Dedication

This thesis is dedicated to my beloved Parents,

Brothers and my Sister,

without their knowledge, wisdom, and guidance, I would not have the goals I have to strive and be the best to achieve my ambitions.

Acknowledgments

First and foremost, I thank God how gave me love and grace upon me and gave me the resolve to finish this thesis.

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Abstract

We investigate nonlinear evolution equations with Lax integrability using the tool of differential forms and exterior differential systems. We restate the Lax equations of the nonlinear evolution equation in the form of an exterior differential system. Therefore, we embark on an attempt to apply the method of Estabrook-Wahlquist EW prolongation structure, which has reviewed and developed the new and extended EW technique. We also extended our analysis to a study of differential systems which defined equations that include the Camassa-Holm and Degasperis-Procesi equations as specific cases.

الخلاصة

في هذه الدراسة، سوف نقوم بدراسة معادلات التطور غير الخطية مع قابلية لاكس للتكامل وذلك باستخدام أدوات الصيغ التفاضلية وأنظمة التفاضل الخارجي. كما شرعنا في المحاولة لمعاينة وتطوير نهج استابروك ووالكوست وهو ما يعرف بهيكل الاطالة. كما تم تمديد تحليل الدراسة للنظام التفاضلي ليشمل معادلتي "كاماسا-هولم" و"ديجاسبيرس-بروسيس" كحالات خاصة.

Introduction

An exterior differential system on a manifold is a natural generalization of a system of partial differential equations. The theory is highly geometrical and computational simultaneously. Created and developed by Cartan-Kähler, exterior differential system theory has been used largely by differential geometrics.

An exterior differential system is defined to be a finite set of homogeneous analytic differential forms on a domain D in a real Euclidean space; analytic functions are considered as differential forms of degree zero. A submanifold D_1 of D is called an integral manifold (or integral, or solution), of the system if the differential forms on D_1 induced by those in the system are always zero forms. The main purpose of the theory of exterior differential systems is, it seems to the writer to find an effective method to construct all integral manifolds and to clarify the structure of the set of all integral manifolds. The theory is essentially a theory of systems of partial differential equations. Namely, given a system of partial differential equations, for instance

$$F(x, y, u, \partial u/\partial x, \partial u/\partial y) = 0,$$

we construct, introducing new variables p and q, a differential system consisting of

$$F(x, y, u, p, q) = 0,$$
$$du - pdx - qdy = 0,$$

on an appropriate domain in the five dimensional Euclidean space (x, y, u, p, q). If D_1 is a two dimensional integral manifold of this system and if dx and dy are linearly independent on D_1, we can express the submanifold in the form $\big(x, y, u(x, y), p(x, y), q(x, y)\big)$. Then the function $u(x, y)$ is a solution of the original equations. Conversely if $u(x, y)$ is a solution of the original equations, the submanifold $F(x, y, u(x, y), \partial u(x, y)/\partial x, \partial u(x, y)/\partial y)$ is an integral manifold of the system.

The above observation suggests the following restriction to the problem. Fix a set of linearly independent Pfaffian forms $dx^1, \dots, dx^p$ on D and restrict our attention to the integral manifolds on which $dx^1, \dots, dx^p$ are linearly independent. In this case, the functions $x^1, \dots, x^p$ will be called independent variables.

The thesis involves five chapters. Firstly, we dealt with exterior differential forms. Some algebraic properties of exterior forms are revealed and a degree decreasing operation called the interior product of a form with a vector field is defined. Then the ideals of the exterior algebra are defined, the exterior derivative of differential forms is introduced as to satisfy certain requirements. After briefly glancing over closed ideals, the Lie derivative of a form with respect to a vector field is introduced.

Secondly, we introduced the concepts of closed exterior differential system, outlined the theory of completely integrable systems (Frobenius theorem) and the Pfaffian equation. Cauchy characteristics of exterior differential systems come up naturally. Some arithmetic invariants are introduced for Pfaffian systems. Then, we present the integral elements and Cartan-Kähler theorem which is gives as generalization of the Cauchy-Kowalewsky theorem, we arrive at the notion of Kähler-regular flag of the integral elements. Moreover, then we introduce the substantial concepts of involution and prolongation which are plays an enormous role on forthcoming chapters.

Eventually, we investigate with Lax integrability using the tool of differential forms and exterior differentials. The relation between complete integrability and Lax integrability will also be discussed. on the other hand, the Lax equation of the NLEE will be restated in the exterior differential form. Last but not least we devote our study about the method of Wahlquist-Estabrook, we attempt to apply prolongation structures to generalized Korteweg-deVries (KdV) equation. Furthermore, we illustrate the method in detail for Lax-integrable versions of the nonlinear generalized (KdV) equation and conservation laws arise and can be expressed in this context will be discussed based on the defining exterior differential system. Finally, this work is extended to a study of a differential system of one-forms which define an equation that includes the Camassa-Holm equation and Degasperis-Procesi equations as specific cases.

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