Sudan University of Science and Technology
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Solution of Partial Differential Equations with Nonlocal Conditions by Combine Homotopy Perturbation Method and Laplace Transform

حل المعادلات التفاضلية الجزئية ذات الشروط اللا موضعية بدمج طريقة ارتجاج الهموتوبيا وتحويل لابلاس

A Thesis Submitted in Fulfillment Requirements for the Degree of Doctor of philosophy in Mathematics

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Dedication

To my Father, my Mother, my Wife, All my colleagues and everyone who shares beliefs, thoughts and efforts.

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I am thankful to Allah the almighty in completing this work. I would like to express my special gratitude and thanks to my advisor **Dr.Tarig M.Elzaki**, who stood behind me. I would like to thank him for encouraging my research and for allowing me to grow as a research scientist. His advice on my research as well as on my career has been invaluable. I would like to thank **Dr. Mohmmed Hassan M.Khabier**, who cooperated in this study.

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Their prayer for me was what sustained me thus far.

Abstract

In this thesis, the homotopy perturbation method (HPM) was presented, and applied for solving some differential and integral equations with non-local conditions (linear and nonlinear). This method provides an analytical approximate solution of the differential equations. A combined form of the Laplace transform method and homotopy perturbation method, called the homotopy perturbation transform method (HPTM) was introduced and used to solve nonlinear equations. The nonlinear terms of the nonlinear equations was easily handled and treated by the use of He's polynomials. One of the significant advantages of this method its ability to find solutions without any discretization or restrictive assumptions avoiding the round-off errors. The fact that the proposed technique solves nonlinear problems without using Adomian's polynomials can also be considered as an additional advantage of this algorithm over the decomposition method. Furthermore, a new approach to solving non-local initial-boundary value problems for linear and nonlinear parabolic and hyperbolic partial differential equations subject to initial and nonlocal boundary conditions of integral type was introduced. The technique of transforming the given non-local initial-boundary value problems of an integral type, into local Dirichlet initial-boundary value problems was implemented for both linear and nonlinear parabolic and hyperbolic partial differential equations, and then the homotopy perturbation method (HPM) was applied to their problems.

الخلاصـــــة

في هذا البحث تم تقديم طريقة الهموتوبيا (HPM) لحل المعادلات التفاضلية والتكاملية ذات الشروط اللا موضعية بنوعيها الخطي واللاخطي .وقد وضح جلياً قدرة هذه الطريقة على ايجاد حلول تقريبية بدقة متناهية وتحليلية لبعض المعادلات اللاخطية والتي كان حلها التحليلي يشكل تحديا للرياضيين عبر قرون مضت بالرغم من أهمية هذه المعادلات في نمذجة الكثير من الظواهر في كافة مناحي الحياة .طبقت كثيرة حدود (هي - He) بكفاءة وسهولة للتعامل مع الحدود اللاخطية .أحد أبرز وأهم مميزات هذه الطريقة هي قدرتها على ايجاد حلول المعادلات التفاضلية الجزئية اللاخطية دون استخدام تجزئة مجال الحل أو وضع فروض وقيود مشددة بالاضافة لقدرتها على الحل دون استخدام كثيرة حدود (أدومين – Adomian) الشيئ الذي تفتقده الطرق المستخدمة من قبل. قدم هذا البحث توجه جديد لحل المعادلات التفاضلية المكافئية واللامكافئية باستخدام شروط حدية ابتدائية ولا موضعية من النوع التكاملي. قبل الشروع في تطبيق (HPM) لحل المعادلات المكافئية والزائدية تم تحويل الشروط الحدية اللاموضعية من النوع التكاملي إلى مسائل القيمة الابتدائية والحدية لدريشلت.

Introduction

The homotopy perturbation method (HPM) is a series expansion method used in the solution of nonlinear partial differential equations. The method employs a homotopy transform to generate a convergent series solution of differential equations. This gives flexibility in the choice of basis functions for the solution and the linear inversion operators (as compared to the Adomian decomposition method), it helps in retaining a simplicity that makes the method easily understandable from the standpoint of general perturbation methods. The HPM was introduced by Ji-Huan He in 1998. He [1–14] developed the homotopy perturbation method (HPM) by marging the standard homotopy and perturbation for solving various physical problems.The authors have applied this method successfully to problems arising in mathematics engineering.

The HPM is a special case of the homotopy analysis method (HAM) developed by Liao Shijunin 1992 [40-43]. The HAM uses a so-called convergence-control parameter to guarantee the convergence of approximation series over a given interval of physical parameters.

The Laplace transform is totally incapable of handling nonlinear equations because of the difficulties that are caused by the nonlinear terms. Various ways have been proposed recently to deal with these nonlinearities such as the Adomian decomposition method [20] and the Laplace decomposition algorithm[15–19]. Furthermore, the homotopy perturbation method is also combined with the well-known Laplace transform method [21] and the variation iteration method [22] to produce a highly effective technique for handling many nonlinear problems.

Over the last few years, various processes in the natural sciences and engineering lead to the non-classical parabolic initial/boundary-value problems. They involve non-local integral terms over the spatial domain. The integral terms may appear in the boundary conditions in which case the boundary condition is called non-local, or in the governing partial differential equation itself, it is referred to as a partial integro-differential equation, or in both. The non-local boundary condition has been studied by several authors [25-39].

Non-local boundary-value problems were first used by [23, 24]. The presence of an integral term in a boundary condition can complicate the application of standard numerical techniques such as finite difference procedures, finite element methods, spectral techniques, boundary integral equation schemes, etc. It is therefore important to convert the non-local boundary value problems to more desirable form. It is a hard task to make them more applicable to the problems of practical interest. The accuracy of the quadrature must be compatible with the discretization of the differential equation.

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CHAPTER ONE

Homotopy Perturbation Method (HPM)

In the last two decades, the rapid development of nonlinear science that has appeared ever-increasing interest of scientists and engineers in the analytical techniques for nonlinear problems. The widely applied techniques are perturbation methods. However, like other nonlinear analytical techniques, perturbation methods have their own limitations. Firstly, most perturbation methods are based on an assumption that a small parameter must exist in the equation. As it is well known, a majority of nonlinear problems have no small parameters at all. Secondly, the determination of small parameters seems to be a special art requiring special techniques. An appropriate choice of small parameters leads to ideal results. Thus, an unsuitable choice of small parameters results in bad effects. Furthermore, the approximate solutions solved by the perturbation methods are valid. Obviously, all these limitations come from the small parameter assumption. Various perturbation methods have been widely applied to solve nonlinear problems. Many new techniques have been proposed recently to eliminate the "small parameter" assumption, such as the artificial parameter method proposed by He [44], the homotopy analysis method proposed by He [45]. A review of recently developed nonlinear analysis method can be found in details in [46]. The homotopy perturbation method (HPM), proposed first by He [1, 2], for solving differential and integral equations linear and nonlinear has been the subject of extensive analytical and numerical studies. The method, which is a coupling of traditional perturbation method and homotopy in topology, reforms continuously to a simple problem which is easily solved. This method, which does not require a small parameter in an equation, has a significant advantage. This HPM yields a very rapid convergence of the solution series in most cases, only a few iterations leading to very accurate solutions. Thus, He's HPM is a universal one which can solve various kinds of nonlinear equations. This chapter is presented the homotopy perturbation method (HPM), the modified homotopy perturbation method. They are applied for solving some partial differential equations (linear and nonlinear) and canceling noise-terms phenomenon. Homotopy perturbation method doesn't require a small parameter in an equations. This method provides an analytical approximate solution for the differential equations.

1.1 Homotopy perturbation method

Definition (1.1.1)

Let X and Y be the topological spaces. If , f and g are continuous maps of the space X into Y, it is said that f is homotopic to g, if there is continuous map $F: X \times I(=[0,1]) \rightarrow Y$ Such that $F(x,0) = f(x)$ *and* $F(x,1) = g(x)$, for each $x \in X$ then the map is called homotopy between f *and* g.

The homotopy perturbation method is a combination of classical perturbation technique and the homotopy map used in topology.

1.1.1 Basic Idea of Homotopy Perturbation Method

To explain the basic ideas of the homotopy perturbation method, we consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, \qquad r \in \Omega \tag{1}$$

With the boundary condition;

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \ \ r \in \Gamma \tag{2}$$

Where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytical function and Γ is the boundary of the domain Ω. Generally speaking, operator A can be divided into two parts which are L and N where L is linear, but N is nonlinear. Therefore, equation (1) can be rewritten in the form:

$$L(u) + N(u) - f(r) = 0 \tag{3}$$

By the homotopy perturbation technique, we construct a homotopy $V(r,p): \Omega \times [0,1] \rightarrow R$ which satisfies:

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0 \quad p \in [0,1], r \in \Omega \quad (4)$$

Or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \quad (5)$$

Where $p\epsilon[0,1]$ is an embedding parameter and u_0 is an initial approximation of equation (1).Obviously, from these definitions we will have:

$$H(v,0) = L(v) - L(u_0) = 0 \quad (6)$$

$$H(v,1) = A(v) - f(u_0) = 0 \quad (7)$$

The changing process of p from zero to one is just that of $v\,(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation and $L(v)$ – $L(u_0)$ and $A\,(v)$ – $f\,(r)$ are called homotopy. According to the HPM, we can first use the embedding parameter p as a “small parameter” and assuming that the solution of Eq. (4) can be written as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + \dots \quad (8)$$

Setting $p = 1$, results in the approximate solution of Eq. (3):

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \dots. \quad (9)$$

1.1.2 Analysis of Convergence

Let us write Eq. (5) in the following form:

$$L(v) = L(u_0) + p[f(r) - N(v) - L(u_0)] \quad (10)$$

Applying the inverse operator, L^{-1} to both sides of Eq. (10), We obtain:

$$v = u_0 + p[L^{-1}f(r) - L^{-1}N(v) - u_0] \quad (11)$$

Suppose that

$$v = \sum_{i=0}^{\infty} p^i\, v_i \quad (12)$$

Substituting Eq. (12) into the right-hand side of Eq. (11), we get;

$$v = u_0 + p\left[L^{-1}f(r) - (L^{-1}N)\left[\sum_{i=0}^{\infty} p^i\, v_i\right] - u_0\right] \qquad (13)$$

If $p \to 1$, the exact solution may be obtained by using Eq. (9)

$$u = \lim_{p\to 1} v = L^{-1}f(r) - (L^{-1}N)\left[\sum_{i=0}^{\infty} v_i\right] = L^{-1}f(r) - \sum_{i=0}^{\infty}(L^{-1}N)\,(v_i) \qquad (14)$$

To study the convergence of the method let us state the following Theorem.

Theorem (1.1.2) (Sufficient Condition of Convergence)

Suppose that X and Y are Banach spaces and $N : X \to Y$ is a contractive nonlinear mapping, that is

$$\forall w;\ w^* \epsilon\, X;\ \|N(w) - N(w^*)\| \leq \|w - w^*\|, 0 < Y < 1 \qquad (15)$$

Then according to Banach's fixed point theorem N has a unique fixed point u, that is $N(u) = u$. Assume that the sequence generated by homotopy perturbation method can be written as;

$$W_n = N(W_{n-1}), \qquad W_{n-1} = \sum_{i=0}^{n-1} w_i \qquad n = 0, 1, 2, 3, \ldots. \qquad (16)$$

And suppose that $W_0 = w_0 \epsilon B_r(w)$ where $B_r(w) = \{w^* \epsilon\, X;\ \|w^* - w\| < r\}$, then we get;

(i) $W_n \epsilon B_r(w)$

(ii) $lim_{n\to} W_n = w$

(i) By inductive approach, for $n = 1$ we get;

$$\|W_1 - w\| = \|N(W_0 - N(w)\| < Y\|w_0 - w\|$$

Assume that

$$\|W_{n-1} - w\| \leq Y^{n-1}\|w_0 - w\|$$

as induction hypothesis, then

$$\|W_n - w\| = \|N(W)_{n-1} - N(w)\| \le Y\|W_{n-1} - w\| \le Y^n\|w_0 - w\|.$$

Using (i), we get;

$$\|W_n - w\| \le Y^n\|w_0 - w\| \le Y^n r \le r \Rightarrow W_n \epsilon B_r(w)$$

(ii) Because of

$$\|W_n - w\| \le Y^n\|w_0 - w\| \text{ and } lim_{n\to\infty} Y^n = 0$$

$$lim_{n\to\infty}\|W_n - w\| = 0 \text{ that is } lim_{n\to\infty} W_n = w.$$

1.2 Application of Homotopy Perturbation Method to Linear Partial Differential Equations

The application of the homotopy perturbation method in linear problems has been devoted by scientists and engineers. The most perturbation methods are based on the assumption that a small parameter exist, which is too over-strict to find wide application. Therefore, many new techniques have been proposed to eliminate the "small parameter" assumption, such as He's homotopy perturbation method. The homotopy perturbation method, which provides analytical approximate solution, is applied to various linear and non-linear equations.

Example (1.2.3) We consider the following inhomogeneous partial differential equation [61],

$$u_x + u_y = x + y \tag{17}$$

With the initial conditions;

$$u(x,0) = 0, \quad u(0,y) = 0 \tag{18}$$

To solve Eq. (17) with initial conditions Eq. (18), according to the homotopy perturbation Eq. (4), we construct the following homotopy:

$$H(v,p) = (1-p)[v_x - (u_0)_x] + p\big[v_x + v_y - (x+y)\big] = 0 \tag{19}$$

Or

$$v_x - (u_0)_x + p\left[(u_0)_x + v_y - (x+y)\right] = 0 \tag{20}$$

Suppose the solution of Eq. (17) has the form:

$$v_j = \sum_{j=0}^{\infty} p^j v_j \tag{21}$$

Substituting Eq. (21) into Eq. (20), and comparing coefficients of the terms with the identical powers of p, we get;

$$\left(\sum_{j=0}^{\infty} p^j v_j\right)_x - (u_0)_x + p\left[(u_0)_x + \left(\sum_{j=0}^{\infty} p^j v_j\right)_y - (x+y)\right] = 0 \tag{22}$$

$$\begin{aligned}
&p^0 : (v_0)_x - (u_0)_x = 0, \qquad v_0(x,y) = 0 \\
&p^1 : (v_1)_x + (u_0)_x + (v_0)_y - (x+y) = 0, v_1(x,y) = \frac{x^2}{2} + xy \\
&p^2 : (v_2)_x + (v_1)_y = 0, \qquad v_2(x,y) = -\frac{x^2}{2} \\
&p^3 : (v_3)_x + (v_2)_y = 0, \qquad v_3(x,y) = 0 \\
&v_k(x,y) = 0, k \geq 3
\end{aligned} \tag{23}$$

Now the solution of Eq. (17) when $p \to 1$ will be reduced to:

$$u(x,y) = xy \tag{24}$$

This solution coincides with the exact one.

Example(1.2.4) We consider the following homogeneous Partial differential equation

$$u_x - u_y = 0 \tag{25}$$

With the initial conditions;

$$u(x,0) = 0, \quad u(0,y) = 0 \tag{26}$$

To solve Eq. (25) with initial conditions Eq. (26), according to the homotopy perturbation Eq. (4) we construct the following homotopy:

$$H(v,p) = (1-p)[v_x - (u_0)_x] + p\big[v_x - v_y - (x+y)\big] = 0 \qquad (27)$$

Or

$$v_x - (u_0)_x + p\big[(u_0)_x - v_y\big] = 0 \qquad (28)$$

Substituting Eq. (9) into Eq. (28), and comparing coefficients of the terms with the identical powers of p, we get;

$$p^0 : \ (v_0)_x - (u_0)_x = 0, \qquad v_0(x,y) = y$$

$$p^1 : \ (v_1)_x + (u_0)_x - (v_0)_y = 0, \qquad v_1(x,y) = x \qquad (29)$$

$$p^2 : \ (v_2)_x - (v_1)_y = 0, \qquad v_2(x,y) = 0$$

$$v_k(x,y) = 0, k \geq 2$$

Now the solution of Eq. (25) when $p \to 1$ will be reduced to:

$$u(x,y) = x + y \qquad (30)$$

This solution coincides with the exact one.

Example (1.2.5) We consider the following homogeneous Partial differential equation [61],

$$xu_x + u_y = 3u \qquad (31)$$

With the initial conditions;

$$u(x,0) = x^2, \quad u(0,y) = 0 \qquad (32)$$

To solve Eq. (31) with initial conditions Eq. (32), according to the homotopy perturbation Eq. (4) we construct the following homotopy:

$$H(v,p) = (1-p)\big[v_y - (u_0)_y\big] + p\big[xv_x + v_y - 3v\big] = 0$$

Or

$$v_y - (u_0)_y + p\big[(u_0)_y + xv_x - 3v\big] = 0 \qquad (33)$$

Substituting Eq. (9) into Eq. (33), and comparing coefficients of the terms with the identical powers of p, we get;

$$p^0 : (v_0)_y - (u_0)_y = 0, \qquad v_0(x,y) = x^2$$
$$p^1 : (v_1)_y + (u_0)_y + (xv_0)_x - 3v_0 = 0, v_1(x,y) = x^2 y \qquad (34)$$
$$p^2 : (v_2)_y + (xv_1)_x - 3v_1 = 0, v_2(x,y) = \frac{x^2y^2}{2}$$

Thus, the solution in series form is given by:

$$u(x,y) = \sum_{k=0}^{\infty} v_k(x,t)$$

$$u(x,y) = x^2(1 + y + \frac{y^2}{2} + \frac{y^3}{3!} + \cdots)$$

The solution for equation (31) in a closed form is given by:

$$u(x,y) = x^2 e^y \qquad (35)$$

Example (1.2.6) We consider the following inhomogeneous Partial differential equation

$$u_x + u_y + u_z = u \qquad (36)$$

With the initial conditions;

$$\begin{cases} u(0,y,z) = 1 + e^y + e^z \\ u(x,0,z) = 1 + e^x + e^z \\ u(x,y,0) = 1 + e^x + e^y \end{cases} \qquad (37)$$

Where $u = u(x,y,z)$

To solve Eq. (36) with initial conditions Eq. (37), according to the homotopy perturbation Eq. (4), we construct the following homotopy:

$$v_x - (u_0)_x + p\left[(u_0)_x + v_y + v_z - v\right] = 0 \qquad (38)$$

Substituting Eq. (9) into Eq. (38), and comparing coefficients of the terms with the identical powers of p, we get;

$$p^0:\ (v_0)_x-(u_0)_x=0,\qquad v_0(x,y,z)=1+e^y+e^z$$

$$p^1:(v_1)_x+(u_0)_x+(v_0)_y+(v_0)_z-v_0=0,\quad v_1(x,y,z)=x$$

$$p^2:\ (v_2)_x+(v_1)_y+(v_1)_z-v_1=0,\qquad v_2(x,y,z)=\frac{x^2}{2!}\qquad(39)$$

$$p^3:\ (v_3)_x+(v_2)_y+(v_2)_z-v_2=0,\qquad v_3(x,y,z)=\frac{x^3}{3!}$$

$$\vdots$$

Thus, the solution in series form is given by:

$$u(x,y,z)=\sum_{k=0}^{\infty}v_k(x,y,z)$$

$$u(x,y,z)=e^y+e^z+\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)\qquad(40)$$

The solution for equation Eq. (40) in a closed form is given by:

$$u(x,y,z)=\quad e^x+e^y+e^z\qquad(41)$$

Example (1.2.7) We consider the following second order Partial differential equation [62],

$$w_t\ =\ w_{xx}-w_x\qquad(42)$$

Subject to the initial condition;

$$w(x,0)=e^x-x\qquad(43)$$

And boundary conditions;

$$w(0,t)=1+t\,,\quad w_x(0,t)=0\qquad(44)$$

The standard HPM, in view of the homotopy Eq. (4), we construct the homotopy in the following form:

$$H(v,p)=(1-p)[v_t-(w_0)_t]+p[v_t-v_{xx}+v_x]=0\qquad(45)$$

Or

$$v_t - (w_0)_t + p[(w_0)_t - v_{xx} + v_x] = 0 \tag{46}$$

Substituting Eq. (9) and the initial condition Eq. (43) into the homotopy Eq. (46) and equating the terms with identical powers of p as follows:

$$w = \sum_{j=0}^{\infty} p^j v_j$$

Then

$$\left(\sum_{j=0}^{\infty} p^j v_j\right)_t - (w_0)_t + p\left[(w_0)_t - \left(\sum_{j=0}^{\infty} p^j v_j\right)_{xx} + \left(\sum_{j=0}^{\infty} p^j v_j\right)_x\right] = 0$$

Comparing the coefficient of like powers of p, we get;

$$p^0 : \ (v_0)_t - (w_0)_t = 0, \qquad v_0(x,t) = e^x - x$$

$$p^1 : \ (v_1)_t + (w_0)_t - (v_0)_{xx} + (v_0)_x = 0, \quad v_1(x,t) = t \tag{47}$$

$$p^2 : \ (v_2)_t - (v_1)_{xx} + (v_1)_x = 0, \quad v_2(x,t) = 0$$

$$v_k(x,t) = 0, k \geq 2$$

Thus, the solution in series form is given by:

$$w(x,t) = \sum_{k=0}^{\infty} v_k(x,t)$$

Or

$$w(x,t) = v_0(x,t) + v_1(x,t) + v_k(x,t), k \geq 2$$

Hence the solution of Eq. (42) with Eqs. (43-44) is given as;

$$w(x,t) = e^x - x + t \tag{48}$$

This solution coincides with the exact one.

Example (1.2.8) We consider linear second order dissipative wave equation [62],

$$w_t + w_{tt} = w_{xx} + w_x + 2(t - x) \tag{49}$$

With initial conditions:

$$w(x,0) = x^2, \;\; w_t(x,0) = 0 \tag{50}$$

And boundary conditions:

$$w(0,t) = t^2, \;\; w_t(0,t) = 0 \tag{51}$$

To find a solution by HPM we construct the homotopy in the following form:

$$H(u,p) = (1-p)[u_t - (w_0)_t] + p[u_t + u_{tt} - u_{xx} - u_x - 2(t-x)] = 0 \tag{52}$$

Or

$$u_t - (w_0)_t + p[(w_0)_t + u_{tt} - u_{xx} - u_x - 2(t-x)] = 0 \tag{53}$$

Substituting Eq. (9) and the initial conditions Eq. (50) into the homotopy Eq. (53) and equating the terms with identical powers of p as follows:

$$\begin{aligned}
&p^0 : \; (u_0)_t - (w_0)_t = 0, \qquad u_0(x,t) = x^2 \\
&p^1 : \; (u_1)_t + (w_0)_t - (u_0)_{xx} - (u_0)_x - 2t + 2x = 0, \quad u_1(x,t) = 2t + t^2 \\
&p^2 : \; (u_2)_t + (u_1)_{tt} - (u_1)_{xx} - (u_1)_x = 0, \;\; u_2(x,t) = -2t \\
&p^3 : \; (u_3)_t + (u_2)_{tt} - (u_2)_{xx} - (u_2)_x = 0, \;\; u_3(x,t) = 0 \\
&u_k(x,t) = 0, k \geq 3
\end{aligned} \tag{54}$$

Thus, the solution in series form is given by:

$$w(x,t) = \sum_{k=0}^{\infty} u_k(x,t)$$

Or

$$w(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_k(x,t), k \geq 3$$

Hence the solution of Eq. (48) with Eqs. (50-51) is given by:

$$w(x,t) = x^2 + t^2 \qquad (55)$$

Which is the exact solution.

Example (1.2.9) We consider the following second order linear homogeneous partial differential equation [62],

$$w_t = w_{xx} - w, \quad 0 \leq x \leq 1 \qquad (56)$$

With initial condition;

$$w(x,0) = x^2, \qquad 0 \leq x \leq 1 \qquad (57)$$

And boundary condition;

$$w_x(0,t) = 0, \quad t > 0 \qquad (58)$$

To find solution by HPM we construct the homotopy in the following form:

$$H(u,p) = (1-p)[u_t - (w_0)_t] + p[u_t - u_{xx} + u] = 0$$

Or

$$u_t - (w_0)_t + p[(w_0)_t - u_{xx} + u] = 0 \qquad (59)$$

Substituting Eq. (9) and the initial condition Eq. (57) into the homotopy Eq. (59) and equating the terms with identical powers of p as follows:

$$p^0: \ (u_0)_t - (w_0)_t = 0, \qquad u_0(x,t) = x^2$$

$$p^1: \ (u_1)_t + (w_0)_t - (u_0)_{xx} + u_0 = 0, \quad u_1(x,t) = -x^2t + 2t$$

$$p^2: \ (u_2)_t - (u_1)_{xx} + u_1 = 0, \quad u_2(x,t) = \frac{t^2x^2}{2} - 2t^2 \qquad (60)$$

$$p^3: \ (u_3)_t - (u_2)_{xx} + u_2 = 0, \quad u_3(x,t) = \frac{t^3x^2}{3!} - t^3$$

$$p^4 : (u_4)_t - (u_3)_{xx} + u_3 = 0, \quad u_4(x,t) = \frac{t^4x^2}{4!} - \frac{8t^4}{4!}$$

Thus, the solution in series form is given by:

$$w(x,t) = \sum_{k=0}^{\infty} u_k(x,t)$$

$$w(x,t) = x^2\left(1 - t + \frac{t^2}{2!} + \cdots\right) + 2t\left(1 - t + \frac{t^2}{2!} + \cdots\right)$$

The solution for equation (56) in a closed form is given by:

$$w(x,t) = x^2e^{-t} + 2te^{-t} \tag{61}$$

Example (1.2.10) The telegraph equations appear in the propagation of electrical signals along a telegraph line, digital image processing, telecommunication, signals and systems .we consider the following telegraph equation [62],

$$3w_t + w_{tt} = w_{xx} + 3(x^2 + t^2 + 1) \tag{62}$$

With initial conditions:

$$w(x,0) = x, \;\; w_t(x,0) = 1 + x^2 \tag{63}$$

And boundary conditions;

$$w(0,t) = t + \frac{t^3}{3}, \;\; w_x(0,t) = 1 \tag{64}$$

To find solution by HPM we construct the homotopy in the following form:

$$H(u,p) = (1-p)[3u_t - 3(w_0)_t] + p[3u_t + u_{tt} - u_{xx} - 3(x^2 + t^2 + 1)] = 0$$

Or

$$3u_t - 3(w_0)_t + p[3(w_0)_t + u_{tt} - u_{xx} - 3(x^2 + t^2 + 1)] = 0 \tag{65}$$

Substituting Eq. (9) and the initial conditions Eq. (63) into the homotopy Eq. (65) and equating the terms with identical powers of p as follows:

$$p^0 : \{\ 3(u_0)_t - 3(w_0)_t = 0, \qquad u_0(x,t) = x$$

$$p^1 : \begin{cases} 3(u_1)_t + 3(w_0)_t + (u_0)_{tt} - (u_0)_{xx} - 3(x^2 + t^2 + 1) = 0 \\ u_1(x,t) = t(x^2+1) + \frac{t^3}{3} \end{cases} \tag{66}$$

$$p^2 : \ \{3(u_2)_t + (u_1)_{tt} - (u_1)_{xx} = 0, \qquad u_2(x,t) = 0$$

$$u_k(x,t) = 0, k \geq 2$$

Thus, the solution in series form is given by:

$$w(x,t) = \sum_{k=0}^{\infty} u_k(x,t)$$

Or

$$w(x,t) = \ u_0(x,t) + \ u_1(x,t) + \ u_k(x,t), k \ \geq 2$$

Hence the solution of Eq. (33) with Eqs.(34-35) is given by:

$$w(x,t) = x + t(x^2+1) + \frac{t^3}{3} \tag{67}$$

Which is the exact solution.

1.3 The Noice terms

In this section, we will present a useful tool that will accelerate the convergence of the homotopy perturbation method.

The noise terms phenomenon provides a major advantage in that it demonstrates a fast convergence of the solution. It is important to note here that the noise terms phenomenon that will be introduced in this section, may appear only for inhomogeneous PDEs. In addition, this phenomenon is applicable to all inhomogeneous PDEs of any order and will be used where appropriate in the coming chapters.

The noise terms, if existed in the components u_0 and u_1, will provide, in general, the solution in a closed form with only two successive iterations.

A useful summary about the noise terms phenomenon can be drawn as follows:

1. The noise terms are defined as the identical terms with opposite signs that may appear in the components u_0 and u_1.
2. The noise terms appear only for specific types of inhomogeneous equations whereas noise terms do not appear for homogeneous equations.
3. Noise terms may appear if the exact solution is part of the zerothcomponent u_0.
4. Verification that the remaining non-canceled terms satisfy the equation is necessary and essential.

The phenomenon of the useful noise terms will be explained by the following illustrative examples.

Example (1.3.11) We consider the following inhomogeneous Partial differential equation

$$u_x + u_y = (1+x)e^y \tag{68}$$

With the initial conditions;

$$u(x,0) = x, \quad u(0,y) = 0 \tag{69}$$

To solve Eq. (68) with initial conditions Eq. (69), according to the homotopy perturbation Eq. (4), we construct the following homotopy:

$$H(v,p) = (1-p)[v_x - (u_0)_x] + p\left[v_x + v_y - (1+x)e^y\right] = 0$$

Or

$$v_x - (u_0)_x + p\left[(u_0)_x + v_y - (1+x)e^y\right] = 0 \qquad (70)$$

Substituting Eq. (9) into Eq. (70), and comparing coefficients of the terms with the identical powers of p, we get;

$$p^0: \; v_x - (u_0)_x = 0, \quad v_0(x,y) = 0$$

$$p^1: \; (v_1)_x + (u_0)_x + (v_0)_y - (1+x)e^y = 0, v_1(x,y) = \left(x + \frac{x^2}{2!}\right)e^y$$

$$p^2: \; (v_2)_x + (v_1)_y = 0 \quad v_2(x,y) = -\left(\frac{x^2}{2!} + \frac{x^3}{3!}\right)e^y \qquad (71)$$

$$p^3: \; (v_3)_x + (v_2)_y = 0, \quad v_3(x,y) = \left(\frac{x^3}{3!} + \frac{x^4}{4!}\right)e^y$$

$$\vdots$$

It is necessary to mention here that the noise terms are those terms who are the same but different in signs .more clearly the noise terms $\frac{x^2}{2!}e^y$ and $-\frac{x^2}{2!}e^y$ between the components $u_1(x,t)$ and $u_2(x,t)$ can be cancelled and the remaining terms of $u_1(x,t)$ still satisfy the equation. The exact solution is therefore

$$u(x,y) = \sum_{k=0}^{\infty} v_k(x,y)$$

$$u(x,y) = xe^y \qquad (72)$$

Notice that the exact solution is verified through substitution in the equation (68) and not only upon the appearance of the noise terms. In addition, the other noise terms that appear between other components will vanish in the limit.

Example (1.3.12) We consider the following inhomogeneous Partial differential equation

$$u_x + u_y = x^2 + 4xy + y^2 \tag{73}$$

With the initial conditions;

$$u(x, 0) = 0, \quad u(0, y) = 0 \tag{74}$$

To solve Eq. (73) with initial conditions Eq. (74), according to the homotopy perturbation Eq. (4), we construct the following homotopy:

$$v_x - (u_0)_x + p\left[(u_0)_x + v_y - (x^2 + 4xy + y^2)\right] = 0 \tag{75}$$

Substituting Eq. (9) into Eq. (75), and comparing coefficients of the terms with the identical powers of p, we get;

$$p^0 : \ v_x - (u_0)_x = 0, \quad v_0(x, y) = 0$$

$$p^1 : (v_1)_x + (u_0)_x + (v_0)_y - (x^2 + 4xy + y^2) = 0,$$

$$v_1(x, y) = \frac{x^3}{3} + 2x^2y + xy^2$$

We can easily observe that the two components u_0 and u_1 do not contain noise terms. This confirms our belief that although the PDE is an inhomogeneous equation, but the noise terms between the first two components did not exist in this problem. Unlike the previous examples, we should determine more components to obtain an insight through the solution. Therefore, other components should be determined. Hence we find

$$p^2 : \ (v_2)_x + (v_1)_y = 0, \quad v_2(x, y) = -\left(\frac{2}{3}x^3 + x^2y\right)$$

$$p^3 : \ (v_3)_x + (v_2)_y = 0, \quad v_3(x, y) = \frac{1}{3}x^3 \tag{76}$$

$$v_k(x, y) = 0, k \geq 4$$

Based on the result we obtained for u_2, other components of $u(x, y)$ will vanish.

Consequently, we find that:

$$u(x,y) = \sum_{k=0}^{\infty} v_k(x,t)$$

$$u(x,y) = x^2y + xy^2 \tag{77}$$

1.4 Application of Homotopy Perturbation Method to Nonlinear Partial Differential Equations

Many linear and nonlinear problems are of fundamental importance in Science and Technology especially in Engineering. The investigation of exact or approximate solution of such problems was one of the challenges before Mathematicians and Engineers. Some valuable contributions have already been made to solving differential equations arising in many scientific and engineering applications using numerical techniques such as Finite Difference Method. The other methods to solve differential equations suggest that in Finite Difference Method discretization of the variables leads to computational complexities while Adomian method narrow down its application due to calculation of complicated Adomian polynomials. Integral transforms such as Laplace and Fourier transforms are commonly used to solve differential equations and usefulness of these integral transforms lies in their ability to transform differential equations into algebraic equations which allows simple and systematic solution procedures. However, using integral transform in nonlinear problems may increase its complexity. Applied Fourier transform to obtain solution of semi linear parabolic equations. The HPM has been employed to solve a large variety of linear and nonlinear problems. The aim of this study is to extend the Homotopy Perturbation Method (HPM) to find numerical solution of some nonlinear partial differential equations.

Example (1.4.13) We consider the first order nonlinear ordinary differential equation

$$y' - y^2 = 1\,, \quad y(0) = 0 \tag{78}$$

To find a solution by HPM we construct the homotopy in the following form:

$$(1-p)[Y' - y_0'] + p[Y' - Y^2 - 1)] = 0 \tag{79}$$

With initial approximation $y(0) = 1$

Substituting Eq. (9) and the initial approximation into the homotopy Eq. (79) and equating the terms with identical powers of p as follows:

$$\begin{aligned}
&p^0 : Y_0' = y_0'\ , && Y_0 = 0\\
&p^1 : Y_1' = 1 - y_0' - \ Y_0{}^2, && Y_1 = x\\
&p^2 : Y_2' = 2Y_0Y_1\ , && Y_2 = 0\\
&p^3 : Y_3' = \ Y_1{}^2 + 2Y_0Y_2\ , && Y_3 = \frac{x^3}{3}
\end{aligned} \tag{80}$$

Consequently, the solution in a series form is given by:

$$y(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots$$

And in a closed form by:

$$y(x) = tanx \tag{81}$$

Example (1.4.14) We consider the first order nonlinear ordinary differential equation

$$y' = \frac{y^2}{1-xy}\ , \qquad y(0) = 1 \tag{82}$$

We first rewrite the equation by:

$$y' = xyy' + y^2$$

To find solution by HPM we construct the homotopy in the following form:

$$(1-p)[Y' - y_0'] + p[Y' - xYY' - Y^2)] = 0$$

Or

$$Y' - y_0' + p[y_0' - xYY' - Y^2)] = 0 \tag{83}$$

With initial approximation $y(0) = 1$

Substituting Eq. (9) and the initial approximation into the homotopy Eq. (83) and equating the terms with identical powers of p as follows:

$$p^0 : Y_0' = y_0' , \qquad Y_0 = 1$$

$$p^1 : Y_1' + y_0' - xY_0Y_0' - Y_0^2 = 0, \qquad Y_1 = x$$

$$p^2 : Y_2' - x(Y_0Y_1' + Y_1Y_0') - 2Y_0Y_1 , \qquad Y_2 = \frac{3}{2}x^2$$

$$p^3 : Y_3' - x(Y_0Y_2' + Y_1Y_1' + Y_2Y_0) \qquad Y_3 = \frac{8}{3}x^3$$

And so on. Based on these calculations, the solution in a series form is given by:

$$y(x) = Y_0 + Y_1 + Y_2 + \cdots$$

$$y(x) = 1 + x + \frac{3}{2}x^2 + \frac{8}{3}x^3 + \frac{125}{24}x^4 + \cdots$$

It is clear that a closed form solution where y is expressed explicitly in terms of x cannot be found. However, the exact solution can be expressed in the implicit expression

$$y = e^{xy} \tag{84}$$

Example (1.4.15) We consider the nonlinear partial differential equation

$$u_t + uu_x = 0, \qquad u(x,0) = x, t > 0 \tag{85}$$

To find solution by HPM we construct the homotopy in the following form:

$$(1-p)[v_t - (u_0)_t] + p[v_t + vv_x] = 0$$

Or

$$v_t - (u_0)_t + p[(u_0)_t + vv_x] = 0 \tag{86}$$

With initial condition $u(x,0) = x$

Substituting Eq. (9) and the initial condition into the homotopy Eq. (86) and equating the terms with identical powers of p as follows:

$$p^0 : (v_0)_t = (u_0)_t\,, \qquad v_0(x,t) = x$$

$$p^1 : (v_1)_t + (u_0)_t + v_0(v_0)_x = 0\,, \qquad v_1(x,t) = -xt$$

$$p^2 : (v_2)_t + v_0(v_1)_x + v_1(v_0)_x = 0\,, \qquad v_2(x,t) = xt^2$$

$$p^3 : (v_3)_t + v_0(v_2)_x + v_1(v_1)_x + v_2(v_0)_x = 0\,, v_3(x,t) = -xt^3$$

$$\vdots$$

And so on. Based on these calculations, the solution in a series form is given by:

$$u(x,t) = x - xt + xt^2 - xt^3 + \cdots.$$

And in a closed form by:

$$u(x,t) = \frac{x}{1+t} \tag{87}$$

Example (1.4.16) We consider the nonlinear partial differential equation

$$u_t - \frac{1}{4}{u_x}^2 = x^2, \qquad u(x,0) = 0, \tag{88}$$

To find solution by HPM we construct the homotopy in the following form:

$$(1-p)[v_t - (u_0)_t] + p\left[v_t - \frac{1}{4}{v_x}^2 - x^2\right] = 0$$

Or

$$v_t - (u_0)_t + p\left[(u_0)_t - \frac{1}{4}{v_x}^2 - x^2\right] = 0 \tag{89}$$

With initial condition $u(x,0) = 0$

Substituting Eq. (9) and the initial condition into the homotopy Eq. (89) and equating the terms with identical powers of p as follows:

$$p^0 : (v_0)_t = (u_0)_t \,, \qquad v_0(x,t) = 0$$

$$p^1 : (v_1)_t + (u_0)_t - \frac{1}{4}{(v_0)_x}^2 - x^2 = 0\,, \quad v_1(x,t) = x^2 t$$

$$p^2 : (v_2)_t - \frac{1}{2}(v_0)_x(v_1)_x = 0\,, \qquad v_2(x,t) = 0$$

$$p^3 : (v_3)_t - \frac{1}{4}{(v_1)_x}^2 + 2(v_0)_x(v_2)_x = 0\,, \quad v_3(x,t) = x^2\frac{t^3}{3}$$

$$\vdots$$

And so on. Based on these calculations, the solution in a series form is given by:

$$u(x,t) = x^2\left(t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{17}{315}t^7 + \cdots\right)$$

And in a closed form by:

$$u(x,t) = x^2 tant \tag{90}$$

Example (1.4.17) We consider the nonlinear partial differential equation

$$u_t + u^2 u_x = 0, \qquad u(x,0) = 2x, t > 0, \tag{91}$$

To find solution by HPM we construct the homotopy in the following form:

$$(1-p)[v_t - (u_0)_t] + p[v_t + v^2 v_x] = 0$$

Or

$$v_t - (u_0)_t + p[(u_0)_t + v^2 v_x] = 0 \tag{92}$$

With initial condition: $u(x,0) = 2x$

Substituting Eq. (9) and the initial condition into the homotopy Eq. (92) and equating the terms with identical powers of p as follows:

$$p^0 : (v_0)_t = (u_0)_t\,, \qquad v_0(x,t) = 2x$$

$$p^1 : (v_1)_t + (u_0)_t + {v_0}^2 (v_0)_x = 0\,, \qquad v_1(x,t) = -8x^2 t$$

$$p^2 : (v_2)_t + {v_0}^2 (v_1)_x + 2v_0 v_1 (v_0)_x = 0\,, v_2(x,t) = 64x^3 t^2$$

$$\vdots$$

And so on. Based on these calculations, the solution in a series form is given by:

$$u(x,t) = 2x - 8x^2 t + 64x^3 t^2 - 640x^4 t^3 \ldots \tag{93}$$

Two observations can be made here. First, we can easily observe that:

$$u(x,t) = 2x\,, \qquad t = 0 \tag{94}$$

That satisfies the initial condition. We next observe that for $t > 0$, the series solution in Eq. (93) can be formally expressed in a closed form by:

$$u(x,t) = \frac{1}{4t}\sqrt{1+16xt} - 1 \tag{95}$$

Combining Eq. (94) and Eq. (95) gives the solution in the form:

$$u(x,t) = \begin{cases} 2x\,, & t = 0 \\ \dfrac{1}{4t}\left(\sqrt{1+16xt} - 1\right), & t > 0 \end{cases} \qquad (96)$$

Example (1.4.18) We consider the nonlinear partial differential equation

$$u_t + \frac{1}{36} x {u_{xx}}^2 \;=\; x^3, \qquad u(x,0) = \; 0, \qquad (97)$$

To find solution by HPM we construct the homotopy in the following form:

$$(1-p)[v_t - (u_0)_t] + p\left[v_t + \frac{1}{36x} {v_{xx}}^2 - x^3\right] = 0$$

Or

$$v_t - (u_0)_t + p\left[(u_0)_t + \frac{1}{36} x {v_{xx}}^2 - x^3\right] = 0 \qquad (98)$$

With initial condition $u(x,0) = \; 0$

Substituting Eq. (9) and the initial condition into the homotopy Eq. (98) and equating the terms with identical powers of p as follows:

$$p^0 : (v_0)_t = (u_0)_t\,, \qquad v_0(x,t) = 0$$

$$p^1 : (v_1)_t + (u_0)_t + \frac{1}{36} x {(v_0)_{xx}}^2 - x^3 = 0\,, \quad v_1(x,t) = x^3 t$$

$$p^2 : (v_2)_t + \frac{1}{18} x (v_0)_{xx} (v_1)_{xx} = 0\,, \qquad v_2(x,t) = 0$$

$$p^3 : (v_3)_t + \frac{1}{36} x\left({(v_1)_{xx}}^2 + 2(v_0)_{xx}(v_2)_{xx}\right) = 0\,, v_3(x,t) = -\frac{1}{3} x^3 t^3$$

$$\vdots$$

And so on. Based on these calculations, the solution in a series form is given by:

$$u(x,t) = x^3(t - \frac{1}{3}t^3 + \frac{2}{15}t^5 - \frac{17}{315}t^7 + \cdots$$

And in a closed form by:

$$u(x,t) = x^3 tanht \qquad (99)$$

Example (1.4.19) We consider the following inhomogeneous PDE

$$u_t + uu_x = 2t + x + t^3 + xt^2, \qquad u(x,0) = 0, \qquad (100)$$

To find solution by HPM we construct the homotopy in the following form:

$$(1-p)[v_t - (u_0)_t] + p[v_t + vv_x - (2t + x + t^3 + xt^2)] = 0$$

Or

$$v_t - (u_0)_t + p[(u_0)_t + vv_x - (2t + x + t^3 + xt^2)] = 0 \quad (101)$$

With initial condition: $u(x,0) = 0$

Substituting Eq. (9) and the initial condition into the homotopy Eq. (101) and equating the terms with identical powers of p as follows:

$$p^0 : \{(v_0)_t = (u_0)_t \,, \qquad v_0(x,t) = 0$$

$$p^1 : \begin{cases} (v_1)_t + (u_0)_t + v_0(v_0)_x - (2t + x + t^3 + xt^2) = 0\,, \\ \quad v_1(x,t) = t^2 + xt + \frac{1}{4}t^4 + \frac{x}{3}t^3 \end{cases}$$

$$p^2 : \begin{cases} (v_2)_t + v_0(v_1)_x + v_1(v_0)_x = 0 \qquad , \\ v_2(x,t) = -\frac{1}{4}t^4 - \frac{x}{3}t^3 - \frac{2}{15}xt^5 - \frac{7}{72}t^6 - \frac{1}{63}xt^7 - \frac{1}{98}t^8 \end{cases}$$

$$\vdots$$

It is important to recall here that the noise terms appear between the components $u_1(x,t)$ and$u_2(x,t)$, where the noise terms are those pairs of terms that are identical but carrying opposite signs. More precisely, the noise terms $\pm\frac{1}{4}t^4 \pm \frac{x}{3}t^3$ between the components $u_1(x,t)$ and $u_2(x,t)$ can be cancelled and the remaining terms of $u_1(x,t)$ still satisfy the equation.

The exact solution is therefore;

$$u(x,t) = t^2 + xt \tag{102}$$

1.5 An Efficient Modification of the Homotopy Perturbation Method

In this section we will introduce a new reliable modification of the HPM. The new modification demonstrates a rapid convergence of the series solution compared with the standard HMP, and therefore it has been shown that to be computationally efficient in several examples in applied fields. In addition the modified algorithm may give the exact solution for nonlinear equation by using two iterations only. The obtained result suggests that this improvement technique introduces a powerful improvement for solving nonlinear problems.

The new modified form of the HPM can be established based on the assumption that the function $f(r)$can be divided into two parts, namely $f_0(r)$ and $f_1(r)$ as:

$$f(r) = f_0(r) + f_1(r) \tag{103}$$

On the assumption that the function $f(r)$ can be replaced by a series of infinite components. Under this assumption we suggest that $f(r)$be expressed in Taylor series:

$$f(r) = \sum_{n=0}^{\infty} f_n(r) \tag{104}$$

According to the first assumption$f(r) = f_0(r) + f_1(r)$, we can construct the homotopy

$$v(r,p)\colon \Omega \times [0,1] \to R,$$

Which satisfies:

$$\mathcal{H}(v,p) = (1-p)[L(v) - L(v_0)] + p[L(v) + N(v) - f_1(r)] = f_0(r), \quad (105)$$

Or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f_1(r)] = f_0(r), \quad (106)$$

Here, a slight variation was proposed only on the component u_0 and u_1. The suggestion was that only the part f_0 be assigned to the zeroscomponent u_0, whereas the remaining part $f_1$be combined with the component u_1. If we set $f_1(r) = f(r)$and $f_0(r) = 0$, then the homotopy Eq. (105) or Eq. (106) reduces to the homotopy Eq. (4) or Eq. (5) respectively. However, the success of the method depends on the proper selection of the functions f_0 and f_1.

According to the second assumption $f(r) = \sum_{n=0}^{\infty} f_n(r)$ we can construct the homotopy $v(r,p): \Omega \times [0,1] \rightarrow \Re$ which satisfies;

$$\mathcal{H}(v,p) = (1-p)[L(v) - L(u_0)] + p[L(v) + N(v)] = \sum_{n=0}^{\infty} p^n f_n(r), \quad (107)$$

Or

$$\mathcal{H}(v,p) = L(v) - L(u_0) + pN(v) = \sum_{n=0}^{\infty} p^n f_n(r), \quad (108)$$

If $f(r)$ consists of two terms only then the homotopy Eq. (107) or Eq. (108) reduces to the homotopy Eq. (105) or Eq. (106), respectively. In this case the term f_0 is combined with the component u_0 and$f_1$is combined with the component u_1 and f_2 is combined with the component u_2 and so on.This suggestion will facilitate the calculations of the terms $u_0, u_1, u_2, \ldots$ and hence accelerate the rapid convergence of the series solution.

It is easily to observe that the algorithm of the new modification of the HPM, based on the homotopy given in the equations Eqs. (105 - 108), reduces the number of terms involved in each component and hence the size of calculations is minimized compared to the standard HPM.

Moreover this reduction of terms in each component facilitates the construction of the homotopy perturbation solution.

To demonstrate the effectiveness of the modified HPM, we have chosen several differential and integral equations.

Example (1.5.20) Consider the nonlinear differential equation [63],

$$u'' + \frac{2}{t}u' + u^3 = t^6 + 6\,, \tag{109}$$

Subject to the initial conditions;

$$u(0) = 0, \qquad u'(0) = 0, \tag{110}$$

The HPM: To solve Eq. (109) by HPM we construct the following homotopy

$$u'' + \frac{2}{t}u' + p(u^3 - t^6 - 6) = 0 \tag{111}$$

Assume that the solution of Eq. (109) has the following form:

$$u = u_0 + pu_1 + p^2u_2 + \ldots. \tag{112}$$

Substituting Eq. (112) and the initial conditions Eq. (110) in to the homotopy Eq. (111) and equation term with identical powers of p, we obtain the following set of linear differential equations:

$$p^0 : {u_0}'' + \frac{2}{t}{u_0}' = 0, \quad u_0(0) = 0, \quad {u_0}'(0) = 0$$

$$p^1 : {u_1}'' + \frac{2}{t}{u_1}' = -{u_0}^3 + t^6 + 6, \quad u_1(0) = 0, \quad {u_1}'(0) = 0 \tag{113}$$

$$p^2 : {u_2}'' + \frac{2}{t}{u_2}' = 3{u_0}^2u_2, \quad u_2(0) = 0, \quad {u_2}'(0) = 0$$

$$p^3 : {u_3}'' + \frac{2}{t}{u_3}' = -3u_0{u_1}^2 - 3{u_0}^2u_2, \quad u_3(0) = 0, \quad {u_3}'(0) = 0$$

$$\vdots$$

Consequently, solving the above equation, we obtain:

$u_0 = 0$

$u_1 = t^2 + \frac{1}{72}t^8$

$u_2 \ = 0$

$u_3 = 0$

$u_4 = -\frac{1}{72}t^8 - \frac{3}{14.15.72}t^{14} - \frac{3}{20.21.(72)^2}t^{20} - \frac{1}{26.27.(72)^3}t^{26}$

$\vdots$

And so, in this manner the rest of HPM can be obtained. The solution for Eq. (109) given by setting $p = 1$ in Eq. (112) we have;

$$u = u_0 \ + u_1 + u_2 + \ldots. \tag{114}$$

The noise terms appear between the components $u_1(x,t)$ and$u_4(x,t)$, where the noise terms are those pairs of terms that are identical but carrying opposite signs. More precisely, the noise terms $\pm\frac{1}{72}t^8$ between the components $u_1(x,t)$ and $u_4(x,t)$ can be cancelled and the remaining terms of $u_1(x,t)$ still satisfy the equation.

The exact solution is therefore;

$$u = t^2 \tag{115}$$

The modified HPM: in view of the homotopy Eq. (106), we construct the following homotopy:

$$u'' + \frac{2}{t}u' + p[u^3 - t^6] = 6 \tag{116}$$

Substituting Eq. (112) in to Eq. (116) and equation term with identical powers of p we obtain the following set of linear differential equations:

$p^0 : {u_0}'' + \frac{2}{t}{u_0}' = 6, \quad u_0(0) = 0, \quad {u_0}'(0) = 0$

$$p^1 : {u_1}'' + \frac{2}{t}{u_1}' = -{u_0}^3 + t^6, \quad u_1(0) = 0, \quad {u_1}'(0) = 0$$

$$p^2 : {u_2}'' + \frac{2}{t}{u_2}' = 3{u_0}^2 u_2, \quad u_2(0) = 0, \quad {u_2}'(0) = 0 \tag{117}$$

$$p^3 : {u_3}'' + \frac{2}{t}{u_3}' = -3u_0{u_1}^2 - 3{u_0}^2 u_2, \quad u_3(0) = 0, \quad {u_3}'(0) = 0$$

$$\vdots$$

Consequently, solving the above equation the first few components of the homotopy perturbation solution of Eq. (109) are derived as follows;

$$u_0 = t^2$$

$$u_k = 0, \quad k \geq 1$$

The exact solution

$$u(t) = t^2 \tag{118}$$

Follows immediately the success of obtaining the exact solution by using two iterations is result of the proper selection of $f_0(r)$ and $f_1(r)$.

Example (1.5.21) Consider the nonlinear partial differetial equation[63],

$$u_{tt} - u_{xx} + u^2 = -xcos(t) + x^2cos^2(t) \tag{119}$$

Subject to the initial conditions;

$$u(x,0) = x, \qquad u_t(x,0) = 0 \tag{120}$$

The modified HPM. In view of the homotopy Eq. (106), we construct the following homotopy:

$$u_{tt} + P[-u_{xx} + u^2 - x^2cos^2(t)] = -xcos(t) \tag{121}$$

Substituting Eq. (112) and the initial conditions Eq. (120) into the homotopy Eq. (121) and equating the terms with identical powers of p, we obtain the following set of linear differential equations:

$$p^0 : (u_0)_{tt} = -xcos(t)\,, \quad u_0(x,0) = x, \quad (u_0)_t(x,0) = 0.$$

$$p^1 : (u_1)_{tt} - (u_0)_{xx} + {u_0}^2 - x^2cos^2(t) = 0, u_1(x,0) = x, (u_1)_t(x,0) = 0 \quad (122)$$

$$p^2 : (u_2)_{tt} - (u_1)_{xx} + 2u_0u_1 = 0, \quad u_2(x,0) = 0, \quad (u_2)_t(x,0) = 0.$$

$$p^3 : (u_3)_{tt} - (u_2)_{xx} + 2u_0u_2 + {u_1}^2 = 0, \quad u_3(x,0) = 0, \quad (u_3)_t(x,0) = 0.$$

$$\vdots$$

Solving the above equations the first few components of the homotopy perturbation solution for Eq. (119) are derived as follows:

$$u_0 = xcos(t)$$

$$u_k = 0, \quad k \geq 1$$

The exact solution

$$u(x,t) = xcos(t) \quad (123)$$

Example (1.5.22) Consider the nonlinear integral equation[64],

$$u(t) = sec(t) + tan(t) - \int_0^t u^2(x)dx \quad (124)$$

The modified HPM in view of the homotopy Eq. (106) we construct the following homotopy

$$u(t) - p\left[tan(t) - \int_0^t u^2(x)dx\right] = sec(t) \quad (125)$$

Substituting Eq. (112) into the homotopy Eq. (125) and equating the terms with identical powers of p, we obtain the following set of linear integral equations.

$$p^0 : u_0 = sec(t),$$

$$p^1 : u_1 = tan(t) - \int_0^t {u_0}^2(x)dx\,,$$

$$p^2 : u_2 = -\int_0^t 2u_0(x)u_1(x)dx, \tag{126}$$

$$p^3 : u_3 = -\int_0^t \left(2u_0(x)u_2(x) - {u_1}^2(x)\right)dx,$$

$$\vdots$$

Solving the above equations, the first few components of the homotopy perturbation solution for Eq. (124) are derived as follows:

$$u_0 = sec(t),$$

$$u_k = 0, \quad k \geq 1$$

The exact solution

$$u(x,t) = sec(t) \tag{127}$$

Follows immediately. It is clear that we used two iterations only to obtain the exact solution.

CHAPTER TWO

Homotopy Perturbation Transform Method (HPTM)

In recent years, it has turned out that many phenomena in engineering, physics, chemistry and other sciences can be described very successfully by using partial differential equations. Hence, great attention has been given to finding solutions of partial differential equations. Most partial differential equations do not have exact analytical solutions, therefore approximate and numerical techniques were used. The homotopy perturbation method (HPM) was first introduced by J.H. He. The HPM was applied to solve the 12th order boundary value problems.

In recent years, many authors have paid attention to studying the solutions of nonlinear partial differential equations by Adomain decomposition method, the tanh method, the sine-cosine method, the differential transform method, the variational iteration method, and the Laplace decomposition method.

He developed the homotopy perturbation method for solving linear, nonlinear, initial and boundary value problems by merging two techniques, the standard homotopy and the perturbation technique.

In this chapter some basic definitions of Laplace transform ,homotopy perturbation method and He's polynomials were presented, also a reliable combination of homotopy perturbation method and Laplace transform was introduced to obtain the solution of linear and nonlinear partial differential equations and system for linear and nonlinear partial differential equations.

Also a combined form of the Laplace transform method with the homotopy perturbation method (HPTM) is proposed to solve both linear and nonlinear partial differential equations. The nonlinear terms can be easily handled by the use of He's polynomials. The proposed method finds the solution without any discretization or restrictive assumptions and avoids the round-off errors. The fact that the proposed technique solves nonlinear problems without using Adomian's polynomials can be considered as a clear advantage of this algorithm over the decomposition method.

2.1 Laplace Transform

The Laplace transform can be helpful in solving ordinary and partial differential equations because it can replace an ODE with an algebraic equation or replace a PDE with an ODE. Another reason that the Laplace transform is useful is that it can help deal with the boundary conditions of a PDE on an infinite domain. In this introductory section, we discuss definitions, theorems, and properties of the Laplace transform

Definition (2.1.1)

Suppose that f is a real- or complex-valued function of the (time) variable $t > 0$ and s is a real or complex parameter. We define theLaplace transform of f as

$$F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t)dt = \lim_{t\to\infty} \int_0^t e^{-st} f(t)dt \qquad (1)$$

If the limit exist and it is finite the above integral is said to be converge otherwise it diverges and there is no Laplace transform defined for f and the limit is the ordinary Riemann integral

The notation $L(f)$ will also be used to denote the Laplace transform of f , and the integral is the ordinary Riemann (improper) integral. The parameter s belongs to some domain on the real line or in the complex plane. We will choose s appropriately so as to ensure the convergence of the Laplace integral Eq. (1). In a mathematical and technical sense, the domain of s is quite important. However, in a practical sense, when differential equations are solved, the domain of s is routinely ignored. When s is complex, we will always use the notation $s = x + iy$. the symbol L is the Laplace transformation, which acts on functions $f = f(t)$ and generates a new function, $F(s) = L(f(t))$.

Definition (2.1.2)

A function f is said to be piecewise continuous on an interval I if I can be subdivided into a finite number of subintervals, in each of which f is continuous and has finite left- and right-hand limits

Definition (2.1.3)

A function f has exponential order α if there exist numbers>0 and α , such that for some $t_0 \geq 0$,

$$|f\ (t)| \leq Me^{\alpha t} \qquad when \ \ t \geq t_0$$

Theorem (2.1.4)

If f is piecewise continuous on $[0, \infty)$ and of exponential order α, then the Laplace transform $L(f\ )$ exists for $Re(s) > \alpha$ and converges absolutely.

Proof: first

$$|f\ (t)| \leq M_1 e^{\alpha t}, \qquad t \geq t_0$$

For some real α. Also, f is piecewise continuous on $[0, t_0]$ and hence bounded there (the bound being just the largest bound over all the subintervals), say

$$|f\ (t)| \leq M_2, \qquad 0 < t < t_0$$

Since $e^{\alpha t}$ has a positive minimum on $[0, t_0]$, a constant M can be chosen sufficiently large so that

$$|f\ (t)| \leq Me^{\alpha t}\ , \qquad t \geq 0$$

Therefore

$$\int_0^t |e^{-st} f\ (t)|\, dt \leq M \int_0^t e^{-(x-\alpha)}\, dt = \frac{M}{x-\alpha} - \frac{Me^{-(x-\alpha)}}{x-\alpha}$$

Letting $\tau \to \infty$ and noting that $Re(s) = x > \alpha$ then

$$\int_0^\infty |e^{-st} f\ (t)|\, dt \le \frac{M}{x-\alpha}$$

Thus the Laplace integral converges absolutely in this instance (and hence converges) for $Re(s) > \alpha$.

Theorem (2.1.5) (Linearity Property)

Let $f_j(t), 1 \le j \le n, \quad 0 \le t < \infty$ be functions whose Laplace transforms exist, and let c_j , $1 \le j \le n$ be real numbers. Then,

$$L[c_1 f_1(t) + \cdots + c_n f_n(t)] = c_1 L[f_1] + \cdots + c_n L[f_n].$$

Proof:

Clearly, we have

$$L[c_1 f_1(t) + \cdots + c_n f_n(t)] = \int_0^\infty e^{-st} \left(c_1 f_1(t) + \cdots + c_n f_n(t)\right) dt$$

$$= c_1 \int_0^\infty e^{-st} f_1(t) dt + \cdots c_n \int_0^\infty e^{-st} f_n(t) dt$$

$$= c_1 L[f_1] + \cdots + c_n L[f_n].$$

This property can be easily extended to more than two functions as shown from the above proof. With the linearity property, Laplace transform can also be called the linear operator.

Theorem (2.1.6) (Uniqueness Property)

If $f(t)$ and $g(t)$ are continuous functions for $0 \le t < \infty$ and if $L[f] = L[g]$, then $f(t) = g(t)$, and conversely.

In fact, if two functions defined on the positive real axis have the same transform, then these functions cannot differ over an interval of positive length, although they may differ at various isolated points. For many applications, it is necessary to recover the function f from its Laplace transform $L[f]$. To this end, if $F(s)$ is the Laplace transform of a function $f(t)$ and if the function $f(t)$ is uniquely defined by the Laplace transform, i.e., if $L[g(t)](s) = F(s)$, then $g(t) = f(t)$ for all $t \ge 0$, then we define the inverse Laplace transform of $F(s)$ as the function $f(t)$ and write

$$L^{-1}[F(s)](t) = f(t)$$

The well-definedness of the inverse Laplace transform plays a central role when we solve some initial value problems using the Laplace transformation theory in the next section.

Theorem (2.1.7) (Inverse Linearity Property)

Let $f_j(t), 1 \le j \le n, 0 \le t < \infty$ be continuous functions, and let $F_j(s), 1 \le j \le n$ be their Laplace transforms. Then

$$L^{-1}[c_1F_1(t) + \cdots + c_nF_n(t)] = c_1L^{-1}[F_1] + \cdots + c_nL^{-1}[F_n].$$

$$= c_1L^{-1}[f_1] + \cdots + c_nL^{-1}[f_n]$$

Theorem (2.1.8) (Transform of the Derivative)

Let $f^{(i)}(t), 0 \le i \le n-1$ be continuous on $[0,\infty)$ and $f^{(n)}(t)$ be piecewise continuous on $[0,\infty)$, with all $f^{(i)}(t),\ 0 \le i \le n$ of exponential order α. Then,

$$L\left[f^{(n)}\right] = s^nL[f] - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0).$$

2.2 Analysis of method

2.2.1 Basic idea

To illustrate the basic idea of this method, we consider a general nonlinear non-homogeneous partial differential equation with initial conditions of the form

$$Du(x,t) + Ru(x,t) + Nu(x,t) = g(x,t), \qquad (2)$$

$$u(x,0) = h(x), u_t(x,0) = f(x)$$

Where D is the second order linear differential operator $D = \partial^2/\partial t^2$, is the linear differential operator of less order than D, N represents the general non-linear differential operator and $g(x,t)$ is the source term.

Taking the Laplace transform (denoted throughout this chpter by L) on both sides of Eq. (2):

$$L[Du(x,t)] + L[Ru(x,t)] + L\,[Nu(x,t)] = L\,[g(x,t)]$$

Using the differentiation property of the Laplace transform, we have;

$$L[u(x,t)] = \frac{h(x)}{s} + \frac{f(x)}{s^2} - \frac{1}{s^2}L[Ru(x,t)] + \frac{1}{s^2}L[g(x,t)] - \frac{1}{s^2}L[Nu(x,t)] \quad (3)$$

Operating with the Laplace inverse on both sides of Eq. (3) gives

$$u(x,t) = G(x,t) - L^{-1}\left[\frac{1}{s^2}\,L[Ru(x,t) + Nu(x,t)]\right] \quad (4)$$

Where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions. Now, we apply the homotopy perturbation method

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \quad (5)$$

And the nonlinear term can be decomposed as;

$$Nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(u) \quad (6)$$

For some He’s polynomials H_n [80, 81] that are given by:

$$H_n(u_0,\ldots,u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\,N\left(\sum_{i=0}^{\infty} p^i u_i\right)_{p=0} \quad ,n\ =\ 0,1,2,3\ldots \quad (7)$$

Substituting Eqs. (6) and (5) in Eq. (4) we get;

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = G(x,t) - p\left(L^{-1}\left[\frac{1}{s^2}\,L\left[\sum_{n=0}^{\infty} p^n u_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right) \quad (8)$$

Which is the coupling of the Laplace transform and the homotopy perturbation method using He's polynomials. Comparing the coefficient of like powers of p, the following approximations are obtained

$$p^0 : u_0(x,t) = G(x,t),$$

$$p^1 : u_1(x,t) = -L^{-1}\left[\frac{1}{s^2} L[Ru_0(x,t) + H_0(u)]\right],$$

$$p^2 : u_2(x,t) = -L^{-1}\left[\frac{1}{s^2} L[Ru_1(x,t) + H_1(u)]\right],$$

$$p^3 : u_3(x,t) = -L^{-1}\left[\frac{1}{s^2} L[Ru_2(x,t) + H_2(u)]\right],$$

$$\vdots$$

Then the solution is

$$u(x,t) = \lim_{p\to 1} u_n(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \quad (9)$$

2.2.2 Homotopy Perturbation Method and He's polynomial:

The homotopy perturbation method is a technique for solving functional equations of various kinds in the form;

$$u - N(u) = f, \quad (10)$$

Where N is nonlinear operator from Hilbert space H to H,u is unknown function, and f is known function in H.

Consider Eq. (1) in the form

$$L(u) = u - f(x) - N(u) \quad (11)$$

With solution $u(x)$. As possible remedy, we can define homotopy $H(u,p)$ as follows:

$$H(u,0) = F(u), \qquad H(u,1) = L(u)$$

Where $F(p)$ is an integral operator with known solution u_0 which can be obtained easily, typically we may choose a convex homotopy in the form

$$H(u,p) = (1-p)F(u) + pL(u) \tag{12}$$

And continuously trace implicitly defined curve from starting point $H(u_0, 0) = F(u)$, to the solution function $H(u, 1) = F(u)$, the embedding parameter p monotonically increase from zero to unit as the trivial problem $F(u) = 0$ is continuously deformed form to original problem $L(u) = 0$, the embedding parameter $p \in [0,1]$ can be considered as an expanding parameter

$$u = u_0 + pu_1 + p^2u_2 + \cdots \tag{13}$$

When $p \to 1$, Eq. (12) corresponds to Eqs. (11) and (13) becomes the approximate of Eq. (11) i.e.

$$u = \lim_{p\to 1} u = u_0 + u_1 + \cdots \tag{14}$$

Theorem (2.2.9) [80], Suppose $N(u)$ is a nonlinear function, and $u = \sum_{i=0}^{\infty} p^i u_i$, then we have;

$$\frac{\partial^n}{\partial p^n} N(u)_{p=0} = \frac{\partial^n}{\partial p^n}\, N\left(\sum_{i=0}^{\infty} p^i u_i\right)_{p=0} = \frac{\partial^n}{\partial p^n}\, N\left(\sum_{i=0}^{n} p^i u_i\right)_{p=0} \tag{15}$$

Proof:

Since

$$u = \sum_{i=0}^{\infty} p^i u_i = \sum_{i=0}^{n} p^i u_i + \sum_{i=n+1}^{\infty} p^i u_i$$

We have such result as following:

$$\frac{\partial^n}{\partial p^n} N(u)_{p=0} = \frac{\partial^n}{\partial p^n}\, N\left(\sum_{i=0}^{\infty} p^i u_i\right)_{p=0}$$

$$= \frac{\partial^n}{\partial p^n}\, N\left(\sum_{i=0}^{n} p^i u_i + \sum_{i=n+1}^{\infty} p^i u_i\right)_{p=0}$$

$$= \frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{n} p^i u_i\right)_{p=0}$$

Therefore, we obtain:

$$\frac{\partial^n}{\partial p^n} N(u)_{p=0} = \frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{\infty} p^i u_i\right)_{p=0} = \frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{n} p^i u_i\right)_{p=0}$$

Theorem (2.2.10) The He's polynomial can be calculated from the formula

$$H_n = \frac{1}{n!}\frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{\infty} p^i u_i\right)_{p=0} \qquad n = 1,2,3,\ldots$$

Proof:

Taking

$$F(u) = u(x) - f(x) - pN(u) = 0,$$

And substituting Eq. (3) into Eq. (4) in chapter1, we get;

$$H(u,p) = u(x) - f(x) - pN(u) = 0 \tag{16}$$

According to Maclaurin expansion of $pN(u)$ with respect to$p$, we get

$$N(u) = N(u)_{p=0} + p\left(\frac{\partial}{\partial p} N(u)_{p=0}\right) + p^2\left(\frac{1}{2!}\frac{\partial^2}{\partial p^2} N(u)_{p=0}\right)$$

$$+p^3\left(\frac{1}{3!}\frac{\partial^3}{\partial p^3} N(u)_{p=0}\right) + \cdots + p^n\left(\frac{1}{n!}\frac{\partial^n}{\partial p^n} N(u)_{p=0}\right) + \cdots$$

Substituting Eq. (9) into the above equation, we get;

$$N(u) = N\left(\sum_{i=0}^{\infty} p^i u_i\right)_{p=0} + p\left(\frac{\partial}{\partial p} N\left(\sum_{i=0}^{\infty} p^i u_i\right)_{p=0}\right)$$

$$+p^2\left(\frac{1}{2!}\frac{\partial^2}{\partial p^2}N\left(\sum_{i=0}^{\infty}p^iu_i\right)_{p=0}\right)+p^3\left(\frac{1}{3!}\frac{\partial^3}{\partial p^3}N\left(\sum_{i=0}^{\infty}p^iu_i\right)_{p=0}\right)$$

$$+\cdots+p^n\left(\frac{1}{n!}\frac{\partial^n}{\partial p^n}N\left(\sum_{i=0}^{\infty}p^iu_i\right)_{p=0}\right)+\cdots$$

According to Theorem (2.2.1)

$$N(u)=N(u_0)+p\left(\frac{\partial}{\partial p}N\left(\sum_{i=0}^{1}p^iu_i\right)_{p=0}\right)$$

$$+p^2\left(\frac{1}{2!}\frac{\partial^2}{\partial p^2}N\left(\sum_{i=0}^{2}p^iu_i\right)_{p=0}\right)+p^3\left(\frac{1}{3!}\frac{\partial^3}{\partial p^3}N\left(\sum_{i=0}^{3}p^iu_i\right)_{p=0}\right)$$

$$+\cdots+p^n\left(\frac{1}{n!}\frac{\partial^n}{\partial p^n}N\left(\sum_{i=0}^{n}p^iu_i\right)_{p=0}\right)+\cdots$$

Substituting Eq. (16) into Eq. (9), and equating the terms with the identical powers of p, we get;

$p^0:\ u_0(x)-f(x)=0\Rightarrow u_0(x)=f(x)$

$p^1:\ u_1(x)-N(u_0)=0\Rightarrow u_1(x)=N(u_0)$

$$p^2:u_1(x)-\frac{\partial}{\partial p}N\left(\sum_{i=0}^{1}p^iu_i\right)_{p=0}=0\Rightarrow u_1(x)=\frac{\partial}{\partial p}N\left(\sum_{i=0}^{1}p^iu_i\right)_{p=0}$$

$\vdots$

$$p^{n+1}:\ u_{n+1}(x)-\frac{\partial^n}{\partial p^n}N\left(\sum_{i=0}^{n}p^iu_i\right)_{p=0}=0\Rightarrow u_{n+1}(x)=\frac{\partial^n}{\partial p^n}N\left(\sum_{i=0}^{n}p^iu_i\right)_{p=0}$$

Definition (2.2.11) [80], The He polynomials are defined as follows:

$$H_n(u_0 + u_1 + \cdots + u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{\infty} p^i u_i\right)_{p=0}, \quad n = 1,2,3,\ldots$$

Therefore, the approximate solution obtained by the homotopy perturbation method can be expressed in the polynomials:

$$u(x) = f(x) + \underbrace{N(u_0)}_{H_0} + \underbrace{\frac{\partial}{\partial p} N\left(\sum_{i=0}^{1} p^i u_i\right)_{p=0}}_{H_1}$$

$$+ \underbrace{\frac{1}{2!}\frac{\partial^2}{\partial p^2} N\left(\sum_{i=0}^{2} p^i u_i\right)_{p=0}}_{H_2} + \cdots \underbrace{\frac{1}{n!}\frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{n} p^i u_i\right)_{p=0}}_{H_n} + \cdots$$

The nonlinear term $N(u)$ can be also expressed in He polynomials:

$$N(u) = \sum_{n=0}^{\infty} H_n\,(u_0 + u_1 + \cdots + u_n)$$

$$= H_0(u_0) + H_1(u_0, u_1) + \cdots + H_n(u_0, u_1, \ldots u_n) + \cdots \quad ,$$

Where

$$H_n(u_0 + u_1 + \cdots + u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{\infty} p^i u_i\right)_{p=0}, \quad n = 1,2,3,\ldots$$

Alternatively, the approximate solution can be expressed as follows:

$$u(x) = f(x) + \sum_{n=0}^{\infty} H_n\,(u_0 + u_1 + \cdots + u_n)$$

It is interesting to point out that we can obtain He polynomials and the solution simultaneously, making the solution procedure much more attractive and fascinating.

2.3 Applications of Homotopy Perturbation Transform Method to Linear and Nonlinear Partial Differential Equations

The aim of this section was to present a homotopy perturbation transform method for solving linear and nonlinear partial differential equations. The homotopy perturbation transform method is a combined form of the homotopy perturbation method and Laplace transform method. The nonlinear terms can be easily obtained by the use of He's polynomials. The technique presents an accurate methodology to solve many types of partial differential equations The approximate solutions obtained by proposed scheme in a wide range of the problem's domain were compared with those results obtained from the actual solutions.

Example (2.3.12) Consider the following homogeneous partial differential equation [61],

$$u_t - xu = 0 \tag{17}$$

Subject to the initial condition;

$$u(x,0) = 1$$

Taking the Laplace transform on both sides of equation (17) subject to the initial condition, we get;

$$u(x,s) = \frac{1}{s} + \frac{1}{s} L[xu] \tag{18}$$

The inverse of Laplace transform implies that:

$$u(x,t) = 1 + L^{-1}\left[\frac{1}{s} L[xu]\right] \tag{19}$$

Now, we apply the homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = 1 + P\left[L^{-1}\left[\frac{1}{s} L\left[x \sum_{n=0}^{\infty} p^n u_n(x,t)\right]\right]\right] \tag{20}$$

Or equivalently:

$$u_0 + pu_1 + p^2u_2 + \cdots = 1 + P\left[L^{-1}\left[\frac{1}{s}L[x(u_0 + pu_1 + p^2u_2 + \cdots)]\right]\right] \tag{21}$$

Comparing the coefficient of like powers of p, the following approximations are obtained;

$$p^0 : \; u_0(x,t) = 1$$

$$p^1 : \; u_1(x,t) = L^{-1}\left[\frac{1}{s}L[xu_0]\right] = xt \tag{22}$$

$$p^2 : \; u_2(x,t) = L^{-1}\left[\frac{1}{s}L[xu_1]\right] = x^2\frac{t^2}{2!}$$

Proceeding in a similar manner, we obtain:

$$p^3 : \; u_3(x,t) = x^3\frac{t^3}{3!}$$

$$p^4 : \; u_4(x,t) = x^4\frac{t^4}{4!}$$

Therefore the solution $u(x,t)$ in series form is given by:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots$$

$$u(x,t) = 1 + xt + x^2\frac{t^2}{2!} + x^3\frac{t^3}{3!} + x^4\frac{t^4}{4!} + \cdots \tag{23}$$

And in closed form given as:

$$u(x,t) = e^{xt} \tag{24}$$

Example (2.3.13) We consider the following second order PDEs [65],

$$u_t = u_{xx} - u_x \tag{25}$$

With the initial condition;

$$u(x,0) = e^x - x \tag{26}$$

And boundary conditions;

$$u(0,t) = 1 + t\,, \quad u_x(0,t) = 0 \tag{27}$$

Taking the Laplace transform on both sides of equation (25) subject to the initial condition Eq. (26), we get;

$$u(x,s) = \frac{e^x - x}{s} + \frac{1}{s}L[u_{xx} - u_x] \tag{28}$$

The inverse of Laplace transform implies that:

$$u(x,t) = e^x - x + L^{-1}\left[\frac{1}{s}L[u_{xx} - u_x]\right] \tag{29}$$

Now, we apply the homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = e^x - x$$

$$+P\left[L^{-1}\left[\frac{1}{s}L\left[\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)_{xx} - \left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)_x\right]\right]\right] \tag{30}$$

Or equivalently:

$$u_0 + pu_1 + p^2u_2 + \cdots = e^x - x$$

$$+P\left[L^{-1}\left[\frac{1}{s}L[(u_0 + pu_1 + \cdots)_{xx} - (u_0 + pu_1 + \cdots)_x]\right]\right] \tag{31}$$

Comparing the coefficients of like powers of p, we have:

$$p^0 :\ u_0(x,t) = e^x - x$$

$$p^1 :\ u_1(x,t) = L^{-1}\left[\frac{1}{s}L[(u_0)_{xx} - (u_0)_x]\right] = t \tag{32}$$

$$p^2 :\ u_2(x,t) = L^{-1}\left[\frac{1}{s}L[(u_1)_{xx} - (u_1)_x]\right] = 0$$

$$u_k(x,t) = 0, k \geq 2$$

Thus, the solution in series form is given by:

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t)$$

Or

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_k(x,t), \qquad k \geq 2$$

Hence the solution of Eq. (25) with Eqs. (26-27) is given as;

$$u(x,t) = e^x - x + t \tag{33}$$

This solution coincides with the exact one.

Example (2.3.14) We consider the following initial-boundary value problem [61],

$$u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0 \tag{34}$$

Subject to the initial condition;

$$u(x,0) = 1 + x + 2\,sin(\pi x) \tag{35}$$

And boundary conditions;

$$u(0,t) = 1, \qquad t \geq 0 \tag{36}$$

$$u(1,t) = 2, \qquad t \geq 0$$

Taking the Laplace transform on both sides of equation (34) subject to the initial condition Eq. (35), we get;

$$u(x,s) = \frac{1 + x + 2sin(\pi x)}{s} + \frac{1}{s} L[u_{xx}] \tag{37}$$

The inverse of Laplace transform implies that:

$$u(x,t) = 1 + x + 2\,sin(\pi x) + L^{-1}\left[\frac{1}{s} L[u_{xx}]\right] \tag{38}$$

Now, we apply the homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = 1 + x + 2\,sin(\pi x) + P\left[L^{-1}\left[\frac{1}{s}L[\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)_{xx}\right]\right] \qquad (39)$$

Or equivalently:

$$u_0 + pu_1 + p^2u_2 + \cdots = 1 + x + 2sin(\pi x)$$

$$+P\left[L^{-1}\left[\frac{1}{s}L[(u_0 + pu_1 + p^2u_2 + \cdots)_{xx}]\right]\right] \qquad (40)$$

Comparing the coefficients of like powers of p, we have:

$$p^0 :\ u_0(x,t) = 1 + x + 2sin(\pi x)$$

$$p^1 :\ u_1(x,t) = L^{-1}\left[\frac{1}{s}L[(u_0)_{xx}]\right] = -2\pi^2 t\,sin(\pi x) \qquad (41)$$

$$p^2 :\ u_2(x,t) = L^{-1}\left[\frac{1}{s}L[(u_1)_{xx}]\right] = 2\pi^4\frac{t^2}{2!}sin(\pi x)$$

And so on. Consequently, the solution in a series form is given by:

$$u(x,t) = \ 1 + x + 2\,sin(\pi x)\left(1 - \pi^2 t + \pi^4\frac{t^2}{2!} - \cdots\right) \qquad (42)$$

Hence the solution of Eq. (34) with Eqs. (35-36) is given as;

$$u(x,t) = 1 + x + 2\,sin(\pi x)\,e^{-\pi^2 t} \qquad (43)$$

This solution coincides with the exact one.

Example (2.3.15) We consider the homogeneous one dimension heat equation [63],

$$u_t = u_{xx} - u\,, \quad 0 \le x \le 1 \tag{44}$$

With initial condition:

$$u(x,0) = x^2, \qquad 0 \le x \le 1 \tag{45}$$

And boundary condition:

$$u_x(0,t) = 0, \quad t > 0 \tag{46}$$

Taking the Laplace transform on both sides of equation (44) subject to the initial condition Eq. (45), we get;

$$u(x,s) = \frac{x^2}{s} + \frac{1}{s} L[u_{xx} - u] \tag{47}$$

The inverse of Laplace transform implies that:

$$u(x,t) = x^2 + L^{-1}\left[\frac{1}{s} L[u_{xx} - u]\right] \tag{48}$$

Now, we apply the homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = x^2 + P\left[L^{-1}\left[\frac{1}{s} L[\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)_{xx} - \sum_{n=0}^{\infty} p^n u_n(x,t)\right]\right] \tag{49}$$

Or equivalently:

$$u_0 + pu_1 + p^2 u_2 + \cdots = x^2$$

$$+P\left[L^{-1}\left[\frac{1}{s} L[(u_0 + pu_1 + \cdots)_{xx} - (u_0 + pu_1 + \cdots)]\right]\right] \tag{50}$$

Comparing the coefficients of like powers of p, we have:

$$p^0 :\ u_0(x,t) = x^2$$

$$p^1 : \; u_1(x,t) = L^{-1}\left[\frac{1}{s} L[(u_0)_{xx} - u_0]\right] = 2t - x^2 t \tag{51}$$

$$p^2 : \; u_2(x,t) = L^{-1}\left[\frac{1}{s} L[(u_1)_{xx} - u_1]\right] = \frac{t^2 x^2}{2} - 2t^2$$

Proceeding in a similar manner, we get;

$$p^3 : \; u_3(x,t) = \frac{t^3 x^2}{3!} - t^3$$

$$p^4 : \; u_4(x,t) = \frac{t^4 x^2}{4!} - \frac{8t^4}{4!}$$

Thus, the solution in series form is given by:

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t)$$

$$u(x,t) = x^2\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right) + 2t\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right)$$

Hence the solution of Eq. (44) with Eqs. (45-46) is given as;

$$u(x,t) = x^2 e^{-t} + 2te^{-t} \tag{52}$$

This solution coincides with the exact one.

Example (2.3.16) We consider the following linear second partial differential equation [62],

$$u_t \; = \; u_{xx} - u_x \tag{53}$$

With initial condition:

$$u(x,0) = e^x - x \tag{54}$$

And boundary condition:

$$u(0,t) = 1 + t \tag{55}$$

Taking the Laplace transform on both sides of equation (53) subject to the initial condition Eq. (54), we get:

$$u(x,s) = \frac{e^x - x}{s} + \frac{1}{s} L[u_{xx} - u_x] \tag{56}$$

The inverse of Laplace transform implies that:

$$u(x,t) = e^x - x + L^{-1}\left[\frac{1}{s} L[u_{xx} - u_x]\right] \tag{57}$$

Now, we apply the homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = e^x - x$$

$$+P\left[L^{-1}\left[\frac{1}{s} L[\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)_{xx} - \left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)_x\right]\right] \tag{58}$$

Or equivalently:

$$u_0 + pu_1 + p^2 u_2 + \cdots = x^2$$

$$+P\left[L^{-1}\left[\frac{1}{s} L[(u_0 + pu_1 + \cdots)_{xx} - (u_0 + pu_1 + \cdots)_x]\right]\right] \tag{59}$$

Comparing the coefficients of like powers of p, we have:

$$p^0 : \ u_0(x,t) = e^x - x$$

$$p^1 : \ u_1(x,t) = L^{-1}\left[\frac{1}{s} L[(u_0)_{xx} - (u_0)_x]\right] = t \tag{60}$$

$$p^2 : \ u_2(x,t) = L^{-1}\left[\frac{1}{s} L[(u_1)_{xx} - (u_1)_x]\right] = 0$$

$$u_k(x,t) = 0, k \geq 2$$

Thus, the solution in series form is given by:

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t)$$

Or

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_k(x,t), \qquad k \geq 2$$

Hence the solution of Eq. (53) with Eq. (54-55) is given as;

$$u(x,t) = e^x - x + t \tag{61}$$

This solution coincides with the exact one.

Example (2.3.17) We consider the following homogeneous advection problem [82],

$$u_t + uu_x = 0, \tag{62}$$

Subject to the initial condition:

$$u(x,0) = -x, \tag{63}$$

By applying the aforesaid method subject to the initial condition, we have

$$u(x,s) = -\frac{x}{s} - \frac{1}{s} L[uu_x] \tag{64}$$

The inverse of Laplace transform implies that:

$$u(x,t) = -x - L^{-1}\left[\frac{1}{s} L[uu_x]\right] \tag{65}$$

Now, we apply the homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = -x - P\left[L^{-1}\left[\frac{1}{s} L[\sum_{n=0}^{\infty} p^n H_n(u)\right]\right] \tag{66}$$

Where $H_n(u)$ are He's polynomials [80, 81] that represent the nonlinear terms.

The first few components of He's polynomials, are given by:

$$H_0(u) = u_0(u_0)_x$$

$$H_1(u) = u_0(u_1)_x + u_1(u_0)_x$$

$$H_2(u) = u_0(u_2)_x + u_1(u_1)_x + u_2(u_0)_x$$

$$\vdots$$

Comparing the coefficients of like powers of p, we have:

$$p^0 : \ u_0(x,t) = -x,$$

$$p^1 : \ u_1(x,t) = -L^{-1}\left[\frac{1}{s}L[H_0(u)]\right] = -xt \tag{67}$$

$$p^2 : \ u_2(x,t) = -L^{-1}\left[\frac{1}{s}L[H_1(u)]\right] = -xt^2$$

Proceeding in a similar manner, we get;

$$p^3 : \ u_3(x,t) = -xt^3,$$

$$p^4 : \ u_4(x,t) = -xt^4,$$

So that the solution $u(x,t)$ is given by

$$u(x,t) = -x(1 + t + t^2 + t^3 + \cdots) \tag{68}$$

In series form, and

$$u(x,t) = \frac{x}{t-1}, \quad |t| > 1 \tag{69}$$

In closed form.

Example (2.3.18) We consider the following non-homogeneous advection problem [66],

$$u_t + uu_x = 2t + x + t^3 + xt^2, \tag{70}$$

Subject to the initial condition:

$$u(x,0) = 0, \tag{71}$$

By applying the aforesaid method subject to the initial condition, we have

$$u(x,s) = \frac{2}{s^3} + \frac{x}{s^2} + \frac{3!}{s^5} + \frac{2x}{s^4} - \frac{1}{s}L[uu_x] \tag{72}$$

The inverse of Laplace transform implies that:

$$u(x,t) = t^2 + xt + \frac{t^4}{4} + x\frac{t^3}{3} - L^{-1}\left[\frac{1}{s}L[uu_x]\right] \tag{73}$$

Now, we apply the homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = t^2 + xt + \frac{t^4}{4} + x\frac{t^3}{3} - P\left[L^{-1}\left[\frac{1}{s}L[\sum_{n=0}^{\infty} p^n H_n(u)\right]\right] \tag{74}$$

Comparing the coefficients of like powers of p, we have:

$$p^0 : u_0(x,t) = t^2 + xt + \frac{t^4}{4} + x\frac{t^3}{3},$$

$$p^1 : u_1(x,t) = -L^{-1}\left[\frac{1}{s}L[H_0(u)]\right]$$

$$u_1(x,t) = -\frac{1}{4}t^4 - \frac{x}{3}t^3 - \frac{2}{15}xt^5 - \frac{7}{72}t^6 - \frac{1}{63}xt^7 - \frac{1}{98}t^8 \tag{75}$$

$$p^2 : u_2(x,t) = -L^{-1}\left[\frac{1}{s}L[H_1(u)]\right]$$

$$u_2(x,t) = \frac{5}{8064}t^{12} + \frac{2}{2079}xt^{11} + \frac{2783}{302400}t^{10} + \frac{38}{2835}xt^9$$

$$+\frac{143}{2880}t^8 + \frac{22}{315}xt^7 + \frac{7}{12}t^6 + \frac{2}{15}xt^5$$

It is necessary to mention here that the noise terms are those terms who are the same but different in signs, more clearly the noise terms $\pm\frac{1}{4}t^4$ *and* $\pm\frac{x}{3}t^3$ between the components $u_0(x,t)$ and $u_1(x,t)$ can be cancelled and the remaining terms of $u_0(x,t)$ still satisfy the equation. The exact solution is therefore

$$u(x,t) = t^2 + xt \tag{76}$$

Example (2.3.19) We consider the following diffusion convection problem [67],

$$u_t \ = u_{xx} - u_x + uu_x - u^2 + u\,, \quad x,t \in R \tag{77}$$

Subject to the initial condition:

$$u(x,0) = e^x, \tag{78}$$

Taking the Laplace transform on both sides of equation (77) subject to the initial condition Eq. (78), we get;

$$u(x,s) = \frac{e^x}{s} + \frac{1}{s} L[(u_{xx} - u_x + u) + (uu_x - u^2)] \tag{79}$$

The inverse of Laplace transform implies that:

$$u(x,t) = e^x + L^{-1}\left[\frac{1}{s} L[(u_{xx} - u_x + u) + (uu_x - u^2)]\right] \tag{80}$$

Now, we apply the homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = \ e^x + p\left(L^{-1}\left[\frac{1}{s^2}\ L\left[\sum_{n=0}^{\infty} p^n u_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right) \tag{81}$$

Where $H_n(u)$ are He's polynomials [80, 81] that represent the nonlinear terms.

The first few components of He's polynomials, are given by

$$\begin{aligned} H_0(u) &= u_0(u_0)_x - {u_0}^2 \\ H_1(u) &= u_0(u_1)_x + u_1(u_0)_x - 2u_0u_1 \\ H_2(u) &= u_0(u_2)_x + u_1(u_1)_x + u_2(u_0)_x - {u_1}^2 \ - 2u_2u_0 \\ &\vdots \end{aligned} \tag{82}$$

Comparing the coefficients of like powers of p, we have:

$$p^0 : \; u_0(x,t) = e^x,$$

$$p^1 : \; u_1(x,t) = L^{-1}\left[\frac{1}{s}L[(u_0)_{xx} - (u_0)_x + u_0 + H_0(u)]\right] = e^x t$$

$$p^2 : \; u_2(x,t) = L^{-1}\left[\frac{1}{s}L[(u_1)_{xx} - (u_1)_x + u_1 + H_1(u)]\right] = e^x \frac{t^2}{2!}$$

$$p^3 : \; u_3(x,t) = L^{-1}\left[\frac{1}{s}L[(u_2)_{xx} - (u_2)_x + u_2 + H_2(u)]\right] = e^x \frac{t^3}{3!}$$

Proceeding in a similar manner, we have

$$p^4 : \; u_4(x,t) = e^x \frac{t^4}{4!},$$

$$p^5 : \; u_5(x,t) = e^x \frac{t^5}{5!},$$

$$\vdots$$

Therefore the series solution $u(x,t)$ is given as;

$$u(x,t) = e^x\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right) \tag{83}$$

In series form, and

$$u(x,t) = e^{x+t} \tag{84}$$

In closed form.

Example (2.3.20) We consider a nonlinear partial differential equation

$$u_t = x^2 + \frac{1}{4}{u_x}^2, \tag{85}$$

Subject to the initial condition:

$$u(x,0) = 0, \tag{86}$$

Taking the Laplace transform on both sides of Eq. (85) subject to the initial condition Eq. (86), we get;

$$u(x,s) = \frac{x^2}{s^2} + \frac{1}{s} L\left[\frac{1}{4} {u_x}^2\right] \tag{87}$$

The inverse of Laplace transform implies that;

$$u(x,t) = x^2 t + L^{-1}\left[\frac{1}{s} L\left[\frac{1}{4} {u_x}^2\right]\right] \tag{88}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = x^2 t + p\left(L^{-1}\left[\frac{1}{s} L\left[\frac{1}{4}\sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right) \tag{89}$$

Where $H_n(u)$ are He's polynomials that represents the nonlinear terms. The first few components of He's polynomials, are given by

$$H_0(u) = {(u_0)_x}^2$$

$$H_1(u) = 2(u_0)_x {(u_1)_x}^2$$

$$H_2(u) = 2(u_0)_x (u_2)_x + {(u_1)_x}^2$$

$$H_2(u) = 2(u_0)_x (u_3)_x + 2(u_1)_x (u_2)_x$$

$$\vdots$$

Comparing the coefficients of like powers of p, we get:

$$p^0 : \ u_0(x,t) = x^2 t,$$

$$p^1 : \ u_1(x,t) = \frac{1}{4} L^{-1}\left[\frac{1}{s} L[H_0(u)]\right] = \frac{1}{3} x^2 t^3 \tag{90}$$

$$p^2 : \ u_2(x,t) = \frac{1}{4} L^{-1}\left[\frac{1}{s} L[H_1(u)]\right] = \frac{2}{15} x^2 t^5$$

Proceeding in a similar manner, we have:

$$p^3: \; u_3(x,t) = \frac{7}{315}x^2t^7,$$

$$\vdots$$

And so on. Combining the results obtained for the components, the solution in a series form is given by:

$$u(x,t) = x^2\left(t + \frac{t^3}{3} + \frac{2}{15}t^5 + \frac{7}{315}t^7 \ldots\right) \tag{91}$$

And in a closed form by;

$$u(x,t) = x^2 \, tan(t) \tag{92}$$

Example (2.3.21) We consider a nonlinear partial differential equation [61],

$$u_{tt} + \frac{1}{4}{u_x}^2 = u \tag{93}$$

Subject to the initial condition;

$$u(x,0) = 1 + x^2 \tag{94}$$

And boundary condition;

$$u_t(x,0) = 1 \tag{95}$$

Taking the Laplace transform on both sides of Eq. (93) subject to the initial condition Eq. (94) and boundary condition Eq. (95), we get;

$$u(x,s) = \frac{1+x^2}{s} + \frac{1}{s^2} + \frac{1}{s^2}L\left[u - \frac{1}{4}{u_x}^2\right] \tag{96}$$

The inverse of Laplace transform implies that;

$$u(x,t) = 1 + x^2 + t + L^{-1}\left[\frac{1}{s^2}L[u - \frac{1}{4}{u_x}^2]\right] \tag{97}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = 1 + x^2 + t$$

$$+p\left(L^{-1}\left[\frac{1}{s^2}\,L\left[\sum_{n=0}^{\infty} p^n u_n(x,t) - \frac{1}{4}\sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right) \qquad (98)$$

Where $H_n(u)$ the homotopy polynomials are represents the nonlinear term ${u_x}^2$.

The first few components of homotopy polynomials are given by:

$$H_0(u) = {(u_0)_x}^2$$

$$H_1(u) = 2(u_0)_x{(u_1)_x}^2$$

$$H_2(u) = 2(u_0)_x(u_2)_x + {(u_1)_x}^2$$

$$\vdots$$

Comparing the coefficients of like powers of p, we have:

$$p^0 : \ u_0(x,t) = 1 + x^2 + t,$$

$$p^1 : \ u_1(x,t) = L^{-1}\left[\frac{1}{s^2}L\left[u_0 - \frac{1}{4}H_0(u)\right]\right] = \frac{t^2}{2!} + \frac{t^3}{3!} \qquad (99)$$

$$p^2 : \ u_2(x,t) = L^{-1}\left[\frac{1}{s^2}L\left[u_1 - \frac{1}{4}H_1(u)\right]\right] = \frac{t^4}{4!} + \frac{t^5}{5!}$$

Proceeding in a similar manner, we get;

$$p^3 : \ u_3(x,t) = \frac{t^6}{6!} + \frac{t^7}{7!},$$

$$p^4 : \ u_4(x,t) = \frac{t^8}{8!} + \frac{t^9}{9!},$$

$$\vdots$$

And so on. Combining the results obtained for the components, the solution in a series form is given by:

$$u(x,t) = x^2 + (1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \cdots \qquad (100)$$

In series form, and

$$u(x,t) = x^2 + e^t \qquad (101)$$

This solution coincides with the exact one.

2.4 Systems of Linear and Nonlinear Partial Differential Equations by (HPTM)

It is well-known that many physical and engineering phenomena such as wave propagation and shallow water waves can be modeled by systems of PDEs Finding accurate and efficient methods for solving nonlinear system of PDEs has long been an active research undertaking. HPTM deforms a difficult problem into a set of problems which are easier to solve without any need to transform nonlinear terms.

The aim of this section is to present an approach based homotopy perturbation transform method for finding series solutions to linear and nonlinear systems of partial differential equations written in an operator form

$$L_t u + R_1(u,v) + N_1(u,v) = g_1 \qquad (102)$$

$$L_t v + R_2(u,v) + N_2(u,v) = g_2$$

Subject to the initial conditions;

$$u(x,0) = f_1(x) \qquad (103)$$

$$v(x,0) = f_2(x)$$

Where L_t is consider as a first-order partial differential operator,R_1 , R_2 and N_1, N_2 are linear and nonlinear operators and g_1 and g_2 are source terms. The method consists of first applying the Laplace transform to both sides of equations in system Eq. (102) and then by using initial conditions Eq. (103), we get;

$$L[L_t u] + L[R_1(u,v)] + L[N_1(u,v)] = L[g_1] \tag{104}$$

$$L[L_t v] + L[R_2(u,v)] + L[N_2(u,v)] = L[g_2] \tag{105}$$

Using the differential property of Laplace transform and initial conditions, we get;

$$u(x,s) = \frac{f_1(x)}{s} + \frac{1}{s}L[g_1] - \frac{1}{s}L[R_1(u,v)] - \frac{1}{s}L[N_1(u,v)] \tag{106}$$

$$v(x,s) = \frac{f_2(x)}{s} + \frac{1}{s}L[g_2] - \frac{1}{s}L[R_2(u,v)] - \frac{1}{s}L[N_2(u,v)] \tag{107}$$

Applying the inverse of Laplace transform on both sides of Eqs. (106-107), we get:

$$u(x,t) = F_1(x) - L^{-1}\left[\frac{1}{s}\left[L[R_1(u,v)] + L[N_1(u,v)]\right]\right] \tag{108}$$

$$v(x,t) = F_2(x) - L^{-1}\left[\frac{1}{s}\left[L[R_2(u,v)] + L[N_2(u,v)]\right]\right] \tag{109}$$

Where $F_1(x)$ and $F_2(x)$ represents the terms arising from source terms and prescribe initial conditions. According to standard homotopy perturbation method the solution$u$ and v can be expanded into infinite series as;

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t)\,, \quad v(x,t) = \sum_{n=0}^{\infty} p^n v_n(x,t) \tag{110}$$

Where $p\epsilon[0,1]$ is an embedding parameter. Also the nonlinear term N_1 and N_2 can be written as;

$$N_1(u,v) = \sum_{n=0}^{\infty} p^n (H_1)_n(u,v) \tag{111}$$

$$N_2(u,v) = \sum_{n=0}^{\infty} p^n (H_2)_n(u,v)$$

Where H_{1n} and H_{2n} are the He's polynomials [80, 81] that represent the nonlinear terms. By substituting Eqs. (110) and (111) in Eqs. (108) and (109), the solutions can be written as;

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = F_1(x) - p\left(L^{-1}\left[\frac{1}{s}\left[L[R_1(u,v)] + L[(H_1)_n]\right]\right]\right) \tag{112}$$

$$\sum_{n=0}^{\infty} p^n v_n(x,t) = F_2(x) - p\left(L^{-1}\left[\frac{1}{s}\left[L[R_2(u,v)] + L[(H_2)_n]\right]\right]\right) \tag{113}$$

In Eqs. (112) and (113),$(H_1)_n$,H_{2n} are He's polynomials can be generated by several means. Here we used the following recursive formulation:

$$H_n(u_0 + u_1 + \cdots + u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{\infty} p^i u_i\right)_{p=0}, \quad n = 1,2,3,\ldots \tag{114}$$

Equating the terms with identical powers in p in Eqs. (112) and (113), we obtained the following approximations:

$$p^0: u_0 = F_1(x)$$

$$p^1: u_1 = -L^{-1}\left[\frac{1}{s} L\big[[R_1(u_0,v_0)] + [(H_1)_0]\big]\right]$$

$$p^2: u_2 = -L^{-1}\left[\frac{1}{s} L\big[[R_1(u_1,v_1)] + [(H_1)_1]\big]\right]$$

$$\vdots$$

Similarly

$$p^0: v_0 = F_2(x)$$

$$p^1: v_1 = -L^{-1}\left[\frac{1}{s}L\big[[R_2(u_0, v_0)] + [(H_2)_0]\big]\right] \tag{115}$$

$$p^2: v_2 = -L^{-1}\left[\frac{1}{s}L\big[[R_2(u_1, v_1)] + [(H_2)_1]\big]\right]$$

$$\vdots$$

The best approximations for the solutions are

$$u = \lim_{p\to 1} u_n = u_0 + u_1 + u_2 + \cdots \tag{116}$$

$$v = \lim_{p\to 1} v_n = v_0 + v_1 + v_2 + \cdots$$

To give a clear overview of the content of this work, several illustrative examples have been selected to demonstrate the efficiency of the method.

Example (2.4.22) We first consider the linear system

$$u_t + v_x = 0, \tag{117}$$

$$v_t + u_x = 0,$$

With the initial conditions;

$$u(x, 0) = e^x \tag{118}$$

$$v(x, 0) = e^{-x}$$

Taking Laplace transform of equation (117) subject to the initial conditions Eq. (118), we get;

$$u(x, s) = \frac{e^x}{s} - \frac{1}{s}L[v_x] \tag{119}$$

$$v(x, s) = \frac{e^{-x}}{s} - \frac{1}{s}L[u_x]$$

The inverse Laplace transform implies that:

$$u(x,t) = e^{x} - L^{-1}\left[\frac{1}{s}L[v_x]\right] \tag{120}$$

$$v(x,t) = e^{-x} - L^{-1}\left[\frac{1}{s}L[u_x]\right]$$

Now applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = e^{x} - pL^{-1}\left[\frac{1}{s}L\left[\left(\sum_{n=0}^{\infty} p^n v_n(x,t)\right)_x\right]\right] \tag{121}$$

$$\sum_{n=0}^{\infty} p^n v_n(x,t) = e^{-x} - pL^{-1}\left[\frac{1}{s}L\left[\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)_x\right]\right]$$

Comparing the coefficients of like power p, we get;

$$\begin{aligned}
&p^0: u_0(x,t) = e^{x} \quad , \quad v_0(x,t) = e^{-x}\\
&p^1: u_1(x,t) = te^{-x} \quad , \quad v_1(x,t) = -te^{x}\\
&p^2: u_2(x,t) = \frac{t^2}{2!}e^{x} \quad , \quad v_2(x,t) = \frac{t^2}{2!}e^{-x}\\
&p^3: u_3(x,t) = \frac{t^3}{3!}e^{-x} \quad , \quad v_3(x,t) = -\frac{t^3}{3!}e^{x}\\
&\quad \vdots
\end{aligned} \tag{122}$$

And so on. Combining the results obtained for the components, the solution in a series form is given by;

$$u(x,t) = e^{x}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) + e^{-x}\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) \tag{123}$$

$$v(x,t) = e^{-x}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) + e^{x}\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) \tag{124}$$

Which has an exact analytical solution of the form;

$$(u,v) = \left(e^{x}cosh(t) + e^{-x}sinh(t)\,,\ e^{-x}cosh(t) + e^{x}sinh(t)\right) \tag{125}$$

Example (2.4.23) We consider the inhomogeneous linear system [68],

$$u_t - v_x - (u - v) = -2, \tag{126}$$

$$v_t + u_x - (u - v) = -2,$$

With the initial conditions;

$$u(x,0) = 1 + e^{x} \tag{127}$$

$$v(x,0) = -1 + e^{x}$$

Taking the Laplace transform on both sides of Eq. (126), then by using the differentiation property of Laplace transform and initial conditions Eq. (127) gives;

$$u(x,s) = \frac{1 + e^{x}}{s} - \frac{2}{s^2} + \frac{1}{s}L[v_x + (u - v)] \tag{128}$$

$$v(x,s) = \frac{-1 + e^{x}}{s} - \frac{2}{s^2} + \frac{1}{s}L[(u - v) - u_x]$$

The inverse Laplace transform implies that:

$$u(x,t) = 1 + e^{x} - 2t + L^{-1}\left[\frac{1}{s}L[v_x + (u - v)]\right] \tag{129}$$

$$v(x,t) = -1 + e^{x} - 2t + L^{-1}\left[\frac{1}{s}L[(u - v) - u_x]\right]$$

Now applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = 1 + e^{x} - 2t$$

$$+pL^{-1}\left[\frac{1}{s}L\left[\left(\sum_{n=0}^{\infty} p^n v_n(x,t)\right)_x + \left(\sum_{n=0}^{\infty} p^n u_n(x,t) - \sum_{n=0}^{\infty} p^n v_n(x,t)\right)\right]\right] \tag{130}$$

$$\sum_{n=0}^{\infty} p^n v_n(x,t) = -1 + e^x - 2t$$

$$+pL^{-1}\left[\frac{1}{s}L\left[\left(\sum_{n=0}^{\infty} p^n u_n(x,t) - \sum_{n=0}^{\infty} p^n v_n(x,t)\right) - \left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)_x\right]\right] \quad (131)$$

Comparing the coefficients of like power p, we get;

$$p^0:\begin{cases} u_0(x,t) = 1 + e^x - 2t \\ v_0(x,t) = -1 + e^x - 2t \end{cases}$$

$$p^1:\begin{cases} u_1(x,t) = te^x + 2t \\ v_1(x,t) = -te^x + 2t \end{cases} \quad (132)$$

$$p^2:\begin{cases} u_2(x,t) = \frac{t^2}{2!}e^x \\ v_2(x,t) = \frac{t^2}{2!}e^x \end{cases}$$

And so on. Combining the results obtained for the components, the solution in a series form is given by;

$$\begin{cases} u(x,t) = 1 + e^x\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) \\ v(x,t) = -1 + e^x\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right) \end{cases} \quad (133)$$

That converges to the exact solutions

$$u(x,t) = 1 + e^{x+t} \quad (134)$$

$$v(x,t) = -1 + e^{x-t}$$

Which has an exact analytical solution of the form;

$$(u,v) = (1 + e^{x+t}, -1 + e^{x-t}) \quad (135)$$

Example (2.4.24) We consider the inhomogeneous nonlinear system [61],

$$u_t + vu_x + u = 1, \tag{136}$$

$$v_t + uv_x - v = 1,$$

With the initial conditions;

$$u(x,0) = e^x \tag{137}$$

$$v(x,0) = e^{-x}$$

Taking the Laplace transform on both sides of Eq. (136), then by using the differentiation property of Laplace transform and initial conditions Eq. (137) gives;

$$u(x,s) = \frac{e^x}{s} + \frac{1}{s^2} - \frac{1}{s} L[vu_x + u] \tag{138}$$

$$v(x,s) = \frac{-e^x}{s} + \frac{1}{s^2} - \frac{1}{s} L[uv_x - v]$$

The inverse Laplace transform implies that:

$$u(x,t) = e^x + t - L^{-1}\left[\frac{1}{s} L[vu_x + u]\right] \tag{139}$$

$$v(x,t) = e^{-x} + t - L^{-1}\left[\frac{1}{s} L[uv_x - v]\right]$$

Now applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = e^x + t - pL^{-1}\left[\frac{1}{s} L\left[\sum_{n=0}^{\infty} p^n (H_1)_n(u,v) + \sum_{n=0}^{\infty} p^n u_n(x,t)\right]\right] \tag{140}$$

$$\sum_{n=0}^{\infty} p^n v_n(x,t) = e^{-x} + t - pL^{-1}\left[\frac{1}{s} L\left[\sum_{n=0}^{\infty} p^n (H_2)_n(u,v) - \sum_{n=0}^{\infty} p^n v_n(x,t)\right]\right] \tag{141}$$

Where $(H_1)_n(u,v)$ and $(H_2)_n(u,v)$ are He's polynomials that represents nonlinear terms vu_x and uv_x respectively. We have a few terms of the He's polynomials for vu_x and uv_x, which are given by:

$$\begin{aligned}
&(H_1)_0(u,v) = v_0(u_0)_x \\
&(H_1)_1(u,v) = v_1(u_0)_x + v_0(u_1)_x \qquad (142)\\
&(H_1)_2(u,v) = v_2(u_0)_x + v_1(u_1)_x + v_0(u_2)_x \\
&\vdots
\end{aligned}$$

$$\begin{aligned}
&(H_2)_0(u,v) = u_0(v_0)_x \\
&(H_2)_1(u,v) = u_1(v_0)_x + u_0(v_1)_x \qquad (143)\\
&(H_2)_2(u,v) = u_2(v_0)_x + u_1(v_1)_x + u_0(v_2)_x \\
&\vdots
\end{aligned}$$

Comparing the coefficients of like power p, we get;

$$p^0: \begin{cases} u_0(x,t) = e^x + t \\ v_0(x,t) = e^{-x} + t \end{cases} \qquad (144)$$

$$p^1: \begin{cases} u_1(x,t) = -L^{-1}\left[\frac{1}{s}L[(H_1)_0(u,v)+u_0]\right] = -t - te^x + \frac{t^2}{2!} + \frac{t^2}{2!}e^x \\ v_1(x,t) = -L^{-1}\left[\frac{1}{s}L[(H_2)_0(u,v)-v_0]\right] = -t - te^{-x} - \frac{t^2}{2!} - \frac{t^2}{2!}e^{-x} \end{cases},$$

By canceling the noise terms between u_0,u_1,$\cdots$ and between v_1,v_1, $\cdots$, we find:

$$\begin{cases} u(x,t) = e^x\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right) \\ v(x,t) = e^{-x}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) \end{cases} \qquad (145)$$

That converges to the exact solutions

$$\begin{cases} u(x,t) = e^{x-t} \\ v(x,t) = e^{-x+t} \end{cases} \qquad (146)$$

Which has an exact analytical solution of the form;

$$(u,v) = (e^{x-t}, e^{-x+t}) \tag{147}$$

Example (2.4.25) We consider the following nonlinear system [61],

$$\begin{aligned} &u_t - v_x w_y = 1, \\ &v_t - w_x u_y = 5, \\ &w_t - u_x v_y = 5, \end{aligned} \tag{148}$$

With the initial conditions;

$$\begin{aligned} &u(x,y,0) = x + 2y \\ &v(x,,y,0) = \ x - 2y \\ &w(x,,y,0) = -x + 2y \end{aligned} \tag{149}$$

Taking the Laplace transform on both sides of Eq. (136), then by using the differentiation property of Laplace transform and initial conditions Eq. (137) gives;

$$\begin{aligned} &u(x,y,s) = \frac{x+2y}{s} + \frac{1}{s^2} - \frac{1}{s} L\left[v_x w_y\right] \\ &v(x,y,s) = \frac{x-2y}{s} + \frac{5}{s^2} - \frac{1}{s} L\left[w_x u_y\right] \\ &w(x,y,s) = \frac{-x+2y}{s} + \frac{5}{s^2} - \frac{1}{s} L\left[u_x v_y\right] \end{aligned} \tag{150}$$

The inverse Laplace transform implies that:

$$\begin{aligned} &u(x,y,t) = x + 2y + t - L^{-1}\left[\frac{1}{s} L\left[v_x w_y\right]\right] \\ &v(x,y,t) = x - 2y + 5t - L^{-1}\left[\frac{1}{s} L\left[w_x u_y\right]\right] \\ &w(x,y,t) = -x + 2y + 5t - \frac{1}{s} L\left[u_x v_y\right] \end{aligned} \tag{151}$$

Now applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n u_n(x,y,t) = x + 2y + t - pL^{-1}\left[\frac{1}{s}L\left[\sum_{n=0}^{\infty} p^n (H_1)_n(u,v,w)\right]\right]$$

$$\sum_{n=0}^{\infty} p^n v_n(x,y,t) = x - 2y + 5t - pL^{-1}\left[\frac{1}{s}L\left[\sum_{n=0}^{\infty} p^n (H_2)_n(u,v,w)\right]\right] \quad (152)$$

$$\sum_{n=0}^{\infty} p^n w_n(x,y,t) = -x + 2y + 5t - pL^{-1}\left[\frac{1}{s}L\left[\sum_{n=0}^{\infty} p^n (H_3)_n(u,v,w)\right]\right]$$

Where $(H_1)_n(u,v,w)$,$(H_2)_n(u,v,w)$ and $(H_3)_n(u,v,w)$ are He's polynomials that represents nonlinear terms $v_x w_y$, $w_x u_y$ and $u_x v_y$ respectively. We have a few terms of the He's polynomials for $v_x w_y$, $w_x u_y$ and $u_x v_y$, which are given by:

$$(H_1)_0(u,v,w) = (w_0)_x(u_0)_y$$

$$(H_1)_1(u,v,w) = (w_0)_x(u_1)_y + (w_1)_x(u_0)_y \quad (153)$$

$$(H_1)_2(u,v,w) = (w_0)_x(u_2)_y + (w_1)_x(u_1)_y + (w_2)_x(u_0)_y$$

$$\vdots$$

$$(H_2)_0(u,v,w) = (v_0)_x(w_0)_y$$

$$(H_2)_1(u,v,w) = (v_0)_x(w_1)_y + (v_1)_x(w_0)_y \quad (154)$$

$$(H_2)_2(u,v,w) = (v_0)_x(w_2)_y + (v_1)_x(w_1)_y + (v_2)_x(w_0)_y$$

$$\vdots$$

$$(H_1)_0(u,v,w) = (u_0)_x(v_0)_y$$

$$(H_1)_1(u,v,w) = (u_0)_x(v_1)_y + (u_1)_x(v_0)_y \quad (155)$$

$$(H_1)_2(u,v,w) = (u_0)_x(v_2)_y + (u_1)_x(v_1)_y + (u_2)_x(v_0)_y$$

Comparing the coefficients of like power p, we get;

$$p^0:\begin{cases} u_0(x,y,t) = x + 2y + t \\ v_0(x,y,t) = x - 2y + 5t \\ w_0(x,y,t) = -x + 2y + 5t \end{cases} \quad (156)$$

$$p^1:\begin{cases} u_1(x,y,t) = 2t \\ v_1(x,y,t) = -2t \\ w_1(x,y,t) = -2t \end{cases}, \quad (157)$$

$$p^2:\begin{cases} u_2(x,y,t) = 0 \\ v_2(x,y,t) = 0 \\ w_2(x,y,t) = 0 \end{cases}, \quad (158)$$

$$\begin{cases} u_k(x,y,t) = 0 \\ v_k(x,y,t) = 0 \\ w_k(x,y,t) = 0 \end{cases}, \forall k \geq 2 \quad (159)$$

Therefore the exact solutions of the above system of inhomogeneous nonlinear PDES as follows:

$$\begin{cases} u(x,y,t) = x + 2y + 3t \\ v(x,y,t) = x - 2y + 3t \\ w(x,y,t) = -x + 2y + 3t \end{cases} \quad (160)$$

Which has an exact analytical solution of the form;

$$(u,v,w) = (x + 2y + 3t, x - 2y + 3t, -x + 2y + 3t) \quad (161)$$

CHAPTER THREE

Homotopy Perturbation Transform Method for Solving Physical Models

The investigation of the exact solutions to nonlinear equations plays an important role in the study of nonlinear physical phenomena. The linear and nonlinear partial differential equations have been widely used in various application areas, e.g, quantum mechanics, optics, seismology and plasma physics. Since analytic approaches to the partial differential equations have limited applicability in science and engineering problems, there is a growing interest in exploring new methods to solve the equation more accurately and efficiently. In recent years, many research workers have paid attention to study the solutions of nonlinear partial differential equations by using various methods. Among these the Adomian decomposition method Hashim, Noorani, Ahmed.Bakar, Ismail and Zakaria, (2006), the tanh method, the homotopy perturbation method Sweilam, Khader (2009), Sharma and GirirajMethi (2011), Jafari, Aminataei (2010), (2011), the differential transform method (2008), homotopy perturbation transform method and the variational iteration method. He[1-5] developed the homotopy perturbation method (HPM) by merging the standard homotopy and perturbation for solving various physical problems. It is worth mentioning that the HPM is applied without any discretization, restrictive assumption or transformation and is free from round off errors. Various ways have been proposed recently to deal with these nonlinearities; one of these combinations is Laplace transform and homotopy perturbation method. Laplace transform is a useful technique for solving linear differential equations, but this transform is totally incapable of handling nonlinear equations because of the difficulties that are caused by the nonlinear terms. So, the solution can be obtained using both Laplace transform and homotopy perturbation method to solve nonlinear problems. This method provides the solution in a rapid convergent series which may lead the solution in a closed form. The advantage of this method that can combine the two powerful methods for obtaining exact solutions for nonlinear equations. Finally, the HPTM is applies in solving the linear and nonlinear partial differential equations to show the simplicity and straight for wardness of the method.

3.1 The Klein-Gordon Equation

The Klein-Gordon equations appear in quantum field theory, relativistic physics, dispersive wave-phenomena, plasma physics, Nonlinear phenomena have important effects on applied mathematics, physics and related to engineering; many such physical phenomena are modeled in terms of nonlinear partial differential equations.

The Klein-Gordon equation has been extensively studied by using traditional methods such as finite difference method, finite element method, or collocation method, finite element method, Backland transformations, and the inverse scattering method were also applied to handle Klein-Gordon equation.

The methods investigated the concepts of existence, a uniqueness of the solution and the weak solution as well. The objectives of these studies were mostly focused on the determination of numerical solutions where a considerable volume of calculations is usually needed. In this section, the homotopy perturbation method will be applied to obtain exact solutions if exist, and approximate to solutions for concrete problems.

3.1.1 Linear Klein-Gordon Equation

The linear Klein-Gordon equation in its standard form is given by:

$$u_{tt} - u_{xx} + au = h(x,t) \tag{1}$$

Subject to the initial conditions;

$$u(x,0) = f(x), \quad u_t(x,0) = g(x) \tag{2}$$

Where a is a constant and $h(x,t)$ is the source term.to solve Eq. (1) with the initial condition Eq. (2), by the HPTM, we Taking the Laplace transform on both sides of equation (1) subject to the initial conditions, we get;

$$u(x,s)=\frac{f(x)}{s}+\frac{g(x)}{s^2}+\frac{1}{s^2}L[u_{xx}-au+h(x,t)] \tag{3}$$

The inverse of Laplace transform implies that;

$$u(x,t)=f(x)+tg(x)+L^{-1}\left[\frac{1}{s^2}L[u_{xx}-au+h(x,t)]\right] \tag{4}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty}p^n u_n(x,t)=f(x)+tg(x)+L^{-1}\left[\frac{1}{s^2}L[h(x,t)]\right]$$

$$+pL^{-1}\left[\frac{1}{s^2}L\left[\left(\sum_{n=0}^{\infty}p^n u_n(x,t)\right)_{xx}-a\sum_{n=0}^{\infty}p^n u_n(x,t)\right]\right] \tag{5}$$

Comparing the coefficients of like powers p, the following approximations are obtained:

$$p^0: u_0(x,t)=f(x)+tg(x)+L^{-1}\left[\frac{1}{s^2}L[h(x,t)]\right]$$

$$p^1: u_1(x,t)=L^{-1}\left[\frac{1}{s^2}L[(u_0)_{xx}-au_0]\right] \tag{6}$$

$$p^2: u_2(x,t)=L^{-1}\left[\frac{1}{s^2}L[(u_1)_{xx}-au_1]\right]$$

$$\vdots$$

Therefore the solution $u(x,t)$ in series form is given by:

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+\cdots \tag{7}$$

In many cases we can obtain inductively the exact solution. Thealgorithm that is discussed above will be explained by the following illustrative examples.

Example (3.1.1) We consider the following linear Klein-Gordon equation [69],

$$u_{tt} - u_{xx} + u = 0 \tag{8}$$

With the initial conditions;

$$u(x,0) = 0, \qquad u_t(x,0) = x. \tag{9}$$

Taking the Laplace transform on both sides of equation (8) subject to the initial conditions Eq. (9), we get;

$$u(x,s) = \frac{x}{s^2} + \frac{1}{s^2} L[u_{xx} - u] \tag{10}$$

The inverse of Laplace transform implies that;

$$u(x,t) = xt + L^{-1}\left[\frac{1}{s^2} L[u_{xx} - u]\right] \tag{11}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = xt + pL^{-1}\left[\frac{1}{s^2} L\left[\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)_{xx} - \sum_{n=0}^{\infty} p^n u_n(x,t)\right]\right] \tag{12}$$

Comparing the coefficients of like powers p, the following approximations are obtained:

$$\begin{aligned} &p^0: u_0(x,t) = xt, \\ &p^1: u_1(x,t) = L^{-1}\left[\frac{1}{s^2} L[(u_0)_{xx} - u_0]\right] = -x\frac{t^3}{3!} \\ &p^2: u_2(x,t) = L^{-1}\left[\frac{1}{s^2} L[(u_1)_{xx} - u_1]\right] = x\frac{t^5}{5!} \end{aligned} \tag{13}$$

Proceeding in a similar manner, we obtain;

$$p^3: u_3(x,t) = -x\frac{t^7}{7!}$$

$$p^4: u_4(x,t) = x\frac{t^9}{9!}$$

Therefore the solution $u(x,t)$ in series form is given by:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots$$

$$u(x,t) = x\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right) \quad (14)$$

And in closed form given as;

$$u(x,t) = xsin(t) \quad (15)$$

Example (3.1.2) We consider the following linear Klein-Gordon equation [69],

$$u_{tt} - u_{xx} + u = 2sinx, \quad (16)$$

With the initial conditions;

$$u(x,0) = sinx, \qquad u_t(x,0) = 1, \quad (17)$$

Applying the Laplace transform on both sides of Eq. (16) subject to the initial conditions Eq. (17), we have;

$$u(x,s) = \frac{2sinx}{s^3} + \frac{sinx}{s} + \frac{1}{s^2} + \frac{1}{s^2}L[u_{xx} - u] \quad (18)$$

The inverse of Laplace transform implies that;

$$u(x,t) = t^2sinx + sinx + t + L^{-1}\left[\frac{1}{s^2}L[u_{xx} - u]\right] \quad (19)$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = t^2sinx + sinx + t$$

$$+pL^{-1}\left[\frac{1}{s^2}L\left[\left(\sum_{n=0}^{\infty}p^n u_n(x,t)\right)_{xx}-\sum_{n=0}^{\infty}p^n u_n(x,t)\right]\right] \qquad (20)$$

Comparing the coefficients of like powers p, the following approximations are obtained:

$$p^0: u_0(x,t) = t^2 sinx + sinx + t,$$

$$p^1: u_1(x,t) = L^{-1}\left[\frac{1}{s^2}L[(u_0)_{xx} - u_0]\right] = -t^2 sinx - \frac{t^3}{3!} - \frac{t^4}{3!}sinx \qquad (21)$$

$$p^2: u_2(x,t) = L^{-1}\left[\frac{1}{s^2}L[(u_1)_{xx} - u_1]\right] = \frac{t^4}{3!}sinx + \frac{t^6}{90}sinx + \frac{t^5}{5!}$$

$$p^3: u_3(x,t) = L^{-1}\left[\frac{1}{s^2}L[(u_2)_{xx} - u_2]\right] = -\frac{t^6}{90}sinx - \frac{t^7}{7!} - \frac{2t^8}{7!}sinx$$

$$\vdots$$

It is necessary to mention here that the noise terms are those terms which are the same but different in signs .more clearly the noise terms $\pm t^2 sinx$ between the components $u_0(x,t)$ and $u_1(x,t)$ can be cancelled and the remaining terms of $u_0(x,t)$ still satisfy the equation. Therefore the solution $u(x,t)$ in series form is given by:

$$u(x,t) = sinx + \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right) \qquad (22)$$

And in closed form given as;

$$u(x,t) = sin(x) + sin(t) \qquad (23)$$

3.1.2 Nonlinear Klein-Gordon Equation

The nonlinear Klein-Gordon equation in its standard form is given by:

$$u_{tt} - u_{xx} + au + F(u) = h(x,t) \qquad (24)$$

Subject to the initial conditions;

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x), \tag{25}$$

Where a is a constant, $h(x,t)$ is the source term and $F(u)$ is a nonlinear function of u. The nonlinear term $F(u)$ will be equated to the infinite series of homotopy polynomials. To solve Eq. (24) with the initial conditions Eq. (25), by the HPTM, we taking the Laplace transform on both sides of Eq. (24) subject to the initial conditions Eq. (25), we get;

$$u(x,s) = \frac{f(x)}{s} + \frac{g(x)}{s^2} + \frac{1}{s^2} L[u_{xx} - au - F(u) + h(x,t)] \tag{26}$$

The inverse of Laplace transform implies that;

$$u(x,t) = f(x) + tg(x) + L^{-1}\left[\frac{1}{s^2} L[u_{xx} - au - F(u) + h(x,t)]\right] \tag{27}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = f(x) + tg(x) + L^{-1}\left[\frac{1}{s^2} L[h(x,t)]\right]$$

$$+pL^{-1}\left[\frac{1}{s^2} L\left[\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)_{xx} - a\sum_{n=0}^{\infty} p^n u_n(x,t)\right] - \sum_{n=0}^{\infty} p^n H_n(u)\right] \tag{28}$$

Where $H_n(u)$ are He's polynomials that represents the nonlinear terms.

Comparing the coefficients of like powers p, the following approximations are obtained

$$p^0 : u_0(x,t) = f(x) + tg(x) + L^{-1}\left[\frac{1}{s^2} L[h(x,t)]\right]$$

$$p^1 : u_1(x,t) = L^{-1}\left[\frac{1}{s^2} L[(u_0)_{xx} - au_0 - H_0(u)]\right] \tag{29}$$

$$p^2 : u_2(x,t) = L^{-1}\left[\frac{1}{s^2} L[(u_1)_{xx} - au_1 - H_1(u)]\right]$$

$$\vdots$$

Therefore the solution $u(x,t)$ in series form is given by:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \tag{30}$$

The following examples will be used to illustrate the algorithm discussed above.

Example (3.1.3) We consider the following nonlinear Klein-Gordon equation [70],

$$u_{tt} - u_{xx} + u^2 = x^2t^2, \tag{31}$$

With the initial conditions;

$$u(x,0) = 0, \qquad u_t(x,0) = x, \tag{32}$$

Applying the Laplace transform on both sides of Eq. (31) subject to the initial conditions Eq. (32), we have:

$$u(x,s) = \frac{2x^2}{s^5} + \frac{x}{s^2} + \frac{1}{s^2}L[u_{xx} - u^2] \tag{33}$$

The inverse of Laplace transform implies that;

$$u(x,t) = \frac{x^2t^4}{12} + xt + L^{-1}\left[\frac{1}{s^2}L[u_{xx} - u^2]\right] \tag{34}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = \frac{x^2t^4}{12} + xt$$

$$+pL^{-1}\left[\frac{1}{s^2}L\left[\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)_{xx} - \sum_{n=0}^{\infty} p^n H_n(u)\right]\right] \tag{35}$$

Where $H_n(u)$ n are He's polynomial that represents the nonlinear terms. The first few components of He's polynomials, are given by:

$$H_0(u) = (u_0)^2$$

$$H_1(u) = 2u_0u_1$$
$$H_2(u) = 2u_0u_2 + (u_1)^2 \qquad (36)$$
$$H_3(u) = 2u_0u_3 + 2u_1u_2$$
$$\vdots$$

Comparing the coefficients of like powers p, the following approximations are obtained:

$$p^0: u_0(x,t) = \left\{ \frac{x^2t^4}{12} + xt, \right.$$

$$p^1: u_1(x,t) = \left\{ L^{-1}\left[\frac{1}{s^2}L[(u_0)_{xx} - H_0(u)]\right] = \frac{t^6}{180} - \frac{x^4t^{10}}{12960} - \frac{x^3t^7}{252} - \frac{x^2t^4}{12}, \right.$$

$$p^2: u_2(x,t) = \begin{cases} L^{-1}\left[\frac{1}{s^2}L[(u_1)_{xx} - H_1(u)]\right] = -\frac{x^2t^{12}}{71280} - \frac{11xt^9}{22680} - \frac{t^6}{180} + \frac{11x^4t^{10}}{45360} \\ \frac{x^6t^{16}}{18662400} + \frac{383x^5t^{13}}{18662400} + \frac{x^3t^7}{252}, \end{cases}$$

$$\vdots$$

It is necessary to mention here that the noise terms are those terms which are the same but different in signs .more clearly the noise terms $\pm\frac{x^2t^4}{12}$ between the components $u_0(x,t)$ and $u_1(x,t)$ can be cancelled and the remaining terms of $u_0(x,t)$ still satisfy the equation. Therefore the solution $u(x,t)$ is given by:

$$u(x,t) = xt \qquad (37)$$

Example (3.1.4) We consider the non-linear Klein Gordon equation [71],

$$u_{tt} - u_{xx} + u + u^3 = 2x + xt^2 + x^3t^6, \qquad (38)$$

With the initial conditions;

$$u(x,0) = 0, \qquad u_t(x,0) = 0, \qquad (39)$$

Applying the Laplace transform on both sides of Eq. (31) subject to the initial conditions Eq. (32), we have;

$$u(x,s)=\frac{2x}{s^3}+\frac{2x}{s^5}+\frac{6!\,x^3}{s^9}+\frac{1}{s^2}L[u_{xx}-u-u^3] \tag{40}$$

The inverse of Laplace transform implies that;

$$u(x,t)=xt^2+\frac{1}{12}xt^4+\frac{1}{56}x^3t^8+L^{-1}\left[\frac{1}{s^2}L[u_{xx}-u-u^3]\right] \tag{41}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty}p^n u_n(x,t)=xt^2+\frac{1}{12}xt^4+\frac{1}{56}x^3t^8$$

$$+pL^{-1}\left[\frac{1}{s^2}L\left[\left(\sum_{n=0}^{\infty}p^n u_n(x,t)\right)_{xx}-\sum_{n=0}^{\infty}p^n u_n(x,t)-\sum_{n=0}^{\infty}p^n H_n(u)\right]\right] \tag{42}$$

Where $H_n(u)$ n are He's polynomial that represents the nonlinear terms. The first few components of He's polynomials, are given by:

$$H_0(u)=(u_0)^3$$

$$H_1(u)=3(u_0)^2u_1+3u_0(u_1)^2$$

$$\vdots$$

Comparing the coefficients of like powers p, the following approximations are obtained:

$$p^0: u_0(x,t)=\left\{xt^2+\frac{1}{12}xt^4+\frac{1}{56}x^3t^8,\right. \tag{43}$$

$$p^1: u_1(x,t)=\left\{L^{-1}\left[\frac{1}{s^2}L[(u_0)_{xx}-u_0-H_0(u)]\right]\right.$$

$$=-\frac{1}{12}xt^4-\frac{1}{56}x^3t^8-\frac{1}{360}xt^6-\frac{1}{840}xt^{10}-\frac{1}{6336}x^3t^{12}-\frac{1}{314}x^3t^{14}$$

$$+\frac{3}{10192}x^5t^{14}-\frac{1}{26880}x^5t^{16}-\frac{1}{822528}x^5t^{18}+\frac{1}{1191680}x^7t^{20}$$

$$-\frac{1}{5795328}x^7t^{22}-\frac{1}{114150400}x^9t^{22}$$

$$\vdots$$

Considering the first two components u_0 and u_1 in Eq. (43), it is easily observed that the noise terms $\pm\frac{1}{12}xt^4$ and $\pm\frac{1}{56}x^3t^8$ appears in u_0 and u_1 respectively. By canceling the noise terms, and by verifying that the remaining non-canceled terms of u_0 satisfy Eq. (38), we find that the exact solution is given by

$$u(x,t) = xt^2 \tag{44}$$

3.2 The Porous Medium Equation

Many of the physical phenomena and processes in various fields of engineering and science are governed by partial differential equations. The nonlinear heat equation describing various physical phenomena called the porous medium equation.

The standard form of Porous Medium equation is given by:

$$u_t = (u^m u_x)_x \tag{45}$$

Where m is a rational number.

Example (3.2.5)[72], Let us take $m = 1$ in equation (45), we get;

$$u_t = (uu_x)_x \tag{46}$$

And

$$u_t = uu_{xx} + (u_x)^2 \tag{47}$$

With initial condition:

$$u(x,0) = x, \tag{48}$$

Applying the Laplace transform on both sides of Eq. (47) subject to the initial conditions Eq. (48), we have;

$$u(x,s) = \frac{x}{s} + \frac{1}{s} L[uu_{xx} + (u_x)^2] \tag{49}$$

The inverse of Laplace transform implies that;

$$u(x,t) = x + L^{-1}\left[\frac{1}{s} L[uu_{xx} + (u_x)^2]\right] \tag{50}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = x + pL^{-1}\left[\frac{1}{s} L \sum_{n=0}^{\infty} p^n H_n(u)\right] \tag{51}$$

Where $H_n(u)$ n are He's polynomial that represents the nonlinear terms. The first few components of He's polynomials, are given by:

$$\begin{aligned}
&H_0(u) = u_0(u_0)_{xx} + {(u_0)_x}^2 \\
&H_1(u) = u_0(u_1)_{xx} + u_1(u_0)_{xx} + 2(u_0)_x(u_1)_x \\
&H_2(u) = u_0(u_2)_{xx} + u_1(u_1)_{xx} + u_2(u_0)_{xx} + 2(u_0)_x(u_2)_x + {(u_1)_x}^2 \\
&\quad\vdots
\end{aligned} \tag{52}$$

Comparing the coefficients of like powers p, the following approximations are obtained:

$$\begin{aligned}
&p^0 : u_0(x,t) = x\,, \\
&p^1 : u_1(x,t) = L^{-1}\left[\frac{1}{s} L[H_0(u)]\right] = t, \\
&p^2 : u_2(x,t) = L^{-1}\left[\frac{1}{s} L[H_1(u)]\right] = 0, \\
&u_k(x,t) = 0, \quad \forall k \geq 2
\end{aligned} \tag{53}$$

Therefore the solution $u(x,t)$ in series form is given by:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_k(x,t), \quad k \geq 2$$

And

$$u(x,t) = x + t \tag{54}$$

Example (3.2.6) Let us take $m = -1$ in equation (45), we get;

$$u_t = (u^{-1}u_x)_x \tag{55}$$

And

$$u_t = u^{-1}u_{xx} - u^{-2}(u_x)^2 \tag{56}$$

With initial condition:

$$u(x,0) = \frac{1}{x}, \tag{57}$$

Applying the Laplace transform on both sides of Eq. (56) subject to the initial conditions Eq. (57), we have;

$$u(x,s) = \frac{1}{sx} + \frac{1}{s}L[u^{-1}u_{xx} - u^{-2}(u_x)^2] \tag{58}$$

The inverse of Laplace transform implies that;

$$u(x,t) = \frac{1}{x} + L^{-1}\left[\frac{1}{s}L[u^{-1}u_{xx} - u^{-2}(u_x)^2]\right] \tag{59}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = \frac{1}{x} + pL^{-1}\left[\frac{1}{s}L\sum_{n=0}^{\infty} p^n H_n(u)\right] \tag{60}$$

Where $H_n(u)$ n are He's polynomial that represents the nonlinear terms. The first two components of He's polynomials, are given by:

$$H_0(u) = (u_0{}^{-1})(u_0)_{xx} - (u_0{}^{-2})(u_0)_x{}^2$$

$$H_1(u) = (u_0{}^{-1})\left(-\frac{u_1}{u_0}(u_0)_{xx} + (u_1)_{xx}\right) - (u_0{}^{-2})\left(-2\frac{u_1}{u_0}(u_0)_x{}^2 + (u_0)_x(u_1)_x\right)$$

$$\vdots$$

Comparing the coefficients of like powers p, the following approximations are obtained:

$$p^0: u_0(x,t) = \frac{1}{x},$$

$$p^1: u_1(x,t) = L^{-1}\left[\frac{1}{s}L[H_0(u)]\right] = \frac{t}{x^2}, \tag{61}$$

$$p^2: u_2(x,t) = L^{-1}\left[\frac{1}{s}L[H_1(u)]\right] = \frac{t^2}{x^3},$$

Proceeding in a similar manner we have;

$$p^3: u_3(x,t) = \frac{t^3}{x^4},$$

$$p^4: u_4(x,t) = \frac{t^4}{x^5},$$

Therefore the solution $u(x,t)$ in series form is given by:

$$u(x,t) = \frac{1}{x} + \frac{t}{x^2} + \frac{t^2}{x^3} + \frac{t^3}{x^4} + \cdots \tag{62}$$

Therefore the solution $u(x,t)$ in series form is given by:

$$u(x,t) = \frac{1}{x-t} \tag{63}$$

3.3 The Gas Dynamics Equation

Gas dynamics is a science in the branch of fluid dynamics concerned with studying the motion of gases and its effects on physical systems, based on the principles of fluid mechanics and thermodynamics.

The science arises from the studies of gas flows, often around or within physical bodies, some examples of these studies include but not limited to choked flows in nozzles and valves, shock waves around jets, aerodynamic heating on atmospheric reentry vehicles and flows of gas fuel within a jet engine. The gas dynamics equation as a nonlinear model is as follows

$$u_t + uu_x - u(1-u) = 0, \qquad 0 \le x \le 1, t > 1 \tag{64}$$

In this section we discuss the analytical approximate solution of the nonlinear gas dynamic equation.

Example (3.3.7) Consider the nonlinear gas dynamic equation [73],

$$u_t + uu_x - u(1-u) = 0, \tag{65}$$

With the following initial condition:

$$u(x,0) = e^{-x}, \tag{66}$$

Applying the Laplace transform on both sides of Eq. (65) subject to the initial conditions Eq. (66), we have;

$$u(x,s) = \frac{e^{-x}}{s} + \frac{1}{s}L[u(1-u) - uu_x] \tag{67}$$

The inverse of Laplace transform implies that;

$$u(x,t) = e^{-x} + L^{-1}\left[\frac{1}{s}L[u - u^2 - uu_x]\right] \tag{68}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = e^{-x} + pL^{-1}\left[\frac{1}{s}L\left[\sum_{n=0}^{\infty} p^n u_n(x,t) - \sum_{n=0}^{\infty} p^n H_n(u)\right]\right] \tag{69}$$

Where $H_n(u)$ n are He's polynomial that represents the nonlinear terms. The first few components of He's polynomials, are given by:

$$H_0(u) = (u_0)^2 + u_0(u_0)_x$$

$$H_1(u) = 2u_0u_1 + u_0(u_1)_x + u_1(u_0)_x$$

$$H_2(u) = 2u_0u_2 + (u_1)^2 + u_0(u_2)_x + u_1(u_1)_x + u_2(u_0)_x$$

$$\vdots$$

Comparing the coefficients of like powers p, the following approximations areobtained:

$$p^0: u_0(x,t) = e^{-x},$$

$$p^1: u_1(x,t) = L^{-1}\left[\frac{1}{s}L[u_0 - H_0(u)]\right] = te^{-x}, \tag{70}$$

$$p^2: u_2(x,t) = L^{-1}\left[\frac{1}{s}L[u_1 - H_1(u)]\right] = \frac{t^2}{2!}e^{-x},$$

$$p^3: u_3(x,t) = L^{-1}\left[\frac{1}{s}L[u_2 - H_2(u)]\right] = \frac{t^3}{3!}e^{-x},$$

Proceeding in a similar manner we have;

$$p^4: u_4(x,t) = \frac{t^4}{4!}e^{-x},$$

$$p^5: u_5(x,t) = \frac{t^5}{5!}e^{-x},$$

Therefore the solution $u(x,t)$ in series form is given by:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots$$

$$= e^{-x}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right) \tag{71}$$

And in closed form given as;

$$u(x,t) = e^{t-x} \tag{72}$$

Example (3.3.8) Consider the nonlinear gas dynamic equation [73],

$$u_t + uu_x - u(1-u) = -e^{t-x}, \tag{73}$$

With the following initial condition;

$$u(x,0) = 1 - e^{-x}, \tag{74}$$

Applying the Laplace transform on both sides of Eq. (73) subject to the initial conditions Eq. (74), we have;

$$u(x,s) = \frac{1-e^{-x}}{s} - \frac{e^{-x}}{s(s-1)} + \frac{1}{s} L[u(1-u) - uu_x] \tag{75}$$

The inverse of Laplace transform implies that;

$$u(x,t) = 1 - e^{t-x} + L^{-1}\left[\frac{1}{s} L[u - u^2 - uu_x]\right] \tag{76}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = 1 - e^{t-x} + pL^{-1}\left[\frac{1}{s} L\left[\sum_{n=0}^{\infty} p^n u_n(x,t) - \sum_{n=0}^{\infty} p^n H_n(u)\right]\right] \tag{77}$$

Where $H_n(u)$ n are He's polynomial that represents the nonlinear terms. The first few components of He's polynomials, are given by:

$$H_0(u) = (u_0)^2 + u_0(u_0)_x$$

$$H_1(u) = 2u_0u_1 + u_0(u_1)_x + u_1(u_0)_x$$

$$H_2(u) = 2u_0u_2 + (u_1)^2 + u_0(u_2)_x + u_1(u_1)_x + u_2(u_0)_x$$

$$\vdots$$

Comparing the coefficients of like powers p, the following approximations are obtained:

$$p^0: u_0(x,t) = 1 - e^{t-x},$$

$$p^1: u_1(x,t) = L^{-1}\left[\frac{1}{s} L[u_0 - H_0(u)]\right] = 0, \tag{78}$$

$$p^2: u_2(x,t) = L^{-1}\left[\frac{1}{s}L[u_1 - H_1(u)]\right] = 0,$$

$$u_k(x,t) = 0, \qquad \forall k \geq 1$$

Therefore the solution $u(x,t)$ in series form is given by:

$$u(x,t) = u_0(x,t) + u_k(x,t), \tag{79}$$

And

$$u(x,t) = 1 - e^{t-x} \tag{80}$$

3.4 The Burgers Equation

The Burgers equation is considered one of the fundamental model equations in fluid mechanics. The equation demonstrates the coupling between diffusion and convection processes.

The standard form of Burgers' equation is given by:

$$u_t + uu_x = vu_{xx}, \qquad t > 0, \tag{81}$$

Where v is a constant that defines the kinematic viscosity. If the viscosity $v = 0$, the equation is called in viscid Burgers equation. The in viscid Burgers equation governs gas dynamics. The gas dynamics equation has been discussed before in section (3.4).

However, it is the intention of this text to effectively apply the homotopy perturbation transform method. We consider the Burgers' equation

$$u_t + uu_x = u_{xx}, \tag{82}$$

With initial condition as;

$$u(x,0) = f(x), \tag{83}$$

Taking the Laplace transform on both sides of Eq. (82) subject to the initial condition Eq. (83), we get;

$$u(x,s) = \frac{f(x)}{s} - + \frac{1}{s} L[u_{xx} - uu_x] \tag{84}$$

The inverse of Laplace transform implies that;

$$u(x,t) = f(x) + L^{-1}\left[\frac{1}{s} L[u_{xx} - uu_x]\right] \tag{85}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = f(x) + pL^{-1}\left[\frac{1}{s} L\left[\sum_{n=0}^{\infty} p^n u_n(x,t) - \sum_{n=0}^{\infty} p^n H_n(u)\right]\right] \tag{86}$$

Where $H_n(u)$ n are He's polynomial that represents the nonlinear terms. The first few components of He's polynomials, are given by:

$$H_0(u) = u_0(u_0)_x$$

$$H_1(u) = u_0(u_1)_x + u_1(u_0)_x$$

$$H_2(u) = u_0(u_2)_x + u_1(u_1)_x + u_2(u_0)_x$$

$$\vdots$$

Comparing the coefficients of like powers p, the following approximations are obtained:

$$p^0 : u_0(x,t) = f(x),$$

$$p^1 : u_1(x,t) = L^{-1}\left[\frac{1}{s} L[(u_0)_{xx} - H_0(u)]\right], \tag{87}$$

$$p^2 : u_2(x,t) = L^{-1}\left[\frac{1}{s} L[(u_1)_{xx} - H_1(u)]\right],$$

$$p^2 : u_2(x,t) = L^{-1}\left[\frac{1}{s} L[(u_2)_{xx} - H_2(u)]\right],$$

Therefore the solution $u(x,t)$ in series form is given by:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots, \tag{88}$$

The following examples will be used to illustrate the algorithm discussed above

Example (3.4.9) We consider the following Burgers equation [74],

$$u_t + uu_x = u_{xx} \tag{89}$$

With initial condition:

$$u(x,0) = x, \tag{90}$$

Taking the Laplace transform on both sides of Eq. (89) subject to the initial condition Eq. (90), we get;

$$u(x,s) = \frac{x}{s} - + \frac{1}{s} L[u_{xx} - uu_x] \tag{91}$$

The inverse of Laplace transform implies that;

$$u(x,t) = x + L^{-1}\left[\frac{1}{s} L[u_{xx} - uu_x]\right] \tag{92}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = x + pL^{-1}\left[\frac{1}{s} L\left[\sum_{n=0}^{\infty} p^n u_n(x,t) - \sum_{n=0}^{\infty} p^n H_n(u)\right]\right] \tag{93}$$

Where $H_n(u)$ are He's polynomial that represents the nonlinear terms. The first few components of He's polynomials, are given by:

$$H_0(u) = u_0(u_0)_x$$

$$H_1(u) = u_0(u_1)_x + u_1(u_0)_x$$

$$H_2(u) = u_0(u_2)_x + u_1(u_1)_x + u_2(u_0)_x$$

$$\vdots$$

Comparing the coefficients of like powers p, the following approximations are obtained:

$$p^0: u_0(x,t) = x,$$

$$p^1: u_1(x,t) = L^{-1}\left[\frac{1}{s}L[(u_0)_{xx} - H_0(u)]\right] = -xt, \quad (94)$$

$$p^2: u_2(x,t) = L^{-1}\left[\frac{1}{s}L[(u_1)_{xx} - H_1(u)]\right] = xt^2,$$

$$p^3: u_3(x,t) = L^{-1}\left[\frac{1}{s}L[(u_2)_{xx} - H_2(u)]\right] = -\, xt^3,$$

Proceeding in a similar manner we have;

$$p^4: u_4(x,t) = xt^4,$$

$$p^5: u_5(x,t) = -xt^5,$$

And so on. Combining the results obtained for the components, the solution in a series form is given by:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots, \quad (95)$$

$$= x(1 - t + t^2 - t^3 + \cdots)$$

And in closed form given as;

$$u(x,t) = \frac{x}{1+t}, \quad |t| < 1 \quad (96)$$

Example (3.4.10) We consider the following Burgers equation [74],

$$u_t + uu_x = u_{xx} \quad (97)$$

With initial condition:

$$u(x,0) = 1 - \frac{2}{x}, \;\; x > 0, \quad (98)$$

Taking the Laplace transform on both sides of Eq. (97) subject to the initial condition Eq. (98), we get;

$$u(x,s) = \frac{1}{s} - \frac{2}{xs} + \frac{1}{s}L[u_{xx} - uu_x] \quad (99)$$

The inverse of Laplace transform implies that;

$$u(x,t) = 1 - \frac{2}{x} + L^{-1}\left[\frac{1}{s}L[u_{xx} - uu_x]\right] \tag{100}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = 1 - \frac{2}{x} + pL^{-1}\left[\frac{1}{s}L\left[\sum_{n=0}^{\infty} p^n u_n(x,t) - \sum_{n=0}^{\infty} p^n H_n(u)\right]\right] \tag{101}$$

Where $H_n(u)$ n are He's polynomial that represents the nonlinear terms. The first few components of He's polynomials, are given by:

$$H_0(u) = u_0(u_0)_x$$

$$H_1(u) = u_0(u_1)_x + u_1(u_0)_x$$

$$H_2(u) = u_0(u_2)_x + u_1(u_1)_x + u_2(u_0)_x$$

$$\vdots$$

Comparing the coefficients of like powers p, the following approximations are obtained:

$$p^0: u_0(x,t) = 1 - \frac{2}{x},$$

$$p^1: u_1(x,t) = L^{-1}\left[\frac{1}{s}L[(u_0)_{xx} - H_0(u)]\right] = -\frac{2}{x^2}t \tag{102}$$

$$p^2: u_2(x,t) = L^{-1}\left[\frac{1}{s}L[(u_1)_{xx} - H_1(u)]\right] = -\frac{2}{x^3}t^2,$$

$$p^3: u_3(x,t) = L^{-1}\left[\frac{1}{s}L[(u_2)_{xx} - H_2(u)]\right] = - \ -\frac{2}{x^4}t^3,$$

Proceeding in a similar manner we have;

$$p^4: u_4(x,t) = -\frac{2}{x^5}t^4,$$

$$p^5: u_5(x,t) = -\frac{2}{x^6}t^5,$$

And so on. Combining the results obtained for the components, the solution in a series form is given by:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \tag{103}$$

$$= 1 - \frac{2}{x}\left(1 + \frac{t}{x} + \frac{t^2}{x^2} - \frac{t^3}{x^3} + \cdots\right)$$

And in closed form given as;

$$u(x,t) = 1 - \frac{2}{x - t} \tag{104}$$

3.5 The Telegraph Equation

The standard form of the telegraph equation is given by:

$$u_{xx} = au_{tt} + bu_t + cu \tag{105}$$

Where $u = u(x,t)$ is the resistance, and a, b and c are constants related to the inductance, capacitance and conductance of the cable respectively. Note that the telegraph equation is a linear partial differential equation. The telegraph equation arises in the propagation of electrical signals along a telegraph line.

We now proceed formally to apply the homotopy perturbation transform method in a parallel manner to the approach used for handling other physical models. Without loss of generality, consider the initial boundary value telegraph equation

$$u_{xx} = u_{tt} + u_t + u \tag{106}$$

With initial conditions;

$$u(x,0) = h(x), \qquad u_t(x,0) = v(x), \tag{107}$$

And boundary conditions;

$$u(0,t) = f(t), \qquad u_x(0,t) = g(t), \tag{108}$$

Taking the Laplace transform on both sides of Eq. (106) and using boundary conditions Eq. (108), we get;

$$u(s,t) = \frac{f(t)}{s} + \frac{g(t)}{s^2} + \frac{1}{s^2} L[u_{tt} + u_t + u] \tag{109}$$

The inverse of Laplace transform implies that;

$$u(x,t) = f(t) + xg(t) + L^{-1}\left[\frac{1}{s^2} L[u_{tt} + u_t + u]\right] \tag{110}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = f(t) + xg(t)$$

$$+pL^{-1}\left[\frac{1}{s^2} L\left[\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)_{tt} + \left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)_t + \sum_{n=0}^{\infty} p^n u_n(x,t)\right]\right] \tag{111}$$

Comparing the coefficients of like powers of p, we get;

$$p^0: u_0(x,t) = f(t) + xg(t),$$

$$p^1: u_1(x,t) = L^{-1}\left[\frac{1}{s^2} L[(u_0)_{tt} + (u_0)_t + u_0]\right], \tag{112}$$

$$p^2: u_2(x,t) = L^{-1}\left[\frac{1}{s^2} L[(u_1)_{tt} + (u_1)_t + u_1]\right],$$

And so on. Combining the results obtained for the components, the solution in a series form is given by;

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \tag{113}$$

Example (3.5.11) We consider the following homogeneous telegraph equation [61],

$$u_{xx} = u_{tt} + u_t - u \tag{114}$$

Subject to the initial conditions;

$$u(x,0)=e^{x}, \qquad u_t(x,0)=-2e^{x}, \tag{115}$$

And boundary conditions;

$$u(0,t)=e^{-2t}, \qquad u_x(0,t)=e^{-2t}, \tag{116}$$

Taking the Laplace transform on both sides of Eq. (114) and using boundary conditions Eq. (116), we get;

$$u(s,t)=\frac{e^{-2t}}{s}+\frac{e^{-2t}}{s^2}+\frac{1}{s^2}L[u_{tt}+u_t-u] \tag{117}$$

The inverse of Laplace transform implies that;

$$u(x,t)=e^{-2t}+xe^{-2t}+L^{-1}\left[\frac{1}{s^2}L[u_{tt}+u_t-u]\right] \tag{118}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty}p^n u_n(x,t)=e^{-2t}+xe^{-2t}$$

$$+pL^{-1}\left[\frac{1}{s^2}L\left[\left(\sum_{n=0}^{\infty}p^n u_n(x,t)\right)_{tt}+\left(\sum_{n=0}^{\infty}p^n u_n(x,t)\right)_{t}-\sum_{n=0}^{\infty}p^n u_n(x,t)\right]\right] \tag{119}$$

Comparing the coefficients of like powers of p, we get;

$$p^0: u_0(x,t)=e^{-2t}+xe^{-2t},$$

$$p^1: u_1(x,t)=L^{-1}\left[\frac{1}{s^2}L[(u_0)_{tt}+(u_0)_t-u_0]\right]=\frac{1}{2!}x^2e^{-2t}+\frac{1}{3!}x^3e^{-2t},$$

$$p^2: u_2(x,t)=L^{-1}\left[\frac{1}{s^2}L[(u_1)_{tt}+(u_1)_t-u_1]\right]=\frac{1}{4!}x^4e^{-2t}+\frac{1}{5!}x^5e^{-2t},$$

Proceeding in similar manner we have;

$$p^3: u_3(x,t)=\frac{1}{6!}x^6e^{-2t}+\frac{1}{7!}x^7e^{-2t},$$

$$p^4 : u_4(x,t) = \frac{1}{8!}x^8 e^{-2t} + \frac{1}{9!}x^9 e^{-2t},$$

And so on. Combining the results obtained for the components, the solution in a series form is given by:

$$u(x,t) = e^{-2t}\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots\right) \qquad (120)$$

And in closed form given as;

$$u(x,t) = e^{x-2t} \qquad (121)$$

Example (3.5.12) We consider the following homogeneous telegraph equation

$$u_{xx} = u_{tt} + 4u_t + 4u \qquad (122)$$

Subject to the initial conditions;

$$u(x,0) = 1 + e^{2x}, \qquad u_t(x,0) = -2, \qquad (123)$$

And boundary conditions;

$$u(0,t) = 1 + e^{-2t}, \qquad u_x(0,t) = 2, \qquad (124)$$

Taking the Laplace transform on both sides of Eq. (122) and using boundary conditions Eq. (124), we get;

$$u(s,t) = \frac{1 + e^{-2t}}{s} + \frac{2}{s^2} + \frac{1}{s^2}L[u_{tt} + 4u_t + 4u] \qquad (125)$$

The inverse of Laplace transform implies that;

$$u(x,t) = 1 + e^{-2t} + 2x + L^{-1}\left[\frac{1}{s^2}L[u_{tt} + 4u_t + 4u]\right] \qquad (126)$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = 1 + e^{-2t} + 2x$$

$$+pL^{-1}\left[\frac{1}{s^2}L\left[\left(\sum_{n=0}^{\infty}p^n u_n(x,t)\right)_{tt}+4\left(\sum_{n=0}^{\infty}p^n u_n(x,t)\right)_{t}+4\sum_{n=0}^{\infty}p^n u_n(x,t)\right]\right] \quad (127)$$

Comparing the coefficients of like powers of p, we get;

$$p^0: u_0(x,t) = 1 + e^{-2t} + 2x,$$

$$p^1: u_1(x,t) = L^{-1}\left[\frac{1}{s^2}L[(u_0)_{tt} + 4(u_0)_t + 4u_0]\right] = 2x^2 + \frac{4}{3}x^3,$$

$$p^2: u_2(x,t) = L^{-1}\left[\frac{1}{s^2}L[(u_1)_{tt} + (u_1)_t - u_1]\right] = \frac{2}{3}x^3 + \frac{4}{15}x^5,$$

And so on. Combining the results obtained for the components, the solution in a series form is given by;

$$u(x,t) = e^{-2t} + \left(1 + 2x + \frac{1}{2!}(2x)^2 + \frac{1}{3!}(2x)^3 + \frac{1}{4!}(2x)^4 + \cdots\right) \quad (128)$$

And in closed form given as;

$$u(x,t) = e^{2x} + e^{-2t} \quad (129)$$

3.6 Schrödinger Equation

In this section, the linear and nonlinear Schrödinger equations will be investigated.It is well-known that this equation arises in the study of the time evolution of the wave function.

3.6.1 The Linear Schrödinger Equation

The initial value problem for the linear Schrödinger equation for a free particle with mass m is given by the following standard form;

$$u_t = iu_{xx}, \quad i^2 = -1, \quad t > 0 \quad (130)$$

And initial condition as;

$$u(x,0) = f(x), \quad (131)$$

Where $f(x)$ is continuous and square integrable. It is to be noted that Schrodinger equation (130) discusses the time evolution of a free particle. Moreover, the function $u(x,t)$ is complex and Eq. (130) is a first order differential equation in t.

The linear Schrödinger equation (130) is usually handled by using the spectral transform technique among other methods.

The homotopy perturbation transform method will be applied here to handle the linear and the nonlinear Schrödinger equations. To achieves this goal.

We taking the Laplace transform on both sides of Eq. (130) subject to the initial condition Eq. (131), we get;

$$u(x,s) = \frac{f(x)}{s} + \frac{1}{s} L[iu_{xx}] \tag{132}$$

The inverse of Laplace transform implies that;

$$u(x,t) = f(x) + iL^{-1}\left[\frac{1}{s} L[u_{xx}]\right] \tag{133}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = f(x) + ipL^{-1}\left[\frac{1}{s} L\left[\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)_{xx}\right]\right] \tag{134}$$

Comparing the coefficients of like powers of p, we get;

$$p^0: u_0(x,t) = f(x),$$

$$p^1: u_1(x,t) = iL^{-1}\left[\frac{1}{s} L[(u_0)_{xx}]\right],$$

$$p^2: u_2(x,t) = iL^{-1}\left[\frac{1}{s} L[(u_1)_{xx}]\right], \tag{135}$$

$$p^3: u_3(x,t) = iL^{-1}\left[\frac{1}{s}L[(u_2)_{xx}]\right],$$

$$\vdots$$

Thus, the exact solution is given by:

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \tag{136}$$

Example (3.6.13) Consider the linear Schrödinger equation

$$u_t = iu_{xx}, \tag{137}$$

And initial condition as;

$$u(x,0) = e^{ix}, \tag{138}$$

We taking the Laplace transform on both sides of Eq. (137) subject to the initial condition Eq. (138), we get;

$$u(x,s) = \frac{e^{ix}}{s} + \frac{1}{s}L[iu_{xx}] \tag{139}$$

The inverse of Laplace transform implies that;

$$u(x,t) = e^{ix} + iL^{-1}\left[\frac{1}{s}L[u_{xx}]\right] \tag{140}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = e^{ix} + ipL^{-1}\left[\frac{1}{s}L\left[\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)_{xx}\right]\right] \tag{141}$$

Comparing the coefficients of like powers of p, we get;

$$p^0: u_0(x,t) = e^{ix},$$

$$p^1: u_1(x,t) = iL^{-1}\left[\frac{1}{s}L[(u_0)_{xx}]\right] = -ite^{ix}$$

$$p^2: u_2(x,t) = iL^{-1}\left[\frac{1}{s}L[(u_1)_{xx}]\right] = -\frac{1}{2!}t^2e^{ix} \quad (142)$$

$$p^3: u_3(x,t) = iL^{-1}\left[\frac{1}{s}L[(u_2)_{xx}]\right] = \frac{1}{3!}it^3e^{ix}$$

$$\vdots$$

Proceeding in a similar manner we have:

$$p^4: u_4(x,t) = -\frac{1}{4!}t^4e^{ix},$$

$$p^5: u_5(x,t) = \frac{1}{5!}it^5e^{ix},$$

And so on. Combining the results obtained for the components, the solution in a series form is given by:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \quad (143)$$

$$= e^{ix}\left(1 - (it) + \frac{1}{2!}(it)^2 - \frac{1}{3!}(it)^3 + \cdots\right)$$

And in closed form given as;

$$u(x,t) = e^{i(x-t)} \quad (144)$$

Example(3.6.14) Consider the linear Schrödinger equation [75],

$$u_t + iu_{xx} = 0, \quad (145)$$

With the initial condition:

$$u(x,0) = 1 + cosh(2x), \quad (146)$$

Taking the Laplace transform on both sides of Eq. (145) subject to the initial condition Eq. (146), we get;

$$u(x,s) = \frac{1 + cosh(2x)}{s} - \frac{1}{s}L[iu_{xx}] \quad (147)$$

The inverse of Laplace transform implies that;

$$u(x,t) = 1 + cosh(2x) - L^{-1}\left[\frac{1}{s^2} iL[u_{xx}]\right] \tag{148}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = 1 + cosh(2x) - ipL^{-1}\left[\frac{1}{s} iL\left[\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)_{xx}\right]\right] \tag{149}$$

Comparing the coefficients of like powers of p, we get;

$$p^0 : u_0(x,t) = 1 + cosh(2x),$$

$$p^1 : u_1(x,t) = -iL^{-1}\left[\frac{1}{s} L[(u_0)_{xx}]\right] = (-4it)cosh(2x)$$

$$p^2 : u_2(x,t) = -iL^{-1}\left[\frac{1}{s} L[(u_1)_{xx}]\right] = \frac{1}{2!}(-4it)^2 \, cosh(2x) \tag{150}$$

$$p^3 : u_3(x,t) = -iL^{-1}\left[\frac{1}{s} L[(u_2)_{xx}]\right] = \frac{1}{3!}(-4it)^3 cosh(2x)$$

$$\vdots$$

Proceeding in a similar manner we have:

$$p^4 : u_4(x,t) = \frac{1}{4!}(-4it)^4 cosh(2x)\,,$$

$$p^5 : u_5(x,t) = \frac{1}{5!}(-4it)^5 cosh(2x)\,,$$

Therefore the solution $u(x,t)$ is given by:

$$u(x,t) = 1 + cosh(2x)\left(1 + (-4it) + \frac{1}{2!}(-4it)^2 - \frac{1}{3!}(-4it)^3 + \cdots\right) \tag{151}$$

In series form, and:

$$u(x,t) = 1 + e^{-4it} \, cosh(2x) \tag{152}$$

In closed form.

3.6.2 The Nonlinear Schrödinger Equation

We now turn to study the nonlinear Schrödinger equation (NLS) defined by its standard form:

$$iu_t + u_{xx} + \gamma|u|^2u = 0, \tag{153}$$

Where γ is a constant and $u(x,t)$ is complex. The Schrödinger equation (153) generally exhibits solitary type solutions. A solution, or solitary wave, is a wave where the speed of propagation is independent of the amplitude of the wave. Solutions usually occur in fluid mechanics.

The nonlinear Schrödinger equations that are commonly used are given by:

$$iu_t + u_{xx} + 2|u|^2u = 0, \tag{154}$$

And

$$iu_t + u_{xx} - 2|u|^2u = 0, \tag{155}$$

Moreover, Other forms of nonlinear Schrödinger equations are used as well depending on the constant γ . The inverse scattering method is usually used to handle the nonlinear Schrödinger equation where solitary type solutions were derived.The nonlinear Schrödinger equation will be handled differently in this section by using the homotopy perturbation transform method. We start our analysis by considering the initial value problem;

$$iu_t + u_{xx} + \gamma|u|^2u = 0, \tag{156}$$

With initial condition:

$$u(x,0) = f(x), \tag{157}$$

Taking the Laplace transform on both sides of Eq. (148) subject to the initial condition Eq. (157), we get;

$$u(x,s) = \frac{f(x)}{s} + \frac{1}{s} iL[u_{xx} + \gamma|u|^2u] \tag{158}$$

The inverse of Laplace transform implies that;

$$u(x,t) = f(x) + L^{-1}\left[\frac{1}{s} iL[u_{xx} + \gamma|u|^2 u]\right] \quad (159)$$

Or

$$u(x,t) = f(x) + L^{-1}\left[\frac{1}{s} iL[u_{xx} + \gamma u^2 \bar{u}]\right] \quad (160)$$

Where $u^2\bar{u} = |u|^2 u$ and $\bar{u}$ is the conjugate of u.

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = f(x) + pL^{-1}\left[\frac{1}{s} iL\left[\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)_{xx} + \gamma \sum_{n=0}^{\infty} p^n H_n(u)\right]\right] \quad (161)$$

Where $H_n(u)$ are He's polynomial that represents the nonlinear terms. The first few components of He's polynomials, are given by:

$$H_0(u) = (u_0)^2 \overline{u_0}$$

$$H_1(u) = (u_0)^2 \overline{u_1} + 2u_1 u_0 \overline{u_0}$$

$$H_2(u) = (u_0)^2 \overline{u_2} + 2u_0 u_1 \overline{u_1} + +2u_0 u_2 \overline{u_0} + (u_1)^2 \overline{u_0}$$

Comparing the coefficients of like powers of p, we get;

$$p^0: u_0(x,t) = f(x),$$

$$p^1: u_1(x,t) = iL^{-1}\left[\frac{1}{s} L[(u_0)_{xx} + \gamma H_0(u)]\right],$$

$$p^2: u_2(x,t) = iL^{-1}\left[\frac{1}{s} L[(u_1)_{xx} + \gamma H_1(u)]\right], \quad (162)$$

$$p^3: u_3(x,t) = iL^{-1}\left[\frac{1}{s} L[(u_2)_{xx} + \gamma H_2(u)]\right],$$

And so on, therefore the solution $u(x,t)$ is given by:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots$$

Example(3.6.15) Consider the nonlinear Schrödinger equation [75],

$$iu_t + u_{xx} + 2|u|^2 u = 0, \tag{163}$$

With initial condition:

$$u(x, 0) = e^{ix}, \tag{164}$$

Taking the Laplace transform on both sides of Eq. (163) subject to the initial condition Eq. (164), we get;

$$u(x, s) = \frac{e^{ix}}{s} + \frac{1}{s} iL[u_{xx} + 2|u|^2 u] \tag{165}$$

The inverse of Laplace transform implies that;

$$u(x, t) = e^{ix} + L^{-1}\left[\frac{1}{s} iL[u_{xx} + 2|u|^2 u]\right] \tag{166}$$

Or

$$u(x, t) = e^{ix} + L^{-1}\left[\frac{1}{s} iL[u_{xx} + 2u^2\bar{u}]\right] \tag{167}$$

Where $u^2\bar{u} = |u|^2 u$ and $\bar{u}$ is the conjugate of u.

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n u_n(x, t) = e^{ix} + pL^{-1}\left[\frac{1}{s} iL\left[\left(\sum_{n=0}^{\infty} p^n u_n(x, t)\right)_{xx} + 2\sum_{n=0}^{\infty} p^n H_n(u)\right]\right] \tag{168}$$

Where $H_n(u)$ are He's polynomial that represents the nonlinear terms. The first few components of He's polynomials, are given by:

$$H_0(u) = (u_0)^2\overline{u_0}$$

$$H_1(u) = (u_0)^2\overline{u_1} + 2u_1 u_0\overline{u_0}$$

$$H_2(u) = (u_0)^2\overline{u_2} + 2u_0 u_1\overline{u_1} + +2u_0 u_2\overline{u_0} + (u_1)^2\overline{u_0}$$

Comparing the coefficients of like powers of p, we get;

$$p^0: u_0(x,t) = e^{ix},$$

$$p^1: u_1(x,t) = iL^{-1}\left[\frac{1}{s}L[(u_0)_{xx} + 2H_0(u)]\right] = (it)e^{ix}$$

$$p^2: u_2(x,t) = iL^{-1}\left[\frac{1}{s}L[(u_1)_{xx} + 2H_1(u)]\right] = \frac{(it)^2}{2!}e^{ix} \quad (169)$$

$$p^3: u_3(x,t) = iL^{-1}\left[\frac{1}{s}L[(u_2)_{xx} + 2H_2(u)]\right] = \frac{(it)^3}{3!}e^{ix}$$

Proceeding in a similar manner we have;

$$p^4: u_4(x,t) = \frac{(it)^4}{4!}e^{ix},$$

$$p^5: u_5(x,t) = \frac{(it)^5}{5!}e^{ix},$$

$$p^6: u_6(x,t) = \frac{(it)^6}{6!}e^{ix},$$

And so on, therefore the solution $u(x,t)$ is given by:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots$$

$$= e^{ix}\left(1 + (it) + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \cdots\right) \quad (170)$$

In series form, and

$$u(x,t) = e^{i(x+t)} \quad (171)$$

In closed form.

Example(3.6.16) Consider the nonlinear Schrödinger equation [61],

$$iu_t + u_{xx} - 2|u|^2 u = 0, \quad (172)$$

With initial condition:

$$u(x,0) = e^{ix}, \quad (173)$$

Taking the Laplace transform on both sides of Eq. (172) subject to the initial condition Eq. (173), we get;

$$u(x,s) = \frac{e^{ix}}{s} + \frac{1}{s} iL[u_{xx} - 2|u|^2 u] \tag{174}$$

The inverse of Laplace transform implies that;

$$u(x,t) = e^{ix} + L^{-1}\left[\frac{1}{s} iL[u_{xx} - 2|u|^2 u]\right] \tag{175}$$

Or

$$u(x,t) = e^{ix} + L^{-1}\left[\frac{1}{s} iL[u_{xx} - 2u^2 \bar{u}]\right] \tag{176}$$

Where $u^2\bar{u} = |u|^2 u$ and $\bar{u}$ is the conjugate of u.

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = e^{ix} + pL^{-1}\left[\frac{1}{s} iL\left[\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)_{xx} - 2\sum_{n=0}^{\infty} p^n H_n(u)\right]\right] \tag{177}$$

Where$H_n(u)$ are He's polynomial that represents the nonlinear terms. The first few components of He's polynomials, are given by:

$$H_0(u) = (u_0)^2 \overline{u_0}$$

$$H_1(u) = (u_0)^2 \overline{u_1} + 2u_1 u_0 \overline{u_0}$$

$$H_2(u) = (u_0)^2 \overline{u_2} + 2u_0 u_1 \overline{u_1} + +2u_0 u_2 \overline{u_0} + (u_1)^2 \overline{u_0}$$

Comparing the coefficients of like powers of p, we get;

$$p^0 \colon u_0(x,t) = e^{ix},$$

$$p^1 \colon u_1(x,t) = iL^{-1}\left[\frac{1}{s} L[(u_0)_{xx} - 2H_0(u)]\right] = -(3it)e^{ix}$$

$$p^2 \colon u_2(x,t) = iL^{-1}\left[\frac{1}{s} L[(u_1)_{xx} - 2H_1(u)]\right] = \frac{(3it)^2}{2!} e^{ix} \tag{178}$$

$$p^3: u_3(x,t) = iL^{-1}\left[\frac{1}{s}L[(u_2)_{xx} - 2H_2(u)]\right] = -\frac{(3it)^3}{3!}e^{ix}$$

Proceeding in a similar manner we have;

$$p^4: u_4(x,t) = \frac{(3it)^4}{4!}e^{ix},$$

$$p^5: u_5(x,t) = -\frac{(3it)^5}{5!}e^{ix},$$

And so on, therefore the solution $u(x,t)$ is given by:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots$$

$$= e^{ix}\left(1 - (3it) + \frac{(3it)^2}{2!} - \frac{(3it)^3}{3!} + \cdots\right) \quad (179)$$

In series form, and

$$u(x,t) = e^{i(x-3t)} \quad (180)$$

In closed form.

Example (3.6.17) Consider the following nonlinear inhomogeneous Schrödinger equation [90],

$$iu_t = -\frac{1}{2}u_{xx} + ucos^2x + |u|^2u\ , t \geq 0 \quad (181)$$

With the initial condition;

$$u(x,0) = \ sinx \quad (182)$$

Taking the Laplace transform on both sides of Eq. (181) subject to the initial condition Eq. (182), we get;

$$u(x,s) = \frac{sinx}{s} - \frac{1}{s}iL\left[-\frac{1}{2}u_{xx} + ucos^2x + |u|^2u\right] \quad (183)$$

The inverse of Laplace transform implies that;

$$u(x,t) = sinx - L^{-1}\left[\frac{1}{s} iL\left[-\frac{1}{2}u_{xx} + ucos^2x + |u|^2u\right]\right] \quad (184)$$

Or

$$u(x,t) = sinx - L^{-1}\left[\frac{1}{s} iL\left[-\frac{1}{2}u_{xx} + ucos^2x + u^2\bar{u}\right]\right] \quad (185)$$

Where $u^2\bar{u} = |u|^2u$ and $\bar{u}$ is the conjugate of u.

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = sinx$$

$$-pL^{-1}\left[\frac{1}{s} iL\left[\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)_{xx} + cos^2x\sum_{n=0}^{\infty} p^n u_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(u)\right]\right] (186)$$

Where$H_n(u)$ are He's polynomial that represents the nonlinear terms. The first few components of He's polynomials, are given by:

$$H_0(u) = (u_0)^2\overline{u_0}\,,$$

$$H_1(u) = (u_0)^2\overline{u_1} + 2u_1u_0\overline{u_0},$$

$$H_2(u) = (u_0)^2\overline{u_2} + 2u_0u_1\overline{u_1} + +2u_0u_2\overline{u_0} + (u_1)^2\overline{u_0},$$

Comparing the coefficients of like powers of p, we get;

$$p^0: u_0(x,t) = sinx,$$

$$p^1: u_1(x,t) = -iL^{-1}\left[\frac{1}{s}L[(u_0)_{xx} + u_0cos^2x + H_0(u)]\right] = \left(\frac{-3it}{2}\right)sinx,$$

$$p^2: u_2(x,t) = -iL^{-1}\left[\frac{1}{s}L[(u_1)_{xx} + u_1cos^2x + H_1(u)]\right] = \frac{1}{2!}\left(\frac{-3it}{2}\right)^2 sinx,$$

$$p^3: u_3(x,t) = -iL^{-1}\left[\frac{1}{s}L[(u_2)_{xx} + u_2cos^2x + H_2(u)]\right] = \frac{1}{3!}\left(\frac{-3it}{2}\right)^3 sinx,$$

Proceeding in a similar manner we have

$$p^4 : u_4(x,t) = \frac{1}{4!}\left(\frac{-3it}{2}\right)^4 sinx,$$

$$p^5 : u_5(x,t) = \frac{1}{5!}\left(\frac{-3it}{2}\right)^5 sinx,$$

And so on, therefore the solution $u(x,t)$ is given by:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots$$

$$= sinx\left(1 + (-3it) + \frac{1}{2!}\left(\frac{-3it}{2}\right)^2 + \frac{1}{3!}\left(\frac{-3it}{2}\right)^3 + \cdots\right) \quad (187)$$

In series form, and

$$u(x,t) = e^{\frac{-3it}{2}} sinx \quad (188)$$

In closed form.

3.7 Korteweg-DeVries Equation

The Korteweg-DeVries (KDV) equation in its simplest form is given by:

$$u_t + auu_x + u_{xxx} = 0, \quad (189)$$

The KDV equation arises in the study of shallow water waves. In particular, the KDV equation is used to describe long waves traveling in canals. It is formally proved that this equation has solitary waves as solutions; hence, it can have any number of solutions.

The KDV equation has received a lot of attention and has been extensively studied. Several numerical and analytical techniques were employed to study the solitary waves that result from this equation.

In this section, the homotopy perturbation transform method will be used to handle the KDV equation. We first consider the initial value problem

$$u_t + auu_x + bu_{xxx} = 0, \tag{190}$$

With initial condition:

$$u(x,0) = f(x), \tag{191}$$

Where a and b are constants.

Taking the Laplace transform on both sides of Eq. (190) subject to the initial condition Eq. (191), we get;

$$u(x,s) = \frac{f(x)}{s} - \frac{1}{s} L[auu_x + bu_{xxx}] \tag{192}$$

The inverse of Laplace transform implies that;

$$u(x,t) = f(x) - L^{-1}\left[\frac{1}{s} L[bu_{xxx} + auu_x]\right] \tag{193}$$

Now, applying the homotopy perturbation method, we get;

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = f(x)$$

$$-pL^{-1}\left[\frac{1}{s} L\left[b\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)_{xx} + a\sum_{n=0}^{\infty} p^n H_n(u)\right]\right] \tag{194}$$

Where $H_n(u)$ are He's polynomial that represents the nonlinear terms. The first few components of He's polynomials, are given by:

$$H_0(u) = u_0(u_0)_x$$

$$H_1(u) = u_0(u_1)_x + u_1(u_0)_x$$

$$H_2(u) = u_0(u_2)_x + u_1(u_1)_x + u_2(u_0)_x$$

Comparing the coefficients of like powers of p, we get;

$$p^0: u_0(x,t) = f(x),$$

$$p^1: u_1(x,t) = -L^{-1}\left[\frac{1}{s}L[b(u_0)_{xxx} + aH_0(u)]\right], \tag{195}$$

$$p^2: u_2(x,t) = -L^{-1}\left[\frac{1}{s}L[b(u_1)_{xxx} + aH_1(u)]\right],$$

$$p^3: u_3(x,t) = -L^{-1}\left[\frac{1}{s}L[b(u_2)_{xxx} + aH_2(u)]\right],$$

And so on, therefore the solution $u(x,t)$ is given by:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots$$

Example (3.6.18) Consider the following homogeneous KDV equation [91],

$$u_t - 6uu_x + u_{xxx} = 0\,, \tag{196}$$

With the initial condition;

$$u(x,0) = 6x, \tag{197}$$

Taking the Laplace transform on both sides of Eq. (196) subject to the initial condition Eq. (197), we get;

$$u(x,s) = \frac{6x}{s} + \frac{1}{s}\;L[6uu_x - u_{xxx}] \tag{198}$$

The inverse of Laplace transform implies that

$$u(x,t) = 6x - L^{-1}\left[\frac{1}{s}\;L[u_{xxx} - 6uu_x]\right] \tag{199}$$

Now, we apply the homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = 6x$$

$$-PL^{-1}\left[\frac{1}{s}\,L\left(\left(\sum_{n=0}^{\infty}p^n u_n(x,t)\right)_{xxx}-\sum_{n=0}^{\infty}p^n H_n(u)\right)\right] \tag{200}$$

Where $H_n(u)$ are He's polynomials that represents the nonlinear terms.

The first few components of He's polynomials, are given by:

$$H_0(u)=u_0u_{0x}$$
$$H_1(u)=u_0u_{1x}+u_1u_{0x} \tag{201}$$
$$H_2(u)=u_0u_{2x}+u_1u_{1x}+u_2u_{0x}$$

Comparing the coefficient of like powers of p, the following approximations are obtained;

$$p^0: u_0(x,t)=6x,$$
$$p^1: u_1(x,t)=-L^{-1}\left[\frac{1}{s}\,L[(u_0)_{xxx}-6H_0(u)]\right]=6^3xt,$$
$$p^2: u_2(x,t)=\ -L^{-1}\left[\frac{1}{s}\,L[(u_1)_{xxx}-6H_1(u)]\right]=6^5xt^2, \tag{202}$$
$$p^3: u_3(x,t)=\ -L^{-1}\left[\frac{1}{s}\,L[(u_2)_{xxx}-6H_2(u)]\right]=6^7xt^3,$$

Therefore the solution $u(x,t)$ is given by:

$$u(x,t)=6x(1+(36t)+(36t)^2+(36t)^3+(36t)^4+\cdots) \tag{203}$$

In series form, and,

$$u(x,t)=\frac{6x}{1-36t}\ ,|36t|<1 \tag{204}$$

In closed form.

Example (3.6.19) Consider the following homogeneous KDV equation [91],

$$u_t+6uu_x+u_{xxx}=0 \tag{205}$$

With the initial condition;

$$u(x,0) = x \tag{206}$$

Taking the Laplace transform on both sides of Eq. (205) subject to the initial condition Eq. (206), we get;

$$u(x,s) = \frac{x}{s} - \frac{1}{s}\, L[6uu_x + u_{xxx}] \tag{207}$$

The inverse of Laplace transformimplies that:

$$u(x,t) = x - L^{-1}\left[\frac{1}{s}\, L[6uu_x + u_{xxx}]\right] \tag{208}$$

Now, we apply the homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = x$$

$$-PL^{-1}\left[\frac{1}{s}\, L\left(\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)_{xxx} + \sum_{n=0}^{\infty} p^n H_n(u)\right)\right] \tag{209}$$

Comparing the coefficient of like powers of p, the following approximations are obtained;

$$p^0: u_0(x,t) = x,$$

$$p^1: u_1(x,t) = -L^{-1}\left[\frac{1}{s}\, L[(u_0)_{xxx} - 6H_0(u)]\right] = -x(6t),$$

$$p^2: u_2(x,t) = -L^{-1}\left[\frac{1}{s}\, L[(u_1)_{xxx} - 6H_1(u)]\right] = x(6t)^2, \tag{210}$$

$$p^3: u_3(x,t) = -L^{-1}\left[\frac{1}{s}\, L[(u_2)_{xxx} - 6H_2(u)]\right] = -x(6t)^3,$$

Therefore the solution $u(x,t)$ is given by:

$$u(x,t) = x(1 - (6t) + (6t)^2 - (6t)^3 + (6t)^4 - (6t)^5 + \cdots) \tag{211}$$

In series form, and,

$$u(x,t) = \frac{x}{1+6t} \tag{212}$$

In closed form.

Example (3.6.20) Consider the following homogeneous KDV equation [91],

$$u_t - 6uu_x + u_{xxx} = 0 \tag{213}$$

With the initial condition;

$$u(x,0) = -2\frac{k^2 e^{kx}}{(1+e^{kx})^2} \tag{214}$$

Taking the Laplace transform of both sides of Eq. (213) subject to the initial condition Eq. (214), we get;

$$u(x,s) = \frac{-2\frac{k^2 e^{kx}}{\left(1+e^{kx}\right)^2}}{s} + \frac{1}{s}\ L[6uu_x - u_{xxx}] \tag{215}$$

The inverse of Laplace transform implies that:

$$u(x,t) = -2\frac{k^2 e^{kx}}{(1+e^{kx})^2} + L^{-1}\left[\frac{1}{s}\ L[6uu_x - u_{xxx}]\right] \tag{216}$$

Now, we apply the homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = -2\frac{k^2 e^{kx}}{(1+e^{kx})^2}$$

$$-PL^{-1}\left[\frac{1}{s}L\left(\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)_{xxx} - \sum_{n=0}^{\infty} p^n H_n(u)\right)\right] \tag{217}$$

Comparing the coefficient of like powers of p, the following approximations are obtained;

$$p^0: u_0(x,t) = -2\frac{k^2 e^{kx}}{(1+e^{kx})^2}$$

$$p^1: u_1(x,t) = -L^{-1}\left[\frac{1}{s}\,L[(u_0)_{xxx} - 6H_0(u)]\right] = -2\frac{k^5 e^{kx}(e^{kx}-1)}{(1+e^{kx})^3}t$$

$$p^2: u_2(x,t) = -L^{-1}\left[\frac{1}{s}\,L[(u_1)_{xxx} - 6H_1(u)]\right] = -\frac{k^8 e^{kx}(e^{2kx}-4e^{kx}+1)}{(1+e^{kx})^4}t^2$$

Therefore, the solution of Eq. (8), when $p \to 1$ will be as:

$$u(x,t) = -2\frac{k^2 e^{kx}}{(1+e^{kx})^2} - 2\frac{k^5 e^{kx}(e^{kx}-1)}{(1+e^{kx})^3}t - \frac{k^8 e^{kx}(e^{2kx}-4e^{kx}+1)}{(1+e^{kx})^4}t^2 + \cdots$$

Using Taylor series, the closed form solution will be as follows:

$$u(x,t) = -2\frac{k^2 e^{k(x-k^2 t)}}{\left(1+e^{k(x-k^2 t)}\right)^2} \tag{218}$$

(It's worth mentioning that these results were publishrd in "Mohannad H. Eljaily, Tarig M. Elzaki, Homotopy Perturbation Transform Method for Solving Korteweg-DeVries (KDV) Equation, Pure and Applied Mathematics Journal 2015; 4(6): 264-268 Published online November 2; 2015 (http://www.sciencepublishinggroup.com/j/pamj); doi: 10.11648/j.pamj.20150406.17; ISSN: 2326-9790 (Print); ISSN: 2326-9812 (Online))

CHAPTER FOUR

The Homotopy Perturbation Method for Solving Partial Differential Equations with Nonlocal Conditions

Recently, much attention has been to partial differential equations with non-local boundary conditions, this attention was driven by the needs of applications both in industry and the sciences. Theory and numerical methods for solving partial differential equations with nonlocal conditions were investigated by many researchers[96-103]. In the last decade, there has been a growing interest in the analytical new techniques for linear and nonlinear initial boundary value problems with non-classical boundary conditions. The widely applied techniques are perturbation methods.

HPM has gained reputation as being a powerful tool for solving linear or nonlinear partial differential equations. This method has been the subject of intense investigation during recent years and many researchers have used it in their works involving differential equations see [35,39]. He [47], applied HPM to solve initial boundary value problems which are governed by the nonlinear ordinary (Partial) differential equations, the results show that this method is efficient and simple. Thus,the main goal of this work is to apply the homotopy perturbation method (HPM) for solving linear and nonlinear initial boundary value problems with nonlocal boundary conditions.

The general form of the equation is given as:

$$u_t = G(x, t, u_x, u_{xx}), \qquad a < x < b, \qquad 0 < t < T \tag{1}$$

Subject to the initial condition;

$$u(x, 0) = f(x), \qquad 0 \le t \le T \tag{2}$$

And the non-local boundary conditions;

$$u(a,t) = \int_a^b \varphi(x,t)\, u(x,t)dx + g_0(t), \qquad 0 < t \leq T \tag{3}$$

$$u(b,t) = \int_a^b \psi(x,t)\, u(x,t)dx + g_1(t), \qquad 0 < t \leq T \tag{4}$$

Where f,g_0, g_1, φ and ψ are sufficiently smooth known functions and T is a given constant.

4.1 Analysis of Homotopy Perturbation Method

To illustrate the basic ideas Let X and Y be the topological spaces. If f and g are continuous maps of the space X into Y , it is said that f is homotopic to g, if there is continuous map

$F: X \times I(= [0,1]) \to Y$ Such that $F(x,0) = f(x)\ and\ F(x,1) = g(x)$, for each $x \in X$ then the map is called homotopy between $f\ and\ g$.

We, consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, \qquad r \in \Omega \tag{5}$$

Subject to the boundary conditions;

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \;\; r \in \Gamma \tag{6}$$

Where A is a general differential operator, B is a boundary operator, $f\ (r)$ is a known analytical function and Γ is the boundary of the domain Ω. Generally speaking, operator A can be divided into two parts which are L and N where L is linear, but N is nonlinear. Therefore, equation (5) can be rewritten as follows;

$$L(u) + N(u) - f(r) = 0 \tag{7}$$

By the homotopy perturbation technique, we construct a homotopy $V(r,p): \Omega \times [0,1] \to R$ which satisfies:

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0 \quad p \in [0,1], r \in \Omega \tag{8}$$

Or equivalently

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \quad (9)$$

Where $p\epsilon[0,1]$ is an embedding parameter and u_0 is an initial approximation of the equation (1).Obviously, from these definitions we will have:

$$H(v,0) = L(v) - L(u_0) = 0 \quad (10)$$

$$H(v,1) = A(v) - f(u_0) = 0 \quad (11)$$

The changing process of p from zero to one is just that of $v\,(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation and $L(v)$ – $L(u_0)$ and $A\,(v)$ – $f\,(r)$ are called homotopy. According to the HPM, we can first use the embedding parameter p as a "small parameter" and assuming that the solution of Eq. (4) can be written as a power series in p:

$$v = v_0 + pv_1 + p^2v_2 + \ldots. \quad (12)$$

Setting $p = 1$, results in the approximate solution of Eq. (3):

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \ldots. \quad (13)$$

4.2 Linear and nonlinear partial differential Equations Subject to a Non-local Boundary Condition

In this section, we have presented Homotopy Perturbation Method (HPM) to solve linear and nonlinear partial differential equations with nonlocal boundary conditions. This method provides an analytical solution by utilizing only the initial condition. The HPM allows for the solution of the nonlinear parabolic equations to be calculated in the form of a series with easily computable components.

In the section we have used the HPM to solve some linear and nonlinear partial differential equations with nonlocal boundary conditions and compare the solution of them with the exact solution

Example (4.2.1) We consider the problem [20],

$$u_t + u_{xx} = -x(x-1)e^{-t}, \quad 0 < x < 1, t > 0 \tag{14}$$

With the initial condition;

$$u(x,0) = x(x-1) - 2, \tag{15}$$

And the non-local boundary conditions;

$$u(0,t) = \int_0^1 \varphi(x,t)\, u(x,t)dx + g_0(t) \tag{16}$$

Where $\varphi(x,t) = \frac{12}{13}$ and $g_0(t) = 0$

$$u(1,t) = \int_0^1 \psi(x,t)\, u(x,t)dx + g_1(t) \tag{17}$$

Where $\psi(x,t) = \frac{12}{13}$ and $g_1(t) = 0$

To solve Eq. (14) with initial conditions Eq. (15) , according to the homotopy perturbation Eq. (8), we construct the following homotopy:

$$H(v,p) = (1-p)[v_t - (u_0)_t] + p[v_t - v_{xx} + x(x-1)e^{-t}] = 0 \tag{18}$$

Or

$$v_t - (u_0)_t + p[(u_0)_t - v_{xx} + x(x-1)e^{-t}] = 0 \tag{19}$$

Substituting Eq. (12) into Eq. (19), and comparing coefficients of the terms with the identical powers of p, we get;

$p^0 : \ (v_0)_t - (u_0)_t = 0, \qquad v_0(x,t) = x(x-1) - 2$

$p^1 : \ (v_1)_t + (u_0)_t - (v_0)_{xx} + x(x-1)e^{-t} = 0,\ v_1(x,t) = x(x-1)e^{-t} - 2t$

$p^2 : (v_2)_t - (v_1)_{xx} = 0,\ v_2(x,t) = -2e^{-t}$

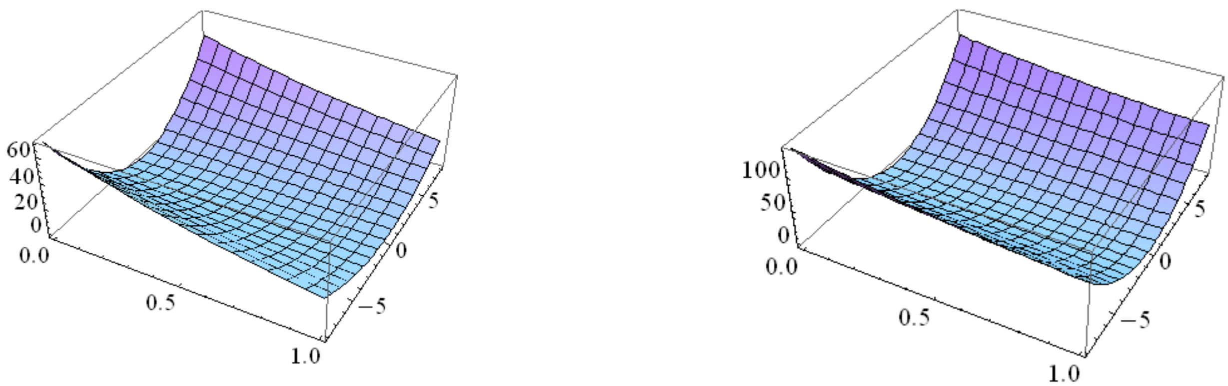
$\vdots$

And so on. Combining the results obtained for the components, the solution in a series form is given by:

$$u(x,t) = \sum_{k=0}^{\infty} v_k(x,t)$$

$$u(x,t) = x(x-1)(1+e^{-t}) - 2(1+t+e^{-t}) + \cdots \qquad (20)$$

Fig (4.2.2) (A) the exact solution, (B) the approximate solution



(A) (B)

The numerical results in both figures are in excellent agreement with the exact solution.

Example (4.2.3) We consider the problem [83],

$$u_t + u_{tt} = u_x + u_{xx} + (4t^3 + 12t^2 - 4x^3 - 12x^2), \qquad (21)$$

$$0 < x < 1, 0 < t < T$$

With the initial conditions;

$$u(x,0) = x^4, \ \ u_t(x,0) = 0\,, 0 < x < 1, 0 < t < T \qquad (22)$$

And the non-local boundary conditions;

$$u(0,t) = \int_0^1 \varphi(x,t)\, u(x,t) dx + g_0(t) = 1 + \frac{1}{5} t^4 \qquad (23)$$

Where $\varphi(x,t) = \frac{1}{5}$ and $g_0(t) = \frac{24}{25}$

$$u(1,t) = \int_0^1 \psi(x,t)\, u(x,t) dx + g_1(t) = 1 + \frac{1}{6} t^4 \qquad (24)$$

Where $\psi(x,t) = \frac{1}{6}$ and $g_1(t) = \frac{29}{30}$

To solve Eq. (20) with initial conditions Eq. (21), according to the homotopy perturbation Eq. (8), we construct the following homotopy:

$$H(v,p) = (1-p)[v_t - (u_0)_t] +$$

$$p[v_t + v_{tt} - v_x - v_{xx} - (4t^3 + 12t^2 - 4x^3 - 12x^2)] = 0 \quad (25)$$

Or

$$v_t - (u_0)_t + p[(u_0)_t + v_{tt} - v_x - v_{xx} - (4t^3 + 12t^2 - 4x^3 - 12x^2)] = 0, (26)$$

Substituting Eq. (12) into Eq. (26), and comparing coefficients of the terms with the identical powers of p, we get;

$p^0 : (v_0)_t - (u_0)_t = 0, \qquad v_0(x,t) = x^4$

$p^1 : (v_1)_t + (u_0)_t + (v_0)_{tt} - (v_0)_x - (v_0)_{xx} - (4t^3 + 12t^2 - 4x^3 - 12x^2) = 0,$

$v_1(x,t) = t^4 + 4t^3$

p^2: $(v_2)_t + (v_1)_{tt} - (v_1)_x - (v_1)_{xx} = 0,\ v_2(x,t) = -4t^3 - 12t^2$

p^3: $(v_3)_t + (v_2)_{tt} - (v_2)_x - (v_2)_{xx} = 0,\ v_3(x,t) = 12t^2 + 24t$

And so on. Combining the results obtained for the components, the solution in a series form is given by:

$$u(x,t) = \sum_{k=0}^{\infty} v_k(x,t)$$

$$u(x,t) = x^4 + t^4 \qquad (27)$$

Which is an exact solution.

Table (4.2.4) Results in different values of x and t

x_i	u_{ex}	$u_{hpm} 3 - iterates$	$\lvert u_{ex} - u_{hpm} \rvert$
0.0	2.56×10^{-2}	-1.92000×10^{-3}	0.0224
0.1	0.0001	-9.2×10^{-5}	8.0×10^{-6}
0.2	0.0016	1.584×10^{-3}	1.6×10^{-5}
0.3	0.0081	7.908×10^{-3}	1.92×10^{-4}
0.4	0.0256	2.5408×10^{-2}	1.92×10^{-4}
0.5	0.0625	6.2308×10^{-2}	1.92×10^{-4}
0.6	0.1296	0.12941	0.00019
0.7	0.2401	0.23991	0.00019
0.8	0.4096	0.40941	0.00019
0.9	0.6561	0.65591	0.00019
1.0	1.0	0.99981	0.00019

Example (4.2.5) We consider the problem [84],

$$u_t = \frac{1}{6}\left(x^2 u_{xx} + y^2 u_{yy} + z^2 u_{zz}\right), \qquad 0 < x, y, z < 1, 0 < t < T \tag{28}$$

Subject to the initial condition;

$$u(x, y, z, 0) = x^2 y^2 z^2 \tag{29}$$

And the non-local boundary conditions;

$$u(0, y, z, t) = \int_0^1 \int_0^1 \int_0^1 u(x, y, z, t)dxdydz + g_1 = \frac{1}{27}e^t, \quad g_1 = 0$$

$$u(1, y, z, t) = \int_0^1 \int_0^1 \int_0^1 u(x, y, z, t)dxdydz + g_2 = \frac{1}{27}e^t + \frac{1}{2}t, \quad g_2 = \frac{1}{2}t$$

$$u(x, 0, z, t) = \int_0^1 \int_0^1 \int_0^1 u(x, y, z, t)dxdydz + g_3 = \frac{1}{27}(e^t + 1), \quad g_3 = \frac{1}{27}$$

$$u(x, 1, z, t) = \int_0^1 \int_0^1 \int_0^1 u(x, y, z, t)dxdydz + g_4 = \frac{1}{27}(e^t + 3), \quad g_4 = \frac{1}{9} \tag{30}$$

$$u(x, y, 0, t) = \int_0^1 \int_0^1 \int_0^1 u(x, y, z, t)dxdydz + g_5 = \frac{1}{27}e^t, \quad g_5 = \frac{1}{6}$$

$$u(x,y,1,t)=\int_0^1\int_0^1\int_0^1 u(x,y,z,t)dxdydz+g_6=\frac{1}{27}e^t,\quad g_6=\frac{1}{5}t$$

To solve Eq. (28) with initial conditions Eq. (29) , according to the homotopy perturbation Eq. (8), we construct the following homotopy:

$$H(v,p)=(1-p)[v_t-(u_0)_t]$$
$$+p\left[v_t-\left(\frac{1}{6}\left(x^2v_{xx}+y^2v_{yy}+z^2v_{zz}\right)\right)\right]=0 \quad (31)$$

Or

$$v_t-(u_0)_t+p\left[(u_0)_t-\left(\frac{1}{6}\left(x^2v_{xx}+y^2v_{yy}+z^2v_{zz}\right)\right)\right]=0 \quad (32)$$

Substituting Eq. (13) into Eq. (32), and comparing coefficients of the terms with the identical powers of p, we get;

$$p^0:\ \{(v_0)_t-(u_0)_t=0,\qquad v_0(x,y,z,t)=x^2y^2z^2$$

$$p^1:\ \begin{cases}(v_1)_t+(u_0)_t-\left(\frac{1}{6}\left(x^2(v_0)_{xx}+y^2(v_0)_{yy}+z^2(v_0)_{zz}\right)\right)=0\\ v_1(x,y,z,t)=x^2y^2z^2t\end{cases},$$

$$p^2:\ \begin{cases}(v_2)_t-\left(\frac{1}{6}\left(x^2(v_1)_{xx}+y^2(v_1)_{yy}+z^2(v_1)_{zz}\right)\right)=0,\\ v_2(x,y,z,t)=x^2y^2z^2\frac{t^2}{2!}\end{cases},$$

$$p^3:\ \begin{cases}(v_3)_t-\left(\frac{1}{6}\left(x^2(v_2)_{xx}+y^2(v_2)_{yy}+z^2(v_2)_{zz}\right)\right)=0\\ v_2(x,y,z,t)=x^2y^2z^2\frac{t^3}{3!}\end{cases},$$

Proceeding in a similar manner, we obtain:

$$p^4:\ v_4(x,y,z,t)=x^2y^2z^2\frac{t^4}{4!}$$

$$p^5 : \; v_5(x, y, z, t) = x^2y^2z^2\frac{t^5}{5!}$$

$$\vdots$$

Therefore the solution $u(x, y, z, t)$, in series form is given by:

$$u(x, y, z, t) = x^2y^2z^2\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) \tag{33}$$

And in closed form given as;

$$u(x, y, z, t) = x^2y^2z^2e^t \tag{34}$$

Which is an exact solution

Table (4.2.6) Results for different values of x and t

x_i	y_i	z_i	u_{ex}	u_{hpm} 5-iterates	$\lvert u_{ex} - u_{hpm}\rvert$
0.0	0.0	0.0	0	0	0
0.1	0.1	0.1	1.004×10^{-6}	1.004×10^{-6}	0
0.2	0.2	0.2	6.4257×10^{-5}	6.4257×10^{-5}	0
0.3	0.3	0.3	7.3192×10^{-4}	7.3192×10^{-4}	0
0.4	0.4	0.4	4.6843×10^{-3}	4.6843×10^{-3}	0
0.5	0.5	0.5	1.5688×10^{-2}	1.5688×10^{-2}	0
0.6	0.6	0.6	4.6843×10^{-2}	4.6843×10^{-2}	0
0.7	0.7	0.7	0.11812	0.11812	0
0.8	0.8	0.8	0.26319	0.26319	0
0.9	0.9	0.9	0.53357	0.53357	0
1.0	1.0	1.0	1.004	1.004	0

Example (4.2.6) We consider the following nonlinear reaction-diffusion equation [85],

$$u_t - u_{xx} = u^2 - (u_x)^2, \qquad 0 < x < 1, 0 < t < T \tag{35}$$

Subject to the initial condition;

$$u(x,0) = e^x, \qquad 0 < x < 1, \tag{36}$$

And the non-local boundary conditions;

$$u(0,t) = \int_0^1 \varphi(x,t)\, u(x,t)dx + g_0(t) = e^{1+t} \tag{37}$$

Where $\varphi(x,t) = 1$ and $g_0(t) = e^t$

$$u(1,t) = \int_0^1 \psi(x,t)\, u(x,t)dx + g_1(t) = \frac{1}{2}e^{1+t} \tag{38}$$

Where $\psi(x,t) = \frac{1}{2}$ and $g_1(t) = \frac{1}{2}e^t$

To solve Eq. (35) with initial conditions Eq. (36), according to the homotopy perturbation Eq. (8), we construct the following homotopy:

$$H(v,p) = (1-p)[v_t - (u_0)_t] + p[v_t - v_{xx} - v^2 + {v_x}^2] = 0 \tag{39}$$

Or

$$v_t - (u_0)_t + p[(u_0)_t - v_{xx} - v^2 + {v_x}^2] = 0 \tag{40}$$

Substituting Eq. (13) into Eq. (40), and comparing coefficients of the terms with the identical powers of p, we have;

$p^0 : (v_0)_t - (u_0)_t = 0, \qquad v_0(x,t) = e^x$

$p^1 : (v_1)_t + (u_0)_t - (v_0)_{xx} - {v_0}^2 + {(v_0)_x}^2 = 0,\ v_1(x,t) = te^x$

$p^2 : (v_2)_t - (v_1)_{xx} - 2v_0v_1 + 2(v_0)_x(v_1)_x = 0,\ v_2(x,t) = \frac{1}{2!}t^2e^x$

$p^3 : (v_3)_t - (v_2)_{xx} - 2v_0v_2 - {v_1}^2 + 2(v_0)_x(v_2)_x + {(v_0)_x}^2 = 0,\ v_3(x,t) = \frac{1}{3!}t^3e^x$

Proceeding in a similar manner, we obtain:

$$p^4 : \; v_4(x,,t) = \frac{1}{4!}t^4 e^x$$

$$p^5 : \; v_5(x,t) = \frac{1}{5!}t^5 e^x$$

And so on. Combining the results obtained for the components, the solution in a series form is given by:

$$u(x,t) = \sum_{k=0}^{\infty} v_k(x,t)$$

$$u(x,t) = e^x\left(1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \cdots\right) \tag{41}$$

In series form, and

$$u(x,t) = e^{x+t} \tag{42}$$

In closed form.

Table (4.2.7) Results for different values of x and t

x_i	u_{ex}	u_{hpm}5-iterates	$\lvert u_{ex} - u_{hpm} \rvert$
0.0	1.004	1.004	0
0.1	1.1096	1.1096	0
0.2	1.2263	1.2263	0
0.3	1.3553	1.3553	0
0.4	1.4978	1.4978	0
0.5	1.6553	1.6553	0
0.6	1.8294	1.8294	0
0.7	2.0218	2.0218	0
0.8	2.2345	2.2345	0
0.9	2.4695	2.4695	0
1.0	2.7292	2.7292	0

4.3 The Wave Equation

Partial differential equations with nonlocal boundary conditions have received much attention in last 20 years. However, most of the articles were directed to the second order parabolic equations, particularly to heat conduction equations. We will deal here with a new type of nonlocal boundary value problem that is the solution of hyperbolic partial differential equations with nonlocal boundary specifications. These nonlocal conditions arise mainly when the data on the boundary cannot be measured directly. Many physical phenomena are modeled by non-classical hyperbolic boundary value problems with nonlocal boundary conditions.

In this section, the following hyperbolic problem is considered with a nonlocal constraint in place of a standard boundary condition:

$$u_{tt} - u_{xx} = q(x,t), \qquad 0 < x < l, \qquad 0 < t < T \tag{43}$$

With initial conditions

$$u(x,0) = f(x), \;\; u_t(x,0) = g(x), 0 < x < l \tag{44}$$

And Dirichlet (Neumann) boundary conditions

$$u(0,t) = h(t), \;\; u_x(0,t) = m(t) \tag{45}$$

Together with the nonlocal condition

$$\int_0^1 u(x,t)\,dx = E(t), \qquad 0 < t < T \tag{46}$$

Where q, f, g, h, m and E are known functions.

It is worth pointing out that f and g satisfies the following compatibility conditions

$$f(0) = h(0), \quad g(0) = h'(0), \int_0^1 f(x)\,dx = E(0), \int_0^1 g(x)\,dx = E'(0) \tag{47}$$

Although there has been considerable interest in the mathematical properties of equations arising in hyperbolic boundary value problems, little attention has been devoted to their numerical solution.

In this section, we present and discuss the numerical results by employing HPM fortwo test examples. The results demonstrate the present method is remarkably effective.

Example (4.3.8) Consider the following wave equation [88],

$$u_{tt} - u_{xx} = 0, \qquad 0 < x < 1, \qquad 0 < t \le 0.5 \tag{48}$$

With the initial conditions;

$$u(x,0) = 0, \;\; u_t(x,0) = \pi\, cos(\pi x)\,, \qquad 0 < x < 1 \tag{49}$$

And Dirichlet (Neumann) boundary condition;

$$u(0,t) = sin(\pi t) \tag{50}$$

Together with the nonlocal condition;

$$\int_0^1 u(x,t)\,dx = 0, \qquad 0 < t \le 0.5 \tag{51}$$

To solve Eq. (48) with boundary condition Eq. (50), according to the homotopy perturbation Eq. (8), we construct the following homotopy:

$$H(v,p) = (1-p)[v_{xx} - (u_0)_{xx}] + p[v_{xx} - v_{tt}] = 0 \tag{52}$$

Or

$$v_{xx} - (u_0)_{xx} + p[(u_0)_{xx} - v_{tt}] = 0 \tag{53}$$

Substituting Eq. (13) in Eq. (53), and comparing coefficients of the terms with the identical powers of p, we have;

$$p^0:\; (v_0)_{xx} - (u_0)_{xx} = 0, \qquad v_0(x,t) = sin(\pi t)$$

$$p^1:\; (v_1)_{xx} + (u_0)_{xx} - (v_0)_{tt} = 0,\;\; v_1(x,t) = -\frac{1}{2!}(\pi x)^2 sin(\pi t)$$

$$p^2: (v_2)_{xx} - (v_1)_{tt} = 0,\ v_2(x,t) = \frac{1}{4!}(\pi x)^4 \sin(\pi t)$$

$$p^3: (v_3)_{xx} - (v_2)_{tt} = 0,\ v_3(x,t) = -\frac{1}{6!}(\pi x)^6 \sin(\pi t)$$

Proceeding in a similar manner, we obtain:

$$p^4 :\ v_4(x,y,z,t) = \frac{1}{8!}(\pi x)^8 \sin(\pi t)$$

$$p^5 :\ v_5(x,y,z,t) = -\frac{1}{10!}(\pi x)^{10} \sin(\pi t)$$

And so on. Combining the results obtained for the components, the solution in a series form is given by:

$$u(x,t) = \sum_{k=0}^{\infty} v_k(x,t)$$

$$u(x,t) = \sin(\pi t)\left(1 - \frac{1}{2!}(\pi x)^2 + \frac{1}{4!}(\pi x)^4 + \cdots\right) \tag{54}$$

In series form, and

$$u(x,t) = \sin(\pi t)\cos(\pi x) \tag{55}$$

In closed form.

Example (4.3.9) Consider the following wave equation with an integral condition [87],

$$u_{tt} - u_{xx} = 0, \qquad 0 < x < 1, \qquad 0 < t \le 0.5 \tag{56}$$

With the initial conditions;

$$u(x,0) = \cos(\pi x), \quad u_t(x,0) = 0, \qquad 0 < x < 1 \tag{57}$$

And Dirichlet (Neumann) boundary condition;

$$u_t(0,t) = 0 \tag{58}$$

Together with the nonlocal condition;

$$\int_0^1 u(x,t)\,dx = 0, \qquad 0 < t \leq 0.5 \tag{59}$$

To solve Eq. (56) with initial conditions Eq. (57), according to the homotopy perturbation Eq. (8), we construct the following homotopy:

$$H(v,p) = (1-p)[v_{tt} - (u_0)_{tt}] + p[v_{tt} - v_{xx}] = 0 \tag{60}$$

Or

$$v_{tt} - (u_0)_{tt} + p[(u_0)_{tt} - v_{xx}] = 0 \tag{61}$$

Substituting Eq. (13) in Eq. (61), and comparing coefficients of the terms with the identical powers of p, we have;

$$p^0 : \ (v_0)_{tt} - (u_0)_{tt} = 0, \qquad v_0(x,t) = cos(\pi x)$$

$$p^1 : \ (v_1)_{tt} + (u_0)_{tt} - (v_0)_{xx} = 0, \ \ v_1(x,t) = -\frac{1}{2!}(\pi t)^2 cos(\pi x)$$

$$p^2 : (v_2)_{tt} - (v_1)_{xx} = 0, \ v_2(x,t) = \frac{1}{4!}(\pi t)^4\, cos(\pi x)$$

$$p^3 : (v_3)_{tt} - (v_2)_{xx} = 0, \quad v_3(x,t) = -\frac{1}{6!}(\pi t)^6\, cos(\pi x)$$

Proceeding in a similar manner, we obtain:

$$p^4 \ : \ v_4(x,t) = \frac{1}{8!}(\pi x)^8\, cos(\pi x)$$

$$p^5 : \ v_5(x,t) = -\frac{1}{10!}(\pi x)^{10}\, cos(\pi x)$$

And so on. Combining the results obtained for the components, the solution in a series form is given by:

$$u(x,t) = \sum_{k=0}^{\infty} v_k(x,t)$$

$$u(x,t) = cos(\pi x)\left(1 - \frac{1}{2!}(\pi x)^2 + \frac{1}{4!}(\pi x)^4 + \cdots\right) \tag{62}$$

In series form, and

$$u(x,t) = cos(\pi x)\,cos(\pi t) \tag{63}$$

In closed form.

CHAPTER FIVE

Solution of Parabolic and Hyperbolic Equations with Nonlocal Conditions by Homotopy Perturbation Method

Various problems arising in heat conduction [48-50], chemical engineering [51], thermo elasticity [52], and plasma physics [53] can be reduced to the nonlocal problems. Boundary value problems with integral conditions constitute a very interesting and important class of problems. Therefore, partial differential equations with nonlocal boundary conditions have received much attention in last 20 years. However, most of articles were directed to the second order parabolic equations, particularly to heat conduction equations.

We will deal here with a new type of nonlocal boundary value problem that is the solution of hyperbolic partial differential equations with nonlocal boundary specifications. These nonlocal conditions arise mainly when the data on the boundary cannot be measured directly. Many physical phenomena are modeled by non-classical hyperbolic boundary value problems with nonlocal boundary conditions.Numerical solution of hyperbolic partial differential equation with an integral condition continues to be a major research area with widespread applications in modern physics and technology. The theoretical aspects of the solutions to the one-dimensional hyperbolic initial-boundary value problems have been studied by several authors [54- 57].

The strong solution of an initial-boundary value problem which combine Neumann and integral conditions for a hyperbolic equation is studied by Bouziani [58]. However, few papers investigate the numerical solutions of this class of equations. Bougoffa presents an adomian method for a class of hyperbolic equations with Dirichlet boundary condition and the nonlocal boundary condition [59]. Dehghan developed several new finite difference schemes for an initial-boundary value problem which combine Neumann and integral conditions for a hyperbolic equation [60].This chapter presents solution for nonlocal initial-boundary value problems for linear and nonlinear parabolic and hyperbolic partial differential equations. We first transform the given nonlocal initial-boundary value problems of integral type for the linear and nonlinear parabolic and hyperbolic partial differential equations into local Dirichlet initial-boundary value problems, and then use a homotopy perturbation method (HPM). Several examples are presented to demonstrate the efficiency of the HPM.

5.1 The linear parabolic Equation with Nonlocal Conditions

In this section, we will employ the homotopy perturbation method to solve the nonlocal initial-boundary value problems for linear variable-coefficient parabolic partial differential equations. For cases, including an input function or additional linear terms, the homotopy perturbation method remains the method of choice to easily and quickly calculate solutions. Several examples will be presented in the sequel.

Now we consider the heat equation:

$$u_t \; - ku_x = 0, \quad a \leq x \leq b, \quad t \geq 0$$

Where k is a constant, which describes motion with constant speed. We specify $u(t,x)$ at the initial time t, which we take to be 0, i.e. $u(0,x)$ equals a given function $u_0(x)$ on $a \leq x \leq b$, and the boundary condition relating the solution of the differential equation to data of the integral type $\int_a^b u(t,x)dx = \beta(t)$, which is called the nonlocal boundary condition of integral type, where $u(t,x)$ denotes the concentration of the pollutant in gr/cm (unit mass per unit length) at time t_0 and $\int_a^b u(t,x)dx$ denotes the amount of pollutant in the interval $[a,b]$ at time t. The problem of determining a solution to a partial differential equation when both initial data and nonlocal boundary conditions are specified is called a nonlocal initial-boundary value problem.

We consider the inhomogeneous linear parabolic partial differential equation

$$u_t - p(t,x)u_{xx} + q(t,x)u = f(t,x), \;\; a \leq x \leq b, t \geq 0, \qquad (1)$$

Subject to the initial condition;

$$u(0,x) = \propto (x) \tag{2}$$

And the nonlocal inhomogeneous boundary conditions of integral type

$$\int_a^b \varphi_1(x)u(t,x)dx = \beta_1(t) and \int_a^b \varphi_2(x)u(t,x)dx = \beta_2(t), \tag{3}$$

Where $\varphi_i(x)$, $\beta_i(t)$, $i = 1,2$ and $\propto (x)$ are specified as continuous functions. we begin our approach by converting Eqs. (1)– (3) To a local initial-boundary value problem by introducing a new function $v(t,x)$ such that:

$$v(t,x) = \int_a^x \varphi(x)u(t,x)dx \tag{4}$$

Where $\varphi(x) = \varphi_1(x) + \varphi_2(x)$

Hence we have:

$$v_x(t,x) = \varphi(x)u(t,x) \tag{5}$$

And

$$u(t,x) = \frac{v_x(t,x)}{\varphi(x)} \tag{6}$$

From Eq. (6) we have:

$$u_t(t,x) = \frac{v_{tx}(t,x)}{\varphi(x)} \tag{7}$$

$$u_x(t,x) = \frac{1}{\varphi(x)} v_x(t,x) + \left(\frac{1}{\varphi(x)}\right)' v_x(t,x) \tag{8}$$

And

$$u_{xx}(t,x) = \left(\frac{1}{\varphi(x)}\right)'' v_x(t,x) + 2\left(\frac{1}{\varphi(x)}\right)' v_{xx}(t,x) + \left(\frac{1}{\varphi(x)}\right) v_{xxx}(t,x) \tag{9}$$

Substituting Eqs. (6)– (9) into Eq. (1) we deduce

$$v_{tx}(t,x)-\left(p(t,x)\left(\frac{1}{\varphi(x)}\right)''\varphi(x)-q(t,x)\right)v_x(t,x)$$

$$-2p(t,x)\left(\frac{1}{\varphi(x)}\right)'\varphi(x)v_{xx}(t,x)-p(t,x)v_{xxx}(t,x)=\varphi(x)f(t,x) \qquad (10)$$

By using Eq. (4) we get;

$$\begin{cases} v_x(0,x)=\varphi(x)u(0,x)=\varphi(x)\propto(x)=h_1(x), \\ v(t,a)=0, \\ v(t,b)=\int_a^b \varphi(x)u(t,x)dx=\beta(t)\,, \end{cases} \qquad (11)$$

Where $\beta(t)=\beta_1(t)+\beta_2(t)$

Thus we deduce:

Lemma (5.1.1) The general nonlocal initial-boundary value problem Eqs. (1)–(3) can always be reduced to a local initial-boundary value problem of the form:

$$\begin{cases} v_{tx}+r(t,x)v_x+s(t,x)v_{xx}-p(t,x)v_{xxx}=g(t,x), \\ v_x(0,x) \qquad\qquad\qquad\qquad =h_1(x), \\ v(t,a)=0 \quad and \quad v(t,b) \qquad =\beta(t), \end{cases} \qquad (12)$$

Where

$$\begin{cases} r(t,x)=-p(t,x)\left(\frac{1}{\varphi(x)}\right)''\varphi(x)+q(t,x), \\ s(t,x) \qquad =-2p(t,x)\left(\frac{1}{\varphi(x)}\right)'\varphi(x), \\ \beta(t) \qquad =\beta_1(t)+\beta_2(t), \\ h_1(x) \qquad =\varphi(x)\propto(x), \\ g(t,x) \qquad =\varphi(x)f(t,x). \end{cases} \qquad (13)$$

A solution of this problem will lead to a solution of the given original problem, where $u(t,x)$ is given by Eq. (6). Based on the the homotopy perturbation method, we write;

$$v_{tx}+r(t,x)v_x+s(t,x)v_{xx}-p(t,x)v_{xxx}=g(t,x)$$

In operator-theoretic notation as;

$$Av = g + Rv \tag{14}$$

Where

$$Av = v_{tx} \quad and \quad Rv = -r(t,x)v_x - s(t,x)v_{xx} + p(t,x)v_{xxx} \tag{15}$$

We conveniently define the inverse linear operator as;

$$A^{-1}{}_{a,tx} = \int_a^x \int_0^t (.)\, dtdx \tag{16}$$

Applying the inverse linear operator $A^{-1}{}_{a,tx}$ to Eq. (14), and taking into account that $v_x(0,x) = h_1(x)$ and $v(t,a) = 0$, we obtain:

$$v(t,x) = \int_a^x h_1(x)dx + A^{-1}{}_{a,tx}\, g + A^{-1}{}_{a,tx} Rv \tag{17}$$

Proceeding as before, applying the inverse linear operator

$$A^{-1}{}_{b,tx}(.) = \int_x^b \int_0^t (.)\, dtdx \tag{18}$$

To both sides of Eq. (14), and taking into account that $v_x(0,x) = h_1(x)$ and $v(t,b) = \beta(t)$, we obtain:

$$v(t,x) = v(t,b) - \int_x^b h_1(x)dx - A^{-1}{}_{b,tx}\, g - A^{-1}{}_{b,tx} Rv \tag{19}$$

Thus

$$v(t,x) = \beta(t) - \int_x^b h_1(x)dx - A^{-1}{}_{b,tx}\, g - A^{-1}{}_{b,tx} Rv \tag{20}$$

Adding the relations in Eq. (17) and Eq. (20) together, and then dividing by two, we obtain the solution as the equal-weight average

$$v(t,x) = \frac{1}{2}\left[\int_a^x h_1(x)dx + \beta(t) - \int_x^b h_1(x)dx + A^{-1}{}_{a,tx}\, g - A^{-1}{}_{b,tx} g\right]$$

$$+\frac{1}{2}\left[A^{-1}{}_{a,tx}Rv - A^{-1}{}_{b,tx}Rv\right] \tag{21}$$

Now, we apply the homotopy perturbation method

$$v(x,t) = \sum_{n=0}^{\infty} p^n v_n(x,t) \tag{22}$$

Substituting Eq. (22) into Eq. (21) we get;

$$\sum_{n=0}^{\infty} p^n v_n(x,t) = \frac{1}{2}\left[\left[\int_a^x h_1(x)dx + \beta(t) - \int_x^b h_1(x)dx + A^{-1}{}_{a,tx}\, g - A^{-1}{}_{b,tx} g\right]\right.$$

$$+\frac{1}{2}p\left[A^{-1}{}_{a,tx} - A^{-1}{}_{b,tx}\right]R\sum_{n=0}^{\infty} p^n v_n(x,t) \tag{23}$$

Comparing the coefficient of like powers of p, we have;

$$p^0: v_0(t,x) = \frac{1}{2}\left[\left[\int_a^x h_1(x)dx + \beta(t) - \int_x^b h_1(x)dx + A^{-1}{}_{a,tx}\, g - A^{-1}{}_{b,tx} g\right]\right.$$

$$p^1: v_1(t,x) = \frac{1}{2}\left[A^{-1}{}_{a,tx} - A^{-1}{}_{b,tx}\right]Rv_0 \tag{24}$$

$$p^2: v_2(t,x) = \frac{1}{2}\left[A^{-1}{}_{a,tx} - A^{-1}{}_{b,tx}\right]Rv_1$$

$$\vdots$$

Proceeding in a similar manner, we have;

$$p^3: v_3(t,x) = \frac{1}{2}\left[A^{-1}{}_{a,tx} - A^{-1}{}_{b,tx}\right]Rv_2$$

$$p^4: v_4(t,x) = \frac{1}{2}\left[A^{-1}{}_{a,tx} - A^{-1}{}_{b,tx}\right]Rv_3$$

$$\vdots$$

So that the solution $v(x,t)$ is given by:

$$v(x,t) = v_0(t,x) + v_1(t,x) + v_1 + (t,x) \ldots \tag{25}$$

Where the term v_0 is to be determined from the initial and boundary conditions.

Once the function $v(t,x)$is calculated, we can return to the original dependent variable $u(t,x)$ by Eq. (6).

Example (5.1.2) We consider the nonlocal linear inhomogeneous initial-boundary value problem [50],

$$u_t - u_{xx} = sinx, \quad 0 \le x \le \pi, \;\; t \ge 0, \tag{26}$$

Subject to the initial condition;

$$u(0,x) = cosx \tag{27}$$

And the nonlocal inhomogeneous boundary conditions of integral type:

$$\begin{cases} \int_0^{\pi} xu(t,x)dx = -(2+\pi)e^{-t} \\ \int_0^{\pi} (k-x)u(t,x)dx = (2+\pi-2k)e^{-t} + 2k - \pi, \end{cases} \tag{28}$$

Where

$$\begin{cases} a = 0, \\ b = \pi, \\ p(t,x) = -1, \\ q(t,x) = 0, \\ f(t,x) = sinx \\ \propto (x) = cosx \\ \beta(t) = 2k(1-e^{-t}) \\ \varphi(x) = k \end{cases} \tag{29}$$

Where k is a constant.

According to Eq. (13), we have;

$$\begin{cases} g(t,x) = ksinx, \\ h_1(x) = kcosx, \\ r(t,x) \qquad = 0, \\ s(t,x) \qquad = 0, \\ Rv \qquad = v_{xxx.} \end{cases} \tag{30}$$

The recursion scheme Eq. (24) produces a rapidly convergent series as;

$$p^0 : v_0(t,x) = \frac{1}{2}\left[\int_a^x h_1(x)dx + \beta(t) - \int_x^b h_1(x)dx + \int_a^x \int_0^t g(t,x)\,dtdx - \int_x^b \int_0^t g(t,x)\,dtdx\right]$$

$$v_0(t,x) = \frac{1}{2}\left[\int_0^x kcosxdx + 2k(1-e^{-t}) - \int_x^\pi kcosxdx + \int_0^x \int_0^t ksinx\,dtdx - \int_x^\pi \int_0^t ksinx\,dtdx\right]$$

Therefore

$$p^0 : \{ \quad v_0(t,x) = ksinx - tkcosx + k(1-e^{-t})$$

$$p^1 : \begin{cases} v_1(t,x) = \frac{1}{2}\left[\int_0^x \int_0^t (v_0)_{xxx}\,dtdx - \int_x^\pi \int_0^t (v_0)_{xxx}\,dtdx\right] \\ \qquad\qquad = -ktsinx + k\frac{t^2}{2!}cosx \end{cases}$$

$$p^2 : \begin{cases} v_2(t,x) = \frac{1}{2}\left[\int_0^x \int_0^t (v_1)_{xxx}\,dtdx - \int_x^\pi \int_0^t (v_1)_{xxx}\,dtdx\right] \\ \qquad\qquad = k\frac{t^2}{2!}sinx - k\frac{t^3}{3!}cosx \end{cases}$$

Proceeding in a similar manner, we have;

$$p^3 : \{ v_3(t,x) = -k\frac{t^3}{3!}sinx + k\frac{t^4}{4!}cosx$$

$$p^4 : \{ v_4(t,x) = k\frac{t^4}{4!}sinx - k\frac{t^5}{5!}cosx$$

$$\vdots$$

And so on. Consequently, the intermediate solution is given as;

$$v(t,x) = k\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right)sinx$$

$$-k\left(t - \frac{t^2}{2!} + \frac{t^3}{3!} - \cdots\right)cosx + k(1 - e^{-t}) \quad (31)$$

Or in a closed form as;

$$v(t,x) = ke^{-t}sinx - k(1 - e^{-t})cosx + k(1 - e^{-t}) \quad (32)$$

Returning to the original dependent variable by Eq. (4), we obtain:

$$u(t,x) = \frac{v_x(t,x)}{\varphi(x)} = \frac{ke^{-t}cosx + k(1 - e^{-t})sinx}{k} \quad (33)$$

Therefore

$$u(t,x) = e^{-t}cosx + (1 - e^{-t})sinx \quad (34)$$

Which is the exact solution to this particular nonlocal initial-boundary value problem.

Example (5.1.3) We consider the nonlocal linear homogeneous initial-boundary value problem

$$u_t - u_{xx} + u = 0, \quad 0 \le x \le \pi, \ \ t \ge 0, \quad (35)$$

Subject to the initial condition;

$$u(0,x) = sinx \quad (36)$$

And the nonlocal inhomogeneous boundary conditions of integral type:

$$\begin{cases} \int_0^\pi xu(t,x)dx = \pi e^{-2t}, \\ \int_0^\pi (1 - x)u(t,x)dx = (2 - \pi)e^{-2t}, \end{cases} \quad (37)$$

Where

$$\begin{cases} a & = 0, \\ b = & \pi, \\ p(t,x) & = 1, \\ q(t,x) & = 1, \\ f(t,x) & = 0, \\ \propto (x) & = sinx, \\ \beta(t) & = 2e^{-2t}, \\ \varphi(x) & = 1. \end{cases} \tag{38}$$

According to Eq. (13), we have;

$$\begin{cases} g(t,x) & = 0, \\ h_1(x) & = sinx, \\ r(t,x) & = 0, \\ s(t,x) & = 0, \\ Rv & = -v_x + v_{xxx} \end{cases} \tag{39}$$

The recursion scheme Eq. (24) produces a rapidly convergent series as;

$$p^0: v_0(t,x) = \frac{1}{2}\left[\int_a^x h_1(x)dx + \beta(t) - \int_x^b h_1(x)dx\right]$$

$$+\frac{1}{2}\left[\int_a^x \int_0^t g(t,x)\, dtdx - \int_x^b \int_0^t g(t,x)\, dtdx\right]$$

$$v_0(t,x) = \frac{1}{2}\left[\int_0^x sinxdx + 2e^{-2t} - \int_x^\pi sinxdx\right]$$

Therefore

$$p^0: \{ \quad v_0(t,x) = -cosx + e^{-2t}$$

$$p^1: \Bigg\{ v_1(t,x) = \frac{1}{2}\left[\int_0^x \int_0^t (-(v_0)_x + (v_0)_{xxx})\, dtdx\right]$$

$$-\frac{1}{2}\left[\int_x^\pi \int_0^t (-(v_0)_x + (v_0)_{xxx})\, dtdx\right] = \quad 2tcosx$$

$$p^2\colon\left\{v_2(t,x) = \frac{1}{2}\left[\int_0^x\int_0^t(-(v_1)_x+(v_1)_{xxx})\,dtdx\right]\right.$$

$$-\frac{1}{2}\left[\int_x^\pi\int_0^t(-(v_1)_x+(v_1)_{xxx})\,dtdx\right] = -2t^2cosx$$

$$p^3\colon\left\{v_3(t,x) = \frac{1}{2}\left[\int_0^x\int_0^t(-(v_2)_x+(v_2)_{xxx})\,dtdx\right]\right.$$

$$-\frac{1}{2}\left[\int_x^\pi\int_0^t(-(v_2)_x+(v_2)_{xxx})\,dtdx\right] = 4\frac{t^3}{3}cosx$$

Proceeding in a similar manner, we have;

$$p^4\colon\{v_4(t,x) = -\,2\frac{t^4}{3}cosx$$

$$p^5\colon\{v_5(t,x) = 4\frac{t^5}{15}cosx$$

And so on. Consequently, the intermediate solution is given as;

$$v(t,x) = -\left(1-(2t)+\frac{(2t)^2}{2!}-\frac{(2t)^3}{3!}+\frac{(2t)^4}{4!}\dots\right)cosx+e^{-2t} \tag{40}$$

Or in a closed form as;

$$v(t,x) = -e^{-2t}cosx+e^{-2t} \tag{41}$$

Returning to the original dependent variable by Eq. (4), we obtain:

$$u(t,x) = \frac{v_x(t,x)}{\varphi(x)} = e^{-2t}cosx \tag{42}$$

Which is the exact solution to this particular nonlocal initial-boundary value problem.

Example (5.1.4) We consider the nonlocal linear homogeneous initial-boundary value problem

$$u_t-u_{xx} = 0,\quad 0\le x\le 1,\;\; t\ge 0, \tag{43}$$

Subject to the initial condition:

$$u(0,x) = x^2 \tag{44}$$

And the nonlocal inhomogeneous boundary conditions of integral type:

$$\begin{cases} \int_0^1 u(t,x)dx & = \frac{1}{3} + 2t, \\ \int_0^1 xu(t,x)dx & = \frac{1}{4} + t, \end{cases} \tag{45}$$

Where

$$\begin{cases} a & = 0, \\ b = & 1, \\ p(t,x) & = 1, \\ q(t,x) & = 0, \\ f(t,x) & = 0, \\ \propto(x) & = x^2, \\ \beta(t) & = \frac{7}{12} + 3t, \\ \varphi(x) & = 1 + x. \end{cases} \tag{46}$$

According to Eq. (13), we have;

$$\begin{cases} g(t,x) & = 0, \\ h_1(x) & = x^3 + x^2, \\ r(t,x) & = \frac{-2}{(1+x)^2}, \\ s(t,x) & = \frac{2}{1+x}, \\ Rv & = v_{xxx}. \end{cases} \tag{47}$$

The recursion scheme Eq. (24) produces a rapidly convergent series as;

$$p^0: v_0(t,x) = \frac{1}{2}\left[\int_a^x h_1(x)dx + \beta(t) - \int_x^b h_1(x)dx + \int_a^x \int_0^t g(t,x)\,dtdx - \int_x^b \int_0^t g(t,x)\,dtdx\right]$$

$$v_0(t,x) = \frac{1}{2}\left[\int_0^x (x^3 + x^2)dx + \frac{7}{12} + 3t - \int_x^1 (x^3 + x^2)dx\right]$$

Therefore

$$p^0 : \left\{ v_0(t,x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{3}{2}t \right.$$

$$p^1 : \begin{cases} v_1(t,x) = \frac{1}{2}\left[\int_0^x \int_0^t (v_0)_{xxx}\, dtdx - \int_x^1 \int_0^t (v_0)_{xxx}\, dtdx\right] \\ \qquad\qquad = x^2 t + 2xt - \frac{5}{2}t \end{cases}$$

$$p^2 : \left\{ v_2(t,x) = \frac{1}{2}\left[\int_0^x \int_0^t (v_1)_{xxx}\, dtdx - \int_x^\pi \int_0^t (v_1)_{xxx}\, dtdx\right] \right.$$
$$= 0$$

$$v_k(t,x) = 0, \quad \forall k \geq 2$$

Thus, the solution is given by:

$$v(t,x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 + x^2 t + 2xt - t \tag{48}$$

Returning to the original dependent variable by Eq. (4), we obtain:

$$u(t,x) = \frac{v_x(t,x)}{\varphi(x)} = \frac{x^2(1+x) + 2t(1+x)}{1+x} = x^2 + 2t \tag{49}$$

Which is the exact solution to this particular nonlocal initial-boundary value problem.

5.2 The Nonlinear parabolic Equation with Nonlocal Conditions

In this section, the homotopy perturbation method will be demonstrated on two examples of nonlinear parabolic equation with nonlocal boundary conditions. For our numerical computation, let us consider nonlinear parabolic partial differential equation of the form:

$$u_t - p(t,x)u_{xx} + q(t,x)u = f(t,x) + F(u), \;\; a \leq x \leq b, \qquad t \geq 0, \tag{50}$$

Subject to the initial condition;

$$u(0,x) = \propto(x) \tag{51}$$

And the nonlocal inhomogeneous boundary conditions of integral type:

$$\int_a^b \varphi_1(x)u(t,x)dx = \beta_1(t) \int_a^b \varphi_2(x)u(t,x)dx = \beta_2(t), \tag{52}$$

Where $\varphi_i(x)$, $\beta_i(t)$, $i = 1,2$ and $\propto(x)$ are specified as continuous functions. we begin our approach by converting Eqs. (50)– (52) To a local initial-boundary value problem by introducing a new function $v(t,x)$ such that:

$$v(t,x) = \int_a^x \varphi(x)u(t,x)dx \tag{53}$$

Where $\varphi(x) = \varphi_1(x) + \varphi_2(x)$

Hence we have:

$$v_x(t,x) = \varphi(x)u(t,x) \tag{54}$$

And

$$u(t,x) = \frac{v_x(t,x)}{\varphi(x)} \tag{55}$$

From Eq. (55) we have:

$$u_t(t,x) = \frac{v_{tx}(t,x)}{\varphi(x)} \tag{56}$$

$$u_x(t,x) = \frac{1}{\varphi(x)} v_x(t,x) + \left(\frac{1}{\varphi(x)}\right)' v_x(t,x) \tag{57}$$

And

$$u_{xx}(t,x) = \left(\frac{1}{\varphi(x)}\right)'' v_x(t,x) + 2\left(\frac{1}{\varphi(x)}\right)' v_{xx}(t,x) + \left(\frac{1}{\varphi(x)}\right) v_{xxx}(t,x) \tag{58}$$

Substituting Eqs.(55)– (58) into Eq. (50) we deduce:

$$\left(\frac{1}{\varphi(x)}\right) v_{tx}(t,x) + \left(-p(t,x)\left(\frac{1}{\varphi(x)}\right)'' \varphi(x) + q(t,x)\right) v_x(t,x)$$
$$-2p(t,x)\left(\frac{1}{\varphi(x)}\right)' v_{xx}(t,x) - \frac{p(t,x)}{\varphi(x)} v_{xxx}(t,x) = f(t,x) + F\left(\frac{v_x(t,x)}{\varphi(x)}\right) \tag{59}$$

Or

$$v_{tx} - \left(p(t,x)\left(\frac{1}{\varphi(x)}\right)'' \varphi(x) - q(t,x)\right) v_x - 2p(t,x)\left(\frac{1}{\varphi(x)}\right)' \varphi(x) v_{xx}$$
$$-p(t,x) v_{xxx} = \varphi(x) f(t,x) + \varphi(x) F\left(\frac{v_x(t,x)}{\varphi(x)}\right) \tag{60}$$

By using Eq. (53) we get;

$$\begin{cases} v_x(0,x) = \varphi(x) u(0,x) = \varphi(x) \propto (x) = h_1(x), \\ v(t,a) = 0, \\ v(t,b) = \int_a^b \varphi(x) u(t,x) dx = \beta(t), \end{cases} \tag{61}$$

Where $\beta(t) = \beta_1(t) + \beta_2(t)$

Thus we deduce:

Lemma (5.2.5) The general nonlocal initial-boundary value problem for the nonlinear parabolic partial differential Eq. (50) subject to Eq. (51) and Eq. (52) can always be reduced to a local initial-boundary value problem of the form:

$$\begin{cases} v_{tx} + r(t,x)v_x + s(t,x)v_{xx} - p(t,x)v_{xxx} = g(t,x) + N(v), \\ v_x(0,x) = h_1(x), \\ v(t,a) = 0 \quad and \quad v(t,b) = \beta(t), \end{cases} \tag{62}$$

Where

$$\begin{cases} r(t,x) = -p(t,x)\left(\frac{1}{\varphi(x)}\right)'' \varphi(x) + q(t,x), \\ s(t,x) = -2p(t,x)\left(\frac{1}{\varphi(x)}\right)' \varphi(x), \\ \beta(t) = \beta_1(t) + \beta_2(t), \\ h_1(x) = \varphi(x) \propto (x), \\ g(t,x) = \varphi(x) f(t,x), \\ N(v) = \varphi(x) F\left(\frac{v_x(t,x)}{\varphi(x)}\right). \end{cases} \tag{63}$$

A solution of this problem will lead to a solution of the given original problem, where $u(t,x)$ is given by Eq. (55). Based on the homotopy perturbation method, we write

$$v_{tx} + r(t,x)v_x + s(t,x)v_{xx} - p(t,x)v_{xxx} = g(t,x) + N(v) \tag{64}$$

In operator-theoretic notation as

$$Av = g + Rv + N \tag{65}$$

Where

$$Av = v_{tx} \quad and \quad Rv = -r(t,x)v_x - s(t,x)v_{xx} + p(t,x)v_{xxx} \tag{66}$$

We conveniently define the inverse linear operator as;

$$A^{-1}{}_{a,tx} = \int_a^x \int_0^t (.)\, dt dx \tag{67}$$

Applying Eq. (67) to Eq. (65), and taking into account that $v_x(0,x) = h_1(x)$ and $v(t,a) = 0$, we obtain:

$$v(t,x) = \int_a^x h_1(x) dx + A^{-1}{}_{a,tx}\, g + A^{-1}{}_{a,tx} Rv + A^{-1}{}_{a,tx} N \tag{68}$$

Proceeding as before, applying the inverse linear operator to both sides of Eq. (65), and taking into account that $v_x(0,x) = h_1(x)$ and $v(t,b) = \beta(t)$, we obtain:

$$A^{-1}{}_{b,tx}(.) = \int_x^b \int_0^t (.)\, dtdx \tag{69}$$

$$v(t,x) = v(t,b) - \int_x^b h_1(x)dx - A^{-1}{}_{b,tx}\, g - A^{-1}{}_{b,tx} Rv - A^{-1}{}_{b,tx} N \tag{70}$$

Thus

$$v(t,x) = \beta(t) - \int_x^b h_1(x)dx - A^{-1}{}_{b,tx}\, g - A^{-1}{}_{b,tx} Rv - A^{-1}{}_{b,tx} N \tag{71}$$

Adding the relations in Eq. (68) and Eq. (71) together, and then dividing by two, we obtain the solution as the equal-weight average

$$v(t,x) = \frac{1}{2}\left[\int_a^x h_1(x)dx + \beta(t) - \int_x^b h_1(x)dx + A^{-1}{}_{a,tx}\, g - A^{-1}{}_{b,tx} g\right] + \frac{1}{2}\left[A^{-1}{}_{a,tx} - A^{-1}{}_{b,tx}\right](Rv + N(v)) \tag{72}$$

Where the nonlinear term $N(v)$ is assumed to be an analytic function and can be expressed by an infinite series given in the form:

$$N\big(v(t,x)\big) = \sum_{n=0}^{\infty} p^n H_n(v) \tag{73}$$

For some He's polynomials H_n (see [80, 81] that are given by:

$$H_n(v_0, \dots, v_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{\infty} p^i v_i\right)_{p=0} \quad , n = 0, 1, 2, 3 \dots. \tag{74}$$

Now, we apply the homotopy perturbation method

$$v(x,t) = \sum_{n=0}^{\infty} p^n v_n(x,t) \tag{75}$$

Substituting Eqs. (73) And Eq. (75) into Eq. (72) we get:

$$\sum_{n=0}^{\infty} p^n v_n(x,t) = \frac{1}{2}\left[\int_a^x h_1(x)dx + \beta(t) - \int_x^b h_1(x)dx + A^{-1}{}_{a,tx}\, g - A^{-1}{}_{b,tx} g\right]$$

$$+\frac{1}{2}p\left[A^{-1}{}_{a,tx} - A^{-1}{}_{b,tx}\right]\left(R\sum_{n=0}^{\infty} p^n v_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(v)\right) \quad (76)$$

Comparing the coefficient of like powers of p, we have;

$$p^0: v_0(t,x) = \frac{1}{2}\left[\int_a^x h_1(x)dx + \beta(t) - \int_x^b h_1(x)dx + A^{-1}{}_{a,tx}\, g - A^{-1}{}_{b,tx} g\right]$$

$$p^1: v_1(t,x) = \frac{1}{2}\left[A^{-1}{}_{a,tx} - A^{-1}{}_{b,tx}\right]\left(Rv_0 + H_0(v)\right) \quad (77)$$

$$p^2: v_2(t,x) = \frac{1}{2}\left[A^{-1}{}_{a,tx} - A^{-1}{}_{b,tx}\right](Rv_1 + H_1(v))$$

$$\vdots$$

Proceeding in a similar manner, we have:

$$p^3: v_3(t,x) = \frac{1}{2}\left[A^{-1}{}_{a,tx} - A^{-1}{}_{b,tx}\right](Rv_2 + H_2(v))$$

$$p^4: v_4(t,x) = \frac{1}{2}\left[A^{-1}{}_{a,tx} - A^{-1}{}_{b,tx}\right](Rv_3 + H_3(v))$$

So that the solution $v(x,t)$ is given by:

$$v(x,t) = v_0(t,x) + v_1(t,x) + v_1 + (t,x) \ldots \quad (78)$$

Where the term v_0 is to be determined from the initial and boundary conditions.

Once the function $v(t,x)$ is calculated, we can return to the original dependent variable $u(t,x)$ by Eq. (55).

Example (5.2.6) We consider the nonlinear nonlocal inhomogeneous initial-boundary value problem [50],

$$u_t - xu_{xx} = -uu_x, \quad 0 \le x \le 1, \ \ t \ge 0, \tag{79}$$

Subject to the initial condition:

$$u(0,x) = x \tag{80}$$

And the nonlocal inhomogeneous boundary conditions of integral type:

$$\begin{cases} \int_0^1 u(t,x)dx & = \dfrac{1}{2(1+t)}, \\ \int_0^1 (e^x - 1)u(t,x)dx & = \dfrac{1}{2(1+t)}, \end{cases} \tag{81}$$

Where

$$\begin{cases} a & = 0, \\ b = & 1, \\ p(t,x) & = x, \\ q(t,x) & = 0, \\ f(t,x) & = 0, \\ \propto (x) & = x, \\ \beta(t) & = \dfrac{1}{1+t}, \\ F(u) = & -uu_x, \\ \varphi(x) & = e^x. \end{cases} \tag{82}$$

According to Eq. (63), we have;

$$\begin{cases} g(t,x) & = 0, \\ h_1(x) & = xe^x, \\ r(t,x) & = -x, \\ s(t,x) & = 2x, \\ Rv \ = & xv_x - 2xv_{xx} + xv_{xxx}. \end{cases} \tag{83}$$

And the nonlinear term:

$$N(v) = -\left(\frac{1}{\varphi(x)}\right)' (v_x)^2 - \frac{1}{\varphi(x)} v_x v_{xx} \tag{84}$$

That is

$$N(v) = e^{-x}((v_x)^2 - v_x v_{xx}) \tag{85}$$

Thus

$$N(v) = \sum_{n=0}^{\infty} p^n H_n(v)$$

The first few components of He's polynomials, are given by:

$$
\begin{aligned}
H_0(v) &= (v_0)^2 - (v_0)_x (v_0)_{xx} \\
H_1(v) &= 2v_0 v_1 - (v_0)_x (v_1)_{xx} \qquad (86) \\
H_2(v) &= (v_1)^2 + 2v_0 v_2 - (v_0)_x (v_2)_{xx} - (v_1)_x (v_1)_{xx} - (v_2)_x (v_0)_{xx} \\
&\vdots
\end{aligned}
$$

The recursion scheme Eq. (77) produces a rapidly convergent series as;

$$p^0 : v_0(t,x) = \frac{1}{2}\left[\int_a^x h_1(x)dx + \beta(t) - \int_x^b h_1(x)dx + \int_a^x \int_0^t g(t,x)\,dtdx - \int_x^b \int_0^t g(t,x)\,dtdx\right]$$

$$v_0(t,x) = \frac{1}{2}\left[\int_0^x xe^x dx + \frac{1}{1+t} - \int_x^1 xe^x dx\right]$$

Therefore

$$p^0 : \{v_0(t,x) = e^x(x-1) + \frac{1}{2}\left(1 + \frac{1}{1+t}\right)$$

$$p^1 : \begin{cases} v_1(t,x) = \begin{array}{c} \frac{1}{2}\int_0^x \int_0^t [(x(v_0)_x - 2x(v_0)_{xx} + x(v_0)_{xxx}) + H_0(v)]\,dtdx \\ -\frac{1}{2}\int_x^1 \int_0^t [(x(v_0)_x - 2x(v_0)_{xx} + x(v_0)_{xxx}) + H_0(v)]\,dtdx \\ = -te^x(x-1) \end{array} \end{cases}$$

$$p^2 : \begin{cases} v_2(t,x) = \begin{array}{c} \frac{1}{2}\int_0^x \int_0^t [(x(v_1)_x - 2x(v_1)_{xx} + x(v_1)_{xxx}) + H_1(v)]\, dtdx \\ -\frac{1}{2}\int_x^1 \int_0^t [(x(v_1)_x - 2x(v_1)_{xx} + x(v_1)_{xxx}) + H_1(v)]\, dtdx \\ = t^2 e^x (x-1) \end{array} \end{cases}$$

Proceeding in a similar manner, we have;

$$p^3 : v_3(t,x) = -t^3 e^x (x-1)$$

$$p^4 : v_4(t,x) = t^4 e^x (x-1)$$

$$\vdots$$

And so on. Consequently, the intermediate solution is given as;

$$v(t,x) = e^x (x-1)(1 - t + t^2 - t^3 + \cdots) + \frac{1}{2}\left(1 + \frac{1}{1+t}\right) \tag{87}$$

Or in a closed form as;

$$v(t,x) = \frac{e^x (x-1)}{1+t} + \frac{1}{2}\left(1 + \frac{1}{1+t}\right) \tag{88}$$

Returning to the original dependent variable by Eq. (55), we obtain:

$$u(t,x) = \frac{v_x(t,x)}{\varphi(x)} = \left(\frac{1}{e^x}\right)\frac{xe^x}{1+t} = \frac{x}{1+t} \tag{89}$$

Which is the exact solution to this particular nonlocal initial-boundary value problem.

Example (5.2.7) We consider the nonlinear nonlocal inhomogeneous initial-boundary value [25],

$$u_t - u_{xx} = -{u_x}^2 + u^2, \quad 0 \le x \le 1, \;\; t \ge 0, \tag{90}$$

Subject to the initial condition:

$$u(0,x) = e^x \tag{91}$$

And the nonlocal inhomogeneous boundary conditions of integral type:

$$\begin{cases} \int_0^1 u(t,x)dx & = e^t(e-1), \\ \int_0^1 \frac{1}{2} u(t,x)dx & = \frac{1}{2} e^t(e-1), \end{cases} \quad (92)$$

Where

$$\begin{cases} a & = 0, \\ b = & 1, \\ p(t,x) & = 1, \\ q(t,x) & = 0, \\ f(t,x) & = 0, \\ \propto (x) & = e^x, \\ \beta(t) & = \frac{3}{2} e^t(e-1), \\ F(u) = & -u_x{}^2 + u^2, \\ \varphi(x) & = e^x. \end{cases} \quad (93)$$

According to Eq. (63), we have:

$$\begin{cases} g(t,x) & = 0, \\ h_1(x) & = \frac{3}{2} e^x, \\ r(t,x) & = 0, \\ s(t,x) & = 0, \\ Rv & = v_{xxx}. \end{cases} \quad (94)$$

And the nonlinear term:

$$N(v) = -\frac{1}{\varphi(x)} (v_{xx})^2 + \frac{1}{\varphi(x)} (v_x)^2 \quad (95)$$

That is

$$N(v) = \frac{2}{3} ((v_x)^2 - (v_{xx})^2) \quad (96)$$

Thus

$$N(v) = \sum_{n=0}^{\infty} p^n H_n(v)$$

The first few components of He's polynomials, are given by:

$$\begin{aligned} H_0(v) &= {(v_0)_x}^2 - {(v_0)_{xx}}^2 \\ H_1(v) &= 2(v_0)_x(v_1)_x - 2(v_0)_{xx}(v_1)_{xx} \\ &\vdots \end{aligned} \tag{97}$$

The recursion scheme Eq. (77) produces a rapidly convergent series as;

$$p^0: v_0(t,x) = \frac{1}{2}\left[\int_a^x h_1(x)dx + \beta(t) - \int_x^b h_1(x)dx + \int_a^x \int_0^t g(t,x)\, dtdx - \int_x^b \int_0^t g(t,x)\, dtdx\right]$$

$$v_0(t,x) = \frac{1}{2}\left[\int_0^x \frac{3}{2}e^x dx + \frac{3}{2}e^t(e-1) - \int_x^1 \frac{3}{2}e^x dx\right]$$

Therefore

$$p^0: \left\{ v_0(t,x) = \frac{3}{2}e^x - \frac{3}{4}(1-e)(e^t+1) \right.$$

$$p^1: \left\{ \begin{aligned} v_1(t,x) &= \frac{1}{2}\int_0^x \int_0^t [(v_0)_{xxx} + H_0(v)]\, dtdx \\ &\quad -\frac{1}{2}\int_x^1 \int_0^t [(v_0)_{xxx} + H_0(v)]\, dtdx \\ &= \frac{3}{2}e^x t - \frac{3}{4}t(1-e) \end{aligned} \right.$$

$$p^2: \left\{ \begin{aligned} v_2(t,x) &= \frac{1}{2}\int_0^x \int_0^t \big[[(v_1)_{xxx} + H_0(v)] + H_1(v)\big]\, dtdx \\ &\quad -\frac{1}{2}\int_x^1 \int_0^t \big[[(v_1)_{xxx} + H_0(v)] + H_1(v)\big]\, dtdx \\ &= \frac{3}{2}e^x \frac{t^2}{2!} - \frac{3}{8}t^2(1-e) \end{aligned} \right.$$

Proceeding in a similar manner, we have:

$$p^3: v_3(t,x) = \frac{3}{2}e^x \frac{t^3}{3!} - \frac{3}{24}t^3(1-e)$$

$$p^4: v_4(t,x) = \frac{3}{2}e^x \frac{t^4}{4!} - \frac{3}{98}t^4(1-e)$$

$$\vdots$$

And so on. Consequently, the intermediate solution is given as;

$$v(t,x) = \frac{3}{2}e^x\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right) + \frac{3}{4}(1-e)\left(1 - t + \frac{t^2}{2!} - \cdots\right) \quad (98)$$

Or in a closed form as;

$$v(t,x) = \frac{3}{2}e^{x+t} + \frac{3}{4}(1-e)e^t \quad (99)$$

Returning to the original dependent variable by Eq. (55), we obtain:

$$u(t,x) = \frac{v_x(t,x)}{\varphi(x)} = \frac{\frac{3}{2}e^{x+t}}{\frac{3}{2}} = e^{x+t} \quad (100)$$

Which is the exact solution to this particular nonlocal initial-boundary value problem.

5.3 The linear Hyperbolic Equation with Nonlocal Conditions

In this section, we will employ the homotopy perturbation method to solve the nonlocal initial-boundary value problems for linear variable-coefficient hyperbolic partial differential equations. For cases, including an input function or additional linear terms, the homotopy perturbation method remains the method of choice to easily and quickly calculate solutions. Several examples will be presented in the sequel.

Now we consider the inhomogeneous linear hyperbolic partial differential equation

$$u_{tt} - p(t,x)u_{xx} + q(t,x)u = f(t,x), \quad a \le x \le b, \qquad t \ge 0, \tag{101}$$

Subject to the initial conditions:

$$u(0,x) = \propto_1 (x), \qquad u_t(0,x) = \propto_2 (x) \tag{102}$$

And the nonlocal inhomogeneous boundary conditions of integral type:

$$\int_a^b \varphi_1(x)u(t,x)dx = \beta_1(t) \int_a^b \varphi_2(x)u(t,x)dx = \beta_2(t), \tag{103}$$

Where $\varphi_i(x)$, $\beta_i(t)$, $i = 1,2$ and $\propto (x)$ are specified as continuous functions. we begin our approach by converting Eqs. (101)– (103) To a local initial-boundary value problem by introducing a new function $v(t,x)$ such that:

$$v(t,x) = \int_a^x \varphi(x)u(t,x)dx \tag{104}$$

Where $\varphi(x) = \varphi_1(x) + \varphi_2(x)$

Hence we have:

$$v_x(t,x) = \varphi(x)u(t,x) \tag{105}$$

And

$$u(t,x) = \frac{v_x(t,x)}{\varphi(x)} \tag{106}$$

From Eq. (106) we have:

$$u_t(t,x) = \frac{v_{tx}(t,x)}{\varphi(x)} \tag{107}$$

$$u_{tt}(t,x) = \frac{v_{ttx}(t,x)}{\varphi(x)} \tag{108}$$

$$u_x(t,x) = \frac{1}{\varphi(x)} v_x(t,x) + \left(\frac{1}{\varphi(x)}\right)' v_x(t,x) \tag{109}$$

And

$$u_{xx}(t,x) = \left(\frac{1}{\varphi(x)}\right)'' v_x(t,x) + 2\left(\frac{1}{\varphi(x)}\right)' v_{xx}(t,x) + \left(\frac{1}{\varphi(x)}\right) v_{xxx}(t,x) \quad (110)$$

Substituting Eqs.(106)– (110) into Eq. (101) we deduce:

$$v_{ttx}(t,x) - \left(p(t,x)\left(\frac{1}{\varphi(x)}\right)'' \varphi(x) - q(t,x)\right) v_x(t,x)$$

$$-2p(t,x)\left(\frac{1}{\varphi(x)}\right)' \varphi(x) v_{xx}(t,x) - p(t,x) v_{xxx}(t,x) = \varphi(x) f(t,x) \quad (111)$$

By using Eq. (104) we get;

$$\begin{cases} v_x(0,x) = \varphi(x)u(0,x) = \varphi(x) \propto_1 (x) = h_1(x), \\ v_{tx}(0,x) = \varphi(x)u_t(0,x) = \varphi(x) \propto_2 (x) = h_2(x) \\ v(t,a) = 0, \\ v(t,b) = \int_a^b \varphi(x)u(t,x)dx = \beta(t)\,, \end{cases} \quad (112)$$

Where $\beta(t) = \beta_1(t) + \beta_2(t)$

Thus we deduce:

Lemma (5.3.8) The general nonlocal initial-boundary value problem (101)–(103) can always be reduced to a local initial-boundary value problem of the form

$$\begin{cases} v_{ttx} + r(t,x)v_x + s(t,x)v_{xx} - p(t,x)v_{xxx} = g(t,x), \\ v_x(0,x) = h_1(x), \quad v_{tx}(0,x) = h_2(x) \\ v(t,a) = 0 \quad and \quad v(t,b) = \beta(t), \end{cases} \quad (113)$$

Where

$$
\begin{cases}
r(t,x) = -p(t,x)\left(\frac{1}{\varphi(x)}\right)'' \varphi(x) + q(t,x), \\
s(t,x) \qquad = -2p(t,x)\left(\frac{1}{\varphi(x)}\right)' \varphi(x), \\
\beta(t) \qquad = \beta_1(t) + \beta_2(t), \\
h_1(x) \qquad = \varphi(x) \propto_1 (x), \\
h_2(x) \qquad = \varphi(x) \propto_2 (x), \\
g(t,x) \qquad = \varphi(x) f(t,x).
\end{cases} \tag{114}
$$

A solution of this problem will lead to a solution of the given original problem, where $u(t,x)$ is given by Eq. (106). Based on the homotopy perturbation method, we write

$$v_{ttx} + r(t,x)v_x + s(t,x)v_{xx} - p(t,x)v_{xxx} = g(t,x) \tag{115}$$

In operator-theoretic notation as;

$$Av = g + Rv \tag{116}$$

Where

$$Av = v_{ttx} \; and \quad Rv = -r(t,x)v_x - s(t,x)v_{xx} + p(t,x)v_{xxx} \tag{117}$$

We conveniently define the inverse linear operator as;

$$A^{-1}{}_{a,ttx} = \int_a^x \int_0^t \int_0^t (.)dt\,dtdx \tag{118}$$

Applying the inverse linear operator $A^{-1}{}_{a,ttx}$ to Eq. (116), and taking into account that $v_x(0,x) = h_1(x)$, $v_{tx}(0,x) = h_2(x)$ and $v(t,a) = 0$, we obtain:

$$v(t,x) = \int_a^x h_1(x)dx + t\int_a^x h_2(x)dx + A^{-1}{}_{a,ttx}\, g + A^{-1}{}_{a,ttx} Rv \tag{119}$$

Proceeding as before, applying the inverse linear operator to both sides of Eq. (116), and taking into account that $v_x(0,x) = h_1(x)$, $v_{tx}(0,x) = h_2(x)$ and $v(t,b) = \beta(t)$, we obtain:

$$A^{-1}{}_{b,ttx}(.) = \int_x^b \int_0^t \int_0^t (.)\, dtdtdx$$

$$v(t,x) = v(t,b) - \int_x^b h_2(x)dx - t\int_x^b h_2(x)dx - A^{-1}{}_{b,ttx}\, g - A^{-1}{}_{b,ttx}Rv.$$

Thus

$$v(t,x) = \beta(t) - \int_x^b h_2(x)dx - t\int_x^b h_2(x)dx - A^{-1}{}_{b,ttx}\, g - A^{-1}{}_{b,ttx}Rv \quad (120)$$

Adding the relations in Eq. (118) and Eq. (120) together, and then dividing by two, we obtain the solution as the equal-weight average

$$v(t,x) = \frac{1}{2}\left[\int_a^x h_1(x)dx + t\int_a^x h_2(x)dx + \beta(t) - \left[\int_x^b h_1(x)dx + t\int_x^b h_2(x)dx\right]\right]$$

$$+\frac{1}{2}\left[A^{-1}{}_{a,ttx}g - A^{-1}{}_{b,ttx}g + A^{-1}{}_{a,ttx}Rv - A^{-1}{}_{b,ttx}Rv\right] \quad (121)$$

Now, we apply the homotopy perturbation method

$$v(x,t) = \sum_{n=0}^{\infty} p^n v_n(x,t) \quad (122)$$

Substituting Eq. (122) into Eq. (121) we get;

$$\sum_{n=0}^{\infty} p^n v_n(x,t) = \frac{1}{2}\left[A^{-1}{}_{a,ttx}g - A^{-1}{}_{b,ttx}g\right]$$

$$+\frac{1}{2}\left[\int_a^x h_1(x)dx + t\int_a^x h_2(x)dx + \beta(t) - \left[\int_x^b h_1(x)dx + t\int_x^b h_2(x)dx\right]\right]$$

$$+\frac{1}{2}p\left[A^{-1}{}_{a,ttx} - A^{-1}{}_{b,ttx}\right]R\sum_{n=0}^{\infty} p^n v_n(x,t) \quad (123)$$

Comparing the coefficient of like powers of p, we have:

$$p^0: v_0(t,x) = \frac{1}{2}\left[\int_a^x h_1(x)dx + t\int_a^x h_2(x)dx + \beta(t)\right]$$

$$-\frac{1}{2}\left[\int_x^b h_1(x)dx + t\int_x^b h_2(x)dx - \left(A^{-1}{}_{a,ttx}g - A^{-1}{}_{b,ttx}g\right)\right]$$

$$p^1: v_1(t,x) = \frac{1}{2}\left[A^{-1}{}_{a,ttx} - A^{-1}{}_{b,ttx}\right]Rv_0 \tag{124}$$

$$p^2: v_2(t,x) = \frac{1}{2}\left[A^{-1}{}_{a,ttx} - A^{-1}{}_{b,ttx}\right]Rv_1$$

Proceeding in a similar manner, we have:

$$p^3: v_3(t,x) = \frac{1}{2}\left[A^{-1}{}_{a,ttx} - A^{-1}{}_{b,ttx}\right]Rv_2$$

$$p^4: v_4(t,x) = \frac{1}{2}\left[A^{-1}{}_{a,tx} - A^{-1}{}_{b,tx}\right]Rv_3$$

So that the solution $v(x,t)$ is given by:

$$v(x,t) = v_0(t,x) + v_1(t,x) + v_1 + (t,x) \dots \tag{125}$$

Where the term v_0 is to be determined from the initial and boundary conditions.

Once the function $v(t,x)$ is calculated, we can return to the original dependent variable $u(t,x)$ by Eq. (106).

Example (5.3.9) We consider the nonlocal linear homogeneous initial-boundary value problem [77],

$$u_{tt} - u_{xx} = 0, \quad 0 \le x \le 1, \ t \ge 0 \tag{126}$$

Subject to the initial conditions

$$u(0,x) = x^2, \qquad u_t(0,x) = 0 \tag{127}$$

And the nonlocal inhomogeneous boundary conditions of integral type:

$$\begin{cases} \int_0^1 u(t,x)dx = \frac{1}{3} + t^2, \\ \int_0^1 xu(t,x)dx = \frac{1}{4} + \frac{t^2}{2}, \end{cases} \tag{128}$$

Where

$$\begin{cases} a = 0, \\ b = 1, \\ p(t,x) = 1, \\ q(t,x) = 0, \\ f(t,x) = 0, \\ \propto_1 (x) = x^2, \\ \propto_2 (x) = 0, \\ \beta(t) = \frac{7}{12} + \frac{3}{2}t^2, \\ \varphi(x) = 1 + x. \end{cases} \tag{129}$$

According to Eq. (113), we have:

$$\begin{cases} g(t,x) = 0, \\ h_1(x) = x^3 + x^2, \\ h_2(x) = 0, \\ r(t,x) = \frac{-2}{(1+x)^2}, \\ s(t,x) = \frac{2}{1+x}, \\ Rv = \frac{2}{(1+x)^2} v_x - \frac{2}{1+x} v_{xx} + v_{xxx}. \end{cases} \tag{130}$$

The recursion scheme Eq. (124) produces a rapidly convergent series as;

$$p^0 : \begin{cases} v_0(t,x) = \frac{1}{2}\left[\int_0^x (x^3 + x^2)dx + \frac{7}{12} + \frac{3}{2}t^2 - \int_x^1 (x^3 + x^2)dx\right] \\ \qquad = \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{3}{4}t^2 \end{cases}$$

$$p^1:\begin{cases} v_1(t,x) = \frac{1}{2}\left[\int_0^x \int_0^t \int_0^t \left[\frac{2}{(1+x)^2}(v_0)_x - \frac{2}{1+x}(v_0)_{xx} + (v_0)_{xxx}\right] dtdtdx\right] \\ -\frac{1}{2}\left[\int_x^1 \int_0^t \int_0^t \left[\frac{2}{(1+x)^2}(v_0)_x - \frac{2}{1+x}(v_0)_{xx} + (v_0)_{xxx}\right] dtdtdx\right] \\ = \frac{1}{2}x^2t^2 + xt^2 \end{cases}$$

$$p^2:\begin{cases} v_2(t,x) = \frac{1}{2}\left[\int_0^x \int_0^t \int_0^t \left[\frac{2}{(1+x)^2}(v_1)_x - \frac{2}{1+x}(v_1)_{xx} + (v_1)_{xxx}\right] dtdtdx\right] \\ -\frac{1}{2}\left[\int_x^1 \int_0^t \int_0^t \left[\frac{2}{(1+x)^2}(v_1)_x - \frac{2}{1+x}(v_1)_{xx} + (v_1)_{xxx}\right] dtdtdx\right] \\ = 0, \end{cases}$$

$$v_k(t,x) = 0, \quad \forall k \geq 2$$

Thus, the solution is given by;

$$v(t,x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2t^2 + xt^2 + \frac{3}{4}t^2 \tag{131}$$

Returning to the original dependent variable by Eq. (106), we obtain:

$$u(t,x) = \frac{v_x(t,x)}{\varphi(x)} = \frac{x^2(1+x) + t^2(1+x)}{1+x} = x^2 + t^2 \tag{132}$$

Which is the exact solution to this particular nonlocal initial-boundary value problem.

Example (5.3.10) We consider the nonlocal linear homogeneous initial-boundary value problem [78],

$$u_{tt} - u_{xx} = 0, \quad 0 \leq x \leq 1, \; t \geq 0 \tag{133}$$

Subject to the initial conditions:

$$u(0,x) = cos(\pi x), \qquad u_t(0,x) = 0, \tag{134}$$

And the nonlocal inhomogeneous boundary conditions of integral type:

$$\begin{cases} \int_0^1 u(t,x)dx = 0, \\ \int_0^1 \frac{1}{2} u(t,x)dx = 0, \end{cases} \tag{135}$$

Where

$$\begin{cases} a = 0, \\ b = 1, \\ p(t,x) = 1, \\ q(t,x) = 0, \\ f(t,x) = 0, \\ \propto_1 (x) = cos(\pi x), \\ \propto_2 (x) = 0, \\ \beta(t) = 0, \\ \varphi(x) = \frac{3}{2}. \end{cases} \tag{136}$$

According to Eq. (113), we have:

$$\begin{cases} g(t,x) = 0, \\ h_1(x) = \frac{3}{2} cos(\pi x), \\ h_2(x) = 0, \\ r(t,x) = 0, \\ s(t,x) = 0, \\ Rv = v_{xxx}. \end{cases} \tag{137}$$

The recursion scheme Eq. (124) produces a rapidly convergent series as;

$$p^0: v_0(t,x) = \frac{1}{2}\left[\int_0^x \frac{3}{2} cos(\pi x)dx - \int_x^1 \frac{3}{2} cos(\pi x)dx\right] = \frac{3}{2} sin(\pi x)$$

$$p^1: v_1(t,x) = \begin{cases} \frac{1}{2}\left[\int_0^x \int_0^t \int_0^t (v_0)_{xxx} dtdtdx - \int_x^1 \int_0^t \int_0^t (v_0)_{xxx} dtdtdx\right] \\ = -\frac{3}{2}\frac{(\pi t)^2}{2!} sin(\pi x) \end{cases}$$

$$p^2: v_2(t,x) = \begin{cases} \dfrac{1}{2}\left[\displaystyle\int_0^x \int_0^t \int_0^t (v_1)_{xxx} dtdtdx - \int_x^1 \int_0^t \int_0^t (v_1)_{xxx} dtdtdx\right] \\ \qquad = \dfrac{3}{2}\dfrac{(\pi t)^4}{4!} sin(\pi x) \end{cases}$$

Proceeding in a similar manner, we have:

$$p^3: v_3(t,x) = -\frac{3}{2}\frac{(\pi t)^6}{6!} sin(\pi x)$$

$$p^4: v_4(t,x) = \frac{3}{2}\frac{(\pi t)^8}{8!} sin(\pi x)$$

And so on. Consequently, the intermediate solution is given as;

$$v(t,x) = \frac{3}{2} sin(\pi x)\left(1 - \frac{(\pi t)^2}{2!} + \frac{(\pi t)^4}{4!} + \frac{(\pi t)^6}{6!} + \cdots\right) \quad (138)$$

Or in a closed form as;

$$v(t,x) = \frac{3}{2} sin(\pi x)\, cos(\pi t) \quad (139)$$

Returning to the original dependent variable by Eq. (106), we obtain:

$$u(t,x) = \frac{v_x(t,x)}{\varphi(x)} = cos(\pi x)\, cos(\pi t) \quad (140)$$

Which is the exact solution to this particular nonlocal initial-boundary value problem.

5.4 The Nonlinear Hyperbolic Equation with Nonlocal Conditions

In this section, we will employ the homotopy perturbation method to solve the nonlocal initial-boundary value problems for nonlinear variable-coefficient hyperbolic partial differential equations. For cases, including an input function or additional nonlinear terms, the homotopy perturbation method remains the method of choice to easily and quickly calculate solutions. Several examples will be presented in the sequel.

Now we consider the nonlinear hyperbolic partial differential equation

$$u_{tt} - p(t,x)u_{xx} + q(t,x)u = f(t,x) + F(u), \quad a \le x \le b, \qquad t \ge 0, \tag{141}$$

Subject to the initial conditions:

$$u(0,x) = \propto_1 (x), \quad u_t(0,x) = \propto_2 (x) \tag{142}$$

And the nonlocal inhomogeneous boundary conditions of integral type

$$\int_a^b \varphi_1(x)u(t,x)dx = \beta_1(t) \int_a^b \varphi_2(x)u(t,x)dx = \beta_2(t), \tag{143}$$

Where $\varphi_i(x)$, $\beta_i(t)$, $i = 1,2$ and $\propto(x)$ are specified as continuous functions. we begin our approach by converting Eqs. (141)– (143) To a local initial-boundary value problem by introducing a new function $v(t,x)$ such that:

$$v(t,x) = \int_a^x \varphi(x)u(t,x)dx \tag{144}$$

Where $\varphi(x) = \varphi_1(x) + \varphi_2(x)$

Hence we have:

$$v_x(t,x) = \varphi(x)u(t,x) \tag{145}$$

And

$$u(t,x) = \frac{v_x(t,x)}{\varphi(x)} \tag{146}$$

From Eq. (106) we have:

$$u_t(t,x) = \frac{v_{tx}(t,x)}{\varphi(x)} \tag{147}$$

$$u_{tt}(t,x) = \frac{v_{ttx}(t,x)}{\varphi(x)} \tag{148}$$

$$u_x(t,x) = \frac{1}{\varphi(x)} v_x(t,x) + \left(\frac{1}{\varphi(x)}\right)' v_x(t,x) \qquad (149)$$

$$u_{xx}(t,x) = \left(\frac{1}{\varphi(x)}\right)'' v_x(t,x) + 2\left(\frac{1}{\varphi(x)}\right)' v_{xx}(t,x) + \left(\frac{1}{\varphi(x)}\right) v_{xxx}(t,x) \qquad (150)$$

Substituting Eqs.(146)– (150) into Eq. (141) we deduce

$$v_{ttx}(t,x) - \left(p(t,x)\left(\frac{1}{\varphi(x)}\right)'' \varphi(x) - q(t,x)\right) v_x(t,x)$$

$$-2p(t,x)\left(\frac{1}{\varphi(x)}\right)' \varphi(x) v_{xx}(t,x) - p(t,x) v_{xxx}(t,x)$$

$$= \varphi(x) f(t,x) + \varphi(x) F(u) \qquad (151)$$

By using Eq. (144) we deduce the initial conditions and boundary conditions as follows;

$$\begin{cases} v_x(0,x) = \varphi(x) u(0,x) = \varphi(x) \propto_1 (x) = h_1(x), \\ v_{tx}(0,x) = \varphi(x) u_t(0,x) = \varphi(x) \propto_2 (x) = h_2(x) \\ v(t,a) = 0, \\ v(t,b) = \int_a^b \varphi(x) u(t,x) dx = \beta(t), \end{cases} \qquad (152)$$

Where $\beta(t) = \beta_1(t) + \beta_2(t)$

Thus we deduce:

Lemma (5.4.11) The general nonlocal initial-boundary value problem (141)–(143) can always be reduced to a local initial-boundary value problem of the form

$$\begin{cases} v_{ttx} + r(t,x) v_x + s(t,x) v_{xx} - p(t,x) v_{xxx} = g(t,x) + N(v), \\ v_x(0,x) = h_1(x), \quad v_{tx}(0,x) = h_2(x) \\ v(t,a) = 0 \quad and \quad v(t,b) = \beta(t), \end{cases} \qquad (153)$$

Where

$$\begin{cases} r(t,x) = -p(t,x)\left(\frac{1}{\varphi(x)}\right)'' \varphi(x) + q(t,x), \\ s(t,x) \qquad = -2p(t,x)\left(\frac{1}{\varphi(x)}\right)' \varphi(x), \\ \beta(t) \qquad = \beta_1(t) + \beta_2(t), \\ h_1(x) \qquad = \varphi(x) \propto_1 (x), \\ h_2(x) \qquad = \varphi(x) \propto_2 (x), \\ g(t,x) \qquad = \varphi(x) f(t,x). \\ N(v) = \qquad \varphi(x) F\left(\frac{v_x(t,x)}{\varphi(x)}\right) \end{cases} \tag{154}$$

A solution of this problem will lead to a solution of the given original problem, where $u(t,x)$ is given by Eq. (146). Based on the homotopy perturbation method, we write

$$v_{ttx} + r(t,x)v_x + s(t,x)v_{xx} - p(t,x)v_{xxx} = g(t,x) + N(v)$$

In operator-theoretic notation as;

$$Av = g + Rv + N \tag{155}$$

Where

$$Av = v_{ttx}\,, \quad Rv = -r(t,x)v_x - s(t,x)v_{xx} + p(t,x)v_{xxx} \tag{156}$$

We conveniently define the inverse linear operator as;

$$A^{-1}{}_{a,ttx} = \int_a^x \int_0^t \int_0^t (.)dt\,dtdx \tag{157}$$

Applying Eq. (157) to Eq. (155), and taking into account that $v_x(0,x) = h_1(x)$, $v_{tx}(0,x) = h_2(x)$ and $v(t,a) = 0$, we obtain:

$$v(t,x) = \left\{\int_a^x h_1(x)dx + t\int_a^x h_2(x)dx + A^{-1}{}_{a,ttx}\, g \right.$$

$$+A^{-1}{}_{a,ttx}Rv + A^{-1}{}_{a,ttx}N \tag{158}$$

Proceeding as before, applying the inverse linear operator to both sides of Eq. (115), and taking into account that $v_x(0,x) = h_1(x)$, $v_{tx}(0,x) = h_2(x)$ and $v(t,b) = \beta(t)$, we obtain:

$$A^{-1}{}_{b,ttx}(.) = \int_x^b \int_0^t \int_0^t (.)\, dtdtdx \tag{159}$$

$$v(t,x) = v(t,b) - \int_x^b h_2(x)dx - t\int_x^b h_2(x)dx - A^{-1}{}_{b,ttx}\, g$$
$$-A^{-1}{}_{b,ttx} Rv - A^{-1}{}_{b,ttx} N$$

Thus

$$v(t,x) = \beta(t) - \int_x^b h_2(x)dx - t\int_x^b h_2(x)dx - A^{-1}{}_{b,ttx}\, g$$
$$-A^{-1}{}_{b,ttx} Rv - A^{-1}{}_{b,ttx} N \tag{160}$$

Adding the relations in Eq. (158) and Eq. (160) together, and then dividing by two, we obtain the solution as the equal-weight average

$$v(t,x) = \frac{1}{2}\left[\int_a^x h_1(x)dx + t\int_a^x h_2(x)dx + \beta(t) - \left[\int_x^b h_1(x)dx + t\int_x^b h_2(x)dx\right]\right]$$
$$+\frac{1}{2}\left[A^{-1}{}_{a,ttx} g - A^{-1}{}_{b,ttx} g\right] + \frac{1}{2}\left[A^{-1}{}_{a,ttx} - A^{-1}{}_{b,ttx}\right][Rv - N(v)] \tag{161}$$

Where the nonlinear term $N(v)$ is assumed to be an analytic function and can be expressed by an infinite series given in the form:

$$N\big(v(t,x)\big) = \sum_{n=0}^{\infty} p^n H_n(v) \tag{162}$$

For some He's polynomials H_n (see [80, 81] that are given by:

$$H_n(v_0, \dots, v_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{\infty} p^i v_i\right)_{p=0} \quad, n = 0, 1, 2, 3 \dots \tag{163}$$

Now, we apply the homotopy perturbation method

$$v(x,t) = \sum_{n=0}^{\infty} p^n v_n(x,t) \tag{164}$$

Substituting Eq. (164) into Eq. (163) we get;

$$\sum_{n=0}^{\infty} p^n v_n(x,t) = \frac{1}{2}\left[A^{-1}{}_{a,ttx}g - A^{-1}{}_{b,ttx}g\right]$$

$$+\frac{1}{2}\left[\int_a^x h_1(x)dx + t\int_a^x h_2(x)dx + \beta(t) - \left[\int_x^b h_1(x)dx + t\int_x^b h_2(x)dx\right]\right]$$

$$+\frac{1}{2}p\left[A^{-1}{}_{a,ttx} - A^{-1}{}_{b,ttx}\right]\left(R\sum_{n=0}^{\infty} p^n v_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(v)\right) \tag{165}$$

Comparing the coefficient of like powers of p, we have:

$$p^0: v_0(t,x) = \frac{1}{2}\left[\int_a^x h_1(x)dx + t\int_a^x h_2(x)dx + \beta(t)\right]$$

$$-\frac{1}{2}\left[\int_x^b h_1(x)dx + t\int_x^b h_2(x)dx - \left(A^{-1}{}_{a,ttx}g - A^{-1}{}_{b,ttx}g\right)\right]$$

$$p^1: v_1(t,x) = \frac{1}{2}\left[A^{-1}{}_{a,ttx} - A^{-1}{}_{b,ttx}\right]\left(Rv_0 + H_0(v)\right) \tag{166}$$

$$p^2: v_2(t,x) = \frac{1}{2}\left[A^{-1}{}_{a,ttx} - A^{-1}{}_{b,ttx}\right](Rv_1 + H_1(v))$$

Proceeding in a similar manner, we have:

$$p^3: v_3(t,x) = \frac{1}{2}\left[A^{-1}{}_{a,ttx} - A^{-1}{}_{b,ttx}\right](Rv_2 + H_2(v))$$

$$p^4: v_4(t,x) = \frac{1}{2}\left[A^{-1}{}_{a,tx} - A^{-1}{}_{b,tx}\right](Rv_3 + H_3(v))$$

$$\vdots$$

So that the solution $v(x,t)$ is given by:

$$v(x,t) = v_0(t,x) + v_1(t,x) + v_1 + (t,x) \dots \quad (167)$$

Where the term v_0 is to be determined from the initial and boundary conditions.

Once the function $v(t,x)$ is calculated, we can return to the original dependent variable $u(t,x)$ by Eq. (156).

Example (5.4.12) We consider the nonlocal linear homogeneous initial-boundary value problem [79],

$$u_{tt} - xu_{xx} = 1 - u^2, \quad 0 \le x \le 1, \ t \ge 0, \quad (168)$$

Subject to the initial conditions:

$$u(0,x) = 1, u_t(0,x) = 0 \quad (169)$$

And the nonlocal inhomogeneous boundary conditions of integral type:

$$\begin{cases} \int_0^1 u(t,x)dx & = 1, \\ \int_0^1 (x-1)u(t,x)dx & = -\frac{1}{2}, \end{cases} \quad (170)$$

Where

$$\begin{cases} a & = 0, \\ b = & 1, \\ p(t,x) & = x, \\ q(t,x) & = 0, \\ f(t,x) & = 1, \\ \propto_1 (x) & = 1, \\ \propto_2 (x) & = 0, \\ \beta(t) & = \frac{1}{2}, \\ \varphi(x) & = x. \end{cases} \quad (171)$$

According to Eq. (154), we have:

$$\begin{cases} g(t,x) & = x, \\ h_1(x) & = x, \\ h_2(x) & = 0, \\ r(t,x) & = \dfrac{-2}{x^2}, \\ s(t,x) & = \dfrac{1}{x}, \\ Rv & = \dfrac{2}{x}v_x - v_{xx} + xv_{xxx}. \end{cases} \tag{172}$$

And the nonlinear term

$$N(v) = -\varphi(x)\left(\frac{v_x}{\varphi(x)}\right)^2 \tag{173}$$

That is

$$N(v) = -\frac{1}{x}(v_x)^2 \tag{174}$$

Thus

$$N(v) = \sum_{n=0}^{\infty} p^n H_n(v)$$

The first few components of He's polynomials, are given by:

$$\begin{aligned} H_0(v) &= (v_0)_x{}^2 \\ H_1(v) &= 2(v_0)_x(v_1)_x \\ H_2(v) &= (v_1)_x{}^2 + 2(v_0)_x(v_2)_x \end{aligned} \tag{175}$$

The recursion scheme Eq. (166) produces a rapidly convergent series as;

$$p^0\colon \begin{cases} v_0(t,x) = \dfrac{1}{2}\left[\displaystyle\int_0^x x dx - \frac{1}{2} - \int_x^1 x dx + \int_x^1 \int_x^1 \int_0^t x dt dt dx - \int_x^1 \int_x^1 \int_0^t x dt dt dx\right] \\ \qquad = \dfrac{1}{2}x^2 - \dfrac{1}{2} \end{cases}$$

$$p^1:\begin{cases} v_1(t,x) = \frac{1}{2}\left[\int_0^x \int_0^t \int_0^t \left[\frac{2}{x}(v_0)_x - (v_0)_{xx} + x(v_0)_{xxx} + H_0(v)\right] dtdtdx\right] \\ \qquad -\frac{1}{2}\left[\int_x^1 \int_0^t \int_0^t \frac{2}{x}(v_0)_x - (v_0)_{xx} + x(v_0)_{xxx} + H_0(v)\right] dtdtdx \\ \qquad = 0 \end{cases}$$

$v_k(t,x) = 0, \quad \forall k \geq 1$

Thus, the solution is given by:

$$v(t,x) = \frac{1}{2}x^2 - \frac{1}{2} \tag{176}$$

Returning to the original dependent variable by Eq. (106), we obtain:

$$u(t,x) = \frac{v_x(t,x)}{\varphi(x)} = 1 \tag{177}$$

Which is the exact solution to this particular nonlocal initial-boundary value problem.

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