Sudan University of Science &Technology

College of graduate studies

The Geometric Extension of Green's Functions and its Application in Global Wave Equations

التمديد الهندسي لدوال قرين و تطبيقها على معادلات الموجة الموسعة

A thesis Submitted In Fulfillment for the Degree of Doctor of Philosophy In Mathematics

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بسم الله الرحمن الرحيم

(وَاللَّهُ أَخْرَجَكُمْ مِنْ بُطُونِ أُمَّهَاتِكُمْ لَا تَعْلَمُونَ شَيْئًا وَجَعَلَ لَكُمُ السَّمْعَ وَالْأَبْصَارَ وَالْأَفْئِدَةَ ۙ لَعَلَّكُمْ تَشْكُرُونَ)

النحل- ٧٨

DEDICATION

To my God for guidance and protection throughout my life

And to my father

Abd Elazeem Abd Elahleem Abobuker

Who spared no effort to bring me up and encouraged me to be who I am.

And to Souls of my beloved mother **Aisha Yaseen Ahmed** and sister **Somia**

And all Brothers and sisters.

With special dedication to my wife

Wafa Ipraheem Mohd Ali.

And Kids

Ahmed

Mozan

Abobuker Alsdeeg

Minat Allah

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Eventually with due respect and honor I have to acknowledge, every single help from many people that made the work fruitful.

Abstract

In this research we considered the geometrical interpretation of the wave equation. We have constricted the geometrical set up for the problem such as fiber bundle and it's coresection . We have also utilized Lorentzian geometry to formulate our problem, where we have described our boundary conditions on Cauchy hyper-surface. This has led to a parallel construction of Green's Function appropriate to a global description of wave equation on differential manifold. The formulation of the solution yields the local solution on space time as known before.

الخلاصة

في هذا البحث تناولنا التفسير الهندسي لمعادلة الموجة . كما قمنا بصياغتها هندسيا كحزمة من الألياف و مقطعها ، و التي استخدمنا فيها هندسة لورنتز لمناسبتها للصياغة الهندسية لمعادلة الموجة ، وكذلك قمنا بوضع الشروط الحدية علي مسألة كوشي علي السطوح الزائدية. مما ساعدنا علي صياغة وتعميم الصورة الموسعة لدالة قرين لمعادلة الموجة علي المانيفولد التفاضلي ، والتي ينتج عنها الحل الموضعي في فضاء الزمان– المكان الرباعي كما هو مُعَرف من قبل.

Introduction:

The thesis developed Green's function techniques for both single and multiple dimension problems in differential equations,[1, 2] then applied these techniques to solve and describe the different sorts of boundaries and boundary conditions which can occur for the N-dimensional problem and write the solution for the N-dimensional problem in terms of the Green's function[3, 4] . As the geometric meaning of Green's function, the thesis discusses the solution theory of geometric wave equations as they arise in Lorentzian geometry[5, 6], for a normally hyperbolic differential operator the existence and uniqueness properties of Green functions and Green operators has been discussed including a detailed treatment of the Cauchy problem on a globally hyperbolic manifold both for the smooth and finite order setting,[7-10] An introduction to the theory of distributions on manifold is also discussed [5, 11], The thesis is composed of five chapters.

In the first chapter the Green's function methods was described and developed to n-dimensional spaces, Section 1.1 described an Inner product and was used to introduce the notation of orthognality in a Hilbert spaces and the geometry of Hilbert spaces is almost in complete agreement with the intention of linear spaces [12, 13] . Also described some important classes of bounded linear operators on Hilbert space, including Projections, adjoint operators, unitary operators, and self-adjoint operators. also the (Riesz representation) Theorem 1.1.8 was proved, which characterizes the bounded linear functional on Hilbert spaces , some general definitions of unbounded operators, adjoin and self-adjoint of linear differential operators was given .

Section 1.2 was aimed to describe the theory of one- dimensional Green' s function for a second order ordinary linear differential equation with a homogeneous boundary conditions, and concerned with a self-ad joint equation (1.13) and gave some properties of the Green Function and generalization to equation of n^{th} order [14, 15].

Section 1.3 extended the study of one- ndimensional Green's functions of linear ODEs to higher dimensional Green's functions of linear PDEs of mathematical physics, and represented a geometrical structure called an m-dimensional manifold or surface in the n-space. The section also considered Cauchy problem[16-18].

In the second chapter the stage for the relevant analysis on manifolds was set, In Section 2.1 the test function and test section spaces was introduced and investigated their locally convex topologies. The central result will be Theorem 2.1.9 establishing the LF topology for compactly supported smooth sections as well as important properties like completeness of this topology. Moreover, continuous linear maps between test section spaces has been studied[19-21], Section 2.2 discussed differential operators and their symbols. In particular, introduced a global symbol calculus based on the usage of covariant derivatives[22, 23]. and showed that differential operators have adjoints for various natural pairings and compute the adjoints explicitly by using the global symbol calculus, in Theorem 2.2.19 . Section 2.3 led to the definition of distributions or, more precisely, of generalised sections. and defined the $weak^{*}$ *topolpgy* and explain the support and singular support of generalized sections[24, 25] .

Chapter 3 contains a rough overview on Lorentz geometry, as Proper actions and locally homogeneous Lorentzian 3-manifolds[11], Deformations, Einstein Universe. Section 3.1 recalled some basic concepts from semi-Riemannian geometry like parallel transport and the exponential map of a connection. Section 3.2 mainly focused on true Lorentzian geometry on aspects related to the causal structure. Also, for the wave equations, recalled some features of general relativity[26, 27]. This gives the notions of time orientability, causality, and ultimately, of Cauchy hypersurfaces[10]. Also discussed the characterization of globally hyperbolic spacetimes by the existence of smooth Cauchy hypersurfaces in Theorem 3.7.22.[28, 29]

Even though Chapter 4 deals with the local construction of Green functions and needs already geometric concepts like parallel transport and the exponential map.Section 4.1 starting with the wave equation on flat Minkowski spacetime and obtaining the advanced and retarded Green functions by constructing an entirely holomorphic family $\{R^{\pm}(\alpha)\}_{\alpha\in C}$ of distributions[5, 6, 30] .Section 4.2 used the exponential map to transfer the Riesz distributions to the curved situation, at least in a small normal neighborhood of a given point. However, the curvature will now cause slightly different features of the Riesz distributions which results in the failure of $R^{\pm}(p,2)$ being a Green function of the scalar d'Alembert operator[31-34]. the Section 4.3 formulated an heuristic Ansatz for the true Green function coefficients. As an application of this general approach, the Hadamard coefficients for the Klein-Gordon equation in flat spacetime explicitly was computed and obtained an explicit formula for the advanced and retarded Green functions in Theorem 4.3.8. Section 4.4 showed how a true Green function with good causal properties can be obtained from the Hadamard coefficients[35]. The result is a parametrix which can be modified in a second step to obtain the Green functions in Theorem 4.4.15. As a first application we use the local Green functions to construct particular solutions of the inhomogeneous wave equation for distributional and smooth inhomogeneities in Section 4.5 in Theorem 4.5.9.[36, 37]

Chapter 5 is then devoted to the global situation. First the notion of the time separation on a Lorentz manifold was recalled in Section 5.1 which is then used to prove uniqueness of solutions in Theorem 5.1.8 with either future or past compact support provided the global causal structure is well-behaved enough[8, 38]. Section 5.2 contains the precise formulation of the global Cauchy problem as well as its solution for globally hyperbolic spacetimes and discusses both the smooth situation as well as certain finite differentiability versions of the Cauchy problem in Theorem 5.2.10.[39-41]The continuous dependence on the initial values in the Cauchy problem follows from general arguments using the open mapping theorem[42, 43]. This feature is then used in Section 5.3 to obtain global Green functions and the corresponding global Green operators[3, 4]. The difference of the advanced and retarded Green operator provides an “inverse” to the wave operator in the sense of a specific exact sequence discussed in Theorem 5.3.16.[44, 45]

The contents:

Chapter(3):Lorentz Geometry and Causality

Chapter (1)

Green's Functions on N-dimensional Spaces

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Green's Functions on N-dimensional Spaces

(1.1) Linear Differential Operators

(1.1.1) Inner Products:

Definition (1.1.1):[46]

An inner product on a complex linear space X is a map $(.,.) : X \times X \to C$,Such that, for all$x, y.z \in X$ and $\lambda, \mu \in C$

(a) (x ,by + μ z) = λ (x, y) + μ (x, z) (linear in the second argument)
(b) (y,x) = ($\overline{x, y}$) (Hermitian symmetric) .
(c) (x , x) ≥ 0 (nonnegative)
(d) (x , x) = 0 if and if x = 0 (positive definite) . We call a linear space with an inner product an inner product space or a pre- Hilbret space .

Definition (1.1.2):

A norm on X if X is a linear space with an inner product (.,.) , then we can define a norm on X by

$$\|x\| = \sqrt{(x, x)} \tag{1.1}$$

Thus, any inner product space is a normed linear space. We will always use the norm defined in (1.1) on an inner product space.

Definition (1.1.3):[12]

A Hilbert space is a complete inner product space.

In particular, every Hilbert space is a Banach space with respect to the norm in (1.1).

(1.1.2) Orthogonality:

If x , y are vectors in a Hilbert space H, then we say that x and y are orthogonal $x \perp y$, if $\langle x, y\rangle = 0$,The subsets A, B are orthogonal, $A \perp B$ if $x \perp y$ for every $x \in A$ and $y \in B$. the orthogonal complement $A^{\perp}$ of a subset A is the set of vectors orthogonal to A. $A^{\perp} = \{x \in H \mid x \perp y \text{ for all } y \in A\}$

Theorem (1.1.4):(Projection) [17]

Let M be a closed linear subspace of a Hilbert space H .

(a) For each $x \in H$ there is a unique closest point $y \in M$ such that

$$\|x - y\| = \min_{z \in \mathcal{M}} \|x - z\| \qquad (1.2)$$

(b) The point $y \in M$ closest to $x \in H$ is the unique element of M with the property that $(x - y) \perp M$.

The decomposition $x = y + z$, with $y \in M$, $z \in N$ is unique if and only if $M \cap N = \{0\}$

Definition (1.1.5):

A Projection on a linear space X is a linear map

$$P: X \to X \text{ such that } P^2 = P \qquad (1.3)$$

$\text{assciated with } x = m + n \text{ with } , y \in M \ , z \in N$

$$\text{hince ran } P = M \text{ and } N$$

Theorem (1.1.6):

let X be a linear space.

(a) if $P: X \to X$ is a Porojection , then $X = \text{ran } P \oplus \ker P$.

(b)if $X = M \oplus N$, where M and N are linear subspaces of X,

then there is a Porojection $P: X \to X$

with $\text{ran } P = M$ and $\ker P = N$

Definition(1.1.7):Orthogonal Projection

Orthogonal Projection on Hilbert space H is a linear map $P: X \to X$ that satisfies

$$P^2 = P . \quad \langle Px, y\rangle = \langle x, Py\rangle \text{ for all } x, y \in H.$$

Orthogonal Projection is necessarily bounded

Theorem (1.1.8) (Riesz Representation)

If φ is abounded linear functional on a Hilbert space H then there is a unique vector y ∈ H Such that:

$$\varphi(x) = \langle y, x\rangle \quad \text{for all } x, y \in H, \qquad (1.4)$$

proof. if $\varphi = 0$ *and* $y = 0$, so let that $\varphi \neq 0$, then $\ker\varphi$ is proper closed subspace of H, and anone zero vector $z \in H$ such that $z \perp \ker\varphi$. we define a linear map $p: H \longrightarrow H$ by

$$p(x) = \frac{\varphi(x)}{\varphi(z)} z\,.$$

then $p^2 = p$, *so theorem (1.1.6) implies that* $H = ran\ P \oplus kerP$. *Morover*

$$ran\ P = \{\alpha z \mid \alpha \in C\}, \ker P = \ker\varphi,$$

so that $ran\ P \perp \ker P$. *It follows that P is an orthognal projection,and*

$$H = \{\alpha z \mid \alpha \in C\} \oplus \ker\varphi$$

is an orthognal direct sum,then

$$x = \alpha z + n, \quad \alpha \in C \text{ and } n \in \ker\varphi$$

taking the inner product of this decompostion with z, we get

$$\alpha = \frac{\langle z, x\rangle}{\|z\|^2}$$

and evaluating φ *on* $x = \alpha z + n$, *we find*

$$\varphi(x) = \alpha\varphi(z).$$

so

$$\varphi(x) = \langle y, x\rangle,$$

where

$$y = \frac{\overline{\varphi(z)}}{\|z\|^2} z.$$

thus, every baunded linear functional is given by the inner product with a fixed vector.

(1.1.3) Bounded Operators: [13]

Definition (1.1.9) The Adjoint of an Operator:

The adjoint of a bounded operator on a Hilbert space. Is defining as adjoint

$A^*\in B$ (H) of an operator $A \in B(x)$ such that

$$\langle x, Ay\rangle = \langle A^*x, y\rangle \quad \text{for all } x, y \in H. \qquad (1.5)$$

The definition implies that

$$(A^*)^* = A \; (AB)^* = B^*A^*$$

Definition (1.1.10) Self-adjoint Operators:[3]

Abounded linear operator $A : H \to H$ on a Hilbert space H is Self-adjoint if $A^* = A$. Equivalently, a bounded linear operator A on H is self-adjoint if and only if

$$\langle x, Ay\rangle = \langle Ax, y\rangle \text{ for all } x, y \in H$$

Example (1.1.11) :[47]

Let k $: L^2([0,1]) \to L^2([0,1])$ be an integral operator of the form $kf(x) = \int_0^1 k(x,y)f(y)dy$, is self- adjoint if and only if $k(x,y) = \overline{k(y,x)}$.

Given linear operator $A : H \to H$ we may define sesquilinear form ,

$$a: H \times H \to \mathbb{C}$$

by $a(x,y) = \langle x, Ay\rangle$. if A is self-adjoint, then this form is Hermitian symmetric, or symmetric, meaning that

$$a(x,y) = \overline{a(y,x)}$$

It follows that the associated quadratic form $q(x) = a(x,x)$,

Or $$q(x) = \langle x, Ax\rangle \qquad (1.6)$$

Is real-valued. We say that A is nonnegative if it is self-adjoint and

$$\langle x, Ax\rangle \geq 0 \text{ for all } x \in H$$

And A is positive, or positive definite , if it is self-adjoint

And $\langle x, Ax\rangle > 0$ *for every nonzero* $x \in H$.

If A is a positive, bounded operator, then

$$(x, y) = \langle x, Ay \rangle$$

Defines an inner product on H .

If, in addition, there is a constant c > 0 such that

$$\langle x, Ax \rangle \quad \geq \ c \, \|x\|^2 \qquad \text{for all } x \in H \tag{1.7}$$

Then we say that A is bounded from below , and norm associated with $(.,.)$ is equivalent to the norm associated with $\langle .,. \rangle$

Corollary (1.1.12):

If A is a bounded operator on a Hilbert space then$\|A^* A\| = \|A\|^2$,If A is self-adjoint , then $\|A\|^2 = \|A^2\|$

Definition (1.1.13) Unitary (Orthogonal) Operators:

A linear map $U : H_1 \to H_2$ between real or complex Hilbert spaces H_1 and H_2 is said to be orthogonal or unitary, respectively, if it is invertible and if $\langle u_x , u_y \rangle_{H_2} = \langle x, y \rangle_{H_1}$ for all $x, y \in H$. Since $H_1$and H_2 are isomorphic as Hilbert spaces if there is a unitary Linear map between them, Thus, a unitary operator is one-to-one and onto, and preserves the inner product.

$U : H \to H$ is unitary if and only if

$$U^*U = UU^* = 1 \tag{1.8}$$

Example (1.1.14):

The operator $U : L^2(T) \to e^2(z)$ that maps a function to its Fourier coefficient is unitary. Explicitly, we have

$$Uf = (c_n)_{n \in z,} \qquad cn = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x) e^{-inx} \ d(x) \tag{1.9}$$

Thus, the Hilbert space of square inferable functions on the circle is isomorphic to the Hilbert space of sequences on Z .

(1.1.4) Unbounded Operators:

Definition (1.1.15):

The definition of an unbounded linear operator $A : D(A) \subset H \to H$, acting on Hilbert space H therefore includes the definition of its domain $D(A)$. An operator $\widetilde{A}$ is an extension of A. or A is a restriction of $\widetilde{A}$, If $D(\widetilde{A}) \supset D(A)$ and $\widetilde{A}x = Ax$ for all $x \in D(A)$, we can write $\widetilde{A} \supset A$ or $A \subset \widetilde{A}$. The adjoint of an unbound operator $A: D(A) \subset H \to H$ is an operator $A^* : D(A^*) \subset H \to H$,then $\langle Ax, y\rangle = \langle x, A^*y\rangle$

$$\text{for all } x \in D(A) \text{ and all } y \in D(A^*) \tag{1 .10}$$

Definition (1.1.16):

Let $A : D(A) \subset H \to H$ is densely defined unbounded linear operator on Hilbert space H the adjoint operator $A^* : D(A^*) \subset H \to H$ is the operator with domain,

$$D(A^*) = \{y \in H \text{ there is a } z \in H\},$$

$$\text{with } \langle Ax, y\rangle = \langle x, z\rangle \text{ for all } x \in D(A)$$

If $y \in D(A^*)$ then we define $A^*y = z$, where z is the unique element such that $\langle Ax, y\rangle = \langle x, z\rangle$ for all $x \in D(A)$, we can say that the adjoint of differential operator is another differential operator ,which we obtain by using integration by parts, the domain $D(A)$ defines adjoint boundary conditions for A, and the domain $D(A^*)$ defines adjoint boundary conditions for A^*, the boundary conditions ensure that the boundary terms arising in the integration by parts vanish.

Definition(1.1.17):

An unbounded operator A is self-adjoint if

$A^* = A$ meaning that $D(A^*) = D(A)$, And $A^* x = Ax$ for all $x \in D(A)$,

An unbounded operator A is symmetric if A^* is an extension of A meaning that , $D(A^*) \supset D(A)$ and $A^* x = Ax$ for all $x \in D(A)$.

Proposition (1.1.18):

if $A : D(A) \subset H \to H$,is a densely defined linear operator on a Hilbert space H with a bounded inverse $A^{-1} : H \to H$, then $(A^*)^{-1} = (A^{-1})^*$

Definition(1.1.19):

The adjoint of a differential operator, we consider differential operators acting on smooth functions,take a linear ordinary differential operator

$$Au = \sum_{j=0}^{n} a_j\, u^{(j)} \qquad (1.11)$$

Where $u^{(j)}$ denotes the jth derivative of u, and the coefficients a_j are real or complex-valued functions. Our goal is to study B V Ps

(boundary value problems),For (ODs) of the form

$$A_u = f\,, \text{ and } \quad B_u = o\,, \qquad (1.12)$$

where $B_u = o$ denotes a set of linear boundary conditions.

(1.2) One-dimensional Green's Functions:

(1.2.1) Introduction: [2, 48]

We are concerned with a self-ad joint equation of the form

$$L(y) = [P(x)y']' + q(x)y = f(x) \qquad (1.13)$$

Where $P(x)$ and $q(x)$are continuous functions of x on a given interval

$$I{:}a \le x \le b \text{ and } p(x) > 0$$

Domain and Range of The Operators (1.2.1)

$L^2[a, b]$ is the Hilbert Space of all square integrable functions on $[a, b]$,

since i.e., $f \in L^2(a, b])$ if $\int_a^b |f|^2 dx < \infty$.

let S be the linear manifold of $L^2[a, b]$

such that for $y(x) \epsilon S$, $L(y) \epsilon L^2[a, b]$ and $y(x)$ satisfies given boundary conditions,then S and $L^2[a, b]$ will be the domain and range spaces of the differential operators L involved in the problems. S will be called the space of testing functions of the operators, from equation (1.13) with

$$B_1: \quad a_1 y(a) + a_2 y'(a) = 0$$

$$B_2: \quad b_1 y(b) + b_2 y'(b) = 0$$

If L^{-1}denotes the operator inverse to equation (1.13) with $B_1 . B_2$

$$y(x) = L^{-1} f(x).$$

Since L is a differential operator, we expect $L^{-1}$to be an integral operator. thus y will be of the form

$$y(x)=\int_a^b G(x,t)f(t)dt \tag{1.14}$$

then $G(x,t)$ defined on $a \le x \le b, a \le t \le b$ is called the Green's Function of the differential problem .

let $y_1(x)$ be a solution of the homogeneous equation satisfying conditionB_1, but not B_2 and $y_2(x)$ a solution satisfying condition B_2 but not B_1, then $y_1(x), y_2(x)$ will be linearly independent, we shall call $y_1(x)$ and $y_2(x)$ respectively the left hand and right hand solutions,their Wronskian is non-vanishing.

i.e. $W(x) = W(y_1,y_2) \neq 0$,where

$W(x) = y_1(x)y_2{'}(x) - y_2(x)y_1{'}(x)$ then

$$y(x) = A(x)y_1(x) + B(x)y_2 \tag{1.15}$$

be the solution of (1.13).Then

$$A'(x)y_1(x) + B'(x)y_2(x) = 0 \tag{1.16}$$

$$A'(x)y_1{'}(x) + B'(x)y_2{'}(x) = \frac{f(x)}{P(x)} \tag{1.17}$$

Solving (1,16) and (1,17) for $A'(x)$ and $B'(x)$ we have

$$A'(x) = -\frac{y_2(x)f(x)}{P(x)W(x)},\ \ B'(x) = \frac{y_1(x)f(x)}{P(x)W(x)} \tag{1.18}$$

we now show that $A(b) = 0, B(a) = 0$

from (1.16) $y'(x) = A(x)y_1{'}(x) + B(x)\, y_2{'}(x)$

hence $\qquad 0 = a_1y(a) + a_2y'(a)$

$$= a_1\{A(a)y_1(a) + B(a)y_2(a)\} + a_2\{A(a)y_1{'}(a) + B(a)y_2{'}(a)$$

$$= B(a)\{a_1y_2(a) + a_2y_2{'}(a)\}$$

Since $a_1y_2(a) + a_2y'(a) \neq 0$, we have $B(a) = 0$. Similarly using condition B_2 we can prove that $A(b) = 0$, using the Abel's formula

$$P(x)W(x) = \text{constant} = C$$

we get $A(x) = \int_x^b \frac{y_2(t)f(t)}{C}\,dt. \qquad C = P(t)W(t)$

$$B(x) = \int_a^x \frac{y_1(t)f(t)}{C}\, dt.$$

using(i) we get

$$y(x) = \int_x^b \frac{y_1(x)y_2(t)}{C} f(t)dt + \int_a^x \frac{y_1(t)y_2(x)}{C} f(t)dt$$

$$= \int_a^b G(x,t)f(t)dt \qquad (1.19)$$

$$\text{where} \quad G(x,t) = \begin{cases} \frac{y_1(x)y_2(t)}{C} & a \leqslant x \leqslant t \leqslant b \\ \frac{y_1(t)y_2(x)}{C} & a \leqslant t \leqslant x \leqslant b \end{cases} \qquad (1.20)$$

from the symmetry of x and t in $G(x,t)$, we can rewrite (1.19) as

$$y(t) = \int_a^b G(x,t)f(x)\, dx \qquad (1.21)$$

(1.2.2) Properties of Green's Function:[43, 49]

The Green's function $G(x,t)$ of a self-adjoint homogeneous boundary-value problem.

$$L(y)=0 \qquad a \leq x \leq b$$

$$B_1 : a_1\, y\, (a) + a_2 y'(a) = 0$$

$$B_2 : \; b_1\, y\, (b) + b_2 y'(b) = 0$$

is characterized by the following properties

(i) $G(x,t)$ is continuous in the domain $a \leq x, \quad t \leq b$.

(ii) $\frac{\partial G(x,t)}{\partial x}$ is discontinuous at $x = t$, with the jump given by

$$\frac{\partial G(t-0,t)}{\partial x} - \frac{\partial G(t+0,t)}{\partial x} = \frac{-1}{p(t)} \qquad (1.22)$$

where $\frac{\partial G(t-0,t)}{\partial x} = [\frac{\partial G(x,t)}{\partial x}]_{x=t}$ when $x \leq t$, and $\frac{\partial G(t+0,t)}{\partial x} = [\frac{\partial G(x,t)}{\partial x}]_{x=t}$ when $x \geq t$

(iii) with x as the independent variable, $G(x,t)$ satisfies the differential equation $L(y) = 0$ except at $x = t$.

(iv) $G(x,t)$ as a function of x satisfies the boundary conditions $B_1 B_2$.

(v) $G(x,t) = G(t,x)$.

the above properties are said to define G(x, t).

Remark (1.2.2)

If $G(x,t)$ is regarded as a function of t, then the jump (1.22) could be expressed in the form

$$\frac{\partial G(x,x+0)}{\partial t} = \frac{\partial G(x,x-0)}{\partial t} = \frac{1}{p(x)}$$

Remark (1.2.3):[50]

(L $G(x,t)$ as a distribution), let $L(y) = f(x)$

on $a = -\infty, b = \infty$, then $y(x) = \int_{-\infty}^{\infty} G(x,t)f(t)dt$

take $y(x) = L^{-1}f(x)$, then $L(y) = (LL^{-1})f(x) = f(x)$

$$\int_{-\infty}^{\infty} LG(x,t)f(t)dt = f(x) \qquad (1.23)$$

where L is a function on x by using *Dirac's* Function

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

Dirac's Function has the following sifting property

$$\int_{-\infty}^{\infty} \delta(x-t)f(t)dt = f(x)\ldots \qquad (1.24)$$

Comparing (1.23) and (1.24) we have

$LG(x,t) = \delta(x-t)$,A solution of $L(y) = \delta(x-t)$ satisfies properties $(i),(ii)$ and (iii) of a *Green's* function and hence is a fundamental solution of the equation$L(y) = 0$.

(1.2.3) Generalization to Equation of nth Order :

Consider the BVP

$$L(y) = P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots P_n(x)y = 0 \qquad \ldots.(M)$$

BC:$\sum_{i=1}^{n} a_{jk}\, y^{(n-i)}(a) + \sum_{i=1}^{n} b_k y^{(n-i)}(b) = 0 (k = 1, \dots, n)$ .... (N)

where the $p_i(x)$are continuous and $p_0 > 0$ on the interval

$$I = a \le x \le b.$$

the $Green$'s function of the BVP (M), (N) is a bivariate function

$$G(x,t), a \le x, t \le b$$

Characterized by the following properties

(i) $G(x,t)$Is continuous in x, t and has continuous derivatives with respect to $x\ or\ t\ up\ to\ order\ n-2\ on\ the\ interval\ I.$

(ii) $\frac{\partial^{n-1} G(x,t)}{\partial x^{n-1}}$ is discontinuous at $x = t$ with a jump given

$$\frac{\partial^{n-1} G(t-o,t)}{\partial x^{n-1}} - \frac{\partial^{n-1} G(t+o,t)}{\partial x^{n-1}} = -1/P(t)$$

(iii) $G(x,t)$ satisfies the equation $L(y) = 0$ except at $x = t$

(iv) $G(x,t)$ as a function of x satisfies the boundary conditions (N).

In addition if L is a self-adjoint differential operator, then$G(x,t)$ is symmetric,

$$G(x,t) = G(t,x)\,, \tag{1.25}$$

Note: $Heavisile's$ unit functio

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases} \;,\quad H'(x) = \delta(x) \tag{1. 26}$$

Theorem (1.2.4)

$Let\ G(x,t)$is the $Green\ 's\ Function$ of the homogeneous problem.

$$L(y) = (py')' + qy = 0 \tag{1.27}$$

$B_1 : a_1(a) + a_2 y'(a) = 0$,$B_2 : b_1(b) + b_2 y'(b) = 0$,

$$p > 0, a \leqslant x \leqslant b.$$

then

$$L(y) = f(x) \tag{1.28}$$

with conditions $B_1 B_2$ is given by $y(x) = \int_a^b G(x,t) f(t)\, dt$

the converse holds.

Green's Function as a Convolution Kernel (1.2.5):[51]

One of the technique of operational calculus, is integral transforms, the following examples illustrate Laplace, Fourier, Mellin and Hankel transforms $= \; (-i\xi)^n F(\xi)$

Definition(1.2.6):

Let $G(x)$ be a frequency function and let

$$f(x) = \int_{-\infty}^{\infty} G(x-t)\ \emptyset(t)\, dt \tag{1.29}$$

then the kernel $G(x)$ is said to be variation diminishing if the number of sign changes in $f(x)$ never exceeds the number of sign changes in

$$\emptyset(x) \text{ on } -\infty < x < \infty.$$

Green's Function as Reproducing Kernel(1.2.7):

Let X be a real or complex inner product space of function on R. A function $K\,(x,t)$ of two variables $x, t\ \in R$ is called reproducing kernel for the space X if

(a) For each fixed $t\ \ K(x,t) \in X$.

(b) For every $f(x) \in X$, the reproducing property

$f(x) = (f(t), k(x,t))$, holds.

The reproducing property of the Green's function actually emanates from the reproducing (i.e., sifting) property of the Dirac's delta function

$\delta(x-t)$.for if we define an inner product in $L^2[0,1]$ by

$$(f, g)_t = \int_0^1 f(t).\, g(t)dt \tag{1.30}$$

The suffix in the bracket indicating variable of integration, then

$$f(x) = \int_{-\infty}^{\infty} f(t)\delta(x-t)dt = (f(x), \delta(x-t)_t) \tag{1.31}$$

The Green's function of an operator of $L(D)$ satisfies

$$L(D)G(x,t) = \delta(x-t).$$

Suppose we define an inner product of f, g by

$$(f,g)=\int_{-\infty}^{\infty} f(x)L(D).\,g(t)dt\ ,$$

Then $(f(t),G(x,t)_t)=\int_{-\infty}^{\infty}[f(t).\,L(D)G(x,t)]\,dt$

$$=\int_{-\infty}^{\infty} f(t)\,\delta(x-t)dt\ =f(x) \tag{1.32}$$

Thus with appropriate inner product. a Green's function can be made to act as a reproducing kernel.

Energy Inner Products (1.2.8):[18]

Some of the specific of inner products are particularly suited to the variational methods in finite elements Energy inner products is an example.

Definition (1.2.9):

A linear operator A in inner product space X is said to be positive definite if $(\mathrm{Ax},\mathrm{x})>$ unless $\mathrm{x}=0$

The operator is said to be symmetric if

$(\mathrm{Ax},\mathrm{y})=(\mathrm{x},\mathrm{Ay})$ for all $\mathrm{x},\mathrm{y},\in\ \mathrm{X}$.

Definition (1.2.10):

Let X=$L^2[0,1)$

where

$$(f,g)_t=\int_0^1 f(t)g(t)dt \tag{1.33}$$

Let A be a positive definite and symmetric operator on X.

Then a new inner product on X be induced byA with the relation

$$(f,g)_A=(Af,g)=\int_0^1 Af(t).\,g(t)dt$$

$$\text{Clearly}\,(f.g)_A=(g.f)_A$$

Such inner products are called energy inner product, for $f=g$ we have the energy norm (or energy integral)

$$\| f \|_A = \left[\int_0^1 A f . f d t \right]^{\frac{1}{2}}$$

They are so called, because they are used to minimize the protential energy of physical systems.

Example (1.2.11):

Let x denote the linear space of function f(x) in $L^2[1,0]$which are n-fold integrals i.e.,

$$F(x) = \int_0^x \frac{(x-t)^{n-1}}{f(n)} f(t) dt.$$

Introduce in X an inner product by means a bilinear differential expression

$$(F, G)_A = \int_0^1 \left[\sum_{i=0}^{n} a i(t) F^{(i)}(t) G^{(i)}(t) \right] dt \tag{1.34}$$

Where $a_o > 0, a_n > 0, a_1 \geq 0, i = 1,2,......,n-1$and A is an operater

$$AF = \sum_{i=1}^{n} (-1)^i \left[a_i(t) F^{(i)}(t) \right]^{(i)} \tag{1.35}$$

integration by parts in(i) leads to

$$(F, G)_A = p + \int_0^1 \left[\sum_{i=1}^{n} (-1)^i \left\{ a^i(t) F^{(i)}(t)^{(i)} \right\} \right] G(t) dt$$

$$= Q + \int_0^1 \left[\sum_{i=1}^{n} (-1)^i \left\{ a_i(t)^n G^{(i)}(t) \right\}^{(i)} \right] G(t) dt$$

where P, Q are constants, If conditions are imposed such that P=Q=0, the

$$(F, G)_A = (AF, G) = (F, AG) = (G, F)$$

A is thus symmetric

also $$(AF, F) = o \Rightarrow \sum_{i=0}^{n} a_i(t) \left[F^i(t) \right]^2 = 0$$

$$i = 1, 2, \ldots, n \Rightarrow F^{(i)}(t) = 0$$

Hence $f = 0 \Rightarrow F \equiv 0.$

Hence A is positive definite. Thus the inner product $(F,G)^A$ is an energy inner product.

Now let, $G(x,t)$ be the Green's function of the BVP,

$$\text{DE: } Ay = 0$$

$$BC_s : \sum_{i=r}^{n} (-1)^i \left[a_i(x) F^{i(x)}\right]^{i-r} = 0 \quad r = 1,2,\ldots,n$$

$$\text{At} \quad x = 0 \ \ and \ \ x = 1 \ .$$

Reproducing kernel for the inner product space.

It can be verified that $G(x,t)$ is the reproducing kernel for the problem.

Definition(1.2.12) (Kernel of an operator):

Let $p_n(D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_n$ be a polynomial operator.

Then

$$\ker p_n(D) = \left\{ f \in C^n : p^n(D) f = 0 \right\}$$

is called the kernel of the operator $p_n(D)$.

Definition(1.2.13) (L-Spline):

Let a function $s(x)$ be defined on [a, b]. We say that $s(x)$ is an L-Spline of order n, if there exist is an nth degree polynomial $p_n(x)$ and a sequence of

knots $x_0 = a \le x_1 \le x_2 \le \ldots\ldots\ldots \le x_k \le k_{k+1} = b$ such that

i. On every non-empty interval $[x_i, x_{i+1}]$ $s(x) \in Ker\, P_n(D)$

ii. If every x_i has multiplicity j, then s has a continuous $(n-1-j)$th derivative in a neighborhood of x_i. If .

Our concern in this note is to define an L-Spline as a Green's function.

Let $\phi \in$ Ker $p^n(D)$ and

$$\phi(0) = \phi'(0) = \ldots\ldots = \phi^{(n-2)}(0) = 0, \phi^{(n-1)}(0) = 1 .$$

$$\text{if } \varphi_+(t) = \begin{cases} \varphi(t) & t \ge 0 \\ 0 & t > 0 \end{cases}$$

Then $\phi_+(t)$ is called the Green's function of the IVP

$$p_n(D)y = 0$$
$$y(0) = y'(0) = \ldots = y^{n-1}(0) = 0. \tag{1.36}$$

If the knots x_i have multiplicity r_i then an L-spline $s(x)$ can be represented as

$$s(x) = g(x) + \sum_{i=1}^{k} \sum_{j=0}^{r_i = 1} a_{ij} \phi^{(j)} + (x - x_i) \tag{1.37}$$

Where a_{ij} are real numbers and $g \in \ker p_n$.

Definition(1.2.14):
A function $f(x)$ is said to be locally integrable on I if
i. $f(x)$ is defined on I with the possible exception of a set s, of isolated points of I called the singularities of f.
ii. $f(x)$ is continuous in I except the singularities The integral of f over any closed sub-interval of I containing no singularities of f in its interior exists either as proper Riemann integral or as an absolutely convergent improper integral. A locally intergrable function f will be written as $f \in Loc(I)$.

(1.3) Higher Dimensional Green's Functions

Let D be a domain in an n-dimensional Euclidean space E_x of points $\bar{x} = (x_1, x_2 ...., x_n)$.

A PDE of order m of the form.

$$\sum_{i_1, i_2, \ldots, i_n} A_{i1, i2 \ldots\ldots in} \frac{\partial^m u(\bar{x})}{\partial x_1^{i1} \partial x_2^{i2} \ldots\ldots \partial x_n^{in}} + Lu(\bar{x}) = f(\bar{x}) \tag{1.38}$$

Where $i_1 + i_2 + ... + i_n = m$ and L is a partial differential operator of order less than m is called

- Quasilinear if it is linear only in its highest derivatives.
- Semilinear if it is linear in all its derivatives.

- Linear if u and all its derivatives are linear and the coefficients are all functions of $\bar{x}$ only,
- Non-linear otherwise.

Given a linear partial differential operator

$$L(u)=\sum_{i,j=1}^{n} a_{ij}\left(\bar{x}\right)\frac{\partial^2 u\left(\bar{x}\right)}{\partial x_i \partial x_j}+\sum_{i=1}^{n} b_i\left(\bar{x}\right)\frac{\partial u\left(\bar{x}\right)}{\partial x_i}+c\left(\bar{x}\right)u\left(\bar{x}\right)=f\left(\bar{x}\right) \qquad (1.39)$$

Where $\bar{x}=(x_1,x_2,\ldots\ldots,x_n)\in D$, $u\left(\bar{x}\right)=u(x_1,x_2,\ldots\ldots,x_n)$

the inverse of $L(u)$ is an integral operator, the kernel of which is a the Green's function. It satisfies the equation.

$$L(u)=\delta\left(\bar{x}\right)=\delta(x_1,x_2,\ldots\ldots,x_n) \qquad (1.40)$$

Where $\delta\left(\bar{x}\right)$ is the n-dimensional Dirac delta function. from (1.39) case n=2 is

$$G\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right)u=\sum_{i,j=1}^{n} a_{ij}\left(\bar{x}\right)\frac{\partial^2 u}{\partial x_i \partial x_j} \qquad (1.41)$$

And

$$Q(\lambda_i,\lambda_j)=\sum_{i,j=1}^{n} a_{ij}\left(\bar{x}\right)\lambda_i\lambda_j \qquad (1.42)$$

$$Q=\sum_{i=1}^{n}\alpha_i\xi_i^2$$

By transformations

$$\lambda_i=\lambda_i(\xi_1,\xi_2,\ldots\ldots\xi_n), i=1,2,\ldots..n.$$

(1.3.1) Partial Differentials of Mathematical Physics:[18, 52]

take the linear PDEs of the form

$$\nabla^2 u=\lambda\frac{\partial^2 u}{\partial t^2}+\mu\frac{\partial u}{\partial t}+\nu u+h \qquad (1.43)$$

Where h is a given function of positive ; λ , μ , ν are certain physical constants and ∇^2 is the Laplace operator in coordinates of the relevant space. For example in Cartesian coordinates.

- In one dimension: $\nabla^2 = \frac{\partial^2}{\partial x^2}$.
- In two dimensions: $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$
-
- In three dimensions: $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
- In n dimensions: $\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$

(1.3.2) Manifolds : [53]

Any consideration of a PDE draws heavily upon geometrical concepts. Hence we explain certain terms related to geometrical structures in a Euclidean n-space. In the Cartesian xy-plane, an equation of the form $y = f(x)$ denotes a curve. For example both functions $y = \pm(1 - x^2)^{1/2}$ represent the same circle $y^2 + x^2 = 1$ In R^3, equations $x = x(t), y = y(t), z = z(t)$ denote a space curve.

A curve $x_i = x_i(t)$, $(i = 1, 2, \ldots, n)$, $a \le t \le b$ in n-space is called a C^n curve if each $x_i(t) \in C^n$. Since the mapping $u = a + (b - a)\ t$ maps the unit interval $I = [0,1]$ onto $[a, b]$ in a one-one continuous manner, there is no loss of generality in assuming the parameter interval to be I i.e. $t \in [0,1]$.A curve in n-space is defined to be a continues map $\bar{x} : I \to R^n$. it is closed curve if $\bar{x}(0) = \bar{x}(1)$, a curve $\bar{x} : I \to R^n$. is plane curve.

In the three-dimensional space R^3, a surface may be represented by two parameters u , v as

$$x = x(u, v), y = y(u, v), z = z(u, v)$$

$$\text{Provided} \quad \frac{\partial(x, y)}{\partial(x, v)} \neq 0, \frac{\partial(x, z)}{\partial(u, v)} \neq 0.$$

S in general if m<n, then for real independent parameters $s_1, s_2, \ldots, s_n$,

the equations

$$\overline{x} = \overline{x}\,(s_1, s_2, \ldots, s_m)$$

i.e., $x_i = x_i\,(s_1, s_2, \ldots, s_m) \qquad (i = 1, 2, \ldots, n)$ (1.44)

represent a geometrical structure called an m-dimensional manifold or surface in the n-space. If m=n-1, the manifold is called a hypersurface. A curve is a one-dimensional manifold. An equation of the form

$f\left(\bar{x}\right) = f\,(x_1, x_2, \ldots, x_n) = c$ represents a hypersurface. If the parameters $s_1, s_2, \ldots, s_n$ appear in the first degree, the manifolds are called planes. In particular:

- A zero dimensional plane is a point
- A one-dimensional plane is a straight line
- An (n-1)- dimensional plane is a hyperplane
- When m=n the plane is the whole space.

If L_m is an m-dimensional subspace of R^n and $\overline{x}_0 \in R^n$ is a fixed vector, then $\overline{x}_0 + L_m$ is an m-dimensional plane parallel to L_m . If $\bar{x} \in R^n$, then the equation

$$a\left(\overline{x}, \overline{x}\right) + 2\,b\,(\overline{x}) + c = 0 \qquad \ldots(1.45)$$

is called a quadratic hypersurface, where $a(\bar{x}, \bar{x})$ is a quadratic form, $b(\bar{x})$ is a linear form and c is a constant.

if the running coordi-nates of the point $\overline{x}$ are $x_1, x_2, \ldots, x_n$,then (1.45) can be written in the form

$$\sum_{i,\,j=1}^{n} a_{ij}\, x_i\, x_j + 2 \sum_{i=1}^{n} b_i\, x_i + c = 0 \qquad (1.46)$$

the following canonical forms are noteworthy:

a) $x_1^2 + x_2^2 + ... + x_n^2 = r^2$ (Sphere)

b) $\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \dots + \frac{x_n^2}{a_n^2} = 1$ (Ellipsoid)

c) $\frac{x_1^2}{a_1^2} + \dots + \frac{xk^2}{ak^2} - \frac{x_{k-1}^2}{a_{k+1}^2} - \dots - \frac{x^2n^2}{an^2}$ (Hyperboloid)

d) $\lambda_1 x_1^2 + \lambda_2 x_2^2 + \dots + \lambda_{n-1} x_{n-1}^2 - 2\mu xn = 0.$ (Paraboloid)

the equation

$$\left|\overline{x} - \overline{a}\right|^2 = (x_1 - a_1)^2 + \dots + (x_n - a_n)^2 = r^2$$

Is the sphere with radius r and centre

$$\overline{a} = (a_1, a_2, \dots, a_n)$$

we can now introduce the notation of a family of curves and surfaces. If t is real parameter, the equation

$$f(x, y; t) = 0 \tag{1.47}$$

represents a single infinity of curves on plane.
the curve will then be represented by

$$x = x(s; t_0), \ y = y(s; t_0)$$

In general the system

$$x_i = x_i(s; t_1, t_2, \dots, t_k)(i = 1, 2, \dots, n)$$

defines a k-parameter or k-fold infinity of curves in n-space. More generally, for m<n, the system

$$x_i = x_i(s_1, s_2, \dots, s_m; t_1, t_2, \dots, t_k)(i = 1, 2, \dots, n)$$

defines a k-fold family of manifolds.

If m=2 the system

$$x_i = x_i(s_1, s_2; t_1, t_2, \dots, t_k)(i = 1, 2, \dots, n)$$

defines a k-fold infinity of surfaces.

(1.3.3) Domain and Range of Operators:[54]

Let dV denote the Euclidean volume element of an n-dimensional region T. for example, in n-space with Cartesian coordinates

$$dV = dx_1 dx_2 \dots dx_n .$$

For n=3 in polar coordinates

$$dV = r^2 \sin\theta dr d\theta d\phi$$

$f \in C^n$ if f is continuous w.r.t $x_1, x_2, \dots, x_n$ of order n ,

A function $u(\bar{x})$ defined on a domain T is said to be squar-integrable in the Lebesgue sense if

$$\int_T u^2 dV < \infty.$$

the space of all such square integrable functions is denoted by $L^2(T)$.

an inner product in $L^2(T)$ is defined by $(u, v) = \int_T uvdV$

Which induces the norm

$$\|u\| = (u, u)^{1/2} = \left(\int_T u^2 dV\right)^{1/2}$$

the operators we are going to discuss are ∇^2, $\frac{\partial}{\partial t}$ and $\frac{\partial^2}{\partial t^2}$

and their domains will be subsets of $L^2(T)$ satisfying additional conditions.

The domain of ∇^2 is a subset of $L^2(T)$ wither its members having piecewise continuous, square integrable second derivatives and satisfy certain boundary data on the boundary of T, the domain of the operator $\frac{\partial}{\partial t}$ is the subset of $L^2(T)$ whose members u are such that $\frac{\partial}{\partial t}$ is piecewise continuous and square integrable over T and satisfies initial condition such as $u(\bar{x},0) = 0$.

the domain of the operator $\frac{\partial^2}{\partial t^2}$ is a subset of $L^2(T)$ such that its members u have piecewise continuous second derivative $\frac{\partial^2 u}{\partial t^2}$.

Also $\frac{\partial^2 u}{\partial t^2}$ are square integrable and initial conditions such as

$$u\left(\overline{x},0\right)=0\ ,\ \frac{\partial u}{\partial t}\left(\overline{x},0\right)=0$$

are satisfied.

the domain of a combination of these operators is the intersection of the domains of the operators involved.

(1.3.4) Boundary and Initial Value Problems:

The problem of finding a solution with conditions related to the boundary is called a boundary value problem (BVP).

Let G be a domain in the (n-1)-dimensional subspace E_{n-1} of the variable, say $x_1, x_2, \ldots, x_n$. then the following is a Cauchy problem.

Find the regular solution u(x) of the equation

$$\sum_{i=1}^{n-1}\frac{\partial^2 u}{\partial x_i^2}-\frac{\partial^2 u}{\partial x_n^2}=0 \tag{1.48}$$

Satisfying the conditions

$$u\left(x_1,x_2,\ldots,x_{n-1},0\right)=f\left(\overline{x}\right) \qquad \text{(i)}$$

$$\left[\frac{\partial u\left(x_1,x_2,\ldots,x_{n-1},x_n\right)}{\partial x_n}\right]_{x_n=0}=g\left(\overline{x}\right) \qquad \text{(ii)}$$

For $\overline{x}=\left(x_1,x_2,\ldots,x_{n-1}\right)\in G$

and f , g are sufficiently smooth functions defined in G.

Conditions (i),(ii) are called Cauchy conditions or initial conditions are called Cauchy data and the system (1.48), (i)-(ii) is called a Cauchy problem or initial value problem (IVP). G is called the initial manifold or initial domain. In the IVP G is the hypersurface obtained by the intersection of the n-dimensional region T and the hyperplane $x_n=0$. An initial domain may not be a proper subset of the boundary.

For example in E_2 consisting of points (x ,t), the initial domain may be $t=0$ or a subset of it.

In general elliptic equations are associated with boundary conditions and hyperbolic and parabolic equations with initial conditions.

Example(1.3.1):

take the PDE:

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0$$

$$\text{ICs}: u(x,0) = f(x), \frac{\partial u}{\partial t} = g(x)$$

The D' Alembert's solution to the Cauchy problem is

$$u(x,t) = \frac{1}{2} f(x+t) + \frac{1}{2} f(x-t) + \frac{1}{2}\int_{x-t}^{x+t} g(s)ds, t > 0$$

The solution exists, is unique and depends continuously on the data $f(x)$ and $g(x)$. Hence the Cauchy problem for the wave equation is well-posed.

*diffention(1.3.2)Vectorial Differentiation :*Inner and Outer Derivatives.

Let $u(\bar{x})$ be a scalar-valued C^1 function of position $\bar{x} = (x_1, x_2, ..., x_n)$ i.e., and $\frac{\partial u}{\partial x_i}$ are continuous.

$P(\bar{a})$ is a given point in space with position vector $\bar{a} = (a_1, a_2, ..., a_n)$ and $\bar{b} = (b_1, b_2, ..., b_n)$ a given unit vector. The vectorial equation of the ray from P parallel to $\bar{b}$ is given by

$$\bar{x} = \bar{a} + s\bar{b}, \quad s \geq 0$$

If $Q(\bar{x})$ is any point on this ray, then the coordinates of Q are given by the equations

$$x_i = a_i + sb_i, \quad (i = 1, 2, ..., n) \tag{1.49}$$

The limit

$$\frac{\partial u}{\partial s} = \lim_{s \to 0} \frac{u(\bar{x}) - u(\bar{a})}{s}$$

If exists, is called the directional derivative of u at P in the direction of the unit vector $\bar{b}$. For example, in 3-space $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ re-present the rates of change of u in the directions of the x- and y- and z-axes respectively. In an n-space

$$\frac{\partial u}{\partial s} = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial x_i}{\partial s} = \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} \tag{1.50}$$

thus $\frac{\partial}{\partial s} = \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}$ denotes differentiation in the direction of the vector b. It is possible to express the directional derivative $\frac{\partial u}{\partial s}$ in (1.50) in terms of vector differential operator Δ defined by

$$grad\ u = \nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n} \right)$$

∇u is called the gradient vector of the scalar function u.

using the inner product notation

$u.v = u_1v_1 + u_2v_2 + \ldots + u_nv_n$ Where $u = (u_1, u_2, \ldots, u_n), v = (v_1, v_2, \ldots, v_n)$

we can express the relation (1.50) in the form

$$\frac{\partial u}{\partial s} = \bar{b}.\nabla u = \frac{d\bar{x}}{ds}.\nabla u \tag{1.51}$$

If the angle between the vector $\bar{b}$ and ∇u is y, then

$$\left| \frac{\partial u}{\partial s} \right| = |\nabla u| \cos \gamma \tag{1.52}$$

$\left| \frac{\partial u}{\partial s} \right|$ is maximum when γ=0, i.e., b is parallel to grad u. this means the vector grad u is in the direction of maximum increase of u.

let $u(\bar{x})$ be scalar C^1 function and $u(\bar{x})=0$ be given surface in n-spase. Then $\frac{d\bar{x}}{ds}$ is a unit tangential vector to S at the point $\bar{x}$.

Now $\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial s}+....+\frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial s} \quad = \nabla u \,.\, \frac{d\,\overline{x}}{ds} = 0\,.$

hence ∇u is a normal vector to the surface u=0, $\overline{n} = \frac{\nabla u}{|\nabla u|}$

If $|\nabla u| \neq 0$, then the directional derivative of f along the unit normal $\overline{n}$ to u=0 is given by $\frac{\partial f}{\partial n} = \overline{n}.\nabla f$, And is called the normal derivative of the scalar function $f(\overline{x})$ at $\overline{x}$ of the surface $u(\overline{x}) = 0$ .Sine $\frac{d\,\overline{x}}{ds}$ is a unit tangent vector to the surface,

$S{:}u(\overline{x})$=0, if $f(\overline{x})$ is a scalar function, then

$$\frac{\partial f}{\partial s} = \frac{d\,\overline{x}}{ds}.\nabla f$$

Is a tangential derivative or inner derivative of f on the surface S.

In case $\frac{d\,\overline{x}}{ds}.\nabla u \neq 0$, then $\frac{d\,\overline{x}}{ds}$ is not a tangent vector to s and $\frac{\partial f}{\partial s}$ will not be a tangential derivative is called an outer derivative. the directional of a vector function can also be defined on the same lines as of a scalar function as given in(1.51)

let $$\overline{w}(\overline{x}) = (w_1, w_n, \dots, w_n)$$

Be a vector function with components $w_i \in C^1$ being functions of $(x_1, x_2, \dots, x_n)$. Then the directional derivative of $\overline{w}$ in the direction of the vector $\overline{b}$ is given by

$$\frac{\partial \overline{w}}{\partial s} = \sum \frac{\partial \overline{w}}{\partial x_i}\frac{dx_i}{ds} == \sum\left(\frac{dx_i}{ds}.\frac{\partial}{\partial xi}\right)\overline{w} = \left(\frac{d\,\overline{x}}{ds}.\nabla\right)\overline{w} \tag{1.53}$$

where $\overline{x}=\overline{a}+s\overline{b}$,the difference in the formula (1.51) and (1.53) should be marked.

Vectorial Integration-Green's theorem, (1.3.3) (Gauss' Divergence Theorem)

Let $\overline{w}$ be a vector field in a region T bounded by a closed smooth surface ∂T such that $\overline{w} \in C^1$ *in* T *and* $\overline{w} \in C$ *in* $T \cup \partial T\ (= T)$ And $\overline{n}$ in the outward drawn unit normal to ∂T, then by the divergence theorem of Gauss.

$$\int_T \nabla\ \overline{w}\ dV = \int_{\partial T} \left(\overline{n}\,.\,\overline{w}\right) dS \tag{1.54}$$

where dv is the volume element of T and ds the surface element of ∂T. the integral on the left of (1.54) denotes multiple integral of the nth order in T and that on the right denotes multiple integral of the (n-1)th order over ∂T .if v is a scalar function and $\overline{n}$ is a unit normal to the surface $v=c$, then normal derivative of v (i.e., the directional derivative of $v$along $\overline{n}$), which we denote by

$$\frac{\partial v}{\partial n} = \overline{n}\,.\nabla\ v \tag{1.55}$$

If we put $\overline{w}=u.\nabla v$ then

$\nabla\overline{w} = \nabla u.\nabla v + u\nabla^2 v$ and $\overline{n}.\overline{w} = \overline{n}.\left(u.\nabla v\right) = u.\left(\overline{n}.\nabla v\right) = u\dfrac{dv}{dn}$

Hence (1.54) leads to the first form of Green' theore

$$\int_T \left(\nabla u.\nabla v + u\nabla^2 v\right) dV = \int_{\partial T} u\frac{\partial v}{\partial n}\,dS \tag{1.56}$$

It is assumed that in (1.56) $u, v \in C^2$ in T and $u, v \in C$ in T,

$v \in C^1$ in $TU\,\partial T$, $v \in C^2$ in T and $u \in C^1$ in T.

Interchanging u and v in (1.56) subtracting we get the second form of Green's theorem

$$\int_T \left(u\nabla^2 v - v\nabla^2 u\right) dV = \int_{\partial T}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS \tag{1.57}$$

where

$u, v \in C^1$ *in* $TU\,\partial T$ and $u,v \in C^2$ *in* T in T .

let $u = v$ be harmonic in T , then (1.56) gives

$$\int_T (\nabla u)^2\, dV = \int_{\partial T} u\,\frac{\partial u}{\partial n}\,dS \tag{1.58}$$

If we take $v=1$ in (1.57) then we get

$$\int_T \nabla^2 u \, dV = \int_{\partial T} u \frac{\partial u}{\partial n}$$

Remark(1.3.4):

(a) for a three-dimensional closed volume T, ∂T is a surface.

In Cartesian coordinates $dV = dxdydz.$ ds is a surface element.

(b) for a two-dimensional closed plane region T, $dv = dx\,dy$ and ∂T is a curve and $dS = ds$.

Example(1.3.5):

in two-dimensional space .Form of the Gauss' theorem (1.54) is $\int_T \int \nabla \overline{w} dxdy = \int_C \overline{n}\overline{\omega} ds$ Here $\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}$, T is a plane region by a closed curve C and ds the element of arc. The unit tangent vector $\bar{t}$ and the unit normal vector n at point $\overline{x} = (x, y)$ of C are given respectively by

$$\bar{t} = i\frac{dx}{ds} + j\frac{dy}{ds} = \left(\frac{dx}{ds}, \frac{dy}{ds}\right), \overline{n} = \left(\frac{dy}{ds}, -\frac{dx}{ds}\right) = (n_1, n_2)$$

If $\overline{\omega}(x,y) = (f(x,y), g(x,y))$ then $\nabla \overline{\omega} = \frac{\partial f}{\partial x} - g\frac{\partial g}{\partial y}$, $\overline{n}\overline{w} = f\frac{dy}{dx} - g\frac{dx}{dx}$

hence we get two-dimensional form of Stoke's theorem

$$\int_T \int \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dxdy = \int_C (fdy - gdx) \qquad (1.59)$$

Example(1.3.6):

In three-dimensional space

$$\overline{n} \cdot \overline{\omega} = (n_1, n_2, n_3) \cdot (\omega_1, \omega_2, \omega_3)$$

$$= (n_1\omega_1 + n_2\omega_2 + n_3\omega_3)$$

$$\nabla \overline{\omega} = \frac{\partial \omega_1}{\partial x} + \frac{\partial \omega_2}{\partial y} + \frac{\partial \omega_3}{\partial z}$$

Hence by (1.54), we have

$$\int_T \int\int \left(\frac{\partial \omega_1}{\partial x} + \frac{\partial \omega_2}{\partial y} + \frac{\partial \omega_3}{\partial z} \right) dx\, dy\, dz$$

$$\int\int_{\partial T} \left(\omega_1\, dy\, dz + \omega_2\, dz\, dx + \omega_3\, dx\, dy \right) \tag{1.60}$$

in the n-dimensional region in Cartesian coordinates we have

$$dV = x_1\, dx_2 \ldots dx_{i-1}\, dx_i\, dx_{i+1} \ldots dx_n$$

$$m\, dS = dS_i = dx_1 . dx_2 \ldots dx_{i-1} dx_{i+1} \ldots dx_n$$

And

$$\int_T \nabla\, \overline{\omega}\, dx_1 dx_2, \ldots dx_n = \int_{\partial T} \left(\sum \omega i\, dS_i \right) \tag{1.61}$$

Diffention (1.3.7) Adjoint Operator and Green's Theorems:

Given $L(u) = f(x)$ then

$$vL(u) = u\, \overline{L}(u) + \nabla\, \overline{\omega} \tag{1.62}$$

when L is linear partial differential operator with n independent variables.

and u, v are continuously differentiable functions,

and

$$vL(u) - u\, \overline{L} = \nabla \overline{\omega} \tag{1.63}$$

Where $\overline{\omega} = p(u, v)$ is an n -dimensional bilinear vector function L and $\overline{L}$ are called adjoint operators.

$$vL(u) - u\overline{L}(v) = \frac{d}{dx}\{p(u,v)\}.$$

The role of the differential operator $\frac{d}{dx}$ is assumed by the operator ∇ in higher dimension.

$$\int_T \{vL(u) - u\, \overline{L}(v)\} dV = \int_T \nabla\, \overline{\omega}\, dV$$

$$= \int_{\partial T} \overline{n}\,\overline{\omega}\, dS \qquad (1.64)$$

(1.64) is the analogue of Green's formula.

Let $u, v \in L^2(T)$ and that $L(u)$ and $L(v)$ be continuous in T ,

then using the inner product

$$(f, g) = \int_T fg dV \quad ,$$

then (1.64) can epressed as

$$(v, L(u)) - (\overline{L}(v), u) = \int_{\partial T} \overline{n}\overline{\omega} dS$$

In case the right hand side of (1.64) vanishes then

$$(v, L(u)) = (\overline{L}(v). u) \qquad (1.65)$$

In view of relation (1.65) ,the use of the term adjoint operators for L and $\overline{L}$ is justified Multi-index. If $p_{1,} p_2, \ldots\ldots p_n$ are non –negative integers , then

$$p = (p_1, p_2, \ldots, p_n)$$

Is called a $multi - index$. The following conventions are adopted in its use :

(a) $|p| = p_1 + p_2 + \ldots + p_n$.

(b) $\overline{x}^p = x_1^{p_1} x_2^{p2} \ldots x_n^{pn}$ where $\overline{x} = (x_1, x_2, \ldots, x_{n)}$

(c) $p\,! = p_1\,!\, p_2\,! \ldots p_n\,!$

(d) $\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} \begin{pmatrix} p_2 \\ q_2 \end{pmatrix} \ldots \begin{pmatrix} p_n \\ q_n \end{pmatrix} = \dfrac{p\,!}{(p - q)!\,q\,!}$

(e) If $Dr = \frac{\partial}{\partial x_i}$. then the $multi - differential\ operator$

$D=(D_1,D_2,....D_n)$ operates as follows

$$D^p = D_1^{p_1} D_2^{p_2} ...D_n^{p_n} = \frac{\partial |p|}{\partial x_1^{p_1} \partial x_2^{p_2} \partial x_n^{p_n}}$$

(f) $a_p = a_{p1 \, p2 \, ... \, pn}$

If we denote unit vectors along the orthogonal coordinate vectors x_i by k then $\sum l_i^2 = 1$ and

$$\nabla = l_1 \frac{\partial}{\partial x_1} + l_2 \frac{\partial}{\partial x_2} + ... + l_n \frac{\partial}{\partial x_n}$$

the multi operator $(D_1, D_2, ..., D_n)$ is not to be confused with the vectorial differential operator

$$\nabla = \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + ... + \frac{\partial}{\partial x_n} \right) = \sum l_i \frac{\partial}{\partial x_1}$$

while powers of ∇ appear in the sense of inner product, a of power D is product of its consituents

. For example, for n=3

$$\nabla^2 u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} \quad \text{but if}$$

$p = (1,0,1)$ and $|p| = 2$ and $D^P u = D_1 D_2^0 D_3 u = \dfrac{\partial^2 u}{\partial x_1 \partial x_3}$

The convention of multi-index enables us to make the following definition of adjoint:

Definition(1.3.8):

A linear partial differential operation of order m can be represented as:

$$L(u) = \sum_{|p| \le m} a_p (\overline{x}) D^p u, \qquad \overline{x} \in T \tag{1.66}$$

Where

$p=(p_1,p_2,...,p_n), D=(D_1,D_2,...,D_n), \overline{x}=(x_1,x_2,...,x_n)$, The adjoint operator to L is given by

$$\overline{L}(v)=\sum_{|p|\le m}(-1)^{|p|}D^{p}(a,v) \tag{1.67}$$

If $L=\overline{L}$, the operator is called self-ad joint.

chapter (2)

Distributions and Differential Operators On Manifolds

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Distributions and Differential Operators On Manifolds

(2.1) Test Functions and Test Sections:

(2.1.1) The Locally Convex Topologies of Test Functions and Test Sections:

Definition(2.1.1):[19]

Let $E \to F$ be a vector bundle of rank N , The dual frame will be denoted by $\{e^{\alpha}\}\ \alpha = 1, \ldots\ldots, N$ where $e^{\alpha} \in \Gamma^{\infty}\left(E^{*}|_{U}\right)$ are the local sections with $e^{\alpha}(e\beta) = \delta_{\beta}{}^{\alpha}$ for $s \in \Gamma^{\infty}(E)$ we have $s^{\alpha} = e^{\alpha} \in \ell^{\infty}(U)$ such that

$$s|_{U} = s^{\alpha}\, e^{\alpha} \tag{2.1}$$

we define the seminorms

$$PU, x, k, e, \{e_{\alpha}\}(s) = \sup_{\substack{p \in k \\ |I| \le e \\ \alpha = 1, \ldots N}} \left| \frac{\partial^{|I|} s^{\alpha}}{\partial x^{I}}(p) \right| , \tag{2.2}$$

where $I = (i_1, \ldots\ldots, i_n) \in N_0{}^{n}$ denotes a multi index of total length $|I| = i_1 + \ldots\ldots + i_n$. Clearly, the integer $\ell \in N_0$ on the choice of the local base sections. In case we have just functions, i.e. sections of the trivial vector bundle $E = M \times C$, we can find

$$PU, x, k, e, (\mathrm{f}) = \sup_{\substack{p \in k \\ |I| \le e}} \left| \frac{\partial^{|I|} s^{\alpha}}{\partial x^{I}}(p) \right| , \tag{2.3}$$

of the seminorm $f \in \ell^{\infty}(M)$

Lemma (2.1.2) : [24]

For all choices of a chart (U, x) , a compact subset $K \subseteq U$, an integer $\ell \in N_0$ and local base sections $\{e_{\alpha}\}$ *of E on* U , the map

$$PU,x,k,e,\{e_\alpha\}:\Gamma^\infty(E|_U)\to R_0^+ \tag{2.4}$$

is a well-defined seminorm .

Definition(2.1.3) Symmetrized covariant differentiation

Let ∇^E be a covariant derivative for a vector bundle $E \to M$ and let ∇ a torsion-free covariant derivative on M . Then

$$D^E : \Gamma^\infty(S^k T^* M \otimes E) \to \Gamma^\infty(S^{k+1} T^* M \otimes E) \tag{2.5}$$

is defined by

$$D^E(\alpha \otimes s)(X_1,......,X_{k+1}) = \sum_{\ell=1}^{k+1}\left(\nabla_{X_\ell}\alpha \otimes s + \alpha \otimes \nabla^E{}_{X_\ell} s\right)\left(X_1,\ldots,\overset{\ell}{\wedge},\ldots,X_{k+1}\right) \tag{2.6}$$

where $\alpha \in \Gamma^\infty(S^k T^* M), s \in \Gamma^\infty(E),\ and\ X_1,......, X_{k+1} \in \Gamma^\infty(TM)$.

Proposition (2.1.4 **)[55]**

The operator D^E is linear, well-defined, and satisfies the following properties:

i.) For $E = M \times C$ with the canonical flat covariant derivative and $f \in \ell^\infty(M)$ we have

$$D f = d f \tag{2.7}$$

ii.) For $\alpha \in \Gamma^\infty(S^k T^* M),\ and\ \beta \otimes s \in \Gamma^\infty(S^\ell T^* M \otimes E)$ we have

$$D^E((\alpha \vee \beta) \otimes s) = (D\alpha \vee \beta) \otimes s + \alpha \vee D^E(\beta \otimes s) \tag{2.8}$$

iii.) Locally in a chart (U, x) we have

$$D^E(\alpha \otimes s)\big|_U = \left(d x^i \vee \nabla_{\frac{\partial}{\partial x^i}} \alpha\right) \otimes s + d x^i \vee \alpha \otimes \nabla^E_{\frac{\partial}{\partial x^i}} s . \tag{2.9}$$

Lemma (2.1.5):

For all choices of a compactum $K \subseteq U$, and $\ell \in N_0$ the map

$$P_{k,e} : \Gamma^\infty(E) \to R_0^+ \tag{2 .10}$$

is a well-defined seminorm ..

Definition (2.1.6) (ℓ^∞ -Topology):[21]

The natural Fréchet topology of $\Gamma^{\infty}(E)$ is called the ℓ^{∞} - topology. Analogously , we call the natural Fréchet topology of $\Gamma^{\infty}(E)$ the ℓ^{k} - topology.

Remark(2.1.7) (ℓ^{∞} -Topology):

i .) A sequence $s_n \in \Gamma^{\infty}(E)$ converges to s with respect to the

(ℓ^{k} **-**Topology) if and only if s_n converges uniformly on all compact subsets of M with all derivatives to s . Similar, the convergence in the ℓ^{k} - topology is the local uniform converges in the first k derivative.

ii.) If M is compact, we can use $K = M$. This shows that the ℓ^{k} -topology is even a Banach topology since we can also take the maximum $0 \le \ell \le k$.

Proposition (2.1.8):[56]

For a vector bundle $E \to M$ the subspace $\Gamma_0^{\infty}(M)$ of compactly supported sections is dense in $\Gamma^{\infty}(M)$ with respect to the

ℓ^{∞} -topology. Analogously, $\Gamma_0^{k}(E)$ is dense in $\Gamma^{k}(E)$ in

ℓ^{k} -topology for all $k \in N_0$.

Theorem (2.1.9) (ℓ_0^{∞} -topology): [20, 57]

Let $k \in N_0 \cup \{+\infty\}$. The inductive limit topology on $\Gamma_0^{k}(E)$ enjoys the following properties:

i.) $\Gamma_0^{k}(E)$ is a Hausdorff locally convex complete and sequentially complete topological vector space.

the topology does not depend on the chosen sequence of exhausting compacta.

ii.) All the inclusion maps $\Gamma_K^{k}(E) \to \Gamma_0^{k}(E)$ are continuous and the

ℓ^{k} -topology is the finest locally convex topology on $\Gamma_0^{k}(E)$ with this property every $\Gamma_K^{k}(E)$ is closed in $\Gamma_0^{k}(E)$ and induced topology on $\Gamma_0^{k}(E)$ is the ℓ_K^{k} - topology .

iii.) A sequence $s_n \in \Gamma_0^{k}(E)$ is a ℓ_K^{k} -Cauchy sequence if and only if there exists a compact subset $K \subseteq M$ with $s_n \in \Gamma_K^k\ (E)$ for all $n\ and\ s_n$ is ℓ_K^k-- Cauchy sequence. An analogous statement holds for convergent sequences.

iv.) If V is a locally convex vector space, then a linear map $\Phi : \Gamma_0^k(E) \to V$ is ℓ_0^k -continuous if and only if each restriction $\Phi\big|_{\Gamma_0^k(E)} : \Gamma_K^k(E) \to V$ is ℓ_K^k continuous, it suffices to consider an exhausting sequence of compact .

v.) If M is non compact $\Gamma_0^k\ (E)$ is not first countable and hence not metrizable.

(2.1.2) Continuous Maps Between Test Section Spaces:

Proposition(2.1.10):[3, 58]

Let $\phi : M \to N$ be a smooth map. Then the pull-back $\phi\ : \ell^{\infty}(N) \to \ell^{\infty}(M)$ is a continuous linear map with respect to the ℓ^{∞} -topology.

Definition (2.1.11) (Proper Map) :

A smooth map $\phi : M \to N$ is called proper if $\phi^{-1}(K) \subseteq M$ is compact for all compact $K \subseteq N$.

Proposition 2.1.12

Let $\phi : M \to N$ be a smooth proper map. Then $\phi^* : \ell_0^{\infty}(N) \to \ell_0^{\infty}(M)$ is continuous in the ℓ_0^{∞} -topology.

Lemma(2.1.12):[59]

Let $\Phi : E \to F$ be a vector bundle morphism and $\omega \in \Gamma^{\infty}(F^*)$ Then $(\Phi^* \omega)\big|_p (s_p) = \omega\big|_{\phi(p)} (\Phi(s_p))$ for $s_p \in E_p$ and $p \in M$ defines a smooth section $\Phi^* \omega \in \Gamma^{\infty}(E^*)$ called the pull-back of $\omega\ by\ \Phi$.

Proposition 2.1.13 Let $\Phi : E \to F$ be a vector bundle morphism. Then

$$\Phi^* : \Gamma^{\infty}(F^*) \to \Gamma^{\infty}(E^*)$$

is continuous with respect to the ℓ^{∞} - topology.

(2.2) Differential Operators

(2.2.1) Differential Operators and Their Symbols:

Let $E \to M$ and $F \to M$ be vector bundles over M .

Definition(2.2.1) (Differential operators) :

Let $D{:}\Gamma^{\infty}(E) \to \Gamma^{\infty}(F)$ be a linear map. Then D is called differential operator of order $k \in N_0$ if the following conditions are fulfilled.

i.) D can be restricted to open subsets $U \subseteq M$, i.e. for any open subset $U \subseteq M$ there exists a linear map

$$D_U : \Gamma^{\infty}(\mathrm{E}|_{\mathrm{U}}) \to \Gamma^{\infty}(F|_{\mathrm{U}})$$

such that

$$D_U(S|_{\mathrm{U}}) = (D_S)|_{\mathrm{U}} \tag{2.11}$$

for all sections $s \in \Gamma^{\infty}(E)$.

ii.) In any chart (U, x) of M and for every local base sections $e_\alpha \in \Gamma^{\infty}(\mathrm{E}|_{\mathrm{U}})$ *and* $f_\beta \in \Gamma^{\infty}(F|_{\mathrm{U}})$ we have

$$Ds\Big|_U = \sum_{r=0}^{k} \frac{1}{r!} D_U^{i1\ldots ir\beta}{}_{\alpha} f_\beta \frac{\partial^r s^\alpha}{\partial x^{i1} \ldots \partial x^{ir}} \tag{2.12}$$

with locally defined functions $D_U^{i1\ldots ir\beta}{}_{\alpha} \in \ell^{\infty}(U)$, , totally symmetric in $i_{1,} \ldots, i_r$. The set of differential operators $D : \Gamma^{\infty}(E) \to \Gamma^{\infty}(F)$ of order $k \in N_0$ is denoted by $Diffop^{K}(E;F)$ and we define

$$Diffop^{\bullet}(E;F) = \bigcup_{K=0}^{\infty} Diffop^{K}(E;F) \tag{2.13}$$

Remark(2.2.2) (Differential operators):

i.) Clearly , $Diffop^{K}(E;F)$ is a vector space and we have

$$Diffp^{k}(E;F)\subseteq Diffp^{k+1}(E;F) \tag{2.14}$$

for all $k \in N_0$. Thus $Diffop^{2}(E;F)$ is a filtered vector space. Note however that (2.13) does not yield a graded vector space.

ii.) The restriction of a differential operator D is important since we also want to apply D to sections which are only locally defined .

iii.) If we are given an atlas of charts and local bases and locally defined functions $D_{U\ \alpha,}^{i_1..i_r\beta}$ then we can define a global differential operator D by specifying its local form as in (2.12), provided the functions $D_{U\ \alpha,}^{i_1..i_r\beta}$ transform in such a way that two definitions agree on the overlap of any two charts in that atlas.

iv.) Differential operators are local, i.e. $(Ds)\subseteq \operatorname{supp}(s)$.

Lemma (2.2.3) (Leading Symbol):[23]

If $D:\Gamma^{\infty}(E)\to\Gamma^{\infty}(F)$ is a differential operator of order $k \in N_0$, locally given by (2.12), then the definition

$$\sigma_K(D)\big|_U=\frac{1}{k!}D_{U\ \ \alpha}^{i1...ik\beta}\,{}_{\alpha}\frac{\partial}{\partial\chi^{i_1}}\vee\ldots\vee\frac{\partial}{\partial\chi^{i_k}}\otimes f_\beta\otimes e^\alpha \tag{2.15}$$

yields a globally well-defined tensor field , called the leading symbol of D

$$\sigma_k(D)\in.\Gamma^{\infty}\left(S^kTM\otimes F\otimes E^*\right) \tag{2.16}$$

we can interpret the leading symbol $\sigma(D)$ also as a section

$$\sigma_k(D)\in.\Gamma^{\infty}\left(S^kTM\otimes Hom(E,F)\right) \tag{2.17}$$

for $k < 0$

$$Diffo\,P^k(\varepsilon;\tau)=\{0\} \tag{2.18}$$

and for $k\geq 0$ inductively

$$DiffoP^k(\varepsilon;\tau)=\left\{D\in Hom_k(\varepsilon;\tau)\middle|[D,L_\alpha]\in Diffo\,p^{K-1}(\varepsilon;\tau)\forall a\in \mathrm{A}\right\}, \tag{2.19}$$

where $L_\propto$ denotes the left multiplication of elements in the module with a.

As before we set

$$DiffoP^{\bullet}(\varepsilon;\tau)=\bigcup_{k\in Z} DiffoP^{k}(\varepsilon;\tau) \tag{2.20}$$

and

$$DiffoP^{k}(\varepsilon;\tau)\subseteq DiffoP^{k+1}(\varepsilon;\tau) \tag{2.21}$$

whence (2.20) is again filtered . Moreover, $DiffoP^{k}(\varepsilon;\tau)$ is a k-vector space and a left $\mathcal{A}$-module via

$$(a\cdot D)(e)=a\cdot D(e). \tag{2.23}$$

where $a\in\mathcal{A}$, $D\in DiffoP^{k}(\varepsilon;\tau)\ and\, e\in\varepsilon$

If g is yet another $\mathcal{A}$ -module then the composition of differential operators is defined and yields again differential operators. In fact,

$$Diffo\ P^{k}(\tau;g)\circ Diffo\ P^{\ell}(\varepsilon;\tau)\subseteq Diffo\ P^{k+\ell}(\varepsilon;g) \tag{2.24}$$

holds for all $k,\ell\in Z$. It follows that

$$Diffo\ P^{\bullet}(\varepsilon)=Diffo\ P^{\bullet}(\varepsilon:\varepsilon) \tag{2.25}$$

is a filtered subalgebra of all k-linear endomorphisms $End_{\kappa}(\varepsilon) of\ \varepsilon$.

Moreover, by definition we have

$$Diffo\ P^{o}(\varepsilon;\tau)=Hom_{A}(\varepsilon,\tau). \tag{2.26}$$

Theorem(2.2.4):

$for\ \mathcal{A}=\ell^{\infty}(M)\ and\ \varepsilon=\Gamma^{\infty}(E), f=\Gamma^{\infty}(F)$ the algebraic definition of $DiffoP^{\bullet}(\varepsilon;\tau)$. yields the usual differential operators $DiffoP^{\bullet}(E;F)$.

(2.2.2) A Global Symbol Calculus for Differential Operators:

For the operator of symmetrized covariant differentiation D^{E} as in Definition 2.1.3 we have in any chart (U,x) and with respect to any local base sections e_{α}

$$\left(D^{E}\right)^{\ell} s\Big|_{U}=\frac{\partial^{\ell} S^{\alpha}}{\partial x^{i1}\ldots\partial x^{i\ell}}dx^{i1}\vee\ldots dx^{i\ell}\otimes e_{\alpha}+(lower\ order\ terms), \tag{2.27}$$

for every section $s \in \Gamma^{\infty}(E)$.[44] and is an easy consequence of the local expression $D^{E}\Big|_{U} = dx^{i} \vee \nabla_{\frac{\partial}{\partial x^{i}}}$ together with a simple induction on ℓ.

Now let $X \in \Gamma^{\infty}(S^{k}TM \otimes Hom(E,F))$ be given. Then locally we can write

$$X\big|_{U} = \frac{1}{k!} X^{i1..ik\beta}{}_{\alpha} \frac{\partial}{\partial \chi^{i1}} \vee \ldots \vee \frac{\partial}{\partial \chi^{ik}} \otimes f_{\beta} \otimes e^{\alpha} \tag{2.28}$$

this indicate how we can define a differential operator out of $X\, and\, D^{E}$. we use the natural pairing $S^{k}TM - part\ of\ (D^{E})^{k}\ s$ and apply the

$Hom(E,F) - part\ of\ X\ to\ the\ E - part\ of\ (D^{E})^{k} s$.This gives a well-defined section of F.

we adopt the following convention, best expressed locally as

$$\left\langle X, (D^{E})^{k} S \right\rangle = K!\, X^{i1\ldots ik\beta}{}_{\alpha} \frac{\partial^{k} S^{\alpha}}{\partial x^{i1} \ldots \partial x^{i\ell}} f_{\beta} + (lower\quad order\quad terms) \tag{2.29}$$

with other words, this is the natural pairing of $V \otimes \ldots \otimes V \quad (k\, times)\ with\ V^{*} \otimes \ldots \otimes V^{*} \quad (k\, times)$ restricted to symmetric tensors without additional prefactors. Indeed, note that the tensor indexes of

$(D^{E})^{k} s$ are given by $(D^{E})^{\ell} s\Big|_{U} = k! \dfrac{\partial^{\ell} S^{\alpha}}{\partial x^{i1} \ldots \partial x^{i\ell}} dx^{i1} \vee \ldots dx^{i\ell} \otimes e_{\alpha} + (lower\ order\ terms)$,

according to our convention for the symmetrized tensor product ∨.

Definition(2.2.5)[3] (Standard Ordered Quantization):

Let $X \in \Gamma^{\infty}\left(S^{\bullet}TM \otimes Hom(E,F)\right)$ be a notnecessarily homogeneous

section and let $h > 0$. Then the standard ordered quantization $estd(\mathrm{X})_{s} : \Gamma^{\infty}(E) \to \Gamma^{\infty}(F)\ \ of\ \ X$ is defined by

$$estd(X)s = \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{h}{i}\right)^{r} \left\langle X^{(r)}, \frac{1}{r!} (D^{E})^{r} s \right\rangle, \tag{2.30}$$

for $s \in \Gamma^{\infty}(E)$, $Wher\ X = \sum_{r} X^{(r)}\ with\ X^{(r)} \in \Gamma^{\infty}\left(S^{r}TM \otimes Hom(E,F)\right)$ are the homogeneous parts of X . note that by definition of the direct sum there are only finitely many $X^{(r)}$ different from zero whence the sum in (2.30) is always finite.

Theorem(2.2.6)(Global Symbol Calculus):[60]

The standard ordered quantization provides a filtration preserving $\ell^{\infty}(M)-$ $linear\ isomorphism$

$$\ell std : \bigoplus_{k=0}^{\infty} \Gamma^{\infty}(S^{k}TM \otimes Hom(E,F)) \to Diffp^{\bullet}(E;F), \tag{2.31}$$

such that for $X \in \Gamma^{\infty}(S^{k}TM \otimes Hom(E,F))$ we have

$$\sigma_{k}(\ell std(\mathrm{X})) = \left(\frac{\hbar}{i}\right)^{k} X \tag{2.32}$$

Proof. From the local expression of $\left(D^{E}\right)^{\ell} s$ it is clear that $estd(X)$ is indeed a differential operator. Note that the sum is finite and for $X = X^{k} \in \Gamma^{\infty}(S^{k}TM \otimes Hom(E,F)) \to Diffp^{\bullet}(E;F),$

the differential operator $estd(X)$ has order k.

$for\, f = \ell^{\infty}(M)$ we clearly have $\ell std(fX) = festd(X)$

since the natural pairing is $\ell^{\infty}(M)$_bilinear. This shows that ℓstd is a filtration preserving $\ell^{\infty}(M)$_linear map.

$$\ell std\,(X)s|_{U} = \frac{1}{k!}\left(\frac{h}{i}\right)^{k}\left\langle X, \frac{1}{k!}\left(D^{E}\right)^{k} s\right\rangle\Big|_{U} =$$

$$\frac{1}{k!k!}\left(\frac{h}{i}\right)^{k} X^{i1\ldots ik\beta}{}_{\alpha}\, \frac{\partial^{k} S^{\alpha}}{\partial x^{i1}\ldots\partial x^{i\ell}} f_{\beta} + (lower\ \ order\ \ terms\,)$$

hence (2.32) is clear by the definition of σ_{k} as in (2.15). Now let

$D \in Diffop^{\bullet}(E;F)$, be given.Then

$$\sigma_{k}\left(D - \left(\frac{i}{h}\right)^{k} estd\,(\sigma_{k}(D))\right) = 0$$

hence $D - \left(\frac{i}{h}\right)^{k} estd(\sigma_{k}(D))$ is a differential operator of order at most $k-1$.

By induction we can find $D_{k} = \sigma_{k}(D),\quad D_{k-1},\ldots,D_{0}$ with $D_{\ell} \in \Gamma^{\infty}(S^{\ell}TM \otimes Hom(E,F))$, such that

$$D = \ell std \left(\sum_{r=0}^{k} \left(\frac{i}{\hbar} \right)^{r} D_{r} \right), \tag{2.33}$$

which proves surjectivity. The injectivity is also clear, as $\sigma_k(D)$ is uniquely determined by D and by induction the above $D_{k-1},........,D_0$ are unique as well.

Remark(2.2.7)(Global Symbol Calculus):

i.) where $E = F = M \times C$ is the trivial line bundle and $\Gamma^{\infty}\left(S^{\bullet}TM\right)$ is identified, there is a unique algebra isomorphism

$$\partial : \bigotimes_{k=0}^{\infty} \Gamma^{\infty}(S^{k}TM) \ni \mathrm{X} \mapsto \partial(\mathrm{X}) \in pol^{*}(T^{*}M) \tag{2.34}$$

with $\partial(f) = \pi^{*} f$ and $\partial(x)(\alpha_p) = \alpha_p(X(p))$ for $f \in \ell^{\infty}(M) = \Gamma^{\infty}\left(S^{0}\,TM\right)$ and $X \in \Gamma^{\infty}(TM)$ where $\alpha_{\rho} \in T_p{}^{*}M$.

ii.) For $X \otimes A \in \Gamma^{\infty}(TM \otimes Hom(E,F))$ with $X \in \Gamma^{\infty}(TM)$ and $A \in \Gamma^{\infty}(Hom(E,F))$ we simply have

$$estd(\mathrm{X} \otimes \mathrm{A})s = \frac{\hbar}{i} \mathrm{A}(\nabla_{X}^{E} S). \tag{2.35}$$

also

$$estd(\mathrm{A}) = \mathrm{A} \tag{2.36}$$

is just a is $\ell^{\infty}(M)$-linear operator.

(2.2.3) Continuity Properties of Differential Operators:

Theorem(2.2.8) (Continuity of Differential Operators):

Let $D \in Diffop^{k}(E;F)$ be a differential operator of order k. Then for all $\ell \in N_0$ the map

$$D : \Gamma^{k+\ell}(E) \to \Gamma^{\ell}(E) \tag{2.37}$$

is well- defined and continuous with respect to the $\ell^{k+e} - and\, \ell^{e} - topology$.

Proof. If (U,x) is a chart and $e_{\alpha} \in \Gamma^{\infty}(E|_{U})$ *and* $f_{\beta} \in \Gamma^{\infty}(F|_{U})$ are local base sections then $$PU,x,k,e,\{f_\beta\}(Ds)=\sup_{\substack{p\in k\\ \alpha,\beta\\ |I|\le e\\ i1...,ir}}\left|\frac{\partial^{|I|}}{\partial x^I}\sum_{r=0}^{\ell}\frac{1}{r!}D_U^{i1...ir\beta}{}_\alpha(p)\frac{\partial^r s^\alpha}{\partial x^{i1}...\partial x^{ir}}(p)\right|,$$

$$\le c\sum_{\substack{i1...jr\\ r}}\sup_{\substack{p\in k\\ |I|\le e\\ \beta,\alpha}}\left|\frac{\partial^{|I|}}{\partial x^I}D_U^{i1...ir\beta}{}_\alpha(p)\right|\sup_{\substack{p\in k\\ |J|\le e\\ ,\alpha}}\left|\frac{\partial^{|J|}}{\partial x^J}\frac{\partial^r s^\alpha}{\partial x^{i1}...\partial x^{ir}}(p)\right|,$$

$$\le c'\max_{\substack{i1........ir\\ \beta,\alpha}}PU,x,k,e\left(D_U^{i1.........ir\beta}{}_\alpha\right)\max_r PU,x,k,e+r,\{e_\alpha\}(s)$$

$$\le c'\ PU,x,k,e,\{e_\alpha\},\{f_\beta\}(D)PU,x,K,\underset{r}{e}+k,\{e_\alpha\}(s),$$

where c' is a combinatorial factor depending only on ℓ $and\ k$, and

$$PU,x,k,\ell,\{e_\alpha\},\{f_\beta\},(D)=\sup_{\substack{p\in k\\ \alpha,\beta\\ |I|\le e\\ i1...,ir}}\left|\frac{\partial^{|I|}D_U^{i1...ir\beta}{}_\alpha}{\partial x^I}(p)\right|,$$

But this is the desired estimate to conclude the continuity with respect to the $\ell^{k+e}-and\,\ell^e-topology$.

Corollary(2.2.9):

A differential operator $D\in Diffop^\bullet(E;F)$ is continuous with respect to the $\ell^e-topology$.

In the proof of Theorem 2.2.8 we have made use of the quantities

$$PU,x,k,\ell,\{e_\alpha\},\{f_\beta\},(D)=\sup_{\substack{p\in k\\ \alpha,\beta\\ |I|\le e\\ i1...,ir}}\left|\frac{\partial^{|I|}D_U^{i1...ir\beta}{}_\alpha}{\partial x^I}(p)\right|, \tag{2.38}$$

which are easily shown to be seminorms on $Diffop^\bullet(E;F)$ For a fixed $k\in N_0$ these make $Diffop^\bullet(E;F)$ Frechet space .then

$$\ell std:\bigoplus_{e=o}^{k}\Gamma^\infty(S^\ell TM\otimes Hom(E,F))\to Diffop^k(E;F). \tag{2.39}$$

is a continuous isomorphism with continuous inverse. However, all differential operators $Diffop^{\bullet}(E;F)$ will have to be equipped with an inductive limit topology similar to the construction of the $\ell_0{}^{e} - topology$.

then consider the restriction of $D \in Diffop^{\bullet}(E;F)$ to compactly supported sections $\Gamma_K^{k+\ell}(E)$, Since $Supp\,(Ds) \subseteq Supp\ s$ we have

$$D : \Gamma_{A}^{k+e}(E) \to \Gamma_{A}^{\ell}(F) \tag{2.40}$$

for all closed subsets $A \subseteq M$. Since in the estimate

$$PU, x, k, e, \{f_\beta\}(Ds) \leq cPU, x, k, e, \{e_\alpha\}, \{f_\beta\}(D)PU, x, k, e, \{e_\alpha\}(s) \tag{2.41}$$

we have the same compactum on both sides, we find that

$$D : \Gamma_K^{k+e}(E) \to \Gamma_K^{e}(F) \tag{2.42}$$

is continuous in the $\ell_K{}^{k+\ell} - topology\ and\ \ell_K{}^{\ell} - topology$.

Theorem(2.2.10):

Let $D \in Diffop^{k}(E;F)$ be a differential operator of order $k \in N_0$ then for all $\ell \in N_0$ restriction

$$D : \Gamma_0^{k+e}(E) \to \Gamma_0^{e}(F) \tag{2.43}$$

is continuous in the $\ell_0{}^{k+\ell} -\ and\ \ell_K{}^{\ell} - topology$. Moreover

$$D : \Gamma_0^{\infty}(E) \to \Gamma_0^{\infty}(F) \tag{2.44}$$

is continuous in the $\ell_0{}^{\infty} - topology$

(2.2.4) Adjoints of Differential Operators:

For a section $s \in \Gamma^{\infty}(E)\ and\ \mu \in \Gamma^{\infty}(E^{*} \otimes |\Lambda^{top}| T^{*}M)$ the natural pairing of $E\ and\ E^{*}$ gives a density $\mu(s) \in \Gamma^{\infty}(E^{*} \otimes |\Lambda^{top}| T^{*}M)$, which we can integrate, provided the support is compact[61]. Therefore we define

$$\langle s, \mu \rangle = \int_M \mu(s) = \int_M S.\mu, \tag{2.45}$$

whenever the support of at least one of s or μ is compact.

Lemma(2.2.11):

The pairing (2.45) is bilinear and non-degenerate. Moreover s,

$$\langle s, f\mu\rangle = \langle fs, \mu\rangle \quad \text{for } f \in \ell^{\infty}(M)$$

Proof.

Let $s \in \Gamma^{\infty}(E)$ be not the zero section and let $p \in M$ be such that $s(p) \neq 0$. Then we find an open neighborhood $U\ of\ p$ and a section $\phi \in \Gamma_0{}^{\infty}(E^*)$ with compact support $\sup p\varphi \subseteq U$ such that

$$\varphi(s) \geq 0\ and\ \varphi(s)\big|_p > 0$$

Using local base sections this is obvious. Now choose a positive density

$v \in \Gamma^{\infty}(\left|\Lambda^{top}\right| T^*M)$ then $\varphi \otimes v \in \Gamma_0{}^{\infty}(E^* \otimes \left|\Lambda^{top}\right| T^*M)$ will satisfy

$\langle s, \varphi, \otimes v\rangle \neq 0$ .This shows that (2.45) is non-degenerate in the first argument. The other non-degeneracy is shown analogously. The second statement is clear.

In particular , $\langle\, .\, ,\, .\, \rangle$ restricts to a non-degenerate pairing

$$\langle\, .\, ,.\, \rangle; \Gamma_0^{\infty}(E) \times \Gamma_0^{\infty}(E^* \otimes \left|\Lambda^{top}\right| T^*M) \to C. \tag{2.46}$$

As an immediate consequence we obtain the following statement. First recall that an operator

$$D: V \to W \tag{2.47}$$

is adjointable with respect to bilinear pairings

$$\langle .,. \rangle_{V,\widetilde{V}}; V \times \widetilde{V} \to C \qquad \text{and} \qquad \langle .,. \rangle_{W,\widetilde{W}}; W \times \widetilde{W} \to C, \tag{2.48}$$

if there is a map

$$D^T : \widetilde{W} \to \widetilde{V}$$

such that

$$\langle Dv, \widetilde{w}\rangle_{W,\widetilde{W}} = \langle v, D^T \widetilde{w}\rangle_{V,\widetilde{V}}. \tag{2.49}$$

If the pairings are non-degenerate then an adjoint D^T is necessarily unique (if it exists at all) and both maps D, D^T are linear maps. Clearly, D^T is adjointable , too with

$$\left(D^T\right)^T = D \quad .$$

Thus in our situation, adjointable maps with respect to the pairing (2.45) or (2.46) have unique adjoints and are necessarily linear.

Proposition(2.2.12):[3]

Let $D \in Diiffop^k(E;F)$ be a differential operator of order k. Then $D : \Gamma_0^\infty(E) \to \Gamma_0^\infty(F)$ is adjointable with respect to (2.45) and the (unique) adjoint

$$D^T : \Gamma^\infty(F^* \otimes \left|\Lambda^{top}\right| T^* M) \to \Gamma^\infty(E^* \otimes \left|\Lambda^{top}\right| T^* M) \tag{2.50}$$

is again a differential operator of order k .

Proof. Let $\{(U_i, x_i)\}_{i \in I}$ be a locally finite atlas and let

$$e_{i\alpha} \in \Gamma^\infty(E \setminus_U) \quad and \quad f_{i\beta} \in \Gamma^\infty(F \setminus_U)$$

be local base sections. Moreover let $\{\mathrm{X}_i\}_{i \in I}$ be a locally finite partition of unity subordinate to the atlas with supp X_i being compact. As usual, we write

$$D\, s\big|_{U_i} = \sum_{r=0}^{k} \frac{1}{r!} D_U^{i1\ldots ir\beta}{}_\alpha f\beta \frac{\partial^r s_i^{\ \alpha}}{\partial x^{i1} \ldots\ldots\ldots \partial x^{ir}}$$

where $S\big|_{U_i} = S_i^{\ \alpha} e_{i\alpha}$ *with* $S_i^{\ \alpha} = e_i^{\ \alpha}(s) \in \ell^\infty(U_i)$ for $\mu \in \Gamma^\infty(F^* \otimes \left|\Lambda^{top}\right| T^* M)$

we write

$$\mu\big|_{Ui} = \in \mu_{i\beta} f^\beta \left| d\, x_i^{\ 1} \wedge \ldots\ldots\wedge d\, x_i^{\ n} \right|$$

with $\mu_{i\beta} \in \ell^\infty(U_i)$. Here $\left| d\, x_i^{\ 1} \wedge \ldots\ldots \wedge d\, x_i^{\ n} \right|$ denotes the unique locally defined density with value 1 when evaluated on the coordinate base vector fields

$\dfrac{\partial}{\partial x^1}, \ldots\ldots\ldots . \dfrac{\partial}{\partial x^n}$. Then we compute

$$\langle D\,s\,,\mu\,\rangle = \int_M \mu\,(Ds) \;\; = \sum_i \int_{xi(Ui)} (\chi i\mu(Ds)) \circ x_i^{\;-1} d^{\,n} x_i$$

$$= \sum_i \int_{xi(Ui)} \left(\chi_i\;\mu_{i\beta} \sum_{r=0}^{k} \;\frac{1}{r!}\; D_U^{i1\ldots ir\beta}{}_{\alpha}\; \frac{\partial^r\; s_i^{\;\alpha}}{\partial\, x^{i1} \ldots\ldots\ldots \partial\, x^{ir}} \right) \circ x_i^{\;-1} d^{\,n} x_i$$

Note that the integrand consists of compactly supported functions only. Thus we can integrate by parts and obtain

$$\langle D\,s\,,\mu\,\rangle = \sum_i \int_{xi(Ui)} \left(\sum_{r=0}^{k} \;\frac{(-1)^r}{r!}\; \frac{\partial^r}{\partial\, x^{i1} \ldots\ldots\ldots \partial\, x^{ir}} \left(\chi_i\;\mu_{i\beta} D_U^{i1\ldots ir\beta}{}_{\alpha}\right) s_i^{\;\alpha} \right) \circ x_i^{\;-1} d^{\,n} x_i$$

Now the function $\chi_i\;\mu_{i\beta} D_U^{i1\ldots ir\beta}{}_{\alpha}$ has compact support in U_i from χ_i. Thus it defines a global function in $\ell_0^{\;\infty}(M)$. It follows that

$$\mu_i \;=\; \sum_i \int_{xi(Ui)} \left(\sum_{r=0}^{k} \;\frac{(-1)^r}{r!}\; \frac{\partial^r}{\partial\, x^{i1} \ldots\ldots\ldots \partial\, x^{ir}} \left(\chi_i\;\mu_{i\beta} D_U^{i1\ldots ir\beta}{}_{\alpha}\right) \right) e_i^{\;\alpha} \otimes \left| d\,x_i^{\;1} \wedge \ldots\ldots\wedge d\,x_i^{\;n} \right|$$

is a global section in $\Gamma_0^{\infty}(E^* \otimes \left|\Lambda^{top}\right| T^*M)$ with compact support in U_i. . Since the χ_i. are locally finite , the sum

$$D^T\;\mu = \sum_i \mu_i$$

is well-defined and yields a global section

$$D^T\;\mu \in \Gamma_0^{\infty}(E^* \otimes \left|\Lambda^{top}\right| T^*M) \;\; \text{such that}$$

$$\langle Ds\,,\mu\rangle = \langle s\;,D^T\;\mu\rangle$$

this shows that D is adjointable. Thus

$$D^T \in Diffop^{\,k}\;(F^* \otimes \left|\Lambda^{top}\right| T^*M\,, E^* \otimes \left|\Lambda^{top}\right| T^*M\;)$$

follows. then

$$D : \Gamma^{\infty}(E) \to \Gamma^{\infty}(F)$$

be a differential operator of order zero. Thus D can be viewed as a section of

$$Hom(E\,,F), i.e.\; D \in \Gamma^{\infty}(Hom(E,F)).$$

Then in $\mu(Ds)$ we can simply apply the pointwise transpose of D to the

$F^* - partof\ \mu$. This defines $D^T \mu$ pointwise in such a way that

$$\left(D^T \mu\right)(s) = \mu(Ds). \text{ Clearly } \langle Ds, \mu\rangle = \langle s, D^T \mu\rangle \quad .$$

for differential operators of order $\ell \le k-1$ the adjoint has order ℓ , too. Thus let $Diffop^k(E;F)$ $and\ f \in \ell^\infty(M)$,then we have ,

$$\langle f\,Ds, \mu\rangle = \langle Ds, f\mu\rangle = \langle s, D^T f\mu\rangle$$

and

$$\langle fDs, \mu\rangle = \langle |f, D| s, \mu\rangle + \langle D(fs), \mu\rangle$$

$$= \left\langle s, |f, D|^T \mu\right\rangle + \left\langle s\ ,\ fD^T\ \mu\right\rangle \quad .$$

hence by the non-degeneracy of $\langle\ .\ ,.\ \rangle$ we conclude that

$$\left[f, D^T\right] = [f, D]^T \in Diffop^{k-1}\ (F^* \otimes \left|\Lambda^{top}\right| T^*M, E^* \otimes \left|\Lambda^{top}\right| T^*M\ )$$

by induction. But this shows

$D^T \in Diffop^k\ (F^* \otimes \left|\Lambda^{top}\right| T^*M, E^* \otimes \left|\Lambda^{top}\right| T^*M\ )$,as wanted.

Corollary (2.2.13):

Let $D \in Diffop^k\ (E;F)$. Then for the leading symbol $\sigma_k\left(D^T\right) \in \Gamma^\infty\left(S^k TM \otimes Hom\,(F^* \otimes \left|\Lambda^{top}\right| T^*M, E^* \otimes \left|\Lambda^{top}\right| T^*M\ )\right)$

we have
$$\sigma_k\left(D^T\right) = (-1)^k \sigma_k(D)^T \otimes id_{\left|\Lambda^{top}\right| T^*M} \tag{2.51}$$

where $\sigma_k\left(D^T\right)$ denotes the pointwise transpose from $Hom(E, F\ )\,to\,Hom\left(F^*, E^*\ \right)$

Proof. From the local computations in the proof of Proposition (2.24) we obtained

$$\mu_i = \sum_{r=0}^{k} \frac{(-1)^r}{r!} \frac{\partial^r}{\partial x^{i1} \ldots\ldots\ldots \partial x^{ir}} \left(\chi_i\ \mu_{i\beta} D_U^{i1\ldots ir\beta}{}_\alpha\right) e_i^{\ \alpha} \otimes \left|d\,x_i^{\ 1} \wedge \ldots\ldots. \wedge d\,x_i^{\ n}\right|$$

$$= \frac{(-1)^k}{k!} \chi_i\ D_U^{i1\ldots ir\beta}{}_\alpha \frac{\partial^k \mu_{i\beta}}{\partial x^{i1} \ldots\ldots\ldots \partial x^{ik}} e_i^{\ \alpha} \otimes \left|d\,x_i^{\ 1} \wedge \ldots\ldots. \wedge d\,x_i^{\ n}\right| + (lower\ order\ term)$$

Since

$D^T\mu = \sum_i \mu i$ *and* $\sum_i \chi i = 1$ We conclude that

$$D^T\mu\big|_{U_i} = \frac{(-1)^k}{k!} D_U^{i1\ldots ir\beta}{}_{\alpha} \frac{\partial^k \mu_{i\beta}}{\partial x^{i1}.........\partial x^{ik}} e_i^{\alpha} \otimes \left| dx_i^1 \wedge ....\wedge dx_i^n \right| + (lower\ order\ term)$$

$$= (-1)^k \sigma_k(D)^T \otimes id_{|\Lambda^{top}|T^*M} + (lower\ order\ term)$$

Remark (2.2.14) (Other Pairings):

i.) There are several variations of the above proposition. On one hand one can consider the natural pairing of $\alpha - and\ (1-\alpha) -$ α- densities for any $\alpha \in C$ to obtain

$$\langle .,. \rangle : \Gamma_0^\infty (E \otimes \left|\Lambda^{top}\right|^{\alpha} T^*M) \times \Gamma_0^\infty (E^* \otimes \left|\Lambda^{top}\right|^{1-\alpha} T^*M) \to C \tag{2.52}$$

then $D = \Gamma_0^\infty (E \otimes \left|\Lambda^{top}\right|^{\alpha} T^*M) \to \Gamma_0^\infty (E^* \otimes \left|\Lambda^{top}\right|^{\beta} T^*M) \to C$ (2.53)

and obtain differential operators

$$D = \Gamma^\infty (F^* \otimes \left|\Lambda^{top}\right|^{1-\beta} T^*M) \to \Gamma^\infty (E^* \otimes \left|\Lambda^{top}\right|^{1-\alpha} T^*M) \to C \tag{2.54}$$

by the same kind of computation as in Proposition 2.2.12. There, we considered the case α = 0 = β.

i.) Another important case is for complex bundles E with a (pseudo-) Hermitian fiber metric h_E Then we can use the C-sesquilinear pairings

$$\langle s, t \otimes \mu \rangle = \int_M h(s,t)\mu, \tag{2.55}$$

where $s,t \in \Gamma^\infty(E)$ *and* $\mu \in \Gamma^\infty \left(\left|\Lambda^{top}\right|^{1-\alpha} T^*M) \right)$ and at least one has compact support. Clearly, this extends to

$$\langle .,. \rangle = \Gamma^\infty (E) \times \Gamma_0^\infty (E \otimes \left|\Lambda^{top}\right|^{1-\beta} T^*M) \to C \tag{2.56}$$

in a C-sesquilinear way. While $D \mapsto D^T$ is C-linear now the adjoint D^* depends on D in an antilinear way.

iii.) A very important situation is obtained by merging the above possibilities. For a Hermitian vector bundle $E \to M$ with Hermitian fiber metric h we consider the sections

$$\Gamma_0^\infty(E\otimes\left|\Lambda^{top}\right|^{\frac{1}{2}}T^*M)$$

On factorizing sections we can define

$$\langle s\otimes\mu, t\otimes\nu\rangle=\int_M h(s,t)\tilde{\mu}\nu, \qquad (2.57)$$

since μν is a 1-density. Then the pairing extends to a

$$\langle .,.\rangle:\Gamma_0^\infty(E\otimes\left|\Lambda^{top}\right|^{\frac{1}{2}}T^*M)\times\Gamma_0^\infty(E\otimes\left|\Lambda^{top}\right|^{\frac{1}{2}}T^*M)\to C \qquad (2.58)$$

which is not only non-degenerate but positive definite. Thus

$$\Gamma_0^\infty(E\otimes\left|\Lambda^{top}\right|^{\frac{1}{2}}T^*M)$$

becomes a pre-Hilbert space. Moreover, taking E to be the trivial line bundle with the canonical fiber metric gives a pre-Hilbert space $\Gamma_0^\infty(E\otimes\left|\Lambda^{top}\right|^{\frac{1}{2}}T^*M)$, For a vector bundle E → M ,we then have the pairing

$$\langle s,\varphi\rangle_\mu=\int_M\varphi(s)\mu, \qquad (2.59)$$

$for\ s\in\Gamma^\infty(E) and\ \ \varphi\in\Gamma^\infty(E^*)$, at least one having compact support. Clearly,

$$\langle s,\varphi\rangle_\mu=\langle s,\varphi\otimes\mu\rangle \qquad (2.60)$$

with the original version (2.45) of the pairing $\langle .,.\rangle$ since $\mu>0$ it easily follows that (2.60) is non-degenerate and satisfies

$$\langle fs,\varphi\rangle_\mu=\langle s, f\varphi\rangle_\mu \qquad (2.61)$$

for all $for\ f\in\ell^\infty(M)$. For the action of differential operators we again have adjoints .

Theorem(2.2.15):

Let $D\in DiffoOp^k(E;F)$ be a differential operator of order $k\in N_0$. Then there exists a differential operator

$$D^T\in Diffop^k(F^*;E^*)$$

such that

$$\langle Ds,\varphi\rangle_{\mu}=\langle s,D^{T}\varphi\rangle_{\mu} \tag{2.62}$$

for all $\textit{for } s\in\Gamma^{\infty}(E)\textit{and}\quad \varphi\in\Gamma^{\infty}(F^{*})$, at least one having compact support

Remark (2.2.16):

i.) Note that D^{T} as in Theorem 2.2.15 depends on the choice of $\mu>0$ while the adjoint as in Proposition 2.2.12 is intrinsically defined, though of course between different vector bundles . However, we shall not emphasize the dependence of D^{T} on μ in our notation. It should become

ii.) Analogously to Corollary 2.2.13 we see that the leading symbol of D^{T}

is given by

$$\sigma_{k}(D^{T})=(-1)^{k}\sigma_{k}(D)^{T} \tag{2.63}$$

where again $\sigma_{k}(D^{T})\in\Gamma^{\infty}(Hom(F^{*},E^{*}))$ is the pointwise adjoint of $\sigma_{k}(D^{T})\in\Gamma^{\infty}(Hom(F,E))$.

then its covariant divergence is defined by

$$div\nabla(X)=tr(Y\mapsto\nabla_{Y}X),$$

in local coordinates (U, x) we have

$$div\nabla(X)\big|_{u}=dx^{i}\left(\nabla_{\frac{\partial}{\partial x^{i}}}X\right) \tag{2.64}$$

Clearly, we have for $f\in\ell^{\infty}(M)$ and $X\in\Gamma^{\infty}(TM)$ the relation

$$div\nabla(fX)=X(f)+fdiv\nabla(X). \tag{2.65}$$

Definition(2.2.17)(Covariant Divergence):

Let ∇ be a torsion-free covariant derivative for M and let ∇^{E} be a covariant derivative for E. For $X\in\Gamma^{\infty}(S^{\bullet}TM\otimes E)$ we define

$$div_{\nabla}^{E}(X)=i_{S}(dx^{i})\nabla_{\frac{\partial}{\partial x^{i}}}X. \tag{2.66}$$

Lemma(2.2.18):

By (2.66) we obtain a globally well-defined operator

$$div_{\nabla}^{E}:\Gamma^{\infty}(S^{*}TM\otimes E)\rightarrow\Gamma^{\infty}(S^{*-1}TM\otimes E), \tag{2.67}$$

which is given on factorizing sections by

$$div_{\nabla}^{E}(X_1\vee\ldots\vee X_1\otimes s)=\sum_{e=1}^{k}X_1\vee\ldots\overset{e}{\wedge}\ldots\vee X_k\otimes(div\nabla(X_k)_s+\nabla_X^E s) \tag{2.68}$$

$$+\sum_{\substack{e,m=1\\e\neq m}}^{k}(\nabla_X X)\vee X_1\vee\ldots\overset{e}{\wedge}\ldots\vee X_K\otimes s, \tag{2.69}$$

where $X_1,\ldots\ldots,X_k\in\Gamma^{\infty}(TM)$ *and* $s\in\Gamma^{\infty}(E)$.

Theorem(2.2.19)(Neumaier):

Let $X\in\Gamma^{\infty}(S^K TM\otimes Hom(F,E))$ and let ∇ *and* ∇^E,∇^F be given then the adjoint operator to $\ell_{std}(X)$ *with respect to* $\langle .,.\rangle_\mu$ is explicitly given by

$$\ell_{std}(X)^T=(-1)^k\ \ell_{std}(N^2 X^T)\ \ where\ N=\exp\left(\frac{h}{2i}div_{\mu}^{Hom(E,F)}\right)$$

and where we use the induced covariant on $Hom(E,F)$ *and* $Hom(E^*,F^*)$

(2.3) Distributions on Manifolds.

(2.3.1) Distributions and Generalized Sections:

For $M=R^n$ we define distributions as continuous linear functionals on the test function spaces:

Definition(2.3.1) (Distribution) :

A distribution u on M is a continuous linear functional

$$u: e_0^{\infty}(M)\rightarrow C \tag{2.70}$$

The space of all distributions is denoted by $e_0^{\infty}(M)'$ *or* $D'(M)$.

Remark(2.3.2)(Distributions):

i.) The continuity of course refers to the LF topology of $e_0^{\infty}(M)$ as introduced in Theorem 2.2.9 In particular, a linear functional is continuous if and only if for all compacta $K\subseteq M$ the restriction

$$u\big|_{e_K^\infty} : e_K^\infty\,(M) \rightarrow C \tag{2.71}$$

is continuous in the e_K^∞ -topology. This is the case if and only if for all $\phi \in e_K^\infty(M)$ we have constant $c>0$ *and* $\ell \in N_0$ such that

$$|u(\varphi)| \leq c \max_{\ell' \leq \ell} PK, \ell'(\varphi) \tag{2.72}$$

Analogously, we could have used the seminorms PU, x, k, ℓ avoiding the usage of a covariant deriva-tive but taking a maximum over finitely many compacta in the domain of a chart. we can combine this to

$$|u(\varphi)| \leq c\, PK, \ell(\varphi) \tag{2.73}$$

In the following, we shall mainly use this version of the continuity.

Since each $e_K^\infty(M)$ is a Fréchet space, u restricted to $e_K^\infty(M)$ is continuous iff it is sequentially continuous. This gives yet another criterion: A linear functional

$u : e_0^\infty\,(M) \rightarrow C$ is continuous iff for all $\varphi_n \in e_0^\infty\,(M)$ with

$\varphi_n \rightarrow \phi$ in the e_0^∞ - topology we have

$$u(\varphi_n) \rightarrow u(\varphi) \tag{2.74}$$

ii.) The minimal $e \in N_0$ such that (2.2.3) is valid is called the local order $ord_K\,(u)$ *of u on K*. Clearly,

this is a quantity independent of the connection used for Pk, ℓ and can analogously be obtained from the seminorms PU, x, k, ℓ as well. The independence follows at once from the various estimates between the seminorms of u is defined as

$$ord\,(u) = \sup_K ord_K(u) \in N_0 \cup \{+\infty\} \tag{2.75}$$

and the distributions of total order $\leq k$ are sometimes denoted by $D'^k(M)$. Their union is denoted by $D'_F(M)$ and called distributions of finite order.

iii.) The distributions $D'(M)$ as well as $D'^k(M)$ and $D'_F(M)$ are vector spaces. We have $D'^k(M) \leq D'^\ell(M)$ *for* $k \leq \ell$. It can be shown that already for $M = R^n$ all the inclusions $D'^k(M) \subseteq D'^\ell(M) \subseteq D'_F(M) \subseteq D'(M)$ are proper.

iv.) If u *has order* $\leq k$ it can be shown that u extends uniquely to a continuous linear function

$$u:\ell_0^\infty(M)\rightarrow C \tag{2.76}$$

with respect to the ℓ_0^∞ -topology provided $\ell \leq k$.

Example(2.3.3) (δ-functional):

For $p \leq M$ the evaluation functional

$$\delta_p : \ell_0^\infty(M) \ni \varphi \mapsto \varphi(p) \in C \tag{2.77}$$

is clearly continuous and has order zero. More generally, if $v_p \in T_p M$ is a tangent vector then

$$v_p : \varphi \mapsto v_p(\varphi) \tag{2.78}$$

is again continuous and has order one .

Definition(2.3.4)(Generalized Section):[62]

Let $E \rightarrow M$ be a smooth vector bundle. Then a generalized section (or: distributional section) of E is a continuous linear functional

$$s:\Gamma_0^\infty\left(E^* \otimes \left|\Lambda^{top}\right| T^* M\right) \rightarrow C\,. \tag{2.79}$$

The generalized sections will be denoted by $\Gamma^{-\infty}(E)$.

Remark(2.3.5)(Module Structure):

The generalized sections $\Gamma^{-\infty}(E)$ become a $\ell^\infty(M)$-module via the definition

$$(f.s)(\omega)=s(f\omega) \tag{2.80}$$

Indeed $\omega \mapsto f\omega$ is ℓ_0^∞ -continuous and hence (2.80) is indeed a continuous linear functional $f.s \in \Gamma^{-\infty}(E)$. The module property is clear.

Remark(2.3.6)(Order of Generalized Sections):[63]

The continuity of $s \in \Gamma^{-\infty}(E)$ is again ex-pressed using the seminorms of $s:\Gamma_0^\infty\left(E^* \otimes \left|\Lambda^{top}\right| T^* M\right) \rightarrow C$ in the following way. For every compactum $K \subseteq M$ there are constants $c > 0$ and ∈ $\ell \in N_0$ such that

$$|s(\omega)| \leq c \max_{\ell' \leq \ell} PK, \ell'(\omega), \tag{2.81}$$

for all $\omega : \Gamma_K^\infty \left(E^* \otimes \left| \Lambda^{top} \right| T^* M \right)$. Again, the local order of $s \, on \, K$ is defined to be the small ℓ such that (2.81) holds. This also defines the global order

$$ord(s) = \sup_K ord_K(s) \tag{2.82}$$

as before. As in the scalar case, a generalized section $s \in \Gamma^{-\infty}(E)$ with global order $ord(s) \leq K$ extends uniquely to a ℓ_0^ℓ -continuous functional

$$s : \Gamma_0^\ell \left(E^* \otimes \left| \Lambda^{top} \right| T^* M \right) \to C \tag{2.83}$$

for $\ell \geq k$. We shall denote the distributional sections of $order \leq \ell \, by \, \Gamma^{-\ell}(E)$ note that $\Gamma^{-0}(E)$ are not just the continuous sections.

We also want to topologize the distributions. Here we use the most simple locally convex topology:

Definition (2.3.7) (Weak∗ Topology):

The $weak^* \; topolpgy$ for $\Gamma^{-\infty}(E)$ is the locally convex topology obtained from all the seminorms

$$P_\omega(s) = |s(\omega)| \tag{2.84}$$

where $\omega : \Gamma_0^\infty \left(E^* \otimes \left| \Lambda^{top} \right| T^* M \right)$ in the following we always use the $weak^* \; topolpgy$ for $\Gamma^{-\infty}(E)$. We have the following properties:

Theorem(2.3.8):

($weak^* \; topolpgy$ of $\Gamma^{-\infty}(E)$)

i.) A sequence $sn \in \Gamma^{-\infty}(E)$ converges to $s \in \Gamma^{-\infty}(E)$ if and only if for all $\omega \in \Gamma_0^\ell \left(E^* \otimes \left| \Lambda^{top} \right| T^* M \right)$

$$s_n(\omega) \to s(\omega) \quad . \tag{2.85}$$

ii.) $\Gamma^{-\infty}(E)$ is sequentially complete, i.e. every $weak^* \; Cauchy$ sequence converges.

iii.) The inclusions $\Gamma^K(E) \subseteq \Gamma^{-\infty}(E)$ are continuous in the $\ell^k - and \, weak^* \; topology$ for all $k \in N_0 \cup \{+\infty\}$.

iv.) The map $\Gamma^{-\infty}(E) \ni s \mapsto f\,s \in \Gamma^{-\infty}(E)$ is $weak^{*}$ *continous for all* $f \in \ell^{\infty}(M)$.

v.) The sections $\Gamma_0^{\infty}(E)$ are sequentially $weak^{*}$ *dense in* $\Gamma^{\infty}(E)$.

(2.3.2) Calculus with Distributions:

Definition(2.3.9) (Restriction and Support):

Let $U \subseteq M$ be open and $s \in \Gamma^{-\infty}(E)$.

i.) The restriction $s|_U$ is defined by

$$s|_U(\omega) = s(w) \tag{2.86}$$

for $\omega \in \Gamma_0^{\infty}\left(E^{*} \otimes \left|\Lambda^{top}\right| T^{*} M|_U\right)$, i.e. for $\omega \in \Gamma_0^{\infty}\left(E^{*} \otimes \left|\Lambda^{top}\right| T^{*} M\right)$ with $Supp\ \ \omega \subseteq U$

ii.) The support of s is defined by

$$Supp\ s = \bigcap_{\substack{A \subseteq M\ closed \\ s|_{M \setminus A = 0}}} A. \tag{2.87}$$

Definition(2.3.10)(Singular support):

Let $s \in \Gamma^{-\infty}(E)$.

i.) S is called regular in $p \in M$ if there is an open neighborhood $U \subseteq M\ of\ p$ such that

$$s|_U \in \Gamma^{\infty}(E|_U).$$

ii.) The singular support of s is

$$\text{sing supps} = \left\{p \in M \mid s\ is\ not\ regular\ in\ p\right\}$$

The singular support of s indeed behaves similar to the support.

Theorem (2.3.11)(Generalized sections with compact support): [64]

Let $s \in \Gamma^{-\infty}(E)$ have compact support. Then we have :

i.) s has finite global order $ord(s) < 0$.

ii.) s has a unique extension to a linear function

$$s:\ \Gamma_0^{\infty}\left(E^{*}\otimes\left|\Lambda^{top}\right|T^{*}M\right)\to C, \qquad (2.88)$$

which is continuous in the ℓ^{∞} -topology.

Conversely, if $s:\ \Gamma_0^{\infty}\left(E^{*}\otimes\left|\Lambda^{top}\right|T^{*}M\right)\to C$, is a continuous linear functional then its restriction to $\Gamma_0^{\infty}\left(E^{*}\otimes\left|\Lambda^{top}\right|T^{*}M\right)$, is a generalized section of E with compact support.

Proposition (2.3.12):[65]

Let $s\in\Gamma^{-\infty}(E)$ be a Generalized sections ,then there exists a unique extension $\tilde{s}$ of s to a linear functional

$$\tilde{s}:\left\{\omega\in\Gamma^{\infty}\left(E^{*}\otimes\left|\Lambda^{top}\right|T^{*}M\right)\middle| Supp\,\omega\cap Supp\,s\ is\ compact\right\}\to C, \qquad (2.89)$$

i) $\tilde{s}$ coincides with s on $\Gamma_0^{\infty}\left(E^{*}\otimes\left|\Lambda^{top}\right|T^{*}M\right)$,

ii) $\tilde{s}\,\omega := if\ Supp\ s\cap\ Supp\ \omega=\phi$

Definition(2.3.13)(Push-forward of Distributions):

Let $\phi:\ M\to N$ be a smooth map. The push-forward of compactly supported generalized densities

$$\phi_{*}:\ \Gamma_0^{\infty}\left(E^{*}\otimes\left|\Lambda^{top}\right|T^{*}M\right)\to,\Gamma_0^{\infty}\left(E^{*}\otimes\left|\Lambda^{top}\right|T^{*}N\right) \qquad (2.90)$$

is defined on $f\in\ell^{\infty}(M)$ by

$$\left(\phi_{*}\mu\right)(f)=\mu\left(\phi^{*}f\right) \qquad (2.91)$$

Proposition(2.3.14)(Push-forward of Distributions):

Let $\phi:\ M\to N$ be a smooth map.

i.) The push-forward $\phi_{*}\mu\ of\ \mu\in\Gamma_0^{-\infty}\left(\left|\Lambda^{top}\right|T^{*}M\right)$ is a well-defined generalized density with compact support

$$\phi_{*}\mu\in\Gamma_0^{-\infty}\left(\left|\Lambda^{top}\right|T^{*}N\right)$$

The map ϕ_{*} is linear and continuous with respect to the $weak^{*}\ topolpgy$.

ii.) Assume ϕ is in addition proper. Then the push-forward extends uniquely to $\Gamma^{-\infty}\left(\left|\Lambda^{top}\right|T^{*}M\right)$ and gives a linear continuous map

$$\phi_{*}:\quad \Gamma^{-\infty}\left(\left|\Lambda^{top}\right|T^{*}M\right)\to\Gamma^{-\infty}\left(\left|\Lambda^{top}\right|T^{*}N\right) \tag{2.92}$$

with respect to the *weak* * *topolpgy* , Explicitly, for all $\varphi\in\ell_{0}^{\infty}(N)$ the push-forward $\phi_{*}\mu$ *of* μ is given by

$$(\phi_{*}\mu)(\varphi)=\mu\left(\phi^{*}\varphi\right), \tag{2.93}$$

iii.) We have

$$(id_{M})_{*}=id_{\Gamma_{0}^{-\infty}\left(\left|\Lambda^{top}\right|T^{*}M\right)}$$

and

$$(\phi\circ\varphi)_{*}=\phi_{*}\circ\varphi_{*} \tag{2.94}$$

Definition(2.3.15)(Differentiation of Generalized Sections):[66, 67]

Let $diffop^{\bullet}(E;F)$ then

$$D:\ \Gamma^{-\infty}(E)\to\Gamma^{-\infty}(F) \tag{2.95}$$

is defined by

$$(Ds)(\mu)=s\left(D^{T}\mu\right) \tag{2.96}$$

for all $s\in\Gamma^{-\infty}(E)$ and $\mu\in\Gamma_{0}^{-\infty}\left(F^{*}\otimes\left|\Lambda^{top}\right|T^{*}M\right)$.

This definition indeed gives a reasonable notion of differentiation of generalized sections as the following theorem shows

Theorem(2.3.16):[22]

i.) For all $s\in\Gamma^{-\infty}(E)$ the definition (2.96) gives a well-defined generalized section $Ds\in\Gamma^{-\infty}(E)$ and the map

$$D:\ \Gamma^{-\infty}(E)\to\Gamma^{-\infty}(F) \tag{2.97}$$

is linear and $weak^{*}$ *continous* . Moreover, we have for all $\ell\in N_{0}$

$$D:\ \Gamma^{-\ell}(E)\to\Gamma^{-\ell-k}(F). \tag{2.98}$$

ii.) The map D is the unique extension of $D:\Gamma^{\infty}(E)\to\Gamma^{\infty}(F)$ which is linear and $weak^*$ *continous* .

iii.) With respect to the $\ell^{\infty}(M)$ -module structure of $\Gamma^{\infty}(E)$ *and* $\Gamma^{\infty}(F)$, the map D as in (2.97) is a differential operator of order k in the sense of the algebraic definition of differential operators, i.e.

$$D \in Diffop^{k}(\Gamma^{-\infty}(E), \Gamma^{-\infty}(F)) \tag{2.99}$$

iv.) We have

$$Supp(Ds) \subseteq Supp s \tag{2.100}$$

and $Sing\ Supp\ (D s) \subseteq Sing\ Supp\ s$ (2.101)

v.) For every open subset $U \subseteq M$ we have

$$D s\big|_{U} = D\big|_{U}\left(s\big|_{U}\right) \tag{2.102}$$

(2.3.3) Tensor Products

Definition(2.3.17)(Vector-valued Generalized Sections):[68]

Let $E \to M$ be a vector bundle and V a finite-dimensional vector space, then a V -valued generalized section of E is a continuous linear map.

$$s:\ \Gamma_0^{\infty}\left(E^{*} \otimes \left|\Lambda^{top}\right| T^{*} M\right) \to V . \tag{2.103}$$

the set of all V -valued generalized sections of E is denoted by $\Gamma^{-\infty}(E;V)$.

Proposition(2.3.18):[69]

For a finite-dimensional vector space V and a vector bundle $E \to M$ we have the canonical isomorphism

$$\Gamma^{-\infty}(E) \otimes V \ni s \otimes v \mapsto (\omega \to s(\omega)v) \in \Gamma^{-\infty}(E;V) \tag{2.104}$$

Remark(2.3.19):

For the external tensor product

$$\oplus : \Gamma^{-\infty}(E) \otimes \Gamma^{-\infty}(F) \to \Gamma^{-\infty}(E \oplus F) \tag{2.105}$$

one immediately obtains

$$Supp\ (s \oplus t) = Supp\ s \times Supp\ t \tag{2.106}$$

whence we also have

$$\oplus : \Gamma_0^{-\infty}(E) \otimes \Gamma_0^{-\infty}(F) \to \Gamma_0^{-\infty}(E \oplus F)\,. \tag{2.107}$$

It can be shown that for compactly supported s and t, the abve equations hold.

chapter(3)

Lorentz Geometry and Causality

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Lorentz Geometry and Causality

(3.1) Basics Concept

In this section ,we introduce elementary notions on 3-dimensions Minkowski space , its relationship to the hyperbolic plane, and its isometries.

(3.1.1) Affine Space and Its Tangent Space:[70]

We define n-dimensional affine space A^n to be the set of all n-tuples of real numbers $(p_1, \ldots, p_n)$. An affine space could be defined over any field,but we will restrict to the field of real numbers. Elements of affine space will be called points.

for $p = (p_1, \ldots, p_n) \in A^n$ and $t = \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix}$,we define :

$$p + t = (p_1 + t_1, \ldots, p_n + t_n) \in A^n \tag{3.1}$$

Thus the vector space R^n, considered as a Lie group, acts transitively on A^n by *translations*; the translation by $t \in R^n$, denoted τ_t, is defined as follows:

$$\tau_t : R^n \times A^n \to A^n$$

$$(t, p) \mapsto p + t \tag{3.2}$$

(3.1.2)The Inner Product and $(2+1)$-dimensional Minkowski Space:[71, 72]

A Lorentzian vector space of dimension 3 is a real 3-dimensional vector space V endowed with an inner product of signature (2,1). The Lorentzian inner product will be denoted:

$$V \times V \to R$$

$$(v, u) \mapsto v.u$$

We also fix an orientation on V. The orientation determines a nondegenerate alternating trilinear form

$$V \times V \times V \xrightarrow{Det} R$$

which takes a positively oriented orthogonal basis e_1, e_2, e_3 with inner products

$$e_1 . e_1 = e_2 . e_2 = 1 \ , e_3 . e_3 = -1$$

to 1. The oriented Lorentzian 3-dimensional vector space determines an alternating bilinear mapping $V \times V \to V$, called the *Lorentzian cross-product,* defined by

$$Det(u, v, w) = u \times v . w. \tag{3.3}$$

We call the affine space modeled on V *Minkowski space* and denote it E. This is an oriented manifold, since V is oriented. Alternatively, E can be defined as a 3-dimensional, oriented, geodesically complete, 1-connected, flat Lorentzian manifold.

(3.1.3) Light, Space and Time. The Causal Structure of Minkowski Space:[73, 74]

The inner product induces a *causal structure* on V: a vector $V \neq 0$ is called

- *timelike* if $V . V < 0$,
- *null* (or *lightlike*) if $V . V = 0$, or
- *spacelike* if $V . V > 0$.

We will call the corresponding subsets of V respectively V_-, V_0, V_+. The set V_0 of null vectors is called the *light cone*.

(3.1.4) Null Frames:

The restriction of the inner product to the orthogonal complement $S^{\perp}$ of a spacelike vector **s** is indefinite having signature (1, 1). The intersection of the light cone with $S^{\perp}$ consists of two null lines intersecting transversely at the origin, for a unit spacelike vector **s**:

$$s \times s^- = s^- .$$

$$s \times s^+ = s^+$$

The basis defines linear coordinates (*a, b, c*) on V: $v := a s + b s^- + c s^-$

so the corresponding Lorentz metric on E is:

$$d a^2 - d b d c$$

(3.1.5) Relationship to The Hyperbolic Plane:

Let $H^2 \subset V$ the set of unit future-pointing timelike vectors, that is

$H^2 = \{v \in V - | v.v = -1|\}$, denote the restriction of the Lorentzian metric to H^2, denoted dH^2, is positive definte for $u, v \in H^2 : \cosh\left(dH^2(u, v)\right) = u.v$

the resulting metric is a Riemannian metric with constant curvature -1, and we identify H^2 with the hyperbolic plane.

Geodesics in the hyperbolic plane correspond to indefine planes in V, which are precisely the planes that intersect H^2, equivalently, these are Lorentzian-perpendicular planes to spacelike vectors. Thus spacelike vector **s** is identified with a geodesic in H.

(3.1.6) Components of The Isometry Group:

The group $O(2,1)$ has four connected components. The identity component $SO^0(2,1)$ consists of orientation-preserving linear isometries preserving time-orientation. It is isomorphic to the group $PSL(2,R)$ of orientation-preserving isometries of the hyperbolic plane. The group $O(2,1)$ is semi direct product

$$O(2,1) \cong (Z/2 \times Z/2) \times SO^0(2,1)$$

(3.1.7) Transvections , Boosts, Homotheties and Reflections:*[75, 76]*

In the null frame coordinates ,the one-parameter group of linear isometries

$$\xi_t := \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$$

(for $t \in R$) fixes **s** and acts on the (indefinite) plane $\mathbf{s}^{\perp}$. These transformations, called *boosts,* constitute the identity component $SO^0(1,1)$of the isometry group of $V^{\perp}$.The one-parameter group R^{+} of *positive homotheties*

$$\eta_s := \begin{bmatrix} e^s & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^s \end{bmatrix}$$

(where $s \in R$) acts conformally on Minkowski space, preserving orientation. The involution

$$\rho := \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

preserves orientation, reverses time-orientation, reverses **s**, and interchanges the two null lines R**s**$^-$ and R**s**$^+$.

(3.2) Proper Actions and Locally Homogeneous Lorentzian 3-Manifolds

(3.2.1) Groups of Isometries :[77]

Definition(3 . 2 . 1) :

Let X be a locally compact space and G a group acting on X. We say that G acts properly discontinuously on X if for every compact $K \subset X$, the set:

$$\{\gamma \in G \,|\, \gamma\, K \cap K \neq 0\} \tag{3.4}$$

is finite,

Theorem (3.2.2):[78]

Let X be a Hausdorff manifold and let G be a group that acts freely and properly discontinuously on X. Then X/G is a Hausdorff manifold

Remark(3.2.3):

A group that acts properly discontinuously on E is discrete. But the converse, which holds for Riemannian manifolds, is false for group actions on E

Definition(3 . 2 . 4) :

Let X be a topological space and G a group acting on X. Let $F \subset X$ be a closed subset with non-empty interior. We say that *F* is a fundamental domain for the *G*-action on *X* if ;

- $X = U_{\gamma \in G}\, \gamma\, F$;
- for all. $\gamma \neq \eta \in G$, $\mathrm{int}(\gamma\, F) \cap \mathrm{int}(\eta\, F) \neq \phi$

Theorem(3.2.5):

Let X be a topological space and G a group acting on X. Suppose there exists a fundamental domain F for the G-action on X. Then G acts properly discontinuously on X and:

$$X/G = F/G \tag{3.5}$$

Definition(3.2.6):

A Margulis spacetime is a Hausdorff manifold E/G

where G is free and non-abelian.

(3.2.2) Examples of Margulis Spacetime ; Crooked Planes :

Definition(3.2.7):[79]

Let $X \in R^{2,1}$ be a future-pointing null vector. Then the closure of the following halfplane:

$$Wing\,(X) = \{u \in X^{\perp} \mid X = u^{+}\} \tag{3.6}$$

is called a positive linear wing.

In the affine setting, given $p \in E, p + Wing\,(X)$ is called a positive

wing, observe that if $u \in R^{2,1}$ is spacelike:

$$u \in Wing\,(u^{+})$$

$$-u \in Wing\,(u^{-})$$

$$u \in Wing\,(u^{+}) \cap u \in Wing\,(u^{-}) = 0$$

The set of positive linear wings is $SO(2,1)$-invariant.

Definition(3.2.8):[80]

Let $u \in R^{2,1}$ be spacelike. Then the following

$$Stem(u) = \{X \in u^{\perp} \mid x.x \leq 0\}$$

is called a linear stem. For $p \in E, p + Stem\,(u)$, is called a stem .

Observe that $Stem(u)$ is bounded by the lines $R\,u^{+}$ *and* $R\,u^{-}$ and thus respectively intersects the closures of $Wing\,(u^{+})$ and $Wing\,(u^{-})$ in these lines.

Definition(3.2.9):

Let $p \in E$ and $u \in R^{2,1}$ be spacelike. The positively extended crooked plane with vertex p and director u is the union of:

- *the stem* $p + Stem(u)$;
- $the\, postivewing + Wing\ (u^{+})$;
- $the\, postivewing + Wing\ (u^{-})$;

It is denoted $C(p,u)$.

(3.2.3) Crooked Halfspaces and Disjointness:

The complement of a crooked plane in $C(p,u) \in E$ consists of two crooked halfspaces, respectively corresponding to u *and* $-u$. A crooked

halfspace will be determined by the appropriate *stem quadrant*,.

Definition(3.2.10):

Let $u \in V$ be spacelike and $p \in E$. The associated

stem quadrant is:

$$Quad\ (p,u) = p + \{a u^{-} - b u^{+} \mid a, b \geq 0\} \tag{3.7}$$

The stem quadrant $Quad\ (p,u)$ is bounded by light rays parallel to u^{-} *and* $-u^{+}$.

Definition(3.2.11):[81]

Let $p \in E$ and $u \in V$ be spacelike,the crooked half-space $H\,(p,u)$ is the component of complement of $C\,(p,u)$ containing int $(Quad\ (p,u))$.

by defintion crooked halfspaces are open. While the crooked planes $C\,(p,u), C\,(p,-u)$are equal, the crooked halfspaces $H\,(p,u), H\,(p,-u)$,are disjoint, sharing $C\,(p,u)$ as a common boundary.

Definition(3.2.12):[82]

Let $\circ \in E$ and $u_1, u_2 \in V$ be spacelike.The vectors are said to be consistently oriented if the closures of the crooked halfspaces $H(\circ, u_1)$ *and* $H(\circ, u_2)$intersect only in o.

Definition(3.2.13):

Let $u_1, u_2 \in V$ be a pair of consistently oriented ultraparallel spacelike vectors.The set of allowable translations

for u_1, u_2 is:

$$A(u_1, u_2) = \operatorname{int}\left(Quad\ (p, u_1) - Quad\ (p, u_2)\right) \subset V$$

where $p \in E$ can be arbitrarily chosen.

(3.3) Deformations.

(3.3.1) Lorentzian Transformations and Affie Deformations:

Let $Isom^{+}(E)$ denote the group of all orientation-preserving affine transformations that preserve the Lorentzian inner product on the space of directions; $Isom^{+}(E)$ is isomorphic to $SO(2,1) \propto R^{2,1}$ then the projection is

$$Isom^{+}(E) \xrightarrow{l} SO(2,1)$$

Definition(3.3.1):[78, 82]

Let $g \in SO^{\circ}(2,1)$ be a non identity element;

- g is hyperbolic if it has three, distinct real eigenvalues;
- g is parabolic if its only eigenvalue is 1;
- g is elliptic otherwise.

We also call $\gamma \in Isom^{+}(E)$ *hyperbolic* (respectively *parabolic*, *elliptic*) if its linear part $L(\gamma)$ is hyperbolic (respectively parabolic, elliptic).

Let $\Gamma_0 \subset SO(2,1)$ be a subgroup. An *affine deformation* of Γ_0 is a representation

$$\rho : \Gamma_0 \to Isom^{+}(E) \tag{3.8}$$

(3.3.2) The Lie Algebra $\varsigma\iota(2,R) as V$:

The Lie algebra $\varsigma\iota(2,R)$ is the tangent space to $PSL(2,R)$ at the identity and consists of the set of traceless 2×2 matrices[83].The three-dimensional vector space has a natural inner product , the Killing form ,defined to be

$$\langle v,w\rangle = \frac{1}{2} Tr(v,w) \tag{3.9}$$

A basis for $\varsigma\iota(2,R)$ is given by

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{3.10}$$

Evidently, $\langle E_1,E_1\rangle = \langle E_2,E_2\rangle = 1$, $\langle E_3,E_3\rangle -1$ and $\langle E_i, E_j\rangle = 0 \; for\, i \neq j$

that is , $\varsigma\iota(2,R)$ is isomorphic to V as a vector

$$\left\{ V = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right\} \leftrightarrow \{ x E_1 + y E_2 + z E_3 = V \}$$

the adjoint action of $PSL(2,R)$ on $\varsigma\iota(2,R)$:

$g(v) = g v g^{-1}$ corresponds to the linear action of $SO^{\circ}(2,1)\; on\, V$ Using these identifications, set:

$G \cong PSL(2,R) \cong SO^{0}(2,1) \;\; g \cong \varsigma\iota(2,R) \cong V$.

(3.3.3) The Margulis invariant:

Let $g \in G$ be a non-elliptic element. Lift g to a representative in $SL(2,R)$; then the following element of g is a g -invariant vector which is independent of choice of lift:

$$F_g = \rho(g)\left(g - \frac{Tr(g)}{2} I \right) \tag{3.11}$$

where $\rho(g)$ is the sign of the trace of the lift.

We define the *non-normalized Margulis invariant* of $\rho(g) \in \rho(\Gamma_0)$ to be:

$$\widetilde{\alpha}_p(g)=\langle u(g), F_g\rangle \tag{3.12}$$

If $p(g)$ is hyperbolic, then F_g is spacelike and

$$X_g^0 = \frac{2\rho(g)}{\sqrt{Tr(g)^2-4}}\left(g-\frac{Tr(g)}{2}I\right)$$

is the unit-spacelike vector, then

$$\alpha_\rho(g)=\langle u(g), X_g^0\rangle \tag{3.13}$$

In Minkowski space, $\alpha_\rho(g)$ is the *signed Lorentzian length* of a closed geodesic in

$$E/\langle\alpha_\rho(g)\rangle.$$

(3.4) Length Changes in Deformations:

Let $\rho_0 : \pi_1(\Sigma)\to\Gamma_0\subset G$ be a holonomy representation and let $\rho:\Gamma_0\to Isom^+(E)$ be an affine deformation of ρ_0, with corresponding cocycle $u\in Z^1(\Gamma_0, g)$.

The affine deformation ρ induces a path of holonomy representations ρ_t as follows:

$$\rho_t : \pi_1(\Sigma)\to G\ ,$$

$$\gamma\to\exp(tu(g))g\,,$$

where $g=\rho_0(\gamma)$, and u is the tangent vector to this path at t.

Conversely, for any path of representations ρ_t

$$\rho_t(\gamma)=\exp\left(tu(g)+O(t^2)\right)g \tag{3.14}$$

where $u \in Z^1(\Gamma_0, g)u$ and $g = \rho_0(\gamma)$ .Suppose g is hyperbolic. Then the length of the corresponding closed geodesic in Σ is

$$l(g) = 2\cosh^{-1}\left(\frac{Tr(\widetilde{g})}{2}\right)$$

where $\widetilde{g}$ is a lift of g to $SL(2,R)$. With ρ, ρ_t since the Margulis invariant of ρ can also be seen to be a function of its corresponding cocycle u ,then

$$\alpha_u(g) := \alpha_\rho(g)$$

and

$$\frac{d}{dt}\Big|_{t=0} i_t(\gamma) = \frac{\alpha_u(g)}{2} \tag{3.15}$$

so we may interpret α_u as the change in length of an affine deformation

Although $i_t(\gamma)$ is not differentiable at 0 for parabolic g ,

$$\frac{d}{dt}\Big|_{t=0} = \frac{\alpha(g)}{2} Tr(\rho_t(\gamma)) = \widetilde{\alpha}_u(g) \tag{3.16}$$

(3.4.1) Deformed Hyperbolic Transformations:[84]

Let $g \in SL(2,R)$ be a hyperbolic element, thus a lift of a hyperbolic isometry of H^2. Given a tangent vector in $V \in \varsigma i(2,R)$, and

$$\pi_V : g \to \exp(V).g \tag{3.17}$$

and

$$\pi'_V : g \to g.\left(\exp(V)^{-1}\right) = g.\exp(-V) \tag{3.18}$$

Therefore,

$$g = \begin{bmatrix} e^s & 0 \\ 0 & e^{-s} \end{bmatrix} = \exp\left(\begin{bmatrix} s & 0 \\ 0 & -s \end{bmatrix}\right)$$

whose trace is $Tr(g)=2\cosh(s)$, the eigenvalue frame for the action of g on $\varsigma i(2,R)$ is

$$X_g^0=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\quad X_g^-=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix},\ X_g^+=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},$$

where

$$g\,X_g^0\,g^{-1}=X_g^0$$

$$g\,X_g^-\,g^{-1}=e^{-2s}X_g^-$$

$$g\,X_g^+\,g^{-1}=e^{2s}X_g^+$$

then $V\in\varsigma i(2,R)$

$$V=a\ X^0(g)+b\,X^-(g)+c\,X^+(g)=\begin{bmatrix}a & b\\ c & -a\end{bmatrix} \tag{3.19}$$

then

$$Tr\left(\pi_V\left(g\right)\right)=2\cosh\ s\cosh\ \sqrt{a^2+bc}\ +\ \frac{2a\ \sinh\ s\sinh\ \sqrt{a^2+bc}}{\sqrt{a^2+bc}} \tag{3.20}$$

(3.4.2) Deformed Parabolic Transformations:

$$p=\begin{bmatrix}1 & r\\ 0 & 1\end{bmatrix},\ =\ \exp\begin{bmatrix}0 & r\\ 0 & 0\end{bmatrix},$$ where $r>0\,and\ Tr(p)=2$

$$X^u(g)=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\qquad X^0(g)=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix},\quad X^c(g)=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} \tag{3.21}$$

then the Trace of the deformation of ρ is

$$Tr\left(\pi_V\left(p\right)\right)=2\cosh\sqrt{a^2+bc}\ +\ \frac{cr}{\sqrt{a^2+bc}}\sinh\sqrt{a^2+bc}$$

(3.5) Einstein Universe.

The Einstein Universe Ein_n can be defined as the projectivisation of the lightcone of $R^{n,2}$ .We will write everything for $n=3$.

Let $R^{3,2}$ denote the vector space R^5 endowed with a symmetric bi-linear form of signature (3, 2). Specifically, for $X=(x_1,\dots,x_5)$ and $Y=(y_1,\dots,y_5)\in R^5$

set: $X.Y = x_1 y_1 + x_2 y_2 + x_3 y_3 - x_4 y_4 - x_5 y_5$

Let $X^{\perp}$ denote the orthogonal hyperplane to X: $X^{\perp}=\{Y\in R^{3,2} \mid X.Y=0\}$

Let $N^{3,2}$ denote the *lightcone* of $R^{3,2}$:

$$N^{3,2}=\{X\in R^{3,2}\setminus 0 \mid X.X=0\} \tag{3.22}$$

the quotient of $N^{3,2}$ under the action of R^* by scaling:

$$Ein_3 = N^{3,2}/R^* \tag{3.23}$$

Denote by $\pi(V)$ the image of $V\in N^{3,2}$ under this projection. Wherever convenient, for $V=(v_1,v_2,v_3,v_4,v_5)$ we will also write:

$$\pi(V)=(v_1 : v_2 : v_3 : v_4 : v_5) \tag{3.24}$$

Denote by $\hat{Ein}$ orientable double-cover of Ein_3.Alternatively:

$$\hat{Ein}_3 = N^{3,2}/R^+ \tag{3.25}$$

Any lift of $\hat{Ein}_3$ to $N^{3,2}$ induces a metric on Ein_3 by restricting $,\cdot$ to the image of the lift. For instance, the intersection with $N^{3,2}$ of the

sphere of radius 2, centered at 0, consists of vectors X such that:

$$x_1^2+x_2^2+x_3^2=1=x_4^2+x_5^2 \tag{3.26}$$

It projects bijectively to $\hat{Ein}_3$ endowing it with the Lorentzian product metric dg^2-dt^2 , where dg^2 is the standard round metric on the 2-spher S^2 , and dt^2 is the standard metric on the circle S^1 .

Thus Ein_3 is conformally equivalent to:

$$S^2 \times S^1 / \sim , where X \sim -X$$

Here $-I$ factors into the product of two antipodal maps.

Any metric on $\hat{Ein}_3$ pushes forward to a metric on Ein_3. Thus Ein_3 inherits a conformal class of Lorentzian metrics from the ambient space- time $R^{3,2}$. The group of conformal automorphisms of Ein_3 is:

$$Conf(Ein_3) \cong PO(3,2) \cong SO(3,2)$$

As $SO(3,2)$ acts transitively on $N^{3,2}$, $Conf(Ein_3)$ acts transitively on Ein_3.

(3.6) Preliminaries on Semi-Riemannian Manifolds

(3.6.1) Parallel Transport and Curvature:[85]

Let ∇^E be a covariant derivative for a vector bundle $E \to M$ as before. Then the curvature tensor R of ∇^E is defined by

$$R(X,Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s \tag{3.27}$$

for $X, Y \in \Gamma^\infty(TM)$ and $s \in \Gamma^\infty(E)$, A simple computation shows that $R \; is \; \ell^\infty(M)$ -linear in each argument and thus defines a tensor field

$$R \in \Gamma^\infty\left(End(E) \otimes \Lambda^2 T^* M\right) \tag{3.28}$$

There are certain contractions we can build out of R. The most important one is the pointwise trace of the $End(E) - Part \; of \; R$. This gives a two-from

$$tr\, R(X,Y) = tr\,(s \mapsto R(X,Y)s), \tag{3.29}$$

i.e. a section $tr\, R \in \Gamma^\infty\left(\Lambda^2 T^* M\right)$.The following lemma gives an interpretation of $tr\, R$:

Lemma(3.6.1):

Let E be a covariant derivative for a vector bundle $E \to M$.

i.) The two-form $tr\; T \in \Gamma^\infty\left(\Lambda^2 T^* M\right)$ is closed , $d\, tr\; R = 0$.

ii.) The two-form $tr\; R$ is exact. In fact,

$$tr\; R = -d\alpha \tag{3.30}$$

where $\alpha \in \Gamma^{\infty}\left(T^{*}M\right)$ is defined by

$$\alpha\left(X\right)=\frac{\nabla_{X}^{E}\mu}{\mu} \tag{3.31}$$

with respect to any chosen positive density $\mu \in \Gamma^{\infty}\left(\left|\Lambda^{top}\right| E^{*}\right)$.

Definition (3.6.2) (UnimodularCcovariant Derivative):[86]

A covariant derivative ∇^{E} is called uni-modular if $tr\ R^{E}=0$.

Let $\gamma : I \subseteq R \to M$ be a smooth curve defined on an open interval I.

then a section $s \in \Gamma^{\infty}\left(\gamma^{\#}E\right)$ with

$$\nabla^{\#}_{\frac{d}{dt}} s = 0 \tag{3.32}$$

if $\{e_{\alpha}\}$ are local base section of E over some open subset $U \subseteq M \ and\ \gamma\left(I\right) \subseteq U$

the (3.32) is equivalent to

$0=\nabla^{\#}_{\dot{\gamma}}\left(s^{\alpha}\left(t\right)e_{\alpha}\left(\gamma(t)\right)\right)= s^{\cdot\alpha}\left(t\right)e_{\alpha}\left(\gamma(t)\right)+s^{\alpha}\left(t\right)A_{\alpha}^{\beta}\dot{\gamma}\left(t\right)e_{\beta}\left(\gamma(t)\right)$

i.e.

$$s^{\cdot\beta}\left(t\right)+A_{\alpha}^{\beta}\left(\dot{\gamma}\left(t\right)\right)s^{\alpha}\left(t\right)=0 \tag{3.33}$$

Proposition(3.6.3):

Let ∇^{E} be a covariant derivative for $E \to M$ and let

$\gamma : I \subseteq R \to M$ be smooth curve. Let $a, b \in I$.

i.) For every initial condition $s_{\gamma(a)} \in E_{\gamma(a)}$ there exists a unique solution $s(t) \in E_{\gamma(a)}$ (3.32).

ii.) The map $s_{\gamma(a)} \mapsto s(b)$ is a linear isomorphism $E_{\gamma(a)} \to E_{\gamma(b)}$ which is denoted by

$$P_{\gamma, a\to b} : E_{\gamma(a)} \to E_{\gamma(b)} \tag{3.34}$$

Definition(3.6.4)(Parallel Transport):

The linear isomorphism $P_{\gamma,a\to b} : E_{\gamma(a)} \to E_{\gamma(b)}$ is called the parallel transport along γ with respect to ∇^E.

(3.6.2) The Exponential Map:

In the case $E = TM$ a covariant derivative has additional features . First, we have another contraction of the curvature tensor R given by

$$Ric(X,Y) = tr\,(z \mapsto R(Z,X)Y) \tag{3.35}$$

or $X, Y \in \Gamma^{\infty}(TM)$. The resulting tensor field

$$Ric \in \Gamma^{\infty}(T^*M \otimes T^*M) \tag{3.36}$$

is called the Ricci tensor of ∇. Note that the trace in (3.35) only can be defined for $E = TM$. The third contraction $tr\,(z \mapsto R(Z,X)Y)$ would give again the Ricci tensor up to a sign. Thus (3.35) is the only additional interesting contraction.

For a covariant derivative ∇ on TM we have yet another tensor field , the torsion

$$Tor(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \tag{3.37}$$

which gives a tensor field

$$Tor \in \Gamma^{\infty}(\Lambda^2 T^*M \otimes TM) \tag{3.38}$$

then ∇ is called torsion-free if $Tor = 0$.

Theorem (3.6.5) (Geodesics) :[87, 88]

Let ∇ be a covariant derivative for $TM \to M$.

i.) For every $v_p \in T_p M$ there exists a unique solution $\gamma = I_{vp} \subseteq R \to M$ of

($\ddot{\gamma}^i(t) + \Gamma^i_{k\ell}(\gamma(t))\dot{\gamma}^k(t)\dot{\gamma}^\ell(t)$ with $\dot{\gamma}(0) = v_p$ and maximal open interval $I_{vp} \subseteq R$ around 0.

ii.) let $\lambda \in R$ and $v_p \in T_p M$ if γ denotes the geodesic with $\dot{\gamma}(0) = v_p$ then $\dot{\gamma}_\lambda(t) = \gamma(\lambda t)$ is the geodesic with $\dot{\gamma}_\lambda(0) = \gamma_{vp}$

iii.) There exists an open neighborhood $\upsilon \subseteq TM$ of the zero section such that for all $v_p \in \upsilon$ the geodesic with $\ddot{\gamma}(0) = v_p$ is defined for all $t \in [0,1]$. We set $\exp_p(vp) = \gamma(1)$ for this geodesic .

iv.) for $v_p \in \upsilon \subseteq TM$ the curve $t \mapsto \exp_p(tv_p)$ is the geodesic γ *with* $\gamma(0) = \gamma_v$

v.) The map exp $\upsilon \subset TM \to M$ is smooth.

vi.) The map $\pi \times \exp\colon \upsilon \subseteq TM \ni v_p \mapsto (p, \exp_p(v_p)) \in M \times M$ (3.39)

is a local diffeomorphism around the zerosection. It maps the zero section onto the diagonal and for all $p \in M$

$$T_{0p} \exp_p = id T_p M \tag{3.40}$$

Definition (3.6.6) ,(Exponential Map):

For a given covariant derivative ∇ the map $\exp\colon \upsilon \subseteq TM \to M$ given by $v.)$ of Theorem 3.6.5 is called the exponential map of ∇.

Definition(3.6.7):

An open subset $U \subseteq M$ is called

i.) geodesically star-shaped with respect to $p \in M$ if there is a star-shaped

$$V \subseteq V_p \; with \exp_p(V) \exp_p|_V : V \cong U$$

ii.) geodesically convex if it is geodesically star-shaped with respect to any point $p \in U$.

Definition(3.6.8)(Geodesic Completeness):

The covariant derivative ∇ plete if all geodesics are defined for all times.

(3.6.3) Levi-Civita Connection and The D'Alembertian:

Definition(3.6.9)(Semi-Riemannian metric): [34]

A section $g \in \Gamma^\infty(S^2 T^* M)$ is called semi-Riemannian metric if the bilinear form

$g_p \in S^2 T_p^* M$ *on* $T_p M$ is non-degenerate for all $p \in M$. If in addition g_p is positive definite for all $p \in M$ then g is called Riemannian metric. If g_p has signature $(+,-,\ldots,-)$ then g is called Lorentz metric.

Remark (3.6.10) (Semi-Riemannian Metrics):

i.) The signature of a semi-Riemannian metric is locally constant and hence constant on a connected manifold, since it depends continuously on p and has only discrete values.

ii.) For Lorentz metrics also the opposite signature $(-,+,\ldots,+)$ is used in the literature.

A semi-Riemannian metric specifies a unique covariant derivative and a unique positive density

Proposition(3.6.11):[89]

Let g be a semi-Riemannian metric on M .

i.) There exists a unique torsion-free covariant derivative ∇ , the Levi-Civita connection, such that

$$\nabla_g = 0. \tag{3.41}$$

ii.) There exists a unique positive density $\mu_g \in \Gamma^\infty(|\Lambda^{top}| T^* M)$ such that

$$\mu_g|_p(v_1, \ldots, v_n) = 1 \tag{3.42}$$

whenever $v_1, \ldots, v_n$ form a basis of $T_p M$ $with \left| g_p(v_i, v_j) \right| = \delta_{ij}$ in a chart (U, x) we have

$$\mu_g | u = \sqrt{|\det(gij)||} \, dx^1 \wedge \ldots \wedge dx^n |, \tag{3.43}$$

iii.) The density μ_g is covariantly constant with respect to the Levi-Civita connection,

$$\nabla_{\mu g} = 0. \tag{3.44}$$

Thus ∇ unimodular.

Remark(3.6.12)(Semi-Riemannian Metrics) :

Let g be a semi-Riemannian metric on M .

i.) For a semi-Riemannian metric we have a notion of geodesics, namely those with respect to the corresponding Levi-Civita connection.

ii.) The covariant divergence $div_\nabla(X)$ of a vector field $X \in \Gamma^\infty(TM)$ and the divergence with respect to the density μ_g , i.e.

$$div\,\mu_g(X) = \frac{\wp \times \mu_g}{\mu_g} \tag{3.45}$$

coincide: we have

$$div \nabla(X) = div \mu_g(X), \tag{3.46}$$

since $\nabla_{\mu g} = 0$ then

$$div(\mathrm{X}) = div\nabla(\mathrm{X}) = div\mu_g(\mathrm{X}) \tag{3.47}$$

on a semi-Riemannian manifold

iii.) Since $g \in \Gamma^{\infty}(S^2T^*M)$ is non-degenerate it induces a musical isomorphism

$$b: T_p\mathrm{M} \ni v_p \mapsto \overset{b}{v_p} = g(v_p,.) \in T_p^*\mathrm{M}, \tag{3.48}$$

which gives a vector bundle isomorphism

$$b: \mathrm{TM} \to T^*\mathrm{M}. \tag{3.49}$$

the inverse of b usually denoted by

$$\#: T^*\mathrm{M} \to \mathrm{TM} \tag{3.50}$$

extending $\# \, and \, b$ to higher tensor powers we get musical isomorphisms also between all corresponding contravariant and covariant tensor bundles. If locally in a chart (U,x)

$$g|_U = \frac{1}{2} g_{ji}\, dx^i \vee dx^j, \tag{3.51}$$

then $v^b = g_{ji} v^i dx^i$, where $v = v^i \dfrac{\partial}{\partial x^i}$, If g^{ij}, denotes the inverse matrix to the g_{ji} from (3.51),i.e. $g^{ij} g_{jk} = \delta_{ik}$ g , then

$$\alpha^{\#} = g^{ij} \alpha_i \frac{\partial}{\partial x_i} = \delta_{ik} \tag{3.52}$$

or a one-form $\alpha = \alpha_i d x^i$. This motivates the notion as b lowers the indexes while # raises them. Finally, we have the dual metric locally given by

$$g^{-1}|_U = \frac{1}{2} g^{ij} \frac{\partial}{\partial x^i} \vee \frac{\partial}{\partial x^j}, \tag{3.53}$$

which is a global section $g^{-1} \in \Gamma^{\infty}(S^2TM)$.

iv.) The metric $g \in \Gamma^{\infty}(S^2T^*M)$ can equivalently be interpreted as a homogeneous quadratic function on TM via the usual canonical isomorphism from Remark 2.2.7, the function

$$\mathrm{T} = \eta(g) \in pol^2(\mathrm{TM}) \tag{3.54}$$

is then usually called the kinetic energy function in the Lagrangian picture of mechanics. Analogously $g^{-1} \in \Gamma^{\infty}(S^2TM)$, gives a homogeneous quadratic function

$$T = \eta\left(g^{-1}\right) \in pol^2(T^*M) \tag{3.55}$$

v.) Using the inverse matrix g^{ij} we have the following local Christoffel symbols of the Levi-Civit connection

$$\Gamma_{ij}^{k} = \frac{1}{2} g^{k\ell} \left(\frac{\partial g_{\ell i}}{\partial x^j} + \frac{\partial g_{\ell j}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\ell}\right). \tag{3.56}$$

Since Proposition (3.6.11, iii.) for a semi-Riemannian manifold (M,g) the Ricci tensor Ric is in fact symmetric

$$Ric \in \Gamma^{\infty}(S^2T^*M), \tag{3.57}$$

we can compute a further "trace" by using the metric g. Note that while Ric can be defined for every covariant derivative this further contraction requires g . One calls the function

$$scal = \langle g^{-1}, Ric\rangle \in \ell^{\infty}(M) \tag{3.58}$$

the scalar curvature. Locally, scal is just

$$scal|_U = g^{ij} Ric_{ij}. \tag{3.59}$$

Definition(3.6.13)(Gradient and D'Alembertian) :[90]

On a semi-Riemannian manifold (M,g) the gradient of a function is defined by

$$grad\, f = (df)^{\#} \in \Gamma^{\infty}(TM) \tag{3.60}$$

and the d'Alembertian of a function $f \in \ell^{\infty}(M)$ is

$$\Box f = div(grad f) \in e^{\infty}(M) \tag{3.61}$$

In case of a Riemannian manifold we write $\Delta f = div\,(grad\ f).$ instead and call Δ the Laplacian.

Proposition(3.6.14):[91]

Let (M,g) be a semi-Riemannian manifold and let (U,x) be a chart of M .

i.) The gradient of $f \in \ell^{\infty}(M)$ is locally given by

$$grad\,(f)\big|_U = g^{ij}\,\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j} \quad . \tag{3.62}$$

ii.) The divergence of $X \in \Gamma^\infty(\mathrm{TM})$ is locally given by

$$div\,(X)\,|_U = \frac{\partial X^i}{\partial x^i} + \Gamma^k_{ki}\,X^i. \tag{3.63}$$

iii.) The d'Alembertian of $f \in \ell^\infty(\mathrm{M})$ is locally given by

$$\Box f\,|_U = g^{ij}\,\Big(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij}\,\frac{\partial f}{\partial x^k}\Big). \tag{3.64}$$

iv.) The d'Alembertian is a second order differential operator with leading symbol

$$\sigma(\Box) = 2g^{-1} \in \Gamma^\infty(S^2TM). \tag{3.65}$$

Moreover, with respect to the global symbol calculus induced by the Levi-Civita connection we have

$$\Box = \Big(\frac{i}{\hbar}\Big)^2 \ell std\,\left(2g^{-1}\right), \tag{3.66}$$

whence

$$\Box f = \frac{1}{2}\langle g^{-1}, D^2 f\rangle. \tag{3.67}$$

Remark(3.6.15)(Hessian):

Sometimes $\frac{1}{2}D^2 f \in \Gamma^\infty(S^2\mathrm{T}^*\mathrm{M})$. is also called the Hessian

$$Hess\,(f) = \frac{1}{2}D^2 f \in \Gamma^\infty(S^2\mathrm{T}^*\mathrm{M}). \tag{3.68}$$

Then the d'Alembertian is the trace of the Hessian with respect to g^{-1}. Moreover, the gradient $grad{:}\,\ell^\infty(M) \to \Gamma^\infty(\mathrm{TM})$. is a differential operator of order one, the same holds for the divergence $div{:}\,\Gamma^\infty(\mathrm{TM}) \to \ell^\infty(M)$.

Remark(3.6.16):

Take the Leibniz rules

$$grad\;(f\,g) = g\;grad\;(f) + f\;grad\;(g) \tag{3.69}$$

$$div(f\,X) = f\,div(x) + X(f), \tag{3.70}$$

$$\Box(f\,g) = g\Box f + grad\,(g)\,f + grad\,(f)\,g + f\Box g$$

$$= g\Box f + 2\langle grad\,(f), grad\,(g)\rangle + f\Box g, \tag{3.71}$$

for $f, g \in \ell^{\infty}(\mathrm{M})$ and $X \in \Gamma^{\infty}(\mathrm{TM})$, they can easily be obtained from the definitions.

Example(3.6.17) (Minkowski Spacetime) : [92, 93]

We consider the n-dimensional Minkowski spacetime. As a manifold we have $M = R^n$ with canonical coordinates $x^0, x^1, \ldots, x^{n-1}$. Then the Minkowski

metric η *on* M is the constant metric

$$\eta = \frac{1}{2}\eta_{ij}dx^i \vee d\,x^j \tag{3.72}$$

with $\eta_{ij} = diag(+1, -1, \ldots, -1)$. One easily computes that in this global chart all Christoffel symbols vanish: (M, η) is flat. Moreover, we have for the above differential operators

$$grad\ \ f = \frac{\partial f}{\partial x^0}\frac{\partial}{\partial x^0} - \sum_{i=1}^{n-1}\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^i}, \tag{3.73}$$

$$div\ X = \frac{\partial X^0}{\partial x^0} + \sum_{i=1}^{n-1}\frac{\partial X^i}{\partial x^i}, \tag{3.74}$$

$$\Box f = \frac{\partial^2 f}{\partial\left(x^0\right)^2} - \sum_{i=1}^{n-1}\frac{\partial^2 f}{\partial\left(x^i\right)^2}. \tag{3.75}$$

This shows that $\Box$is indeed the usual wave operator or d'Alembertian as known from the theory of special relativity, . Finally, the Lorentz density with respect to η is just the usual Lebesgue measure

$$\mu_g = |\,d\,x^0 \wedge \ldots \wedge d\,x^{n-1}\,|. \tag{3.76}$$

(3.6.4) Normally Hyperbolic Differential Operators:

Definition (3.6.18) (Normally Hyperbolic Operator):[94, 95]

Let $E \to M$ be a vector bundle over a Lorentz manifold (M, g). A differential operator $D : \Gamma^{\infty}(E) \to \Gamma^{\infty}(E)$ is called normally hyperbolic if it is of second order and

$$\sigma_2(D) = 2g^{-1} \otimes id_E. \tag{3.77}$$

Recall that $\sigma_2(D) \in \Gamma^{\infty}\left(S^2TM \otimes End(E)\right)$. which explains the second tensor factor in (3.77) .

Example(3.6.19)(Connection D'Alembertian):

Let E be a covariant derivative for $E \to M$ and let ∇ be the Levi-Civita connection. This yields a global symbol calculus whence by

$$\Box^{\nabla} = \left(\frac{i}{\hbar}\right)^2 \ell std(2g^{-1} \otimes id_E) = \frac{1}{2}\langle 2g^{-1} \otimes id_E, \frac{1}{2}(D^E)^2 \rangle \tag{3.78}$$

a second order differential operator is given with leading symbol

$$\sigma\left(\Box^{\nabla}\right) = \left(\frac{i}{\hbar}\right)^2 \sigma_2(\ell std(2g^{-1} \otimes id_E)) = 2g^{-1} \otimes id_E \tag{3.79}$$

by Theorem 2.2.6. Thus $\Box^{\nabla}$ is normally hyperbolic for any choice of ∇^E called the connection d'Alembertian with respect to E .

(3.7) Causal Structure on Lorentz Manifolds:

(3.7.1) Some Motivation from General Relativity:

In general relativity the spacetime is described by a four-dimensional manifold M equipped with a Lorentz metric g subject to Einstein's equation [96, 97]. One defined the Einstein tensor

$$G = Ric - \frac{1}{2} scal.g, \tag{3.80}$$

which is a symmetric covariant tensor field

$$G \in \Gamma^{\infty}(S^2 T^* M). \tag{3.81}$$

It can be shown that the covariant divergence of G vanishes,

$$div\, G = 0, \tag{3.82}$$

while G itself needs not to be covariant constant at all. Physically, (3.82) is interpreted as a conservation law. Einstein's equation is then given by

$$G = k\mathrm{T}, \tag{3.83}$$

where $T \in \Gamma^{\infty}(S^2 T^* M)$. is the so-called energy-momentum tensor of all matter and interaction fields on M excluding gravity. The constant k is up to numerical constants Newton's constant of gravity. More generally, Einstein's equation with cosmological constant are

$$G + \lambda g = k\, T, \tag{3.84}$$

where $\lambda \in R$ is a constant,

(3.7.2) Future and Past on a Lorentz Manifold:

Having a fixed Lorentz metric g on a spacetime manifold M we can now transfer the notions of special relativity, to (M,g)[98]. In fact, each tangent space (T_pM, g_p) is isometrically isomorphic to Minkowski spacetime (R^n, η) with $\eta = diag(+1,-1,\ldots,-1)$, by choosing a Lorentz frame: there exist tangent vectors $e_i \in T_pM$ with $i = 1,\ldots,n$ such that

$$g_p(e_i, e_j) = \eta_{ij} = \pm\delta_{ij} \tag{3.85}$$

Remark(3.7.1)(Local Lorentz frame):

The pointwise isometry from (T_pM, g_p) to (R^n, η) can be made to depend smoothly on p at least in a local neighborhood : For every $p \in M$ there exists a small open neighborhood U of p and local sections $e_1, \ldots, e_n \in \Gamma^\infty(E|_U)$. such that for all $q \in U$

$$g_p(e_i(q), e_j(q)) = \eta_{ij} \tag{3.86}$$

in general the frame $\{e_i\}_i = 1,\ldots,n$ can not be chosen to come from a chart $x \ on \ U$, then there exists a unique smooth function $\Lambda : U \to O(1, n-1)$ such that

$$e_i(p) = \Lambda_i^j(p)\tilde{\ell}_j(p), \tag{3.87}$$

since the Lorentz transformations $O(1, n-1)$ are precisely the linear isometries of (R^n, η) . As in special relativity one states the following definition:

Definition(3.7.2):

Let (M,g) be a Lorentz manifold and $v_p \in T_pM$ a non-zero vector. Then v_p called

i.) timelike if $g_p(v_p, v_p) > 0$,

ii.) lightlike or null if $g_p(v_p, v_p) = 0$,

iii.) spacelike if $g_p(v_p, v_p) < 0$.

Non-zero vectors with $g_p(v_p, v_p) \geq 0$ are sometimes also called causal. To the zero vector,.

Definition(3.7.3)(Time- orientability):

Let (M, g) be a Lorentz manifold.

i.) (M, g) is called a time- orientable if there exist a timelike vector field $X \in \Gamma^{\infty}(TM)$.

ii.) The choice of a timelike vector field $X \in \Gamma^{\infty}(TM)$ is called a time-orientation.

iii.) With respect to a time-orientation, a timelike vector $v_p \in T_p M$ is called future directed if v_p is in the same connected component as $X_{(p)}$. It is called past directed if $-v_p$ is future directed.

Definition(3.7.4):

Let (M, g) be a time-oriented Lorentz manifold and $p, q \in M$ [74]. Then we define

i.) $p \langle\langle q$ if there exists a future directed, timelike smooth curve from p *to* q .

ii.) $p \leq q$ if either $p = q$ or there exists a future directed, causal smooth curve from p *to* q .

iii.) $p < q$ if $p \leq q$ but $p = q$.

Definition(3.7.5)(Chronological and Causal Future and Past):[99]

Let (M, g) be a time-oriented Lorentz manifold and $p \in M$.

i.) The chronological future of p is

$$I^{+}(p) = \{q \in M \mid p \langle\langle q\}. \tag{3.88}$$

ii.) The chronological past of p is

$$I^{-}(p) = \{q \in M \mid q \langle\langle p\}. \tag{3.89}$$

iii.) The causal future of p is

$$J^{+}(p) = \{q \in M \mid p \leq q\}. \tag{3.90}$$

v.) The causal past of p is

$$J^{-}(p) = \{q \in M \mid q \leq p\}. \tag{3.91}$$

Definition(3.7.6)(Future and past compactness):

Let (M,g) be a time-oriented Lorentz manifold. Then a subset $A \subseteq M$ is called future compact if $J_M^+(p) \cap A$ is compact for all $p \in M$ and past compact if $J_M^-(p) \cap A$ is compact for all $p \in M$.

Definition(3.7.7)(Causal Compatibility):

Let (M,g) be a time-oriented Lorentz manifold and $U \subseteq M$ open. Then U is called causally compatible if for all $p \in M$ we have

$$J_U^{\pm}(p) = J_U^{\pm}(p) \cap U \qquad (3.92)$$

(3.7.3) Causality Conditions and Cauchy-Hypersurfaces:

Definition(3.7.8)(Causal Subsets):[29]

Let $U \subseteq M$ be an open subset. Then U is called causal if there is a geodesically convex open subset $U' \subseteq M$ such that $U^{c1} \subseteq U'$ and for any two points $p, q \in U^{c1}$,the diamond $J_{U'}(p,q)$ is compact and contained in U^{c1} .

Definition(3.7.9)(A Causal and Achronal Subsets):[100]

Let $A \subseteq M$ be a subset of a time-oriented Lorentz manifold. Then A is called

i.) a chronal if every timelike curve intersects A in at most one point.

ii.) a causal if every causal curve intersects A in at most one point

Theorem(3.7.10)(A Chronal Hypersurfaces):

Let (M,g) be a time-oriented Lorentz manifold and $A \subseteq M$ achronal. Then A is a topological hypersurface in M if and only if A does not contain any of its edge points.

Definition(3.7.11)(Cauchy hypersurface):[29, 101]

Let (M,g) be a time-oriented Lorentz manifold. A subset $\Sigma \subseteq M$ is called a Cauchy hypersurface if every inextensible timelike curve meets Σ in exactly one point.

Definition(3.7.12) (Cauchy Development):[102]

Let $A \subseteq M$ be a subset. The future Cauchy development $D_M^+(A) \subseteq M$ of A is the set of all those points $p \in M$ for which every past-inextensible causal curve through p also meets A, Anolgously, one defines the past Cauchy development $D_M^-(A)$ and we call

$$D_M(A) = D_M^+(A) \cap D_M^-(A) \tag{3.93}$$

the Cauchy development of A.

Remark(3.7.13)(Cauchy Development):

Let $A \subseteq M$ be a subset, the physical interpretation of $D_M^+(A)$ is that $D_M^+(A)$ is predictable from Analogously, $D_M^-(A)$ consists of those points which certainly influence A in their future. We have $A \subseteq D_M^{\pm}(A)$.

Remark(3.7.14):

For $A \subseteq M$ we clearly have

$$D_M^{\pm}(D_M^{\pm}(A)) = D_M^{\pm}(A) \tag{3.94}$$

and hence

$$D_M(D_M(A)) = D_M(A) \tag{3.95}$$

Moreover, for $A \subseteq B \subseteq M$ we have

$$D_M^{\pm}(A) \subseteq D_M^{\pm}(B) \tag{3.96}$$

and

$$D_M(A) \subseteq D_M(B) \tag{3.97}$$

Definition(3.7.15)(Causality Condition):

Let (M, g) be a time-oriented Lorentz manifold.

i.) M is called causal if there are no closed causal curves in M .

ii.) An open subset $U \subseteq M$ is called causally convex if no causal curve intersects with U in a disconnected subset of U .

iii.) M is called strongly causal at $p \in M$ if every open neighborhood of p contains an open causally convex neighborhood.

iv.) M is called strongly causal if M is strongly causal at every point $p \in M$.

(3.7.4) Globally Hyperbolic Spacetimes:

Definition(3.7.16)(Globally Hyperbolic Spacetime):[103]

A time-oriented Lorentz manifold (M,g) is called globally hyperbolic if

i.) (M,g) is causal,

ii.) all diamonds $J_M(p,q)$ are compact for $p,q \in M$.

Definition(3.7.17)(Time Function):[38]

Let (M,g) be a time-oriented Lorentz manifold and $t: M \to R$ a continuous function. Then t is called a

i.) time function if t is strictly increasing along all future directed causal curves.

ii.) temporal function if t is smooth and grad t is future directed and timelike.

i.) Cauchy time function if t is a time function whose level sets are Cauchy hypersurfaces.

v.) Cauchy temporal function if t is a temporal function such that all level sets are Cauchy hyper-surfaces.

Remark(3.7.18)(Time Functions):

i.) With the other sign convention for the metric a temporal function has past directed gradient.

ii.) If t is temporal, its level sets are (if nonempty) embedded smooth submanifolds since the gradient is non-zero everywhere and hence every value is a regular value. Note that they do not need to be Cauchy hypersurfaces at all , In fact, remove a single point from Minkowski spacetime then the usual time function is temporal but there is no Cauchy hypersurface at all

iii.) The gradient flow of t gives a diffeomorphism between the different level sets of t . Since every timelike curve intersects a Cauchy hypersurface precisely once we see that this gives a diffeomorphism

$$M \approx t(M) \times \Sigma_{t_0}, \tag{3.98}$$

and all Cauchy hypersurfaces are diffeomorphic to a given reference Cauchy hypersurface Σ_{t_0} , this gives a very strong implication on the structure of M .

iv.) By rescaling t we can always assume that the image of t is the whole real line as the image of t is necessarily open and connected (for connected M).

Theorem(3.7.19):

Let (M,g) be a connected time-oriented Lorentz manifold. Then the following statements are equivalent:

i.) (M,g) is globally hyperbolic.

ii.) There exists a topological Cauchy hypersurface.

iii.) There exists a smooth spacelike Cauchy hypersurface.

In this case there even exists a Cauchy temporal function t and (M,g) is isometrically diffeomorphic to the product manifold

$$R\times\Sigma \text{ with metric } g=\beta\, dt^2 - g_t \qquad (3.99)$$

where $\beta\in\ell^{\infty}(R\times\Sigma)$ is positive and $g_t \in\Gamma^{\infty}(S^2\, T^*\Sigma)$ is a Riemannian metric on Σ depending smoothly on t. Moreover, each level set

$$\Sigma_t = \{(t,\sigma)\in R\times\Sigma\}\, g \subseteq M \qquad (3.100)$$

of the temporal function t is a smooth spacelike Cauchy hypersurface.

Example(3.7.20)(Minkowski strip):

We consider $\Sigma=(a,b)$ an open interval with $-\infty<a<b<+\infty$ and $M=R\times\Sigma\subseteq R^2$ as open subset of Minkowski space. Then Σ_t is not a Cauchy hypersurface for any t. This is clear from the observation that there are inextensible timelike geodesics not passing

through Σ_t . In fact, M is not globally hyperbolic at all: while M is causal (and even strongly causal) , the metric is of the very simple form

$$g = dt^2 - dx^2 \qquad (3.101)$$

Proposition(3.7.21):

Let $M = R\times\Sigma$ with Lorentz metric

$$g = \frac{1}{2} dt \vee - dt - f(t) g\Sigma \qquad (3.102)$$

where $g\Sigma$ is a Riemannian metric on Σ and $f\in\ell^{\infty}(R)$ is positive. The time-orientation is such that $\frac{\partial}{\partial t}$ is future directed. Then (M,g) is globally hyperbolic if and only if $g\Sigma$ is geodesically complete.

Theorem(3.7.22):

Let (M, g) be globally hyperbolic and let $\Sigma \subseteq M$ be a smooth spacelike Cauchy hypersurface. Then there exists a Cauchy temporal function t such that the $t=0$ Cauchy hypersurface coincides with Σ.

(3.8)The Cauchy Problem and Green's Functions .

Having the notion of a Cauchy hypersurface we are now in the position to formulate the Cauchy problem for a normally hyperbolic differential operator [104]. Here we still be rather informal only fixing the principal ideas

Thus let (M, g) be globally hyperbolic and $\Sigma \subseteq M$ a smooth Cauchy hypersurface which we assume to be spacelike throughout the following. At a given point $p \in \Sigma \subseteq M$ the tangent plane $T_p\Sigma \subseteq T_p M$ is spacelike whence there exists a unique vector $n_p \in T_p M$ which satisfies

$$g_p(n_p, T_p\Sigma) = 0 \tag{3.103}$$

$$g_p(n_p, n_p) = 1 \tag{3.104}$$

$$n_p \text{ is future directed} \tag{3.105}$$

This vector is called the future directed normal vector of Σ *at* p. Taking all points $p \in \Sigma$ we obtain the future directed normal vector field of Σ, i.e. the vector field

$$n \in \Gamma^\infty\left(TM\big|_\Sigma\right) \tag{3.106}$$

such that (3.103), (3.104), and (3.105) hold for every $p \in \Sigma$. Since Σ is a smooth submanifold, n is smooth itself. We consider now a normally hyperbolic differential operator $D \in Diffop(E)$ on some vector bundle $E \to M$. Then this operator gives the homogeneous wave equation

$$D_u = 0 \tag{3.107}$$

or more generally

$$D_u = v \tag{3.108}$$

where $v \in \Gamma^\infty(E)$ is a given inhomogeneity and $u \in \Gamma^\infty(E)$ is the field we are looking for. Having specified the inhomogeneity which physically corresponds to a source term, we can try to find a solution u which has specified initial values and initial velocities on Σ. More precisely, we want

$$u\big|_\Sigma = u_0 \in \Gamma^\infty\left(E\big|_\Sigma\right) \tag{3.109}$$

and $$\nabla_n^E u\big|_\Sigma = \dot{u}_0 \in \Gamma^\infty\left(E\big|_\Sigma\right) \tag{3.110}$$

for $u_0, \dot{u}_0$ are given the fundamental solutions $F_p \in \Gamma^\infty(E) \otimes E_p^* \otimes \left|\Lambda^{top}\right| T_p^* M = u_0 \in \Gamma^\infty\left(E\big|_\Sigma\right)$ such that

$$DF_p = \delta_p \tag{3.111}$$

where δ_p is the δ-distribution at $p \in M$ viewed as $E_p^* \otimes \left|\Lambda^{top}\right| T_p^* M$-valued generalized section of E, i.e. for a test section $\mu \in \Gamma_0^\infty(E^* \otimes \left|\Lambda^{top}\right| T^* M) = u_0 \in \Gamma^\infty\left(E\big|_\Sigma\right)$ we have

$$\delta_p(\mu) = \mu\,(p) \in E_p^* \otimes \left|\Lambda^{top}\right| T_p^* M \tag{3.112}$$

Definition(3.8.1)(Green's functions):

Let $p \in M$. A generalized section $F_p\ of\ E$ which satisfies (3.111) is called fundamental solution of $D\, at\, p$. If a fundamental solution $F_p^\pm$ in addition satisfies $$Supp F_p^\pm \subseteq J_M^\pm(p), \tag{3.113}$$

then $F_p^\pm$ is called advanced or retarded Green function of $D\, at\, p$, respectively.

Chapter(4)

The Local Theory of Wave Equations

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The Local Theory of Wave Equations

(4.1) The D'Alembert Operator on Minkowski Spacetime

We consider the d'Alembert operator on flat Minkowski spacetime.

(4.1.1) The Riesz Distributions:[105]

Let
$$\Box = \frac{\partial^2}{\partial t^2} - \Delta \tag{4.1}$$

with $t = x^0$ and $\vec{x} = \left(x^1, \ldots, x^n\right)$ by using Minkowsky metric η we have

$$\eta(x) = \eta(x, x) \tag{4.2}$$

on R^n clearly $\eta \in Pol^2\left(R^2\right)$ is a homogeneous quadratic polynomial then

$$\eta\left(x^0, \ldots, x^{n-1}\right) = \left(x^0\right)^2 - \sum_{i=1}^{n-1}\left(x^i\right)^2 = t^2 - \left(\vec{x}\right)^2 \tag{4.3}$$

Definition(4.1.1):

Let $\alpha \in C \; have \operatorname{Re}(\alpha) > n$ then

$$R^{\pm}(\alpha)(x) = \begin{cases} c(\alpha, n)\eta(x)^{\frac{\alpha-n}{2}} & for \, \alpha \in I^{\pm}(0) \\ 0 & else \end{cases} \tag{4.4}$$

where the coefficient is

$$c(\alpha, n) = \frac{2^{1-\alpha}\pi^{\frac{2-n}{2}}}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha-n}{2}+1\right)} \tag{4.5}$$

Remark(4.1.2)(Gamma Function)[35, 106]:

The Gamma function

$$\Gamma : C \setminus \{0, -1, -2, \ldots.\} \to C \tag{4.6}$$

is known to be a holomorphic function with simple poles at $-n \; for \; n \in N_0$

we have the following properties:

i.) The residue at $n \in N_0$ is given by

$$res_{-n}\Gamma = \frac{(-1)}{n!} \tag{4.7}$$

ii.) For $z \in C \setminus \{0,-1,-2,...\}$ one has the functional equation

$$\Gamma(z+1) = z\Gamma(z) \quad \text{with} \quad \Gamma(1) = 1 \tag{4.8}$$

iii.) For $n \in N_0$ we obtains from (4.8) immediately

$$\Gamma(n+1) = n! \tag{4.9}$$

iv.) For $\operatorname{Re}(z) > 0$ we have Euler's integral formula

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \tag{4.10}$$

in the sense of an improper Riemann integral.

v.) For all $z \in C \setminus \{0,-1,-2,...\}$ we have Legendre's duplication formula

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z) \tag{4.11}$$

Thus

$$c(\alpha,n) = 0 \;\; iff \;\; \alpha \in \{-2k \mid k \in N_0\} \cup \{n-2k \mid k \in N_0\} \tag{4.12}$$

since the nominator has clearly no zeros.

Lemma(4.1.3):

Let $\Lambda \in L^{\uparrow}(1,1-n)$ be an orthochronous Lorentz transformation and $\operatorname{Re}(\alpha) > n$ Then

$$\Lambda^* R^{\pm}(\alpha) = R^{\pm}(\alpha) \tag{4.13}$$

If $T \in L(1,1-n)$is the time-reversal $x^0 \mapsto -x^0$ then

$$T^* R^{\pm}(\alpha) = R^{\pm}(\alpha) \tag{4.14}$$

Lemma(4.1.4):

Let $\operatorname{Re}(\alpha) > n$.

i.) For every $x \in R^n$ the function

$$\alpha \mapsto R^{\pm}(\alpha)(x) \tag{4.15}$$

is holomorphic.

ii.) For every test function $\varphi \in \ell_0^\infty(R)$ the function

$$\alpha \mapsto R^{\pm}(\alpha)(\varphi) \tag{4.16}$$

is holomorphic.

Lemma(4.1.5):

In the sense of continuous functions we have:

i.) For $\operatorname{Re}(\alpha) > n$ we have

$$\eta R^{\pm}(\alpha) = \alpha(\alpha - +2)R^{\pm}(\alpha + 2)$$

ii.) For $\operatorname{Re}(\alpha) > n + 2k$ the function $R^{\pm}(\alpha)\, is\, \ell^k$ and we have

$$\frac{\partial}{\partial x^i} R^{\pm}(\alpha) = \frac{1}{\alpha - 2} R^{\pm}(\alpha - 2)\eta_{ij}x^j \tag{4.17}$$

iii.) For $\operatorname{Re}(\alpha) > n$ we have

$$grad\,.\eta.R^{\pm}(\alpha) = 2\alpha gradR^{\pm}(\alpha + 2) \tag{4.18}$$

iv.) For $\operatorname{Re}(\alpha) > n + 2$ we have

$$\Box R^{\pm}(\alpha + 2) = R^{\pm}(\alpha) \tag{4.19}$$

Proof. At the first part We have

$$\alpha(\alpha + 2 - n)R^{\pm}(\alpha + 2) = \alpha(\alpha + 2 - n)c(\alpha + 2, n)\eta^{\frac{\alpha + 2 - n}{2}}$$

$$= \alpha(\alpha + 2 - n)c(\alpha + 2, n)\eta^{\frac{\alpha - n}{2}}\eta$$

$$= \frac{\alpha(\alpha + 2 - n)c(\alpha + 2, n)\eta}{c(\alpha, n)}\eta R^{\pm}(\alpha)$$

And

$$\frac{c(\alpha + 2, n)}{c(\alpha, n)} = \frac{2^{1-2-\alpha}\pi^{\frac{2-n}{2}}\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha - n}{2} + 1\right)}{\Gamma\left(\frac{\alpha+2}{2}\right)\Gamma\left(\frac{\alpha+2-n}{2} + 1\right)2^{1-\alpha}\pi^{\frac{2-n}{2}}} = \frac{2^{-2}\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha - n}{2} + 1\right)}{\frac{\alpha}{2}\Gamma\left(\frac{\alpha}{2}\right)\frac{\alpha + 2 - n}{2}\Gamma\left(\frac{\alpha - n}{2} + 1\right)} = \frac{1}{\alpha(\alpha + 2 - n)} \tag{4.20}$$

For the second part we recall that in $I^{\pm}(0)$ the function $R^{\pm}(\alpha)$ is smooth as well as in $R^{\pm} \setminus J^{\pm}(0)$.In $I^{\pm}(0)$ we compute

$$\frac{\partial}{\partial x^i} R^{\pm}\alpha\bigg|_{I^{\pm}(0)} c(\alpha, n)\frac{\partial}{\partial x^i}\eta(x)\frac{\alpha - n}{2} = c(\alpha, n)\frac{\alpha - n}{2}\eta(x)^{\frac{\alpha - n}{2} - 1}\frac{\partial}{\partial x^i}\eta(x)$$

$$=\frac{c(\alpha,n)}{c(\alpha-2,n)}(\alpha-n)R^{\pm}(\alpha-2)\eta_{ij}x^{j}\overset{(4.20)}{=}\frac{1}{(\alpha-2)(\alpha-n)}(\alpha-n)R^{\pm}(\alpha-2)\eta_{ij}x^{j}$$

$$=\frac{1}{(\alpha-2)}R^{\pm}(\alpha-2)\eta_{ij}x^{j}$$

Now if $\operatorname{Re}(\alpha)>n+2k$ then $\operatorname{Re}(\alpha-2)>n+2k-2$ is still larger than n for positive $k\in N$. Thus the partial derivative $\frac{\partial}{\partial x^i}R^{\pm}\alpha\Big|_{I^{\pm}(0)}$ is the continuous function $\frac{1}{(\alpha-2)}R^{\pm}(\alpha-2)\eta_{ij}x^{j}\, in\, I^{\pm}(0)$ (α−2) which continuously extends to R^n by setting it zero outside of $I^{\pm}(0)$. We have

$$grad\,\eta=\left(\frac{\partial\eta}{\partial x^i}dx^i\right)^{\#}=\frac{\partial\eta}{\partial x^i}\eta^{ij}\frac{\partial\eta}{\partial x^j}=2\eta_{ik}x^k\eta^{ij}=2x^i\frac{\partial}{\partial x^j}=2\xi$$

Thus grad η is twice the Euler vector field on R^n , Using (4.17) we compute for $\operatorname{Re}(\alpha)>n$

$$2\alpha gradR^{\pm}(\alpha+2)=2\alpha\eta^{ij}\frac{\partial R^{\pm}(\alpha+2)}{\partial x^i}\frac{\partial}{\partial x^j}=2\alpha\eta^{ij}\frac{1}{\alpha+2-2}R^{\pm}(\alpha)\eta_{ik}x^k\frac{\partial}{\partial x^j}$$

$$=R^{\pm}(\alpha)2x^k\frac{\partial}{\partial x^k}=R^{\pm}(\alpha)grad\eta$$

For the last part we use (4.17) twice and obtain

$$\Box R^{\pm}(\alpha+2)=\eta^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}R^{\pm}(\alpha+2)$$

$$=\eta^{ij}\frac{\partial}{\partial x^i}\left(\frac{1}{\alpha+2-2}R^{\pm}(\alpha+2-2)\right)\eta_{jk}x^k$$

$$=\eta^{ij}\frac{1}{\alpha x^i}R^{\pm}(\alpha)\eta_{jk}x^k+\eta^{ij}R^{\pm}(\alpha)\eta_{jk}\frac{\partial}{\partial x^i}x^k$$

$$=\frac{1}{\alpha}\eta^{ij}\frac{1}{\alpha-1}\eta_{il}x^lR^{\pm}(\alpha-2)\eta_{jk}x^k+\frac{1}{\alpha}R^{\pm}(\alpha)\eta^{ij}\delta_i^k$$

$$=\frac{1}{\alpha}\frac{1}{\alpha-1}\eta_{il}x^lx^iR^{\pm}(\alpha-2)+\frac{n}{\alpha}R^{\pm}(\alpha)$$

$$=\frac{1}{\alpha(\alpha-2)}\eta.R^{\pm}(\alpha-2)+\frac{n}{\alpha}R^{\pm}(\alpha)$$

$$\overset{i.)}{=}\frac{(\alpha-2)(\alpha-2-n+2)}{\alpha(\alpha-2)}R^{\pm}(\alpha-2)+\frac{n}{\alpha}R^{\pm}(\alpha)$$

$$=\frac{\alpha-n+n}{\alpha(\alpha-2)}R^{\pm}(\alpha)=R^{\pm}(\alpha)\tag{4.21}$$

Since $\alpha \mapsto R^{\pm}(\alpha)$ is a holomorphic family of distributions for $\mathrm{Re}(\alpha) > n$ by Lemma 4.1.4 ii.) the equation (4.21) and the previous Definition 4.1.1 coincide as they coincide for $\mathrm{Re}(\alpha) > n+2$ by Lemma 4.1.5, iv.). Thus we can define inductively for $\mathrm{Re}(\alpha + 2k) > n$

$$R^{\pm}(\alpha) = \Box R^{\pm}(\alpha + 2k) \tag{4.22}$$

Lemma(4.1.6):

Let $\alpha \in C$ and define $R^{\pm}(\alpha)$ by

$$R^{\pm}(\alpha) = \Box^{k} R^{\pm}(\alpha + 2k) \tag{4.23}$$

where $k \in N_0$ is such that $\mathrm{Re}(\alpha + 2k) > n$. Then (4.23) does not depend on the choice of k and yields an entirely holomorphic family of distributions which extends the family

$\{R^{\pm}(\alpha)\}_{\mathrm{Re}(\alpha) > n}$.

Definition(4.1.7)(Riesz Distributions):

For $\alpha \in C$ the distributions $R^{+}(\alpha)$ are called the advanced Riesz distributions and the $R^{-}(\alpha)$ are called the retarded Riesz distributions.

Theorem(4.1.8)(Green's Function of $\Box$):

The Riesz distributions $R^{\pm}(2)$ are advanced and retarded Green functions for the scalar d'Alembert operator $\Box$ on Minkowski spacetime [5, 107]

(4.1.2)The Riesz Distributions in Dimension n = 1, 2:

Propostion(4.1.9):

Let $n = 1$ then the advanced and retarded Green functions of

$\Box = \frac{\partial^2}{\partial t^2}$ are explicitly given as the continuous functions

$$R^{+}(2)(t) = \begin{cases} t & \mathit{for}\ t > 0 \\ 0 & \mathit{else} \end{cases} \tag{4.24}$$

and

$$R^{-}(2)(t) = \begin{cases} |t| & \mathit{for}\ t < 0 \\ 0 & \mathit{else} \end{cases} \tag{4.25}$$

moreover , for $\mathrm{Re}(\alpha) > 1$, we have

$$R^{\pm}(\alpha)(t)= \begin{cases} \frac{1}{\Gamma(\alpha)}|t|^{\alpha-1} & for\, t \in R^{\pm} \\ 0 & else \end{cases} \tag{4.26}$$

Remark(4.1.10)(Riesz Distribution in One Dimension):

i.) take $\frac{\partial^2}{\partial t^2} R^{\pm}(2)$ in the sense of distributions directly to show that

$$\frac{\partial^2}{\partial t^2} R^{\pm}(2) = \delta_0 \tag{4.27}$$

ii.) The functions $R^{\pm}(\alpha)\, for\, \mathrm{Re}(\alpha)>1$ then

$$R^{\pm}(\alpha)(t)= \begin{cases} \frac{1}{\Gamma(\alpha)}|t|^{\alpha-1} & for\, t \in R^{\pm} \\ 0 & else \end{cases} \tag{4.28}$$

in case $n=1+1$. by using the coordinates $(t,x)\in R^2$ with

$$\eta(t,x) = t^2 - x^2 \tag{4.29}$$

then $c(\alpha, n)\, for\, n=2$. We have

$$c(\alpha,2) = \frac{2^{1-\alpha}}{\Gamma\left(\frac{\alpha}{2}\right)^2} \tag{4.30}$$

In order to evaluate $\eta^{\frac{\alpha-2}{2}}$ we introduce new coordinates on R^2 . We pass to the light cone coordinates

$$u = \frac{1}{\sqrt{2}}(t-x) \quad \text{and} \quad v = \frac{1}{\sqrt{2}}(t+x) \tag{4.31}$$

i.e.

$$t = \frac{1}{\sqrt{2}}(u+v) \quad \text{and} \quad x = \frac{1}{\sqrt{2}}(v-u) \tag{4.32}$$

Since this is clearly a global diffeomorphism we can evaluate $R^{\pm}(\alpha)$in these new coordinates. the function η in these coordinates is

$$\eta(u,v) = \frac{1}{2}(u+v)^2 - \frac{1}{2}(v-u)^2 = \frac{1}{2}\left(u^2 + 2uv + v^2 - u^2 + 2uv - v^2\right) = 2uv \tag{4.33}$$

Moreover, the future and past $I^{\pm}(0) of\ 0$ can be described by

$$I^{+}(0)=\{(u,v)\in R^{2}|u,v>0\} \tag{4.34}$$

And

$$I^{+}(0)=\{(u,v)\in R^{2}|u,v<0\} \tag{4.35}$$

then we have for $\operatorname{Re}(\alpha)>2$

$$R^{\pm}(\alpha)(u,v)=\begin{cases}\dfrac{2^{1-\alpha}}{\Gamma\left(\dfrac{\alpha}{2}\right)^{2}}(2uv)^{\frac{\alpha-2}{2}} & for\ u,v\in R^{\pm}(0)\\ 0 & else,\end{cases} \tag{4.36}$$

$$=\begin{cases}\dfrac{2^{1-\alpha}}{\Gamma\left(\dfrac{\alpha}{2}\right)^{2}}\left|\sqrt{2u}\right|^{\frac{\alpha-2}{2}}\left|\sqrt{2v}\right|^{\frac{\alpha-2}{2}} & for\, u,v\in R^{\pm}\\ 0 & else,\end{cases} \tag{4.37}$$

whence $R^{\pm}(\alpha)$ is factorizing in these coordinates. This suggests to consider the following functions

$$r^{\pm}(\alpha)(u)=\begin{cases}\dfrac{2^{1-\alpha}}{\Gamma\left(\dfrac{\alpha}{2}\right)}|u|^{\frac{\alpha-2}{2}} & for\, u\in R^{\pm}\\ 0 & else,\end{cases} \tag{4.38}$$

for $\operatorname{Re}(\alpha)>2$. Since the prefactor is still holomorphic for all $\alpha\in C$.

lemma(4.1.11):

In the light cone coordinates the d'Alembert operator is

$$\Box=\frac{\partial^{2}}{\partial u\partial v} \tag{4.39}$$

Proposition (4.1.12):[108]

Let u,v be the light cone coordinate on R^{2} then the distributions

$$R^{\pm}(2)(u,v)=r^{\pm}(2)(u)\, r^{\pm}(2)(v)\ . \tag{4.40}$$

are advanced and retarded Green functions of $\Box$ of order zero .

(4.2)The Riesz Distributions on a Convex Domain:

We pass now from Minkowski spacetime to a general Lorentz manifold (M,g) and try to find analogs of the Riesz distributions at least locally around a point $p\in M$. The main idea is to use the Riesz distributions on the tangent space $T_p M$, which is isometric to Minkowski space , and push forward the Riesz distributions via the exponential map[108].

(4.2.1) The Functions ℓ_p and η_p:

From Proposition 3.6.11, ii.), we can use this density to identify functions and densities once and for all. In particular, this results in an identification of the generalized sections $\Gamma^{\infty}(E)$ of a vector bundle $E\to M$ with the topological dual of $\Gamma_0^{\infty}(E^*)$ and not of $\Gamma_0^{\infty}\left(E\otimes\left|\Lambda^{top}\right|T^*M\right)$ as we did before. for $s\in\Gamma^{-\infty}(E)$, and a test section $\varphi\in\Gamma_0^{\infty}(E^*)$ we first map $\varphi\, to\, \varphi\otimes\mu_g\in\Gamma_0^{\infty}\left(E\otimes\left|\Lambda^{top}\right|T^*M\right)$ and then apply s we set

$$s(\varphi)=s(\varphi\otimes\mu_g), \tag{4.41}$$

and drop the explicit reference to μ_g to simplify our notation. Since

$$\Gamma_0^{\infty}(E^*)\ni\varphi\mapsto\varphi\otimes\mu_g\in\Gamma_0^{\infty}\left(E^*\otimes\left|\Lambda^{top}\mathrm{T}^*M\right|\right) \tag{4.42}$$

is indeed an isomorphism of LF spaces, we have an induced isomorphism of the topological duals which is (4.48). Let $V_p\subseteq T_pM$ be a suitable open star-shaped neighborhood of 0_p and, let $U_p=e_p(V_p)\subseteq M$ be the corresponding open neighborhood of P such that

$$\exp_p: V_p\to U_p \tag{4.43}$$

is a diffeomorphism. Then we define the function

$$\ell_p=\frac{\mu_g\left|U_p\right.}{\exp_{p*}(\mu_g(p))\left|U_p\right.} \tag{4.44}$$

Proposition 4.2.1:

Let (M,g) be a time-oriented Lorentz manifold and $p\in M$. Moreover, let $U\subseteq M$ be geodesically star-shaped with respect to p .

i.) The gradient of $\eta_p\in\ell^{\infty}(N)$ is given by

$$grad\eta_p\big|_q = 2\mathrm{T}_{\exp_p^{-1}(q)} \exp_p\left(\exp_p^{-1}(q)\right) \tag{4.45}$$

for $q \in U$.

ii.) and we have

$$g\left(grad\,\eta_p, grad\,\eta_p\right) = 4\eta_p \tag{4.46}$$

iii.) On $I_U^{\pm}(p)$ the gradient of η_p is a future resp. past directed timelike vector field.

iv.) and we have

$$\eta_p = 2n + g\left(grad\ \log \ell_p, grad\,\eta_p\right) \tag{4.47}$$

(4.2.2) Construction of Riesz Distributions $R_U^{\pm}(\alpha, p)$:

For $\mathrm{Re}(\alpha) > n$ the Riesz distributions $R_U^{\pm}$ are even continuous functions on Minkowski space , as such we can simply push-forward via $\exp_p$ at least on the star-shaped $V \subseteq T_p M$ continuous functions on $U \subseteq M$ continuous functions defines a distribution after multiplying with the density μ_g .

Remark(4.2.2):

Let $f \in \ell^0(T_P M)$ be continuous functions on the tangent space of p, we view f as a distribution as usual via

$$f(\varphi) = \int_{T_p M} f(v)\varphi(v)\mu_g(p) \tag{4.48}$$

for $\varphi \in \ell_0^{\infty}(T_P M)$,using $\exp_p$ we can write this as follows ,let $\varphi \in \ell_0^{\infty}(M)$with $Supp\varphi \subseteq U$ then the continuous function $\exp_{p^*}\left(f|_V\right) \in \ell^0(U)$can be viewed as a distribution on U

$$\exp_{p^*}\left(f|_V\right)(\varphi) = \int_M \exp_{p^*}\left(f|_V\right)(q)\varphi(q)\mu_g(q) \tag{4.49}$$

$$\exp_{p^*}\left(f|_V\right)(\varphi) = \int_M \exp_{p^*}\left(f|_V\right)\exp_{p^*}(\exp_{p^*}\varphi)(q)\,\ell_p(q)(\exp_{p^*}\mu_g(p))(q) \tag{4.50}$$

$$= \int_M \exp_{p^*}\left(f|_V \exp_p^* \varphi \exp_p^* \ \ell_p \mu_g(p)\right)(q) \tag{4.51}$$

$$= \int_{T_p M} f \exp_p^* \varphi\, \widetilde{\ell}_p\, \mu_g(p) \tag{4.52}$$

$$= (\widetilde{\ell}_p f)\ (\exp_p^* \varphi) \tag{4.53}$$

Definition(4.2.3)(Riesz Distributions on U):

Let $p \in M$ and let $U \subseteq M$ be a geodesically star-shaped open neighborhood of p. Moreover, let $V = e_p^{-1}(U) \subseteq T_p M$ be the corresponding star-shaped open neighborhood of $0 \in T_p M$. Then the advanced and retarded Riesz distributions $R_U^{\pm}(\alpha, p) \in \ell_0^{\infty}(U)'$ are defined by0

$$R_U^{\pm}(\alpha, p)(\varphi) = e_{p^*}\left(R^{\pm}(\alpha)\big|_V\right)(\varphi) = R^{\pm}(\alpha)\big|_V\left(\tilde{\ell}_p\, e_p^*\, \varphi\right) \tag{4.54}$$

for $\alpha \in C$ and $\varphi \in \ell_0^{\infty}(U)$.

Proposition(4.2.4):

Let $U \subseteq M$ be a geodesically star-shaped around $p \in M$, then Riesz distributions $R_U^{\pm}(\alpha, p)$ have the following properties :

i.) If $\operatorname{Re}(\alpha) > n$ then $R_U^{\pm}(\alpha, p)$ is continuous on U and given by

$$R_U^{\pm}(\alpha, p)(q) = \begin{cases} c(\alpha, n)(\eta_n(q))^{\frac{\alpha - n}{2}} & \text{for } q \in I_U^{\pm}(p) \\ 0 & \text{else.} \end{cases}$$

ii.) for $\operatorname{Re}(\alpha) > n + 2k$ then the function $R_U^{\pm}(\alpha, p)$ is even ℓ^k on U .

iii.) for all $R_U^{\pm}(\alpha, p)\Big|_{I_U^{\pm}(p)} = c(\alpha, n)\eta_n^{\frac{\alpha - n}{2}} \in \ell^{\infty}\left(I_U^{\pm}(p)\right)$ and

$0 = R_U^{\pm}(\alpha, p)\Big|_{U \setminus J_U^{\pm}(p)} \in \ell^{\infty}\left(U \setminus J_U^{\pm}(p)\right)$

Proposition 4.2.5 (Symmetry of $R_U^{\pm}(\alpha, p)$*)* Let $U \subseteq M$ be geodesically convex and $\alpha \in C$.[90, 108]

i.) If $\operatorname{Re}(\alpha) > n$ then

$$R_U^{\pm}(\alpha, p)(q) = R_U^{\pm}(\alpha, p)(q) \tag{4.55}$$

for all $p, q \in U$.

ii.) For all $\Phi \in \ell_0^{\infty}(U \times U)$ we have

$$\int_U R^{\pm}(\alpha, p)(\Phi(p, .))\mu_g(p) = \int_U R^{\pm}(\alpha, q)(\Phi(., q))\mu_g(q) \tag{4.56}$$

(4.3) The Hadamard Coeffcients:

Differently from the flat situation, the Riesz distribution $R_U^{\pm}(2, p)$ does not yield a fundamental solution for □ . we had to exclude the value of α needed for $R_U^{\pm}(2,p)$ explicitly. Instead, from

$$\Box R_U^{\pm}(\alpha+2,p)=\left(\frac{\Box\eta_p-2n}{2\alpha}+1\right)R_U^{\pm}(\alpha,p),$$

(4.3.1) The Ansatz for The Hadamard Coeffcients:[109]

We consider a normally hyperbolic differential operator $D=\Box^{\nabla}+B$ on some vector bundle $E\to M$ over M with induced connection ∇^{E} and $B\in\Gamma^{\infty}(End(E))$ as in Section 3.6.4. Moreover, for $p\in M$ we choose a geodesically star-shaped open neighborhood $U\subseteq M$ on which $R_U^{\pm}(\alpha,p)$ is defined as before. According to our convention for distributions, the Green functions are now generalized sections

$$\Re^{\pm}(p)\in\Gamma^{-\infty}(E)\otimes E_p^{*}, \tag{4.57}$$

as we take care of the density part using μ_g . The pairing with a test section $\varphi\in\Gamma_0^{\infty}(E^{*})$ yields then an element in E_p^{*} The equation to solve is

$$D\Re^{\pm}(p)=\delta_p \tag{4.58}$$

where δ_p is viewed as E_p^{*} -valued distribution on $\Gamma_0^{\infty}(E^{*})$ and $D\Re^{\pm}(p)$ is defined as usual.The Ansatz for $\Re^{\pm}(p)$ is now the following. Since the $R_U^{\pm}(\alpha,p)$ have increasing regularity for increasing $\mathrm{Re}(\alpha)$ we try a series

$$\Re^{\pm}(p)=\sum_{k=0}^{\infty}V_p^{k}R_U^{\pm}(2+2k,p) \tag{4.59}$$

With smooth section

$$V_p^{k}\in\Gamma^{\infty}(E|_U)\otimes E_p^{*} \tag{4.60}$$

Then (4.59) should be thought of as an expansion with respect to regularity.

First we note that a scalar distribution like $R_U^{\pm}(\alpha,p)$ can be multiplied with a smooth section like V_p^{k} and yields a distributional section

$$V_p^{k}R_U^{\pm}(2+2k,p)\in\Gamma_0^{\infty}(E^{*})'\otimes E_p^{*}=\Gamma^{-\infty}(E)\otimes E_p^{*} \tag{4.61}$$

In Remark 2.3.5 it is only necessary that one factor of the product is actually smooth. We compute now (4.58). First we assume that the series (4.59) converges at least in the $Weak^{*}\ topo\log y$ so that we can apply D componentwise. This yields

$$D\Re^{\pm}(p) = D\sum_{k=0}^{\infty} V_p^k R_U^{\pm}(2+2k, p)$$

$$\begin{aligned} &= \sum_{k=0}^{\infty} D\left(V_p^k R_U^{\pm}(2+2k, p)\right) \\ &= \sum_{k=0}^{\infty}\left(D\left(V_p^k\right)R_U^{\pm}(2+2k, p) + 2\nabla^E_{gradR_U^{\pm}(2+2k,p)} V_p^k + V_p^k R_U^{\pm}(2+2k, p)\right) \end{aligned} \tag{4.62}$$

by the Leibniz rule of a normally hyperbolic differential operator , Inserting the properties of $R_U^{\pm}(\alpha, p)$ from Proposition 4.2.4 yields then

$$D\Re^{\pm}(p) = D\left(V_p^0\right)R_U^{\pm}(2,2p) + 2\nabla^E_{gradR_U^{\pm}(2,2p)} V_p^0 + V_p^0 R_U^{\pm}(2,2p)$$

$$+\sum_{k=1}^{\infty}\left(D\left(V_p^k\right)R_U^{\pm}(2+2k, p) + 2\nabla^E_{\frac{1}{4}R_U^{\pm}(2k,p)grad\eta_p} V_p^k + V_p^k\left(\frac{\eta_p - 2n}{4k} + 1\right)R_U^{\pm}(2k, p)\right)$$

$$= 2\nabla^E_{gradR_U^{\pm}(2,2p)} V_p^0 + V_p^0 R_U^{\pm}(2,p) + \sum_{k=0}^{\infty} D\left(V_p^k\right)R_U^{\pm}(2+2k, p) + \sum_{k=1}^{\infty}\left(2\nabla^E_{\frac{1}{4k}R_U^{\pm}(2k,p)grad\eta_p} V_p^k + V_p^k\left(\frac{\eta_p - 2n}{4k} + 1\right)\right)R_U^{\pm}(2k, p)$$

$$= 2\nabla^E_{gradR_U^{\pm}(2,p)} V_p^0 + V_p^0 R_U^{\pm}(2,p) +$$

$$\sum_{k=1}^{\infty}\left(D\left(V_p^{k-1}\right) + 2\nabla^E_{\frac{1}{4k}grad\eta_p} V_p^k + \left(\frac{\eta_p - 2n}{4k} + 1\right)V_p^k\right)R_U^{\pm}(2k, p) \tag{4.63}$$

We view (4.63) as an expansion with respect to regularity. Thus, we ask for (4.62) in each “order” i.e. (4.62) should be fulfilled for each component in front of the $R_U^{\pm}(\alpha, p)$. This yields the following equations. In lowest order we have for V_p^0 the equation

$$2\nabla^E_{gradR_U^{\pm}(2,p)} V_p^0 + V_p^0 R_U^{\pm}(2,p) = \delta_{p,} \tag{4,64}$$

while for $K \geq 1$ we have the recursive equations

$$\frac{1}{2k}\nabla^E_{grad\eta_p} V_p^k + \left(\frac{\square\eta_p - 2n}{4k} + 1\right)V_p^k = -D\left(V_p^{k-1}\right) \tag{4.65}$$

for V_p^K . Equivalently, we can write this for $K \geq 1$ as

$$\nabla^E_{grad\eta_p} V_p^k + \left(\frac{1}{2}\square\eta_p - n + 2k\right)V_p^k = -2kD\left(V_p^{k-1}\right) \tag{4.66}$$

Since (4.66) also makes sense for $K = 0$ it seems tempting to unify (4.64) and (4.66). To this end, we take (4.66) for $K = 0$ and multiply this by $R_U^{\pm}(\alpha, p)$ yielding

$$\nabla^{E}_{grad\eta_p R_U^{\pm}(\alpha=2,p)} V_p^0 + \left(\frac{1}{2}\Box\eta_p - n\right) V_p^0\, R_U^{\pm}(\alpha, p) = 0 \tag{4.67}$$

which is equivalent to

$$\nabla^{E}_{2\,gradR_U^{\pm}(\alpha+,2\,p)} V_p^0 + \alpha\left(\Box R_U^{\pm}(\alpha+2, p) - R_U^{\pm}(\alpha, p)\right) V_p^0 = 0 \tag{4.68}$$

we obtain the condition

$$2\nabla^{E}_{gradR_U^{\pm}(2,p)} V_p^0 + \left(\Box R_U^{\pm}(2, p) - R_U^{\pm}(0, p)\right) V_p^0 = 0 \tag{4.69}$$

whose limit $\alpha \to 0$ exists and is given by

$$2\nabla^{E}_{gradR_U^{\pm}(2,p)} V_p^0 + \left(\Box R_U^{\pm}(2, p) - R_U^{\pm}(0, p)\right) V_p^0 = 0 \tag{4.70}$$

since $R_U^{\pm}(\alpha, p)$ is holomorphic in α for all $\alpha \in C$. Since moreover $R_U^{\pm}(0, p) = \delta_p$ we can evaluate the condition (4.65) further and obtain

$$2\nabla^{E}_{gradR_U^{\pm}(2,p)} V_p^0 + V_p^0 \Box R_U^{\pm}(2, p) = V_p^0 \delta_p \tag{4.71}$$

Thus we conclude that (4.66) for $K = 0$ implies (4.65) iff $V_p^0(p) = id_{Ep}$. This motivates that we want to solve (4.65) with the additional requirement

$$V_p^0(p) = id_{Ep} \tag{4.72}$$

Definition(4.3.1)(Transport Equations):[110]

Let $k \in N$ and let $D \in Diffop^{2}(E)$ be normally hyper-bolic. Then the recursive equations

$$\nabla^{E}_{grad\eta_p} V_p^k + \left(\frac{1}{2}\Box\eta_p - n + 2k\right) V_p^k = -2\,k\,D\,V_p^{k-1} \tag{4.73}$$

together with the initial condition

$$V_p^0(p) = id_{Ep} \tag{4.74}$$

are called the transport equations for $V_p^k \in \Gamma^{\infty}\left(E|_U\right) \otimes E_p^*$ corresponding to D

Remark(4.3.2)(Transport Equations):

Let $D \in Diffop^{2}(E)$ be normally hyper-bolic.

i.) According to our above computation, the transport equation for $k = 0$ implies

$$2\nabla^{E}_{gradR_U^{\pm}(2,p)} V_p^0 + V_p^0 \Box R_U^{\pm}(2, p) = \delta_p \tag{4.75}$$

ii.) The transport equations are the same for the advanced and retarded $\Re^{\pm}(p)$ thus we only have to solve them once and can us the same coefficients V_p^k for both the Green Functions.

Definition (4.3.3)(Hadamard Coeffcients):

Let $D \in Diffop^2(E)$ be normally hyperbolic and $U \subseteq M$ geodesically star-shaped around $p \in M$ as before. Solutions $V_p^k \in \Gamma^\infty(E|_U) \otimes E_p^*$ of the transport equations are then called Hadamard coeffcients for D at the point p.

(4.3.2) Uniqueness of The Hadamard Coeffcients:

Since on U we have unique geodesics joining p with any other point $q \in U$, namely

$$\gamma_{p\to q}(t) = \exp_p\left(t \exp_p^{-1}(q)\right), \tag{4.76}$$

For abbreviation, we set

$$P_{p\to q} = P_{\gamma p\to q,0\to 1} : E_p \to E_q \tag{4.77}$$

Lemma(4.3.4):

The parallel transport along geodesics in yields a smooth map

$$U \ni q \mapsto P_{p\to q} \in E_q \otimes E_p^* \tag{4.78}$$

which we can view as a smooth section

$$P_{p\to .} \in \Gamma^\infty(E|_U) \otimes E_p^* \tag{4.79}$$

Theorem(4.3.5) (Uniqueness of the Hadamard Coeffcients):

Let $U \subseteq M$ be geodesically star-shaped around p and let $D \in Diffop^2(E)$ be normally hyperbolic.

Then the Hadamard coeffcients for D at p are necessarily unique. In fact, they satisfy

$$V_p^0 = \frac{1}{\sqrt{\ell p}} P_{p\to .} \tag{4.80}$$

and for $k \geq 1$ and $q \in U$

$$V_p^k(q) = -k\frac{1}{\sqrt{\ell p(q)}}P_{p\to q}\left(\int_0^1 \sqrt{\ell p}(\gamma_{p\to q}(\tau))\tau^{k-1}P_{\gamma_{p\to q},0\to\tau}D(V_p^{k-1})(\gamma_{p\to q})(\tau)\right)d\tau.$$

(4.81)

(4.3.3) Construction of The Hadamard Coefficients:

Using (4.80) and (4.81) we recursively define V_p^k for $k \geq 0$ by

$$V_p^0(q) = \frac{1}{\sqrt{\ell p(q)}}P_{p\to q} \tag{4.82}$$

$$V_p^k(q) = -\frac{k}{\sqrt{\ell p(q)}}P_{p\to q}\int_0^1 \sqrt{\ell p}(\gamma_{p\to q}(\tau))\tau^{K-1}P^{-1}_{\gamma p\to q,0\to\tau}(D(V_P^{K-1})\ \exp_p(\tau \exp_p^{-1}(q)))\ d\tau \tag{4.83}$$

for $q \in U$.

Proposition(4.3.6)(Smoothness of V^k):

Let $O \subseteq U \subseteq M$ be open subsets such that U is geodesi-cally star-shaped around all $p \in O$. Then the recursive definitions (4.82) and (4.83) yield smooth sections

$$V^k \in \Gamma^\infty(E^* \otimes E|_{O\times U}) \tag{4.84}$$

via the definition

$$V^k(p,q) = V_p^k(q) \tag{4.85}$$

for $(p,q) \in O \times U\ and\ k \geq 0$.

(4.3.4) The Klein-Gordon Equation:

Consider the flat Minkowski spacetime (R^n,η) but now the Klein-Gordon[111] equation

$$(\Box + m^2)\varphi = 0 \tag{4.86}$$

m^2 denotes a positive constant. compute the Hadamard coefficients at a single point $p \in R^n$, choose $p = 0$, and $\exp_0$ is just the addition with p whence

$$\exp_0 : T_0 R^n = R^n \to R^n \tag{4.87}$$

is simply the identity map. Also the density function ℓ_P becomes very simple as we have

$$\ell_P = 1 \tag{4.88}$$

for all p . Thus the recursion for the Hadamard coefficients simplifies drastically. Finally, we note that the Klein-Gordon operator $\Box+m^2$ has already the normal form with $B=m^2$. Therefor we have

$$V_p^0=\frac{1}{\sqrt{\ell P}}P_{p\to .}=id$$

And $$V_p^k(q)=-\frac{k}{\sqrt{\ell p(q)}}P_{p\to q}\int_0^1\sqrt{\ell p}(\gamma_{p\to q}(\tau))P^{-1}_{\gamma p\to q,0\to\tau}\left(D(V_p^{k-1})(\gamma_{P\to Q}(\tau))\right)\tau^{k-1}d\tau$$

$$=-k\int_0^1 D(V_p^{k-1})(p+\tau(q-p))\tau^{k-1}d\tau$$

Now V_p^0 is constant. We claim that, since m^2 is constant as well, all Hadamard coefficients are constant, too. Indeed, assuming this for $k-1$ shows that

$$V_p^k(q)=-k\int_0^1 D(V_p^{k-1})(p+\tau(q-p))\tau^{k-1}dt$$

$$=-kD(V_p^{k-1})\int_0^1\tau^{k-1}d\tau$$

$$=-D(V_p^{k-1})$$

$$=-m^2V_p^{k-1},$$

which is again constant. Thus by induction we conclude the following:

Lemma(4.3.7):[112]

The Hadamard coefficients for the Klein-Gordon operator $\Box+m^2$ on Minkowski spacetime are constant and explicitly given by

$$V_p^k=\left(-m^2\right)^k \tag{4.89}$$

for $k\in N_0$ and all points p ∈ $p\in R^n$.

then

$$R^\pm(2+2K)(\chi)=\frac{2^{1-(2+2K)}\pi^{\frac{2-n}{2}}}{\Gamma^{(\frac{2+2k}{2})}\Gamma^{(\frac{2+2k-n}{2}+1)}}\eta(x)^{\frac{2+2k-n}{2}}=\frac{\pi^{\frac{2-n}{2}}}{2^{2k-1}k!\Gamma(k+2-\frac{n}{2})}\eta(x)^{k+1-\frac{n}{2}} \tag{4.90}$$

for $x\in I^\pm(0)$ and 0 elsewhere. We want to estimate $R^\pm(2k)$ and its derivatives over a compactum $K\subseteq R^n$. To this end we compute the first partial derivatives of $R^\pm(\alpha)$ explicitly. We know already

$$\frac{\partial}{\partial x^{i1}} R^{\pm}(\alpha) = \frac{1}{\alpha-2} R^{\pm}(\alpha-2)\eta_{i1j} x^j = \frac{1}{\alpha-2} R^{\pm}(\alpha-2)\, x_{i1}, \tag{4.91}$$

where we use the notation

$$x_i = \eta_{ij}\, x^j. \tag{4.92}$$

Thus we get

$$\frac{\partial^2}{\partial x^{ij} \partial x^{i2}} R^{\pm}(\alpha) = \frac{R^{\pm}(\alpha-4)}{(\alpha-2)(\alpha-4)} x_{i1} x_{i2} + \frac{R^{\pm}(\alpha-2)}{\alpha-2} \eta_{i_1 i_{,_2}} \eta_{i1i2} \tag{4.93}$$

since clearly $\frac{\partial}{\partial x^{i2}} x_{i1} = .\eta_{i1i2}$.

we get

$$\frac{\partial^3}{\partial x^{i1} \partial x^{i2} \partial x^{i4}} R^{\pm}(\alpha) = \frac{R^{\pm}(\alpha-6)}{(\alpha-2)(\alpha-4)(\alpha-6)} x_{i1}\, x_{i2}\, x_{i3}$$

$$+ \frac{R^{\pm}(\alpha-4)}{(\alpha-2)(\alpha-4)} (\eta_{i1i3} x_{i2} + \eta_{i2i3} x_{i1} + \eta_{i1i2} x_{i4}\ ) \tag{4.94}$$

and

$$\frac{\partial^4}{\partial x^{i1} \partial x^{i2} \partial x^{i4} \partial x^{i4}} R^{\pm}(\alpha) = \frac{R^{\pm}(\alpha-8)}{(\alpha-2)(\alpha-4)(\alpha-6)(\alpha-8)} x_{i1}\, x_{i2}\, x_{i3} x_{i4}$$

$$+ \frac{R^{\pm}(\alpha-6)}{(\alpha-2)(\alpha-4)(\alpha-6)} (\eta_{i1i4} x_{i2} x_{i3} + \eta_{i2i4} x_{i1} x_{i3} + \eta_{i3i4} x_{i1} x_{i2} + \eta_{i1i2} x_{i3} x_{i4} + \eta_{i1i3} x_{i2} x_{i4}\ )$$

$$+ \frac{R^{\pm}(\alpha-4)}{(\alpha-2)(\alpha-4)} (\eta_{i1i2}\eta_{i3i4} + \eta_{i1i3}\eta_{i2i4} + \eta_{i1i4}\eta_{i2i3}) \tag{4.95}$$

Theorem(4.3.8)(Green's Functions of The Klein-Gordon Operator):

Let $p \in R^n$. Then the series

$$R^{\pm}(p) = \sum_{k=0}^{\infty} \left(-m^2\right)^k R^{\pm}(2+2k, p) \tag{4.96}$$

converges in the $weak^*\ topo\log y$ to the advanced and retarded Green function of the Klein-Gordon operator $\Box + m^2$ respectively. More precisely, for $2+2k > 2\ell + n$ the series.

$$\sum_{2+2k>2\ell+n} \left(m^2\right)^k R^{\pm}(2+2k, p) \tag{4.97}$$

converges in the $\ell^{\ell} - topo\log y$ to $\ell^{\ell} - function$ on R^n. Finally, on $I^{\pm}$ the series (4.96) converges in the $\ell^{\infty}-topo\log y$ to a smooth function given by

$$R^{\pm}(0)\Big|_{I^{\pm}} = \sum_{K=0}^{\infty} \frac{\pi^{\frac{2-n}{2}}\left(-m^{2}\right)^{k}}{2^{2k-1}k!\Gamma(k+2-\frac{n}{2})}\eta^{k+1-\frac{n}{2}} \tag{4.98}$$

for $p=0$ from which the other $R^{\pm}(p)$ can be obtained by translation.

(4.4) The Fundamental Solution on Small Neighborhoods [113].

Take the Hadamard coefficients as smooth sections

$$V^{k} \in \Gamma^{\infty}(E^{*} \otimes E\big|_{U'\times U'}). \tag{4.99}$$

out of which we obtain the formal fundamental solution

$$R^{\pm}(p) = \sum_{k=0}^{\infty} V_{p}^{k} R_{U'}^{\pm}(2+2k, p) \tag{4.100}$$

on U'.

Of course, there is no reason to believe that (4.100) converges in general, even not in the weak* sense. However, the Riesz distributions $R_{U'}^{\pm}(2+2k,p)$ are continuous functions if k is large enough.In fact, by Proposition 4.2.4 we know that $R_{U'}^{\pm}(2+2k,p)$ is at least continuous if $k \geq \frac{n}{2}$

(4.4.1) The Approximate Fundamental Solution:[114]

The idea is now that the finite sum

$$\sum_{k=0}^{n-1} V_{p}^{k} R_{U'}^{\pm}(2+2k, p) \in \Gamma_{0}^{\infty}\left(E^{*}\big|_{U'}\right)' \tag{4.101}$$

take a cutoff function

$\chi \in \ell_{0}^{\infty}(R)$ with

$$\sup p\, \chi \subseteq [-1,1],\ 0 \leq \chi \leq 1 \ and\ \chi\Big|_{\left[-\frac{1}{2},\frac{1}{2}\right]} = 1 \tag{4.102}$$

Lemma(4.4.1):

Let $\ell \in N \, and \, \ell' > \ell + 1$ then there are universal constant $c(\ell,\ell')$ such that for all $0 < \varepsilon \leq 1$

$$Pk,0\left(\frac{d^{\ell}}{dt^{\ell}}\left(\chi\left(\frac{t}{\varepsilon}\right)t^{\ell'}\right)\right) \leq \varepsilon\, c(\ell,\ell')Pk,\ell(\chi), \tag{4.103}$$

where K is any compactum containing $[-1,1]$

Lemma(4.4. 2):

Let $g:U\subseteq R^n\to R$ and $f:R\to R$ be smooth , then for every multi-index $I\in N_0^n$

$$\frac{\partial^{|I|}}{\partial x^I}(f\circ g)=\sum_{\substack{r=1..|I|\\ j1,\dots jr\leq I}} c^r_{j1\dots jr}\,\frac{d^r f}{dt^r}\circ g\,\frac{\partial^{|j1|}g}{\partial x^{j1I}}\cdots\frac{\partial^{|jr|}g}{\partial x^{j1r}} \tag{4.104}$$

with some universal constants $c^r_{j1..,jr}\in Q$.

Lemma(4.4. 3):

Let $\ell,k\in N_0$ and j large enough such that $j-N\geq k$ .then we have

$$Pk_e\times k_{e,k}\left(\chi_j^{\pm}\right)\leq\varepsilon_j c\left(k,\ell,j\right)\ , \tag{4.105}$$

with constant $c\left(k,\ell,j\right)>0$ independent of ε_j satisfying

$$c\left(k,\ell,j\right)\leq c\left(k',\ell\ \ ,j\right) \tag{4.106}$$

for $\ell\leq\ell'\,and k\leq k'$

Lemma(4.4.4):

Let $\ell,k\in N_0$ and $j\geq N+k$ then the $j-th$ term of the series (4.104) satisfies estimate

$$Pk_e\times k_{e,k}\left(\chi\left(\frac{\eta}{\varepsilon_j}\right)V^j\,R^{\pm}_{U'}\left(2+2_j\,,.\right)\right)\leq\varepsilon_j c\left(k,\ell,j\right)c\left(2+2_j\,,n\right)Pk_e\times k_{e,k}\left(V^j\right) \tag{4.107}$$

(4.4.2) Construction of the Local Fundamental Solution:[35]

Having an open subset $U\subseteq U'$ such that

$$U^{c1}\subseteq U' \tag{4.108}$$

is compact , consider the $K^{\pm}_{U'}\in\Gamma\infty\left(E^*\otimes E\big|_{U'\times U'}\right)$ and φ a section of E^*defined on U^{c1} then we can naturally pair $K^{\pm}_{U'}\,(p,q\,).\varphi(q)$and integrate , this gives

$$(\kappa^{\pm}_U\varphi)(p)=\int_{U^{c1}}K^{\pm}_{U'}(p,q).(q)\mu_g(q) \tag{4.109}$$

Depending on the properties of φ the integral will be well-defined and yields a rather nice section of E^* defined on U' .

Definition(4.4.5):

With respect to some auxiliary positive fiber metric on E^* we define

$$\Gamma_b(E^*\big|_U) = \{U \to E^* \mid (q) \in E_q^* \text{and is bounded and measurable}\} \tag{4.110}$$

here the fiber metric is used to define a norm on each fiber .

Lemma(4.4.6)(The Banach space $\Gamma_b(E^*\big|_U))$:

Let $U \subseteq M$ be open with compact closure.

i.) The definition of $\Gamma_b(E^*\big|_U)$ does not depend on the auxiliary smooth fiber metric.

ii.) The vector space $\Gamma_b(E^*\big|_U)$ becomes a Banach space via the norm.

$$P_{u,o}(\varphi)=\sup_{q\in U}\|\varphi(q)\|_{E_q^*} \tag{4.111}$$

iii.) Different choices of positive fiber metrics on E^* yield equivalent Banach norms(4.111)

iv.) The restriction map

$$\Gamma^k(E^*) \ni \varphi \mapsto \varphi\big|_U \in \Gamma_b(E^*\big|_U) \tag{4.112}$$

is continuous for all $k \in N_0 \cup \{+\infty\}$.

Lemma(4.4.7):

Let $k \in N_0$ and $U \subseteq U'$ open with compact closure $U^{c1} \subseteq U'$. .

i.) For $\varphi \in \Gamma_b(E^*\big|_U)$ we have $K_U^{\pm}\varphi \in \Gamma^\infty(E^*\big|_{U'})$

ii.) We have an estimate of the form

$$P_{K},(K_U^{\pm}\varphi) \le vol(U^{c1})P_{K\times U^{c1},k}(K_{U'}^{\pm})P_{U,0}(\varphi) \tag{4.113}$$

for all $\varphi \in \Gamma_b(E^*\big|_U)$ and compact $K \subseteq U'$.

Proof. We first proof continuity. Thus let $P \in U'$ be fixed and consider $P_n \to p.$. Since the integrand $K_U^{\pm}(P_n, q).\varphi(q)$ is bounded by some integrable function, namely by the constant function $P_{K\times U^{c1},0}(K_U^{\pm})P_{U,0}(\varphi)$ where K is any compactum containing the convergent sequence P_n we can apply Lebesgue's dominated convergence and find

$$\lim_{n\to\infty}(K_U^{\pm}\varphi)(P_n)=\lim_{n\to\infty}\int_{U^{c1}}K_{U'}^{\pm}(P_n,q)\varphi(q)\mu_g(q)$$

$$\overset{lebesgue}{=}\int_{U^{c1}}\lim_{n\to\infty}K_{U'}^{\pm}(P_n,q).\varphi(q)\mu_g(q)$$

$$\int_{U^{c1}}K_{U'}^{\pm}(P_n,q).\varphi(q)\mu_g(q)$$

$$=(K_U^{\pm}\varphi)(p)$$

which is the continuity of $K_U^{\pm}\varphi$. then

$$\frac{\partial}{\partial x^i}(K_U^{\pm}\varphi)=\int_{U^{c1}}\frac{K_{U'}^{\pm}(P_n,q)}{\partial x^i}.\varphi(q)\mu_g(q) \tag{4.114}$$

all with respect to some local trivialization of E^* . Thus $K_U^{\pm}\varphi$ turns out to be ℓ^1 and by induction we get $K_U^{\pm}\varphi\in\Gamma^{\infty}(E^*\big|_{U'})$. . This shows the first part. For the second, we use a local trivialization and

(4.114) to obtain

$$\frac{\partial^{|I|}}{\partial x^I}(K_U^{\pm}\varphi)\Big|_p=\int_{U^{c1}}\frac{\partial^{|I|}K_{U'}^{\pm}}{\partial x^I}(P_n,q).\varphi(q)\mu_g(q)$$

from which we get

$$P_{U^{c1},k}(K_U^{\pm}\varphi)\le\sup_{\substack{p\in U^{c1}\\|I|\le k}}=\int_{U^{c1}}\left\|\frac{\partial^{|I|}K_{U'}^{\pm}}{\partial x^I}(P_n,q)\right\|\|\varphi(q)\|\mu_g(q).$$

$$\le vol(U^{c1})_{PK\times U^{c1},K}(K_{U'}^{\pm})_{PU,0}(\varphi).$$

Thus we have

$$\Gamma_b(E^*\big|_{U'})\ni\varphi\mapsto K_U^{\pm}\varphi\big|_U\in\Gamma_b(E^*\big|_{U'}). \tag{4.115}$$

By some slight abuse of notation we denote the composition $\varphi\mapsto K_U^{\pm}\varphi\mapsto K_U^{\pm}\varphi\big|_{\mathrm{U}}$ again simply by $K_U^{\pm}\varphi$.

Lemma(4.4.8):

The linear operator

$$K_U^{\pm}:\Gamma_b(E^*\big|_U)\ni\varphi\mapsto K_U^{\pm}\varphi\big|_U\in\Gamma_b(E^*\big|_U). \tag{4.116}$$

is continuous with operator norm

$$\|K_U^{\pm}\| \leq vol(U^{c1})_{PU^{c1}\times U^{c1},0}(K_{U'}^{\pm}). \qquad (4.117)$$

Proof. From Lemma 4.4.7 We know that for all $\varphi \in \Gamma_b(E^*|_U)$ we have

$$P_{U,0}(K_U^{\pm}\varphi)=P_{U^{c1},0}(K_U^{\pm}\varphi)\leq vol(U^{c1})_{PU^{c1}\times U^{c1},0}(K_U^{\pm}).P_{U,0}(\varphi)$$

which gives the continuity as well as the estimate on the operator norm (4.115).

Corollary(4.4.9):

If the open subset $U \in U'$ is sufficiently small in the sense that

$$vol(U^{c1})_{PU^{c1}\times U^{c1},0}(K_U^{\pm}).<1 \qquad (4.118)$$

then the operator

$$id + K_U^{\pm} : \Gamma_b(E^*|_U) \to \Gamma_b(E^*|_U) \qquad (4.119)$$

is invertible with continuous inverse given by the absolutely norm-convergent geometric series

$$\left(id+K_U^{\pm}\right)^{-1} = \sum_{j=0}^{\infty}\left(-K_U^{\pm}\right)^j \qquad (4.120)$$

Definition(4.4.10)(The space $\Gamma^b(E^*|_{U^{c1}})$ *:*

Let $k \in N_0$ then a section $\varphi \in \Gamma^0(E^*|_{U^{c1}})$ is called ℓ^k on U^{c1} if it can be approximated by sections $\varphi_n|_{U^{c1}}$ with $\varphi_n \in \Gamma^0(E^*|_{U^{c1}})$ with respect to the norm $P_{U^{c1},k,}$ where $U_n \supseteq U^{c1}$ is open. The set of all such section is denoted by $\Gamma^k(E^*|_{U^{c1}}) = \left\{\varphi \in \Gamma^0(E^*|_{U^{c1}}) \middle| \ \varphi \ is \ \ell^k\right\}$.

Lemma(4.4.11):

The operator $\kappa_U^{\pm}(p):\Gamma^0\left(E^*|_{U^{c1}}\right)\to\Gamma^0\left(E^*|_{U^{c1}}\right)$ restricts to a continuous linear operator

$$\kappa_U^{\pm} :\Gamma^k\left(E^*|_{U^{c1}}\right)\to\Gamma^k\left(E^*|_{U^{c1}}\right) \qquad (4.121)$$

for all $k \in N_0$ whose image are restrictions of smooth sections of E^* defined on U' the operator norm of (4.121) is bounded by

$$\|\kappa_U^{\pm}\| \leq Vol\left(U^{c1}\right)_{pU^{c1}\times U^{c1},k}\left(\kappa_{U'}^{\pm}\right)$$

Definition(4.4.12)(Local Fundamental Solution):

Let $U \subseteq M$ be geodesically convex and $U \subseteq U'$ be open with compact closure $U^{c1} \subseteq U'$ such that the volume of U^{c1} is small enough . then for $p \in U$ we define

$$F_U^{\pm}(p) : \Gamma_0^{\infty}\left(E^*|_U\right) \ni \varphi \mapsto \left(id + K_U^{\pm}\right)^{-1}\left(\widetilde{R}_U^{\pm}(.)(\varphi)\right)\Big|_p \in E_p^*$$

Theorem(4.4.13)(local fundamental solution):

Let $U \subseteq M$ be geodesically convex and let $U \subseteq U'$ be open with compact closure $U^{c1} \subseteq U'$ such that the volume of U^{c1} is small enough . then for $p \in U$ the map

$$F_U^{\pm}(p) : \Gamma_0^{\infty}\left(E^*|_U\right) \to E_p^*$$

is local fundamental solution of D at p such that for every $\varphi \in \Gamma_0^{\infty}\left(E^*|_U\right)$

$$F_U^{\pm}(.)\varphi : p \mapsto F_U^{\pm}(p)(\varphi)$$

is a smooth section of E^* over U, in fact ,

$$F_U^{\pm} : \Gamma_0^{\infty}\left(E^*|_U\right) \ni \varphi \mapsto F_U^{\pm}(.)(\varphi) \in \Gamma^{\infty}\left(E^*|_U\right) \tag{4.122}$$

is a continuous liner map .

(4.4.3) Causal Properties of $F_U^{\pm}$:

Lemma(4.4.14)):

Let $U \subseteq U'$ be in addition causal . Then for $\varphi \in \Gamma^0(E^*|_{U^{c1}})$ we have

$$Supp\left(K_U^{\pm}\varphi\right) \subseteq J_{U^{c1}}^{\pm}(Supp\varphi)$$

Theorem(4.4.15)(Local Green's Functions):[115]

Let $U \subseteq U'$ be small enough and causal. Then the fundamental solutions $F_U^{\pm}(p)$ from Theorem 4.4.13 are advanced and retarded Green functions, i.e.

we have

$$SuppF_U^{\pm}(p) \subseteq J_U^{\pm}(p) \tag{4.123}$$

Proof. Let $\varphi \in \Gamma_0^{\infty}(E^*|_U)$ be a test section. Then

$$Supp\, F_U^{\pm}(\,.\,)(\varphi) = Supp\,\left(\left(id + K_U^{\pm}\right)^{-1}\widetilde{R}_{U'}^{\pm}(.)(\varphi)\right)$$

$$\subseteq J^{\pm}_{U^{c1}}\left(\widetilde{R}^{\pm}_{U'}(.)(\varphi)\right)$$

$$\subseteq J^{\pm}_{U^{c1}}\left(J^{\pm}_{U^{c1}}(Supp\varphi)\right) = J^{\pm}_{U^{c1}}(Supp\varphi) \tag{4.124}$$

since $Supp\,\widetilde{R}^{\pm}_{U'}(p)\big|_U \subseteq J^{\pm}_{U^{c1}}(p)$ whence for $Supp\ \varphi \cap J^{\pm}_{U^{c1}}(p) = \phi$ we conclude $\widetilde{R}^{\pm}_{U'}(p)(\varphi) = 0$ Thus $p \notin J^{\pm}_{U^{c1}}(Supp\ \varphi)$ implies $\widetilde{R}^{\pm}_{U'}(p)(\varphi) = 0$.

Corollary(4.4.16):

Let $D \in Diffop^{2}(E)$ be normally hyperbolic . Then every point in M has small enough neighborhood $U \subseteq M$ such that on U we have advanced and retarded Green functions $F^{\pm}_{U}(p)$ at $p \in U$, i.e.

$$D\,F^{\pm}(p) = \delta_p \tag{4.125}$$

and

$$Supp\ F^{\pm}_{U}(p) \subseteq J^{\pm}_{U}(p) \tag{4.126}$$

such that in addition

$$F^{\pm}_{U} : \Gamma^{\infty}_{0}(E^{*}\big|_{U}) \ni \varphi \mapsto \left(p \mapsto F^{\pm}_{U}(p) \in \Gamma^{\infty}(E^{*}\big|_{U})\right) \tag{4.127}$$

is a continuous linear map .

(4.5) Solving the Wave Equation Locally:

In this section we show how the Green functions $F^{\pm}_{U}$ can be used to obtain solutions to the wave equation

$$Du = \upsilon \tag{4.128}$$

with a prescribed source term υ[116]. The main idea is that a suitable υ can be written as a superposition of δ-functionals. Since $F^{\pm}_{U}(P)$ solves (4.128) for $\upsilon = \delta_p$ we get a solution to (4.128) for arbitrary υ by taking the corresponding superposition of the fundamental solutions $F^{\pm}_{U}(P)$. Of course, at the moment we are restricted to υ having compact support in U .

Then we are interested in two extreme cases: for a distributional υ we can only expect to obtain distributions u as solutions. However, if υ has good regularity then we can expect u to be regular as well.

(4.5.1) Local Solutions for Distributional Inhomogeneity:[117]

Let $\upsilon \in \Gamma^{-\infty}_{0}(E\big|_{U})$ be a generalized section of E with compact support in U . We want to solve

$$D u^{\pm} = \upsilon \tag{4.129}$$

with some $u^{\pm} \in \Gamma^{-\infty}\left(E\big|_U\right)$

Lemma(4.5.1):

Let $U \subseteq M$ be a small enough open subset such that the construction of $F_U^{\pm}$ as in Section 4.4 applies.

i.) The map $F_U^{\pm} : \Gamma_0^{\infty}(E^*\big|_U) \to \Gamma^{\infty}(E^*\big|_U)$ induces a linear map

$$(F_U^{\pm})' : \Gamma_0^{-\infty}(E\big|_U) \to \Gamma^{-\infty}(E\big|_U) \tag{4.130}$$

by dualizing, i.e. for $\upsilon \in \Gamma_0^{-\infty}(E\big|_u)$ and $\varphi \in \Gamma_0^{-\infty}(E^*\big|_U)$ we defines

$$\left(\left(F_U^{\pm}\right)'(v)\right)(\varphi) = v\left(F_U^{\pm}(\varphi)\right) \tag{4.131}$$

ii.) The map $(F_U^{\pm})'$ is *weak * continuous* .

iii.) We have

$$D(F_U^{\pm})'(\upsilon) = \upsilon \tag{4.132}$$

for all $\upsilon \in \Gamma_0^{-\infty}(E\big|_U)$.

Proof. For the first part we recall that we have the identification

$$\Gamma_0^{\infty}(E^*\big|_U) \ni \varphi \mapsto \varphi \otimes \mu_g \in \Gamma_0^{\infty}(E^*\big|_U \otimes \left|\Lambda^{top}\right| T^* M)$$

from which we obtain the identification

$$\Gamma^{-\infty}(E\big|_U) \ni u \mapsto (\varphi \mapsto u(\varphi \otimes \mu_g)) \in \Gamma_0^{\infty}(E^*\big|_U)'. \tag{4.133}$$

Since tensoring with $\mu_g > 0$ does not change the supports we can dualize the continuous map

$$F_U^{\pm} : \Gamma_0^{\infty}(E^*\big|_U) \to \Gamma^{\infty}(E^*\big|_U)$$

to a map

$$(F_U^{\pm})' : \Gamma^{\infty}(E^*\big|_U)' \to \Gamma_0^{\infty}(E^*\big|_U)' \tag{4.134}$$

Using (4.133) and the fact that the dual space of all test sections are the compactly supported generalized sections, see Theorem 2.3.11, we get

$$\Gamma_0^{-\infty}(E\big|_U) \xrightarrow{(*)} \Gamma^{\infty}(E^*\big|_U)' \xrightarrow{(F_U^{\pm})'} \Gamma_0^{\infty}(E^*\big|_U)' \xrightarrow{(*)} \Gamma^{-\infty}(E\big|_U)$$

whose composition we denote by $(F_U^{\pm})'$ as well. This is the map (4.130). Dualizing yields a $weak^*\, continuous$ map in (4.134). Finally, the identifications (4.133) are $weak^*\, continuous$ as well, hence it results in a $weak^*\, continuous$ map (4.130). This shows the first and second part. For the third part we unwind the definition of $D F_U^{\pm}$. Let $\Gamma_0^{\infty}(E^*\big|_U)$ be a test section and compute

$$\begin{aligned}(D((F_U^{\pm})'(\upsilon)))(\varphi) &= ((F_U^{\pm})'(\upsilon))(D^T\varphi)\\ &= \upsilon\left(p \to F_U^{\pm}(D^T\varphi)\big|_p\right)\\ &= \upsilon\left(p \to (F_U^{\pm}(p))(D^T\varphi)\right)\\ &= \upsilon(p \to \varphi(p))\\ &= \upsilon(\varphi),\end{aligned}$$

using the definition of the dualized map and the feature $DF_U^{\pm}(p) = \delta_p$. But this means (4.132).

(4.5.2) Local Solution for Smooth Inhomogeneity:

Let $U \subset M$ be open with U^{c1} compact and let $U^{c1} \subseteq U'$ with U' open. Moreover , let $K \in \Gamma^{\infty}\left(E^* \otimes E\big|_{U'\times U'}\right)$ be a smooth kernel on the larger open subset $U'\times U'$. For sections $\varphi \in \Gamma_b(E^*\big|_U)$ we consider the integral operator

$$(\kappa\varphi)(p) = \int_{U^{c1}} \kappa(p,q).\varphi(q)\mu_g(q) \tag{4.135}$$

analogously to (4.102), where $p \in U'$ [118] . Repeating the arguments from Lemma 4.4.7 and Lemma 4.4.11we obtain the following general result:

Lemma(4.5.2):

Let $U \subseteq U^{c1} \subseteq U'$ with U, U' open and U^{c1} compact. For the integral operator κ corresponding to a smooth kernel $K \in \Gamma^{\infty}(E^* \otimes E\big|_{U'\times U'})$ as in (4.121) the following statements are true:

i.) For $\varphi \in \Gamma_b(E^*\big|_U)$ one has $\kappa\varphi \in \Gamma^k(E^*\big|_{U^{c1}})$ for all $k \in N_0$ and $\kappa\varphi\big|_U \in \Gamma^{\infty}(E^*\big|_U)$.

ii.) The maps (all denoted by κ)

$$\kappa : \Gamma_b(E^*\big|_U) \ni \varphi \mapsto \kappa\varphi \in \Gamma^k(E^*\big|_{U^{c1}}) \tag{4.136}$$

And

$$\kappa : \Gamma_b(E^*\big|_U) \ni \varphi \mapsto \kappa\varphi\big|_U \in \Gamma^k(E^*\big|_{U^{c1}}) \tag{4.137}$$

are continuous. In fact, for $k \in N_0$ we have

$$P_{U^{c1},K}(\kappa\varphi) \le cP_{U^{c1},0}(\varphi) \tag{4.138}$$

for some $c > 0$ depending on k.

Lemma(4.5.3):[119]

Let $U \subseteq U^{c1} \subseteq U' \subseteq M$ be as in Section 4.4 with U small enough and let $\kappa_U^{\pm}$ be the integral operator from (4.116).

i) For every $k \in N_0$ there is a $c > 0$ such that for $\varphi \in \Gamma_b(E^*\big|_{U'})$ we have

$$P_{U^{c1},K}\left(\left(\left(id + \kappa_U^{\pm}\right)^{-1} \circ \kappa_U^{\pm}\right)(\varphi)\right) \le cP_{U^{c1},0}(\varphi) \tag{4.139}$$

ii.) For $\varphi \in \Gamma^k(E^*\big|_{U'})$ there is a $\tilde{c} > 0$ such that

$$P_{U^{c1},K}\left(\left(\left(id + \kappa_U^{\pm}\right)^{-1}\right)(\varphi\big|_{U^{c1}})\right) \le \tilde{c}P_{U^{c1},k}(\varphi) \tag{4.140}$$

Proof. we know that the operator $\left(id + \kappa_U^{\pm}\right)^{-1} \circ \kappa_U^{\pm}$ has a smooth kernel in $\Gamma^\infty(E^* \otimes E\big|_{U' \times U'})$. Thus the previous Lemma 4.5.3, ii.) applies and (4.138) gives (4.139). For the second part we note that

$$\left(id + \kappa_U^{\pm}\right)^{-1}(\varphi\big|_{U^{c1}})\Big|_{U^{c1}} = \varphi\big|_{U^{c1}} - \left(id + \kappa_U^{\pm}\right)^{-1} \circ \kappa_U^{\pm}(\varphi)\big|_{U^{c1}}$$

then

$$= P_{U^{c1},K}\left(\left(\left(id + \kappa_U^{\pm}\right)^{-1}\right)(\varphi\big|_{U^{c1}})\right)$$

$$p_{U^{c1},K}\left(id + \kappa_U^{\pm}\right)^{-1}(\varphi\big|_{U^{c1}})\Big|_{U^{c1}} = p_{U^{c1},K}\left(\varphi - \left(id + \kappa_U^{\pm}\right)^{-1} \circ \kappa_U^{\pm}(\varphi)\right) \le p_{U^{c1},K}(\varphi) + cp_{U^{c1},0}(\varphi)$$

with $c > 0$ from (4.139). Since $P_{U^{c1},K}(\varphi) \ge P_{U^{c1},0}(\varphi)$ we take $\tilde{c} = 1 + c$ to obtain (4.140).

Proposition(4.5.4):

Let $U \subseteq U^{c1} \subseteq U^{c1}$ be as before and let $F_U^{\pm} = \left(id + \kappa_U^{\pm}\right)^{-1} \circ \widetilde{R}_{U'}^{\pm}(.)$, Then for all compacta $K \subseteq U$ and all $k \in N_0$ we have a such that

$$p_{U^{c1},K}\left(F_U^{\pm}(\varphi)\right) \le c_{K,k}\, p_{K,k+n+1}(\varphi) \tag{4.141}$$

for all $\varphi \in \Gamma_k^{\infty}(E^*\big|_U)$.

Proof. We know already from the proof of Theorem 4.4.13 that the operator $F_U^{\pm}$ is continuous but (4.141) gives a more precise statement of this. We have by (4.140) .

$$\begin{aligned} p_{U^{c1},K}\left(F_U^{\pm}(\varphi)\right) &= p_{U^{c1},K}\left(id + \kappa_U^{\pm}\right)^{-1}\left(\widetilde{R}_{U'}^{\pm}\right)(\varphi) \\ &\le \tilde{c}\, p_{U^{c1},K}\left(\widetilde{R}_{U'}^{\pm}\right)(\varphi) \\ &\le \tilde{c}\, c_{K,U^{c1}k+n+1}\, p_{K,k+n+1}(\varphi) \end{aligned}$$

which is (1.41).

Corollary(4.5.5):

The operator $F_U^{\pm}$ has a continuous extension to an operator

$$F_U^{\pm} : \Gamma_0^{k+n+1}(E^*\big|_U) \to \Gamma^k(E^*\big|_U) \tag{4.142}$$

for all $k \ge 0$, and the estimate (4.141)also holds for $\varphi \in \Gamma_K^{k+n+1}(E^*\big|_U)$.

Lemma(4.5.6):

Let $\widetilde{R}_{U'}^{\pm}$ be as before and let $k \in N_0$ there for all $u \in \Gamma_0^{k+n+1}(E\big|_{U'})$ we have

i.) $\widetilde{R}_{U'}^{\pm}$ dualizes to a *weak * continuous* linear map

$$(\widetilde{R}_{U'}^{\pm})' : \Gamma_0^{-k}(E\big|_{U'}) \to \Gamma^{-k-n-1}(E\big|_{U'}) \tag{4.143}$$

ii.) We have $(\widetilde{R}_{U'}^{\pm})'(u) \in \Gamma^k(E\big|_{U'})$ explicitly given by

$$\left((\widetilde{R}_{U'}^{\pm})'(u)\right)(q) = \sum_{j=0}^{\infty}\left(\widetilde{V}_q^{\,j}\right)^T \widetilde{R}_{U'}^{\pm}(2+2j,q)(u) \tag{4.144}$$

where

$$\widetilde{V}^{j} = V^{j} \ for \ j \le N-1 \text{ and } \widetilde{V}^{j} = V^{j}\chi\left(\frac{\eta}{\varepsilon_j}\right) for \ j \ge N$$

for abbreviation and

$$^{T}:\Gamma^{\infty}(E^{*} \times E\big|_{U' \times U'}) \to \Gamma^{\infty}(E \times E^{*}\big|_{U' \times U'})$$

is the canonical transportation also flipping the arguments .

Lemma(4.5.7):

Let $K \subseteq U'$ be compact and $k \in N_0 \cup \{+\infty\}$

1.) Assume $u \in \Gamma^{k+n+1}(E^{*}\big|_{U'})$ has support in $J^{\pm}_{U'}(K)$.then

$$(\widetilde{R}^{\pm}_{U'})^{T}(u) = \sum_{j=0}^{\infty} \left(\widetilde{V}^{j}_{.}\right)^{T} \widetilde{R}^{\pm}_{U'}(2+2j,.)(u) \tag{4.145}$$

converges in ℓ^{k} *topdogy* .

ii.) Assume $\varphi \in \Gamma^{k+n+1}(E^{*}\big|_{U'})$ has support in $J^{\pm}_{U'}(K)$.then

$$\widetilde{R}^{\pm}_{U'}(p)(\varphi) = \sum_{j=0}^{\infty} \widetilde{V}^{j}_{p} R^{\pm}(2+2j,p)(\varphi) \tag{4.146}$$

converges in ℓ^{k} *topdogy*

Lemma(4.5.8):

Let $u \in \Gamma^{\infty}_{0}(E\big|_{U'})$ then

$$\left(F^{\pm}_{U'}\right)'(u) = (\widetilde{R}^{\pm}_{U})^{T}\left(q \mapsto u(q) - \int_{U} u(p).L^{\pm}_{U}(p.q)\mu_{g}(p)\right) \tag{4.147}$$

with $L^{\pm}_{U}$ being the smooth integral kernel of $\left(id + \kappa^{\pm}_{U}\right)^{-1} \circ \widetilde{R}^{\pm}_{U}$.

thus $\left(F^{\pm}_{U'}\right)'(u) \in \Gamma^{\infty}_{0}(E\big|_{U})$.

Theorem(4.5.9):

Let $k \in N_0 \cup \{+\infty\}$ and $u \in \Gamma^{k+n+1}_{0}(E\big|_{U'})$ then $\left(F^{\pm}_{U'}\right)'(u)$, explicitly given by (4.146) is a ℓ^{k} −sec*tionof* $E\big|_{U}$ with

$$Supp\left(F^{\pm}_{U'}\right)'(u) \subseteq J^{\pm}_{U}(Suppu) \ and \ D\left(F^{\pm}_{U'}\right)'(u) = u \tag{4.148}$$

In particular ,we have a smooth local of solution of the wave equation for a smooth and compactly supported inhomogeneity .

Chapter(5)

The Global Theory Of Geometric Wave Equations

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The Global Theory Of Geometric Wave Equations

The topic in this chapter is now to globalize the (small) neighborhoods to the whole Lorentz manifold. Here the global causal structure yields obstructions of various kinds. Here the best situation will be obtained for globally hyperbolic Lorentz manifolds. On such spacetimes we can then also formulate and solve the Cauchy problem for the wave equation. This nice solutions theory allows to treat the wave equation essentially as an (in finite - dimensional) Hamiltonian dynamical system. We will illustrate this point of view by determining the relevant Poisson algebra of observables.

(5.1) Uniqueness Properties of Fundamental Solutions

(5.1.1) Time Separation:[11]

The time separation function τ on M will be the Lorentz analogue of the Riemannian distance d .However, in various aspects it behaves quite differently . It will help us to formulate appropriate conditions on M to ensure uniqueness properties for the fundamental solutions.

Definition(5.1.1)(Arc Length):[37]

Let $\gamma:[a,b]\rightarrow M$ be a piecewise e^1 curve in a semi-Riemannian manifold (M,g). Then its arc length is defined by

$$l(\gamma)=\int_a^b \sqrt{\left|g_{\gamma(t)}\left(\dot{\gamma}(t),\dot{\gamma}(t)\right)\right|}\,dt \tag{5.1}$$

Clearly, the definition makes sense for piecewise e^1curve as well. The following is obvious:

Lemma(5.1.2):

The arc length of a piecewise e^1curve γ is invariant under monotonous piecewise e^1reparametrization. Unlike in Riemannian geometry, for different points p and $q$there may still be curves γ joining p and q which have arc length 0, namely if γ is timelike. This makes the concept of a "distance" more complicated.

Definition(5.1.3)(Time Separation):

The time separation function $\tau: M\times M\rightarrow R\cup\{+\infty\}$in a time-oriented Lorentz manifold (M,g) is defined by

$$t(p,q)=\sup\{L(\gamma)|\ \gamma\ is\ \text{a future directed causal curve form p to q}\} \quad (5.2)$$

$if\ q \in J_M^+(p) and\ \tau(p,q)=0\ if\ q \notin J_M^+(p)$

In contrast to the Riemannian situation where one uses the infimum over all arc lengths of curves joining p and q to define the Riemannian distance, the time separation τ has some new features:

first it is clear that $\tau(p,q)=0$ may happen even for $p \neq q$; this is possible already in Minkowski space time.

Moreover, in general $\tau(p,q)$ is not a symmetric function as it involves the choice of the time-orientation. Again, this can easily be seen for Minkowski space time and points $p \neq q$ with $q \in I_M^+(p)$ In this case $\tau(p,q)$ is the Minkowski length of the vector $\overrightarrow{pq}=q-p$. The fact that all other future directed causal curves from $\tau(p,q)$ $p\ to q$ are shorter is the mathematical fact underlying the so - called twin paradoxon. In the more weird examples of Lorentz manifolds it may happen that τ $\tau(p,q)=+\infty$ for some or even all pairs of points .

Recall that a light like curve γ from $p\ to q$ is called maximizing if there is no time like curve from $p\ to q$. Then we have the following useful Lemma:

Lemma(5.1.4):

If there is a causal curve γ from $p\ to q$ which is not a maximizing lightlike curve then there also exists a time like curve from $p\ to q$.

Theorem(5.1.5)(Time separation):[120]

$Let(M,g)$ be a time-oriented Lorentz manifold and $p,q,r \in M$.

i.) Then $\tau(p,q)>0 \quad iff \quad p \ll q$.

ii.) If there exists a time like closed curve through p then we have $\tau(p,q)=+\infty$ Otherwise $\tau(p,q)=0$.

iii.) $If\ 0<\tau(p,q)<+\infty\ then \quad \tau(p,q)=0$.

iv.) For $p \le q \le r$ we reverse triangle inequality, i.e.

$$t(p,q)+t(q,r) \le t(p,r) \quad (5.3)$$

v.) Suppose $p,q \in U \subseteq M$ with an open geodesically convex U . $if\ q \in I_U^+$ then the geodesic $\gamma(t)=\exp_p\left(t\exp_p^{-1}(q)\right)$ maximizes the arc length of all causal curves from p to q which are entirely in U *and* $\tau_U(p,q)=\sqrt{g_p\left(\exp_p^{-1}(q)\exp_p^{-1}(q)\right)}$.

vi.) The time separation function τ is lower semi continuous, i.e. for convergent sequence $p_n \to p$ and $q_n \to q$ one has

$$\lim_{n\to\infty} \inf \; t(p_n , q_n) = t(q , r) \tag{5.4}$$

(5.1.2) Uniqueness of Solutions to The Wave Equation:[30, 35]

In general, the wave equation

$$Du=0 \tag{5.5}$$

has many solutions $u \in \Gamma^{-\infty}(E)$, we know that the causal relation $\leq$ is called closed if for any sequence $p_n \to p$ and $q_n \to q$ with $p_n \leq q_n$ we have $p \leq q$ as well. Equivalently, this means that

$$J_M^{+}=\{(p,q)\in M\times M \,|\, p\leq q\}\subseteq M\times M \tag{5.6}$$

is a closed subset of $M \times M$.

We consider now the following three properties which will turn out to be sufficient to guarantee the uniqueness of the solutions to (5.5) with future or past compact support.

i.) (M , g) is causal .

ii.) J_M^{+} is closed.

iii.) The time separation $\tau$is finite and continuous.

Concerning the relation among these three properties some remarks are in due:

Remark(5.1.6)(Causally Simple Space Times):

A time-oriented Lorentz manifold (M , g) which satisfies the causality condition i.) is called causally simple if in addition $J_M^{+}(p)$ are closed for all $p \in M$, One can show that this is equivalent to being causal and J_M^{+} being closed which is equivalent to being causal and $J_M^{+}(K)$ being closed for all compact subsets $K \subseteq M$.

Thus i.) And ii.) just say that (M , g) is causally simple.

Remark(5.1.7):

i.) The finiteness of τ clearly implies that there are no timelike loops.

ii.) There are examples of causally simple spacetimes which do not satisfy iii.). So this is indeed an additional requirement.

iii.) Convex spacetimes satisfy all three requirements.

iv.) Also globally hyperbolic spacetimes satisfy all three conditionsWith these conditions we can now prove the following theorem:

Theorem(5.1.8):

Assume that a time-oriented Lorentz manifold (M,g) satisfies the three conditions i.), ii.), iii.). $Let D \in DiffOp^2(E)$ be a normally hyperbolic differential operator on some vector bundle $E \to M$ and $let u \in \Gamma^{-\infty}(E)$ be a distributional section. If u has either past or future compact support and satisfies the homogeneous wave equation

$$Du = 0 \tag{5.7}$$

then $u = 0$.

Corollary(5.1.9):[103]

$Let (M,g)$ be a causally simple Lorentz manifold with finite and continuous time separation. Then for every normally hyperbolic differential operator $D \in DiffOp^2(E)$ there exists at most one fundamental solution at $p \in M$ with past compact support and at most one with future compact support.

Proof. Indeed if $DF = \delta_p = D\widetilde{F}$ then $F - \widetilde{F}$ solves the homogeneous wave equation and has still past (or future) compact support .Thus $F - \widetilde{F} = 0$ by the preceding theorem .Now we pass to a globally hyperbolic space time (M,g) .On one hand we know from Remark 5.1.7 that (M,g) satisfies the hypothesis of Theorem 5.1.8 . On the other hand on a globally hyperbolic space time the sub set $J_M^{\pm}(p)$ are always past/future compact: indeed, by the very definition of global hyper bolicity, $J_M^{+}(p) \cap J_M^{-}(p) = J_M(p,q)$ is a compact diamond for all $p,q \in M$. This is just the statement that $J_M^{+}(p)$ is past compact and $J_M^{-}(p)$ is future compact. This gives immediately the following result:

Corollary (5.1.10):[9]

Let (M,g) be a globally hyperbolic Lorentz manifold. Then for every normally hyperbolic differential operator $D \in DiffOp^2(E)$ there exists at most one advanced and at most one retarded Green function at $p \in M$.

Example(5.1.11)(Uniqueness of Green's Functions):[6]

Let (R^n, η) be the flat Min kowski space time as before. Since this is a globally hyperbolic space time we have the following global and unique Green functions:

i.) The Riesz distributions $R^{\pm}(2)$ are the unique advanced and retarded Green functions for $\Box$ at 0.Their translates to arbitrary $p \in R^n$ are the unique advanced and retarded Green functions for $\Box$ at p.

ii.) The distributions $\widetilde{\Re}^{\pm}(p) = \sum_{k=0}^{\infty} (-m^2)^k R^{\pm}(2+2k, p)$ are the unique advanced and retarded Green functions at $p \in R^m$ of the Klein-Gordon operator $\Box + m^2$ on Minkowski space time,

Finally, we mention that on convex domains we cannot conclude the uniqueness of advanced and retarded Green functions without further assumptions. Even though geodesically convex domains satisfy the hypothesis of Theorem 5.1.8 it may not be true that $J_U^{\pm}(p)$ is past or future compact, respectively. if in this situation we take the Green function $R^{\pm}(2)$ of on (R^n, η) and restrict them to U we obtain advanced and retarded Green functions $\{R^{\pm}(2)(p)|_U\}_{p \in U}$ for all points $p \in U$. Taking now a point $r \in R^n$ and adding $R^{\pm}(2)(r)|_U$ to $R^{\pm}(2)(q)|_U$ we still have an advanced Green function since $R^{\pm}(2)(r) = 0$ on U. However, as sing supp $R^{+}(2)(r) = C^{+}(r)$ by Proposition 3.1.12 for n even, we see that this new advanced Green function differs from $R^{\pm}(2)(q)|_U$ on the intersection $C^{+}(r) \cap U$, even in an essential way. Thus we cannot hope for uniqueness of advanced and retarded Green functions in general.

(5.2) The Cauchy Problem:

In order to pose the Cauchy problem[50, 95] we have to assume that we have a Cauchy hyper surface on which we can specify the initial values[95]. Thus in this section we assume that (M, g) is a globally hyperbolic space time and $\iota : \Sigma \mapsto M$ is a smooth space like Cauchy hyper surface in M whose existence is guaranteed by Theorem 3.7.19**.** Furthermore, the future directed time like normal vector field of Σ will be denoted by $n \in \Gamma^{\infty}(TM|_{\Sigma})$ as in Section 3.8 .

Remark (5.2.1) :

When solving the wave equation $Du = v$ in a distributional sense for $u, v \in \Gamma^{-\infty}(E)$ one might be tempted to ask for the initial conditions of u on Σ. However, since $\iota : \Sigma \mapsto M$ is far from being a submersion the restriction $\iota^* u$ is not at all well-defined. To see the problem one should try to define $\iota^* \delta$ for the δ distribution on R and $\iota : \{0\} \mapsto R$. Thus for the Cauchy problem to make sense we either have to specify conditions on u *and* v which ultimately allow to define $\iota^* u$ etc., or we restrict ourselves directly to regular initial conditions and solutions of some ℓ^k -regularity. As usual, the most convenient situation will be the ℓ^∞ -case.

Given an in homogeneity $v \in \Gamma^\infty(E)$ we want to find a solution $u \in \Gamma^\infty(E)$ of

$$Du = v \tag{5.8}$$

for given initial conditions $u_0, \dot{u}_0 \in \Gamma_0^\infty(\iota^\# E)$, i.e.

$$\iota^\# u = u_0 \tag{5.9}$$

Here ∇^E will always be the covariant derivative on E determined by D as usual. Note that the left hand side of (5.9) is indeed well-defined as for $p \in \Sigma$ the value $\nabla^E_{n(p)} u \in E_p$ is defined as ∇^E is function linear in the tangent vector field argument. Thus we can interpret $p \mapsto \nabla^E_{n(p)} u$ indeed as a section of $\iota^\# E$.

(5.2.1) Uniqueness of The Solution to The Cauchy Problem:

For the Cauchy problem the uniqueness will be easier to show than the existence. We start with some preparatory material on the ad joint D^T of D. Recall from Theorem 2.2.15 that $D^T \in DiffOp^2(E^*)$ is determined by

$$\int_M \varphi(Dv)\mu_g = \int_M (D^T \varphi) v \mu_g \tag{5.10}$$

For $\varphi \in \Gamma^\infty(E^*)$ and $u \in \Gamma^\infty(E)$ with at least one of them having compact support. We want to compute now D^T explicitly.

Lemma(5.2.2):

Let $D \in DiffOp^2(E)$ normally hyperbolic differential operator written as $D = \Box^\nabla + B$ with $B \in \Gamma^\infty(End(E))$ and the connection d'Alembertion $\Box^\nabla$ build out of the connection ∇^E defined by D.

i.) The transported operator $D^T \in DiffOp^2(E^*)$ is given by

$$D^T = \Box^\nabla + B^T \tag{5.11}$$

where $\Box^{\nabla}$ is the connection d'Alembertion with respect to the induced connection ∇^{E^*} for E^* coming from ∇^E

ii.) for $s \in \Gamma^{\infty}(E)$ and $\psi \in \Gamma^{\infty}(E^*)$ we have

$$\Box(\psi(s)) = (\Box^{\nabla}\psi)(s) + \psi(\Box s) + \langle g^{-1}, (D^{E^*}\psi) \vee (D^E s)\rangle \tag{5.12}$$

iii.) for $s \in \Gamma^{\infty}(E)$ and $\psi \in \Gamma^{\infty}(E^*)$ we have

$$(D^T\psi)(s) - \psi(Ds) + = div\left(\left((D^{E^*}\psi)(s) - \psi(D^E s)\right)^{\#}\right) \tag{5.13}$$

Lemma(5.2.3):

Assume $u \in \Gamma^{\infty}(E|_U)$ is a solution to the homogeneous wave equation $D_u = 0$ and let $\varphi \in \Gamma_0^{\infty}(E^*|_U)$ then we have

$$\int_U \varphi(p).u(p)\mu_g(p) = \int_{\Sigma}\left(\left(\nabla_n^{E^*} G_U'(\varphi)\right)u_0(\sigma) - G_U'(\varphi)(\sigma).\dot{u}_0(\sigma)\right)\mu_{\Sigma}(\sigma) \tag{5.14}$$

where $u_0 = \iota^{\#}u$, $\dot{u}_0 = \iota^{\#}\nabla_n^E u \in \Gamma^{\infty}(\iota^{\#}E)$ *are the initial values of* u *on* Σ.

Lemma(5.2.4):

Assume $u \in \Gamma^{\infty}(E|_U)$ is a solution to the homogeneous wave equation $D_u = 0$ and let $u_0, \dot{u}_0 \in \Gamma^{\infty}(\iota^{\#}E)$, denote the initial values of u on Σ,

Then

$$\operatorname{supp} u \subseteq J_U(\operatorname{supp} u_0 \cup \operatorname{supp} \dot{u}_0) \tag{5.15}$$

Theorem (5.2.5) :[121]

Let (M, g) be globally hyperbolic and let *let* $\iota : \Sigma \mapsto M$ be a smooth space like Cauchy hyper surface with future directed normal vector field $n \in \Gamma^{\infty}(\iota^{\#}TM)$. *and Let* $u \in \Gamma^{\infty}(E)$ is a solution to the wave equation $D_u = 0$

Then u is uniquely determined by its initial conditions

$$u_0 = 0 = \dot{u}_0 . \tag{5.16}$$

Then

$$u = 0 \tag{5.17}$$

Theorem 5.2.6:

Let $Let\,(M,g)$ be globally hyperbolic and let $let\ \iota:\Sigma\mapsto M$ be a smooth space like Cauchy hyper surface with future directed normal vector field $n\in\Gamma^{\infty}(\iota^{\#}TM)$. $Let\ v\in\Gamma^{0}(E)$ be a continuous section and $u\in\Gamma^{2}(E)\ a\ \ell^{2}$-section satisfying the inhomogeneous wave equation

$$D_{u}=v. \tag{5.18}$$

Then

u is uniquely determined by its initial conditions $u_0=\iota^{\#}u\ and\ \dot{u}_0=\iota^{\#}\nabla_n^E u\ \ on\,\Sigma$.

(5.2.2) Existence of Local Solutions to The Cauchy Problem:

After the uniqueness we pass to the existence of solutions to the Cauchy problem. We will assume that the Cauchy data as well as the inhomogeneity of the wave equation have compact support.

The first statement is still a local result to the Cauchy problem:

Proposition(5.2.7 [122]:

Let (M,g) be a time-oriented Lorentz manifold with a smooth spacelike hypersur -face $\iota:\Sigma\mapsto M$ with future directed normal vector field n. Moreover, let $U\subseteq U^{c1}\subseteq U'$ be a sufficiently small causal open subset of M such that $\Sigma\cap U\mapsto U$ is a Cauchy hypersurface for U . Then there exists a unique solution $u\in\Gamma^{\infty}(E\big|_U)$ for given initial values $u_0,\dot{u}_0\in\Gamma_0^{\infty}(\iota^{\#}E\big|_U)$ and given inhomogeneity $\upsilon\in\Gamma_0^{\infty}(E\big|_U)$ of the inhomogeneous wave equation

$$Du=\upsilon \tag{5.19}$$

with $\iota^{\#}v=u_0$, $and\ \iota^{\#}\nabla_n^E u=\dot{u}_0$. in addition we have

$$\text{supp } u\subseteq J_M(\text{supp } u_0\cup\text{supp } \dot{u}_0\cup\text{supp } v\ ) \tag{5.20}$$

Proposition(5.2.8):

Let $k\geq 2$. Under the same general assumptions as in proposition 5.2.7 we assume to have initial values $u_0\in\Gamma_0^{2(k+n+1)+2}(i^{\#}E\big|_U)$, $\dot{u}_0\in\Gamma_0^{2(k+n+1)+1}(i^{\#}E\big|_U)$ and inhomogeneity $v\in\Gamma_0^{2(k+n+1)}(E\big|_U)$.

Then there exists a unique solution $u\in\Gamma^{k}(E\big|_U)$ of The inhomogeneous wave equation

$$Du = \upsilon \qquad (5.21)$$

with initial conditions $\iota^{\#} v = u_0$, *and* $\iota^{\#} \nabla_n u = \dot{u}_0$ .for the support we still have

$$\operatorname{supp} u \subseteq J_M (\operatorname{supp} u_0 \cup \operatorname{supp} \dot{u}_0 \cup \operatorname{supp} v \ ) \qquad (5.22)$$

(5.2.3) Existence of Global Solutions to The Cauchy Problem:[41]

To approach the global existence of solutions we assume M is globally hyperbolic with a smooth spacelike Cauchy hypersurface Σ . Now we again use the splitting theorem $\mathrm{M} \cong R \times \Sigma$ with the first coordinate being the Cauchy temporal function and Σ_t the Cauchy hypersurface of constant time t where we shift the origin to $\Sigma_0 = \Sigma$. For every $p \in \mathrm{M}$ we have a unique time t with $p \in \sum_t$ On each Σ_t we have a Riemannian metric g_t such that $g = \beta\, d\, t^2 - g_t$. This allows to speak of the open balls around $P \in \Sigma_t$ of radius r > 0 with respect to this metric g_t . We denote these by $B_r(p)$ without explicit reference to t. Note tha $B_r(p) \subseteq \Sigma_t$ is open in Σ_t but not in M , Here we use the Riemannian distance d_{gt} in Σ_t with respect to g_t for defining the ball, i.e.

$$d_{gt}(p,q) = \inf\left\{\int_a^b gt(\dot{\gamma}(\tau), \dot{\gamma}(\tau)) d\tau \mid \gamma(a) = p, \gamma(b) = q, \gamma(\tau) \in \Sigma_t \right\} \qquad (5.23)$$

where γ is an at least piecewise ℓ^1 curve joining $p,q \in \Sigma_t$ inside Σ_t .

Having such a ball we consider its Cauchy development $D_M(B_r(p)) = D_M^+((p)) \cup D_M^-(B_r(p))$ in M according to Definition 3.7.12, We now want to find r small enough that $D_M(B_r(p))$ is a nice open neighborhood f p allowing a local fundamental solution we call an open neighborhood

a relatively a compact causal open neighborhood of small valume or short RCCSV for abbreviation.

Lemma(5.2.9):

The function $\rho : M \to (0, +\infty]$ defined by

$$\rho(p) = \sup\{r > 0 | D(B_r(p))\, is RCCSV\} \qquad (5.24)$$

is well-defined and lower semi-continuous.

Theorem(5.2.10):

Let (M,g) be a globally hyperbolic spacetime with smooth spacelike Cauchy hyper-surface $\iota:\Sigma\to M$.

i.) $for\ u_0, \dot{u}_0 \in \Gamma_0^\infty(i^\# E)\ and\ v\in\Gamma_0^\infty(E)$, there exists a unique global solution $u\in\Gamma^\infty(E)$ of the inhomogeneous wave equation $Du=v$ with initial conditions

$\iota^\# u=u_0\ and \iota^\# \nabla_n^E u=\dot{u}_0$, We have

$$\operatorname{supp} u \subseteq J_M(\operatorname{supp} u_0 \cup \operatorname{supp}\dot{u}_0 \cup \operatorname{supp} v)$$

ii.) For $k\geq 2$ and $u_0\in\Gamma_0^{2(k+n+1)+2}(i^\# E)$,$\dot{u}_0\in\Gamma_0^{2(k+n+1)+1}(i^\# E)and\ v\in\Gamma_0^{2(k+n+1)}(E)$ there

exists a unique global solution $u\in\Gamma^k(E)$ of the inhomogeneous wave equation $Du=v$ with initial conditions $\iota^\# u=u_0\ and \iota^\# \nabla_n^E u=\dot{u}_0$. It also satisfies (5.22)

(5.2.4) Well- posedness of The Cauchy Problem:

Theorem(5.2.11)(Open Mapping Theorem):

Let $\varepsilon,\tilde{\varepsilon}$ be Fréchet spaces and let

$\phi:\varepsilon\to\tilde{\varepsilon}$ be a continuous linear map. If ϕ is surjective then ϕ is an open map.

As usual, a map ϕ is called open if the images of open subsets are again open

Corollary(5.2.12):

Let $\phi:\varepsilon\to\tilde{\varepsilon}$ be a continuous linear bijection between Fréchet spaces. Then ϕ^{-1} is continuous as well.

Indeed, let $U\subseteq\varepsilon$ be open. Then the set-theoretic $(\phi^{-1})^{-1}(U)$, i.e. the pre-image of U under ϕ^{-1} , coincides simply with $\phi(U)$ which is open by the theorem. Thus ϕ^{-1} is continuous. Note that for general maps between topological spaces a continuous bijective map needs not have a continuous inverse at all.

We are now interested in the following situation: the result of Theorem 5.2.10 can be viewed as a map

$$\Gamma_0^\infty(i^\# E)\otimes\Gamma_0^\infty(i^\# E)\otimes\Gamma_0^\infty(E)\to\Gamma^\infty(E), \tag{5.25}$$

Theorem(5.2.13)(Well-posed Cauchy Problem I):

Let (M,g) be a globally hyperbolic spacetime with smooth spacelike Cauchy hypersurface $\iota:\Sigma\to M$. Then the linear map (5.25) sending the initial conditions and the inhomogeneity to the corresponding solution of the Cauchy problem is continuous.

Theorem(5.2.14)(Well-posed Cauchy Problem II):

Let (M,g) be a globally hyperbolic spacetime with smooth spacelike Cauchy hypersurface $\iota:\Sigma\to M$ and let $k\geq 2$. Then the linear map

$$\Gamma_0^{2(k+n+1)+2}(\iota^{\#}E)\otimes\Gamma_0^{2(k+n+1)+1}(\iota^{\#}E)\otimes\Gamma_0^{2(k+n+1)}(E)\to\Gamma^k(E) \tag{5.26}$$

sending $(u_0,\dot{u}_0,v)$ to the unique solution u of the inhomogeneous wave equation $Du=v$ with initial˙ $\iota^{\#}u=u_0$ $and\,\iota^{\#}\nabla_n^E u=\dot{u}_0$ continuous.

Thus we have in both cases a well-posed Cauchy problem. There are, however, some small drawbacks of the above theorems:

First ,we are limited to inhomogeneities v with compact support in M . Physically more appealing would be an inhomogeneity with compact support only in spacelike direction, i.e. the “eternally moving electron”. Note that this is clearly an intrinsic concept on a globally hyperbolic spacetime. Moreover, the control of derivatives in Theorem 5.2.10 and hence in Theorem 5.2.13 seems not to be optimal. In particular, it would be nice to show that the map (5.26) has some fixed order independent of k.

(5.3) Global Fundamental Solutions and Green's Operators

(5.3.1) Global Green's Functions:*[123, 124]*

We first consider the smooth version. Here we start with the following theorem

Theorem(5.3.1):

Let (M,g) be a globally hyperbolic spacetime and $D\in Diffiop^2(E)$ a normally hyperbolic differential operator. For every point $P\in\mathrm{M}$ there is a unique advanced and retarded fundamental solution $F_M^{\pm}(P)$ $of\ D\ at\ p$. Moreover, for every test section $\varphi\in\Gamma_0^{\infty}(E^*)$ the section.

$$\mathrm{M}\ni P\mapsto F_M^{\pm}(P)\varphi\in E_p^{\pm} \tag{5.28}$$

is a smooth section of E^* which satisfies the equation

$$D^T F_M^{\pm}(.)\varphi = \varphi. \qquad (5.29)$$

Finally, the linear map

$$F_M^{\pm} : \Gamma_0^{\infty}(E^*) \ni \varphi \mapsto F_M^{\pm}(.)\varphi \in \Gamma^{\infty}(E^*) \qquad (5.30)$$

is continuous.

Theorem(5.3.2):

Let (M, g) be a globally hyperbolic spacetime and $D \in Diffiop^2(E)$ a normally hyperbolic differential operator. Then the unique advanced and retarded Green functions $F_{\mathrm{M}}^{\pm}(p)$ of D at p are of global order

$$ord\ F_{\mathrm{M}}^{\pm}(p) \le 2n + 6. \qquad (5.31)$$

More precisely, the linear map (5.30) extends to a continuous linear map

$$F_{\mathrm{M}}^{\pm} : \Gamma_0^{2(k++1)}\ (E^*) \ni \varphi \mapsto F_{\mathrm{M}}^{\pm}(.)\ \varphi \in \Gamma^k\ (E^*) \qquad (5.32)$$

for all k ≥ 2 such that we still have

$$D^T\ F_{\mathrm{M}}^{\pm}(.)\ \varphi = \varphi \qquad (5.33)$$

(5.3.2) Green's Operators:

The fundamental solutions $F_{\mathrm{M}}^{\pm}(p)$ were constructed as the map $\varphi \mapsto (p \mapsto F_{\mathrm{M}}^{\pm}(p)\ \varphi)$ being a map $\Gamma_0^{\infty}\ (E^*) \to \Gamma^{\infty}(E)$, i.e. the solution map from the Cauchy problem. We shall now investigate this map more closely as it provides almost an inverse to D. In general, one defines the following operators.

Definition(5.3.3)(Green's Operators):

Let (M, g) be a time-oriented Lorentz manifold and $D \in Diffop^2(E)$ a normally hyperbolic differential operator. Then a continuous linear map

$$G_{\mathrm{U}}^{\pm} : \Gamma_0^{\infty}(E) \to \Gamma^{\infty}(E) \qquad (5.34)$$

with

i.) $D\,G_{\mathrm{M}}^{\pm} = id\,\Gamma_0^{\infty}(E)$,

ii.) $G_{\mathrm{M}}^{\pm} D\big|_{\Gamma_0^{\infty}} = id\,\Gamma_0^{\infty}(E)$,

iii) $Supp\left(G_{\mathrm{M}}^{\pm}u\right) \subseteq J_{\mathrm{M}}^{\pm}(Suppu)^{c1}\ forall\ u \in \Gamma_0^{\infty}(E)$,

is called an advanced and retarded Green operator for D respectivly .

Proposition (5.3.4)(Green's Operators and Fundamental Solutions):[125]

Let (M, g) be a time-oriented Lorentz manifold and $D \in Diffop^{2}(E)$ a normally hyperbolic differential operator.

i.) Assume $\{G_{\mathrm{M}}^{\pm}(p)\}$ is a family of global advanced or retarded fundamental solutions of D^{T} at every point $P \in \mathrm{M}$ with the following property: for every test section $u \in \Gamma_{0}^{\infty}(E)$ the section $p \mapsto G_{\mathrm{M}}^{\pm}(p)u$ is a smooth section of E depending continuously on u and satisfying $D\, G_{\mathrm{M}}^{\pm}(.)u = u$.

Then

$$(G_{\mathrm{M}}^{\pm} u)(p) = G_{\mathrm{M}}^{\mp}(p)u \tag{5.35}$$

yield advanced or retarded Green operator for D, respectively.

ii.) Assume $G_{\mathrm{M}}^{\pm}$ are advanced or retarded Green operator for D, respectively. Then $G_{\mathrm{M}}^{\pm}(p) : \Gamma_{0}^{\infty}(E) \to C$ defined by

$$(G_{\mathrm{M}}^{\pm})(p)\, u = (G_{\mathrm{M}}^{\mp} u)(p) \tag{5.36}$$

defines a family of advanced and retarded fundamental solutions of D^{T} at every point $P \in \mathrm{M}$ with the properties described in i.), respectively.

Proof. For the first part we assume to have a family $\{G_{\mathrm{M}}^{\pm}(p)\}_{p \in \mathrm{M}}$ of advanced or retarded funda-mental solutions of D^{T} with the above properties. By assumption, the resulting linear map (5.35) is continuous. It satisfies $D\, G_{\mathrm{M}}^{\pm} = id_{\Gamma_{0}^{\infty}(E)}$ also by assumption. Since the $G_{\mathrm{M}}^{\pm}(p)$ are fundamental solutions of D^{T} we have

$$(G_{\mathrm{M}}^{\pm}\, Du)\,(p) = G_{\mathrm{M}}^{\mp}(p)(Du) = (D^{T}\ G_{\mathrm{M}}^{\mp}(p))(u)\ \ \delta_{p}\ = u(p)$$

for all $P \in \mathrm{M}$ and $u \in \Gamma_{0}^{\infty}(E)$. Thus $G_{\mathrm{M}}^{\pm}\ D = id_{\Gamma_{0}^{\infty}(E)}$ as well. Finally, we have to check the support properties thereby explaining the flip from $\pm to \mp$ in (5.35). Thus let $P \in \mathrm{M}$ be given such that $0 \neq (G_{\mathrm{M}}^{\pm} u)(p) = G^{\mp}(p)u$. Since the support of the distributions $G_{\mathrm{M}}^{\mp}(p)$ is in $J_{\mathrm{M}}^{\mp}(p)^{cl}$ this implies that supp u has to intersect $J_{\mathrm{M}}^{\mp}(p)^{cl}$. Since $J_{\mathrm{M}}^{\mp}(p)^{cl} = J_{\mathrm{M}}^{\mp}(p)^{cl}$, and since $Supp\, u$ has an open interior which is non-empty, we see that supp u also has to intersect $I_{\mathrm{M}}^{\mp}(p)$. But then $p \in I_{\mathrm{M}}^{\mp}$ (supp u) whence supp $(G_{\mathrm{M}}^{\pm} u) \subseteq I_{\mathrm{M}}^{\pm}(\text{supp u})^{\text{cl}} = J_{\mathrm{M}}^{\pm}(\text{supp u})^{\text{cl}}$ follows, proving the first part for the second part assume $G_{\mathrm{M}}^{\pm}$ is given and difine $G_{\mathrm{M}}^{\pm}(p) = \delta_{p} \circ G_{\mathrm{M}}^{\mp}$ according to (5.36) This is clearly a distribution since δ_{p} is continuouse and $G_{\mathrm{M}}^{\mp}$ is continuous by assumption . by construction, the section section

$p \mapsto G_M^{\pm}(p)u = (G_M^{\mp}u)(p)$ is smooth and depends continuously on u. We have

$$DG_M^{\mp}(.)u = D(p \mapsto G_M^{\mp}(p)u) = DG_M^{\pm}u = u$$

as well as

$$(D^T G_M^{\mp}(p))(u) = G_M^{\mp}(p)(Du) = (G_M^{\pm}(Du))(p) = u(p),$$

whence $G_M^{\mp}(p)$ is a fundamental solution satisfying also $D\, G_M^{\pm}(.)u = u.$. Finally, for the support we can argue as before in part i.).

Remark(5.3.5)(Green's Operators):

i.) If the causal relation "$\leq$" is closed then the definition of a Green operator simplifies and also the above proof simplifies. This will be the case for globally hyperbolic spacetimes.

ii.) At first glance, a Green operator of D looks like an inverse on the space of compactly supported sections. However, this is not quite correct as $G_M^{\pm}$ maps into $\Gamma^{\infty}(E)$ and not into $\Gamma^{\infty}(E)$. Nevertheless, the Green operator behaves very much like an inverse of $D\big|_{\Gamma_0^{\infty}(E)}$.

iii.) In general, Green operators do not exist: if e.g. M is a compact Lorentz manifold and $D = \Box$ is the scalar d'Alembertian then the constant function 1 has compact support but satisfied $\Box\, 1 = 0$ Thus $G\, \Box\, 1 = 1$ is impossible for a linear map G .

In the case of a globally hyperbolic spacetime our construction of advanced and retarded fundamental solutions in Theorem 5.3.1 gives immediately advanced and retarded Green operators:

Corollary (5.3.6):

On a globally hyperbolic spacetime any normally hyperbolic differential operator has unique advanced and retarded Green operators.

Proof. Indeed, the fundamental solutions were precisely constructed as in the proposition with the operator coming from the solvability of the Cauchy problem in Theorem 5.3.1.Having related the Green operators of D to the fundamental solutions of D^T we can also relate the Green operators of D and D^T directly. First we notice that, the Green operators allow for dualizing:

Proposition(5.3.7):

Let (M,g) be globally hyperbolic and let $D \in Diffop^{2}(E)$ be a normally hyperbolic differential operator with advanced and retarded Green operators $G_{\mathrm{M}}^{\pm} : \Gamma_0^{\infty}(E) \to \Gamma^{\infty}(E)$.

i.) The dual map $(G_{\mathrm{M}}^{\pm})' : \Gamma_0^{-\infty}(E^*) \to \Gamma^{-\infty}(E^*)$ is weak**continuous* and satisfies

$$D^{T}\,(G_{\mathrm{M}}^{\pm})'(\varphi) = \varphi = (G_{\mathrm{M}}^{\pm})'\,D^{T}\,\varphi \tag{5.37}$$

for all generalized sections $\varphi \in \Gamma_0^{-\infty}(E^*)$ with compact support .

ii.) for generalized section $\varphi \in \Gamma_0^{-\infty}(E^*)$ with compact support we have

$$Supp\,(G_{\mathrm{M}}^{\pm})'(\varphi) \subseteq J_{\mathrm{M}}^{\pm}(Supp\,\varphi). \tag{5.38}$$

Lemma(5.3.8):

Let (M,g) be globally hyperbolic and let $D \in Diffop^{2}(E)$ be a normally hyperbolic differential operator with advanced and retarded Green operators $G_{\mathrm{M}}^{\pm}$, Moreover , denote the corresponding Green operator of $D^{T} \in Diffop^{2}(E^*)$ by $F_{\mathrm{M}}^{\pm}$ .Then we have for $\varphi \in \Gamma_0^{\infty}(E^*)$ *and* $u \in \Gamma_0^{\infty}(E)$

$$\int_M \left(F_{\mathrm{M}}^{\pm}\varphi\right).u\,\mu_g = \int_M \varphi.\left(G_{\mathrm{M}}^{\pm}\,u\right)\mu_g\ . \tag{5.39}$$

Theorem (5.3.9):

Let (M,g) be a globally hyperbolic and $D \in Diffop^{2}(E)$ be a normally hyperbolic differential operator. Denote the global advanced and retarded Green operator of D by $G_{\mathrm{M}}^{\pm}$ and those of D^{T} by $F_{\mathrm{M}}^{\pm}$ respectively .

i.) For the dual operators we have

$$\left(G_{\mathrm{M}}^{\pm}\right)'\Big|_{\Gamma_0^{\infty}(E^*)} = F_{\mathrm{M}}^{\pm} \tag{5.40}$$

$$\left(F_{\mathrm{M}}^{\pm}\right)'\Big|_{\Gamma_0^{\infty}(E^*)} = G_{\mathrm{M}}^{\pm} \tag{5.41}$$

ii.) The duals of the Green operators restrict to maps

$$\left(G_{\mathrm{M}}^{\pm}\right)' ; \Gamma_0^{\infty}\left(E^*\right) \to \Gamma^{\infty}\left(E^*\right)\ , \tag{5.42}$$

$$\left(F_{\mathrm{M}}^{\pm}\right)' ; \Gamma_0^{\infty}\left(E\right) \to \Gamma^{\infty}\left(E\right)\ , \tag{5.43}$$

which are continuous with respect to the $\ell_0^{\infty} - and\ \ell^{\infty} - topo\log y, respectively.$

iii.) The Green operators have unique $weak^{*}$continuous extensions to operators

$$G_{\mathrm{M}}^{\pm}; \Gamma_0^{-\infty}(E) \to \Gamma^{-\infty}(E)\ , \tag{5.44}$$

$$F_{\mathrm{M}}^{\pm}; \Gamma_0^{-\infty}(E^{*}) \to \Gamma^{-\infty}(E^{*})\ , \tag{5.45}$$

satisfying

$$Supp\ \left(G_{\mathrm{M}}^{\pm}\ u\ \right) \subseteq\ J_{\mathrm{M}}^{\pm}\ (Supp\ u\ ), \tag{5.46}$$

$$Supp\ \left(F_{\mathrm{M}}^{\pm}\ \varphi\ \right) \subseteq\ J_{\mathrm{M}}^{\pm}\ (Supp\ \varphi\ ), \tag{5.47}$$

respectively. for these extensions one has

$$G_{\mathrm{M}}^{\pm} = \left(F_{\mathrm{M}}^{\pm}\Big|_{\Gamma_0^{\infty}(E^{*})}\right)' \tag{5.48}$$

$$F_{\mathrm{M}}^{\pm} = \left(G_{\mathrm{M}}^{\pm}\Big|_{\Gamma_0^{\infty}(E^{*})}\right)' \tag{5.49}$$

Remark(5.3.10):

With some slight abuse of notation we do not distinguish between the Green's Operators and their canonical extension to generalized sections. This gives the short hand version

$$G_{\mathrm{M}}^{\pm} = \left(F_{\mathrm{M}}^{\pm}\right)' \tag{5.50}$$

of (5.48) and (5.49). In particular, the Green operators of D^{T} are completely determined by those of D and vice versa.

Theorem(5.3.11):

Let (M, g) be a globally hyperbolic spacetime and $D \in Diffiop^{2}(E)$ normally hyperbolic with advanced and retarded Green operators $G_{\mathrm{M}}^{\pm}$.

i.) The Green's Operators $G_{\mathrm{M}}^{\pm}; \Gamma_0^{-\infty}(E) \to \Gamma^{-\infty}(E)$, satisfy

$$DG_{\mathrm{M}}^{\pm} = id_{\Gamma_0^{-\infty}(E)} = G_{\mathrm{M}}^{\pm} D\Big|_{\Gamma_0^{-\infty}(E)} \tag{5.51}$$

ii.) For every $v \in \Gamma_0^{-\infty}(E)$, every smooth spacelike Cauchy hypersurface $\iota : \Sigma \mapsto M$ with

$$Supp\ v \subseteq I_{M}^{+}(\Sigma) \tag{5.52}$$

and all $u_0, \dot{u}_0 \in \Gamma_0^{-\infty}\left(\iota^{\#} E\right)$, there exists a unique generalized section $u \in \Gamma^{-\infty}(E)$, with

$$D u_{+} = v, \tag{5.53}$$

$$Suppu_{+} \subseteq J_{M}\left(Suppu_{0} \cup Supp\dot{u}_{0} \cup J_{M}^{+}\left(Suppv\right)\right) \tag{5.54}$$

$$Sing\ Supp\ u_{+} \subseteq J_{M}^{+}(Supp\ v) \tag{5.55}$$

$$\iota^{\#} u_{+} = u_{0}\ and\ \iota^{\#} \nabla_{n}^{E} u = \dot{u}_{0}. \tag{5.56}$$

The section u_{+} *depends weak* continuously on v and continuously on* $u_0, \dot{u}_0$.

iii.) An analogous statement holds for the case $Supp\ v \subseteq I_{M}^{-}(\Sigma)$.

(5.3.3) The Image of The Green's Operators:[5, 125]

In this section we want to characterize the image of the Green operators $G_{M}^{\pm}\ in\ \Gamma^{\infty}$ in some more detail.

Defintion(5.3.12) (The Space $\Gamma_{sc}^{K}(E)$*):*

Let $k \in N \cup \{+\infty\}$,For a time - oriented Lorentz manifold we denote by $\Gamma_{sc}^{K}(E) \cup \Gamma^{k}(E)$ those section u for which there exists a compact subset $K \subseteq M$ with $Supp\, u \subseteq J_{M}(K)$

We are mainly interested in the globally hyperbolic case. The notion “sc” refers to spacelike compact support. We want to endow the subspace $\Gamma_{SC}^{K}(E) \subseteq \Gamma^{K}(E)$ with a suitable topology analogous to the one of $\Gamma_{0}^{K}(E)$. Indeed, $\Gamma_{SC}^{K}(E)\ is\ dense\ in\ \Gamma^{k}(E)$ for the $e^{\infty} - topolpgy$ as $\Gamma_{0}^{k}(E) \subseteq \Gamma_{SC}^{k}(E) \subseteq \Gamma^{K}(E)$

is already dense. Thus we need a finer topology for $\Gamma_{Sc}^{k}(E)$ to have good completeness properties. Since $J_{M}(K)$ is closed in M on a globally hyperbolic spacetime we can construct a LF topology for $\Gamma_{SC}^{K}(E)$ as follows: For $K \subseteq K'$ we have $J_{M}(K) \subseteq J_{M}(K)$ whence

$$\Gamma_{JM(K)}^{K}(E) \to \Gamma_{JM(K')}^{K}(E) \tag{5.57}$$

is continuous in the $\ell_{JM(k)}^{K}\ and\ \ell_{JM(k')}^{K} - topolpgy$ and we have a closed image

Theorem(5.3.13)(LF Topology for $\Gamma_{SC}^{K}(E)$):

Let (M,g) be a time-oriented Lorentz manifold with closed causal relation and let $K \in N_O \cup \{+\infty\}$. Endow $\Gamma_{SC}^{K}(E)$ with the inductive limit topology coming from (5.57)

i.) $\Gamma_{SC}^{K}(E)$ is a Hausdorff locally convex complete and sequentially complete topological vector space.

ii.) All inclusions $\Gamma_{jm(k)}^{k}(E) \to \Gamma_{sc}^{k}(E)$ are continuous and the $\ell_{sc}^{k} - topo\log y$ is the finest locally convex topology on $\Gamma_{sc}^{k}(E)$ with this property. Every $\Gamma_{Jm\ (k)}^{k}(E)$ is closed in $\Gamma_{jM}^{K}(E)$ and the induced topology from the $\ell_{sc}^{k} - topo\log y$ is again the $\ell_{JM(K)}^{k} - topo\log y$.

iii.) A sequence un $u_n \in \Gamma_{sc}^{K}(E)$ is a ℓ_{sc}^{k} *cauchy sequence* iff there is a compact subset $K \subseteq M$ with un $u_n \in \Gamma_{Jm(k)}^{k}(E)$ and is a ℓ_{JM}^{k} *cauchy sequence* . An analogous statement holds for convergent sequences

iv.) If V is a locally convex vector space then a linear map
$\Phi : \Gamma_{sc}^{k} \to V$ *is* $\ell_{sc}^{k} - continous$ iff all restriction
$\Phi\Big|_{\Gamma_{JM(K)}^{k}} : \Gamma_{JM(K)}^{k}(E) \to V$ *are* $\ell_{JM(K)}^{k} - continous$ iff all exhausting sequence of compacta .

v.) If in addition M is globally hyperbolic with a smooth spacelike Cauchy hypersurface Σ then $\Gamma_{SC}^{K}(E) = \Gamma^{k}(E)$ is compact in which case the ℓ_{SC}^{K} *and* $\ell_{0}^{K} - topolpgycoincide.$,otherwise the $\Gamma_{SC}^{K} - topolpgy$ is strictly finer . in fact

$$\iota^{\#} : \Gamma_{SC}^{K}(E) \to \Gamma_{0}^{K}(\iota^{\#} E) \tag{5.58}$$

is a surjective linear map which is continuous in the ℓ_{SC}^{K} *and* $\ell_{0}^{K} - topolpgy.$. It furthermore has a continuous right inverses.

Remark (5.3.14)(The e_{sc}^{K}-Topology):

We can repeat the discussion of continuous maps also for $e_{SC}^{K} - topop\log y$ in complete analogy to the case of the $e_{0}^{K} - topop\log y$ as in Subsection 2.1.2 and Subsection 2.2.3. In particular, any differential operator $D \in DiFFOp^{K}(E;F)$ of order k gives a continuous linear map $D:\Gamma_{SC}^{k+\ell}(E)\to\Gamma_{SC}^{\ell}(F)$ (5.59)

with respect to the e_{sc}^{K+e}- and the e_{sc}^{ℓ}-topology for all $e\in N_0\cup\{+\infty\}$. we also have approximation theorems resulting The space $\Gamma_{sc}^{\infty}(E)\subseteq\Gamma\infty(E)$ is the natural target space for the Green operators $G_{M}^{\pm}$ since the causality requirement

$$Supp\left(G_{M}^{\pm}(u)\right)\subseteq J_{M}(Supp\ u) \tag{5.60}$$

immediately implies $G_{M}^{\pm}(u)\in\Gamma_{sc}^{K}(E)$ The continuity of $G_{M}^{\pm}$ with respect to the $e\infty$-topology on $\Gamma\infty(E)$ implies also the continuity with respect to the in general strictly finer e_{sc}^{∞} topology

Proposition(5.3.15):

Let (M,g) be a time-oriented Lorentz manifold with closed causal relation. Assume that $G_{M}^{\pm}$ are advanced or retarded Green operators for a normally hyperbolic differential operator $D\in Diffiop^{2}(E)$

$$G_{M}^{\pm}:\Gamma_{0}^{\infty}(E)\to\Gamma_{SC}^{\infty}(E) \tag{5.61}$$

is continuous with respect to the $e_{sc}^{\infty}-and\ e_{0}^{\infty}-topo\log y$.

Proof. We know that $G_{M}^{\pm}:\Gamma_{K}^{\infty}(E)\to\Gamma^{\infty}(E)$ is continuous by definition. Thus let $K\subseteq M$ be compact then $G_{M}^{\pm}:\Gamma_{K}^{\infty}(E)\to\Gamma^{\infty}(E)$ is continuous in the $\ell_{K}^{\infty}-and\ \ell_{0}^{\infty}-topo\log y$ be Theorem 2.1.9, iv.). Since the image is in $\Gamma_{JM(K)}^{\infty}(E)$ and th $\ell_{JM(k)}^{\infty}-topo\log y\ of\ \Gamma_{JM(K)}^{\infty}$ is the subspace topology inherited from $\Gamma^{\infty}(E)$ we have continuity of

$$G_{M}^{\pm}:\Gamma_{K}^{\infty}(E)\to\Gamma_{JM(k)}^{\infty}(E)$$

for all compact subsets $K\subseteq M$. By Theorem 5.3.13, ii.) we conclude that also

$$G_{M}^{\pm}:\Gamma_{K}^{\infty}(E)\to\Gamma_{Sc}^{\infty}(E)$$

is continuous. Since K was arbitrary, by Theorem 2.1.9 , iv.) we have the continuity of (5.61).

then we find the main result of this section which describes the image of the difference of the advanced and the retarded Green operator: as already in the local case we consider the propagator

$$G_{M}=G_{M}^{+}-G_{M}^{-}:\Gamma_{0}^{\infty}(E)\to\Gamma_{SC}^{\infty}(E) \tag{5.62}$$

if $G_{M}^{\pm}$ are advanced and the retarded Green operators for normally hyperbolic differential operator D.

Theorem(5.3.16):

Let (M,g) be a time-oriented Lorentz manifold with closed causal relation. Assume that a normally hyperbolic differential operator $D \in Diffiop^2(E)$ has advanced or retarded Green operators $G_M^{\pm}$.

i.) The sequence of linear maps

$$0 \to \Gamma_0^{\infty}(E) \xrightarrow{D} \Gamma_0^{\infty}(E) \xrightarrow{G_M} \Gamma_{SC}^{\infty}(E) \xrightarrow{D} \Gamma_{SC}^{\infty}(E) \tag{5.63}$$

is a complex of continuous linear maps.

ii.) The complex (5.63) is exact at the first $\Gamma_0^{\infty}(E)$.

iii.) if (M,g) is globally hyperbolic then (5.63) is exact everywhere.

Theorem(5.3.17):

Let (M,g) be a globally hyperbolic spacetime and let $\iota:\Sigma \to M$ be a smooth spacelike Cauchy hypersurface. Let $D \in DiFFOp^2(E)$ be normally

hyperbolic and let $F_M^{\pm}$ be the advanced and retarded Green operators of D^T. Then the solution $u \in \Gamma_{sc}^{\infty}(E)$ of the homogeneous wave equations $D_u = 0$ with initial values $\iota^{\#}u = u_0$ *and* $\iota^{\#}\nabla_n^E u = \dot{u}_0$ *on* Σ is determined by

$$\int_M \varphi(p).u(p)\mu_g(p) = \int_{\Sigma}\Big(\big(\nabla_n^E F_M(\varphi)\big)(\sigma).u_0(\sigma) - F_M(\varphi)(\sigma).\dot{u}_0(\sigma)\Big)\,\mu_{\Sigma} \tag{5.64}$$

for $\varphi \in \Gamma_0^{\infty}(E^*)$.

(5.4) A Poisson's Algebra:

In this section we describe first attempt to establish a Hamiltonian picture for the wave equation based on a certain Poisson algebra of observables coming from the canonical symplectic structure on the space of initial conditions. Throughout this section, (M,g) will be globally hyperbolic. For the vector bundle $E \to M$ we have to be slightly more specific, we choose E to be a real vector bundle.

(5.4.1) Symmetric Differential Operators:

Now we equip the vector bundle E with an additional structure, namely a fiber metric h. In most applications this fibre metric will be positive definite, a fact which we shall not use though. In any case, the fibre metric induces a musical isomorphism $b: E \to E^*$ with inverse $\# : E^* \to E$. then we have

$$b : \Gamma^\infty(E) \ni u \mapsto u^b = h(u\ ,.) \in \Gamma^\infty(E^*). \tag{5.65}$$

Definition(5.4.1)(Symmetric Differential Operators):

Let (E,h) be a real vector bundle with fibre metric and $D \in DiffOp^*(E)$ Then the adjoint of D with respect to h is $D^* \in DiffOp^*(E)$) with

$$\int_M h(D^* u\ ,v)\mu_g = \int_M h(u, Dv)\mu_g \tag{5.66}$$

$for\ all\ u, v \in \Gamma_0^\infty(E).$ The operator D is called Symmetric differential operators

$$D = D^* \tag{5.67}$$

Remark(5.4.2)(Symmetric Differential Operators):[126]

i.) The definition of the adjoint D^* with respect to h is well-defined indeed. Namely, $if\ \ D \in DiffOp^k(E)$ then

$$D^* u = (D^T, u^b)^\# \tag{5.68}$$

with the adjoint operator $D^T \in DiFFOp^K(E^*)$ from Theorem 2.2.15. This follows from the simple computation

$$\int_M h((D^T u^b)^\#, v)\mu_g = \int_M (D^T u^b)\ .\mu_g = \int_M u^b\ ..Dv\ \mu_g = \int_M h(u, Dv)\mu_g, \tag{5.69}$$

which shows that (5.68) solves the condition (5.66). It is clear that D^* is again a differential operator of the same order as D and it is necessarily unique since the inner product is non-degenerate.

ii.) The adjoint D^* depends on h but also on the density μ_g in the integration (5.66). The map $D \mapsto D^*$ is a linear involutive anti-automorphism, i.e. we have

$$(D^*)^* = D\ and\ \ (D\tilde{D})^* = \tilde{D}^* D^* \tag{5.70}$$

$for\ \ D, \tilde{D} \in DiffOp^*(E),$

iii.) In the case of a complex vector bundle one proceeds similarly: for a given (pseudo -) Hermitian fibre metric one defines the adjoint D^* by the same condition (5.66). Now $D \mapsto D^*$ is antilinear in addition to (5.70) and $DiFFOp^\bullet(E)$ becomes $a^* -$ algebra over C by this choice. Differential operators with $D = D^*$ are now called Hermitian. A particular case is obtained for a complexified vector bundle $E_C = E \otimes C$. If h is a fibre metric on E then it induces a Hermitian fibre metri on E_C by setting

$$h_C(u \otimes z, v \otimes w) = h\ (u,v)\,\bar{z}\,w \tag{5.71}$$

for $u, v \in E_p$ *and* $z, w \in C$. . Then a symmetric operator $D \in DiFFOp^{\bullet}(E)$ yields a Hermitian operator $D_C \in DiffOp^{\bullet}(E_C)$ which commutes in addition with the complex conjugation of sections.

Proposition(5.4.3)(Symmetry of Green's Operators):

Let (M, g) be globally hyperbolic and let $D \in DiffOp^{2}(E)$ be a normally hyperbolic Differential operators on the real vector bundle E.Assume that D is Symmetric with respect to fiber metric h *on* E.

i.) For the Green's operators of D *and* D^T *and* $u \in \Gamma_0^{\infty}(E)$ we have

$$(G_{\mathrm{M}}^{\pm}\, u)^{b} = F_{\mathrm{M}}^{\pm}\, u^{b} \tag{5.72}$$

ii.) For $u, v \in \Gamma_0^{\infty}(E)$ we have

$$\int_{\mathrm{M}} h(u, G_{\mathrm{M}}^{\pm} v)\,\mu_g = \int_{\mathrm{M}} h(G_{\mathrm{M}}^{\pm}\, u, v)\,\mu_g\,. \tag{5.73}$$

iii.) The Green's Operators of The Canonical [125]

$C-$*linear extension of* D *to* $E_C = E \otimes C$ *are coninical* $C-$*linear extension of the Green operator* $G_M^{\pm}$ *of* D

They still satisfy (5.72) ,

$$\int_{\mathrm{M}} h_C\,(u, G_{\mathrm{M}}^{\pm}\, v)\mu_g = \int_{\mathrm{M}} h_C\,(G_{\mathrm{M}}^{\pm} u, v)\mu_g \tag{5.74}$$

for $u, v \in \Gamma_0^{\infty}(E_C)$ and the reality condition

$$\overline{G_{\mathrm{M}}^{\pm} u} = G_{\mathrm{M}}^{\pm}\, \bar{u} \tag{5.75}$$

Proof. Clearly, $u \in \Gamma_0^{\infty}(E)$ has compact support iff u^{b} has compact support, making (5.72) meaningful. We comput for $\varphi \in \Gamma_0^{\infty}(E^{*})$

$$D^T (G_{\mathrm{M}}^{\pm}\, \varphi^{\#})^{b} \overset{(5.68)}{=} \left(D\, G_{\mathrm{M}}^{\pm}\, \varphi^{\#}\right)^{b} = (\varphi^{\#})^{b} = \varphi,$$

since $G_{\mathrm{M}}^{\pm}$ is a Green's operator of $G_{\mathrm{M}}^{\pm}$. Analogously,

$$\left(G_{\mathrm{M}}^{\pm}\left(D^{T}\varphi\right)^{\#}\right)^{b} \overset{(5.68)}{=} \left(G_{\mathrm{M}}^{\pm}\, D\, \varphi^{\#}\right)^{b} = (\varphi^{\#})^{b} = \varphi,$$

Now $\varphi \mapsto (G_{\mathrm{M}}^{\pm}\, \varphi^{\#})^{b}$ is clear linear and continuous since $(\#, b)$, as well as $G_{\mathrm{M}}^{\pm}$ are continuous. Finally,since $\#$ *and* b preserve supports we have supp

$(G_{\mathrm{M}}^{\pm}\, \varphi^{\#})^{b} \subseteq J_{\mathrm{M}}^{\pm}(Supp\ \varphi)$ This shows that the map $\varphi \mapsto (G_{\mathrm{M}}^{\pm}\, \varphi^{\#})^{b}$ is indeed an advanced and retarded Green operator for D^T , respectively. By uniqueness according to Corollary 5.3.6 we get (5.73). Using this, we compute

$$\int_M h(u, G_{\mathrm{M}}^{\pm} v)\,\mu_g = \int_M u^b \,.\, \left(G_{\mathrm{M}}^{\pm}\, v\right)\mu_g$$

$$\overset{(5.39)}{=} \int_M \left(F_{\mathrm{M}}^{\pm}\, u^b\right).v\,\mu_g \quad \overset{(5.73)}{=} \int_M \left(G_{\mathrm{M}}^{\pm}\, u\right)^b .v\,\mu_g$$

$$= \int_M h\left(G_{\mathrm{M}}^{\pm}\, u, v\right)\mu_g$$

for $u, v \in \Gamma^{\infty}(E)$. Now consider $u, v \in \Gamma_0^{\infty}(E_C)$. Then $\overline{D u} = D\,\overline{u}$ yields the hermiticity $D = D^*$ with respect to h_C. With the same kind of uniqueness argument we see that the Green operators $G_{\mathrm{M}}^{\pm}$ *of* D canonically extended to $G_M^{\pm} : \Gamma_0^{\infty}(E_C) \to \Gamma^{\infty}(E_C)$, yield the Green operators of the extension

$D \in DiffOp^2(E_C)$. Moreover, we clearly have (5.75) by construction. But then (5.74) follows from (5.76) and (4.4.73) at once.

Remark(5.4.4):

Extending our notation of the adjoint to more general operators we can rephrase the result of (5.73) or (5.74) by saying

$$\left(G_{\mathrm{M}}^{\pm}\right)^* = G_{\mathrm{M}}^{\pm} \tag{5.76}$$

Note that Proposition 5.4.3, iii.) still holds for arbitrary Hermitian $D = D^*$ on arbitrary complex vector bundles except for (5.75). In both cases, it follows that the propagator $G_M = G_M^+ - G_M^-$ is antisymmetric

$$G_M^* = -G_M \tag{5.77}$$

or anti-Hermitian in the complex case, respectively. In the complex case we can rescale G_M by i to obtain a Hermitian operator

$$\left(i\,G_M\right)^* = i\,G_M \;. \tag{5.78}$$

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