}$$

b) Derivation

$$V(\omega \wedge \theta) = V(\omega) \wedge \theta + \omega \wedge V(\theta) \tag{2.34}$$

c) Communication with the differential

$$V(d\omega) = dV(\omega) \tag{2.35}$$

Thus we have the use full formula

$$V(W \lrcorner\, \omega) = [V, W] \lrcorner\, \omega + W \lrcorner\, V(\omega), \tag{2.36}$$

For vector fields V and W and ω a differential form.

In local coordinates, the Lie derivative of differential form determined as follows. If

$$V = \sum_{i=1}^{m} \xi^i(x) \frac{\partial}{\partial x^i}$$

Then

$$V(dx^i) = dV(x^i) = d\xi^i = \sum_{j=1}^{m} \frac{\partial \xi^i}{\partial x^j} \; . \; \partial x^j$$

Therefore, we have the general formula

$$V\left(\sum_i \alpha_i(x) dx^i\right) = \sum_l \left\{V(\alpha_l) dx^l + \sum_{i=1}^{k} \alpha_i dx^{i_1} \wedge \dots \wedge d\xi^{i_1} \wedge \dots \wedge dx^{i_k}\right\} \quad (2.37)$$

*Note, the three properties (2.31), (2.33) along with its action on smooth functions sever to define the Lie derivative operation uniquely.

Example (6):

Let $M = \Re^2$ and

$$V = \xi(x,y)\partial x + \eta(x,y)\partial y$$

Then the Lie derivative of a two form is

$$V(\gamma(x,y)dx\Lambda dy) = V(\gamma)dx\Lambda dy + \gamma d\xi\Lambda dy + \gamma dx\Lambda d\eta = \{\xi\gamma_x + \eta\gamma_y + \gamma\xi_x\}dx\Lambda dy.$$

(2.3.3) Proposition:

A differential k-form on M is invariant under the flow of a vector field V:

$$\omega|_{exp(\varepsilon v)x} = exp(-\varepsilon v)^*(\varepsilon/x),$$

if and only if $v(\omega) = 0$ (A similar result holds for vector fields).

Proof:

Applying $\phi_\varepsilon^* = exp(\varepsilon v)^*$ to (2.30) and using the basic group property of the flow, we find

$$exp(\varepsilon v)^*\left(v(\omega)|_{exp(\varepsilon v)x}\right) = \frac{d}{d\varepsilon}\left\{exp(\varepsilon v)^*\left(\omega|_{exp(\varepsilon v)x}\right)\right\} \quad (2.38)$$

For all ε where defined:

(2.3.4) Proposition:

Let ω be a differential form and V be vector field on M. then

$$V(\omega) = d(V \lrcorner\, \omega) + V \lrcorner\, (d\omega) \qquad (2.37)$$

Proof:

Define the operator $\mathcal{g}_v(\omega)$ by the right hand side of (2.37). Since the Lie derivative is uniquely determined by its action on function and the properties (2.31), (2.33) it suffice to check that $\mathcal{g}_v$enjoy the same properties.

First:

$$\mathcal{g}_v(f) = v\lrcorner\, df = \langle df; v\rangle = v(f),$$

So the action function is the same. Linearly of $\mathcal{g}_v$ is clear while the closure property of d immediately proves the communication property:

$$\mathcal{g}_v(d\omega) = d\left(v\lrcorner\, d\omega\right).$$

Finally, if ω is a k-form and θ an 1-form, we use (1.52) (1.54) to prove that

$$\begin{aligned}\mathcal{g}_v(\omega \wedge \theta) = d\left[\left(v\lrcorner\, \omega\right) \wedge \theta + (-1)^k \omega \wedge \left(v\lrcorner\, \theta\right)\right] + v\lrcorner\, [(d\omega) \wedge \theta(-1)^k \omega \wedge (d\theta)] \\ = d\left(v\lrcorner\, \omega\right) \wedge \theta(-1)^{k-1}\left(v\lrcorner\, \omega\right) \wedge d\theta + (-1)^k (d\omega) \wedge \left(v\lrcorner\, \theta\right) + (-1)^{2k}\omega \\ \wedge d\left(v\lrcorner\, \theta\right) + \left(v\lrcorner\, d\omega\right) \wedge \theta + (-1)^{k+1} d(\omega) \wedge \left(v\lrcorner\, \theta\right) + (-1)^k \left(v\lrcorner\, \omega\right) \wedge d\theta \\ + (-1)^{2k}\omega \wedge \left(v\lrcorner\, d\theta\right)\end{aligned}$$

$$= \mathcal{g}_v(\omega) \wedge \theta + \omega \wedge \mathcal{g}_v(\theta),$$

The remaining terms cancelling.

Chapter Three

The Lie - Poisson Structure

Chapter Three

The Lie - Poisson Structure

The guiding concept of a Hamiltonian system of differential equations forms the basis of much of more advanced work in classical mechanics, including motion of rigid bodies, celestial mechanics, quantization theory.

(3.1) Poisson Brackets

On a smooth manifold M, a Poisson bracket assigns to each pair of smooth , real - valued functions $F, H: M \to R$ another smooth real-valued function, which are denote by $\{F, H\}$. there are certain basic properties that such a bracket operation must satisfy in order to qualify as a Poisson bracket .

(3.1.1) Definition:

A Poisson bracket on smooth manifold M is an operation that assigns a smooth real - valued function $\{F, H\}$ $on\ M$ to each pair F, H of smooth, real valued functions, with the basic properties :

a) Bilinearity:

$$\{cF + c'P, H\} = c\{F, H\} + c'\{P, H\},$$

$$\{F, cH + c'P\} = c\{F, H\} + c'\{F, P\}, \text{ for constants } c, c' \in \Re$$

b)Skew - symmetry :

$$\{F, H\} = -\{H, F\}$$

c)Jacobi Identity:

$$\big\{\{F, H\}, P\big\} + \big\{\{P, F\}, H\big\} + \big\{\{H, P\}, F\big\} = 0$$

d)Leibniz's Rule :

$$\{F, H.P\} = \{F, H\}.P + H.\{F, P\}$$

(here. Denotes the ordinary multiplication of real- valued functions) in all these equations F, H and P are arbitrary smooth real-valued functions on M.

A manifold M with a Poisson bracket is called a Poisson manifold, the bracket defining a Poisson structure on M.

The notion of a Poisson manifold is slightly more general then that of a symplectic manifold, or manifold, or manifold with Hamiltonian structure; in particular, the underlying manifold M need not be even - dimensional.

Example (1):

Let M be the even - dimensional Euclidean space $\Re^{2n}$ with coordinates $(p,q) = (p^1,\dots,p^n,q^1,\dots,q^n)$. (in physical the p's represent momenta and q's positions of the mechanical objects.) if $F(p,q)$ and $H(p,q)$ are smooth functions, we define their Poisson bracket to be the function

$$\{F,H\} = \sum_{i=1}^{n}\left\{\frac{\partial F}{\partial q^i}\frac{\partial H}{\partial p^i} - \frac{\partial F}{\partial p^i}\frac{\partial H}{\partial q^i}\right\} \tag{3.1}$$

This bracket is clearly satisfying the basic properties of the Poisson bracket. We note that the particular identities

$$\{p^i,p^j\} = 0, \quad \{q^i,q^j\} = 0, \quad \{q^i,p^j\} = \delta_j^i \tag{3.2}$$

In which i and j run from 1 to n and δ_j^i is the kroneck symbol, which is 1 if $i = j$ and 0 otherwise.

More general, we can determine a Poisson bracket on any Euclidan space $\Re^m$. Just let $(p,q,z) = (p^1,\dots,p^n,q^1,\dots,q^n,z^1,\dots z^l)$ be the coordinates so 2n+L= m and define the Poisson bracket between two functions $F(p,q,z),H(p,q,z)$ by the formula (3.1). In particular, if the function $F(z)$ depends on the z's only, then $\{F,H\} = 0$ for all functions H. Such functions in particular the $z^{k\prime}s$ themselves, are known as distinguished functions, or cassimere functions and are characterized by the property that their Poisson bracket with any other function is always zero. We suplement the basic coordinate bracket (3.2) by the additional relations

$$\{p^i,z^k\} = \{q^i,z^k\} = \{z^j,z^k\} = 0, \tag{3.3}$$

For all $i = 1,\dots,n$, and $j,k = 1,\dots,l$.

(3.1.2) Definition:

Let M be a Poisson manifold. A smooth, real-valued function $C: M \to \Re$ is called a distinguished if the Poisson bracket of C with any other real valued function vanishes identically, i.e $\{C,H\} = 0$ for $H: M \to \Re$.

In the case of canonical Poisson bracket (3.1) on $\Re^{2n}$, the only distinguished functions are the constants, which always satisfy the requirements of the definition. At the other extreme if the Poisson bracket is completely trivial i.e .$\{F,H\} = 0$ for every F,H then every function is distinguished.

(3.2) Hamiltonian Vector Field

(3.2.1) Definition:

Suppose that $H(p,q)$ is a smooth function of its arguments for p and $q \in \Re^n$.Then the dynamical system

$$\dot{p}_\iota = \frac{\partial H}{\partial q} \tag{3.4}$$

$$\dot{q}_\iota = -\frac{\partial H}{\partial p_i} \tag{3.5}$$

where $(\, i = 1,2,\dots,n)$ is called a Hamiltonian system and H is the Hamiltonian function (or just the Hamiltonian) of the system. Equation (3.4) are called Hamilton's equations.

(3.2.2) Definition:

Let M be a Poisson manifold and $H: M \to \Re$ a smooth function. The Hamiltonian vector field associated with H is the unique smooth vector field $\hat{V}_H\ on\ M$ satisfying

$$\hat{V}_H(F) = \{F,H\} = -\{H,F\} \tag{3.6}$$

for every smooth function $F: M \to \Re$ the equations governing the flow of $\hat{V}_H(F)$ are referred to as Hamilton's equation for the "Hamiltonian" function H.

Example (2):

In the case of the canonical Poisson bracket (3.1) on $\Re^m, m = 2n + 1$, the Hamiltonian vector field corresponding to $H(p,q,z)$ is clearly

$$\hat{V}_H = \sum_{i=1}^{n} \left\{ \frac{\partial H}{\partial p^i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p^i} \right\} \tag{3.7}$$

The corresponding flow is obtained by integrating the system of ordinary differential equations

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p^i}\ , \frac{\partial p^i}{dt} = -\frac{\partial H}{\partial q^i} \quad i = 1,\dots,n \tag{3.8}$$

$$\frac{dz^j}{dt} = 0, \quad j = 1,\dots,l, \tag{3.9}$$

which are Hamilton's equations in this case. In the nondegenerate case $m = 2n$ we have just (3.8), which is the canonical form Hamilton's equations in classical mechanics. More generally (3.9) just add in the constancy of the distinguished coordinates z^j under the flow. In particular, if H depends only on the distinguished coordinates z, its Hamiltonian flow is completely trivial. This remark hold in general: A function C on a Poisson manifold is distinguished if and only if its Hamiltonian vector field $\hat{V}_C = 0$ vanishes everywhere

(3.2.3) Proposition:

Let M be a Poisson manifold, let $F, H: M \to \mathfrak{R}$ be smooth function with corresponding Hamiltonian vector field $\widehat{V}_F$, $\widehat{V}_H$. The Hamiltonian vector field associated with the Poisson bracket of F and H is, up to sign, the Lie bracket of two Hamiltonian vector fields:

$$\widehat{V}_{\{F,H\}} = -[\widehat{V}_F, \widehat{V}_H] = [\widehat{V}_H, \widehat{V}_F]. \tag{3.10}$$

Proof

Let $P: M \to \mathfrak{R}$ be any other smooth function. Using the commutator definition of the Lie bracket, we find

$$\begin{aligned}
[\widehat{V}_H, \widehat{V}_F]P &= \widehat{V}_H \cdot \widehat{V}_F(P) - \widehat{V}_F \cdot \widehat{V}_H(P) \\
&= \widehat{V}_H\{P, F\} - \widehat{V}_F\{P, H\} \\
&= \{\{P, F\}, H\} - \{\{P, H\}, F\} \\
&= \{P, \{F, H\}\} \\
&= \widehat{V}_{\{F,H\}}(P),
\end{aligned}$$

Where we have made use of the Jacobi identity, the skew- symmetry of the Poisson bracket , and the definition (3.2.2) of a Hamiltonian vector field . since P is arbitrary.

Example (3):

Let $M = \mathfrak{R}^2$ with coordinates (p, q) and canonical Poisson bracket $\{F, H\} = F_q H_p - F_p H_q$. For a function $H(p, q)$ the corresponding Hamiltonian vector field is $\widehat{V}_H = H_p \partial_q - H_q \partial_p$. Thus for $H = \frac{1}{2}(p^2 + q^2)$ we have $\widehat{V}_H = p\partial_q - q\partial_p$, whereas for $F = pq$, $\widehat{V}_F = q\partial_q - p\partial_p$. the Poisson bracket of F and H is $\{F, H\} = p^2 - q^2$, which has Hamiltonian vector field $\widehat{V}_{\{F,H\}} = 2p\partial_q - 2q\partial_p$. this agrees with the commutator $[\widehat{V}_H, \widehat{V}_F]$.

1. *The structure functions*

(3.2.4)Definition:

The general local coordinate picture for a Poisson manifold, at the Hamiltonian vector fields. Let $x = (x^1, \dots, x^m)$ be local coordinates on M and $H(x)$ a real - valued function. The associated Hamiltonian vector field will be of the general form $\widehat{V}_H = \sum_{i=1}^{m} \xi^i(x) \frac{\partial}{\partial x^i}$, where the coefficient function $\xi^i(x)$ which depend on H are to be determined. Let $F(x)$ be a second smooth function. Using (3.6) we find

$$\{F, H\} = \hat{V}_H(F) = \sum_{i=1}^{m} \xi^i(x) \frac{\partial F}{\partial x^i}$$

But, by (3.6)

$$\xi^i(x) = \hat{V}_H(x) = \{x^i, H\},$$

So this formula becomes

$$\{F, H\} = \sum_{i=1}^{m} \{x^i, H\} \frac{\partial F}{\partial x^i} \quad (3.11)$$

Using the skew - symmetry of the Poisson bracket, we can compute the letter set of Poisson bracket in term of the particular Hamiltonian vector fields $\hat{V}_i = \hat{V}_{x^i}$ associated with the local coordinate functions x^i, n- amely

$$\{x^i, H\} = -\{H, x^i\} = -\hat{V}_i(H) = -\sum_{j=1}^{m} \{x^j, x^i\} \frac{\partial H}{\partial x^j},$$

The last equality following form a second application of (3.11),with H replacing F and x^i replacing H. thus we obtain the basic formula

$$\{F, H\} = \sum_{i=1}^{m} \sum_{j=1}^{m} \{x^i, x^j\} \frac{\partial F}{\partial x^i} \frac{\partial H}{x^j} \quad (3.12)$$

For the Poisson bracket. In other words, to compute the Poisson bracket of any pair of functions in some given set of local coordinates it suffices to know the Poisson brackets between the coordinate function themselves. These basic brackets

$$J^{ij} = \{x^i, x^j\}, \quad i, j = 1, \dots m \quad (3.13)$$

Are called the structure functions of the Poisson manifold M relative to the given local coordinates, for convenience, we assemble the structure functions in to a skew-symmetric $m \times m$ matrix $J(x)$, called the structure matrix of M. Using ∇H to denote the ‘’column’’ gradient vector for H, the local coordinate from (3.12) for the Poisson bracket takes the form :

$$\{F, H\} = \nabla F \cdot J \nabla H \quad (3.14)$$

Example (4):

In the case of the canonical bracket (3.1) on $\Re^m = 2n + 1$ the structure matrix has the simple form

$$J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

relative to the (p,q,z) – coordinates. Where I is the $n \times n$ identity matrix. The Hamiltonian vector field associated with $H(x)$ has the form

$$\hat{V}_H = \sum_{i=1}^{m} \left(\sum_{j=1}^{m} J^{ij}(x) \frac{\partial H}{\partial x^j} \frac{\partial}{x^i} \right), \tag{3.15}$$

or in matrix notation $\hat{V}_H (J\nabla H) \cdot \partial_x$, ∂_x being the "vector" with entries $\partial / \partial x^i$. Therefore in the given coordinate chart, Hamilton's equations take the form

$$\frac{dx}{dt} = J(x) \nabla H(x) \tag{3.16}$$

Alternatively, using (3.11), we could write this in the " bracket form"

$$\frac{dx}{dt} = \{x, H\}$$

The i-th component of the right-hand side being $\{x^i, H\}$.

Any system of the first order ordinary differential equations is said to be a Hamiltonian system if there is a Hamiltonian function $H(x)$ and a matrix of functions $J(x)$ determining a Poisson bracket (3.15) whereby the system takes the form (3.16). of course, we need to know which matrices $J(x)$ are the structure matrices for Poisson brackets.

(3.2.5)Proposition:

Let $J(x) = \left(J^{ij}(x)\right)$ be an $m \times m$ matrix of functions of $x = (x^i, \dots, x^m)$ defined over an open subset $M \in \Re^m$. Then $J(x)$ is the structure matrix for a Poisson bracket $\{F, H\} = \nabla F \cdot \nabla \mathrm{J} H$ over M if and only if it has the properties of :

a) Skew- symmetry

$$J^{ij}(x) = -J^{ij}(x) \quad i, j = 1, \dots, m$$

b) Jacobi identity:

$$\sum_{i=1}^{m} \{ J^{il} \partial_l J^{jk} + J^{kl} \partial_l J^{ij} + J^{jl} \partial_l J^{ki} \} = 0, \;\; i, j, 1, \dots, m \tag{3.17}$$

For all $x \in M$. (Here, as usual $\partial_I = \frac{\partial}{\partial x^I}$.)

Proof

In its basic form (3.14) the Poisson bracket is automatically bilinear and satisfies Leibniz's rule. The skew-symmetry of the structure matrix is clearly equivalent to the skew- symmetry of the bracket. Thus we need only verify the equivalence of (3.17) with Jacobi identity. That by (3.12) and (3.13)

$$\left\{\{x^i, x^j\}, x^k\right\} = \sum_{l=1}^{m} J^{lk}(x)\partial_l J^{ij}(x),$$

So (3.17) is equivalent to the Jacobi identity for the coordinate functions x^i, x^j and x^k. More generally, for $F, H, P: M \to \Re$,

$$\left\{\{F,H\},P\right\} = \sum_{k,l=1}^{m} J^{lk}\frac{\partial}{\partial x^l}\left\{\sum_{i,j=1}^{m} J^{ij}\frac{\partial F}{\partial x^i}\frac{\partial H}{x^j}\right\}\frac{\partial P}{\partial x^k} = \sum_{i,j,k=1}^{m}\{J^{lk}\frac{\partial J^{ij}}{\partial x^l}\frac{\partial F}{\partial x^i}\cdot\frac{\partial H}{\partial x^j}\frac{\partial P}{\partial x^k} + J^{lk}J^{ij}(\frac{\partial^2 F}{\partial x^l x^i}\cdot \frac{\partial H}{\partial x^j}\frac{\partial P}{\partial x^k} + \frac{\partial F}{\partial x^j}\cdot\frac{\partial^2 H}{\partial x^l x^j}\frac{\partial P}{\partial x^k})\}$$

Summing cyclically on F, H, P we find that the first set of terms vanishes by virtue of (3.17), while the remaining term can conform to skew- symmetry of structure matrix.

(3.3) The Lie –Poisson Structure

(3.3.1) Definition:

Let g be r - dimensional Lie algebra, and C_{ij}^k, $i,j,k = 1,\dots,r$, be the structure constants of g relative to a basis$\{v_1,\dots,v_r\}$,let V be another r - dimensional vector space, with coordinates $x = (x^1,\dots,x^r)$ determined by a basis $\{w_1,\dots,w_r\}$. We define the Lie- Poisson bracket between two functions $F, H: V \to \Re$,

$$\{F,H\} = \sum_{i,j,k=1}^{r} C_{ij}^k\, x^k \frac{\partial F}{\partial x^i}\frac{\partial H}{x^j} \tag{3.18}$$

This takes the form (3.12) with linear structure functions $J^{ij}(x) = \sum_{k=1}^{r} C_{ij}^k x^k$

(3.3.2) Definition:

Let V be any vector space and $F: V \to \Re$ smooth, real - valued function, then the gradient $\nabla F(x)$ at any point $x \in V$ is naturally an element of the dual vector space V^* consisting of all (continuous) linear functions on V defined by

$$\langle \nabla F(x); y\rangle = \lim_{\varepsilon\to 0}\frac{F(x+\varepsilon y) - F(x)}{\varepsilon} \quad for\ any\ \ y \in V$$

Where $\langle;\rangle$ is the natural pairing between V and its dual V^*, we identify the vector space V used in our initial construction of the Lie Poisson bracket with dual space g^* to the Lie algebra $g, \{w_1,\dots,w_r\}$ begin the dual basis to $\{V_1,\dots,V_r\}$. If $F: g^* \to \Re$ is any smooth function, then its gradient $\nabla F(x)$ is an element of $(g^*)^* \simeq g$ (since g is finite dimensional). Then the Lie Poisson bracket has the coordinate free form

$$\{F,H\}(x) = \langle x; [\nabla F(x), \nabla H(x)]\rangle, \ \ x \in g^* \tag{3.19}$$

Where $[,]$ is the ordinary Lie bracket on the Lie algebra g if $H: g \to \Re$ is any function, the associated system of Hamilton's equation take the form

$$\frac{dx^i}{dt} = \sum_{j,k=1}^{r} C_{ij}^k x^k \frac{\partial H}{\partial x^j} \quad i = 1, \dots, r,$$

In which the coordinates x^k themselves appear explicitly.

Example (5):

Consider the 3 - dimensional Lie algebra So(3) of the rotation group So(3). Using the basis $V_1 = y\partial_z - z\partial_y, V_2 = z\partial_x - x\partial_z, V_3 = x\partial_y - y\partial_x$ of infinitesimal rotation around the x, y and z axes of $\Re^3$ (or their matrix counter parts), we have the commutation relations $[V_1, V_2] = -V_3, [V_3, V_1] = -V_2, [V_2, V_3] = -V_1$ let w_1, w_2, w_3 be a dual basis for $So(3)^* \simeq \Re^3$ and $u = u^1 w_1 + u^2 w_2 + u^3 w_3$ at typical point therein. If $F: So(3)^* \to \Re$, then the gradient is the vector

$$\nabla F = \frac{\partial F}{\partial u^1} V_1 + \frac{\partial F}{\partial u^2} V_2 + \frac{\partial F}{\partial u^3} V_3 \in So(3)$$

Thus from (3.19) we find the Lie - Poisson bracket on $So(3)^*$ to be

$$\{F, H\} = u^1 \left(\frac{\partial F}{\partial u^3}\frac{\partial H}{\partial u^2} - \frac{\partial F}{\partial u^2}\frac{\partial H}{\partial u^3}\right) + u^2 \left(\frac{\partial F}{\partial u^1}\frac{\partial H}{\partial u^3} - \frac{\partial F}{\partial u^3}\frac{\partial H}{\partial u^1}\right) + u^3 \left(\frac{\partial F}{\partial u^2}\frac{\partial H}{\partial u^1} - \frac{\partial F}{\partial u^1}\frac{\partial H}{\partial u^2}\right) = -u\nabla F \times \nabla H,$$

using the standard cross product on $\Re^3$. Thus the structure matrix is

$$J(u) = \begin{pmatrix} 0 & -u^3 & u^2 \\ u^3 & 0 & -u^1 \\ -u^2 & u^1 & 0 \end{pmatrix}, \quad u \in So(3)^*$$

Hamilton's equations corresponding to Hamiltonian function $H(u)$ are therefore

$$\frac{du}{dt} = u \times \nabla H(x)$$

1. The correspondence Between one forms and vector fields:

A Poisson structure on a manifold M sets up a correspondence between smooth function $H: M \to \Re$ and their associated Hamiltonian vector field $\hat{V}_H$ $on\, M$. In local coordinates this correspondence is determined by multiplication of the gradient ∇F by the structure matrix $J(x)$ determined by the Poisson bracket. This can given a more intrinsic formulation if we recall that the coordinate - free version of the gradient of real - valued function H is its differential dH. Thus the Poisson structure determines a correspondence between differential one - forms

dH on M and their associated Hamiltonian vector fields $\hat{V}_H$ which in fact extends to general one - forms.

2. Rank of a Poisson structure:

(3.3.3) Definition:

Let M be a Poisson manifold and $x \in M$. The rank of M at x is the rank of the linear map $B|_x : \mathrm{T}^*\mathrm{M}|_x \to \mathrm{TM}|_x$.

In local coordinates $B|_x$ is the same as multiplication by the structure matrix $J(x)$, so the rank of M at x equals the rank of $J(x)$, independent of the choice of coordinates.

3. Symplectic Manifolds:

(3.3.4)Definition:

Poisson manifold M of dimension m is symplectic if it Poisson structure has maximal rank m everywhere.

4. Maps between Poisson Manifolds:

(3.3.5) Definition :

If M and N are Poisson Manifold map is a smooth map $\phi: M \to N$ preserving the Poisson brackets:

$$\{F \circ \phi, H \circ \phi\}_M = \{F, H\}_N \circ \phi \quad \text{for all } F, H: N \to \Re.$$

- In the case of symplectic manifolds these are canonical maps of classical mechanics .

(3.3.6) Proposition:

Let M be a Poisson manifold and $\hat{V}_H$ a Hamiltonian vector field . For each t , the flow $exp(t\hat{V}_H): M \to M$ determines a (local) Poisson map from M to itself .

Proof

Let F and P be real - valued functions, and let $\phi_t = exp(t\hat{V}_H)$. If we differentiate the Poisson condition $\{F \circ \phi_t, P \circ \phi_t\} = \{F, P\} \circ \phi$ with respect to t we find the infinitesimal version

$$\{\hat{V}_H(F), P\} + \{F, \hat{V}_H(P)\} = \hat{V}_H(\{F, P\})$$

at the point $\phi_t(x)$. By (3.6) this is the same as the Jacobi identity . At $t = 0$, ϕ_o is the identity, and trivially Poisson, so a simple integration proves the Poisson condition for general t.

Example (6):

If $M = \Re^2$ with canonical coordinates (p, q), then the function $H = \frac{1}{2}(p^2 - q^2)$ generates the group of rotations in the plane, determined by $\hat{V}_H = p\partial_q - q\partial_p$. Thus each rotation in $\Re^2$ is a canonical map . Since any Hamiltonian flow preserves the Poisson bracket on , in particular it preserves its rank.

(3.3.7) Corollary:

If $\hat{V}_H$ is a Hamiltonian vector field on a Poisson manifold M, then the rank of M at $exp(t\hat{V}_H)\,x$ is the same as the rank of M at x for any $t \in \Re$.

- For instance, the origin in $So(3)^*$, being the only point of rank 0 ,is a fixed point of any Hamiltonian system with the given Lie - Poisson structure. In fact, any point of rank 0 on a Poisson manifold is fixed point for any Hamiltonian system there.

5. Poisson submanifolds:

(3.3.8) Definition:

A submanifold $N \subset M$ is a Poisson submanifold if its defining immersion $\phi: \hat{N} \to M$ is a Poisson map.

- An equivalent way of stating this definition is that for any pair of $F, H: M \to \Re$ which restrict to functions $\hat{F}, \hat{H}: N \to \Re$ on N, their Poisson bracket $\{F, H\}_M$ naturally restricts to a Poisson bracket $\{\hat{F}, \hat{H}\}_N$.

(3.3.9) Proposition:

A submanifold N of a Poisson manifold M is a Poisson submanifold if and only if $TN|_y \supset H|_y$ for all $y \in N$, meaning every Hamiltonian vector field on M is everywhere tangent to N.

In particular, if $TN|_y = H|_y$ for all $y \in N$ is a symplectic submanifold of M.

Proof

Since a Poisson bracket is determined by its local character , we can without loss of generality assume that N is a regular submanifold of M and use flat local coordinates

$(y,w) = (y^1, \dots, y^n, w^1, \dots, w^{n-n})$ with $N = \{(y,w) := w = 0\}$. First suppose that N is a Poisson submanifold , and let $\widehat{H}: N \to \mathfrak{R}$ be any smooth function. Then we can extend $\widehat{H}$ to a smooth function $H: M \to \mathfrak{R}$ defined in a neighbourhood of N, with $\widehat{H} = H \backslash N$. In our local coordinates, $\widehat{H} = \widehat{H}(y) and\ H(y,w)$ is any function so that $H(y,0) = \widehat{H}(y). if\ \hat{F}: N \to \mathfrak{R}$ has a similar extension F, then by definition the Poisson bracket between $\hat{F}\ and\ \widehat{H}$ on N is obtained by restricting that of F and H to N.

$$\{\hat{F}, \widehat{H}\}_N = \{F, H\}/N.$$

In particular, for any choice of $\hat{F}, \widehat{H}$, the bracket $\{F, H\}/N$ cannot depend on the particular extensions F and H which are selected. Clearly, this is possible if and only if $\{F, H\}/N$ contains on partial derivatives of either F or H with respect to the normal coordinates w^i, so

$$\{F,H\}|N = \sum_{i,j} J^{ij}(y,0) \frac{\partial F}{\partial y^i} \frac{\partial H}{\partial y^j} \equiv \sum_{i,j} \hat{J}^{ij}(y) \frac{\partial \hat{F}}{\partial y^i} \frac{\partial \widehat{H}}{\partial y^j} \quad , \tag{3.20}$$

but then the Hamiltonian vector field $\hat{V}_{H,}$ restricted to N, takes the form

$$\hat{V}_H|N = \sum_{i,j} \hat{J}^{ij}(y) \frac{\partial H}{\partial y^j} \frac{\partial}{\partial y^i}, \tag{3.21}$$

and is thus tangent to N everywhere.

Conversely, if the tangency condition $H|_y \subset TN|_y$ hold for all $y \in N$, any Hamiltonian vector field, when restricted to N must be combination of the tangential basis vectors $\partial / \partial y^i$ only, and hence of the form (3.21) if $F(w)$ depends on w alone, then $\{F, H\} = \hat{V}_H(F)$ must therefore vanish when restricted to N.

In particular ,

$$\{y^i, w^j\} = \{w^k, w^j\} = 0 \quad \text{on } N \text{ for all } i, j, k,$$

And hence the Poisson bracket on N takes the form (3.2) in which $\hat{J}^{ij}(y) = J^{ij}(y,0) = \{y^i, y^j\}|N$. The fact that the structure function $\hat{J}^{ij}(y)$ of the induced Poisson bracket on N satisfy the Jacobi identity easily follows (3.17) since on restriction on . All w-terms vanish. Thus N is a Poisson submanifold.

Note that the rank of the Poisson structure on N at $y \in N$ equals the rank of the Poisson structure on M at the same point.

Example (7):

For the Lie - Poisson structure on $\mathrm{So}(3)^*$, the subspace $H|u$ at $u \in \mathrm{So}(3)^*$ is spanned by the elementary Hamiltonian vectors $\hat{V}_1 = u^3\partial_2 - u^2\partial_3$, $\hat{V}_2 = u^1\partial_3 - u^3\partial_1$, $\hat{V}_3 = u^2\partial_1 - u^1\partial_2$, $(\partial_i = \partial \backslash \partial u^i$, corresponding to coordinate functions u^1, u^2, u^3 respectively. If $u \neq 0$, there vectors span a two-dimensional subspace of $T\mathrm{So}(3)^*|_u$ which coincides with the tangent space to the sphere $S_\rho^2 = \{u: |u| = \rho\}$ passing through $u = H|_u = TS_\rho^2|_u$, $|u| = \rho$. Proposition (3.3.8) therefore implies that each such sphere is a symplectic submanifold of $\mathrm{So}(3)^*$. In terme of spherical coordinates $u^1 = \rho\cos\theta\sin\varphi$, $u^2 = \rho\sin\theta\sin\varphi$, $u^3 = \rho\cos\varphi$ $on\ S_\rho^2$, the Poisson bracket between $\hat{F}(\theta,\varphi) and\ \hat{H}(\theta,\varphi)$ computed by extending them to a neighbourhood of S_ρ^2, set $F(\rho,\theta,\varphi) = \hat{F}(\theta,\varphi)$, $H(\rho,\theta,\varphi) = \hat{H}(\theta,\varphi)$, computing the Lie- Poisson bracket $\{F,H\}$ and then restricting to S_ρ^2. However, according to (3.12) , $\{\hat{F},\hat{H}\} = \{\theta,\varphi\}(\hat{F}_\theta\hat{H}_\varphi - \hat{F}_\varphi\hat{H}_\theta)$ so we only really need compute the Lie-Poisson bracket between the spherical angles θ,φ

$$\{\theta,\varphi\} = -u.(\nabla_u\theta \times \nabla_u\varphi) = \frac{-1}{\rho\sin\varphi}.$$

Thus

$$\{\hat{F},\hat{H}\} = \frac{-1}{(\rho\sin\varphi)}\left(\frac{\partial\hat{F}}{\partial\theta}\frac{\partial\hat{H}}{\partial\varphi} - \frac{\partial\hat{F}}{\partial\varphi}\frac{\partial\hat{H}}{\partial\theta}\right)$$

is the induced Poisson bracket on $S_\rho^2 \subset \mathrm{So}(3)^*$

- Thus if $N \subset M$ is a Poisson submanifold , any Hamiltonian vector field $\hat{V}_H\ on\ M$ is everywhere tangent to N and thereby naturally restricts to Hamiltonian vector field $\hat{V}_H\ on\ N$, where $\hat{H} = H|N$ is the restriction of H to N and we are using the induced Poisson structure on N to compute $\hat{V}_H$.

If we are only interested in solution to the Hamiltonian system corresponding to H on M with initial conditions $x_\circ\ on\ N$ we can restrict $t_\circ$ the Hamiltonian system corresponding to $\hat{H}\ on\ N$ without loss of information, thereby reducing the order of the system.

In particular as far as finding particular solutions of the Hamiltonian system goes, we may as all restrict to the minimal Poisson submanifolds of M, these are always symplectic submanifolds so every Hamiltonian system can be reduced to one in which the Poisson bracket is symplectic .

(3.3.10)Theorem:

Let M be an m - dimensional Poisson manifold of constant rank $2n \leq m$ everywhere. At each $x_\circ \in M$ there exist canonical local coordinates $(p,q,z) =$

$(p^1, \dots, p^n, q^1, \dots, q^n, z^1, \dots, z^l)$, $2n + L = m$, in terms of which the Poisson bracket takes the form

$$\{F, H\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q^i} \frac{\partial H}{\partial p^i} - \frac{\partial F}{\partial p^i} \frac{\partial H}{\partial q^i} \right)$$

The leaves of the symplectic foliation intersect the coordinate chart in the slices $\{z^1 = c_1, \dots, z^l = c_l\}$ determined by the distinguished coordinate .

Poof:

If the rank of the Poisson structure is 0 everywhere there is nothing to prove. Indeed, the Poisson bracket is trivial : $\{F, H\} = 0$ for all F, H, and any set of local coordinates $z = (z^1, \dots, z^l), l = m$ satisfies the condition of the theorem. Otherwise, we proceed by induction on the "half - rank" n . since the rank at $x_\circ$ is non zero, we can find real - valued functions F and P on M whose Poisson bracket does not vanish at $x_\circ$:

$$\{F, P\}(x_\circ) = \hat{V}_P(F)(x_\circ) \neq 0.$$

In particulare, $\hat{V}_P|_{x_\circ} \neq 0$, so proposition (1.4.11) to straighten out $\hat{V}_P$ in a neighbourhood U of $x_\circ$ and thereby find a function $Q(x)$ satisfying

$$\hat{V}_P(Q) = \{Q, P\} = 1 \quad \text{for all} \quad x \in U$$

(In notation of proposition (1.4.11), Q would be the coordinate y^l), since $\{Q, P\}$ is constant, (3.10) and (3.15) imply that

$$[\hat{V}_P, \hat{V}_Q] = \hat{V}_{\{Q,P\}} = 0$$

For all $x \in U$. on the other hand $\hat{V}_Q(Q) = \{Q, Q\} = 0$, so $\hat{V}_P$ and $\hat{V}_Q$ form a commuting, linearly independent pair of vector fields defined on U. If we set $p = P(x), q = Q(x)$, then allows us to complete p, q to form a system of local coordinates $(p, q, y^3, \dots, y^m)$ on possibly smaller neighborhood $\tilde{U} \subset U$ of $x_\circ$ with $\hat{V}_P = \partial_q, \hat{V}_q = -\partial_p$ therefore the bracket relations $\{p, q\} = 1, \{p, y^i\} = 0 = \{q, y^i\}, \quad i = 3, \dots, m$ imply that the structure matrix takes the form

$$J(p, q, y) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \tilde{J}(p, q, y) \end{pmatrix}$$

Where $\tilde{J}$ has entries $\tilde{J}^{ij} = \{y^i, y^j\}, i, j = 3, \dots, m$ finally we prove that $\tilde{J}$ is actually independent of p and q, and hence form the structure matrix of a Poisson bracket in the y variable of rank two less than that of J, from which the induction step is clear. To prove the claim, we just use the Jacobi identity and the above bracket relations, for instance

$$\frac{\partial J^{ij}}{\partial q} = \{\tilde{J}^{ij}, P\} = \left\{\{y^i, y^j\}, P\right\} = 0$$

and similarly for P.

6. The coadjoint representation:

(3.3.11)Definition:

Let G be a Lie group with Lie algebra g. The co-adjoint action of a group element $g \in G$ is the linear map $Ad^*g: g^* \to g^*$ on the dual space satisfying

$$\langle Ad^*g(\omega); w\rangle = \langle \omega; Adg^{-1}(w)\rangle \tag{3.22}$$

for all $\omega \in g^*, w \in g$. Here $\langle\ ;\ \rangle$ is the natural pairing between g and g^*, and Adg the adjoint action of G on .

- If we identity the tangent space $Tg^*|_\omega$, with g^* itself and similarly for g, then the infinitesimal generators of the co-adjoint action are determined by differentiating (3.22):

$$\langle ad^*v|_\omega; w\rangle = -\langle \omega; adv|_w\rangle = \langle \omega; [v, w]\rangle, \tag{3.23}$$

For $v, w \in g, \omega \in g^*$

(3.3.12)Theorem:

Let G be connected Lie group with co-adjoint representation $Ad^*\ G\ on\ g^*$. Then the orbits of Ad^*G are precisely the leaves of the symplectic foliation induced by the Lie Poisson bracket on g^*. Moreover, for each $g \in G$, the co- adjoint map Ad^*g is Poisson mapping on g^* preserving the leaves of the foliation.

Proof

Let $v \in g$ and consider the linear function $H(\omega) = H_v(\omega) = \langle \omega, v\rangle\ on\ g^*$. Note that for $\omega \in g^*$, the gradient $\nabla H(\omega)$, considered as an element of $T\ \ g^*|_\omega \simeq g$, is just v, itself. Using the intrinsic definition of the Lie Poisson bracket , we find

$$\hat{V}_H(F)(\omega) = \{F, H\}(\omega) = \langle \omega; [\nabla F(\omega), \nabla H(\omega)]\rangle$$

$$= \langle \omega; [\nabla F(\omega), v]\rangle = \langle \omega, adv(\nabla F(\omega))\rangle$$

$$= -\langle ad^*v(\omega); \nabla F(\omega)\rangle$$

for any $F: g^* \to \mathfrak{R}$ on the other hand,

$$\hat{V}_H(F)(\omega) = \langle \hat{V}_H|_\omega; \nabla F(\omega)\rangle$$

is uniquely determined by its action on all such functions. We conclude that Hamiltonian vector field determined by linear function $H = H_v$ coincides, up to sign, with the infinitesimal generator of the co-adjoint action determined by $v \in g$: $\hat{V}_H = -adv$. Thus the corresponding one – parameter groups satisfy

$$exp\left(t\hat{V}_H\right) = Ad^*[exp(-tv)].$$

proposition (3.3.6) and the usual connectivity arguments show that Ad^*g is a Poisson mapping for each $g \in G$.

Moreover, the subspace $\mathcal{H}|_\omega, \omega \in g^*$, is spanned by the Hamiltonian vector fields $\hat{V}_H$ corresponding to all such linear functions $H = H_v$, $v \in g$, hence $\mathcal{H}|_\omega = ad^*{}_g|_\omega$coincides with the space spanned by the corresponding infinitesimal generators $ad^*V|_\omega$ .Since $ad^*g|_\omega$ precisely the tangent space to the co-adjoint orbit of G through ω, which is connected, we immediately conclude that this co-adjoint orbit is the corresponding integral submanifold of $\mathcal{H}$.

(3.3.13) Corollary:

The orbits of the co-adjoint representation of G are even - dimensional submanifolds of g^*.

7. Hamiltonian Transformation Groups:

(3.3.14)Definition:

Let M be a Poisson manifold. Let G be Lie group with structure constants $C_{ij}^k, i, j, k = 1, \dots, r$, relative to some basis of its Lie algebra g. The functions $P_1, \dots, P_r: M \to \Re$, generate a Hamiltonian action of G on M provided their Poisson bracket satisfy the relations.

$$\left\{P_i, P_j\right\} = -\sum_{k=1}^{r} C_{ij}^r P_k, \quad i, j = 1, \dots, r$$

Note that by (3.10), the corresponding Hamiltonian vector field $\hat{V}_i = \hat{V}_{p_i}$satisfy the same commutation relation (up to sign)

$$[\hat{V}_l, \hat{V}_j] = \sum_{k=1}^{r} c_{ij}^k v_k,$$

And therefore generate a local action of G on M by theorem (1.4.11). Given a Hamiltonian system on M, will say that G is Hamiltonian symmetry group if each generate functions $P_i$is first integral $\{P_i, H\}$ =0, i= 1,…,r , Which implies that each $\hat{V}_i$ generates a one-parameter symmetry group. Any first order system of differential equations on a manifold M which admits a regular symmetry group G reduces to a first order system on the quotient manifold M/G (of course, if G is not solvable, we will not be able to reconstruct the solutions to the original system from those of the reduced system by quadrature , but we ignore this point at the moment.) In the case M is a Poisson manifold and, G a Hamiltonian group of transformations, the quotient manifold naturally inherits a Poisson structure, relative to which the reduced system is a Hamiltonian. Moreover, the degree of degeneracy of the Poisson bracket on M/G will determine how much further we can reduce the system using any distinguished functions on the quotient space.

(3.3.15) Theorem:

let G be a Hamiltonian group of transformation acting regularly on the Poisson manifold M. then the quotient manifold M/G inherits a Poisson structure so that whenever $\widetilde{F}.\ \widetilde{H} : M/G \to \mathbb{R}$ correspond to the G-invariant function $F, H : M \to \mathbb{R}$, their Poisson bracket $\{\tilde{F}, \widetilde{H}\}_{M/G}$ correspond to the G-invariant function $\{F, H\}_M$ Moreover, if G is Hamiltonian symmetry group for Hamiltonian system on M, then there is a reduced Hamiltonian system on M/G whose solutions are just the projections of the solution system on M.

PROOF:

First note that the fact that the Poisson bracket $\{F, H\}$ of two G-invariant function remains G-invariant is a simple consequence of the Jacobi identity and the connectivity of G; we find, for i = 1,…,r,

$$\hat{V}_i(\{F, H\}) = \{\{F, H\}, P_i\} = \{\{F, P_i\}, H\} + \{F, \{H, P_i\}\} = 0$$

Since F and H and invariant, verifying the infinitesimal invariance condition . thus the Poisson bracket well defined on M/G; the verification that it satisfy the properties of definition(3.1.1) is trivial .

Now if $H : M \to \mathbb{R}$ has G as Hamiltonian symmetry group, then H is automatically a G-invariant function:$\hat{V}_i(H) = \{H, P_i\} = 0$ since each P_i is by assumption, first integral. Let $\widetilde{H}: M/G \to \mathbb{R}$ be the corresponding function on the quotient manifold . to prove the corresponding Hamiltonian vector field are related , $d\pi(\tilde{F}) =$, $\pi: M \to M/G$ the natural Projection, it suffices to note that by (1.25)

$$d_\pi(\hat{V}_H)(\tilde{F})^\circ \pi = \hat{V}_H[\tilde{F} \circ \pi] = \{\ \circ \pi, H\}_M$$

For any $\widetilde{F} : M/G \to \mathbb{R}$ but this equals

$$\{\tilde{F},\tilde{H}\}_{M/G^\circ\pi} = \hat{V}_{\tilde{H}}(\tilde{F})^\circ\pi$$

By the definition of Poisson bracket on M/G , and hence proves correspondence.

Example(8)

consider the Euclidean space $\mathbb{R}^6$ with canonical coordinates (p, q) = $(p^1,p^2,p^3,q^1,q^2,q^3)$. The functions

$$p_1 = q^2p^3 - q^3 p^2, \qquad p_2 = q^3p^1 - q^1p^3, \quad p_3 = q^1p^2 - q^2p^1$$

Satisfy the bracket relations

$$\{p_1, p_2\} = p_3, \qquad \{p_2, p_3\} = p_1, \qquad \{p_3, p_1\} = p_2$$

And hence generate a Hamiltonian action of the rotation group SO (3) on $\mathbb{R}^6$, which is , in fact, given by (p, q) $\to (R_p, R_q)$, $R \in SO(3)$.this action is regular on the open subset $M = \{(p,q): p,q$ are linearly independent$\}$, with three dimensional orbits and global invariants

$$\xi(p,q) = \frac{1}{2}|p|^2 \qquad \eta(p,q) = p.q, \qquad \zeta(p,q) = \frac{1}{2}|q|^2$$

We can thus identify the quotient manifold with the subset $M/G \backsimeq \{(x,y,z): x > 0, z > 0\ y^2 < 4xz\}$ of $\mathbb{R}^3$, where $x = \xi$, $y = \eta$, $z = \zeta$ are the new coordinates.

How do we compute the reduced Poisson bracket on M/G? According to (3.12), we need only compute the basic Poisson brackets between the corresponding invariants $\xi,\eta,$....using the Poisson bracket on M itself, and re-expressing them in terms of the invariants themselves. For instance, since

$$\{\xi,\eta\} = \sum_{i=1}^{3}\left(\frac{\partial\xi}{\partial q^i}\frac{\partial\eta}{\partial p^i} - \frac{\partial\xi}{\partial p^i}\frac{\partial\eta}{\partial q^i}\right) = -\sum_{i=1}^{3}(p^i)^2 = 2\xi,$$

We have $\{x,y\}_{M/G} = -2x$ Similarly the bracket relations $\{\xi,\zeta\} = -\eta, \{\eta,\zeta\} = 2\zeta$ on M lead to the structure functions $\{x,y\}_{\mathrm{M/G}} = -2z$ On M/G . the structure matrix on M/G is thus

$$J/\mathrm{G} = \begin{pmatrix} 0 & -2x & -y \\ 2x & 0 & -2z \\ y & 2z & 0 \end{pmatrix}$$

With Poisson bracket

$$\{\tilde{F},\tilde{H}\} = -2x(\hat{F}_x\hat{H}_y - \hat{F}_y\hat{H}_x) - y(\hat{F}_x\hat{H}_z - \hat{F}_z\hat{H}_x) - 2z(\hat{F}_y\hat{H}_z - \hat{F}_z\hat{H}_y)$$

Any Hamiltonian system on M admitting angular momenta P_i as first integrals will reduce to a Hamiltonian system on M/G For example, the general kepler problem of a mass moving in a central force field with potential v(r) is such a candidate. Here the Hamiltonian function is the energy $H(p,q) = \frac{1}{2}|p|^2 + V(|q|)$. The reduced system on M/G is obtained by rewriting H in terms of the invariants and then using the given Poisson bracket to reconstruct the Hamiltonian vector field. We find reduced Hamiltonian $\widetilde{H}(x,y,z) = x + \ldots$ where $\tilde{v}(z)=$ v($\sqrt{2z}$), and reduced system

$$x_t = -y\,\tilde{v}(z) \quad y_t = 2x - 2z\tilde{v}(z). \quad z_t = y. \qquad (3.24)$$

(The reader many enjoy deriving this directly from Hamilton's equations on M.)

Now M/G is three-dimensional, so there is at least one distinguished function. This is easily seen to be $C(x,y,z) = 4xz - y^2$, which is an invariant of any Hamiltonian system M/G (In the original variables, $C = |p \times q|^2$.)The hyperboloids $4xz - y^2 = k^2$, being the level sets of C, are the leaves of the symplectic foliation, and hence we can restrict (3.24) to any such leaf Using (x, y) as coordinates , we find the fully reduced system

$$x = -\sqrt{4xz - k^2}\,\tilde{V}(z), \qquad z = -\sqrt{4xz - k^2}\,, \qquad (3.25)$$

Which is Hamiltonian relative to induced Poisson bracket $\{\tilde{F}, \widetilde{H}\} = -\sqrt{4xz - k^2}\,(\tilde{F}_x\widetilde{H}_z - \tilde{F}_z\widetilde{H}_x)$on the hyperboloid. This final two-dimensional system can be solved by method of Proposition (4.2.12). so can solve the reduced system (3.24) by quadrature. however, at this stage we cannot use the solution to integrate the original central force problem because SO(3) is not a solvable group. But, as we will soon see, this difficulty can be circumvented by an alternative approach to the reduction procedure .

8.The Momentum Map

The above approach to the reduction problem, while geometrically appealing, leaves something to be desired form a computational standpoint. The problem is that we are concentrating initially on the more complicated aspect of Hamiltonian symmetry group, namely the group transformations and ignoring the first integrals, which are also present, until after the symmetry reduction has been effected, at which point they manifest their presence as distinguish functions. A more logical approach would be to use the first integrals at the outset , restricted they system to common level set thereof , and then completing the reduction by using any residual symmetry Properties of the resulting system . this turns out to be equivalent to the above procedure, but now we stand a better chance of being able to reconstruct the solution to the original system by quadratures alone.

The first step here is to organize the first integrals furnished by a Hamiltonian group of symmetries in more natural framework. It is here that the dual to the lie algebra of symmetry group and, subsequently, the co-adjoint action makes its appearance.

(3.3.16) Definition:

let G be a Hamiltonian group of transformation acting on the Poisson manifold M, generated by the real-valued functions $P_1, \dots, P_r$. The momentum map for G is the smooth map $P: M \to g^*$ given by

$$P(x) = \sum_{i=1}^{r} p_1(x)\omega_i,$$

In which $\{\omega_1, \dots, \omega_r\}$are the dual basis to $g^*$for the basis $\{\hat{V}_1, \dots, \hat{V}_r\}$ of g relative to which the structure constants c_{ij}^k were computed.

Explains why we allowed it to take values in g^*, is its invariance (or more correctly, " equivariance ") with respect to the co-adjoint representation of G on g^*.

(3.3.17) Proposition:

let $P : M \to g^*$ be the momentum map determined by a Hamiltonian group action of G on M. Then

$$p(g.\, x) = Ad^*\, g\,(p(x)) \qquad (3.26)$$

for all $x \in M$, g$\in G$.

PROOF .

As usual, it suffices to prove the infinitesimal form of this identity which is

$$dP(\hat{v}_j|_x) = ad^*\, \hat{v}_j|_{p(x)}, \qquad x \in M \qquad (3.27)$$

For any generator $\hat{V}_j \in g$, $j = 1, \dots, r$ of G. if we identify $Tg^*|_{p(x)}$ with g^* itself, Then

$$dP(\hat{v}_j|_x) = \sum_{i=1}^{r} \hat{V}\,(P_i)\omega_i = \sum_{i=1}^{r}\{P_i, P_j\}(x)\omega_i = -c_{ij}^k P_k \omega_i$$

Cf. (1.25), (3.6). by (3.23) this expression is the same as the right-hand

To prove (3.26), we note that if g= $\exp(\varepsilon\hat{v}_j)$ and we differentiate with respect to ε ,then we recover (3.27) at $\tilde{x} = \exp(\varepsilon\hat{v}_j)x$ since this holds at all $\tilde{x}$ the usual connectivity arguments prove that (3.26) holds in general.

Example (9):

consider the Hamiltonian action of SO (3) on $\mathbb{R}^6$ presented in Example (8). The momentum map is

$$P(p,q) = (q^2p^3 - q^3\,p^2)\omega_1 + (q^3p^1 - \,q^1p^3)\omega_2 + (q^1p^2 - q^2p^1)\omega_3\,,$$

Where $\{\omega_1, \omega_2, \omega_3\}$ are the basis of so $(3)^*$ of Example (5). note that if we identify so$(3)^*$ with $\mathbb{R}^3$, $P(p,q) = q \times p$ is the same as the cross product of vector in $\mathbb{R}^3$. In this case. SO(3) acts on so $(3)^*$ by rotations, and the equivariance of the momentum map is just a restatement of the rotational invariance of the cross product: $R(q,p) = R(q) \times R(p)$ for $R \in$ SO(3) .

Now , as remarket earlier, any Hamiltonian system with G as a Hamiltonian symmetry group naturally restricts to system of ordinary differential equations on the common level set $\{P_i(x) = c_i\}$ of the given first integrals.

Note that these common level sets of momentum map, denoted $\varphi_\alpha = \{x: P(x) = \alpha\}$ where $\alpha = \sum c_i\omega_i \in g^*$. Moreover, the reduced system will automatically remain invariant under the residual symmetry group

$$G_\propto \equiv \{\text{g} \in G\text{:g}.\varphi_\alpha \subset \varphi_\alpha\}$$

Of group elements leaving the chosen level set invariant, there is any easy characterization of this residual group.

(3.3.18)Proposition:

let $P: M \rightarrow g^*$ be the momentum map associated with a Hamiltonian group action. Then the residual symmetry group of a level set $\varphi_\alpha = \{x: P(x) = \alpha\}$ is the isotropy subgroup of element $\propto \in g^*$:

$$G_\propto = \{\text{g} \in \text{G: Ad}^*\,\text{g}(\alpha) = \alpha\}.$$

Moreover if g$\in \mathsf{G}_\propto$ has the property that it takes one point $x \in \varphi_\propto$ to point g. $x \in \varphi_\propto$, then property for all $x \in \varphi_\propto$

Proof . By definition, g$\in \mathsf{G}_\propto$ if and only if P(g. x)= α whenever P(x) α. But, by the equivariance of P,

α = P(g. x)= Ad* **g**$(P(x))$ = Ad* g(α),

So g is in the isotropy subgroup of α. The second statement easily follows from this identity.

Not that the residual Lie algebra corresponding to $G_\propto$ is the isotropy subgroup $g_\propto \equiv \{V \in g$:ad$^*V\,|_\alpha$ =0}, which is readily computable. In particular, the dimension of $G_\propto$can be computed as the dimension of its Lie algebra $g_\propto$. For instance, if $G_\propto$ is an abelian Lie group, its co-adjoint representation is trivial, Ad* g(α) = α for all g$\in G$, $\alpha \in g^*$, hence $G_\alpha = G$ For every α. Therefore any Hamiltonian system admitting an Aeolian Hamiltonian symmetry group remains invariant under the full group , even on restriction to a common level set $\varphi_\propto$ This will imply that we can always reduce such a system in order by $2r$, twice the dimension. on the group. As a second example, consider the two-parameter solvable group of Example 6.40. Here there momentum map is

$P(p, q, \tilde{p},\ \tilde{q}) = p\omega_1 + (pq + \tilde{p})\omega_2,$

Where $\{\omega_1, \omega_2\}$ are a basis of g^*dual to the basis {v, w} of g. The co-adjoint representation of g = exp($\varepsilon_1 V + \varepsilon_2 w$) is found to be

Ad* g$(c_1\omega_1 + c_2\omega_2 = e^{-\varepsilon_2}\, c_1\omega_1 + (\varepsilon_1\varepsilon_2^{-1}(e^{-\varepsilon_2} - 1)c_1 + c_2)\omega_2$

(with appropriate limiting values if $\varepsilon_2 = 0$). Thus the isotropy subgroup of $\propto = c_1\omega_1 + c_2\omega_2$ is just {e} unless $c_1 = 0$, in which case it is all of G. Thus we expect that the restriction of Hamiltonian system with symmetry group G to a level set $\varphi_\propto$ = $\{p = c_1\,, pq + \tilde{p} = c_2\}$ will retain no residual symmetry group unless $c_1 = 0$, in which case the entire group G will remain . This is precisely what we observed.

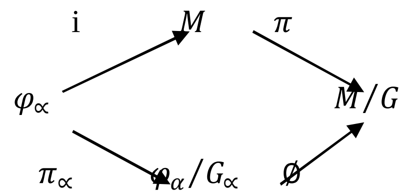
Once we have restricted the Hamiltonian system to the level set $\varphi$the idea is then to utilize the methods of Section5 in chapter2 to reduce further using the residual symmetry group $G_\propto$. Under certain regularity assumptions on the group action, the quotient manifold$\varphi_\propto / G_\propto$, on which the fully reduced system will live, has a natural identification as a Poisson submanifold of M/G. Thus the fully reduced system inherits a Hamiltonian structure itself .

In particular , if the residual group $G_\propto$ is solvable (rather than G itself being solvable) we can reconstruct the solution to the original system on $\varphi_\propto/G_\propto$ the general result follows:

(3.3.19) Theorem :

let M be a Poisson manifold and G a regular Hamiltonian group of transformations. Let $\propto \in g^*$. Assume that the momentum map $P: M \rightarrow g^*$ is of maximal rank everywhere on the level set $\varphi_\propto = P^{-1}\{\alpha\}$, and that the residual symmetry group G_α acts regularly on the submanifold

$\varphi_\propto$. Then there is a natural immersion $\emptyset$ making φ_α/G_α into Poisson submanifold of M/G IS such a way that the diagram



(6.39)

Commutes (Here $\pi\ and\ \pi_\alpha$ are the natural projection and i the immersion realizing $\varphi_\propto$ as submanifold of M.) Moreover, any Hamiltonian system on M which admits G as a Hamiltonian symmetry group naturally restricts to systems on the other space in (6.39), which are Hamiltonian on M/G and$\varphi_\propto/G_\propto$, and which are related by the appropriate maps. In particular, we obtain a Hamiltonian system on $\varphi_\propto/G_\propto$ by first restricting to $\varphi_\propto$ and

Then projecting using π_α.

Proof :

Assume G is a global group of transformations, although the proof is easily modified to incorporate the local case. According to the diagram if $z = \pi_\propto(x) \in \varphi_\propto/G_\propto$ then we should define $\emptyset(z) = \pi(x) \in M/G$. Note that $\pi_\alpha(x) = \pi_\alpha(\tilde{x})$ if and only if x= g . $\tilde{x}$ for some g ∈ $G_\propto$, but this means $\pi(x) = \pi(\tilde{x})$ and hence $\emptyset$ is well defined. Similarly, $\emptyset$ is one-to-one since if $x, \tilde{x} \in \varphi_\propto$ and $\pi(x) = \pi(\tilde{x})$, then $x =$ g . $\tilde{x}$ for some g ∈ G; according to Proposition (3.3.18), g ∈ $G_\propto$, and hence $\pi_\alpha(x) = \pi_\alpha(\tilde{x})$. Finally, $\emptyset$ is an immersion, meaning $d\emptyset$ has maximal rank everywhere, since $d\emptyset \circ d\pi_\alpha = d\pi \circ di$, and by Proposition (3.3.18)

Ker $d\pi_\alpha = g_\alpha = g \cap T\varphi_\propto$ =ker($d\pi \circ di$).

Let $\widetilde{H} : M/G \to \mathbb{R}$ correspond to the G-invariant function $H : M \to \mathbb{R}$, so by theorem (3.3.14) the corresponding Hamiltonian systems are related : $\tilde{V}_{\widetilde{H}} = d\pi(\tilde{V}_H)$. We also know that $\tilde{V}_H$ is everywhere tangent to the level set $\varphi_\propto$ And hence there is a reduced vector field $\check{V}$ on $\varphi_\propto$ with $\tilde{V}_H = di(\tilde{V})$ there Moreover, as $\tilde{V}_H$ has G as a symmetry group …. Retain $G_\propto$ as a residual symmetry group and there is thus a well-defined vector field V^*= $d\pi_\alpha(V^*)$on the quotient manifold $\varphi_\propto/G_\propto$ furthermore , this vector field agrees with the restriction of $\tilde{v}_{\widetilde{H}}$ to submanifold $\emptyset(\varphi_\propto/G_\propto)$ since

$$d\phi(V^*) = d\phi \circ d\pi_\propto(V) = d\pi \circ di(\tilde{V}) = d\pi(\hat{V}_{\widetilde{H}}) = \hat{V}_{\widetilde{H}}$$

there.

This last argument proves that every Hamiltonian vector field on

M/G is everywhere tangent to $\phi(\varphi_\propto/G_\propto)$. Proposition (3.3.9)then implies that ϕ makes $\varphi_\propto/G_\propto$ into a Poisson submanifold of M/G and, moreover, the restriction of Hamiltonian vector field $\hat{V}_{\tilde{H}}$ on M/G to$\varphi_\propto/G_\propto$(i.e. V^*) is with Hamiltonian respect to the induced Poisson restriction. This completes the proof of the theorem and hence the reduction procedure.

If M is symplectic, then it is not true that M/G is necessarily symplectic However, it is possible to show that the submanifolds $\varphi/G_\propto$form the leaves of the symplectic foliation of M/G.

Example (9).

consider the abelian Hamiltonian symmetry group G acting on $\mathfrak{R}^6$, with canonical coordinates $(p,q) = (p^1,p^2,p^3,q^1,q^2,q^3)$, generated by the functions $P = p^3$ $Q = q^1p^2 - q^2p^1$ .The corresponding Hamiltonian vector fields

$$V_1 = \frac{\partial}{\partial q^3} \quad \text{and} \quad v_2 = p^1\frac{\partial}{\partial p^2} - p^2\frac{\partial}{\partial p^1} + q^1\frac{\partial}{\partial q^2} - q^2\frac{\partial}{\partial q^1}$$

Generate a two-parameter abelian group of transformation. Any Hamiltonian function of the form $H(\rho,\sigma,\gamma,\xi,\tau)$, where $\rho = \sqrt{(q^1)^2 + (q^2)^2}$ $\sigma = \sqrt{(p^1)^2 + (p^2)^2}$ $\gamma = q^1p^2 - q^2p^1$ $\xi = p^3$, has G as a symmetry group; In particular $H = \frac{1}{2}|p|^2 + V(\rho)$, a cylindrically energy potential is such a function.

The method of proposition (3.3.19) will allow us to reduce the order of such a Hamiltonian system by four.(and H does not depend on τ , we can integrate the entire system by quadratures .) First we restrict to the level set $\varphi = \{P = \xi, \quad Q = \gamma\}$ for ξ , γ constant. If we use the $p = (\sigma \cos\psi, \sigma\sin\psi, \xi)$, for q and p, then

$$\gamma = \rho\sigma\sin(\psi - \theta) = \rho\sigma\sin\phi,$$

Where $\phi = \psi - \theta$. In term of the variables ρ, θ, ϕ, z, the Hamiltonian system , when restricted to φ, takes the from

$$\rho_\tau = \cos\phi\,.\mathrm{H}_\sigma \qquad \phi_1 = \sin\phi\left(\sigma^{-1}H_\rho - \rho^{-1}H_\sigma\right) \qquad (3.28a)$$

$$\theta_\tau = \rho^{-1}\sin\phi H_\sigma + H_\gamma \qquad z_\tau = \mathrm{H}_\xi \qquad (3.28b)$$

The subscripts on H denote partial derivatives. These are also designed so φ ,$V_1 = \partial_z$, $V_2 = \partial_\theta$ Theorem (3.3.19) guarantees that (3.28a), (3.28b) is invariant under the reduced symmetry group of φ Which owing abelian character of G, is all of G itself. This is reflected in the fact that neither z nor θ appears explicitly on the right-hand sides of (3.28a). Thus once we have determined $\rho(t)$ and $\phi(t)$ to solve the first two equations, $\theta(t)$ and $z(t)$ are determined by quadrature.

Moreover, Theorem (3.3.19) says that (3.28a) forms a Hamiltonian system in its own right. Fixing γ and ξ, let

$$\widehat{H}(\rho, \phi, t) = H(\rho, \gamma/(\rho \sin\phi), \gamma, \xi, t)$$

be the reduced Hamiltonian . Note that

$$\{\rho, \phi\} = -\gamma\rho^{-1}\sigma^{-2} = -\gamma^{-1}\rho \sin^2\phi.$$

An easy computation using the chain rule shows that (3.28a)is the same as

$$\rho_t = -\gamma^{-1}\rho \sin^2\phi \grave{H}_\phi, \qquad \phi_t = \gamma^{-1}\rho \sin^2\phi \grave{H}_\phi, \qquad (3.29)$$

Which is indeed Hamiltonian. In particular, if H(and hence $\widehat{H}$) is independent of t we can, in principle, integrate (3.29) by quadrature and hence solve the original system. (In practice, however, even for simple functions H, the intervening algebraic manipulations may prove to be overly complex.)

In general, if a Hamiltonian system is invariant under an r-parameter abelian Hamiltonian symmetry group, one can reduce the order by $2r$. This is because the residual symmetry group is always the entire abelian group itself owing to the triviality of the co-adjoint action. A $2n$ –th order Hamiltonian system with an n-parameter abelian Hamiltonian symmetry group, or, equivalently possessing n first integrals $P_1(x), \ldots, P_n(x)$ which are in involution

$$\{P_i, P_j\} = 0 \text{ for all } i, j,$$

Is called a completely integrable Hamiltonian system since, in principle, it's solutions can be determined by quadrature alone. Actually, much more can be said about such completely integrable system and the topic forms a significant chapter in classical theory of Hamiltonian mechanics.

Chapter Four

Integrability of Hamiltonian Systems

Chapter Four

Integrability of Hamiltonian Systems

(4.1) Introduction :

In this chapter we consider the notion of complete integrability of Hamiltonian system . In spite of the recent development of the methods and technique of complete inegrability that have been invented in the last three decades (i.e the method of Lax pairs, the bi-Hamiltonian method etc -) the classical 19th century approach to complete integrability via the Hamilton - Jacobi method of separation of variables is being revived.

One of the main impetuses for the renewed interest in this method was Cartan's discovery that geodesic equation in Kerr black holes space-time can be integrated by separation of variables .Remarkably , in the course of the last ten years ,this classical method has been effectively linked with the method of the lax representation and the bi-Hamiltonian method , thus leading to new theories in the area of integrable Hamiltonian system.

The key idea behind the method of separation of variables is to see k - set of special coordinates $q \coloneqq (q^1, \dots, q^n)$ in which the corresponding Hamilton Jacobi partial differential equation

$$\tfrac{1}{2} g^{ij}\, \partial_i w\, \partial_j w + V = E \tag{4.1}$$

admits a complete integral of the form

$$w(q, c) = w_1(q^1, c) + \cdots + w_n(q^n, c) \tag{4.2}$$

The above Hamilton-Jacobi equation (4.1) in fact corresponds to the Hamiltonian

$$H_0 = \tfrac{1}{2} g^{ij}(q) p_i\, p_j + V(q),\ \ i, j = 1, \dots, n \tag{4.3}$$

where $c = (c_1, \dots c_n)$ are the constants of integration . These constants are the n first integrals in involution with respect to $\omega_\circ = \sum_{i=1}^{n} dp^i \wedge dq_i$ ($\omega_\circ$is canonical symplectic structure) or $p_\circ = \sum_{i=1}^{n} \partial_i \wedge \partial^i$ ($p_\circ$ is Poisson bi-vector) that guarantee the complete integrability of (4.3).A complete integral w can be interpreted as an n-dimensional Lagrangian submanifold on M lying on the level surface H_0=const. The coordinates $(q^1, \dots, q^n)$ in (4.2) are called separable coordinates. Moreover, if the metric g of (4.3) is diagonal in these coordinates, they are also said to be orthogonal and the system defined by the Hamiltonian (4.3) is said to be orthogonally separable. In what follows , we concentrate our attention on this type of separable Hamiltonian system. We note that the orthogonal case has been extensively studied in the past in numerous articles by such famous scholars as Dall'Acqua, Eisenhart, Levi-Civita, Ricci, Stackel and others. Major advances in the area have been achieved in recent years by Bbenenti, Klnins and Miller Shapovalov , as well as many others.

The main objective of this chapter is to combine the theory of orthogonally separable Hamiltonian system and the method of moving frames .The method has been extensively studies and successfully applied under different names (for instance, "the method of quasi-coordinates'', '' … non-coordinate basis'',"…orthogonal enuples'') in such areas of mathematics and physics as differential geometry, general relativity and theory of Lie groups. Introduced by Darboux and developed by Cartan , the method has been chiefly used in two cases: As an alternative method to the classical tensor calculus to avoid ,in Cartan's , the "debauch d'indices" and as an effective tool to study geometrical invariants of submanifolds under the action of transformation Lie groups . In the present work, we are mainly concerned with the former case, when the application of the moving frames method can significantly alleviate the complications of dealing with tensorial geometrical quantities, in this chapter defined in a Riemannian manifold $(\widetilde{M}, g)$. We note that an equivalent version of the method of moving frames based on the frame of vectors, unlike Cartan's approach via co-vectors, was effectively used by Eisenhart .

(4.2) Hamiltonian Systems

(4.2.1) Definition:

Let $H(x,p)$ and $L(x,p)$ be differentiable functions of their arguments for x and $p \in \Re^n$ the Poisson bracket of H with L,

$$\{H, L\} = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p^i} \frac{\partial L}{\partial x^i} - \frac{\partial H}{\partial x^i} \frac{\partial L}{\partial p^i} \right)$$

A quantity L is called a first integral of Hamiltonian system if it's a constant of motion (i.e if i=0 under the flow implied by Hamilton's equation).

(4.2.2) Corollary:

The quantity L is a first integral of Hamiltonian system with Hamiltonian H if $\{H, L\} = 0$.

(4.2.3) Definition:

A Hamiltonian system is said to be completely integrable if it has n first integrals (including the Hamiltonian itself), where n is the number of degrees of freedom .

(4.2.4) Proposition:

A function $P(x,t)$ is first integral for the Hamiltonian system if and only if

$$\frac{\partial P}{\partial t} + \{P, H\} = 0 \qquad (4.4)$$

for all x , t. Particular, a time - independent function $P(x)$ is a first integral if and only if $\{P, H\} = 0$ every where.

Proof

Let $\widehat{V}_H$ be the Hamiltonian vector field , then if $x(t)$ is any solution to Hamitlon's equation,

$$\frac{d}{dt}\{P(x(t), t)\} = \frac{\partial P}{\partial t}(x(t), t) + \widehat{V}_H(P)(x(t), t),$$

Thus $\frac{dP}{dt} = 0$ along solution if and only if (4.4) hold everywhere.

(4.2.5) Corollary:

If $\dot{x}_i = J\nabla H$ is any Hamiltonian system with time - independent Hamiltonian function $H(x)$,then $H(x)$ itself is automatically a first integral.

(4.2.6) Corollary:

If $\dot{x}_i = J\nabla H$ is a Hamiltonian system , then any distinguished function $C(x)$ for the Poisson bracket determined by J automatically a first integral .

1. ***Hamiltonian symmetry groups:***

(4.2.7) Definition:

The first integral arise from variational symmetry groups; For Hamiltonian systems this role is played by the one - parameter Hamiltonian symmetry groups whose infinitesimal generators (in evolutionary form) are Hamiltonian vector fields .Any first integral leads to such a symmetry group.

(4.2.8) Proposition:

Let $p(x, t)$ be a first integral of a Hamiltonian system. Then the Hamiltonian vector field $\hat{V}_P$ determined by P generates a one parameter symmetry group of the system.

Proof

Note first that since the structure matrix $J(x)$ does not depend on t the Hamiltonian vector field associated with $\frac{\partial P}{\partial t}$ is just t-derivative $\frac{\partial \hat{V}_P}{\partial t}$ of that associated with P. Thus the Hamiltonian vector field associated with the combination $\frac{\partial P}{\partial t} + \{P, H\}$ occurring in (4.4) using (3.8) in chapter 3

$$\frac{\partial \hat{V}_P}{\partial t} + \{\hat{V}_H, \hat{V}_P\} .$$

If P is a first integral, this last vector field vanishes, which is just condition $\frac{\partial \hat{V}_Q}{\partial t} + \{\hat{V}_P, \hat{V}_Q\} = 0$ that $\hat{V}_P$ generate a symmetry group.

In particular ,if $H(x)$ is time - independent, the associated symmetry group is generated by $\hat{V}_H$ which is equivalent to the generator ∂_t of the symmetry group of time translations reflecting the autonomy of the Hamiltonian system .A distinguished function $C(x)$, the corresponding symmetry is trivial : $\hat{V}_c \equiv 0$

Example (1):

Consider the equation of a harmonic oscillator $p_t = -q,\ q_t = p$ which form a Hamiltonian system on $M = \Re^2$ relative to the canonical Poisson bracket. The Hamiltonian function $H(q, p) = \frac{1}{2}(p^2 + q^2)$ is thus a first integral, reflecting the fact that the solution move on the circles $p^2 + q^2$ = constant.

(4.2.9) Corollary:

Every Hamiltonian symmetry group corresponds directly to first integral.

(4.2.10) Theorem:

A vector w generates a Hamiltonian symmetry group of a Hamiltonian system of ordinary differential equations if and only if there exist a first integral $P(x,t)$ so that $w = \widehat{V}_p$ is the corresponding Hamiltonian vector field. A second function $\widetilde{P}(x,t)$ determines the same Hamiltonian symmetry if and only if $\widetilde{P} = P + C$ for some time - dependent distinguished function $\tilde{C}(x,t)$.

Proof:

The second statement follows immediately from definition (3.2.4) in chapter 3 of a distinguished function applied to the difference $P - \widetilde{P}$ .To prove the first part, let $w = \widehat{V}_{\widetilde{P}}$ for some function $\widetilde{P}(x,t)$. The symmetry condition says that ($\frac{\partial V_Q}{\partial t} + \{V_P, V_Q\} = 0$) implies that the Hamiltonian vector field associated with the function $\frac{\partial \widetilde{P}}{\partial t} + \{\widetilde{P}, H\}$ vanishes everywhere, and hence this combination must be a time - dependent distinguished function $\tilde{C}(x,t)$:

$$\frac{d\widetilde{P}}{dt} = \frac{\partial \widetilde{P}}{\partial t} + \{\widetilde{P}, H\} = \widetilde{C}.$$

Set $C(x,t) = \int_0^t \tilde{C}(x,\tau)d\tau$, so that C is also distinguished. Moreover for solution $x(t)$ of the Hamiltonian system ,

$$\frac{dC}{dt} = \frac{\partial C}{\partial t} + \{C, H\} = \tilde{C}.$$

The modified function $P = \widetilde{P} - C$ has the same Hamiltonian vector field , $\widetilde{V}_P = w$,and provides a first integral $\frac{dP}{dt} = 0$ on solution.

2. Reduction of order in Hamiltonian systems:

(4.2.11) Theorem:

Suppose $\widehat{V}_P \neq 0$ generates a Hamiltonian symmetry group of the Hamiltonian system $x^{\cdot} = J\nabla H$ corresponding to time - independent first integral $P(x)$. Then there is reduced Hamiltonian system involving two fewer variables with the property that every solution of the original system can be determined using one quadrature from those of reduced system .

Proof:

Let $p = P(x), q = Q(x)$, $y = (y^1, \ldots, y^{m-1}) = Y(x)$ which straighten out the symmetry , so $\widehat{V}_P = \partial q$ in the (p, q, y) - coordinates . In terms of these coordinates the structure matrix has the form

$$J(p, q, y) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & a \\ 0 & -a^T & \tilde{J} \end{bmatrix}$$

Where (p, q, y) is a row vector of length m-2 and $\tilde{J}(p, y)$ is an (m-2) ×(m-2) skew - symmetric matrix , which is independent of q, and for each fixed value of p is the structure matrix for a Poisson bracket in the y variables. (if $= y^1, \ldots, y^{m-2}$) are chosen as flat coordinates as in the proof of Darboux theorem then $a = 0$ and $\tilde{J}(y)$ is independent of p also ,as we saw earlier . However, to effect the reduction procedure this is not necessary, and indeed may be impractical to achieve, the proofs of the above statements on the form of the structure matrix follow as in the “flat” case.

The reduced system will be Hamiltonian with respect to the reduced structure matrix $\tilde{J}(p, y)$ for any fixed value of the first integral $P = P(x)$. Note that in terms of the (p, q, y) coordinates.

$$0 = \{P, H\} = -\tilde{V}_p(H) = -\partial H/\partial q\,,$$

Hence $\tilde{J}H = H(p, y)$ also only depends on p and y. Therefore Hamilton's equations takes the form

$$\frac{dp}{dt} = 0 \qquad (4.5)$$

$$\frac{dq}{dt} = -\frac{\partial H}{\partial p} + \sum_{j=1}^{m-2} a^j(p, y)\frac{\partial H}{\partial y^j}\quad , \qquad (4.6)$$

$$\frac{dy^i}{dt} = \sum_{j=1}^{m-2} \tilde{J}^{ij}(p, y)\frac{\partial H}{\partial y^j}\,,\ i = 1, \ldots, m-2 \qquad (4.7)$$

The first equation says that p is constant (as should be). Fixing a value of p, we see that the (m-2) equations (4.7) form a Hamiltonian system relative to reduced structure matrix $\tilde{j}(p, y)$ and the Hamiltonian function $H(p, y)$; this is the reduced system referred to in the statement of the theorem. Finally (4.6), which governs the time evolution of the remaining coordinate q, can be integrated by a single quadrature once we know the solution to the reduced system (4.7) since the right- hand side does not depend on q.

Example (2):

Let $M = \Re^4$ with canonical Poisson bracket and consider a Hamiltonian function of the form

$$H(p_1, p_2, q_1, q_2) = \tfrac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2).$$

The corresponding Hamiltonian system

$$\frac{dq_1}{dt} = p_1\,,\ \ \frac{dq_2}{dt} = p_2,\ \frac{dp_1}{dt} = V^1(q_1, q_2), \frac{dp_2}{dt} = V^1(q_1, q_2) \qquad (4.8)$$

Determines the motion of two particles of unit mass on a line whose interaction comes from a potential $V(r)$ depending on their relative displacements .This system admits an obvious translational invariance $V = \partial q_1 + \partial q_2$; the corresponding first integral is the linear momentum p_1, p_2. According to the theorem above we can reduce the order of the system by two if we introduce new coordinates.

$$p = p_1 + p_{2,} \quad q = q_{1,} \quad y = p_1, \quad r = q_1 - q_2,$$

Which straighten out $V = \partial q$. In these variables, the Hamiltonian function is

$$H(p, y, r) = y^2 - py + \tfrac{1}{2}p^2 + V(r), \tag{4.9}$$

and the Poisson bracket is

$$\{F, H\} = \frac{\partial F}{\partial q}\frac{\partial H}{\partial y} + \frac{\partial F}{\partial r}\frac{\partial H}{\partial y} + \frac{\partial F}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial F}{\partial y}\frac{\partial H}{\partial q} - \frac{\partial F}{\partial y}\frac{\partial H}{\partial r} - \frac{\partial F}{\partial p}\frac{\partial H}{\partial q}$$

Further, the Hamiltonian system splits into

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q} = 0, \quad \frac{dq}{dt} = \frac{\partial H}{\partial p} + \frac{\partial H}{\partial y} = -y,$$

and

$$\frac{dy}{dt} = -\frac{\partial H}{\partial q} - \frac{\partial H}{\partial r} = -V^1(r), \quad \frac{dr}{dt} = \frac{\partial H}{\partial y} = 2y - p \tag{4.10}$$

The solution to the first pair

$$p = a, \quad q = \int y(t)dt + b,$$

(a , b are constant) can be determined from the solutions to the second pair (4.10). These form a reduced Hamiltonian system relative to the reduced Poisson bracket $\{\tilde{F}, \tilde{H}\} = \tilde{F}_r\tilde{H}_y - \tilde{F}_y\tilde{H}_r$ for function of y and r, with the energy (4.9) obtained by fixing $p = a$. presently, we will see how the two-dimensional system (4.10) can be explicitly integrated ,thereby solving the original two-particle system (4.8).

(4.2.12) Proposition:

Let $\dot{x} = J\nabla H$ be a Hamiltonian system in which $H(x)$ does not depend on t. Then there is a reduced, time –dependent Hamiltonian system in two fewer variables, from whose solutions those of the original system can be found by quadrature.

Proof :

The reduction in order by two per se is easy. First, since H is constant, we can restrict to a level set $H(x) = c$, reducing the order by one. Furthermore, the resulting system remains autonomous and so can be reduced in order once more using the method in example (2.67) the problem is that unless we choose our coordinates more astutely, the system resulting from this reduction will not be of Hamiltonian form in any obvious way. The easiest way to proceed is to first introduce the coordinates (p, q, y), relative to which the original system takes the form

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}, \qquad \frac{dq}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dy^i}{dt} = \sum_{j=1}^{m-2} \tilde{J}^{ij}(y)\,\frac{\partial H}{\partial y^i}, \quad i = 1, \dots, m-2.$$

Assume that $\partial H/\partial p \neq 0$, so that we can solve the equation $w = H(p, q, y)$ locally for $p = K(w, q, y)$. (If $\partial H/\partial p = 0$ everywhere, q is a first integral and we can use the previous reduction procedure). We take t, w and y to be the new dependent variables and q the new independent variable, in terms of which the system takes the form

$$\frac{dt}{dq} = \frac{1}{\partial H/\partial p} = \frac{\partial K}{\partial w}, \qquad \frac{dw}{dq} = 0, \tag{4.11}$$

$$\frac{dy^i}{dt} = \sum_{j=1}^{m-2} \tilde{J}^{ij}(y) \frac{\partial H/\partial y^i}{\partial y^i/\partial p} = \sum_{j=1}^{m-2} \tilde{J}^{ij}(y) \frac{\partial K}{\partial y^i} \tag{4.12}$$

The system (4.11) is Hamiltonian using the reduced Poisson bracket corresponding value of w, once we have solved (4.12) we can determine the remaining variable t(q) from (12) by a single quadrature. This completes the procedure.

Example(3):

In case of an autonomous Hamiltonian system

$$\dot{q} = \frac{\partial \mathrm{H}}{\partial \mathrm{p}}, \qquad \dot{p} = -\frac{\partial H}{\partial q},$$

in the plane, we can use this method to explicitly integrate it. We first solve $w = H(p, q)$ for one of the coordinates, say p, in terms of q and w, which is constant. The first equation, then, leaves an autonomous equation for q, which we can solve by quadrature. For example, in the case of a single pendulum $H(p, q) = \frac{1}{2}p^2 + (1 - \cos q)$, so on the level curve $H = \omega + 1$, $p = \sqrt{2(\omega + \cos q)}$. The remaining equation

$$\frac{dq}{dt} = p = \sqrt{2(\omega + \cos q)}$$

Can be solved in terms of Jacobi elliptic functions

$$q(t) = 2\sin^{-1}\{\mathrm{sn}(k^{-1}(t+\delta), k)\},$$

where sn has modulus $k = \sqrt{2/(\omega+1)}$.

Similarly, in the case of the two-particle system on the line from example (2), setting $H(y,r) = \omega + \frac{1}{4}p^2$, we find

$$y = \frac{1}{2}p \pm \sqrt{\omega - V(r)}.$$

Thus we recover the solution just by integrating

$$\frac{dr}{dt} = 2y - p = \pm 2\sqrt{\omega - V(r)}.$$

Example(4):

Consider the equations of rigid body motion (4.20a), which were realized as a Hamiltonian system on $so(3)^*$. The distinguished function $C(u) = |u|^2$ naturally reduces the order by one by restriction to a level set or co-adjoint orbit. Provided the moments of inertia I_1, I_2, I_3 are not all equal, the Hamiltonian itself provides a second independent first integral. We conclude that the integral curves of this the Hamiltonian vector field are determined by the intersection of a sphere $\{C(u) = |u|^2 = c\}$ and an ellipsoid $\{H(u) = \omega\}$ forming the common level set of these two first integrals. The explicit solutions can be determined by eliminating two of the variables, say u^2 and u^3, from the pair of equations $C(u) = c, H(u) = \omega$. Proposition (2.4.12) then guarantees that the one remaining equation for $u^1 = y$ is autonomous, and hence can be integrated. It turns out to be of the form

$$\frac{dy}{dt} = \sqrt{\alpha(\beta^2 - y^2)(\gamma^2 - y)},$$

and hence the solutions can be written in terms of elliptic functions.

(4.3) The geometrical machinery:

This section will be devoted to the geometrical setup needed for the rest of the chapter. Mainly we shall develop the theory of moving frame due to Cartan. The essence of the method of moving frames can be briefly described as follows .

In a given n-dimensional pseudo-Riemannian manifold $(\widetilde{M}, g)$ at each point $p \in \widetilde{M}$ we replace for the natural basis of the cotangent space $T\widetilde{M}_p^*$: $(dq^1, \dots, dq^n)$arising from a coordinate system $(q^1, \dots, q^n)$by a basis of n pointwise linearly independent one-forms (co-vector) $E^1, \dots, E^n \in T\widetilde{M}_p^*$, that can be adapted to the geometric situation. In the considerations that follow the natural choice is that in which the metric tensor g takes itis algebraic canonical form. In other words, with respect to the basis $E^a, a = 1, \dots, n$, we have .

$$g_{ab} = \mathrm{diag}(1, \dots, 1, -1, \dots, -1) \tag{4.13}$$

The co-frame of one-frames $E^1, \dots, E^n$ is said to be rigid in this case. One can new proceed to study the relations between the one-forms $E^a \in T\widetilde{M}_p^*$ their exterior derivatives $dE^a$and the dual basis $(E_1, \dots, E_n)$ of the tangent space $T\widetilde{M}_p$ independently of local coordinates. Thus, we can consider an open set $A \ni p$ and (orthonormal) moving co-frame $E^1, \dots, E^n$ of one-forms defined in *A* for which the metric tensor g takes the form (4.13). We note that the elements of the moving co-frame E^a and their counterparts E_a are connected with the natural basis associated to local coordinates $(q^1, \dots, q^n)$ about $p \in A$ as follows

$$E^a = h_i^a dq^i, \quad E_a = h_i^a \frac{\partial}{\partial q^i}, \tag{4.14}$$

The structure functions C_{ab}^c are defined by

$$[E_a, E_b] = C_{ab}^c E_c \text{ or } dE^a = -\frac{1}{2} C_{bc}^a E^b \wedge E^c \tag{4.15}$$

Now by (4.14) $C_{ab}^c = h_i^c (h_{a^j} h_{b,j^i} - h_{b^j} h_{a,j^i}), a, b, c, i, j = 1, \dots, n$. Here and below, we denote the usual partial derivative with respect to the coordinate. We introduce the connection coefficients Γ_{jk}^i corresponding to the Levi-Civita connection ∇ associated to g_{ab} as follows:

$$\nabla_{E_c} E_b = \Gamma_{ab}^c E_c, \quad \nabla_{E_c} E^b = -\Gamma_{cd}^b E^d$$

The vanishing of the torsion tensor of ∇ is expressed by

$$\Gamma_{bc}^a - \Gamma_{cb}^a - C_{bc}^a = 0 \tag{4.16}$$

while the curvature tensor of ∇ is given by

$$R_{bcd}^a = E_c \Gamma_{db}^a + \Gamma_{db}^e \Gamma_{ce}^a - E_d \Gamma_{cb}^a - \Gamma_{cb}^e \Gamma_{de}^a - C_{cd}^e \Gamma_{eb}^a \tag{4.17}$$

We now define a one - form valued matrix ω_b^a called the connection one - form by

$$\omega_b^a \coloneqq \Gamma_{cb}^a E^c. \tag{4.18}$$

Further, we define

$$\omega_{ab} \coloneqq g_{ac} \omega_b^c .$$

On account of the above connection one-forms, ω_{ab} are obviously skew – symmetric . The condition (4.16) and the definition (4.17)may be expressed in the language of differential forms as

$$dE^a + \omega^a_b \wedge E^b = 0 \tag{4.19}$$

and

$$d\omega^a_b + \omega^a_c \wedge \omega^c_b = \Theta^a_b, \tag{4.20}$$

Where $\wedge$ is exterior multiplication, d the exterior derivative an $\Theta^a_b := \frac{1}{2} R^a_{bcd} E^c \wedge E^d$ the curvature two-form. Taking the exterior derivative of (4.19) and (4.20) yields the first and second Bianchi identities , respectively

$$\Theta^a_b \wedge E^b = 0 \tag{4.21}$$

and

$$d\Theta^a_b + \omega^a_c \wedge \Theta^c_b - \Theta^a_c \wedge \omega^c_b = 0 \tag{4.22}$$

Finally, the equations satisfied by a valence two, symmetric, covariant killing tensor K can be written in frame components as

$$K_{(ab;c)} = 0 \tag{4.23}$$

Where ; denotes the covariant derivative defined by

$$K_{ab;c} := E_c K_{ab} - K_{db} \Gamma^d_{ca} - K_{ad} \Gamma^d_{cb} \tag{4.24}$$

This is all the geometric machinery that we need in the forthcoming sections to study integrability of Hamiltonian systems by the method of separation of variables.

(4.4) Orthogonal Separation

The orthogonal separability of Hamiltonian systems (4.3) has a long history.

It was Stackel who first found the necessary and sufficient conditions for the system (4.3) to be orthogonally separable. In spite of their complicated form these fundamental conditions are still being used today by many mathematicians to study orthogonal separability.

Levi-Civita established (local) criterion of separability (not necessarily orthogonal) of the Hamilton - Jacobi equation associated with a general Hamiltonian system defined by (4.3) in local coordinates $(q^1, \dots, q^n, p_1, \dots, p_n)$ consisting of the $1/2n(n-1)$ equations

$$\partial^i \partial^j H \partial_i H \partial_j H - \partial_i \partial^j H \partial^i H \partial_j H + \partial_i \partial_j H \partial^i H \partial^j H - \partial^i \partial_j H \partial_i H \partial^j H = 0 \quad (4.25)$$

The next breakthrough was by Eisenhart who presented in turn necessary and sufficient conditions for a Hamiltonian system defined by the geodesic Hamiltonian.

$$H_g = \tfrac{1}{2} g^{ij} p_i p_j \quad (4.26)$$

To be of the Stackel type and thus orthogonally integrable. The result was based on the fact that the n first integrals involution (including the Hamiltonian) are necessarily quadratic in momenta, when the system defined by (4.26) is considered in natural position - momenta coordinates. Moreover, the involution of any of these n-1 first integrals $F_1, \dots, F_{n-1}$:

$$F_r \coloneqq \tfrac{1}{2} K_r^{ij} p_i p_j, \quad r = 1, \dots, n-1$$

with the Hamiltonian (4.24)

$$\{H_g, F_r\} = 0, \qquad r = 1, \dots, n-1$$

entails the killing tensor equation

$$[g, K_r] = 0, \quad r = 1, \dots, n-1$$

which is equivalent to

$$K_{r(ab;c)} = 0, \quad r = 1, \dots n-1,$$

where the indices of $K_1, \dots, K_{n-1}$ have been lowered. Hence, the first integrals $F_1, \dots, F_{n-1}$ are defined by the n-1 valence two killing tensors $K_1, \dots, K_{n-1}$ that share, in view of Eisenhart's result, certain geometrical properties. In particular, they must possess the same eigenvectors and these eigenvectors are normal which means that each eigenvector is normal to an (n-1) - dimensional hypersurface .

Kalnins and Miller have further improved the results of Eisenhart. In particular, they have studied the n-dimensional Abelian Lie algebra of killing tensors of order 2, $\widetilde{K}_1, \dots, \widetilde{K}_n$, where $\widetilde{K}_1 = g, \dots, \widetilde{K}_n = K_{n-1}$ in the notation above. Indeed, we note that the Schouten bracket satisfies the Jacobi identity in the space of two - contravant tensors (symmetric or otherwise). Moreover, they concluded that every killing tensor $\widetilde{K}_i$, $i = 2, \dots, n$ that is linearly independent of $g = \widetilde{K}_1$ and defines (locally) a separable coordinate system for the Hamilton-Jacobi equation (4.1) on $(\widetilde{M}, g)$, and conversely, every separable coordinate system arises in this way.

We note, however, that the complications arising from dealing with the n killing tensors (including the metric) connected via certain algebraic and differential conditions makes this result difficult to apply.

Finally generalized the result above and obtained a characterization of orthogonal separability in terms of a single killing tensor. The result is the following theorem.

(4.4.1) Theorem:

A Hamiltonian system defined by (4.3) is orthogonally separable if and only if there exists a valence two killing tensor *K* with pointwise simple and real eigenvalues, orthogonally integrable eigenvectors and such that $d(\widetilde{K}dV) = 0$, where the linear operator $\widetilde{K}$ is given by $\widetilde{K} := Kg$ (or in the index for $\widetilde{K}^i_j := K^{il}g_{lj}$) .

(4.4.2) Remark:

We note that starting with one *K* that satisfies the conditions of theorem (4.4.1) one can reconstruct the n-dimensional Abelian Lie algebra of killing tensors (including the metric) by either finding the sets of separable coordinates or using the intrinsic iterative process described in which does not require having separable coordinates. Conversely, having the n-dimensional Abelien Lie algebra , we can easily obtain the killing tensor $\widetilde{K}$ of theorem (4.4.1) by considering the total sum of elements. Another way to see this is the following :The killing equation (4.41) for $\widetilde{K}$ is equivalent to a system of n linear partial differential equation , the general solution of which naturally depends on n constants of integration, where in turn can be viewed as the dimension of the corresponding Abelian Lie algebra of killing tensors. Further, the killing tensor *K* does not define a single set of separable coordinates, for example, by varying its eigenvalues (i.e, intrinsic invariants) or otherwise, we can extract all the sets of orthogonally separable coordinates for a given Hamiltonian system defined by (4.3).

(4.4.3)Remark:

The statement of theorem (4.4.1) implies that there exists an additional first integral quadratic in momenta (say):

$$F(q,p) = \tfrac{1}{2}K^{ij}(q)p_i p_j + U(q), \tag{4.27}$$

where the matrix K^{ij} is that of K the involutiveness $\{H_0, F\} = 0$ yields the killing equation $[g, K] = 0$, and the condition $d(\widetilde{K}dV) = 0$(which entails locally that $dU = \widetilde{K}dV$).

Theorem (4.4.1) offers the advantage of working with a single geometrical quantity instead of n such quantities. However, in general it is still very difficult to check whether or not a given killing tensor K has normal eigenvector .This is rather non-trivial task even in three-dimensional pseudo-Riemannian manifolds $(\widetilde{M}, g)$. The main difficulty is the computational effect required by the straightforward approach. To solve the killing equation in this case in given position momenta coordinates yields six functions(i.e.,$K^{11}, K^{22}, K^{33}, K^{12}, K^{13}, K^{23}$) depending upon 20 constants of integration that represent the dimension of the space of (2,0) killing tensor in $\mathfrak{R}^n$. Conceivably, for n=4,where n=dim$\widetilde{M}$ the problem of finding the normal eigenvectors of K is practically insurmountable without employing computer algebra.

Therefore, in this chapter , we propose the use of the moving frame approach where the frame vectors are chosen to be a set of suitably normalized eigenvectors of K. it appears that the method not only results in a significant algebraic simplification, but also allows one to consider the problem in much more general setting, namely without any restrictions at all on the curvature of the pseudo-Riemannian manifold $(\widetilde{M}, g)$.

To demonstrate how the method works and give a flavor of its applications, we begin by proving the following criterion for orthogonal separability in cartesian coordinates.

(4.4.4) theorem:

The Hamiltonian system (4.3) is orthogonally separable with respect to cartesian coordinates if the associated pseudo-Riemannian manifold $(\tilde{M}, g)$ admits a valence two covariant killing tensor *K* with pointwise simple eigenvalues and vanishing Nijenhuis tensor $N_{\tilde{K}}$.

Proof

Consider a C^{∞}pseudo-Riemannian manifold $(\tilde{M}, g)$ associated to the Hamiltonian (4.3) which possesses a symmetric C^{∞}tensor field *K* of type (0,2). The eigenvalue equation

$$K_{ij}E^{j} = \lambda g_{ij}E^{j} \tag{4.28}$$

admits n pointwise simple eigenvalues $\lambda_1, \dots, \lambda_n$. We note that since $(\tilde{M}, g)$

is the Riemannian eigenvalues are necessarily real. Let $E_1, \dots, E_n$be a set of eigenvectors of *K* corresponding to the eigenvalues $\lambda_1, \dots, \lambda_n$. It can be shown that the eigenvectors are real, mutually orthogonal and that none of them is a null vector. Thus, the eigenvectors can be normalized such that

$$g(E_a, E_a) = 1 \tag{4.29}$$

The above set of eigenvalues is uniquely determined up to sign.

Since g and *K* are C^{∞} tensor fields, and the operations of solving for the eigenvalues and eigenvectors and normalizing the eigenvectors are rational operations it follows that the eigenvectors $E_1, \dots, E_n$ define a set of C^∞ pointwise linearly independent vector fields on some open set $A \subset \tilde{M}$. hence, we may choose these vectors as a rigid moving frame on A with respect to which the components of g and K are given by

$$g_{ab} = \mathrm{diag}(1, \dots, 1) \tag{4.30}$$

and

$$K_{ab} = \mathrm{diag}(\lambda_1, \dots, \lambda_n) \tag{4.31}$$

it follows that the metric tensor has the form

$$ds^2 = (dx^1)^2 + \cdots + (dx^n)^2 \tag{4.32}$$

A rigid co-frame can thus be chosen as follows:

$$E^1 = dx^1, \dots, E^n = dx^n$$

with corresponding dual frame being

$$E_1 = \partial_i, \dots, \partial_n \tag{4.33}$$

It is obvious that the frame vector fields are orthogonally integrable. Consider now the (0,2) tensor K, the components of which in the above co-frame are given by

$$K_{ab} = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$$

with λ_a are constants satisfying $\lambda_a \neq \lambda_b$ for all a,b =1,…,n, a≠b. it is clear that E_a is an eigenvector corresponding to the eigenvalue λ_a for each a=1,…,n. Since the connection coefficients for frame (4.33) are zero, E_q (4.24) has the form

$$K_{ab;c} = \partial_c K_{ab} .$$

It thus easy to verify that the tensor (4.31) satisfies the killing equation (4.23) we conclude that the tensor defined by (4.31) is the killing tensor, the existence of which is guaranteed by theorem (4.4.1). it follows from (4.30) and (4.31)that

$$\widetilde{K} = \mathrm{diag}(\lambda_1, \dots, \lambda_n) \tag{4.34}$$

and that $\widetilde{K}$ has a trivially vanishing Nijenhuis tensor .this fact may be established from the following expression of $N_{\widetilde{k}}$ in local coordinates:

$$N^i_{\widetilde{k}jk} = \partial_l B^i_k B^l_j - \partial_l B^i_j B^l_k + \partial_k B^l_j B^i_l - \partial_j B^l_k B^i_l = 0 \tag{4.35}$$

where i ,j, k=1,…,n. note that $N^i_{\widetilde{k}jk} = -N^i_{\widetilde{B}kj}$

let K be a (0,2) killing tensor with pointwise simple and real eigenvalues and vanishing Nijenhuis tensor. In the rigid moving frame of eigenvectors $E_1, \dots, E_n$ of K the condition (4.33) reads

$$N_{\widetilde{k}}(E_a, E_b) = (\widetilde{K} - \lambda_a)(\widetilde{K} - \lambda_b) C^c_{ab} E_c + (\lambda_a - \lambda_b)(E_a(\lambda_b) E_b + E_b(\lambda_a) E_a) = 0 \tag{4.36}$$

and taking into account (4.34) can be decomposed into the following system of equations :

$$C^c_{ab} = 0, \quad a, b, c \text{ are distinct.} \tag{4.37}$$

$$E_a(\lambda_b)(\lambda_a - \lambda_b) = 0 \quad a, b \text{ are distinct.} \tag{4.38}$$

Concurrently, the killing equation(4.23) for K with lower indices decomposes as follows:

$$K_{(aa;a)} = 0 \Leftrightarrow E_a K_{aa} = 0 \quad n \text{ equations,} \tag{4.39}$$

$$K_{(aa;b)} = 0 \Leftrightarrow E_b(\lambda_a) = 2\Gamma_{aab}(\lambda_b - \lambda_a) \quad 2\binom{n}{2} \text{ equations,} \tag{4.40}$$

$$K_{(ab;c)} = 0 \binom{n}{3} \text{ equations} \tag{4.41}$$

where a, b and c are distinct. Therefore, since $\lambda_1 - \lambda_n$ are distinct, the connection coefficients Γ^a_{bc} vanish. Hence, the Riemannian space $(\widetilde{M}, g)$ is flat and the eigenvalues of *K* are constants. This implies that the Hamiltonian system defined by (4.3) is separable only with respect to Cartesian coordinates.

(4.4.5) Remark :

It is instructive to contrast the above result with an analogous result for Poisson - Nijenhuis manifolds. Recall that in the case of two compatible Poisson bi-vectors P_1 and P_2, the linear operator $A := P_2 P_1^{-1}$ with the components $A^i_j = P_2^{im} P^{-1}_{1mj}$ (if P_1 is non-degenerate) has a vanishing Nijenhuis tensor $N_A = 0$ we observe that the killing tensor equation [g, $K] = 0$ satisfied by the two killing tensors g and *K* resembles the condition $[P_1, P_2]$=0 of Compatibility of the two Poisson bi-vectors in the theory of bi - Hamiltonian systems. However, as may be seen from the proof of theorem (4.4.4) the killing tensor equation is not equivalent to the vanishing of the Nijenhuis tensor of the corresponding linear operator $\widetilde{K} := Kg$. Moreover, as we have just seen, the vanishing of the tensor $N_{\widetilde{K}}$ appears to be a very restrictive additional condition on $\widetilde{K}$.

(4.5) Moving Frame in a Surface and Separability

We start our considerations in arbitrary Riemannian manifold $(\widetilde{M}, g)$, dim$\widetilde{M}$ =2 defined by (4.3) making a priori no assumptions on its curvature. Using the techniques presented in the previous two sections, we introduce a rigid moving frame of co-vectors E^1, E^2 with respect to which the metric g and killing tensor k of theorem (4.4.1) take the following forms:

$$g_{ab} = \delta_{ab}E^a \odot E^b \tag{4.42}$$

$$k_{ab} = \lambda_a\delta_{ab}E^a \odot E^b, \tag{4.43}$$

where $\odot$ is the symmetric tensor product and $a, b = 1,2$ and λ_1, λ_2 along with the dual vectors E_1, E_2 are the eigenvalues and eigenvectors of K respectively. In this case we have two independent connection coefficients Γ_{112} and Γ_{212} and one component of the Riemann curvature tensor R_{1212} . For convenience we write $\alpha \coloneqq \Gamma_{112}$ and $\beta \coloneqq \Gamma_{212}$ then the formulas (4.15),(4.17) and (4.24) become

$$[E_1, E_2] = \alpha E_1 - \beta E_2 \tag{4.44}$$

$$dE^1 = \alpha E^1 \wedge E^2, \quad dE^2 = \beta E^1 \wedge E^2, \tag{4.45}$$

$$R_{1212} = -E_1\beta + E_2\alpha - \alpha^2 - \beta^2, \tag{4.46}$$

$$E_1\lambda_1 = 0, \;\; E_2\lambda_1 = 2\alpha(\lambda_2 - \lambda_1), \;\; E_1\lambda_2 = 2\beta(\lambda_2 - \lambda_1), E_2\lambda_2 = 0 \tag{4.47}$$

where (4.14) has been used. Our next observation is that in a two-dimensional Riemannian manifold the conditions of orthogonal intergrability for E_1 and E_2,

$E^a \wedge dE^a = 0$, $a = 1,2$ are automatically satisfied. Hence, by Frobenius' theorem, there exist functions f, g, u and v such that

$$E^1 = f du, \quad E^2 = g dv. \tag{4.48}$$

we choose (u , v) as coordinates, while the functions f and g remain to be determined by the condition of problem. Clearly with respect to (u, v) we have $\alpha = \alpha(u , v), \beta = \beta(u , v)$ and the eigenvectors E_1, E_2 of K are given by

$$E_1 = (f)^{-1}\,\partial_u, \quad E_2 = (g)^{-1}\,\partial_v. \tag{4.49}$$

substituting (4.48) into (4.45), yields

$$\alpha = -(fg)^{-1}\partial_u f, \quad \beta = (fg)^{-1}\partial_v g. \tag{4.50}$$

Consider again the Hamiltonian function (4.3) in natural (position- momenta, say) coordinates :

$$H = \tfrac{1}{2}g^{ij}p_i p_j + v$$

in a rigid moving frame in view of the above, we have

$$H = \tfrac{1}{2}g^{ab}p_a p_b + V \tag{4.51}$$

where $g^{ab} = g^{ij}h_i^a h_j^b$ and $p_a = h_a^k p_k$, where h_a^i is defined in (4.14) and V is a function of u and v. next we apply the vector field $[E_1, E_2]$ to λ_1 and λ_2 to obtain the following integrability conditions :

$$E_1\alpha = -3\alpha\beta, \tag{4.52}$$

$$E_2\beta = 3\alpha\beta, \tag{4.53}$$

Now it is natural to analyze the following three cases defined with respect to α and β.

C1 $\alpha = \beta = 0 \Leftrightarrow \lambda_1$ and λ_2 constant,

C11 $\alpha = 0, \beta \neq 0 (\alpha \neq 0, \beta = 0) \Leftrightarrow \lambda_1$ constant (λ_2 constant),

C111 $\alpha\beta \neq 0 \Leftrightarrow \lambda_1$ and λ_2 both non – constant.

This classification is intrinsic since the rigid moving frame we are using is defined up to a sign. The general forms of the separable metric

$$ds^2 = (E^1)^2 + (E^2)^2, \tag{4.54}$$

and the corresponding killing tensor K (4.43) will be derived in each case. Having found the killing tensor, we shall derive the form of the most general separable potential $V(u,v)$ admitted by original Hamiltonian (4.3). To accomplish this, we take into consideration the condition $d(BdV) = 0$ of theorem (4.4.1) which may be written in terms of the moving frame as

$$E_1E_2V + 3\beta E_2V - 2\alpha E_1V = 0 \tag{4.55}$$

Once the potential V is found, we derive the second first integral of the Hamilonian system defined by (4.3) given by $F = K^{ab}p_ap_b + U$ or

$$F(u,v,p_1,p_2) = \lambda_1 p_1^2 + \lambda_2 p_2^2 + U(u,v) \tag{4.56}$$

In the moving frame, by solving the equation $dU = 2BdV$. writing this condition in the moving frame, we immediately obtain the following system

$$E_1U = 2\lambda_1 E_1V, \tag{4.57}$$

$$E_2U = 2\lambda_2 E_2V, \tag{4.58}$$

Case 1: $\alpha = \beta = 0$

It follows immediately form (4.48) that $f = f(u)$and $g =g(v)$ therefore, $E^1 = f(u)du, E^2 = g(v)dv$, and the metric takes the form

$$ds^2 = f^2(u)du^2 + g^2(v)dv^2$$

We observe that there exist coordinate transformations $(u,v) \rightarrow (\tilde{u},\tilde{v})$,such that

$$E^1 = f(u)du = d\tilde{u}, \quad E^2 = g(v)dv = d\tilde{v} \tag{4.59}$$

Where

$$\tilde{u} = \int f(u)du, \quad \tilde{v} = \int g(v)dv.$$

The remaining coordinate freedom is

$$\tilde{u} = \tilde{u} + u_0, \quad \tilde{v} = \tilde{v} + v_{0.}$$

Thus, for C1 we have

$$E^1 = du, \quad E^2 = dv \tag{4.60}$$

Where the tilders have been dropped. Thus, the metric (4.54) has the form

$$ds^2 = du^2 + dv^2 \tag{4.61}$$

We conclude that the separable coordinates in this case are Cartesian. We also observe, by (4.46), that $R_{1212} = 0$, in C1, which means that the case when both eigenvalues of K are constant is compatible with only a flat two-dimensional Riemannian space. Now taking into account the above facts along with the killing equation, we easily recover that $\lambda_1 = c_1$ and $\lambda_2 = c_2$ where $c_1$and c_2 are co-nstants .hence

$$K = diag(c_1, c_2) \tag{4.62}$$

And in view of (4.53), we have

$$V(u, v) = V_1(u) + V_2(v) \tag{4.63}$$

Similarly, by making use of (4.57) and (4.58), we find the corresponding U to be

$$U(u, v) = 2kV_1(u) + 2lV_2(v) \tag{4.64}$$

We conclude that a second first integral F that is functionally independent of the Hamiltonian H is

$$F(u, v, p_u, p_v) = p_v^2 + 2V_2(v) \tag{4.65}$$

We note that the class of Hamiltonian systems just described has the properties of being bi-Hamiltonian in the separable coordinates (u, v) with respect to the constant Poisson bi-vectors P_0 and P_1:

$$P_0 = \partial_u \wedge \partial_{p_u} + \partial_v \wedge \partial_{p_v}, \quad P_1 = \partial_u \wedge \partial_{p_u} - \partial_v \wedge \partial_{p_v}, \tag{4.66}$$

and having a Lax representation defined by matrices L and M of the form

$$L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}, \qquad M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}, \tag{4.67}$$

where

$$L_i = \begin{pmatrix} \frac{1}{\sqrt{2}} p_j & 2\omega_j \\ \frac{f_i(\omega_j)}{\omega_j} & -\frac{1}{\sqrt{2}} p_j \end{pmatrix}, \qquad M_i = \frac{1}{2\omega_j} \begin{pmatrix} 0 & 0 \\ \frac{d}{dt}\left(\frac{p_j}{\sqrt{2}}\right) & -2p_j \end{pmatrix}, \tag{4.68}$$

here $i, j = 1,2, i \neq j, \omega_1 = u, \omega_2 = v$ and $f_1, f_2 \in C^1(\mathbb{R})$

are arbitrary functions. We note that the separable coordinates (u, v)in this case are simply the Darboux – Nijenhuis coordinates, defining bi-Hamiltonian structure (4.66).

Case 11: $\alpha = 0, \beta \neq 0 (\alpha \neq 0, \beta = 0)$

The condition $\alpha = 0$ in (4.50) immediately yields $f = f(u)$ and by an appropr- iate coordinate transformation, we may set $f = 1$ .similarly, we use (4.53) to conclude $\beta = \beta(u)$, which entails in turn after solving (4.53) that $g = C(u)D(v)$, where $C(u)$ and $D(v)$ are arbitrary functions. We may absorb $D(v)$ by a further coordinate transformation to obtain $g = g(u)$. Hence, the metric in this case is given by

$$ds^2 = du^2 + g^2(u)dv^2 \tag{4.69}$$

Where $g(u)$ is an arbitrary function. To solve the killing equation and find the corresponding K, we observe that in view of the above $\beta = \partial_u g/g$ .now (4.47) transform into the following system of partial differential equations

$$\partial_u\lambda_1 = \partial_v\lambda_1 = \partial_v\lambda_2 = 0, \quad \partial_u\lambda_2 = \partial_u g g^{-1}(\lambda_2 - \lambda_1), \tag{4.70}$$

Solving for λ_1 and λ_2 we find $\lambda_1 = k$, $\lambda_2 = lg^2(u) + k$, where l, k are arbitrary constants. Hence, the killing tensor in this case takes the form:

$$K = \mathrm{diag}(k, lg^2(u) + k) = kg + lK_1, \tag{4.71}$$

where $K_1 = \mathrm{diag}\left(0, g^2(u)\right)$ and $g, K_1,$ span two dimensional Abelian Lie algebra of killing tensors. We note that, since the variable v is ignorable, the killing tensor $K_1$is simply the square of the corresponding killing vector corresponding to the first integral linear in the momenta.

(4.5.1) Remark

This observation illustrates the fact that Beneti's approach is in fact equivalent to the approach due to Eisenhart and Kanlnins and miller. In the most general case the killing tensor K of theorem (4.4.1) is simply a linear combination of the n killing tensor (including the metric) $g, \ldots, K_{n-1}$

Next, taking into account that $\alpha = 0$ and $f = 1$, we solve equation (4.55) for V to obtain

$$V(u,v) = V_1(u) + \frac{V_2(v)}{g^2(u)}, \tag{4.72}$$

where V_1 and V_2 are arbitrary functions. it follows by (4.55) that

$$U(u,v) = 2kV_1(u) + 2lV_2(v) + \frac{2kV_2(v)}{g^2(u)}, \tag{4.73}$$

Finally, substituting (4.72) and (4.73) in (4.56) and removing the expression for the Hamiltonian we find a second first integral F for this family of separable Hamiltonian systems just described, namely

$$F(u,v,p_1,p_2) = kg^2(u)p_2^2 + lV_1(u) + 2lV_2(v) + \frac{kV_2(v)}{g^2(u)}, \tag{4.74}$$

or, in terms of the separable coordinates:

$$F(u,v,p_u,p_v) = c_2(u)p_v^2 + c_1V_1(u) + 2c_2V_2(v) + \frac{c_1V_2(v)}{g^2(u)}, \tag{4.75}$$

We note that (4.46) in this case becomes

$$R_{1212} = -\partial_u\left(\frac{\partial_u g}{g}\right) - \left(\frac{\partial_u g}{g}\right)^2 = -\frac{g''}{g}, \tag{4.76}$$

or, simply

$$g'' + ag = 0, \tag{4.77}$$

where $a(u) = R_{1212}(u)$. The case $\alpha \neq 0, \beta = 0$ corresponds to metric $ds^2 = f^2(v)du^2 + dv^2$, which can be obtained from (4.69) in an obvious way.

Case 111: $\alpha\beta \neq 0$

We begin by proving first that in this case the functions f and g may be assume equal. equation (4.52) and (4.53) imply that

$$E_1\alpha = -E_2\beta,$$

Which, on account of (4.50), may be written as

$$\partial_u \partial_v \left(\ln\left(\frac{f}{g}\right)\right) = 0.$$

It follows $\ln(f/g) = G(u) + H(v)$, where G and H are arbitrary functions, from which we obtain

$$f = g(u,v)C(u)D(v) \tag{4.78}$$

Where $C(u) = e^{G(u)}$ and $D(v) = e^{H(v)}$. After appropriate coordinate transforations applied to the metric, we get

$$f(u,v) = g(u,v), \tag{4.79}$$

We now proceed to determine the general form of metric. In view of (4.79), either of (4.52) and (4.53), yields

$$\partial_u \partial_v f^2(u,v) = 0.$$

Therefore,

$$f^2(u,v) = A(u) + B(v) \tag{4.80}$$

where A and B are arbitrary functions. It follows that the metric has the form

$$ds^2 = \big(A(u) + B(v)\big)(du^2 + dv^2) \tag{4.81}$$

(4.5.2) Remark:

We note immediately that the metric (4.81) is that of the well-known Liouville surface. Hence, in this case the dynamics of (4.3) can be viewed as the motion of a liouville surface under the action of a conservative force with potential energy $V(u, v)$.

We proceed to find the corresponding killing tensor K. substituting (4.50) along with (4.80) in to (4.47) leads to the following system of partial differential equations with respect to λ_1 and λ_2:

$$\partial_u \lambda_1(u, v) = \partial_v \lambda_2(u, v) = 0, \qquad \partial_v \lambda_1(u, v) = \frac{B'(v)}{A(u) + B(v)} (\lambda_1(u, v) - \lambda_1(u, v)$$

$$\partial_u \lambda_2(u, v) = \frac{A'(u)}{A(u)+B(v)} \left(\lambda_2(u, v) - \lambda_1(u, v)\right) \tag{4.82}$$

Solving (4.80), we obtain $\lambda_1 = KB(v) + L$ and $\lambda_2 = KA(u) + L$, where K and L are arbitrary constants. Thus

$$K = \mathrm{diag}(kB(v) + L, -kA(u) + L) = Lg + kK_1 \tag{4.83}$$

where $K_1 = \mathrm{diag}(B(v) - A(u))$ (remark (4.5.1)). Equation (4.55) for $V(u, v)$.

may be written as

$$\partial_u \, \partial_v \left[\left(A(u) + B(v)\right) V(u, v)\right] = 0$$

which has the solution

$$V(u, v) = \frac{V_1(u, v) + V_2(u, v)}{A(u) + B(v)}, \tag{4.84}$$

where V_1 and V_2 are arbitrary functions. It follows that (4.57) and (4.58) may be solved to obtain

$$U(u,v) = 2lV(u,v) + 2k\frac{B(v)V_1(u,v) - A(u)V_2(u,v)}{A(u) + B(v)} \tag{4.85}$$

We conclude that the second first integral independent of H has the form

$$F(u,v,p_u,p_v) = B(v)p_1^2 - A(u)p_2^2 + 2\left(\frac{V_1(u,v) + V_2(u,v)}{A(u) + B(v)}\right), \tag{4.86}$$

Noting that $h_1^1 = f^{-1}, h_2^2 = f^{-1}, h_1^2 = h_2^1 = 0$, we may rewrite (4.84) in terms of the coordinates as

$$F(u,v,p_u,p_v) = \frac{B(v)\left(p_u^2 + 2V_1(u)\right) - A(u)(p_v^2 + 2V_2(v))}{A(u) + B(v)} \tag{4.87}$$

We note that the form of the Hamiltonian H (4.3) in the coordinates (u, v) becomes

$$H(u,v,p_u,p_v) = \frac{p_u^2 - p_v^2}{2(A(u) + B(v))} + \frac{V_1(u) + V_2(v)}{A(u) + B(v)}, \tag{4.88}$$

The forms (4.87) and (4.88) demonstrate that the Hamiltonian system under consideration is a Liouville system in the separable coordinates (u, v). Conversely, it is easy to see that the Hamilton-Jacobi equation corresponding to(4.88) separates in the coordinates (u, v). Indeed in this case (4.1) takes the following form:

$$\frac{1}{2\left(A(u) + B(v)\right)}\left((\partial_u W)^2 + (\partial_v W)^2 + 2\left(V_1(u) + V_2(v)\right)\right) = E.$$

Now, putting $W(u,v) = W_1(u) + W(v)$, we find the complete integral W to be

$$W(u,v) = \int \sqrt{\beta - 2V_1(u) + EA(u)}\,du + \int \sqrt{-\beta - 2V_2(v) + EB(v)}\,dv.$$

Differentiating W with respect to β and E, we can find the solutions for specific choices of $A(u), B(v), V_1(u)$, and $V_2(v)$. Hence, without imposing any restriction on the curvature of the corresponding pseudo-Riemannian manifold we have proven the following criterion of separability.

(4.5.3) Theorem

The following conditions are equivalent.

1- The Riemannian manifold $(\widetilde{M}, g)$ defined by (4.3) admits a valence two killing tensor K with distinct eigenvalues;

2- There exist coordinates (u, v) with respect to which the metric takes the form (4.81);

3- The Hamiltonian system defined by the Hamiltonian

$$H = \frac{1}{2} g^{ij}(q) p_i p_j + V(q), \quad i, j = 1,2 \qquad (4.89)$$

in the Riemannian manifold $(\widetilde{M}, g)$ of an arbitrary curvature can be integrated by separation of variables.

Having derived the explicit formula (4.87) for the second first integral F, we can now investigate whether or not the Liouville system (4.88) admits a bi-Hamiltonian representation with respect to the coordinates (u, v). Recall that the bi-Hamiltonian property is a combination of algebraic and differential conditions, which can be quite restrictive for low-dimensional Hamiltonian systems. Indeed, it is easy to see that the symplectic ω_1 corresponding to F: $i_{X_H}\omega_1 = -dF$ is given by

$$\omega_1 = 2B(v) du \wedge dp_u - 2A(u) dv \wedge dp_v, \qquad (4.90)$$

Clearly, (4.91) satisfies the differential conditions $d\omega_1 = 0$ and $L_{X_H}(\omega_1) = 0$ iff $A(u) = B(v) = cont$. in this case ω_1 is equivalent to P_1in (4.66). we answer the question of whether there exists a second Hamiltonian representation with respect to F by the following result.

(4.5.4) Proposition

The Liouville system defined by (4.88) admits a bi-Hamiltonian representation in the separable coordinates (u, v) iff the coordinates are Cartesian.

Finally, we note that the formula (4.46) assumes in this case the following form

$$R_{1212} = -\frac{1}{f}\left[\partial_u\left(\frac{\partial_u f}{f^2}\right) + \partial_v\left(\frac{\partial_v f}{f^2}\right)\right] - \left(\frac{\partial_u f}{f^2}\right)^2 - \left(\frac{\partial_v f}{f^2}\right)^2, \qquad (4.91)$$

Where $f^2(u, v) = A(u) + B(v)$.

Chapter Five

Hamiltonian Systems on Some Surfaces

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Hamiltonian Systems on Some Surfaces

(5.1) Surface Theory:

We shall review surface theory in this section.

We first introduce parameterized surface in Euclidean three- dimensional space. Then we study the shape operator that we shall utilize to introduce the normal curvature, Gauss curvature and mean curvature.

A surface M in E^3 (Euclidean three dimensional space).

May be parameterized by a differentiable $X(u,v)$of two variable u and v .we wired a point p in M as :

$$X(u,v) = \big(\mathrm{x}(u,v), y(u,v), z(u,v)\big) = p$$

1- Carton Method of the moving frame:

(5.1.1) Definition:

A smooth 1-from $\emptyset$ on $\mathfrak{R}^n$ is a real - valued Function on the set or all tangent vectors to $\mathfrak{R}^n$, ie

$$\varphi: \mathfrak{R}^n \to \mathfrak{R} \tag{5.1}$$

with the properties that :

1- $\emptyset$ is linear on the tangent space $T_x\mathfrak{R}^n$ for each x $\in \mathfrak{R}^n$
2- For any smooth vector field $V = V(x)$ the function
$\emptyset V(x) = : \mathfrak{R}^n \to \mathfrak{R}$

Given a 1-from $\emptyset$, for each $x \in \mathfrak{R}^n$ the map

$$\emptyset_x : T_x\emptyset\, \mathfrak{R}^n \to \mathfrak{R} \tag{5.2}$$

Is an element for the dual space $(T_x\emptyset\mathfrak{R}^n)$, wgen we entend the nation all of $\mathfrak{R}^n$ .we see that the space of 1-from on $\mathfrak{R}^n$ is dual to the space or vector field on $\mathfrak{R}^n$.
In particular, the 1-form dx^1, …,dx^n are denied by the property that for any vector.

$$V = (V^1, \dots, V^n) \in T_x\Re^n, dx^i(v) = v^i \quad (5.3)$$

The $dx^{i,s}$ from a basis for 1-from on $\Re^n$, so any other 1-from $\emptyset$ may be enpressed in the from

$$\emptyset = \sum_{i=1}^{n} f_i(x)\, dx^i \quad (5.4)$$

If a vector field von $\Re^n$ has the from

$$V(x) = (V^i(x), \dots, V^n(x))$$

Then at any point $x \in \Re^n$

$$\emptyset = \sum_{i=1}^{n} f_i(x)\, V^i(x) \quad (5.5)$$

(5.1.2) Definition:

A definition 1-from $\emptyset$ defined on a domain $\Re$ is said to be closed if

$$\frac{da_i}{dx_j}(p) = \frac{da_j}{dx_i}(p) \quad \forall\, i, j \text{ and } x \in \Re \quad (5.5)$$

We say that a differential 1-from $\emptyset$ is enat it there enist a smooth function $F \approx \Re$ such that

$$\emptyset = dF \quad (5.6)$$

(5.1.3) Definition:

A smooth a differential x from won M is collection of smoothly varying alternating k-linear maps

$w/x \in A_k T^{n*} I_x$ for each $x \in M$, where we require that for all smooth vector field $V_1, \dots, V_k$.

$\langle w, V_1, \dots, V_k\rangle$(n) = $\langle wI_n, V_1 I_x, \dots, V_n I_x\rangle$ is a smooth read-valued function of X.

We reviewing cartons formulation of local differential geometry in terms of moving frames.

Let $S \subset \Re^3$ be a surface and let the dot product of $\Re^3$. Be given $\langle .,.\rangle$ let a local chart for S be given by the map $X: V \to \Re^2$ where U, V an open set in a neighborhood of a point we choose a local orthonormal frame, smooth vector fields $\{e_1, e_2, e_3\}$ such that

$$\langle e_i, e_j\rangle = \delta_{ij} = 1, 2, 3 \quad (5.7)$$

We choose the from a dappled in such a way that e_3 is the unit normal vector and e_1 and e_2 span the tangent space $T_P S$. The corresponding coframe field of one forms $\{W^i\}$ is defined by the differential

$$dx = w^1 e_1 + w^2 e_2 \tag{5.8}$$

In local coordinates $(u^1, u^2) \in U$ the one forms are linear functional of the form.

$$w(.) = P_1(u^1, u^2)du^1 + P_2(u^1, u^2)du^2 \tag{5.9}$$

Where P is asmooth function in U, $du^1$are the differentials for the coordinate functions $u^i = U \to \Re$ which from a basis for the linear functional on the vector spaces T $(u^1, u^2)^U$. The vectors in local coordinals have the expression

$$v = v^1 (u^1, u^2)\frac{d}{du^1} + v^2 (u^1, u^2)\frac{d}{du^2}$$

The one form (5.3) acts by

$$w(v) = v^1(u^1, u^2)P_1(u^1, u^2) + v^2(u^1, u^2)P_2(u^1, u^2)$$

The usual identifications between $X: U \to X(U)$ or $= T\ (u^1, u^2)U \to T_p s$, the one forms can be interpreed as linear functional on T_ps as well . for example if choose vector fields E_i in U such that $dx(E_i) = e_i$ then we may set

$$w(e_i) = w(E_i).$$

In particular $w^i(e_i)= \delta^i_j$. it also means that metric takes the form $ds^2 = (w^1)^2 + w^2)^2$ then w^i is aciform and the vector fields e_i determined by duality $w^i(e_i)= \delta^i_j$. are corresponding orthonormal frame .

One form can be integrated on curves in the usual way.

If $\propto: \{o, L\} \to U$ is a piecewise smooth curve where $\propto (t) = (u^1 t), (u^2 t\ )$ then

$$\int w_{\propto(\{o,L\}} := \int_0^L w(\propto (t)dx = \int \propto (\{o, L\}^{P_1 \mathrm{d}u^1} + P_2 \mathrm{d}u^2$$

Is the usual line integral. Two one forms may be multiplied (wedged) to give a Two form, which is askew symmetric bilinear from on the tangent space. For example if θ and where one forms then vector field X, Y we have the form

$$(\theta \wedge \omega)\ (X, Y) := \theta(X)\omega(Y) - \theta(Y)\omega(X)$$

in local coordinates this gives

$$(p_1 du^{\,1} + p_2 du^{\,2}) \wedge (q_1 du^{\,1} + q_2 du^{\,2}) := (p_1 q_1 + p_2 q_2) du^1 \wedge du^2$$

Because there vectors are dependent there are no skew symmetric three forms in $\Re^2$ and the most general two forms is

$$\beta = A\,(u^1, u^2) du^1 \wedge du^2$$

When evaluated on the vectors

$$V = v^{\,1}(u^{\,1}, u^{\,2}) \frac{d}{du^{\,1}}(u^{\,1}, u^{\,2}) + v^{\,2}(u^{\,1}, u^{\,2}) \frac{d}{du^{\,2}}(u^{\,1}, u^{\,2}),$$

$$Z = z^{\,1}(u^{\,1}, u^{\,2}) \frac{d}{du^{\,1}}(u^{\,1}, u^{\,2}) + z^{\,2}(u^{\,1}, u^{\,2}) \frac{d}{du^{\,2}}(u^{\,1}, u^{\,2})$$

The two forms gives

$$\beta(V, Z) = A(u^1, u^2)\, v^1(u^1, u^2)\, z^2(u^1, u^2) - v^2\,(u^1, u^2)\, z^1(u^1, u^2)$$

A two from say, may be integrated over a region $\Re \subset V$ by the:

$$\int_{\Re} \beta = \int_{\Re} A(u^{\,1}, u^{\,2})\, du^1 d\, u^{\,2}$$

Where $du^1 du^2$ denotes lebesgue measure on U

The first fundamental form the metric, has expression form

$$ds^2 = \langle dX, dX \rangle$$

$$= \langle w^1 e_1 + w^2 e_2, w^1 e_1 + w^2 e_2 \rangle \qquad (*)$$

$$= (w^1)^2 + (w^2)^2$$

The area from is $w^{\,1} \wedge w^{\,2}$. That is because by using (5.8) to write in terms of the e_1 basis, the area of the parallelogram spanned by $dx(d/du)$ and $dx(d/dv)$ is

$$w^1 \left(\frac{d}{du}\right) w^2 \left(\frac{d}{dv}\right) - w^1\left(\frac{d}{dv}\right) w^2 \left(\frac{d}{du}\right) = w^1 \wedge w^2\left(\frac{d}{du}, \frac{d}{dv}\right)$$

The Weingarten equations express the rotation of the frame when moored along the surfaces

$$de_i = \int w_i^j e_j \tag{5.10}$$

this equation derived the 3×3 matrix of one forms W_A^B which is called the matrix of connect ion forms or E^3 . the fact that the frame is or thonormal implies that when $\delta_{ij} = \langle e_i , e_j \rangle$ is different tiated using (5.10)

$$\begin{aligned} 0 &= d\delta_{ij} \\ &= d\langle e_i , e_j \rangle \\ &= \langle de_i , e_j \rangle + \langle e_i , de_j \rangle \\ &= \langle \sum w_i^k e_k , e_\beta \rangle + \langle e_i, \sum w_j^k e_k \rangle \\ &= w_i^j + w_j^i \end{aligned} \tag{5.11}$$

This equation says that the matrix of connection forms is skew so there are only three distinct w_i^j. Geometrically it says that the motion of the vectors is already determined in large part by the motion or the vectors in the frame.

The forms w_3^i determine the motion or the normal vector and hence define second fundamental form is given using (5.9), (5.10) and (5.11)

$$\begin{aligned} IL(.,.) &= \langle de_3, \mathrm{dx} \rangle \\ &= (w_3^1 e_1 + w_3^2 e_2 , w^1 e_1 + w^2 e_2) \\ &= -w_3^1 \otimes w^1 \neq <-w_3^2 \otimes w^2 \\ &= w_1^3 \otimes w^1 + w_2^3 \otimes w^2 \end{aligned} \tag{5.12}$$

We may express the connection forms using the basis

$$w_3^1 = h_{11} w^1 + h_{12} w^2 \tag{5.13}$$

$$w_2^3 = h_{21} w^1 + h_{22} w^2$$

Thus Inserting into (5.12)

$$\text{II}(.,.) = \sum h_{ij}\, w^{i}(\text{x})\, w^{j}$$

In particular, if one searches through all unit tangent vectors

$$V_{\Phi} := \cos(\Phi)e_1 + \sin(\Phi)e_2$$

for which II (V_{Φ}, V_{Φ}) is maximum and minimum, one finds that the extreme occur as Eigen vector h_{ij} and the principle curvatures k_{i} are the corresponding Eigen values. The Gaub and mean curvatures are

$$n = k_1 k_2 = \det(h_{\text{ij}}) = h_{11h_{22}} - h_{12h_{21}} \tag{5.14}$$

$$II = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(h_{11}) + (h_{11} + h_{22})$$

2- Covariant differentiation:

Of a vector field y in the direction or anther vector field $V = \sum v^{i} \frac{d}{du^{i}}$ on U is a vector field denoted $\nabla_{v}.y$. It is determined by orthogonal projection to the tangent space $\nabla_{v}y = proj\ (dy(v))$, Hence , in the local frame .

$$\nabla_{v}\text{e}_{i} := proj\left(d\text{e}_{i}(V)\right) = \sum w_{i}^{j}\,(\text{v})\text{e}_{i},$$

Covariant differentiation extends to all smooth vectors fields v, w on U, and y, z on S and smooth functions Φ, φ by the formulas

1) $\nabla\Phi\text{v} + \varphi wz = \Phi\nabla vz + \varphi\nabla_{w}z$ (Linearity)
2) $\nabla_{v}(\Phi\text{y} + \varphi z) = X(\Phi)\text{y} + \Phi xy + X(\varphi)\text{z} + \varphi\nabla_{v}z$ (Leibnitz)
3) $v\langle y, z\rangle = \langle\nabla_{v}y, z\rangle + \langle y, \nabla_{v} z\rangle$ (Metric compatibility).

With these formulas one can deduce $\nabla_{v}\ (\sum y^{i}\,\text{e}_{i})$. As

3- Gauss equation and intrinsic geometry:

We will have to differentiate (5.2) and (5.6) once more. The exterior derivative d is the differential on functions. The exterior derivative of a one from (5.3) is a two form given by

$$dw = \frac{d\text{p}_1}{du^1}(u^1, u^2) - (\frac{d\text{p}_1}{du^2}(u^1, u^2)$$

Thus $d^2 = 0$ because, for functions f,

$$D(df) = d\frac{df}{du^1}(u^1, u^2)\, d^1 \frac{df}{du^2}(u^1, u^2)\, du^2$$

$$= \frac{d^2 f}{du^1 du^2} u^1, u^2) - \frac{d^2 f}{du^2 du^1} u^1, u^2) \;\; du^1 \wedge du^2 = 0$$

The formula implies that if f (u^1, u^2) is a function and w a one form then

$$D(dw) = df \wedge w + f dw, \text{ but } d(fw) = dwf - w \wedge df$$

Greens formula becomes particular hyelegant:

$$\int_{\Re} w = \int_{d\Re} p_1 du^1 + p_2 du^2 = \int_{\Re} (\frac{dp_2}{du^1} - (\frac{dp_1}{du^2}(u^1, u^2) = \int_{\Re} dw$$

Differentiating (5.8) and (5.10)

$$0 = d^2 x = d(\sum w^i e_i) = \sum w^i e_i - \sum w^i \wedge de_i$$

$$= \sum w^i e_i + \sum w^i \wedge w_i^L e_L$$

$$0 = d^2 e_j = d(\sum w_j^k e_k) = \sum w_j^k e_k - \sum w_j^k de_k$$

$$= \sum w_j^k e_k - \sum w_j^k \wedge w_k^L e_L$$

Now collect coefficients for the basis vectors e_L

$$0 = \mathrm{d}\, w^j - \sum w^i \wedge w_j^k$$

$$0 = w_j^k - \sum w_j^k \wedge w_j^k \tag{5.15}$$

Now are called the first and second structure equations: by taking also the e_3 coefficient of $d^2 x$

$$0 = \sum w^j \wedge w_j^k$$

so follow that $h_{ij} = h_{ji}$ is a symmetric matrix. In the tent, we saw this when we proved the shape operator. de_3 was self a djoint

The Second structure equation enables to complete Gauss curates form the connection matrix indeed b by (5.13)

$$dw_1^2 = \sum w_1^i \wedge w_i^2 = w_1^3 \wedge w_3^2 = w_1^3 \wedge w_2^3$$

$$= -(h_{11}h_{22} - h_{12}^2)\, w^1 \wedge w^2 = -\mathrm{k}\, w^1 \wedge w^2 \qquad (5.16)$$

The remarkable thing is that the conditions (5.11) and (5.8)

$$\mathrm{d}w^i = \sum w^i \wedge w_j^i$$

$$w_j^i + w_j^i = 0 \qquad (5.17)$$

determine w_1^2 unequal. Since w^i is known once the metric is known by (*) this says that w_1^2 and thus k can be determined form the metric alone.

Computation of curvature from the matrix.

Let us compute the curvature of a metric in orthogonal coordinates, for simplicity sake, I take coefficients to squares. Thus we are given the metric

$$\mathrm{d}\, s^2 = E^2\, \mathrm{d}u^2 + G^2 \mathrm{dv}$$

Where $E(u, v), G(u, v) > o$ are smooth functions in U. It is natural to guess that

$$w^1 = Edu, \qquad w^2 = Gdv$$

$$w^1 = E_{\mathrm{v}} \wedge dvdu = w^2 \wedge w_2^1 = Gdv\, \frac{Ev}{G}\, \mathrm{d}u_1$$

$$w^2 = E_{\mathrm{u}} \wedge dudv = w^1 \wedge w_1^2 = Edu\, \frac{Gu}{E}\, dv$$

Thus we may take

$$w_1^2 = -\, w_2^1 = \frac{Gu}{E} dv - \frac{Ev}{G} du.$$

Hence by differentiating a gain

$$kw^1 \wedge w^2 = -kEGdu \wedge dv = d\, w_1^2 = (\frac{d}{du}(\frac{Gu}{E}) + \frac{d}{dv}(\frac{Ev}{G})\, du \wedge dv$$

form which it following that

$$K = (\frac{\mathrm{d}}{\mathrm{du}}(\frac{\mathrm{Gu}}{\mathrm{E}}) + \frac{\mathrm{d}}{\mathrm{dv}}(\frac{\mathrm{Ev}}{\mathrm{G}}).$$

(5.2) On The Gaussian and Mean Curvature of Certain Surfaces

The Gaussian and mean curvatures of surfaces are real valued functions of two real variables. We apply our software for differential geometry to represent the Gaussian and mean curvatures of various types of surfaces.

1. Introduction and notations:

Throughout this chapter we assume that $D \subset \Re^2$ is a domain and surfaces are given by a parametric representation

$$\vec{x}_1(u^i) = \left(x^1(u^1, u^2), x^2(u^1, u^2), x^3(u^1, u^2)\right)\left((u^1, u^2) \in D\right) \qquad (5.18)$$

where the component functions $x^j: D \rightarrow \Re$ ($j = 1,2,3$) have continuous partial derivatives of order $r \geq 1$, denoted as usual by $\vec{x} \in C^r(D)$, and the vectors $\vec{x}_k = \partial\vec{x}/\partial u^k$ (k = 1,2) satisfy $\vec{x}_1 \times \vec{x}_2 \neq 0$. If we denote the surface normal vectors, the first and second fundamental coefficients of a surface S given by (5.18) by

$$\vec{N}(u^i) = \frac{\vec{x}_1(u^i) \times \vec{x}_2(u^i)}{\|\vec{x}_1(u^i) \times \vec{x}_2(u^i)\|}, g_{jk}(u^i) = \vec{x}_j(u^i) \bullet \vec{x}_k(u^i) \quad and$$

$$L_{jk}(u^i) = \vec{N}(u^i) \bullet \vec{x}_{jk} \;\; where \;\; \vec{x}_{jk}(u^i) = \frac{\partial^2\vec{x}}{\partial u^j \partial u^k} \;\; for\, j, k = 1,2,$$

respectively, then the functions $K : D \rightarrow \Re$ and $H : D \rightarrow \Re$ with

$$K = \frac{L}{g} \; and \; H = \frac{1}{2g}(L_{11}g_{12}g_{12} + L_{22}g_{11}),$$

Where $g = det(g_{jk})$ and $L = det(L_{jk})$, are the Gaussian curvature· and the mean curvature of S. We use our software to give a graphical representation of the Gaussian and mean curvatures of some interesting surfaces.

2. Pseudo-Spheres:

Pseudo-spheres are surfaces of revolution with constant Gaussian curvature. Let 1 be a curve with parametric representation $\vec{x}$ (s) = (r(s), 0, h(s)) and r(s) > 0 ($s \in I \subset \Re$), where s is the arc length along γ and RS be the surface of revolution generated by the rotation of γ about the x^3-axis. Putting $u^1 = s$ and writing u^2 for the angle of rotation, we obtain the following parametric representation for RS on $D = I \times (0,2\pi)$

$$\vec{x}(u^i) = \left(r(u^1)\cos u^2, r(u^1)\sin u^2, h(u^1)\right)\left((u^1, u^2) \in D\right) \qquad (5.19)$$

Omitting the argument u^1, we find that the fundamental coefficients of RS are given by $g_{11} = (r')^2 + (h')^2 = 1$, since u^1 is the arc length along γ, $g_{12} = 0$, $g_{22} = r^2$, $L_{11} = r'\,h'' - r''\,h'$, $L_{12} = 0$ and $L_{22} = rh'$. So the Gaussian curvature of RS is given by $K = r^{-1}(r'h'' - r''h')$. Since $(r')^2 + (h')^2 = 1$ implies $r'r'' + h'h'' = 0$, we obtain $K = r^{-1}(r'h''h' - r''(h')^2) = -r^{-1}((r')^2 + (h')^2)r'' = -r''/r$ and consequently

$$r''(u^1) + K(u^1)r(u^1) = 0. \qquad (5.20)$$

first, we assume $K = 0$. Then $r = c_1u^1 + c^2$ with the constants c_1 and c_2. If we choose $c_1 = 0$ then $h' = \pm 1$ implies $h = \pm u^1 + d$ with some constant d, and we obtain a circular cylinder. If $c_1 \neq 0$ then $(r')^2 + (h')^2 = 1$ implies $|c_1| \leq 1$. For $|c_1| = 1$, we have $h' \equiv 0$, hence $h \equiv const$, and we obtain a plane. For $0 < |c_1| < 1$ and a suitable choice of the coordinate system, we have $r = c_1u^1$ and $h = d_1u^1$ for some constant d_1 with $c_1^2 + d_1^2 = 1$, and we obtain a circular cone.

Let $K \neq 0$. Then we may assume $K = \pm 1$.

Let $K = 1$. Then the general solution of (5.20) is given by $r(u^1) = C \cdot \cos(u^1 + u_0^1)$. By a suitable choice of the arc length, we may assume that $C > 0$ and $u_0^1 = 0$. Now $(r')^2 + (h')^2 = 1$ implies

$$h(u^1) = \int \sqrt{1 - C^2 \sin^2(u^1)}\, du^1. \qquad (5.21)$$

The choice $C = 1$ yields the unit sphere. For $C \neq 1$, the integral in (5.21) is elliptic. It exists on $(-\pi/2, \pi/2)$ if $C < 1$, on $(-arc\sin(1/C), arc\sin(1/C))$ if $C > 1$.

finally, let $K = -1$. Then the general solution of (5.21) is given by $r(u^1)$ $C_1 \cosh u^1 + C_2 \sinh u^1$. In the special case $C_1 = 1/2 = -C_2$, we obtain

$$r(u^1) = e^{-u^1} and\ h(u^1) = \int \sqrt{1 - e^{-2u^1}}\, du^1\ for\ u^1 > 0.$$

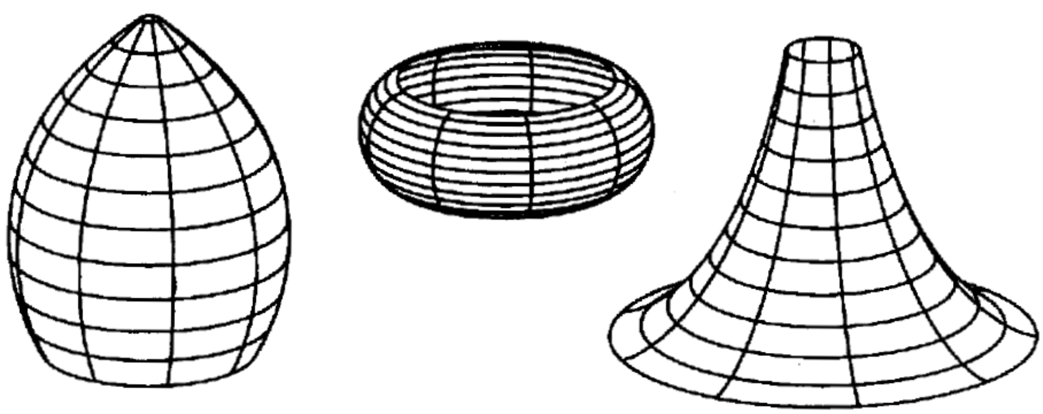

Figure 5.1: Pseudo-spheres

$K = 1$, C = 0.75; $K = 1$, C = 1.5; and $K = -1, C_1 = 1/2 = -C_2$.

3. Exponential Cones:

Let h: $\mathbb{C} \rightarrow \mathbb{C}$ be an analytic function and $f = |h| : \Re^2 \rightarrow \Re$. We write $z = u^1 + i.u^2$. Then the function h generates an explicit surface with the parametric representation

$$\vec{x}(u^i) = \left(u^1, u^2, f(u^1, u^2)\right)\left((u^1, u^2) \in \Re^2\right) \tag{5.22}$$

in a very natural way, and represents the modulus of h. A classification of surfaces of this kind with Gaussian curvature K of constant sign. The surfaces generated by the function h defined by $h(z) = z^{\alpha+i.\beta}$ for real constants $\alpha$and $\beta$are called exponential cones. Here the cases $\alpha \geq 1$ and $a < 1$ correspond to $K > 0$ and $K < 0$, respectively. Using the representation of complex numbers by polar coordinates $z = \rho e^{i\phi}$for $\rho > 0$ and $\phi \in (0{,}2\pi)$, we obtain $f(z) = \rho^{\alpha} e^{-\beta\phi}$. We put $u^1 = \rho$ and $u^2 = \phi$. Then exponential cones on $D = (0,\infty) \times (0{,}2\pi)$are given by

$$\vec{x}(u^i) = \left(u^1 \cos u^2, u^1 \sin u^2, (u^1)^{\alpha} e^{-\beta u^2}\right)\left((u^1, u^2) \in D\right);$$

are special cases of screw surfaces given by

$$\vec{x}(u^i) = (u^1 \cos u^2, u^1 \sin u^2, f(u^1, u^2)). \tag{5.23}$$

since the first and second fundamental coefficients of exponential cones are

$$g_{11} = 1 + \alpha^2 (u^1)^{2\alpha - 2\beta u^2}, \quad g_{12} = -\alpha\beta (u^1)^{2\alpha - 2\beta u^2},$$

$$g_{22} = (u^1)^2 \left(u + \beta^2 (u^2)^{2\alpha-2} e^{-2\beta u^2}\right),$$

$$g = (u^1)^2 \left(1 + (\alpha^2 + \beta^2)^2 (u^1)^{2\alpha-2} e^{-2\beta u^2}\right)$$

$$L_{11} = \frac{1}{\sqrt{g}} \alpha(\alpha - 1)(u^1)^{\alpha-1} e^{-\beta u^2},$$

$$L_{12} = \frac{1}{\sqrt{g}}(1-\alpha)\beta(u^1)^{\alpha-1}e^{-\beta u^2},$$

$$L_{22} = \frac{1}{\sqrt{g}}(\alpha+\beta^2)(u^1)^{\alpha+1}e^{-\beta u^2},$$

we obtain, putting $\delta = \alpha^2 + \beta^2$ *and* $\gamma = (\alpha-1)\delta$,

$$K(u^i) = (\alpha-1)\delta\frac{(u^1)^{2\alpha}e^{-2\beta u^2}}{g^2} = \gamma\frac{(u^1)^{2\alpha-4}e^{-2\beta u^2}}{\left(1+\delta(u^1)^{2\alpha-2}e^{-2\beta u^2}\right)^2}$$

and similarly

$$H(u^i) = \delta\frac{(u^1)^{\alpha-2}e^{-\beta u^2}\left(1+\alpha(u^1)^{2\alpha-2}e^{-2\beta u^2}\right)}{2\left(1+\delta(u^1)^{2\alpha-2}e^{-2\beta u^2}\right)^{3/2}}.$$

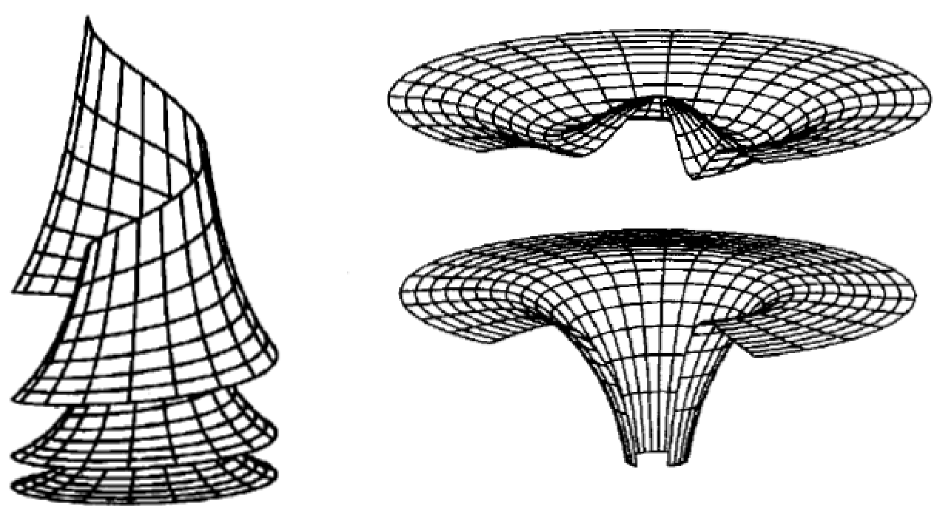

Figure 5.2: Exponential cone,

α= -1, $\beta = -0.1$ and its Gaussian and mean curvature.

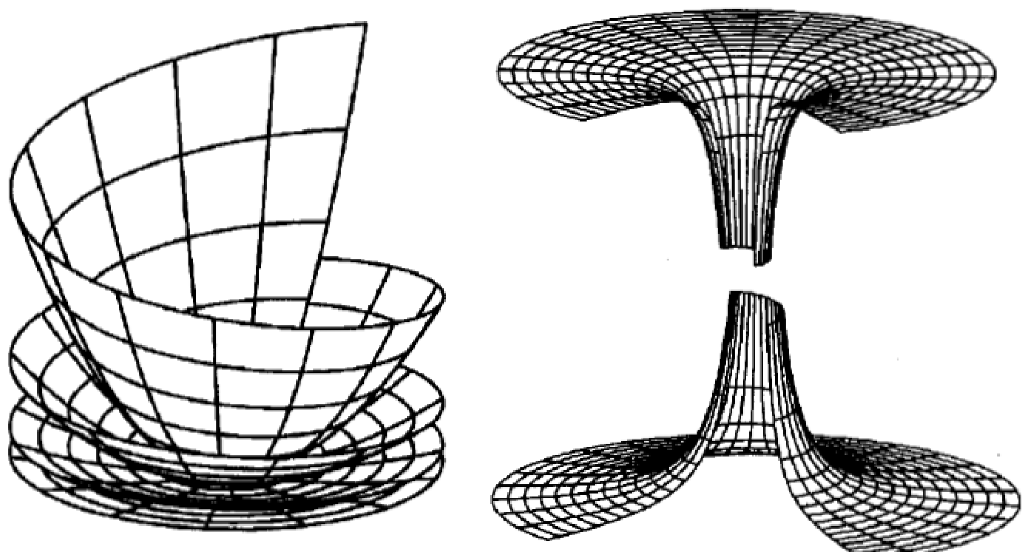

Figure 5.3: Exponential cone,

α= 0.5, $\beta = -0.05$ and its Gaussian and mean curvature.

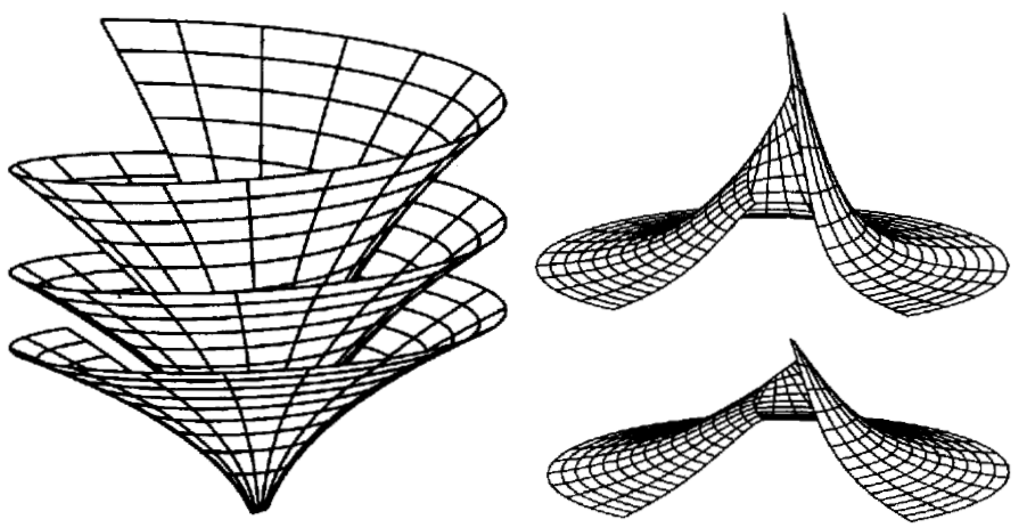

Figure 5.4: Exponential cone,

$\alpha = 2, \beta$ and its Gaussian and mean curvature.

4. Minimal surfaces :

Surfaces with identically vanishing mean curvature are called minimal surfaces. It is well known (cf. e. g. that if S is a surface the boundary of which is a closed curve such that the surface area of S is less than or equal to the surface area of any other "neighboring" surface with the same boundary then S has identically. vanishing mean curvature.

The mean curvature of surfaces of revolution with parametric representation (5.19) is given by $H = (r^2(r'h''-r''h')+rh')/(2r^2)$. Now $H = 0$ is equivalent with $r(r' h'' - r'' h') + h' = 0$. If $h' = 0$, we obtain a plane. If $h'' = 0$ then, multiplying by h' and using $h''h' = -r'r''$ and $(r')^2+(h')^2 = 1$, we obtain $r''r = (h')^2$. This yields $(r^2)'' = 2$, since $r''r = 1/2(r^2)''-(r')^2$ and $(r')^2 + h')^2 = 1$. By a suitable choice of the parameter u^1, we obtain $r(u^1) = \sqrt{(u^1)^2 + c^2}(u^1 \in \mathbb{R})$ where c is a constant. Ifc $\neq 0$, then $r'(u^1) = u^1$ since $r(u^1) \geq 0$, and then $h'(u^1) = 0$, and we obtain a plane. If c $\neq 0$, then $r'(u^1) = u^1((u^1)^2 + c^2)^{-1/2}$, and $(r')^2 + (h')^2 = 1$ yields $(h')^2 = c^2((u^1)^2 + c^2)^{-1}$, hence $h'(u^1) = |c|/\sqrt{(u^1)^2 + c^2}$. Therefore $h(u^1) = C.\ \mathrm{arcsinh}(u1/c)$ for a suitable choice of the coordinate system. Putting $u^{*1} = h(u^1)$ and $u^{*2} = u^2$, we obtain

$$\vec{x}(u^{*i}) = (|c| \cosh u^{*1} \cos u^{*2}, |c| \cosh u^{*1} \sin u^2, u^{*1})$$

$$\left((u^{*1}, u^{*2}) \in \Re \times (0.2\pi)\right).$$

Thus the minimal surfaces of revolution are planes and catenoids.

Another minimal surface is Scherk's surface, given by a parametric representation

$$\vec{x}(u^i) = \left(u^1, u^2, \log\left(\frac{\cos u^2}{\cos u^1}\right)\right)\left((u^1, u^2) \in \Re_{kj}\right)$$

where, for $k, j \in \mathbb{Z}$ 7L, with $k + j \in 2 \cdot \mathbb{Z}$

$$\Re_{Kj} = I_k \times I_j = \left(\left(k - \frac{1}{2}\right)\pi, \left(k + \frac{1}{2}\right)\pi\right) \times \left(\left(j - \frac{1}{2}\right)\pi, \left(j + \frac{1}{2}\right)\pi\right).$$

It is easy to see that the Gaussian curvature of Scherk's minimal surface is given by $K(u^1, u^2) = -\cos^2 u^1 \cos^2 u^2 (1 - \sin^2 u^1 \sin^2 u^2)^{-2}$.

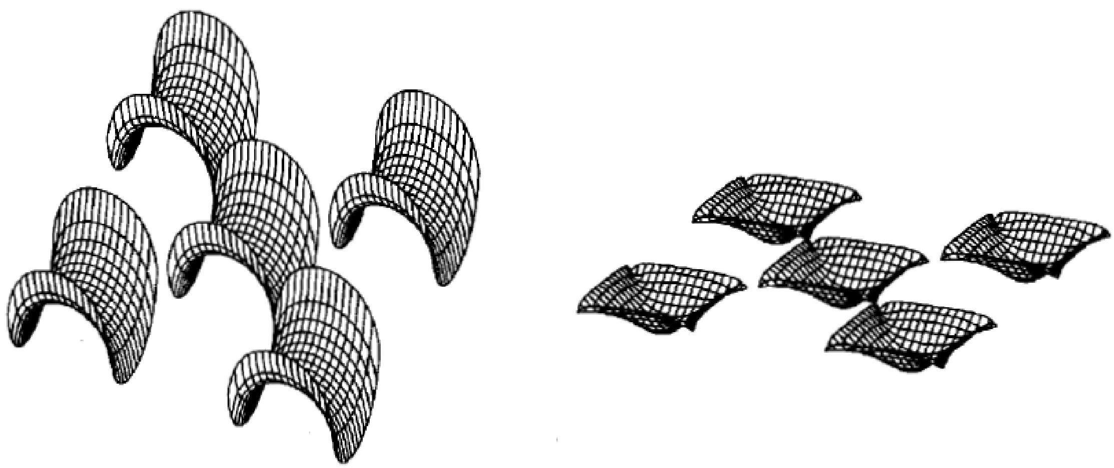

Figure 5.5: Scherk's minimal surface and its Gaussian curvature.

5. Surfaces generated by the modulus of analytic functions:

The Gaussian and mean curvatures of surfaces with parametric representation (5), where $f = |h|$ and h is an analytic function, are given by

$$K = \frac{|h''|^2}{g^2}\left(Re\left(\frac{(h')^2}{h''h}\right) - 1\right) \; where \; g = 1 + |h'|^2 \quad and$$

$$H = \frac{2}{g^{\frac{3}{2}}}|h|\left(\left|\frac{h'^2}{h}\right| g - |h''|^2 Re\left(\frac{(h')^2}{h''h}\right)\right).$$

we consider the function h defined by $h(z) = 1/\sin \pi z$ $(z \in \mathbb{Z})$, and put $w = w(z) = \cos 2\pi z$ and $\psi(z) = (h'(z))^2(h''(z)h(z))^{-1}$. Since $2\cos^2 \pi z = 1 + \cos 2\pi z = 1 + w$ and $2 \sin^2 \pi z = 1 - \cos 2\pi z = 1 - w$, we obtain

$$h'(z) = -\pi \frac{\cos \pi z}{\sin^2 \pi z},$$

$$h''(z) = \pi^2 \frac{\sin^2 \pi z + 2\cos^2 \pi z}{\sin^3 \pi z} = \frac{\pi^2}{2} \frac{w+3}{\sin^3 \pi z}, \qquad and\ so$$

$$\psi = \frac{2\pi^2 \cos^2 \pi z}{\pi^2 (w+3)} = \frac{w+1}{w+3}.$$

Therefore

$$\Re e\big(\psi(w)\big) - 1 = \Re e(\psi(w) - 1) = \Re e\left(\frac{w+1}{w+3} - 1\right) = -2\Re e\left(\frac{1}{w+3}\right)$$

$$= -\left(\frac{1}{w+3} + \frac{1}{w+1}\right) = -2\frac{\frac{1}{2}(\overline{w} + w) + 3}{|w+3|^2} = -\frac{2\big(3 + \Re e(w)\big)}{|w+3|^2}.$$

Furthermore

$$|h''|^2 = \frac{\pi^4}{4} \frac{|w+3|^2}{|\sin^2 \pi z|^2}, and\ with\ \phi(w) = |w-1|^2 + 2\pi |w+1|$$

$$g = 1 + |h'|^2 = 1 + \frac{\pi^2}{2} \frac{|1+w|^2}{|\sin^2 \pi z|^2}$$

$$= \frac{1}{4|\sin^2 \pi z|^2} (|2\sin^2 \pi z|^2 + 2\pi^2 |w+1|) = \frac{1}{4|\sin^2 \pi z|^2} \phi(w),$$

and so

$$K = \frac{|h''|^2}{g^2} \Re e(\psi(w) - 1) = \frac{4\pi |w-1| \big(3 + \Re e(w)\big)}{\phi^2(w)}.$$

Finally putting

$$w_1(u^i) = \Re e(w) = \cosh 2\pi u^2 \cos 2\pi u^1,$$

$$w_2(u^i) = |w - 1| = \frac{1}{\sqrt{2}}(\cosh 4\pi u^2 + \cos 4\pi u^1 + 4w_1(u^i) + 2)^{\frac{1}{2}},$$

$$w_3(u^i) = |w + 1| = \frac{1}{\sqrt{2}}(\cosh 4\pi u^2 + \cos 4\pi u^1 + 4w_1(u^i) + 2)^{\frac{1}{2}}$$

and $w_4(u^i) = \phi(w) = \left(w_2(u^i)\right)^2 + 2\pi^2 w_3(u^i)$, *we have*

$$K(u^i) = -\frac{4\pi^2 w_2(u^i)\left(3+w_1(u^i)\right)}{\left(w_4(u^i)\right)^2} \tag{5.24}$$

similarly, putting

$$w_5(u^i) = \left(w_3(u^i)\right)^2, w_6(u^i) = w_3^2(u^i) w_4(u^i) - 2\pi^2 w_5(u^i) \quad and$$

$$f(u^i) = |h(z)| = \frac{\sqrt{2}}{\sqrt{\cosh 2\pi u^2 - \cos 2\pi u^1}}, \tag{5.25}$$

We represent the Gaussian and mean curvatures of exponential cones and explicit surfaces as screw surfaces and explicit surfaces by putting $f = K$ and $f = H$ in (5.23) and (5.22), respectively.

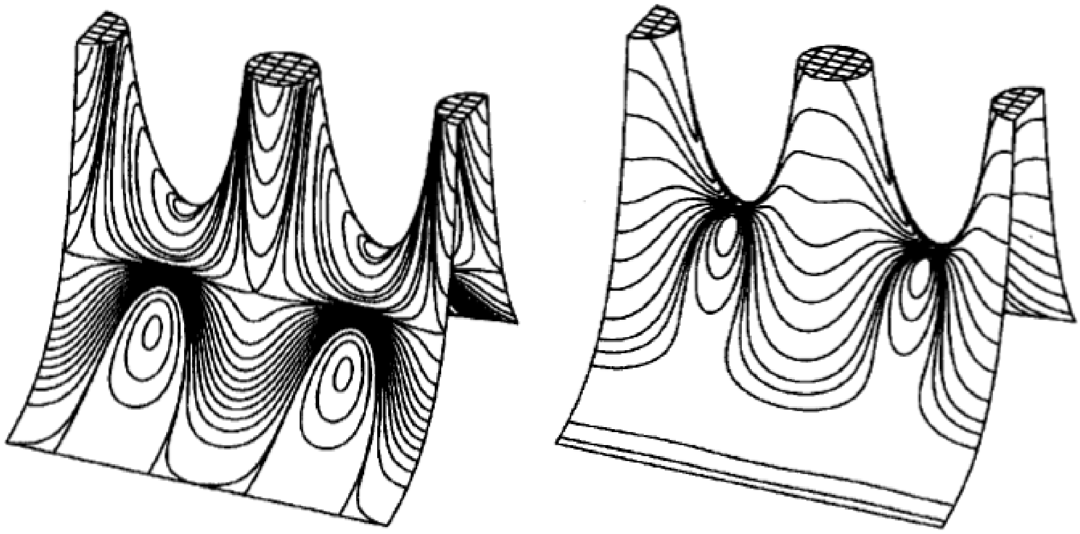

Figure 5.6**:** Lines of constant Gaussian and lines of constant mean curvature.

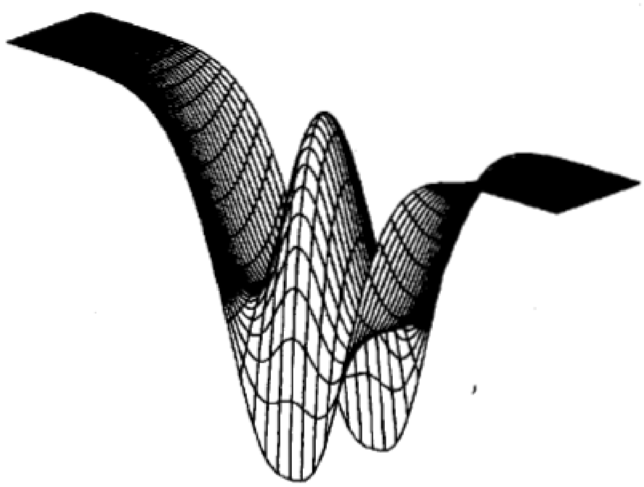

Figure 5.7**:** Gaussian curvature given by (5.24).

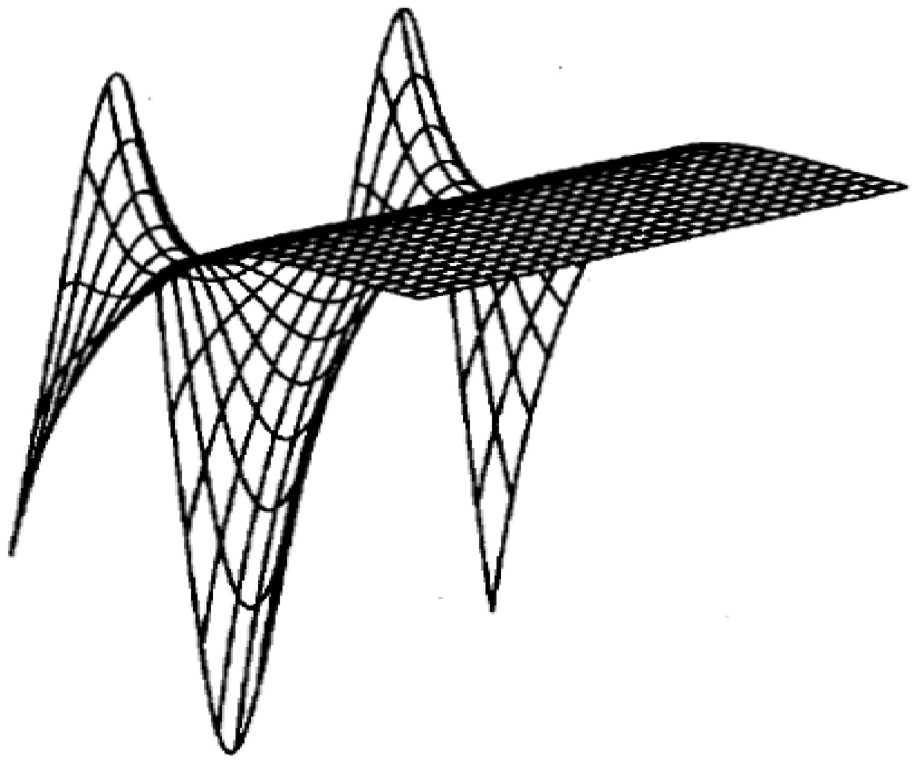

Figure 5. 8: Mean curvature given by (5.25).

(5.3) Applications to Hamiltonian Systems

We now apply the classifications of the previous section to Hamiltonian systems defined in particular Riemannian spaces.

1. Two-dimensional Euclidean space E^2:

In this case $R_{1212} = 0$, which entails

$$E_2\alpha - E_1\beta = \alpha^2 + \beta^2.$$

Consider now the following three separable cases (SC), defined with respect to the functions α and β.

$$SCI: \alpha = \beta = 0.$$

in this case the separable coordinates are obviously Cartesian and $R_{1212} = 0$, is automatically satisfied.

Solving Eq. (4.75) we obtain that the metric can be written as follows:

$$\mathrm{d}s^2 + \mathrm{d}u^2 + u^2\mathrm{d}v^2, \quad (5.26)$$

which we immediately recognize as the Euclidean metric in polar coordinates.$SCIII: \alpha\beta \neq 0$.

Employing (4.44.) ($\alpha\beta \neq 0, R_{1212} = 0$)to find the functions *A(u)* and *B(v)* defining the formula for the metric of a Liouville surface, we arrive at the following equation.

$$(A(u) + B(v))(A''(u) + B''(v) = \left(A'(u)\right)^2 + \left(B'(v)\right)^2,$$

which after taking partial derivatives reduces to

$$\frac{A'''}{A'(u)} + \frac{B'''(v)}{B'(v)} = k^2 \quad (5.27)$$

for some constant $k \geq 0$. Solving (5.27) separately for $k = 0$ and $k \neq 0$ yields the metrics

$$ds^2 = (u^2 + v^2)(du^2 + dv^2), \quad (5.28)$$

and

$$ds^2 = a^2(\cosh^2(u) - \cos^2(v))(du^2 + dv^2), \quad (5.29)$$

Respectively, where *a* is a scaling parameter. We note that the expressions (5.28) and (5.29) represent the Euclidean metric in parabolic and elliptic–hyperbolic coordinates, where *a* represents half the distance between the foci. Hence, we have extracted the four separable systems of coordinates in the Euclidean space by employing the method of moving frames.

The corresponding Killing tensors, second first integrals and potential functions can be recovered by making use of the formulas derived in Section (4.5).

2. Surfaces of rotation:

A surface of rotation is the surface generated by the rotation of a plane curve C around an axis in its plane. If C is parameterized by the equations $\rho = \rho(u)$ and $z = z(u)$, the position vector of the surface of rotation is $\mathbf{r} = \{\rho(u) \cos v, \rho(u) \sin v, z(u)\}$, where u is the parameter of the curve C, ρ is the distance between a point on the surface and the axis z of rotation and v is the angle of rotation, which is the ignorable (cyclic) coordinate. The metric of the surface of rotation is

$$ds^2 = ((\rho')^2 + (z')^2)du^2 + \rho^2 dv^2. \tag{5.30}$$

Clearly, the metric (5.30) can be reduced to the form (4.67) by an appropriate coordinate transformation. Once the curvature $\Re_{1212}(u)$ is known, the function(s) $g(u)$ and the corresponding

metric(s) may be recovered from (4.75) and vice versa. Consider an example. The metric

$$ds^2 = a^2 du^2 + \backslash\ell^2 \left(1 + \frac{a}{\ell}\cos u\right)^2 dv^2 \tag{5.31}$$

defines the surface of a two-dimensional torus T^2, where a and ℓ are the radii of the rotating and axial circles, respectively. We note that in this paper we do not consider global properties of two-dimensional pseudo-Riemannian manifolds; hence here T^2 is not a topological torus. Locally, the metric (5.31) yields one system of separable coordinates with $g(u) = \ell(1 + (a/\ell) \cos(u/a)$,

$\Re_{1212}$ = cos*(u/a)/(aℓ+ a* cos*(u/a))* and the other quantities as in Case II of Section(4.3) corresponding to the given *g(u)*.

3. Surfaces of constant curvature:

In this section, we assume the curvature $R_{1212} = \epsilon a^2$, where $\epsilon = \pm 1$ and $a > 0$ is constant. Let us consider again the two cases: $\alpha = 0$, $\beta \neq 0$ and $\alpha\beta \neq 0$.

Case I: $\alpha = 0$, $\beta \neq 0$. In this case the coordinate v is ignorable (cyclic). Solving (4.75) for *a(v)* = const, yields:

$$g(u) = c_1 \cos au + c_2 \sin au = \tilde{c}_1 \cos au + \tilde{c}_2 \frac{1}{a} \sin u, \ \epsilon = 1,$$

$$g(u) = c_3 e^{au} + c_4 e^{au} = \tilde{c}_3 \cosh au + \tilde{c}_4 \frac{1}{a} \sinh au, \ \epsilon = -1.$$

Now varying the constants of integration we recover four distinct solutions for *g(u)* corresponding to the following metrics.

$$ds^2 = \frac{1}{a}(du^2 + \sin au \, dv^2), \qquad \epsilon = 1, \tag{5.32}$$

$$ds^2 = du^2 + \cosh^2 au \, dv^2, \quad \epsilon = -1, \tag{5.33}$$

$$ds^2 = du^2 + \left(\frac{\sinh au}{a}\right)^2 dv^2, \tag{5.34}$$

$$ds^2 = du^2 + e^{-2au} dv^2. \tag{5.35}$$

using the explicit expression for the function *g(u)* above and the formulas (4.69), (4.70) and (4..72) we can write down in each case the corresponding potentials, Killing tensors and second first integrals.

Case II: $\alpha\beta \neq 0$. Again, assume $\Re_{1212} = \epsilon a^2$. Then (4.89) reads

$$(A+B)(A''+B'')-(A')^2-(B')^2=-2\epsilon a^2(A+B)^3, \qquad (5.36)$$

where $A = A(u)$ and $B = B(v)$. Eq. (5.36) can be separated as follows:

$$\frac{A'''}{A'}+12\epsilon a^2 A=-\frac{B'''}{B'}-12\epsilon a^2 B=\lambda.$$

Hence, we arrive at the following two equations for A and B, respectively,

$$A'''+12\epsilon a^2 AA'=\lambda A', \quad B'''+12\epsilon a^2 BB'=-\lambda B'. \qquad (5.37)$$

Assuming $\lambda \neq 0$ and solving (5.37) with respect to u and v, we get

$$\pm du=\frac{dA}{(-4\epsilon a^2 A^3+\lambda A^2+2\ell A+2m)^{\frac{1}{2}}}, \qquad (5.38)$$

$$\pm dv=\frac{dB}{(-4\epsilon a^2 B^3-\lambda B^2+2\tilde{\ell}B+2\tilde{m})^{\frac{1}{2}}}, \qquad (5.39)$$

where $p^3(x) = x^3 + px + q$ with arbitrary coefficients p and q. Note that we have derived the metric (5.40) without solving (5.38) and (5.39) for A and B, respectively. Comparing the metrics (4.79) and (5.40) we see that the latter metric is not in the Liouville form and so we cannot complete the analysis by deriving the corresponding first integrals, potentials and Killing tensors. However, since the functions A and B and their derivatives in (5.38) and (5.39) essentially parametrize appropriate elliptic curves, clearly it can be done by expressing A and B in terms of the Weierstrass function $\wp$. Indeed, by appropriate linear transformations Eqs. (5.38) and (5.39) can be transformed into the corresponding form of the Weierstrass differential equation

$$\left(\frac{d\wp}{dz}\right)^2=4\wp^3-g_2\wp-g_3, \qquad (5.40)$$

thus leading to the following solutions for the functions *A(u)* and *B(v)*, respectively:

$$A(u) = \wp\left(a\sqrt{-\epsilon}u + c_1; \omega_1, \omega_2\right) - \lambda, \tag{5.41}$$

$$B(v) = \wp\left(a\sqrt{-\epsilon}v + c_2; \omega_1, -\omega_2\right) + \lambda, \tag{5.42}$$

where c_1, c_2, λ are arbitrary functions and ω_1, ω_2 define the periods of the meromorphic, doubly periodic function $\wp$. Now we can use the expressions (5.41) and (5.42) and the analysis of Section 3 to derive in each case the corresponding separable potential (formula (4.82)), Killing tensor (formula (4.81)), as well as the second first integral (formula (4.84)).

Let x_1, x_2 and x_3 be the roots of p_3: $p_3(x) = (x - x_1)(x - x_2)(x - x_3)$.Without loss of generality we impose the condition $A > B$. To extract all the metrics depending on different choices of x_1, x_2 and x_3, we impose the condition that the right-hand side of (5.40) must be positive definite. When $\epsilon = 1$ there is only one possibility for A and B for which (5.40) is positive definite, while $\epsilon = -1$ leads to six different possibilities:

$$x_1 < B < x_2 < A < x_3, \quad \epsilon = 1, \tag{5.43}$$

$$x_1 < x_2 < B < x_3 < A, \quad \epsilon = -1, \tag{5.44}$$

$$B < x_1 < x_2 < x_3 < A, \tag{5.45}$$

$$B < x_3 < A, \quad x_1 = \bar{x}_2, \tag{5.46}$$

$$x_1 = x_2 < B < x_3 < A, \tag{5.47}$$

$$B < x_1 = x_2 < x_3 < A, \tag{5.48}$$

$$B < x_1 = x_2 = x_3 < A. \tag{5.49}$$

we observe that these separable cases were first derived by Olevsky, while studying separability of Laplace–Beltrami's operator in the spaces of constant curvature. He used Eisenhart's (coordinate) approach to the problem. The moving frame method applied to two-dimensional separable Hamiltonian systems yields the same results without considering initially a particular system of coordinates. We note that the separable coordinates *(A,B)* are essentially the eigenvalues of the Killing tensor $\mathbf{K}_1$ in (4.81).

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