Chapter 4
Symmetries and Lie algebras

4.1 Introduction

Symmetry is a characteristic of geometrical shapes, equations and other objects; we say that such an object is symmetric with respect to a given operation if this operation, when applied to the object, does not appear to change it. The three main symmetrical operations are reflection, rotation and translation. A reflection "flips" an object over a line, inverting it as if in a mirror. A rotation rotates an object using a point as its center. A translation "slides" an object from one area to another by a vector. Even more complex operations on a geometric object, like shrinking or shape warping, can be reduced to the operation of translation of every point within the object. Symmetry occurs in geometry, mathematics, physics, biology, art, literature (palindromes), etc.\footnote{\textcopyright file:localhost/Noether’s 20%/Symmetric group/Wikipedia/20%/free/com” Wikipedia paper.}

Although two objects with great similarity appear the same, they must logically be different. For example, if one rotates an equilateral triangle around its center 120 degrees, it will appear the same as it was before the rotation to an observer. In theoretical Euclidean geometry, such a rotation would be unrecognizable from its previous form. In reality however, each corner of any equilateral triangle composed of matter must be composed of separate molecules in separate locations. Symmetry therefore, is a matter of similarity instead of sameness. The difficulty for an intelligence to differentiate such a seemingly exact similarity might for a responsible for the mild altered state of consciousness one gets by observing intricate patterns based on symmetry.

4.1.1 Symmetry in Geometry

The object with the most symmetry is empty space because any part of it can be rotate, reflected or translated without apparent change.

The most familiar and conventionally taught type of symmetry is the left-right or mirror image symmetry exhibited for instance by the letter \( T \): when this letter is reflected along a vertical axis, it appears the same. An equilateral triangle exhibits such a reflection symmetry along three axes, and in addition it shows rotational symmetry: if rotated by 120 or 240 degrees, it remains unchanged. An instance of a shape which exhibits only rotational but no reflectional symmetry is the swastika. The German geometer Felix Klein enunciated a very influential Erlangen programme in 1872, suggesting symmetry as unifying and organizing principle in geometry (at a time when that was read 'geometries'). This is a broad rather than
deep principle. Initially it led to interest in the group attached to geometries, and the slogan transformation geometry (an aspect of the New Math, but hardly controversial in modern mathematical practice). By now it has been applied in numerous forms, as kind of standard attack on problems.

A fractal, coined by Mandelbrot is symmetry involving scale. For example an equilateral triangle can be shrunk so that each of its sides is one third the length of the original's sides. These smaller triangles can be rotated and translated until they are adjacent and in the center of each of the larger triangle's lines. The smaller triangles can repeat the process, resulting in even smaller triangle's on their sides. Fascinating intricate structures can be created by repeating such scaling symmetrical operations many times.

4.1.2. Symmetry in Mathematics

An example of a mathematical expression exhibiting symmetry is \( a^2 c + 3ab + b^2 c \). If \( a \) and \( b \) are exchanged, the expression remains unchanged due to the commutatively of addition and multiplication.

In mathematics, one studies the symmetry of a given object by collecting all the operations that leave the object unchanged. These operations form a group. For a geometrical object, this is known as its symmetry group; for an algebraic object one uses the term auto Orphism group. The whole subject of galios Theory deals with well-hidden symmetries of fields.

4.1.3. Generalization of Symmetry

If we have a given set of objects with some structure, then it is possible for symmetry to merely convert only one object into another instead of acting upon all possible objects simultaneously. This requires a generalization from the concept of symmetry groups to that of a groupoid.

4.1.4. Symmetry in Physics

The generalization of symmetry in physics to mean invariance under any kind of transformation has become one of the most powerful tools of theoretical physics. See Noether's theorem for more details. This has led to group theory being one of the areas of mathematics most studied by physicists.

4.1.5. Symmetry in the arts and crafts

You can find the use of symmetry across a wide variety of arts and crafts. Symmetry has long been a predominant design element in architecture; prominent examples include the Leaning
Tower of Pisa, Monticello, the Astrodome, the Sydney Opera House, Gothic church windows, and the Pantheon. Symmetry is used in the design of all over floor plan of buildings as well as the design of individual building elements such as doors, windows, floors, frieze work, and ornamentation; many facades adhere to bilateral symmetry.

4.1.6. Symmetry Group

The symmetry group of an object (image, signal, etc) is the group of all isometries under which it's invariant with composition as the operation. It's a subgroup of the isometry group of the space concerned. If not stated otherwise; this article considers symmetry groups in Euclidean geometry.

The symmetry group is sometimes also called full symmetry group in order to emphasize that it includes the orientation-reversing isometries like reflections, glide reflections and improper rotations under which the figure is invariant.

The subgroup of orientation-preserving isometries (i.e. transformations, rotations, and composition of these) which leave the figure invariant is called its proper symmetry group. The proper symmetry group of an object is equal to it's full symmetry group if and only if the object is chiral (and thus there are no orientation-reversing isometries under which it's invariant). Any symmetry group whose elements have a common fixed point, which is true for all finite symmetry groups and also for the symmetry groups of bounded figures, can be represented as a subgroup of orthogonal $O(0)$ by choosing the origin to be a fixed point. The proper symmetry group is a subgroup of the special orthogonal group $SO(n)$ then, and therefore also called rotation group of the figure.

4.1.7. Discrete symmetry groups

Finite point groups, which include only rotations, reflections, inversion and roto inversion-they are in fact just the finite subgroups of $O(0)$.

Infinite lattice groups which include only translations, and infinite space groups which combines elements of both previous types, and may be also include extra transformations like screw axis and glide reflection . There are also continuous symmetry groups, which contains of arbitrary small angles or translations of arbitrary small distance.

The group of all symmetries of a sphere $O(3)$ is an example of this, and in general such continuous symmetry groups are studied a Lie groups. With a categorization of subgroups of the Euclidean group corresponds a categorization of symmetry groups.
Two geometry figures are considered to be of the same symmetry type if their symmetry groups are conjugate subgroups of the Euclidean group $E(n)$ (the isometry group of $R^2$). Where two subgroups $H_1, H_2$ of a group $G$ are conjugate, if there exists $g \in G$ such that

$$H_1 = g^{-1}H_2 g$$

**Examples 4.1.1.**

i. Two 3D figures have mirror symmetry, but with respect to different mirror planes.

ii. Two 3D figures have 3-fold rotational symmetry, but with respect to different axes.

iii. Two 2D patterns have translational symmetry, each in one direction; the two translation vectors have the same length but a different direction.

**4.2 Symmetric Group**

In Mathematics, the symmetric group on a set is the group consisting of all bijections of the set (all one-to-one and on to functions from the set to itself) with function composition as the group operation.

**Definition 4.2.1.**

The symmetric group on a set $X$ is the group whose underlying set is the collection of all bijections from $X$ to $X$ and whose group operation is that of function composition. The symmetric group of degree $n$ is the symmetric group on the set $[X = 1, 2, ..., n]$.

The symmetric group on a set $X$ is denoted in various ways including $S_X, G_X, \Sigma_X$ and $\text{sym}(X)$. If $X$ is the set $[1, 2, ..., n]$ when the symmetry group on $X$ is also denoted $S_n, G_n, \Sigma_n$ and $\text{sym}(n)$.

i. **Elements**

The elements of symmetric group on a set $X$ are the permutations of $X$.

ii. **Multiplication:**

The group operation in a symmetric group is function composition, denoted by juxtaposition of the permutations. The composition $f \circ g$ of permutations $f$ and $g$, pronounced $f$ after $g$, maps any element $x$ of $X$ to $F(g(x))$. Concretely, let

$$f = (1 \ 3)(4 \ 5) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}$$
And
\[ g = (1 \ 2 \ 5)(3 \ 4) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 3 & 1 \end{pmatrix} \]

Applying \( f \) and \( g \) maps 1 first to 2 and then 2 to itself, 2 to 5 and then to 4,3 to 4 and then to 5, and so on. So composing \( f \) and \( g \) gives
\[ fg = f \circ g = (1 \ 2 \ 4 \ 3) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{pmatrix} \]

A cycle of length \( L = K \cdot m \), taken to \( k \)-th power, will decompose into \( k \) cycles of length \( m \).
For example: \((k = 2, m = 3)\)
\[ (1 \ 2 \ 3 \ 4 \ 5 \ 6)^2 = (1 \ 3 \ 5)(4 \ 5 \ 6) \]

### 4.3 Symmetry Sets

#### Reflections and Symmetries

Let \( G \) be a group with identity 1 and let \( R \) be a subset of \( G \) containing 1. Then \( R \) has a partial product \( \pi(a, b) = ab \) defined on \( D = \{(a, b) \in R \times R | ab \in R\} \), and \( R \) together with \( \pi \) is a product set in the sense of the following definition.

**Definition 4.3.1.**

A product set is a set \( R \) together with a function \( \pi \) from a subset \( D \) of \( R \times R \) to \( R \), denoted \( \pi(a, b) = ab \) and called the partial product of \( R \), such that:

1. \( R \) has an identity element 1 such that \((1, a), (a, 1) \in D \) and \( 1a = a1 = a \) for all \( a \in R \);
2. \( ab = ac \) implies \( b = c \) for all \((a, b), (a, c) \in D \).

If \((a, b) \in D \) implies \((b, a) \in D \) and \( ab = ba \) for all \( b \in R \), \( R \) is abelian.

Here we will denote \( R \) as a finite product set with identity 1, partial product \( \pi \) and the domain \( D \) of \( \pi \). The motivating example is that of a subset \( R \) of a group \( G \), such as the set \( R \) of roots of a Lie algebra.

We define \( a^i b \) for \( a, b \in R, i \in \mathbb{Z} \) as follows:

1. \( a^0 b = b \)
2. \( a^i b = a(a^{i-1} b) \) for all \( i \geq 1 \) for which \( a^{i-1} b \) and \( a(a^{i-1} b) \) are defined:

3. $a^{-i}b = d$ if $= a^i d$ ($i \geq 0$).

We let $a^i b \in R$ indicate that $a^i b$ is defined, and we let $a^i b \notin R$ indicate that $a^i b$ is not defined (an abuse of language). Note that $a^i (a^j b) = a^{i+j} b$ for all $i, j \in \mathbb{Z}$ for which $a^i b$, $a^i (a^j b) \in R$.

### 4.4 Homomorphisms of product sets

**Definition 4.4.1.**

Let $R_1, R_2$ are mapping $f : R_1 \rightarrow R_2$ such that $f(ab) = f(a)f(b)$ for all $a, b ab \in R_1$. A homomorphism of product sets $f : R_1 \rightarrow R_2$ is an isomorphism (automorphism $R_1 = R_2$) if $f$ is bijective and both $f$ and $f$ are homomorphism. The set of homomorphisms from $R_1$ to $R_2$ is denoted $\text{Hom}(R_1, R_2)$, and $\text{Aut } R$ denotes the group of automorphism of $R$.

Note that $f(a^i b) = f(a)^i b$ for $f \in \text{Hom}(R_1, R_2), i \in \mathbb{Z}, a, b ab \in R_1$.

For $a \in R$ and $S \subset R$, the relation $\{(x, y) \in S \mid y = ax\}$ generates an equivalence relation on $S$. The corresponding equivalence class of $b \in S$ is the string $S_b(a) = \{a^{-r} b, ..., b, ..., a^q b\}$.

If $a^{-(r+1)} b, a_{q+1} b \notin S$, we say that the string $S_b(a)$ is bounded of length $q + r$.

If all strings $S_b(a)$ ($b \in S$) are bounded, we introduce the reflection $r_a : S \rightarrow S$, which is the bijection from $S$ to itself reversing each string $S_b(a) = \{a^{-r} b, ..., a^q b\}$: $r_a(a^i b) = a^{q+r-i} b$.

In particular, $r_a(b) = a^{-a \cdot (b)} b$, where $a \cdot (b) = r - q$, the Cartan integer of $b$ at $a$. Clearly, $r_a$ is a symmetry of $S$ at $a$ in the following sense.

**Definition 4.4.2.**

A symmetry of $S$ at $a$ is a bijection $s : S \rightarrow S$ such that

(i) $s S_b(a) = S_b(a)$ for all $b \in S$

(ii) $s(a^i b) = a^{-i} s(b)$ for all $b \in S$ and all $a^i b \in S_b(a)$ ($-r \leq i \leq q$).

Clearly, a symmetry $s$ of $S$ at $a$ has period 2. Moreover, if all strings $S_b(a)$ ($b \in S$) are bounded, $r_a$ is the only symmetry of $S$ at $a$.

### 4.5 Reflection Sets and Symmetry Sets

**Definition 4.5.1.**

A reflection set is a finite product set $R$ such that:
1. The strings $R_b(a)$ ($a, b \in R, a \neq 1$) are all bounded.

2. The reflections $r_a(a \in R, a \neq 1)$ are automorphisms of $R$.

3. The subgroup $W(R)$ of $Aut R$ generated by the reflections $r_a(a \in R, a \neq 1)$ is called the Weyl group of $R$.

Clearly, a reflection set is a symmetry set in the following sense,

**Definition 4.5.2.**

A symmetry set is a finite product set $R$ such that $Aut R$ contains a symmetry $s_a$ of $R$ at $a$ for all $a \in R, a \neq 1$.

In a symmetry set $R$, each element $a \in R$ has a unique inverse $b \in R$ such that $ab = ba = 1$. This follows from the equations

$$s_a(a) = s_a(a1) = a^{-1}s_a(1) = a^{-1}1,$$

$$1 = a(a^{-1}1)$$

$$1 = s_a(a(a^{-1}1)) = s_a(a)s_a(a^{-1}1) = (a^{-1}1) (a$s_a(1)) = (a^{-1}1)a$$

Clearly, the inverse of $a$ is $a^{-1}$, which we denote henceforth by $a^{-1}$. Clearly, $s_a(a) = a^{-1}$

If $s_a$ is the reflection $r_a(b) = a^{-a * (b)} b$, we have $a * (a) = 2$. For example,

$$R_a(a) = \{a^{-r}a, ..., a^q a\} = \{a^{1-r}1, ..., a^{-1}1, a1, ..., a^{q+1}1\}$$

And $r_a(1) = 1$, implies that $-(1-r) = q + 1$ and $a * (a) = r - q = 2$. This is needed for (Theorem 4.5.1)

**Definition 4.5.3.**

A root system is a finite product set $R$ such that for all $a \in R, a \neq 1$, there exists

$a \in Hom(R, \mathbb{Z})$ such that $a * (a) = 2$ and $s_a(b) = a^{-a * (b)} b$ defines an automorphism and symmetry $s_a$ of $R$ at $a$.

**Theorem 4.5.1.**

The following conditions are equivalent:

1. $R$ is a root system
2. $R$ is a symmetry set and $\text{Hom}(R, \mathbb{Z})$ separates $R$

3. $R$ is reflection set and $a^* \in \text{Hom}(R, \mathbb{Z})$ for all $a \in R$, $a \neq 1$

4. $R$ is isomorphic to a system of roots in the sense of Bourbaki with 0 added.

**Proof:**

We show that (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (1). Suppose first that $R$ is a root system, $b, c \in R$, $b \neq c$ and $a(b) = a(c)$ for all $a \in R$, $a \neq 1$. Consider, first the possibility that $c^{-1}b \in R - \{1\}$. Then $c \in R$, $c^{-1}b \in R$ and $c(c^{-1}b) = b \in R$ implies that

$$a(b) = a(c^{-1}b) = a(c) + a(c^{-1}b),$$

so that $a(c^{-1}b) = a(b) - a(c) = 0$ for all $a \in R$, $a \neq 1$.

In particular, $2 = (c^{-1}b)(c^{-1}b) = 0$, a contradiction.

Thus, $c^{-1}b \notin R - \{1\}$, so that $R_b(c) = \{b, ..., c^q b\}$ and $R_b(c)$ is bounded of length $q$. It follows that $s_c(b) = c^q b$, so that $s_c(b) = c^{-c(b)}b$ implies that $-q = c'(b) = c'(c) = 2$, a contradiction. We must conclude that $\{c^* | a \in R, a \neq 1\}$ separates $R$, so that (1) $\implies$ (2).

For (2) $\implies$ (3), suppose that $R$ is symmetry set and $\text{Hom}(R, \mathbb{Z})$, let $R''$ be the group $\text{Hom}(R'', \mathbb{Z})$ and define $\hat{a} \in R''$ for $a \in R$ by $\hat{a}(f) = f(a)(f \in R'')$:

$$\hat{a}(f + g) = (f + g)(a) = f(a) + g(a) = \hat{a}(f) + \hat{a}(g)$$

Then $\Lambda: a \mapsto \hat{a}$ is a homomorphism from $R$ to $\hat{R} = \{\hat{a} | a \in R\}$ which is bijective, since $\text{Hom}(R, \mathbb{Z})$ separates $R$. Since $\hat{R}$ is contained in the torsion free group $R''$, the strings $\hat{R}_b(\hat{a})$ are bounded for all $\hat{a}, \hat{b} \in \hat{R}, \hat{a} \neq \hat{1}$. It follows that the strings $R_b(a)$ are bounded for all $a, b \in R, a \neq 1$, so that the symmetries $s_a$ are the reflection set. Finally, for $b, c, bc \in R$ we have $\alpha^{*}(b)bc = r_a(b)c = r_a(b)r_a(c) = (\alpha^{*}(b)c)((\alpha^{*}(c)c)$, so that

$$\hat{b} + \hat{c} - \alpha^{*}(bc)\hat{a} = \hat{b} - \alpha^{*}(c)\hat{a},$$

and $\alpha^{*}(b) = \alpha^{*}(b) + \alpha^{*}(c)$, since the additive group $R''$ is torsion free. Thus, $\alpha^{*} \in \text{Hom}(R, \mathbb{Z})$ for all $a \in R, a \neq 1$. Clearly, (4) $\implies$ (1). It remains to show that (3) $\implies$ (4). For this assume that $R$ is a reflection set and $a^* \in \text{Hom}(R, \mathbb{Z})$ for all $a \in R, a \neq 1$. As in (1) $\implies$ (2), $\{a^* | a \in R, a \neq 1\}$ separates $R$. Therefore, as in (2) $\implies$ (3), $R \rightarrow \hat{R}$ is a bijective homomorphism with $\hat{R}$ abelian. But his homomorphism is an isomorphism by (Theorem 4.5.2) below, and $\hat{R}$ is isomorphic to a system of roots with 0 added by the discussion below.
The strategy in the above is to pass from a symmetry set $R$ to its image $R$ under the closure homomorphism $R : R \mapsto R^{**}$ where $\tilde{a}$ is defined by $\tilde{a}(f)(f \in R^*)$ for $a \in R$. Here $R^* = \text{Hom}(R, \mathbb{Z})$ and $R^{**} = \text{Hom}(R, \mathbb{Z})$ are torsion free additive groups. Since automorphisms $s$ of $R$ determine adjoints $s^* \in \text{Aut} R^*$ and $s^{**} = (s^*)^* \in \text{Aut} R^{**}$ such that $s^{**}(\tilde{a}) = \tilde{s}(a)$, the image $\tilde{R}$ of the symmetry set $R$ is a reflection set called the closure of $R$. Here $s^*$ is defined, as one would expect, by $s^*(f) = f \ast s$ for $f \in R^*$. More specifically, a symmetry $s_a \in \text{Aut} R$ of $R$ at $a \neq 1$ determines a symmetry $\tilde{s}_a = (s_a)^*\ast R \in \text{Aut} \tilde{R}$ at $\tilde{a}$ which, since $R^{**}$ is torsion free, is the reflection of $R^*$ at $\tilde{a} : \tilde{s}_a = r_{\tilde{a}}$. As in the above proof, we have $\tilde{a}^* \in \text{Hom}(\tilde{R}, \mathbb{Z})$ for all $\tilde{a} \in \tilde{R}$, $\tilde{a} \neq 0$. Since $r_{\tilde{a}} = \tilde{b} - \tilde{a}^*(\tilde{b})\tilde{a}(\tilde{a} \in \tilde{R}, \tilde{a} \neq 0)$, $\tilde{R}$ is isomorphic to a system of roots in the sense of Bourbaki\textsuperscript{3} with 0 added, the system of roots being the subset $(\tilde{R} - \{0\}) \otimes 1$ of the subspace $V$ which it generates in $R^{**} \otimes_\mathbb{R} R$.

The closure homomorphism $\rightarrow \tilde{R}$ is an isomorphism if and only if it is injective, that is if and only if $\text{Hom}(R, \mathbb{Z})$ separates points, by the following theorem.

**Theorem 4.5.2.**

Let $s : R \rightarrow R'$ be a surjective homomorphism of product sets, denoted $s(b) = b'$. Suppose that all strings $R_b(a)(a, b \in R, a \neq 1)$ and $R'_b(a')(a', b' \in R', a' \neq 1)$ are bounded and $r_a'(b') = r_a(b')$ for all $a, b \in R, a \neq 1, a' \neq 1'$. Then

$$a'^*(b') = a^*(b)$$

for all $a, b \in R, a \neq 1, a' \neq 1'$

For any $a \in R, c' \in R', a' \neq 1'$, there exists $b \in R$ such that $b' = c'$ and $s$ maps $R_b(a)$ bijectively to $R_{b'}(a')$. $s$ is an isomorphism if and only if $s$ is bijective

**Proof:**

For (1), we have $r_{a'}(b') = r_a(b') = (a^{-a^{-1}(b')}) = a'^{-a'^{-1}(b')}$ and $r_{a'}(b') = a'^{-a^{-1}(b')}$.

Since $s$ is a homomorphism, we have $-r' \leq -r \leq -a^*(b) \leq q \leq q'$. where $R_b(a) = \{a^{-r}b, \ldots, a^q b\}$ and $R'_b(a') = \{a^{-r'} b', \ldots, a'^{q'} b'\}$. This implies that $a^*(b) = a'^*(b')$.

proving (1). For (2), let $R'_c((a') = \{a'^{-r'} c', \ldots, a'^{q'} c'_n\} = \{c'_0, \ldots, c'_n\}$ with $n' = q' + r'$ and $c'_i = a'^{-1} c'_0$ for $0 \leq i \leq n'$.

Let $R_c(a) = \{b_0, \ldots, b_n\}$, where $b_j = a' b_0 \quad (0 \leq j \leq n)$, and note that $c_0 = b_0$, since $c_0 = a' b_0$ for some $j$ and $a'^{-1} c'_0 \notin R'$. Then

\[ c'_n = r_a (c'_0) = r_a (c_0) = r_a (b_0) = b'_n = a^n b_0 = a^n c_0 = c'_n \]

Thus, \( n' = n \) and \( b'_j = (a^j b_0)' = a^j c'_0 = c'_j \) for \( 0 \leq j \leq n \), so that \( s \) maps \( R_{c_0}(a) = \{ b_0, \ldots, b_n \} \) bijectively to \( \{ c'_0, \ldots, c'_n \} = R_{c'}(a') \). Clearly, \( R_{c_0}(a) = R_b(a) \), where \( b = b_i \), and \( i \) is chosen with \( 1 \leq i \leq n \) so that \( b'_i = c' \) and \( b' = c' \).

For (3), let \( s \) be bijective and \( a, b \in R, a \neq 1 \) such that \( a'b' \in R' \). With notation as in (2) above, we then have \( R_a(a') = \{ b_0, \ldots, b_n \}' = \{ b'_0, \ldots, b'_n \} = R_b(a') \). \( b = b_r = a^r b_0 \) for some \( r \), \( 0 \leq r \leq n \), \( b' = (a^r b_0)' = a^r b_0' \) and

\[ a' b' = a^{r+1} b_0 = b_{r+1}' = (a^{r+1} b_0)' = (a a^r b_0)' = (ab)' \]

Thus, \( ab \in R \) and \( s^{-1}(a' b') = ab = s^{-1}(a') s^{-1}(b') \). This shows that \( s^{-1} \in Hom(R', R) \), so that \( s \) is an isomorphism.

**Definition 4.5.4. (Direct Sums)**

Let \( R_1, \ldots, R_n \) be product sets. Resituate them (up to isomorphism) so that their identities all coincide (call it 1) and \( R_i \cap R_j = \{ 1 \} \) for all \( i \neq j \). Then the outer direct sum

\[ R = R_1 \oplus \cdots \oplus R_n \] of \( R_1, \ldots, R_n \) is the product set \( R = R_1 \cup \cdots \cup R_n \) such that \( ab = c \) in \( R \) if and only if there exists \( i \) such that \( a, b, c \in R_i \) and \( ab = c \) in \( R_i \).

Conversely, for any product set \( R \) with identity 1, \( R \) is the inner direct sum \( R = R_1 + \cdots + R_n \) of subsets \( R_1, \ldots, R_n \) of \( R \) if \( R = R_1 \cup \cdots \cup R_n \), \( R_i \cap R_j = \{ 1 \} \) for all \( i \neq j \) and \( ab = c \) in \( R_i \).

Note that if \( R \) is an inner direct sum \( R = R_1 + \cdots + R_n \), then \( R_1, \ldots, R_n \) are product sets whose outer direct sum is \( R \).

\[ R_b(a) = R_{jb}(a) \text{ for } a \in R, b \in R_j \text{, and } R_b(a) = \{ b \} \text{ if } b \in R_j \text{ and } a \in R - R_j. \]

\( R \) is symmetry set (respectively reflection set) if and only if the \( R_1, \ldots, R_n \) are also a symmetry (respectively reflection) \( s_a \) of \( R \) at \( a \in R_i \) is the same as a symmetry (respectively reflection) \( s_{ia} \) of \( R_i \) at extended to \( R \) by fixing the elements of \( R - R_j \).

Let \( R \) be a symmetry set and let \( S \), \( T \subset R \). We let \( = \{ ab|a \in S, b \in T, ab \in R \} \),

\[ S^{-1} = \{ a^{-1}|a \in S \} \text{ and } \hat{S} = S - \{ 1 \}. \text{ We say that } S \text{ is } \pi \text{-closed if } S \subset S \text{ and symmetric if } S^{-1} \subset S. \]

A subset \( \hat{S} \) of \( \hat{R} \) is closed in \( \hat{R} \) if \( R = S + \hat{S} \) (inner direct sum), where \( S = \hat{S} \cup \{ 1 \} \) and \( \hat{S} = R - \hat{S} \). For \( \hat{S} \) is closed, \( S \) and \( \hat{S} \) are clearly \( \pi \)-closed and symmetric, and they can
viewed as symmetry sets. Moreover, $\tilde{S}$ is closed and $\tilde{S} = S$. For $\tilde{S}$ and $\tilde{T}$ closed in $R$, the inner direct sum.

$R = (S \cup \tilde{S}) \cap (T \cup \tilde{T}) = S \cap T + S \cap \tilde{T} + \tilde{S} \cap T + \tilde{S} \cap \tilde{T}$, can be used to show that $S \cap \tilde{T}$ and $\tilde{S} \cup T$ are closed in $\hat{R}$. Thus, the set $\mathcal{D} = \{\tilde{S} | \tilde{S} \text{ is closed in } \hat{R}\}$ is a topology for $\hat{R}$. Since $\tilde{S}$ is closed if and only if $\tilde{S}$ is open, the connected components of $\hat{R}$ are the minimal elements $\tilde{S}_1, \ldots, \tilde{S}_n$ of $\mathcal{D}$ and $R = S_1 + \cdots + S_n$ is the unique inner direct sum decomposition of $R$ which cannot further be refined. If $n = 1$, we say that $R$ is irreducible. The irreducible symmetry sets $S_i$ are the components of $R$.

If $R$ is a subset of a group, then the topology $\mathcal{D}$ introduced here for $\hat{R}$ coincides with the symmetric $G$–topology. For any finite subset $\hat{R}$ of a group $G$ by the $S \subset \hat{R}$ is closed if $S$ and $\hat{R} - S$ are both $\pi$–closed and symmetric.

**Examples of Symmetry sets  4.5.1.**

1. Symmetry sets which are not root systems are symmetry sets having a component which is a nonzero finite vector space $V = (\mathbb{Z}_p)^d$, a symmetry $s_a(v)$ at $a$ being defined for any $a \in Hom(V, \mathbb{Z}_p)$ such that:

$$\hat{a}(a) = 2, \text{ by } s_a(v) = v - \hat{a}(v)a \quad (a \in V, a \neq 0)$$

2. Any two symmetries $s_{1a}, s_{2a}$ are conjugate in $Aut V$ by a unipotent $u_a \in Aut V$ of $V$ at $a: u_a(a + b) = a + u_a(b)$ for all $b \in V$ ($p > 2$). In fact, it the conjugate of $2 - S$–Sylow groups in the dihedral group $\langle s_{1a}, s_{2a} \rangle$. This unicity can be generalized along the lines of Winter$^4$.

**Definition 4.5.4. (Reduced Symmetry Sets in an Abelian Group)**

Let $R$ be a sub set containing 0 of an additive abelian group $G$ having no 2,3,5,7 torsion, and assume that $R$ is a symmetry set which is reduced, that is $2a \notin R$ for all $a \in R, a \neq 0$.

Using simple modifications of techniques of Seligman, we show that $R$ is root system. We first note that a string $R_b(a)$ cannot contain $b, b + a, b + 2a, b + 3a, b + 4a$. For suppose otherwise.

Since $\pm 2a \notin R$, the elements $b, b + a, b + 2a, b + 3a, b + 4a$ are nonzero. Since $2(b + a) \notin R$ and $-2a \notin R, R_b(b + 2a) = \{b\}$, so that $s_{b+2a}(b) = b$. Since $2(b + 3a) \notin R$.

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and $2a \not\in R$, \quad $R_{b+4a}(b + 2a) = \{b + 4a\}$ so that $s_{b+2a}(b + 4a) = b + 4a$. Letting $s = s_{b+2a}$, we then have $b + 4a = s(b + 4a) = s(b) + 4s(a) = b + 4s(a)$. Thus, $4a = 4s(a)$ and $a = s(a)$. But then $-(b + 2a) = s_{b+2a}(b + 2a) = s(b + 2a) = b + 2a$, so that $2(b + 2a) = 0$ and $-2a \not\in R$, a contradiction.

It follows that the strings $R_b(a)$ are bounded of length at most 3. Therefore, the symmetries $s_a(b)$ are the reflections $r_a(b) = b - a^*(b)a$, $a^*(b) = r - q$.

Since $r_a(b + c) = r_a(b) + r_a(c)$, we have $a^*(b + c)a = (a^*(b) + a^*(c))a$ for all $b, c, b + c \in R$. Since $G$ has no 2,3,5,7 torsion, the order of $a$ is at least 11, so that the restrictions on the Cartan integers $r - q$ that $r, q \geq 0$ and $r + q \leq 3$ lead us to conclude that $a^*(b + c) = a^*(b) + a^*(c)$. It follows that $a^*Hom(R, \mathbb{Z})$ for all $a \in R, a \neq 0$, so that $R$ is a root system.

4.6 Classical Lie Algebras

Let $\mathfrak{g}$ be a classical Lie algebra with classical Cartan subalgebra $H$ and corresponding set of roots $R = R(\mathfrak{g}, H)$. Suppose that the characteristic of $\mathfrak{g}$ is not 2,3,5,7. Then $R = R(\mathfrak{g}, H)$ is a reduced reflection set with reflections $r_a(b) = b - 2(b(h_a)/a(h_a))a$, by the beautiful results of Lemma II2.2, II3.1, IIe.2, II4.2 of Seligman$^5$. Thus, $R = R(\mathfrak{g}, H)$ is isomorphic to a reduced system of roots with 0 added, by (Theorem 4.1). since $\mathfrak{g}$ together with $H$ is determined uniquely up to isomorphism by $R = R(\mathfrak{g}, H)$, by a version of (Theorem 3.7.4.9) of winter, the classical Lie algebras $\mathfrak{g}$ with $H$ are classified by systems of roots in the sense of Bourbaki with zero added by the correspondence $(\mathfrak{g}, H) \mapsto R(\mathfrak{g}, H)$. In this approach to the classification, we transfer $R$ together with its combinatorial structure from the hostile environment of characteristic $p$ to real Euclidean space by passing to $\hat{R} \subset \mathbb{R}^{**}$ rather than classify $R$ "on the spot" using orderings, fundamental systems of roots and their Cartan matrices.

**Definition 4.6.1. (Relation Sets and Symmetry Sets)**

A relation set is a set $R$ and a subset $\pi$ of $R^3$, such that $R$ has an identity element 1 such that $(x, 1, x), (1, x, x) \in R$ for all $x \in R$. We let $ax = y$ indicate that $(a, x, y) \in \pi$. Note that $y$ need not be uniquely determined by $a$ and $x$.

Clearly, a product set is just a relation set such that :

1. $ax = y$ and $ax = z$ implies $y = z$

---

2. $ay = x$ and $az = x$ implies $y = z$ for all $a, x, y, z \in R$.

In this section we let $R$ be a finite relation set. Our objective is to:

i. Indicate briefly how the above theory of symmetries generalizes from product sets to relation sets

ii. Describe the root system $S(R)$ which $R$ determines.

A homomorphism of relation sets $R_1, R_2$ is a mapping $f : R_1 \rightarrow R_2$ such that $ax = y$ implies $f(a)f(x) = f(y)$ for all $a, x, y \in R$. And $f$ is an isomorphism (automorphism if $R_1 = R_2$) if $f$ is bijective and $f, f^{-1}$ are homomorphisms. We let $\text{Hom}(R_1, R_2)$ denote the set of homomorphism from $R_1$ to $R_2$ and we let $\text{Aut} R$ denote the automorphism group of $R$.

For $a \in R$ and $S \subset R$, $a$ determines the equivalence relation on $S$ generated by the relation $(x, y) \in S \times S | ax = y$. We let $S_b(a)$ denote the corresponding equivalence class of $b \in S$.

A symmetry of $S$ at $a$ is a bijection $s_a : S \rightarrow S$ such that $s_a S_b(a) = S_b(a)$ for all $b \in S$

$ax = y$ implies $a s_a(y) = s_a(x)$ for all $x, y \in S$

Note that if $R$ is a product set, condition (2) is equivalent to $s_a(\alpha^i x) = \alpha^{-i} s_a(x)$ for all $\alpha^i x \in R$.

A symmetry set is a finite relation set $R$ such that $\text{Aut} R$ contains a symmetry $s_a$ of $R$ at $a$ for every $a \in R$. Clearly, this coincides with our earlier definition if $R$ is a product set.

Consider the additive group $R^* = \text{Hom}(R, \mathbb{Z})$, where $f + g = h$ is the function defined by $h(a) = f(a) + g(a) (a \in R)$ for $f, g \in R^*$. Similarly, consider the additive group $R^{**} = \text{Hom}(R', \mathbb{Z})$. As in previous section we get the closure homomorphism $\Lambda : R \rightarrow R^{**}$ with $\hat{a}(f) = f(a)(f \in R^*)$ for $a \in R$:

$xy = z$ in $R \Rightarrow f(x) + f(y) = f(z) \quad \forall f \in R^*$

$\Rightarrow \hat{x}(f) + \hat{y}(f) = \hat{z}(f) \quad \forall f \in R^*$

$\Rightarrow \hat{x} + \hat{y} = \hat{z}$ in $R^{**}$

The subset $\hat{R} = \{ \hat{a} | a \in R \}$ of torsion free additive group $R^{**}$ is a product set called the closure of $R$. 

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If $R$ is a symmetry set in the sense of this section, then the product set $\hat{R}$ is a symmetry set since, as one easily checks, the double adjoint $s^*_a$ of $s_a$ restricted to a symmetry $\hat{s}_a = s^*_a |_{\hat{R}}$ of $\hat{R}$ at $\hat{a}$ in $\text{Aut} \hat{R}$. In fact, $\hat{R}$ is a system of roots with 0 added, as we now show in greater generality.

For any relation set $R$ we let $S(R) = \{a \in R | \text{there exists a symmetry } s_a \in \text{Aut } R \text{ of } R \text{ at } a \}$.

Clearly, $S(R)$ is stable under $\text{Aut } R$, so that $s_a S(R) = S(R)$ for all $a \in S(R)$. Thus, $S(R)$ is a symmetry set in $R$ in the following sense.

**Definition 4.6.2.**

A symmetry set in $R$ is a subset $S$ of $R$ containing 1 such that for each $a \in S$, there exists a symmetry $s_a \in \text{Aut } R$ of $R$ at $a$. The closure of $S$ in $R$ is the image $\hat{S} = \{s | s \in S \text{ of under the closure mapping } \Lambda: R \rightarrow \hat{R} \text{ of } R \}$. For $a \in S(R)$, the double adjoint $s^*_a$ is an automorphism of the torsion free group $R^*$ and $\hat{s}_a = s^*_a |_{R} \in \text{Aut } (\hat{R})$ satisfies $\hat{s}_a(b) \equiv b (\text{mod } \mathbb{Z}\hat{a})$ for $b \in \langle R \rangle$. Thus, there exists $\alpha \in \text{Hom}(\langle \hat{R} \rangle, \mathbb{Z})$ such that $\hat{s}_a(b) = b - \alpha (b) a \ (b \in \langle \hat{R} \rangle)$. Clearly, $\alpha (\hat{a}) = 2$ since $-\hat{a} = \hat{s}_a(\hat{a})$.

**Theorem 4.6.1.**

$S(\hat{R})$ is a system of roots with 0 added.

**Proof:**

By the above discussion, $S(\hat{R})$ has the required reflections $r_\hat{a}(\hat{b}) = \hat{b} - \alpha (\hat{b}) \hat{a} \ (\hat{a} \in S\hat{R}, a \neq 0$.

In taking the closure $S(R)$ of $S(R)$ in $R$ rather than in $S(\hat{R})$ in the above discussion, we assure the validity of the otherwise unwarranted step $\hat{s}_a(b) \equiv b (\text{mod } \mathbb{Z}\hat{a})$ for $b$ in the closure of $S(R)$ in $S(R)$. (the symmetry $s_a$ might not preserve the sets $S(R)_a$ ($a$)).

4.7 Root Systems

**Definition 4.7.1.**

A root system is a pair $(V, R)$, where $V$ is a vector space and $R$ is a finite subset of $V$ containing 0 which has a symmetry $r_a(v) = v - a^0(v) a \ (v \in V)$ for each $a \in R - \{0\}$: $a^0 \in \text{Hom}_k(V, k)$ and $a^0(a) = 2$.

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We also assume that $R$ spans $V$.

The rank of $(V, R)$ is the dimension of the span $V = kR$ of $R$. The $\mathbb{Z}$-rank of $R$ is the rank of the groupoid dual $\text{Hom}(R, \mathbb{Z})$.

We let $Ra = R \cap \mathbb{Z}a$ and define the set $R^* = \{a \in R - \{0\}|Ra = R^0 \cup R^* \text{ and } R_b(a) \text{ has } 1 \text{ or } p - 1 \text{ or } p \text{ elements for every } a \in R^0 - \{0\}, b \in R.$

We call the orbits $R_b(a)$ $(a \in R^* b \in R)$ classical orbits, and the orbits $R_b(a)$ $(a \in R^0 - \{0\}, b \in R)$ Witt orbits. Accordingly, a root system is a Lie root system if all are roots are either classical or Witt and every Witt orbit has 1 or $p - 1$ or $p$ elements.

Note that if $2a, 2(b + a), 2(b + 3a) \not\in R$. Then $R_b(a)$ does not contain $a, b + a, b + 2a, b + 3a, b + 4a$ and therefore, is bounded of length at most 3. Thus, orbits $R_b(a)$ of length greater than there exist only when $R^0$ contains one of $a, b + a, b + 2a, b + 3a$. It follows that if $R^0$ is a group, that is, $R^0 + R^0 = R^0$. Then $R_b(a)$ with $b \in R^0$ has length greater than three only for $a \in R^0$.

**Proposition 4.7.1.**

Let $R^0$ be a group, $b \in R^0$ and $R_b(a) \neq \{b\}$. Then:

For $p > 5$, $a \in R^0$ if and only if $R_b(a)$ has $p - 1$ or $p$ elements. For $p = 5$, $a \in R^0$ if $R_b(a) = \mathbb{Z}a + b$.

By the same argument, it follows that $R = R^* \cup \{0\}$ if and only if every orbit $R_b(a)$ $(a \in R^0 - \{0\}, b \in R)$ is bounded, in which case the Lie root-system $R$ is a reduced symmetry set.

**Definition 4.7.2.**

A classical root system is a Lie root system all of whose nonzero roots are classical, that is $R = R^* \cup \{0\}$.

**Theorem 4.7.2.**

A root system $(V, R)$ is classical if and only if $R$ is isomorphic as groupoid to a reduced root system in the sense of Bourbaki with 0 added.

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$^3$David J. Winter, Symmetric Lie algebras, Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109, Communicated by N. Jacobson, Received April 10, 1982.
In the next section, we classify the rank two Lie root systems $Rab = R \cap (\mathbb{Z}a + \mathbb{Z}b)$ up to isomorphism. All turn to be symmetry sets. At the same time, only two rank 1 Lie root system are symmetry sets, namely $A_1$ and $W_1$ as the Lie root systems of rank 1 defined over $\mathbb{Z}_p$. The general situation for rank one is as follows.

**Theorem 4.7.3.**

Let $R$ be a rank one Lie root system. Then either $R = R^0$ and $R$ is a group, or $R = \{-a, 0, a\}$.

**Proof:**

Let $a \in R^0 - \{0\}$, $b \in R - \{0\}$ and write $b = ma$, where $m \in k$ (which is possible since $R$ is of rank 1). If $R_b(a)$ is bounded, then $a^0(b) = 2m$ is in the prime field $\mathbb{Z}_p = \mathbb{Z}1$, so that $b = ma$ is in $\mathbb{Z}1 = R_0(a)$.

But then $R_b(a) = R_0(a) = \mathbb{Z} a$ and $R_b(a)$ is not bounded, a contradiction.

Thus, $R_b(a) = \mathbb{Z}a + b$. Iterating, we have $R^0 + R \subset R$, $R^0 + R^0 + R \subset R$, ..., $G = R^0 + \cdots + R^0$ ($n$ times) $\subset R$. Since $R$ is finite, $G$ is a group for some $n$.

Since every subgroup of $R$ is in $R^0$, $G = R^0$ and $R^0$ is a group. Next, let $a \in R^0 - \{0\}$, $b \in R - \{0\}$ and consider $R_a(b)$. If it is unbounded, we have $b \in R^0$, by (Prop 4.7.1) Suppose that it is bounded: $R_a(b) = \{a - rb, \ldots, a + qb\} = R_c(b) = \{c, \ldots, c + (r + q)b\}$. Then $b^0(c) = -(r + q) \in \mathbb{Z}_p$. We may write $c = sb$ with $s \in k$, by invoking "rank 1", so that $2s = r + q$ and $s \in \mathbb{Z}_p$. Then we have $b \in R^0: c = sb \in \mathbb{Z}_p \Leftrightarrow a = c + rb \in \mathbb{Z}_p b = \mathbb{Z}_p a \subset R^0 \Rightarrow b \in R^0$. Thus, an element $b \in R - \{0\}$ is in $R^0$ in all cases, so that $R = R^0$ if and only if $R^0$ is nonzero. Suppose, finally that $R^0 = \{0\}$, so that $R = R - \{0\}$. By the remarks preceding (Def 4.7.2), each orbit $R_b(a)$ is bounded:

$$R_b(a) = \{b - ra, \ldots, b + qa\} \quad (a \in R^0) = R - \{0\}, b \in R).$$

Then $b + qa = r_a(b - ra)$, $a^0(b) = r - q$ and $2s = a^0(sa) = a^0(b) = r - q$, where $b = sa$ ($s \in k$), so that $s \in \mathbb{Z}_p$ and $b \in s \cap \mathbb{Z}_p = ra = \{-a, 0, a\}$. It follows that $R = \{-a, 0, a\}$.

Any finite subgroup $G$ of $k^+$ determines the Lie root system $(k, G)$. For the others, define $SVT = \{(s, t) \in S \times T | s = 0 \text{ or } t = 0\}$ and $S \oplus T = \{(s, t) | s \in S, t \in T\}$, where $S$ and $T$ are sets with a distinguished element : $0 \in S, 0 \in T$. Identify $s = (s, 0), t = (0, t)$ and write $s + t = (s, t)$. Let $A = \{-a, 0, a\} \subset ka$ and $W = \{-a, 0, \ldots, p - 2\} \subset ka$, the rank 1 Lie root systems defined over $\mathbb{Z}_p$. 

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Next construct $AVA, AVW$, $WVV$, the reducible rank 2 Lie root systems defined over $\mathbb{Z}_p$.

Let $A_2 = \{(0,0), \pm(1,0), \pm(0,1), \pm(1,1)\}, B_2 = \{(0,0), \pm(1,0), \pm(0,1), \pm(1,1), (1,1), (1,1)\}, G_2 = \{(0,0), \pm(1,0), \pm(0,1), \pm(1,1), \pm(1,2), \pm(2,1), \pm(1,-1)\}$, which are symmetry sets in $\mathbb{Z}_p^2$ whose groupoid reflections $r_a(b) = b - a(a)(b)$ determine $a^* \in Hom(R, \mathbb{Z})$ and $a^0 \in Hom(\mathbb{Z}_p^2, \mathbb{Z}_p)$ ($a^* \text{ reduced modulo } p$) for $a \in R - \{0\}$ and $B = A_2, B_2, G_2$. These are the irreducible rank 2 classical root systems. Note that the irreducible nonreduced classical symmetry sets

$2A = \{0, \pm a, \pm 2a\}, BC_2 = \{(0,0), \pm(1,0), \pm(0,1), \pm(1,1), \pm(1,-1), \pm(0,2), \pm(2,0)\}$ are not Lie root systems, due to the occurrence of $a, 2a \in R, 3a \notin R$.

Finally, construct $W_2 = W \oplus W$, $W \oplus A, S_2 = \{(i,j) \in \mathbb{Z}_p^2 | i,j \neq 0\} \cup \{(0,0)\}, T = T_2(n) = S_2 \cup \{\pm(n,-n)\} = S_2 \cup A = S_2 + A$, where $A = \{(0,0), \pm(n,-n)\}(1 \leq n \leq p - 1)$, the irreducible rank 2 nonclassical Lie root systems defined over $\mathbb{Z}_p$.

To see that $W \oplus A = \mathbb{Z}a - b \cup \mathbb{Z}a + 0 \cup \mathbb{Z}a + b , S_2$ and $T_2(n)$ are Lie root systems, we define the symmetries $r_a(b) = b - c_0(b)(c)$ in the three cases $W \oplus A$. $S_2, T_2(n)$ by specifying the appropriate Cartan function $c^0 \in Hom(\mathbb{Z}_p^2, \mathbb{Z}_p)$:

$(ia)^0(ia) = 2 \quad (ia)^0(b) = 0$

$(\pm b + ja)^0(\pm b + ja) = 2 \quad (\pm b + ja)^0(a) = 0$

$(i,j)^0(r, s) = 2 \frac{r+s}{i+j} \quad ((i,j) \in S_2, (r, s) \in \mathbb{Z}_p^2)$

$(n, -n)^0(n, -n) = 2 \quad (n, -n)^0(1,0) = \text{anything}$

**Definition 4.7.3.**

A root system $(V, R)$ is defined over $\mathbb{Z}_p$ if $R$ is contained in some $\mathbb{Z}_p$-form $V_{\mathbb{Z}_p}$ of $V$: any $\mathbb{Z}_p$-basis for $V_{\mathbb{Z}_p}$ is a $k$-basis for $V$.

The following propositions are straight forward. In the first proposition, $Ra_1, \ldots, a_n = R \cap (\mathbb{Z}a_1, \ldots, \mathbb{Z}a_n)$.

**Proposition 4.7.2.**

A Lie root system $R$ of rank $n$ is defined over $\mathbb{Z}_p$ if and only if $R = Ra_1, \ldots, a_n$ for some $a_1, \ldots, a_n \in R$.  

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4.8 Classification of Lie Root System of Rank

We now determine Recognition properties for all possible for all possible $k$-independent pairs $a, b$ of roots in a Lie root system $R$, and classify all corresponding Lie root systems $R_{ab}$ of rank 2 up to isomorphism. The results are given in Table 1, the irreducible root systems of rank 2 defined over $\mathbb{Z}_p$ being $A_2, B_2$.

<table>
<thead>
<tr>
<th>No.</th>
<th>Diagram</th>
<th>Recognition conditions on $a$ and $b$</th>
<th>Type of $R_{ab}$</th>
<th>$a^0(b)$ $b^0(a)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td><img src="" alt="Diagram" /></td>
<td>$a, b \in R, R_{R(a)} = {b}$</td>
<td>$A \vee A$</td>
<td>0 0</td>
</tr>
<tr>
<td>2.</td>
<td><img src="" alt="Diagram" /></td>
<td>$a, b \in R, a^<em>(b) b^</em>(a) = 1$</td>
<td>$A_2$</td>
<td>$-1 -1$</td>
</tr>
<tr>
<td>3.</td>
<td><img src="" alt="Diagram" /></td>
<td>$a, b \in R, a^<em>(b) b^</em>(a) = 2$</td>
<td>$B_2$</td>
<td>$-2 -1$</td>
</tr>
<tr>
<td>4.</td>
<td><img src="" alt="Diagram" /></td>
<td>$a, b \in R, a^<em>(b) b^</em>(a) = 3$</td>
<td>$G_2$</td>
<td>$-3 -1$</td>
</tr>
<tr>
<td>5.</td>
<td><img src="" alt="Diagram" /></td>
<td>$a \in R^0_i, b \in R, R_{R(a)} = {b}$</td>
<td>$W \vee A$</td>
<td>0 0</td>
</tr>
<tr>
<td>6.</td>
<td><img src="" alt="Diagram" /></td>
<td>$a, b \in R^0_i, R_{R(a)} = {b}$</td>
<td>$W \vee W$</td>
<td>0 0</td>
</tr>
<tr>
<td>7.</td>
<td><img src="" alt="Diagram" /></td>
<td>$a, b \in R^0_i, R_{R(a)} = 2a + b$</td>
<td>$W_2$</td>
<td>$-m -n$</td>
</tr>
<tr>
<td>8.</td>
<td><img src="" alt="Diagram" /></td>
<td>$a \in R^0, b \in R, a + b \in R$</td>
<td>$W \oplus A$</td>
<td>$-m 0$</td>
</tr>
<tr>
<td>9.</td>
<td><img src="" alt="Diagram" /></td>
<td>$a, b \in R^0, a^<em>(b) b^</em>(a) = 4, a^<em>(b) b^</em>(a) = 1$</td>
<td>$W \oplus A$</td>
<td>$-2 -2$</td>
</tr>
<tr>
<td>10.</td>
<td><img src="" alt="Diagram" /></td>
<td>$a, b \in R^0, a^<em>(b) b^</em>(a) = 4$</td>
<td>$S_2$</td>
<td>$-m -4m$</td>
</tr>
<tr>
<td>11.</td>
<td><img src="" alt="Diagram" /></td>
<td>$a, b \in R^0, a + b \in R^0$</td>
<td>$T_2$</td>
<td>0 $-m$</td>
</tr>
</tbody>
</table>

$G_2, W_2, W \oplus A, S_2, T_2$. In this table, diagrams are introduced to represent each of the 11 classes of $k$-independent root pairs $a, b$. Classical and Witt roots are denoted by black and white nodes, respectively. The number of solid lines is the product $a^0(b) b^0(a)$. No lines indicates that $a$ and $b$ are orthogonal: $a \pm b$ are not roots and $a^0(b) b^0(a) = 0$. A dotted line indicates that $a$ and $b$ are not orthogonal and $a^0(b) b^0(a) = 0$, which occurs for types $S_2, T_2$ and for type $W_2$ if $m = 0$ or $n = 0$.

Orientation indicates which root is shorter, for types $B_2, B_2$. Orientation indicates that $a + b$ is classical or Witt, for types $W \oplus A$ and $T_2$, depending on whether the black or white nodes is "less" (which suffices to distinguish between types $W \oplus A$ and $T_2$). Actual values for $m, n$ in 7, 8, 10, 11 are suppressed in the diagrams. Adjustments in $a, b$ would lead to default values $-1, -1$ in 7, $-1, 0$ (or $-p$) in 8, $-2, -2$ in 10 and 0 (or $-p$), $-1$ in 11, which bring the use of orientation (or lack thereof) in these diagrams in line with its conventional use in the diagrams of the classical root systems 1, 2, 3, 4.
Here and in the sequel, we decompose a root system into its irreducible components as follows. We say that \( S \subset R \) is closed if \( 0 \in S \), \( S = -S \) and \( a + b \in S \) whenever \( a, b \in S \) and \( a + b \in R \). If \( (R - S) \cup \{0\} \) is closed, we say that \( S \subset R \) is open. Then \( \{S - \{0\}\} \) is open and closed in \( R \) is a topology for \( R - \{0\} \), whose connected components \( R_1 - \{0\}, \ldots, R_n - \{0\} \) determine the irreducible components \( R_1, \ldots, R_n \) of \( R \):

\[
R = R_1 \cup \ldots \cup R_n \text{ with } R_i \cap R_j = \{0\} \text{ for } i = j
\]

\( a = a_1 + \cdots + a_n \in R \) with \( a_i \in R_i \) \((1 \leq i \leq n)\) implies \( a = a_i \in R_i \) for some \( i \).

We use the notation \( R = S \cup R = S \cup T = \{0\} \), where \( S, T \) are open and closed in \( R \). Then \( R \) is irreducible if only if \( R = R_1 \) if and only if \( R = \{0\} \) is connected if and only if \( R = S \cup T \) implies \( R = S \) or \( R = T \).

**Proposition 4.8.1.**

The irreducible components \( R_i \) of a Lie root system are Lie root systems.

We begin with the following theorem, which establishes Recognition Conditions 1, 5, 6. It is proved in the cases of \( a \in R^0, b \in R \) needed for the ensuing rank 2 classification. The case \( a \in R^- \) then follows from the classification.

**Theorem 4.8.1.**

Let \( R \) be a Lie root system and let \( a, b \in R - \{0\} \). Then \( Rab = Ra \cup Rb \) with

\[
Ra \cup Rb = \{0\} \text{ if only if } R_b(a) = \{b\}.
\]

**Proof:**

For one direction, note that \( a \pm b \in R_b(a) \subset Ra \subset Za \), say implies that \( Rab = \{-a, 0, a\} \) or \( Za \) and \( R_b(a) = Rab \).

For the other direction, suppose first that \( a \in R^0, b \in R^- \) and \( R_b(a) = \{b\} \). Thus, we know that \( a \pm b \not\in R \). We claim that \( = Ra \cup Rb, Ra \cup Rb = \{0\} \). Suppose, otherwise that there exist \( b' = ra + sb \in R \) with \( r, s \neq 0 \). We claim that \( s \in \{(p - 1)/2, (p + 1)/2\} \), and that \( R = ((Ra + Rb) - Rb) \cup Rb = s(W + A) - A \cup A \). Where \( Ra = W = Za \), \( Rb = A = \{-b, 0, b\} \). For this, note first that \( r_a(b) = b \) and \( a^0(b) = 0 \), so \( a^0(b') \neq 0 \) and \( r_a(b') \neq b' \). It follows that \( R_b(a) \neq \{b'\} \) and therefore, has \( p \) or \( p - 1 \) elements, by (Def 4.7.1), since \( a \in R^0 \). We refer to Table II in what follows.
Note that $R_b'(a)$ is contained in Column $C_s$, among the columns $C_0 = C_0$, $C_{p-1} = C_p - 1$ of Table II, each of which has at most $p$-elements. Since $R \cap C_{\pm 1}$ exclude $\pm (b - a), \pm (b + a)$, we must have $s \neq \pm 1$. Thus, since $b \in R$, we have $sb \notin R_b'(a)$. These constraints force $R_b'(b) = \mathbb{Z}(b) + sb - \{sb\}$. Moreover, the constraint $s \neq \pm 1$ forces $R_b'(b)$ to have fewer than $p$-elements, so that $b'' = \text{def. } r_b(b') = ra - sb \in R$. Repeating the above argument, we have $R_b''(a) = \mathbb{Z}(a) - sb - \{-sb\}$. Thus,

$$R \ni ((\mathbb{Z}a + s(-b, 0, b)) - s(-b, 0, b)) \cup \{-b, 0, b\} = ((W + sA) - sA) \cup A$$

$$= s((W + A) - A) \cup A.$$

It follows that $s(b - a) \in R$. Since $(b - a) \notin R$, $s$ can take on only two values, by (Prop 4.7.1). It follows that $R = s((W + A) - A) \cup A$. Interchanging signs, if necessary we have $2 \leq s \leq \cdots \leq (p - 1)s \leq \cdots \leq p - 2$. It follows that $R_{s(b+a)} = \{sb + a, (s+1)b + a\}$ and $R_{s(b+a)} = \{b, \frac{1}{2}b, 0 + 2a, -\frac{1}{2}b + 3a\}$. Then $(sb + a)^0(sb + a) - (sb + a)^0(b) = -3$ and $(sb + a)^0(sb) = \frac{2}{3}$. Thus,

$$(sb + a)^0(a) = (sb + a)^0(sb + a) - (sb + a)^0(sb) = 2 - \frac{2}{3} = \frac{1}{2}.$$

But $R_a(sb + a) = \{a, sb + 2a\}$ implies, to the contrary that $(sb + a)^0(a) = -1$. We must therefore conclude that $R = Ra \cup R_b$ for $b \in R'$ and $R_b(a) = \{b\}$.

Next, suppose that $b \in R^0$ and $R_b(a) = \{b\}$. We begin claim that $Rab = Ra \cup Rb$,

$$R a \cup R b = \{0\} .$$

Note that $C_1 \cap R$ exclude $b \pm a$ and therefore no $b' = b + ra$ ($r = 0$) is in $R$: $b' \in R \Rightarrow R_{b'}(a) = \{b'\} \Rightarrow a^*(b') = 0 \Rightarrow 0 = a^*(ra) = 2r \Rightarrow r = 0$

\textbf{Table II}

The Roots of $W_2 \supset T_2(n) \supset S_2$ as Successively Generated in $a$-orbits from the Nonroots $m(b - a)$ of $S_2$

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>$p - 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$b - a$</td>
<td>$2b - 2a$</td>
<td>$3b - 3a$</td>
<td>$4b - 4a$</td>
<td>$(p - 1)b + a$</td>
</tr>
<tr>
<td>1</td>
<td>$b$</td>
<td>$b - a$</td>
<td>$3b - 2a$</td>
<td>$4b - 2a$</td>
<td>$(p - 1)b + 2a$</td>
</tr>
<tr>
<td>2</td>
<td>$2a$</td>
<td>$b + a$</td>
<td>$b$</td>
<td>$3b - a$</td>
<td>$(p - 1)b + 3a$</td>
</tr>
<tr>
<td>3</td>
<td>$3a$</td>
<td>$b + 2a$</td>
<td>$2b + a$</td>
<td>$3b$</td>
<td>$(p - 1)b + 4a$</td>
</tr>
<tr>
<td>4</td>
<td>$4a$</td>
<td>$b + 3a$</td>
<td>$2b + 2a$</td>
<td>$3b + a$</td>
<td>$(p - 1)b + 5a$</td>
</tr>
<tr>
<td>5</td>
<td>$5a$</td>
<td>$b + 4a$</td>
<td>$2b + 3a$</td>
<td>$3b + 2a$</td>
<td>$(p - 1)b + 6a$</td>
</tr>
<tr>
<td>6</td>
<td>$6a$</td>
<td>$b + 5a$</td>
<td>$2b + 4a$</td>
<td>$3b + 3a$</td>
<td>$(p - 1)b + 7a$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$p - 1$</td>
<td>$(p - 1)a$</td>
<td>$b + (p - 2)a$</td>
<td>$2b + (p - 3a)$</td>
<td>$3b + (p - 4)a$</td>
<td>$(p - 1)b$</td>
</tr>
</tbody>
</table>
Similarly, no \( a' = a + sb \ (s \neq 0) \) is in \( R \). But then every column \( C_s \) excludes \( sb \pm a \), so that each \( R_b' (a) \) is \( \{b'\} \) for \( b' \in C_s - \{sb\} \), which is impossible since:

\[
0 = a^*(b') = a^*(sb + ra) = 2r \text{ implies } r = 0. \text{ Thus, } = Ra \cup Rb .
\]

The remaining case \( a, b \in R' \) is not needed for the rank 2 classification given below. Applying this classification, we need only observe that one of the conditions \( R_b(a) = \{b\} \) or \( a^*(b) = 0 \) is not met for each pair \( 2,3,4,9 \), to complete the proof.

**Theorem 4.8.2.**

Let \( R \) be a Lie root system and let \( a \in R^0, b \in R' \) with \( Rab \) irreducible. Then \( b - a \in R \) and \( Rab \supset Ra + Rb = W \oplus A \).

**Proof:**

Suppose that \( b - a \notin R \). Since \( Rab \) is irreducible, \( b + a \in R \), by (Theorem 4.8.1). Thus, \( R_b(a) = b, ..., b + (p - 2)a \), by (Def 4.7.1). It follows that \( a^0(b) = 2 = a^0(a) \) and \( a^0(b - a) = 0 \). Thus, \( a^0(2b - 4a) = 2a^0(b - a) - 2a^0(a) = -4 \) and

\[
r_a(2b - 4a) = 2b - 4a - a^0(2b - 4a)a = 2b \notin R .
\]

It follows that \( 2b - 4a \notin R \). Since \( 2b, 2b - 4a \notin R \) and \( -2b, -2b + 4a \notin R \), \( R \) contains no root \( \pm(2b + ra) \) otherwise \( R_{2b+ra}(a) \), say has fewer than \( p - 1 \) elements, so \( R_{2b+ra}(a) = \{2b + ra\} \), which contradicts the irreducibility of \( Rab \). It follows that \( R_a(b) = \{a, a + b\} \), whereas \( R_{2a}(b) = \{-b + 2a, 2a, b + 2a\} \). Thus, \( -1 = b^0(a) \) and \( -2 = b^0(-b + 2a) = -2 + 2b^0(a) = -1 \) and \( 2 = 0 \), a contradiction. We conclude that \( b - a \in R \) for all \( b \in R' \). If \( Za + b \in Rab \), we are done.

Otherwise, choose \( r \) such that \( b' = b - ra \in R \) and \( b' - a \notin R \). From our discussion above, we conclude that \( b' \in R^0 \). But then the classification below, which does not depend on this case, implies that \( Rab' \) is \( W_2, S_2 \), or \( T_2 \): Nos. 7, 10, 11 of Table I. But then \( Rab' = T_1 \), since \( R' \) is empty for \( R = W_2 \) or \( S_2 \) in which case \( b \in T_2 \) and \( T_2 = R \supset Ra + Rb = W \oplus A \) as asserted.

**Theorem 4.8.3.**

Every non classical irreducible rank 1 or 2 Lie root system defined over \( \mathbb{Z}_p \), \( R \) is one of \( W, W \oplus A, W_2, S_2, T_2 \).
Proof:

Let $R$ be non classical, irreducible and not one of $W, W \oplus A, W_2$. We claim that $R$ is $S_2$, or $T_2$, as asserted. Since $R$ is not classical it is not reduced, so that there exist $a \in R^0$ by (Def 4.7.1). We know that $R$ is an irreducible rank 2 root system defined over $\mathbb{Z}_p$. Letting $V$ be the $\mathbb{Z}_p \oplus$-span of $R$, we have $R \subset \mathbb{Z}a + \mathbb{Z}b = V$ for any $b \in V - \mathbb{Z}a$.

Suppose first that $(b + \mathbb{Z}a) \cap R = \emptyset$ for some $b \in V - \mathbb{Z}a$. Then $c \in R - \mathbb{Z}a$ implies that $c \not\in R^0$, so that $c \in R$. But then it follows from the proven part of (Theorem 4.8.2) that $c + \mathbb{Z}a \subseteq R$, so that $R \ni \mathbb{Z}a + \{-c, 0, c\} = W \oplus A$. Moreover, any element $d \in R - \mathbb{Z}a$ has the form $d = r(c + ta)$ with $c + ta \in R$ (just as in the above case $r = 1, t = 0$), so that $r = \pm 1$ and $d \in \pm c + \mathbb{Z}a$. It follows that $R = W \oplus A$.

Suppose next that $(b + \mathbb{Z}a) \cap R \neq \emptyset$ for every $b \in V - \mathbb{Z}a$. Then $(b + \mathbb{Z}a) \cap R$ has more than one element for every $b \in V - \mathbb{Z}a$, by (Theorem 4.8.1) and the irreducibility of $R$. But then $(b + \mathbb{Z}a) \cap R$ has at least $p - 1$ elements for every $b \in V - \mathbb{Z}a$, by (Def 4.7.1). Take $c \in V - \mathbb{Z}a$ with $\mathbb{Z}c \cap R$ as small as possible. Then $R \ni \mathbb{Z}a + \mathbb{Z}c$ and $\mathbb{Z}c \cap R$ has one or three or $p$ elements. If $\mathbb{Z}c \cap R$ has $p$ elements, then $R = W_2$. If $\mathbb{Z}c \cap R$ has only one element, namely $0$, then each set $(\mathbb{Z}a + jc) \cap R$ ($j \neq 0$) has exactly $p - 1$ elements in it and $R = \left( (\mathbb{Z}a + \mathbb{Z}c) - \mathbb{Z}c \right) \cup \{0\} = S_2$. In the remaining case, $\mathbb{Z}c \cap R$ has three elements, which may take to be $\{-c, 0, c\}$ with no loss in generality. Then $R$ contains $ia + jc$ when $i \neq 0$ and $j \neq \pm 1$. It follows that $R$ contains four or more, hence all $p$, multiples of every $b \in V - \mathbb{Z}c$. Thus, $R = (\mathbb{Z}a + \mathbb{Z}c - \mathbb{Z}c) \cup \{-c, 0, c\} = T_2$.

4.9 Base and Closure

Let $(V, R), (W, S)$ be root systems and consider $R \oplus S = \{a \oplus b | a \in R, b \in S\} \subset V \oplus W$.

Introduce

$$r_{a \oplus b} (c \oplus d) = c \oplus d - (a \oplus b)^0 (c \oplus d) (a \oplus b)$$

specifying $(a \oplus b)^0 \in \text{Hom}(V \oplus W, \mathbb{Z}_p)$ as follows:

$$(a \oplus 0)^0 (c \oplus d) = a^0 (c)$$

$$(a \oplus b)^0 (c \oplus d) = b^0 (d) \quad (b \neq 0)$$

Note that $R + S_{c \oplus d} (a \oplus 0) = R_c (a) \oplus d$ and $r_{a \oplus 0} (c \oplus d) = c \oplus d - a^0 (c) (a \oplus 0) = r_a (c) \oplus d$.

Next, suppose that $R$ is a group, that is , $R + R = R$, and note that

$$R + S_{c \oplus d} (a \oplus b) = \{c \oplus d - r(a \oplus b), ..., c \oplus d + q(a \oplus b)\},$$

where
\[ S_d(b) = \{d - rb, \ldots, d + qb\}, \text{ so that } d + qb = r_b(d - rb) = (d - rb) - b^0(d - rb) b \text{ implies } e \oplus d + q(a \oplus b) = r_{a+b}(e \oplus d - r(a \oplus b)). \] It follows that \( V \oplus W, R \oplus S \) is a root system, provided that \( R \) is a group (Table III).

Next, let \( V = \mathbb{Z}_p^n \) with basis \( a_1, \ldots, a_n \). Consider \( S_n = \{0\} \cup \{r_1a_1 + \cdots + r_n a_n | r_1 + \cdots + r_n \neq 0\} \) and note that \( S_n = \{v \in V | f(v, v) \neq 0\} \), where \( f \) is the symmetric bilinear form \( f(a, a) = 1 \) for all \( i, j \). Then \( S_n \) is a Lie root system

<table>
<thead>
<tr>
<th>Rootsystems Constructed from a Given Rootsystem ( R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G \oplus R )</td>
</tr>
<tr>
<td>( W_n \oplus R )</td>
</tr>
<tr>
<td>( S_n + R = S_n \cup R = S_n(R) \leq \mathbb{Z}_p^n )</td>
</tr>
</tbody>
</table>

With the symmetries \( r_a(b) = b - 2(f(a, b)/f(a, a))a = b - 2(\sum s_i/\sum r_i)a \) for \( a = \sum r_i a_i \), \( b = \sum s_i a_i \). The condition \( 0 = f(a, b) = (\sum r_i)^2 \) defines the hyper-plane \( V - S_n = W \) of dimension \( n - 1 \). Let \( (W, R) \) be any Lie root system in \( W \). Then \( S_n + R = S_n \cup R \) is a Lie root system:

\[
\begin{align*}
(S_n + R)_b(a) &= \mathbb{Z}a + b \text{ for } a \in S_n - \{0\}, b \in R \\
(S_n + R)_a(b) &= a + \mathbb{Z}b \text{ for } a \in S_n - \{0\}, b \in R - \{0\} \\
(S_n + R)_c(b) &= R_c(b) \text{ for } R, b \in R - \{0\}. \\
(S_n + R)_b(a) &= (S_n)_b(a) \text{ or } b + \mathbb{Z}a \text{ for } a, b \in S_n - \{0\}, \text{ (use the latter if } b + ia \in R \text{ for some } i). 
\end{align*}
\]

**Definition 4.9.1.**

We define base for a Lie root system \( (V, R) \) to be a basis \( \pi = \{a_1, \ldots, a_n\} \) for \( V \) contained in \( R \) such that

1. If \( Ra_i a_j \) is type \( A_2, B_2 \) or \( G_2 \) then the diagram for \( a_i, a_j \) is

\[
\text{Fig (4.1)}
\]

2. \( R = R^-(\pi) \cup \{0\} \cup R^+(\pi) \), where \( R^-(\pi) = -R^+(\pi) \) and \( R^+(\pi) \) is the set of those \( a \in R \) for which there exist \( a_{i_1}, \ldots, a_{i_s} \in \pi \) such that \( \sum_{j=1}^r a_{i_j} \in R \) for \( 1 \leq r \leq s \) and \( \sum_{j=1}^s a_{i_j} \).
In characteristic 0, this is the usual concept of base, since condition (1) implies that 
\( (a_i, a_j) \leq 0 \) for all \( i \neq j \) and condition (1) implies condition (2).

If \( R \) has base \( a_1, \ldots, a_n \), then \( W_m \oplus R \) has base \( b_1, \ldots, b_m, a_1, \ldots, a_n \), where \( b_1, \ldots, b_m \) is a basis for \( W_m \) as a group. And \( S_m + R(m > n) \) has base \( a_1, \ldots, a_n, a_{n+1}, \ldots, a_m \) obtained by 
taking any \( \in S_m \), forming the independent set \( a + a_1, \ldots, a + a_n, a \), showing that it is part of a base \( a + a_1, \ldots, a + a_n, a = a_{n+1}, \ldots, a_m \) for \( S_m \) and then returning to 
\( a_1, \ldots, a_n, a_{n+1}, \ldots, a_m \).

**Definition 4.9.2:**

A Lie root system \((V, R)\) which has a base is said to be regular. A Lie root system \((V, R)\) need 
not be regular. In fact, a classical root system \((V, R)\) of type \( A_r \) where \( p|\(r+1\) \) need not be 
regular, since it is possible that \( \dim V' < r \). We illustrate this by describing two root systems 
\((V, R), (V', R')\) of type \( A_r \) (\( p|\(r+1\) \)) with \((V, R)\) regular and \((V', R')\) not regular.

For this, let \( k \) be a field of characteristic \( p > 0 \), let \( e_0, \ldots, e_r \) be the basis of \( k^{r+1} \) with 
coordinate conditions \( (e_i)_j = \delta_{ij} \). Let \( R = \{e_i - e_j | i \neq j, 0 \leq i, j \leq r\} \), let \( \pi = \{a_i | 1 \leq i \leq r \} \) with \( a_j = e_{j-1} - e_j \) and let \( V \) be the \( k \)-span of \( R \). Then \((V, R)\) is a root system of type \( A_r, \pi \) is a basis for \((V, R)\) in the sense of (Def 4.9.1) and \((V, R)\) is regular.

Next assuming that \( p|\(r+1\) \), note that \( V \) contains \( e_0 + \ldots + e_r \) and pass to quotients of \( V \) 
m modulo \( k(e_0 + \ldots + e_r) \). Let \( f(v) = v + k(e_0 + \ldots + e_r) \) be the quotient map and define 
\( v' = f(v) \), \( V' = \{v' | v \in V\} \), \( V' = \{a' | a \in R\} \). Then \((V', R')\) is a root system of type \( A_r \) and 
\( f: R \rightarrow R' \) is an isomorphism of groupoids. Since a dimension is lost in passing from \( V \) to \( V' \), 
\( \pi' \) is not a basis for \( V' \) and \((V', R')\) is not regular.

To by pass this pathology for classical root systems \((V, R)\), we pass to their \( k \)-closures 
\((H^*, R)\), described below. This passage corresponds, for certain Lie algebras \( g \) to passage 
certain Lie algebras \( H^* + \text{ad} \ g^\infty \) of derivations, \( H^* = \text{Hom}(R, k) \), where \( [h^*, x] = h^*(a)x \) for 
\( a \in R, x \in g, h^* \in H^* \). When \( g \) is classical, this Lie algebra is \( \text{Der} \ g = H^* + \text{ad} \ g \), which 
since \( g \) is center less and idempotent.

**Definition 4.9.3.**

\( H = \text{Hom}(R, K) = \{f: R \rightarrow K | f(a + b) = f(a) + f(b) \text{ for all } a, d, a + b \in R\} \) is called the 
Cartan space of \((V, R)\) and \( H_\infty \) denotes the \( k \)-span of its subset \( \{a^0 | a \in R - \{0\}\} \). The 
groupoid homomorphism \( R \rightarrow H^* = \text{Hom}_k(H, k) \) which sends \( a \in R \) to \( \bar{a} \in H^* \) defined by
\(\tilde{a}(f) = f(a)\) is called the \(k\)-closure homomorphism, and \(\tilde{R} = \tilde{a}|a \in R\) is the set of roots of the Cartan space \(H\).

We identify \(a^0\) and \(a^0/R\). Then the \(k\)-closure \((H^*, \tilde{R})\) of \((V, R)\) is a root system with Cartan functions \(\tilde{a}^0(b) = a^0(b)\) and symmetries \(\tilde{r}_a(b) = r_a(b)\) \((a, b \in R, \tilde{a} \neq 0)\).

Using \(k\)-closures and regular functions, we show in (Theorem 4.9.2) that we can pass from a classical root system \((V, R)\) that may not be regular to an isomorphic root system \((H^*, \tilde{R})\) which is regular.

If the \(k\)-closure homomorphism \(R \rightarrow \tilde{k}\) is an isomorphism of groupoids, and if it can be extended to an isomorphism of vector-spaces from \(V\) to \(H^*\), then we say that \((V, R)\) coincides, up to identification, with its \(k\)-closures.

**Definition 4.9.4**

\((V, R)\) is \(k\)-closed if \((V, R)\) coincides, up to identifications, with its \(k\)-closure. It is convenient to have passed from a root system \((V, R)\) to its \(k\)-closure \((H^*, \tilde{R})\), in order to have realized all latent independence among roots, as in the case of passage from \(A_r(k)(p|r + 1)\) covered by (Theorem 4.9.2) below.

Moreover, \(R\) is isomorphic to \(\tilde{R}\) in the absence of the root system \(S_2\), in which case we may simplify notation and work with the closed root system \((H^*, R)\) with \(R \subset H^*\),

\[a^0 =_{def} h_a \in H_\infty\] and \(a^0(b) = b(h_a) \in k\) for all \(a \in R - \{0\}\).

We recall from Winter, the \(\mathbb{Z}\)-closure homomorphism \(R \rightarrow \tilde{R} = \{\tilde{a}|a \in R\}\) from \(R\) into \(H_\mathbb{Z} = Hom(R, \mathbb{Z}) \subset H_Q = Hom(R, Q) \subset H_\mathbb{R} = Hom(R, \mathbb{R})\), defined by \(\tilde{a}(f) = f(a)\), is an isomorphism of groupoids to a \(\mathbb{Z}\)-root system, provided that \(R\) is reduced. Thus, there exists a regular function on \(R\) that is a function \(f \in H_\mathbb{R}\) such that \(f(a) \neq 0\) for all \(a \in R - \{0\}\), and we then define \(R^\pm = R^\pm(f) = \{a \in R|f(a) > 0\}\) and \(\pi^\pm = \pi^\pm(f) = \{a \in R^\pm(f)|a\) is not contained in \(R^\pm(f) = R^\pm(f)\). Conversely, let there exist a regular function \(f \in H_\mathbb{R}\) for \(R\). Then \(f|_{Rab}\) is a regular function on the rank 2 root system \(Rab\). A look at the possibilities for \(Rab\),

**Theorem 4.9.1.**

A root system \((V, R)\) is classical if and only if there is a regular function \(f \in H_\mathbb{R}\) on \(R\). For any regular function \(f \in H_\mathbb{R}, \pi^+(f)\) is base for \(R\) if and only if \(\pi^+(f)\) is linearly independent.
We observe that the closure \((H^*, \overline{R})\) of a classical root system \(R\) is a regular root system \(\overline{R}\) isomorphic to \(R\) as groupoid such that:

\[
\overline{\pi^+(f)} = \{\overline{a}_1, \ldots, \overline{a}_r\}\text{ is a base for } \overline{R}\text{ for any regular function } f \in H_\overline{R}\text{ on } \overline{R}.
\]

For this, define \( \hat{d}_i \in Hom(\overline{R}, \mathbb{Z}) \) such that \( \hat{d}_i(\overline{a}_j) = \delta_{ij} \) \((1 \leq i, j \leq r)\). This is possible since \( \overline{\pi^+(f)} = \{\overline{a}_1, \ldots, \overline{a}_r\}\) is a base for \( \overline{R}\), where \( f \in Hom(\overline{R}, \mathbb{Z}) \) is defined by \( f(\overline{a}) = f(a)(a \in R) \). Then define \( d_i: R \rightarrow K \) by taking \( d_i(a) \) to be \( \hat{d}_i(\overline{a}) \) reduced modulo \( p \). Then \( d_i(\overline{a}_j) = \delta_{ij} \), so that \( \overline{\pi^+(f)} \) is a basis for \( \overline{R} \) and \( \overline{R} \) is a regular root system isomorphic to \( R \).

**Theorem 4.9.2.**

The closure \((H^*, \overline{R})\) of a classical root system \((V, R)\) is a regular root system with \( R \) isomorphic to \( \overline{R} \) as groupoid and base \( \pi = \overline{\pi^+(f)} \).

### 4.10 Rigidity and Collapse under Core

In studying a root system \( R \) of Lie algebra \( \mathfrak{g} \), it is important to understand the passage from \( R \) to Core \( R \) and from \( \mathfrak{g} \) to Core \( \mathfrak{g} = \mathfrak{g}^\infty/Nil \mathfrak{g}^\infty \). For this, we further develop concepts introduced in (Def 4.3.1).

**Definition 4.10.1**

The core of a root system \((V, R)\) is \((H_\infty^*, R_\infty)\), where core \( R = R_\infty = \{a_\infty | a \in R\} \) and \( a_\infty = \overline{a}|_{H_\infty} \). Here, \( a \mapsto \overline{a} \) is the closure map and \( H_\infty \) is the \( k \)-span of \( \{a^0 | a \in R - \{0\}\} \). We call \( R \mapsto R_\infty \), sending \( a \) to \( a_\infty (a \in R) \), the core map. If the core map is bijective, we say that \( R \) is rigid.

The following proposition is evident.

**Proposition 4.10.1.**

If \( R \) is rigid, the closure mapping is an isomorphism.

We say that a set \( \{a_1, \ldots, a_n\} \) \((n \geq 2)\) of \( n \) distinct roots collapse if \( a_{1,\infty} = \cdots = a_{n,\infty} \). The following theorem on collapse shows that \( R \) is rigid (has no collapse) if \( R \) has no root system \( Rab \) of type \( S_2 \).

**Theorem 4.10.1.**

Let \( \{a_1, \ldots, a_n\} \) collapse. Then \( R a_1 \ldots a_n = R \cap (\mathbb{Z}a_1 + \cdots + \mathbb{Z}a_n) \) is a root system of type \( S_m \) for some \( m \).
Proof:

Without loss of generality, take $a_1, \ldots, a_n$ linearly independent.

First, take distinct elements $a, b \in \{a_1, \ldots, a_n\}$. Note that $b - a \not\in R$: for otherwise

$$2 = (b - a)^0 (b - a) = (b - a)^0 (b_{\infty} - a_{\infty}) = 0.$$  Thus, $R_b(a) = \{b, \ldots, b + qa\}$, where $-q = a^0(b) = a^0(b_{\infty}) = a^0(b) = a^0(b) = 0$, and $R_b(a) = \{b, \ldots, b - 2a\}$.

Similarly, $R_b(a) = \{a, \ldots, a - 2b\}$. As in section 4.8, we now proceed to cogenerate roots and non-roots of $S_2$, as in Table II. Note that no difference $mb - ma = m(b - a)$ is a root.

For this, observe that $2b - a, a \in R$ with $2b - a \not\in R$: for otherwise

$$(2b - 2a)^0 (2b - 2a) = (2b - 2a)^0 (2b_{\infty} - 2a_{\infty}) = 0.$$  Similarly, $2b, b \in R$ with $2a - 2b \not\in R$. This generalizes easily to $mb - (m - 1)a, a \in R$, with $m(b - a) \not\in R$.

Therefore, this cogeneration leads to $Rab = \{ra + sb | r + s \neq 0\} = S_2$. Next, suppose that we have $m \geq 2$ such that $a = \sum^m r_i a_i \in R$ for $\sum^m r_i \neq 0$. For any such $a$, consider $b = (\sum^m r_i) a_{m+1}$ and note that $b - a \not\in R$; otherwise

$$2 = (b - a)^0 (b - a) = (b - a)^0 (b_{\infty} - a_{\infty}) = \sum^m r_i \ (b - a)^0 (b_{m+1\infty} - a_{\infty}) = 0.$$  Thus, $R_b(a) = \{b, \ldots, b + qa\}$, where $-q = a^0(b) = \text{etc} = a^0(b) = 2$, and $R_b(a) = \{b, \ldots, b - 2a\}$. Similarly, $R_b(a) = \{a, \ldots, a - 2b\}$. It follows that

$$a = \sum^m r_i a_i \in R,$$  provided that $\sum^{m+1} r_i \neq 0$. By induction, therefore, $R$ contains $S_n$.

Finally, let $a = \sum^n r_i a_i \in R$ and suppose that $\sum^n r_i = 0$. Then $a_{\infty} = 0$, which is impossible:

$$2 = (a)^0 (a) = (a)^0 (a_{\infty}) = 0.$$  It follows that $Ra_1 \ldots a_n = S_n$.

4.11 Lie Root System Excluding $T_2$

A Lie root system $R$ has a closure $\hat{R} = \{\hat{a} | a \in R\}$ over the field $\mathbb{R}$ of real numbers, and a closure homomorphism $R \rightarrow \hat{R}$ sending $a \in R$ to $\hat{a} \in H^* = Hom_{\mathbb{R}}(H, \mathbb{R})$, where $\hat{a}$ is defined by $\hat{a}(f) = f(a)$ for $f \in H = Hom_{\text{Groupoid}}(H, \mathbb{R}^+)$. $\mathbb{R}^+ = (\mathbb{R}, +): a + b = \hat{a} + \hat{b}$ for $a, b, a + b \in R$. We let $R\hat{R}$ denote the $\mathbb{R}$-span of $\hat{R}$ in $H^*$. The closure $(R\hat{R}, \hat{R})$ of $(V, R)$ is a root system over $\mathbb{R}$ in the sense of Winter:

1. Each $\hat{a} \in \hat{R} - \{\hat{0}\}$ has an associated $\hat{a}^0 \in Hom_{\mathbb{R}}(R\hat{R}, \mathbb{R})$, with $\hat{a}^0(\hat{a}) = 2$ defined by the condition $r^\hat{a}_a(b) = \hat{b} - \hat{a}^0(\hat{b}) \hat{a}$, where $r^\hat{a}_a$ is defined as in (2) below;
2. Each $\hat{a} \in \hat{R} - \{\hat{0}\}$ has an associated symmetry $r_{\hat{a}} \in Aut_R(\mathbb{R}\hat{R})$ defined as $r_{\hat{a}} = \hat{r}_a$, where $\hat{s} \in Aut_R(\mathbb{R}\hat{R})$ is as defined below for $s \in Aut\ R$.

Here, we define $\hat{s} = s^*\lfloor \mathbb{R}\hat{R}$, where $s^* \in Aut_{\mathbb{R}}H$ and $s^{**} \in Aut_{\mathbb{R}}H^*$ are the adjoints of $s, s^*(f) = f \circ (s(f \in h)), s^{**}(g) = g \circ s^*(g \in H^*), \hat{s}(\hat{a}) = s^{**}(\hat{a}) = \hat{a} \circ s^*$, $\hat{s}(\hat{a})(f) = \hat{a} \circ s^*(f) = (s^*(f))9a = (f \circ s)(a) = f(s(a)) = s(a)(f)(a \in R, f \in H)$.

Thus, $\hat{s}(\hat{a})s(\hat{a})$ and $r_{\hat{a}}(b) = r_{\hat{a}}(b)$.

Note that $\hat{a} \in \hat{R} - \{\hat{0}\}$ implies that $\hat{R}_b(\hat{a})$ is bounded, thus that $R_b(\hat{a})$ is bounded and $a^0(b) = a^*(b) \mod p$, where $a^*(b)$ is the Cartan integer in the sense of Bouraki with 0 added. The following theorem is needed for the proof of the main theorem. It is a variation of (Theorem 4.5.2).

**Theorem 4.11.1.**

Let $R$ be a Lie root system. Then

1. $R_a(b)$ is bounded and $a^*(\hat{b}) = a^*(b)$ for all $a, b \in R, \hat{a} \neq \hat{0}$;

2. For any $a, b \in R, \hat{a} = \hat{0}$, there exists $c \in R$ such that the closure mapping maps $R_c(a)$ bijective onto $R_b(\hat{a})$;

3. The closure mapping $R \to \hat{R}$ is an isomorphism (of groupoids) if and only if it is bijective.

$R_a(b)$ is bounded for $\hat{a} \neq \hat{0}$, as noted above, so that $r_{\hat{a}}$ can be written in terms of the Cartan integers $a^*(b) = r_a(b) = b - a^*(b)a$. Hence forth, we assume that no pair of type $T_2$ occurs in $R$. We then proceed to prove the Decomposition Theorem announced in the introduction.

**Theorem 4.11.2.**

Let $R$ be a Lie root system excluding $T_2$ and let $\hat{a}_1, \ldots, \hat{a}_p$ be a base for the classical root system $\hat{R}$. Then $S = \{n_1a_1 + \cdots + n_r a_r | n_i \in \mathbb{Z}, \hat{n}_1 \hat{a}_1 + \cdots + \hat{n}_r \hat{a}_r \in \hat{R}\}$ is a classical root system isomorphic to $\hat{R}$ and $R \subset R_0 + S$, where $R_0$ is a Witt root system given by $R_0 = \{a \in R | \hat{a} = \hat{0}\}$.

**Proof:**

From Table 1, $b^*(a) = 0, -1, -1, -1, -2, -3, 0, 0, -2$ in types $AVA, A_2, B_2$ long $G_2$ long $B_2$ short $B_2$ short, $WVA, W \oplus A$, mixed $W \oplus A$ classical, since $R$ excludes $T_2$. These are the
Cartan integers, up to sign, for all classical $b$ and all classical or Witt $a$. Since $b^0(c + d) = b^0(c) + b^0(d)(c, d \in kR)$, it follows that $b^∗(c + d) = b^∗(c) + b^∗(d)(c, d \in kR)$ for $b \in R$: this verified for $p > 7$ by considering the integers $b^∗(c)$ modulo $p$:

$$b^∗(c + d) = b^∗(c) + b^∗(d)(c, d \in kR) \mod p$$

with $−3 \leq b^∗(c), b^∗(d), b^∗(c + d) \leq 3$ implies $b^∗(c + d) = b^∗(c) + b^∗(d)$. For $p = 5$ and 7, it follows from the characteristic 5 and 7 theory developed in (Section 4.1).

Consequently, the groupoid dual $H = \text{Hom}(R, \mathbb{R})$ of $R$ over $\mathbb{R}$ contains $b^∗(b \in R)$.

Consider the $\mathbb{Z}$-closure mapping $R \to \hat{R} = \{ \hat{a} | a \in R \}$ described in (section 4.9), which is a groupoid homomorphism from $R$ to the classical root system $\hat{R}$. Note that $\hat{a} = \hat{b} \iff f(a) = f(b)$ for all $f \in H = \text{Hom}(R, \mathbb{R})$. We claim that $R^0 = \{ a \in R | a$ is Witt root $\}$ is the kernel { $a \in R | \hat{a} = \hat{0}$ } of $R \to \hat{R}$. Since $\hat{a}, 2\hat{a}, \ldots, (p − 1)\hat{a} \in \hat{R}$ for $a \in R^0$, and since $\hat{R}$ is classical, we have $\hat{a} = \hat{0}$ for $a \in R^0$.

Next, let $b \in R$; then $b^∗ \in H = \text{Hom}(R, \mathbb{R})$, as observed above, so that $\hat{b}(b^∗) = b^∗(b) = 2$.

It follows that $\hat{b} \neq \hat{0}$. Thus, $R^0 = R − R^∗$ is the kernel { $a \in R | \hat{a} = \hat{0}$ } is a Lie root system, $R^0$ is Witt Lie root system. We now construct a copy $\hat{S}$ of the classical root system $\hat{R}$ in $R$ such that $\hat{R} \subset R^0 + S$. A part of this construction was done in collaboration with $M$. Let $\hat{f} \in H = \text{Hom}_R(\mathbb{R}\hat{R}, \mathbb{R})$ be regular on the root system $\hat{R}$, that is $\hat{f}(\hat{a}) \neq \hat{0}$ for $\hat{a} \in \hat{R} − \{ 0 \}$.

Define $f: R \to \mathbb{R}$ by $f(a) = \hat{f}(\hat{a})$. Let $\langle \hat{a}, \hat{b} \rangle$ be a positive definite symmetric bilinear form on $\mathbb{R}\hat{R}$ such that $\hat{a}^∗(\hat{b}) = 2(\langle \hat{a}, \hat{b} \rangle/\langle \hat{a}, \hat{a} \rangle)$. define $\langle a, b \rangle = \langle \hat{a}, \hat{b} \rangle (a \in R, b \in R)$ and note that $a^∗(b) = \hat{a}^∗(\hat{b}) = \langle \hat{a}, \hat{b} \rangle = \langle a, b \rangle (a \in R, b \in R)$.

Observe, accordingly that if $\langle a, b \rangle > 0$, then $b − a \in R (a \in R, b \in R)$.

Let $\hat{R}^+ = \{ a \in \hat{R} | \hat{f}(\hat{a}) > 0 \}$ and $R^+ = \{ a \in R | f(a) > 0 \}$.

Let $a_1, \ldots, a_r$ be any elements of $R$ such that $\hat{R} = \{ \hat{a}_1, \ldots, \hat{a}_r \}$ is the set of simple roots $\hat{R}^+$ in the classical root system $\hat{R}$. We claim first that $S^+ = \{ n_1a_1, \ldots, n_r a_r | n_i \in \mathbb{Z}, n_1 \hat{a}_1, \ldots, n_r \hat{a}_r \in \hat{R}^+ \}$ is contained in $R^+$ and that

$$\hat{R}_b(\hat{a}_i) = R_b(\hat{a}_i) \text{ for } \hat{b} = \Sigma n_i \hat{a}_i \in \hat{R}^+$$. We proceed by induction on the height $h(\hat{b}) = \Sigma n_i$ of $\hat{b}$. If $h(\hat{b}) = 1$, $\hat{b} = \hat{a}_i$ and $a_i \in S^+$ for some $i$. Moreover, $a_i − a_j \notin R$, since $\hat{a}_i − \hat{a}_j \notin \hat{R}$: $a_i − a_j \in R \Rightarrow \hat{a}_i − \hat{a}_j = \hat{a}_i + (−\hat{a}_j) = a_i − a_j \in R$. Since $a_j^∗(a_i) = \hat{a}_j^∗(\hat{a}_i)$ and
Next, let \( h(\hat{b}) > 1 \) and suppose that our assertion has been establish for \( h - 1 \). Since \( \hat{a}_i, \ldots, \hat{a}_r \), \( \hat{b} \) are lineary dependent elements of \( \hat{R}^+ \) and \( \langle \hat{a}_i, \hat{a}_j \rangle < 0 \) for all \( i \neq j \), we have \( \langle \hat{b}, \hat{a}_j \rangle > 0 \) for some \( i \). But then \( \hat{b} - \hat{a}_i = \sum_n \hat{a}_j - \hat{a}_i \in \hat{R}^+ \). With \( h(\hat{b} - \hat{a}_i) = h(\hat{b}) - 1 \).

By induction, therefore \( c = \sum n_j a_j - a_i \) is in \( R \) and

\[
R_c(a_i) = R_c(\hat{a}_i) = R_{\hat{b} - \hat{a}_i}(\hat{a}_i) = R_b(\hat{a}_i) \ni \hat{b} = \hat{c} + \hat{a}_i.
\]

We do not yet know that \( \sum n_j a_j \in R \). However, we know that some element of \( R_c(a_i) \) maps to \( \hat{c} + \hat{a}_i \), and this element, by virtue of its f-value, must be \( c + a_i = \sum n_j a_j =_{def} b \). This said, we may conclude that \( b = \sum n_j a_j \in R^+ \) and moreover, that

\[
R_b(\hat{a}_i) = R_{\hat{b} - \hat{a}_i}(\hat{a}_i) = R_c(\hat{a}_i) = R_b(\hat{a}_i) \text{ by what are shown above.}
\]

It remains to show that \( R_b(a_j) = R_b(\hat{a}_i) \) for \( j \neq i \). If \( \hat{b} - \hat{a}_j \in \hat{R}^+ \), we argue by induction, just as in the case \( j = i \) above. We conclude, by induction that \( S^+ \subset R^+ \) and

\[
R_b(a_j) = R_b(\hat{a}_j), \text{ since } \hat{a}_j(\hat{b}) = \hat{a}_j * (\hat{b}), \text{ with details as in the similar case encountered above. We conclude, by induction that } S^+ \subset R^+ \text{ and } R_b(a_j) = R_b(\hat{a}_j) = R_b(a_j) \text{ for } \hat{b} \in S^+, \text{ as asserted.}
\]

Implicitly derived in the above considerations is the decisive identity \( \hat{b} = \sum n_i \hat{a}_i \), valid for any \( \sum n_i a_i \in \hat{R}^+ \) and \( b = \sum n_i \hat{a}_i \in S^+ \). This is based on the implicit iterative construction/reconstruction of elements \( b \in S^+/\hat{b} \in \hat{R}^+ \) as \( b = a_{i_1}, \ldots, a_{i_k} \) and \( \hat{b} = \hat{a}_{i_1}, \ldots, \hat{a}_{i_k} \), where all partial sums are roots in \( S^+ \), respectively in \( \hat{R}^+ \).

We claim next that \( R \subset R^0 + S \). Suppose not, and take \( b \in R^+ - (R^0 + S) \) with \( f(b) \) minimal, noting that \( \hat{b} \in \hat{R}^+ \). We claim, firstly that \( b - a_i \in R \) for some \( 1 \leq i \leq r \). For this, note that \( \delta_{i_1}, \ldots, \delta_r, \hat{b} \in \hat{R}^+ \) are linearly dependent and \( \langle \delta_{i_i}, \delta_{i_j} \rangle < 0 \) for all \( 1 \leq i \neq j \leq r \), so that \( \langle \hat{b}, \hat{a}_i \rangle > 0 \) for some \( i \). But then \( \langle b, a_i \rangle > 0 \) for some \( i \), so that \( b - a_i \in R \).

Next, we observe that \( b \in R^0 + S \), contrary to assumption. In the case \( \hat{b} = \hat{a}_i \), we have \( b - a_i \in R \) (see above) and \( \hat{b} - \hat{a}_i = 0 \), so that \( b - a_i \in R^0 \) and \( b = (b - a_i) + a_i \in R^0 + S \). In the case \( b \neq a_i, b - a_i \in R \), we have \( b - a_i \in R \) (as above), \( \hat{b} - \hat{a}_i \in \hat{R}^+ \). But then
\(b - a_i \in R^+\) with \(f(b - a_i) < f(b)\). By minimality of \(f(b)\), \(b - a_i \in R^0 + S\). But then \(b \in R^0 + S\).

To see this, write \(b - a_i = a + \sum n_j a_j\), where \(a \in R^0\) and \(\sum n_j a_j \in R^+\).

Then \(\hat{b} - \hat{a}_i = \sum n_i \hat{a}_i \in \hat{R}^+\) and \(\hat{b} = n_1 \hat{a}_1 + \cdots + n_i \hat{a}_i + (n_i + 1) \hat{a}_i + n_{i+1} \hat{a}_i + \cdots + n_r \hat{a}_r\). Thus, \(b = a + n_1 a_1 + \cdots + n_i a_i + (n_i + 1)a_i + n_{i+1}a_i + \cdots + n_r a_r\).

And \(b \in R^0 + S\), by the definition of \(S\), contrary to assumption. Thus, \(R \subset R^0 + S\) as asserted.

Finally, \(S\) has at most as many elements as \(\hat{R}\), by its constructional definition, and \(\hat{R} \subset R^0 + S = \hat{S}\), so that \(S \rightarrow \hat{S}\) is surjective from \(S\) to \(\hat{R}\). It follows that \(S \rightarrow \hat{S}\) is bijective from \(S\) to \(\hat{R}\), so that \(S \rightarrow \hat{R}\) is an isomorphism by (Theorem 4.11.1), which completes the proof of (Theorem 4.11.2), since the root system \(\hat{R}\) is classical.

Finally, we briefly consider subsystems. Consider a subset \(\sigma\) of \(\pi = \{a_1, \ldots, a_r\}\), where \(\hat{\sigma} = \{\hat{a}_1, \ldots, \hat{a}_r\}\) is a simple system of \(\hat{R}\), \(\sigma = \{a_1, \ldots, a_k\}\). Construct \(R_\sigma = R_{a_1 \ldots a_k} = \{n_1 a_1 + \cdots + n_k a_k | n_1 \hat{a}_1 + \cdots + n_k \hat{a}_k \in \hat{R}\}\). Then \(\hat{R}_\sigma = \hat{R} \cap (\mathbb{Z} \hat{a}_1 + \cdots + \mathbb{Z} \hat{a}_k) = R \hat{a}_1 \ldots \hat{a}_k\) is a classical root system and \(\hat{R}_\sigma^{-1} = \{a \in R | \hat{a} \in \hat{R}_\sigma\}\) is a root system \(R_\sigma\). Relative to the global closure maps \(R \rightarrow \hat{R}\), the same arguments as above show that \(R_\sigma \subset R_\sigma^0 + R_\sigma\), where \(R_\sigma\), as defined above is a classical subsystem of \(R_\sigma\). Both the global closure map and the closure map defined relative to \(R_\sigma\) map \(R_\sigma\) isomorphically to the image (closure in either sense) of \(R_\sigma\).

Taking \(\hat{\sigma} = \hat{\sigma}_1 \cup \ldots \cup \hat{\sigma}_n\) to be the decomposition of \(\hat{\sigma}\) into connected components, and letting \(R^i = R^{\hat{\sigma}_i}\) and \(R_i = R_{\hat{\sigma}_i}\), one now easily sees that:

1. \(R = R^1 \cup \ldots \cup R^n\)

2. \(a, b \in R^i, a + b \in R \Rightarrow a + b \in R^i\)

3. \(R^i \subset R_0 + R_i\) with \(R_i\) irreducible and classical and \(R_0^0\) Witt.

Next, take any \(b \in R - R^0\), so that \(\hat{b} \neq \hat{0}\), and take a simple system \(\hat{\sigma} = \{\hat{a}_1, \ldots, \hat{a}_r\}\) for \(\hat{R}\) such that \(\hat{b} = \hat{a}_1\). This is possible, since \(\hat{R}\) is reduced and a classical root system. In this case, \(R_b\) as defined above is \(R_b = \{-b, 0, b\}\).

We write \(R^b = R(b) = \hat{R}_b^{-1} = \{a \in R | \hat{a} \in \hat{R}_b\}\). Then \(R^b \subset R_0^0 + R_b\).
Let \( a \in R^{b_0} \). If \( a + b \in R \), then \( Rab = R \cap (\mathbb{Z}a + \mathbb{Z}b) \) is type \( W \oplus A \), by the exclusion of \( T_2 \) and consequently, \( a - b \) is in \( R^b \) as well. It follows that \( a + R_b \subset R^b \) if \( a + b \in R \), so that
\[
R^b = U_{a \in R^0, b \in R; a+b \in R} a + R_b , \quad \text{where } R^1 = R_1 \text{ and } a + R_b = a + [-b, 0, b].
\]

By the above arguments and the inclusion \( R \subset R^0 + R^1 \) established in (Theorem 4.11.2).

4.12 Lie Algebras

i. Preliminaries

Throughout previous part of this chapter, \( k \) denotes a field of characteristic \( p > 3 \). The following results of part I on Lie root systems \( (V, R) \) and Cartan functions \( a^0 \in Hom_k(V, k) \) play key roles in part II.

Proposition 4.12.1.

A reduced nonzero symmetry set in \( \mathbb{Z}_p \) whose symmetries are the reversals \( r_a(b) = -b \) must be \( \{-a, 0, a\} \) for some \( a \in \mathbb{Z}_p \).

Theorem 4.12.1.

Let \( R \) be a classical Lie symmetry system or reduced symmetry set. Then \( R \) is isomorphic as groupoid to a root system in the sense of Bouraki.

Theorem 4.12.2.

Let \( R \) have rank 1. Then \( R = \{-a, 0, a\} \) or \( R \) is a subgroup of \( k^+ \).

Theorem 4.12.3.

Let \( c, b \in R \) be \( k \)-linearly independent. Then \( R \subset D \) is classical or one of \( W \) (irreducible rank), \( W \cup A, W \cup W \) (reducible rank 2), \( W \oplus A, W \oplus W, S_2, T_2 \) (irreducible rank 2), there \( W = \{0, a, \ldots, (p-1)a\} = \mathbb{Z}a, A = \{-b, 0, b\}, W \oplus A = \{ia+jb|j = \pm 1 \text{ or } 0\}, W \oplus W = \mathbb{Z}_p^2, S_2 = \{(i,j)|i+j \neq 0\}, T_2 = S_2 \cup \{(m,-m),(0,0),(-m,m)\} \).

Theorem 4.12.4.

Let \( R \rightarrow R|_{H_{\infty}} = R_{\infty} \) be the core map and let \( a_1, \ldots, a_n \) (\( n \geq 2 \)) be \( k \)-independent elements of \( R \) such that \( a_1_{\infty} = \cdots = a_n_{\infty} \). Then \( Ra_1 \ldots a_n = R \cap (\mathbb{Z}a_1 + \cdots + \mathbb{Z}a_n) \) is type of \( S_n \).

We need the following theorems on representations of 3 dimensional Lie algebra \( L = ke + kh + kf \) with \([e, f] = h, [h, e] = e, [h, f] = -f \). Suppose that the set of characteristic roots is \( \{b, b+1, \ldots, b+j\} \). Then \( j = p-1 \) or \( 2b = -j \).
In any \( g \)-module \( V, g \) a Lie algebra over \( k \), we define \((x - a)v = xv - av \) and \((x - a)^{n+1}v = (x - a)(x - a)^iv \) recursively for \( x \in g, a \in k, v \in V \); and we let \( V_a(x) = \{x \in V | (x - a)^iv = 0\} \).

**Theorem 4.12.5.**

Let \( g = ke + kh + kf \), where \([e, f] = h \) and \([h, g] = 0 \). Let \( V \) be an \( g \)-module such that \( e^{p-1}V = 0 \). Then \( h^n V = 0 \) for some \( n \).

**Proof:**

Let \( b \in k \), be an eigen value for \( h \) on \( V \). Choose \( v \in V - \{0\} \) satisfying \( hv = bv \), subject to the constraint that the corresponding integer \( n \) such that \( e^n v \neq 0 \) and \( e^{n+1}v = 0 \) is maximal. Note that \( n + 1 \leq p - 1 \). Define \( [e^n, f]v = e^n(fv) - f(e^n v) \), and note that \([e^n, f]v = 0 \), since \( f(fv) = bv \) and consequently, \( e^{n+1}(fv) = 0 \) by the constraint on \( v \). Once can show, by induction, that \( 0 = [e^{n+1}, f]v = (n + 1)b e^n v \). Thus, \( 0 = (n + 1)b \) and \( b = 0 \). Consequently, \( h \) is nilpotent on \( V \).

**ii. Reductive Lie algebras**

Let \( g \) be a Lie algebra. Let \( g = g_1 \supseteq g_2 \supseteq \ldots \supseteq g_{n+1} = 0 \) be a maximally refined chain of ideals of \( g \) and \( g_2 \oplus \ldots \oplus g_n \), where \( g_i = g_i / g_{i+1} \) (\( 1 \leq i \leq n \)). Then the ideal \( Nil g \) consists of nilpotent elements and is called the nil radical of \( g \). Note that \( g/Nil g \) has the faithful completely reducible module \( \bar{g} = \bar{g}_2 \oplus \ldots \oplus \bar{g}_n \) it follows, as in the proof of (Theorem 4.12.6) below, that \( Nil g \) contains every other ideal \( I \) of \( g \) such that \( ad_g I \) consist of nilpotent elements : \( I \bar{g} = \{0\} \) and therefore \( I \subset Nil g \). That is \( Nil g \) is the unique maximal ideal such that \( ad Nil g \) consists of nilpotent elements.

**Definition 4.12.1.**

\( g \) is reductive if \( Nil g \) is central in \( g \).

**Theorem 4.12.6.**

\( g \) is reductive if and only if \( ad g \) has a faithful completely reducible representation which preserves nilpotency of elements of \( ad L \).

**Proof :**

If \( g \) is reductive, the representation afforded to \( ad g \) by the \( g \)-module \( \bar{g} = \bar{g}_2 \oplus \ldots \oplus \bar{g}_n \) is such a faithful completely reducible representation. Conversely, let \( V = N_1 \oplus \ldots \oplus V_n \) be a
representation for ad \( g \) with nonzero irreducible submodules \( V_1, \ldots, V_n \). Let ad \( N \) be an ideal of ad \( g \) consisting of nilpotent elements, and assume that ad \( N \) acts by nilpotent transformations on \( V \). Since ad \( N \) is an ideal of ad \( g \), \( V_{io} = \{ v \in V_i \mid \text{ad } N, v = 0 \} \) is a nonzero ad \( g \)-submodule of \( V_i \), so that \( V_i = V_{io} \) for \( 1 \leq i \leq n \). Thus, (ad \( N \))\( V = 0 \). It follows that \( N = \{ 0 \} \) and \( N \) is central in \( g \) if \( V \) is faithful. Thus, \( g \) is reductive.

**Theorem 4.12.7.**

\( g \) is reductive if and only if every solvable ideal is central in \( g \).

**Proof:**

One direction is trivial, since Nil \( g \) is nilpotent and therefore solvable. For the other, suppose that \( g \) is reductive and let \( I \) be a solvable ideal of \( g \). We show by induction on the dimension of \( I \) that \( I \) is central.

Suppose first that \( I \) is nilpotent. Then ad \( I \) is an ideal consisting of nilpotent elements since \( [I, \ldots [I, g] \ldots] \subset I^n = \{ 0 \} \). Thus, \( I \) is central, by (Def 4.12.1). Next suppose that the assertion is true for solvable ideals of lower dimension than that of \( I \) and let \( J \) be the ideal \( J = [I, I] \) by induction, \( J \) is central in \( g \). Thus, \( I \) is nilpotent. But then \( I \) is central, as shown above.

**Definition 4.12.2.**

\( g \) is semisimple if every solvable ideal of \( g \) is 0.

**Corollary 4.12.1.**

\( g \) is semisimple if and only if \( g \) is reducible with center 0 if and only if Nil \( g = 0 \).

**Proof:**

One direction of the first implication is clear. For the other, suppose that \( g \) is reductive with center 0, and let \( I \) be a solvable ideal of \( g \). Then \( I \) is central, by (Theorem 4.12.7). Thus, \( I = \{ 0 \} \). The remaining implication follows easily.

**Definition 4.12.3.**

Core \( g = g^\infty / \text{Nil } g^\infty \), where \( g^\infty = \bigcap_{i=1}^\infty g^i \). Since Center \( g \) is the kernel of ad : \( g \to \text{Der } g \), and since \( [d, \text{ad } x] = \text{ad } d(x) \) for \( d \in \text{Der } g, x \in g \), \( C = \text{Center } g \) is stabilized by \( \text{Der } g \). It follows that \( \text{Der } (g, C) = \{ d \in \text{Der } g | d(g) \subset C \} = \text{Hom } (g/g(1), C) \) is an ideal in \( \text{Der } g \). d \in \text{Der } (g, C), e \in \text{Der } g \Rightarrow [d, e] = de - ed \) maps \( g \) to 0. Note that \( \text{Der } (g, C) \cap \)
ad \mathfrak{g} = \text{Center } ad \mathfrak{g} : ad x (\mathfrak{g}) \subset C \iff [x,\mathfrak{g}] \subset C \iff [ad x, ad \mathfrak{g}] = 0. \text{ We can now easily prove the following theorem.}

**Theorem 4.12.8.**

Let \mathfrak{g} be reductive. Then the solvable radical of Der \mathfrak{g} is contained in Der (\mathfrak{g}, C).

**Proof:**

Let \mathcal{I} be a solvable ideal of Der \mathfrak{g}, so that \mathcal{I}, ad \mathfrak{g} = ad \mathcal{I}(\mathfrak{g}) \subset \mathcal{I} \cap ad \mathfrak{g} is solvable ideal of ad \mathfrak{g}. Then \mathcal{I}(\mathfrak{g}) + C is central in \mathfrak{g}, since \mathfrak{g} is reductive, so that \mathcal{I}(\mathfrak{g}) \subset C and \mathcal{I} \subset Der (\mathfrak{g}, C).

**Corollary 4.12.2.**

Suppose that \mathfrak{g} is semisimple or that \mathfrak{g} is reductive and idempotent in the sense that \mathfrak{g} = \mathfrak{g}^2. Then Der \mathfrak{g} is semisimple.

**Proof :**

In either case, Der (\mathfrak{g}, C) = \text{Hom}(\mathfrak{g}/\mathfrak{g}^{(1)}, C) = 0.

Now consider Cartan decomposition \mathfrak{g} = \sum_{a \in R} \mathfrak{g}_a of \mathfrak{g} with Cartan sub-algebra \mathcal{H} = \mathfrak{g}_0. Note that \mathcal{H}_\infty = \mathcal{H} \cap \mathfrak{g}^\infty = \sum_{a \in R - \{0\}} [\mathfrak{g}_{-a}, \mathfrak{g}_a] is a Cartan sub-algebra of \mathfrak{g}^\infty if and only if \mathcal{H}_\infty, \mathfrak{g}_b = \mathfrak{g}_b for all \ b \in R - \{0\}, in which case ad \mathcal{L} \mathcal{H}_\infty contains Center ad \mathfrak{g}^\infty.

**Theorem 4.12.9.**

Let Der (\mathfrak{g}^\infty, C) denote the set of derivatives of \mathfrak{g} mapping \mathfrak{g}^\infty into C. Then Der (\mathfrak{g}^\infty, C) is deal of Der \mathfrak{g} and Der (\mathfrak{g}^\infty, C) \cap ad \mathfrak{g}^\infty = Center ad \mathfrak{g}^\infty.

**Proof:**

We have Der \mathfrak{g} = D \triangleright ad \mathfrak{g} \triangleright ad \mathcal{H} \text{ with } [D, ad \mathcal{H}] \subset ad \mathfrak{g}, \text{ so that } D = D_0 \text{ ad } \mathfrak{g}^\infty \text{ where } D_0 = D_0(\text{ ad } ad \mathcal{H}). \text{ We then have } D = \text{Der } \mathfrak{g} = \sum_{b \in R} D_b \text{ with } D_b = ad \mathfrak{g}_b (b \in R - \{0\}).

It suffices to show that \{D_0, \text{Der } (\mathfrak{g}^\infty, C)\} \subset \text{Der } (\mathfrak{g}^\infty, C), \text{ since } [\text{ad } \mathfrak{g}^\infty, \text{ Der } (\mathfrak{g}^\infty, C)] = 0. \text{ Thus, take } d_0 \in D_0, d \in \text{Der } (\mathfrak{g}^\infty, C) \text{ and observe that } [d_0, d] = d_0 d - dd_0 \text{ maps :}

\mathfrak{g}_a (a \in R - \{0\}) \text{ to } C: d_0 d(\mathfrak{g}_a) \subset d_0(\mathfrak{C}) \subset C \text{ and } dd_0(\mathfrak{g}_a) \subset d(\mathfrak{g}_a) \subset C. \text{ Thus, } [d_0, d] \text{ maps } \mathfrak{g}^\infty \text{ to } C \text{ and } [d_0, d] \in \text{Der } (\mathfrak{g}^\infty, C).

**Definition 4.12.4.**

A Lie algebra \mathfrak{g} is complete if \mathfrak{g} has center 0 and Der \mathfrak{g} = ad \mathfrak{g}.

**Theorem 4.12.10.**
Let \( \mathfrak{g} \) be a Lie algebra over any field such that \( \mathfrak{g} = \mathfrak{g}^2 \) and Center \( \mathfrak{g} = 0 \). Then \( \text{Der} \mathcal{L} \) is complete semisimple.

**Corollary 4.12.3.**

Let \( \mathfrak{g} \) be semisimple with \( \mathfrak{g} = \mathfrak{g}^2 \). Then \( \text{Der} \mathfrak{g} \) is complete semisimple.

**Corollary 4.12.4.**

Let \( \mathfrak{g} \) be simple. Then \( \text{Der} \mathfrak{g} \) is complete semisimple.

**Theorem 4.12.11.**

Let \( \mathfrak{g} \) be semisimple. Then \( \text{Der} \mathcal{I} \) is complete semisimple and \( \text{Der} \mathcal{I} \geq \mathfrak{g} \geq \mathcal{I} \) (up to identifications) for \( \mathcal{I} = \mathfrak{g}^{(\infty)} \), in fact, for any idempotent ideal \( \mathcal{I} \) of \( \mathfrak{g} \) containing Scole \( \mathfrak{g} \). If \( \mathcal{I} \) is \( \text{Der} \mathfrak{g} \)-stable, then \( \text{Der} \mathfrak{g} \) is normalize of \( \mathfrak{g} \) in \( \text{Der} \mathcal{I} \).

### 4.13 Reflective and Toral Lie Algebras

Let \( \mathcal{L} \) be a finite dimensional Lie algebra over \( \mathbb{k} \) with split Cartan sub-algebra \( \mathcal{H} = \mathcal{L}_0 \) and root space decomposition \( \mathfrak{g} = \sum_{\alpha \in \mathcal{R}} \mathfrak{g}_\alpha \). Thus, \( \mathfrak{g}_\alpha = \{ x \in \mathfrak{g} \mid (ad h - a(h))^{\dim \mathcal{L}} x = 0 \) for all \( h \in \mathcal{H} \} \) for \( a \) in the additive group \( \mathcal{H}^* \) of functions from \( \mathcal{H} \) to \( \mathbb{k} \), and \( \mathcal{R} = \mathcal{R}(\mathfrak{g}, \mathcal{H}) = \{ a \in \mathcal{H}^*|\mathfrak{g}_a \neq \{0\} \} \) is the set of roots of \( \mathcal{H} \) on \( \mathfrak{g} \).

Let \( V \) be an \( \mathfrak{g} \)-module, \( V_b = \{ v \in V | (h - b(h))^{\dim V} v = 0 \) for all \( h \in \mathcal{H} \} \) and \( S(V, \mathcal{H}) = \{ b \in \mathcal{H}^* | V_b = \{0\} \} \), the set of weights of \( \mathcal{H} \) in \( V \).

**Theorem 4.13.1.**

Let \( a \in \mathcal{R} \) and let \( S_b(a) = \{ b - ra, ..., b + qa \} \) be a bounded \( a \)-orbit in \( S = S(V, \mathcal{H}) \). Suppose that \( h = [e, f], [h, e] = a(h) e, [h, f] = -a(h)f \) with \( e \in \mathfrak{g}_a, f \in \mathfrak{g}_{-a} \). Then \( 2b(h) = (r - q)a(h) \). If \( a(h) \neq 0 \), then the reflection \( r_a \) reversing \( S_b(a) \) is given by \( r_a(c) = c - 2(c(h)/a(h)) a(c \in S_b(a)) \).

**Proof:**

First suppose that \( a(h) \neq 0 \) and let \( h' = a(h)^{-1}h \). Then \( 2(b(h)^{-1} - r = -(q + r)), \) so that \( 2b(h)^{-1} = r - q \) and \( 2b(h) = (r - q)a(h) \). Suppose next that \( a(h) = 0 \). Then \( [e, f] = h, [h, e] = 0, [h, f] = 0 \) and \( W = \sum_{c \in S_b(a)} V_c \) is a module for \( N = ke + kh + kf \) such that \( e^{p-1}W = 0 \). It follows from theorem 7.7 that \( h^nW = 0 \) for some \( n \), so that \( b(h) = 0 \), for \( a(h) = 0 \).
Let \( c = b + ia \) and observe that: \( c - 2(c(h)/a(h))a = (b + ia) - 2((b(h)/a(h)) + i)a \).

Since \( 2(b(h)/a(h)) = r - q \), it follows that \( c - 2(c(h)/a(h))a = b + (q - r - i)a = r_a(b + ia) \).

We let \( R_* \) be the set of those \( a \in R \) such that \( R_b(a) \) is bounded for all \( b \in R \). We also define \( g_a^1 = \{ x \in g_a | [h, x] = a(h)x \text{ for all } h \in H \} \).

**Theorem 4.13.2.**

Let \( a \in R_* \) and suppose that \( h = [e, f] \) with \( e \in g'_a, f \in g_{-a} \). Then

1. If \( a(h) = 0 \), \( \text{ad } h \) is nilpotent
2. If \( a(h) \neq 0 \), then \( r_a(c) = c - 2(b(h)/a(h))a \) is symmetry of \( R \) at \( a \);
3. If \( a(h) \neq 0 \), then \( 2a \not\in R \).

**Proof:**

For (1), suppose that \( a(h) = 0 \). Since the \( R_b(a) \) are bounded \( (b \in R) \), \( k \) has characteristic \( p \neq 2 \). Since \( 2b(h) = (r - q) \ a(h) = 0 \), by (Theorem 4.13.1), \( b(h) = 0 \) for all \( b \in R \). Thus, \( \text{ad } h \) is nilpotent. Note that (2) follows directly from (Theorem 4.13.1). For (3), consider \( = kf + H + g_a + \cdots g_{qa} \), where \( a, \ldots, qa \in R \) and \( (q + 1)a \not\in R \). Let \( S = S(V, H) \), so that \( S_0(a) = \{ -a, 0, a, \ldots, qa \} \). Then \( qa = r_a(-a) = a \), by (2). Thus, \( S_0(a) = \{-a, 0, a\} \) and \( 2a \not\in R \).

We recall the definition of classical Lie algebra.

**Definition 4.13.1.**

A Lie algebra \( g \) with split Cartan subalgebra \( H \) is classical if \( g \) has center \( 0 \), \( g = g^{(1)} \), \( \text{ad } H \) is diagonalizable on \( g \), \( [g_a, g_{-a}] \) is one dimensional and \( R_b(a)(b \in R) \) is bounded for all \( a \in R - \{0\} \).

We now introduce the reflective Lie algebra as diagonalizations of classical Lie algebras. In our definition, \( [g_a^1, g_a^1] \) denotes the span of \( \{ e, f | e \in g_a^1, f \in g_{-a}^1 \} \).

**Definition 4.13.2.**

A Lie algebra \( g \) with split Cartan subalgebra \( H \) is reflective if \( \text{ad } [g_a^1, g_{-a}^1] \) has some nonnilpotent element and \( R_b(a)(b \in R) \) is bounded for all \( a \in R - \{0\} \).
Note that there are no reflective Lie algebras in characteristics 2 and 3, by the boundedness condition.

We let \( g^∞ = \cap_{r=1}^{∞} g^i \) and Core \( g = g^∞/\text{Nil} g^∞ \). If \( g = g^{(1)} = g^∞ \), Recall that \( g \) is idempotent. For any Cartan subalgebra \( H \) of \( g \), we let \( H_∞ = H \cap g^∞ \). Then \( g = H + g^∞ \) and

\[
H_∞ = \sum_{a ∈ R−(0)}[g_a, g_{−a}],
\]
where \( g = \sum_{a ∈ R−(0)} g_a \) is the Cartan decomposition of \( g \) with \( g_0 = H \).

The following theorem shows that reflective Lie algebras \( (g, H) \) are roughly classified by corresponding classical root systems \( R(g, H) \), defined and described before. For classical Lie algebras \( (g, H) \) this classification \( (g, H) → R(g, H) \) is "up to isomorphism" e.g., by version Theorem 3.7.4.9 of Winter : \( (g_1, H_1) \cong (g_2, H_2) \) if and only if \( R(g_1, H_1) \cong R(g_2, H_2) \) for \( (g_1, H_1), (g_2, H_2) \) classical.

**Theorem 4.13.3.**

Let \( g \) be a reflective Lie algebra with split Cartan sub-algebra \( g_0 = H \) and Cartan decomposition \( g = \sum_{a ∈ R} g_a \). Then

\( R \) is a classical root system and \( \text{dim} g_a = \text{dim}[g_a, g_{−a}] = 1 \) for all \( a ∈ R − \{0\} \) and \([g_a, g_b] = g_{a+b} \) for all \( a, b, a + b ∈ R − \{0\} \);

**Proof:**

Consider the set \( \{(e, f) | e ∈ g^1_a, e ∈ g_{−a}^1 \} \) and note that \( W \) is commutative :

\[
[[x, y], [e, f]] = [[x, y], e, f] + [e, [x, y], f] = a((x, y))[e, f] − a((x, y))[e, f] = 0.
\]

Since the span \( \text{ad} [g^1_a, g_{−1}^a] \) of \( \text{ad} W \) has some non-nilpotent element, \( \text{ad} W \) must therefore contain a non-nilpotent element \( \text{ad} h_a \). Let \( h_a = [e_a, f_a] \) with \( e_a ∈ g^1_a, f_a ∈ g_{−1}^1 \). When the context is clear, we abbreviate \( h = h_a \), \( e = e_a \), \( f = f_a \). Note that \( a(h) ≠ 0 \). (by Theorem 4.13.2), so that \( r_a(c) = c - 2(c(h)/a(h))a \) is a symmetry of \( R \) at \( a \) by the same (Theorem 4.13.2). Note that \( [H, f] = kf, \) since \( f ∈ g^1_a \) and \( [h, f] = a(h)f ≠ 0 \). It follows that \( 2a \notin R \). Thus, \( R \) is reduced symmetry set . Since \( R \) is reduced with bounded orbits, the characteristic of \( k \) is not 2 or 3. Thus, \( R \) is reduced root system by (Theorem 4.12.1).

Consider \( g^∞ = H_∞ + \sum_{a ∈ R−(0)} g_a \) and \( H_∞ = H ∩ g^∞ = \sum_{a ∈ R−(0)}[g_a, g_{−a}] \). Note that \( [g_a, g_a] = \{0\} \), since \( 2a \notin R \). Take \( h_a = h = [e, f] ∈ [g^1_a, g_{−1}^1] \) as above , with \( a(h) ≠ 0 \). For
\[ \in L_\alpha \] , note that 

\[ -\alpha(h)u = [u, h] = [u, [e, f]] = [e, [u, f]] + 0 = -\alpha([u, f])e \quad \text{and} \quad u = (a([u, f])/\alpha(h)) e \in ke. \]

Thus, \( g_\alpha^1 = ke. \) We claim that \( g_\alpha = ke. \) Suppose that \( g_\alpha \supsetneq ke. \)

Since \((\text{ad} - \alpha(h))^d \) \( g_\alpha = 0 \)

There exists \( u \in g_\alpha - ke \) such that \((\text{ad} - \alpha(h))u = ce \) and \([h, u] - \alpha(h)u = ce \) for some \( c \in k. \) But then \(-\alpha(h)u - ce = [u, h] = [u, [e, f]] = [e, [u, f]] + 0 = -\alpha([u, f])e \) and \( u = (a([u, f]) - c/a(h)) e \in ke, \) a contradiction.

Thus, \( g_\alpha = ke. \) This establishes that \( g_\alpha \) and \([g_\alpha, g_{-\alpha}] \) are one dimensional for all \( \alpha \in R - \{0\}. \)

Now let \( \alpha, \beta, \alpha + \beta \in R - \{0\}, \) \( S_\beta(\alpha) = \{b - \alpha a, \ldots, b + qa\}, \) \( T = \{b - \alpha a, \ldots, b\}, V = \sum_{e \in T} g_e. \) If \([g_\alpha, g_\beta] = 0, \) then \( V \) is a module for \( ke + k\alpha + k\beta = g_\alpha + [g_\alpha, g_{-\alpha}] + g_{-\alpha}, \) so that \( r_\alpha(b - \alpha a) = b + qa \) with \( q \geq 1. \) Thus \([g_\alpha, g_\beta] \neq 0, \) so that \([g_\alpha, g_{-\alpha}] = g_\alpha + b \). 

Next, we introduce toral Lie algebras as generalizations of reflective Lie algebras.

**Definition 4.13.3.**

A Lie algebra \( g \) with split Cartan sub algebra \( H \) is toral if \( \dim g_\alpha = 1 \) and \( \alpha([g_\alpha, g_{-\alpha}]) \neq 0 \) for all \( \alpha \in R - \{0\}. \)

The algebras of Block are those toral Lie algebras \((g, H)\) which are idempotent, have center 0 and have \( \text{ad} H \) diagonalizable. The algebras of Block are classified in Block\(^8\) for \( p > 5. \)

**Proposition 4.13.1.**

Let \( g \) be toral. Then \( g^\infty \cap \text{Center } g = \text{Center } g^\infty \cap \text{Center } H, \) and \( H^\infty \) is a Cartan sub algebra of \( g^\infty. \) Moreover, \( \text{ad } g^\infty H^\infty \) is diagonalizable.

**Proof:**

Each \( \text{ad } h \in \text{ad } H \) is diagonalizable on the \( g_\alpha (\alpha \in R - \{0\}), \) therefore on the algebra \( g^\infty \) generated by them. Thus, \([H, H^\infty] = 0. \) But then \( \text{ad } H^\infty \) is diagonalizable on \( g^\infty \) and 0 on \( H. \)

We now determine \( \text{Nil } g. \) For this , observe that \( \text{Kern } R = \{h \in H | \alpha(h) = 0 \forall \alpha \in R\} \) is contained in the centralizer \( C_g(g^\infty) = \{x \in g | [x, g^\infty] = 0\}. \) For \( \text{Kern } R \) centralizes the

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generators \( g_a (a \in R - \{0\}) \) for \( g^\infty \), so that \( \text{Kern} R \subset C_g (g^\infty) \). Conversely, any \( x \in C_g (g^\infty) \) centralizes the \( g_a (a \in R - \{0\}) \). Writing \( x = \sum_{b \in R} x_b \) with \( x_b \in g_b \),

\[
0 = [x, e_a] = \sum_{b \in R} [x_b, e_a],
\]
which implies that \( 0 = [x_b, g_a] \) for \( g_a = k e_a \ (a \in R - \{0\}, b \in R) \). Thus, \( [x_b, g_{-a}] = 0 \), which implies that \( x_b = 0 \ (b \in R - \{0\}) \) and, therefore, that \( x = x_0 \in H \). Thus, \( C_g (g^\infty) \subset H \) and, therefore, \( C_g (g^\infty) \subset \text{Kern} R \). Thus, \( \text{Kern} R = C_g (g^\infty) \) is an ideal of \( g \) contained in \( H \) centralizing \( g^\infty \). As such, \( \text{Kern} R \subset \text{Nil} g \). Conversely, \( \text{Nil} g \) is \( \text{ad} H \)-stable and is, therefore, a sum of \( (\text{Nil} g) \cap H \) and certain of the one dimensional spaces \( g_a \). But \( [g_a, g_{-a}] \not\subset \text{Nil} g \), since \( a \ (\{[g_a, g_{-a}]\} \neq 0 \), whereas \( ([\text{Nil} g], g_{-a}] \subset \text{Nil} g \) for \( a \in R - \{0\} \). It follows that \( \text{Nil} g \subset H \) and therefore, that \( \text{Nil} g \subset \text{Kern} R \). This establishes the following theorem.

**Theorem 4.13.4.**

Let \( g \) be toral. Then \( \text{Nil} g = \text{Kern} R = C_g (g^\infty) \), where \( \text{Kern} R = \{H \in H | a(h) = 0 \ \forall a \in R\} \) and \( C_g (g^\infty) \) is centralizer in \( g \) of \( g^\infty \).

**Corollary 4.13.1.**

Let \( g \) be toral and idempotent. Then \( g \) is reductive.

**Corollary 4.13.2.**

Let \( g \) be toral and \( H \) abelian. Then \( g \) is reductive.

**Proposition 4.13.2.**

Let \( g \) be toral with center \( 0 \). Then \( H \) is abelian, \( g \) is reductive and \( \text{Core} g = g^{(1)} \), that is \( g^{(1)} = g^\infty \) and \( g^{(1)} \) has center \( 0 \).

**Proof:**

Suppose that \( H \) is not abelian, and choose a nonzero element \( h \in H^{(1)} \cap \text{Center} H \). Then \([h, g_a] = 0 \) for all \( a \in R \) and \( h \in \text{Center} g \), so Center \( g \neq \{0\} \) in contradiction to the hypothesis. Thus, \( H \) is abelian. It follows that \( g \) is reductive, by (Corollary 4.13.2). Finally \( g^\infty \) has center \( 0 \), since \( g = H + g^\infty \) has center \( 0 \) and \( H \) is abelian. Thus, \( g^{(1)} = H^{(1)} + g^\infty = g^\infty \).

**Proposition 4.13.3.**

Let \( g \) be toral and reductive. Then \( \text{Core} \text{ad} = (\text{ad} g)^{(1)} \).
Proof:

Since $H^{(1)} \subset \text{Kern} \, R = \text{Nil} \, g$ and $\text{Nil} \, g$ is central, $0 = \text{ad} \, H^{(1)} = [\text{ad} \, H, \text{ad} \, H]$. Thus, $\text{ad} \, H$ is abelian and $(\text{ad} \, g)^{(1)} = \text{ad} \, H^{(1)} + \text{ad} \, g^{\infty} = \text{ad} \, g^{\infty}$. Since $\text{ad} \, g^{\infty}$ is idempotent, it suffices to show that it had center $0$.

Let $\text{ad} \, h$ be central in $\text{ad} \, g^{\infty}$, so that $h \subset \text{Kern} \, R \subset \text{Nil} \, g \subset \text{Center} \, g$. Then $\text{ad} \, h = 0$.

The following theorem shows that the rough classification of reflective Lie algebras $(g, H)$ by their root systems, discussed in the paragraph preceding is equivalent to a rough classification of reflective Lie algebras $g$ by their (classical) cores. The latter classification is independent of a split Cartan sub algebra $H$ of $g$.

Theorem 4.13.5.

Let $g$ be reflective. Then Core $g = g^{\infty}/\text{Nil} \, g^{\infty}$ is classical and isomorphic to $(g/\text{Nil} \, g)^{(1)}$, and the root systems of $g$ and Core $g$ are canonically isomorphic.

Proof:

Let $\tilde{g} = g/\text{Nil} \, g, \tilde{H} = (H + \text{Nil} \, g)/\text{Nil} \, g$. Then $\tilde{H}$ is a split Cartan sub algebra of $\tilde{g}, \text{dim} \, \tilde{g}_a = 1$ and $a(\tilde{g}_a, \tilde{g}_{-a}) \neq 0$ for all nonzero roots $a$ of $\tilde{g}$ and $\tilde{g}$ is toral, by (Theorem 4.13.3) and (Def 4.13.3). By (Theorem 4.13.4) $\tilde{g}$ has center $0$, since any central element $x$ would lie in $\{ \tilde{x} \in \tilde{H} | a(x) = 0, \forall \text{ roots } a \} \subset \text{Nil} \, g/\text{Nil} \, g = \{ 0 \}$. Thus, Core $\tilde{g} = \tilde{g}^{(1)}$, by (Prop 4.13.2).

Since $g^{\infty}$ is idempotent, the homomorphism $g^{\infty} \rightarrow \tilde{g}$ has image $\tilde{g}^{(1)} = \text{Core} \, \tilde{g}$ and therefore, Kernel $\text{Nil} \, g^{\infty}$. Thus, Core $g$ is isomorphic to $R_{g^{\infty}} \rightarrow R_{\text{Core} \, g}$ (reduction mod Nil ), where $R_{g}, R_{g^{\infty}}, R_{\text{Core} \, g}$ are sets of roots for $(g, H), (g^{\infty}, H_{\infty}), (\text{Core} \, g, H_{\infty}/\text{Nil} \, g^{\infty})$, respectively. We know, that $\{ a^{\oplus} | a \in R - \{ 0 \} \}$ separates $R$. Since $a^{\oplus} (b) = 2(b(h_a)/a(h_a))$ with $h_a \in g^{\infty}$, it follows that $\{ a^{\oplus} | a \in R - \{ 0 \} \}$ separates $R_{\infty}$ and , moreover, $g^{\infty}$ is reflective and Core $g$ classical, since $a(h_a) \neq 0 (a \in R - \{ 0 \})$. Thus, the mappings $R_{g} \rightarrow R_{g^{\infty}} \rightarrow R_{\text{Core} \, g}$ are bijections. It therefore follows from Winter, (Theorem 4.5.2), that they are isomorphisms of groupoids.

We say that $g = \sum_{a \in R} g_a$ is weakly reflective if $R_{g}(a)$ is bounded and $[g_{-a}, g_a]$ has a non nilpotent element for all $a, b \in R$ such that $g_a \not\subset \text{Nil} \, g^{\infty}$ and $g_b \not\subset \text{Nil} \, g^{\infty}$. The above results lead easily to the following version of part of them which, by the conjugacy of Cartan sub

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algebras of classical Lie algebras is an "invariant characterization". The proof is based on passage from \( \mathfrak{g} \) to \( \mathfrak{g}^\infty \).

Theorem 4.13.6.

A Lie algebra \( \mathfrak{g} \) is weakly reflective with respect to some split Cartan sub algebra \( H \) if and only if \( \text{Core} \mathfrak{g} \) is classical.

4.14 The Nilpotent Roots of \( \mathfrak{g} \)

For a Lie algebra \( \mathfrak{g} = \sum_{a \in R} \mathfrak{g}_a \) with orbits \( R_b(a) (a \in R - \{0\}, b \in R) \) bounded, \( \mathfrak{g} \) is reflective if and only if the set \( \text{Nil}^1 R \triangleq \{ c \in R - \{0\}\mid [\mathfrak{g}_{-c}, \mathfrak{g}_c] \text{ consists of ad-nilpotent elements}\} \cup \{0\} \) of nilpotent roots of \( \mathfrak{g}, H \) is \( \{0\} \). Note, in this connection, that \( 0 \in \text{Nil}^1 R \) is not an anomaly, since \( [\mathfrak{g}_0, \mathfrak{g}_0] = 0 \). For \( \text{ad} L_0 \) diagonalizable \( \text{Nil}^1 R = \text{Nil} R \), where \( \text{Nil}^1 R \triangleq \{ c \in R - \{0\}\mid [\mathfrak{g}_{-c}, \mathfrak{g}_c] \text{ consists of ad-nilpotent elements}\} \cup \{0\} \).

Without assuming a condition that \( \text{ad} L_0 \) be diagonalizable, we now show that the sub algebra \( \mathfrak{g}_{\text{Nil} R} \) generated by \( \{ \mathfrak{g}_c\mid c \in \text{Nil} R - \{0\} \) is ad-nilpotent on \( \mathfrak{g} \) provided that the orbits \( R_b(a) (a \in R - \{0\}, b \in R) \) are bounded. Note in this connection, that \( 0 \in \text{Nil} R \) is an anomaly for certain Lie algebras \( \mathfrak{g} \), even when \( \mathfrak{g} = \mathfrak{g}_0 \bigoplus \mathfrak{g}^\infty = \mathfrak{g}_0 \bigoplus \mathfrak{g}_a \), where \( \mathfrak{g}^\infty \) is abelian and \( \text{ad} \mathfrak{g}_0 \) is irreducible on \( \mathfrak{g}^\infty = \mathfrak{g}_a; [\mathfrak{g}_0, \mathfrak{g}_0] \) need not be ad-nilpotent.


Suppose that the orbits \( R_b(a) (a \in R - \{0\}, b \in R) \) are bounded. Then \( \mathfrak{g}_{\text{Nil} R} \) is ad-nilpotent on \( \mathfrak{g} \).

Proof:

Let \( S = \text{Nil} R - \{0\} \). Observe that the sub algebra \( H_{\text{Nil} R} \) generated by the "weakly closed" st \( \cup_{c \in S} [\mathfrak{g}_{-c}, \mathfrak{g}_c] \) is ad-nilpotent on \( \mathfrak{g} \), by the Jacobson-Engel theorem (Jacobson).\(^{10}\)

We claim that the "weakly closed" set \( W = \bigcup_{n=1}^{\infty} W_n \) of commutators \([x_1, \ldots, x_n] \in W_n\) (with any legal arrangements of brackets) \((n = 1, 2, \ldots, c_i \in S, x \in \mathfrak{g}_{c_i})\) consists of ad-nilpotent elements. Consider \( x = [x_1, \ldots, x_n] \in W_n \) of weight \( \sum c_i = 0, x_i \in \mathfrak{g}_{c_i}, c_i \in S \).

After successive factorizations \( x = [[x_1, \ldots, x_m], [x_{m+1}, \ldots, x_n]] \) of \( x \) and generated terms thereof, and successive use of the Jacobi identity in conjunction there with \( x \) can be written as a linear combination of terms of \( W_n \) of the form \( x' = [x'_1, [x'_2, \ldots, x'_n]] \) of weight

0 = \sum_i^n c'_i \cdot x_i \in \mathfrak{g}_c, \ c'_i \in S. But then \( x' \in [\mathfrak{g}_{c'_1}, ..., \mathfrak{g}_{c'_i}] \subset H^S \), so that \( x \in H^S \) as a linear combination of the generated terms \( x' \), and \( \text{ad} \, x \) is nilpotent on \( \mathfrak{g} \), as an element of \( \text{ad} \, H^S \).

Finally, consider an element \( x = [x_1, ..., x_m] \in W_0 \) of weight \( c = \sum_i^n c_i \neq 0 \). Then \( \text{ad} \, x \) is nilpotent on \( \mathfrak{g} \), by the boundedness of orbits \( R_b(c)(b \in R) \). Since \( \text{ad} \, W \) is a weakly closed set of nilpotent linear transformations of \( \mathfrak{g} \), it follows that \( \text{ad} \, \mathfrak{g}^S \) is nilpotent on \( \mathfrak{g} \), by the Jacobson-Engle Theorem.

We now identify \( \text{Nil} \, R \) precisely in the case of Lie algebras \( \mathfrak{g} \) of characteristic 0.

**Theorem 4.14.2.**

Let \( \mathfrak{g} = \sum_{a \in R} \mathfrak{g}_a \) be a Lie algebra of characteristic 0. Then:

\[ \text{Nil} \, R = \{ c \in R - \{0\} | \mathfrak{g}_c \subset \text{Nil} \, \mathfrak{g} \} \cup \{0\} \]

**Proof:**

Since \( \mathfrak{g}/\text{Nil} \, \mathfrak{g} \) is reductive, the theory of reductive Lie algebras of characteristic 0 implies that \( \mathfrak{g}_c \subset \text{Nil} \, \mathfrak{g} \) of \( c \in \text{Nil} \, R \). For the other direction, let \( h \in [\mathfrak{g}_{-c}, \mathfrak{g}_c] \), where \( \text{ad} \, h \) is not nilpotent on \( \mathfrak{g} \). By (Theorem 3.5.1) of Winter¹¹ we have \( c(h) \neq 0 \), \( \text{Tr} \, (\text{ad} \, h)^2 \neq 0 \) and \( h \notin \mathfrak{g}^\perp \) when \( \mathfrak{g}^\perp \) is the radical of the Killing form on \( \mathfrak{g} \). It follows that \( \bar{h} = h + \text{Rad} \, \mathfrak{g} \) is non zero in \( \bar{\mathfrak{g}} = \mathfrak{g}/\text{Rad} \, \mathfrak{g} \), where \( \text{Rad} \, \mathfrak{g} \) is the solvable radical of \( \mathfrak{g} \). Since \( \bar{h} \in [\bar{\mathfrak{g}}_{-c}, \bar{\mathfrak{g}}_c] \) and \( c(h) \neq 0 \), we have shown that \( c \notin \text{Nil} \, R \) implies that \( \mathfrak{g}_c \not\subset \text{Nil} \, \mathfrak{g} \).

The following corollary to (Theorem 4.14.2) is straightforward. In it, \( \langle \mathfrak{g}_{\text{Nil} \, R} \rangle \) denotes the ideal of \( \mathfrak{g} \) generated by \( \mathfrak{g}_{\text{Nil} \, R} \) and \( \text{Core} \, R = R - \text{Nil} \, R \cup \{0\} \).

**Corollary 4.14.1.**

Let \( \mathfrak{g} = \sum_{a \in R} \mathfrak{g}_a \) be a split Lie algebra of characteristic 0. Then \( \mathfrak{g}/\langle \mathfrak{g}_{\text{Nil} \, R} \rangle \) is reductive with root system canonically isomorphic to \( \text{Core} \, R \).

**Definition 4.14.1.**

Given a Lie algebra \( \mathfrak{g} = \sum_{a \in R} \mathfrak{g}_a \), we let \( \mathfrak{g}^1_a = \{ x \in \mathfrak{g}_a | [h, x] = a(h)x \text{ for all } h \in \mathfrak{g}_0 \} \) \( (a \neq 0) \) and \( \mathfrak{g}^0 = \mathfrak{g}_0 \), and we let \( \mathfrak{g}^1 \) be the sub algebra \( \mathfrak{g}^1 = \sum_{a \in R} \mathfrak{g}^1_a \). We say that \( \mathfrak{g} \) is symmetric if \( a([\mathfrak{g}^1_{-a}, \mathfrak{g}^1_0]) \neq 0 \) for all \( a \in R - \{0\} \).

Let $\mathfrak{g} = \sum_{a \in R} g_a$ be symmetric and let $a \in R - \{0\}, \ b \in R$ with $R_b(a)$ bounded. Then $+b \in R \Rightarrow [g_a^1, g_b^1] \neq 0$.

Proof:
Let $R_b(a) = \{b - ra, \ldots, b + qa\}$ and suppose that $[g_a^1, g_b^1] = 0$. Let $T = \{b - ra, \ldots, b\}$ and consider $\sum_{c \in T} g_c^1$. Then $V$ is a module for $g^{(1)} = kf_a + kh_a + ke_a$ with $0 \neq h_a = [e_a, f_a] \in [g_a^1, g_{-a}^1]$, so that $r_b(c) = c - (c(h_a)/a(h_a))$ maps $b - ra$ to $b + qa$ and to $b$. Thus, $q = 0$ and $a + b \notin R$.

Theorem 4.14.3.
Let $\mathfrak{g} = \sum_{a \in R} g_a$ be symmetric. Then $\mathbb{Z}_p a \cap R$ is either $\{-a, 0, a\}$ or $\{-a, 0, a, \ldots, (p-2)a\}$ for any $a \in R - \{0\}$, that is, $R = R^o \cup R^r$.

Proof:
Suppose that $S = \mathbb{Z}_p a \cap R$ is not $\{-a, 0, a, \ldots, (p-2)a\}$. Then $S_b(c) = R_a(c)$ is bounded for all $b \in S - \{0\}, c \in S$.

Let $b \in S - \{0\}$. Then the orbits $R_b(c)$ are stable under the reversal

$$R_c(b) = b - 2 (b(h_c)/c(h_c))c = ic - 2(ic(h_c)/c(h_c))c = -ic = -b.$$ 

Thus, $S$ is symmetry set in $\mathbb{Z}_p$, all of whose orbits $S_b(a)$ are bounded. Let $a \in S - \{0\}$ and consider the module $g_{ra} + \cdots + g_0 + ke_a$ for $f_a + kh_a + ke_a$, where $S_0(a) = \{-ra, \ldots, ra\}$. Then $T = \{-ra, \ldots, 0, a\}$ is stable under $r_a$, so that $a = r_a(-ra) = ra$ and $r = 1$. It follows that $S$ is reduced. But then $S = \{-a, 0, a\}$.

Let $\mathfrak{g} = \sum_{a \in R} g_a$ be symmetric and let $g_0$. Then:

1. $R \subset \text{Hom}_k(H, k)$
2. $\text{ad} H$ is triangulable on $g$
3. $g^\infty$ is symmetric with Cartan aub algebra $H_\infty$
4. Core $g$ is symmetric
5. \( g/(\text{Nil}) g \) has center 0

6. \( g^1 = \sum_{a \in R} g^1_a \) is symmetric and the Cartan sub algebra \( \bar{H}^1 \) of \( \bar{g}^1 = g^1/\text{Nil} g^1 \) is ad-diagonalizable.

**Proof :**

For (1) and (2), observe that \( \sum_{a \in R - \{0\}} g^1_a \) is a module for \( \text{ad} H \) which is annihilated by \( (\text{ad} H)^{(1)} \). Thus, \( (\text{ad} H)^{(1)} \) is upper triangular with only zeros on the diagonal, by Engle’s Theorem. It follows that \( \text{ad} H \) is triangular on \( g \). For (3), note that \( [H^\omega, g_a] = g_a \), so that \( H^\omega \) is a Cartan sub algebra of \( g^{\omega} \). For (4), note that if \( h \in [g^1_a, g^1_b] \) with \( a(h) \neq 0 \), then \( h \in g^{\omega} - \text{Nil} g \); for otherwise \( \text{Nil} g \) contains \( g^1_{-a} + kh + g^1_a \), since \( \text{Nil} g \) is an ideal which would contain \( h \), contradicting the nilpotence of \( \text{Nil} g \). Similarly, \( h \in g^{\omega} - \text{Nil} g^{\omega} \).

Thus, \( \text{Core} g = g^{\omega}/\text{Nil} g^{\omega} \) is symmetric. If \( g \) is toral, \( g^{\omega} \to g/\text{Nil} g \) has image \( (g/\text{Nil} g)^{(1)} \) and kernel \( \text{Nil} g^{\omega} \), by a straightforward verification. For (5), let \( h + \text{Nil} g \in \text{Center} g/\text{Nil} g \).

Then \( [h, g_a] \subset \text{Nil} g (a \in R - \{0\}) \). Since \( g \) is symmetric, \( g_a \subset \text{Nil} g (a \in R - \{0\}) \). Thus, \( a(h) = 0 \), for otherwise, \( g_a = [h, g_a] \subset \text{Nil} g (a \in R - \{0\}) \).

It follows that the ideal \( kh + \text{Nil} g \) is ad-nilpotent on \( g \), so that \( h \in \text{Nil} g \), by the maximality of \( \text{Nil} g \). Thus, \( g/\text{Nil} g \) has center 0.

For (6), note that if \( g \) has center 0 and \( \text{ad} H \) is diagonalizable on the \( g_a \ (a \in R - \{0\}) \), then \( \text{ad} H \) is diagonalizable since \( H \) is then abelian:

\[
h \in H^{(1)} \cap \text{Center} H \Rightarrow h \text{ central in } g \Rightarrow h = 0.
\]

**Theorem 4.14.5.**

Let \( g = \sum_{a \in R} g_a \) be symmetric with \( 0 \neq \mathbb{Z}_p a \subset R \). Then \( g_1(a)/\text{Solv} g_1^{(a)} \) is the Witt algebra \( W_1 \) for \( g_1^{(a)} = H + \sum_{i=1}^{p-1} g^1_a \).

**Proof :**

Since \( g_1^{(a)} \) is symmetric \( \bar{g} = \bar{g}_1^{(a)}/\text{Nil} g_1^{(a)} \) is symmetric with center 0, by (Theorem 4.14.4). Let \( \bar{H} \) be the image of \( H \) in \( \bar{g} \). It follows that \( \bar{H} \) is abelian, for otherwise any \( h \in \bar{H}^{(1)} \cap \text{Center} \bar{H} \) is central in \( \bar{g} \). But then \( \bar{H} \) is ad-diagonalizable on \( \bar{g} \).

Since \( \text{Center} \bar{g} = 0 \), it follows that \( \bar{H} \) has dimension 1.

Let \( S \) be a maximal proper ideal of \( g_1^{(a)} \) containing \( \text{Nil} g_1^{(a)} \). If \( \bar{H} \cap \bar{S} \), then \( \bar{g} = \bar{g}_0(\text{ad} \bar{H}) + \bar{S} = \bar{H} + \bar{S} = \bar{S} \) and \( g = S \). Thus, \( \bar{H} \not\subset S \). Since \( \dim \bar{H} = 1 \), it follows that \( \bar{H} \cap \bar{S} = \bar{0} \) and
\[ S = \sum_{i=1}^{p-1} S_{ia}, \] where \( S_{ia} = \tilde{S}_{ia} (\text{ad} \tilde{H}) \). It then follows that \( \text{ad} S_{ia} \) is nil \((1 \leq i \leq p - 1)\) and \( S \) is nilpotent. Thus, \( S = \text{Solv}_a \) 1(a). It follows that \( g^{1(a)}/\text{Solv}_a \) \( g^{1(a)} \) is simple of rank 1 and toral rank 1, so that it is \( W_1 \) by Kaplansky.12

**Theorem 4.14.6.**

Let \( g = \sum_{a \in \mathbb{R}} g_a \) be symmetric Lie algebra. Then \((g_0, R)\) is a Lie root system.

**Proof:**

Let \( a \in R - \{0\} \) and choose \( h_a \in [g_{-a}, g_a] \) with \( \alpha(h_a) \neq 0 \). Define \( a^0 \in \text{Hom}_K(g_0, K) \) by \( a^0(v) = 2(v(h_a)/\alpha(h_a)) \), let \( r_a(v) = v - a^0(v) a (v \in g_0^1) \), and note that \( a^0(a) = 2 \) and \( r_a R_b(a) = R_b(a) \) for every bounded \( a \)-orbit \( R_b(a) (b \in R) \), by (Theorem 4.8.1). Thus, \((g_0, R)\) is a root system in the sense of (section 4.7), and it remains to verify the supplemental "Lie conditions that \( R = R^0 \cup R^* \) and each "Witt orbit" \( R_b(a) (a \in R^0 - \{0\}, b \in R) \) has 1 or \( p - 1 \) or \( p \) elements. The condition \( R = R^0 \cup R^* \) was proved in (Theorem 4.10.1).

Next, consider a "Witt orbit" \( R_b(a) (a \in R^0 - \{0\}, b \in R) \). we must show that \( R_b(a) \) has 1 or \( p - 1 \) or \( p \) elements. Accordingly, we may, without loss of generality, assume that \( 1 < |R_b(a)| \leq p - 1 \). To show that \( |R_b(a)| = p - 1 \), we may replace \( g = \sum_{a \in \mathbb{R}} g_a \) by another symmetric Lie algebra having corresponding \( a \in R^w, b \in R \) and \( R_b(a) \) of the same length. It follows that we can, successively, replace \( g \) by \( \sum_{i,j=0}^{p-1} g_{ia+jb}^1 \) with \( g_0 \defeq \sum_{i,j=0}^{p-1} [g_{ia+jb}^1, g_{ia-jb}^1], g/\text{Center} g \). Consequently, we may assume, with no loss of generality, that \( g = \sum_{i,j=0}^{p-1} g_{ia+jb}^1 \) and Center \( g = \{0\} \).

For each \( 1 \leq i \leq p - 1 \), choose \( e_i \in g_{ia}^1, f_i \in g_{ia}^1 \), \( h_i = [e_i, f_i] \) such that \( \alpha(h_i) = 1 \) and define \( r_i(a)(v) = v - 2(v(h_i)/\alpha(h_i)) ia = v - 2v(h_i) a (v \in L_0) \).

Note that \( r_i(a) = -a (1 \leq i \leq p - 1) \). By (Theorem 4.13.1) and the assumption \( |R_b(a)| \leq p - 1 \), the \( ia \)-orbits \( T_{b'}(ia)(b' \in T) \) of \( T = R \cap (b + \mathbb{Z}_p a) \) are \( r_i \) stable for any \( 1 \leq i \leq p - 1 \). It follows, in particular, that \( T \) contains \( r_i(b) = b - 2b(h_i)a \), so that \( \mathbb{Z}_p \) contains \( b(h_i) \). Define \( r_i(b) = b - b(h_i) a \in b + \mathbb{Z}_p a \) and note that \( r_i(a) = c_i \) since \( c_i(h_i) = 0 \), for \( 1 \leq i \leq p - 1 \). Since, for \( 1 \leq i \leq p - 1 \), we have \( r_i(c_i) = c_i \) and \( r_i T_{b'}(ia) = T_{b'}(ia) (b' \in T) \), one can easily verify that :

1. \( T \) has either 1 or 2 \( ia \)-orbits.

---

2. If $T$ has $2$ $ia$-orbits $T', T''$, then an odd number of elements and $T'' = T - T'$, and then one of $T', T''$ has an odd number of elements and contains $c_i$, and the other has an even number of elements.

We claim, for each $1 \leq i \leq p - 1$ and each $b' \in T$, that the $ia$-orbit $T'_b (ia)$ is stable under each $r_i (1 \leq j \leq p - 1)$. By (1) and (2) above, $T = T'_b (ia)$ (case of one $ia$-orbit) or $T' = T'_b (ia)$ and the $ia$-orbits of $T$ are $T'$ and $T'' = T - T'$, where one of $T', T''$ has odd number of elements and the other has an even number of elements. Since $T = \eta (T) = \eta (T') \cup \eta (T'')$ and since $r_i (ia) = -ia$, one can easily verify that $\eta (T'), \eta (T'')$ are $ia$-orbits of $T$, thus that they are $T', T''$ in one of the orders $T', T''$ or $T'', T'$. But $r_j$ preserves "odd" and "even" numbers of elements. It follows that $\eta (T') = T'$ and $\eta (T'') = T''$ for $1 \leq j \leq p - 1$.

Take one fixed $ia$-orbit $T'_b (a)$ of $T$. Since it is stable under $r_1, ..., r_p - 1$ and $\eta (a) = -a (1 \leq j \leq p - 1)$, each of the $r_1, ..., r_p - 1$ reverse the $a$-orbit $T'_b (a)$.

It follows that $r_1 (b') = ... = r_p - 1 (b')$ and $b' (h_1) = ... = b' (h_{p - 1})$ for suppose that all $b' \in T$. Consequently, we have $b (h_i - h_j) = b (h_i - h_j) = 0$ for all $1 \leq j \leq p - 1$.

Since $g = \sum_{i,j=0}^{p - 1} g_{ia + jb}$ with $g_{ia}^0 \equiv \sum_{i,j=0}^{p - 1} [g_{ia + jb} , g_{ia - jb}]$, it follows that:

$h_i - h_j \in \text{Center } g = \{0\}$ and $h_i = h_j$ for all $1 \leq i, j \leq p - 1$.

Finally, we let $h$ denote $h_1$, so that $h = h_i$ for $1 \leq i \leq p - 1$ and $a(h) = 1$. By the flexibility in the choice of $h_i$ above, it follows that $e \in g\|a, f \in g\|a$ with $a[e, f] = 1$ implies that $h = [e, f]$, for $1 \leq i \leq p - 1$. We claim that:

$e' \in g\|a, f' \in g\|a \cdot a([e', f']) = 0$ implies that $[e', f'] = 0$ for $1 \leq i \leq p - 1$.

To see this, let $1 \leq i \leq p - 1$, choose $e \in g\|a, f \in g\|a$ such that $h = [e, f]$ and $e' \in g\|a, f' \in g\|a \cdot h' = [e', f], a(h') = 0$. We claim that $h' = 0$.

To see this, let $h'' = [e', f], h''' = [e', f']$. Consider first the case where $a(h'') = a(h''') = 0$ then $1 = a(h + h''') = a([e + e', f])$, so that $h = [e + e', f]$ as observed above. But then

Fig(2.4)
\( h = h + h' \) and \( h'' = 0 \). Similarly, \( h''' = 0 \). It follows that \( [e + e', f + f'] = h + h' + h'' + h''' = h + h' \) and \( h'' = 0 \). Since \( 1 = a(h) = (a(h + h')) = a((e + e', f + f')) \), it follows from the discussion above that \( h = h + h' \) and \( h'' = 0 \). Thus, \( [e', f'] = 0 \) in the present case.

Next, consider the case where one of \( a(h''), a(h'') \) is not zero. We may then assume with no less of generality that \( a(h'') \neq 0 \), for otherwise we can interchange \( h'' \), \( h''' \). By replacing \( e' \) by \( (1/a(h'')) e' \), we may also assume that \( 1 = a(h'') = a(e', f) \). But then \( h' = h \), by our earlier discussion. But then \( h + h' = [e', f] + [e', f'] + [e', f + f'] \) and \( 1 = a([e', f + f']) \) implies that \( [e', f + f'] = h \), by our earlier discussion, so that \( h + h' = h \) and \( h = 0 \).

By the preceding paragraph, we have \( g_0^1 = \text{def} \sum_{i,j=0}^{p-1} g_{ia + jb}^1 = kh \), that is, the Cartan sub algebra \( H = g_0 \) of \( g \) is one dimensional. Let \( g = \sum_{i=0}^{p-1} g_{ia} = \sum_{i=0}^{p-1} g_{ia}^1 \) and let \( S = \text{Solv} g^a \).

We observed in (Theorem 4.14.5) that \( g/S \) is the Witt algebra \( W_1 \). Since \( S \) is a proper ideal of \( g \), we have \( H \not\subset S \). Since \( \dim H = 1 \), it follows that \( H \cap S = \{0\} \). Consequently, \( = \sum_{i=0}^{p-1} S_{ia} \).

Regard \( V = \sum_{c \in T} g_c \) as \( g \)-module via adjoints, where \( T = R \cap (b + \mathbb{Z}_p a) \not\subset b + \mathbb{Z}_p a \). Let \( f: g \rightarrow \text{Hom} V \) be the associated representation. Since \( g_{b \not\subset b + \mathbb{Z}_p a} \), \( f(S_{ia}) \) consists of nilpotent transformations of \( V \) for \( 1 \leq i \leq p - 1 \). By the theorem of Jacobson\(^{13}\) on weakly closed sets of linear transformations, it follows that \( f(S) \) consists of nilpotent linear transformations of \( V \). Letting \( \bar{V} \) be any irreducible subquotient \( V_i / V_{i+1} \), where \( V_1, \ldots, V_r \) is a composition series for \( V \), and letting \( f: g \rightarrow \text{Hom} \bar{V} \) be the associated representation of \( g \), we claim that \( f(S) \bar{V} = \{0\} \). We use the notation \( \bar{v} = v + V_{i+1} \in \bar{V} \) for \( v \in V_i \).

Note that since \( f(S) \) is a Lie algebras of nilpotent linear transformations of \( \bar{V} \),

\[ \bar{V} = \{ v \in \bar{V} | f(S) \bar{v} = 0 \} \]

is nonzero. One sees easily that \( \bar{V}_0 \) is an \( g \)-sub module of \( \bar{V} \), since \( S \) is an ideal of \( g \):

\[ f(S) \bar{v} = 0 \Rightarrow f(S)[f(g)] \bar{v} = 0. \]

Since \( \bar{V} \) is irreducible, \( \bar{V} = \bar{V}_0 \) and \( f(S) \bar{V} = \{0\} \).

Since \( f(S) \bar{V} = \{0\} \), we may regard \( \bar{V} \) as a module for \( W_1 = g/S \) where the module action given by

\[ (x + s)\bar{v} = [x, v] \]

For \( x + S \in g/S, \bar{v} = v + V_{i+1} \in \bar{V}, [x, v] = [x, v] + V_{i+1} \in \bar{V} \). We let \( \bar{V} = \sum_{c \in T} \bar{V}_c \) be the root decomposition of \( \bar{V} \) with respect to \( H \subset g \) and \( H + S/S = H \) in \( g/S = W_1 \), and regard

The restricted irreducible $W_1$-modules have dimensions $1, p - 1$ or $p$ and one-dimensional weight spaces. It follows that $|T'| = 1, p - 1$ or $p$. Since $1 < |R_b(a)|$, one of $b - a, b + a$ is in $R$. Consequently, we can choose $\bar{V}$ such that $\bar{V}_c \neq \{\bar{0}\}$ or $V_{b+a} \neq \{\bar{0}\}$, by (Prop 4.14.1). for this, $|T'| \neq 1$, so that $|T'| = p - 1$ or $p$. But then $p - 1 \leq |T'| \leq |R_b(a)| \leq p - 1$, so that $|R_b(a)| = p - 1$ as was to be proved.

**Corollary 4.14.2.**

Let $\mathfrak{g}$ be a symmetric Lie algebra. Then $\mathfrak{g}$ is reflective if and only if $\mathbb{Z}_p a \not\subset R$ for all $a \in R - \{0\}$ if and only if $R$ is classical : $R = R \cup \{0\}$.

**Proof:**

The latter condition is equivalent, by theorem 11.6, to the condition that the Lie root system $R$ is classical, that is $R = R \cup \{0\}$, which in turn is equivalent, by the results stated Section 4.12, to the condition that the orbits $R_b(a)$ ($a \in R - \{0\}, b \in R$) of the Lie root system $R$ are all bounded.

**Corollary 4.14.3.**

Let $\mathfrak{g} = \sum_{a \in R} g_a$ be symmetric. Then $\dim g_a = 1$ for all $a \in R$.

**4.15 Exclusion of sub types of $R$ and $\mathfrak{g}$**

Let $R$ be a Lie root system and / or $\mathfrak{g} = \sum_{a \in R} g_a$ a symmetric Lie algebra. We have observed that $\mathfrak{g}$ is reflective and Core $\mathfrak{g}$ is classical if and only if $R$ is classical : $R$ has no $Ra = R \cap \mathbb{Z}a$ of type $W_1$. The latter condition can be restricted "$R$ excludes $W_1$" in the following language.

**Definition 4.15.1.**

Let $S$ be a root system of rank $r$. Then $R$ excludes $S$ if $Ra_1, \ldots, a_r = R \cap (\mathbb{Z}a_1, \ldots, \mathbb{Z}a_r)$ is not isomorphic to $S$ for any $a_1, \ldots, a_r \in R$. $\mathfrak{g} = \sum_{a \in R} g_a$ excludes $S$ if $R$ excludes $S$.

**Theorem 4.15.2.**

Let $R$ be an irreducible Lie root system. Then :

1. $R$ is classical or $R = R^0$ if and only if $R$ excludes $W \oplus A$ and $T_2$;
2. $R$ is classical or rank 1 if and only if $R$ excludes $W \oplus A, W \oplus W$ and $S_2$. 

Proof:

One direction for both (1) and (2) is clear. For the other, suppose that $R$ excludes $W \oplus A$ and $T_2$. Suppose that $a \in R^0, b \in R^*$ and consider $Rab$. If $a \neq 0$, then $Rab = W \cup A$ and $a + b \notin R$, since the possibilities $W \oplus A, T_2$ are exclude. It follows that $a, a' \in R^0$ and $a + a' \in R$ implies $a + a' \in R^0; a + a' =_\text{def} -b \in R^* \Rightarrow a + b \in R \Rightarrow a = 0 \Rightarrow a + a' = a' \in R^0$.

Similarly, $b, b' \in R^*$ and $b + b' \in R$ implies $b + b' \in R \cup \{0\}; b + b' = -a \in R^0 \Rightarrow a + b \in R \Rightarrow a = 0 \Rightarrow b + b' = 0 \in R \cup \{0\}$. Since $R$ is irreducible, it follows that $R = R^0$ or $R = R^* \cup \{0\}$. This proves (1).

For (2), suppose that $R$ excludes $W \oplus A, W \oplus W$ and $S_2$. Then $R$ also excludes $T_2$, so that $R$ is classical or $R = R^0$. Let $a \in R - \{0\}$ and defines $S = R \cap ka, T = (R - S) \cup \{0\}$. Take $b \in T$ and note that $b \neq 0$ implies $Rab = W \cup W$ and $a + b \notin R$, by exclusion of $W \oplus W$. It follows that $a, a' \in S, a + a' \in R$ implies $a + a' \in S$ and $b, b' \in T, b + b' \in R$ implies $b + b' \in T$, as in earlier arguments.

By irreducibility of $R$, therefore, $R = S$ and $R$ has rank 1.