Chapter One

(1-1) History of Quantum Mechanics

In 1900 Max Planck introduced the concept of quantum energy. He argued that the energy exchange between an electromagnetic wave of certain frequency and matter occurs only in integer multiples of a quantities which is proportional to the frequency, this energy is called quanta [1,2].

In 1905 Einstein provided a powerful consolidation to Blanck's quantum concept. In trying to understand the photoelectric effect, he posited that the light is made of discrete bits of energy called photons [3,4].

After discovering atomic nucleus by Rutherford's experimental in 1911, and combining Rutherford's atomic model, Blanck's quantum concept, and Einstein's photons, Bohr introduced in 1913 his model of the hydrogen atom. Compton made an important discovery in 1923 that gave the most conclusive confirmation for the corpuscular aspect of light. By scattering X-rays with electrons, he confirmed that the X- ray photons behave like. De Broglie introduced in 1923 another powerful new concept that classical physics could not reconcile: he postulate that not only does radiation exhibit particle like behavior but, conversely, material particles themselves display wave-like behavior. This concept was confirmed experimentally in 1927 by Davisson and Germer; they showed that interference patterns, a property of waves, can be obtained with material particles such as electrons [5,6,7].

Heisenberg present first formulation called matrix mechanics to develop atomic structure in 1925, the second formulation, called wave mechanics, was due to Schrödinger 1926. In 1927 Max Born proposed his probabilistic interpretation of wave mechanics: he took the square moduli of wave functions that are solutions of Schrödinger equation and he interpreted them as probability density [8,9,10,11].

Combining special relativity with quantum mechanics, Dirac derived in 1928 an equation which describes the motion of electrons [5,12,13].

(1-2) Research Problem

The research problem is related to the fact that the effect of vacuum on atomic electrons is not widely studied. The behavior of bulk matter on atomic electrons does not studied within the frame work of fluid mechanics by treating bulk matter as a viscous fluid.

(1-3) Literature Review

Different attempts were made to account for the effect of vacuum or bulk matter on free or atomic electrons. They either adds a potential term or friction term or vacuum term to quantum equations [14,15,16,17].

In some of these attempts Maxwell's equations having damping term was used to derive the quantum mechanics. The damping term in the quantum equations recognize the effect of the medium on the particle behavior [18,19].

Other attempt relies on the ordinary expression for harmonic oscillator subjected to friction. The friction energy is added to the ordinary energy expression. Then by treating a particles as a wave packet the new modified Schrödinger equation is derived

[20,21,22,23].

(1-4) The Aim of Work

The aim of the present study is to modify energy of some quantum particles in Hydrogen atom, rigid body, free particle and harmonic oscillator using Stock's force to drive an expression for energy lost by the particles due to viscosity. This expression of energy is used to derive quantum equation that accounts for the effect of viscosity.

(1-5) Thesis Content

In this thesis, the introduction is in chapter one. Chapter two contain basic theory of quantum mechanics like De Broglie relation, uncertainty principle, wave packet, operator and Schrödinger equation, it also contain application of quantum theory like Hydrogen atom, rigid rotator body, free particle and harmonic oscillator. The details of modified Schrödinger equation which accounts for energy loses due to viscous media, are in chapter three. Chapter four contains discussion, conclusion and suggestions for further works.

Chapter Two

Quantum Mechanical Laws and Applications

(2-1) Introduction

The behavior of atomic and subatomic particles can be described by the laws of quantum mechanics. This means derivation of Schrödinger equation simple applications for physical systems were needed. All these will be done in this chapter.

(2-2) The De Broglie Relation

At the beginning of the 20^{th} century, experimentation revealed that electromagnetic radiation has particle-like properties (as an example, photoelectric and Compton’s effect), and as a result, it was theorized that all particles must also have wavelike properties (electron diffracted).

De Broglie summarized the universal *duality* of particles and waves; he proposed all matter has an associated wave with a wavelength that is inversely proportional to the momentum, p, of the particle (verified experimentally by Davison and Germer) [24]

$$p = \frac{h}{\lambda} \tag{2.2.1}$$

The constant of proportionality, h, is Planck's constant.

The de Broglie relation fuses the ideas of particle-like properties (i.e. momentum) with wave-like properties (i.e. wavelength). The de Broglie relationship also begins to hint at the idea of quantization in mechanics. If a particle is in an orbit, the only allowed radii and momenta are those where the waves associated with the particle will interfere non-destructively as they wrap around each orbit [25].

(2-3) The Uncertainty Principle

The formulation of classical mechanics implies that the position and momentum of a particle are assumed to have well defined values and can be determined simultaneously with perfect accuracy. But the wave particle duality compels us to a band on the idea of simultaneous determination of position and momentum with perfect accuracy [26].

In his 1927 paper Heisenberg introduced the concept that position and momentum could not be measured with unlimited precision. The principle may be stated mathematically as

$$\Delta x \Delta p \geq \hbar \qquad (2.3.1)$$

where Δx and Δp are the uncertainties in the measurements of position and momentum, respectively.

Similarly if ΔE is the uncertainty in energy and Δt is uncertainty in time then

$$\Delta E \Delta t \geq \frac{\hbar}{2} \qquad (2.3.2)$$

It is important to understand that the uncertainties, Δx, Δp or ΔE and Δt, are not the result of any flow in our measurement technique or apparatus. They are consequences of the wave–particle duality that, as we have seen above, is inherent in nature [27].

These principles indicate that, although it is possible to measure for example momentum or position of a particle accurately, it is not possible to measure these two observables simultaneously to an arbitrary accuracy. If we try to measure the position of particle with almost accuracy *i.e.*, $\Delta x \to 0$, the corresponding uncertainty in momentum becomes very large *i.e.*, $\Delta p \to 0$ and vice versa [28].

(2-4) Wave Packet

In classical physics, a particle well localize in space, for its position and momentum can be calculated simultaneously to arbitrary precision. As for quantum mechanics, it describes a material particle by a wave function corresponding to the matter wave associated with the particle (de Broglie's conjecture). Wave functions, however, depend on the whole space, hence they cannot be localized. To localize the wave function around the region of space within which the particle is confined uses the wave packet. It is formed by superposition of a large number of waves of different wave numbers, with phases and amplitudes so chosen that they interfere constructively over small region of space and destructively elsewhere. Mathematically, we can construct the packet by superposing plan waves of different frequencies by means of Fourier transforms

$$\psi(r,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi(k)e^{i(kr-\omega t)}\,dk \qquad (2.4.1)$$

$\varphi(k)$ is the amplitude of the wave packet, and the inverse

Fourier transform is

$$\varphi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(r,t)e^{-i(kr-\omega t)}\,dr \qquad (2.4.2)$$

This means the particle in quantum mechanics is represented not by a single de Broglie wave of well-defined frequency and wave length, but by a wave packet that is obtained by adding a large number of waves of different frequencies [27].

(2-5) Wave Function

In quantum mechanics the motion of a particle can be described by using a wave function $\psi(r,t)$, the wave function is a mathematical complex form to describe the quantum particle (wave-particle) and it has all information about its motion [29]. But the wave function cannot be measured directly because it is complex quantity and have no physical meaning, then we cannot use $\psi(r,t)$ to represent the particle directly, easiest way to guarantee a real value is to measure its intensity $\psi_i^*\psi_j = |\psi|^2$or called probability density, hence $|\psi(r,t)|^2d^3r$ is evidently a probability density (of dimensions probability/ length), which means a probability of finding particle at position r in volume element d^3r of space at time t and must be integrated to obtain a finite probability, then the probability density is a physical meaning of $\psi(r, t)$ [30].

The wave function is a solution of Schrödinger equation must satisfy the conditions: (i) they must be single-valued, continuous and differentiable at every point in space.

(ii) they must be finite for all values $of\, x, y$ and z, (iii) the wave functions must be normalized i.e. $\int_{-\infty}^{\infty} |\psi|^2 d^3r = 1$, (iv) any two solutions to the wave equation must be orthogonal $\int \psi_i^* \, \psi_j d^3r = 0$. The orthonormality condition requires $\int \psi_i^* \, \psi_j d^3r = \delta_{ij}$ where δ_{ij} is Kronecker or Dirac delta function and $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$ [31].

(2-6) Operator

An operator is a rule or mathematical operation which transforms a function to another function.

Every observable or dynamical variable (energy, momentum, angular momentum and position coordinates) in quantum mechanics is represented by a linear Hermitian operator [31].

The role of linearity properties of operator is transform a function to other function and satisfy the linearity condition like distributive property

$$\hat{A}(a\psi + b\Omega) = a\hat{A}\psi + b\hat{A}\Omega \qquad (2.6.1)$$

In other word, linearity means that an operator has no effect on constants (numbers), and that its action on states obeys the distributive property [32].

The most important of all linear operators in quantum physics is the Hermitian operator. Because for Hermitian operator the possible results of a measurement (eigen values) are real.

Quite simply, an operator is Hermitian if it is equal to its Hermitian conjugate.

That is, $\hat{A}$ is Hermitian if [27]

$$\int \psi_i^* \hat{A}\psi_j d^3r = \int \hat{A}^* \psi_i^* \psi_j d^3r \qquad (2.6.2)$$

<u>(2-7) Schrödinger Equation</u>

We know the information about the state of a particle is described by a wave function; the equation that describes its time evolution is called the Schrödinger equation.

This equation is a homogenous liner in $\psi(x, t)$, but it is first-order in the time derivative, which differ from other equations of motion.

Consider a particle of mass m subject to a potential $V(r, t)$ is described by the following partial differential equation:

$$i\hbar\frac{\partial\psi(r,t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi(r,t) + V(r,t)\psi(r,t) \qquad (2.7.1)$$

This equation is known as the time-dependent Schrödinger equation of a particle in three dimensions, but in case of a potential independent of time $V(r)$ equation (2.7.1) become

$$-\frac{\hbar^2}{2m}\nabla^2\psi(r) + V(r)\psi(r) = E\psi(r) \qquad (2.7.2)$$

The solution of this equation is stationary solution or stationary state, because its probability density is independent of time. A particle whose state is described by wave function [33]

$$\psi(r,t) = \psi(r)e^{-i\omega t} \qquad (2.7.3)$$

The time-dependent Schrödinger equation in polar coordinate is [34]

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi(r,\theta,\phi)$$

$$+\frac{2m}{\hbar^2}(E-V)\psi(r,\theta,\phi)=0 \tag{2.7.4}$$

Therefore the time independent Schrödinger equation has the form of an eigen value equation:

$$\hat{H}\psi(r)=E\psi(r) \tag{2.7.5}$$

Where the eigen values of operator $\hat{H}$ are the possible measurements of energies E and $\psi(r)$is the eigen functions of$\hat{H}$.

The expectation value of energy measurements or the average energy eigen values is

$$\langle\hat{H}\rangle=\int\psi(r)^*\hat{H}\psi(r)\,d^3r \tag{2.7.6}$$

By above integration can get expectation value of any dynamic variable [35].

(2-8) Hydrogen Atom

Hydrogen atom is a system consisting of electron and proton moving under their mutual interaction.

The Schrödinger equation for the system is

$$\nabla^2\psi+\frac{2\mu}{\hbar^2}(E-V)\psi=0 \tag{2.8.1}$$

The task of finding the solution of Schrödinger equation becomes easier in spherical polar coordinates. The Schrödinger equation in polar coordinates is

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2 sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi(r,\theta,\phi)$$

$$+\frac{2\mu}{\hbar^2}(E-V)\psi(r,\theta,\phi)=0 \qquad (2.8.2)$$

Multiplying Eqn. (2.8.2) by $r^2 sin^2\theta$ we obtain

$$\left[sin^2\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+sin\theta\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)+\frac{\partial^2}{\partial\phi^2}\right]\psi(r,\theta,\phi)+$$

$$\frac{2\mu r^2 sin^2\theta}{\hbar^2}(E-V)\psi(r,\theta,\phi)=0 \qquad (2.8.3)$$

Assume the wave function $\psi(r,\theta,\phi)$ can be written as the product of functions R(r),Θ(θ),Φ(ϕ)

$$\psi(r,\theta,\phi)=R(r),\Theta(\theta),\Phi(\phi) \qquad (2.8.4)$$

Substitute equation (2.8.4) in equation (2.8.3)

$$\Theta\Phi sin^2\theta\frac{\partial}{\partial r}\left(r^2\frac{dR}{dr}\right)+R\Phi sin\theta\frac{\partial}{\partial\theta}\left(sin\theta\frac{d\Theta}{d\theta}\right)+R\Theta\frac{d^2\Phi(\phi)}{d\phi^2}$$

$$+\frac{2\mu r^2 sin^2\theta}{\hbar^2}(E-V)R\Theta\Phi=0 \qquad (2.8.5)$$

Divided equation (2.8.5) by $R(r)\Theta(\theta)\Phi(\phi)$

$$\frac{sin^2\theta}{R}\frac{\partial}{\partial r}\left(r^2\frac{dR}{dr}\right)+\frac{sin\theta}{\Theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}$$

$$+\frac{2\mu r^2 sin^2\theta}{\hbar^2}(E-V)=0 \qquad (2.8.6)$$

Or

$$\frac{sin^2\theta}{\mathrm{R}}\ \frac{\partial}{\partial r}\left(r^2\frac{d\mathrm{R}}{dr}\right)+\frac{sin\theta}{\Theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{d\Theta}{d\theta}\right)++\frac{2\mu r^2 sin^2\theta}{\hbar^2}\ (E-V)=$$

$$-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} \qquad (2.8.7)$$

Left hand side of this equation is function of r and θ whereas the right hand side is function of ϕ only. This equality can hold only if each side is equal to the same constant. Let the separation constant denoted by m_l^2.

Thus, we have

$$\frac{d^2\Phi(\phi)}{d\phi^2}=-m_l^2\Phi(\phi) \qquad (2.8.8)$$

And

$$\frac{sin^2\theta}{\mathrm{R(r)}}\ \frac{\partial}{\partial r}\left(r^2\frac{d\mathrm{R(r)}}{dr}\right)+\frac{sin\theta}{\Theta(\theta)}\frac{\partial}{\partial\theta}\left(sin\theta\frac{d\Theta(\theta)}{d\theta}\right)+\frac{2\mu r^2 sin^2\theta}{\hbar^2}\ (E-V)=m_l^2 \qquad (2.8.9)$$

Dividing Eqn. (2.8.9) by $sin^2\theta$ and transferring the r-dependent terms on left hand side and θ-dependent terms on the right hand side of equality sign, we have

$$\frac{1}{\mathrm{R(r)}}\ \frac{\partial}{\partial r}\left(r^2\frac{d\mathrm{R(r)}}{dr}\right)+\frac{2\mu r^2}{\hbar^2}(E-V)$$

$$= \frac{m_l^2}{sin^2\theta} - \frac{1}{\Theta(\theta)sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\,\frac{d\Theta(\theta)}{d\theta}\right) \qquad (2.8.10)$$

Again the equality of two functions of independent variables demands that each side be equal to the same constant. The equations obtained by equating both sides to a constant were already solved in classical physics where the separation constant was chosen of the form $\ell(\ell+1)$ and the two equations obtained were

$$\frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) + \frac{2\mu r^2}{\hbar^2}(E - V) = \ell(\ell + 1) \qquad (2.8.11)$$

$$\frac{m_l^2}{sin^2\theta} - \frac{1}{\Theta(\theta)sin\theta}\frac{d}{d\theta}\left(sin\theta\,\frac{d\Theta(\theta)}{d\theta}\right) = \ell(\ell + 1) \qquad (2.8.12)$$

After solved Eqn. (2.8.8), its wave function is

$$\Phi(\phi) = \frac{1}{\sqrt{2\pi}}e^{im_l\phi} \qquad (2.8.13)$$

And the wave function of Eqn. (2.8.12) after solved is

$$\Theta(\theta) = (-1)^m\sqrt{\frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}}P_l^m(x) \qquad (2.8.14)$$

The total angular wave function is obtained as

$$Y_{lm} = (-1)^m\sqrt{\frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}}P_l^m(x).\frac{1}{\sqrt{2\pi}}e^{im_l\phi} \qquad (2.8.15)$$

Now we solve radial equation (2.8.11)

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right)+\frac{2\mu}{\hbar^2}\left(E-V-\frac{\ell(\ell+1)\hbar^2}{2\mu r^2}\right)R(r)=0 \qquad (2.8.16)$$

Notice that an extra term $\frac{\ell(\ell+1)\hbar^2}{2\mu r^2}$ appears as addend in potential energy V(r). It is often called centrifugal potential energy since its negative gradient is equal to the centrifugal force experienced by the particle while moving in a circular orbit of radius r. The kinetic energy associated with the rotational motion is $\frac{L^2}{2I}$ $=\frac{\ell(\ell+1)\hbar^2}{2\mu r^2}$. So this term may be interpreted as the centrifugal energy of the particle. Also we substitute the electrostatic potential energy of the system $V = \frac{-Ze^2}{4\pi\varepsilon_0 r}$ and for bound state, the energy E is negative so the radial equation becomes

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\mathrm{R(r)}}{dr}\right)+\left(\frac{2\mu(-E)}{\hbar^2}+\frac{2\mu Ze^2}{4\pi\varepsilon_0\hbar^2 r}-\frac{\ell(\ell+1)}{r^2}\right)\mathrm{R(r)} = 0 \qquad (2.8.17)$$

Equation (2.8.17) can be written in a convenient form making use of transformation

$$\rho = \alpha r \qquad (2.8.18)$$

where α is constant and is so chosen that resulting equation look simpler. When Eqn. (2.8.18) is substituted in Eqn. (2.8.17), we get

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)+\left(\frac{-2\mu E}{\hbar^2\alpha^2}+\frac{2\mu Ze^2}{4\pi\varepsilon_0\hbar^2\alpha\rho}-\frac{\ell(\ell+1)}{\rho^2}\right)R(r)=0 \qquad (2.8.19)$$

Now α is chosen to make the first term in square bracket equal to 1/4. So

$$\alpha=\sqrt{\frac{-8\mu E}{\hbar^2}} \qquad (2.8.20)$$

In the second term in square bracket the coefficient of 1/ ρ is put equal to ¥

$$¥=\frac{Ze^2}{4\pi\hbar\varepsilon_0}\sqrt{\frac{\mu}{-2E_0}} \qquad (2.8.21)$$

Equation (2.8.19) now becomes

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)+\left(\frac{¥}{\rho}-\frac{1}{4}-\frac{\ell(\ell+1)}{\rho^2}\right)R(r)=0 \qquad (2.8.22)$$

For large ρ, the first term of Eqn. (2.8.22) reduces to $\frac{d^2R}{d\rho^2}$ and Eqn. (2.8.22) becomes

$$\frac{d^2R}{d\rho^2}-\frac{1}{4}R=0 \qquad (2.8.23)$$

The solution of Eqn. (2.8.23) is

$$R(\rho)=e^{\pm\rho/2}$$

Positive sign leads to an unacceptable solution. So we choose the negative sign.

$$R(\rho) = e^{-\rho/2} \tag{2.8.24}$$

To determine the nature of solution near origin (for small ρ) we put

$$R(\rho) = \frac{1}{\rho}\mathrm{F}(\rho) \tag{2.8.25}$$

Where F(ρ) is unknown function. Substituting Eqn. (2.8.25) in (2.8.22), we have

$$\frac{d^2\mathrm{F}(\rho)}{d\rho^2} + \left(\frac{\yen}{\rho} - \frac{1}{4} - \frac{\ell(\ell+1)}{\rho^2}\right)\mathrm{F}(\rho) = 0 \tag{2.8.26}$$

For $\ell=1$ ($\ell \neq 0$) the last term $\frac{\ell(\ell+1)}{\rho^2}$ is large near origin ($\rho \to 0$), Eqn. (2.8.26) reduces to

$$\frac{d^2\mathrm{F}(\rho)}{d\rho^2} - \frac{\ell(\ell+1)}{\rho^2}\mathrm{F}(\rho) = 0 \tag{2.8.27}$$

Solution of Eqn. (2.8.27) can be obtained in form

$$\mathrm{F}(\rho) = constant\ \rho^s \tag{2.8.28}$$

Substituting Eqn. (2.8.28) in Eqn. (2.8.27), we have

$$s(s-1) - \ell(\ell+1) = 0 \tag{2.8.29}$$

This gives $s = -\ell$ or $s = \ell + 1$

$R(\rho) = \frac{1}{\rho} F(\rho) = \frac{\rho^s}{\rho} = \rho^{s-1} = \rho^{-\ell-1}$ or ρ^{ℓ}

Near origin $\rho \to 0, \frac{1}{\rho^{\ell+1}} \to \infty$ this is not acceptable.

However near origin $\rho \to 0$, $\rho^{\ell} \to 0$, this is acceptable. So we can assume the solution of Eqn. (2.8.22) of the form

$$R(\rho) = e^{-\rho/2}.\rho^{\ell}.L(\rho) \qquad (2.8.30)$$

Where L(ρ) is unknown function to be determined.

Substituting Eqn. (2.8.30) in Eqn. (2.8.22) , we get

$$\rho \frac{d^2 L(\rho)}{d\rho^2} + [2(\ell + 1) - \rho] \frac{dL(\rho)}{d\rho} + [¥ - (\ell + 1)]L(\rho) = 0 \qquad (2.8.31)$$

Let us assume the power series solution of Eqn. (2.8.31) of the form

$$L(\rho) = a_0 + a_1\rho + a_2\rho^2 + \cdots = \sum_{r=0}^{\infty} a_r \rho^r \qquad (2.8.32)$$

Substituting Eqn. (2.8.32) in (2.8.31) and equating the coefficient of ρ^r equal to zero, we get

$$a_{r+1} = \frac{r+\ell+1-¥}{(r+1)(r+2\ell+2)} a_r \qquad (2.8.33)$$

For large value of r

$$\frac{a_{r+1}}{a_r} \approx \frac{1}{r} \quad as\, r \to \infty$$

The ratio of successive coefficients of series

$$e^{\rho} = 1 + \rho + \frac{\rho^2}{2!} + \cdots + \frac{\rho^r}{r!} + \frac{\rho^{r+1}}{(r+1)!} + \cdots$$

Is $\frac{b_{r+1}}{b_r} = \frac{1}{r+1} \approx \frac{1}{r}\ as\ r \to \infty$

So for large value of ρ the function L(ρ) behaves like e^{ρ}.

Hence

$$R(\rho \to \infty) = e^{-\rho/2}.\rho^{\ell}.e^{\rho} \to \quad e^{\rho/2}.\rho^{\ell} \to \infty$$

This form of R(ρ) is not acceptable. If infinite series L(ρ) terminates after finite number of terms *i.e.*, it becomes a polynomial, then it will be valid solution of Eqn. (2.8.31) . From the recursion relation (2.8.33) we see that this requirement is met if r reaches some integer, say n' (= r), given by

n' +(ℓ+1)- ¥= 0

Or ¥= n' +(ℓ+1)

then a_{r+1} (= $a_{.n'.+1}$) and all higher coefficients become zero and L(ρ) becomes a polynomial of degree n'. Since n' is a non-negative integer, so is ¥. The integer ¥ is denoted by *n* and is called principal quantum number

$$n = ¥ = n' + (\ell + 1) \qquad (2.8.34)$$

Since $n' > 0$

$n - \ell - 1 > 0$

$$\ell < n - 1 \qquad (2.8.35)$$

This puts restriction on the values that ℓ can take on for a given value of n. Putting the value of ¥ in (2.8.21), we have

$$E = \frac{-1}{2}\left[\frac{1}{4\pi\varepsilon_0}\right]^2 \frac{\mu e^4}{\hbar^2}\frac{Z^2}{n^2} \qquad (2.8.36)$$

The wave function

$$R_{nl}(\rho) = -\left(\frac{2Z}{na_0}\right)^{3/2}\sqrt{\frac{(n-l-1)!}{2n[(n+l)!]^3}}\, e^{-\rho/2}\rho^{\ell} \qquad (2.8.37)$$

The total wave function is [26]

$$\psi(r,\theta,\phi) = -\left(\frac{2Z}{na_0}\right)^{3/2}\sqrt{\frac{(n-l-1)!}{2n[(n+l)!]^3}}\, e^{-\rho/2}\rho^{\ell}\, Y_{lm} \quad (2.8.38)$$

(2-9) Bohr Model's

In 1913, Bohr proposed an atomic model, which explained with amazing accuracy the main features of the spectra of hydrogenic atoms. His model was based on the following postulates:

(*i*) The electron in hydrogen atom moves in circular orbit around the nucleus. The dynamics of the electron is governed by the Newtonian mechanics *i.e.*, the centripetal force required for circular motion is provided by Coulomb attraction of nucleus on the electron [36].

$$\frac{\mu v^2}{r} = \frac{Ze^2}{4\pi\varepsilon_0 r^2} \tag{2.9.1}$$

Or

$$\mu v^2 = \frac{Ze^2}{4\pi\varepsilon_0 r} \tag{2.9.2}$$

(*ii*) In contrast to classical physics where the radius of electronic orbit can assume any magnitude, Bohr asserted that only those orbits are allowed in which the angular momentum of electron is integral multiple of $\hbar$

$$L = n\hbar = \mu v r \tag{2.9.3}$$

(*iii*) Since the revolving electron around the nucleus is not a stable system under the laws of classical electrodynamics. Bohr assumed that the classical laws do not apply, at least, to the atomic phenomena. That is, the electron revolving in any one of the allowed orbits does not radiate. These non-radiating orbits are called stationary orbits. However, while making transition from a stationary orbit of higher energy to that of lower energy it does radiate. The electron may also go over from orbit of lower energy to that of higher energy by absorbing energy. If E_i and E_f are the energies of electron in the initial and final orbit, the frequency is given by

$$E_i - E_f = hf \qquad (2.9.4)$$

Radius of orbit**:** Eliminating ν from Eqns. (2.9.1) and Eqn. (2.9.4) and solving the resulting equation for r, we have

$$r = \frac{4\pi\varepsilon_0 h^2 n^2}{\mu Z e^2} \qquad (2.9.5)$$

For hydrogen atom Z = 1, the radius of the first orbit (n = 1), called Bohr orbit a_0 comes out to be

$$r_1 = a_0 = \frac{4\pi\varepsilon_0 h^2}{\mu\, e^2} \tag{2.9.6}$$

In terms of Bohr radius a_0, the radius of the *n*th orbit is given by

$$r_\mathrm{n} = a_0 \frac{n^2}{Z} \tag{2.9.7}$$

From Eqn. (2.9.6) the energy of electron in Bohr level is

$$E = \frac{-1}{2}\left[\frac{1}{4\pi\varepsilon_0}\right]^2 \frac{\mu e^4}{\hbar^2}\frac{Z^2}{n^2} = -\frac{13.6}{n^2}\ eV \tag{2.9.8}$$

Equation (2.9.8)can be rearranged as

$$E = -\left[\left[\frac{1}{4\pi\varepsilon_0}\right]^2 \frac{\mu e^4}{4\pi c\hbar^3}\right](2\pi c\hbar)\frac{Z^2}{n^2} = -2\pi c\hbar R_H \frac{Z^2}{n^2} \tag{2.9.9}$$

Where

$$R_H = \left[\left[\frac{1}{4\pi\varepsilon_0}\right]^2 \frac{\mu e^4}{4\pi c\hbar^3}\right] \tag{2.9.10}$$

The first orbit (*n* = 1) of hydrogen atom (Z = 1) has energy equal to

$$E = -13.6\, eV$$

Frequency of emitted radiation: If the electron makes transition from an orbit of quantum number n_i to the orbit of quantum number n_i, the frequency ν of the emitted radiation is given by [26]

$$E_i - E_f = 2\pi c\hbar\, Z^2 R_H \left[\frac{1}{{n_f}^2} - \frac{1}{{n_i}^2}\right] \tag{2.9.11}$$

(2-10) Rigid Rotator Body

Let m_1 and $m_2$masses of two particles are rotate about their center of mass perpendicular to the line joining them, r_1 and r_2 be their distances from the center of mass and r their separation. From the definition of center of mass [37]

$$m_1 r_1 = m_2 r_2 \tag{2.10.1}$$

$$r_1 + r_2 = r \tag{2.10.2}$$

From these equations we can find r_1 and r_2 in terms of r. Thus

$$r_1 = \frac{m_2}{m_1+m_2} r \ , \ r_2 = \frac{m_1}{m_1+m_2} r \tag{2.10.3}$$

The moment of inertia of the rigid rotator about an axis passing through the center of mass and perpendicular to the line joining the particles is

$$I = m_1 {r_1}^2 + m_2 {r_2}^2 = \frac{m_1 m_2}{m_1 + m_2} r^2 = \mu r^2 \tag{2.10.4}$$

Where $\mu = \frac{m_1 m_2}{m_1+m_2}$ is called reduced mass of the system.

The kinetic energy of particle of mass *m* moving in space in Cartesian coordinates is given by

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \tag{2.10.5}$$

In spherical polar coordinates (r, θ, φ) the expression for kinetic energy is

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\varphi}^2 sin^2\theta\right) \qquad (2.10.6)$$

If r is fixed (*i.e.*, the particle is moving on the surface of a sphere) then $\dot{r} = 0$ and the expression for kinetic energy simplifies to

$$T = \frac{1}{2}mr^2\left(\dot{\theta}^2 + \dot{\varphi}^2 sin^2\theta\right) \qquad (2.10.7)$$

The kinetic energy of rotation of a rigid rotator is equal to the sum of the kinetic energies of the constituent particles.

$$T = T_1 + T_2 = \frac{1}{2}m_1{r_1}^2\left(\dot{\theta}^2 + \dot{\varphi}^2 sin^2\theta\right) +$$

$$\frac{1}{2}m_2{r_2}^2\left(\dot{\theta}^2 + \dot{\varphi}^2 sin^2\theta\right) \qquad (2.10.8)$$

For a rigid rotator moving in free space, potential energy is zero, hence the total energy of the rotator is

$$E = T = \frac{1}{2}(\,m_1{r_1}^2 + m_2{r_2}^2)\left(\dot{\theta}^2 + \dot{\varphi}^2 sin^2\theta\right)$$

$$E = T = \frac{1}{2}I\left(\dot{\theta}^2 + \dot{\varphi}^2 sin^2\theta\right) \qquad (2.10.9)$$

Where

$I = m_1{r_1}^2 + m_2{r_2}^2$

Comparison of Eqns. (2.9.10) and (2.9.10) shows that a rigid rotator behaves like a single particle of mass I moving on the surface of a sphere of fixed radius, equal to unity.

The Schrödinger equation for a particle, in polar coordinates, is expressed as

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi(r,\theta,\phi) + \frac{2m}{\hbar^2}(E - V)\psi(r,\theta,\phi) = 0 \qquad (2.10.10)$$

To write the equation for rigid rotator, put $r = 1$, $V = 0$ and replace *m* by I in above equation. Doing so, we obtain

$$\left[\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{2IE}{\hbar^2}\right]\psi(\theta,\phi) = 0 \qquad (2.10.11)$$

Equation (2.10.11) can also be obtained as follows. The Hamiltonian of a rigid rotator, in absence of potential field, is

$$H = T = \frac{L^2}{2I} \qquad (2.10.12)$$

The corresponding operator is

$$\hat{H} = \frac{\hat{L}^2}{2I} = -\frac{\hbar^2}{2I}\left[\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \qquad (2.10.13)$$

The energy eigen value equation for rigid rotator is

$$\widehat{H}\psi = E\psi$$

Or

$$-\frac{\hbar^2}{2I}\left[\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi = \mathrm{E}\psi \qquad (2.10.14)$$

This is the same as Eqn. (2.10.11).

We assume the solution of Eqn. (2.10.14) to be of the form

$$\psi(\theta,\phi) = \Theta(\theta)\Phi(\phi) \qquad (2.10.15)$$

Substituting Eqn. (2.10.15) in (2.10.14) , we get

$$\frac{sin\theta}{\Theta(\theta)}\frac{d}{d\theta}\left(sin\theta\frac{d\Theta}{d\theta}\right) + \mathfrak{B}sin^2\theta + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0 \qquad (2.10.16)$$

Where

$$\mathfrak{B} = \frac{2IE}{\hbar^2} \qquad (2.10.17)$$

Transposing the ϕ dependent terms on the right hand side in

Eqn. (2.10.16), we get

$$\frac{sin\theta}{\Theta(\theta)}\frac{d}{d\theta}\left(sin\theta\frac{d\Theta}{d\theta}\right) + \mathfrak{B}sin^2\theta = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} \qquad (2.10.18)$$

The left hand side of Eqn. (2.10.18) depends on θ alone whereas right hand side on Φ alone and both the sides remain equal for all values of independent variables θ and Φ ; this can happen only when each side is equal to the same constant, say m^2. So the Eqn. (2.10.18) separates into two equations:

$$\frac{1}{sin\theta}\frac{d}{d\theta}\left(sin\theta\frac{d\Theta}{d\theta}\right)+\left(\mathfrak{B}-\frac{m^2}{sin^2\theta}\right)\Theta=0 \qquad (2.10.19)$$

$$\frac{d^2\Phi}{d\phi^2}+m^2\Phi=0 \qquad (2.10.20)$$

Solution of Eqn. (2.10.20) is of the form

$$\Phi=Ce^{im\phi} \qquad (2.10.21)$$

Since$\Phi(\phi)=\Phi(\phi+2\pi)$, then$e^{\pm 2\pi im}=1\rightarrow m=0,\pm1,\pm2,..$

The constant C in Eqn. (2.10.21) can be obtained making use of normalization condition

$$\int_0^{2\pi} \phi\phi^* \, d\phi = 1 \qquad (2.10.22)$$

Thus $\int_0^{2\pi} Ce^{2\pi im} \, Ce^{-2\pi im} d\phi$

$$C = \frac{1}{\sqrt{2\pi}}$$

So the solution of equation can be written as

$$\phi = \frac{1}{\sqrt{2\pi}} e^{im\phi} \; m = 0, \pm 1, \pm 2, \ldots \qquad (2.10.23)$$

Now let us return to Eqn. (2.10.19). This equation can be transformed into a convenient form by change of independent variable θ to x as follows:

$$x = \cos\theta \,, \sin\theta = \sqrt{1 - x^2}$$

$$\frac{d\Theta}{d\theta} = \frac{d\Theta}{dx}\frac{dx}{d\theta} = -\sin\theta \, \frac{d\Theta}{dx} \rightarrow \frac{d}{d\theta} = \frac{d}{dx}\frac{dx}{d\theta} = -\sin\theta \, \frac{d}{dx}$$

$$\sin\theta \frac{d\Theta}{d\theta} = -sin^2\theta \, \frac{d\Theta}{dx} = -(1 - x^2)\frac{d\Theta}{dx}$$

Making use of these results, Eqn. (2.10.19). becomes making use of these results, Eqn. (2.10.19). becomes

$$\frac{d}{dx}\left[(1-x^2)\frac{d\Theta}{dx}\right]+\left(\mathfrak{B}-\frac{m^2}{1-x^2}\right)\Theta=0, -1<x<1 \quad (2.10.24)$$

Equation (2.10.24) is similar to the famous associated Legendre's equation:

$$\frac{d}{dx}\left[(1-x^2)\frac{dF}{dx}\right]+\left(\ell(\ell+1)-\frac{m^2}{1-x^2}\right)\mathrm{F}=0 \quad (2.10.25)$$

Writing the constant $\mathfrak{B}$ appearing in Eqn. (2.10.25). as $\mathfrak{B} = \ell(\ell+1)$ where ℓ is another constant, we have

$$\frac{d}{dx}\left[(1-x^2)\frac{d\Theta}{dx}\right]+\left(\ell(\ell+1)-\frac{m^2}{1-x^2}\right)\Theta=0 \quad (2.10.26)$$

$$\mathfrak{B}=\ell(\ell+1)=\frac{2IE}{\hbar^2} \quad (2.10.27)$$

Equation (2.10.26) has single-valued and finite solutions only for certain values of parameter $\mathfrak{B}$ given by

$$\mathfrak{B}=\ell(\ell+1)=0,2,6,12,20,\ldots \quad (2.10.28)$$

$$\ell = 0,1,2,3,4,\ldots. \quad (2.10.29)$$

Substituting the value $\mathfrak{B}$ in Eqn. (2.10.28)., we find

$$E=\frac{\ell(\ell+1)\hbar^2}{2I}, \ell = 0,1,2,3,4,\ldots. \quad (2.10.30)$$

The wave function of a rigid rotator is [26]

$$\psi(\theta, \phi) = \sqrt{\frac{2l + 1}{4\pi} \frac{(l - m)!}{(l + m)!}} e^{im\varphi} \sin^m\theta \ \frac{d^m}{d(\cos\theta)^m} P_l(\cos\theta) \quad (2.10.31)$$

(2-11) Free Particle

For free particle the potential equal zero everywhere. Classically this would just mean motion at constant velocity, but in quantum mechanics the problem is surprisingly subtle and tricky. The time independent Schrödinger equation reads [38]

$$\frac{-h^2}{2m} \frac{d^2\psi(x)}{dx^2} = E\psi(x) \quad (2.11.1)$$

Or

$$\frac{d^2\psi(x)}{dx^2} + k^2 \psi(x) = 0 \quad (2.11.2)$$

Where

$$k = \sqrt{\frac{2m}{h^2} E} \quad (2.11.3)$$

We note there are no boundary conditions restrict the possible value of k (and hence of E); the free particle can carry any positive energy.

The general solution of Eqn. (2.11.2) is

$$\psi(x) = Ae^{ikx} + Be^{-ikx} \quad (2.11.4)$$

The first term of Eqn. (2.11.4) represents a wave traveling to the right, and the second represents a wave (of the same energy) going to the left, suppose a free particle propagating to the right so that $B = 0$ we might as well write [39]

$$\psi(x) = Ae^{ikx} \tag{2.11.5}$$

We see immediately that the wave function for such a particle cannot be normalized. This state represents a stationary state, but $\psi_i^* \psi_j = |\psi|^2 = |A|^2$ which, because

$$\int_{-\infty}^{\infty} \psi_j(x.t) \psi_i^*(x, t)\, dx = \infty \tag{2.11.6}$$

cannot be normalized. This is consistent with the uncertainty principle because the free particle, by its very nature, has definite momentum so $\Delta p = 0$ and, commensurately, $\Delta x = \infty$. The fact that the wave function cannot be normalized should not be a cause for concern. This entire means that the wave function does not represent a bound state which, by definition, is confined by some potential energy function to a specific region of space. Only bound state wave functions must be normalizable, square integrable.

From equation (2.11.3) the eign value of energy is [27]

$$E = \frac{h^2 k^2}{2m} \tag{2.11.7}$$

(2-12) Harmonic Oscillator

The Schrödinger equation for the harmonic oscillator in one dimension is [40]

$$\frac{-h^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x) \tag{2.12.1}$$

Where $V(x)$ is potential energy and define as

$$V = \frac{1}{2}m\omega^2 x^2 \tag{2.12.2}$$

$$\frac{d^2\psi(x)}{dx^2} + \frac{2m}{h^2}\left[E - \frac{1}{2}m\omega^2 x^2\right]\psi(x) = 0 \tag{2.12.3}$$

We shall transform this equation into a convenient form by introducing a new independent variable ξ defined by

$$\xi = \acute{\alpha} x \tag{2.12.4}$$

The parameter ά will be chosen in such a way that the new equation looks simple.

$$\frac{d\psi}{dx} = \frac{d\psi}{d\xi}\frac{d\xi}{dx} = \acute{\alpha}\frac{d\psi}{d\xi}$$

$$\frac{d^2\psi(x)}{dx^2} = \acute{\alpha}\frac{d}{d\mathrm{x}}\left(\frac{d\psi}{d\xi}\right) = \acute{\alpha}\frac{d}{d\xi}\left(\frac{d\psi}{d\xi}\right)\frac{d\xi}{d\mathrm{x}} = \acute{\alpha}^2\frac{d^2\psi}{d\xi^2}$$

In terms of new variable Eqn. (2.12.3) becomes

$$\frac{d^2\psi}{d\xi^2} + \left[\frac{2mE}{h^2\acute{\alpha}^2} - \frac{\mathrm{m}^2\omega^2\xi^2}{h^2\acute{\alpha}^4}\right]\psi = 0 \qquad (2.12.5)$$

Let us choose

$$\acute{\alpha}^2 = \frac{m\omega}{h} \qquad (2.12.6)$$

Equation (2.12.5) now becomes

$$\frac{d^2\psi}{d\xi^2} + \left[\frac{2E}{h\omega} - \xi^2\right]\psi = 0 \qquad (2.12.7)$$

Introducing the dimensionless parameter $\mathcal{B}$ defined by

$$\mathcal{B} = \frac{2E}{h\omega} \qquad (2.12.8)$$

Eqn. (2.12.7) becomes

$$\frac{d^2\psi}{d\xi^2} + [\mathcal{B} - \xi^2]\psi = 0 \qquad (2.12.9)$$

Asymptotic solution ($\xi \to \pm\infty$) : The wave function $\psi(\xi)$ must satisfy the condition

$$\psi(\xi \pm \infty) = 0$$

In the limit $\xi \to \infty, \mathcal{B}$ may be neglected. Eqn. (2.12.9) assumes the form

$$\frac{d^2\psi}{d\xi^2} - \xi^2\psi = 0 \qquad (2.12.10)$$

The solution to this equation is

$$\psi(\xi) = e^{\pm\xi^2/2}$$

We omit the positive sign because it does not satisfy the condition $(\xi \pm \infty) \rightarrow 0$. Hence the asymptotic solution of Eqn. (2.12.9) has the form

$$\psi(\xi) = e^{-\xi^2/2} \qquad (2.12.11)$$

Let us verify that Eqn. (2.12.11) satisfies Eqn. (2.12.10). From Eqn. (2.12.11)

$\frac{d\psi}{d\xi} = -\xi\psi,\ \ \frac{d^2\psi}{d\xi^2} = -\left(\psi + \xi\frac{d\psi}{d\xi}\right) = (\xi^2 - 1)\psi = \xi^2\psi$ for $\xi \rightarrow \infty$.

This ensures that Eqn. (2.12.11) is an asymptotic solution of Eqn. (2.12.9) . We may now assume that the solution of Eqn. (2.12.9) is of the form

$$\psi(\xi) = H(\xi)e^{-\xi^2/2} \qquad (2.12.12)$$

Where $H(\xi)$, is unknown function to be determined.

Substituting Eqn. (2.12.12) in (2.12.9) we obtain

$$\frac{d^2\mathrm{H}(\xi)}{d\xi^2} - 2\xi\frac{d\mathrm{H}(\xi)}{d\xi} + (\mathcal{B} - 1)\mathrm{H}(\xi) = 0 \qquad (2.12.13)$$

The unknown function $\mathrm{H}(\xi)$ obeying the differential Eqn. (2.12.13) is known as Hermit function.

Solution of Eqn. (2.12.13) is obtained in the form of power series. We assume the solution of the form

$$H(\xi) = \sum_{n=0}^{\infty} a_n\xi^n$$
$$= a_0 + a_1\xi + a_2\xi^2 + \cdots + a_n\xi^n + \cdots \qquad (2.12.14)$$

Now

$$\frac{d\mathrm{H}(\xi)}{d\xi} = \sum_{n=0}^{\infty} na_n\xi^{n-1} = 0 + a_1 + 2a_2\xi + \cdots + na_n\xi^{n-1}$$

$$\xi\frac{d\mathrm{H}}{d\xi} = \sum_{n=0}^{\infty} 2n\, a_n\xi^n \qquad (2.12.15)$$

$$\frac{d^2\mathrm{H}(\xi)}{d\xi^2} = \sum_{n=0}^{\infty} n(n-1)\, a_n\xi^{n-2} = 0 + 0 + 2a_2 + \cdots + n(n-1)\, a_n\xi^{n-2} + \ldots \qquad (2.12.16)$$

Or

$$\frac{d^2\mathrm{H}(\xi)}{d\xi^2} = \sum_{n=0}^{\infty} (n+1)(n+2)\, a_{n+2}\xi^n$$

Substituting Eqns. (2.12.14)(2.12.15), (2.12.16) in (2.12.13)

$$\sum_{n=0}^{\infty}[(n+1)(n+2)\,a_{n+2}+(\mathcal{B}-1-2n)a_n]\xi^n=0 \quad (2.12.17)$$

Equation (2.12.17) holds for all values of ξ. Hence the coefficient of each power of ξ must vanish separately. Hence

$$a_{n+2}=\frac{2n+1-\mathcal{B}}{(n+1)(n+2)} \qquad (2.12.18)$$

This equation is called the recurrence formula for the coefficients a_n. Since the recurrence formula determines the coefficients a_{n+2} in terms of a_n, the power series (2.12.14) contains only with even or only odd powers of ξ.

All steps like that step to obtain wave function of harmonic oscillator function until define Hermit polynomials by Rodrigues formula

$$H_n(\xi)=(-1)^n e^{\xi^2}\frac{d^n}{d\xi^n}\left(e^{-\xi^2}\right) \qquad (2.12.19)$$

From Eqn. (2.12.18) it is evident that our requirement is met if

$$\mathcal{B}=\frac{2E}{h\omega}=2n+1 \qquad (2.12.20)$$

Then harmonic oscillator can have only a discrete set of energies given by

$$E_n = \left(n + \frac{1}{2}\right) = h\omega \qquad (2.12.21)$$

The wave functions $\psi(x)$ of the harmonic oscillator are [26]

$$\psi_n(x) = \sqrt{\frac{\acute{\alpha}}{2^n n! \sqrt{\pi}}} H_n(\xi) e^{-\frac{1}{2}x^2 \acute{\alpha}^2} \qquad (2.12.22)$$

Chapter Three

Literature Review

(3-1) Introduction

In many physical systems friction plays an important role. Thus there is a need to construct quantum mechanical model that can recognize this effect. Many attempts were done to account for friction effect [41,42,43,15,16,20].

(3-2) Quantum Schrödinger String Theory for Frictional Medium and Collision

Maxwell's equation for decaying wave due to friction, beside a classical and quantum expression for oscillating string energy are used to derive a useful expression for particle energy in frictional resistive media. This expression is used to derive Schrödinger equation for oscillating string in resistive media. This new quantum reduces to the ordinary. Schrödinger equation in the absence of friction it also gives collision probability similar to that obtained by transport equation. This new equation is used to derive an expression for energy lost by friction by the vibrating string. This energy is shown to be quantized [44,45,46].

(3-2-1) Maxwell's Equations for Time Decaying Wave in Resistive Medium

Consider an electromagnetic wave enters a medium of conductivity σ, and electric polarization ṗ. Maxwell's equations for this medium are given by

$$-\nabla^2 E - \text{ᾒ}\sigma\, \partial E/\partial t + \text{ᾒ}\varepsilon\, \partial^2 E/\partial t^2 = -\text{ᾒ}\, \partial^2 \dot{\mathrm{P}}/\partial t^2 \quad (3.2.1.1)$$

The electric field intensity decays in this case and can be described by the relation

$$E = E_0 e^{-\mu_k t} e^{i(kx-\omega t)} \qquad (3.2.1.2)$$

The corresponding displacement is given by

$$x = x_0 e^{-\mu_k t} e^{i(kx-\omega t)} = \frac{x_0}{E_0} E \qquad (3.2.1.3)$$

The electric polarization terms is defined to be

$$\dot{\mathrm{P}} = enx = en\frac{x_0}{E_0} E \qquad (3.2.1.4)$$

With the aid of equations (3.2.1.2) and (3.2.1.4) of equation (3.2.1.1) becomes

$$K^2 + \frac{{\mu_k}^2}{c^2} - \frac{\omega^2}{c^2} - \text{ᾒ}\sigma\tilde{\alpha} + \frac{2i\mu_k\,\omega}{c^2} - i\text{ᾒ}\sigma\omega = \frac{en\omega^2\text{ᾒ}_0\, x_0}{E_0} \qquad (3.2.1.5)$$

Equation (3.2.1.5) can be simplified by using the relation

$$K = \frac{2\pi}{\lambda} = \frac{2\pi f}{\lambda f} = \frac{\omega}{c} \qquad (3.2.1.6)$$

And by assuming

$$\mu_k \ll c \qquad (3.2.1.7)$$

Where c is the speed of light in vacuum which is large. Thus equation (3.2.1.5) becomes

$$-\text{ᾓ}\sigma\mu_k + \frac{2i\mu_k\,\omega}{c^2} - i\text{ᾓ}\sigma\omega = en\omega^2\text{ᾓ}\frac{x_0}{E_0} \qquad (3.2.1.8)$$

Comparing real parts on both sides of equation (3.2.1.8) yields

$$\sigma\mu_k = -en\omega^2\mu\text{ᾓ}\frac{x_0}{E_0} \qquad (3.2.1.10)$$

(3-2-2) Friction Coefficient and Relaxation

It is quite natural to relate frictional coefficient μ_k to the relaxation time τ.

This due to the fact that by physical intuition are one can deduce that shorter the relaxation time, the bigger frictional coefficient. This can show also mathematics, by using the expression of energy dissipated by friction, which for oscillation particle is given by

$$E_f = \frac{m}{\tau} = V_e \qquad (3.2.2.1)$$

$$E_f = \frac{m\omega A^2}{2\tau} = \frac{m\omega^2 A^2}{2\tau\omega} = \frac{E}{\tau\omega} \qquad (3.2.2.2)$$

Where the effective displacement and velocity , are related to the maximum displacement by A by

$$x_e = \frac{A}{\sqrt{2}} \qquad V_e = \omega x_e = \frac{\omega A}{\sqrt{2}}$$

Where the classical energy of the oscillator is

$$E = \frac{1}{2}\omega^2 A^2 \qquad (3.2.2.3)$$

The oscillator frequency for classical and quantum system is the same. Thus one can write the quantum oscillator energy as

$$E = \hbar\omega \qquad (3.2.2.4)$$

Thus inserting (3.2.2.3) in (3.2.2.1) yields

$$E_f = \frac{\hbar E}{\tau\hbar\omega} = \frac{\hbar}{\tau} \qquad (3.2.2.5)$$

(3-2-3) Derivation of frictional Schrödinger equation on the basis of frictional energy equation

Ordinary Schrödinger is based on the postulates. The first postulate is related to the nature of micro particles. In this case the wave function takes the form

$$\psi = Ae^{i(kx-\omega t)} \qquad (3.2.3.1)$$

Using the fact that

$$P = \hbar k \quad \text{And } E_0 = \hbar\omega \qquad (3.2.3.2)$$

Therefore

$$\psi = Ae^{\frac{i}{\hbar}(Px-Et)} \qquad (3.2.3.3)$$

The second postulate is based on the classical expression of energy

$$E = \frac{P^2}{2m} + V \qquad (3.2.3.4)$$

Using these two postulates one can derive Schrödinger equation, where

$$E_0\psi = \frac{P^2}{2m}\psi + V\psi$$

$$i\hbar\frac{\partial\psi}{\partial t} = E_0\psi$$

$$\frac{P^2}{2m}\psi = \frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}$$

Thus in three dimensions

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{\hbar^2}{2m}\nabla^2\psi + V\psi \qquad (3.2.3.5)$$

The expression of energy Eqn. (3.2.3.4) in the presence of friction is given, with the aid of equation (3.2.3.2) to be

$$E = E_0 + E_f = \frac{P^2}{2m} + V - \frac{i\hbar}{\tau} \qquad (3.2.3.6)$$

Multiply both sides by ψ, one gets

$$E\psi = \frac{P^2}{2m}\psi + V\psi - \frac{i\hbar}{\tau}\psi \qquad (3.2.3.7)$$

But

$$\frac{\partial\psi}{\partial t} = -\frac{i}{\hbar}E\psi$$

$$i\hbar\frac{\partial\psi}{\partial t} = E\psi$$

$$\frac{\partial\psi}{\partial x} = \frac{i}{\hbar}P\psi$$

$$\frac{\partial^2\psi}{\partial x^2} = -\frac{P^2}{\hbar^2}\psi$$

In three dimensions

$$-\hbar^2\nabla^2\psi = P^2\psi \tag{3.2.3.8}$$

Inserting Eqn. (3.2.3.8) in Eqn. (3.2.3.7), one gets

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi - i\frac{\hbar}{\tau}\psi \tag{3.2.3.9}$$

Which is the Schrödinger equation or resistive media.

Let equation for harmonic oscillator defined as

$$\psi = f(t)u(r) \tag{3.2.3.10}$$

Inserting in Eqn. (3.2.3.9) yields

$$i\frac{\hbar}{\mathrm{f}}\frac{\partial f}{\partial t} = -\frac{\hbar^2}{2mu}\nabla^2 u + V - i\frac{\hbar}{\tau} = E_0$$

Hence

$$i\hbar\frac{\partial f}{\partial t} = E_0 f \tag{3.2.3.11}$$

For harmonic oscillator string vibrating in one dimension the potential is given by

$$V = \frac{1}{2}Kx^2$$

Thus Eqn. (3.2.3.9) reads

$$-\frac{\hbar^2}{2m}\nabla^2 u + \frac{1}{2}Kx^2 u = \left(E_0 + i\frac{\hbar}{\tau}\right)u = Eu \tag{3.2.3.12}$$

This is the ordinary harmonic oscillator equation, with have quantized energy

$$E = \left(n_1 + \frac{1}{2}\right)\hbar\omega \tag{3.2.3.13}$$

For Eqn. (3.2.3.9) one can suggest the solution

$$f = Ae^{-i\mu_{k_0} t} \tag{3.2.3.14}$$

$$\mu_{k_0} \hbar = E_0$$

The periodicity condition requires

$$f(t + T) = f(t) \tag{3.2.3.15}$$

Hence from Eqn. (3.2.3.13)

$$Ae^{-i\mu_{k_0} (t+T)} = Ae^{-i\mu_{k_0} t}$$

$$Ae^{-i\mu_{k_0} t} = Ae^{-i\theta} = cos\theta - i\ sin\theta = 1$$

$$\theta = \mu_{k_0} T = 2n_2\pi$$

$$\mu_{k_0} = \frac{2}{T}\pi \frac{2\pi n_2}{T} n_2 = n_2\omega$$

Eqn. (3.2.3.14)

$$E_0 = n_2\omega\hbar \tag{3.2.3.16}$$

In view of equation (3.2.3.12)

$$E = E_0 - i\frac{\hbar}{\tau} \tag{3.2.3.17}$$

Using Eqns. (3.2.3.13), (3.2.3.16) and (3.2.3.17) yields

$$\left(n_1 + \frac{1}{2}\right)\hbar\omega = n_2\omega\hbar + i\frac{\hbar}{\tau}$$

$$\frac{i}{\tau} = \left[(n_2 - n_1) - \frac{1}{2}\omega\right] = \left[-n - \frac{1}{2}\right]\omega$$

$$\tau^{-1} = -\frac{\left(n + \frac{1}{2}\right)}{i}\omega = i\omega\left(n + \frac{1}{2}\right) \tag{3.2.3.18}$$

The physical meaning of complex relaxation can be known from

Eqn. (3.2.3.6)

$$E = E_0 + i\frac{\hbar}{\tau} = E_0 + E_f$$

Thus the energy lost due to friction is given by

$$E_f = -i\frac{\hbar}{\tau} = \left(n + \frac{1}{2}\right)\hbar\omega \qquad (3.2.3.19)$$

The minus sign indicates that the energy is lost by the particle [16].

(3-3) The Quantum Expression of the Role of Relaxation Time on Optical Absorption in semiconductors

In this work theoretical method based on classical mechanics. Maxwell equations and quantum mechanics is utilized to find the mathematical expression for relaxation time, this expression play an important role in the simulation of quantum semiconductor devices. One has studied the relaxation time in the semi classical approximation, the optical potential, perturbation method and generalized Schrödinger equation for relaxation time. Theoretical relation between the relaxation time and absorption coefficient is obtained. At last one has studied the quantum mechanical absorption coefficient, absorption coefficient for Maxwell's equations and quantum mechanics, and parameters affecting absorption coefficient [47,48,49].

(3-3-1) To Relate Quantum Relaxation Time with Absorption Coefficient

One needs along math derivation, when a radiation beam or particle beam enters a medium its intensity decreases due to the decrease in its energy per particle and due to decrease of the number of particles. This indicates the existence of a resistive force which causes these changes. The time taken by a particle on the beam to stop is called relaxation time τ, τ is defined as a time between two successive collisions. This relaxation time can be obtained by different quantum methods. For instance one can use perturbation method to find the change particle energy due to a resistive potential. For simplicity assume the perturbing part of Hamiltonian $\hat{H}_1$to be a constant i,e:

$$\hat{H}_1 = -V_0 \qquad (3.3.1.1)$$

In this case the perturbation, i,e change in energy is given by

$$E_1 = (\hat{H}_1)_{KK} = \int \bar{U}_K\,(-V_0)U_K dr = -V_0 \int \bar{U}_K\,U_K dr$$
$$= -V_0 \qquad (3.3.1.2)$$

Where the total energy E is given by

$$E = E_K + E_1 \qquad (3.3.1.3)$$

The energy change is given by

$$\Delta E = |E_1| = V_0 \qquad (3.3.1.4)$$

According to uncertainty principle the relaxation time is given by

$$\tau = \Delta t = \frac{\hbar}{\Delta E} = \frac{\hbar}{V_0} \quad (3.3.1.5)$$

Here relaxation time means the time taken by the particle in the perturbed excited state. This equation indicates that in strong resistive forceV_0 → ∞, causes the relaxation time to be short τ → ∞ which is in conformity with commonsense.

Relaxation time can also be obtained by utilizing the optical potential V_1which is introduced to a count for particle energy loses in inelastic scattering where the wave function of the free particle takes the form

$$\psi = Ae^{\frac{V_1}{\hbar}t}e^{-i\frac{E}{\hbar}t} \quad (3.3.1.6)$$

For constant optical potential

$$V_1 = constant \quad (3.3.1.7)$$

The current density $\hat{j}$ and hence the beam intensity I are given by

$$I = nevx\hbar\omega \quad (3.3.1.8)$$

Thus using (3.3.1.7) and (3.3.1.8) yields

$$I = hf|\psi|^2\nu = A^2\nu e^{\frac{2V_1}{\hbar}hft} = I_0e^{\frac{2V_1}{\hbar}t} \quad (3.3.1.9)$$

The time τ taken by the beam to decrease can be defined to be

$$\frac{I}{I_0} = e^{-\frac{2V_1}{\hbar}\tau} = e^{-1}$$

$$\frac{2V_1}{ħ}\tau = 1$$

$$\tau = \frac{ħ}{2V_1} \qquad (3.3.1.10)$$

again the expression for the relaxation time τ resembles that obtained perturbation method. The two expressions coincide if one sets

$$V_1 = \frac{1}{2}V_0$$

In this case

$$\tau = \frac{ħ}{V_0} \qquad (3.3.1.11)$$

The relaxation time can also be obtained quantum mechanically, if one treats atoms and electrons as harmonic oscillators subjected to resistive force $\mu_k v$ beside the restoring force$-K_0 x$ in this case the total force is given by

$$ma = -m\omega^2 x = -K_0 x + \mu_k v \qquad (3.3.1.12)$$

But

$$x = x_0 e^{i\omega t} \ldots\ldots..v = \frac{v_0}{x_0} x_0 e^{i\omega t} \ldots v = \frac{v_0}{x_0} x \qquad (3.3.1.13)$$

Thus

$$ma = -m\omega^2 x = -K_0 x + \mu_k \frac{v_0}{x_0} x = \left(\mu_k \frac{v_0}{x_0} - K_0\right) x \qquad (3.3.1.14)$$

$$-m\omega^2 x = (\mu_{k0} - K_0)$$

Where

$$\mu_{k_0} = \mu_k \frac{\nu_0}{x_0}$$

Hence

$$\omega^2 = \frac{\mu_{k_0} - K_0}{m} \qquad (3.3.1.15)$$

$$\omega = \pm\sqrt{\frac{\mu_{k_0} - K_0}{m}}$$

Treating atoms as harmonic oscillators, the force is given by

$$F = -Kx = -(\mu_{k_0} - K_0)x \qquad (3.3.1.16)$$

And utilizing Schrödinger equation for harmonic oscillator the energy is given by

$$E_n = n + \frac{1}{2}\hbar\sqrt{\frac{\mu_{k_0} - K_0}{m}} \qquad (3.3.1.17)$$

When the resistive force $F_r = \eta_0 x$ dominate i,e:

$$F_r > F \ldots.\mu_{k_0} x > K_0 x \ldots\ldots \mu_{k_0} > K_0$$

Thus the energy can be written as

$$E_n = \pm\left(n + \frac{1}{2}\right)\hbar\sqrt{\frac{\mu_{k_0} - K_0}{m}}\ i \ldots\ldots.for\ n = 0 \qquad (3.3.1.18)$$

$$\omega_0 = \sqrt{\frac{\mu_{k_0} - K_0}{m}} \ldots\ldots E_n = \pm\frac{1}{2}\hbar\omega_0 i \qquad (3.3.1.19)$$

In this case the time dependent part of Schrödinger equation becomes

$$\psi = Ae^{-i\frac{E_n}{\hbar}t} = Ae^{-i\frac{(\pm\hbar\omega_0 i)}{\hbar}t} = Ae^{\pm\omega_0 t} \qquad (3.3.1.20)$$

For decaying one chooses the minus sign to get

$$\psi = Ae^{\pm\omega_0 t} \qquad (3.3.1.21)$$

Thus the intensity of the beam becomes

$$I = n\hbar\omega\nu = |\psi|^2\hbar\omega\nu \qquad (3.3.1.22)$$

$$I = A^2\nu e^{-2\omega_0 t}\hbar\omega = I_0 e^{-2\omega_0 t}$$

The relaxation time is given by

$$\frac{I}{I_0} = e^{-2\omega_0 t} = e^{-1}$$

Hence

$$2\omega_0\tau = 1$$

Then

$$\tau = \frac{1}{2\omega_0} \qquad (3.3.1.23)$$

$$\tau = \sqrt{\frac{2m}{\mu_{k_0} - K_0}} \qquad (3.3.1.24)$$

This is the quantum expression for relaxation time which is related to friction force F_r by [17]

$$F_r = \frac{mv}{\tau} = \mu_{k_0} v$$

Where

$$\mu_{k0} = \frac{m}{\tau}$$

$\mu_k \, \mu_{k_0}$

(3-4) Quantization of Friction for Nano Isolated Systems

A wide variety of materials have mechanical friction. The friction plays an important role in determine the mechanical properties and the electrical properties of the matter. The most popular physical theory that is used to describe the physical properties of matter is quantum mechanics. Recently quantum laws found to be incapable of describing the behavior of some new materials. This may be attributed to the fact that quantum laws have no terms sensitive to friction. This work aims to derive Schrödinger quantum equation having frictional term. This equation is used to solve the problem of particle in a box. The solution shows quantized frictional energy [50,51,52].

(3-4-1) Schrödinger Equation in Presence of Thermal and Resistive Energy

The energy of ordinary Schrödinger equation includes kinetic and potential energy. However, there are other energy types which should be considered, for example the energy lost by friction E_{f_k} for oscillating system which is given by

$$\int f_k\, dx \qquad (3.4.1.1)$$

The thermal energy is given in terms of temperature T and Boltzman constant K as

$$E_T = K_B T \qquad (3.4.1.2)$$

Where there is no room in ordinary conventional Schrödinger equation for feeling the effect of friction and heating does not recognize these energy types.

The total energy of one particle that include thermal no friction energy becomes can be written in the form

$$E = \frac{₱^2}{2m} + V + K_B T - i\frac{m\omega x^2}{2\tau} \qquad (3.4.1.3)$$

Where ₱ stands for the force, thermal pressure.

This expression stands for the total energy of a single particle, which consists beside kinetic energy, additional terms. The third term represents the thermal energy, while the forth term stands for the frictional energy.

To derive Schrödinger equation, for this new energy expression, equation (3.4.1.3) must be multiplied by a wave function ψ to get

$$E\psi = \frac{₱^2}{2m}\psi + V\psi + K_B T\psi - i\frac{m\omega x^2}{2\tau}\psi \qquad (3.4.1.4)$$

The wave function for a free particle is given by

$$\psi = Ae^{\frac{i}{\hbar}(Px - Et)} \tag{3.4.1.5}$$

Differentiating ψ with respect to t and x yields

$$\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar} E\psi$$

$$i\hbar \frac{\partial \psi}{\partial t} = E\psi$$

$$\frac{\partial \psi}{\partial x} = \frac{i}{\hbar} P\psi$$

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{P^2}{\hbar^2}\psi \tag{3.4.1.6}$$

Substitute Eqn. (3.4.1.6) in Eqn. (3.4.1.4)

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{\hbar^2 \upsilon^2}{2m}\psi + V\psi + K_B T\psi - i\frac{m\omega x^2}{2\tau}\psi \tag{3.4.1.7}$$

If one rewrite the frictional term in the form

$$E_{fk} = i\frac{m\omega A^2}{2\tau} = i\frac{m\omega^2 A^2}{2\tau\omega} = i\frac{m^2 \upsilon^2}{2\tau m} = i\frac{m^2 \upsilon^2 \hbar}{2\tau m \hbar \omega}$$

$$= i\frac{P^2 \hbar}{2\tau m^2 c^2} = i\frac{\hbar}{2\tau m^2 c^2}P^2 \tag{3.4.1.8}$$

Where

$$\hbar\omega = m^2 c^2$$

And

$$\upsilon^2 = |\upsilon^2| = \upsilon.\upsilon^* = (-i\omega x)(i\omega x) = \omega^2 x^2$$

Substitute Eqn. (3.4.1.8) in Eqn. (3.4.1.4)

$$E\psi = \frac{P^2}{2m}\psi + V\psi + K_B T\psi - i\frac{\hbar}{2\tau m^2 c^2}P^2\psi \tag{3.4.1.9}$$

Substitute Eqn. (3.4.1.8) in Eqn. (3.4.1.7)

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}\psi + V\psi + K_BT\psi - i\frac{\hbar^3}{2\tau m^2c^2}\frac{\partial^2\psi}{\partial x^2}\psi$$

In three dimension

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi + K_BT\psi - i\frac{\hbar^3}{2\tau m^2c^2}\nabla^2\psi \qquad (3.4.1.10)$$

Which is Schrödinger equation for thermal resistive media.

(3-4-2) Harmonic Oscillator

This equation can be used for harmonic oscillator. Harmonic oscillator is characterized by the potential

$$V = \frac{1}{2}Kx^2 \qquad (3.4.2.1)$$

For time independent Schrödinger equation, Eqn. (3.4.1.10) become

$$Eu = -\frac{\hbar^2}{2\mathrm{m}}\nabla^2u + Vu + K_BTu - i\frac{\hbar^3}{2\tau m^2c^2}\nabla^2u \qquad (3.4.2.2)$$

Inserting Eqn. (3.4.2.1) in Eqn. (3.4.2.2)

$$-\frac{\hbar^2}{2\mathrm{m}}\nabla^2u - \frac{1}{2}Kx^2u + K_BTu - i\frac{\hbar^3}{2\tau m^2c^2}\nabla^2u = Eu \qquad (3.4.2.3)$$

Consider the solution

$$u = Ae^{-kx^2}$$

$$\nabla u = -2kxu$$

$$\nabla^2u = -2ku - 2kx\nabla u = -2ku + 4k^2x^2u \qquad (3.4.2.4)$$

A direct substitution of Eqn. (3.4.2.4) in Eqn. (3.4.2.3)

$$\frac{ħ^2}{2\mathrm{m}}[2k - 4k^2x^2]u + \frac{1}{2}Kx^2u + K_BTu - i\frac{ħ^2}{2\mathrm{m}}\left[\frac{2k - 4k^2x^2}{\tau mc^2}\right]ħu = Eu \qquad (3.4.2.5)$$

Equating the coefficient of u and ux^2 on both side yields:

$$-\frac{4ħ^2}{2\mathrm{m}}\left[1 + \frac{iħ}{\tau mc^2}\right]k^2 - \frac{1}{2}K = 0$$

$$-2\frac{ħ^2}{2\mathrm{m}}\left[1 + \frac{iħ}{\tau mc^2}\right]k + K_BT = E$$

By ignoring temperature term, one get

$$\mathrm{E} = \frac{ħ^2}{\mathrm{m}}\left[1 + \frac{iħ}{\tau mc^2}\right]k \qquad (3.4.2.6)$$

The energy quantization can be obtained from Eqn. (3.4.1.10)

by separating variables, and assuming ψ to be

$$\psi = \omega(t)v(x) \qquad (3.4.2.7)$$

To get

$$\frac{\mathrm{i}ħ}{\omega}\frac{d\omega}{dt} = \frac{1}{u}\left(-\frac{ħ^2}{2\mathrm{m}}\left[1 + \frac{iħ}{\tau mc^2}\right]\right)\nabla^2u + V = E$$

Thus

$$\frac{\mathrm{i}ħ}{\omega}\frac{d\omega}{dt} = E$$

$$L_n\omega = C_1 - i\frac{E}{ħ}t$$

$$\int\frac{d\omega}{\omega} = i\frac{E}{ħ}\int dt + C_1$$

$$\omega = e^{C_1} e^{i\frac{E}{\hbar}t} = C_2 e^{i\frac{E}{\hbar}t} \qquad (3.4.2.8)$$

The periodicity condition requires

$$\frac{\omega(t+T) = \omega(t)}{e^{i\frac{E}{\hbar}t} = 1}$$

$$\cos\frac{E}{\hbar}T = 1$$

$$\frac{E}{\hbar}T = 2n\pi$$

$$e^{i\frac{E}{\hbar}(t+T)} = e^{i\frac{E}{\hbar}t}$$

$$\cos\frac{E}{\hbar}T - i\,\sin\frac{E}{\hbar}T = 1$$

$$\sin\frac{E}{\hbar}T = 0$$

$$E = \frac{n\hbar}{T} = nf\hbar \qquad (3.4.2.9)$$

Therefore energy is quantized according to Eqn. (3.4.2.6) and Eqn. (3.4.2.7) beside Eqn. (3.4.2.8), one get [15]

$$nf\hbar = \frac{\sqrt{mk}}{2\hbar\sqrt{1+\frac{i\hbar}{\tau mc^2}}}$$

$$E = nf\hbar = \frac{\hbar}{2}\sqrt{\frac{i\hbar}{\tau mc^2}} = \frac{1}{2}\hbar\omega c\sqrt{1+\frac{i\hbar}{\tau mc^2}} \qquad (3.4.2.10)$$

(3-5) Quantum Radioactive Decay Law and Relaxation time

Relaxation time plays an important role in nuclear excitation. This is because excitation energy is related to the relaxation time according to uncertainty principle. This relation between relaxation time and energy excitation was deducing from classical and quantum laws of harmonic oscillator. It resembles that of uncertainty principle, except that existence of imaginary term which is related to the energy lost by excitation and friction. This relation is used to find the wave function for frictional media. This wave function is used to derive radioactive decay law [53,54,55].

(3-5-1) Relaxation time and friction

For any particle have mass m and velocity υ the force F exerted on it can be describe by the equation

$$m\frac{d\upsilon}{dt} = F \qquad (3.5.1.1)$$

Considering the particle as harmonic oscillator the velocity υ is given by

$$\upsilon = \upsilon_0 e^{i\omega_0 t} \qquad (3.5.1.2)$$

Where

ω_0 is the angular frequency

υ_0 is the maximum velocity

Substitute Eqn. (3.5.1.2) in Eqn. (3.5.1.1)

$$i\omega_0 t = F \tag{3.5.1.3}$$

If the particle moves in a resistive medium of coefficient μ_k the equation of motion becomes

$$m\frac{dv}{dt} = F - \mu_k\, mv \tag{3.5.1.4}$$

Assuming that the frictional force affects the frequency only, one can assume

$$v = v_0 e^{i\omega t} \tag{3.5.1.5}$$

This obvious if we treat the particle as a harmonic oscillator, where

$$E_0 = \hbar\omega_0, \qquad E = \hbar\omega \tag{3.5.1.6}$$

But the force can be define from Eqn. (3.5.1.2) and Eqn. (3.5.1.5) as

$$F = im\omega v_0 \qquad and\; F_0 = im\omega_0 v_0 \tag{3.5.1.7}$$

Substitute Eqn. (3.5.1.7) in Eqn. (3.5.1.4) yields

$$im\omega v_0 = im\omega_0 v_0 - \mu_k\, mv_0$$

Cancelling similar terms and multiplying both sides by i as

$$\omega - \omega_0 = -i\mu_k = \frac{1}{\tau} \tag{3.5.1.8}$$

Thus the energy loss is given by

$$\Delta E = \hbar\omega_0 - \hbar\omega = -i\hbar\mu_k = \frac{-i\hbar}{\tau} \tag{3.5.1.9}$$

Thus the energy of the system is

$$E = E_0 + \Delta E = E_0 - i\mu_k \qquad (3.5.1.10)$$

Thus the wave equation can be written

$$\psi = Ae^{\frac{i}{\hbar}(Px-[E_0-i\mu_k]t)}$$

$$\psi = Ae^{\frac{i}{\hbar}(Px-E_0t)}e^{-\frac{\mu_k\hbar t}{\hbar}} \qquad (3.5.1.11)$$

$$\psi = Ae^{-\mu_k t}e^{\frac{i}{\hbar}(Px-E_0t)} \qquad (3.5.1.12)$$

It is very striking to observe that the imaginary friction term in Eqn. (3.5.1.8) and Eqn. (3.5.1.9) appears in Eqn. (3.5.1.12) to make the amplitude of ψ decay with time.

Therefore the average energy which is equal total classical value, i.e.

$$\bar{E} = \int \bar{\psi}\, E\psi dt = Ee^{-2\mu_k t} \qquad (3.5.1.13)$$

Indicates that the energy decay with time. This agrees with the fact that friction cause particle energy to decrease.

The relation time from uncertainty principle is given by

$$\Delta E \Delta t = \hbar \qquad (3.5.1.14)$$

$$\Delta E = \frac{\hbar}{\Delta t} = \frac{\hbar}{\tau} \qquad (3.5.1.15)$$

$$\Delta t = \tau$$

It’s very interesting to note that Eqn. (3.5.1.9) and Eqn. (3.5.1.15) give the same numerical values. But the expression

(3.5.1.9) is more convenient, as far as it is insertion in the wave function predicts time decaying energy.

This means that treating particles as harmonic oscillators gives quantum results similar to the classical one.

A simple expression for radioactive decay law can be obtained by using Eqn. (3.5.1.11). The number of particle is given by

$$ŋ = \bar{\psi}\psi = Ae^{-2\mu_k t} \quad (3.5.1.16)$$

Since at $t = 0 \quad and \quad ŋ = ŋ_0$

Thus Eqn. (3.5.1.16) gives

$$ŋ_0 = A$$

Thus Eqn. (3.5.1.16) becomes

$$ŋ = ŋ_0 e^{-2\mu_k t} \quad (3.5.1.17)$$

by setting

$$2\mu_k = \tilde{\alpha}$$

$$ŋ = ŋ_0 e^{-\tilde{\alpha} t} \quad (3.5.1.18)$$

Which is the ordinary radioactive decay law.

The radiation emitted by unstable nuclei is due to the fact that these nuclei are in excited state. This can be shown with the aid of Eqns. (3.5.1.9), (3.5.1.15) and (3.5.1.17), where

$$\Delta E = \frac{ħ}{\tau} = ħ\mu_k \quad (3.5.1.19)$$

Thus

$$ŋ = ŋ_0 e^{i\frac{\Delta E}{ħ}t} \quad (3.5.1.20)$$

This shown that nuclear decay is due to excitation, i.e. the existence of the nucleus in an excited state [16].

(3-6) Summary and Critique

Most of the work done to describe medium and vacuum effect on quantum particle motion accounts for the effect of friction by considering the medium as a solid [56,57]. Unfortunately they do not recognize the effect of vacuum explicitly. They are not also treating the medium or vacuum as a fluid having certain viscosity.

Chapter Four

Modified Schrӧdinger Equation Due to Vacuum and Medium Viscosity

(4-1) Introduction

In this chapter, one gets the energy loses by electrons in hydrogen atom due to viscous media around nucleus. This viscous media may result from vacuum, which was recently proved to exist by having a well known energy as shown by casimer and others [58]. It may also be related to the crystal field [59] or to the bulk matter which can be considered as a viscous fluid [60].

(4-2) Electron Energy Loss Due to Viscosity of Fluid Around Nuclear of Hydrogen Atom

To get electron energy loss due to viscosity of fluid around nuclear of hydrogen atom E_{vis} one must obtain the Stock's force acting on the electron on its orbit for laminar and non laminar flow [61].

The Stock's force acting on an electron on its orbit for non laminar flow is given by

$$F_{Stock} = 6\pi\gamma\eta\upsilon = \beta_0\upsilon \tag{4.2.1}$$

Where

$$\beta_0 = 6\pi\gamma\eta \tag{4.2.2}$$

Where η is viscosity coefficient, γ radius of electron orbit and υ is the velocity of electron in hydrogen atom.

For laminar flow the Stock's force becomes [62]

$$F_{Stock} = \frac{1}{4}\pi\eta\, N_g\gamma C_D\upsilon = \beta_1\upsilon \tag{4.2.3}$$

Where

$$\beta_1 = \frac{1}{4}\pi\eta\, N_g\gamma C_D \tag{4.2.4}$$

Then let the Stock's force in general has the form

$$F_{Stock} = \beta\upsilon \tag{4.2.5}$$

And the electron energy loss due to viscosity of fluid around nuclear of hydrogen atom takes the form

$$E_{vis} = -\int \beta\upsilon\,.\,ds \tag{4.2.6}$$

But the velocity in spherical coordinates which were described by $r, \theta, \phi -$ direction is given by [63]

$$\upsilon = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r sin\theta\, \dot{\phi}\widehat{\phi} \tag{4.2.7}$$

And the displacement along the path in $r, \theta, \phi -$ direction becomes

$$ds \;=\; dr\hat{r} \;+ rd\theta\,\hat{\theta} + rsin\theta\; d\phi\hat{\phi} \qquad (4.2.8)$$

Sub Eqn. (4.2.7) and Eqn. (4.2.8) in Eqn. (4.2.6) to obtain electron energy by a viscose media as

$$E_{vis} = -\int \beta\left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + rsin\theta\;\dot{\phi}\hat{\phi}\right).\left(dr\hat{r} \;+ rd\theta\;\hat{\theta} + rsin\theta\; d\phi\hat{\phi}\right) \qquad (4.2.9)$$

Let us consider an electron move near at one ends of orbit($\dot{r} \to 0$) in θ- direction, by approximately uniform velocity $\dot{\theta}$. Also one can consider the motion of electron is parallel to the earth and almost parallel to azimuth coordinate, then $\dot{r} \approx \dot{\phi} \approx 0$ and Eqn. (4.2.9) become

$$E_{vis} = -\int \beta(r\dot{\theta}\hat{\theta}).(dr\hat{r} \;+ rd\theta\;\hat{\theta} + rsin\theta\; d\phi\hat{\phi})$$

Or

$$E_{vis} = -\int_0^{\pi}\;\beta\left(r^2\dot{\theta}\right)d\theta \qquad (4.2.10)$$

The value of angular momentum is usually constant in electron orbit; by tacking above consideration one can obtain it as

$$L = r \times \mu\, v = \begin{pmatrix} \hat{r} & \hat{\theta} & \widehat{\Phi} \\ r & 0 & 0 \\ 0 & r\dot{\theta} & 0 \end{pmatrix} = \mu r^2 \dot{\theta} \qquad (4.2.11)$$

Since $\dot{\theta}$ must equal to

$$\dot{\theta} = \frac{L}{\mu\, r^2} \qquad (4.2.12)$$

Substitute Eqn. (4.2.12) in Eqn. (4.2.11) to get

$$E_{vis} = -\int_0^{\pi} \beta(r^2 . \frac{L}{r^2 \mu}) d\theta$$

Or

$$E_{vis} = -\int_0^{\pi} \frac{\beta . L}{\mu} d\theta = -\frac{\beta . L}{\mu} \int_0^{\pi} d\theta = -\frac{\pi \beta . L}{\mu} \qquad (4.2.13)$$

This is the electron energy loss on its orbit due to viscosity of fluid around nuclear of hydrogen atom due to the work done by Stock's force on the electron of hydrogen atom, in terms of conserved angular momentum. For non laminar flow this relation can be obtained by substituting equation(4.2.2) in Eqn. (4.2.13) to find

$$E_{vis} = -\frac{6\pi^2 \gamma \eta}{\mu} L = -\frac{\pi \beta_0}{\mu} L \qquad (4.2.14)$$

Thus the energy loss operator is written in the form

$$\hat{E}_{\text{vis}} = -\frac{6\pi^2\gamma\eta}{\mu}\hat{L} = -\frac{\pi\beta_0}{\mu}\hat{L} \qquad (4.2.15)$$

Also the energy loses and its operator for laminar flow is given by substituting Eqn. (4.2.4) in Eqn. (4.2.13) to get

$$E_{vis} = -\frac{1}{4}\frac{\pi^2\eta\ N_g\gamma C_D}{\mu}L = -\frac{\pi\beta_1}{\mu}L \qquad (4.2.16)$$

$$\hat{E}_{\text{vis}} = -\frac{1}{4}\frac{\pi^2\eta\ N_g\gamma C_D}{\mu}L = -\frac{\pi\beta_1}{\mu}\hat{L} \qquad (4.2.17)$$

In general the form the energy loss of electron due to viscous media around nuclear of hydrogen atom is given as

$$\hat{E}_{\text{vis}} = -\frac{\pi\beta}{\mu}\hat{L} \qquad (4.2.18)$$

(4-3) Effect of Viscosity on Total Energy of Electron in Hydrogen Atom

To obtain the effect of viscosity on total energy of electron in hydrogen atom, it is convenient to write the Hamiltonian of this electron in classical mechanics as

$$H = \frac{P^2}{2m} + V + E_{vis} \qquad (4.3.1)$$

In quantum mechanics the Hamiltonian becomes

$$\hat{H} = \frac{\hat{P}^2}{2m} + \hat{V} + \hat{E}_{\text{vis}} \qquad (4.3.2)$$

The eigen equation of electron in Hydrogen atom is given by

$$\hat{H}\psi(r,\theta,\phi) = \frac{\hat{P}^2}{2m}\psi(r,\theta,\phi) + \left[\hat{V} + \hat{E}_{\text{vis}}\right]\psi(r,\theta,\phi) \qquad (4.3.3)$$

Substitute Eqn. (4.2.18) in Eqn. (4.3.3) to get

$$\hat{H}\psi(r,\theta,\phi) = \frac{\hat{P}^2}{2m}\psi(r,\theta,\phi) + \left[\hat{V} - \frac{\pi\beta}{\mu}\hat{L}\right]\psi(r,\theta,\phi) \qquad (4.3.4)$$

When the potentials are independent of time, the above equation can be expresses as time independent Schrödinger equation, which reads

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi(r,\theta,\phi) +$$

$$\frac{2\mu}{\hbar^2}\left(E - V + \frac{\pi\beta}{\mu}\hat{L}\right)\psi(r,\theta,\phi) = 0 \qquad (4.3.5)$$

Where β is either β_0 or β_1 according to the type of flow

Thus wave function can thus be separated as
`

$$\psi(r,\theta,\phi) = R(r)Y(\theta,\phi) \qquad (4.3.6)$$

The result of applying energy loses operator in Eqn. (4.2.18) on the wave function in Eqn. (4.3.6), produces

$$\frac{\pi\beta}{\mu}\hat{L}\,\psi(r,\theta,\phi) = \frac{\pi\beta}{\mu}R\hat{L}\,Y(\theta,\phi) = \frac{\pi\beta}{\mu}R\sqrt{\ell(\ell+1)}\,\hbar Y(\theta,\phi)$$

$$= \frac{\pi\beta}{\mu}\sqrt{\ell(\ell + 1)}\,\hbar\psi(r, \theta, \phi) \qquad (4.3.7)$$

Substituting equation (4.3.7) in (4.3.5) one gets

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2}\frac{\partial^2}{sin^2\theta\,\partial\phi^2}\right]\psi +$$

$$\frac{2\mu}{\hbar^2}\left(E - V + \frac{\pi\beta}{\mu}\sqrt{\ell(\ell + 1)}\hbar\right)\psi = 0 \qquad (4.3.8)$$

But the terms E and $\frac{\pi\beta}{\mu}\sqrt{\ell(\ell + 1)}\hbar$ are constants with respect

to ψ, thus one can consider

$$E_0 = E + \frac{\pi\beta}{\mu}\sqrt{\ell(\ell + 1)}\hbar \qquad (4.3.9)$$

Then equation (4.3.8) become

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi(r, \theta, \phi) +$$

$$\frac{2\mu}{\hbar^2}(E_0 - V)\psi(r, \theta, \phi) = 0 \qquad (4.3.10)$$

Where this equation resemble Eqn. (2.8.2), thus energy eigen values are given by

$$E_0 = \frac{-1}{2}\left(\frac{1}{4\pi\varepsilon_0}\right)^2\frac{\mu e^4}{\hbar^2}\frac{Z^2}{n^2} \qquad (4.3.11)$$

By inserting equation (4.3.9) in (4.3.11), one gets

$$E + \frac{\pi\beta}{\mu}\sqrt{\ell(\ell + 1)}\hbar = \frac{-1}{2}\left(\frac{1}{4\pi\varepsilon_0}\right)^2\frac{\mu e^4}{\hbar^2}\frac{Z^2}{n^2}$$

Or

$$E = \frac{-1}{2}\left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{\mu e^4}{\hbar^2}\frac{Z^2}{n^2} - \frac{\pi\beta}{\mu}\sqrt{\ell(\ell + 1)}\hbar \qquad (4.3.12)$$

For non laminar flow β equals β_0 given by Eqn. (4.2.2), thus Eqn. (4.3.12) becomes

$$E = \frac{-1}{2}\left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{\mu e^4}{\hbar^2}\frac{Z^2}{n^2} - \frac{6\pi^2\gamma\eta}{\mu}\sqrt{\ell(\ell + 1)}\hbar \qquad (4.3.13)$$

The second term is the energy loss of electron due to viscous media around nuclear of hydrogen.

Coefficient of viscosity of fluid around nuclear of hydrogen is in the form

$$\eta = \left[\,|E| - \frac{1}{2}\left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{\mu e^4}{\hbar^2}\frac{Z^2}{n^2}\right].\frac{\mu}{6\pi^2\gamma\sqrt{\ell(\ell+1)}\,\hbar} \qquad (4.3.14)$$

The first term E represent energy of electron move in viscous energy level but the second term is the electron energy when viscosity disappear.

In view of Eqn. (4.3.14) Planck's constant is given by

$$\hbar = \left[\,|E| - \frac{1}{2}\left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{\mu e^4}{\hbar^2}\frac{Z^2}{n^2}\right].\frac{\mu}{6\pi^2\gamma\eta\sqrt{\ell(\ell + 1)}} \qquad (4.3.15)$$

From De Broglie relation Eqn. (2.2.1), the De

Broglie wave length λ is given to be

$$\lambda = \frac{\hbar}{P} = \left[|E| - \frac{1}{2}\left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{\mu e^4}{\hbar^2}\frac{Z^2}{n^2}\right] . \frac{\mu}{6\pi^2\gamma\eta\, P\sqrt{\ell(\ell + 1)}}$$

Or

$$\lambda = \left[|E| - \frac{1}{2}\left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{\mu e^4}{\hbar^2}\frac{Z^2}{n^2}\right] . \frac{\mu}{6\pi^2\gamma\eta\sqrt{\ell(\ell+1)}\hbar P} \qquad (4.3.16)$$

For laminar flow one replace the value of βin Eqn. (4.3.12) by β_1 in Eqn. (4.2.3), to get

$$E = \frac{-1}{2}\left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{\mu e^4}{\hbar^2}\frac{Z^2}{n^2} - \frac{1}{4}\frac{\pi^2\eta\, N_g\gamma C_D}{\mu}\sqrt{\ell(\ell + 1)}\hbar \qquad (4.3.17)$$

The coefficient of viscosity from Eqn. (4.3.17) can be defined as

$$\eta = \left[|E| - \frac{1}{2}\left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{\mu e^4}{\hbar^2}\frac{Z^2}{n^2}\right] . \frac{4\,\mu}{\pi^2\gamma N_g C_D\sqrt{\ell(\ell+1)}\hbar} \qquad (4.3.18)$$

From Eqn. (2.2.1), De Broglie wave length is thus takes the form

$$\lambda = \left[|E| - \frac{1}{2}\left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{\mu e^4}{\hbar^2}\frac{Z^2}{n^2}\right] . \frac{4\mu}{\pi^2\gamma\eta N_g C_D\sqrt{\ell(\ell + 1)}\hbar P} \qquad (4.3.19)$$

(4-4) Effect of Viscosity on Total Energy of Electron in Hydrogen Atom According to Bohr Model

One can add the energy loss of electron due to viscous media around nuclear of hydrogen atom for laminar and non laminar flow in Eqn. (4.2.13) to Eqn. (2.8.2) as

$$\mu v^2 - \frac{kze^2}{r} - \frac{\pi\beta L}{\mu} = 0$$

Or

$$\mu v^2 = \frac{kze^2}{r} + \frac{\pi\beta L}{\mu} \qquad (4.4.1)$$

According to Eqn. (2.8.3) the angular momentum can be define as

$$L = n\hbar = \mu v r \qquad (4.4.2)$$

Or

$$v = \frac{L}{\mu r} = \frac{n\hbar}{\mu r} \qquad (4.4.3)$$

By inserting Eqn. (4.4.3) in Eqn. (4.4.1), one obtains

$$\frac{n^2\hbar^2}{\mu r^2} = \frac{kze^2}{r} + \frac{\pi\beta n\hbar}{\mu} \qquad (4.4.4)$$

Or

$$\beta = \frac{n\hbar}{\pi r^2} - \frac{kze^2\mu}{\pi r n\hbar} \qquad (4.4.5)$$

For non laminar flow, one substitute β in Eqn. (4.4.5) by the β_0

in Eqn. (4.2.2) to get

$$6\pi\gamma\eta = \frac{n\hbar}{\pi r^2} - \frac{kze^2\mu}{\pi n\hbar r} \tag{4.4.6}$$

From Eqn. (4.4.6) one can get the static viscosity coefficient as

$$\eta = \left(\frac{n\hbar}{r^2} - \frac{kze^2\mu}{n\hbar r}\right)\frac{1}{6\pi^2\gamma} \tag{4.4.7}$$

The energy loss of electron due to viscous media can be defined from Eqn. (4.4.4) after inserting the value of β from Eqn. (4.2.2) as

$$E_{vis} = \frac{6\pi^2\gamma\eta n\hbar}{\mu} \tag{4.4.8}$$

According to equation (4.4.6) and Eqn. (2.2.1) the De Broglie wave length is given by

$$\lambda = \left(\frac{n\hbar^2}{r^2} - \frac{kze^2\mu}{nr}\right)\frac{1}{12\pi^2\gamma\eta P} \tag{4.4.9}$$

For laminar flow replace β in Eqn. (4.4.5) by β_1 in Eqn. (4.2.3) to obtain the static viscosity coefficient as

$$\eta = \left(\frac{n\hbar}{r^2} - \frac{kze^2\mu}{rn\hbar}\right)\frac{4}{12\pi^2\gamma C_D N_g} \tag{4.4.10}$$

The energy loss for laminar flow can be defined from(4.4.4) after exchange β by its value Eqn. (4.2.4) as

$$E_{vis} = \frac{\pi^2\gamma\eta C_D N_g n\hbar}{4\mu} \tag{4.4.11}$$

From Eqn. (2.2.1) the wave length of De Broglie is thus

$$\lambda = \left(\frac{n\hbar^2}{r^2} - \frac{kze^2\mu}{nr}\right)\frac{4}{12\pi^2\gamma C_D N_g \eta P} \qquad (4.4.12)$$

(4-5) Friction of Electron in Hydrogen Atom

One can find a surface at which the centrifugal force is equal to centripetal force. This surface has kinetic friction force F_k equal to coefficient of kinetic friction μ_k times the press force on a surface (centrifugal force). But centrifugal force F_c is equal to electrostatic force F_e and also equal to reaction force F_r, i.e

$$F_r = F_c = F_e = \frac{Ze^2}{4\pi\varepsilon_0 r^2} \qquad (4.5.1)$$

But the friction force F_k is related to the reaction force according to the relation

$$f_k = \mu_k F_r = \mu_k \cdot \frac{Ze^2}{4\pi\varepsilon_0 r^2} \qquad (4.5.2)$$

The energy corresponding to Kinetic friction force is given for constant r by

$$E_{f_k} = -\int f_k \, ds = -\int_0^{\pi} \mu_k \cdot \frac{Ze^2}{4\pi\varepsilon_0 r^2} \, r d\theta = -\pi\mu_k \cdot \frac{Ze^2}{4\pi\varepsilon_0 r}$$

$$E_{f_k} = \pi\mu_k \, . V \qquad (4.5.3)$$

The operator of energy loss by friction is

$$\hat{E}_{f_k} = \pi\mu_k \hat{V} \qquad (4.5.4)$$

Where $\hat{V}$ is the electrostatic potential energy. To obtain the effect of friction and viscosity on total energy of electron in hydrogen atom, the Hamiltonian in classical mechanics can be written as

$$H = \frac{P^2}{2m} + V + E_{vis} + E_{f_k} \qquad (4.5.5)$$

In quantum mechanics the Hamiltonian becomes

$$\hat{H} = \frac{\hat{P}^2}{2m} + \hat{V} + \hat{E}_{\text{vis}} + \hat{E}_{f_k} \qquad (4.5.6)$$

The eigen equation of electron in Hydrogen atom with friction and viscosity energy losses is given by

$$\hat{H}\psi(r,\theta,\phi) = \left[\frac{\hat{P}^2}{2m} + \hat{V} + \hat{E}_{f_k} + \hat{E}_{\text{vis}}\right] \psi(r,\theta,\phi) \qquad (4.5.7)$$

Substitute Eqns. (4.2.18) and (4.5.4) in Eqn. (4.5.7) to get

$$\hat{H}\psi(r,\theta,\phi) = \left[\frac{\hat{P}^2}{2m} + \hat{V} + \pi\mu_k \hat{V} - \frac{\pi\beta}{\mu}\hat{L}\right] \psi(r,\theta,\phi) \quad (4.5.8)$$

The potentials are independent of time, since the above equation can be solved by independent time Schrödinger equation as

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2 sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi(r,\theta,\phi)+$$

$$\frac{2\mu}{\hbar^2}\left(E-\hat{V}-\pi\mu_k\hat{V}+\frac{\pi\beta}{\mu}\hat{L}\right)\psi(r,\theta,\phi)=0 \qquad (4.5.9)$$

One can get the effect of applying energy loss by viscous media operator on wave function from equation (4.3.7), and substitute it in equation (4.5.9) to get

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2 sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi(r,\theta,\phi)+$$

$$\frac{2\mu}{\hbar^2}\left(E-\hat{V}-\pi\mu_k\hat{V}+\frac{\pi\beta}{\mu}\sqrt{\ell(\ell+1)}\hbar\right)\psi(r,\theta,\phi)=0 \quad (4.5.10)$$

Similar to Eqn. (4.3.9) let

$$E_0=E+\frac{\pi\beta}{\mu}\sqrt{\ell(\ell+1)}\hbar \qquad (4.5.11)$$

Then equation (4.5.10) become

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2 sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi(r,\theta,\phi)+$$

$$\frac{2\mu}{\hbar^2}\left[E_0-(1+\pi\mu_k)\hat{V}\right]\psi(r,\theta,\phi)=0 \qquad (4.5.12)$$

Using the same procedures as in section (4-3) by definingE_0 as in Eqn. (4.4.9) and by using the solution (4.3.11), beside defining ¥ as

$$¥ = \frac{Ze^2(1+\pi\mu_k)}{4\pi\hbar\varepsilon_0}\sqrt{\frac{\mu}{-2E_0}} \tag{4.5.13}$$

One gets

$$E_0 = \frac{-1}{2}\left[\frac{1+\pi\mu_k}{4\pi\varepsilon_0}\right]^2\frac{\mu e^4}{\hbar^2}\frac{Z^2}{n^2} \tag{4.5.14}$$

Sub Eqn. (4.5.11) in Eqn. (4.5.14) to obtain

$$E + \frac{\pi\beta}{\mu}\sqrt{\ell(\ell+1)}\hbar = \frac{-1}{2}\left[\frac{1+\pi\mu_k}{4\pi\varepsilon_0}\right]^2\frac{\mu e^4}{\hbar^2}\frac{Z^2}{n^2} \tag{4.5.15}$$

From Eqn. (4.5.11) the total energy is

$$E = \frac{-1}{2}\left[\frac{1+\pi\mu_k}{4\pi\varepsilon_0}\right]^2 . \frac{\mu e^4}{\hbar^2}\frac{Z^2}{n^2} - \frac{\pi\beta}{\mu}\sqrt{\ell(\ell+1)}\hbar \tag{4.5.16}$$

Hence the coefficient of kinetic friction becomes

$$\mu_k = \frac{1}{\pi}\left[\sqrt{\left(|E| - \frac{\pi\beta}{\mu}\sqrt{\ell(\ell+1)}\hbar\right) . \frac{32\pi^2{\varepsilon_0}^2\hbar^2}{\mu e^4} . \frac{n^2}{Z^2}} - 1\right] \tag{4.5.17}$$

The De Broglie wave length in term of Stock and friction force becomes

$$\lambda = \frac{\hbar}{\pi P\mu_k}\left[\sqrt{\left(|E| - \frac{\pi\beta}{\mu}\sqrt{\ell(\ell+1)}\hbar\right) . \frac{32\pi^2{\varepsilon_0}^2\hbar^2}{\mu e^4} . \frac{n^2}{Z^2}} - 1\right] \tag{4.5.18}$$

(4-6) Effect of Friction and Viscosity of Electron in Hydrogen Atom According to Bohr Model

The work done is produced by friction force in Eqn. (4.5.3) can be added to Eqn. (4.4.4) as

$$\frac{n^2\hbar^2}{\mu r^2} = \frac{kze^2}{r} + \mu_s \frac{kze^2}{r} + \frac{\pi\beta n\hbar}{\mu} \tag{4.6.1}$$

Or

$$\mu_s = \frac{r}{kze^2}\left[\frac{n^2\hbar^2}{\mu r^2} - \frac{kze^2}{r} - \frac{\pi\beta n\hbar}{\mu}\right] \tag{4.6.2}$$

This expression can determine the coefficient of static friction for either flow.

(4-7) Effect of Viscosity for A Rigid Rotator Body

The electron in an atom and rotation of on atom in molecules be described by dynamics of a rigid body.

If the motion of a rotator body does not effected by any forces except Stock force, hence its classical Hamiltonian can be written as

$$H = \frac{P^2}{2m} + E_{vis} \tag{4.7.1}$$

But the Hamiltonian in quantum mechanics becomes

$$\hat{H} = \frac{\hat{P}^2}{2m} + \hat{E}_{\text{vis}} \tag{4.7.2}$$

The eigen equation of rigid rotator body is given by

$$\hat{H}\psi(r,\theta,\phi) = \left[\frac{\hat{P}^2}{2m} + \hat{E}_{\text{vis}}\right] \psi(r,\theta,\phi) \tag{4.7.3}$$

By comparing a rigid rotator body with Hydrogen atom, the energy loss due to viscous media for either flow of rigid body can be described by Eqn. (4.2.18), thus Eqn. (4.7.3) can be written as

$$\hat{H}\psi(r,\theta,\phi) = \left[\frac{\hat{P}^2}{2m} - \frac{\pi\beta}{\mu}\hat{L}\right] \psi(r,\theta,\phi) \tag{4.7.4}$$

The energy loss due to viscous media is independent of time, since the above equation can be solved by independent time Schrödinger equation as

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi(r,\theta,\phi) + \frac{2\mu}{\hbar^2}\left(E + \frac{\pi\beta}{\mu}\hat{L}\right)\psi(r,\theta,\phi) = 0 \tag{4.7.5}$$

To write the equation for rigid rotator, put $r = 1$ and replace m by I in the above equation. Doing so, one can obtain

$$\left[\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{sin\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi(\theta,\phi)+$$

$$\frac{2I}{\hbar^2}\left(E+\frac{\pi\beta}{I}\hat{L}\right)\psi(\theta,\phi)=0 \qquad (4.7.6)$$

The result of applying energy loss operator on the wave function $\psi(\theta,\phi)$ produces

$$\frac{\pi\beta}{I}\hat{L}\,\psi(\theta,\phi)=\frac{\pi\beta}{I}R\sqrt{\ell(\ell+1)}\,\hbar\psi(\theta,\phi) \qquad (4.7.7)$$

But E and $\frac{\pi\beta}{I}\sqrt{\ell(\ell+1)}\hbar$ are constant with respect to $\psi(\theta,\phi)$, then one can consider

$$E_0=E+\frac{\pi\beta}{I}\sqrt{\ell(\ell+1)}\hbar \qquad (4.7.8)$$

By substitute Eqn. (4.7.8) in Eqn. (4.7.6), one finds

$$\left[\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{sin\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi+\frac{2IE_0}{\hbar^2}\psi=0 \qquad (4.7.9)$$

Following the steps as shown in section (2.10) one obtains the value of $\mathfrak{B}$ in Eqn. (2.10.27) as

$$\mathfrak{B}=\ell(\ell+1)=\frac{2IE_0}{\hbar^2}$$

Or

$$E_0=\frac{\ell(\ell+1)\hbar^2}{2I}\ ,\ \ell\ =\ 0,1,2,3,4,\ldots. \qquad (4.7.10)$$

By inserting Eqn. (4.7.8) in Eqn. (4.7.10), the total energy of general flow becomes

$$E + \frac{\pi\beta}{I}\sqrt{\ell(\ell+1)}\hbar = \frac{\ell(\ell+1)\hbar^2}{2I}$$

Or

$$E = \frac{\ell(\ell+1)\hbar^2}{2I} - \frac{\pi\beta}{I}\sqrt{\ell(\ell+1)\hbar^2} \qquad (4.7.11)$$

For non laminar flow, replace β in Eqn.(4.7.11) by β_0 from Eqn.(4.2.2) to obtain

$$E = \frac{\ell(\ell+1)\hbar^2}{2I} - \frac{6\pi^2\gamma_a\eta_a}{I}\sqrt{\ell(\ell+1)\hbar^2} \qquad (4.7.12)$$

Where γ_a *and* η_a are radius and viscosity coefficient of rigid rotator body respectively.

The viscosity coefficient of a rigid rotator is thus given by

$$\eta_a = \left[E + \frac{\ell(\ell+1)\hbar^2}{2I}\right]\frac{I}{6\pi^2\gamma_a\sqrt{\ell(\ell+1)\hbar^2}} \qquad (4.7.13)$$

De Broglie wave length can be obtained from above equation as

$$\lambda = \left[E + \frac{\ell(\ell+1)\hbar^2}{2I}\right]\frac{I}{6\pi^2\eta_a\, P\gamma_a\sqrt{\ell(\ell+1)}} \qquad (4.7.14)$$

To get the total energy for laminar flow, replace β in Eqn.(4.7.11) by β_1 from Eqn.(4.2.4) to get

$$E = \frac{\ell(\ell+1)\hbar^2}{2I} - \frac{1}{4}\frac{\pi^2\eta_a\, N_{ga}\gamma_a C_{Da}}{I}\sqrt{\ell(\ell+1)}\hbar \qquad (4.7.15)$$

Where$N_{ga}$is Reynold's number and C_{Da} is the coefficient of air

Resistance.

The viscosity coefficient for laminar flow is thus given by

$$\eta_a = \left[E + \frac{\ell(\ell + 1)\hbar^2}{2I}\right]\frac{4\,I}{\pi^2 N_{ga}\gamma_a C_{Da}\sqrt{\ell(\ell + 1)\hbar^2}} \qquad (4.7.16)$$

The De Broglie wave length is given as

$$\lambda = \left[E + \frac{\ell(\ell + 1)\hbar^2}{2I}\right]\frac{4I}{\pi^2\, \eta_a P N_{ga} C_{Da}\gamma_a\sqrt{\ell(\ell + 1)}} \qquad (4.7.17)$$

(4-8) Effect of Viscosity of A Particle in Free Space

If the particle move in x-direction without effected by any forces except Stock force, hence the classical Hamiltonian can be found as

$$H = \frac{P^2}{2m} + E_{vis} \qquad (4.8.1)$$

But the Hamiltonian in quantum mechanics becomes

$$\hat{H} = \frac{\hat{P}^2}{2m} + \hat{E}_{\text{vis}} \qquad (4.8.2)$$

The eigen equation of a free particle in space is given by

$$\hat{H}\psi(x) = \left[\frac{\hat{P}^2}{2m} + \hat{E}_{\text{vis}}\right]\psi(x) \qquad (4.8.3)$$

The energy loss due to viscosity of vacuum for a free particle move in $x -$ direction is given according to Eqn. (4.2.6)

$$E_{vis} = -F_{stock}.x = -\beta\frac{px}{m} \tag{4.8.4}$$

Where, one consider υ to be constant

But the energy loss due to vacuum viscosity operator in general form is

$$\hat{E}_{vis} = -\frac{\beta\hat{p}\hat{x}}{m} \tag{4.8.5}$$

The energy loss due to viscous media is independent of time, since the above equation can be solved by independent time Schrödinger equation. By substitute Eqn. (4.8.5) in Eqn. (4.8.3) one can write Eqn. (4.8.3) as

$$\frac{-h^2}{2m}\frac{d^2\psi(x)}{dx^2} - \frac{\beta\hat{p}\hat{x}}{m}\psi(x) = E\psi(x) \tag{4.8.6}$$

But from uncertainty relation

$$\hat{p}\hat{x}\psi(x) = \Delta\hat{p}\Delta\hat{x}\psi(x) = \hbar\psi(x) \tag{4.8.7}$$

Substitute equation (4.8.7) in equation (4.8.6) one obtains

$$\frac{-h^2}{2m}\frac{d^2\psi(x)}{dx^2} - \frac{\beta\hbar}{m}\psi(x) = E\psi(x) \tag{4.8.8}$$

Or

$$\frac{d^2\psi(x)}{dx^2} + \left(\frac{2m}{h^2}E + \frac{2\beta}{\hbar}\right)\psi(x) = 0 \tag{4.8.9}$$

Let $k^2 = \frac{2m}{h^2}E + \frac{2\beta}{\hbar}$ (4.8.10)

Thus equation (4.8.9) became

$$\frac{d^2\psi(x)}{dx^2} + k^2\,\psi(x) = 0 \tag{4.8.11}$$

The solution of equation (4.8.11) is

$$\psi(x) = e^{ikx} + e^{-ikx} \tag{4.8.12}$$

Assume the motion in positive direction of x, the wave function became

$$\psi(x) = e^{ikx} \tag{4.8.13}$$

From equation (4.8.10) the eign value of energy is

$$E = \frac{h^2k^2}{2m} - \frac{\beta ħ}{m} \tag{4.8.14}$$

Substitute Eqn. (4.2.2) in Eqn. (4.8.14) to get energy eigen value for non laminar flow as

$$E = \frac{P^2}{2m} - \frac{6\pi\eta_f\gamma_f ħ}{m} \tag{4.8.15}$$

Then the coefficient of viscosity can be writing as

$$\eta_f = \frac{P^2}{12\pi\gamma_f ħ} - \frac{mE}{6\pi\gamma_f ħ} \tag{4.8.16}$$

De Broglie relation (2.2.1), the wave length is given as

$$\lambda = \frac{P^2}{12\pi\gamma_f\eta_f P} - \frac{mE}{6\pi\gamma_f\eta_f P} \tag{4.8.17}$$

Also substitute Eqn. (4.2.4) in Eqn. (4.8.14) for laminar flow, the eigen value became

$$E = \frac{P^2}{2m} - \frac{\pi\, \eta_f \gamma_f\, N_{gf} C_{Df} ħ}{4\, m} \tag{4.8.18}$$

And the coefficient of viscosity is given as

$$\eta_f = \frac{4m}{\pi\, \gamma_f\, N_{gf} C_{Df} ħ}\left[\frac{P^2}{2m} - E\right] \tag{4.8.19}$$

Also from De Broglie relation (2.2.1), the wave length in this case becames

$$\lambda = \frac{4m}{\pi\, \gamma_f \eta_f P\, N_{gf} C_{Df}}\left[\frac{P^2}{2m} - E\right] \tag{4.8.20}$$

(4-9) Work Done of Stock Force in Oscillation System

The energy loss due to harmonic oscillator vibrating in viscous media is obtained in terms of Stock's force acting on it as

$$E_{vis} = -\int F_{Stock}\, dx = -\int \beta\, \upsilon dx \tag{4.9.1}$$

The position of oscillation system is

$$x = x_0 e^{i\omega t} \tag{4.9.2}$$

But the linear velocity can be found from first derivative of position as

$$\upsilon = \frac{dx}{dt} = i\omega x$$

Or

$$\upsilon = i\omega x \tag{4.9.3}$$

Sub Eqn. (4.9.3) in Eqn. (4.8.1) to get

$$E_{vis} = -i\omega\beta \int x\, dx = -\frac{i\omega\beta}{2} x^2$$

$$E_{vis} = -\frac{i\omega\beta}{2} x^2 \tag{4.9.4}$$

Thus the energy loss due to viscous media operator is given by

$$\hat{E}_{vis} = -\frac{i\omega\beta}{2} \hat{x}^2 \tag{4.9.5}$$

<u>(4-10) The Effect of Viscosity on Oscillation System</u>

The Hamiltonian of harmonic oscillator along x-direction can be described in classical mechanics as

$$H = \frac{P^2}{2m} + V + E_{vis} \tag{4.10.1}$$

But the Hamiltonian in quantum mechanics becomes

$$\hat{H} = \frac{\hat{P}^2}{2m} + \hat{V} + \hat{E}_{\mathrm{vis}} \tag{4.10.2}$$

By substitute Eqn. (4.9.5) and Eqn. (2.11.2) in Eqn. (4.10.2), the eigen equation of harmonic oscillator is thus given by

$$\hat{H}\psi(x) = \left[\frac{\hat{P}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 - \frac{i\omega\beta}{2} \hat{x}^2\right] \psi(x) \tag{4.10.3}$$

Since the potential is independent of time, Eqn. (4.10.3) can be written as independent time Schrödinger equation in the form

$$\frac{d^2\psi(x)}{dx^2} + \frac{2m}{\hbar^2}\left[E - \frac{1}{2}m\omega^2x^2 + \frac{i\omega\beta}{2}x^2\right]\psi(x) = 0 \qquad (4.10.4)$$

Transform this equation into a variable ξ defined by

$$\xi = \acute{\alpha}x \qquad (4.10.5)$$

The parameter $\acute{\alpha}$ will be chosen in such a way that the new equation looks simple, thus

$$\frac{d\psi}{dx} = \frac{d\psi}{d\xi}\frac{d\xi}{dx} = \acute{\alpha}\frac{d\psi}{d\xi}$$

$$\frac{d^2\psi(x)}{dx^2} = \acute{\alpha}\frac{d}{d\mathrm{x}}\left(\frac{d\psi}{d\xi}\right) = \acute{\alpha}\frac{d}{d\xi}\left(\frac{d\psi}{d\xi}\right)\frac{d\xi}{d\mathrm{x}} = \acute{\alpha}^2\frac{d^2\psi}{d\xi^2}$$

In terms of new variable Eqn. (4.10.5) becomes

$$\frac{d^2\psi}{d\xi^2} + \left[\frac{2mE}{h^2\acute{\alpha}^2} - \left(\frac{\mathrm{m}^2\omega^2}{h^2} - \frac{i\omega\beta m}{h^2}\right)\frac{\xi^2}{\acute{\alpha}^4}\right]\psi = 0 \qquad (4.10.6)$$

Assume $\acute{\alpha}^4$ to be defined as

$$\acute{\alpha}^4 = \frac{\mathrm{m}^2\omega^2 - i\omega\beta m}{h^2} \qquad (4.10.7)$$

Equation(4.10.6) now becomes

$$\frac{d^2\psi}{d\xi^2} + \left[\frac{2mE}{h^2\acute{\alpha}^2} - \xi^2\right]\psi = 0 \qquad (4.10.8)$$

Introducing the dimensionless parameter $\mathcal{B}$ defined by

$$\mathcal{B} = \frac{2mE}{h^2\acute{\alpha}^2} \tag{4.10.9}$$

Eqn. (4.10.8) becomes

$$\frac{d^2\psi}{d\xi^2} + [\mathcal{B} - \xi^2]\psi = 0 \tag{4.10.10}$$

Using the same steps of solution in section (2-11) one obtains the value of $\mathcal{B}$ in the form

$$\mathcal{B} = \frac{2mE}{h^2\acute{\alpha}^2} = 2n + 1 \tag{4.10.11}$$

The energy eigen value is thus given by

$$E_n = \left(n + \frac{1}{2}\right)\frac{h^2\acute{\alpha}^2}{m} = \left(n + \frac{1}{2}\right)\frac{h^2}{m}\sqrt{\frac{\mathrm{m}^2\omega^2 - i\beta\omega m}{h^2}}$$

Or

$$E_n = \left(n + \frac{1}{2}\right)h\omega\sqrt{1 - \frac{i\beta}{\omega m}} \tag{4.10.12}$$

Or

$$\beta = \omega m \frac{\left[\left(n + \frac{1}{2}\right)h\omega\right]^2 - E_n^{\ 2}}{i\left[\left(n + \frac{1}{2}\right)h\omega\right]^2} \tag{4.10.13}$$

For non laminar flow substitute the value of β from Eqn. (4.2.2) in the above equation to obtain viscosity coefficient as

$$\eta_h = \frac{\omega m}{i6\pi\gamma_h}\frac{\left[\left(n + \frac{1}{2}\right)h\omega\right]^2 - E_n^{\ 2}}{\left[\left(n + \frac{1}{2}\right)h\omega\right]^2} \tag{4.10.14}$$

From De Broglie relation (2.2.1), the wave length is given as

$$\lambda = \frac{\omega m}{i6\pi\gamma_h hP\eta_h}\frac{\left[\left(n+\frac{1}{2}\right)h\omega\right]^2 - E_n^{\,2}}{\left[\left(n+\frac{1}{2}\right)\omega\right]^2} \qquad (4.10.15)$$

But for laminar flow viscosity coefficient can be obtained by substituting Eqn. (4.2.4) in Eqn. (4.10.13) to get

$$\eta_h = \frac{4\omega m}{i\pi\gamma_h N_{gh} C_{Dh}}\frac{\left[\left(n+\frac{1}{2}\right)h\omega\right]^2 - E_n^{\,2}}{\left[\left(n+\frac{1}{2}\right)h\omega\right]^2} \qquad (4.10.16)$$

From De Broglie relation (2.2.1), the wave length is given as

$$\lambda = \frac{4\omega m}{i\pi\eta_h\gamma_h N_{gh} C_{Dh} hP}\frac{\left[\left(n+\frac{1}{2}\right)h\omega\right]^2 - E_n^{\,2}}{\left[\left(n+\frac{1}{2}\right)\omega\right]^2} \qquad (4.10.17)$$

(4-10) Discussion

The quantum equation which describes particle motion in a viscous medium requires an expression for energy lost by viscous medium, this relation is shown by equation (4.2.13). It relates angular momentum to viscosity as shown in this equation. The corresponding operator is exhibited in equation (4.2.18). The new Schrödinger equation accounts for the effect of viscosity as shown by Eqn. (4.3.5). It was derived by replacing the viscous classical energy, which depends on L, by the corresponding operator after adding it to the energy part in conventional Schrödinger equation [see Eqn. $(4.3.5)$]. By separating variables to angular and radial parts, the viscous energy adds to Schrödinger equation additional terms in equation $(4.3.8)$ and $(4.3.12)$ which shows viscous energy quantization. This viscous energy is proportional to the viscosity coefficient η. It reduces to the ordinary energy of Hydrogen atom in the absence of viscosity. The Renold's number obtained in equation $(4.3.17)$ is consistent with the observed values. It is very interesting to note that the electron De Broglie wave length depends on viscosity as Eqn. $(4.3.16)$ indicate. The fluid pressure which may represent vacuum one is quantized as shown in Eqn. $(4.3.18)$. It depends on angular quantum number. This is obvious as far as electrons wave in a circular path. Bohr model is also used to quantize viscosity, in section $(4-4)$. By using relations between forces of centrifugal, electric and viscous force, viscosity coefficient is shown to be quantized in Eqn. $(4.4.7)$. The electron De Broglie wave length is quantized also as shown by Eqn. (4.4.9). It is very interesting to note that both η and λ depends on principle quantum number n. This is since in Bohr model the angular momentum L depends on principle quantum number n and coefficient of viscosity η results from electron circular motion. For a rigid body Eqn. (4.7.13) and (4.7.14) shows that both η and λ depends on orbital quantum number ℓ. Using uncertainty principle the De Broglie wave length λ and η for free particles are shown to be correlated, as Eqn. (4.8.16) shows as η increase λ decrease, which conforms with common sense. According to equation (4.8.6) the energy lost due to viscosity is dependent on x^2, ω as well as viscosity coefficient β. This is obvious, since increasing x and ω besides β, increases losses. Using classical and quantum expressions of oscillator [see Eqn. (4.8.3), (4.9.7) and (4.9.8)], the viscous energy is constant which depends on β as well as h and m. The Hamiltonian and Schrödinger equation are given by Eqns. (4.10.1) – (4.10.4). Equation (4.10.4) reduces to ordinary Schrödinger equation in the absence of friction. The energy in equation (4.10.12) consists of an additional term consisting of viscosity coefficient β. This expression reduces to that of ordinary harmonic oscillator in the absence of friction. Equation (4.10.14) in dicates that the friction is quantized.

(4-11) Conclusion

The quantum viscous model reduces to Schrödinger equation in the absence of friction. This model shows that viscosity is quantized, it also shows that electron De Broglie wave length depends on viscosity. For harmonic oscillator it shows that the total energy is quantized, including friction energy.

Suggestion for Future Work;

1- Modified Schrödinger equation for other quantum systems do not study in this research by energy loses due to viscous media.
2- Modified Schrödinger equation by energy loses due to Archimedes principle.
3- Quantization of electron mass in Hydrogen atom.

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Appendix A

The term viscosity is commonly used in the description of fluid flow to characterize the degree of internal friction in the fluid. This internal friction, or *viscous force*, is associated with the resistance that two adjacent layers of fluid have to moving relative to each other. Viscosity causes part of the kinetic energy of a fluid to be converted to internal energy. This mechanism is similar to the one by which an object sliding on a rough horizontal surface loses kinetic energy.

The viscosity coefficient is define as the ratio of shear stress to rate of shear

$$\eta = \frac{F/A}{\upsilon/L} \qquad (1)$$

The dominator of equation (1) is pressure

$$P = \frac{F}{A} \qquad (2)$$

For spherical particle the viscosity coefficient is

$$\eta = \frac{F}{6\pi\gamma} \qquad (3)$$

Where F is the force act on a particle from fluid in opposite direction of motion which called Stock's force is define as

$$F_{Stock} = 6\pi\gamma\eta \qquad (4)$$

When a particle moving in fluid, the flow of fluid can be characterized as being one of two main types. The flow is said to be steady, or non laminar, if each particle of the fluid follows a smooth path and can be describe by equation (4) and the laminar flow, if the particle of the fluid disturbed, but here the Stock's force proportional to the square power of velocity and can be define by equation

$$F_{Stock} = \frac{1}{2}\rho C_D A \upsilon^2 \qquad (5)$$

Renold's number is the critical value determine the laminar non laminar flow, and it's define as

$$N_g = \frac{\rho \upsilon d}{\eta} \qquad (\mathbf{6})$$

Where d is specific dimension for type of flow and it represent diameter for spherical particle or

$$d = 2\gamma \qquad (7)$$

Substitute Eqn. (6) and (7) in Eqn. (5) to obtain

$$F_{Stock} = \frac{1}{4}\pi\eta \, N_g \gamma C_D \upsilon \qquad (8)$$