Chapter 1

Operators with Singular Continuous Spectrum and Rank One Perturbations

If $[\alpha, \beta] \subset \operatorname{spec}(A)$ and A has no a.c. spectrum, we show that $A + \lambda P$ has purely singular continuous spectrum on (α, β) for a dense G_δ of λ's. Our purpose is to show that most results of Gesztesy, Kiselev, and Simon are valid for rank one perturbations of self-adjoint operators, which are not necessarily semibounded. We use the fact that rank one perturbations constitute self-adjoint extensions of an associated symmetric operator. The use of so-called Q-functions facilitates the descriptions. In the special case that ω belongs to the scale space $\boldsymbol{H}_{-1}$ associated with $\boldsymbol{H}_{+2} = \operatorname{dom} |A|^{1/2}$ the limiting perturbation $A(\infty)$ is shown to be the generalized Friedrichs extension.

Section (1.1): Rank One Operators:

The subject of rank one perturbations of self-adjoint operators and the closely related issue of the boundary condition dependence of Sturm-Liouville operators on $[0,\infty)$ has a long history. We're interested here in the connection with Borel-Stieltjes transforms of measures ($\operatorname{Im} z > 0$):

$$F(z) = \int \frac{d\rho(x)}{x - z} \tag{1}$$

where ρ is a measure with

$$\int (|x|+1)^{-1} d\rho(x) < \infty \tag{2}$$

In two fundamental papers Aronszajn [2] and Donoghue [3] related F to spectral theory with important later input by Simon-Wolff [4]. In all three works, as in ours, the function (y real)

$$G(y) = \int \frac{d\rho(x)}{(x-y)^2}$$

plays an important role. Note we define G to be $+\infty$ if the integral diverges. Note too if $G(y)<\infty$, then the integral defining F is finite at $z = y$ and so we can and will talk about $F(y)$.

Donoghue studied the situation

$$A_\lambda = A_0 + \lambda P,$$

Where $P\psi = (\varphi,\psi)\varphi$ with φ a unit vector cyclic for A. $d\rho$ is then taken to be spectral measure for φ, that is,

$$(\varphi, e^{isA_0}\varphi) = \int e^{isx}\, d\rho(x)$$

Aronszajn studied the situation

$$H_{formal} = \frac{d^2}{dx^2} + V(x)$$

on $[0, \infty)$, where V is such that the operator is limit point at ∞. Then, there is a one-parameter family of operators, H_θ, with boundary condition

$$u(0)\cos 0 + u`(0)\sin\theta = 0.$$

ρ is the conventional Weyl-Titchmarsh-Kodaira spectral measure for a fixed boundary condition, θ_0

An important result of the Aronszajn-Donoghue theory is

Theorem (1.1.1)[1]: E is an eigenvalue of A_λ (resp. H_θ) if and only if

(i) $G(E) < \infty$,

(ii) $F(E) = -\lambda^{-1}$ (resp. $\cot(\theta-\theta_0)$)

Our goal here is to prove the following pair of Theorem:

Theorem (1.1.2)[1]: $\{E|G(E) = \infty\}$ is a dense G_δ in $\operatorname{spec}(A_0)$ (resp. H_{θ_0}).

Theorem (1.1.2) is a generalization of del Rio [5]. Gordon [6,7] has independently obtained these results by different methods.

Theorem (1.1.3)[1]: let $d\rho$ be a measure obeying (2). Let

$$G(y) = \int \frac{d\rho(x)}{(x-y)^2}$$

Then, $\{y|G(y) = \infty\}$ is a dense G_δ in $\operatorname{supp}(d\rho)$, the support of $d\rho$.

Proof. The following are fundamental facts about Borel-Stieltjes transforms and their relation to $d\rho$ (see [8]).

(*i*) $\lim_{\varepsilon\downarrow 0} F(E + i\varepsilon) \equiv F(E+i0)$ exists and is finite for Lebesgue a.e.E.

(*ii*) $d\rho_{ac}$ is supported on $\{E|\mathrm{Im}F(E + i0)>0\}$.

(*iii*) $d\rho_{\sin g}$ is supported on $\{E \mid \lim_{\varepsilon\downarrow 0} \ \mathrm{Im}\ F(E+i0) = \infty\}$.

If $G(y)< \infty$, it is easy to see that $\lim_{\varepsilon\downarrow 0} F(E + i0)$ exists, is finite and real. Thus, if $G(y)< \infty$ on an interval $(\alpha,\beta) \subset IR$, $d\rho\,(\alpha,\beta)= 0$, that is, $((\alpha,\beta)\cap \mathrm{supp}(d\rho) = \phi$. Thus, $\{y|G(y)=\infty\}$ is dense in $\mathrm{supp}(d\rho)$.

That $\{y|G(y) = \infty\}$ is a G_δ follows from the fact that G is lower semi-continuous. To be explicit, let

$$G_m(y) = \int \frac{d\rho(x)}{(x-y)^2 + (m^{-1})^2}$$

Which is a C^∞ function by (2) and $G(y) = \sup_m G_m(y)$. Thus

$$\{y|G(y) = \infty\}= \{ y|\forall n, \exists m G_{\mathbf{m}}(y)>n\}$$

$$= \bigcap_n \bigcup_m \{y \mid G_m(y) > n\}$$

is a G_δ .

Example (1.1.4)[1]: Let $A \subset [0, 1]$ be a nowhere dense set of positive measure (e.g., remove the middle open $\frac{1}{4}$ from [0, 1], the middle $\frac{1}{9}$ from the remaining two pieces, the middle $\frac{1}{16}, \dots, \frac{1}{n}$ at the $(n – 1)^{st}$ step).Let

$$\widetilde{F}(y) = | A \cap [0, y] |,$$

where $|.|$ is Lebesgue measure. Then $\widetilde{F}$ is Lipschitz; indeed, if $x < y$, $|\widetilde{F}(x) - \widetilde{F}(y)| = | A \cap [x,y] | \le | x-y |$. But $\widetilde{F}[A]=[0,|A|]$ has non-empty interior. Thus for our F, we need more than just Lipschitz properties (our F is certainly not Lipschitz but $F|\{y|G(y) < \alpha\}$ is the restriction of a Lipschitz function to that set).

The idea of the proof will be to break up $\{y \mid G(y) < \infty, y\in \mathrm{supp}(d\rho)\}$ into a countable union of nowhere dense sets, $A_{\mathbf{n}}$, so that F is a homeomorphism on each of those sets. On each $A_{\mathbf{n}}$, G will be close to constant. We'll use:

Lemma (1.1.5)[1]: Let $B \subset \mathbf{R}$ be a nowhere dense set and let $F: B\to \mathbf{R}$ be a function obeying for $x < y$, with $x,y\in B$:

$$\alpha(y-x) < F(y) – F(x) < \beta(y – x) \qquad (3)$$

For fixed α, $\beta > 0$. Then F[B] is nowhere dense.

Proof. By (3) F has a unique continuous extension to $\overline{B}$ obeying (3). $\mathbf{R} \setminus \overline{B}$ is a union of intervals (x_i, y_i) with x_i, $y_i \in \overline{B}$. Extend F to the interval by linear interpolation using slope $\frac{1}{2}(\alpha+\beta)$ on any semi-infinite subintervals of $\mathbf{R}\setminus\overline{B}$. The extended F also obeys (3) and so defines a homeomorphism of **R** to **R**. As a homeomorphism, it takes nowhere dense sets to nowhere dense sets.

Lemma (1.1.6)[1]: Let $d\rho$ obey (2) Then

$$\{F(y|G(y)< \infty \text{ and } y \in \operatorname{supp}(d\rho)\}$$

is a countable union of nowhere dense subsets of **R**.

Note that $G(y)<\infty$ implies the integral defining $F(y)$ is absolutely convergent and $F(y)$ is real. The proof will depend critically on the fact that F is the boundary value of an analytic function. That such considerations must enter is seen by.

Proof . We first break $A = \{y\in\operatorname{supp}(d\rho)|G(y)< \infty\}$ into a countable family of sets A_n so that for each n, there is $a_n > 0, \delta_n > 0$ so that

(i) for $y \in A_n$, $\frac{8a_n}{9} < G(y) \le a_n$;

(ii) for $y \in A_n$, $\int_{|x-y|<\delta_n} \frac{d\rho(x)}{|x-y|^2} \le \frac{a_n}{21}$;

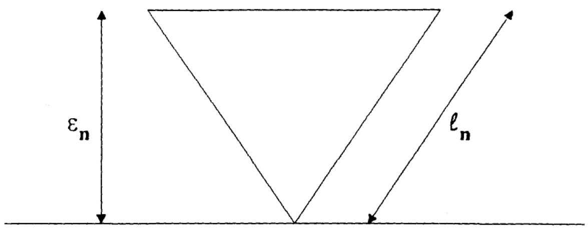
(iii) $\bigcup_{y \in A_n} [y-\beta\delta_n, y+\beta\delta_n]$ is connected where $\beta = \frac{1}{18}\left(\frac{3}{4}\right)^{3/2}$

Such a breakup exists for we can first break **R** into intervals $\left[\left(\frac{8}{9}\right)^{m+1}, \left(\frac{8}{9}\right)^{m}\right]$ and pigeonhole G by its values. Since $G(y) < \infty$ implies $\lim_{\delta\downarrow 0}$

$$\int_{|x-y|<\delta} \frac{d\rho(x)}{(x-y)^2} = 0,$$

we can break each such set into countably many sets where (ii) holds. Then we can break each such set into countably many sets so that (iii) holds by looking for gaps of size longer than $\delta_n\beta$.

Operators with Singular Continuous Spectrum



y

Fig.(1)

Each A_n is nowhere dense by Theorem (1.1.2) and we'll show that on A_n, $y > x$ implies that

$$\frac{1}{3}a_n(y-x) < F(y) - F(x) < \frac{5}{3}a_n(y-x), \tag{4}$$

so that the lemma follows from Lemma (1.1.11).

Define $\ell_n = \beta\delta_n$ and $\varepsilon_n = \sqrt{\tfrac{3}{4}}\ell_n$. *For* $y \in A_n$, *let* $\Delta_n(y)$ be t he triangle in **C**(see Fig (1))

$$\Delta_n(y) = \left\{ z \mid 0 < \operatorname{Im} z \le \varepsilon_n, \left| \arg(z-y) - \frac{\pi}{2} \right| \le \frac{\pi}{6} \right\}.$$

This is the equilateral triangle of side ℓ_n with one side parallel to the real axis at distance ε_n from that axis and the opposite vertex at y. For $z \in \Delta_n(y)$, define

$$G(z) = \int \frac{dp(x)}{(x-z)^2} = \frac{dF}{dz}.$$

We claim that for $z \in \Delta_n$

$$|G(z) - a_n| \le \frac{a_n}{3}. \tag{5}$$

Accepting (5) for the moment, let us prove (4). Boy the fundamental theorem of calculus, (5) implies for $z, z' \in \Delta_n(y)$:

$$|F(z) - F(z') - a_n(z-z')| \le \frac{a_n}{3}|z-z'| \tag{6}$$

By hypothesis (iii) on A_n, $\bigcup_{y \in A_n} \Delta_n(y)$ is connected and so, given $y < y' \in A_n$ we can find a finite sequence $y_0 = y < y_1 < \ldots < y_n = y'$ and $z_1, \ldots, z_n$ so that (see Fig(1))

$$z_j \in \Delta_n(y_{j-1}) \cap \Delta_n(y_j) and \left|z_j - y_{j-1}\right| = \left|z_j - y_i\right| = \left|y_j - y_{j-1}\right|, \tag{7}$$

By (6) and (7)

$$\left|F(y) - F(y)' - a_n(y - y')\right| \le \frac{2a_n}{3}(y - y')$$

which is (4).

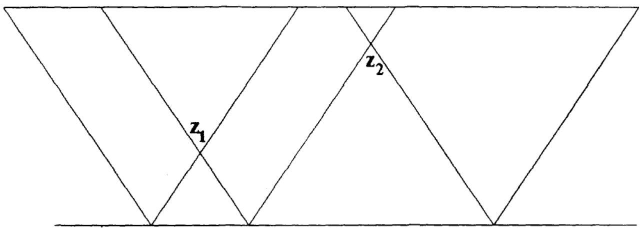


Fig.(2)

Thus we need only prove (5). We write

$$|G(z)-a_{\mathbf{n}}| \le |b_{\mathbf{0}}| + |b_{\mathbf{1}}| + |b_{\mathbf{2}}| + |b_{\mathbf{3}}|,$$

Where

$$b_0 = G(y) - a_{n,}$$
$$b_1 = \int_{|x-y|<\delta_n} \frac{dp(x)}{|x-z|^2},$$
$$b_2 = \int_{|x-y|<\delta_n} \frac{dp(x)}{|x-y|^2},$$
$$b_3\, \widetilde{G}(z) - \widetilde{G}(y),$$

With

$$\widetilde{G}(z) = \int_{|x-y|\ge\delta_n} \frac{dp(x)}{(x-z)^2}$$

By hypothesis (i) on $A_n, |b_0| \le \frac{a_n}{9}$.

By hypothesis (ii) on $A_n, |b_2| \le \frac{a_n}{21}$. By elementary trigonometry,

$$z \in \Delta(y)\, and\ x \in \mathbf{R} \Rightarrow |z - y| \ge \sqrt{\frac{3}{4}}|x - y|. \tag{8}$$

Thus

$$|b_1| \le \frac{4}{3}|b_2|,$$

so

$$|b_1|+|b_2| \le \frac{7}{3}\frac{a_n}{21} = \frac{a_n}{9}$$

Finally, using the fundamental theorem of calculus and (8)

$$\begin{aligned}|b_3| &\le 2|z-y|\left(\frac{4}{3}\right)^{3/2}\delta_n^{-1} \int_{|x-y|\ge\delta_n} \frac{dp(x)}{(x-y)^2} \\ &\le 2\delta_n^{-1}\ell_n \left(\frac{4}{3}\right)^{3/2} a_n \\ &= 2\beta\left(\frac{4}{3}\right)^{3/2} a_n = \frac{a_n}{9}\end{aligned}$$

by definition of the constant β. Thus (5) hold

Theorem (1.1.7)[1]: $\{\lambda|$ A_λ has no eigenvalues in spec $(A_0)\}$ (resp. $\{\theta|H_\theta$ has no eigenvalues *in* spec $(H_{\theta_0}\}$ *is a dense* $\mathbf{G}_\delta$ in **R** (resp. $[0, 2\pi]$)).

Proof. The maps M_1: $\mathbf{R}\backslash\{0\}\to \mathbf{R}\backslash\{0\}$ by $M_1(\lambda) = -\lambda^{-1}$ and M_2: $[0, \pi)\to\mathbf{R} \cup\{\infty\}$by $M_2(\theta) =\cot(\theta - \theta_0)$ are homomorphisms. Thus, by Lemma (1.1.6)

$$\{\lambda|\ \exists\ E \text{ s.t. } G(E) < \infty, E\in\text{space } (A_0), F(E) = -\lambda^{-1}\}$$

and

$$\{\theta|\ \exists\ E \text{ s.t. } G(E) < \infty, E\in\text{space } (A_0), F(E) = \cot (\theta- \theta_1)\}$$

are countable unions of nowhere dense sets. Its omplement is thus a dense set by Baire category theorem. But by Theorem (1.1.1), this is precisely $\{\lambda|A_\lambda$ has no eigenvalues on spec $(A_0)\}$, which we conclude is dense. By general principles [9], it as also a G_δ .

Here are some simple corollaries of Theorem (1.1.7). We state them in the rank one case but they hold in the cot $(\theta - \theta_0)$B.C. case also.

Corollary (1.1.8)[1]: Suppose that A_0 is an operator with no a.c. spectrum and P is a rank one projection whose range is cyclic for A. Then for a dense G_δ of λ's, $A_\lambda = A + \lambda P$ has only singular continuous spectrum in spec $(A_0)^{\text{int}}$

Proof. A_λ has no a.c .spectrum since the a.c. spectrum is left invariant by finite rank perturbations. space(A_0) has no eigenvalues for adense G_δ of λ there can be eigenvalues on $\mathbf{R}\backslash$spec (A_0) and so point spectrum on ∂ (spec(A_0)).But there cannot be point spectrum in spec $(A_0)^{int}$

Corollary (1.1.9)[1]: Suppose that A_0 is an operator with no a.c. spectrum and an interval $[\alpha,\beta] \subset$ spec (A_0). Let P be a rank one projection whose range is a cyclic vector for A_o. Then for a dense G_δ of λ's, $A_0 + \lambda P$ has singular contionuous spectrum on all of (α,β) and only singular continuous spectrum there .

Theorem (1.1.10)[1]: Let V(x) be a locally L_1 function on $[0,\infty)$ and let $H_\theta = \frac{d^2}{dx^2} + V(x)$ with θ boundary conditions. Suppose there is some θ_0 and $\alpha < \beta$ so that

(i) $[\alpha,\beta]$ spec(H_{θ_0})

(ii) for Lebesgue a.e., $E_0 \in [\alpha,\beta]$, there exists a function φE_0 obeying

$$- \varphi''(x)+V(x)\, \varphi(x) = E_0 \varphi(x), \tag{9}$$

$$\int_0^\infty |\varphi(x)|^2 \, dx < \infty \tag{10}$$

Then:

(i) For a dense G_δ of E's in $[\alpha,\beta]$, there is no solution of (9) obeying (10).

(ii) For Lebesgue a.e. θ, H_θ has only point spectrum in (α,β).

(iii)For a dense G_δ of θ, H_θ has only singular continuous spectrum in (α,β).

Proof. If E is such that (9) has a solution obeying (10), then φ_E obeys some boundary condition at x = 0 and so E is an eigenvalue of some H_θ.Thus (i) follows from Theorem (1.1.2).

To prove (ii), note that if E_0 has a solution and E_0 is not an eigenvalue of H_{θ_0}, $\lim_{\varepsilon\downarrow 0} \int_0^\infty |G(0,x;E+i\varepsilon)|^2 dx < \infty$.Now apply the ideas of Kotani [10] and Simon- Wolff [4].(iii) follows from Theorem (1.1.7).

Example (1.1.11)[1]: Suppose that $[a,b] \subset spec\left(-\frac{d^2}{dx^2}+V(x)\right)$ and that for a.e. E $\subset$ [a, b], $\lim_{x\to\infty} \cdot \frac{1}{|x|} In\|T_E(x)\| = \gamma(E)$ and is positive. Here T is the standard transfer matrix, that is,

$$T_E(x) = \begin{pmatrix} \varphi_1(x) & \varphi_2(x) \\ \varphi_1'(x) & \varphi_2'(x) \end{pmatrix},$$

where φ_i obeys - u″ + Vu = Eu with $\varphi_1(0) = \varphi'_2(0) = 1$ and $\varphi'_1(0) = \varphi_2 = (0)$. Then (i) implies there must be a dense G_δ of E where either $\lim . \frac{1}{|x|}\|T_E(x)\|$ fails to exist or is zero. Thus, a positive limit can never exist for all E in an interval. Results of this genre have been found previously by Goldsheid [11] and Carmona [12].

Example (1.1.12)[1]: Consider a one-dimensional random model with localization, for example, the GMP model [13,12]. Then for almost every E in $[\alpha,\infty)$, one knows $\gamma(E)$ exists and is positive. It follows from Theorem (1.1.10) that for a locally uncountable set of boundary conditions (a Lebesgue typical set), one has pure point spectrum, while for a distinct set of locally uncountable boundary conditions (a Baire typical set), one has singular spectrum. Each spectral type is unstable to change to the other spectral type.

Example (1.1.13)[1]: Let $H = \frac{d^2}{dx^2} + \cos(\sqrt{x})$ on $L^2(0,\infty)$, a model studied by Stolz [14]. As proven by him for any boundary condition θ:

$$\text{Spec } (H_\theta) = [-1, \infty).$$

Spec (H_θ) is purely absolutely continuous on $(1, \infty)$. Krisch et al. [15] prove that for a.θ.H_θ has pure point spectrum in [-1, 1] only. Our results show that for a dense G_δ of θ, the spectrum is purely singular continuous. Once again you have intertwined purely pure point and purely singular-continuous spectrum.

Finally, we consider the case of Anderson model:

Section (1.2): Self-Adjoint Operators and Rank One Perturbations

Let A be a self-adjoint operator in a Hilbert space H. Its rank one perturbations $A + \tau\,(.,\omega)\,\omega$, $\tau\in\mathbf{R}$, are studied when ω belongs to the scale space H_{-2} associated with H_{+2}= dom A and (.,.) is the corresponding duality. If A is nonnegative and ω belongs to the scale space H_{-1}, Gesztesy and Simon [17] prove that the spectral measures of $A(\tau), \tau\in R$, converge weakly to the spectral measure of the limiting perturbation $A(\infty)$. In fact $A(\infty)$ can he identified as a Friedrichs extension. Further results for nonnegative operators A were obtained by Kiselev and Simon [18] by allowing $\omega\in H_{-2}$. Our purpose is to show that most results of Gesztesy, Kiselev, and Simon are valid for rank one perturbations of self-adjoint operators, which are not necessarily semi-bounded. We use the fact that rank one perturbations constitute self-adjoint extensions of an associated symmetric operator.

The use of so-called Q-functions [19,20] facilitates the descriptions. In the special case that ω belongs to the scale space H associated with H_{+1} =dom $|A|^{1/2}$, the limiting perturbation $A(\infty)$ is shown to be the generalized Friedrichs extension [21].

Let A be a self-adjoint operator in a Hilbert space H with inner product [.,.], Associate with A the Hilbert space H_{+2}, which is dom A provided with the inner product $[f,g]_{+2} = [f,g]+[Af,Ag]$, $f,g \in$ dom A. Define the dual space H_{-2} in the usual way, denote the duality between H_{+2} and δ_{-2} by (.,), and extend the form $(.,\omega)$ to dom δ^*, cf. [20]. For an element ω in the scale space H_{-2}, consider the rank one perturbations of A:

$$A(\tau) = A + \tau\,(.,\omega)\omega, \tau \in \mathbf{R} \tag{11}$$

In this formula A stands for the unique continuation of the original operator A acting from H to H_{-2}, cf. [20]. When $\omega \in$ H, no continuation of A is needed and $(.,\omega)$ can be replaced by $[.,\omega]$. The family $A(\tau),\tau\in\mathbf{R}$, in (11) must be augmented by a certain self-adjoint operator or, in general, relation $A(\infty)$ to account for all possible “perturbations” of A. F. Gesztesy and B. Simon [17] prove that the spectral measures of $A(\tau)$ converge weakly to the spectral measure of $A(\infty)$, when A is a nonnegative operator and ω belongs to the scale space H_{-1} associated with H_{+1}=dom

$A^{1/2}$. The case where A is nonnegative and $\omega \in H_{-2}$ is studied by A. Kiselev and B. Simon [18]. The main ingredient which Gesztesy, Kiselev, and Simon use, is the "basic formula" (1) in [17] which makes it necessary to distinguish between $\tau \in \mathbf{R}$ and $\tau = \infty$

We take another point of vie by associating a symmetric operator with rank one perturbations [22,19,18,23]. Let A be a self-adjoint operator, not necessarily semibounded, and let $\omega \in H_{-2}$. Introduce

$$S = \{\{f,g\} \in A : (f,\omega) = 0\}. \tag{12}$$

Then S is a closed, symmetric operator with defect numbers (1, 1), cf. [20]. The perturbation formula (11) augmented with $A(\infty)$ parametrizes all self-adjoint extensions $A(\tau), \tau \in \mathbf{R} \cup \{\infty\}$, of S.

The operator is densely defined and the perturbation $A(\infty)$ is a self-adjoint operator, only when $\omega \in H_{-2} \setminus H$. When $\omega \in H$ the condition $(f,\omega) = 0$ reads as $[f,\omega] = 0$, so that S is not densely defined and $A(\infty)$ has the form

$$A(\infty) = S + (\{0\} \oplus \operatorname{mul} S^*), \tag{13}$$

where $\operatorname{mul} S^* = (\operatorname{dom} S)^{\perp}$ is the multivalued part of S^*. In particular, $A(\infty)$ is a self-adjoint relation with the same multivalued part as S^*. The notion of generalized Friedrichs extension of S occurs when the element ω belongs to the scale space H_{-1} associated with $H_{+1} = \operatorname{dom} |A|^{1/2}$.

Proposition (1.2.1)[16]: Let A be a self-adjoint operator, let $\omega \in H_{-1}$, and let S be defined by (12)

$$A(\infty) = \{\{f,g\} \in H^* : f \in H_{+1}\}, \tag{14}$$

Moreover, $A(\infty)$ is the only self-adjoint extension H of S such that dorm $H \subset H_{+1}$.

This description reduces to (13) when $\omega \in H$. When S is semibounded $A(\infty)$ is precisely the usual Friedrichs extension [24,21,19].

Our interpretation of rank one perturbations of A in (11) as self-adjoint extensions of a symmetric operator, shows that many of the results of Gcsztesy, Kiselev, and Simon remain valid without the condition that A is nonnegative. We discuss spectral measures for self-adjoint extensions of closed symmetric operators with defect numbers (1,1). Various descriptions of self-adjoint extensions of a symmetric operator. We shown that $A(\tau)$ converges to $A(\tau_0)$ as $\tau\to\tau_0$ in the graph sense. That the spectral measures of $A(\tau)$ converge weakly to the spectral measure of $A(\tau_0)$ consider self-adjoint operators whose resolvents differ by a rank one operator, answering a question of B. Simon. We contains a discussion of rank one perturbations by means of Q- functions.

Let S be any closed symmetric relation with defect numbers (1,1). Let A be a self-adjoint extension of S. Choose for $\mu\in\mathbf{C}\setminus\mathbf{R}$ a nontrivial defect vector $\chi(\mu)\in$ ker $(S^*-\mu)$. Then for $\ell\in\rho(A)$

$$(\ell)=(I+(\ell-\mu)(A-\ell)^{-1}\chi(\mu)) \tag{15}$$

defines a holomorphic basis for ker $(S^*-\ell)$. The Q-function $Q(\ell)$ of A and S is defined (uniquely tip to a real constant) as a solution of the equation

$$\frac{Q(\ell)-\overline{Q(\lambda)}}{\ell-\lambda}=[\chi(\ell),\chi(\lambda)]. \tag{16}$$

The function $Q(\ell)$ belongs to the class **N** of Nevanlinna functions. Recall that a fnnction $Q(\ell)$ belongs to **N** precisely when

$$Q(\ell)=\alpha+\beta\ell+\int_{\mathbf{R}}\left(\frac{1}{t-\ell}-\frac{t}{t^2+1}\right)d\sigma(t) \tag{17}$$

here $\alpha\in\mathbf{R},\beta\geq\mathbf{0}$ the function $\sigma(t)$ is nondecreasing on **R** and satisfies

$$\int_{\mathbf{R}}\frac{d\sigma}{t^2+1}<\infty \tag{18}$$

Another way of writing (17) is

$$Q(\ell)=\alpha+\ell\left(\beta+\int_{\mathbf{R}}\frac{d\sigma(t)}{t^2+1}\right)+(\ell^2+1)\int_{\mathbf{R}}\frac{1}{t-\ell}\frac{d\sigma(t)}{t^2+1} \tag{19}$$

we will only consider Nevanlinna functions $Q(\ell)$ which do not reduce to real constants or equivalently, which do not take real values off the real axis. The spectral measure which we associate with A and S is the measure $d\sigma(t)$ in (17), when $Q(\ell)$ is the Q-function of A and S. It can be recovered from $Q(\ell)$ by means of the Stieltjes inversion formula. It follows from (15) and (16) that

$$Q(\ell)=Q(\mu)^*+(\ell-\bar{\mu})[(I+(\ell-\mu)(A-\ell)^{-1})\chi(\mu),\chi(\mu)],\quad \ell\in\rho(A). \tag{20}$$

Let **R** be the orthogonal projection onto H θ mul A = $\overline{dom}$ A and let E(t) be the spectral family of the operator part A_s of A in that space. For the following connection see [19].

Proposition (1.2.2)[16]: The connection between the operator representation (20) and the integral representation (17) is as follows:

(*i*) $\alpha = Q(i)^*+i[\chi(i),\chi(i)]=\operatorname{Re}Q(i)$,

(*ii*) $\beta=[(1-R)\chi(i),(I-R)\chi(i)]=\lim_{y\to\infty}\dfrac{lmQ(iy)}{y}$,

(*iii*) $\dfrac{d\sigma(t)}{t^2+1}=d([E(t)R\chi(i),R\chi(i)])$.

A consequence of (ii) is that β = 0 if and only $\chi(\ell)\in\overline{dom}$ A for some (and hence for all) $\ell\in\mathbf{C}\setminus\mathbf{R}$. In particular, if S is an operator then β = 0 if and only if A is an operator [25].

Since A is a self-adjoint extension of S there exists a pair $\{(\varphi,\psi)\in H^2\setminus A\}$ such that

$$S=\{\{f,g\}\in A:(\{f,g\},\{\varphi,\psi\})=0. \tag{21}$$

Here we have used the notation

$$(\{f,g\},\{h,k\}) = [g,h]-[f,k],\quad \{f,g\},\{h,k\}\in H$$

This pair {φ,ψ} is determined uniquely modulo A. In terms of {φ,ψ} the Q-function of A and S can be expressed by

$$Q(\ell)=[\chi(\ell),\bar{\ell}\varphi-\psi]+[\varphi,\psi],\quad \chi(\ell)=(A-\ell)^{-1}(\ell\psi-\psi)+\varphi. \tag{22}$$

If χ(μ) is given, then {φ,ψ} may be chosen as $\{\varphi,\psi\}=\{\chi(\mu),\mu\chi(\mu)\}$. Now suppose in addition that A (and hence also S) is an operator and denote the continuation of A from H to H_{-2} by $\tilde{A}$. Let ω be defined by

$$\omega=\tilde{A}\varphi-\psi,. \tag{23}$$

then $\omega\in H_{-2}$ and (12) and (21) define the same symmetric operator S. Each $\omega\in H_{-2}$ is of the form (23). The relation between the element $\omega\in H_{-2}$ in (12) and the functions $\chi(\ell)$ and $Q(\ell)$ is given by (see [20])

$$\chi(\ell)=(\tilde{A}-\ell)^{-1}\omega,\ Q(\ell)=((\tilde{A}-\ell)^{-1}\omega-\varphi,\omega)+(\psi,\varphi) \tag{24}$$

The subclass $\mathbf{N}_1$ is the set of functions $Q(\ell)$ with $\int_1^\infty \frac{\operatorname{Im} Q(iy)}{y} dy < \infty$.

It was introduced by I.S. Kac [26]. The function $Q(\ell)$ belongs to $\mathbf{N_1}$ precisely when

$$Q(\ell) = \gamma + \int_{\mathbf{R}} \frac{d\sigma(t)}{t-\ell}, \int_{\mathbf{R}} \frac{d\sigma(t)}{|t|+1} < \infty, \quad \gamma \in \mathbf{R} \tag{25}$$

It follows from (25) that

$$\gamma = \lim_{y \to \infty} Q(\mathrm{iy}). \tag{26}$$

The Q-function of A and S belongs to $\mathbf{N_1}$ if and only if A is an operator and $\chi(\ell) \in$ H_{+1} dom $|A|^{1/2}$. This last condition is equivalent to $\varphi \in H_{+1}$ and to $\omega \in H_{-1}$. The subclass $\mathbf{N_0}$ is the set of all functions $Q(\ell)$ with $\sup_{y>0}$ y $\operatorname{Im} Q(\mathrm{iy}) < \infty$. The function $Q(\ell)$ belongs to $\mathbf{N_0}$ precisely when

$$Q(\ell) = \gamma + \int_{\mathbf{R}} \frac{d\sigma(t)}{t-\ell}, \quad \int_{\mathbf{R}} d\sigma(t) < \infty, \qquad \gamma \in \mathbf{R}. \tag{27}$$

The Q-function of A and S belongs to $\mathbf{N_0}$ if and only if A is an operator and $\chi(\ell) \in \mathrm{H}$. This last condition is equivalent to $\varphi \in \operatorname{dom} A$ and to $\omega \in \mathrm{H}$. We refer for these classes and their integral representations to [26,27,19].The further characterizations can be found in [21,19,25,20].

Assume that $\omega \in H_{-2}$ is given by (23), so that also (24) is valid. If $\omega \in H_{-1}$ or $\omega \in \mathrm{H}$, the formulas (23) and (24) are still valid when the continuation $\tilde{A}$ and the duality (.,.) are correctly interpreted. The norm of ω can be expressed in terms of the spectral measure $d\sigma(t)$ as follows [21,20].

Lemma (1.2.3)[16] : For $\omega \in H_{-2}$ we have

$$\|\omega\|_{-2}^2 = \int_{\mathbf{R}} \frac{d\sigma(t)}{t^2+1} < \infty. \tag{28}$$

If $\omega \in H_{-1}$ then

$$\|\omega\|_{-1}^2 = \int_{\mathbf{R}} \frac{d\sigma(t)}{|t|+1} < \infty. \tag{29}$$

If $\omega \in \mathrm{H}$ then

$$\|\omega\|^2 = \int_{\mathbf{R}} d\sigma(t) < \infty \tag{30}$$

Let S be a closed, symmetric relation with equal defect numbers (1,1). Since the defect numbers of S are equal there are self-adjoint extensions of S in H and there is no need for exit spaces. We fix one such self-adjoint extension A of S to describe the others Krein's formula.

Let $\chi(\ell)$ and $Q(\ell)$ be defined by (15) and (16). Then the resolvent operators of self-adjoint extensions $A(\tau), \tau \in \mathbf{R} \cup \{\infty\}$, of S are given by

$$(A(\tau)-\ell)^{-1} = (A-\ell)^{-1} - \chi(\ell)\frac{1}{Q(\ell)+1/\tau}[., \chi(\bar{\ell})], \quad \ell \in \mathbf{C} \setminus \mathbf{R} \tag{31}$$

one-dimensional graph perturbations.

If S is defined by (31), then the selfadjoint extensions $A(\tau), \tau \in \mathbf{R} \cup \{\infty\}$ are given for $1/\tau + [\psi, \varphi] \neq 0$ by

$$A(\tau) = \left\{ \{f, g\} - \frac{\langle \{f, g\}, \{\varphi, \psi\} \rangle}{1/\tau + [\psi, \varphi]} [\varphi, \psi] : \{f, g\} \in A \right\}, \tag{32}$$

And for $1/\tau + [\psi, \varphi] = 0$ by

$$A(\tau) = S + \text{span } \{\varphi, \psi\}. \tag{33}$$

In fact, the resolvent operators of all self-adjoint extensions $A(\tau)$, $\tau \in \mathbf{R} \cup \{\infty\}$, in (32) and (33) are parametrized precisely by (31), when $Q(\ell)$ and $x(\ell)$ are given by (22), see [25].

When $\varphi = 0$ the condition in (21) reduces to $[f, \psi] = 0$ and mul $S^* = \text{span}$ $\{\psi\}$. The formula (32) now reads as

$$A(\tau) = \{\{f, g\} - \tau [f, \psi] \{0, \psi\} : \{f, g\} \in A\}, \quad \tau \in \mathbf{R}. \tag{34}$$

Rank one perturbations and triplet spaces. Under the assumption that A is an operator the expressions (32) and (33) are equivalent to:

$$A(\tau) = \{\{h, \tilde{A}h + c\omega\} \in H^2 : c(1/\tau + [\varphi, \psi]) = (h, \omega), c \in \mathbf{C}\} \tag{35}$$

where $(h, \omega) = [S^*h, \varphi] - [h, \psi]$, see [20]. When $\varphi = 0$ this formula reduces to (34) and when $[\varphi, \psi] = 0$ we have (11). The formula (35) can be interpreted as the compression of usual rank one perturbations in a larger Hilbert or Pontryagin space, see [20].

An analytic description. Let $Q_\tau(\ell)$ denote the Q-function of the self-adjoint extension $A(_\tau)$ in (32) and (33) normalized by $\operatorname{Re} Q_\tau(\mu)=0$. It follows from Krein's formula (31), cf. [19], that

$$Q_\tau(\ell) = \frac{Q(\ell) - \tau(\operatorname{Im} Q(\mu))^2}{\tau Q(\ell)+1}, \qquad \tau \in \mathbf{R} \cup \{\infty\}. \tag{36}$$

The corresponding defect vectors are given by

$$\chi_\tau(\ell) = \frac{Q(\mu)+1/\tau}{Q(\ell)+1/\tau}\chi(\ell), \tag{37}$$

so that

$$\frac{Q_\tau(\ell) - \overline{Q_\tau(\lambda)}}{\ell - \overline{\lambda}} = [\chi_\tau(\ell), \chi_\tau(\lambda)]. \tag{38}$$

When $\tau = \infty$, by (36) and (37) we mean

$$Q_\infty(\ell) = -\frac{\operatorname{Im} Q(\mu)}{Q(\ell)}, \qquad \chi_\infty(\ell) = \frac{Q(\mu)}{Q(\ell)}\chi(\ell). \tag{39}$$

If the Q-function $Q(\ell)$ of A and S belongs to $\mathbf{N_1}$ or $\mathbf{N_0}$, then also the Q-functions $Q_\tau(\ell)$ have this property when $1/\tau + \gamma \neq 0$, where $\gamma \in \mathbf{R}$ satisfies (37). For the exceptional value of $\tau \in \mathbf{R} \cup \{\infty\}$, i.e. when $1/\tau + \gamma = 0$, the function $Q_\tau(\ell)$ belongs to $\mathbf{N} \setminus \mathbf{N}_1$, cf. [19].

There is a simple relation between different extensions. Let $Q_\tau(\ell)$, $\tau \in \mathbf{R}$, be defined by (36) and let $\eta \in \mathbf{R}$. Since $Q_\tau(\ell)$ is a Nevanlinna function, the expression $(Q_\tau)_\eta(\ell)$ is well-defined, and

$$(Q_\tau)_\eta(\ell) = Q_\zeta(\ell), \qquad \ell \in \mathbf{C} \setminus \mathbf{R} \tag{40}$$

where η, $\tau, \zeta \in R \cup \{\infty\}$ are connected by

$$\zeta = \frac{\eta + \tau}{1 - \eta\tau(\operatorname{Im} Q(\mu))^2}$$

If $\tau = \infty$ or $\eta = \infty$, then (40) still holds with a limiting interpretation.

Since for each $\tau \in \mathbf{R} \cup \{\infty\}$ the function $Q_\tau(\ell)$ in (36) belongs to $\mathbf{N}$, there exist $\alpha_\tau \in \mathbf{R}, \beta_\tau \geq 0$, and a nondecreasing function $\sigma_\tau(t)$ on $\mathbf{R}$, such that

$$Q_\tau(\ell)=\alpha_\tau+\beta_\tau\ell+\int_{\mathbf{R}}\left(\frac{\mathbf{1}}{\mathbf{t}-\ell}-\frac{\mathbf{t}}{\mathbf{t}^2+1}\right)d\sigma_\tau(t),\qquad \int_{\mathbf{R}}\frac{d\sigma_\tau(t)}{t^2+1}<\infty \tag{41}$$

For the value $\tau = 0$ we will write α,β and $\sigma(t)$ as in (17). In terms of these data the identity (38) reads as

$$[\chi_\tau(\ell),\chi_\tau(\lambda)]=\beta_\tau+\int_{\mathbf{R}}\left(\frac{\mathbf{1}}{\mathbf{t}-\ell}\frac{\mathbf{1}}{\mathbf{t}-\bar{\lambda}}\right)d\sigma_\tau(t). \tag{42}$$

Let S be a closed symmetric relation with defect numbers (1,1).Then its self-adjoint extensions $A(\tau),\tau\in\mathbf{R}\cup\{\infty\}$, are continuous in τ in a sense to be explained below.

We will say that closed linear relations A_τ tend to a closed linear relation $A_{\tau 0}$ as $\tau\to\tau_0$ in the graph sense, denoted by $A_\tau\to A_{\tau 0}$, if for each $\{f,g\}\in A_{\tau 0}$ there are elements $\{f_\tau,g_\tau\}\in$ g A_τ such that $\{f_\tau,g_\tau\}\to\{f,g\}$.When, for instance, A_τ and $A_{\tau 0}$ are all self-adjoint, this definition is equivalent to the strong convergence of $(A_\tau-\ell)^{-1} to (A_{\tau_0}-\ell)^{-1}$ for some (and, hence, for all) $\ell\in\mathbf{C}\setminus\mathbf{R}$. For the case of operators this is proved in [28, 29];

Proposition (1.2.4)[16]: Let S be a closed symmetric relation with defect numbers (1, 1) of the form (21). Let its self-adjoint extensions $A(\tau),\tau\in R\cup\{\infty\}$, be given by (32) and (33). Let $\tau_0\in\mathbf{R}\cup\{\infty\}$.Then for $\tau\to\tau_0$, we have in the graph sense

$$A(\tau)\to A(\tau_0),$$

Proof. We use the representations (32) and (33) of the selfadjoint extensions $A(\tau)$. When $1/\tau_0+[\psi,\varphi]\neq 0$, then the proposition follows directly from the definition and (32). Now consider $1/\tau_0+[\psi,\varphi]=0$. By means of Krein's formula (31) we observe that

$$(A(\tau)-\ell)^{-1}h-(A(\tau_0)-\ell)^{-1}h=\chi(\ell)\frac{\tau_0-\tau}{(\tau Q(\ell)+1)(\tau_0 Q(\ell)+1)}[h,\chi(\bar{\ell})],$$

for $\ell\in\mathbf{C}\setminus\mathbf{R}$. When suitably interpreted for the case $\tau_0=\infty$, this shows that for each $h\in H$ we have $(A(\tau)-\ell)^{-1}h\to(A(\tau_0)-\ell)^{-1}h$ in H,as $\tau\to\tau_0$.

Let the Q-function $Q(\ell)$ of A and S belong to $\mathbf{N_1}$ or $\mathbf{N_0}$, and let $1/\tau_0+\gamma=0$ Then $A(\tau_0)$ is the only self-adjoint extension of S whose Q-function does not belong to $\mathbf{N_1}$. It is the generalized Friedrichs extension given by the right side of (14); in particular, if $Q(\ell)$ belongs to N_0, then $A(\tau_0)$ is a true relation given by the right side of (13), see [21]. However, according to Proposition (1.2.4), the operators $A(\tau)$ tend to $A(\tau_0)$ in the graph sense, as $\tau\to\tau_0$.

Let S be a closed symmetric operator with defect numbers (1, 1) as in (21). Let $A(\tau), \tau\in\mathbf{R}\cup\{\infty\}$, be its self-adjoint extensions as given by (23) and (33), with corresponding spectral measures $d\sigma_\tau(t)$ in (41). We are interested in the limiting behaviour of these spectral measures. Note that at most one self-adjoint extension of S is not an operator.

Lemma (1.2.5)[16]: For all $\ell,\lambda\in\mathbf{C}\setminus\mathbf{R}$

$$\lim_{\tau\to\tau_0}\int_{\mathbf{R}}\frac{1}{t-\ell}\frac{1}{t-\bar{\lambda}}d\sigma_\tau(t)=\beta_{\tau_0}+\int_{\mathbf{R}}\frac{1}{t-\ell}\frac{1}{t-\bar{\lambda}}d\sigma_{\tau_0}(t). \tag{43}$$

Proof. It follows from (37) and (39) that $\chi_\tau(\ell)$ converges to $\chi_{\tau_0}(\ell)$ in the norm of H. Hence, in particular, we obtain

$$\lim_{\tau\to\tau_0}[\chi_\tau(\ell),\chi_\tau(\lambda)]=\left[\chi_{\tau_0}(\ell),\chi_{\tau_0}(\lambda)\right]. \tag{44}$$

Now we rewrite this result by means of (42). Observe that $\beta_\tau>0$ if and only if $A(\tau)$ is not an operator. Hence $\beta_\tau=0$ for all $\tau\in\mathbf{R}\cup\{\infty\}$ with the possible exception of at most one τ. Therefore we obtain (43).

The weak convergence of the spectral measures is one of the consequences of Lemma (1.2.5) see [17].

Theorem (1.2.6)[16]: Let S be a closed symmetric operator with defect numbers (1,1) as in (21). Let $A(\tau)$, $\tau\in\mathbf{R}\{\infty\}$, be its self-adjoint extensions as given by (32) and (33), with corresponding spectral measures $d\sigma_\tau(t)$ in (41). For each continuous function f with compact support

$$\lim_{\tau\to\tau_0}\int_{\mathbf{R}}f(t)d\sigma_\tau(t)=\int_{\mathbf{R}}f(t)d\sigma_{\tau_0}(t).$$

Proof. The theorem can be proved in a classical way as in [30]. Here we will use an approximation argument as suggested in [17]. Let [a, b] be a compact interval. Then

C([a, b]) is equal to the closed linear span of the functions $\frac{1}{t-\ell}, \ell \in \mathbf{C}\setminus\mathbf{R}$, in the norm $\|\ .\|_\infty$ of C([a, b]). To see this, we use the Stone-Weierstrass theorem, cf. [31] and the fact that the identity function on $[a, b]$ can be uniformly approximated by these functions. Let f(t) be a continuous function with support in $[a, b]$, then the function $f(t)(t-i)^2$ can be uniformly approximated on [a, b] by the above functions. Now

$$\left|\int_{\mathbf{R}} f(t)d\sigma_t(t) - \int_{\mathbf{R}} f(t)d\sigma_{\tau_0}(t)\right| \le \left|\int_{\mathbf{R}}\left(f(t)(t-i)^2 - \sum_{k=1}^{n}\frac{1}{t-\ell_k}\right)\frac{d\sigma_\tau(t)}{(t-i)^2}\right|$$
$$+\left|\int_{\mathbf{R}}\sum_{k=1}^{n}\frac{1}{t-\ell_k}\frac{d\sigma_\tau(t)}{(t-i)^2} - \int_{\mathbf{R}}\sum_{k=1}^{n}\frac{1}{t-\ell_k}\frac{d\sigma_{\tau_0}(t)}{(t-i)^2}\right|$$
$$+\left|\int_{\mathbf{R}}\left(f(t)(t-i)^2 - \sum_{k=1}^{n}\frac{1}{t-\ell_k}\right)\frac{d\sigma_{\tau_0}(t)}{(t-i)^2}\right|.$$

The middle term in the right side can be made as small as possible by differentiating (43). The remaining terms in the right side are estimated by

$$\left\|f(t)(t-i)^2 - \sum_{k=1}^{n}\frac{1}{t-\ell_k}\right\|_\infty \int_R \frac{d\sigma_\tau(t)}{t^2+1},$$

and

$$\left\|f(t)(t-i)^2 - \sum_{k=1}^{n}\frac{1}{t-\ell_k}\right\|_\infty \int_R \frac{d\sigma_{\tau_0}(t)}{t^2+1}.$$

Now the term

$$\left\|f(t)(t-i)^2 - \sum_{k=1}^{n}\frac{1}{t-\ell_k}\right\|_\infty$$

can be made as small as possible by the Stone-Weierstrass argument, while the integrals $\int_{\mathbf{R}} \frac{d\sigma_\tau(t)}{t^2+1}$ ‘K are uniformly bounded in T by Lemma (1.2.5).

Let A be a self-adjoint operator extension of S. We will interpret Lemma (1.2.5) in terms of $\omega \in H_{-2}$ and similar elements $\omega_\tau \in H_{-2}(A(\tau))$ corresponding to the self-adjoint operator extensions $A(\tau), \tau \in \mathbf{R} \cup \{\infty\}$.

If $\omega \in H_{-2} \backslash H_{-1}$ then (28) holds. Moreover,S is densely defined and each self-adjoint extension $A(\tau), \tau \in \mathbf{R} \cup \{\infty\}$. is an operator. By means of $\chi_\tau(\ell)$ in (37) we define ω_τ by

$$\omega_\tau = (\tilde{A}(\tau) - \ell)\chi_\tau(\ell)$$

which is independent of ℓ. Here $\tilde{A}(\tau)$ is the continuation of $A(\tau)$ to all of H relative to the scale space $H_{-2}(A(\tau))$ associated with $A(\tau)$ Then $\omega_\tau \in H_{-2}(A(\tau))$ and

$$S = \{\{f, g\} \in A(\tau) : (f, \omega_\tau) = 0\}, \tag{45}$$

where (.,.) denotes the appropriate duality. In particular, **S** is independent of τ.

If $\omega \in H_{-1} \backslash H$, then (29) holds. We may repeat the arguments as given above. Furthermore, in the present case the topological spaces H_{-1} $(A(\tau))$ do not depend on τ, $1/\tau + \gamma \neq 0$, see [21] and [20]. Although the norms are equivalent, they may still depend on τ This motivates the following result.

Lemma (1.2.7)[16]: The elements ω and ω_τ are related by

$$\omega_\tau = \frac{Q(\mu) + 1/\tau}{\gamma + 1/\tau}\omega \in H_{-1}, \qquad 1/\tau + \gamma \neq 0. \tag{46}$$

Proof. The function $Q(\ell)$ in (24) may be written as

$$Q(\ell) = \gamma + ((\tilde{A} - l)^{-1}\omega, \omega), \qquad \gamma = [\varphi, \psi] + [\psi, \varphi] - (\tilde{A}\varphi, \varphi) \in \mathrm{R},$$

since $\omega \in H_{-1}$ cf. (24). Using this together with the expression for $\chi(\ell)$ in (35), (37), and Krein's formula for the continuations (see [21]) we obtain

$$\begin{aligned}(Q(\ell) + 1/\tau)\omega_\tau &= (Q(\mu) + 1/\tau)(\tilde{A}(\tau) - \ell)(\tilde{A} - e)^{-1}\omega \\ &= (Q(\mu) + 1/\tau)(\tilde{A}(\tau) - \ell)\left[(\tilde{A}(\tau) - \ell)^{-1}\omega + \frac{(\omega, \chi(\tilde{\ell}))}{Q(\ell) + 1/\tau}\chi(\ell)\right] \\ &= (Q(\mu) + 1/\tau)\omega + (Q(\ell) - \gamma)\omega_\tau.\end{aligned}$$

Now solve ω_τ to complete the proof.

If $\omega \in H$, then (30) holds and S is not densely defined. Again (46) can be shown to hold.

Proposition (1.2.8)[16]: Let A be a self-adjoint operator and let S be defined by (12) with $\omega\in H_{-2}$.

(i) If $\omega\in H_{-2}\backslash H_{-1}$, then $\lim_{\tau\to\tau_0}\|\omega_\tau\|_{-2} = \left\|\omega_{\tau_0}\right\|_{-2} (<\infty)$.

(ii) If $\omega\in H_{-1}\backslash H$, then $\lim_{\tau\to\tau_0}\|w_\tau\|_{-1} = \left\|w_{\tau_0}\right\|_{-1}$ *when* $A(\infty)$ is not the generalized Friedrichs extension; otherwise the limit is ∞.

(iii) If $\omega\in H$, then $\lim_{\tau\to\tau_0}\|\omega_\tau\| = \left\|\omega_{\tau_0}\right\|$, *when* $A(\infty)$ is not the generalized Friedrichs extension; otherwise the limit is ∞.

Proof. For the proof of (i), we take $\ell=\lambda=i$ in Lemma (1.2.5) and apply (28). To prove (ii), choose a compact interval Δ of **R** and apply Theorem (1.2.6) with $f(t) = (|t|+1)^{-1}$. Then take $\Delta\to\mathbf{R}$ and interchange the limits. The value of the limit in (ii) is ∞ if and only if ω_{τ_0} does not belong to H_{-1}, in which case $A(\tau_0)$ is the generalized Friedrichs extension. Finally, to show (iii) we take f(t) = 1 and proceed as in the proof of (ii) or we use (46) both for $1/\tau+\gamma\neq 0$ and for $1/\tau+\gamma=0$.

If $A(\tau_0)$ is not an operator, then its multivalued part is equal to mul S*. In fact, $A(\tau_0)$ is reduced by mul S*. Observe that the self-adjoint operator

$$A(\tau_0)\cap(H\ominus mul\ s^*)^2$$

is the (orthogonal) operator part of $A(\tau_0)$. It is not difficult to see that it is equal to $RA|_{H\ominus mul\ s^2}$, cf. [17]. These details are worked out in [25].

In [18] Kiselev and Simon raise the question how to characterize two self-adjoint operators when their resolvents differ by a rank one operator. We use the idea of graph perturbations to give such a characterization. For a different approach, see [32].

Proposition (1.2.9)[16]: Let A and B be self-adjoint relations, such that for some $\ell\in\rho(A)\cap\rho(B)$

$$\operatorname{rank}((B-\ell)^{-1}-(A-\ell)^{-1})=1 \tag{47}$$

Then (47) holds for all ℓ ρ (A) $\cap$ ρ(B). Moreover, there exists a closed symmetric restriction S with defect numbers (1, 1) of the form (21), such that for some $\tau \in \mathbf{R} \cup \{\infty\}$, B = A($\tau$) as defined in (32) and (33).

Proof. Note that any closed linear relation A with nonempty resolvent set can be written as

$$A = \{\{(A-\ell)^{-1} h, (I+\ell(A-\ell)^{-1})h\}: \ h \in H\},$$

for some $\ell \in \rho(A)$. By assumption

$$(B-\ell_0)^{-1} - (B-\ell_0)^{-1} = c[.,\eta]\zeta, \tag{48}$$

for some $\ell_0 \in \rho(A) \cap \rho(B)$, some $\eta, \zeta \in H$ and $c \in \mathbf{C}$. Define

$$S = \{\{(A-\ell_0)^{-1}h, (I+\ell_0(A-\ell_0)^{-1})h\}: h \in H, [h, \eta]=0\}. \tag{49}$$

Clearly, S is a closed symmetric restriction of A, and therefore S has equal defect numbers. Moreover,

$$\operatorname{ran}(S-\ell_0) = \{h \in H: [h, \eta] = 0.$$

It follows that ran (S $-\ell_0$) is closed; moreover, it follows from the definition that ℓ_0 is not an eigenvalue of S. Hence ℓ_0 is a point of regular type for S. As the set of points of regular type of S is open and consists of at most two connected components (including $\mathbf{C}^+$ and $\mathbf{C}^-$), we see that the defect numbers of S are (1, 1).

It follows from (48) with $h \in H$, $[h, \eta] = 0$, that $(B-\ell_0)^{-1} h = (A-\ell_0)^{-1}h$. Hence, (48) and (49) show that B is also a self-adjoint extension of S.

Let $\{\varphi, \psi\}$ be a pair in (the graph of) S*\A. Then S is given by (21) and so B = A(τ) for some $\tau \in \mathbf{R} \cup \{\infty\}$, as defined in (32) and (33). The resolvent operators of the self-adjoint extensions A(τ) are parametrized precisely by (31). Therefore, the condition (47) holds for all $\ell \in \rho(A) \cap \rho(B)$.

It is clear from the above proof that the symmetric relation S in (49) is in general not an operator; in fact mul S=$\{h \in \text{mul}A: [h,\eta]=0\}$. Clearly, S is an operator if and only if A or B is an operator. We have concluded that if (47) holds, then B and A are self-adjoint extensions of S in (49), and B is a graph perturbation of A. This approach allows us to obtain some of the results of Kiselev and Simon in a different way, cf. [18].

Proposition (1.2.10)[16]: Let A be a self-adjoint operator and let B be a self-adjoint relation, such that for some $\ell \in \rho(A) \cap \rho(B)$ (47) holds. Then there exists an element $\omega \in H_{-2}(A)$, such that for some $\tau \in \mathbf{R} \cup \{\infty\}$, $B = A(\tau)$ as defined in (35).

The previous result is just a restatement of Proposition (1.2.9), since (32), (33) and (35) describe the same self-adjoint extensions. We can say more when we know the difference $(B-\ell)^{-1} - (A-\ell)^{-1}$ explicitly.

Corollary (1.2.11)[16]: Let A be a self-adjoint operator and let B be a self-adjoint relation, such that for some $\ell \in \rho(A) \cap \rho(B)$

$$(B-\ell)^{-1} - (A-\ell)^{-1} = c[\,.\,,\eta]\zeta,$$

with elements $\eta, \zeta \in H$ and $c \in \mathbf{C}$.

(i) Assume one of the equivalent conditions $\eta \in \operatorname{dom}|A|^{\frac{1}{2}} \setminus \operatorname{dom} A$, $\xi \in \operatorname{dom}|A|^{\frac{1}{2}} \setminus \operatorname{dom} A$, or $\omega \in H_{-1} \setminus H$. Then S is densely defined and B is an operator. Moreover, B is the generalized Friedrichs extension of S if and only if $\eta \in H \setminus \operatorname{dom}|B|^{\frac{1}{2}}$.

(ii) Assume one of the equivalent conditions $\eta \in \operatorname{dom} A$, $\zeta \in \operatorname{dom} A$, or $\omega \in H$. Then S is not densely defined. Moreover, B is the generalized Friedrichs extension of S (i.e. B is not an operator) if and only if $\eta \in H \setminus \operatorname{dom} B$.

Let A be a self-adjoint operator arid let $\omega \in H_{-2}$. Then the rank one perturbations of A, defined by

$$A(\tau) = \tilde{A} + \tau(.,\omega)\omega, \qquad \tau \in \mathbf{R}, \tag{50}$$

are of the form (35). Hence, we may interpret the rank one perturbations of A as self-adjoint extensions of S. In this sense the convergence results for the self-adjoint extensions carry over to the rank one perturbations. We present a brief discussion of (50) in terms of special properties of the element $\omega \in H_{-2}$ from the point of view of Q-functions.

If ω belongs to H, then the Q-functions of $A(\tau), \tau \in \mathbf{R}$, all belong to $\mathbf{N_0}$ and the corresponding self-adjoint extensions are rank one perturbations of A in the usual sense. The limit in the graph sense of $A(\tau)$ as $\tau \to \infty$ is the generalized Friedrichs extension of S, given by (13). It has a Qfunction $Q(\ell)$ with the property that $\lim_{y\to\infty} \frac{\operatorname{Im} Q(iy)}{y} > 0$; hence it belongs to **N** \ **N₁**.

If $\omega \in H_{-1} \backslash H$, then the Q-functions of $A(\tau)$, $\tau \in \mathbf{R}$, all belong to $\mathbf{N_1}$. The limit in the graph sense of $A(\tau)$ as $\tau \to \infty$ is the generalized Friedrichs extension of S, given by $A(\infty)$ in Proposition (1.2.1). Moreover, its Q-function belongs to $\mathbf{N} \setminus \mathbf{N_1}$. When we consider the continuations of the self-adjoint extensions to $H_{+1} \times H_{-1}$, then the description of the self-adjoint extensions is formally the same as in the case $\omega \in H$), cf. [21].

The situation is quite different when $\omega \in H_{-2} \backslash H_{-1}$. Then all Q-functions of S belong to $\mathbf{N} \backslash \mathbf{N_1}$. They may all behave in the same way and there is no exceptional self-adjoint extension: in [20] there is even an example where all self-adjoint extensions have the same Q-function.

If A is semibounded, then the Q-function of A and S belongs to $\mathbf{N_1}$ if and only if $\omega \in H_{-1}$. This case is considered in [17], [18], [33] and [34]. If the Q-function of A does not belong to $\mathbf{N_1}$, i.e. if $\omega \in H_{-2} \backslash H_{-1}$, then A is necessarily the Friedrichs extension of S, cf. [19], [21] and [18], but now all the other Q-functions belong to $\mathbf{N_1}$. For further literature about these cases, see [34]. A treatment of positive operators in Pontryagin and Krein spaces appears in [35].

The case that A is not an operator is studied in [25]. The Q-functions of all other self-adjoint extensions belong to $\mathbf{N_0}$.The spectral measure of the exceptional Q-function is arbitrary. More specific information can be given by a subdivision of $\mathbf{N_0}$ into subclasses $\mathbf{N_{-k}}$, according to $\omega \in \operatorname{dom} |A|^{K/2}$, $k \in \mathbf{N} \cup \{0\}$, cf. [36].

Chapter 2

Operators with Singular Continuous Spectrum and Smooth Rank One Perturbations

In this chapter we consider smooth perturbations, i.e. we consider $\omega = \mathrm{dom}\,|A|^{k/2}$ for some $k \in \boldsymbol{N} \cup \{0\}$. Function-theoretic properties of their so-called Q-functions and operator-theoretic consequences will be studied. While we're interested in the abstract theory of rank one perturbations, we're especially interested in those rank one perturbations obtained by taking a random Jacobi matrix and making a Baire generic perturbation of the potential at a single point.

Section (2.1): Self-adjoint Operators and Smooth Rank One Perturbations:

Let A be a self-adjoint operator in a Hilbert space **H** with inner product $[\cdot, \cdot]$. For a nontrivial element $\omega \in$ H the rank one perturbations of A are defined by

$$A(\tau) = A + \tau\, [\cdot, \omega]\omega, \tau \in \boldsymbol{R}, \tag{1}$$

cf. [38]. Let S be the restriction of A to the orthogonal complement of span$\{\omega\}$:

$$dom\ S = \{h \in dom\ A : [h, \omega] = 0\}. \tag{2}$$

Then S is a nondensely defined, closed symmetric operator with defect numbers (1,1). Clearly, the perturbations $A(\tau)$ in (1) are self-adjoint extensions of S and dom $A(\tau)$ = dom A, $\tau \in \mathbf{R}$. Since all self-adjoint extensions of S are parametrized over $\mathbf{R} \cup \{\infty\}$, one self-adjoint extension of S is not of the form (1). It is given by

$$A(\infty) = S \dot{+} (\{0\} \oplus \mathrm{span}\{\omega\}), \tag{3}$$

which is a self-adjoint relation (multivalued operator), whose ultivalued part mul A is given by mul A = span$\{\omega\}$. In fact, $A(\infty)$ is the generalized Friedrichs extension of S [39], [21], [40]. There is a more general interpretation of (1) by allowing ω to belong to the scale spaces $H_{-1}(A)$ and $H_{-2}(A)$, associated with $H_{+1}(A) = \mathrm{dom}|A|^{1\backslash 2}$ and $H_{+2}(A)$ = dom A, respectively, [22], [17], [21], [20], [18], [34]. In the present section our interest is in the spectral properties of smooth perturbations of A, i.e. perturbations for which $\omega \in \mathrm{dom}|\ |A|^{k/2}$ for some $k \in \mathbf{N} \cup \{0\}$.

The main emphasis is on a function-theoretic description of the corresponding Q-functions. These functions belong to the class N of Nevanlinna functions. A subdivision of N was originated by I.S. Kac [26], [27] and further extended in [41]. In this section a complete subdivision of **N** is presented. We show by means of asymptotic expansions how these subclasses of Nevanlinna functions (and their moments) behave under certain linear fractional transformations. For this purpose we need an extension of asymptotic results due to Hamburger and Nevanlinna; cf. [42]. Finally, we connect the function-theoretic results to rank one perturbations.

A function $Q(\ell)$ belongs to the class N of Nevanlinna functions if $Q(\ell)$ is holomorphic on C \ R, $\overline{Q(\ell)} = Q(\bar{\ell})$, and Im $Q(\ell)$/ Im $\ell \geq 0$ for $\ell \in$ C\R. It is well known that a function $Q(\ell)$ belongs to N if and only if there exist α ∈ R. β ≥ 0, and a nondecreasing function σ(t) on R with $\int_R d\sigma(t) \backslash (t2+1) < \infty$, such that

$$Q(\ell) = \alpha + \beta\ell + \int_R \left(\frac{1}{t-\ell} - \frac{t}{t^2+1} \right) d\sigma(t) \tag{4}$$

Clearly (4) implies that

$$\frac{\operatorname{Im} Q(iy)}{y} = \beta + \int_R \frac{1}{t^2+y^2} d\sigma(t). \quad y \neq 0 \tag{5}$$

A function $Q(\ell)$ belongs to the Kac class N1 if and only if

$$Q(\ell) \in N \text{ and} \int_1^\infty \frac{\operatorname{Im} Q(iy)}{y} dy < \infty.$$

It follows from $\int_R (t^2+y^2)^{-1} dy = 1/|t|(\pi/2 - \arctan 1/|t|).t \neq 0$ that $Q(\ell)$ belong to N_1 if and only if there exist γ∈R and a nondecreasing function σ(t) on R with $\int_R d\sigma(t) \backslash (|t|+1) < \infty$, such that

$$Q(\ell) = \gamma \int_R \frac{d\sigma(t)}{t-\ell} \tag{6}$$

$$\gamma = \lim_{y\to\infty} Q(iy) = \lim_{y\to\infty} \operatorname{Re} Q(iy) \tag{7}$$

cf. [26], [27]. Observe that the constant γ is given by A function $Q(\ell)$ belongs to the class N_0 if and only if

$$Q(\ell) \in \text{N and} \sup_{y>0} y \operatorname{Im} Q(iy) < \infty$$

or equivalently, if there exist γ ∈ R and a nondecreasing function σ(t) on R with

$$\int_R d\sigma(t) < \infty \tag{8}$$

such that (6) holds. Clearly, $N_0 \subset N_1 \subset N$.

Let the function $Q(\ell)$ belong to N and fix $\mu \in C \setminus R$. For $\tau \in R \cup \{\infty\}$ we define a linear fractional transformation $Q_\tau(\ell)$ of $Q(\ell)$. When $\tau \in R$ we define

$$Q_\tau(\ell) = \frac{Q(\ell) - \tau(\operatorname{Im} Q(\mu))^2}{\tau Q(\ell) + 1} = \frac{1}{\tau} - \frac{1 + (\tau \operatorname{Im} Q(\mu))^2}{\tau^2} \frac{1}{\frac{1}{\tau} + Q(\ell)} \tag{9}$$

and when $\tau = \infty$ we define

$$Q_\infty(\ell) = -\frac{(\operatorname{Im} Q(\mu))^2}{Q(\ell)} \tag{10}$$

For each $\tau \in \mathbf{R} \cup \{\infty\}$ the function $Q_\tau(\ell)$ belongs to N. Moreover, if $Q(\ell)$ belongs to N_1 or N_0, then for all but one $\tau \in \mathbf{R} \cup \{\infty\}$, the corresponding function $Q_\tau(\ell)$ belongs to N_1 or N_0, respectively. The exceptional value of $\tau \in \mathbf{R} \cup \{\infty\}$, $\tau \neq 0$, is given by $1/\tau + \gamma = 0$, where γ is the limit in (7); cf. [40]. If $Q(\ell)$ reduces to a real constant c, then the exceptional value τ is given by $1/\tau + c = 0$ and the corresponding linear fractional transform is interpreted as ∞. We will tacitly exclude this situation. Finally, note that $\beta\tau = \lim_{y\to\infty} \operatorname{Im} Q_\tau(iy)/y$, exists for $\tau \in R \cup \{\infty\}$ and that if $\tau \in N_0$, then

$$\beta_\tau = 0, \frac{1}{\tau} + \gamma \neq 0, \text{ and } \beta_\tau =: \beta > 0, \frac{1}{\tau} + \gamma = 0 \tag{11}$$

For any function $Q(\ell)$ in $\mathbf{N}$ we define $Q_{[0]}(y) = \dfrac{\operatorname{Im} Q(iy)}{y}$. If $Q(\ell) \in N_0$ we define

$$Q^{[2]}(y) = \sup_{y>0} y^2 Q^{[0]}(y) - y^2 Q^{[0]}(y).$$

According to [41], $Q(\ell)$ belongs to $\mathbf{N}_{-1}$ if and only if

$$Q(\ell) \in N_0 \text{ and } \int_1^\infty Q^{[2]}(y)dy < \infty$$

and $Q(\ell)$ belongs to N_{-2} if and only if

$$Q(\ell) \in \mathbf{N}_0 \text{ and } \sup_{y>0} y^2 Q^2(y) < \infty$$

Therefore, $\mathbf{N}_{-2} \subset \mathbf{N}_{-1} \subset \mathbf{N}$. Now we proceed by induction. Assume that $Q(\ell)$ belongs to $N_{-2}k$ for some $k \in \mathbf{N} \cup \{0\}$ and that the function $Q^{[2k]}(y)$ has been given with $\sup_{y>0} y_2 Q^{[2k]}(y) < \infty$. Then we define

$$Q^{[2K+2]}(y) = \sup_{y>0} y^2 Q^{[2k]}(y) - y^2 Q^{[2k]}(y) \tag{12}$$

The function $Q(\ell)$ belongs to N_{-2k-1} if and only if

$$Q(\ell) \in N_{-2k} \text{ and } \int_1^{\infty} Q^{[2k+2]}(y)dy < \infty.$$

and the function $Q(\ell)$ belongs to N_{-2k-2} if and only if

$$Q(\ell) \in N_{-2k} \text{ and } \sup_{y>0} y^2 Q^{[2k+2]}(y) < \infty$$

Clearly,

$$\cdots \subset \mathbf{N}_{-2k-2} \subset \mathbf{N}_{-2k-1} \subset \mathbf{N}_{-2k} \subset \cdots \subset \mathbf{N}_{-2} \subset \mathbf{N}_{-1} \subset \mathbf{N}_0.$$

We give an equivalent description of the classes N_{-k}.

Theorem (2.1.1)[37]: Assume that $Q(\ell) \in N_0$ has the integral representation (6) with $\gamma \in R$ and $\sigma(t)$ as in (8). Let $k \in N \cup \{0\}$. Then $Q(\ell) \in N_{-k}$ if and only if

$$\int_R (|t|^k + 1) d\sigma(t) < \infty \tag{13}$$

Proof. We begin with the case of even indices. We claim that

$$Q(\ell) \in N_{-2k} \quad \text{if and only if } \int_R (t^{2k} + 1) d\sigma(t) < \infty \tag{14}$$

in which case

$$Q^{[2k+2]}(y) = \int_R \frac{t^{2k+2}}{t^2 + y^2} d\sigma(t) \tag{15}$$

If k = 0, (14) is clear and (15) follows as

$$Q^{[2]}(y) = \int_R d\sigma(t) - \int_R \frac{y^2}{t^2 + y^2} d\sigma(t) = \int_R \frac{t^2}{t^2 + y^2} d\sigma(t)$$

Now assume that (14) and (15) hold for some $k \in N \cup \{0\}$. Then

$$\sup_{y>0} y^2 Q^{[2k+2]}(y) = \int_R t^{2k+2} d\sigma(t) \quad (\le \infty).$$

Hence, if $Q(\ell) \in N_{-2k}$, then

$$Q(\ell) \in N_{-2k-2} \text{ if and only if } \int_R (t^{2k+2} + 1) d\sigma(t) < \infty,$$

in which case

$$Q^{[2K+4]}(y)=\int_R t^{2k+2}d\sigma(t)-\int_R \frac{t^{2k+2}y^2}{t^2+y^2}d\sigma(t).=\int_R \frac{t^{2k+4}}{t^2+y^2}\,d\sigma(t).$$

Therefore, (14) and (15) hold with k ∈ N ∪ {0} replaced by k +1

We now take care of the case of odd indices. For $Q(\ell)$ ∈ N_{-2k} , (15) implies that

$$\int_1^\infty Q^{[2k+2]}(y)dy=\int_R t^{2k+2}\,\frac{1}{|t|}\left(\frac{\pi}{2}-\arctan\frac{1}{|t|}\right)\,d\sigma(t)\ (\le\infty).$$

Hence, if $Q(\ell) \in$ N_{-2k}, we conclude that

$$Q(\ell)\in \mathrm{N}{-2k-1}\ \text{if and only if}\ \int(|t|^{2k+1}+1)\,d\sigma(t)<\infty.$$

Let $Q(\ell)\in$ N_{-k} , k ∈ N ∪ {0}. Then according to Theorem (2.1.1) the moments

$$m_i=\int_R t^i d\sigma(t),\quad i=0,\ldots\ldots\ldots k.$$

are well defined as absolutely convergent integrals. The following theorem with k even is well known; cf. [42].

Theorem (2.1.2)[37]: Let $Q(\ell)$ be a function in N_0 and assume that it has the integral representation (6) with γ ∈ R and σ(t) as in (8). If $Q(\ell)\in$ N_{-k} for some k ∈ N ∪ {0}, then $Q(\ell)$ has the asymptotic expansion

$$\ell^{k+1}\left(Q(\ell)-\gamma+\sum_{i=0}^{k}\frac{m_i}{\ell^{i+1}}\right)=0(1),\quad \ell\to\infty \tag{16}$$

uniformly for $\delta \le \arg\ \ell\le\pi-\delta$ with any $0<\delta<\frac{1}{2}\pi$ Moreover, if k is odd, the function in the left side of (16) belongs to N_1.

Proof. For $Q(\ell)\in$ N_{-k} the moments mi, i = 0,... , k, are well defined, and

$$\ell^{k+1}\left(Q(\ell)-\gamma+\sum_{i=0}^{k}\frac{m_i}{\ell^{i+1}}\right)=\int_R\frac{t^{k+1}}{t-\ell}d\sigma(t) \tag{17}$$

As in [42], it follows that the function in the right side is o(1) as $\ell\to\infty$. Moreover, for $\ell=iy$, the right side of (17) is equal to

$$\int_R\frac{t^{k+2}}{t^2+y^2}d\sigma(t)+i\int_R\frac{t^{k+1}y}{t^2+y^2}d\sigma(t).$$

Hence, if k is odd, the function in the left side of (16) is a Nevanlinna function, which even belongs to N_1.

Conversely, the class N_{-k} can be described in terms of these asymptotic expansions. For k even, the statement of the following result may be found in [42].

Theorem (2.1.3)[37]: Let $k \in N \cup \{0\}$, and let $\hat{\gamma}$ and $\hat{m}_i$, $i = 0,\dots, k$, be real numbers. Let the function $Q(\ell) \in N$ have the asymptotic expansion

$$\ell^{k+1}\left(Q(\ell) - \hat{\gamma} + \sum_{i=0}^{k} \frac{\hat{m}_i}{\ell^{i+1}}\right) = 0(1) \tag{18}$$

for $\ell = iy$, $y \to \infty$. Then the function in the left side of (18) belongs to N if k is odd. If k is even, or if k is odd and the function in the left side of (18) belongs to N_1, then $Q(\ell) \in N_{-k}$ and $\hat{\gamma} = \gamma$, $\hat{m}_i = m_i$, $i = 0,\dots, k$.

Proof. For k even, we refer to [42]. For k odd, (18) implies that

$$\ell^{k}\left(Q(\ell) - \hat{\gamma} + \sum_{i=0}^{k-1} \frac{\hat{m}_i}{\ell^{i+1}}\right) = -\frac{\hat{m}k}{\ell} + o(\frac{1}{\ell}) = o(1), \qquad \ell = iy \to \infty.$$

As k_{-1} is even, we conclude that $Q(\ell) \in N_{-k+1}$ and $\hat{\gamma} = \gamma$, $\hat{m}_i = m_i$, $i = 0,\dots, k-1$. Hence, the left side of (18) is given by

$$\ell^{k+1}\left(Q(\ell) - \gamma + \sum_{i=0}^{k-1} \frac{m_i}{\ell^{i+1}}\right) + \hat{m}_k = \int_{R} \frac{t^k \ell}{t-\ell}\, d\sigma(t) + \hat{m}_k$$

For $\ell = iy$ this is equal to

$$-\int_{R} \frac{t^k y^2}{t^2 + y^2}\, d\sigma(t) + i \int_{R} \frac{t^{k+1} y}{t^2 + y^2}\, d\sigma(t) + \hat{m}_k \tag{19}$$

Therefore, for k odd, the function in the left side of (18) is a Nevanlinna function. Under the further assumption that the function in the left side of (18) belongs to $\mathbf{N_1}$, it follows that $\int_R (|t|^{k}+1)d\sigma(t) < \infty$, so that the moment m_k is well defined and $Q(\ell) \in N_{-k}$. By taking $y \to \infty$ in (19), we obtain

$$m_k = \hat{m}_k .$$

Note that for k odd, (18) only implies that $Q(\ell) \in N_{-k+1}$ and that

$$\ell^{k+1}\left(Q(\ell)-\gamma+\sum_{i=0}^{k-1}\frac{m_i}{\ell^{i+1}}\right)+\hat{m}_k=o(1) \tag{20}$$

for ℓ=iy→ ∞. If, in this case, the function σ(t) in (6) has support in [0,∞), it follows from (19) that $Q(\ell)\in N_{-k}$ and $\hat{m}_k = m_k$. Moreover, then the function in the left side of (18) belongs to $\mathbf{N}_1$. In general, for k odd, the function in the left side of (18) does not belong to $\mathbf{N}_1$ and $\hat{m}_k$ in (20) cannot be interpreted as an absolutely convergent moment. We give an example for k = 1.

Example (2.1.4)[37]: Let σ(t) be a nondecreasing function on **R** such that

$$\int_{\mathbf{R}} d\sigma(t)<\infty, \qquad \int_{\mathbf{R}} |t|\, d\sigma(t)=\infty,$$

and for which the function

$$F(\ell)=\int_{\mathbf{R}}\frac{1+t\ell}{t-\ell}\,d\sigma(t)$$

belongs to **N\N₁**, while $\lim_{y\to\infty}F(iy)=0$. The essential part in the construction of such a function is that the support of σ(t) is unbounded in each direction; cf. [43]. Clearly, the function

$$H(\ell)=\int_{\mathbf{R}}\frac{t\ell}{t-\ell}\,d\sigma(t)$$

also belongs to $\mathbf{N}\backslash\mathbf{N}_1$ and $\lim_{y\to\infty}H(iy)=0$. Now define

$$m_0=\int_{\mathbf{R}} d\sigma(t), \qquad Q(\ell)=-\frac{m_0}{\ell}+\frac{H(\ell)}{\ell^2}$$

Then $Q(\ell)$ has the representation (6) with γ = 0, $Q(\ell)\in\mathbf{N}_0 \setminus N_{-1}$, and

$$\ell^2\left(Q(\ell)+\frac{m_0}{\ell}\right)=0(1), \qquad \ell=iy\to\infty$$

A similar example for positive definite functions is due to A. Wintner; see [44].

In order to see how the class N_{-k}, $k\in\mathbf{N}\cup\{0\}$, behaves under the linear fractional transformation (9),(10) we state and prove the following simple lemma.

Lemma (2.1.5)[37]: Let c_i, d_i, i = −1, 0, 1,... , k, be real numbers satisfying

$$c_{-1}\,d_{-1}=-1 \quad \sum_{j=0}^{i} c_{j-1}\,d_{i-j-1}=0, \qquad i=1,\ldots,k+1$$

And let $C(\ell)=0(1), \ell \to \infty.$

$$-\left(c_{-1}+\frac{c_0}{\ell}+\ldots+\frac{c_k}{\ell^{k+1}}+\frac{C(\ell)}{\ell^{k+1}}\right)^{-1}=d_{-1}+\frac{d_0}{\ell}+\ldots+\frac{d_k}{\ell^{k+1}}+\frac{D(\ell)}{\ell^{k+1}}$$

Where $D(\ell) = 0\,(1)$, $\ell \to \infty$. Moreover

$$D(\ell)=\frac{1}{(c_{-1})^2}C(\ell)+O\left(\frac{1}{\ell}\right), \quad \ell \to \infty.$$

Proof. It follows from the assumption about the convolution products that

$$-D(\ell)=\frac{\alpha(\ell)+\delta(\ell)C(\ell)}{\gamma(\ell)+C(\ell)}.$$

where $\alpha(\ell)$, $\gamma(\ell)$, and $\delta(\ell)$ are polynomials of the form

$$\alpha(\ell)=\sum_{j=o}^{k} a_j \ell^j, \quad \gamma(\ell)=\sum_{j=0}^{k+1} c_{j-1}\,\ell^{k-j+1}, \quad \delta(\ell)=\sum_{j=0}^{k+1} d_{j-1}\,\ell^{k-j+1}$$

Hence, we may write

$$-D(\ell)=\frac{d_{-1}}{c_{-1}}C(\ell)+\frac{\alpha(\ell)}{0(\ell)+C(\ell)}+\left(\frac{\delta(\ell)}{O(\ell)+C(\ell)}-\frac{d_{-1}}{c_{-1}}\right)C(\ell)$$

The degrees of $\alpha(\ell)$ and of $c_{-1}\delta(\ell) - d_{-1}\gamma(\ell)$ are at most k, so the second term in the right side is $O(1/\ell)$ and the third term in the right side is $o(1/\ell)$.

According to Lemma (2.1.5), there is a constant $A > 0$, such that

$$\frac{|\operatorname{Im} D(iy)|}{y}\le\frac{1}{(c_{-1})^2}\frac{|\operatorname{Im} C(iy)|}{y}+\frac{A}{y^2}, \qquad y \ge 1.$$

Hence, if $C(\ell)\in \mathbf{N_1}$, then $|\operatorname{Im} D(iy)|/y$ is integrable over $[1, \infty)$. If, in addition, $D(\ell) \in \mathbf{N}$, it therefore automatically belongs to $\mathbf{N_1}$.

Theorem (2.1.6)[37]: Assume that the function $Q(\ell)$ belongs to N_{-k} for some $k \in \mathbf{N}\cup\{0\}$.

Then

(i) $Q_\tau(\ell) \in N_{-k}$ for $1/\tau + \gamma \neq 0$,

(ii) $Q_\tau(\ell) - \beta\ell \in N_{-k+2}$ for $1/\tau + \gamma = 0$.

Proof. Without loss of generality we may assume that $\gamma=0$, so that the exceptional value of τ corresponds to $\tau=\infty$. Due to (9) and (10), it suffices to show that

$$-\frac{1}{\frac{1}{\tau}+Q(\ell)} \in N_{-k} for \tau \in \mathbf{R} \setminus \{0\}, \tag{21}$$

and

$$-\frac{1}{Q(\ell)}-\frac{1}{m_0}\ell \in N_{-k+2} for\ \tau=\infty. \tag{22}$$

For the formulation of (22) we used that β in (11) satisfies $\beta = \ell \operatorname{Im} Q(\mu))^2/m_0$ when $\gamma = 0$; see [40]. As the function $Q(\ell)$ belongs to N_{-k} for some $k \in \mathbf{N} \cup \{0\}$, it follows from Theorem (2.1.2) and the assumption $\gamma = 0$ that

$$Q(\ell) = -\sum_{i=0}^{K} \frac{m_i}{\ell^{i+1}} + \frac{C(\ell)}{\ell^{k+1}} \tag{23}$$

where $C(\ell) = o(1)$, $\ell \to \infty$, and $C(\ell)$ belongs to $\mathbf{N}_1$ when k is odd.

We now prove (21) for $\tau \in \mathbf{R} \setminus \{0\}$. From (23) and Lemma (2.1.5) we obtain with real numbers $\hat{m}_i$, i = 0,... , k, the asymptotic expansion

$$-\frac{1}{\frac{1}{\tau}+Q(\ell)} = -\tau - \sum_{i=0}^{k} \frac{\hat{m}_i}{\ell^{i+1}} + \frac{D(\ell)}{\ell^{k+1}}$$

where $D(\ell) = \tau^2 C(\ell) + O(1/\ell) = o(1)$, $\ell \to \infty$. It follows from Theorem (2.1.3) and the asymptotic estimate of $D(\ell)$ that (21) holds.

Next we prove (22). The statement for k = 0 is obvious, so assume that $k \geq 1$. From (23) and Lemma (2.1.5)) (with k instead of k + 1) we obtain with real numbers $\hat{m}_i$,, i = 1,... , k, the asymptotic expansion

$$\frac{1}{\ell Q(\ell)} = \frac{1}{m_0} - \sum_{i=1}^{k} \frac{\hat{m}_i}{\ell^i} + \frac{D(\ell)}{\ell^k},$$

with $D(\ell) = C(\ell)/m_0^2 + O(1/\ell), \ell \to \infty$. *Hence*

$$-\frac{1}{Q(\ell)} - \frac{1}{m_0}\ell = -\sum_{i=1}^{k} \frac{\hat{m}_i}{\ell^{i-1}} + \frac{D(\ell)}{\ell^{k-1}}.$$

Again we apply Theorem (2.1.3) and the asymptotic estimate of $D(\ell)$. Hence, (22) holds for k = 1 and for $k \geq 2$.

Let $Q(\ell) \in N_{-k}$ for some $k \in \mathbf{N} \cup \{0\}$. Assume that $1/\tau + \gamma \neq 0$. Then it follows from Theorems (2.1.2) and (2.1.6) that

$$Q_\tau(\ell)=\gamma(\tau)-\sum_{i=0}^{K}\frac{m_i(\tau)}{\ell^{i+1}}+0\left(\frac{1}{\ell^{k+1}}\right),\qquad \ell\to\infty, \tag{24}$$

where $\gamma(\tau)=\lim_{y\to\infty}Q_\tau(iy)$ and $m_i(\tau)$, i=0,... , k, are the corresponding moments of $Q_\tau(\ell)$. Now assume that $1/\tau+\gamma=0$. For k=0 the function $Q_\tau(\ell)-\beta\ell$ belongs to **N**, where β is given by (11). For $k\geq 1$ it belongs to $\mathbf{N_1}$, in which case

$$\gamma(\tau)=\lim_{y\to\infty}(Q_\tau(iy)-i\beta y) \tag{25}$$

is a real number. For $k\geq 2$ the function $Q_\tau(\ell)-\beta\ell$ belongs to N_{-k+2}, and it follows from Theorems (2.1.2) and (2.1.6) that

$$Q_\tau(\ell)-\beta\ell=\gamma(\tau)-\sum_{i=0}^{k-2}\frac{m_i(\tau)}{\ell^{i+1}}+0\left(\frac{1}{\ell^{k-1}}\right).\qquad \ell\to\infty, \tag{26}$$

where $\gamma(\tau)$ is given by (25) and $m_i(\tau)$, $i=0,\ldots,k-2$, are the moments of $Q_\tau(\ell)-\beta\ell$. The constants β and $\gamma(\tau)$ and the moments in (24) and in (26) can be expressed in terms of the corresponding data of the expansion (16) of $Q(\ell)$.

Corollary (2.1.7)[37]: Assume that the function $Q(\ell)$ belongs to N_{-k}, $k\in\mathbf{N}\cup\{0\}$.

For $1/\tau+\gamma\neq 0$ and $\tau\in\mathbf{R}$, the constant $\gamma(\tau)$ is given by

$$\gamma(\tau)=\frac{\gamma-\tau(\operatorname{Im}Q(\mu))^2}{1+\tau\gamma} \tag{27}$$

and the moments $m_i(\tau)$, $i=0,\ldots,k$, in (24) are given by

$$\begin{pmatrix} m_0(\tau)\\ m_1(\tau)\\ \vdots\\ m_{k-1}(\tau)\\ m_k(\tau)\end{pmatrix}=\frac{1+(\tau\operatorname{Im}Q(\mu))^2}{(1+\tau\gamma)^2} \tag{28}$$

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -\frac{\tau m_0}{1+\tau\gamma} & 1 & \cdots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -\frac{\tau m_{k-2}}{1+\tau\gamma} & & \ddots & 1 & 0 \\ -\frac{\tau m_{k-1}}{1+\tau\gamma} & -\frac{\tau m_{k-2}}{1+\tau\gamma} & \cdots & -\frac{\tau m_0}{1+\tau\gamma} & 1 \end{pmatrix}^{-1} \begin{pmatrix} m_0 \\ m_1 \\ \vdots \\ m_{k-1} \\ m_k \end{pmatrix}$$

The case $1/\tau + \gamma \neq 0$ and $\tau = \infty$ is obtained as a limiting case of (27) and (28).

For $1/\tau + \gamma = 0$ and $\tau \in \mathbf{R}$, the constant β is given by

$$\beta = \frac{1+(\tau\,\mathrm{Im}Q(\mu))^2}{\tau^2 m_0} \tag{29}$$

and when $k \geq 1$, the constant $\gamma(\tau)$ is given by

$$\gamma(\tau) = \frac{1}{\tau} - \frac{1+(\tau\,\mathrm{Im}Q(\mu))^2}{\tau^2}\frac{m_1}{m_0^2} \tag{30}$$

Moreover, when $k \geq 2$ the moments $m_i(\tau)$, $i = 0,\ldots,k-2$, in (26) are given by

$$\begin{pmatrix} m_0(\tau) \\ m_1(\tau) \\ \vdots \\ m_{k-3}(\tau) \\ m_{k-2}(\tau) \end{pmatrix} = \frac{1+(\tau\,\mathrm{Im}Q(\mu))^2}{\tau^2 m_0} \tag{31}$$

$$\begin{pmatrix} m_0 & 0 & \cdots & 0 & 0 \\ m_1 & m_0 & \cdots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ m_{k-3} & & \ddots & m_0 & 0 \\ m_{k-2} & m_{k-3} & \cdots & m_1 & m_0 \end{pmatrix}^{-1} \begin{pmatrix} m_2 - \frac{m_1}{m_0}m_1 \\ m_3 - \frac{m_1}{m_0}m_2 \\ \vdots \\ m_{k-1} - \frac{m_1}{m_0}m_{k-2} \\ m_k - \frac{m_1}{m_0}m_{k-1} \end{pmatrix}$$

The case $\gamma = 0$ and $\tau = \infty$ is obtained as a limiting case of (29), (30) and (31).

Proof. From (9) and (10) we obtain

$$Q_\tau(\ell) - Q(\ell) + \tau Q(\ell)Q_\tau(\ell) + \tau(\mathrm{Im}\,Q(\mu))^2 = 0,\ \ \tau \in \mathbf{R} \tag{32}$$

$$Q(\ell)Q_\infty(\ell) + (\mathrm{Im}\,Q(\mu))^2 = 0 \tag{33}$$

We will substitute the asymptotic expansions (16) for $Q(\ell)$ and (24) or (26) for $Q_\tau(\ell)$ in (32) and (33), and calculate the coefficients of the powers of ℓ.

For $1/\tau + \gamma \neq 0$ and $\tau \in R$ we use the expansion (24) for $Q_\tau(\ell)$ in (32). The coefficient of ℓ gives (27), and the coefficient of ℓ^{-1} gives

$$m_0(\tau) = \frac{1-\tau\gamma(\tau)}{1+\tau\gamma} m_0 = \frac{1+(\tau \operatorname{Im} Q(\mu))^2}{(1+\tau\gamma)^2} m_0.$$

Moreover, the coefficients of ℓ^{-i-1}, $i = 1,\dots, k$, give

$$(1 + \tau\gamma)m_i(\tau) = m_i(1 - \tau\gamma(\tau)) + \tau(m_0 m_{i-1}(\tau) + \cdots + m_{i-1}m_0(\tau)).$$

This leads to (28). For $1/\tau+\gamma \neq 0$ and $\tau = \infty$, we use the expansion (24) for $Q_\tau(\ell)$ in (33) and obtain the limiting case of (27) and (28) as $\tau \to \infty$.

For $1/\tau+\gamma= 0$ we substitute the expansion (26) in (32). Note that the coefficient of ℓ is automatically 0. The coefficient of ℓ^0 gives (29) (cf. [40]), and the coefficient of ℓ^{-1} gives $1-\tau\gamma(\tau) = \tau m_1\beta/m_0$, so that (30) follows. Similarly, the coefficients of ℓ^{-i-1} then give

$$m_0 m_{i-1}(\tau) + \dots + m_{i-1} m_0(\tau) = \beta\left(m_{i+1} - \frac{m_1}{m_0} m_i\right),$$

for $i = 1,\dots,k-2$. Moreover, the identity also holds for $i = k-1$. This leads to (31). For $\gamma=0$ and $\tau=\infty$, we use the expansion (26) for $Q_\tau(\ell)$ in (33) and obtain the limiting case of (29), (30) and (31) as $\tau \to \infty$.

Let $H(\ell)$ be a Nevanlinna function with $\beta = \lim_{y\to\infty} \operatorname{Im} H(iy)/y > 0$. We have seen that $H(\ell) - \beta\ell$ belongs to $\mathbf{N}$. Define the function $Q(\ell)$ by

$$Q(\ell) = -\frac{|H(\mu)|^4}{(\operatorname{Im} H(\mu))^2 H(\ell)} \tag{34}$$

Clearly, $Q(\ell) \in \mathbf{N}_0$ and $\lim_{y\to\infty} Q(iy)=0$. Hence, $H(\ell)=Q_\infty(\ell)$ is the exceptional function corresponding to the exceptional value $\tau = \infty$ of $Q(\ell)$; cf. [41].

Theorem (2.1.8)[37]: Let $H(\ell)$ belong to $\mathbf{N}$ with $\beta = \lim_{y\to\infty} \operatorname{Im} H(iy)/y > 0$. Assume that $H(\ell) - \beta\ell$ belongs to N_{-k+2} for some $k \in \mathbf{N} \cup \{0\}$. Then the function $Q(\ell)$ in (34) belongs to N_{-k}.

Proof. It is sufficient to assume that k ≥ 1. Then

$$H(\ell)-\beta\,\ell=\gamma-\sum_{i=0}^{k-2}\frac{m_i}{\ell^{i+1}}+\frac{C(\ell)}{\ell^{K-1,}}$$

where $\gamma=\lim_{y\to\infty}(Q\,(iy)-i\beta y)$ and m_i, i= 0,... ,k−2, are the moments of $H(\ell)-\beta\ell$ (absent for k=1). Moreover, if k is odd, the function $C(\ell)$ belongs to N_1. Therefore, by Lemma (2.1.5) with k+1 replaced by k, we find real numbers $d_{-1},\ldots, d_{k-1}$ and a function $D(\ell)=o(1)$, $\ell\to\infty$, such that

$$-\frac{1}{H(\ell)}=\frac{1}{\ell}\left(-\frac{1}{\beta+\frac{\gamma}{\ell}\sum_{i=0}^{k-2}\frac{m_i}{\ell^{i+2}}+\frac{C(\ell)}{\ell^k}}\right)=\sum_{i=0}^{k}\frac{d_{i-1}}{\ell^{i+1}}+\frac{D(\ell)}{\ell^{k+1}}$$

with $D(\ell)$ in N_1 when k is odd. Hence $Q\,(\ell)\in N_{-k}$ by Theorem (2.1.3).

The relation between the data for the functions $H(\ell)-\beta\ell$ and $Q\,(\ell)$ may be recovered from Corollary (2.1.7) by inversion of the case $\gamma=0$ and $\tau=\infty$; cf. [45] for a special case.

Let A be a self-adjoint relation in a Hilbert space H. For $\mu\in C\setminus R$ we choose a nontrivial element $\chi(\mu)\in H$ and define

$$\chi(\ell)=(I+(\ell-\mu)(A-\ell)^{-1})\chi(\mu).$$

Let S be the restriction of A given by

$$S=\{\ \{f,g\}\in A:\{\,g-\ell\ f,\chi(\bar{\ell})]=0\ \}\tag{35}$$

Clearly, this definition is independent of $\ell\in\mathbf{C}\backslash\mathbf{R}$, and S is a closed symmetric relation with defect numbers (1, 1). The relation S is completely nonself-adjoint if and only if $H=\overline{span}\{\ \chi(\ell):\ \ell\in\mathbf{C}\setminus\mathbf{R}\ \}$, in which case S is necessarily an operator. A function $Q\,(\ell)$ is a Q -function of A and S if

$$\frac{Q(\ell)-Q(\lambda)^*}{\ell-\bar{\lambda}}=[\ \chi(\ell).\chi(\lambda)]$$

Hence, a Q -function is determined up to a real constant and belongs to the Nevanlinna class N. If S is completely nonself-adjoint, the Q -function uniquely determines, up to isometric isomorphisms, the relation A and its restriction S. All self-adjoint extensions $A(\tau)$, $\tau \in \mathbf{R} \cup \{\infty\}$, of S are parametrized by means of Kreın's formula

$$(A(\tau)-\ell)^{-1}=(A-\ell)^{-1}-\chi(\ell)\frac{1}{Q(\ell)+1/\tau}\ [.,\chi(\bar{\ell})]$$

The Q -functions $Q_{\tau}(\ell)$ of $A(\tau)$, $\tau \in \mathbf{R} \cup \{\infty\}$, are related to $Q(\ell)$ via (9) and (10); see [40].

In the following we assume that A is a self-adjoint operator. The restriction S in (35) coincides with (2) if and only if $\chi(\ell) \in \operatorname{dom} A$ for some (and hence for all) $\ell \in \rho(A)$. Then $\chi(\ell) = (A - \ell)^{-1}\omega$, and $Q(\ell)$ can be chosen as

$$Q(\ell)=[(A-\ell)^{-1}\omega,\omega].$$

This choice of $Q(\ell) \in N_0$ gives $\gamma = 0$, so that the exceptional value in (9) and (10) is $\tau = \infty$. The self-adjoint extensions of S in (2) are now the rank one perturbations $A(\tau)$, $\tau \in \mathbf{R}$, of A given in (1) (cf. [38]), and the exceptional extension $A(\infty)$ in (3). If $E(t)$, $t \in \mathbf{R}$, is the spectral family of A, and $Q(\ell)$ is given by (6) with $\gamma = 0$ and (8), then $d\sigma(t) = d([E(t)\omega, \omega])$. We denote the polar decomposition of A by $A = U|A|$. The following result is clear.

Theorem (2.1.9)[37]: Let $k \in \mathbf{N} \cup \{0\}$ Then $Q(\ell) \in N_{-k}$ if and only if $\omega \in \operatorname{dom} |A|^{k/2}$ · In this case, the moments m_j, $j = 0,\dots, k$, are given by

$$m_j = [A^{j/2}\omega, A^{j/2}\omega],\quad j \text{ even},\qquad m_j = [U|A^{j/2}\omega, |A^{j/2}\omega],\quad j \text{ odd}.$$

Note that if $A \in L(H)$, then each $\omega \in H$ has the property that $\omega \in \operatorname{dom} |A|^{k\backslash 2}$,for all $k \in \mathbf{N} \cup \{0\}$. In particular this applies when the closed symmetric operator S is bounded and, consequently, $A(\tau) \in L(H)$, $\tau \in \mathbf{R}$; see also [46].

Theorem (2.1.10)[37]: Assume that the Q -function $Q(\ell)$ of S and A belongs to N_{-k} for some $k \in \mathbf{N} \cup \{0\}$. Then

$$\operatorname{dom} |A(\tau)|^{k/2+1} = \operatorname{dom} |A|^{k/2+1},\ \tau \in \mathbf{R} \cup \{\infty\},\ 1/\tau + \gamma \neq 0 \qquad (36)$$

Proof. The statement is true for k = 0, 1, 2; cf. [20]. We proceed by induction. Let $g \in \operatorname{dom} |A(\tau)|^{k/2+1}$, so that $g = (A(\tau) - \ell)^{-1} f$ for some $f \in \operatorname{dom} |A(\tau)|^{k/2}$. By Kreın's formula.

$$g=(A-\ell)^{-1}f-\frac{\left[f,\chi(\bar{\ell})\right]}{1/\tau+Q(\ell)}\chi(\ell) \qquad (37)$$

Since $Q(\ell)\in N_{-k}$, Theorem (2.1.9) shows that $\chi(\ell)=(A-\ell)^{-1}\omega\in$ dom$|A|^{k/2+1}$. Moreover, $N_{-k}\subset N_{-k+2}$, and since $1/\tau+\gamma\neq 0$, we conclude by an induction argument that

$$f\in \text{dom } |A(\tau)|^{k/2}=\text{dom } |A|^{k/2}.$$

Hence, $(A-\ell)^{-1}f\in$ dom $|A|^{k/2+1}$. It follows from (37) that $g\in$ dom $|A|^{k/2+1}$ and there fore

$$\text{dom } |A(\tau)|^{k/2+1}=\text{dom } |A|^{k/2+1},\quad 1/\tau+\gamma\neq 0.$$

According to Theorem (2.1.6), A and $A(\tau)$, $1/\tau+\gamma\neq 0$, both have a Q-function belonging to N_{-k}, so the reverse inclusion follows by symmetry.

Theorem (2.1.11)[37]: Let S be a closed symmetric operator in H with defect numbers (1, 1) and let $\alpha\geq 0$, $k\in \mathbf{N}\cup\{0\}$. If for two different self-adjoint operator extensions A_1 and A_2 of S the inclusion

$$\text{dom } |A_1|^{k/2+1}\supset \text{dom } |A_2|^{k/2+1+\alpha}$$

is satisfied, then for all but one self-adjoint extension $A(\tau)$ of S we have

$$\text{dom } |A(\tau)|^{k/2+1}=\text{dom } |A_1|^{k/2+1}.$$

Moreover, the Q-functions of these extensions of S all belong to N_{-k}.

Proof. The statements hold for k = 0, 1, 2; cf. [20]. Let $R_1(\ell)$ and $R_2(\ell)$ be the resolvent operators of A_1 and A_2, respectively. Let $h\in$ dom$|A_2|^{k/2+\alpha}$, $k\geq 2$, be such that $[h,\chi(\bar{\ell})]\neq 0$. By Kreın's formula

$$\frac{\left[h,\chi(\bar{\ell})\right]}{1/\tau+Q(\ell)}\chi(\ell)=R_1(\ell)h-R_2(\ell)h,\qquad h\in H\qquad \tau\neq 0 \qquad (38)$$

Since $R_1(\ell)h\in$ dom A_1 and

$$R_2(\ell)h\in \text{dom } |A_2|^{k/2+1+\alpha}\subset \text{dom } |A_1|^{k/2+1}\subset \text{dom } A_1,$$

it follows from (38) and the selection of h that $\chi(\ell)\in$dom A_1. Hence, we may write $\chi(\ell)=(A-\ell)^{-1}\omega$ for some $\omega\in$ H. Since A_1 and A_2 both are operator extensions of S, Theorems (2.1.6), (2.1.9) and (2.1.10) imply that dom A_1 = dom

A_2. Hence, $h \in$ dom A_1, and thus (38) shows that $\chi(\ell) \in$ dom A_1^2 or $\omega \in$ dom A_1. Repeating this argument, we finally observe that, in fact,

$$\chi(\ell) = (A1 - \ell)^{-1}\omega \in \text{dom } |A_1|^{k/2+1},$$

or, equivalently, $\omega \in$dom $|A_1|^{k/2}$. According to Theorem (2.1.9) the Q -function $Q(\ell)$ of A_1 and S belongs to N_{-k}. Now apply Theorems (2.1.10) and (2.1.6).

Section (2.2): Rank One Perturbations, and Localization:

Although concrete operators with singular continuous spectrum have proliferated recently [1, 48, 49, 50, 51, 52, 53, 54], we still don't really understand much about singular continuous spectrum. In part, this is because it is normally defined by what it isn't — neither pure point nor absolutely continuous. An important point of view, going back in part to Rodgers and Taylor [55,56] and studied recently within spectral theory by Last [57] (also see references therein), is the idea of using Hausdorff measures and dimensions to classify measures. Our main goal is to look at the singular spectrum produced by rank one perturbations (and discussed in [1,48,58]) from this point of view.

A Borel measure μ is said to have exact dimension $\alpha \in [0, 1]$ if and only if $\mu(S) = 0$ if S has dimension $\beta < \alpha$ and if μ is supported by a set of dimension α. If $0 < \alpha < 1$, such a measure is, of necessity, singular continuous. But, there are also singular continuous measures of exact dimension 0 and 1 which are "particularly close" to point and a.c. measures, respectively. Indeed, as we'll explain, we know of "explicit" Schrödinger operators with exact dimension 0 and 1, but, while they presumably exist, we don't know of any with dimension $\alpha \in (0, 1)$.

While we're interested in the abstract theory of rank one perturbations, we're especially interested in those rank one perturbations obtained by taking a random Jacobi matrix and making a Baire generic perturbation of the potential at a single point. It is a disturbing fact that the strict localization (dense point spectrum with $\|xe^{-itH}\delta_0\|^2 = (e^{-itH}\delta_0, x^2e^{-itH}\delta_0)$ bounded in t), that holds a.e. for the random case, can be destroyed by arbitrarily small local perturbations [1,48]. We'll see that, the spectrum is always of dimension zero, albeit sometimes pure point and sometimes singular continuous. And we'll show that not only does the set of couplings with singular continuous spectrum has Lebesgue measure zero, it has Hausdorff dimension zero., We'll also see that while $\|xe^{-itH}\delta_0\|$ may be unbounded after the local perturbation, it never grows faster than $C\ln(t)$.

We'll review some basic facts about Hausdorff measures that we'll use later. We relate these to boundary behavior of Borel transforms. We use these ideas to present relations between spectra produced by rank one perturbations and the behavior of the spectral measure of the unperturbed operator. We'll relate Hausdorff dimensions of some energy sets to the dimensions of some coupling constant sets.

We use the results to present examples (some related to those in [59]) that show that the Hausdorff dimension under perturbation can be anything.

We turn to systems with exponentially localized eigenfunctions, and show that under local perturbations the spectrum remains of Hausdorff dimension zero. Some of the lemmas in this section on the nature of localization are of independent interest. Finally, we prove that "physical" localization is "almost stable," that is, suitable decay of $(\delta_n, e^{-itH}\delta_m)$ in $|n-m|$ uniform in t implies that $\|x\exp(-it(H+\lambda\delta_0))\delta_0\|$ grows at worst logarithmically.

Given a Borel set S in R and $\alpha \in [0, 1]$, we define

$$Q_{\alpha,\delta}(S) = \inf\left\{ \sum_{\nu=1}^{\infty} |b_\nu|^{\alpha} \,\middle|\, |b_\nu| < \delta;\ S \subset \bigcup_{\nu=1}^{\infty} b_\nu \right\},$$

the inf over all δ-covers by intervals b_ν of size at most δ. Obviously, as δ decreases, Q increases since the set of covers becomes fewer, and

$$h^{\alpha}(S) = \lim_{\delta\downarrow 0} Q_{\alpha,\delta}(S)$$

is called α-dimensional Hausdorff *measure*. It is a non-sigma-finite measure on the Borel sets. Note that h^0 coincides with the counting measure (i.e., assigns to each set the number of points in it), and h^1 coincides with Lebesgue measure. Clearly, if $\beta < \alpha < \gamma$,

$$\delta^{\alpha-\gamma} Q_{\gamma,\delta}(S) \le Q_{\alpha,\delta}(S) \le \delta^{\alpha-\beta} Q_{\beta,\delta}(S),$$

so if $h^{\alpha}(S) < \infty$, then $h^{\gamma}(S) = 0$ for $\gamma > \alpha$ and if $h^{\alpha}(S) > 0$, then $h^{\beta}(S) = \infty$ for $\beta<\alpha$. Thus, for any S, there is a unique α_0, called its Hausdorff dimension, dim(S), so $h^{\alpha}(S) = 0$ if $\alpha > \alpha_0$ and $h^{\alpha}(S) = \infty$ if $\alpha < \alpha_0$. $h^{\alpha_0}(S)$ can be zero, finite, infinite, or so infinite S isn't even h^{α_0}-sigma-finite.

In what follows, we shall use Hausdorff measures and dimensions to classify measures. Unless pointed otherwise, by "a measure" (equivalently, "a measure on R "; usually denoted by μ) we mean a positive sigma-finite Borel measure on R . however, we discuss more restricted classes of measures, such as finite measures.

Definition (2.2.1)[47]: A measure μ on R is said to be of exact dimension α for $\alpha \in [0, 1]$ if and only if

(i) For any $\beta \in [0, 1]$ with $\beta < \alpha$ and S a set of dimension β, $\mu(S) = 0$.

(ii) There is a set S_0 of dimension α which supports μ in the sense that $\mu(R \setminus S_0) = 0$ [55].

Every measure is of some exact dimension; indeed, the sum of measures of exact distinct dimensions is not of any exact dimension. But, most of our examples will involve measures of some exact dimension. Last [57], following Rodgers-Taylor [55,56], discusses many different decompositions of any measure into a part of dimension less than α, equal to α, and larger than α. The piece of exact dimension α can be further decomposed in terms of its relation to h^{α}.

Definition (2.2.2)[47]: Given any measure μ and any $\alpha \geq 0$, we define

$$D_{\mu}^{\alpha}(x) = \overline{\lim_{\delta \downarrow 0}} = \frac{\mu(x-\delta, x+\delta)}{\delta^{\alpha}} \tag{39}$$

Note that if $D_{\mu}^{\alpha_0}(x_0) < \infty$ for some x, then $D_{\mu}^{\beta}(x_0) = 0$ for all $\beta < \alpha_0$ and if $D_{\mu}^{\alpha_0}(x_0) > 0$ for some x_0 , then $D_{\mu}^{\beta}(x_0) = \infty$ for all $\beta > \alpha_0$. In particular, for each x_0 , there is an $\alpha(x_0)$ so $D_{\mu}^{\alpha}(x_0) = 0$ if $\alpha < \alpha(x_0)$ and $= \infty$ if $\alpha > \alpha(x_0)$. Indeed,

$$\alpha(x_0) = \lim_{\delta \downarrow 0} = \frac{\ln \mu(x_0-\delta, x_0+\delta)}{\ln \delta} \tag{40}$$

We'll sometimes write $\alpha_{\mu}(x_0)$ if we want to be explicit about the μ involved; and if we have a one-parameter family μ_{λ}, we'll use α_{λ} for $\alpha_{\mu_{\lambda}}$.

The following is a result of Rodgers-Taylor [55,56] (also see [60]):

Theorem (2.2.3)[47]: Let μ be any measure and $\alpha \in [0, 1]$. Let $T_\alpha = \{x \backslash D_{\mu}^{\alpha}(x) = \infty\}$ and let χ_α be its characteristic function. Let $d\mu_{\alpha s} = \chi_\alpha \, d\mu$ and $d\mu_{\alpha c} = (1 - \chi_\alpha) \, d\mu$. Then $d\mu_{\alpha s}$ is singular with respect to h^α (i.e., supported on a set of h^α-measure zero) and $d\mu_{\alpha c}$ is continuous with respect to h^α (i.e., gives zero weight to any set of h^α-measure zero).

Corollary (2.2.4)[47]: A measure μ is of exact dimension $\alpha_0 \in [0, 1]$ if and only if

(i) For any $\beta > \alpha_0$, $D_{\mu}^{\beta}(x) = \infty$ a.e. x w.r.t. μ.

(ii) For any $\beta < \alpha_0$, $D_{\mu}^{\beta}(x) = 0$ a.e. x w.r.t. μ.

Equivalently, if $\alpha(x) = \alpha_0$ a.e. x w.r.t. μ). More generally, if (i) holds (equivalently, $\alpha(x) \le \alpha_0$ a.e. w.r.t. μ), then μ is supported on a set of dimension α and if (ii) holds (equivalently, $\alpha(x) \ge \alpha_0$ a.e. w.r.t. μ), then μ gives zero weight to any set S of dimension $\beta < \alpha_0$.

Corollary (2.2.5)[47]: Let μ be a measure on R, let $S \subset R$ be a Borel set with $\mu(S) > 0$, and suppose that $\alpha_0 \in [0, 1]$ and

$$D_{\mu}^{\alpha_0}(x) < \infty$$

for μ-a.e. x in S. Then $\dim(S) \ge \alpha_0$.

Proof. $\alpha_0 = 0$ is trivial, so suppose $\alpha_0 > 0$. Let v be the measure $\mu(S \cap \cdot)$. Then, since $v \le \mu$, the hypothesis implies that

$$D_{v}^{\alpha_0}(x) < \infty$$

for a.e. x w.r.t. v. Thus, by Theorem (2.2.3), v gives zero weight to sets of h^{α_0}-measure zero, and so, since $v(S) \neq 0$, we must have $h^{\alpha_0}(S) > 0$, which implies $\dim(S) \ge \alpha_0$.

It is often easier to deal with power integrals, so we note:

Proposition (2.2.6)[47]: Let μ be a finite measure, and *let* $\tilde{G}_\alpha(x_0)=\int\frac{d\mu(y)}{|x_0-y|^\alpha}$.

Then

(i) $\tilde{G}_\alpha(x_0)<\infty$ implies $D_\mu^\alpha(x_0)<\infty$.

(ii) $D_\mu^\alpha(x_0)<\infty$ implies $\tilde{G}_\beta(x_0)<\infty$ for any $0\le\beta<\alpha$.

Proof: (i) Looking at the contribution to the integral of the set where $|x_0-y|<\delta$, we see that

$$\mu(x_0-\delta,x_0+\delta)\le\delta^\alpha\tilde{G}\alpha(x_0)$$

so

$$D_\mu^\alpha(x_0)\le\tilde{G}\alpha(x_0).$$

(ii) Let $M_\mu^\delta(x_0)=\mu(x_0-\delta,x_0+\delta)$. Then (with λ = Lebesgue measure)

$$\tilde{G}_\beta(x_0)=(\mu\otimes\lambda)((y,\ t)\mid 0\le t\le|x_0-y|^{-\beta})$$

$$=\int_0^\infty M_\mu^{t-1/\beta}(x_0)dt$$

$$=\beta\int_0^\infty M_\mu^\delta(x_0)^{\delta-\beta-1}d\delta.$$

The integral always converges for δ large since M_μ^δ is bounded; and if $\beta<\alpha$, and $D_\mu^\alpha(x_0)<\infty$, then it converges for small δ.

Consider the set

$$w_\alpha=\left\{x\,\middle|\,\overline{\lim_{\delta\downarrow0}}\frac{\mu(x-\delta,x+\delta)}{\delta^\alpha}\ne\lim_{\delta\downarrow0}\frac{\mu(x-\delta,x+\delta)}{\delta^\alpha}\right\}\tag{41}$$

For $\alpha=0$, W_α is empty; and for $\alpha=1$, the theorem of de la Vallée-Poussin (see [61], [62]) says that $\mu(W_1)=0$. For $0<\alpha<1$, however, the situation is quite different: A result going back to Besicovitch [63] (also see [64]) is that if μ is the restriction of h^α to a set of finite positive h^α-measure, then μ is supported on W_α. Moreover, there are even examples of μ's where for a.e. x w.r.t. μ,

$$\overline{\lim_{\delta\downarrow 0}}\frac{\ln\mu(x-\delta,x+\delta)}{\ln\delta}=1 \text{ and } \underline{\lim_{\delta\downarrow 0}}\frac{\ln\mu(x-\delta,x+\delta)}{\ln\delta}=0.$$

Given a measure μ with $\int(|x|+1)^{-1}d\mu(x)<\infty$, we define its Borel transform by

$$F_\mu(z)=\int\frac{d\mu(x)}{x-z}$$

for Im $z > 0$. These play a crucial role in the theory of rank one perturbations as originally noticed by Aronszajn-Donoghue [65,66]; see [58] for their properties and this theory. We'll translate Theorem (2.2.3) into Borel transform language.

Definition (2.2.7)[47]: Fix $\gamma\le 1$ and x. Let

$$Q_\mu^\gamma(x)=\overline{\lim_{\varepsilon\downarrow 0}}\ \varepsilon^\gamma\ \operatorname{Im}F_\mu(x+i\varepsilon)$$

$$R_\mu^\gamma(x)=\overline{\lim_{\varepsilon\downarrow 0}}\ \varepsilon^\gamma\ |F_\mu(x+i\varepsilon)|.$$

Theorem (2.2.8)[47]: Fix μ and x_0. Fix $\alpha\in[0,1)$ and let $\gamma=1-\alpha$. Then $D_\mu^\alpha(x_0)$, $Q_\mu^\gamma(x_0)$, and $R_\mu^\gamma(x_0)$ are either all infinite, all zero, or all in $(0,\infty)$.

Lemma (2.2.9)[47]: For any $\gamma\le 1$,

$$D_\mu^{1-\gamma}(x_0)\le 2Q_\mu^\gamma(x_0)\le 2R_\mu^\gamma(x_0).$$

Proof. Let $M_\mu^\delta(x_0)=\mu(x_0-\delta,x_0+\delta)$. Then looking at the contribution of $(x_0-\varepsilon, x_0+\varepsilon)$ to Im $F_\mu(x_0+i\varepsilon)$, we see that

$$\operatorname{Im}F_\mu(x_0+i\varepsilon)=\varepsilon\int_{-\infty}^{\infty}\frac{d\mu(y)}{(y-x_0)^2+\varepsilon^2}\ge\frac{1}{2\varepsilon}M_\mu^\varepsilon(x_0), \tag{42}$$

so

$$\varepsilon^\gamma\operatorname{Im}F_\mu(x_0+i\varepsilon)\ge\frac{1}{2}\frac{1}{\varepsilon^{1-\gamma}}M_\mu^\varepsilon(x_0),$$

so the first inequality in the lemma holds. $Q_\mu^\gamma(x_0)\le R_\mu^\gamma(x_0)$ is, of course, trivial.

Lemma (2.2.10)[47]: Let $\alpha<1$. If $D_\mu^\alpha(x_0)<\infty$ (resp.= 0), $R_\mu^{1-\alpha}(x_0)<\infty$ (resp. = 0).

Proof. Suppose first that $D_{\mu}^{\alpha}(x_0) < \infty$. Let $M_{\mu}^{\delta}(x_0) = \mu(x_0 - \delta, x_0 + \delta)$. The case $\alpha=0$ is trivial so we'll suppose $\alpha > 0$. By hypothesis,

$$M_{\mu}^{\delta}(x_0) \leq C\delta^{\alpha}, \tag{43}$$

so with $\gamma = 1 - \alpha$:

$$\overline{\lim_{\varepsilon\downarrow 0}}\, \varepsilon^{\gamma} \mid F_{\mu}(x_0 + i\varepsilon \mid \leq \overline{\lim_{\varepsilon\downarrow 0}}\, \varepsilon^{\gamma} \int_{-\infty}^{\infty} \frac{d\mu\,(y)}{\left[(x_0 - y)^2 + \varepsilon^2\right]^{1/2}}$$

$$= \overline{\lim_{\varepsilon\downarrow 0}}\, \varepsilon^{\gamma} \int_0^1 \frac{1}{(\varepsilon^2 + \delta^2)^{1/2}} [d_{\delta} M_{\mu}^{\delta}(x_0)]$$

$$= \overline{\lim_{\varepsilon\downarrow 0}}\, \varepsilon^{\gamma} \int_0^1 \frac{\delta}{(\varepsilon^2 + \delta^2)^{3/2}} M_{\mu}^{\delta}(x_0)\, d\delta$$

$$\leq \lim_{\varepsilon\downarrow 0} C\varepsilon^{\gamma} \int_0^1 \frac{\delta^{\alpha+1}}{(\varepsilon^2 + \delta^2)^{3/2}}\, d\delta$$

$$= \lim_{\varepsilon\downarrow 0} C \int_0^{\varepsilon^{-1}} \frac{\delta^{\alpha+1}}{(\delta^2 + 1)^{3/2}}\, d\delta$$

$$< \infty.$$

The first equality comes from noting that since $\gamma > 0$,

$$= \lim_{\varepsilon\downarrow 0} \varepsilon^{\gamma} \int_{|y-x_0|>1} d\mu(y) \big/ |x_0 - y - i\varepsilon| = 0.$$

The second equality is an integration by parts. The boundary term at zero vanishes since $\alpha > 0$. The term at 1 has a zero limit since $\gamma > 0$. The final equality comes by noting that since $\alpha < 1$, the integral is finite as $\varepsilon^{-1} \to \infty$.

If $D_{\mu}^{\alpha}(x_0)=0$, then (43) holds for $\delta \leq \delta_0$ where C can be taken arbitrarily small (by taking δ_0 small). The above calculation (with 1 as the upper integrand replaced by δ_0) shows that

$$R_{\mu}^{1-\alpha}(x_0) \le C \int_0^{\infty} \frac{\delta^{\alpha+1}}{(\delta^2+1)^{3/2}}\, d\delta .$$

Since C is arbitrarily small, R is zero.

Corollary (2 . 2 . 11)[47] : Let $\gamma \in [0, 1]$. Let $S \subset R$ be a Borel set with $\mu(S) > 0$. suppose $Q_{\mu}^{\gamma}(x) < \infty$ for μ-a.e. $x \in S$. Then, $\dim(S) \ge 1-\gamma$.

Theorem (2.2.12)[47]: Suppose that

$$\sup_{\varepsilon>0} \varepsilon^{s} \int_a^b \left|\operatorname{Im} F_{\mu}(x+i\varepsilon)\right|^2 dx < \infty$$

for some $s < 1$. Then $\mu(a, b)$ gives zero weight to sets of dimension less than $1-s$.

Proof. We'll prove that for any $\beta < 1-s$ and any closed interval $I \subset (a, b)$, we have

$$\int_{\substack{x\in I\\ y\in I}} \frac{d\mu(x)d\mu(y)}{|x-y|^{\beta}} < \infty . \tag{44}$$

This implies $\tilde{G}_{\beta}(x) = d\mu(y) \int \frac{d\mu(y)}{|x-y|^{\beta}} < \infty$ for μ-a.e. $x \in I$, and the theorem thus follows from Proposition (2.2.6) and Corollary (2.2.5).

Replacing μ by μ I and noting that $\operatorname{Im}\left(\int_{x\in I} \frac{d\mu(x)}{x-z}\right) \le \operatorname{Im} F_{\mu}(z)$, we an suppose μ is supported in I. Since $I \subset (a, b)$ and $|\operatorname{Im} F_{\mu,I}(z)| \le \frac{C|\operatorname{Im} z|}{dist(z,I)^2}$, we can suppose that

$$\sup_{\varepsilon>0} \varepsilon^{s} \int_{-\infty}^{\infty} |\operatorname{Im} F_{\mu}(x+i\varepsilon)|^2 dx < \infty. \tag{45}$$

By a straightforward calculation,

$$\int_{-\infty}^{\infty} |\operatorname{Im} F_{\mu}(x+i\varepsilon)|^2\, dx = 2\pi\varepsilon \int_{\substack{x\in I\\ y\in I}} \frac{d\mu(x)\, d\mu(y)}{(x-y)^2+4\varepsilon^2}$$

So (45) says that

$$\int_{\substack{x\in I\\ y\in I}} \frac{d\mu(x)\, d\mu(y)}{(x-y)^2+\varepsilon^2} \le C\varepsilon^{-1-s}. \tag{46}$$

Let

$$M_\mu^{(2)}(\delta) = \int_{\substack{|x-y|<\delta \\ x\in I \\ y\in I}} d\mu(x)\, d\mu(y).$$

$$\int_{\substack{x\in I \\ y\in I}} \frac{d\mu(x)d\mu(y))}{(x-y)^2+\epsilon^2} \le C\, \epsilon^{-1-\delta}$$

Then (46) with $\varepsilon = \delta$ says that

$$M_\mu^{(2)}(\delta) \le 2C\,\delta^{1-s}$$

Thus, if $\beta < 1-s$,

$$\int_{\substack{|x-y|\le 1 \\ x\in I \\ y\in I}} \frac{d\mu(x)d\mu(y)}{|x-y|^\beta} \le \sum_{n=0}^{\infty} M_\mu^{(2)}(2^{-n})2^{(n+1)/\beta} < \infty$$

and (44) is proven.

Let μ be a normalized finite measure. Let A be the operator of multiplication x on $L^2(R, d\mu)$. Let φ be the unit vector $\varphi(x) \equiv 1$. Let $A_\lambda = A + \lambda(\varphi, \cdot)\varphi$, and let $d\mu_\lambda$ be the spectral measure for φ and the operator A_λ. Let $F_\lambda(z) = \int \frac{d\mu_\lambda(x)}{x-z}$ and denote $F(z)$ for $F_0(z)$. Then [58]

$$F_\lambda(z) = \frac{F(z)}{1+\lambda F(z)} \tag{47}$$

$$\operatorname{Im} F_\lambda(z) = \frac{\operatorname{Im} F(z)}{|1+\lambda F(z)|^2} \tag{48}$$

$$d\mu_\lambda(x) = \lim_{\varepsilon\downarrow 0} \frac{1}{\pi} \operatorname{Im} F_\lambda(x+i\varepsilon)dx \tag{49}$$

$$\mu_{\lambda,\text{sing}} \text{ is supported by } \{x \mid F(x+i0) = -\frac{1}{\lambda}\} \tag{50}$$

Theorem (2.2.13)[47]: Let $\alpha \in [0,1]$. Let $S_\alpha = \{x \mid \underline{\lim}\, \varepsilon^{-(1-\alpha)} \operatorname{Im} F(x+i\varepsilon) > 0\}$. If $\mu_\lambda([a, b]\backslash S_\alpha) = 0$ for some $\lambda \ne 0$, then μ_λ gives zero weight to any subset of $[a, b]$ of dimension $\beta < \alpha$.

Proof. Suppose $\underline{\lim}\ \varepsilon^{-(1-\alpha)}\mathrm{Im}\, F(\mathbf{x_0} + i\varepsilon) > 0$ (i.e., $x_0 \in S\alpha$). By (48),

$$\mathrm{Im}\, F_\lambda(x_0 + i\varepsilon) \le \frac{1}{\lambda^2 \mathrm{Im} F(x_0 + i\varepsilon)}$$

So

$$Q^{1-\alpha}_{\mu_\lambda}(x_0) = \overline{\lim}\, \varepsilon^{(1-\alpha)}\, \mathrm{Im}\, F_\lambda(x_0 + i\varepsilon) < \infty.$$

Theorem (2.2.14)[47]: Let $0 \le \alpha < 1$. Suppose μ is purely singular. Let $\hat{S}_\alpha = \{x \mid \lim \varepsilon^{-(1-\alpha)}\ \mathrm{Im}\, F(x + i\varepsilon) < \infty\}$. If $\mu_\lambda\ (R \setminus \hat{S}_\alpha) = 0$ for some $\lambda \neq 0$, then μ_λ is supported on a set of dimension α.

Proof. Suppose $\overline{\lim}\ \varepsilon^{-(1-\alpha)}\mathrm{Im}\, F(x_0 + i\varepsilon) < \infty$ (i.e., $x_0 \in \hat{S}_\alpha$) and that $F(x_0 + i0) = -\frac{1}{\lambda}$. By (42),

$$M^{\epsilon}_{\mu}(x_0) \le C \epsilon^{2-\alpha}$$

and

$$\begin{aligned} |1 + \lambda\, \mathrm{Re}\, F(x_0 + i\varepsilon)| &= |\lambda|\,|\mathrm{Re}\, F(x_0 + i\varepsilon) - \mathrm{Re}\, F(x_0 + i0)| \\ &= |\lambda| \left| \int \left[\frac{(y - x_0)}{(y - x_0)^2} - \frac{(y - x_0)}{(y - x_0)^2 + \varepsilon^2} \right] d\mu(y) \right| \\ &= |\lambda| \left| \int \frac{\varepsilon^2}{(y - x_0)[(y - x_0)^2 + \varepsilon^2]} d\mu(y) \right| \\ &\le |\lambda| \int \frac{\varepsilon^2}{\delta(\delta^2 + \varepsilon^2)} [d_\delta M^{\delta}_{\mu}(x_0)]. \end{aligned}$$

We can integrate by parts, use the bound on M^{ε}_{μ}, and integrate by parts again to bound this last integral by

$$|\lambda|(2-\alpha) \int_0^\infty \frac{\varepsilon^2 \delta^{1-\alpha}\, d\delta}{\delta(\delta^2 + \varepsilon^2)} = |\lambda|(2-\alpha)\varepsilon^{1-\alpha} \int_0^\infty \frac{dy}{y^\alpha (y^2 + 1)}$$

And note the integrand is finite.

Thus, $|1 + \lambda F(x_0 + i\varepsilon)| \le C\varepsilon^{1-\alpha}$ and so $\underline{\lim}\ \varepsilon^{1-\alpha} |1 + \lambda F(x_0 + i\varepsilon)|^{-1} > 0$. Thus, by (47), if $x_0 \in \hat{S}_\alpha \cap \{x \mid F(x_0 + i\varepsilon) = -\frac{1}{\lambda}\}$, $\overline{\lim}\ \varepsilon^{(1-\alpha)} |F\lambda(x_0 + i\varepsilon)| > 0$. Since μ_λ is supported on $\{x \mid F(x_0 + i\varepsilon) = -\frac{1}{\lambda}\}$, if $\mu_\lambda(R \backslash \hat{S}_\alpha) = 0$, then by Theorem ((2.2.8), $\alpha_\lambda(x) \leq \alpha$ a.e. and so by Corollary (2.2.4), μ is supported on a set of dimension α.

In addition to the functions $F_\lambda(z), F(z)$ of (47), an important role is played by

$$G(x) = \int \frac{d\mu(y)}{(x-y)^2} \tag{51}$$

in that

$$\{x \mid G(x) < \infty, F(x + i0) = -\lambda^{-1}\} = \text{set of eigenvalues of } A_\lambda. \tag{52}$$

Note that $G(x) = \lim \varepsilon^{-1} \operatorname{Im} F(x + i_\varepsilon)$, so (52) follows from (50) and the $\alpha = 0$. Moreover, if $\lambda < \infty$ (see [58]):

$$d\mu_\lambda^{PP}(y) = \sum_{\{x \mid G(x) < \infty, F(x+io) = -\lambda^{-1}\}} \frac{1}{\lambda^2 G(x)} d\delta_x(y). \tag{53}$$

Note that $G(x) < \infty$ implies $F(x + i_\varepsilon)$ has a real limit so

$$M = \{x \mid G(x) < \infty\} = \bigcup_{0 < |\lambda| \leq \infty} \{eigenvalue\ s\ of\ A_\lambda\}$$

In [1] del Rio, Markov, and Simon prove that

$$M = \bigcup_{n=1}^{\infty} M_n$$

where M_n is such that there exists C_n with (59) for all $x < y$ both in M_n.

Let $L_n = \{\lambda \mid -\lambda-1 \in F[M_n]\}$. It follows from (54) that $\dim(M_n) = \dim(L_n)$. Thus, since $\dim(\bigcup_{n=1}^{\infty} A_n) = \sup \dim(A_n)$, we see that

Theorem (2.2.15)[47]: Fix a Borel set I. Then the Hausdorff dimension of the set of λ's where A_λ has some eigenvalues in I is the same as the Hausdorff dimension of the set of $x \in I$ where $G(x) < \infty$.

There is also a result on the other side:

We'll need a lemma that could have many other applications to the theory of rank one perturbations:

Lemma (2.2.16)[47] : Let η be a finite measure on $\mathbb{R}$ and define a measure ν on $\mathbb{R}$ by

$$\nu(A) = \int \mu_\lambda(A) d\eta(\lambda). \tag{55}$$

Let $F_\kappa(z) = d\kappa(x)/x - z$ be the Borel transform of the measure κ. Then

$$F_\nu(z) = F_\eta(-1/F_\mu(z)). \tag{56}$$

Proof. By the definition (55):

$$F_\nu(z) = \int d\eta(\lambda) F_{\mu_\lambda}(z).$$

Equation (47) implies the result.

Lemma (2.2.17)[47]: Let $0 \le \alpha < 2$ and let μ be a measure obeying $\mu(x - \delta, x + \delta) \le C\delta^\alpha$ for some C and x and all $\delta > 0$. Then there exists C_1 so that $\operatorname{Im} F_\mu(x + i\varepsilon) \le C_1\varepsilon^{-(1-\alpha)}$ for all $\varepsilon > 0$. Moreover, if $\mu(x - \delta, x + \delta) \le C\delta^\alpha$ holds for some fixed C and all x and $\delta > 0$, then there exists C_1 so that $\operatorname{Im} F_\mu(x + i\varepsilon) \le C_1\varepsilon^{-(1-\alpha)}$ for all x and $\varepsilon > 0$.

Proof.

$$\operatorname{Im} F_\mu(x + i\varepsilon) = \int \frac{\varepsilon\, d\mu(y)}{(x-y)^2 + \varepsilon^2}$$

$$\int_{|x-y|<\varepsilon} \frac{d\mu(y)}{(x-y)^2+\varepsilon^2} + \sum_{n=0}^{\infty} \int_{2^n\varepsilon \le |x-y| < 2^{n+1}\varepsilon} \frac{\varepsilon\, d\mu(y)}{(x-y)^2+\varepsilon^2}$$

$$\le \frac{C\varepsilon^\alpha}{\varepsilon} + \sum_{n=0}^{\infty} \frac{\varepsilon C(2^{n+1}\varepsilon)^\alpha}{(2^n\varepsilon)^2+\varepsilon^2}$$

$$\le \frac{C\varepsilon^\alpha}{\varepsilon}\left(1 + 2^\alpha \sum_{n=0}^{\infty} 2^{n(\alpha-2)}\right)$$

so we see that the claim holds.

Proof. The $\alpha = 0$ case is trivial, so suppose $0 < \alpha \le 1$ and $h^\alpha(S) > 0$. Let $T_1 = \{x \mid G(x) = \infty, \lim_{\varepsilon \downarrow 0} F(x + i\varepsilon)$ exists and is finite and nonzero$\}$. We'll show

$h^{\alpha}(T_1) > 0$, so we can conclude that $h^{\alpha}(T) > 0$. For each $\lambda \in S_1 \equiv S\backslash\{0, \pm\infty\}$, μ_{λ}^{sc} is supported on T_1 so $\mu_{\lambda}(T_1) > 0$. Since $h^{\alpha}(S_1) > 0$, it is well known ([64]) that we can find a measure η so that η is supported by S_1, $\eta(S_1) > 0$, and

$$\eta(x-\delta, x+\delta) \le C\delta^{\alpha} \tag{57}$$

for all x and $\delta > 0$. Let v be given by (55). Then $v(T_1) > 0$.

By (57) and Lemma (2.2.16) there exists C_1 so that

$$\operatorname{Im} F_{\eta}(x + i\varepsilon) \le C_1 \varepsilon^{-(1-\alpha)}$$

for all x and $\varepsilon > 0$. It follows from (56) that for $x \in T_1$,

$$\overline{\lim_{\varepsilon\downarrow 0}}\, \varepsilon^{(1-\alpha)} \operatorname{Im} F_{v}(x+i\varepsilon) \le C_1 \overline{\lim_{\varepsilon\downarrow 0}}\, \varepsilon^{(1-\alpha)} [\operatorname{Im}(-1/F_{\mu}(x+i\varepsilon))]^{-(1-\alpha)}. \tag{58}$$

since $G(x) = \infty$, we have

$$\lim_{\varepsilon\downarrow 0} \frac{\operatorname{Im} F_{\mu}(x+i\varepsilon)}{\varepsilon} = G(x) = \infty$$

and since $\pm\infty \notin S_1$, $F_{\mu}(x + i\varepsilon) \to -\lambda^{-1} \neq 0$ so $\varepsilon\, [\operatorname{Im}(-1/F_{\mu}(x + i\varepsilon))]^{-1} \to 0$. Thus, we see from (58) that for all $x \in T_1$,

$$Q_{V}^{1-\alpha}(x) < \infty$$

and if $\alpha < 1$, then $Q_{V}^{1-\alpha}(x) = 0$. Since $v(T_1) > 0$, Corollary (2.2.19) (along with its remark) implies that $h^{\alpha}(T_1) > 0$. The fact that in the $\alpha < 1$ case T_1 is not h^{α}-sigma finite follows from Lemma (2.2.11).

Theorem (2.2.18)[47]: Suppose μ is purely singular. Let $S = \{\lambda | A_{\lambda}$ has some continuous spectrum$\}$. Let $T = \{x \,|\, G(x) = \infty\}$. Then

$$\dim(S) \le \dim(T).$$

In particular, if T has Hausdorff dimension zero, so does S.

Rank one perturbations can be described by a measure μ given by

$$(\varphi, (A-z)^{-1}\varphi) = \int \frac{d\mu(x)}{x-z}$$

where $A+\lambda(\varphi, \cdot)\,\varphi$ is the rank one perturbation, so we'll phrase our examples in this section in terms of $d\mu$. To make things operator theoretic, one can always take **H** = $L^2(\mathbf{R}, d\mu)$, A= multiplication by x, and φ the function $\varphi(x) \equiv 1$.

We'll discuss four classes of examples in this section:

(i) Point measures with rapidly decreasing weights for which we'll show that the perturbed spectrum is supported by a set of Hausdorff dimension zero. This class is relevant for our study of localization.

(ii) Point measures where for a.e. λ, $d\mu_\lambda$ has exact dimension α_0. These are variants of the measures in [59].

(iii) A family of singular continuous measures where one can calculate many distinct dimensions.

(iv) A set of examples that show $\{x \mid G(x) < \infty\}$ can have any dimension and that have point spectrum embedded in singular continuous spectrum.

Example (2.2.19)[47]: Point spectrum with decaying weights Given a sequence of sets A_n, we call $A_\infty = \bigcap_{n=1}^{\infty}\bigcap_{m=n}^{\infty} A_m$,, the lim sup($A_n$) consisting of points in infinitely many A_n 's.

Lemma (2.2.20)[47]: Suppose that for a family of intervals A_n, we have for each j > 0

$$|A_n| \le C_j n^{-j}. \tag{59}$$

Then A_∞ = lim sup(A_n) is a set of Hausdorff dimension zero.

Proof. $|A_n| \to 0$ so given δ, choose N_0 so $|A_n| \le \delta$ for $n \ge N_0$. Then for $m \ge N_0$, $\bigcup_{n=m}^{\infty} A_n$ is a δ-cover of A_∞. Thus,

$$Q_{\alpha,\delta}(A_\infty) \le C_j^\alpha \sum_{n=m}^{\infty} n^{-j\alpha}$$

a fixed α > 0, pick j so jα > 1. Then the sum is finite and clearly,

$$Q_{\alpha,\delta}(A_\infty) \le C_j^\alpha \inf_{m\ge N_0} \sum_{n=m}^{\infty} n^{-j\alpha} = 0.$$

Thus, $h^\alpha(A_\infty) = 0$ if α > 0 and so A_∞ has dimension zero as claimed.

Theorem (2.2.21)[47]: Suppose $d\mu(E) = \sum_{n=1}^{\infty} a_n\, d\delta_{E_n}(E)$ where a_n obeys the condition that for all j, there is a C_j with

$$|a_n| \le C_j\, n^{-j}\,. \tag{60}$$

Then for every λ, $d\mu_\lambda$ is supported on a set of Hausdorff dimension zero. Moreover, $d\mu_\lambda$ is pure point except for a set of λ's of Hausdorff dimension zero.

Proof. Let $G(x)$ be defined by (51) and let $S = \{x \mid G(x) = \lambda^{-2}, x \notin \{E_i\}_{i=1}^{\infty}\}$. Then the Aronszajn-Donoghue theory [58] says that for any $\lambda \neq 0$, $d\mu_\lambda^{sc}$, the singular continuous measure for A_λ is supported by S. Thus, the spectral measure $d\mu_\lambda$ is supported by $S \cup \{\text{eigenvalues of } A_\lambda\}$. Since the set of eigenvalues is a zero-dimensional set, it suffices to prove that S is zero-dimensional. The final assertion then follows from Theorem (2.2.18).

Let $b_n = \sqrt[3]{a_n}$ and let $A_n = [E_n - b_n, E_n + b_n]$. Then

$$|A_n| \le 2C_j^{1/3} n^{-j/3}$$

for any j, so A_n obeys (59). Thus, $A_\infty \equiv \limsup(A_n)$ has dimension zero.

We claim $S \subset A_\infty$. To prove this, we need only show if $x \notin A_\infty$ and $x \notin \{E_i\}_{i=1}^{\infty}$ then $G(x) < \infty$. But if $x \notin A_\infty$, then for some N_0, $x \notin \bigcup_{n=N_0}^{\infty} A_n$ so

$$\sum_{n=N_0}^{\infty} \frac{a_n}{|x-E_n|^2} \le \sum_{n=N_0}^{\infty} \frac{a_n}{b_n^2} = \sum_{n=N_0}^{\infty} a_n^{1/3} < \infty$$

by (60). Since $x \notin \{E_i\}_{i=1}^{\infty}$,

$$\sum_{n=1}^{N_0-1} \frac{a_n}{|x-E_n|^2} < \infty$$

so $G(x) < \infty$ as required.

Example (2.2.22)[47]: Perturbed measures of prescribed exact dimension our second class of examples is intended to show that it can happen that for any $\alpha_0 \in [0, 1]$, there is a rank one perturbation situation where μ_λ [0, 1] is a measure of exact dimension α_0 for a.e. λ (w.r.t. Lebesgue measure). All our unperturbed measures in this example will live on [0, 1] and be point measures. The third set of examples will be similar but the unperturbed measures will be continuous. For each $n = 0, 1, 2, \ldots$ let

$$d\mu_n = \frac{1}{2^n}\sum_{j=0}^{2^n} d\delta_{j/2^n}, \tag{61}$$

and for $\alpha \in (0, 1)$ define

$$d\nu_\alpha = \sum_{n=0}^{\infty} 2^{-n(1-\alpha)} d\mu_n. \tag{62}$$

For any $x_0 \in [0, 1]$ and n, there is $j/2^n$ within 2^{-n-1} of x_0, so

$$\nu_\alpha\left(\left[x_0 - \frac{1}{2^{n+1}}, x_0 + \frac{1}{2^{n+1}}\right]\right) \geq 2^{-n(1-\alpha)}.$$

Thus for any $\varepsilon > 1$, $\nu_\alpha(x_0 - \varepsilon, x_0 + \varepsilon) \geq \varepsilon^{2-\alpha}$ so by (42), for $x_0 \in [0, 1]$ and $0 < \varepsilon$ Im $F_{\nu_\alpha}(x_0 + i\varepsilon) \geq \frac{1}{2}\varepsilon^{1-\alpha}$. So the set S_α of Theorem (2.2.13) is all of [0, 1], and so (by Theorem (2.2.13)):

Theorem (2.2.23)[47]: Fix $0<\alpha<1$. Let $d\nu_\alpha$ be the measure (61),(62) and let $d\nu_{\alpha;\lambda}$ be its rank one perturbations. Then for any $\lambda \neq 0$, $d\nu_{\alpha;\lambda}$ gives zero weight to any $S \subset [0, 1]$ of dimension $\beta < \alpha$.

On the other hand, suppose (for $\frac{j}{2^n}$ closest to x_0)

$$|x_0 - \frac{j}{2^n}| > \varepsilon_n \equiv 2^{-n(1+\eta)}\delta_0 \tag{63}$$

for some $\eta, \delta_0 > 0$. Pick $1 < \gamma < (2-\alpha)/(1+\eta)$. Then

$$\begin{aligned}\int \frac{d\mu_n(y)}{|x_0 - y|^\gamma} &\leq \varepsilon_n^{-\gamma} 2^{-n} + \int_{2^{-n-1} \leq |x-y| \leq 1} \frac{dy}{|x-y|^\gamma} \\ &\leq C\left[\varepsilon_n^{-\gamma} 2^{-n} 2^{n(\gamma-1)}\right].\end{aligned}$$

Thus, by (61),(62)

$$\int \frac{d\nu_\alpha(y)}{|x_0 - y|^\gamma} \leq C\left(\sum_{n=0}^{\infty} 2^{-n(2-\alpha-\gamma)} + \sum_{n=0}^{\infty} \delta_0^{-\gamma} 2^{-n[-\gamma(1+\eta)+1+1-\alpha]}\right) < \infty$$

by the choice of γ and $\alpha + \gamma < 2$.

The measure of the set of $x_0 \in [0, 1]$ where (63) fails $\sum_{n=0}^{\infty} 2^{-n\eta}\delta_0$ and is arbitrarily small if δ_0 gets small. Thus,

Lemma (2.2.24)[47]: For any $\gamma < 2 - \alpha$ and a.e. $x_0 \in [0, 1]$,

$$\int \frac{d\nu_\alpha(y)}{|x_0 - y|^\gamma} < \infty.$$

Since γ can be taken arbitrarily close to $2 - \alpha$, we see by Proposition (2.2.6) and Lemma (2.2.17) that the set $\hat{S}_\beta$ of Theorem (2.2.14) has Lebesgue measure 1 if $\beta > \alpha$. Thus, $\left|[0,1] \setminus \bigcap_{\beta>\alpha} \hat{S}_\beta\right| = 0$. By the result of Simon-Wolff (4), $\mu_\lambda\left([0,1] \setminus \bigcap_{\beta>\alpha} \hat{S}_\beta\right) = 0$ for a.e.λ. Thus, by Theorem (2.2.14):

Theorem (2.2.25)[47]: Fix $0<\alpha<1$. Then for a.e. λ, $\nu_{\alpha;\lambda}$ is supported on a set of dimension α. In particular, $\nu_{\alpha;\lambda}[0, 1]$ is of exact dimension α.

If we take $d\nu_1 = \sum_{n=1}^{\infty} n^{-2}\, d\mu_n$, it is not hard to see that for all $\lambda \neq 0$, $\nu_{1;\lambda} \restriction [0, 1]$ is of exact dimension one. Thus, we see that for any $\alpha \in [0, 1]$ there are examples with singular spectrum of exact dimension α (in $[0, 1]$) for a.e. λ (and for $\alpha = 0$, for all λ).

Example (2.2.26)[47]: Some number theoretic examples

Our third class of examples illustrates change of dimension from singular continuous to singular continuous spectrum. Details will be presented in [47].

These examples will depend critically on the binary expansion of a number x in [0, 1]. Given such an x, we can expand it, viz.

$$x = \sum_{n=0}^{\infty} \frac{\alpha_n(x)}{2^n} \tag{64}$$

We deal with the non-uniqueness for binary decimals (e.g., numbers of the form $\frac{j}{2^n}$) by requiring $a_m(x) = 0$ for m large for such x (except for $x = 1$). Thus, (64) defines a map of $\{0, 1\}^N \xrightarrow{F} [0, 1]$, and $x \to \{a_n(x)\}$ defines a left inverse.

Any measure λ on $\{0, 1\}^N$ defines a measure μ on [0, 1] by $\mu(A) =$

$\lambda(F^{-1}[A])$. For any p with $0 < p < 1$, let A_p be the product measure on $\{0,1\}^N$ with each factor giving weights p to 0 and $(1-p)$ to 1, that is, the a_n's are i.i.d.'s with density $pd\delta_0 + (1-p)d\delta_1$. Let μ_p be the corresponding measure on [0, 1].

Two dimensions will arise below:

$$H(p) \equiv -\frac{p\ln p + (1-p)\ln(1-p)}{\ln 2} \tag{65}$$

$$L(p) \equiv 2 + \frac{\ln p\,(1-p)}{2\ln 2} = 2 - \gamma(p) \tag{66}$$

We note that

$$L(p) < H(p) < 1, \qquad p \neq \frac{1}{2}$$

(but in fact H(p)− L(p) $\cong 0((p - \frac{1}{2})^4)$ for p near ½ so they are very close for most p's). Notice also that $H(p) > 0$ and that

$$p \in \left(\frac{2-\sqrt{3}}{4}, \frac{2+\sqrt{3}}{4}\right) \equiv I_0 \Leftrightarrow L(p) > 0$$

($\mathbf{I}_0$ is about (0.07, 0.93)).

Theorem (2.2.27)[47]: (I) $d\mu_p$ has exact dimension H(p).

(ii) Suppose $p \in I_0$. Then for a.e. λ w.r.t. Lebesgue measure, the restriction to [0, 1] of the rank one perturbation of $d\mu_p$ has exact dimension L(p).

(iii) If $p \notin \bar{I}_0$, then for a.e. λ, the rank one perturbation of $d\mu_p$ is pure point

(iv) If $p \in (¼, ¾)$, $p \neq ½$, then for all λ, the restriction to [0, 1] of the rank one perturbation of $d\mu_p$ is purely singular continuous (so we have an example with singular continuous spectrum for all λ).

Example (2.2.28)[47]: Examples with pure point spectrum

Our last class of examples will show $\{x|G(x)<\infty\}$ can have any Hausdorff dimension, and also provid examples where $d\mu_\lambda$ has a singular continuous component for all $\lambda \neq 0$ but sometimes mixed with embedded point spectrum. In this example, dμ will be a measure fixed once and for all with supp(μ) = [0,1] and

$$G_\mu(x) \equiv \int \frac{d\mu(y)}{(x-y)^2} = \infty$$

on [0, 1]. Three possibilities to keep in mind are:

(i) $\chi_{[0,1]}(x)\,dx$ which is absolutely continuous.

(ii) $d\mu_p$, the measure of Example (2.2.26) with $p \in (\frac{1}{4}, \frac{1}{2})$ where $G(x) = \infty$ by Theorem (2.2.27).

(iii) Any of the point measures $d\nu_\alpha$ of Example (2.2.22) having

$$G(x_0) = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \operatorname{Im} F_{\nu_\alpha}(x_0 + i\varepsilon) = \infty \quad \text{for all } x_0 \in [0, 1].$$

These show there are such μ with any spectral type.

Theorem (2.2.29)[47]: Let C be an arbitrary closed nowhere dense set in [0, 1]. Let μ be a Borel measure on [0, 1] with $G_\mu(x) = \infty$ on [0, 1] and $\int d\mu(x) = 1$. Let:

$$d\nu(x) = \operatorname{dist}(x, C)^2 \; d\mu(x).$$

Then, $\operatorname{supp}(\nu) = [0, 1]$, $G_\nu(x) = \infty$ on $[0, 1]\backslash C$ and $G_\nu(x) \le 1$ on C.

Proof. If $x \notin C$, $\operatorname{dist}(x, C) = \delta > 0$ since C is closed. Thus,

$$G_\nu(x) \ge \left(\frac{\delta}{2}\right)^2 \int_{|x-y| \le \delta/2} \frac{d\mu(y)}{(x-y)^2} = \infty$$

since $G_\mu(x) = \infty$. On the other hand, if $x \in C$,

$$G_\nu(x) = \int \frac{\operatorname{dist}(y,C)^2}{\operatorname{dist}(x,y)^2} d\mu(y) \le \int d\mu(y) = 1$$

since $\operatorname{dist}(x, y) \ge \operatorname{dist}(C, y)$. Finally, since $[0, 1]\backslash C$ is dense, $\operatorname{supp}(d\nu) = [0, 1]$.

Now let $\tilde{\nu}$ be $\nu/[\int d\nu]$. Then for every $x \in C$, $G_{\tilde{\nu}}(x) \le \frac{1}{N}$ for $N = \int d\nu$. Consider now the rank one perturbation $d\tilde{\nu}_\lambda$ of $d\tilde{\nu}$. From (53), we see each pure point has weight at least $\frac{N}{\lambda^2}$ so there are at most $\frac{\lambda^2}{N}$ pure points (since $d\tilde{\nu}_\lambda$ is normalized in (53)). Thus,

Proposition (2.2.30)[47]: If $N = \int d\nu(x)$ for the measure ν of Theorem (2.2.29), then $A_\lambda \equiv A + \lambda(\varphi, \cdot)\varphi$ has at most $\frac{\lambda^2}{N}$ eigenvaluesin [0, 1]. In particular, if $\lambda^2 < N$, A_λ has purely singular continuous spectrum in [0, 1]; and for any λ, $\sigma_{sc}(A_\lambda) = [0, 1]$.

One of our goals in this section is to prove that local perturbations of random Hamiltonians in the Anderson localization regime, while they may produce singular continuous spectrum, always produce zero-dimensional spectrum, in the sense that the spectral measures are all supported on a set of Hausdorff dimension zero. We'll use Theorem (2.2.21). Naively, one might confuse exponential decay of eigenfunctions in Z^{ν} (as in $|\varphi_n(m)| \le C_n e^{-A|m|}$) with exponential decay in eigenfunction label (as in $|\varphi_n(0)| \le C\ e^{-B|n|}$) which allows one to apply Theorem (2.2.21). In fact, they are distinct — indeed, if $\nu \ge 2$, we will not prove that $|\varphi_n(0)| \le Ce^{-B|n|}$ but only $|\varphi_n(0)| \le C\exp(-B|n|^{1/\nu})$, also see [47].

Throughout this section, n is an eigenvalue label and m is a Z^{ν} point. It will be convenient to take the norm $|m| = \max_{j=1,\dots,\nu} |m_j|$ on $\mathbf{Z}^{\nu}$.

Definition (2.2.31)[47]: Let H be a self-adjoint operator on $\ell^2(Z^{\nu})$. We say that H has semi-uniformly localized eigenfunctions (SULE), pronounced "operators with a soul," if and only if H has a complete set $\{\varphi_n\}_{n=1}^{\infty}$ of orthonormal eigenfunctions, there is $\alpha > 0$ and $m_n \in Z^{\nu}$, $n = 1, \dots,$ and for each $\delta > 0$, a C_δ so that

$$|\varphi_n(m)| \le C_\delta\, e^{\delta|m_n| - \alpha|m - m_n|} \tag{67}$$

for all $m \in Z^{\nu}$ and $n = 1, 2, \dots$.

Thus, eigenfunctions are "localized about" points m_n. We use the "semi" in SULE because one can define ULE by requiring the bound with $\delta = 0$. The theory below extends to this case, but we'll restrict ourselves to the SULE case. We'll show that large classes of models, including the Anderson model in any dimension and the almost Mathieu operator, do not have ULE.

Below we'll first prove a result about the number of m_n in a box of side L, essentially proving that the number grows like L^{ν} as $L \to \infty$. This will show that local perturbations of SULE operators have zero-dimensional spectrum. Then, we'll relate SULE to dynamics and to Green's function localization; essentially, SULE always implies dynamical localization, and if the spectrum is simple, dynamical localization implies SULE. This will imply that Anderson-model Hamiltonians have SULE.

[47] has an example to show that a Jacobi matrix can have localized

eigenfunctions which are not (semi) uniformly localized.

Let

$$\sum_m |\varphi_n(m)|^2 = 1 \qquad = 1, 2, \dots, \tag{68}$$

$$\sum_m |\varphi_n(m)|^2 = 1 \qquad \text{each } m \in Z^\nu . \tag{69}$$

Lemma (2.2.32)[47]: For each ε > 0, there is a D_ε so that for each n and L:

$$\sum_{|m-m_n|\geq \varepsilon(|m_n|+L)} |\varphi_n(m)|^2 \leq D_\varepsilon e^{-\alpha\varepsilon L} e^{-\alpha\varepsilon|m_n|/2}$$

Proof: By hypothesis, we can find $C_\varepsilon^{(1)}$ so

$$|\varphi_n(m)| \leq C_\varepsilon^{(1)} e^{\alpha[\varepsilon|m_n|/2-|m-m_n|]} .$$

If $|m-m_n| \geq \varepsilon(|m_n|+L)$, then $|m-m_n| \geq \frac{1}{2}|m-m_n| + \frac{\varepsilon}{2}|m_n| + \frac{\varepsilon}{2}L$ so in that regime

$$|\varphi_n(m)| \leq C_\varepsilon^{(1)} e^{-\varepsilon\alpha L/2} e^{-\alpha|m-m_n|/2} .$$

so

$$\sum_{|m-m_n|\geq \varepsilon(|m_n|+L)} |\varphi_n(m)|^2 \leq [C_\varepsilon^{(1)}]^2 e^{-\alpha\varepsilon L} \sum_{|k|\geq\varepsilon|m_n|} e^{-\alpha|k|} \leq D_\varepsilon e^{-\varepsilon\alpha L} e^{-\alpha\varepsilon|m_n|/2}$$

as claimed.

Theorem (2.2.33)[47]: Suppose H has SULE. For each L, $\#\{n \| m_n| \leq L\}$ is finite and

$$\lim_{L\to\infty} \frac{1}{(2L+1)^\nu} \#\{n \mid |m_n| \leq L\} = 1.$$

Proof. To get the upper bound, we'll use the fact that functions localized in a box of side $2L$ contribute most of their norm to a box of side $2(1+\varepsilon)L$. By the lemma, if $|m_n| \leq L$, then

$$\sum_{|m|\geq(1+2\varepsilon)L} |\varphi_n(m)|^2 \leq \sum_{|m-m_n|\geq\varepsilon(L+|m_n|)} |\varphi_n(m)|^2 \leq D_\varepsilon e^{-\alpha\varepsilon L}$$

and so by (68),

$$\sum_{|m|\leq(1+2\varepsilon)L} |\varphi_n(m)|^2 \geq 1 - D_\varepsilon e^{-\alpha\varepsilon L} .$$

Thus by (69),

$$[2(1+2\varepsilon)L+1]^{\nu} \geq \sum_{\substack{all\ n \\ |m| \leq (1+2\epsilon)L}} |\phi_n(m)|^2$$

$$\geq \sum_{\substack{n\, so\, that\ |m_n| \leq L \\ |m_n| \leq (1+2\varepsilon)L}} |\phi_n(m)|^2$$

$$\geq \#\{n \mid |m_n| \leq L\}(1-D_{\varepsilon}e^{-\alpha\varepsilon L}).$$

Thus, $\#\{n \mid |m_n| \leq L\}$ is finite and

$$\overline{\lim}\ (2L+1)^{-\nu}\#\{n \mid |m_n| \leq L\} \leq 1. \tag{70}$$

In particular,

$$\#\{n \mid |m_n| \leq L\} \leq c_0 L^{\nu} \tag{71}$$

for some c_0 and all $L \geq 1$.

To get the lower bound, we'll use the fact that wave functions localized far outside a box of side 2L cannot contribute much to the wave function sum inside that box. Explicitly, suppose that

$$|m_n| \geq \tfrac{1+\varepsilon}{1-\varepsilon} L \text{ and } |m| \leq L.$$

Then we claim

$$|m - m_n| \geq \varepsilon(|m_n| + L)$$

for

$$|m-m_n| \geq |m_n| - L \geq |m_n|\left(1-\frac{1-\varepsilon}{1+\varepsilon}\right) = \varepsilon\left(1+\frac{1-\varepsilon}{1+\varepsilon}\right)|m_n| \geq \varepsilon(|m_n|+L).$$

Thus by Lemma (2.2.32), if

$$|m_n| \geq \frac{1+\varepsilon}{1-\varepsilon} L,$$

then

$$\sum_{|m| \leq L} |\varphi_n(m)|^2 \leq D_{\varepsilon}\, e^{-\alpha\varepsilon L}\, e^{-\alpha\varepsilon |m_n|/2}$$

so

$$\sum_{\substack{n\, so\, that\, |m_n| \geq \frac{1+\epsilon}{1-\epsilon}L \\ |m| \leq L}} |\phi_n(m)|^2 \leq \sum_{k=0}^{\infty} \#\{n \mid |m_n| \leq (k+1)L\} D_{\varepsilon}\, e^{-\alpha\varepsilon L}\, e^{-\alpha\varepsilon kL/2} \leq \tilde{D}_{\varepsilon}\, e^{-\alpha\varepsilon L/2}$$

by (71).

Thus by (69),

$$(2L+1)^{\nu} = \sum_{\substack{all\ n \\ |m|\le L}} |\phi_n(m)|^2 \le \#\{n \mid |m_n| < \frac{1+\varepsilon}{1-\varepsilon} L\} + \tilde{D}_{\varepsilon}\, e^{-\alpha\varepsilon L/2},$$

from which one immediately sees that

$$\underline{\lim}\,(2L+1)^{-\nu}\,\#\{n \mid |m_n| \le L\} \ge 1.$$

Combining this with (70) yields the theorem.

Corollary (2.2.34)[47]: Suppose that H has SULE. Then there are C and D and a labeling of eigenfunctions so that

$$|\varphi_n(0)| \le C \exp\left(-D_n^{1/\nu}\right). \tag{72}$$

Proof. Reorder the eigenfunctions so $|m_n|$ is increasing. By Theorem (2.2.33), $|m_n|/\frac{1}{2} n^{1/\nu} \to 1$ as $n \to \infty$ so $|m_n| \ge \frac{1}{3} n^{1/\nu} - C_0$ for some constant C_0. By (67), we get (72); indeed, we see D can be taken arbitrarily close to $\frac{1}{2}\alpha$.

Combining this corollary with Theorem (2.2.21), we see:

Theorem (2.2.35)[47]: Suppose that H has SULE. Let $H_\lambda = H + \lambda(\delta_0, \cdot)\delta_0$. Then for every λ, the spectral measures for H_λ are supported on a set of Hausdorff dimension zero. Moreover, H_λ has pure point spectrum except for a set of λ's of Hausdorff dimension zero.

Next, we relate SULE to other conditions. We'll suppose H has simple spectrum, although one can easily extend this to examples with spectrum having a uniform finite upper bound on multiplicity.

Definition (2.2.36)[47]: Let H be a self-adjoint operator on $\ell^2(Z^\nu)$. We say that H has semi-uniform dynamical localization (SUDL) if and only if there is $\alpha > 0$ and for each $\delta > 0$, a C_δ so that for all $q, m \in \mathbf{Z}^\nu$:

$$\sup_t |(\delta_q, e^{-itH}\delta_m)| \le C_\delta\, e^{\delta|m| - \alpha|q-m|}. \tag{73}$$

We say that H has semi-uniformly localized projections (SULP) if and only if H has a complete set of normalized eigenfunctions and there is $\alpha > 0$ and for each $\delta > 0$, a C_δ so that for all $q, m \in Z^\nu$:

$$|(\delta_q, P_{\{E\}}\, \delta_m)| \le C_\delta e^{\delta|m| - \alpha|q-m|}$$

for all spectral projections $P_{\{E\}}$ onto a single point (uniformly in E).

Theorem (2.2.37)[47]: Let H be a self-adjoint operator on $\ell^2(Z^\nu)$ with simple spectrum. Then the following are equivalent:

(i) H has SUDL.

(ii) H has SULP.

(iii) H has SULE.

Proof: (i) $\Rightarrow$ (ii): Follows immediately from

$$P_{\{E\}} = s-\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{iEs} e^{-iHs} ds.$$

(ii) $\Rightarrow$ (iii): Label the eigenvalues of H: $E_1, E_2, \ldots$. For each $E_n \in \mathrm{spec}(H)$, pick an eigenfunction $\varphi_n(\cdot)$, unique up to phase. Then by (ii):

$$|\varphi_n(q)\varphi_n(m)| \le C_\varepsilon e^{\delta|m|} e^{-\alpha|q-m|}. \tag{74}$$

Since $\varphi_n \in \ell^2$, it takes its maximum value so choose m_n so that

$$|\varphi_n(m_n)| = \sup_m |\varphi_n(m)|. \tag{75}$$

Then by (74)(75),

$$|\varphi_n(q)|^2 \le |\varphi_n(q)| \sup_m |\varphi_n(m)| \le |\varphi_n(q)||\varphi_n(m_n)|$$

$$\le C_\delta e^{\delta|m_n|} e^{-\alpha|q-m_n|}$$

so H has SULE by taking square roots.

(iii) $\Rightarrow$ (i): Let φ_n be the eigenfunctions and E_n eigenvalues. Then

$$(\delta_q, e^{-itH}\delta_m) = \sum_n \overline{\varphi_n(q)} e^{-itE_n} \varphi_n(m)$$

so, assuming SULE,

$$\sup_t |(\delta_q, e^{-itH}\delta_m)| \le \sum_n \left|\overline{\phi_n(q)}\phi_n(m)\right| \le C_\delta^2 \sum_n e^{2\delta|m_n|} e^{-\alpha(|q-m_n|+|m-m_n|)} \tag{76}$$

Now,

$$|q - m_n| + |m - m_n| \geq |q - m|$$

and

$$|q - m_n| + |m - m_n| \geq |m_n| - |m|.$$

Thus,

$$e^{-\alpha(|q-m_n|+|m-m_n|)} \leq e^{-3\delta|m_n|} e^{3\delta|m|} e^{-(\alpha-3\delta)|m-q|}.$$

So, by (76)

$$\sup_t |(\delta_q, e^{-itH}\delta_m)| \leq C_\delta^2 e^{3\delta|m|} e^{-(\alpha-3\delta)|m-q|} A_0$$

where

$$A_0 = \sum_n e^{-\delta|m_n|}.$$

By (71) which follows from SULE, A_0 is finite.

One can prove by the above means a result that shows that if H has simple spectrum and $\sup_t |(\varphi, e^{-itH}\delta_n)| \leq C e^{-\alpha|n|}$, then the spectral measure for φ can be written $\sum_{n=1}^{\infty} a_n d\delta_{E_n}$ where $|a_n| \leq De^{-\beta n^{1/v}}$ if the E_n's are properly labeled. That is, one can prove a result that requires less uniformity than the full-blown theory assumes.

Finally, we turn to when any, and hence all, of the conditions of Theorem (2.2.37) hold in the context of the Anderson model. We're dealing here with models depending on a random parameter so we first reduce SUDL to a requirement on expectations. General considerations [69,70,71] imply that the spectrum is simple in the localized regime.

Theorem (2.2.38)[47]: Let (Ω, μ) be a probability measure space and $E(\cdot)$ its expectation. Let $\omega \to H_\omega$ be a strongly measurable map from Ω to the self-adjoint operators on $\ell^2(Z^v)$ which is translation invariant in the sense that for each $m \in \mathbf{Z}^v$, there is a measure preserving $T_m : \Omega \to \Omega$ so $H_{T_m\omega} = U_m H_\omega U_m^{-1}$ where $(U_m\varphi)(q) = \varphi(q - m)$. Suppose that

$$E(\sup_t |(\delta_q, e^{-itH_\omega}\delta_0)|) \leq C_1 e^{-\alpha|q|} \tag{77}$$

for some $\alpha > 0$ and that H_ω has simple spectrum for a.e. ω. Then for each

$\beta < \alpha$, for a.e. ω, there is a $C_\omega < \infty$ so that for all $0 < \varepsilon \leq 1$

$$\sup_t |(\delta_q, e^{-itH\omega}\delta_m)| \leq \frac{C_\omega}{\varepsilon^{\nu+1}} e^{\varepsilon|m|} e^{-\beta(m-q)}$$

In particular, a.e. H_ω has SULE.

Proof. Let

$$Q(\omega) = \sum_{m,q} (1+|m|)^{-(\nu+1)} e^{\beta|m-q|} \sup_t \left|\delta_q, e^{-itH\omega}\delta_m\right|.$$

Then by (77),

$$E(Q(\omega)) < \infty$$

so $Q(\omega) < \infty$ for a.e. ω. But for such ω,

$$\sup_t \left|\delta_q, e^{-itH\omega}\delta_m\right| \leq C_\omega (1+|m|)^{\nu+1} e^{-\beta|m-q|}.$$

The result now follows from the trivial bound $(1+x)^\nu \leq \nu^\nu e^{\varepsilon x} \varepsilon^{-\nu}$ for $\varepsilon \leq 1$.

Delyon-Kunz-Souillard [72] have proven this bound for a general class of one-dimensional random potentials.

Theorem (2.2.39)[47]: (Aizenman's theorem) Let $V_\omega(n)$ be a family of independent identically distributed random variables (indexed by $n \in Z^\nu$; $\omega \in \Omega$ is the probability parameter). Suppose H_0 is an operator on $\ell^2(Z^\nu)$ commuting with translations and $H_\omega = H_0 + V_\omega$ with V_ω viewed as a diagonal matrix. Suppose $V_\omega(n)$ has a distribution $g(\lambda)\,d\lambda$ with $g \in L^\infty$ and has compact support. Suppose

$$E\left(\int_a^b \left|(\delta_n, (H_\omega - \lambda - i0)^{-1}\delta)\right|^s d\lambda\right) \leq Ce^{-\mu|n|} \tag{78}$$

for some $s \in (0,1)$. Then

$$E(\sup_t |(\delta_n, e^{-itH_\omega} P_{[a,b]}(H_\omega)\delta_0)| \leq \tilde{C} e^{-\mu|n|/(2-s)} \tag{79}$$

where $\tilde{C}$ only depends on s and the distribution g.

Combining this result with those of Aizenman-Molchanov [73,74], we see that the strongly coupled multi-dimensional Anderson model has SULE.

Anderson localization (at least as proven in [209]) implies that if $\vec{x}$ is the operator

$$(x_i\,\psi)(m) = m_i\psi(m_i) \qquad i = 1,\ldots,\nu,$$

then in the localized regime,

$$\sup_t\,(e^{-itH}\,\delta_0\,,\,x^2\;e^{-itH}\,\delta_0\,) < \infty. \tag{80}$$

It follows from the RAGE theorem (see, e.g., [57,75]) that (80) implies that H has pure point spectrum.

For operators H with dense pure point spectrum, it is proven in [1,48] that for a Baire generic set of λ, $H_\lambda = H + \lambda(\delta_0,\cdot)\delta_0$ has only singular continuous spectrum and so for such H_λ's, (80) must fail. Our purpose in this section is to show that the failure is only very mild. $\langle x^2\rangle(t)\equiv(e^{-itH}\delta_0, x^2 e^{-itH}\delta_0)$ is unbounded but grows at worst logarithmically!

Theorem (2.2.40)[47]: Suppose that H is a self-adjoint operator on) $\ell^2(Z^\nu)$ with SULE. Let $H_\lambda = H + \lambda(\delta_0\,,\cdot\,)\delta_0$. Then

$$\left\langle x^{2n}\right\rangle(t) \equiv (e^{-itH_\lambda}\delta_0,(x^2)^n e^{-itH_\lambda}\,\delta_0)$$

obeys

$$\left\langle x^{2n}\right\rangle(t) \le C_n(\ln|t|)^{2n}$$

for $|t|$ large.

Proof. Write a DuHamel expansion:

$$(\delta_m,\, e^{-itH_\lambda}\,\delta_0\,) = (\delta_m\,,\,e^{-itH}\,\delta_0\,) - i\lambda\int_0^t\,(\delta_m\,,\,e^{-isH}\,\delta_0)(\delta_0\,,\,e^{-i(t-s)H_\lambda}\,\delta_0)\,ds. \tag{81}$$

Since H has SULE, by Theorem (2.2.37),

$$\sup_t|(\delta_m,\,e^{-itH}\,\delta_0\,)|\le Ce^{-\alpha|m|}$$

for suitable C and α. Plugging this into (81) and using $|(\delta_0\,,\,e^{-itH_\lambda}\,\delta_0\,)|\le 1$, we see that

$$|(\delta_m,\,e^{-itH_\lambda}\,\delta_0\,)|\le Ce^{-\alpha|m|}[1+|\lambda|\,|t|]. \tag{82}$$

This would seem to give linear growth in t for $\left\langle x^{2m}\right\rangle^{1/2m}$ trivial bound but we'll combine it with the

$$\sum_m\,|(\delta_m,\,e^{-itH_\lambda}\,\delta_0\,)|^2 = 1. \tag{83}$$

Use (82) only if $|m| > 2\ln(1+|\lambda|\,|t|)/\alpha \equiv G(t)$. In that regime (82) says that

$$|(\delta_m, e^{-itH_\lambda}\delta_0)| \leq C\, e^{-\alpha|m|/2}.$$

Thus,

$$\sum_{|m|\geq G(t)} (m^2)^n |(\delta_m, e^{-itH_\lambda}\delta_0)|^2 \leq C_n$$

and obviously by (83),

$$\sum_{|m|\leq G(t)} (m^2)^n |(\delta_m, e^{-itH_\lambda}\delta_0)|^2 \leq (G(t))^{2n},$$

so $\langle x^{2n}\rangle(t) \leq (G(t))^{2n} + C_n$, as claimed.

In fact, the proof shows that

$$\overline{\lim_{|t|\to\infty}} (\ln|t|)^{-2n} \langle x^{2n}\rangle(t) \leq \left(\frac{\alpha}{2}\right)^{-2n}.$$

Chapter 3

Generalization of Projection Constants and Minimal- Volume

We show some characterization of sufficiently Enlargements. Our main result is that for some subspaces there exist minimal-volume shadows that are far from parallelepipeds with respect to the Banach–Mazur distance.

Section (3.1): Sufficient Enlargements:

Let X be a Banach space and let Y be a finite dimensional subspace. We denote the unit ball of X by B(X). Let P: $X \to Y$ be some continuous linear projection. Then $P(B(X)) \supset B(Y)$ and P(B(X)) is a convex, symmetric with respect to 0, bounded subset of Y.

Let X be a finite dimensional normed space.

Definition (3.1.1)[76]: Asymmetric with respect to 0 bounded, closed convex body $A \subset X$ will be called a sufficient enlargement for X (or of B(X)) if for arbitrary isometric embedding $X \subset Y$ there exists a projection P: $Y \to X$ such that $P(B(Y)) \subset A$.

Convention (3.1.2)[76]: We shall use the term ball for symmetric with respect to 0, bounded, closed convex body with nonempty interior in a finite dimensional linear space.

We use standard definitions and notation of Banach space theory (see [77], [78]).

Let A be a ball in a finite dimensional space X. The space X normed by the gauge functional of A will be denoted by X_A.

We start with some simple observations. Their proofs are straightforward and we omit them. By γ_∞ we denote the L_∞-factorable norm (see [78]).

Proposition (3.1.3)[76]: A ball A is a sufficient enlargement for X if and only if $\gamma_\infty(I) \leq 1$, where I is the natural identity mapping $I : X \to X_A$.

Corollary (3.1.4)[76]: If X and Y are $\mathbf{R}^n$ with different norms and $B(X) \subset B(Y)$ then every sufficient enlargement for Y is a sufficient enlargement for X.

Corollary (3.1.5)[76]: Let $T: X \to Z$ be an invertible linear operator between finite dimensional normed spaces. Then:

$$\gamma_\infty(T) . T^{-1}(B(Z))$$

is a sufficient enlargement for X.

Corollary (3.1.6)[76]: A symmetric with respect to 0 parallelepiped containing B(X) is a sufficient enlargement for X.

Proposition (3.1.7)[76]: [79,80,81,82] Convex combination of sufficient enlargements for X is a sufficient enlargement for X.

The same is true for integrals with respect to probability measures. In order to make this statement precise we need to introduce a notion of integral of function, whose values are convex subsets in $\mathbf{R}^n$

I introduce the notion of integral for convex body-valued functions as some mixture of Riemann and Lebesgue integrals. This definition of integral is somewhat unnatural, but it is sufficient for our purposes and at the moment I do not want to overcome difficulties which appear for more general notions of integral.

Let M be a compact metric space with a regular Borel probability measure μ. (The main example for us is the group of orthogonal matrices in $\mathbf{R}^n$ or its closed subgroups with the normalized Haar measures).

The set of all compact convex subsets of $\mathbf{R}^n$ will be denoted by C(n). We shall consider C(n) as a metric space with the Hausdorff metric:

$$d(A,B) = \max\{\sup_{a \in A} \operatorname{dist}(a,B), \sup_{b \in B} \operatorname{dist}(b,A)\}$$

Recall the following well-known fact: C(n) is complete with respect to d. For this and other results on convex bodies we refer to [83,84,85].

Let $f: M \to C(n)$ be a continuous function.

Definition (3.1.8)[76]: The integral of f with respect to measure μ is defined to be:

$$\int_M f(m)d\mu(m) := \lim_{\operatorname{diam}\Delta \to 0} \sum_{i=1}^{k(\Delta)} f(a_i(\Delta))\,\mu(M_i(\Delta)), \tag{1}$$

where Δ is a pair consisting of a partition of M onto a finite number of measurable subsets $\{M_i(\Delta)\}_{i=1}^{k(\Delta)}$ and a family $\{a_i,(\Delta)\}_{i=1}^{k(\Delta)}$ of points for which $\{a_i(\Delta) \in M_i(\Delta)$. Diameter of Δ is defined to be the maximum of the diameters of the sets $M_i(\Delta)$ (i = 1, ... ,k(Δ)) in the metric space M. The limit in (1) is considered in the Hausdorif metric.

A proof that the integral exists can be obtained in the same way as the proof of existence of Riemann integral in classical analysis.

Proposition (3.1.9)[76]: [82,86,87,88] Let X = ($\mathbf{R}^n$, ||.||) be a normed space and M be a compact metric space with a probability measure μ. Suppose that a mapping f :M $\to$ C(n) is continuous and that f(m) is a sufficient enlargement for X for all m $\in$M. Then:

$$\int_M f(m)d\mu(m)$$

is also a sufficient enlargement for X.

Corollary (3.1.6) and Propositions (3.1.7) and (3.1.9) supply us with the following family of sufficient enlargements for a space X: parallelepipeds containing B(X), their convex combinations and integrals with respect to probability measures. It is natural to ask: is it true that any sufficient enlargement contains some sufficient enlargement of the described type?

The answer to this question is negative. The first example was found by V.M.Kadets (1993). In his example X is a two-dimensional space, whose unit ball is a regular hexagon. The space X can be isometrically embedded into l_∞^3.

Let P : l_∞^3.$\to$ X be the orthogonal projection. It is clear that A: = P(B(l_∞^3)) is a sufficient enlargement for X. V. M. Kadets proved that A does not contain any integral with respect to a probability measure of parallelograms containing B(X).

Our purpose is to prove that analogous examples can be constructed even for two dimensional Euclidean space.

Theorem (3.1.10)[76]: There exists a sufficient enlargement for l_2^2. which does not contain any integral with respect to a probability measure of parallelograms containing $B(l_2^2)$.

Proof. Let us denote by S_1 and S_2 the operators of counterclockwise rotation of l_2^2 onto $2\pi/3$ and $4\pi/3$ respectively. Let e_1 and e_2 be the unit vector basis of l_2^2 and e_1^* and e_2^* be its biorthogonal functionals .

It is easy to verify that for all x, y $\in \mathbf{R}^2$, $\|y\|_2 = 1$ we have

$$x = \frac{2}{3}(\langle x, y\rangle y + \langle x, S_1 y\rangle S_1 y + \langle x, S_2 y\rangle S_2 y).$$

Let $y = e_2$. We have the following factorization of the identity operator on l_2^2:

$$I = RQ,\ l_2^2 \xrightarrow{Q} l_\infty^3 \xrightarrow{R} l_2^2,$$

where

$$Q(x) = \{\langle x, e_2\rangle, \langle x, S_1 e_2\rangle, \langle x, S_2 e_2\rangle\},$$

$$R(\{a_0, a_1, a_2\}) = \frac{2}{3}(a_0 e_2 + a_1 S_1 e_2 + a_2 S_2 e_2).$$

Hence the Minkowski sum of the line segments

$$A = \frac{2}{3}([-e_2, e_2] + [-S_1 e_2, S_1 e_2] + [-S_2 e_2, S_2 e_2])$$

is a sufficient enlargement for l_2^2.

It is easy to verify that A is a regular hexagon with

$$\sup\{ e_1^*(x) : x \in A\} = \frac{2}{\sqrt{3}}.$$

We need the following lemma.

Lemma (3.1.11)[76]: Let P be a parallelogram containing $B(l_2^2)$. Then

$$\sup\{ e_1^*(x) : x \in \frac{1}{3}(P + S_1 P + S_2 P)\} > \frac{2}{\sqrt{3}}.$$

Proof. We represent P as a sum of two line segments: $P[-f_1, f_1] + [-f_2, f_2]$. We introduce the notation

$$a := \sup\{ e_1^*(x) : x \in \tfrac{1}{3}(P + S_1P + S_2P)\}.$$

We have

$$\mathbf{a = \tfrac{1}{3}\ (| e_1^*(f_1)| + | e_1^*(f_2)| + | e_1^*(S_1f_1)| + | e_1^*(S_1f_2)| + | e_1^*(S_2f_1)| + | e_1^*(S_2f_2)|).}$$

Set

$$t(f_1) := \tfrac{1}{3}(| e_1^*(f_1)| + | e_1^*(S_1f_1)| + | e_1^*(S_2f_1)|).$$

Let us show that

$$t(f_1) \geq \frac{\|f_1\|}{\sqrt{3}},$$

and the equality is attained if and only if the angle between f_1 and e_2 is a multiple of $\pi/3$.

It is easy to see that in order to prove this statement it is sufficient to consider the case when the angle α between f_1 and e_2 is in the interval $[0, \frac{\pi}{3}]$.

We have

$$\begin{aligned} t(f_1) &= \frac{\|f_1\|}{3}(|\sin\alpha| + |\sin(\alpha + \frac{2\pi}{3})| + |\sin(\alpha + \frac{4\pi}{3}|) \\ &= \frac{\|f_1\|}{3}(\sin\alpha + \sin(\alpha + \frac{2\pi}{3}) - \sin(\alpha + \frac{4\pi}{3})) \\ &= \frac{\|f_1\|}{3}(\sin\alpha + \sqrt{3}\cos\alpha). \end{aligned}$$

It is clear that for-vectors of the same norm this product is minimal if and only if $\alpha = 0$ or $\alpha = \pi/3$. In both cases we have $t(f_1)=\|f_1\|/\sqrt{3}$. So we have proved the assertion about $t(f_1)$.

Since $a = t(f_1) + t(f_2)$, then:

$$a \geq \frac{\|f_1\| + \|f_2\|}{\sqrt{3}},$$

and the equality is attained if and only if the angles between f_1, f_2 and e_2 are multiples of $\pi/3$. On the other hand since $[-f_1, f_1] + [-f_2, f_2] \supset B(l_2^2)$, then $\|f_1\|$, $\|f_2\| \geq 1$ and if the angles between f_1, f_2 and e_2 are multiples of $\pi/3$, then

$$\|f_1\| + \|f_2\| > 2.$$

Hence $a > 2/\sqrt{3}$.

We return to the proof of the theorem. Suppose the contrary. Let M be a metric space with a probability measure μ and let F: M → C(n) be a uniformly continuous function for which F(m) is a parallelogram containing $B(l_2^2)$ for each m ∈ M and

$$\int_M F(m)\, d\mu(m) \subset A.$$

Since A is invariant under action of S_1 and S_2, then

$$\int_M \tfrac{1}{3}(F(m)+S_1F(m)+S_2F(m)d\mu(m) \subset A. \qquad (2)$$

Hence

$$\sup\{e_1^*(x): x \in \int_M \tfrac{1}{3}(F(m)+S_1F(m)+S_2F(m)d\mu(m)\} \le \frac{2}{\sqrt{3}}.$$

This supremum equals to

$$\int_M \sup\{e_1^*(x): x \in \tfrac{1}{3}(F(m)+S_1F(m)+S_2F(m))\}d\mu(m).$$

By the lemma the integrand is $> \frac{2}{\sqrt{3}}$ for each m. Hence the integral is $> \frac{2}{\sqrt{3}}$ This contradicts (2).

It is natural to consider an"isomorphic" version of the question above. I mean the following. If a sequence $\{X_n\}_{n=1}^{\infty}$ of finite dimensional normed spaces is such that for some sufficient enlargements A_n (n ∈ **N**) for X_n, arbitrary $0 < C < \infty$ and arbitrary integrals I_n with respect to probability measures of parallelepipeds containing B (X_n) we have

$$\exists n \in \mathbf{N},\ In \not\subset C \,.\, A_n,$$

then we shall say that $\{X_n\}$has property N.

Section (3.2): Shadows of Cubes:

Let $K^m \subset \mathbf{R}^m$ be defined by $K^m = \{(x_1, ..., x_m): |x_i| \leq 1$ for every $i \in \{1, ..., m\}\}$. We refer to K^m as an m-cube. Let L be a linear subspace in $\mathbf{R}^m$ and P: $\mathbf{R}^m \rightarrow L$ be a linear projection onto L. The set $P(K^m)$ will be called a shadow of K^m in L. Using a compactness argument it can be proved that for every $m \in \mathbf{N}$ and for every subspace $L \subset \mathbf{R}^m$ there exists a linear projection that minimizes the volume of $P(K^m)$. In such a case the set $P(K^m)$ will be called a minimal-volume shadow of K^m in L.

It may happen that K^m has many different minimal-volume shadows in L. we study the shape of minimal-volume shadows of cubes. It is known that among minimal-volume shadows in an arbitrary subspace there is always a parallelepiped (see Theorem (3.2.1)). Our main result is that there exist minimal-volume shadows that are far from parallelepipeds with respect to the BanachMazur distance. Such shadows can be found by a simple and explicit construction; see the beginning of the proof of Theorem (3.2.3).

Initially this study was motivated by the study of sufficient enlargements (see [90]). Here we do not discuss this connection, because it is also a natural geometric problem.

The following result is essentially known. It is implicitly contained in [91]. We prove it because our proof is more direct than the proof in [91] and we use our proof in further considerations.

Theorem (3.2.1)[89]: Let L be a linear subspace in $\mathbf{R}^m$. Let **M** be the set all minimal-volume shadows of K^m in L. The set **M** contains a parallelepiped.

Proof. Denote by $\{e_i\}_{i=1}^m$ the unit vector basis in $\mathbf{R}^m$.. Let n = dim L and let

$$E = lin\{e_{i(1)}, ..., e_{i(m-n)}\}$$

where $\{i(1), ..., i(m-n)\}$ is a subset of $\{1, ..., n\}$, be such that $L \cap E = [0]$. Let P be the projection of $\mathbf{R}^m$ onto L with kernel E. Then $P(K^m)$ is a parallelepiped. We endow $\mathbf{R}^m$ with the standard inner product and compute all volumes with the corresponding normalization. Let $z_1, ..., z_m \in \mathbf{R}^m$ be such that $z_j = \sum_{i=1}^m z_{i,j} e_i$. By $\det[z_1, ..., z_m]$ we mean the determinant of the matrix $[z_{i,j}]_{i,j=1}^m$.

Let $\{x_1, ..., x_n\}$ be some orthonormal basis in L. Then

$$Vol\ \ P(K^m) = \frac{2^n}{|\det[x_1,..., x_n, e_{i(1)},..., e_{i(m-n)}]|}.$$

Suppose that E is chosen in such a way that

$$|\det[x_1,..., x_n, e_{i(1)},..., e_{i(m-n)}]|$$

takes the maximal possible value.

Let Q: $\mathbf{R}^m \to L$ be another projection. Let $q_1, ..., q_{m-n}$ be an orthonormal basis in its kernel. We have

$$Vol\, Q\,(K^m) = \frac{2^n}{|\det[x_1,...,x_n, q_1,...,q_{m-n}]|}$$

$$\times \sum_{\{j(1),...,j(n)\} \subset \{1,...,m\}} |\det[q_1,....,q_{m-n}, e_{j(1)},....,e_{j(n)}]|,$$

where the sum is over all n-element subsets of [1, ..., m]. (To prove this formula we first project the cube onto the orthogonal complement of the kernel of Q and use the well-known formula for the volume of a zonotope, see [92]. Then we use the previous formula.)

In order to prove the theorem it is enough to show that

$$\text{vol } P(K^m) \le \text{vol } Q(K^m) \tag{3}$$

Inequality (3) is equivalent to the following inequality:

$$|\det[x_1, ..., x_n, q_1, ..., q_{m-n}]|$$
$$\le |\det[x_1, ..., x_n, e_{i(1)}, ..., e_{i(m-n)}]|$$
$$\times \sum_{\{j(1),...,j(n)\} \subset \{1,...,m\}} |\det[q_1,....,q_{m-n}, e_{j(1)},....,e_{j(n)}]|. \tag{4}$$

By the Laplacian expansion (see [93]) the determinant

$$\det[x_1, ..., x_n, q_1, ..., q_{m-n}]$$

can be represented as

$$\sum_{I \subset \{1,...,m\}, \neq I = n} \theta_I \det X_I \det Q_I,$$

where X_I is the $n\times n$-submatrix of $[x_1, ..., x_n]$ corresponding to I, Q_I is the corresponding (complementary) $(m-n)\times(m-n)$-submatrix of $[q_1, ..., q_{m-n}]$, and $\{\theta_I\}$ are some signs.

It is easy to see that by the choice of {i(1), ..., i(m-n)} we have

$$|\det[x_1,....,x_n, e_{j(1)},....,e_{j(m-n)}]| . \max_I |\det X_I|.$$

It is easy to see also that

$$\sum_{\{j(1),...,j(n)\}\subset\{1,...,m\}} |\det[q_1,....,q_{m-n}, e_{j(1)},....,e_{j(n)}]| = \sum_I \left|\det Q_I\right|.$$

The inequality (4) follows.

Our next purpose is to show that there exist minimal-volume shadows that are far from parallelepipeds.

Observe that each shadow is convex, closed, bounded and symmetric with respect to 0. A shadow of K^m in L has a non-empty interior in L. Hence it is a unit ball of some norm on L. With some abuse of terminology we define the Banach-Mazur distance between a shadow and a parallelepiped as the Banach-Mazur distance between the normed space correspondent to the shadow and l_∞^d, where d is the dimension of the shadow. We refer to [78] for basic facts on the Banach- Mazur distance.

Convention (3.2.2)[89]: We use the term ball for a symmetric-with-respect-to-0, bounded, closed, convex body with nonempty interior in a finite dimensional linear space.

We say that two balls are affinely equivalent if there exists a linear operator between the corresponding spaces that is a bijection of the balls.

A Minkowski sum of (finitely many) line segments in $\mathbf{R}^n$ is called a zonotope (see [279] for basic facts on zonotopes). We shall consider zonotopes that are sums of line segments of the form [-x, x]. Such zonotopes are balls according to our convention. Let $a_1, ..., a_m$ be some collection of vectors in $\mathbf{R}^n$. The Minkowski sum

$$\sum_{i=1}^{m} [-a_i, a_i]$$

will be called the zonotope spanned by $a_1, ..., a_m$.

Construction. Subspaces L satisfying the condition of the theorem can be found in the following way. Let G_n be a two-dimensional discrete torus with n vertices. (It means that $G_n=\mathbf{Z}_k\times\mathbf{Z}_k$, where $\mathbf{Z}_k$ is the group of residue classes of integers modulo k; vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if either $x_1=x_2$ and $y_1=y_2 \pm 1$ (in $\mathbf{Z}_k$) or $y_1 = y_2$ and $x_1=x_2 \pm 1$ (in $\mathbf{Z}_k$). We can visualize this graph drawing 2k circles on a usual torus; k of the circles are meridians and k are parallels.)

We consider G_n as a directed graph, edges are directed in an arbitrary way.

Let M_n be the incidence matrix of G_n, that is, an nx(2n) matrix whose rows and columns are indexed by the vertices and edges of G_n, respectively, and the column corresponding to an edge e has exactly two non-zero entries: -1 in the row corresponding to the starting vertex of e and 1 in the row corresponding to the end vertex of e.

We consider rows of M_n as vectors in $\mathbf{R}^{2n}$. LetL be the subspace of $\mathbf{R}^{2n}$ spanned by the rows of M_n.

Definition (3.2.3)[89]: A matrix A with real entries is called totally unimodular if determinants of all submatrices of A are equal to -1, 0 or 1.

Totally unimodular matrices is a very important object in integer programming. There exists a vast literature devoted to them (see [95]). We need only the following observation that goes back to H.Poincare : an incidence matrix of any directed graph is totally unimodular. (See [95] for historical notes and a very short proof.) So, M_n is totally unimodular.

Lemma (3.2.4)[89]: Let A be a totally unimodular $r \times m$ matrix of rank l. Let L be the subspace in $\mathbf{R}^m$ spanned by rows of A. Let P_L be the orthogonal projection onto L. Then

(i) $P_L(K^m)$ is a minimal-volume shadow of K^m in L.

(ii) $P_L(K^m)$ is affinely equivalent to the zonotope in $\mathbf{R}^r$ spanned by columns of A.

Proof. We rearrange the rows of A in order to get a matrix whose first l rows are linearly independent. It is clear that the zonotope spanned by the columns of the obtained matrix is affinely equivalent to the zonotope spanned by the columns of A.

Hence without loss of generality we may assume that the first l rows of A are linearly independent.

By A^T we denote the transpose of A. Let C be an upper-triangular $r \times r$ matrix such that the first l columns of the product A^TC form an orthonormal basis in L and the remaining columns contain zeros only. The existence of such matrices can be shown using the Gram-Schmidt orthonormalization process. We denote by D the $l \times l$ submatrix of C correspondent to the first l rows and the first l columns. It is easy to see that D is invertible.

Straightforward verification shows that the product A^TCC^TA is the matrix of P_L with respect to the unit vector basis of $\mathbf{R}^m$.

Let $\{x_1, ..., x_m\}$ be an orthonormal basis in $\mathbf{R}^m$ satisfying the following condition: vectors $\{x_1, ..., x_l\}$ are the first l columns of A^TC. Writing $[x_k, ..., x_s]$ we mean the matrix with columns $x_k, ..., x_s$.

We use results on compound matrices. We refer to [93] for necessary definitions and results.

Let $u = \{u_i\}$ be an $\binom{m}{l}$-dimensional vector, where u_i are $l \times l$ minors of $[x_1, ..., x_l]$. Since a compound matrix of an orthogonal matrix is orthogonal (see [93]), then u is normalized (with respect to the Euclidean norm). For the same reason the vector $v=[v_i]$ in the $\binom{m}{m-l}$-dimensional space, where v_i are $(m-l)\times(m-l)$ minors of $[x_{l+1}, ..., x_m]$, is also normalized.

Since the matrix $[x_1, ..., x_m]$ is orthogonal, its determinant is equal to ± 1. On the other hand, by the Laplacian expansion (see [93]) the determinant is equal to

$$\sum_{i=1}^{\binom{m}{l}} \theta_i u_i v_i$$

for proper signs θ_i and for proper ordering of u_i and v_i. (Observe that $\binom{m}{l} = \binom{m}{m-l}$.) Since u and v are normalized, it implies that either $u_i = \theta_i v_i$ for every i or $u_i = \theta_i v_i$ for every i.

Now we let n=l, $Q = P_L$, and $\{q_1, ..., q_{m-l}\} = [x_{l+1}, ..., x_m]$ in the argument of Theorem (3.2.1). We get: vol $P(K^m)$=vol $P_L(K^m)$ is equivalent to

$$|\det[x_1, ..., x_m]| = |\det[x_1, ..., x_l, e_{i(1)}, ..., e_{i(m-l)}]|$$

$$\times \sum_{\{j(1),...,j(l)\} \subset \{1,...,m\}} |\det[x_{l+1}, ..., x_m, e_{j(1)}, ..., e_{j(l)}]|.$$

where $\{i(1), ..., i(m-l)]$ are chosen to maximize

$$|\det[x_1, ..., x_l, e_{i(1)}, ..., e_{i(m-l)}]|.$$

In terms of u_i and v_i this equality is

$$1 = \max_i |u_i| \sum_{i=1}^{\binom{m}{l}} |v_i|. \tag{5}$$

Let E be the matrix consisting of the first l rows of A. It is clear that E is totally unimodular. It is easy to see that $[x_1, ..., x_l] = E^T D$. Therefore u_i is equal to det D, 0 or - det D for every i.

To prove equality (5) we observe that $\max_i |u_i| = |\det D|$. Assume that $u_i = \theta_i v_i$ for every i (the case $u_i = -\theta_i v_i$ is similar). Then $1 = |\sum_i \theta_i u_i v_i| = |\det D|^2 \omega$, where ω is the number of non-zero u_i 's (=the number of non- zero v_i 's). On the other hand,

$$|\det D|^2 . \omega = \max_i |u_i| \sum_{i=1}^{\binom{m}{l}} |v_i|.$$

It proves that $P_L(K^m)$ is a minimal-volume shadow.

The statement (ii) can be proved in the following way. Consider A as an operator from $\mathbf{R}^m$ to $\mathbf{R}^r$. The image of K^m under A coincides with the zonotope spanned by the columns of A in $\mathbf{R}^r$. This zonotope spans a subspace of dimension l in $\mathbf{R}^r$ (because l= rank A). The operator $P_L = A^T C C^T A: \mathbf{R}^m \to \mathbf{R}^m$ also has l-dimensional image. Therefore the restriction of $A^T C C^T$ to the range of A is an isomorphic embedding. Therefore the image of K^m under P_L is a finely equivalent to the zonotope.

Lemma (3.2.5)[89]: If $\ln k \geq 2\pi^3C^2+3$, then the Banach-Mazur distance between the zonotope spanned by the columns of M_n and the parallelepiped of the same dimension is $\geq C$.

Proof. Observe that the linear space spanned by the columns of M_n in $\mathbf{R}^n$ is (n-1)-dimensional and it consists of all vectors whose sum of the coordinates is equal to 0.

Let X_n be this space normed by the gauge functional of the zonotope.

Observe that vertices of the zonotope spanned by $a_1, \ldots, a_n$ are contained in the set $\left\{\sum_{i=1}^{n}\theta_i a_i : \theta_i = \pm 1\right\}$ and that this set is contained in the zonotope. Therefore the maximal value of a functional f over the zonotope is equal to $\sum_{i=1}^{n}\left|f(a_i)\right|$ .Using this observation we can identify the dual space X_n^* of X_n with the space of functions on the set of vertices of G_n with zero average and with the norm

$$\|f\|_* = \sum_{u \sim v}|f(u)-f(v)|$$

where $u \sim v$ means that u and v are adjacent in G_n.

We need to estimate the Banach-Mazur d (X_n, l_∞^{n-1}) from below. Since d(X, Y) = d(X*, Y*) for every finite-dimensional spaces X and Y, then

$$d(X_n, l_\infty^{n-1}) = d((X_n^*, l_1^{n-1}).$$

To estimate the distance $d(X_n^*, l_1^{n-1})$) we use the approach that goes back to J. Lindenstrauss and A. Petczynski (see [96]).

Recall that the 2-summing norm of an operator T: X→Y is defined to be the smallest constant C satisfying the condition

$$\left(\sum_{i=1}^{n}\|Tx_i\|^2\right)^{1/2} \leq C\ \sup\left\{\left(\sum_{i=1}^{n}(\zeta(x_i))^2\right)^{1/2} : \zeta \in X^*, \|\zeta\| \leq 1\right\}$$

for every collection $\{x_1, \ldots, x_n\} \in X$. The 2-summing norm of T is denoted by $\pi_2(T)$.

Let T: Z→ H be a non-zero operator, where H is a Hilbert space and dim Z = n. The dual form of the ``little Grothendieck theorem'' (see [79]; this form of the Grothendieck theorem [97] was proved in [91]) implies that

$$d(Z, l_1^n) \geq \left(\frac{2}{\pi}\right)^{1/2} \frac{\pi_2(T)}{\|T\|} \tag{6}$$

So we need to find a Hilbert space H and an operator T: $X_n^* \to$ H with ``large" ratio $\pi_2(T)/\|T\|$.

With this purpose in mind we introduce the norm

$$\|f\|_2 = \left(\sum_v (f(v))^2\right)^{1/2}$$

on the space of all functions on the set of vertices of G_n. We denote the obtained normed space by $l_2(G_n)$.

Let I_n be the identical embedding of X_n^* into $l_2(G_n)$.

To estimate the norm of this embedding from above we use a Sobolev type inequality due to F. R. K. Chung and S.-T. Yau [98] (see, also, [99]).

We need the following definitions.

Definition (3.2.6)[89]: Let G be a graph. By d_v we denote the degree of a vertex v. Let X be some set of vertices of a graph G. Let vol X $:= \sum_{v \in X} d_v$. The number of edges joining X and its complement $\tilde{X}$ is denoted by $|E(X, \tilde{X})|$. We say that G has isoperimetric dimension δ with isoperimetric constant c_δ if

$$|E(X, \tilde{X})| \geq c_\delta \, (\text{vol } X)^{(\delta-1)/\delta}$$

Whenever vol $X \leq$ vol $\tilde{X}$. The constant c_δ depends on δ only.

Definition (3.2.7)[89]: A graph G is called k-regular if $d_v = k$ for every v.

We need the following special case of [98].

Theorem (3.2.8)[89]: Let G be a connected k-regular graph with isoperimetric dimension 2 and isoperimetric constant c_2. Let f be a function on the set of vertices of G with zero average. Then

$$\sum_{u \sim v} |f(u) - f(v)| \geq c_2 \frac{k^{1/2}}{2} \left(\sum_v (f(v))^2\right)^{1/2}$$

Observe that G_n is 4-regular. To apply Theorem (3.2.8) to G_n we need to estimate the isoperimetric constant c_2 for this graph. Since the author has not found a proper reference, we present such an estimate (with the best possible constant).

Sublemma (3.2.9)[89]: The graph G_n has isoperimetric dimension 2 with constant $\sqrt{2}$.

Proof. Let X be a set of vertices of G_n with $\neq X \leq k^2/2$. Sets of vertices of the form

$$\{(x, 0), (x, 1), (x, 2), ..., (x, k-1)\}$$

will be called meridians and sets of the form

$$[(0, y), (1, y), (2, y), ..., (k-1, y)]$$

will be called parallels.

Let m_1 be the number of meridians contained in X and let m_2 be the number of meridians intersecting X. Let p_1 be the number of parallels contained in X and let p_2 be the number of parallels intersecting X. It is easy to see that

$$|E(X, \widetilde{X})| \geq 2(m_2 - m_1) + 2(p_2 - p_1).$$

We have also $\neq X \geq m_1 k$ and $\neq X \geq p_1 k$. Hence $m_1 \leq k/2$ and $p_1 \leq k/2$. We have three possibilities:

(i) Both m_1 and p_1 are nonzero.

(ii) Exactly one of the numbers m_1 and p_1 is nonzero.

(iii) $m_1 = p_1 = 0$.

(i) In this case $m_2 = p_2 = k$. Hence

$$|E(X, \widetilde{X})| \geq 2(m_2 - m_1) + 2(p_2 - p_1) \geq 2\left(k - \frac{k}{2}\right) + 2\left(k - \frac{k}{2}\right)$$

$$= 2k \geq 2\sqrt{2}\left(\frac{k^2}{2}\right)^{1/2} \geq 2\sqrt{2}(\neq X)^{1/2} = \sqrt{2}(vol\, X)^{1/2}.$$

(ii) We consider the case $m_1 \neq 0$ and $p_1 = 0$ (the case $p_1 \neq 0$ and $m_1 = 0$ is similar). In this case $p_2 = k$ and

$$|E(X, \widetilde{X}) \geq 2p_2 = 2k \geq \sqrt{2}\ (\text{vol } X)^{1/2}.$$

(iii) In this case $\neq X \leq m_2 p_2$ and

$$|E(X, \widetilde{X})| \geq 2(m_2 + p_2) \geq 4\sqrt{m_2 p_2} \geq 4(\neq X)^{1/2} = 2(vol\ X)^{1/2}$$

Remark (3.2.10)[89]: If k is even and X is the union of $\frac{k}{2}$ meridians (or $\frac{k}{2}$ parallels), then

$$|E(X, \widetilde{X})| = 2k = \sqrt{2}\ (\mathrm{vol}\ X)^{1/2}.$$

So the constant $\sqrt{2}$ is the best possible.

By Theorem (3.2.8) we get $\|f\|_* \geq \sqrt{2}\frac{2}{2}\|f\|_2$ for every f with average 0.

Hence $\|I_n\| \leq \frac{1}{\sqrt{2}}$.

To estimate $\pi_2(I_n)$ from below we use the approach developed by S. V. Kislyakov [287] for continuous case.

Let p be the integer part of $\frac{k-1}{2}$. We introduce a family $\{f_{s,t}\}_{s,t=1}^{p}$ of functions on G_n in the following way. We consider G_n as {0, ..., k-1} ×{0, ..., k-1} and let

$$f_{s,t}(x,y) = \frac{1}{k(s+t)} \sin\left(\frac{2\pi s}{k} x\right) \sin\left(\frac{2\pi t}{k} y\right)$$

Observe that

$$\pi_2(I_n) \geq \frac{(\Sigma_{s,t} \|f_{s,t}\|_2^2)^{1/2}}{\sup\{\Sigma_{s,t} |\zeta(f_{s,t})|^2)^{1/2} : \zeta \in (X_n^*)^*, \|\zeta\| \leq 1\}}$$

So we need to estimate the quantity

$$\sup\{(\sum_{s,t} |\zeta(f_{s,t})|^2)^{1/2} : \zeta \in (X_n^*)^*, \|\zeta\|, \leq 1\} \tag{7}$$

from above and the quantity

$$(\sum_{s,t} \|f_{s,t}\|_2^2)^{1/2} \tag{8}$$

from below.

To estimate (8) we observe that

$$\|f_{s,t}\|_2 = \frac{1}{k(s+t)}\left(\sum_{x,y=0}^{k-1} \sin^2\left(\frac{2\pi s}{k}x\right)\sin^2\left(\frac{2\pi t}{k}y\right)\right)^{1/2}$$

$$= \frac{1}{k(s+t)}\left(\sum_{x=0}^{k-1} \sin^2\left(\frac{2\pi s}{k}x\right)\sum_{y=0}^{k-1} \sin^2\left(\frac{2\pi t}{k}y\right)\right)^{1/2}$$

(we use the fact that $1 \leq s,t < \frac{k}{2}$)

$$= \frac{1}{k(s+t)}\left(\frac{k}{2}\frac{k}{2}\right)^{1/2} = \frac{1}{2(s+t)}.$$

Hence

$$\left(\sum_{s,t=1}^{P}\left\|f_{s,t}\right\|_2^2\right)^{1/2} \geq \frac{1}{2}\left(\frac{1}{2^2}+\frac{2}{3^2}+\ldots+\frac{p}{(p+1)^2}\right)^{1/2} \geq \frac{1}{2}(\ln k-3)^{1/2}$$

To estimate (7) we observe that

$$\sum_{u\sim v}\left|f(u)-f(v)\right| \leq \sqrt{2k}\left(\sum_{u\sim v}(f(u)-f(v))^2\right)^{1/2}.$$

The right-hand side in this inequality is a hilbertian norm on (X_n^*) induced by the inner product

$$\langle f,g\rangle = 2k^2\left(\sum_{x,y=0}^{k-1}(f(x+1,y)-f(x,y))(g(x+1,y)-g(x,y))\right.$$
$$\left.+\sum_{x,y=0}^{k-1}(f(x,y+1)-f(x,y))(g(x,y+1)-g(x,y))\right), \tag{9}$$

where (k-1) + 1 = 0.

We denote by H_n the corresponding Hilbert space. We shall use (6) for $H = H_n$. Since the natural embedding of H_n into X_n^* has norm 1, then the supremum in (7) is not greater than

$$\sup\left\{\left(\sum_{s,t}|\zeta(f_{s,t})|^2\right)^{1/2} : \xi\in(H_n)^*,\left\|\xi\right\|\leq 1\right\}, \tag{10}$$

where the norm is in $(H_n)^*$.

To estimate this supremum we show that the functions $\{f_{s,t}\}_{s,t=1}^{P}$ are orthogonal with respect to the inner product (9).

We have

$$\left\langle f_{s,t}, f_{s',t'}\right\rangle = 2k^2\left(\sum_{x,y=0}^{k-1}(f_{s,t}(x+1,y)-f_{s,t}(x,y))\right.$$
$$\times(f_{s',t'}(x+1,y)-f_{s',t'}(x,y))$$
$$+\sum_{x,y=0}^{k-1}(f_{s,t}(x,y+)-f_{s,t}(x,y))$$
$$\left.\times(f_{s',t'}(x,y+1)-f_{s',t'}(x,y))\right).$$

We shall show that the first sum is equal to zero (the same argument works for the second sum also).

$$\sum_{x,y=0}^{k-1}(f_{s,t}(x+1,y)-f_{s,t}(x,y))((f_{s',t'}(x+1,y)-f_{s',t'}(x,y))$$

$$=\frac{1}{k(s+t)}\frac{1}{k(s'+t')}\sum_{x,y=0}^{k-1} 2\sin\left(\frac{2\pi s}{k}\frac{1}{2}\right)\cos\left(\frac{2\pi s}{k}\left(x+\frac{1}{2}\right)\right)\sin\left(\frac{2\pi t}{k}y\right)$$

$$\times 2\sin\left(\frac{2\pi s'}{k}\frac{1}{2}\right)\cos\left(\frac{2\pi s'}{k}\left(x+\frac{1}{2}\right)\right)\sin\left(\frac{2\pi t'}{k}y\right)$$

$$=\frac{4}{k^2(s+t)(s'+t')}\sin\left(\frac{2\pi s}{k}\frac{1}{2}\right)\sin\left(\frac{2\pi s'}{k}\frac{1}{2}\right)$$

$$\times\sum_{x=0}^{k-1}\cos\left(\frac{2\pi s}{k}(x+\frac{1}{2})\right)\cos\left(\frac{2\pi s'}{k}\left(x+\frac{1}{2}\right)\right)$$

$$\times\sum_{y=0}^{k-1}\sin\left(\frac{2\pi t}{k}y\right)\sin\left(\frac{2\pi t'}{k}y\right).$$

By use of the fact that $1\leq s, s', t, t' < \frac{k}{2}$ it is easy to show that if $s \neq s'$, then the first sum in the last product is equal to 0, and if $t \neq t'$, then the second sum is equal to 0.

Since the functions $\{f_{s,t}\}_{s,t=1}^{P}$ are orthogonal with respect to the inner product (9), then the supremum in (10) is not greater than

$$\max_{s,t}\sqrt{\langle f_{s,t}, f_{s,t}\rangle}.$$

Using the computation above we get

$$\sqrt{\langle f_{s,t}, f_{s,t}\rangle}=2k^2\left(\frac{1}{k(s+t)}\right)^2\left(\sum_{x,y=0}^{k-1}4\sin^2\left(\frac{2\pi s}{k}\frac{1}{2}\right)\right.$$

$$\times\cos^2\left(\frac{2\pi s}{k}\left(x+\frac{1}{2}\right)\right)\sin^2\left(\frac{2\pi t}{k}y\right)$$

$$+\sum_{x,y=0}^{k-1}4\sin^2\left(\frac{2\pi t}{k}\frac{1}{2}\right)\cos^2\left(\frac{2\pi t}{k}\left(y+\frac{1}{2}\right)\right)\sin^2\left(\frac{2\pi s}{k}x\right)$$

(we use the inequality $|\sin z| \leq |z|$)

$$\leq \frac{8}{(s+t)^2}\left(\left(\frac{\pi s}{k}\right)^2 \sum_{x=0}^{k-1} \cos^2\left(\frac{2\pi s}{k}\left(x+\frac{1}{2}\right)\right)\sum_{y=0}^{k-1}\sin^2\left(\frac{2\pi t}{k}y\right)\right.$$

$$\left.+\left(\frac{\pi t}{k}\right)^2 \sum_{x=0}^{k-1} \sin^2\left(\frac{2\pi s}{k}x\right)\sum_{y=0}^{k-1}\cos^2\left(\frac{2\pi t}{k}\left(y+\frac{1}{2}\right)\right)\right)$$

(each sum is equal to $\frac{k}{2}$ (since $1 \leq s, t < \frac{k}{2}$))

$$= \frac{8}{(s+t)^2}\left(\left(\frac{\pi s}{k}\right)^2\frac{k^2}{4}+\left(\frac{\pi t}{k}\right)^2\frac{k^2}{4}\right) \leq 2\pi^2.$$

Hence the suprema in (10) and (7) are not greater than $\sqrt{2\pi}$. Therefore

$$\pi_2(I_n) \geq \frac{\frac{1}{2}(\ln k-3)^{1/2}}{\sqrt{2}\,\pi}$$

and

$$d(X_n, l_\infty^{n-1}) = d(X_n^*, l_1^{n-1}) \geq \left(\frac{2}{\pi}\right)^{1/2}\frac{\pi_2(I_n)}{\|I_n\|} \geq \frac{(\ln k-3)^{1/2}}{\sqrt{2}\,\pi^{3/2}} \geq C.$$

Theorem (3.2.11)[89]: Let $1 < C < \infty$. If $\ln k \geq 2\pi^3 C^2 + 3$, $k \in \mathbf{N}$ and $n=k^2$, then there exists an (n-1)-dimensional subspace L of $\mathbf{R}^{2n}$ such that the shadow $P(K^{2n})$, where P is the orthogonal projection onto L, is a minimal-volume shadow of K^{2n} in L; and its Banach-Mazur distance to an (n-1)-dimensional parallelepiped is $\geq C$.

Proof . Consider the subspace L in $\mathbf{R}^{2n}$ spanned by the rows of M_n. By Lemma (3.2.4) the image of K^{2n} under the orthogonal projection onto L is a minimal-volume shadow. By the same lemma this shadow is a finely equivalent to the zonotope spanned by the columns of M_n. By Lemma (3.2.5) the Banach-Mazur distance between the zonotope and a parallelepiped is $\geq C$.

Chapter 4

Rank-One Perturbations of Diagonal Normal Operators

We show that two well known results about the eigenvalues of rank-one perturbations and one-codimension compressions of self-adjoint compact operators are equivalent. Sufficient conditions are given for existence of nontrivial invariant subspaces for this class of operators. It is shown that if $T \notin \mathbf{C}1$ and the vectors u and v have Fourier coefficients $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ with respect to an orthonormal basis that diagonalizes D that satisfy $\sum_{n=1}^{\infty}\left(|\alpha_n|^{2/3} + |\beta_n|^{2/3}\right) < \infty$, then T has a nontrivial hyperinvariant subspace. This partially answers an open question of at least 30 years duration.

Section (4.1): Diagonal Operators and Rank-One Perturbations:

We let H be a separable, infinite dimensional, complex Hilbert space, and let L(H) denote the algebra of all bounded linear operators on H If u, v $\in$ H, we shall write u $\otimes$ v for the operator of rank one defined by

$$(u \otimes v)\, x = < x\, ,v > u, \qquad x \in H$$

where <, > denotes the inner product of the Hilbert space H. The class N of operators T in L(H) which can be written in the form T= N + (u $\otimes$ v),where N is a normal operator and (u$\otimes$ v) $\neq 0$ is still not very well understood. Indeed, even the smaller class of operators of the above form, where N is a diagonalizable normal operator, is not in a much better situation, despite the structural simplicity of diagonalizable operators. In this section we are interested in this second class of operators which will be denoted simply by D.

Similar problems concerning operators in the class N, or rank-one perturbations of different classes f operators such as isometries, self-adjoint compact operators, self-adjoint Toeplitz operators, shift restriction operators, cyclic operators, differential perators, (or Volterra operator) have been studied in a series of sections of which we cite only a few of them :[102], [103], [104]-[107], [108]-[111], [112]-[34], [53], [114]. It is worth mentioning that the class of rank- one perturbations of bounded (or unbounded) self-adjoint operators has been extensively studied and many interesting spectral properties have been established in various works (see for instance [115]-[128], [116], [117], [16], [37], [34], [53]).

We let $\{e_n\}_{n=1}^{\infty}$ denote an orthonormal basis for H which will remain fixed throughout the section. We also let $\{\lambda_n\}_{n=1}^{\infty}$ be an arbitrary bounded sequence of complex numbers and throughout the remainder of the section we shall write Diag $(\{\lambda_n\})$ for the unique operator D satisfying $De_n=\lambda_n e_{n,}$, $n\in$ **N**. We shall denote henceforth by D_O the subset of L(H) consisting of all operators T which can be written in the form

$$T = \mathrm{Diag}(\{\lambda_n\}) + u\otimes v, \qquad u \neq 0 , v\neq 0 \tag{1}$$

We shall suppose that u and v are nonzero vectors in H and their expansions with respect to the (ordered, orthonormal) basis $\{e_n\}$ are

$$u = \sum_{n=1}^{\infty} \alpha_n e_n , \quad v = \sum_{n=1}^{\infty} \beta_n e_n . \tag{2}$$

Note that up to unitary equivalence, D_0 consists exactly of all sums N + R, where N is a normal operator whose eigenvectors span H and R is an operator of rank one. Note also that the inclusion $D \subset N$ is a strict one. One way to see this is to make use of Kato and Rosenblum' s result (cf. [118]) stating that the absolutely continuous parts of a self-adjoint operator and its self-adjoint trace class perturbation are unitarily equivalent.

Observe that the expression for T in (1) is not necessarily unique. If we restrict our study, though, to the class D_1 of those operators in D_0 which admit a representation as in (1) with u and v having nonzero components α_n and β_n for all $n \in$ **N**, we have uniqueness in the following sense.

Proposition (4.1.1)[101]: If $T \in D_1$ then the representation (1) for T is unique in the sense that if $T = \mathrm{Diag}(\{\lambda_n\}) + (u \otimes v) = \mathrm{Diag}(\{\lambda'_n\}) + (u' \otimes v')$, then $\mathrm{Diag}(\{\lambda_n) = \mathrm{Diag}(\{\lambda'_n)$ and $(u\otimes v) = (u' \otimes v')$.

Proof. We may assume T = Diag($\{\lambda_n\}$)+(u⊗v)=Diag($\{\lambda_n\}$)+ (u'⊗v'). Where all the Fourier coefficients of u and v in (2) are not zero. This means that Diag ($\{\lambda_n\}$) - Diag($\{\lambda'_n\}$) = Diag($\{\lambda_n - \lambda'_n\}$)=(u' ⊗ v') - (u⊗v) has rank at most two. Thus, there exist different positive integers n_1, n_2 such that $\lambda_k = \lambda'_k$ for all k ∈ **N** \\{n_1,n_2}. Moreover the range of S = Diag$(\{\lambda_n - \lambda'_n\})$ is contained in V $\{e_{n_1}, e_{n_2}\}$, and so we may have three essentially different situations. If the range of S is (0) we are done. If the range of S is one- dimensional-say, spanned by e_{n_1}, then since (u'⊗v')–(u⊗v) would have a two-dimensional range if {u, u' } and {v, v' } are linearly independent sets of vectors, we get that either u and u' are linearly dependent or v and v' are. Let us suppose that u and u' are linearly dependent. Then u = α_{n_1}, e_{n_1} and u' = β_{n_1}, e_{n_1}. But this cannot happen since we have assumed that < $u_1 e_k$ > ≠ 0 for all k ∈**N** . Similarly the case in which v and v' are linearly dependent is ruled out. If the range of S were two-dimensional, then V{u, u'} = V$\{e_{n_1}, e_{n_2}\}$, and again we would have a contradiction.

The next two propositions show that when looking for nontrivial invariant subspaces for operators in D_0, one can then restrict his attention to the subset D_2 of D_I consisting of those operators T = D + (u ⊗ v) in D_I such that D has uniform multiplicity one (i.e., if D = Diag($\{\lambda_n\}$), then all of the numbers λ_n, n ∈ **N**, are pairwise distinct).

Proposition (4.1.2)[101]: Suppose T=Diag($\{\lambda_n\}$) + (u ⊗ v) ∈ D_0 is not a normal operator, and for some n_0 ∈ **N**, α_{n_0} = 0 or β_{n_0} = 0. Then T* [resp. T] has point spectrum and T and T* have nontrivial hyperinvariant subspaces (n.h.s).

Proof. In case α_{n_0} = < u, e_{n_0} > = 0, we have

$$T^* e_{n_0} = \bar{\lambda}_{n_0} e_{n_0} + (v \otimes u)\, e_{n_0} = \bar{\lambda}_{n_0} e_{n_0} + < e_{n_0}, u > v = \bar{\lambda}_{n_0} e_{n_0},$$

which shows that σρ(T*), the point spectrum of T*, is nonempty, and since T* is non-normal, the eigenspace associated with $\overline{\lambda}_{n_0}$ is a n.h.s, for T*. Its orthogonal complement is thus hyperinvariant for T. The case $\beta_{n_0} = 0$ is handled similarly.

For a diagonal operator D = Diag $(\{\lambda_n\})$ we denote by A(D) the set of all its eigenvahies λ_n.

Proposition (4.1.3)[101]: If T = D + (u ⊗ v) ∈ D_1 and at least one λ ∈ A(D) has multiplicity larger than 1, then T has λ in its point spectrum.

Proof. Suppose $\lambda = \lambda_{n_0} = \lambda_{n_1}$, $n_0 \neq n_1$. Then $(T - \lambda)e_{\lambda_{n_0}} = \langle e_{\lambda_{n_0}}, v\rangle u = \bar{\beta}_{n_0} u$, and $(T - \lambda)\, e_{\lambda_{n_1}} = \langle e_{\lambda_{n_1}}, v\rangle u = \bar{\beta}_{n_1} u$. Hence, if $\beta_{n_0} \neq 0$ and $\beta_{n_1} \neq 0$ then

$$(T - \mu)(\bar{\beta}_{n_1} e_{\lambda_{n_0}} - \bar{\beta}_{n_0} e_{\lambda_{n_1}}) = 0.$$

In any case T - λ is not injective, and then $\lambda \in \sigma_\rho(T)$.

For an operator T ∈ D_1 given by (1), an interesting phenomenon happens with the isolated eigenvalues of Diag(λ_n): they are not in the spectrum of T. The following theorem gives necessary and sufficient conditions for a point μ in σ(D) (T = D + (u ⊗ v) ∈ D_0) to be in g(T) (resolvent set).

Theorem (4.1.4)[101]: Suppose we have T= D + (u ⊗ v) ∈ D_0 and μ ∈σ (D). Then μ∈ g(T) if and only if the following two conditions are satisfied:

(i) μ is an isolated eigenvalue of D, λ_{n_0} of multiplicity one,

(ii) $\beta_{n_0} = \langle v, e_{n_0}\rangle \neq 0$ and $\alpha_{n_0} = \langle u, e_{n_0}\rangle \neq 0$.

Proof. For the necessity part of this theorem, let us assume first that (i) is not satisfied. We have three cases: (I)μ is not an eigenvalue; (II)μ is an eigenvaiue but is not isolated, and (III)μ is an isolated eigenvalue but has multiplicity larger than 1. In the cases (I) and (II), there exists a sequence of distinct eigenvalues $\{\lambda_{n_k}\}_{k\geq 1}$ such that $\lambda_{n_k} \to \mu$. Then, since

$$(T - \mu)\, e_{\lambda_{n_k}} = (\lambda_{n_k} - \mu)\, e_{\lambda_{n_k}} + \langle e_{\lambda_{n_k}} , v \rangle u$$

we have

$$\|(T- \mu)\, e_{\lambda_{n_k}} \| \leq |\lambda_{n_k} - \mu| + |\langle e_{\lambda_{n_k}} ,v\rangle| \, \| u \| \to 0,$$

as k goes to infinity. This says in particular that T - μ is not bounded below (if it is injeetive), and then it cannot be invertible. In other words, $\mu \in \sigma(T)$. In the case (III), if we have $\mu = \lambda_{n_0} = \lambda_{n_1}$, then $(T - \mu)e_{\lambda_{n_k}} = \langle e_{\lambda_{n_k}} , v \rangle u = \bar{\beta}_{n_0}, u$ and

$$(T - \mu)e_{\lambda_{n_1}} = \langle e_{\lambda_{n_1}} , v \rangle u = \bar{\beta}_{n_1} u \ .$$

Hence, if $\bar{\beta}_{n_0}, \neq 0$ and $\bar{\beta}_{n_1}, \neq 0$ then

$$(T - \mu)(\bar{\beta}_{n_1} e_{\lambda_{n_1}} - \bar{\beta}_{n_0} e_{\lambda_{n_1}}) = 0.$$

In any case T - μ is not injective, and then again $\mu \in \sigma(T)$.

Suppose now that (i) holds but (ii) doesn't. First, if $\beta_{n_0} = 0$, we get as above $(T - \mu)\, e_{\lambda_{n_0}} = 0$, and so $\mu \in \sigma(T)$. If $\alpha_{n_0} = 0$, then $(T^* - \bar{\mu})\, e_{\lambda_{n_0}} = 0$, and then $\bar{\mu} \in \sigma(T^*)$, or equivalently, $\mu \in \sigma(T)$.

For the sufficiency, we assume now that (i) and (ii) hold. We want to show that T - μ is invertible. Since μ is an isolated point in σ (D) and D is normal, D - μ and hence T- μ, is Fredholm with index zero. Thus it suffices to show that μ is not an eigenvalue for T. If $(T - \mu)x = (D - \lambda_{n_0})\, x + \langle x, v \rangle u = 0$, then by our hypothesis, $\alpha_{n_0} \neq 0$, it follows that $\langle x, v \rangle = 0$. So, $x = \gamma e_{n_0}$ with $\gamma \neq 0$, and this contradicts the hypothesis $\beta_{n_0} \neq 0$.

We characterize now the point spectrum of an operator T in D_1 [resp. D_2].

Proposition (4.1.5)[101]: For $\lambda \in C$, λ is an eigenvalue for $T = D + (u \otimes v) \in D_1$ if and only if

(i) $\mu \in$ Rang $(D - \lambda)$, and

(ii) $\langle x, v \rangle + 1 = 0$ for at least one vector $x \in H$ satisfying $u = (D - \lambda)x$. Equivalently λ is an eigenvaIue for $T = \mathrm{Diag}(\{\lambda\}) + (u \otimes v) \in D_2$ if and only if

(iii) $\lambda \notin A(D)$,

(iv) $\sum_{\lambda_n \in A(D)} \frac{|\alpha_n|^2}{|\lambda - \lambda_n|^2} < \infty, and$

(v) $\sum_{\lambda_n \in A(D)} \frac{\alpha_n \beta_n}{\lambda - \lambda_n} = 1$

Proof. For the necessity part, let $\lambda \in C$ be an eigenvalue for T and $x \in H \setminus \{0\}$, such that $Tx = \lambda X$ Then $\langle x, v \rangle u = (\lambda - D)x$. We cannot have $\langle x, v \rangle = 0$ because we obtain then $\lambda = \lambda_{i_0}$ $x = \zeta e_{i_0}, \zeta \in \mathbf{C} \setminus \{\mathbf{0}\}$ and then $\bar{\beta}_{i_0}, = \langle e_{i_0}, v \rangle = \frac{1}{\zeta} \langle x, v \rangle = 0$ which is not possible since $T \in D_1$. Hence, if we write $\tilde{x} = -\frac{1}{\langle x, v \rangle} x$, then $u = (D - \lambda)\tilde{x}$ and $\langle \tilde{x}, v \rangle + l = 0$.

For the sufficiency part, we can assume that there exists $x \in H$ such that $u = (D-\lambda)x$ and $\langle x, v \rangle + l = 0$. Then $x \neq 0$ and $Tx = Dx + \langle x, v \rangle u = u + \lambda x - u = \lambda x$.

Finally, suppose (i) is valid and $\lambda \in A(D)$. Then $u = (D - \lambda I)x = (D - \lambda_{n_0})x$ for some $x \in H$, and so $\alpha_{n_0} = 0$ which contradicts that $T \in \mathbf{D_1}$. It follows that $\lambda \notin A(D)$ and the rest of the equivalence between (i) together with (ii) and (iii)-(v) is now obvious.

For $T = D + u \otimes v \in D_1$, the diagonal operator D and the rank-one operator are uniquely determined by T and so we can define the fuction $F_{\mathbf{T}}(z) = \langle (zI - D)^{\mathbf{-1}} u, v \rangle$, for $z \in \mathbf{C} \setminus \overline{A(D)}$. This function is clearly an analytic fuction and it can be written as a Borel series ([119]) :

$$f_T(z) = \sum_{n=1}^{\infty} \frac{\alpha_n \bar{\beta}_n}{z - \lambda_n}, \quad z \in \mathbf{C} \setminus \overline{A(D)} \tag{3}$$

Corollary (4.1.6)[101]: Assume $T = D + u \otimes v \in D_1$ and $\lambda \in C \setminus \overline{A(D)}$. Then λ is an eigenvalue for T if and only if $f_{\mathbf{T}}(\lambda) = 1$

Proof. Since $\lambda \in \mathbf{C} \setminus \overline{A(D)}$.part (i) in Proposition (4.1.5) is satisfied. Taking x = $(D-\lambda I)^{-1}$ u in part (ii) of Proposition (4.1.5) we obtain the corollary. The next corollary describes the spectrum of an operator $T \in D_2$.

Corollary (4.1.7)[101]: If $T = D + (u \otimes v) \in D_2$ then

$$\sigma(T) = A(D)' \cup \{ z \in \mathbf{C} \setminus \overline{A(D)}, f_T(z) = 1 \} \tag{4}$$

Where A(D) ′ denotes the derived set of A(D)

Proof. In general for an operator $A \in L(H)$, $\sigma(A) = \sigma_e(A) \cup \sigma_p(A) \cup \sigma_p(A^*)^*$, where if $\Delta \subset \mathbf{C}$, $\Delta^* = \{\bar{z} : z \in \Delta\}$ (cf. [120]). Since $T \in D_2$,we have $\sigma_e(T) = \sigma_e(D) = A(D)`$. and so by Corollary (4.1.6), one inclusion necessary to establish (4) follows. For the other inclusion, let us assume $\lambda \in \sigma(T) = \sigma_e(T) \cup \sigma_P(T) \cup \sigma_P(T^*)^*$ Since $\sigma_e(T) = A(D)'$, we can assume that $\lambda \notin \sigma_e(T)$. Suppose then that $\lambda \in \sigma_P(T)$. If $\lambda \in \sigma_P(T) \cap \overline{A(D)}$, by Proposition (4.1.5), $\lambda \notin A(D)$ and so $\lambda \in A(D)' = \sigma_e(T)$ which contradicts our assumption. It follows that $\lambda \in \sigma_p(T) \setminus \overline{A(D)}$ and so by Corollary (4.1.6) , $f_T(\lambda) = 1$. Since $f_T(z) = \overline{f_{T^*}(\bar{z})}$ for all $z \in \mathbf{C} \setminus \overline{A(D)}$, one takes care likewise of the case $\lambda \in \sigma p(T^*)^*$.

Example (4.1.8)[101]: ([121]) Let $T = \mathrm{Diag}(\lambda_n) + u \otimes u$ where $D = \mathrm{Diag}(\lambda_n)$ and u are constructed in the following way. First we consider a family of open disjoint (and non tangent) disks $\{D_n\}_{n \in \mathbf{N}}$ (D_n is centered at λ_n and has radius r_n) contained in the unit disk $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$ and such that the set $\mathbf{D} \setminus \bigcup_{n \in \mathbf{N}} \overline{D_n}$ has Lebesgue measure zero. Such a family can be constructed using an induction argument, covering at each step a closed set of whose measure is a fixed nonzero fraction of the measure of the open set uncovered by the disks constructed at previous steps. Moreover, one can refine the argument in order to satisfy the condition $\sum_{n \in \mathbf{N}} r_n < \infty$.

The diagonal operator D is defined by the sequence $\{\lambda_n\}$ constructed above and u is given as in (2) where $\alpha_n = r_n$, $n \in \mathbf{N}$. We want to compute the point spectrum of T. In order to do this let us observe that the essential spectrum of T is

$A(D)' = \overline{\mathbf{D}} \setminus U_{n\in\mathbf{N}} D_n$. Also we need the following formula which can be proved easily by a change of variables to polar coordinates:

$$\int_D \int_{(a,r)} \frac{dx\,dy}{z-(x+yi)} = \begin{cases} \dfrac{\pi r^2}{z-a} & \text{if } |z-a|<r, \\ \pi(\bar{z}-\bar{a}), & \text{if } |z-a| \le r, \end{cases} \tag{5}$$

for every a $\in$ **C** and r > 0. Then if z $\notin$ **D**, by (3) and (5), we have

$$f_T(z) = \sum_{k\in\mathbf{N}} \frac{r_k^2}{z-\lambda_k} = \frac{1}{\pi}\sum_{k\in\mathbf{N}} \int_{D(\lambda_k,r_k)}\int \frac{dx\,dy}{z-(x+yi)} = \frac{1}{\pi}\int\int_{\mathbf{D}} \frac{dx\,dy}{z-(x+yi)} = \frac{1}{z}.$$

Hence, by Proposition (4.1.5) T does not have any eigenvahes z$\in$**C**$\setminus\overline{\mathbf{D}}$ Let us suppose that $z\in\mathbf{D}\setminus U_{n\in\mathbf{N}}\overline{D}_n$ In this case if z were an eigenvalue for T then by Proposition (4.1.5), the sum $\sum_{k\in\mathbf{N}} \frac{r_k^2}{z-\lambda_k}$ would be absolutely convergent and it would be equal to 1. But using again (5), we have

$$\sum_{k\in\mathbf{N}} \frac{r_k^2}{z-\lambda_k} = \frac{1}{\pi}\sum_{k\in\mathbf{N}} \int_{D(\lambda_k,r_k)}\int \frac{dx\,dy}{z-(x+yi)} = \frac{1}{\pi}\int\int_{\mathbf{D}} \frac{dx\,dy}{z-(x+yi)} = \bar{z}$$

This implies that the only possible point which may be an eigenvalue is z = 1. In fact, under our hypothesis, z =1 is indeed an eigenvalue because $\sum_{k\in\mathbf{N}} \frac{r_k^2}{|1-\lambda_k|} < \sum_{n\in\mathbf{N}} r_n < \infty$

Suppose z $\in \overline{\mathbf{D_n}}\setminus\{\lambda_n\}$ for some n $\in$ **N** and let us assume that z is an eigenvalue for T. Then using (5) again we can compute

$$\sum_{k\in\mathbf{N}} \frac{r_k^2}{z-\lambda_k} = \frac{1}{\pi}\sum_{k\in\mathbf{N},k\neq\lambda_n} \int_{D(\lambda_k,r_k)}\int \frac{dx\,dy}{z-(x+yi)} + \frac{r_n^2}{z-\lambda_n} = \bar{z}-(\bar{z}-\bar{\lambda}_n) + \frac{r_n^2}{z-\lambda_n} = \bar{\lambda}_n + \frac{r_n^2}{z-\lambda_n}$$

This shows that $z = \lambda_n + \frac{r_n^2}{1-\lambda_n}$ is the only possible eigenvalue for T in this case. In fact, it is easy to see that these values are indeed eigenvalues for T. Hence, $\sigma_P(T)$ $= \{\lambda_n + \frac{r_n^2}{1-\lambda_n} : n\in\mathbf{N}\}\ U\ \{1\}$.

A natural question which arises at this point is whether or not there exist operators T $\in D_0$ with empty point spectrum . An example of such an operator was first constructed by J. G. Stampfli in [113], for the case when the spectrum of T is a square. Given an arbitrary nonempty compact subset of the plane K, it is interesting to know if there are examples of operators $T \in D_0$ with empty point spectrum and such that $\sigma(T) = K$. Next, we put together some information about the resolvent of operators T in D_2 around points which are isolated in A(D).

Proof. If $< A^{-1} u, v > +1 = 0$, then $u \neq 0$ and since $S(A^{-1} u) = 0$, it is clear that S is not invertible. On the other hand, if $\langle A^{-1} u, v \rangle +1 \neq 0$, then it is enough to cheek that (6) gives the inverse of S:

$$[A+(u\otimes v)][\ A^{-1}-\frac{1}{<A^{-1}u,v>+1}(A^{-1}u\otimes(A^*)^{-1}v)]=I+(u\otimes(A^*)^{-1}v)-$$

$$\frac{1}{\langle\ A^{-1}u,v\ \rangle+1}(u\otimes(A^*)^{-1}v)-\frac{\langle\ A^{-1}u,v\ \rangle}{\langle\ A^{-1}u,v\ \rangle+1}(u\otimes(A^*)^{-1}v)=I$$

The second part of the lemma clearly follows from the first part.

For $T = D + u \otimes v \in D_0$ we define the function $F_T(Z) = \langle (ZI - T)^{-1} u, v \rangle$ for $z \in \mathbf{C}\backslash\sigma(T)$. We have the following relation between the functions F_T and f_T.

Proposition (4.1.9)[101]: Assume $T = D+(u \otimes v) \in D_1$ Then for all $z \in C\backslash(\sigma(T)\cup\sigma(D))$ we have

$$F_T(z)=\frac{f_T(z)}{1-f_T(z)} \tag{6}$$

Moreover ,if $\zeta \in A(D)\backslash A(D)'(\zeta=\lambda_{n_0})$, then $F_T(\zeta) = -1, \frac{dF_T}{dx}(\zeta)=-\left(\alpha_{n_0}\bar{\beta}_{n_0}\right)^{-1}$,

and if $T\in D_2$ *we have*

$$(T-\zeta)^{-1}=\tilde{D}-\frac{1}{\alpha_{n_0}}\tilde{D}u\otimes e_{n_0}-\frac{1}{\bar{\beta}_{n_0}}e_{n_0}\otimes\tilde{D}^*v+ \tag{7}$$

$$(\alpha_{n_0}\bar{\beta}_{n_0})^{-1}(\sum_{k\neq n_0}\frac{\alpha_k\bar{\beta}_k}{\lambda_k-\zeta}+1)e_{n_0}\otimes e_{n_0},$$

where $\tilde{D}=\sum_{k\neq n_0}(\lambda_k-\zeta)^{-1}e_k\otimes e_k.$

Proof. Formula (8) can be easily derived from (7). Each $\zeta \in A(D)\backslash A(D)'$ is an isolated eigenvalue of multiplicity one for D, and hence by Theorem (4.1.4), $T - \zeta$ is invertible. We have $\zeta - D = \zeta - T + (u \otimes v)$ and then by Lemma (4.1.9), $\langle (\zeta - T)^{-1}$

u, v ⟩ +1 = 0, which proves that $F_T(\zeta) = -1$. To compute $\frac{dF_T}{dx}(\zeta)$ we differentiate (8) at a point z different of ζ and take the limit as $z \to \zeta$

$$\frac{dF_T}{dz}(\zeta) = \lim_{z\to\zeta} \frac{f_T'(z)}{(1-f_T(z))^2} = \lim_{z\to\zeta} \frac{(z-\zeta)^2 f_T'(z)}{[z-\zeta-(z-\zeta)f_T(z)]^2} = -\left(\alpha_{n_0}\bar{\beta}_{n_0}\right)^{-1}$$

The equality (9) follows from (7) by a similar argument of passing to the limit as $z \to \zeta$.

As an application to formula (9) we will show the equivalence of two interesting facts from the theory of self-adjoint compact operators. The first result appears in [114] (see also [104]) and the second result was proved independently by several authors (cf. [121], [122] and [123]).

Theorem (4.1.10): (i) Let $\{v_k\}_{k\in\mathbf{N}}$ and $\{\mu_k\}_{k\in\mathbf{N}}$ be two distinct monotone in- creasing sequences of real numbers, each having zero as the limit point. Further assume that $(\mu_k\}$ belongs to (v_k, v_{k+1}) for each $k \in \mathbf{N}$. Then if A is a self-adjoint compact operator on a separable Hilbert space H having the sequence v_k ($k \in \mathbf{N}$) as its eigenvalues (with multiplicity one), there exists a vector $x \in H$ such that

$A + x \otimes x$ has precisely the eigenvaIues $\{\mu_k\}_{k\in\mathbf{N}}$

(ii) Let $\{v_k\}_{k\in\mathbf{N}}$ and $\{\mu_k\}_{k\in\mathbf{N}}$ be two distinct monotone decreasing sequences of real numbers, each having zero as the limit point and such that $\{\mu_k\}$ belongs to (v_{k+1}, v_k) for each $k \in\mathbf{N}$. Then if A is a self-adjoint compact operator on a Hilbert space H having the eigenvalues v_k ($k \in \mathbf{N}$) (with multiplicity one), there exists a vector $y\in H$ such that if P denotes the orthogonal projection on the one-dimensional space spanned by the vector y, the compact operator $(I - P)A(I - P)_{|(I-P)(H)}$ has exactly as its eigenvalues the sequence $\{\mu_k\}_{k\in\mathbf{N}}$

Proof. For the implication (i) $\Rightarrow$(ii) we assume that $\{v_k\}_{k\in\mathbf{N}}$, $\{\mu_k\}_{k\in\mathbf{N}}$ and A are as in (ii) and let us take the diagonal operator D on H whose eigenvalues are $\{\lambda_k\}_{k\in\mathbf{N}}$ where $\lambda_1 = -1$, $\lambda_{k+1} = (1+\mu_k)^{-1} - 1$ for $k \in\mathbf{N}$. Then by (i) we can find x such that $T = D + x \otimes x$ has exactly the eigenvalues $\{(1 + v_k)^{-1} - 1\}_{k\in\mathbf{N}}$. We take $\zeta = \lambda_1$ and apply formula (7) for D, u = v = x and $n_0 = 1$. Let Q be the orthogonal projection on el.

We see that $(I - Q)(T - \zeta I)^{-1}(I - Q)|_{(I-Q)(H)}$ is a diagonal whose eigenvalues are precisely $\left\{\frac{1}{\mu_k - \zeta}\right\} k \geq 2 = \{1 + \mu_k\}_{k \in \mathbf{N}}$ Hence, by spectral mapping theorem the operator S $= (T - \zeta I)^{-1} - I$ is compact and has the eigenvalues $\{v_k\}_{k \in \mathbf{N}}$. Thus, we can find an unitary operator U such that $U^*SU = A$. To finish the proof we take $y = U^*e_1$ and observe that $(I - P)A(I - P) = U^*(I - Q)S(I-Q)U$, where P is the orthogonal projection on the one-dimensional space spanned by y. For the implication (ii)~(i), let $\{v_k\}_{k \in \mathbf{N}}$, $\{\mu_k\}_{k \in \mathbf{N}}$ and A be as in (i). Without loss of generality, we can assume that A is a diagonal operator with respect to the basis $\{e_k\}_{k \in \mathbf{N}}$ and $v_1 = -1$. Let B be an arbitrary compact operator on H which has $\{(\mu_k + 1)^{-1} - 1\}_{k \in \mathbf{N}}$ has its only eigenvalues (multiplicity one). Using (ii) we can find $y := y_1 \in H$ such that $(I-P)A(I-P)_{\backslash (I-p) H)}$ has precisely $\{(v_{k+1} + l)^{-1} - 1\}_{k \in \mathbf{N}}$ as its eigenvalues. Let $\{y_{k+1}\}_{k \in \mathbf{N}}$ be an orthonormal basis in $(I - P)(H)$ with respect to which $(I - P)A(I - P)_{\backslash (I-P)(H)}$ diagonalizes. Then the matrix of $B + I$ with respect to the basis $\{y_k\}_{k \in \mathbf{N}}$ looks exactly as the right hand side of (7) (for $D = A$, $\lambda_k = v_k$ ($k \in \mathbf{N}$), $\zeta = v_1$, $u = v$ and $e_{n_0} = 1$). We shall show that we can determine the coefficients of u such that these two matrices coincide (which will give a unitarily equivalence between the operators which admit this same representation matrix in different orthonormal basis). Let us write the representation of B as follows

$$B + I = b_1 e_1 \otimes y_1 + \sum_{k \geq 2} b_k y_1 \otimes y_k + \sum_{k \geq 2} \bar{b}_k y_k \otimes y_1 + \sum_{k \geq 2} (v_{k+1} + 1)^{-1} y_k \otimes y_k.$$

If we compare this with (7) we obtain that $\alpha_k = -\alpha_1(v_k + 1)b_k$, $(k \geq 2)$ and then

$$\frac{1}{|\alpha_1|^2} = b_1 - \sum_{k \geq 2} (v_k + 1)|b_k|^2 \qquad (8).$$

This will allow us to solve for α_1 if the right hand side of (8) is not zero. Suppose by way of contradiction that this is not true. Then a simple computation shows that $(B + I)z = 0$ where $z = y_1 - \sum_{k \geq 2} (v_k + 1)\overline{b_k}$ and so $B + I$ admits the value 0 as one of its eigenvalues but by our assumption the only eigenvalues of $B + I$ are the elements of the sequence $\{(\mu_k + 1)^{-1}\}_{k \in \mathbf{N}}$. This proves that we have a solution for $u \in H$ and so by spectral theorem $A + u \otimes u$ has precisely the eigenvalues $\{\mu_k\}_{k \in \mathbf{N}}$.

Proposition (4.1.11)[101]: Let $T = N + (u \otimes v) \in L(H)$ where N is a normal operator and u, v are nonzero vectors in H. Then T is a normal operator if and only if either

(i) u and v are linearly dependent and u is an eigenvector for $.\Im(\alpha N^*)$, where $\frac{\langle u, v\rangle}{\| v\|^2}$, *or*

(ii) u,v are linearly independent vectors and there exist $\alpha, \beta \in \mathbf{C}$ such that

$$(N^* - \bar{\alpha} I)u = \|u\|^2 \beta v \quad and \qquad (9)$$
$$(N - \alpha I)v = \|v\|^2 \bar{\beta} u,$$

where $R(\beta) = -1/2.$

Proof: We observe that the equation $T^*T = TT^*$ is equivalent to

$$N^*u \otimes v + v \otimes N^*u + \|u\|^2 V \otimes V = \qquad (10)$$
$$Nv \otimes u + u \otimes Nv + \|v\|^2 u \otimes u.$$

It is a simple computation to check that (10) is satisfied if (i) or (ii) is true.

Let us assume that T is a normal operator. We distinguish two distinct cases.

Case I: We assume that u, v are linearly dependent. Thus, there exists $\alpha \in C$ such that $u = \alpha v$ ($\alpha = \langle u,v\rangle/\|v\|^2$). Since $\|v\|^2 u \otimes u = |a|^2\|v\|^2 v \otimes v = \|u\|^2 v \otimes v$, if we write $\omega = (\alpha N^* - \bar{\alpha} N)v$ ($= 2.\Im(\alpha N^*)v$), (12) becomes $\omega \otimes v = -v \otimes \omega$. This last equality holds if and only if $\omega = itv$ for some $t \in \mathbf{R}$ and (i) is proved.

Case II: We assume that u, v are linearly independent vectors. From (10) we get that $\langle x, N^*u\rangle v = \langle x, Nv\rangle u$, $x \in (V\{u,v\})$

Hence $\langle N^*u, x\rangle = \langle Nv, x\rangle = 0$ for every $x \in (V\{u, v\})^{\perp}$. which means that

$$N^*u = a_{11}u + a_{12}v,\ Nv = a_{21}u - a_{22}v, \qquad (11)$$

for some $a_{ij} \in \mathbf{C}$. Substituting in (10) we obtain that the aij satisfy the following relations:

$$a_{11} = \bar{a}_{22},\ a_{12} + \bar{a}_{12} + \|u\|^2 = a_{21} + \bar{a}_{21} + \|v\|^2 = 0.$$

So, if we write $a_{11} = \bar{\alpha}$ and $a_{12} = -\|u\|^2/2 + is_1$, $a_{21} = -\|v\|^2/2 + is_2$, where $s_1, s_2 \in \mathbf{R}$, (13) implies that $(N^* - \bar{\alpha} I)u = (-\|u\|^2/2 + is_1)v$ and $(N - \alpha I)v = (-\|v\|^2/2 + is_2)u$. Thus $(N - \alpha I)^*(N - \alpha I)u = (-\|u\|^2/2 + is_1)(-\|u\|^2/2 + is_2)u$ which implies that $s_1/\|u\|^2 = -s_2/\|v\|^2$. if we write $t = 1/4 + (1/\|u\|^4)s_1^2$ and $\beta = -1/2 + \mathrm{sign}(s_1)\, i\sqrt{t-\frac{1}{4}}$ then clearly u and v satisfy (11). (Here, we used the notation sign for the real valued function defined by sign(x) = 1 if x > 0, sign(x) = -1 if x < 0 and sign(0) = 0.)

Corollary (4.1.12)[101]: $T = D + u \otimes v \in :D_1$ is normal if and only if either

(a) there exist $\alpha \in \mathbf{C}$ and $t \in \mathbf{R}$ such that A(D) lies on the line $\{z \in \mathbf{C} : \Im(\alpha\bar{z}) = t\}$, and $u = \alpha v$, or

(b) there exist $\alpha \in \mathbf{C}$ and $t \in \mathbf{R}$ such that A(D) lies on the circle $\{z \in \mathbf{C} : |z - \alpha| = t\}$, $t \in \mathbf{R}$, and

$$tu/\|u\| = e^{i\vartheta} (D - \alpha I)(v/\|v\|),$$

where $\vartheta \in [0, \pi)$ is determined by the equation $R(te^{i\vartheta}/\|u\|\|v\|) = -1/2$.

Proof. Suppose that (a) or (b) holds. Then either $\Im(\alpha D^*) = t\,I$ or $|D-\alpha I| = tI$. If (a) holds then (i) in Proposition (4.1.12) holds and hence T is a normal operator. If (b) holds then an easy computation shows that (9) holds for $\beta = te^{i\vartheta}/\|u\|\|v\|$. The two relations in (9) alone imply that (10) holds and so T is normal.

On the other hand if T is normal then, by Proposition (4.1.11), (i) or (ii) holds. In case (i) is true then $\Im(\alpha D^*)u = tu$ for some $t \in \mathbf{R}$. Thus $\Im(\alpha\bar{\lambda}_n\alpha_n) = t\alpha_n$ for all $n \in \mathbf{N}$ and since $a_n \neq 0$ for every n in $\mathbf{N}$ we obtain that A(D) is a subset of the line $\{z \in C: \Im(\alpha\bar{z}) = t\}$ and (a) follows. If (ii) holds, we get from (9) that $(D-\alpha I)^*(D-aI)v = \|u\|^2\|v\|^2 |\beta|^2 v$, and by a similar argument as above, we get that A(D) is a subset of the circle $\{z \in \mathbf{C}: |z- \alpha| = t\}$, where $t = \|u\|\|v\||\beta|$. Then, the other part of (b) follows easily from (9).

It is worth mentioning that actually if A(D) is a subset of a line or of a circle then $T = D + u \otimes v$ is a decomposable operator (cf. [124]). Moreover, T has the property (Triang_0) (cf.[119]), i.e., for any pair $S_1 \subset S_2$ of invariant subspaces for T such that $\dim(S_1/S_2) > 1$ there exists another invariant subspace S_3 of T verifying

$$S_1 \underset{\neq}{\subset} S_2 \underset{\neq}{\subset} S_3$$

Another interesting question about the class D_0 is whether we have the decomposability property for operators in D_0 whose spectrum is not necessarily an arc of an analytic curve. It is known ([124]) that every decomposable operator has the following property.

Definition (4.1.13)[101]: We say that an operator $T\in L(H)$ has the single valued extension property (notation: SVEP) if the only vector-valued analytic function $f : G \to H$, where G is an arbitrary open connected subset of **C**, which satisfies the equality

$$(T- z\,I)\, f(z) = 0, \qquad z \in G,$$

is the function identically equal to zero.

Proposition (4.1.14)[101]: Every operator $T: D+(u\otimes v)\in D_1$ for which the set $\mathbf{C}\setminus\overline{A(D)}$ is connected has the SVEP.

Proof. Let $f: G \to H$ be an analytic function such that $(T - z\,I)\, f(z) = 0$ for every $z \in G$. If $G\cap(\mathbf{C}\setminus \overline{A(D)} \neq \theta$ then by Corollary (4.1.6), $T - z\,I$ is invertible for all $z\in(G\setminus \overline{A(D)}))\setminus\{z \in\mathbf{C}\setminus \overline{A(D)},\ f_T(z) = 1\}$ and so $f(z) = 0$ for all $z \in (G\setminus\overline{A(D)})\setminus\{z \in\setminus\mathbf{C}\setminus A(D);\ f_T(z) = 1\}$.

The function f_T cannot be identically equal to 1 on the connected set $\mathbf{C}\setminus \overline{A(D)}$ because $\lim_{|z|\to\infty} f_T(z) = 0$. Hence the set $\{z;\ f_T(z) = 1\}$ is discrete and since G is connected it follows that f is identically zero.

We may assume that actually $G \subset \overline{A(D)}$. If we expand f in the basis $\{e_n\}$ as $\sum_{n=1}^{\infty} f_n e_n$, where $f_n : G \to C$ are scalar-valued analytic functions, we get

$$(\lambda_2 - z)\, f_n(z) + < f(z), v > \alpha_n = 0, \qquad z \in G,\ n\in\mathbf{N} \tag{12}$$

If we take $z = \lambda_n \in G \cap A(D)$ in the above equation, we obtain that $< f(\lambda_n), v > = 0$ for all $\lambda_n\in G \cap A(D)$. Since the set A(D) is dense in $\overline{A(D)}$ and $G \subset \overline{A(D)}$, the set $G \cap A(D)$ is clearly dense in G. Hence $< f(z), v >= 0$ for all $z \in G$. Thus (14) implies that for every integer $n \in \mathbf{N}$, $f_n(z) = 0$ for all $z \in G\setminus A(D)$. Since each f_n is a continuous function and $G\setminus A(D)$ is dense in G, it follows that f_n is identically equal to zero on G for every $n \in \mathbf{N}$ and so is f.

We consider the class $D_0(\mathbf{D})$ of the operators $T= D + u\otimes v\in D_0$ for which $A(D)\subset \overline{\mathbf{D}}$ We will characterize the contraction operators in $D_0(\mathbf{D})$. The following proposition provides one such characterization and leads us to Corollary (4.1.18) which gives a simple sufficient condition for an operator T∈ $D_0(\mathbf{D})\cap D_2$ to be a contraction.

Proposition (4.1.15)[101]: $T = D + u\otimes v \in D_0(\mathbf{D})$ is a contraction operator if and only if

$$\frac{|1-s\langle \tilde{u}(s), D\tilde{v}(s)\rangle|}{\|\tilde{u}(s)\|\,\|\tilde{v}(s)\|} > \sqrt{s}, \qquad s\in(0,1), \tag{13}$$

where $\tilde{u}(s)=(I-sD^*D)^{-\frac{1}{2}}u$ and $\tilde{v}(s)=(I-sD^*D)^{-\frac{1}{2}}v$, or equivalently, in case $T \in D_0(\mathbf{D})\cap D_2$, if and only if

$$\left|1-s\sum_{k=1}^{\infty}\frac{\alpha_k\overline{\beta_k\lambda_k}}{(1-s|\lambda_k|^2)}\right|^2 > s\left(\sum_{k=1}^{\infty}\frac{|\alpha_k|^2}{(1-s|\lambda_k|^2)}\right)\left(\sum_{k=1}^{\infty}\frac{|\beta_k|^2}{(1-s|\lambda_k|^2)}\right), \quad s\in(0,1) \tag{14}$$

Proof. Clearly T is a contraction if and only if T*T is a contraction. Since T*T is a positive self-adjoint operator, T*T is a contraction if and only if its spectrum is contained in the interval [0, 1]. A simple computation shows that

$$T^*T = D^*D + (D^*u + \|u\|^2 v)\otimes v + v\otimes D^*u.$$

Hence, $\sigma_e(T^*T) = \sigma_e(D^*D) \subset \sigma(D^*D) \subset [0,1]$ and so T*T ($\sigma(T^*T) = \sigma_e(T^*T)\cup\sigma_P(T^*T)$) has its spectrum contained in the interval [0, 1] if and only if its point spectrum does not intersect the interval $(1,\infty)$. We need the following lemma .

Lemma (4.1.16) [101] : Let $A \in L(H)$ be invertible and $S = A + (a\otimes b) + (c\otimes d)$ for some vectors $a, b, c, d \in H$. Then the following are equivalent :

(i) S is not invertible,

(ii) $\ker(S) \neq \theta$,

(ii)the determinant of the matrix $\begin{bmatrix} 1+<A^{-1}a,b> & <A^{-1}c,b> \\ <A^{-1}a,d> & 1+<A^{-1}c,d> \end{bmatrix}$ *is zero.*

Lemma (4.1.17)[101]: Let $A \in L(H)$ be an invertible operator, and let $S = A + (u\otimes v)$. Then S is invertible if and only if $< A^{-1}u, v > +1 \neq 0$, and its inverse is given by the formula

$$S^{-1} = A^{-1} - \frac{1}{\langle A^{-1}u, v\rangle + 1}(A^{-1}u\otimes(A^*)^{-1}v) \tag{15}$$

In particular, if $T = D + (u\otimes v) \in D_1$ and $f_T(\lambda)\neq 1$ for some $\lambda \in \mathbf{C}\backslash\overline{A(D)}$, we have

$$(\lambda - T)^{-1} =$$
$$(\lambda - D)^{-1} - (f_T(\lambda) - 1)^{-1}((\lambda - D)^{-1}u \otimes ((\lambda - D)^*)^{-1}v) \quad (16)$$

Proof. Since S = A (I + (A^{-1} a ⊗ b) + (A^{-1} c ⊗ d)), S is not invertible if and only if I + (A^{-1} a ⊗ b) + (A^{-1} c ⊗ d) is not invertible. Using the Fredholrn theory, this latter operator being Fredholrn of index zero, it is not invertible if and only if its kernel is not the (0) subspace. Hence (i) and (ii) are equivalent. For the equivalence of (ii) with (iii), let x ∈ H be a vector such that Sx = 0. This implies that x+ < x, b > A^{-1} a + < x, d > A^{-1} c = 0. Taking the inner product of this equation with b and d respectively, we get the following system of equations with the the unknowns < x, b > and < x, d >:

$$\begin{cases} (1+<A^{-1}a,b>)<x,b>+<A^{-1}c,b><x,d>=0 \\ <A^{-1}a,d><x,b>+(1+<A^{-1}c,d>)<x,d>=0. \end{cases}$$

Therefore if we assume that (ii) is true, then

$$x = -<x, b> A^{-1} a - <x, d> A^{-1} c \neq 0$$

and so at least one of the numbers < x, b > or < x, d > is not zero. This implies that the above homogeneous system has a nontrivial solution. This fact is equivalent with the statement (iii). Let us assume that (iii) is true. Then there is a nontrivial solution of the above homogeneous system of equations-say < x, b > = α and < x, d > = β. Hence x = – αA^{-1}a – β A^{-1}c is not the zero vector and a simple calculation shows that (I + (A^{-1}a ⊗ b) + (A^{-1} c ⊗ d)) x = 0 or Sx = 0.

We apply Lemma (4.1.17) for the case A = D*D – t I, a = D*u + $\|u\|^2$ v, b = c = v, and d = D*u, where t ∈ **R**, t > 1. Hence, T*T is a contraction if and only if the determinant of the matrix

$$\begin{bmatrix} 1+<(D^*D-tI)^{-1}(D^*u+\|u\|^2 v),v> & <(D^*D-tI)^{-1}v,v> \\ <(D^*D-tI)^{-1}(D^*u+\|u\|^2 v),D^*u> & 1+<(D^*D-tI)^{-1}v,D^*u> \end{bmatrix}$$

equals zero for no t ∈ (1, ∞). If we multiply the second column of this matrix by $\|u\|^2$ and subtract it from the first column, the determinant is the same as the determinant of the resulting matrix

$$\begin{bmatrix} 1+<(D^*D-tI)^{-1}D^*u,v> & <(D^*D-tI)^{-1}v,v> \\ <(D^*D-tI)^{-1}D^*u,D^*u>-\|u\|^2 & 1+<(D^*D-tI)^{-1}v,D^*u> \end{bmatrix}$$

The (2, 1) entry can be written differently as follows :

$$< (D^*D - tI)^{-1} D^*u, D^*u > - \|u\|^2 = < (D^*D - tI)^{-1} D^*Du, u > - \|u\|^2 =$$
$$< (D^*D - tI)^{-1}(D^*D - tI)u, u > - \|u\|^2 + t < (D^*D - tI)^{-1} u, u >$$
$$= t < (D^*D - tI)^{-1} u, u >.$$

If we observe that the (1, 1) entry is the complex conjugate of the (2, 2) entry, we obtain that T*T is a contraction operator if and only if the equation (in t)

$$|1+ < (D^*D - tI)^{-1} D^*u,v >|^2 - t < (D^*D - tI)^{-1} u, u > < (D^*D - tI)^{-1} v, v > = 0$$

has no solution in the interval $(1, \infty)$. Finally, if we change variables by setting $s = 1/t$, $s \in (0, 1)$, the above equation becomes

$$\frac{\left|1 - s < (I - sD^*D)^{-1} D^*u, v >\right|^2}{< (I - sD^*D)^{-1} u, u >< (I - sD^*D)^{-1} v, v >} = s,$$

which implies (15) since both members of the above equality are continuous functions of s and the sign of the inequality is determined when s = 0. The inequality (16) follows form (15) taking into account the explicit form of the operator D.

Corollary (4.1.18)[101]: Assume that for $T = D + (u \otimes v) \in D_0(ID) \cap D_2$ the coordinates of u and v satisfy the inequality

$$\left(\sum_{k=1}^{\infty} \frac{|\alpha_k|^2}{(1 - |\lambda_k|^2)}\right)\left(\sum_{k=1}^{\infty} \frac{|\beta_k|^2}{(1 - |\lambda_k|^2)}\right) \le 3 - 2\sqrt{2} \cong 0.171572876 \qquad (17)$$

Then T is a contraction operator.

Proof. Using Proposition (4.1.15) we get that T is a contraction operator if and only if

$$s \|\tilde{u}(s)\|^2 \|\tilde{v}(s)\|^2 <$$
$$1 - 2s \operatorname{Re} < \tilde{u}(s), D\tilde{v}(s) > + s^2 |< \tilde{u}(s), D\tilde{v}(s) >|^2, \qquad (18)$$

for every $s \in (0, 1)$. We observe that (18) is satisfied if $\|\tilde{u}(1)\|$ and $\|\tilde{v}(1)\|$ are finite numbers satisfying

$$\|\tilde{u}(1)\|^2 \tilde{v}(1)\|^2 + 2\|\tilde{u}(1)\| \, \|\tilde{v}(1)\| \le 1.$$

This last inequality is clearly satisfied if we have (17).

Corollary (4.1.19)[101]: Assume that $T = D + (u \otimes v) \in Do(ID) \cap D_2$ is a contraction operator. Then the following inequality holds for every $s \in (0, 1)$:

$$\left(\sum_{k=1}^{\infty}\frac{|\alpha_k|^2}{(1-s|\lambda_k|^2)}\right)\left(\sum_{k=1}^{\infty}\frac{|\beta_k|^2}{(1-s|\lambda_k|^2)}\right)<\frac{1}{s(1-\sqrt{s})^2} \tag{19}$$

Proof. If T is a contraction operator then we have (18), which implies that

$$s\|\widetilde{u}(s)\|^2\|\widetilde{v}(s)\|^2 < 1+2\,\|\widetilde{u}(s)\|\;\|\widetilde{v}(s)\| + s^2\|\widetilde{u}(s)\|^2\|\widetilde{v}(s)\|^2,\; s\in(0,1).$$

This last inequality is equivalent to (19) by simple computations.

If $A \in L(H)$ and $x \in H$ we write $C_x(A) = V_{n=0}^{\infty}\{A^n x\}$ A vector $x \in H$ is called cyclic for A if $G_x(A) = H$. The following proposition characterizes those operators $T = D + (u \otimes v) \in D_0$ for which $Lat(T) \cap Lat(D) \neq (0)$.

Proposition (4.1.20)[101]: If $T : D+ (u \otimes v) \in \mathbf{D_0}$ then D and $u \otimes v$ have a common n.i.s if and only if $C_{\mathbf{u}}(D) \neq H$ or $C_{\mathbf{v}}(D^*) \neq H$.

Proof. One can easily find all the invariant subspaces of $u\otimes v$. Namely, a subspace S is invariant for $u \otimes v$ if and only if $u \in S$ or $v \perp S$. Let us assume that S is a common n.i.s. for D and $u \otimes v$. If $u \in S$ we get that $C_{\mathbf{u}}(D) \neq H$ and if $v\perp S$, $S^{\perp}$ is nontrivial invariant for D^* containing v. Hence in this case $C_{\mathbf{v}}(D^*) \neq H$. This proves the necessity. For the sufficiency, we just have to observe that $C_{\mathbf{u}}(D)$ and $(C_v(D^*))^{\perp}$ are common invariant subspaces for D and $u \otimes v$.

The following proposition is a particular case of Bram's result [120] and answers the natural question whether an arbitrary diagonal operator admits a cyclic vector. For completeness we include here a simple proof of this fact which is a simplified version of the proof of Bram's result given in [120].

Proposition (4.1.21)[101]: Let $D=Diag(\{\lambda_{\mathbf{n}}\})\in L(H)$ such that every value in A(D) has multiplicity one. Then there exits a cyclic vector for D.

Proof. We consider the operator $M_{\mathbf{z}}$, the multiplication with the variable on $L^{\mathbf{2}}(X,\eta)$, where $X = \overline{A(D)}$ and $\eta = \sum_{n=1}^{\infty}\frac{1}{n^2}\delta_{\lambda_n}$. Define $V : H \to L^{\mathbf{2}}(X, \eta)$ by $Vx = f_x$ where $f_{\mathbf{x}}(z) = nx_n$ if $z=\lambda_n$ and zero otherwise, $x= x_{\mathbf{1}}e_{\mathbf{1}} + x_{\mathbf{2}}e_{\mathbf{2}} + ... \in H$. We have for each $x \in H$, $\|Vx\|^{\mathbf{2}}= \|f_x\|^{\mathbf{2}} = \int_X | f_x(z)|^{\mathbf{2}}\, d\eta(z) = \sum_{n=1}^{\infty}\frac{1}{n^2}|f_x(\lambda_n)|^2=\sum_{n=1}^{\infty}|x_n|^2=\|x\|^2$.

Clearly, V is an unitary operator and $VDV^{\mathbf{-1}} = M_z$, which implies that it suffices to show that M_z has a cyclic vector. For each n $\in$ **N**, denote $K_n = \{\lambda_1, \lambda_2, ..., \lambda_n\}$. Since all the eigenvalues λ_n are assumed to be distinct, the following system of linear equations has a unique solution in

$c_0, c_l, ..., c_n$:

$$\overline{\lambda}_j = c_0 + c_1\lambda_j + ... c_n \lambda_j^n, \quad j = 1, 2, ..., n. \tag{20}$$

Let $p_n(z) = c_0, c_l z + \ldots + c_n z^n$, where the coefficients $c_o, c_l, ..., c_n$ are satisfying (20). Using this notation, (20) can by written as $\bar{z} = p_n(z)$ on K_n. We now construct a Borel measure v on X with the following properties:

(i) v is a measure absolutely continuous with respect to η

(ii) $\frac{dv}{d\eta} := \phi$ is essentially bounded ([η]),

(iii) the function 1(z) = 1 is a cyclic vector for M_z acting on L^2 (X, v).

First we choose $\alpha_n = (\max_{l \le k \le n.} [\sup_{z \in X} |p_k(z)|^2)])^{-1}$ for each n $\in$ **N**, and let then $v = \delta_{\lambda_1} + \sum_{n=2}^{\infty} \frac{1}{n^2} a_{n-1} \delta_{\lambda_n}$. Clearly, $a_l \geq a_2 \geq ... \geq a_n, > 0$. It is easy to observe that (i) is satisfied, and in order to check the second property we take $\phi(z) = a_{n-1}$ if $z = \lambda_n$, $n \geq 2$, 1 if $z = \lambda_1$ and zero anywhere else. Hence, $0 \leq \phi(z) \leq \max\{a_1, 1\} = a_0$ for every $z \in X$. To check the third property, we want to show that p_n converges in L^2 (X, v) to the function

$z \to \bar{z}$:

$$\int_X |\bar{z} - p_n(z)|^2 dv(z) = \int_{X \setminus K_n} |\bar{z} - p_n(z)|^2 dv(z) \leq 2 \int_{X \setminus K_n} |\bar{z}|^2 dv(z) +$$

$$2 \int_{X \setminus K_n} |p_n(z)|^2 dv(z) \leq 2 \int_{x \setminus k_n} \varphi d\eta + 2\|p_n\|_\infty^2 \int_{X \setminus K_n} \varphi d\eta \leq$$

$$2a_0\eta(X \setminus K_n) + 2\|p_n\|_\infty^2 a_0\eta\ (X \setminus K_n) \leq\ (2a_0 + 1)\eta\ (X \setminus K_n) \to 0,$$

as n $\to \infty$ In other words, this means that the sequence of functions $z \to p_n(M_z)l(z)$ converges in $L^2(X, v)$ to the function $z \to \bar{z}$. From here, we obtain that for any polynomial q $\in \mathbf{C}$ |z| , the sequence $(qp_n)(M_z)$ 1 (.) converges to $z \to q(z)\bar{z}$. Thus, the function $z \to (\bar{z})^2$ is in $C_{1(.)}(M_z)$. Inductively, we can show that $z \to (\bar{z})^n \in C_{1(.)}(M_z)$ for every n $\in \mathbf{N}$. Finally, $p(z,\bar{z}) \in C_{1(.)}(M_z)$ for every polynomial in two variables $p(z,\bar{z})$,, and by Stone-Weierstrass theorem we get that any continuous function on X is in $C_{I(.)}(M_z)$. This shows that the property (c) holds.

Now, we want to show that $\phi^{1\backslash 2}$ is a cyclic vector for M_z acting on $L^2(X,\eta)$. If $f \in L^2(X, \eta)$ then clearly $\frac{f}{\phi^{1/2}}$ is in L^2.(X, v) and hence it can be approximated by a sequence of polynomials q_n in $L^2(X, u)$. Therefore,

$$\int_X | q_n(z)\phi^{1/2} - f(z)|^2 \, d\eta(z) = \int_X \phi(z) | q_n(z) - \frac{f(z)}{\phi^{1\backslash 2}(z)}|^2 \, d\eta(z) =$$

$$\int_X | q_n(z) - \frac{f(z)}{\phi^{1\backslash 2}(z)}|^2 \, dv(z) \to 0,$$

by our assumption. This proves that $q_n(M_z)\,\phi^{1\backslash 2}$ converges to f in $L^2(X, \eta)$ which finishes the proof.

Let us observe that if $T = D + (u \otimes v) \in D_2$ we have $\pi(T) = \pi(D)$, and hence, since $\pi(D)$ is normal in the Calkin algebra, we have that $\sigma_{l_e}(D) = \sigma_{r_e}(D)$ and consequently $\sigma_{l_e}(T)=\sigma_{r_e}(T)=\sigma_e(T)=\sigma_e(D)=A(D)'$. Hence, well-known reductions of the invariant subspace problem (see [124] for part (iii)) applied to our particular case and together with what we have proved so far give the following proposition.

Proposition (4.1.22)[101]: If $T=D + (u \otimes v) \in D_2$, and

(i) $\sigma(T) \neq A(D)'$ (equivalently $1 \in f_T^{-1}(C \setminus \overline{A})$), or

(ii) A(D)' is not connected, or

(iii) A(D)' is a singleton, or

(iv) u[resp. v] is not cyclic for D[resp. D*], then T has a n.h.s.

When one searches for invariant subspaces for an operator T it is useful to have a description of its commutant $\{T\}' := \{A \in L(H) : AT = TA\}$.

Proposition (4.1.23)[101]: Let T= D + (u ⊗ v) ∈ D_2, and A ∈ L(H) Then A ∈ $\{T\}'$ if and only if there exist a sequence of complex numbers $\{t_n\}_{n\in N}$ and a positive constant C such that

(i) for every square-summable sequence $\{\zeta_k\}_{k\geq 1}$ we have

$$\sum_n |\alpha_n|^2 \left| \sum_{k\geq 1, k\neq n}^{\infty} \zeta_k \bar{\beta}_k \gamma_{k,n} \right|^2 \leq C \sum_n |\zeta_n|^2, \tag{21}$$

where $\gamma_{k,n} := \dfrac{t_k - t_n}{\lambda_k - \lambda_n}$,   for k ≠ n, (k, n ∈ IN),

(ii) for every k ∈IN,

$$Ae_k = s_k e_k + \bar{\beta}_k \sum_{n\geq 1, n\neq k} \alpha_n \gamma_{k,n} e_n, \tag{22}$$

where the sequence defined bY

$$s_k = t_k - \sum_{n\geq 1, n\neq k} \alpha_n \bar{\beta}_n \gamma_{k,n}, \quad k \in \mathbf{N}, \tag{23}$$

is a bounded sequence.

Proof. The equality AT = TA can be written equivalently as

$$AD - DA = (u \otimes A^*v) - (Au \otimes v), \tag{24}$$

For the necessity part, let $\{t_k\}$ be defined by the equation $Au = \sum_{k=1}^{\infty} t_k \alpha_k e_k$. For every integer k ≥1, we have $< (AD - DA)e_k, e_k >= 0$ and then from (24) we obtain

$$< e_k, A^*v > < u, e_k > - < e_k, v > < Au, e_k > = 0,$$

which in turn implies that $< e_k, A^*v >= t_k \bar{\beta}_k$. Hence, using (24) again, we get

$$(\lambda_k - D)Ae_k = \bar{\beta}_k (t_k u - Au) = \bar{\beta}_k \sum_{n\geq 1, n\neq k}^{\infty} \alpha_n (t_n - t_k) e_n, \ k \geq 1, \tag{25}$$

which implies that we can express Ae_k as in (22). Taking the inner product of both sides of (22) with v, we obtain that s_k is given by (23). To obtain the inequality (21) we first need to observe that $s_k =< Ae_k, e_k >$ (by (22) and so $\{s_k\}$ is a bounded sequence. Thus, the inequality (21) follows easily from the boundedness of the operator $A - \tilde{D}$, where $\tilde{D}$ is the diagonal operator defined by $\tilde{D}e_k = s_k e_k$, k ∈ **N**.

For Sufficiency, we observe that the linear operator A defined by (22) is bounded because of (21) and the hypothesis that $\{s_k\}$ is bounded. Then from (22) and (23) we get that $< e_k, A^*v > = t_k\overline{\beta}_k$ and $Au = \sum_{k=1}^{\infty} t_k \alpha_k e_k$. Using these two relations and (22), we obtain (25) which is equivalent to (24).

Next we would like to combine Proposition (4.1.23) with Lomonosov's theorem (cf. [125]) to obtain sufficient conditions for existence of n.i.s, for operators in D_2. For this purpose we introduce some more notation. Let H(U) be the set of analytic functions on the open set U($\subset$ C). For a fixed w $\in$ U we define a linear transformations on H(U), $\psi \to \Gamma(\psi)(.,w), by$

$$\Gamma(\psi)(z,w) = \begin{cases} \dfrac{\psi(z)-\psi(w)}{z-\omega} & if\ z \neq w \\ \\ \varphi`(w) & if\ z = w, \end{cases} \qquad z \in U,\ \ \psi \in H(U) \tag{26}$$

For $T \in D_2$ given by (1), and U such that $\overline{A(D)} \subset U$ we define another linear transformation on H(U) by

$$B_T(\psi)(z) = \int_{A(D)} \Gamma(\psi)(z,w)dv(w), \qquad z \in U \cup A(D),\ \psi \in H(U), \tag{27}$$

where v is the atomic measure supported on A(D) given by $v = \sum_{n \geq 1} \alpha_n \overline{\beta}_n \delta_{\lambda n}$.

Theorem (4.1.24)[101]: Let T $\in$ D_2 given by (1) and B_T defined by (27). Suppose there exists a function $\psi \in$ H(U), with U $\supset$ $\overline{A(D)}$, such that $B_T\psi = \psi$ and ψ is not zero on $\overline{A(D)}$. Then T has a nontriviaI invariant subspace.

Proof. Let us consider $t_n = \psi(\lambda_n)$ n $\in$ **N**, and let A_ψ be the operator A defined as in (22) and (23). We will show that A_ψ satisfies (21) and it is a nonzero compact operator. By Proposition (4.1.23), T commutes with a nonzero compact operator and then using Lomonosov's theorem T admits a n.i.s.

Suppose that $A_\psi = 0$. Then, from the proof of Proposition (4.1.23), we have $A_\psi u = \sum_{n \in \mathbf{N}} \alpha_n t_n e_n$, and so $t_n = 0$ for all n $\in$ **N**. By Proposition (4.1.22) we can assume that $\sigma(T) = A(D)'$ and $A(D)'$ is connected. Thus we can consider $\widetilde{U}$ to be the connected component of U containing $A(D)'$. Hence, $\psi = 0$ on $\widetilde{U}$ since A(D) must have an accumulation point in $A(D)' \subset \widetilde{U}$ ($\widetilde{U}$ is connected). $\mathbf{C}\setminus\widetilde{U}$ cannot contain but finitely many points of A(D) where ψ must be zero because $t_n = 0$, $n\in\mathbf{N}$,. This contradicts our assumption on ψ and so A_ψ is not zero.

Since $\psi \in H(U)$ and $\overline{A(D)} \subset U$, there exists a constant $\mathbf{C_1} > 0$ such that $|\Gamma(\psi)(z,w)| \leq \mathbf{C}_1$ for all $z,w \in \overline{A(D)}$ and so, with the notation from Proposition (4.1.2) $|\gamma_{\mathbf{k,n}}| \leq \mathbf{C}_1$ for every $k, n \in \mathbf{N}$, $k \neq n$. Then, using Cauchy's inequality, we have

$$\sum_n |\alpha_n|^2 \left| \sum_{k \geq 1, k \neq n} \zeta_k \overline{\beta}_k \gamma_{k,n} \right|^2 \leq C_1^2 \sum_n \left|\alpha_n\right|^2 \sum_{k \geq 1, k \neq n} \left|\zeta_k\right|^2 \sum_{k \geq 1, k \neq n} \left|\beta_k\right|^2 \leq C \sum \left|\zeta_k\right|^2$$

where $C = C_1^2 \|u\|^2 \|v\|^2$. This proves that inequality (21) is satisfied. Also, the sequence defined by (23) is bounded since $\{t_n\}$ is clearly bounded and for every $k \in \mathbf{N}$

$$\left| \sum_{n\geq 1, n\neq k} \alpha_n \overline{\beta}_n \gamma_{k,n} \right| \leq C_1 \|u\| \|v\|.$$

Then, by Proposition (4.1.23) , A_ψ commutes with T. From (23), for every $k \in\mathbf{N}$ we have

$$s_k = \psi(\lambda_k) - \sum_{n\geq 1, n\neq k} \alpha_n \overline{\beta}_n \gamma_{k,n} = \psi(\lambda_k) - B_T(\psi)(\lambda_k) + \alpha_k \overline{\beta}_k \psi'(\lambda_k)$$

which simplifies to $s_k = \alpha_k \overline{\beta}_k \psi'(\lambda_k)$ because of our hypothesis on ψ. Clearly, $\lim_{k\to\infty} s_k = 0$ and so the diagonal operator $\widetilde{D}(\widetilde{D}e_k = s_k e_k, k \in \mathbf{N})$ is a compact operator. Since $A_\psi = \widetilde{D} + B$ where B is defined by

$$Be_k = \overline{\beta}_k \sum_{n\geq 1, n\neq k} \alpha_n \gamma_{k,n} e_n, \qquad k \in \mathbf{N},$$

it suffices to show that B is a compact operator. In fact, B is a Hilbert-Schmit operator since

$$\sum_{k\in N} \left\|Be_k\right\|^2 = \sum_{k\in N} | B_k |^2 \sum_{n\geq 1, n\neq k} | \alpha_k |^2 \, | \gamma_{k,n} |^2 < C,$$

which finishes our proof.

Corollary (4.1.25)[101]: Let $T \in D_2$ given by (1) such that $A(D) \in \overline{\cap}$. Suppose that f_T (cf. (3)) is bounded on **C****D** and let T_ϕ be the Toeplitz operator on $H^2(\mathbf{D})$ of symbol $\phi(\zeta) = \overline{f_T(\overline{\zeta})}$ for $\zeta \in \partial$ **D**. In addition we assume that the equation $T_\phi(\psi) = \psi$ has a solution $\psi \in H^2(\mathbf{D})$ which is analytic on an open set $U \supset (\overline{\cap})$ and not zero on $\overline{A(D)}$. Then there exists a n.i.s for T.

Proof. The assumption on f_T insures that ϕ is in $L^\infty(\partial\, \mathbf{D})$. and so the Toeplitz operator T_ϕ is well defined. Indeed, for $z \in$ **D** we have f_T $(\frac{1}{Z}) = z\sum_{n=1}^{\infty} \frac{\alpha_n \overline{\beta_n}}{1 - z\lambda_n} = \sum_{K=0}^{\infty} m_k z^{k+1}$, Where m_k are the moments of the measure v (

i.e., $m_k = \int_{\mathbf{D}} \zeta^k dv(\zeta), k \in \mathbf{N} \cup \{0\}$). So,

$z \to f_T(1/z)$ is a bounded analytic function on **D**, and thus $\phi \in L^\infty(\partial\, \mathbf{D})$ In fact, T_ϕ is a co-analytic Toeplitz operator. We want to show that B_T and T_ϕ act the same way on functions $\psi \in H^2(\mathbf{D})$ which are analytic on open neighborhoods of $\overline{\cap}$. Forsooth, if $\psi(z) = \sum_{k=0}^{\infty} a_k z^k \in H^2(\mathbf{D})$ is such a function, we have

$$T_\varphi(\psi)(e^{i\theta}) = P_{H^2}(\varphi(e^{i\theta})\psi(e^{i\theta})) = P_{H^2}(\sum_{k=0}^{\infty} m_k e^{-i(k+1)\theta} \sum_{l=0}^{\infty} a_l e^{il\theta})$$
$$= P_{H^2}(\sum_{k,l\ge 0} m_k a_l e^{i(l-k-1)\theta}) = \sum_{n=0}^{\infty}(\sum_{k=0}^{\infty} m_k a_{n+k+1}) e^{in\theta} in\ H^2(\mathbf{D}). \tag{28}$$

On the other hand, if $z \in \mathbf{D} \backslash A(D)$ we have

$$B_T(\psi)(z) = \sum_{k=1}^{\infty} \frac{\psi(z) - \psi(\lambda_k)}{z - \lambda_k} \alpha_k \overline{\beta}_k = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a_n (z^{n-1} + \dots + \lambda_k^{n-1}) \alpha_k \overline{\beta}_k =$$
$$\sum_{n=1}^{\infty} a_n (z^{n-1} + m_1 z^{m-2} + \dots + m_{n-1}) = \sum_{k=0}^{\infty}(\sum_{n=0}^{\infty} m_n a_{n+k+1}) z^k. \tag{29}$$

The assumptions on ψ allows one to do the computations in (28) and (29). Moreover, if $T_\phi(\psi) = \psi$, comparing (28) with (29) we have $B_T(\psi)(z) = (\psi)(z)$ for $z \in \mathbf{D}$, and so we can apply Theorem (4.1.24) to conclude the corollary.

Section (4.2): Normal Operators and Rank-one Perturbations:

Let H be a separable, infinite-dimensional, complex Hilbert space, and denote by L(H) the algebra of all bounded linear operators on H. For T in L(H), we write $\{T\}'$ for the commutant of T (i.e., for the algebra of all $S \in L(H)$ such that $TS = ST$) and $\{T\}'' = (\{T\}')'$ for the double commutant of T . As usual in what follows, **N, R, C,** and **T** will denote the sets of positive integers, real numbers, complex numbers, and complex numbers of modulus one, respectively.

We now choose an ordered orthonormal basis $\{e_n\}_{n\in \mathbf{N}}$ for H which will remain fixed throughout the section. If $A = \{\lambda_n\}_{n\in \mathbf{N}}$ is any bounded sequence in **C**, we write D_A for the normal operator in L(H) determined by the equations

$$DA\,(e_n) = \lambda_n e_n, \quad n \in \mathbf{N}. \tag{30}$$

This notation for $A = \{\lambda_n\}_{n\in \mathbf{N}}$ and D_A will also remain fixed throughout, as well the notation A' the derived set of A. By definition, we shall say that an operator T in L(H) is a rank-one perturbation of a diagonal normal operator if there exist nonzero vectors $u = \sum_{n\in \mathbf{N}} \alpha_n e_n$ and $v = \sum_{n\in \mathbf{N}} \beta_n e_n$ in H and a bounded sequence $A = \{\lambda_n\}_{n\in \mathbf{N}}$ in **C** such that T is unitarily equivalent to the operator $D_A + u \otimes v$, where, as usual, $u \otimes v$ is the operator of rank one defined by

$$(u \otimes v)(x) = \langle x, v\rangle\, u, \quad x \in H. \tag{31}$$

The notation $\{\alpha_n\}_{n\in \mathbf{N}}$ and $\{\beta_n\}_{n\in \mathbf{N}}$ for the Fourier coefficients of u and v, respectively, will also remain fixed throughout this section. There is a vast literature devoted to the study of this class of operators and its various subclasses (cf., e.g., the bibliography of [101]), but almost all of these studies are concerned with the special case in which the sequence Alies either on **R** or **T**. In fact, very little is known about the structure of operators $T = D_A + u \otimes v$ when no restriction is placed on the location of the eigenvalues λ_n of D_A, and one of the most annoying unsolved problems in operator theory (on Hilbert space) is the following.

(I) Does every rank-one perturbation $T = D_A + u \otimes v \in L(H)\backslash \mathbf{C}1_H$ of a diagonal normal oper ator D_A have a nontrivial invariant subspace (n.i.s.),or better yet,a nontrivial hyperinvariant subspace (n.h.s.)?

Despite the fact that Problem (I) is at least thirty years old (cf., for example, [129] where it is explicitly posed, but probably not for the first time), it has remained stubbornly intractable, although E. Ionascu [101] addressed the problem. It is thus natural to regard this section as a sequel to [101], some results from which we use below.

The purpose of this article is to provide a partial solution to Problem (I) by exhibiting a rather substantial subset of operators of the form $T = D_A + u \otimes v$ each of which has an n.h.s. More precisely, our main result is as follows.

Theorem (4.2.1)[128]: Let $T = D_A + u \otimes v$ be any rank-one perturbation of a diagonal normal operator Such that $T \notin \mathbf{C}1_H$ and $\sum_{n\in\mathbf{N}}(|\alpha_n|^{2/3} + |\beta_n|^{2/3}) < +\infty$.Then T has an n.h.s.

To prove this theorem, we first treat some rather easy cases and thereby reduce the proof of Theorem (4.2.1) to the derivation of the following technical result.

Theorem (4.2.2)[128]: With the notation as introduced above, suppose $T = D_A + u \otimes v$ is such that

(i) the map $n \to \lambda_n$ of $\mathbf{N}$ onto A is injective and A' is not a singleton,

(ii) for every $n \in \mathbf{N}$, $\alpha_n\beta_n \neq 0$, and

(iii) $\sum_{n\in\mathbf{N}}(|\alpha_n|^{2/3} + |\beta_n|^{2/3}) < +\infty$ (the on trivial assumption).

Then either

(i) there exists an idempotent F with $0 \neq F \neq I_H$ such that $F \in \{T\}''$, and consequently, T has a complemented n.h.s. (i.e., there exist n.h.s. M and N of T with $M \cap N = (0)$ and $M + N = H$), or

(ii) there exists an uncountable set $\{\mu: \mu \in P\}$ of eigenvalues of T and an associated family $\{u\mu\}_{\mu \in P}$ of linearly independent eigenvectors (with $Tu_\mu = \mu u_\mu$) such that $M = V_{\mu \in P} \{u_\mu\}$ is an n.h.s. for T and $H \ominus M$ is infinite-dimensional.

The techniques and results herein also allow us to show, in as equal [130] to this section, that the operators $T= D_A+u \otimes v$ satisfying (i)–(iii) above but not (II) are decomposable in the sense of [124].

We introduce some needed notation and set forth some known results from [101] bearing on Problem (I). The ideal of compact operators in L(H) will be denoted by **K** and the Calkin map $L(H) \rightarrow L(H)/\mathbf{K}$ by π. For T in L(H) we denote by $\sigma(T)$ the spectrum of T, by $\sigma_{le}(T)$ $[\sigma_{re}(T)]$ the left essential [right essential] spectrum of T, and

$$\sigma_e(T) = \sigma(\pi(T)) = \sigma_{le}(T) \cup \sigma_{re}(T), \qquad \sigma_{lre}(T) = \sigma_{le}(T) \cap \sigma_{re}(T).$$

Moreover, we write, as usual, $\sigma_p(T)$ for the point spectrum of T.

We first take note of some cases treated in [101].

Proposition (4.2.3)[128]: (See [101].) If $T = D_A + u \otimes v \in L(H)\backslash \mathbf{C}1_H$ and there exists $n_0 \in \mathbf{N}$ such that $\alpha_{n_0}\beta_{n_0} = 0$, then either $\lambda_{n_0} \in \sigma_p(T)$ or $\bar{\lambda}_{n_0} \in \sigma_p(T^*)$. Moreover, if there exist $m_0, n_0 \in \mathbf{N}$ with $m_0 \neq n_0$ such that $\lambda_{m_0} = \lambda_{n_0}$, then $\lambda_{n_0} \in \sigma_p(T)$. Finally, if A' is a singleton, then $\{T\}'$ contains a nonzero compact operator. Consequently, in all cases T has an n.h.s.

Thus in what follows we restrict our attention to the class (RO) consisting of all operators $T = D_A + u \otimes v$ in L(H) for which all coefficients α_n and β_n are nonzero, $A = \{\lambda_n\}_{n \in \mathbf{N}}$ is a one- to-one map of **N** into **C**, and A' is not a singleton. We remark that it follows easily that if $T_1 = D_{A1} + u_1 \otimes v_1$ and $T_2 = D_{A2} + u_2 \otimes v_2$ belong to (RO) with $T_1 = T_2$, then the sequences A_1 and A_2 coincide and $u_1 \otimes v_1 = u_2 \otimes v_2$ [2, Proposition 1.1]. It is also clear that for all $T = D_A + u \otimes v \in$ (RO), we have $\sigma_e(T) = \sigma_{lre}(T) = \sigma_{lre}(D_A) = A'$.

The following proposition gives very useful necessary and sufficient conditions that a number $\lambda \in \mathbf{C}$ belong to $\sigma_p(T)$.

Proposition (4.2.4)[128]: (See [2].) Let $T = D_A + u \otimes v \in (RO)$. Then a point $\mu \in \mathbf{C}$ is an eigenvalue of T if and only if

(i) $\mu \notin A$,

(ii) $\sum_{n \in \mathbf{N}} \frac{|\alpha_n|^2}{|\mu - \lambda_n|^2} < +\infty$ (which implies by the Schwarz inequality that

$\sum_{n \in \mathbf{N}} \frac{|\alpha_n \overline{\beta}_n|}{|\mu - \lambda_n|} < +\infty$, and

(iii) $\sum_{n \in \mathbf{N}} \frac{\alpha_n \overline{\beta}_n}{\mu - \lambda_n} = +1$

Moreover, if $\mu \in \sigma_p(T)$ [respectively $\bar{\mu} \in \sigma_p(T^*)$], then the eigenspace associated with μ [respectively $\bar{\mu}$] is spanned by the single vector $\sum_{n \in \mathbf{N}} (\frac{\alpha_n}{\mu - \lambda_n}) e_n$ [respectively

$$\sum_{n \in \mathbf{N}} (\frac{\beta_n}{\overline{\mu} - \lambda_n}) e_n],$$

and so is one-dimensional. Finally, $(A \backslash A') \cap \sigma(T) = \varnothing$ (i.e., all isolated points λ_n of the set Λ lie outside of $\sigma(T)$).

We observe that the last statement of Proposition (4.3.4) can be proved in two lines by noting that if λ_n is isolated in Λ, then $(D_A - \lambda_n)$ (and thus $(T - \lambda_n)$) is a Fredholm operator of index zero, and hence necessarily either $\lambda_n \in \sigma_p(T)$ (which is impossible by (a)) or $\lambda_n \in \mathbf{C} \backslash \sigma(T)$.

One might expect that an arbitrary T in (RO) would satisfy $\sigma_p(T) \cup \sigma_p(T^*) \neq \varnothing$ (and thus trivially have an n.h.s.), but that this is false has been known (in the case $D_A = D^*_A$) for at least fifty years (cf., e.g., [3]). Perhaps the first example of an operator $T \in (RO)$ such that A' has positive planar Lebesgue measure and $\sigma_p(T) = \varnothing$ was given by Stampfli [131].

Before turning to more serious business, there is one more easy case to dispose of by using the Riesz–Dunford functional calculus and elementary Fredholm theory.

Proposition (4.2.5)[128]: If $T = D_A + u \otimes v \in (RO)$ and either $\sigma_e(T)(= \sigma_{lre}(T) = A')$ is not connected or $\sigma(T) \neq \sigma_e(T)$, then either conclusion (I) or (II) of Theorem (4.2.2) obtains.

Proof. Suppose first that $\sigma_e(T)$ is not connected. Then, either (1) $\sigma(T)$ is not connected, in which case the well-known argument consisting of integrating the resolvent of T about a curve surrounding a separated part of $\sigma(T)$ produces an idempotent $0 \neq E \neq 1_H$ in $\{T\}''$, or (2) $\sigma(T)$ is connected, from which one deduces, since $\sigma_e(T)$ is not a singleton, that $\sigma(T)$ must fill at least one hole H in $\sigma_e(T)$, and (via the normality of D_A) H necessarily has associated Fredholm index zero. Thus every point $\mu \in H$ lies in $\sigma p(T)$ and $\bar{\mu} \in \sigma_p(T^*)$. It follows easily (see Proposition (4.2.9) where the needed notation is available) that conclusion (II) of Theorem (4.2.2) holds.

Now suppose that $\sigma_e(T)$ is connected but $\sigma(T) \neq \sigma_e(T)$. Then clearly either $\sigma(T)$ contains an isolated point, in which case $\{T\}''$ contains a nonzero idempotent as above, or $\sigma(T)$ is connected but fills at least one hole in $\sigma e(T)$, in which case (II) of Theorem (4.2.2) holds (again via Proposition (4.2.9).

Our first order of business is to delineate a class of operators of the form $T = D_A + u \otimes v$ with which we shall be concerned in the remainder of the section. In view of Proposition (4.2.5), to establish Theorems (4.2.1) and (4.2.2), it suffices to deal with those T in the subset $(RO)_1$ defined as follows.

Definition (4.2.6)[128]: Suppose $T = D_A + u \otimes v \in (RO) \subset L(H)$. If $\sigma(T) = \sigma_e(T)$ $(= A')$, $\sigma(T)$ is a (perfect) connected subset of $\mathbf{C}$, and the sequences $\{\alpha_n\}_{n\in\mathbf{N}}$ and $\{\beta_n\}_{n\in\mathbf{N}}$ satisfy

$$\Sigma_{n\in\mathbf{N}}|\alpha_n|^{2/3} < +\infty, \quad \Sigma_{n\in\mathbf{N}}|\beta_n|^{2/3} < +\infty, \tag{32}$$

then T will be said to belong to the class $(RO)_1$. Note that for

$$T \in (RO)_1,\ \sigma_p(T) \subset \sigma(T) = A'.$$

The development of the techniques and results that will eventually yield the remainder of the proof of Theorems (4.2.1) and (4.2.2) now begins.

Definition (4.2.7)[128]: For $T = D_A + u \otimes v$ in $(RO)_1$, we define $\gamma_n = \max\{|\alpha_n|, |\beta_n|\}$, $n \in \mathbf{N}$, and set

$$c_1^2 = \sum_{n\in\mathbf{N}} \gamma_n^{2/3} \quad (< +\infty). \tag{33}$$

Moreover, for $\zeta \in \mathbf{C}$ and $s > 0$, we define the open disc $D(\zeta, s)$ by

$$D(\zeta, s) := \{\lambda \in \mathbf{C}: |\lambda - \zeta| < s\},$$

and, in particular, we set, for every $r > 0$,

$$A_r := \bigcup_{n\in\mathbf{N}} D(\lambda_n, \gamma_n^{2/3} r). \qquad \Delta_r := \mathbf{C} \setminus A_r, \tag{34}$$

and

$$\Delta_0 := \bigcup_{r>0} \Delta_r.$$

Denoting planar Lebesgue measure on $\mathbf{C} = \mathbf{R}^2$ by m_2, we obtain that

$$m_2(A_r) \le \sum_{n\in\mathbf{N}} \pi \gamma_n^{4/3} r^2 = \pi r^2 \sum_{n\in\mathbf{N}} \gamma_n^{4/3}$$

Proposition (4.2.8)[128]: Suppose $T \in (RO)$ has the property that $\sigma_p(T) \cap \Delta_0$ is uncountable (which, of course, is true if $\sigma(T)$ fills a hole in $\sigma e(T)$). Then T satisfies conclusion (II) of Theorem (4.2.2)

Proof. Since $\sigma_p(T) \cap \Delta_0$ is uncountable, there exists $r_0 > 0$ such that $\sigma_p(T) \cap \Delta_{r0}$ is also uncountable, and thus contains a perfect set P. For $\mu \in P$, u_μ spans the eigenspace of T corresponding to μ (by Proposition (4.2.4), and since $\langle u_\mu, v\rangle = -1$, by taking complex conjugates we get $\langle \bar{v}_\mu, u\rangle = -1$. Thus by another application of Proposition (4.2.4), we see that $\bar{\mu} \in \sigma_p(T^*)$ and $\bar{v}_\mu$ spans the associated eigenspace. Partition P as $P = P_1 \cup P_2$, where P_1 is countably infinite and P_2 is uncountable, and set $M = \bigvee_{\mu\in P_2} \{u_\mu\}$. Note that since each one-dimensional space $\mathbf{C}u_\mu$ is an n.h.s. for T, so is M. Moreover, the computation

$$\mu_1\langle u_{\mu_1},\bar{v}_{\mu_2}\rangle=\langle Tu_{\mu_1},\bar{v}_{\mu_2}\rangle=\langle u_{\mu_1},T^*\bar{v}_{\mu_2}\rangle=\mu_2\langle u_{\mu_1},\bar{v}_{\mu_2}\rangle,$$

valid for all $\mu_1 \in P_1$, $\mu_2 \in P_2$, shows that $u_{\mu_1} \perp \bar{v}_{\mu_2}$ for all such μ_1, μ_2. Thus, for $\mu \in P_1$, $\bar{v}_\mu \in H\ominus M$, and since these $\bar{v}_\mu$ with $\mu \in P_1$ are linearly independent, we see that $H\ominus M$ is infinite-dimensional, and thus T does, indeed, satisfy (II) of Theorem (4.2.2).

Note that this result also completes the proof of Proposition (4.2.5) Because of the frequency with which notation such as $(D_A - \lambda 1_H)$ or $(D_A - \lambda 1_H)^{-1}$ occurs below, we shall henceforth simply use the slightly simplified notation $(D_A - \lambda)$ for $(D_A - \lambda 1_H)$, $(D_A - \lambda)^{-1}$ for $(D_A - \lambda 1_H)^{-1}$, etc., where the inverse maps make sense (as possibly unbounded, densely defined, linear transformations) whenever the respective maps are injective.

Lemma (4.2.9)[128]: Suppose $T = D_A + u \otimes v \in (RO)_1$ and $r > 0$ is fixed. Then for every $\lambda \in \Delta_r$, we have $u, v \in \operatorname{ran}(D_A - \lambda) \cap \operatorname{ran}(D_A^* - \bar{\lambda})$, the vectors

$$u_\lambda := (D_A - \lambda)^{-1}u, \qquad v_\lambda := (D_A - \lambda)^{-1}v,$$
$$\bar{u}_\lambda := (D_A^* - \bar{\lambda})^{-1}u, \qquad \bar{v}_\lambda := (D_A^* - \bar{\lambda})^{-1}v, \tag{35}$$

are nonzero and satisfy

$$\max\{\|u_\lambda\|, \|v_\lambda\|, \|\bar{u}_\lambda\|, \|\bar{v}_\lambda\| \le c_1/r, \quad \lambda\in\Delta_r \tag{36}$$

Proof. Calculations show that, providing the two series converge, we have

$$\|u_\lambda\|^2=\|\bar{u}_\lambda\|^2=\sum_{n\in\mathbf{N}}\frac{|\alpha_n|^2}{|\lambda-\lambda_n|^2}>0, \qquad \|v_\lambda\|^2=\|\bar{v}_\lambda\|^2=\sum_{n\in\mathbf{N}}\frac{|\beta_n|^2}{|\lambda-\lambda_n|^2}>0,$$

and the result thus follows immediately from the inequality

$$\sum_{n\in\mathbf{N}}\frac{\max\{|\alpha_n|^2|\beta_n|^2\}}{|\lambda-\lambda_n|^2}\le\sum_{n\in\mathbf{N}}\frac{\gamma_n^2}{r^2\gamma_n^{4/3}}={c_1^2}/{r^2}, \qquad \lambda\in\Delta_r$$

Lemma (4.2.10)[128]: With $T= D_A+u\otimes v\in(RO)_1, r> 0$ fixed, and u_λ, v_λ, $\bar{u}_\lambda$, $\bar{v}_\lambda$ as in Lemma (4.2.9), each of these four functions (of λ) is strongly continuous on Δ_r.

Consequently, functions of the form $\lambda \to \langle u_\lambda, \bar{v}_\lambda \rangle$ are also continuous on Δ_r.

Proof. The equality

$$\sum_{n \geq N} \frac{|\alpha_n|^2}{|\lambda - \lambda_n|^2} \leq \frac{1}{r^2} \sum_{n \geq N} \gamma_n{}^{2/3} \qquad N \in \mathbf{N},$$

shows that the partial sums $\sum_{n=1}^{N} \left(\frac{\alpha_n}{\lambda - \lambda_n} \right) e_n$ (which are clearly strongly continuous functions of λ on Δ_r) converge uniformly there to u_λ. This establishes the strong continuity of u_λ, and the arguments for the other functions are similar.

Definition (4.2.11)[128]: We write$(RO)_2$ for the set of all $T \in (RO)_1$ such that $\sigma_p(T) \cap \Delta_0$ is a countable set, and note that to complete the proofs of Theorems (4.3.1) and (4.3.2), it suffices to show that each $T \in (RO)_2$ has the appropriate properties.

Proposition (4.2.12)[128]: Suppose $T = D_A + u \otimes v \in (RO)_2, r > 0$ is fixed, u_λ, v_λ, $\bar{u}_\lambda$, and $\bar{v}_\lambda$ are as in Lemma (4.3.11), and we define

$$\varphi_\lambda := 1 + \langle u_\lambda, v \rangle = 1 + \langle (D_A - \lambda)^{-1} u, v \rangle, \qquad \lambda \in \Delta_r . \tag{37}$$

Then for every compact subset $K \subset \Delta_r$ such that $\phi\lambda$ does not vanish on K,

$$\lambda \in K, \quad u, v \in \operatorname{ran}(T - \lambda) \cap \operatorname{ran}(T^* - \bar{\lambda}),$$

each of the four function

$$u_\lambda^T := (T - \lambda)^{-1} u, \qquad v_\lambda^T := (T - \lambda)^{-1} v,$$
$$\bar{u}_\lambda^T := (T^* - \bar{\lambda})^{-1} u, \qquad \bar{v}_\lambda^T := (T^* - \bar{\lambda})^{-1} v,$$

is strongly continuous on K (where here again, the linear transformations $(T - \lambda)^{-1}$ and $(T^* - \bar{\lambda})^{-1}$ are possibly unbounded but densely defined), and there exists $\varepsilon_{\mathbf{K,r}} > 0$ such that

$$|\varphi_\lambda| \geq \varepsilon_{K,r} \qquad \|u_\lambda^T\|, \|v_\lambda^T\|, \|\bar{u}_\lambda^T\|, \|\bar{v}_\lambda^T\| \leq c_1 / r\varepsilon_{K,r}, \qquad \lambda \in K, \tag{38}$$

Proof. We treat only the case of u_λ^T ; the arguments for the other three functions are similar. Clearly

$$(T - \lambda) u_\lambda = (D_A - \lambda) u_\lambda + \langle u_\lambda, v \rangle u = \varphi_\lambda u, \qquad \lambda \in \Delta_r , \tag{39}$$

and we know from Proposition (4.2.12) that φ_λ is continuous on Δ_r . Since φ_λ does not vanish on K, there exist $0 < \varepsilon_{K,r} < M_{K,r} < \infty$ such that $\varepsilon_{K,r} \le |\varphi_\lambda| \le M_{K,r}$ on K. Moreover, (39) yields

$$u_\lambda^T = (T-\lambda)^{-1} u = (1/\phi_\lambda) u_\lambda, \qquad \lambda \in K, \tag{40}$$

which shows, via the continuity of φ_λ^{-1} and strong continuity of u_λ on K, that u_λ^T is strongly continuous there and also, via (45), that

$$\| u_\lambda^T \| \le c_1 / r\varepsilon_{K,r} \text{ for all } \lambda \in K.$$

The following result is established by some calculations closely resembling those in Lemmas (4.2.9), (4.3.10), and Proposition (4.2.12), so we only sketch the proof.

Definition (4.2.13)[128]: For $T = D_A + u \otimes v \in (RO)_2$ and $\mathbf{r} > 0$ fixed, we write $D_A = \int \lambda dE$ (so E is the spectral measure of D_A), and for every $x \in H$, we define the extended real number $c_x \in [0, +\infty]$, by

$$c_x^2 := \sum_{n \in \mathbf{N}} \left(\left| \langle x, e_n \rangle \right|^2 \Big/ \gamma_n^{4/3} \right)$$

and the set $L \subset H$ as

$$L := \{ x \in H : \ c_x < +\infty \}. \tag{41}$$

Theorem (4.2.14)[128]: For $T \in (RO)_2$ and $\mathbf{r} > 0$ fixed, the set L in (50) is a dense linear manifold in H, invariant under D_A, D^*_A , T , T*, $(D^*_A - \lambda)^{-1}$, and $(D^*_A - \bar{\lambda})^{-1}$ for every $\lambda \in \Delta_\mathbf{r}$. Moreover L contains u, v, the basis vectors $\{e_n\}_{n \in \mathbf{N}}$, and is invariant under every value of E. Furthermore, for every compact subset $K \subset \Delta_\mathbf{r}$ on which $\varphi\lambda$ does not vanish,

$$L \subset \bigcap_{\lambda \in K} (\operatorname{ran}(D_A - \lambda) \cap \operatorname{ran}(T - \lambda) \cap \operatorname{ran}(D_A^* - \bar{\lambda}) \cap \operatorname{ran}(T^* - \bar{\lambda})),$$

and, upon defining, for each $x \in L$ and $\lambda \in \Delta_\mathbf{r}$,

$$\begin{aligned} x_\lambda &:= (D_A - \lambda)^{-1} x, & \bar{x}_\lambda &:= (D_A^* - \bar{\lambda})^{-1} x, \\ x_\lambda^T &:= (T - \lambda)^{-1} x, & \bar{x}_\lambda^T &:= (T^* - \bar{\lambda})^{-1} x, \end{aligned} \tag{42}$$

we obtain, for all $x \in L$ and $\lambda \in K$, that the four functions in (51) take values in L, that

$$x_\lambda^T := x_\lambda - \langle x, \bar{v}_\lambda \rangle u_\lambda^T = x_\lambda - \phi_\lambda^{-1} \langle x, \bar{v}_\lambda \rangle u_\lambda, \qquad \lambda \in K \tag{43}$$

where φ_λ is as in (37), that

$$\|x_\lambda\|, \|\bar{x}_\lambda\| \le c_x / r, \qquad \lambda \in \Delta_r, \tag{44}$$

and that

$$\|x_\lambda^T\|, \|\bar{x}_\lambda^T\| \le (c_x / r) + (c_1 / r)^2 (\|x\| / \varepsilon_{K,r}) \qquad \lambda \in \Delta_r, \tag{45}$$

where $\varepsilon_{K,r}$ is a lower bound on $|\varphi_\lambda|$ on K. Finally, for every $x \in L$, each of the functions appearing in (42) is bounded and weakly continuous on K.

Sketch of proof : It is clear that L is a linear manifold invariant under every value of E, and that $L^{-}=H$ follows because every $x \in H$ with only finitely many nonzero Fourier coefficients $\langle x, e_n \rangle$ belongs to L. Thus $\{e_n\}_{n \in \mathbf{N}} \subset L$ and earlier calculations showed that $u, v \in L$. For each $x \in L$, we calculate

$$\begin{aligned} \|x_\lambda\|^2 &= \|(D_A - \lambda)^{-1} x\|^2 = \|\bar{x}_\lambda\|^2 = \|(D_A^* - \bar{\lambda})^{-1} x\|^2 \\ &= \sum_{n \in \mathbf{N}} \frac{|\langle x, e_n \rangle|^2}{|\lambda - \lambda_n|^2} \le \sum_{n \in \mathbf{N}} \frac{|\langle x, e_n \rangle|^2}{r^2 \gamma_n^{4/3}} = \frac{c_x^2}{r^2} < +\infty, \qquad \lambda \in \Delta_r \end{aligned} \tag{46}$$

so $x \in \operatorname{dom}(D_A - \lambda)^{-1}$, L is invariant under $(D_A - \lambda)^{-1}$ and $(D_A^* - \bar{\lambda})^{-1}$ (for $\lambda \in \Delta_r$), $x \in \operatorname{ran}(D_A - \lambda) \cap \operatorname{ran}(D_A^* - \bar{\lambda})$, and $\|x_\lambda\|$, $\|\bar{x}_\lambda\|$ are bounded by c_x / r, as desired. Moreover, for $\lambda \in K$ almost the same calculation shows that L is invariant under D_A , D_A^* ,T, and T *, and the weak continuity on K of the four functions in (42) is established by an argument like that in Lemma (4.2.9). Next, (43) is verified by a calculation similar to (40). Then, by (43) and what has already been shown, for $\lambda \in K$, L is also invariant under $(T - \lambda)^{-1}$ and $(T^* - \bar{\lambda})^{-1}$. Finally, (46) follows from (43), (46), (36), and (3113), where $\varepsilon_{K,r}$ is as in (38).

We shall need one additional easy lemma.

Lemma (4.2.15)[128]: Let $T = D_A + u \otimes v \in (RO)_2$, $A \in \{T\}'$, $r > 0$ be fixed, and let $\emptyset \ne K \subset \Delta_r$ be a compact subset on which φ does not vanish. Then for every $x \in L$

and $\lambda \in K$, $(T-\lambda)^{-1}Ax$ is well-defined and $(T-\lambda)^{-1}Ax = A(T-\lambda)^{-1}x$. Consequently, $(T-\lambda)^{-1}Ax$ is bounded and weakly continuous on K.

Proof. We know from Theorem (4.2.14) that for $x \in L$, and $\lambda \in K$, $x_\lambda^T = (T-\lambda)^{-1}x$ is well-defined, bounded, and weakly continuous on K, and therefore so is $A(x_\lambda^T) = A(T-\lambda)^{-1}x$. Moreover, since $A(\mathrm{ran}(T-\lambda)) \subset \mathrm{ran}(T-\lambda)$ for every $\lambda \in K$, $(T-\lambda)^{-1}Ax$ is also well-defined, and a trivial calculation shows that

$$(Ax)_\lambda^T := (T-\lambda)^{-1}Ax = A(T-\lambda)^{-1}x = A(x_\lambda^T), \quad x \in L,\ \lambda \in K. \tag{47}$$

We are now almost prepared to write down some integrals that will be needed to complete the proof of Theorems (4.2.1)and (4.2.2) (for an arbitrary T in $(RO)_2$). We will use without further comment the notation, definitions, and results of this section, and we shall need some additional notation and a standing convention. Recall that if $\Gamma \subset \mathbf{C}$ is a simple, closed Jordan curve in $\mathbf{C}$, then according to the Jordan curve theorem, $\mathbf{C}\backslash\Gamma$ consists of exactly two disjoint open connected sets which we shall denote by $\mathrm{Int}(\Gamma)$ and $\mathrm{Ext}(\Gamma)$, with $\mathrm{Ext}(\Gamma)$ being unbounded.

Standing Conventions (4.2.16)[128]: Thus far, for $T = D_A + u \otimes v \in (RO)_2$, no assumption has been made concerning the size of $\|T\|$ or the location of $\sigma(T)$, and the significance of this work is that none is needed. Nevertheless, to simplify the notation in the plane geometry to be undertaken below, it will be convenient in what follows to, first, recall that $\sigma(T) = A'$ is a perfect connected set, and thus has diameter $d > 0$, and then to replace D_Λ and T by ζD_Λ and ζT for a suitable $\zeta \in \mathbf{C}\backslash(0)$ (which will have no effect on the validity of Theorems (4.2.1) and (4.2.2) or any other result to follow), so that every $T \in (RO)_2$ under consideration satisfies the following standing conventions: $\|D_A\|$, $\|T\| < 1$,

$$-1 < a := \min\{\mathrm{Re}(\lambda): \lambda \in \sigma(T)\} < b := \max\{\mathrm{Re}(\lambda): \lambda \in \sigma(T)\} < 1,$$

and all $r>0$ under consideration satisfy $r \in (0, \min\{1-\|T\|, (b-a)/(4c_1^2)\})$. Note that these conventions ensure that $A_r \subset \mathbf{D}$. Moreover, we write $\rho(T) := \mathbf{C}\backslash\sigma(T)$, the resolvent set of T, and since

$$\lim_{|\lambda|\to\infty}\left\|(T-\lambda)^{-1}\right\| = 0,$$

one knows that the function $\lambda \to (T-\lambda)^{-1}$ is analytic and norm-continuous on $\rho(T)$ and bounded in **C****D**. Also we shall denote by P the projection of $\mathbf{C} = \mathbf{R}^2$ onto $\mathbf{R} \subset \mathbf{C}$. For $r > 0$ fixed, it follows immediately from the definition of the set A_r in (4334), the connectedness of $\sigma(T) = A'$, and the standing conventions, that $P(\sigma(T)) = [a, b]$ and that $P(A_r)$ is a union of open subintervals of **R** of total length at most $2r\sum_{n\in\mathbf{N}}\gamma_n^{2/3}\ (= 2rc_1^2)$. Therefore

$$\Pi_r := (a, b) \setminus [P(A_r) \cup (\sigma_P(T) \cap \Delta_0)]$$

and Π_r has (linear, Lebesgue) measure larger than $(b-a)(1-r c_1^2) > (b-a)/2$ (since $\sigma_P(T) \cap \Delta_0$ is a countable (perhaps void) set). We note that an important and needed property of Π_r is that for every $s \in \Pi_r$, the vertical line $x = s$ lies entirely in Δ_r. We also will use the fact that the subset Π_r' consisting of all points of Π_r with Lebesgue density 1 has the same linear measure as does Π_r. Consequently, Π_r' is dense in Π_r, and for each $s \in \Pi_r'$, there exist monotone sequences $\{s_n^-\}_{n\in\mathbf{N}}$ and $\{s_n^+\}_{n\in\mathbf{N}}$ in Π_r, with $a < s_n^- < s < s_n^+ < b$, such that $s_n^- \nearrow s$ and $s_n^+ \searrow s$.

The following result, whose proof is long and is, in particular, given in a sequence of five steps, implies (what remains to be proved to establish) Theorems (4.2.1) and (4.2.2).

Theorem (4.2.17)[128]: Let $T = D_A + u \otimes v \in (RO)_2$. Then, with T and $r > 0$ as in the standing conventions, for every $s \in \Pi_r'$, there exist two nonzero idempotents $F_j^s \in \{T\}''$, $j = 1, 2$, such that $F_1^s + F_2^s = 1_H$. Furthermore, for all $s, s' \in \Pi_r'$ with $s \neq s'$, and for $j = 1, 2$, $F_j^s \neq F_j^{s'}$.

Proof. The proof will be given in several steps.

Step I: Since T satisfies the standing conventions, we have $\sigma(T) \cup \sigma(D_A) = A' \cup A \subset \mathbf{D}$. We fix an arbitrary $s \in \pi_r' \subset (a, b)$, so the vertical line segment $l_s \subset \mathbf{D}^-$ defined by

$$l_s = \{s + it : -(1 - s^2)^{1/2} \le t \le (1 - s^2)^{1/2}\}$$

lies entirely in $\Delta_r \cap \mathbf{D}^-$ and has endpoints on $\mathbf{T}$.

We next construct two positively oriented, piecewise smooth, simple closed, Jordan curves $\Gamma_1^s, \Gamma_2^s \subset \mathbf{T} \cup l_s$ as follows. Let $\Gamma_j^S, j = 1,2$ consist of the line segment l_s together with an arc a_j^S of $\mathbf{T}$ (each properly oriented), where

$$a_1^S = \{e^{i\theta} \in \mathbf{T} : \mathrm{Re}(e^{i\theta}) \le s\}, \quad a_2^S = \{e^{i\theta} \in \mathbf{T} : s \le \mathrm{Re}(e^{i\theta})\}.$$

Note that both Γ_1^S and Γ_2^S contain l_s (with opposite orientations) as a subarc and are compact sets. Thus $\mathbf{T} = a_1^S \cup a_2^S \subset \rho(T) \cap \rho(D_A)$, so the resolvents $R_\lambda(T) = (\lambda - T)^{-1}$ and $R_\lambda(D_A)$ are analytic in a neighborhood of $\mathbf{T} = a_1^S \cup a_2^S$. Since $l_s \cup \mathbf{T}$ (= $\Gamma_1^S \cup \Gamma_2^S$) is a compact set on which φ_λ does not vanish, we see that for every $x \in L$ (the dense linear manifold of Theorem (4.2.14)), the functions x_λ, $\bar{x}_\lambda$, x_λ^T, $\bar{x}_\lambda^T$ and $\langle x, \bar{v}_\lambda \rangle u_\lambda^T$ from Theorem (4.2.14), as well as all functions $(Ax)_\lambda^T$ as in (56) where $A \in \{T\}'$, are bounded and weakly continuous on $l_s \cup \mathbf{T}$. Therefore, these functions are weakly measurable and (since H is separable) strongly measurable on $\Gamma_1^S \cup \Gamma_2^S$. Consequently, the vector-valued integrals

$$E_j^s x := \frac{1}{2\pi i} \int_{\Gamma_j^s} (\lambda - D_A)^{-1} x \, d\lambda \quad \left(= -\frac{1}{2\pi i} \int_{\Gamma_j^s} x_\lambda \, d\lambda \right), x \in L, j = 1,2, \tag{48}$$

and

$$F_j^S x := \frac{1}{2\pi i} \int_{\Gamma_j^s} (\lambda - T)^{-1} x \, d\lambda \quad \left(= -\frac{1}{2\pi i} \int_{\Gamma_j^s} x_\lambda^T \, d\lambda \right), x \in L, j = 1,2, \tag{49}$$

exist in the strong topology on H, and from (57), (58), and (52) we get

$$F_j^S x = E_j^S x + \frac{1}{2\pi i} \int_{\Gamma_j^s} \langle x, \bar{v}_\lambda \rangle u_\lambda^T d\lambda$$

$$= E_j^S x + \frac{1}{2\pi i} \int_{\Gamma_j^s} \phi_\lambda^{-1} \langle x, \bar{v}_\lambda \rangle u_\lambda \, d\lambda \quad x \in L,\ j = 1,2. \tag{50}$$

Moreover, with $D_A = \int \lambda \, dE$, as in Definition (4.2.13), so E is the (purely atomic) spectral measure of D_A, it is easy to check (for example, by computing the weak integrals $\langle E_j^s x, e_n \rangle$ for $x \in L$) that

$$F_j^S x = E(Int(\Gamma_j^S))x, \quad \left\| E_j^S x \right\| \le \|x\|, \qquad x \in L, \ j = 1,2, \tag{51}$$

and hence from (50), (51), (36), and (38), we obtain that

$$\left\| F_j^S x \right\| \le (1 + c_1^2 / (r^2 \varepsilon_{k,r})) \, \|x\|, \qquad x \in L, \ j = 1,2, \tag{52}$$

where $|\varphi_\lambda| \ge \varepsilon_{K,r}$ on $K = \Gamma_1^S \cup \Gamma_2^S$ as in Proposition (4.2.12). Since it is now obvious from (48) – (52) that E_j^s and F_j^s, j = 1, 2, are bounded linear transformations defined on L, we may extend them by continuity (without changing the notation) to be elements of L(H) (but the equalities (48)–(50) obtain only for $x \in L$), so

$$E_j^s = E(Int(\Gamma_j^s)), \qquad j = 1,2,$$

and since $A \subset Int(\Gamma_1^S) \cup Int(\Gamma_2^S)$ and $Int(\Gamma_1^S) \cap Int(\Gamma_2^S) = \phi$,

$$E_1^S + E_2^S = 1_H, \qquad E_1^S . E_2^S = 0. \tag{53}$$

Since, by Theorem (4.3.16) , $E_j^s L \subset L$, we also get from (62) that

$$L = E_1^S L + E_2^S L, \qquad E_1^S L \perp E_2^S L, \tag{54}$$

the direct sum of the indicated mutually orthogonal linear manifolds. Moreover, since for $x \in L$, in the integral $(F_1^s + F_2^s)x$ the integrations along l_S cancel one another, we get immediately that

$$F_1^S + F_2^S \, x = \frac{1}{2\pi i} \int_{\mathbf{T}} (\lambda - T)^{-1} x \, d\lambda, \ \ x \in L,$$

and since $\sigma(T) \subset \mathbf{D}$, we see that also, by the Riesz–Dunford functional calculus,

$$F_1^S + F_2^S = 1_H. \tag{55}$$

Therefore to show that F_1^s and F_2^s are idempotents, it clearly suffices to show that

$F_1^s \cdot F_2^s = F_2^s . F_1^s = 0.$

Step II: We expand the set of $x \in H$ for which (49) is valid, as follows.

Lemma (4.2.18)[128]: With $T \in (RO)_2$ and r and s fixed as in Theorem (4.2.17), let $L' \supset L$ denote the set of all x in H for which the function $x_\lambda^T = (T - \lambda)^{-1} x$ is well-defined, bounded, and weakly continuous on $\Gamma_1^s \cup \Gamma_2^s$. Then $A(x_\lambda^T) \in L'$ for every $x \in L$ and every $A \in \{T\}'$. Moreover,

$$F_j^S x = \frac{1}{2\pi i} \int_{\Gamma_j^s} (\lambda - T)^{-1} x d\lambda \quad x \in L',\ j = 1,2, \tag{56}$$

and

$$F_j^s A = A F_j^s, \quad A \in \{T\}', \qquad j = 1,2.$$

Proof. Obviously the hypotheses guarantee that the integral in (56) exists, so we fix $x_0 \in L'$ and, via the density of L in H, let $\{x_n\}_{n \in \mathbf{N}} \subset L$ be such that $\|x_n - x_0\| \longrightarrow 0$. Then from (49), we have

$$F_j^S x_n = \frac{1}{2\pi i} \int_{\Gamma_j^s} (\lambda - T)^{-1} x_n d\lambda, \quad n \in \mathbf{N}, \qquad j = 1,2,$$

and since $F_j^S \in L(H)$, clearly $\|F_j^s x_n - F_j^s x_0\| \to 0$ for j=1,2 (so the sequence $\{\| F_j^s x_n \|_{n \in \mathbf{N}}\}$ is bounded). Thus it suffices to show that

$$\langle F_j^S x_n, y \rangle \to \left\langle \frac{1}{2\pi i} \int_{\Gamma_j^s} (\lambda - T)^{-1} x_0 d\lambda, \lambda \right\rangle, \quad y \in L, \qquad j = 1,2.$$

But, for j = 1, 2, and $y \in L$,

$$\langle F_j^S x_n, y \rangle = \left\langle \frac{1}{2\pi i} \int_{\Gamma_j^s} (\lambda - T)^{-1} x_n d\lambda, y \right\rangle$$

$$= \frac{1}{2\pi i} \int_{\Gamma_j^s} \left\langle (\lambda - T)^{-1} x_n d\lambda, y \right\rangle d\lambda, \qquad n \in \{0\} \cup \mathbf{N},$$

since the integrals in question are limits of finite (Riemann–Stieltjes) sums, and moreover, the convergence

$$\int_{\Gamma_j^s} \left\langle (\lambda - T)^{-1} x_n, y \right\rangle d\lambda = -\int_{\Gamma_j^s} \left\langle x_n, \bar{y}_\lambda^T \right\rangle d\lambda \to -\int_{\Gamma_j^s} \left\langle x_0, \bar{y}_\lambda^T \right\rangle d\lambda$$

$$= \langle \int_{\Gamma_j^s} (\lambda - T)^{-1} x_0 \, d\lambda, y$$

now follows from the fact that the sequence of continuous functions $\{\langle x_n, \bar{y}_\lambda^T \rangle\}_{n\in\mathbf{N}}$ (on F_j^S) con- verges uniformly on F_j^S to $\langle x_0, \bar{y}_\lambda^T \rangle$. Next, note that by Lemma (4.2.15) $(=(Ax)_\lambda^T) \in L'$, and thus from (56) we obtain that

$$F_j^S Ax = \frac{1}{2\pi i}\int_{\Gamma_j^s} (\lambda - T)^{-1} Ax \, d\lambda = \frac{1}{2\pi i}\int_{\Gamma_j^s} A(\lambda - T)^{-1} x \, d\lambda$$
$$= AF_j^S x, \qquad x \in L,\ A \in \{T\}' \qquad j = 1,2, \tag{57}$$

so F_j^S commutes with $\{T\}'$ as desired.

Step III. We formulate the penultimate step of the proof as follows.

Lemma (4.2.19)[128]: With $T \in (RO)_2$ = and r, s fixed as in Theorem (4.2.17) we have that for j = 1, 2, and each fixed $\zeta \in \mathrm{Ext}(\Gamma_j^s)$ there exist operators $B_j^s(\zeta)$, $A_j^s(\zeta)$ in L(H) with $B_j^s(\zeta) \in \{D_A\}'$, $A_j^s(\zeta) \in \{T\}'$ such that

$$B_j^s(\zeta)(D_A - \zeta) = E_j^s(= E(Int(\Gamma_j^s))), \quad A_j^s(\zeta)(T - \zeta) = F_j^s, \zeta \in Ext(\Gamma_j^s), j = 1,2.$$

Moreover, for each x ∈ L and j = 1, 2,

$$B_j^s(\zeta)x = \frac{1}{2\pi i}\int_{\Gamma_j^s} (\zeta - \lambda)^{-1} x_\lambda d\lambda, \tag{58}$$

$$A_j^s(\zeta)x = \frac{1}{2\pi i}\int_{\Gamma_j^s} (\zeta - \lambda)^{-1} x_\lambda^T \, d\lambda, \tag{59}$$

and $B_j^s(\cdot)x$, $A_j^s(\cdot)x$: $\mathrm{Ext}(\Gamma_j^s) \to H$ are analytic (vector-valued) functions. Furthermore, F_j^s is an idempotent different from 0 and 1_H, M_j^s := ran(F_j^s) is a nontrivial hyperinvariant subspace for T, and

$$\sigma(T \mid M_j^s) \subset Int(\Gamma_j^s) \cup \Gamma_j^s \quad (= \mathbf{C} \setminus Ext(\Gamma_j^s)), j = 1,2. \tag{60}$$

Proof. We give the argument for j = 1; the other argument is essentially the same. Fix $\zeta \in \mathrm{Ext}(\Gamma_1^s)$. It is clear that the functions $\lambda \to (\zeta - \lambda)^{-1} x_\lambda$ and $\lambda \to (\zeta - \lambda)^{-1} x_\lambda^T$ are bounded and weakly continuous on Γ_1^s, so the integrals in (58) and (59) are well-defined for each x $\in$ L, and thus we define $B_1^s(\zeta)$ and $A_1^s(\zeta)$ on L by (58) and (59). (We note here that since $D_A = \int \lambda \, d\, E$, one could also define $B_1^s(\zeta)$ by using the functional calculus for the normal operator D_A, but we need both $B_1^s(\zeta)$ and $A_1^s(\zeta)$ to be written as line integrals so we can compare them later in the proof.) We shall first show that $B_1^s(\zeta)$ is bounded on L, and thus extends to an element of L(H), and then use this fact to show that $A_1^s(\zeta)$ is also bounded on L. First, since L = E_1^s L + E_2^s L and $(D_A - \zeta)\, E_1^s\, L \subset E_1^s\, L \subset L$, j = 1, 2 (via Theorem (4.2.14) and (51)), we compute, with x = $x_1 + x_2 \in E_1^s L + E_2^s L$ arbitrary in L,

$$
\begin{aligned}
B_j^s(\zeta)(D_A - \zeta)xk &= \frac{1}{2\pi i}\int_{\Gamma_j^s} (\zeta-\lambda)^{-1}(D_A-\lambda)^{-1}(D_A-\zeta)xk\, d\lambda,\\
&= \frac{1}{2\pi i}\int_{\Gamma_j^s} (\zeta-\lambda)^{-1}(D_A-\lambda+\lambda-\zeta)(D_A-\lambda)^{-1}xk\, d\lambda,\\
&= E_1^s xk - \frac{1}{2\pi i}\int (\lambda-\zeta)^{-1} xk\, d\lambda,\\
&= E_1^s xk - 0, \quad k = 1,2,\\
&= \begin{cases} x_1, & \text{if } k = 1,\\ 0, & \text{if } k = 2, \end{cases}
\end{aligned} \tag{61}
$$

where the next-to-last equality results because the function $\lambda \to (\lambda - \zeta)^{-1} x$ is analytic on a neighborhood of the simply connected region $\Gamma_1^s \cup \mathrm{Int}(\Gamma_1^s)$. Since $(D_A - \zeta)|\, E_1^s\, \mathrm{H}$ is clearly invertible

$$(D_A - \zeta)[\, B_1^s(\zeta) - (D_A|E_1^s - \zeta)^{-1} E_1^s\,]\, x = 0, \quad x \in L.$$

Therefore for all $\zeta \in \mathrm{Ext}(\Gamma_1^s)\backslash A$, we have

$$B_1(\zeta)x = (D_A|E_1^s - \zeta)^{-1} E_1^s\, x, \qquad x \in L.$$

But from (58) the left-hand side of this last equality is analytic (in ζ) on $\mathrm{Ext}(\Gamma_1^s)$, and obviously so is the right-hand side. Therefore that equality holds for all ζ in $\mathrm{Ext}(\Gamma_1^s)$. In particular, $B_1^s(\zeta)$ extends to a bounded operator in L(H) satisfying the same equation for each ζ in $\mathrm{Ext}(\Gamma_1^s)$ and every x in H.

We now show that As $(\zeta)|$ L is bounded on L, first by computing, using (43), (58), and (59):

$$A_j^s(\zeta)x = \frac{1}{2\pi i}\int_{\Gamma_j^s}(\zeta-\lambda)^{-1}xk\,d\lambda - \frac{1}{2\pi i}\int(\zeta-\lambda)^{-1}\phi_\lambda^{-1}(x,\bar{u}\lambda)u\lambda\,d\lambda,$$

$$= B_j^s(\zeta)x = \frac{1}{2\pi i}\int_{\Gamma_j^s}(\zeta-\lambda)^{-1-1-1}\phi_\lambda^{-1}(x,\bar{u}\lambda)u\lambda\,d\lambda, \qquad x \in L$$

Then, using (45), we obtain (with K = Γ_1^s)

$$\left\|A_1^s(\zeta)\right\| \le \{\left\|B_1^s(\zeta)\right\| + c_1^2/(r^2\varepsilon k, r\,\mathrm{dist}(\zeta,\Gamma_1^s))\}\|x\|, \qquad x \in L.$$

Thus $A_1^s(\zeta)$ is bounded on L, and extends by continuity to an operator in L(H). Recall that from (58) and (59) we also obtain that for x $\in$ L, the functions A_1^s , B_1^s: Ext(Γ_1^s) $\rightarrow$ H are analytic on Ext(Γ_1^s). Moreover, that A_1^s $(\zeta) \in \{T\}'$ is immediate from the computation

$$\begin{aligned} A_j^s(\zeta)Tx &= \frac{1}{2\pi i}\int_{\Gamma_j^s}(\zeta-\lambda)^{-1}(T-\zeta)^{-1}Tx\,d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_j^s}T(\zeta-\lambda)^{-1}x_\lambda^T\,d\lambda \\ &= TA_1^s(\zeta)x, \qquad x \in L, \zeta \in \mathrm{Ext}(\Gamma_1^s) \end{aligned} \tag{62}$$

which is valid since $T\,L \subset L \subset \bigcap_{\lambda\in K}$ ran(T $-\lambda$). Next, we calculate

$$\begin{aligned} A_j^s(\zeta)(T-\zeta)x &= \frac{1}{2\pi i}\int_{\Gamma_j^s}(\zeta-\lambda)^{-1}(T-\zeta)^{-1}x\,d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_j^s}(\zeta-\lambda)^{-1}(T-\lambda+\lambda-\zeta)(T-\lambda)^{-1}x\,d\lambda \\ &= F_1^s x - \frac{1}{2\pi i}\int_{\Gamma_1^s}(\lambda-\zeta)^{-1}x\,d\lambda \\ &= F_1^s x, \qquad x \in L, \zeta \in Ext(\Gamma_1^s), \end{aligned}$$

since the function $\lambda \rightarrow (\lambda-\zeta)^{-1}x$ is analytic in a neighborhood of the simply connected region $\Gamma_1^s \cup$ Int (Γ_1^s) Hence

$$(T-\zeta)\,A_1^s(\zeta) = A_1^s(\zeta)(T-\zeta) = F_1^s\,, \qquad \zeta \in \mathrm{Ext}\,(\Gamma_1^s), \tag{63}$$

and we observe that this (together with its counterpart for $j = 2$) shows that both F_1^s and F_2^s are nonzero. For instance, if $F_2^s = 0$, then $F_1^s = 1H$ and (62) give that $\sigma(T) \cap$ Ext(F_1^s) $=\emptyset$, which we know to be false since s $\in \Pi'_r$ and there exists a point $\lambda_0 \in \sigma$ (T) $\subset$ D such that s < P (λ_0)< b.

Step IV. In this step we show that $F_1^s F_2^s$ (which equals $F_1^s F_2^s$ by Lemma (4.2.18)) is the zero operator, which simultaneously shows that F_1^s and F_2^s are idempotents. To accomplish this, however, and also for use in the sequel [130] to obtain the decomposability of the operators in $(RO)_2$, we must introduce some additional machinery.

Since $T \in (RO)_2, T$ *has the single-valued-extension property* (SVEP); i.e., if $\emptyset \neq G \subset C$ is a connected open set and $w : G \to H$ is an analytic (vector-valued) function such that $(T - \lambda)\omega(\lambda) \equiv 0$ on G, then $w \equiv 0$ on G. (Indeed, if $G \cap \sigma(T) = \emptyset$, this is trivial.

Otherwise, let l be a vertical line with $l \cap G \neq \emptyset$ and $P(l) \in \Pi'_r$. Since $\sigma p(T) \cap \Pi_r' = \emptyset$, $w \equiv 0$ on $l \cap G$, which contains an open interval, and thus $w \equiv 0$ on G via the analyticity of w.) This makes it possible to define for every x in H, the local spectrum $\sigma T(x) \subset \sigma(T)$ of T at x to be the (compact) set $C \backslash \rho T(x)$, where $\rho_T(x)$, the local resolvent of T at x, is defined as the (open) set consisting of all $\lambda_0 \in C$ such that there exists an open neighborhood $N_{\lambda 0}(x)$ of λ_0 and an analytic function $x_{\lambda 0}$: $N_{\lambda 0}(x) \to H$ satisfying $(T - \lambda)x_{\lambda 0}(\lambda) \equiv x$ on $N_{\lambda 0}(x)$. The SVEP guarantees the uniqueness of $x_{\lambda 0}$, and therefore one has an analytic function $x_T(\lambda)$ defined on $\rho T(x)$ such that $(T - \lambda)_{xT}(\lambda) \equiv x$ on $\rho T(x)$. It is well known (cf. [124]) that $\sigma_T(x) = \emptyset$ if and only if $x = 0$ and also that $\sigma_T(Ax) \subset \sigma_T(x)$ for every $A \in \{T\}'$. In particular,

$$(F_j^s x) \subset \sigma_T(x) \subset \sigma(T), \qquad x \in H, \qquad j = 1,2, \tag{64}$$

and using Lemma (4.2.19) (see also (63)), we obtain

$$A_j^s(\zeta)(T-\zeta)x = (T-\zeta)A_j^s(\zeta)x = F_j^s x, \qquad \zeta \in Ext(\Gamma_j^s), x \in (\Gamma_j^s), x \in H, \quad j = 1,2$$

The analyticity of $A^S(\cdot)x$ on $Ext(\Gamma^S)$, together with the definition of local spectrum, gives

$$\sigma_T(F_j^s x) \subset l_s \cup Int\ (\Gamma_j^s), \qquad x \in H, \qquad j = 1,2, \tag{65}$$

and putting (64) and (65) together, we get

$$\sigma_T(F_j^s x) \subset \sigma_T(x) \cap (l_s \cup \mathrm{int}(\Gamma_j^s)), \qquad x \in H, \qquad j = 1,2, \tag{66}$$

To complete the argument that F_1^s and F_2^s are idempotents (for each $s \in \Pi_r'$), it is convenient now to fix s $\in \Pi_r'$ and introduce monoton sequences $\{s_n^-\}_{n\in N}$ and $\{s_n^+\}_{n\in N}$ in Π_r' such that $s_n^- \nearrow s$ and $s_n^+ \searrow s$. Since s was completely arbitrary in Π_r', all of the preceding results are valid for s Thus we obtain from (66) that

$$\begin{aligned}\sigma_T(F_1^s F_2^{S_n{}^+}x) &\subset \sigma_T(F_1^s F_2^{S_n{}^+}x)\cap(l_s\cup Int\ (\Gamma_1^s)),\\ &\subset (l_{S_n{}^+}\cup Int\ (\Gamma_2^{S_n{}^+})),\cap(l_s\cup Int\ (\Gamma_1^s))\\ &=\phi, x\in H, n\in N\end{aligned}$$

Hence, by what was said above, $F_1^S F_2^{s_n^+}=0$ for each n$\in$N, and to complete the argument that $F_1^S F_2^s=0$ we shall show that the sequence $\left\{F_2^{S_n^+}\right\}_{n\in N}=0$ converges to F_2^s in the weak operator topology (WOT). For this purpose we note, that $\tilde{K}=\Gamma_2^s\cup(\bigcup_{n\in N}\Gamma^{S_n^+})$ is a compact set, and thus (52) with K replaced by $\widetilde{K}$ gives that the sequence $F_2^{S_n^+}$ is uniformly bounded. Thus it suffices to show that

$$\left\langle (F_2^s F_2^{S_n^+})e_m\ ,e_k)\right\rangle \to 0, k, m\in N.$$

Next we use (50) and (51) to write

$$F_2^s e_m = E(Int(\Gamma_2^s))e_m + G_2^s e_m, \qquad m\in N,$$

Where

$$G_2^s e_m=\frac{1}{2\pi i}\int\varphi_\lambda^{-1}\left\langle (D_A-\lambda)^{-1}e_m, u\right\rangle u_\lambda\, d\lambda=\frac{\overline{\beta}_m}{2\pi i}\int\varphi_\lambda^{-1}(\lambda_m-\lambda)^{-1}u_\lambda d\lambda,$$

and similarly for $F_2^{S_n^+}e_m, n\in N$ Since it is obvious from the definitions of the Jordan loops $\Gamma_2^{S_n^+}$ that

$$\bigcup_{n\in N} Int\,(\Gamma_2^{S_n^+}) = Int(\Gamma_2^s)$$

the regularity of the spectral measure E gives us that $E_2^{S_n^+}\to E_2^S$ in the strong operator topology, and thus what remains is to show that

$$\frac{1}{2\pi i}\Big(\int_{\alpha_2^s}-\int_{\alpha_2^{s_n^+}}\Big)(\alpha_k\overline{\beta}_m\phi_\lambda^{-1}(\lambda-\lambda_m)^{-1}(\lambda-\lambda)(T-\lambda)^{-1}x\ d\lambda$$

$$+\frac{1}{2\pi i}\int_{L_S}-\int_{l_{s_n^+}}\Big)(\alpha_k\overline{\beta}_m\phi_\lambda^{-1}(\lambda-\lambda_m)^{-1}(\lambda-\lambda_k)^{-1}\,d\lambda$$

$$=((G_2^S-G_2^{S_n^+})e_m,e_k)\to 0. \qquad k.m\in N, \tag{67}$$

where the arcs $a_2^s, a_2^{s_n^+}, l_s$, and $l_{s_n^+}$ are all properly oriented to agree with their definitions at the beginning of Section this . Moreover, since $s_n^+ > s, s_n^+ \to s, >$ and

$$(\alpha_k\bar{\beta}_m\phi_\lambda^{-1}(\lambda-\lambda_m)^{-1}(\lambda-\lambda_k)^{-1}$$
$$\le(\left|\alpha_k\bar{\beta}_m\right|/\varepsilon_{\tilde{K},r})(1/\min\{dist(\lambda_k,\Gamma_2^S),dist(\lambda_m,\Gamma_2^S)\})^2,\lambda\in\alpha_2^S \quad k,m\in N, \tag{68}$$

it is obvious that the first term on the left side of (67) tends to zero as $s_n^+ \to s$. On the other hand, if the line segments l_s and $l_{s_n^+}$ are parameterized as at the beginning of the proof of Theorem (4.2.17) the second term on the left-hand side of (67) becomes

$$\frac{\alpha_k\bar{\beta}_m}{2\pi}\left[\int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}}\phi_{s+it}^{-1}(s+it-\lambda_m)^{-1}(s+it-\lambda_k)^{-1}dt\right.$$
$$\left.\int_{-\sqrt{1-s_n^{+2}}}^{\sqrt{1-s_n^{+2}}}\phi_{s^++it}^{-1}(s_n^++it-\lambda_m)^{-1}(s_n^++it-\lambda_k)^{-1}dt\right]$$
$$\frac{\alpha_k\bar{\beta}_m}{2\pi}\int_{-\sqrt{1-s_n^{+2}}}^{\sqrt{1-s_n^{+2}}}(\psi(t)-\chi_{[-\sqrt{1-s_n^{+2}}.\sqrt{1-s_n^{+2}}}(t)\psi_n(t))dt,$$

Where

$$\psi(t)=\varphi_{s+it}^{-1}(s+it-\lambda_m)^{-1}(s+it-\lambda_k)^{-1},$$

and the functions ψ_n (t) are defined analogously. Since ψ and the ψ_n (for n large enough) are uniformly bounded as in (68), and $\{\psi_n\}_{\mathbf{n\in N}}$ converges pointwise on $-\left[\sqrt{1-s^2},\sqrt{1-s^2}\right]$ to ψ convergence in the WOT of $\left\{F_2^{s_n^+}\right\}_{n\in N}$ to F_2^S follows, for example, from the Lebesgue bounded convergence theorem.

Step V. To complete the proof of Theorem (4.2.17), we first notice that from Lemmas (4.2.18), (4.2.19), and Step IV, we know that for each s $\in \Pi$`r , $F_1^S$ and $F_2^S$ are nonzero idempotents in {T }″, and therefore that for all such s, ran( $F_1^S$ ) and ran( $F_2^S$ ) are n.h.s. for T . Thus it only remains to show that for s, s' $\in \Pi$`r, $F_j^S \neq F_j^S$.

Thus suppose that $F_1^S = F_1^S$ s < s^t. Then $1H = F_1^S + F_1^{S`}$ and therefore for every x ∈ H, $\sigma_T(x) \subset \sigma_T(F_1^S x) \cup \sigma_T(F_1^S x) \subset \mathrm{Int}((\Gamma_1^S)^- \cup \mathrm{Int}((\Gamma_1^S)^-$. Hence for every λ ∈ C such that s < Re(λ) < s^t, and every x ∈ H, we have $(T-\lambda)_{xT}(\lambda) = x$. Thus $(T-\lambda)H = H$, and it follows that $\sigma(T) \subset \mathrm{Int}(\Gamma_1^S)^- \cup \mathrm{Int}((\Gamma_1^S)^-$, which contradicts the fact that σ (T) is a connected set.

Chapter5

Characteristic Functions for Infinite Sequences of

As the main result of this chapter, we obtain a model for a completely non-co isometric (c.n.c) sequence T (in our notation $T \in C^{(1)}$) in which the "characteristic function" θ occurs explicitly. We obtain criteria for joint similarity of n-tuples of operators to Cuntz row isometries. In particular, we show that a completely non-coisometric row contraction T is jointly similar to a Cuntz row isometry if and only if the characteristic function of T is an invertible multi-analytic operator.

Section (5.1): Sequences of Non-commuting Operators:

This section with the "characteristic function" of an infinite sequence $\mathscr{T} = \{T_n\}_{n=1}^{\infty}$ of noncommuting operators on a Hilbert space H , when the matrix $\{T_1, T_2, \dots\}$ is a contraction. in connexion with this, we extend to our setting the results from [133] for two operators and many of the results from [134] for one operator.

As the main result of this note, we obtain a model for a completely non-co-isometric c.n.c.) sequence $\mathscr{T}$ (in our notation $\mathscr{T} \in C^{(1)}$) in which the "characteristic function" θ occurs explicitly.

Further, it is shown when an operator $\theta: \mathscr{E} \rightarrow \ell^2(\mathscr{F}, \mathscr{E}_*)(\mathscr{E}, \mathscr{E}_*$ generates a c.n.c. sequence Fas above. Using these theorems, we prove that two c.n.c. sequences $\mathscr{T}$and $\mathscr{T}'$ are unitarily equivalent if and only if their characteristic functions coincide.

Finally, by using the above-mentioned-model and the Sz.-Nagy—Foias lifting theorem [140], [134], [131], [133], [137], we give explicit forms for the commutants of an infinite sequence $\mathscr{T}$of noncommuting operators.

We point out that an important role in this section is played by a sequence $S = \{S_1, S_2, \cdots\}$ of unilateral shifts on a Hubert space $\ell^2(\mathscr{F}, \mathscr{H})$with orthogonal final spaces and such that the operator matrix $[S_1, S_2, \cdots]$ is nonunitary.

Let us mention that A. F. Frazho uses (in [138]) a countable number of shifts in a Fock space, in an algebraic setting, to solve a realization problem. Reference [139] also uses two shifts on an ℓ^2 space to solve certain problem in stochastic processes.

Although the Fock space setting is natural for transfer functions of certain system, as explained in [140], or to certain problems in control, it is not the best space to use in dilation theory. The framework of this paper is that of an $\ell^2(\mathscr{F}, \mathscr{H})$ space.

To put our work in perspective, let us recall from [134], [133], [141], [137], some facts from dilation theory for an infinite sequence $\mathscr{T} = \{T_n\}_{n=1}^{\infty}$ of noncommuting operators on a Hilbert space H when the matrix $[T_1, T_2, \dots]$ is a contraction.

Let Λ be the set $\{1,2, \dots k\} (k \in N)$ or $N = \{1,2,4, \dots\}$ and, for every $n \in N$ let $F(n, \Lambda)$ be the set of all functions from the set $\{1,2, \dots, n\}$ to the set Λ. Denote the set $\bigcup_{\epsilon=1}^{\infty} F(n, \Lambda)$ by F where $F(0, \Lambda) = \{0\}$.

A subspace $\mathscr{L}$ of $\mathscr{H}$ will be called a wandering subspace for the selfadjoint sequence $\mathscr{V} = \{V_\lambda\}_{\lambda \in \Lambda}$ of isometries on H if for any distinct functions $f, g \in \mathscr{F}$ we have $V_f \mathscr{L} \perp V_g \mathscr{L}$ where for each $f \in \mathscr{F}$, V_f stands for the product $V^*_{f_m(1)} V^*_{f_m(2)} \dots V^*_{f_m(n)}$ and $V_0^* = I_{\mathscr{H}}$ (the identity on $\mathscr{H}$).

We say that $\mathscr{V}$ is a Λ-orthogonal shift on $\mathscr{H}$ if there exists a subspace $\mathscr{L} \subset \mathscr{H}$ which is wandering for V and

$$\mathscr{H} = M_{\mathscr{F}}(\mathscr{L}) \stackrel{\text{def}}{=} \bigoplus_{f \in \mathscr{F}} V_f \mathscr{L}. \tag{1}$$

Now let $S = \{S_\lambda\}_{\lambda \in \Lambda}$ be the Λ-orthogonal shift with the wandering subspace $\mathscr{H}$ defined on the Hubert space

$$\ell^2(\mathscr{F}, \mathscr{H}) = \left\{ (h_f)_{f \in \mathrm{F}} \;\; ; \sum_{f \in \mathscr{F}} \|h_f\|^2 < \infty, h_f \in \mathscr{H} \right\} \tag{2}$$

as follows.

For each $\lambda \in \Lambda$ we put $S_\lambda\left((h_{f_m})_{f_m \in \mathscr{F}} \right) = (h'_{f_{m+1}})_{f_{m+1} \in \mathscr{F}}$,where $h'_0 = 0$ and for $f_{m+1} \in F(1+\epsilon, \Lambda)$ $(\epsilon \geq 0)$

$$h'_g = \begin{cases} h_0; & if \;\; g \in F(1, \Lambda) \; and \; g(1) = \lambda, \\ h_f; & if \;\; g \in F(n, \Lambda), (n \geq 2), f \in F(n-1, \Lambda) \; and \; g(1) = \lambda, \\ & \quad g(2) = f(1), g(3) = f(2), \dots, g(n) = f(n-1), \\ 0; & otherwise \end{cases}$$

This model will play an important role in our investigation.

We can easily see how acts the Λ-orthogonal shift with the wandering sub- space $\mathcal{H}$ if we consider another model. For this, let us form the Hilbert space of all formal power series with noncommuting indeterminates $X_\lambda(\lambda\in\Lambda)$

$$S^2(\mathcal{F},\mathcal{H}) = \left\{ \sum_{f\in\mathcal{F}} a_f X_f \,;\, a_f \in \sum_{f\in\mathcal{F}} \|a_f\|^2 < \infty \right\},$$

With the inner product

$$\langle \sum_{f\in\mathcal{F}} a_f X_f \,, \sum_{f\in\mathcal{F}} b_f X_f \rangle = \sum_{f\in\mathcal{F}} (a_f, b_f)$$

Where for any $f \in F(n,\Lambda), X_f$ stands for $X_{f(1)}\, X_{f(2)} \cdots X_{f(n)}$.

Define the Λ-orthogonal shift $S = \{S_\lambda\}_{\lambda\in\Lambda}$ on S^2 $(\mathcal{F},\mathcal{H})$ by setting

$$S_\lambda\left(\sum_{f\in\mathcal{F}} a_f X_f \right) = \sum_{f\in\mathcal{F}} a_f X_\lambda X_f, \qquad (\lambda \in \Lambda)$$

When $\Lambda = \{1\}$ we find again the unilateral shift S defined by

$$S\left(\sum_{n=0}^{\infty} a_{1+\epsilon} X^n \right) = \sum_{n=0}^{\infty} a_n X^{n+1}$$

which is unitarily equivalent with the usual unilateral shift on the Hardy space $H^2(D,\mathcal{H})$ where $D = \{z \in \boldsymbol{C}: |z| < 1\}$.

In the case when $\Lambda = \{1,2\}$ the Λ-orthogonal shift $S = \{S_1, S_2\}$ will be unitarily equivalent with the shifts $\{S, E\}$ defined in [133] on a Fock space.

We recall from [137] that for any sequence $\mathcal{T} = \{T_\lambda\}_{\lambda\in\Lambda}$ of noncommuting operators on a Hilbert space $\mathcal{H}$ such that $\sum_{\lambda\in\Lambda} T_\lambda T_\lambda^* \le I_{\mathcal{H}}$, there exists a minimal isometric dilation (m.i.d.) $\mathcal{V} = \{V_\lambda\}_{\lambda\in\Lambda}$ on a Hilbert space $\mathcal{K} \supset \mathcal{H}$, which is uniquely determined up to an isomorphism, i.e., the following conditions hold

$$\begin{cases} \text{(i)} \ \ \textit{Each operator } V_\lambda(\lambda\in\Lambda) \textit{ is an isometry,} \\ \text{(ii)} \ \ \sum_{\lambda\in\Lambda} (V_\lambda V_\lambda^*)^2 \le I_{\mathcal{H}} \\ \text{(iii)} \ \ \textit{For each, } \lambda\in\Lambda, V_\lambda^*(\mathcal{H}) \subset \mathcal{H} \ \textit{and } V_\lambda^*|\mathcal{H} = T_\lambda^*, \\ \text{(iv)} \ \mathcal{K} = \bigvee_{f\in\mathcal{F}} V_f \mathcal{H}, \end{cases} \tag{3}$$

(see [137]).

if we consider the following subspaces of $\mathscr{K}$

$$\mathscr{L} = \bigvee_{\lambda\in\Lambda}(V_\lambda - T_\lambda)\ \mathscr{H}\ ;\ \mathscr{L}_* = \overline{\left(I_{\mathscr{K}} - \sum_{\lambda\in\Lambda} V_\lambda T_\lambda^*\right)\mathscr{H}}\ , \qquad (4)$$

we have the orthogonal decompositions

$$\mathscr{K} = \mathscr{R} \oplus M_{\mathscr{T}}(\mathscr{L}_*) = \mathscr{H} \oplus M_{\mathscr{T}}(\mathscr{L}) \qquad (5)$$

and $\mathscr{R}$ reduces each operator $V_\lambda(\lambda\in\Lambda)$.

Moreover, $\mathscr{R} = \{0\}$ if and only if $\mathscr{H}_0 = \{0\}$, where

$$\mathscr{H}_0 = \left\{h \in \mathscr{H},\ \lim_{\epsilon\to\infty} \sum_{f\in F(n,\Lambda)} \left\|T_f^* h\right\|^2 = 0\right\}. \qquad (6)$$

Further, we have

$$\mathscr{L} \cap \mathscr{L}_* = \{0\} \qquad (7)$$

and

$$M_{\mathscr{T}}(\mathscr{L}) \bigvee M_{\mathscr{T}}(\mathscr{L}_*) = \mathscr{K} \ominus \mathscr{H}_1\ , \qquad (8)$$

where

$$\mathscr{H}_1 = \left\{h \in \mathscr{H},\ \sum_{f\in F(1+\epsilon,\Lambda)} \left\|T_f^* h\right\|^2 = \|h\|^2\ for\ every\ n \in N\right\}. \qquad (9)$$

For any sequence $\mathscr{T} = \{T_\lambda\}_{\lambda\in\Lambda}$ of operators on $\mathscr{H}$ with $\sum_{\lambda\in\wedge} T_\lambda T_\lambda^* \le I_{\mathscr{H}}$ we have the following orthogonal decomposition ([137])

$$\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1 \oplus \mathscr{H}_2,$$

where $\mathscr{H}_0, \mathscr{H}_1$ are given by (6), (7) and $\mathscr{H}_2 = \mathscr{H} \ominus (\mathscr{H}_0 \oplus \mathscr{H}_1)$.

We shall say that $\mathscr{T} \in C^{(k)}$ $\left(\mathscr{T} \in C_{(k)}\right)$ if $\mathscr{H}_{\mathrm{k}} = \{0\}$ $\left(\mathscr{H}, \mathscr{H}_{\mathrm{k}}\right)$, where $k\in\{0,1,2\}$. A sequence $\mathscr{T} \in C^{(1)}$ will be called a completely non-coisometric (c.n.c.) sequence.

Let $\mathscr{E}$, $\mathscr{E}_*$ be Hilbert spaces and $S = \{S_\lambda\}_{\lambda\in\Lambda}$ the Λ-orthogonal shift acting on $\ell^2(\mathscr{F}, \mathscr{E})$ or $\ell^2(\mathscr{F}, \mathscr{E}_*)$.

An operator $A: \ell^2(\mathscr{F}, \mathscr{E}) \to \ell^2(\mathscr{F}, \mathscr{E}_*)$ which commutes with each $S_\lambda(\lambda\in\Lambda)$ is uniquely determined by theoperator

$\theta : \mathcal{E} \to \ell^2(\mathcal{F}, \mathcal{E}_*)$, $\theta = A|\mathcal{E}$. This follows because for every $f \in \mathcal{F}, h \in \mathcal{E}$ we have $AS_f h = S_f \theta h$ and $\bigvee_{f \in \mathcal{F}} S_f \mathcal{E} = \ell^2(\mathcal{F}, \mathcal{E})$.

Now, let us consider an operator $\theta : \mathcal{E} \to \ell^2(\mathcal{F}, \mathcal{E}_*)$, We define M_θ : $\ell^2(\mathcal{F}, \mathcal{E}) \to \ell^2(\mathcal{F}, \mathcal{E}_*)$ by the relation

$$M_\theta S_f h = S_f \theta h = S_f M_\theta h \qquad (h \in \mathcal{H}, f \in \mathcal{F}).$$

In this section we only work with θ such that M_θ is a contraction. One can show that

$$M_\theta\left(\left(h_f\right)_{f \in \mathcal{F}}\right) = \sum_{f \in \mathcal{F}} S_f \theta h_f \ for \left(h_f\right)_{f \in \mathcal{F}} \in \ell^2(\mathcal{F}, \mathcal{E}).$$

Throughout this paper an Operator θ: $\mathcal{E} \to \ell^2(\mathcal{F}, \mathcal{E}_*)$ will be called

(i) inner if M_θ is an isometry,

(ii) outer if $\overline{M_\theta \ell^2(\mathcal{F}, \mathcal{E})} = \ell^2(\mathcal{F}, \mathcal{E}_*)$

(iii) purely contractive if $\|P_{\mathcal{E}_*} \theta h\| < \|h\|$ for every $h \in \mathcal{H}, h \neq 0$.

Proposition (5.1.1)[132] Let. θ : $\mathcal{E} \to \ell^2(\mathcal{F}, \mathcal{E}_*)$, be an operator such that M_θ is a contraction.

(i) θ is inner if and only if θ is an isometry and θ$\mathcal{E}$ is a wandering subspace for S.

(ii) θ is outer if and only $\theta\mathcal{E}$ is cyclic for S, i.e.,

$$\bigvee_{f \in \mathcal{F}} S_f(\theta \mathcal{E}) = \ell^2(\mathcal{F}, \mathcal{E}_*)$$

(iii) θ is inner and outer if and only if θ is a unitary operator from $\mathcal{E}$ to $\mathcal{E}_*$.

The version of the Beurling-Lax theorem [134], [133] in to our setting is.

Theorem (5.1.2)[132]: A subspace $\mathcal{M} \subset \ell^2(\mathcal{F}, \mathcal{E})$ is invariant for each $S_\lambda (\lambda \in \Lambda)$ if and only if there exists a Hilbert space S and an inner operator θ : $h \to \ell^2(\mathcal{F}, \mathcal{E})$such that

$$\mathcal{M} = M_\theta \ell^2(\mathcal{F}, \mathcal{Y})$$

Proof. Using the Wold decomposition for an infinite sequence $\mathcal{V} \doteq \{V_\lambda\}_{\lambda \in \Lambda}$ of isometries with orthogonal final spaces. ([137]).

Let $\mathcal{V} \doteq \{V_\lambda\}_{\lambda \in \Lambda}$ be a Λ-orthogonal shift acting on a Hilbert space $\mathcal{K}$such that $\mathcal{L} \subset \mathcal{K}$ is wandering subspace for $\mathcal{V}$ that is,

$$\mathcal{K} = M_{\mathcal{F}}(\mathcal{L}) = \bigoplus_{f \in \mathcal{F}} V_f \mathcal{L}.$$

Denote by $\Phi^{\mathcal{L}}$ the unitary operator from $M_{\mathcal{F}}(\mathcal{L})$ to $\ell^2(\mathcal{F},\mathcal{L})$ defined by

$$\Phi^{\mathcal{L}}\left(\sum_{f \in \mathcal{F}} V_f l_f\right) = \sum_{f \in \mathcal{F}} S_f l_f \quad \left(l_f \in \mathcal{L};\ \sum_{f \in \mathcal{F}} \|l_f\|^2 < \infty\right),$$

where $S = \{S_\lambda\}_{\lambda \in \Lambda}$ is the Λ-orthogonal shift acting on $\ell^2(\mathcal{F},\mathcal{L})$.

Then for any $\lambda \in \Lambda$ we have

$$\Phi^{\mathcal{L}} V_\lambda^* = S_\lambda \Phi^{\mathcal{L}}.$$

The following extension in [9] will be used in the sequel. We omit the proof which is simple to deduce.

Lemma (5.1.3)[132] Let $\mathcal{V} = \{V_\lambda\}_{\lambda \in \Lambda}$ and $\mathcal{V}' = \{V'_\lambda\}_{\lambda \in \Lambda}$ be Λ-orthogonal shifts on the Hilbert spaces $\mathcal{K}$ and $\mathcal{K}'$, with the wandering subspaces $\mathcal{L}$ and $\mathcal{L}'$, respectively.

Let Q be a contraction of $\mathcal{K}$ into $\mathcal{K}'$ such that for any $\lambda \in \Lambda$

$$QV_\lambda = V'_\lambda Q.$$

Then there exists θ a contraction of L into $\ell^2(\mathcal{F},\mathcal{L}')$ such that

$$\Phi^{\mathcal{L}'} Q = M_\theta \Phi^{\mathcal{L}}.$$

In order that θ be

(a) purely contractive,
(b) inner,
(c) outer,
(d) a unitory from $\mathcal{L}$ *to* $\mathcal{L}'$,

it is necessary and sufficient that the following conditions hold, respectively:

(a) $\|P_{\mathcal{L}'} Ql\| < \|l\|$ *for every* $l \in \mathcal{L}$, $l \neq 0$,
(b) Q is an isometry,
(c) $\overline{Q\mathcal{K}} = \mathcal{K}'$,
(d) Q is a unitary.

Let $\mathcal{T} = \{T_\lambda\}_{\lambda \in \Lambda}$ be a sequence of noncommuting operators on a Hilbert space $\mathcal{H}$ such that the matrix $[T_1, T_2, \ldots]$ is a contraction. Let us recall from [137] that the defect operators of $\mathcal{T}$ are

$$D_* = \left(I_{\mathscr{H}} - \sum_{f \in \Lambda} T_\lambda T_\lambda^* \right)^{1/2} \quad ; D = D_T,$$

where T^* stands for the matrix $[T_1, T_2, \dots]$ and $D_T = (I - T^*T)^{1/2}$.

The defect spaces of $\mathscr{T}$ are

$$\mathscr{D}_* = \overline{D_* \mathscr{H}}; \quad \mathscr{D} = \overline{D \left(\underset{\lambda \in \Lambda}{\oplus} \mathscr{H}_\lambda \right)},$$

where each $\mathscr{H}_\lambda$ $(\lambda \in \Lambda)$ is a copy of $\mathscr{H}$.

We define the characteristic function of $\mathscr{T}$ as the operator $\theta_{\mathscr{T}} \colon \mathscr{D} \to \ell^2(\mathscr{F}, \mathscr{D}_*)$ by

$$\theta_{\mathrm{T}}\ (h) = -\sum_{\lambda \in \Lambda} T_\lambda P_\lambda h + \sum_{\lambda \in \Lambda} S_\lambda \left(\left(D_* T_f^* P_\lambda D h \right)_{f \in \mathscr{F}} \right) \ (h \in \mathscr{D}),$$

where P_λ stands for the orthogonal projection of $\mathscr{D} \subset \underset{\lambda \in \Lambda}{\oplus} \mathscr{H}_\lambda$ onto $\mathscr{H}_\lambda$ and $S = \{S_\lambda\}_{\lambda \in \Lambda}$ is the Λ-orthogonal shift acting on $\ell^2(\mathscr{F}, \mathscr{D}_*)$.

It is easy to see that $\theta_{\mathscr{T}}$ is a contraction and moreover $\theta_{\mathscr{T}}$ is purely contractive.

Let us remark that if $\mathscr{T} = \{T^*\} (\|T^*\| \leq 1)$ the "characteristic function" of $\mathscr{T}$ is the operator $\theta_{\mathscr{T}} \colon \mathscr{D}_{\mathrm{T}} \to \ell^2(N, \mathscr{D}_{\mathrm{T}^*})$ given by the following matrix

$$\begin{vmatrix} -T \\ D_{T^*} D_T \\ D_{T^*} T^* D_T \\ D_{T^*} T^{*2} D_T \\ \vdots \end{vmatrix}$$

We remark that $M_{\theta_{\mathscr{T}}}$ is unitarily equivalent to $(\theta_T)_+ \colon L_+^2(\mathscr{D}_{\mathrm{T}^*}) \to L_+^2(\mathscr{D}_{\mathrm{T}'^*})$, where θ_T is the classical characteristic function of the contraction T and $(\theta_T)_+$ is defined in [134].

Let us consider another sequence $\mathscr{T}' = \{T'_\lambda\}_{\lambda \in \Lambda}$ on a Hilbert space $\mathscr{H}'$ such that the matrix $[T'_1, T'_2, \dots]$ is a contraction.

We say that the characteristic functions $\theta_{\mathscr{T}}$ and $\theta_{\mathscr{T}'}$ coincide if there exists two unitary operator

$$W \colon \mathscr{D} \to \mathscr{D}', \qquad W_* \colon \mathscr{D}_* \to \mathscr{D}'_*$$

Such that

$$M_{W_*}\theta_{\mathcal{T}} = \theta_{\mathcal{T}'}W,$$

One can easily show that if $\mathcal{T}$ and $\mathcal{T}'$ are unitarily quivalent, i.e., $T'_\lambda = UT_\lambda U^*$ for any $\lambda \in \Lambda$, where U is a unitary operator [from $\mathcal{H}$ to $\mathcal{H}'$, then their characteristic functions coincide. The converse is not true, at least not in this generality. Notice also that if $\mathcal{T} \in C_{(1)}$ then $\theta_{\mathcal{T}} = 0$.

We are now going to show that the definition of the characteristic function for $\mathcal{T}$ arises in a natural way in the context of the theory of isometric dilation of asequence $\mathcal{T} = \{T_\lambda\}_{\lambda\in\Lambda}$ of noncommuting operators on $\mathcal{H}$ such the matrix $[T_1, T_2, \ldots]$ is a contraction (see [137]).

Let $\mathcal{V} = \{V_\lambda\}_{\lambda\in\Lambda}$ be m.i.d of T on the Hilbert space $\mathcal{K} \supset \mathcal{H}$.

By (5) we have that $\{V_\lambda | M_{\mathcal{F}}(\mathcal{L}_*)\}_{\lambda\in\Lambda}$ and $\{V_\lambda | M_{\mathcal{F}}(\mathcal{L})\}_{\lambda\in\Lambda}$ are Λ-orthogonal shifts acting on $M_{\mathcal{F}}(\mathcal{L}_*)$ and $M_{\mathcal{F}}(\mathcal{L})$,respectively .

Moreover, for each $\lambda \in \Lambda$

$$\left(P^{\mathcal{L}_*}\middle|M_{\mathcal{F}}(\mathcal{L})\right)\left(V_\lambda^*|M_{\mathcal{F}}(\mathcal{L})\right) = \left(V_\lambda^*|M_{\mathcal{F}}(\mathcal{L}_*)\right)\left(P^{\mathcal{L}_*}|M_{\mathcal{F}}(\mathcal{L})\right),$$

where $P^{\mathcal{L}_*}$ stands for the orthogonal projection of $\mathcal{K}$ onto $M_{\mathcal{F}}(\mathcal{L}_*)$.

Setting $Q = P^{\mathcal{L}_*}|M_{\mathcal{F}}(\mathcal{L})$, we can apply Lemma (5.1.3) and we obtain that there exists a contraction $\theta_{\mathcal{L}}\, \mathcal{L} \to \ell^2(\mathcal{F}, \mathcal{L}_*)$ such that

$$\Phi^{\mathcal{L}_*}Q = M_{\theta_{\mathcal{L}}}\Phi^{\mathcal{L}},$$

Hence we deduce that

$$\theta_{\mathcal{L}} = \Phi^{\mathcal{L}_*}\left(P^{\mathcal{L}_*}\middle|\mathcal{L}\right)\left(\Phi^{\mathcal{L}}\right)^*|\mathcal{L}. \qquad (10)$$

We remark first that $\theta_{\mathcal{L}}$ is purely contractive. Indeed, if $P_{\mathcal{L}_*}$ denotes the orthogonal projection onto $\mathcal{L}_*$, we have $\|P_{\mathcal{L}_*}P_*^{\mathcal{L}}l\| < \|l\|$ for every $l \in \mathcal{L}$, $l \neq 0$. Otherwise there would exist an $l \in \mathcal{L}, l \neq 0$ such that, $l = P_{\mathcal{L}_*}P_*^{\mathcal{L}}l, i.e.,$ $l \in \mathcal{L}_*$, and this contradicts the relation (7).

Let us recall from [137] that the operator Φ_* defined from $\mathcal{L}_*$ to $\mathcal{D}_*$ by

$$\Phi_*\left(I_{\mathcal{K}} - \sum_{\lambda\in\Lambda} V_\lambda^* T_\lambda^*\right)h = D_*h; (h \in \mathcal{H}) \qquad (11)$$

is unitary and the operator Φ defined form $\mathcal{L}$ to $\mathcal{D}$ by

$$\Phi\left(I_{\mathscr{K}} - \sum_{\lambda\in\Lambda}(V_\lambda^* - T_\lambda^*)h_\lambda\right) = D((h_\lambda)_{\lambda\in\Lambda});\quad (h_\lambda)_{\lambda\in\Lambda} \in \bigoplus_{\lambda\in\Lambda}\mathscr{H}_\lambda \ (12)$$

is unitary too.

We are ready for proving the following theorem which is a generalization in [134].

Theorem (5.1.4)[132]: the characteristic function $\theta_{\mathscr{T}}$ for $\mathscr{T}$ coincides with $\theta_{\mathscr{L}}$.

Proof. we show that

$$M_{\Phi_*}\theta_{\mathscr{L}} = \theta_{\mathscr{T}}\Phi, \qquad (13)$$

Where Φ_*, Φ are the unitary operators in (11), (12), respectively. For this, it is necessarily to prove that

$$P_{\mathscr{L}_*}S_f^* M_{\Phi_*}\theta_{\mathscr{L}} = P_{\mathscr{L}_*}S_f^*\theta_{\mathscr{T}}\Phi \quad (f \in \mathscr{F}), \qquad (14)$$

Where $P_{\mathscr{D}_*}$ stands for the orthogonal projection of $\ell^2(\mathscr{F}.\mathscr{D}_*)$ onto $\mathscr{D}_*$.

By (10) and by the Wold decomposition (5), the relation (14) is equivalent to

$$\Phi_* P_{\mathscr{L}_*}V_f^*|\mathscr{L} = P_{\mathscr{D}_*}S_f^*\theta_{\mathscr{T}}\,\Phi \quad (f \in \mathscr{F}), \qquad (15)$$

In what follows we shall prove this relation. First let us notice that

$$P_{\mathscr{D}_*}\theta_{\mathscr{T}} = -\sum_{\lambda\in\Lambda} T_\lambda P_\lambda, \qquad (16)$$

$$P_{\mathscr{D}_*}S_f^* S_\lambda\theta_{\mathscr{T}} = D_* T_f^* P_\lambda D \qquad (\lambda\in\Lambda, f \in \mathscr{F}).$$

For $f = 0$ the relation (15) holds true. Indeed for

$$l = \sum_{\lambda\in\Lambda}(V_\lambda - T_\lambda)h_\lambda = \Phi^* D\left(\bigoplus_{\lambda\in\Lambda} h_\lambda\right) \ \left(\sum_{\lambda\in\Lambda}\|h_\lambda\|^2 < \infty\right) \ (17)$$

we have that $l + (I - \sum_{\lambda\in\Lambda} V_\lambda T_\lambda^*)\sum_{\lambda\in\Lambda} T_\lambda h_\lambda \in \bigoplus_{\lambda\in\Lambda} V_\lambda\mathscr{K}$ and by (15) we obtain that

$$P_{\mathscr{L}_*} l = -\left(I - \sum_{\lambda\in\Lambda} V_\lambda T_\lambda^*\right)\sum_{\lambda\in\Lambda} T_\lambda h_\lambda.$$

Hence, by (16) we have

$$\Phi_* P_{\mathscr{L}_*} l = -D_*[T_1, T_2, \ldots]\left(\bigoplus_{\lambda\in\Lambda} h_\lambda\right) = [T_1, T_2, \ldots]D\left(\bigoplus_{\lambda\in\Lambda} h_\lambda\right) = -[T_1, T_2, \ldots]\Phi l$$
$$= P_{\mathscr{D}_*}\theta_{\mathscr{T}}\,\Phi l.$$

It remains to show that for any $f \in \mathscr{F}, \lambda \in \Lambda$

$$\Phi_* P_{\mathscr{L}_*} V_f^* V_\lambda^* l = P_{\mathscr{D}_*} S_f^* S_\lambda^* \theta_{\mathscr{T}} \Phi l \quad (l \in \mathscr{L}). \qquad (18)$$

Let l be as in (17); then according to (16) the relation (18) becomes

$$\Phi_* P_{\mathscr{L}_*} V_f^* V_\lambda^* l = D_* T_f^* P_\lambda D^2 \left(\bigoplus_{\lambda \in \Lambda} h_\lambda \right).$$

Since

$$D_* T_f^* P_\lambda D^2 \left(\bigoplus_{\lambda \in \Lambda} h_\lambda \right) = \Phi_* \left(I - \sum_{\lambda \in \Lambda} V_\lambda T_\lambda^* \right) T_f^* P_\lambda D^2 \left(\bigoplus_{\lambda \in \Lambda} h_\lambda \right).$$

we have only to show that

$$P_{\mathscr{L}_*} V_f^* V_\lambda^* l = \left(I - \sum_{\lambda \in \Lambda} V_\lambda T_\lambda^* \right) T_f^* P_\lambda D^2 \left(\bigoplus_{\lambda \in \Lambda} h_\lambda \right). \qquad (19)$$

Let us notice that for any $\lambda \in \Lambda$

$$P_\lambda D^2 \left(\bigoplus_{\lambda \in \Lambda} h_\lambda \right) = - \sum_{\substack{\mu \in \Lambda \\ \mu \neq \lambda}} T_\lambda^* T_\mu h_\mu + D_{T_\lambda}^2 h_\lambda.$$

Consequently, the relation (19) holds if and only if the following relations hold

$$P_{\mathscr{L}_*} V_f^* V_\lambda^* (V_\lambda h_\lambda - T_\lambda h_\lambda) = - \left(I - \sum_{\lambda \in \Lambda} V_\lambda T_\lambda^* \right) T_f^* D_{T_\lambda}^2 h_\lambda \ \ (\lambda \in \Lambda)$$

and

$$P_{\mathscr{L}_*} V_f^* V_\lambda^* \left(V_\mu h_\mu - T_\mu h_\mu \right) = - \left(I - \sum_{\lambda \in \Lambda} V_\lambda T_\lambda^* \right) T_f^* T_\lambda^* T_\mu h_\mu \ \ (\lambda \neq \mu).$$

These relations hold since the element

$$V_f^* V_\lambda^* (V_\lambda h_\lambda - T_\lambda h_\lambda) - \left(I - \sum_{\lambda \in \Lambda} V_\lambda T_\lambda^* \right) T_f^* D_{T_\lambda} h_\lambda \ \ (\lambda \in \Lambda)$$

and

$$V_f^* V_\lambda^* \left(V_\mu h_\mu - T_\mu h_\mu \right) + \left(I - \sum_{\lambda \in \Lambda} V_\lambda T_\lambda^* \right) T_f^* T_\lambda^* T_\mu h_\mu \ \ (\lambda \neq \mu)$$

are orthogonal on $\mathscr{L}_*$. This follows by simple computation .The proof is complete.

Remark (5.1.5)[132]: if $\mathscr{T} \in C_{(0)}$ then $\theta_{\mathscr{T}}$ is inner.

Proof. Taking into account [137], it follows that the m.i.d. $\mathscr{V}$ of $\mathscr{T}$ is pure, i.e., $\mathscr{K} = M_{\mathscr{T}}(\mathscr{L}_*)$. By relation (10) and Theorem(5.1.4) it follows that $\theta_{\mathscr{T}}$ is inner.

In this Section we make the additional assumption that $\mathscr{T}$ is c.n.c. on $\mathscr{H}$. Then the relation (8) implies

$$\mathscr{K} = M_{\mathscr{T}}(\mathscr{L}) \bigvee M_{\mathscr{T}}(\mathscr{L}_*)$$

and consequently,

$$\left(I - P^{\mathscr{L}_*}\right)|M_{\mathscr{T}}(\mathscr{L}) = \mathscr{R} \qquad \left(\text{cf. } (5)\right).$$

Consider the operator $\Delta_{\mathscr{L}}$ defined on $\ell^2(\mathscr{T},\mathscr{L})$ by

$$\Delta_{\mathscr{L}} = \left(I - M_{\theta_{\mathscr{L}}}^* M_{\theta_{\mathscr{L}}}\right)^{1/2},$$

where $\theta_{\mathscr{L}}$ is given by (10).

For $k \in M_{\mathscr{T}}(\mathscr{L})$ we have

$$\begin{aligned}\left\|\left(I - P^{\mathscr{L}_*}\right)k\right\|^2 &= \|k\|^2 - \left\|P^{\mathscr{L}_*}k\right\|^2 = \left\|\Phi^{\mathscr{L}} k\right\|^2 - \left\|\Phi^{\mathscr{L}_*}P^{\mathscr{L}_*}k\right\|^2 \\ &= \left\|\Phi^{\mathscr{L}} k\right\|^2 - \left\|M_{\theta_{\mathscr{L}}} \Phi^{\mathscr{L}} k\right\|^2 = \left\|\Delta_{\mathscr{L}}\Phi^{\mathscr{L}} k\right\|^2.\end{aligned}$$

We can define the unitary operator $\Phi_{\mathscr{R}}$ from $\mathscr{R}$ onto $\overline{\Delta_{\mathscr{L}}\ell^2(\mathscr{T},\mathscr{L})}$ by the relation

$$\Phi_{\mathscr{R}}\left(I - P^{\mathscr{L}_*}\right)k = \Delta_{\mathscr{L}}\Phi^{\mathscr{L}} k \quad \left(k \in M_{\mathscr{T}}(\mathscr{L})\right).$$

Consequently,

$$\Phi = \Phi^{\mathscr{L}_*} \oplus \Phi_{\mathscr{R}}.$$

is a unitary operator from space $\mathscr{K} = M_{\mathscr{T}}(\mathscr{L}_*) \oplus \mathscr{R}$ to the Hilbert Space

$$K = \ell^2(\mathscr{T},\mathscr{L}_*) \oplus \overline{\Delta_{\mathscr{L}}\ell^2(\mathscr{T},\mathscr{L})}.$$

Let us find the image of space $\mathscr{H}$ under the operator Φ . Since $\mathscr{H} = \mathscr{K} \ominus M_{\mathscr{T}}(\mathscr{L})$ and for each $k \in M_{\mathscr{T}}(\mathscr{L})$

$$\Phi k = \Phi^{\mathscr{L}_*}P^{\mathscr{L}_*}k \oplus \Phi_{\mathscr{R}}\left(I - P^{\mathscr{L}_*}\right)k = M_{\theta_{\mathscr{L}}} \Phi^{\mathscr{L}} k \oplus \Delta_{\mathscr{L}}\Phi^{\mathscr{L}} k$$

we have

$$\Phi\mathscr{H} = H = \left[\ell^2(\mathscr{T},\mathscr{L}_*) \oplus \overline{\Delta_{\mathscr{L}}\ell^2(\mathscr{T},\mathscr{L})}\right] \ominus \left\{M_{\theta_{\mathscr{L}}} u \oplus \Delta_{\mathscr{L}} u;\ u \in \ell^2(\mathscr{T},\mathscr{L})\right\}.$$

Because $P^{\mathscr{L}_*}$ commutes with each V_λ($\lambda \in \Lambda$) it follows that

$$\Phi_{\mathscr{R}}V_\lambda\left(I - P^{\mathscr{L}_*}\right)k = \Phi_{\mathscr{R}}\left(I - P^{\mathscr{L}_*}\right)V_\lambda k = \Delta_{\mathscr{L}}\Phi^{\mathscr{L}} V_\lambda k = \Delta_{\mathscr{L}} S_\lambda \Phi^{\mathscr{L}} k$$

for every $k \in M_{\mathscr{T}}(\mathscr{L})$, where $S = \{S_\lambda\}_{\lambda \in \Lambda}$ is the Λ-orthogonal shift on $\ell^2(\mathscr{T},\mathscr{L})$

Therefore,

$$\Phi_{\mathscr{R}} V_\lambda^* \Phi_{\mathscr{R}}^* (\Delta_{\mathscr{L}} v) = \Delta_{\mathscr{L}} S_\lambda v \ \left(v \in \ell^2(\mathscr{F}, \mathscr{L})\right)$$

and

$$\Phi V_\lambda^* \Phi^* = V_\lambda^* = S_\lambda \oplus C_1 \ \ for\ every\ \ \lambda \in \Lambda,$$

where each operator C_λ is an isometry defined on $\overline{\Delta_{\mathscr{L}} \ell^2(\mathscr{F}, \mathscr{L})}$ by the relation

$$C_\lambda(\Delta_{\mathscr{L}} v) = \Delta_{\mathscr{L}} S_\lambda v \ \ for\ v \in \ell^2(\mathscr{F}, \mathscr{L})$$

Now , sinse $(\sum_{\lambda \in \Lambda} (V_\lambda^*)^2 - I)|\mathscr{R} = 0$ we have

$$\sum_{\lambda \in \Lambda} C_\lambda C_\lambda^* = I_{\overline{\Delta_{\mathscr{L}} \ell^2(\mathscr{F}, \mathscr{L})}}, whence\ \ \overline{\Delta_{\mathscr{L}} \ell^2(\mathscr{F}, \mathscr{L})} = \overline{\Delta_{\mathscr{L}} (\ell^2(\mathscr{F}, \mathscr{L}) \ominus \mathscr{L})}.$$

It is easy to see that for every $v \in \ell^2(\mathscr{F}, \mathscr{L})$ and $\lambda, \mu \in \Lambda$

$$C_\lambda^*\left(\Delta_{\mathscr{L}} S_\mu v\ \right) = \begin{cases} \Delta_{\mathscr{L}} v & if \quad \lambda = \mu \\ 0 & if \quad \lambda \neq \mu, \end{cases}$$

According to (3) we have $T_\lambda^* = V_\lambda^* | H$, where T_λ is the transform of T_λ by Φ.

Therefore, for $u \oplus \Delta_{\mathscr{L}} S_\mu v \in H$ we can write that

$$T_\lambda^*\left(u \oplus \Delta_{\mathscr{L}} S_\mu v\ \right) = \begin{cases} S_\lambda^* u \oplus \Delta_{\mathscr{L}} v & if \quad \lambda = \mu \\ S_\lambda^* u \oplus 0 & if \quad \lambda \neq \mu, \end{cases}$$

Where $\lambda, \mu \in \Lambda$.

The above results permit us to construct a model for a c.n.c. sequence $\mathscr{T}$, in which the characteristic function occurs explicitly. We obtain a generalization in [134], namely:

Theorem(5.1.6)[132]: Every completely non-isometric sequence $\mathscr{T} = \{T_\lambda\}_{\lambda \in \Lambda}$ on the Hilbert space $\mathscr{H}$ is unitarily equivalent to a sequence $T^* = \{T_\lambda\}_{\lambda \in \Lambda}$ on the Hilbert space

$$H = \left[\ell^2(\mathscr{F}, \mathscr{D}_*) \oplus \overline{\Delta_{\mathscr{L}} \ell^2(\mathscr{F}, \mathscr{D})}\right] \ominus \left\{M_{\theta_{\mathscr{T}}} u \oplus \Delta_{\mathscr{L}} u; u \in \ell^2(\mathscr{F}, \mathscr{D})\right\},$$

where $\Delta_{\mathscr{T}} = \left(I - M_{\theta_{\mathscr{T}}}^* M_{\theta_{\mathscr{T}}}\right)^{1/2}$.

For each $\lambda \in \Lambda$ the operator T_λ is defined by

$$\boldsymbol{T}_\lambda^*\left(u \oplus \Delta_{\mathscr{T}} S_\mu v\ \right) = \begin{cases} S_\lambda^* u \oplus \Delta_{\mathscr{T}} u & if \quad \mu = \lambda, \\ S_\lambda^* u \oplus 0 & if \quad \mu \neq \lambda, \end{cases}$$

Where $S = \{S_\lambda\}_{\lambda \in \Lambda}$ is the Λ-orthogonal shift acting on $\ell^2(\mathscr{F}, \mathscr{D})$or $\ell^2(\mathscr{F}, \mathscr{D}_*)$.

If $\mathscr{T} \in C_{(0)}$, and only in this case, $\theta_{\mathscr{T}}$ is inner, and this model reduces to

$$H = \ell^2(\mathcal{F}, \mathcal{D}_*) \ominus M_{\theta_{\mathcal{T}}} \ell^2(\mathcal{F}, \mathcal{D}); \;\; T_\lambda^* u = S_\lambda^* u \quad (u \in H).$$

Proof. By virtue of the relation (13) it follows that

$$M_{\Phi_*} M_{\theta_{\mathcal{L}}} = M_{\theta_{\mathcal{T}}} M_\Phi.$$

Hence we obtain that $\Delta_{\mathcal{T}} = M_\Phi \Delta_{\mathcal{L}} M_{\Phi_*}$

On the other hand the operators Φ and Φ_* defined by (11) and (12) generate the unitary operator

$$U = M_{\Phi_*} \oplus M_\Phi$$

from the space $\ell^2(\mathcal{F}, \mathcal{L}_*) \oplus \overline{\Delta_{\mathcal{L}} \ell^2(\mathcal{F}, \mathcal{L})}$ to the space

$\ell^2(\mathcal{F}, \mathcal{D}_*) \oplus \overline{\Delta_{\mathcal{T}} \ell^2(\mathcal{F}, \mathcal{D})}$ Such that

$$U\{M_{\theta_{\mathcal{L}}} u \oplus \Delta_{\mathcal{L}} u \,;\, u \in \ell^2(\mathcal{F}, \mathcal{L})\} = \{M_{\theta_{\mathcal{T}}} v \oplus \Delta_{\mathcal{T}} v \,;\, v \in \ell^2(\mathcal{F}, \mathcal{D})\}.$$

By means of this unitary operator we can re write the result obtained before this theorem and; in this way, we complete the proof.

Let us remark that for $\mathcal{T} = \{T\}$, we find a model for completely non-co isometric contractions, which coincides with the Sz-Nagy-Foias model. Indeed, if T is a completely non-coisometric contraction, that is, if there is no non-zero invariant subspace for T^* on which T^* is an isometry, then it is easy to see that

$$\Delta_T H^2(\mathcal{D}_T) = \Delta_T L^2(\mathcal{D}_T)$$

Let us note that the Sz.-Nagy-Foias, model is given for a larger class of contraction, namely for completely non-unitary contractions.

Now show that any contraction $\theta: \mathcal{E} \to \ell^2(\mathcal{F}, \mathcal{E}_*)$ ($\mathcal{E}$, $\mathcal{E}_*$ Hilbert spaces) such that M_θ is contraction generates, a c.n.c. sequences $T = \{T_\lambda\}_{\lambda \in \Lambda}$

In the case when θ is purely contractive and

$$\overline{\Delta_\theta \ell^2(\mathcal{F}, \mathcal{E})} = \overline{\Delta_\theta [\ell^2(\mathcal{F}, \mathcal{E}) \ominus \mathcal{E}]} \qquad (20)$$

we shall show that θ coincides with the characteristic function of $\mathcal{T}$.

The main result of this section is the following generalization of in [134].

Theorem (5.1.7)[132]: Let θ be a contraction from $\mathcal{E}$ to $\ell^2(\mathcal{F}, \mathcal{E}_*)$ such that M_θ is a contraction. Setting $\Delta_\theta = (I - M_\theta^* M_\theta)^{1/2}$ the sequences $T = \{T_\lambda\}_{\lambda \in \Lambda}$ of operator defined on the Hilbert space

$$H = \left[\ell^2(\mathcal{F}, \mathcal{E}_*) \oplus \overline{\Delta_\theta \ell^2(\mathcal{F}, \mathcal{E})}\right] \ominus \{M_\theta w \oplus \Delta_\theta w \,;\, w \in \ell^2(\mathcal{F}, \mathcal{E})\}.$$

by

$$T_\lambda(u \oplus \Delta_\theta v) = S_\lambda^* u \oplus C_\lambda^*(\Delta_\theta v) \qquad (\lambda \in \Lambda),$$

where each operator C_λ is defined by $C_\lambda(\Delta_\theta g) = \Delta_\theta S_\lambda g\big(g \in \ell^2(\mathcal{F}, \mathcal{E})\big)$ and $S = \{S_\lambda\}_{\lambda\in\Lambda}$ is the Λ-orthogonal shift action on $\ell^2(\mathcal{F}, \mathcal{E})$ or $\ell^2(\mathcal{F}, \mathcal{E}_*)$ is completely Non-coisometric.

If θ is purely contractive and (20) holds, then θ coincides with the characteristic function of $\mathcal{T}$ In this case, considering H as a subspace of

$$K = \ell^2(\mathcal{F}, \mathcal{E}_*) \oplus \overline{\Delta_\theta \ell^2(\mathcal{F}, \mathcal{E})}$$

we have that the sequence $V = \{V_\lambda\}_{\lambda\in\Lambda}$ of operator defined on K by

$$V_\lambda = S_\lambda \oplus C_\lambda \quad (\lambda \in \Lambda)$$

is the minimal isometric dilation of $\boldsymbol{T}^*$

Proof. Let us consider the following Hilbert space

$$K = \ell^2(\mathcal{F}, \mathcal{E}_*) \oplus \overline{\Delta_\theta \ell^2(\mathcal{F}, \mathcal{E})}$$

$$G = \{M_\theta w \oplus \Delta_\theta w \,;\, w \in \ell^2(\mathcal{F}, \mathcal{E})\}$$

and let $V = \{V_\lambda\}_{\lambda\in\Lambda}$ be a sequence of isometrics defined on K by $V_\lambda = S_\lambda \oplus C_\lambda$ ($\lambda \in \Lambda$), where each C_λ is given by

$$C_\lambda(\Delta_\theta g) = \Delta_\theta S_\lambda g \quad for \;\; g \in \ell^2(\mathcal{F}, \mathcal{E})$$

It is easy to see that

$$\sum_{\lambda\in\Lambda} V_\lambda V_\lambda^* \le I_K$$

and that G is invariant for each$V_\lambda (\lambda \in \Lambda)$.

Setting $H = K \ominus G$ and $\boldsymbol{T}_\lambda^* = \boldsymbol{V}_\lambda^* | H (\lambda \in \Lambda)$ we see that V is an isometric dilation of $T = \{T_\lambda\}_{\lambda\in\Lambda}$.

Let us show that T is c.n.c. For this, let $u \oplus \Delta_\theta v \in H$ such that for every $n \in N$ we have

$$\sum_{f\in F(n,\Lambda)} \left\| T_f^*(u \oplus \Delta_\theta v) \right\|^2 = \|u \oplus \Delta_\theta v\|^2. \tag{21}$$

Since

$$\lim_{n\to\infty} \sum_{f\in F(1+\epsilon,\Lambda)} \left\| S_f^* u \right\|^2 = 0 \;\; and \quad \sum_{f\in F(n,\Lambda)} \left\| C_f^* \Delta_\theta v \right\|^2 \le \|\Delta_\theta v\|^2$$

it follows that $u = 0$. But, $(0 \oplus \Delta_\theta v, M_\theta w \oplus \Delta_\theta w) = 0$ for any $w \in \ell^2(\mathcal{F}, \mathcal{E})$ implies $\Delta_\theta = 0$.

Thus T is c.n.c.

We assume from now on that θ is purely contractive and that (20) holds.

Let us show that V is m.i.d. of T i.e.

$$K = \bigvee_{f \in \mathscr{F}} V_f^* H.$$

First we note that (20) implies

$$\sum_{\lambda \in \Lambda} C_\lambda C_\lambda^* = I_{\overline{\Delta_\theta \ell^2(\mathscr{F}, \mathscr{E})}}. \tag{22}$$

Suppose $u \oplus \Delta_\theta v \in K$ and for every $f \in \mathscr{F}$, $u \oplus \Delta_\theta v \perp \boldsymbol{V}_f^* H$ i.e., $\boldsymbol{V}_f^*(u \oplus \Delta_\theta v) \in G$.

This means that for each $f \in \mathscr{F}$ there exists $w_{(f)} \in \ell^2(\mathscr{F}, \mathscr{E})$ such that

$$\boldsymbol{V}_f^*(u \oplus \Delta_\theta v) = M_\theta w_{(f)} \oplus \Delta_\theta w_{(f)}.$$

Therefore, for each $\lambda \in \Lambda, f \in \mathscr{F}$ there exists $w_{(f,\lambda)} \in \ell^2(\mathscr{F}, \mathscr{E})$ such that

$$\boldsymbol{V}_\lambda^*\left(M_\theta w_{(f)} \oplus \Delta_\theta w_{(f)}\right) = M_\theta w_{(f,\lambda)} \oplus \Delta_\theta w_{(f,\lambda)}.$$

By using the information of $V_\lambda (\lambda \in \Lambda)$ we obtain

$$\left(\sum_{\lambda \in \Lambda} S_\lambda S_\lambda^*\right) M_\theta w_{(f)} \oplus \left(\sum_{\lambda \in \Lambda} C_\lambda C_\lambda^*\right) \Delta_\theta w_{(f)} = M_\theta \left(\sum_{\lambda \in \Lambda} S_\lambda\, w_{(f,\lambda)}\right) \oplus \Delta_\theta \left(\sum_{\lambda \in \Lambda} S_\lambda\, w_{(f,\lambda)}\right).$$

Hence according to (22), we have

$$M_\theta \omega_{(f)} = P_{\mathscr{E}_*} M_\theta w_{(f)} \;\; and \;\; \Delta_\theta \omega_{(f)} = 0, \tag{23}$$

Where $\omega_{(f)}$ stands for $w_{(f)} - \sum_{\lambda \in \Lambda} S_\lambda\, w_{(f,\lambda)}$

Since M_θ commutes with each $S_\lambda (\lambda \in \Lambda)$ it follows that

$$P_{\mathscr{E}_*} M_\theta w_{(f)} = P_{\mathscr{E}_*} M_\theta P_{\mathscr{E}} w_{(f)}$$

and (23) gives

$$\omega_{(f)} = M_\theta^* P_{\mathscr{E}_*} M_\theta P_{\mathscr{E}} w_{(f)}, \tag{24}$$

Hence $P_{\mathscr{E}} w_{(f)} = P_{\mathscr{E}} \omega_{(f)} = P_{\mathscr{E}} M_\theta^* P_{\mathscr{E}_*} M_\theta P_{\mathscr{E}} w_{(f)}$.

Consequently, $\left\| P_{\mathscr{E}} w_{(f)} \right\| = \left\| P_{\mathscr{E}_*} M_\theta P_{\mathscr{E}} w_{(f)} \right\|$ and since θ is purely contractive it follows that

$$P_{\mathscr{E}} w_{(f)} = 0. \tag{25}$$

Now, the relation (24) implies $\omega_{(f)} = 0$, i.e.

$$w_{(f)} = \sum_{\lambda\in\Lambda} S_\lambda w_{(f,\lambda)} \quad for \quad f \in \mathscr{F}$$

Hence, we obtain that

$$w_{(\theta)} = \sum_{\lambda\in\Lambda} S_\lambda w_{(\lambda)} = \sum_{\lambda\in\Lambda} S_\lambda \left(\sum_{\mu\in\Lambda} S_\mu w_{(\lambda,\mu)}\right) = \sum_{g\in F(2,\Lambda)} S_{f_g} w_{(g)} = \cdots$$
$$= \sum_{f\in F(n,\Lambda)} S_f w_{(f)} \quad for\ any\ n \in N.$$

We deduce that $S_f^* w_{(\theta)} = w_f$ for every $f \in \mathscr{F}$. By (25) we find $P_{\mathscr{E}} S_{(f)}^* w_{(\theta)} = P_{\mathscr{E}} w_{(f)} = 0$ for every $f \in \mathscr{F}$.

It follows that $w_{(\theta)} = 0$ and $u \oplus \Delta_\theta v = M_\theta w_{(\theta)} \oplus \Delta_\theta w_{(\theta)} = 0$, which implies the minimality of V.

Our next step is to determine

$$L_* = \overline{\left(I_H - \sum_{\lambda\in\Lambda} V_\lambda \boldsymbol{T}_\lambda^*\right) H}.$$

Taking into account (22), for $u \oplus \Delta_\theta v \in H$ we have

$$\left(I_H - \sum_{\lambda\in\Lambda} V_\lambda \boldsymbol{T}_\lambda^*\right)(u \oplus \Delta_\theta v) = P_{\mathscr{E}_*} u \oplus 0$$

and hence $L_* \subset \mathscr{E}_* \oplus \{0\}$.

Let$e_* \in \mathscr{E}_*$ and let us choose $u = (I - M_\theta M_\theta^*) e_*$ and

$\Delta_\theta v = -\Delta_\theta M_\theta^* e_*$. Since $M_\theta^* u = \Delta_\theta^3 v = 0$ it follows that $u \oplus \Delta_\theta v \in H$.

Thus

$$\left(I_H - \sum_{\lambda\in\Lambda} V_\lambda T_\lambda^*\right)(u \oplus \Delta_\theta v) = \left(I_{\mathscr{E}_*} - P_{\mathscr{E}_*} M_\theta M_\theta^*\right) e_* \oplus 0.$$

Now the element of the form $\left(I_{\mathscr{E}_*} - P_{\mathscr{E}_*} M_\theta M_\theta^*\right) e_*$, $(e_* \in \mathscr{E}_*)$, are dense in $\mathscr{E}_*$.

Otherwise there exist an $e_*' \in \mathscr{E}_*$, $e_*' \neq 0$, such that $e_*' = P_{\mathscr{E}_*} M_\theta M_\theta^* e_*'$ and hence $\|e_*'\| = \|M_\theta^* e_*'\| = \left\|P_{\mathscr{E}_*} M_\theta M_\theta^* e_*'\right\|$; $e_*' = M_\theta M_\theta^* e_*'$. Since $M_\theta^* e_*' \in \mathscr{E}_*$ and θ is purely contractive it follows that $M_\theta^* e_*' = 0$ and $e_*' = 0$ which is a contradiction.

Thus

$$L_* = \mathscr{E}_* \oplus \{0\} \tag{26}$$

and $M_{\mathscr{F}}(L_*) = \ell^2(\mathscr{F}, \mathscr{E}_*)\oplus\{0\}$.

Denoting by P^{L_*} the orthogonal projection of K onto $M_{\mathscr{F}}(L_*)$ we have for $u\oplus \Delta_\theta v \in K$

$$P^{L_*}(u\oplus\Delta_\theta v) = u\oplus 0, \qquad (27)$$

$$\Phi^{L_*}P^{L_*}(u\oplus\Delta_\theta v) = \Phi^{\mathscr{E}_*}u\oplus 0 = u\oplus 0.$$

Next we show that

$$L = \bigvee_{\lambda\in\Lambda}(V_\lambda - T_\lambda)H = \{M_\theta e\oplus\Delta_\theta e\,;\, e \in \mathscr{E}\}.$$

Notice that an element $u\oplus\Delta_\theta v$ in K belongs to H if and only if

$$M_\theta^* u\oplus\Delta_\theta^2 v = 0. \qquad (28)$$

For $u\oplus \Delta_\theta v \in H$ and $\lambda \in \Lambda$ we have

$$T_\lambda(u\oplus\Delta_\theta v) = P_H V_\lambda(u\oplus\Delta_\theta v) = (S_\lambda u\oplus\Delta_\theta S_\lambda v) - (M_\theta w_\lambda\oplus\Delta_\theta w_\lambda),$$

where each $w_\lambda \in \ell^2(\mathscr{F}, \mathscr{E})$ is defined by

$$\langle (S_\lambda u - M_\theta w_\lambda)\oplus(\Delta_\theta S_\lambda v - \Delta_\theta w_\lambda), M_\theta w'\oplus\Delta_\theta w'\rangle = 0$$

For every $w' \in \ell^2(\mathscr{F}, \mathscr{E})$.

Hence, we find that

$$w_\lambda = M_\theta^* S_\lambda u + \Delta_\theta^2 S_\lambda v$$

and

$$(V_\lambda - T_\lambda)(u\oplus\Delta_\theta v) = M_\theta w_\lambda\oplus\Delta_\theta w_\lambda.$$

By (28) an easy computation shows that $\langle w_\lambda, S_f e^*\rangle = 0$ for every $e_* \in \mathscr{E}_*$, $f \in \mathscr{F}, f \neq 0$.Consequently, $w_\lambda \in \mathscr{E}$.

Let us show that if $u\oplus\Delta_\theta v$ varies over H and λ over Λ, then the corresponding elements w_λ vary over a set dense in $\mathscr{E}$.

It is easy to see that for $e \in \mathscr{E}$ and $\lambda \in \Lambda$ the element

$w_\lambda = M_\theta^* S_\lambda S_\lambda^* M_\theta e + \Delta_\theta C_\lambda C_\lambda^* \Delta_\theta e$ is the corresponding element of $S_\lambda^* M_\theta e\oplus C_\lambda^*\Delta_\theta e \in H$.

Thus, for $e \in \mathscr{E}$ we have

$$\sum_{\lambda\in\Lambda} w_\lambda = M_\theta^*(I - P_{\mathscr{E}_*})M_\theta e + \Delta_\theta^2 e = e - M_\theta^* P_{\mathscr{E}_*} M_\theta e \in \mathscr{E}.$$

It remains to prove that the set

$$\{(I_{\mathcal{E}} - M_\theta^* P_{\mathcal{E}_*}\theta)e\ ;\ e \in \mathcal{E}\}$$

is dense in $\mathcal{E}$.

Indeed, otherwise there exists $e' \in \mathcal{E}, e' \neq 0$ such that $e' = M_\theta^* P_{\mathcal{E}_*} M_\theta\ e'$ .It follows that $\|e'\| = \|P_{\mathcal{E}_*} M_\theta\ e'\|$, which contradicts that θ is purely contractive.

The last step is to prove that the characteristic function of T coincides with θ.

It is easy to see that the operator ω defined from $\mathcal{E}$ to L by $\omega(e) = M_\theta e \oplus \Delta_\theta e\ (e \in \mathcal{E})$ is a unitary one.

On the other hand, from (26) it follows that the operator ω_* defined from $\mathcal{E}_*$ to L_* by $\omega_*(e_*) = e_* \oplus 0 (e_* \in \mathcal{E}_*)$ is a unitary too.

According to (40), for $l = M_\theta e \oplus \Delta_\theta e\ (e \in \mathcal{E})$ we have

$$\Phi^{L_*} P^{L_*}(M_\theta e \oplus \Delta_\theta e) = \Phi^{L_*}(M_\theta e \oplus 0) = M_\theta e \oplus 0 = M_{\omega_*} M_\theta e = M_{\omega_*}\theta\omega^{-1}e.$$

Hence, using Theorem (5.1.4), we deduce that characteristic function of T coincides with θ.

The proof is completed.

Proposition (5.1.8)[132]:. Let $\theta: \mathcal{E} \to \ell^2(\mathcal{F}, \mathcal{E}_*)$ and $\theta': \mathcal{E}' \to \ell^2(\mathcal{F}, \mathcal{E}'_*)$ be some operators such that M_θ and $M_{\theta'}$ be contractions.

If θ and θ' coincide, then the sequences T and T' which they generate in the sense of Theorem (5.1.7) are unitary equivalent.

Proof. If $\chi: \mathcal{E} \to \mathcal{E}'$ and $\chi_*: \mathcal{E}_* \to \mathcal{E}'_*$ are unitary operators such that

$$M_{\chi_*}\theta = \theta'\chi$$

then $U = M_{\chi_*} \oplus M_\chi$ is a unitary operator from H to H' such that

$T'_\lambda = UT_\lambda U^*$ for every $\lambda \in \Lambda$.

The proof is just the same as in the particular case considered in the proof of Theorem (5.1.6).

Applying this result to characteristic function and by using Theorem (5.1.6) we obtain a generalization in [134] and [133], namely:

Theorem (5.1.9)[132]: Tow completely non-coisometric sequences $\mathcal{T}$ and $\mathcal{T}'$ are unitarily equivalent if and only if their characteristic function coincide.

Finally, let us show when the characteristic function is outer.

Proposition (5.1.10)[132]: For a c.n.c. sequence $\mathcal{T}$ we have that $\theta_{\mathcal{T}}$ is outer if and only if $\mathcal{T} \in C_{(2)}$.

Proof. It suffices to prove our assertion for the functional model of $\mathcal{T}$.

Accordingly, let $T = \{T_\lambda\}_{\lambda\in\Lambda}$ be the sequences defined in Theorem (2.1.7). For every $u \oplus \Delta_{\mathcal{T}} \in H$ we have

$$\lim_{\epsilon\to\infty} \sum_{f\in F(n,\Lambda)} \left\|T_f^*(u \oplus \Delta_{\mathcal{T}} v)\right\|^2 = \|\Delta_{\mathcal{T}} v\|^2.$$

This shows that $T \in C_{(2)}$. if and only if $u \oplus 0 \in H$ implies $u = 0$. On the other hand, $u \oplus 0 \in H$ means $u \perp M_{\theta_{\mathcal{T}}} \ell^2(\mathcal{F}, \mathcal{D})$.

The last condition implies $u = 0$ if and only if

$$\overline{M_{\theta_{\mathcal{T}}} \ell^2(\mathcal{F}, \mathcal{D})} = \ell^2(\mathcal{F}, \mathcal{D}_*),$$

i.e., $\theta_{\mathcal{T}}$ is outer.

Using our functional model for a c.n.c. Sequences $\mathcal{T} = \{T_\lambda\}_{\lambda\in\Lambda}$ and the lifting theorem [135], [134], [136], [133] to our setting [137], we provide explicit forms for the commutants of T .

For the sake of simplicity we only consider the case when $\mathcal{T} \in C_{(0)}$. Thus, assume that $\theta: \mathcal{E} \to \ell^2(\mathcal{F}, \mathcal{E}_*)$ is a purely contractive inner operator.

Let $T = \{T_\lambda\}_{\lambda\in\Lambda}$ be a sequence of operators defined on the Hilbert space

$$H = \ell^2(\mathcal{F}, \mathcal{E}_*) \ominus M_\theta \ell^2(\mathcal{F}, \mathcal{E}),$$

By

$$\boldsymbol{T}_\lambda^* u = S_\lambda^* u \qquad (u \in H)$$

for every $\lambda \in \Lambda$.

By Theorem (2.1.7), the Λ-orthogonal shift $S = \{S_\lambda\}_{\lambda\in\Lambda}$ acting on $K = \ell^2(\mathcal{F}, \mathcal{E}_*)$ is a minimal isometric dilation of T.

Let $H', \boldsymbol{T}'$ etc. corresponding similarly to an operator

$\theta': \mathcal{E}' \to \ell^2(\mathcal{F}, \mathcal{E}'_*)$ the same kind.

We have that every operator

$$Y: \ell^2(\mathcal{F}, \mathcal{E}'_*) \to \ell^2(\mathcal{F}, \mathcal{E}_*)$$

such that

$$S_\lambda Y = Y S_\lambda \quad (\lambda \in \Lambda)$$

can be represented in the form $Y = M_\chi$,where χ: $\mathscr{E}'_* \to \ell^2(\mathscr{F}, \mathscr{E})$ is an operator such that M_χ is bounded.

Combining this fact in [137], we obtain a generalization.

Theorem (5.1.11)[132]: every operator $X: H' \to H$ satisfying

$$T_\lambda X = X T'_\lambda \quad for\ every\ \lambda \in \Lambda. \qquad (29)$$

can be represented in the form

$$Xu = P_H M_\chi u \quad (u \in H'), \qquad (30)$$

where P_H is the orthogonal projection of $\ell^2(\mathscr{F}, \mathscr{E}_*)$ onto H, and χ: $\mathscr{E}'_* \to \ell^2(\mathscr{F}, \mathscr{E}_*)$ is an operator such that the following condition hold

a) M_χ is a bounded operator,
b) $M_\chi M_{\theta'} \ell^2(\mathscr{F}, \mathscr{E}') \subset M_\theta \ell^2(\mathscr{F}, \mathscr{E})$.

Conversely, every χ satisfying the above- mentioned condition yields, by (30), and solution X of (29).

Theorem (5.1.12)[211]: A subspace $\mathscr{M} \subset \ell^2(\mathscr{F}, \mathscr{E})$ is invariant for each $S_{(\lambda^2-1)}((\lambda^2 - 1) \in \Lambda)$ if and only if there exists a Hilbert space $\mathscr{Y}$ and the sequence of inner operators θ_j : $\mathscr{Y} \to \ell^2(\mathscr{F}, \mathscr{E})$ such that

$$\mathscr{M} = M_{\sum_{j=1}^q \theta_j} \ell^2(\mathscr{F}, \mathscr{Y})$$

Proof. Using the Wold decomposition for an infinite sequence $\mathscr{V} = \{V^*_{(\lambda^2-1)}\}_{(\lambda^2-1)\in\Lambda}$ of isometries with orthogonal final spaces. ([137]) this proof is a simple extension of that of Theorem 3.3 in [134] or Theorem 2 in [133].

Let $\mathscr{V} = \{V^*_{(\lambda^2-1)}\}_{(\lambda^2-1)\in\Lambda}$ be a Λ-orthogonal shift acting on a Hilbert space $\mathscr{K}$ such that $\mathscr{L} \subset \mathscr{K}$ is wandering subspace for, $\mathscr{V}$ that is,

$$\mathscr{K} = M_{\mathscr{F}}(\mathscr{L}) = \bigoplus_{f_m \in \mathscr{F}} V^*_{f_m} \mathscr{L}.$$

Denote by $\Phi^{\mathscr{L}}$ the unitary operator from $M_{\mathscr{F}}(\mathscr{L})$ to $\ell^2(\mathscr{F}, \mathscr{L})$ defined by

$$\Phi^{\mathscr{L}}\left(\sum_{f_m \in \mathscr{F}} V^*_{f_m} l_{f_m}\right) = \sum_{f_m \in \mathscr{F}} S_{f_m} l_{f_m} \left(l_{f_m} \in \mathscr{L}; \sum_{f_m \in \mathscr{F}} \left\|l_{f_m}\right\|^2 < \infty\right),$$

where $\mathscr{Y} = \{S_{(\lambda^2-1)}\}_{(\lambda^2-1)\in\Lambda}$ is the Λ-orthogonal shift acting on $\ell^2(\mathscr{F}, \mathscr{L})$

Then for any $(\lambda^2 - 1) \in \Lambda$ we have

$$\Phi^{\mathscr{L}} V^*_{(\lambda^2-1)} = S_{(\lambda^2-1)} \Phi^{\mathscr{L}}.$$

The following extension [132] of Lemma 3.2 in [134] will be used in the sequel.

Theorem (5.1.13)[211]: The sequence of the characteristic functions $(\theta_j)_{\mathscr{T}}$ for $\mathscr{T}$ coincides with the sequence of $(\theta_j)_{\mathscr{L}}$.

Proof. We show that

$$M_{\Phi_*}\left(\sum_{j=1}^{q} \theta_j\right)_{\mathscr{L}} = \left(\sum_{j=1}^{q} \theta_j\right)_{\mathscr{T}} \Phi, \tag{31}$$

Where Φ_*, Φ are the unitary operators in (11), (12), respectively. For this, it is necessarily to prove that

$$P^2_{\mathscr{D}_*} S^*_{f_m} M_{\Phi_*}\left(\sum_{j=1}^{q} \theta_j\right)_{\mathscr{L}} = P^2_{\mathscr{D}_*} S^*_{f_m}\left(\sum_{j=1}^{q} \theta_j\right)_{\mathscr{T}} \Phi \quad (f_m \in \mathscr{F}), \tag{32}$$

Where $P^2_{\mathscr{D}_*}$ stands for the orthogonal projection of $\ell^2(\mathscr{F}, \mathscr{D}_*)$ onto $\mathscr{D}_*$.

By (10) and by the Wold decomposition (5), the relation (32) is equivalent to

$$\Phi_* P^2_{\mathscr{L}_*} V^*_{f_m} | \mathscr{L} = P^2_{\mathscr{D}_*} S^*_{f_m}\left(\sum_{j=1}^{q} \theta_j\right)_{\mathscr{T}} \Phi \quad (f_m \in \mathscr{F}). \tag{33}$$

In what follows we shall prove this relation. First let us notice that

$$P^2_{\mathscr{D}_*}\left(\sum_{j=1}^{q} \theta_j\right)_{\mathscr{T}} = -\sum_{(\lambda^2-1)\in\Lambda} T^*_{(\lambda^2-1)} P^2_{(\lambda^2-1)}, \tag{34}$$

$$P^2_{\mathscr{D}_*} S^*_{f_m} S_{(\lambda^2-1)}\left(\sum_{j=1}^{q} \theta_j\right)_{\mathscr{T}} = D_* T^*_{f_m} P^2_{(\lambda^2-1)} D \quad \left((\lambda^2-1) \in \Lambda, f_m \in \mathscr{F}\right).$$

For $f_m = 0$ the relation (33) holds true. Indeed, for

$$l = \sum_{(\lambda^2-1)\in\Lambda} \left(V^*_{(\lambda^2-1)} - T^*_{(\lambda^2-1)}\right) h_{(\lambda^2-1)}$$

$$= \Phi^* D\left(\bigoplus_{(\lambda^2-1)\in\Lambda} h_{(\lambda^2-1)}\right) \left(\sum_{(\lambda^2-1)\in\Lambda} \left\|h_{(\lambda^2-1)}\right\|^2 < \infty\right) \tag{35}$$

we have that $l + \left(I - \sum_{(\lambda^2-1)\in\Lambda} V^*_{(\lambda^2-1)} T^*_{(\lambda^2-1)}\right) \sum_{(\lambda^2-1)\in\Lambda} T^*_{(\lambda^2-1)} h_{(\lambda^2-1)} \in \bigoplus_{(\lambda^2-1)\in\Lambda} V^*_{(\lambda^2-1)} \mathscr{K}$ and by (33) we obtain that

$$P^2_{\mathscr{L}_*} l = -\left(I - \sum_{(\lambda^2-1)\in\Lambda} V^*_{(\lambda^2-1)} T^*_{(\lambda^2-1)}\right) \sum_{(\lambda^2-1)\in\Lambda} T^*_{(\lambda^2-1)} h_{(\lambda^2-1)}.$$

Hence, by (34) we have

$$\begin{aligned}\Phi_* P^2_{\mathscr{L}_*} l &= -D_*[T^*_1, T^*_2, \ldots]\left(\bigoplus_{(\lambda^2-1)\in\Lambda} h_{(\lambda^2-1)}\right)\\ &= [T^*_1, T^*_2, \ldots] D\left(\bigoplus_{(\lambda^2-1)\in\Lambda} h_{(\lambda^2-1)}\right)\\ &= -[T^*_1, T^*_2, \ldots]\Phi l = P^2_{\mathscr{D}_*}\left(\sum_{j=1}^{q}\theta_j\right)_{\mathscr{F}} \Phi l.\end{aligned}$$

It remains to show that for any $f_m \in \mathscr{F}$, $(\lambda^2-1)\in\Lambda$

$$\Phi_* P^2_{\mathscr{L}_*} V^*_{f_m} V^*_{(\lambda^2-1)} l = P^2_{\mathscr{D}_*}\ S^*_{f_m} S^*_{(\lambda^2-1)}\left(\sum_{j=1}^{q}\theta_j\right)_{\mathscr{F}} \Phi l \quad (l \in \mathscr{L}). \tag{36}$$

Let l be as in (35); then according to (34) the relation (36) becomes

$$\Phi_* P^2_{\mathscr{L}_*} V^*_{f_m} V^*_{(\lambda^2-1)} l = D_* T^*_{f_m} P^2_{(\lambda^2-1)} D^2\left(\bigoplus_{(\lambda^2-1)\in\Lambda} h_{(\lambda^2-1)}\right).$$

Since

$$\begin{aligned}&D_* T^*_{f_m} P_{(\lambda^2-1)} D^2\left(\bigoplus_{(\lambda^2-1)\in\Lambda} h_{(\lambda^2-1)}\right)\\ &\qquad = \Phi_*\left(I - \sum_{(\lambda^2-1)\in\Lambda} V^*_{(\lambda^2-1)} T^*_{(\lambda^2-1)}\right) T^*_{f_m} P^2_{(\lambda^2-1)} D^2\left(\bigoplus_{(\lambda^2-1)\in\Lambda} h_{(\lambda^2-1)}\right),\end{aligned}$$

we have only to show that

$$P^2_{\mathscr{L}_*} V^*_{f_m} V^*_{(\lambda^2-1)} l = \left(I - \sum_{(\lambda^2-1)\in\Lambda} V^*_{(\lambda^2-1)} T^*_{(\lambda^2-1)}\right) T^*_{f_m} P^2_{(\lambda^2-1)} D^2\left(\bigoplus_{(\lambda^2-1)\in\Lambda} h_{(\lambda^2-1)}\right). \tag{37}$$

Let us notice that for any $(\lambda^2-1)\in\Lambda$

$$P^2_{(\lambda^2-1)}D^2\left(\bigoplus_{(\lambda^2-1)\in\Lambda} h_{(\lambda^2-1)}\right)$$
$$= -\sum_{\substack{(\lambda^2-1)+\epsilon\in\Lambda\\ \epsilon>0}} T^*_{(\lambda^2-1)}\,T^*_{(\lambda^2-1)+\epsilon}h_{(\lambda^2-1)+\epsilon} + D^2_{T^*_{(\lambda^2-1)}}h_{(\lambda^2-1)}.$$

Consequently, the relation (37) holds if and only if the following relations hold

$$P^2_{\mathscr{L}_*}V^*_{fm}V^*_{(\lambda^2-1)}\left(V^*_{(\lambda^2-1)}h_{(\lambda^2-1)} - T^*_{(\lambda^2-1)}h_{(\lambda^2-1)}\right)$$
$$= \left(I - \sum_{(\lambda^2-1)\in\Lambda} V^*_{(\lambda^2-1)}T^*_{(\lambda^2-1)}\right)T^*_{fm}D^2_{T^*_{(\lambda^2-1)}}h_{(\lambda^2-1)} \quad \left((\lambda^2-1)\in\Lambda\right)$$

and

$$P^2_{\mathscr{L}_*}V^*_{fm}V^*_{(\lambda^2-1)}\left(V^*_{(\lambda^2-1)+\epsilon}h_{(\lambda^2-1)+\epsilon} - T^*_{(\lambda^2-1)+\epsilon}h_{(\lambda^2-1)+\epsilon}\right)$$
$$= -\left(I - \sum_{(\lambda^2-1)\in\Lambda} V^*_{(\lambda^2-1)}T^*_{(\lambda^2-1)}\right)T^*_{fm}T^*_{(\lambda^2-1)}T^*_{(\lambda^2-1)+\epsilon}h_{(\lambda^2-1)+\epsilon} \quad (\epsilon > 0).$$

These relations hold since the element

$$V^*_{fm}V^*_{(\lambda^2-1)}\left(V^*_{(\lambda^2-1)}h_{(\lambda^2-1)} - T^*_{(\lambda^2-1)}h_{(\lambda^2-1)}\right)$$
$$-\left(I - \sum_{(\lambda^2-1)\in\Lambda} V^*_{(\lambda^2-1)}T^*_{(\lambda^2-1)}\right)T^*_{fm}D_{T^*_{(\lambda^2-1)}}h_{(\lambda^2-1)} \quad \left((\lambda^2-1)\in\Lambda\right)$$

and

$$V^*_{fm}V^*_{(\lambda^2-1)}\left(V^*_{(\lambda^2-1)+\epsilon}h_{(\lambda^2-1)+\epsilon} - T^*_{(\lambda^2-1)+\epsilon}h_{(\lambda^2-1)+\epsilon}\right)$$
$$+\left(I - \sum_{(\lambda^2-1)\in\Lambda} V^*_{(\lambda^2-1)}T^*_{(\lambda^2-1)}\right)T^*_{fm}T^*_{(\lambda^2-1)}T^*_{(\lambda^2-1)+\epsilon}h_{(\lambda^2-1)+\epsilon} \quad (\epsilon > 0)$$

are orthogonal on $\mathscr{L}_*$. This follows by simple computation .

The proof is complete.

Remark (5.1.14)[211]: if $\mathscr{T}\in C_{(0)}$ then the sequence$\left(\theta_j\right)_{\mathscr{T}}$is inner.

Proof. Taking into account Theorem 2.8 in [137], it follows that the m.i.d. $\mathscr{V}$ of T is pure, i.e., $\mathscr{K} = M_{\mathscr{T}}(\mathscr{L}_*)$. By relation (10) and Theorem (5.1.13) it follows that the sequence $\left(\theta_j\right)_{\mathscr{T}}$is inner.

Theorem(5.1.15)[211]: Every completely non-isometric sequence $\mathscr{T}=\{T^*_{(\lambda^2-1)}\}_{(\lambda^2-1)\in\Lambda}$ on the Hilbert space $\mathscr{H}$ is unitarily equivalent to a sequence $\boldsymbol{T}^* = \{\boldsymbol{T}^*_{(\lambda^2-1)}\}_{(\lambda^2-1)\in\Lambda}$ on the Hilbert space

$$H = \left[\ell^2(\mathscr{F}, \mathscr{D}_*)\oplus \overline{\Delta_{\mathscr{T}}\ell^2(\mathscr{F}, \mathscr{D})}\right] \ominus \left\{M_{\left(\sum_{j=1}^q \theta_j\right)_{\mathscr{T}}} u\oplus \Delta_{\mathscr{T}}u; u \in \ell^2(\mathscr{F}, \mathscr{D})\right\},$$

where $\Delta_{\mathscr{T}}= \left(I - M^*_{\left(\sum_{j=1}^q \theta_j\right)_{\mathscr{T}}} M_{\left(\sum_{j=1}^q \theta_j\right)_{\mathscr{T}}}\right)^{1/2}$.

For each $(\lambda^2-1) \in \Lambda$ the operator $\boldsymbol{T}^*_{(\lambda^2-1)}$ is defined by

$$\boldsymbol{T}^*_{(\lambda^2-1)}\left(u\oplus\Delta_{\mathscr{T}}S_{(\lambda^2-1)+\epsilon}v\right) = \begin{cases} S^*_{(\lambda^2-1)}u \oplus\Delta_{\mathscr{T}}u & \text{if} \quad \epsilon > 0, \\ S^*_{(\lambda^2-1)}u \oplus 0 & \text{if} \quad \epsilon > 0, \end{cases}$$

where $\mathscr{S}= \{S_{(\lambda^2-1)}\}_{(\lambda^2-1)\in\Lambda}$ is the Λ-orthogonal shift acting on $\ell^2(\mathscr{F}, \mathscr{D})$or $\ell^2(\mathscr{F}, \mathscr{D}_*)$.

If $\mathscr{T}\in C_{(0)}$, and only in this case, the sequence $(\theta_j)_{\mathscr{T}}$ is inner, and this model reduces to

$$H = \ell^2(\mathscr{F}, \mathscr{D}_*) \ominus M_{\left(\sum_{j=1}^q \theta_j\right)_{\mathscr{T}}} \ell^2(\mathscr{F}, \mathscr{D}); \;\; \boldsymbol{T}^*_{(\lambda^2-1)}u = S^*_{(\lambda^2-1)}u \quad (u \in H).$$

Proof. By virtue of the relation (13) it follows that

$$M_{\Phi_*}M_{\left(\sum_{j=1}^q \theta_j\right)_{\mathscr{L}}} = M_{\left(\sum_{j=1}^q \theta_j\right)_{\mathscr{T}}} M_{\Phi}.$$

Hence we obtain that $\Delta_{\mathscr{T}}= M_{\Phi}\Delta_{\mathscr{L}}M_{\Phi_*}$

On the other hand the operators Φ and Φ_* defined by (11) and (12) generate the unitary operator

$$U = M_{\Phi_*}\oplus M_{\Phi}$$

from the space $\ell^2(\mathscr{F}, \mathscr{L}_*)\oplus \overline{\Delta_{\mathscr{L}}\ell^2(\mathscr{F}, \mathscr{L})}$ to the space $\ell^2(\mathscr{F}, \mathscr{D}_*)\oplus \overline{\Delta_{\mathscr{T}}\ell^2(\mathscr{F}, \mathscr{D})}$ Such that

$$U\left\{M_{\left(\sum_{j=1}^q \theta_j\right)_{\mathscr{L}}} u\oplus \Delta_{\mathscr{L}}u \,; u \in \ell^2(\mathscr{F}, \mathscr{L})\right\}$$
$$= \left\{M_{\left(\sum_{j=1}^q \theta_j\right)_{\mathscr{T}}} v\oplus \Delta_{\mathscr{T}}v \,; v \in \ell^2(\mathscr{F}, \mathscr{D})\right\}.$$

By means of this unitary operator we can rewrite the result obtained before this theorem and; in this way, we complete the proof.

Let us remark that for $\mathscr{T} = \{T^*\}$, we find a model for completely non-coisometric contractions, which coincides with the Sz-Nagy-Foias model. Indeed, if T^* is a completely non-coisometric contraction, that is, if there is no non-zero invariant subspace for T^* on which T^* is an isometry, then it is easy to see that

$$\Delta_{T^*} H^2(\mathscr{D}_{\mathrm{T}^*}) = \Delta_{T^*} L^2(\mathscr{D}_{\mathrm{T}^*})$$

(see Theorem2.3 in [134]).

Note that the Sz.-Nagy-Foias, model is given for a larger class of contractions, namely for completely non-unitary contractions.

Theorem (5.1.16)[211]: Let the sequence θ_j be a contraction from $\mathscr{E}$ to $\ell^2(\mathscr{F}, \mathscr{E}_*)$ such that $M_{\sum_{j=1}^q \theta_j}$ is a contraction. Setting $\Delta_{\sum_{j=1}^q \theta_j} = \left(I - M^*_{\sum_{j=1}^q \theta_j} M_{\sum_{j=1}^q \theta_j}\right)^{1/2}$ the sequences $\boldsymbol{T}^* = \left\{\boldsymbol{T}^*_{(\lambda^2-1)}\right\}_{(\lambda^2-1)\in\Lambda}$ of operator defined on the Hilbert space

$$H = \left[\ell^2(\mathscr{F}, \mathscr{E}_*) \oplus \overline{\Delta_{\sum_{j=1}^q \theta_j} \ell^2(\mathscr{F}, \mathscr{E})}\right] \ominus \left\{M_{\sum_{j=1}^q \theta_j} w \oplus \Delta_{\sum_{j=1}^q \theta_j} w \,; w \in \ell^2(\mathscr{F}, \mathscr{E})\right\}$$

by

$$\boldsymbol{T}^*_{(\lambda^2-1)}\left(u \oplus \Delta_{\sum_{j=1}^q \theta_j} v\right) = S^*_{(\lambda^2-1)} u \oplus C^*_{(\lambda^2-1)}\left(\Delta_{\sum_{j=1}^q \theta_j} v\right) \qquad \left((\lambda^2-1) \in \Lambda\right),$$

where each operator $C_{(\lambda^2-1)}$ is defined by $C_{(\lambda^2-1)}\left(\Delta_{\sum_{j=1}^q \theta_j} f_{m+1}\right) = \Delta_{\sum_{j=1}^q \theta_j} S_{(\lambda^2-1)} f_{m+1}\left(f_{m+1} \in \ell^2(\mathscr{F}, \mathscr{E})\right)$ and $\mathscr{S} = \left\{S_{(\lambda^2-1)}\right\}_{(\lambda^2-1)\in\Lambda}$ is the Λ-orthogonal shift action on $\ell^2(\mathscr{F}, \mathscr{E})$ or $\ell^2(\mathscr{F}, \mathscr{E}_*)$ is completely Non-coisometric.

If the sequence θ_j is purely contractive and (20) holds, then the sequence θ_j coincides with the characteristic functions of $\mathscr{T}$ In this case, considering H as a subspace of

$$K = \ell^2(\mathscr{F}, \mathscr{E}_*) \oplus \overline{\Delta_{\sum_{j=1}^q \theta_j} \ell^2(\mathscr{F}, \mathscr{E})}$$

we have that the sequence $\boldsymbol{V}^* = \left\{\boldsymbol{V}^*_{(\lambda^2-1)}\right\}_{(\lambda^2-1)\in\Lambda}$ of operator defined on K by

$$\boldsymbol{V}^*_{(\lambda^2-1)} = S_{(\lambda^2-1)} \oplus C_{(\lambda^2-1)} \quad \left((\lambda^2-1) \in \Lambda\right)$$

is the minimal isometric dilation of $\boldsymbol{T}^*$.

Proof. A. Let us consider the following Hilbert spaces

$$K = \ell^2(\mathscr{F}, \mathscr{E}_*) \oplus \overline{\Delta_{\sum_{j=1}^q \theta_j} \ell^2(\mathscr{F}, \mathscr{E})},$$

$$G = \left\{M_{\sum_{j=1}^q \theta_j} w \oplus \Delta_{\sum_{j=1}^q \theta_j} w \,; w \in \ell^2(\mathscr{F}, \mathscr{E})\right\},$$

and let $\boldsymbol{V}^* = \{\boldsymbol{V}^*_{(\lambda^2-1)}\}_{(\lambda^2-1)\in\Lambda}$ be a sequence of isometries defined on K by $\boldsymbol{V}^*_{(\lambda^2-1)} = S_{(\lambda^2-1)} \oplus C_{(\lambda^2-1)}$ $((\lambda^2-1)\in\Lambda)$, where each $C_{(\lambda^2-1)}$ is given by

$$C_{(\lambda^2-1)}\left(\Delta_{\sum_{j=1}^{q}\theta_j} f_{m+1}\right) = \Delta_{\sum_{j=1}^{q}\theta_j} S_{(\lambda^2-1)} f_{m+1} \qquad for\ f_{m+1} \in \ell^2(\mathscr{F},\mathscr{E}).$$

It is easy to see that

$$\sum_{(\lambda^2-1)\in\Lambda} \left(\boldsymbol{V}^*_{(\lambda^2-1)}\right)^2 \le I_K$$

and that G is invariant for each $\boldsymbol{V}^*_{(\lambda^2-1)}((\lambda^2-1)\in\Lambda)$.

Setting $H = K \ominus G$ and $\boldsymbol{T}^*_{(\lambda^2-1)} = \boldsymbol{V}^*_{(\lambda^2-1)}|H\big((\lambda^2-1)\in\Lambda\big)$ we see that $\boldsymbol{V}^*$ is an isometric dilation of $\boldsymbol{T}^* = \{\boldsymbol{T}^*_{(\lambda^2-1)}\}_{(\lambda^2-1)\in\Lambda}$.

Let us show that $\boldsymbol{T}^*$ is c.n.c. For this, let $u\oplus\Delta_{\sum_{j=1}^{q}\theta_j} v \in H$ such that for every $\epsilon \ge 0$ we have

$$\sum_{f_m\in F(1+\epsilon,\Lambda)} \left\| T^*_{f_m}\left(u\oplus\Delta_{\sum_{j=1}^{q}\theta_j} v\right)\right\|^2 = \left\|u\oplus\Delta_{\sum_{j=1}^{q}\theta_j} v\right\|^2. \tag{38}$$

Since

$$\lim_{\epsilon\to\infty} \sum_{f_m\in F(1+\epsilon,\Lambda)} \left\|S^*_{f_m} u\right\|^2 = 0 \ \ and \qquad \sum_{f_m\in F(1+\epsilon,\Lambda)} \left\|C^*_{f_m}\Delta_{\sum_{j=1}^{q}\theta_j} v\right\|^2 \le \left\|\Delta_{\sum_{j=1}^{q}\theta_j} v\right\|^2$$

it follows that $u = 0$. But, $\left(0\oplus\Delta_{\sum_{j=1}^{q}\theta_j} v, M_{\sum_{j=1}^{q}\theta_j} w \oplus \Delta_{\sum_{j=1}^{q}\theta_j} w\right) = 0$ for any $w \in \ell^2(\mathscr{F},\mathscr{E})$ implies $\Delta_{\sum_{j=1}^{q}\theta_j} = 0$.

Thus $\boldsymbol{T}^*$ is c.n.c.

B. We assume from now on that the sequence θ_j is purely contractive and that (20) holds.

Let us show that $\boldsymbol{V}^*$ is m.i.d. of $\boldsymbol{T}^*$, i.e.

$$K = \bigvee_{f_m\in\mathscr{F}} V^*_{f_m} H.$$

First we note that (20) implies

$$\sum_{(\lambda^2-1)\in\Lambda} C_{(\lambda^2-1)} C^*_{(\lambda^2-1)} = I_{\overline{\Delta_{\sum_{j=1}^{q}\theta_j}\ell^2(\mathscr{F},\mathscr{E})}}. \tag{39}$$

Suppose $u\oplus\Delta_{\sum_{j=1}^{q}\theta_j} v \in K$ and for every $f_m \in \mathscr{F}$, $u\oplus\Delta_{\sum_{j=1}^{q}\theta_j} v \perp \boldsymbol{V}^*_{f_m} H$ i.e.,

$\boldsymbol{V}^*_{f_m}\left(u\oplus\Delta_{\sum_{j=1}^q\theta_j}v\right)\in G.$

This means that for each $f_m\in\mathscr{F}$ there exists $w_{(f_m)}\in\ell^2(\mathscr{F},\mathscr{E})$ such that

$$\boldsymbol{V}^*_{f_m}\left(u\oplus\Delta_{\sum_{j=1}^q\theta_j}v\right)=M_{\sum_{j=1}^q\theta_j}w_{(f_m)}\oplus\Delta_{\sum_{j=1}^q\theta_j}w_{(f_m)}.$$

Therefore, for each $(\lambda^2-1)\in\Lambda, f_m\in\mathscr{F}$ there exists $w_{\left(f_m,(\lambda^2-1)\right)}\in\ell^2(\mathscr{F},\mathscr{E})$ such that

$$\boldsymbol{V}^*_{(\lambda^2-1)}\left(M_{\sum_{j=1}^q\theta_j}w_{(f_m)}\oplus\Delta_{\sum_{j=1}^q\theta_j}w_{(f_m)}\right)=M_{\sum_{j=1}^q\theta_j}w_{\left(f_m,(\lambda^2-1)\right)}\oplus\Delta_{\sum_{j=1}^q\theta_j}w_{\left(f_m,(\lambda^2-1)\right)}.$$

By using the definition of $\boldsymbol{V}^*_{(\lambda^2-1)}((\lambda^2-1)\in\Lambda)$ we obtain

$$\left(\sum_{(\lambda^2-1)\in\Lambda}S_{(\lambda^2-1)}S^*_{(\lambda^2-1)}\right)M_{\sum_{j=1}^q\theta_j}w_{(f_m)}\oplus\left(\sum_{(\lambda^2-1)\in\Lambda}C_{(\lambda^2-1)}C^*_{(\lambda^2-1)}\right)\Delta_{\sum_{j=1}^q\theta_j}w_{(f_m)}$$

$$=M_{\sum_{j=1}^q\theta_j}\left(\sum_{(\lambda^2-1)\in\Lambda}S_{(\lambda^2-1)}\,w_{\left(f_m,(\lambda^2-1)\right)}\right)\oplus\Delta_{\sum_{j=1}^q\theta_j}\left(\sum_{(\lambda^2-1)\in\Lambda}S_{(\lambda^2-1)}\,w_{\left(f_m,(\lambda^2-1)\right)}\right).$$

Hence, according to (39), we have

$$M_{\sum_{j=1}^q\theta_j}\omega_{(f_m)}=P^2_{\mathscr{E}_*}M_{\sum_{j=1}^q\theta_j}w_{(f_m)}\ \ and\ \ \Delta_{\sum_{j=1}^q\theta_j}\omega_{(f_m)}=0,\qquad(40)$$

where $\omega_{(f_m)}$ stands for $w_{(f_m)}-\sum_{(\lambda^2-1)\in\Lambda}S_{(\lambda^2-1)}\,w_{\left(f_m,(\lambda^2-1)\right)}$.

Since $M_{\sum_{j=1}^q\theta_j}$ commutes with each $S_{(\lambda^2-1)}((\lambda^2-1)\in\Lambda)$ it follows that

$$P^2_{\mathscr{E}_*}M_{\sum_{j=1}^q\theta_j}w_{(f_m)}=P^2_{\mathscr{E}_*}M_{\sum_{j=1}^q\theta_j}P^2_{\mathscr{E}}\,w_{(f_m)}$$

and (40) gives

$$\omega_{(f_m)}=M^*_{\sum_{j=1}^q\theta_j}P^2_{\mathscr{E}_*}M_{\sum_{j=1}^q\theta_j}P^2_{\mathscr{E}}\,w_{(f_m)},\qquad(41)$$

hence $P^2_{\mathscr{E}}\,w_{(f_m)}=P^2_{\mathscr{E}}\,\omega_{(f_m)}=P^2_{\mathscr{E}}\,M^*_{\sum_{j=1}^q\theta_j}P^2_{\mathscr{E}_*}M_{\sum_{j=1}^q\theta_{\square}}P^2_{\mathscr{E}}w_{(f_m)}$.

Consequently, $\left\|P^2_{\mathscr{E}}\,w_{(f_m)}\right\|=\left\|P^2_{\mathscr{E}_*}M_{\sum_{j=1}^q\theta_j}P^2_{\mathscr{E}}\,w_{(f_m)}\right\|$ and since the sequence θ_j is purely contractive it follows that

$$P^2_{\mathscr{E}}\,w_{(f_m)}=0.\qquad(42)$$

Now, the relation (41) implies $\omega_{(f_m)}=0$, i.e.

$$w_{(f_m)} = \sum_{(\lambda^2-1)\in\Lambda} S_{(\lambda^2-1)}\, w_{\left(f_m,(\lambda^2-1)\right)} \quad for \quad f_m \in \mathscr{F}.$$

Hence, we obtain that

$$\begin{aligned} w_{\left(\sum_{j=1}^{q}\theta_j\right)} &= \sum_{(\lambda^2-1)\in\Lambda} S_{(\lambda^2-1)}\, w_{(\lambda^2-1)} \\ &= \sum_{(\lambda^2-1)\in\Lambda} S_{(\lambda^2-1)} \left(\sum_{(\lambda^2-1)+\epsilon\in\Lambda} S_{(\lambda^2-1)+\epsilon}\, w_{\left((\lambda^2-1),(\lambda^2-1)+\epsilon\right)} \right) \\ &= \sum_{f_{m+1}\in F(2,\Lambda)} S_{f_{m+1}}\, w_{(f_{m+1})} = \cdots \\ &= \sum_{f_m\in F(1+\epsilon,\Lambda)} S_{f_m}\, w_{(f_m)} \quad \text{for any } \epsilon \geq 0. \end{aligned}$$

We deduce that $S_{f_m}^{*} w_{\left(\sum_{j=1}^{q}\theta_j\right)} = w_{(f_m)}$ for every $f_m \in \mathscr{F}$. By (42) we find

$P_{\mathscr{E}}^{2} S_{f_m}^{*} w_{\left(\sum_{j=1}^{q}\theta_j\right)} = P_{\mathscr{E}}^{2} w_{f_m} = 0$ for every $f_m \in \mathscr{F}$.

It follows that $w_{\left(\sum_{j=1}^{q}\theta_j\right)} = 0$ and
$u \oplus \Delta_{\sum_{j=1}^{q}\theta_j} v = M_{\sum_{j=1}^{q}\theta_j} w_{\left(\sum_{j=1}^{q}\theta_j\right)} \oplus \Delta_{\sum_{j=1}^{q}\theta_j} w_{\left(\sum_{j=1}^{q}\theta_j\right)} = 0$, which implies the minimality of $\boldsymbol{V}^{*}$.

C. Our next step is to determine

$$L_{*} = \overline{\left(I_H - \sum_{(\lambda^2-1)\in\Lambda} \boldsymbol{V}_{(\lambda^2-1)}^{*} \boldsymbol{T}_{(\lambda^2-1)}^{*} \right) H}.$$

Taking into account (39), for $u \oplus \Delta_{\sum_{j=1}^{q}\theta_j} v \in H$ we have

$$\left(I_H - \sum_{(\lambda^2-1)\in\Lambda} \boldsymbol{V}_{(\lambda^2-1)}^{*} \boldsymbol{T}_{(\lambda^2-1)}^{*} \right) \left(u \oplus \Delta_{\sum_{j=1}^{q}\theta_j} v \right) = P_{\mathscr{E}_*}^{2} u \oplus 0$$

and hence $L_{*} \subset \mathscr{E}_{*} \oplus \{0\}$.

Let $e_{*} \in \mathscr{E}_{*}$ and let us choose $u = \left(I - M_{\sum_{j=1}^{q}\theta_j} M_{\sum_{j=1}^{q}\theta_j}^{*} \right) e_{*}$ and

$\Delta_{\sum_{j=1}^{q}\theta_j} v = -\Delta_{\sum_{j=1}^{q}\theta_j} M_{\sum_{j=1}^{q}\theta_j}^{*} e_{*}$. Since $M_{\sum_{j=1}^{q}\theta_j}^{*} u + \Delta_{\sum_{j=1}^{q}\theta_j}^{3} v = 0$ it follows that $u \oplus \Delta_{\sum_{j=1}^{q}\theta_j} v \in H$.

Thus,

$$\left(I_H - \sum_{(\lambda^2-1)\in\Lambda} \boldsymbol{V}^*_{(\lambda^2-1)}\boldsymbol{T}^*_{(\lambda^2-1)}\right)\left(u\oplus\Delta_{\sum_{j=1}^q\theta_j}v\right)$$
$$= \left(I_{\mathscr{E}_*} - P^2_{\mathscr{E}_*}M_{\sum_{j=1}^q\theta_j}M^*_{\sum_{j=1}^q\theta_j}\right)e_*\oplus 0.$$

Now the element of the form $\left(I_{\mathscr{E}_*} - P^2_{\mathscr{E}_*}M_{\sum_{j=1}^q\theta_j}M^*_{\sum_{j=1}^q\theta_j}\right)e_*, (e_*\in\mathscr{E}_*)$, are dense in $\mathscr{E}_*$.

Otherwise there exist an $e'_*\in\mathscr{E}_*, e'_*\neq 0$, such that $e'_* = P^2_{\mathscr{E}_*}M_{\sum_{j=1}^q\theta_j}M^*_{\sum_{j=1}^q\theta_j}e'_*$ and hence $\|e'_*\| = \left\|M^*_{\sum_{j=1}^q\theta_j}e'_*\right\| = \left\|P^2_{\mathscr{E}_*}M_{\sum_{j=1}^q\theta_j}M^*_{\sum_{j=1}^q\theta_j}e'_*\right\|$; $e'_* = M_{\sum_{j=1}^q\theta_j}M^*_{\sum_{j=1}^q\theta_j}e'_*$. Since $M^*_{\sum_{j=1}^q\theta_j}e'_*\in\mathscr{E}_*$ and the sequence θ_j is purely contractive it follows that $M^*_{\sum_{j=1}^q\theta_j}e'_* = 0$ and $e'_* = 0$ which is a contradiction.

Thus

$$L_* = \mathscr{E}_*\oplus\{0\} \tag{43}$$

and $M_{\mathscr{F}}(L_*) = \ell^2(\mathscr{F},\mathscr{E}_*)\oplus\{0\}$.

Denoting by P^{2L_*} the orthogonal projection of K onto $M_{\mathscr{F}}(L_*)$, we have for $u\oplus\Delta_{\sum_{j=1}^q\theta_j}v\in K$

$$P^{2L_*}\left(u\oplus\Delta_{\sum_{j=1}^q\theta_j}v\right) = u\oplus 0, \tag{44}$$

$$\Phi^{L_*}P^{2L_*}\left(u\oplus\Delta_{\sum_{j=1}^q\theta_j}v\right) = \Phi^{\mathscr{E}_*}u\oplus 0 = u\oplus 0.$$

D. Next we show that

$$L = \bigvee_{(\lambda^2-1)\in\Lambda}\left(\boldsymbol{V}^*_{(\lambda^2-1)} - \boldsymbol{T}^*_{(\square^2-1)}\right)H = \left\{M_{\sum_{j=1}^q\theta_j}e\oplus\Delta_{\sum_{j=1}^q\theta_j}e\,;e\in\mathscr{E}\right\}.$$

Notice that an element $u\oplus\Delta_{\sum_{j=1}^q\theta_j}v$ in K belongs to H if and only if

$$M^*_{\sum_{j=1}^q\theta_j}u\oplus\Delta^2_{\sum_{j=1}^q\theta_j}v = 0. \tag{45}$$

For $u\oplus\Delta_{\sum_{j=1}^q\theta_j}v\in H$ and $(\lambda^2-1)\in\Lambda$ we have

$$\boldsymbol{T}^*_{(\lambda^2-1)}\left(u\oplus\Delta_{\sum_{j=1}^q\theta_j}v\right) = P^2_H\boldsymbol{V}^*_{(\lambda^2-1)}\left(u\oplus\Delta_{\sum_{j=1}^q\theta_j}v\right)$$
$$= \left(S_{(\lambda^2-1)}u\oplus\Delta_{\sum_{j=1}^q\theta_j}S_{(\lambda^2-1)}v\right) - \left(M_{\sum_{j=1}^q\theta_j}w_{(\lambda^2-1)}\oplus\Delta_{\sum_{j=1}^q\theta_j}w_{(\lambda^2-1)}\right),$$

where each $w_{(\lambda^2-1)}\in\ell^2(\mathscr{F},\mathscr{E})$ is defined by

$$\langle\left(S_{(\lambda^2-1)}u - M_{\sum_{j=1}^q \theta_j} w_{(\lambda^2-1)}\right) \oplus \left(\Delta_{\sum_{j=1}^q \theta_j} S_{(\lambda^2-1)} v - \Delta_{\sum_{j=1}^q \theta_j} w_{(\lambda^2-1)}\right), M_{\sum_{j=1}^q \theta_j} w' \oplus \Delta_{\sum_{j=1}^q \theta_j} w'\rangle = 0$$

for every $w' \in \ell^2(\mathscr{F}, \mathscr{E})$.

Hence, we find that

$$w_{(\lambda^2-1)} = M^*_{\sum_{j=1}^q \theta_j} S_{(\lambda^2-1)} u + \Delta^2_{\sum_{j=1}^q \theta_j} S_{(\lambda^2-1)} v$$

and

$$\left(\boldsymbol{V}^*_{(\lambda^2-1)} - \boldsymbol{T}^*_{(\lambda^2-1)}\right)\left(u \oplus \Delta_{\sum_{j=1}^q \theta_j} v\right) = M_{\sum_{j=1}^q \theta_j} w_{(\lambda^2-1)} \oplus \Delta_{\sum_{j=1}^q \theta_j} w_{(\lambda^2-1)}.$$

By (45) an easy computation shows that $\langle w_{(\lambda^2-1)}, S_{f_m} e_*\rangle = 0$ for every $e_* \in \mathscr{E}_*$, $f_m \in \mathscr{F}, f_m \neq 0$. Consequently, $w_{(\lambda^2-1)} \in \mathscr{E}$.

Let us show that if $u \oplus \Delta_{\sum_{j=1}^q \theta_j} v$ varies over H and $(\lambda^2 - 1)$ over Λ, then the corresponding elements $w_{(\lambda^2-1)}$ vary over a set dense in $\mathscr{E}$.

It is easy to see that for $e \in \mathscr{E}$ and $(\lambda^2 - 1) \in \Lambda$ the element $w_{(\lambda^2-1)} = M^*_{\sum_{j=1}^q \theta_j} S_{(\lambda^2-1)} S^*_{(\lambda^2-1)} M_{\sum_{j=1}^q \theta_j} e + \Delta_{\sum_{j=1}^q \theta_j} C_{(\lambda^2-1)} C^*_{(\lambda^2-1)} \Delta_{\sum_{j=1}^q \theta_j} e$ is the corresponding element of $S^*_{(\lambda^2-1)} M_{\sum_{j=1}^q \theta_j} e \oplus C^*_{(\lambda^2-1)} \Delta_{\sum_{j=1}^q \theta_j} e \in H$.

Thus, for $e \in \mathscr{E}$ we have

$$\begin{aligned}\sum_{(\lambda^2-1)\in\Lambda} w_{(\lambda^2-1)} &= M^*_{\sum_{j=1}^q \theta_j}\left(I - P^2_{\mathscr{E}_*}\right) M_{\sum_{j=1}^q \theta_j} e + \Delta^2_{\sum_{j=1}^q \theta_j} e \\ &= e - M^*_{\sum_{j=1}^q \theta_j} P^2_{\mathscr{E}_*} M_{\sum_{j=1}^q \theta_j} e \in \mathscr{E}.\end{aligned}$$

It remains to prove that the set

$$\left\{\left(I_{\mathscr{E}} - M^*_{\sum_{j=1}^q \theta_j} P^2_{\mathscr{E}_*}\left(\sum_{j=1}^q \theta_j\right)\right) e \,;\, e \in \mathscr{E}\right\}$$

is dense in $\mathscr{E}$.

Indeed, otherwise there exists $e' \in \mathscr{E}, e' \neq 0$ such that

$e' = M^*_{\sum_{j=1}^q \theta_j} P^2_{\mathscr{E}_*} M_{\sum_{j=1}^q \theta_j} e'$ .It follows that $\|e'\| = \left\|P^2_{\mathscr{E}_*} M_{\sum_{j=1}^q \theta_j} e'\right\|$, which contradicts that the sequence θ_j is purely contractive.

E. The last step is to prove that the characteristic functions of $\boldsymbol{T}^*$ coincides with the sequence θ_j.

It is easy to see that the operator ω defined from $\mathscr{E}$ to L by $\omega(e) = M_{\sum_{j=1}^q \theta_j} e \oplus \Delta_{\sum_{j=1}^q \theta_j} e$ $(e \in \mathscr{E})$ is a unitary one.

On the other hand, from (43) it follows that the operator ω_* defined from $\mathscr{E}_*$ to L_* by $\omega_*(e_*) = e_* \oplus 0 (e_* \in \mathscr{E}_*)$ is a unitary too.

According to (44), for $l = M_{\sum_{j=1}^q \theta_j} e \oplus \Delta_{\sum_{j=1}^q \theta_j} e$ $(e \in \mathscr{E})$ we have

$$\Phi^{L_*} P^{L_*} \left(M_{\sum_{j=1}^q \theta_j} \mathrm{e} \oplus \Delta_{\sum_{j=1}^q \theta_j} e \right) = \Phi^{L_*} \left(M_{\sum_{j=1}^q \theta_j} \mathrm{e} \oplus 0 \right) = M_{\sum_{j=1}^q \theta_j} \mathrm{e} \oplus 0$$
$$= M_{\omega_*} M_{\sum_{j=1}^q \theta_j} e = M_{\omega_*} \left(\sum_{j=1}^{q} \theta_j \right) \omega^{-1} e.$$

Hence, using Theorem (5.1.13) we deduce that characteristic functions of $\boldsymbol{T}^*$ coincides with the sequence θ_j.The proof is completed.

Proposition (5.1.17)[211]: Let the sequences $\theta_j : \mathscr{E} \to \ell^2(\mathscr{T}, \mathscr{E}_*)$ and $\theta_j' : \mathscr{E}' \to \ell^2(\mathscr{T}, \mathscr{E}_*')$ be some operators such that $M_{\sum_{j=1}^q \theta_j}$ and $M_{\sum_{j=1}^q \theta_j'}$ be contractions.

If the sequences θ_j and θ_j' coincide, then the sequences $\boldsymbol{T}_j^*$ and $\boldsymbol{T}_j'$ which they generate in the sense of Theorem (5.1.16) are unitary equivalent.

Proof. If $\chi_j : \mathscr{E} \to \mathscr{E}'$ and $\chi_{*_j} : \mathscr{E}_* \to \mathscr{E}_*'$ are unitary operators such that

$$M_{\chi_{*_j}} \left(\sum_{j=1}^{q} \theta_j \right) = \left(\sum_{j=1}^{q} \theta_j' \right) \chi_j$$

then $U = \bigoplus_{j=1}^{q} \left(M_{\chi_{*_j}} \oplus M_{\chi_j} \right)$ is a unitary operator from H to H' such that

$\sum_{j=1}^q (\boldsymbol{T_j})'_{(\lambda^2-1)} = U \sum_{j=1}^q (\boldsymbol{T_j})^*_{(\lambda^2-1)} U^*$ for every $(\lambda^2 - 1) \in \Lambda$.

The proof is the same as in the proof of Theorem (5.1.15)

Applying this result to the series of the characteristic functions and by using Theorem (5.1.15) we obtain a generalization of Theorem3.4 in [134] and Corollary 2 in [133], namely (see [132]):

Proposition (5.1.18)[211]: For a c.n.c. sequence $\mathscr{T}$ we have that the sequence $(\theta_j)_{\mathscr{T}}$ is outer if and only if $\mathscr{T} \in C_{(2)}$.

Proof. It suffices to prove our assertion for the functional model of $\mathscr{T}$.

Accordingly, let $\boldsymbol{T}^* = \{\boldsymbol{T}^*_{(\lambda^2-1)}\}_{(\lambda^2-1)\in\Lambda}$ be the sequences defined in Theorem (5.1.16) For every $u \oplus \Delta_{\mathscr{T}} v \in H$ we have

$$\lim_{\epsilon\to\infty} \sum_{f_m\in F(1+\epsilon,\Lambda)} \left\|T^*_{f_m}(u\oplus\Delta_{\mathscr{T}} v)\right\|^2 = \|\Delta_{\mathscr{T}} v\|^2.$$

This shows that $\boldsymbol{T}^* \in C_{(2)}$ if and only if $u \oplus 0 \in H$ implies $u = 0$. On the other hand, $u \oplus 0 \in H$ means $u \perp M_{\left(\sum_{j=1}^{q}\theta_j\right)_{\mathscr{T}}} \ell^2(\mathscr{F}, \mathscr{D})$.

The last condition implies $u = 0$ if and only if

$$\overline{M_{\left(\sum_{j=1}^{q}\theta_j\right)_{\mathscr{T}}} \ell^2(\mathscr{F}, \mathscr{D})} = \ell^2(\mathscr{F}, \mathscr{D}_*),$$

i.e., the sequence $(\theta_j)_{\mathscr{T}}$ is outer.

Section (5.2): Joint Invariant subspaces:

In the classical case of a single operator, the connection between the invariant subspaces of an operator and the corresponding characteristic function was first considered, for certain particular classes of operators, in the work of Livšic, Potapov, Šmulyan, Brodskii, etc. (see the references from [143,144]). One of the fundamental results in the Sz.-Nagy–Foia¸s theory of contractions [134] states that the invariant subspaces of a completely non-unitary (c.n.u.) contraction T on a (separable) Hilbert space are in "one-to-one" correspondence with the regular factorizations of the characteristic function associated with T . This general result, although influenced in part by the work of the authors cited above, was obtained by Sz.-Nagy and Foia¸s in [143,144], following an entirely different approach based on the geometric structure of the unitary dilation and the corresponding functional model for c.n.u. contractions.

The main goal of this section is to obtain a multivariable version of the above-mentioned result, for n-tuples of operators, and to provide a functional model for the joint invariant subspaces in terms of the regular factorizations of the characteristic function. This comes as a natural continuation of our program to develop a free analogue of Sz.-Nagy–Foia¸s theory, for row contractions.

An n-tuple $T := [T_1, . . . , T_n]$ of bounded linear operators acting on a common Hilbert space H is called row contraction if

$$T_1T_1^* + \ldots + T_nT_n^* \leq I.$$

A distinguished role among row contractions is played by the n-tuple $S := [S_1, . . . , S_n]$ of left creation operators on the full Fock space with n generators, $F^2(H_n)$, which satisfies the noncommutative von Neumann inequality [145] (see also [146,147])

$$\|p(T_1,\ldots,T_n)\| \leq \|p(S_1,\ldots,S_n)\|$$

for any polynomial $p(X_1, . . . , X_n)$ in n noncommuting indeterminates. For the classical von Neumann inequality [148] (case n = 1) and a nice survey, we refer to Pisier's book [149]. Based on the left creation operators and their representations, a noncommutative dilation theory and model theory for row contractions was developed in [150,151,152-153,132,154], etc. In this study, the role of the unilateral shift is played by the left creation operators and the Hardy algebra $H^\infty(\mathbf{D})$ is replaced by the noncommutative analytic Toeplitz algebra F_n^∞ . We recall that F_n^∞ was introduced in [145] as the algebra of left multipliers of $F^2(H_n)$ and can be identified with the weakly closed (or w^*-closed) algebra generated by the left creation operators $S_1, \ldots, S_n$ and the identity.

In [132], we defined the standard characteristic function Θ_T of a row contraction (a multi analytic operator acting on Fock spaces) which, as in the classical case (n = 1) [134], turned out to be a complete unitary invariant for completely non-coisometric row contractions (c.n.c.). We also constructed a model for c.n.c. row contractions, in which the characteristic function occurs explicitly. In a very recent paper [155], Ball and Vinnikov introduced an additional invariant L_T so that the pair (L_T, Θ_T) is a complete unitary invariant for the more general case when T is a completely non-unitary (c.n.u.) row contraction.

In 2000, Arveson [156] introduced and studied the curvature and Euler characteristic associated with a row contraction with commuting entries. Noncommutative analogues of these numerical invariants were defined and studied by the author [157] and, independently, by D. Kribs [158]. We showed in [159] that the curvature invariant and Euler characteristic associated with a Hilbert module generated by an arbitrary (respectively commuting) row contraction $T := [T_1, \ldots, T_n]$ can be expressed only in terms of the (respectively constrained) characteristic function of T. We also proved in [159,160] that the constrained characteristic function is a complete unitary invariant for the class of constrained c.n.c. row contractions, and we provided a model.

We continue the study of the characteristic function Θ_T associated with a row contraction $T := [T_1, \ldots, T_n]$ in connection with joint invariant subspaces under the operators $T_1, \ldots, T_n$, and the joint similarity of T to a Cuntz row isometry $W := [W_1, \ldots, W_n]$, i.e., $W_1, \ldots, W_n$ are isometries with

$$W_1W_1^*+,\ldots,W_nW_n^*=I$$

We establish the existence of a “one-to-one” correspondence between the joint invariant subspaces under $T_1, \ldots, T_n$, and the regular factorizations of the characteristic function Θ_T associated with a completely non-coisometric row contraction $T := [T_1, \ldots, T_n]$. In particular, we prove that there is a non-trivial joint invariant subspace under the operators $T_1, \ldots, T_n$, if and only if there is a non-trivial regular factorization of Θ_T. Using the model theory for c.n.c. row contractions, we provide a functional model for the joint invariant subspaces in terms of the regular factorizations of the characteristic function(see Theorem (5.2.5)). An important question related to the main result, Theorem (5.2.4), is to what extent a joint invariant subspace determines the corresponding regular factorization of the characteristic function. We address this problem in Theorem (5.2.10).

We prove the existence of a unique triangulation of type

$$\begin{pmatrix} c_{.0} & 0 \\ * & c_{.1} \end{pmatrix}$$

for any row contraction $T := [T_1, \ldots, T_n]$ (see Theorem (5.2.11)), and prove the existence of nontrivial joint invariant subspaces for certain classes of row contractions. We show that there is a non-trivial joint invariant subspace under $T_1, \ldots, T_n$ whenever the inner–outer factorization of the characteristic function associated with T is non-trivial (see Theorem (5.2.18)). We also consider some examples that explicitly illustrate the correspondence between joint invariant subspaces and factorizations of the characteristic function.

We obtain criteria for joint similarity of n-tuples of operators to Cuntz row isometries. In particular, we prove that a completely non-coisometric row contraction T is jointly similar to a Cuntz row isometry if and only if the characteristic function of T is an invertible multi-analytic operator (see Theorem (5.2.20)). Moreover, in this case, we provide a model Cuntz row isometry for similarity. This is a multivariable version of a result of Sz.-Nagy and Foias [160], concerning the similarity to unitary operators.

Extending some results obtained by Sz.-Nagy [161], Sz.-Nagy and Foias [134], and the author [152,163], we prove, in particular, that a one-to-one power bounded n-tuple $[T_1, \ldots, T_n]$ of operators on a Hilbert space H is jointly similar to a Cuntz row isometry if and only if there exists a constant
$c > 0$ such that

$$\sum_{\alpha\in F_n^+,|\alpha|=k} \|T_\alpha{}^* h\|^2 \geq c\|h\|^2, \quad h\in H$$

for any $k = 1, 2, \ldots$ (see next section for notation).

[164], Muhly and Solel extended the results from [132] to c.n.c. representations of the Hardy algebra $H^\infty(E)$ and their characteristic functions. We believe that all the results can be generalized to their setting.

The existence of a non-trivial joint invariant subspace for $T_1, \ldots, T_n$ is equivalent to the existence of non-trivial regular factorizations for the characteristic function Θ_T . This raises the following natural question: does any contractive multi-analytic operator have a non-trivial regular factorization? While this remains an open problem even in the one-variable case, it will be interesting to find, as in the classical case, sufficient conditions for the existence of non-trivial regular factorizations in our multivariable setting.

Another natural open problem worth mentioning is the problem of extending , concerning c.n.c. row contractions, to the case of c.n.u. row contractions by using the complete invariant (L_T, Θ_T) from [155].

Recently [159,160] we developed a dilation theory on noncommutative varieties determined by row contractions $[T_1, \ldots, T_n]$ subject to constraints such as $p(T_1, \ldots, T_n) = 0$, $p \in P$, where P is a set of noncommutative polynomials. It would be interesting to see to what extent the results of this paper can be extended to constrained row contractions and their constrained characteristic functions.

Let H_n be an n-dimensional complex Hilbert space with orthonormal basis $e_1, e_2, \ldots, e_n$ where $n \in \{1, 2, \ldots\}$ or $n=\infty$. We consider the full Fock space of H_n defined by

$$F^2(H_n) := \bigoplus_{k\geq 0} H_n^{\otimes k}$$

where $H_n^{\otimes 0} = C_1$ and $H_n^{\otimes k}$ is the (Hilbert) tensor product of k copies of H_n. Define the left creation operators $S_i : F^2(H_n) \to F^2(H_n)$, $i = 1, \ldots, n$, by

$$S_i\varphi := e_i \oplus \varphi, \quad \varphi \in F^2(H_n).$$

The noncommutative analytic Toeplitz algebra F_n^∞ and its norm closed version, the noncommutative disc algebra *An*, were introduced by the author [12] in connection with a multivariable noncommutative von Neumann inequality F_n^∞ is the algebra of left multipliers of $F^2(H_n)$ and can be identified with the weakly closed (or w^* -closed) algebra generated by the left creation operators $S_1, \ldots, S_n$ acting on $F^2(H_n)$, and the identity. When $n = 1$, F_1^∞ can be identified with $H^\infty(D)$, the algebra of bounded analytic functions on the open unit disc. The algebra F_n^∞ can be viewed as a multivariable noncommutative analogue of $H^\infty(D)$. There are many analogies with the invariant subspaces of the unilateral shift on $H^2(D)$*)*, inner–outer factorizations, analytic operators, Toeplitz operators, $H^\infty(D)$-functional calculus, bounded (respectively spectral) interpolation, etc.

Let F_n^+ be the unital free semigroup on n generators $g_1, \ldots, g_n$, and the identity g_0. The length of $\alpha \in F_n^+$ is defined by $|\alpha| := k$, if $\alpha = g_{i_1} g_{i_2} \cdots g_{i_k}$, and $|\alpha| := 0$, if $\alpha = g_0$. e also define $e_\alpha := e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$ and $e_{g_0} = 1$. It is clear that $\{e_\alpha \colon \alpha \in F_n^+\}$ is an orthonormal basis of $F^2(H_n)$ .If $T_1, \ldots, T_n \in B(H)$, the algebra of all bounded linear operators on a Hilbert space H, we define $T_\alpha := T_{i_1} T_{i_2} \ldots T_{i_k}$ and $T_{g_0} := I_H$.

We need to recall from [132,165,146,166] a few facts concerning multi-analytic operators on Fock spaces. We say that a bounded linear operator A acting from $F^2(H_n) \otimes K$ to $F^2(H_n)) \otimes G$ is multi-analytic if

$$A(S_i \otimes I_k) = (S_i \otimes I_G)A \qquad \text{for any } i = 1, \ldots, n. \tag{46}$$

Notice that A is uniquely determined by the operator $\theta : K \to F^2(H_n) \otimes G$, which is defined by $\theta k := A(I \otimes k)$, $k \in K$, and is called the symbol of A. We denote $A = A_\theta$. Moreover, A_θ is uniquely determined by the "coefficients" $\theta_{(\alpha)} \in B(K,G)$, which are given by

$$\left\langle \theta_{(\tilde{\alpha})} x, y \right\rangle := \left\langle \theta x, e_\alpha \otimes y \right\rangle = \langle A_\theta (I \otimes x), e_\alpha \otimes y \rangle, \quad x \in K,\ y \in G, \alpha \in F_n^+,$$

where $\tilde{\alpha}$ is the reverse of α, i.e., $\tilde{\alpha} = g_{i_k} \cdots g_{i_1}$ if $\alpha = g_{i_1} \cdots g_{i_k}$. We can associate with A_θ a unique formal Fourier expansion

$$A_\theta \sim \sum_{\alpha \in F_n^+} R_\alpha \otimes \theta_{(\alpha)},$$

where $R_i := U^* S_i U$, $i = 1,\dots,n$, are the right creation operators on $F^2(H_n)$ and U is the unitary operator on $F^2(H_n)$ mapping $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$ into $e_{i_k} \otimes \dots \otimes e_{i_2} \otimes e_{i_1}$. Based on the noncommutative von Neumann inequality [146], we proved that

$$A_\theta = SOT - \lim_{r \to 1} \sum_{k=0}^{\infty} \sum_{|\alpha|=k} r^{|\alpha|} R_\alpha \otimes \theta_{(\alpha)},$$

where, for each $r \in (0, 1)$ the series converges in the uniform norm. The set of all multi-analytic operators in $B(F^2(H_n) \otimes K, F^2(H_n) \otimes G)$ coincides with $R_n^\infty \overline{\otimes} B(K,G)$, the WOT closed algebra generated by the spatial tensor product, where $R_n^\infty := U^* F_n^\infty U$ (see [166,167]). The multianalytic operator A_θ is called:

(i) inner if A_θ is an isometry,

(ii) outer if $\overline{(A_\theta(F^2(H_n) \otimes \varepsilon)} = F^2(H_n) \otimes \varepsilon_*$,

(iii) purely contractive if $\|P_{\varepsilon_*} \theta\, h\| < \|h\|$ for every $h \in \varepsilon$, $h \neq 0$,

(iv) unitary constant if $A_\theta = I \otimes W$ for some unitary operator $W \in B(K,G)$.

If $A_\varphi : F^2(H_n) \otimes M \to F^2(H_n) \otimes N$ is another multi-analytic operator, we say that A_θ coincides with A_φ if there exist two unitary operators

$$W : K \to M, \qquad W_* : G \to N$$

such that

$$(I \otimes W_*) A_\theta = A_\varphi (I \otimes W).$$

For simplicity, throughout this paper, $T := [T_1, \dots, T_n]$, $n = 1, \dots, \infty$, denotes either the n-tuple$(T_1, \dots, T_n)$ of bounded linear operators on a Hilbert space $\boldsymbol{H}$ or the row operator matrix $[T_1 \cdots T_n]$ acting from $H^{(n)}$ to $\boldsymbol{H}$, where $\boldsymbol{H}^{(n)} := \oplus_{i=1}^n H$ is the direct sum of n copies of $\boldsymbol{H}$. Assume that

$T := [T_1, \dots, T_n]$ is a row contraction, i.e.,

$$T_1 T_1^* + \dots + T_n T_n^* \leq I.$$

The defect operators of T are

$$\Delta_{T^*} := \left(I_H - \sum_{i=1}^{n} T_i T_i^* \right)^{1/2} \in B(H) \quad \text{and} \quad \Delta_T := (I_{H^{(n)}} - T^* T)^{1/2} \in B(H^{(n)}),$$

and the defect spaces of T are defined by

$$D_* := \overline{\Delta_{T^*} H} \qquad \text{and} \qquad D := \overline{\Delta_T H^{(n)}}\,.$$

The characteristic function of the row contraction T := [T_1, . . . , T_n] is the multi-analytic operator $\Theta_T : F^2(H_n) \otimes D \to F^2(H_n) \otimes D_*$ with symbol Θ_T is given by

$$\Theta_T(h) := -\sum_{i=1}^{n} T_i P_i h + \sum_{i=1}^{n} (S_i \otimes I_{D^*}) \left(\sum_{\alpha \in F_n^+} e_\alpha \otimes \Delta_{T^*} T_\alpha^* P_i \Delta_T h \right) ,h \in D,$$

where P_i denotes the orthogonal projection of $H^{(n)}$ onto the i-component of $H^{(n)}$, and S :=[S_1, . . . , S_n] is the model multi-shift of left creation operators acting on the full Fock space $F^2(H_n)$.

Using the characterization of multi-analytic operators on Fock spaces (see [166,167]), one can easily see that the characteristic function of T is a multi-analytic operator with the formal Fourier representation

$$-I \otimes T + (I \otimes \Delta_{T^*}) \left(I - \sum_{i=1}^{n} R_i \otimes T_i^* \right)^{-1} \left[R_1 \otimes I_H, \ldots, R_n \otimes I_H \right] (I \otimes \Delta_T),$$

where R_1, . . . , R_n are the right creation operators on the full Fock space $F^2(H_n)$.

The definition of the characteristic function of T arises in a natural way in the context of the theory of noncommutative isometric dilations for row contractions (see [153,132]). Let V :=[V_1, . . . , V_n], $V_i \in B(K)$, be the minimal isometric dilation of T on a Hilbert space $K \supset \boldsymbol{H}$.

Therefore,

(i) $V_1, \ldots, V_n$ are isometries with orthogonal ranges;

(ii) $T_i^* = V_i^* |_H, i = 1, \ldots, n;$

(iii) $K = \bigvee_{\alpha \in F_n^*} V_\alpha H.$

Consider the following subspaces of K:

$$L := \bigvee_{i=1}^{n} (V_i - T_i) H, L_* := \overline{\left(I_k - \sum_{i=1}^{n} V_i T_i^* \right) H}.$$

According to [153], we have the following orthogonal decompositions of the minimal isometric dilation space of T:

$$K = R \oplus M_V(L_*) = H \oplus M_V(L) \tag{47}$$

where R reduces each operator V_i , $i = 1, \ldots, n$,

$$M_V(L_*) = \bigoplus_{\alpha \in F_n^*} V_\alpha L_* \quad \text{and} \quad M_V(L) = \bigoplus_{\alpha \in F_n^*} V_\alpha L .$$

Denote by Φ^L the unitary operator from M_V (L) to $F^2(H_n) \otimes L$ defined by

$$\Phi^L\left(\sum_{\alpha \in F_n^+} V_\alpha \ell_\alpha \right) := \sum_{\alpha \in F_n^+} e_\alpha \otimes \ell_\alpha, \quad \ell_\alpha \in L, \sum_{\alpha \in F_n^+} \|\ell_\alpha\|^2 < \infty.$$

One can view Φ^L as the Fourier representation of M_V *(L)* on Fock spaces. Then, for any $i =1, \ldots, n$, we have

$$\Phi^L V_i = (S_i \otimes I_L)\Phi^L,$$

where $S := [S_1, \ldots, S_n]$ is the model multi-shift of left creation operators acting on the full Fock space $F^2(H_n)$. Similarly, one can define the unitary operator (Fourier representation) $\Phi^{L_*} : M_V(L_*) \to F^2(H_n) \otimes L_*$. We proved in [132] that the characteristic function Θ_T coincides with the multi-analytic operator $\Theta_L : F^2(H_n) \otimes L \to F^2(H_n) \otimes L_*$ defined by

$$\Theta_L := \Phi^{L_*}(P_{M_V(L_*)|M_V(L)})(\Phi^L)^*,$$

where $P_{M_V(L_*)}$ denotes the orthogonal projection of K onto $M_V(L_*)$.

Let $T := [T_1, \ldots, T_n]$, $n = 1, \ldots, \infty$, be a row contraction with $T_i \in B(\boldsymbol{H})$ and consider the subspace $\boldsymbol{H}_c \subset \boldsymbol{H}$ defined by

$$H_c := \left\{ h \in H : \sum_{|\alpha|=k} \|T_\alpha^* h\|^2 = \|h\|^2 \quad \text{for any k} = 1, 2, \ldots \right\}$$

We call T a completely non-coisometric (c.n.c.) row contraction if $\boldsymbol{H}_c = \{0\}$. We proved in [153] that $\boldsymbol{H}_c$ is a joint invariant subspace under the operators $T_1^*, \ldots, T_n^*$, and it is also the largest subspace in $\boldsymbol{H}$ on which T^* acts isometrically. Consequently, we have the following triangulation with respect to the decomposition $\boldsymbol{H} = \boldsymbol{H}_c \oplus \boldsymbol{H}_{cnc}$:

$$T_i = \begin{pmatrix} A_i & 0 \\ * & B_i \end{pmatrix}, \qquad i = 1,\ldots n,$$

where $[A_1, \ldots, A_n]$ is a coisometry, i.e., $A_1 A_1^* + \ldots + A_n A_n^* = I_{H_c}$, , and $[B_1, \ldots, B_n]$ is a c.n.c row contraction. We say that T is of class $C_{\cdot 0}$ (or pure row contraction) if

$$\lim_{k\to\infty} \sum_{|\alpha|=k} \left\| T_\alpha^* h \right\|^2 = 0 \quad \text{for any } h \in \boldsymbol{H}.$$

In [132], we constructed the following model for c.n.c. row contractions, in which the characteristic function occurs explicitly.

Theorem (5.2.1)[142]: Every completely non-coisometric row contraction $T := [T_1, \ldots, T_n]$, n = 1, 2, . . . ,∞ , on a Hilbert space **H** is unitarily equivalent to a row contraction $\mathbf{T} := [\mathbf{T}_1, \ldots, \mathbf{T}_n]$ on the Hilbert space

$$\mathbf{H} = [(\mathbf{F}^2(H_n) \otimes D_* \overline{\Delta\Theta_T (F^2(H_n) \otimes D)}] \ominus \{\Theta_T f \oplus \Delta\Theta_T f : f \in F^2(H_n) \otimes D\}$$

Where $\Delta\,\Theta_T := (I - \Theta_T^*\ \Theta_T\ )^{1/2}$ and operator $\mathbf{T}_i$, i = 1, . . . , n, is defined by

$$\mathbf{T}_i^*[f \oplus \Delta\Theta_T (S_j \otimes I_{D_*})g] := \begin{cases} (S_i \otimes I_{D_*}) f \oplus \Delta_{\Theta_T}\ g & \text{if i = j.} \\ (S_i^* \otimes I_{D_*}) f \oplus 0 & \text{if i} \neq \text{j.} \end{cases}$$

i,j=1,…,n and $S_1, \ldots, S_n$ are the left creation operators on the full Fock space $F^2(H_n)$.

Moreover, T is a pure row contraction if and only if Θ_T is an inner multi-analytic operator.In this case the model reduces to

$$\mathbf{H} = (\mathbf{F}^2(H_n) \otimes D_*) \ominus \Theta_T(\mathbf{F}^2(H_n) \otimes D) \qquad \mathbf{T}_i^* f = (S_i^* \otimes I_{D_*}) f, \quad f \in \mathbf{H}$$

Any contractive multi-analytic operator $\Theta : F^2(H_n) \otimes \varepsilon \to F^2(H_n) \otimes \varepsilon_*$ ($\varepsilon, \varepsilon_*$are Hilbertspaces) generates a c.n.c. row contraction $\mathbf{T} := [\mathbf{T}_1, \ldots, \mathbf{T}_n]$. More precisely, we proved in [132] the following result.

Theorem (5.2.2)[142]: Let $\Theta : F^2(H_n) \otimes \varepsilon \to F^2(H_n) \otimes \varepsilon_*$ be a contractive multi-analytic operator and set $\Delta_\Theta := (I - \Theta^* \Theta)^{1/2}$. Then the row contraction $\mathbf{T} := [\mathbf{T}_1, \ldots, \mathbf{T}_n]$ defined on the Hilbert space

$$\mathbf{H} = [(\mathbf{F}^2(H_n) \otimes \varepsilon_*) \oplus \overline{\Delta_\Theta (F^2(H_n) \otimes \varepsilon)}] \ominus \{\Theta_g \oplus \Delta_\Theta g : g \in F^2(H_n) \otimes \varepsilon\}$$

by

$$\mathbf{T}_i^* (f \oplus \Delta_\Theta g) := (S_i^* \otimes I_{\varepsilon_*}) f \oplus C_i^* (\Delta_\Theta g), \; i = 1,\ldots,n,$$

where each operator C_i is defined by

$$C_i(\Delta_\Theta g) := \Delta_\Theta(S_i \otimes I_\varepsilon)g, \ g \in F^2(H_n)\otimes\varepsilon,$$

and $S_1, \ldots, S_n$ are the left creation operators on $F^2(H_n)$, is completely non-coisometric.

If Θ is purely contractive and

$$\overline{\Delta_\Theta(F^2(H_n)\otimes\varepsilon)} = \overline{\Delta_\Theta((F^2(H_n)\otimes\varepsilon)\ominus\varepsilon)},$$

then Θ coincides with the characteristic function of the row contraction $\mathbf{T} := [\mathbf{T}_1, \ldots, \mathbf{T}_n]$. In this case, considering $\mathbf{H}$ as a subspace of

$$\mathbf{K} = (F^2(H_n)\otimes\varepsilon_*)\oplus\overline{\Delta_\Theta(F^2(H_n)\otimes\varepsilon)},$$

we have that the sequence of operators $\mathbf{V} := [\mathbf{V}_1, \ldots, \mathbf{V}_n]$ defined on $\mathbf{K}$ by

$$\mathbf{V}_i := (S_i \otimes I_{\varepsilon_*})\oplus C_i, \qquad i = 1, \ldots, n,$$

is the minimal isometric dilation of $T := [T_1, \ldots, T_n]$.

We establish the existence of a "one-to-one" correspondence between the joint invariant subspaces under $T_1, \ldots, T_n$, and the regular factorizations of the characteristic function Θ_T associated with a completely non-coisometric row contraction $T := [T_1, \ldots, T_n]$. In particular,we prove that there is a non-trivial joint invariant subspace under the operators $T_1, \ldots, T_n$,if and only if there is a non-trivial regular factorization of Θ_T. Using the model theory for c.n.c row contractions, we provide a functional model for the joint invariant subspaces in terms of the regular factorizations of the characteristic function.

Let $\Theta : F^2(H_n)\otimes\varepsilon \to F^2(H_n)\otimes\varepsilon_*$ be a contractive multi-analytic operator and assume that it has the factorization

$$\Theta = \Theta_2\Theta_1,$$

Where $\Theta_1 : F^2(H_n)\otimes\varepsilon \to F^2(H_n)\otimes F$ and $\Theta_2 : F^2(H_n)\otimes F \to F^2(H_n)\otimes\varepsilon_*$ are contractive multi-analytic operators. Define the operator

$$X_\Theta : \overline{\Delta_\Theta(F^2(H_n)\otimes\varepsilon)} \to \overline{\Delta_2(F^2(H_n)\otimes F)}\oplus\overline{\Delta_1(F^2(H_n)\otimes\varepsilon)}$$

by setting

$$X_\Theta(\Delta_\Theta f) := \Delta_2\Theta_1 f \oplus \Delta_1 f, f \in F^2(H_n)\otimes\varepsilon, \tag{48}$$

where $\Delta_\Theta := (I - \Theta^*\Theta)^{1/2}$ and $\Delta_j := (I - \Theta_j^*\Theta_j)^{1/2}$, $j = 1, 2$. Notice that X_Θ is an isometry. Indeed, since

$$I - \Theta^*\Theta = I - \Theta_1^*\Theta_2^*\Theta_2\Theta_1 = \Theta_1^*(I - \Theta_2^*\Theta_2)\Theta_1 + (I - \Theta_1^*\Theta_1),$$

we have

$$\begin{aligned}\|\Delta_2\Theta_1 f \oplus \Delta_1 f\|^2 &= \|\Delta_2\Theta_1 f\|^2 + \|\Delta_1 f\|^2 \\ &= \left\langle \Theta_1^*(I - \Theta_2^*\Theta_2)\Theta_1 f + (I - \Theta_1^*\Theta_1) f, f\right\rangle \\ &= \left\langle (I - \Theta^*\Theta) f, f\right\rangle = \|\Delta_\Theta f\|^2.\end{aligned}$$

As in the classical case (see [134]), we say that the factorization $\Theta = \Theta_1 + \Theta_2$ is *regular* if X_Θ is a unitary operator, i.e.,

$$\{\Delta_2\Theta_1 f \oplus \Delta_1 f : f \in F^2(H_n)\otimes\varepsilon\}^- = \overline{\Delta_2(F^2(H_n)\otimes F)} \oplus \overline{\Delta_1(F^2(H_n)\otimes\varepsilon)}.$$

Now let us prove the following technical result which will be very useful in what follows.

Lemma (5.2.3)[142]: Let $\Theta : F^2(H_n)\otimes\varepsilon \to F^2(H_n)\otimes\varepsilon_*$ be a contractive multi-analytic operator and let C := [C1, . . . , Cn] be the row isometry defined on $\overline{\Delta_\Theta(F^2(H_n)\otimes\varepsilon)}$ by setting

$$C_i\Delta_\Theta f := \Delta_\Theta(S_i \otimes I_\varepsilon) f, \qquad f \in F^2(H_n)\otimes\varepsilon,$$

for each i = 1, . . . , n, where $\Delta_\Theta := (I - \Theta^*\Theta)^{1/2}$. Then C is a Cuntz row isometry, i.e., $C_1C_1^* + \ldots C_nC_n^* = I$, if and only if

$$\overline{\Delta_\Theta(F^2(H_n)\otimes\varepsilon)} = \overline{\Delta_\Theta((F^2(H_n)\otimes\varepsilon)\ominus\varepsilon)}\,. \tag{49}$$

Assume that Θ has the factorization

$$\Theta = \Theta_2\Theta_1,$$

where $\Theta_1 : F^2(H_n)\otimes\varepsilon \to F^2(H_n)\otimes F$ *and* $\Theta_2 : F^2(H_n)\otimes F \to F^2(H_n)\otimes\varepsilon_*$ *are* contractive multi-analytic operators and let $E := [E_1, \ldots, E_n]$ *and* $F := [F_1, \ldots, F_n]$ be the corresponding row isometries defined on $\overline{\Delta_1(F^2(H_n)\otimes\varepsilon)}$ *and* $\overline{\Delta_2(F^2(H_n)\otimes F)}$., respectively. Then

$$X_\Theta C_i = \begin{pmatrix} F_i & 0 \\ 0 & E_i \end{pmatrix} X_\Theta \quad ,i=1,\ldots,n \tag{50}$$

where the operator X_Θ is defined by relation (48). Moreover, if the factorization $\Theta = \Theta_2\Theta_1$ is regular, then C is a Cuntz row isometry if and only if E and F are Cuntz row isometries.

Proof. First, notice that since Θ is a multi-analytic operator, i.e.,

$$\Theta\ (S_i \otimes I_\varepsilon) = (S_i \otimes I_{\varepsilon_*})\,\Theta, \quad i = 1, \ldots, n,$$

we have

$$\begin{aligned}\left\langle C_i\Delta_\Theta f, C_j\Delta_\Theta g\right\rangle &= \left\langle (S_j^* \otimes I_\varepsilon)(I-\Theta^*\Theta)(S_i \otimes I_\varepsilon)f, g\right\rangle \\ &= \left\langle \delta_{ij}(I-\Theta^*\Theta)f, g\right\rangle = \delta_{ij}\left\langle \Delta_\Theta f, \Delta_\Theta g\right\rangle\end{aligned}$$

for any f, g ∈ $F^2(H_n)\otimes\varepsilon$ and i, j = 1, . . . , n. This shows that the operators $C_1,\ldots,C_n$ are isometries with orthogonal spaces. Due to the definition of
C_i , it is clear that $C_1C_1^* + \ldots + C_nC_n^* = I$ if and only if the range of the operator
$[C_1, \ldots, C_n]$ coincides with $\overline{\Delta_\Theta(F^2(H_n)\otimes\varepsilon}$,which is equivalent to (49).

On the other hand, for each $i = 1, \ldots, n$, and $f \in F^2(H_n)\otimes E$, we have

$$\begin{aligned}X_\Theta C_i(\Delta_\Theta f) &= X_\Theta\Delta_\Theta(S_i \otimes I_\varepsilon)f = \Delta_2\Theta_1(S_i \otimes I_\varepsilon)f \oplus \Delta_1(S_i \otimes I_cE)f \\ &= \Delta_2(S_i \otimes I_F)\Theta_1 f \oplus \Delta_1(S_i \otimes I_\varepsilon)f = F_i\Delta_2\Theta_1 f \oplus E_i\Delta_1 f \\ &= \begin{pmatrix} F_i & 0 \\ 0 & E_i\end{pmatrix}(\Delta_2\Theta_1 f \oplus \Delta_1 f) = \begin{pmatrix} F_i & 0 \\ 0 & E_i\end{pmatrix} X_\Theta\Delta_\Theta f,\end{aligned}$$

which proves relation (35). If the factorization $\Theta = \Theta_2\,\Theta_1$ is regular, then X_Θ is a unitary operator. Consequently, we have

$$X_\Theta\ (\sum_{i=1}^n C_iC_i^*)X_\Theta^* = \begin{pmatrix} \sum_{i=1}^n F_iF_i^* & 0 \\ 0 & \sum_{i=1}^n E_iE_i^*\end{pmatrix},$$

which implies that $C := [C_1, \ldots, C_n]$ is a Cuntz row isometry if and only if $E := [E_1, \ldots, E_n]$ and $F := [F_1, \ldots, F_n]$ are Cuntz row isometries. This completes the proof.

Theorem (5.2.4)[142]: Let T := $[T_1, \ldots, T_n]$, T_i ∈ B(**H**), be a completely non-coisometric row contraction and let $\Theta\ : F^2(H_n)\otimes\varepsilon \to F^2(H_n)\otimes\varepsilon_*$ be a contractive multi-analytic operator which coincides with the characteristic function of T . If $\mathbf{H}_1 \subset$ **H** is a joint invariant subspace under the operators $T_1, \ldots, T_n$, then there exists a regular factorization $\Theta = \Theta_2 \Theta_1$, where $\Theta_1 : F^2(H_n) \otimes \varepsilon \to F^2(H_n) \otimes F$ and Θ_2: $F^2(H_n) \otimes F \to F^2(H_n) \otimes \varepsilon_*$ are contractive multi-analytic operators such that T := [T_1, . . . , T_n] is unitarily equivalent to a row contraction **T** := [$\mathbf{T}_1$, . . . ,$\mathbf{T}_n$] defined on the Hilbert space

$$H := [(F^2(H_n) \otimes \varepsilon_*) \oplus \overline{\Delta_2(F^2(H_n) \otimes F)} \oplus \overline{\Delta_1(F^2(H_n) \otimes \varepsilon)}] \ominus$$

$$\left\{ \Theta_2 \Theta_1 f \oplus \Delta_2 \Theta_1 f \oplus \Delta_1 f : f \in F^2(H_n) \otimes \varepsilon \right\}$$

by setting

$$T_i^*(f \oplus \varphi \oplus \psi) := (S_i^* \otimes I_{\varepsilon_*}) f \oplus F_i^* \varphi \oplus E_i^* \psi, \qquad f \oplus \varphi \oplus \psi \in H,$$

for any i = 1, . . . , n, where the operators F_i and E_i are defined in Lemma (5.2.3) and $S_1,...,S_n$ are the left creation operators on $F^2(H_n)$. Moreover, the subspaces corresponding to $\mathbf{H}_1$ and $\mathbf{H}_2 := \mathbf{H} \ominus \mathbf{H}_1$ are

$$H_1 := \left\{ \Theta_2 f \oplus \Delta_2 f \oplus g : f \in F^2(H_n) \otimes F, g \in \overline{\Delta_1(F^2(H_n) \otimes \varepsilon)} \right\} \ominus$$

$$\left\{ \Theta_2 \Theta_1 f \oplus \Delta_2 \Theta_1 f \oplus \Delta_1 f : f \in F^2(H_n) \otimes \varepsilon \right\}$$

and

$$H_2 := [(F^2(H_n) \otimes \varepsilon_*) \oplus \overline{\Delta_2(F^2(H_n) \otimes F)} \oplus \{0\}] \ominus \left\{ \Theta_2 f \oplus \Delta_2 f \oplus \{0\} : f \in F^2(H_n) \otimes F \right\},$$

respectively. Conversely, every regular factorization $\Theta = \Theta_2 \Theta_1$ generates via the above formulas the subspaces $\mathbf{H}_1$ and $\mathbf{H}_2$ with the following properties:

(i) $\mathbf{H}_1$ is invariant under each operator $\mathbf{T}_i$, i = 1, . . . , n;

(ii) $\mathbf{H}_2 = \mathbf{H} \ominus \mathbf{H}_1$.

Under the above identification, $\mathbf{H}_1$ corresponds to a subspace $\mathbf{H}_1 \subset \mathbf{H}$ which is invariant under each operator T_i , i = 1, . . . , n.

Proof. Part I. Let T := [T_1, . . . , T_n], $T_i \in B(\mathbf{H})$, be a row contraction and let V :=[V_1, . . . , V_n],$V_i \in B(K)$, be its minimal isometric dilation on a Hilbert space $K = V_{\alpha \in F_n^+} v_\alpha H$. Since $V_1,...,V_n$ are isometries with orthogonal ranges, the noncommutative Wold decomposition [153] provides the orthogonal decomposition

$$K = R \oplus M_V(L_*), \tag{51}$$

where

Moreover, R is the maximal subspace of K which is reducing for the operators $V_1,\ldots,V_n$ and the row contraction $[V_1|_R, \ldots, V_n|_R]$ is a Cuntz row isometry.

Let $H_1 \subset H$ be an invariant subspace under the operators $T_1,\ldots,T_n$. Since $V_i^*\big|_H = T_i^*$, i = 1,…, n, we deduce that the subspace $\mathbf{H}_2 = \mathbf{H} \ominus \mathbf{H}_1$ is invariant under the operators $V_1^*,\ldots,V_n^*$. Therefore, the subspace $G := K \ominus \mathbf{H}_2$ is invariant under $V_1,\ldots,V_n$. Applying again the noncommutative Wold decomposition to the row isometry $[V_1|_G, \ldots, V_n|_G]$, we obtain the orthogonal decomposition

$$G = R_1 \oplus \mathrm{M}_v(\mathrm{Q}), \tag{52}$$

where

$$R_1 := \bigcap_{k=o}^{\infty}\left[\bigoplus_{|\alpha|=k} V_\alpha G\right] \qquad \text{and} \qquad \mathrm{Q} := G\,\theta\left(\bigoplus_{i=1}^{n} V_i G\right).$$

Since R_1 reduces the operators $V_1, \ldots, V_n$ and $[V_1|_{R_1}, \ldots, V_n|_{R_1}]$ is a Cuntz row isometry, we deduce that $R_1 \subset R$. Notice that $R_2 := R \ominus R_1$ is also a reducing subspace for $V_1, \ldots, V_n$ and $[V_1|_{R_2}, \ldots, V_n|_{R_2}]$ is a Cuntz row isometry. Using relations (51) and (52), we infer that

$$H_2 = k \ominus G = [R \oplus \mathrm{M}_V(L_*)]\ \theta\ [R_1 \oplus \mathrm{M}_V(\mathrm{Q})] = [R_2 \oplus \mathrm{M}_V(L_*)]\ \theta\ \mathrm{M}_V(\mathrm{Q}).$$

Hence, we deduce that

$$\mathrm{M}_V(\mathrm{Q}) \subset R_2 \oplus \mathrm{M}_V(L_*). \tag{53}$$

On the other hand, due to (47), we have

$$K = R \oplus \mathrm{M}_V(L_*) = H \oplus \mathrm{M}_V(L).$$

Hence, we obtain

$$H = [R \oplus \mathrm{M}_V(L_*)] \ominus \mathrm{M}_V(L).$$

Since $H_2 \subset H$, the above representations of H and H_2 imply

$$[R_2 \oplus \mathrm{M}_V(L_*)] \ominus \mathrm{M}_V(\mathrm{Q}) \subset [R \oplus \mathrm{M}_V(L_*)] \ominus \mathrm{M}_V(L).$$

Taking into account that $R = R_1 \oplus R_2$, we have

$$[R_2 \oplus M_V(L_*)] \ominus M_V(\mathrm{Q}) = [R_2 \oplus M_V(L_*)] \ominus [R_1 \oplus M_V(\mathrm{Q})].$$

Consequently, we deduce that

$$M_V(L) \subset R_1 \oplus M_V(\mathrm{Q}) \tag{54}$$

and

$$H_1 = H \ominus H_2 = [R_1 \oplus M_V(\mathrm{Q})] \ominus \mathrm{M}_V(L) = G \ominus \mathrm{M}_V(L).$$

Let $P_{M_V(L_*)}, P_{M_V(\mathrm{Q})}, P_R, P_{R_1}$, *and* P_{R_2}, be the orthogonal projections onto the corresponding spaces. According to relations (53) and (54), for any x $\in M_V(\mathrm{Q})$ and y $\in M_V(L)$, we have

$$\mathrm{x} = P_{R_2}x + P_{M_V(L_*)}x \quad \text{and} \qquad \mathrm{y} = P_{R_1}y + P_{M_V(\mathrm{Q})}y \ . \tag{55}$$

In particular, if x := $P_{M_V(\mathrm{Q})}y$ and y $\in M_V(L)$, we deduce that

$$y = P_{R_1}y + P_{R_2}P_{M_V(\mathrm{Q})}y + P_{M_V(L_*)}P_{M_V(\mathrm{Q})}y \tag{56}$$

Hence and taking into account that the subspace $R_1 \oplus R_2 = R$ is orthogonal to $M_V(L_*)$, we deduce that

$$P_{M_V(L_*)}y = P_{M_V(L_*)}P_{M_V(\mathrm{Q})}y \ \textit{and} \ P_R y = P_{R_1}y + P_{R_2}P_{M_V(\mathrm{Q})}y \tag{57}$$

for any y $\in M_V$ (L). Due to relation (51), we have

$$P_R f = \left(1 - \mathrm{P}_{\mathrm{M_V}(L_*)}\right)f \ , \qquad f \in k \ . \tag{58}$$

On the other hand, relations (54) and (53) imply

$$P_{R_1}y = \left(1 - \mathrm{P}_{\mathrm{M_V}(\mathrm{Q})}\right)y \ , \quad y \in M_V(L) \tag{59}$$

and

$$P_{R_2}x = \left(1 - \mathrm{P}_{\mathrm{M_V}(L_*)}\right)x, \quad x \in M_V(\mathrm{Q}) \tag{60}$$

Assume now that $[T_1, \ldots, T_n]$ is a c.n.c. row contraction. In this case, we have (see [153])

$$\mathrm{K} = M_V(L) \vee M_V(L_*) = R \oplus M_V(L_*),$$

which implies

$$\overline{P_R M_V(L)} = \overline{(1 - P_{M_V(L_*)} M_V(L)} = R \ . \tag{61}$$

Hence and using the second relation in (57), we deduce that

$$\overline{P_{R_1}M_V(L)} = R_1 \quad \textit{and} \quad \overline{P_{R_2}P_{M_V(\mathrm{Q})}M_V(L)} = R_2,$$

and, consequently,

$$\overline{P_{R_1}M_V(L)} = R_1 \quad \textit{and} \quad \overline{P_{R_2}M_V(\mathrm{Q})} = R_2 \ . \tag{62}$$

Part II. Consider the following contractions:

$$
\begin{aligned}
&Q := P_{M_V(L_*)|M_V(L)} : M_V(L) \to M_V(L_*),\\
&Q_1 := P_{M_V(\mathrm{Q})|M_V(L)} : M_V(L) \to M_V(\mathrm{Q}),\ \ and\\
&Q_2 := P_{M_V(L_*)|M_V(\mathrm{Q})} : M_V(\mathrm{Q}) \to M_V(L_*),
\end{aligned}
$$

Since $M_V(L_*), M_V(L\ )$, and $M_V(\mathrm{Q}\varsigma)$ are reducing subspaces for the operators $V_1, \ldots,$ V_n, we deduce that, for each $i = 1, \ldots, n$,

$$
\begin{aligned}
&Q(V_i|_{M_V(L)}) = (V_i|_{M_V(L_*)})\,Q,\\
&Q_1(V_i|_{M_V(L)}) = (V_i|_{M_V(\mathrm{Q})})\,Q_1,\ \ and\\
&Q_2(V_i|_{M_V(\mathrm{Q})}) = (V_i|_{M_V(L_*)})\,Q_2.
\end{aligned}
$$

Let $\Phi^{L_*} : M_V(L_*) \to F^2(H_n) \otimes L_*$ be the Fourier representation of the subspace $M_V(L_*)$,i.e.,

$$
\Phi^{L_*}\left(\sum_{\alpha\in F_n^+} V_\alpha \ell_\alpha\right) := \sum_{\alpha\in F_n^+} e_\alpha \otimes \ell_\alpha,
$$

where $\ell_\alpha \in L_*$ $\ \ and\ \ \sum_{\alpha\in F_n^+} \|\ell_\alpha\|^2 < \infty$ Notice that

$$
\Phi^{L_*}(V_i|_{M_V(L_*)}) = (S_i \otimes I_{L_*})\Phi^{L_*}, \quad i = 1, \ldots, n,
$$

where $S_1, \ldots, S_n$ are the left creation operators on $F^2(H_n)$. Similarly, we define the Fourier representations of the subspaces $M_V(L)$ and $M_V(\mathrm{Q})$, respectively. Now, due to the above intertwining relations satisfied by Q , Q_1, and Q_2, the operators

$$
\begin{aligned}
&\Theta_L : F^2(H_n)\otimes L \to F^2(H_n)\otimes L_*, \quad \Theta_L := \Phi^{L_*}Q(\Phi L)^*,\\
&\psi_1 : F^2(H_n)\otimes L \to F^2(H_n)\otimes \mathrm{Q}, \quad \psi_1 := \Phi^{\mathrm{Q}}Q_1(\Phi^L)^* \ \ and\\
&\psi_2 : F^2(H_n)\otimes \mathrm{Q} \to F^2(H_n)\otimes L_*, \quad \Psi_2 := \Phi^{L_*}Q_2(\Phi^{\mathrm{Q}})^*
\end{aligned} \tag{63}
$$

are contractive and multi-analytic. Hence and using the first equation in (42), we have

$$
\begin{aligned}
\Theta_L &= \Phi^{L_*}Q(\Phi^L)^* = \Phi^{L_*}(P_{M_V(L_*)}|_{M_V(L)})(\Phi^L)^*\\
&= \Phi^L(P_{M_V(L_*)}P_{M_V(\mathrm{Q})}|_{M_V(L)})(\Phi^L)^*\\
&= \left[\Phi^{L_*}(P_{M_V(L_*)}|_{M_V(\mathrm{Q})})(\Phi^{\mathrm{Q}})^*\right]\left[\Phi^{\mathrm{Q}}(P_{M_V(\mathrm{Q})}|_{M_V(L)})(\Phi^L)^*\right]\\
&= \left[\Phi^{L_*}Q_2(\Phi^{\mathrm{Q}})^*\right]\left[\Phi^{\mathrm{Q}}Q_1(\Phi^L)^*\right]\\
&= \Psi_2\Psi_1.
\end{aligned}
$$

Due to (58) and (61), there exists a unique unitary operator $\Phi_R : R \to \overline{\Delta_L(F^2(H_n)\otimes L)}$ such that

$$\Phi_R P_R \Psi := \Delta_L \Phi^L \Psi, \quad \Psi \in M_V(L), \tag{64}$$

where $\Delta_L := (I - \Theta_L^* \Theta_L)^{1/2}$ Indeed, we have

$$\begin{aligned}\left\| \left(I - P_{M_V(L_*)}\right) \Psi \right\|^2 &= \|\Psi\|^2 - \left\| P_{M_V(L_*)} \Psi \right\|^2 \\ &= \left\|\Phi^L \Psi\right\|^2 - \left\|\Phi^{L_*} P_{M_V(L_*)} \Psi\right\|^2 \\ &= \left\|\Phi^L \Psi\right\|^2 - \left\|\Theta_L \Phi^L \Psi\right\|^2 \\ &= \left\|\Delta_L \Phi^L \Psi\right\|^2\end{aligned}$$

Consequently,

$$\Phi := \Phi^{L_*} \oplus \Phi_R \tag{65}$$

is a unitary operator from the dilation space K $=M_V$ $(L_*)\oplus$R onto the Hilbert space

$$\widetilde{K} := \left(F^2(H_n)\otimes L_*\right) \oplus \overline{\Delta_L\left(F^2(H_n)\otimes L_*\right)}.$$

The image of the space H = K $\ominus$ $M_V(L)$. M_V (L) under the operator Φ is

$$\Phi \mathrm{H} = \tilde{\mathrm{H}} := \left[\left(F^2(H_n)\otimes L_*\right) \oplus \overline{\Delta_L(F^2(H_n)\otimes L)}\right] \ominus \left\{\Theta_L f \oplus \Delta_L f : f \in F^2(H_n)\otimes L\right\}.$$

The row contraction T := [$T_1, \ldots, T_n$] is transformed under the unitary operator Φ into the row contraction $\widetilde{T} := [\widetilde{T}_1, \ldots, \widetilde{T}_n]$ where

$$\widetilde{T}_i^*(f \oplus \Delta_L g) := (S_i^* \otimes I_{L_*}) f \oplus \widetilde{C}_i^*(\Delta_L g), \quad i = 1\ldots, n,$$

and each operator $\widetilde{C}_i$ is defined by

$$\widetilde{C}_i(\Delta_L g) = \Delta_L(S_i \otimes I_L) g \;\; , \quad g \in F^2(H_n)\otimes L.$$

Notice that, using relations (59), (60), and (62), one can show that there are some unitary operators

$$\Phi_{R_1} : R_1 \to \overline{\Delta_{\Psi_1}(F^2(H_n)\otimes L)} \quad \textit{and} \quad \Phi_{R_2} : R_2 \to \overline{\Delta_{\Psi_2}(F^2(H_n)\otimes \mathrm{Q})}$$

uniquely defined by the relations

$$\begin{aligned}&\Phi_{R_1} P_{R_1} x := \Delta_{\psi_1} \Phi^L x, \quad x \in M_V(L), \\ &\Phi_{R_2} P_{R_2} y := \Delta_{\psi_2} \Phi^{\mathrm{Q}} y, \quad y \in M_V(\mathrm{Q}),\end{aligned} \tag{66}$$

where $\Delta_{\Psi_j} := (I-\Psi_j^*\Psi_j)^{1/2}$ for j=1,2. Consequently, since R=$R_2 \oplus R_1$ and due to relation (64), the operator

$$x_L : \overline{\Delta_L(F^2(H_n)\otimes L)} \to \overline{\Delta_{\Psi_2}(F^2(H_n)\otimes \mathrm{Q})} \oplus \overline{\Delta_{\Psi_1}(F^2(H_n)\otimes L)}$$

defined by

$$x_L := (\Phi_{R_2} \oplus \Phi_{R_1})\Phi_R^* \tag{67}$$

is unitary. Due to relations (64), (57), (66), and (63), we deduce that

$$\begin{aligned} x_L\Delta_L\Phi^L y = x_L\Phi_R P_R y &= (\Phi_{R_2} \oplus \Phi_{R_1})P_R y \\ &= (\Phi_{R_2} \oplus \Phi_{R_1})(P_{R_2}P_{M_V(\mathrm{Q})}y \oplus P_{R_1}y) \\ &= \Delta_{\Psi_2}\Phi^{\mathrm{Q}}P_{M_V(\mathrm{Q})}y \oplus \Delta_{\Psi_1}\Phi^L y \\ &= \Delta_{\Psi_2}\Psi_1\Phi^L y \oplus \Delta_{\Psi_1}\Phi^L y \end{aligned}$$

for any y $\in M_V$ (L). Hence, we have

$$x_L\Delta_L f = \Delta_{\psi_2}\Psi_1 f \oplus \Delta_{\Psi_1} f, \quad f \in F^2(H_n)\otimes L. \tag{68}$$

Since X_L is a unitary operator, we also deduce that

$$\{\Delta_{\Psi_2}\Theta_1 f \oplus \Delta_{\Psi_1} f,\ \ f \in F^2(H_n)\otimes L\} = \overline{\Delta_{\Psi_2}(F^2(H_n)\otimes \mathrm{Q})} \oplus \overline{\Delta_{\Psi_1}(F^2(H_n)\otimes L)}.$$

Due to (65) and (67), we have

$$\Phi = \Phi^{L*} \oplus X_L^*(\Phi_{R_2} \oplus \Phi_{R_1}).$$

Now, we need to find the images $\widetilde{H}_1$ *and* $\widetilde{H}_2$ *of* H_1 *and* H_2 respectively, under the unitary operator Φ. To find $\widetilde{H}_2$, notice first that, due to relation (67), we have

$$\Phi_R z = X_L^*(\Phi_{R_2} \oplus \Phi_{R_1})(z \oplus 0) = X_L^*(\Phi_{R_2} z \oplus 0) \tag{69}$$

for any z $\in R_2$. Hence and using (64), we infer that

$$\begin{aligned} \Phi(M_V(L_*) \oplus R_2) &= \Phi^{L*}M_V(L_*) \oplus \Phi_R R_2 \\ &= (F^2(H_n)\otimes L_*) \oplus X_L^*(\overline{\Delta_{\Psi_2}(F^2(H_n)\otimes \mathrm{Q})} \oplus \{0\}) \end{aligned}$$

and, due to (55),

$$\Phi M_V(\mathrm{Q}) = \{\Phi^{L*}P_{M_V(L_*)}f \oplus \Phi_R P_{R_2} f : f \in M_V(\mathrm{Q})\}$$

Hence, and using relations (48), (51), and (54), we obtain

$$\Phi M_V(\mathrm{Q}) = \{\Psi_2 u \oplus X_{\mathrm{L}}^*(\Delta_{\Psi_2}u \oplus 0) : u \in F^2(H_n)\otimes \mathrm{Q}\}.$$

Now, using the representation of H_2 from part I, i.e.,

$$H_2 = \left[M_V(L_*) \oplus R_2\right] \ominus M_V(\mathrm{Q}).$$

We obtain

$$\tilde{\mathrm{H}}_2 = \left[(F^2(H_n)\otimes L_*) \oplus X_L^* \overline{(\Delta_{\Psi_2}(F^2(H_n)\otimes \mathrm{Q}))} \oplus \{0\}\right] \ominus$$

$$\left\{\Psi_2 f \oplus X_L^*(\Delta_{\Psi_2} f \oplus 0) : f \in F^2(H_n)\otimes \mathrm{Q}\right\}.$$

Since $\tilde{\mathrm{H}}_1 = \tilde{\mathrm{H}} \ominus \tilde{\mathrm{H}}_2$, we deduce that

$$\tilde{\mathrm{H}}_1 = \left\{\Psi_2 f \oplus X_L^*(\Delta_{\Psi_2} f \oplus g) : f \in F^2(H_n)\otimes \mathrm{Q}, g \in \overline{\Delta_{\Psi_1}(F^2(H_n)\otimes L)}\right\} \ominus$$

$$\left\{\Theta_L w \oplus \Delta_\Theta w : w \in F^2(H_n)\otimes L\right\}.$$

The characteristic function Θ_T of the row contraction T coincides with Θ_L, and therefore with Θ. Via this identification, the regular factorization $\Theta_L = \Psi_2\Psi_1$ corresponds to a regular factorization $\Theta = \Theta_2\Theta_1$, where $\Theta_1 : F^2(H_n)\otimes \varepsilon \to F^2(H_n)\otimes F$ and $\Theta_2 : F^2(H_n)\otimes F \to F^2(H_n)\otimes \varepsilon_*$ are contractive multi-analytic operators. Now, it is easy to see that, under the above identification, the subspaces $\tilde{\mathrm{H}}_1$ and $\tilde{\mathrm{H}}_2$ correspond to the subspaces

$$\mathrm{H}_2 = \left[(F^2(H_n)\otimes \varepsilon_*) \oplus X_\Theta^* \overline{(\Delta_2(F^2(H_n)\otimes F))} \oplus \{0\}\right]$$

. $$\ominus \left\{\Theta_2 f \oplus X_\Theta^*(\Delta_2 f \oplus 0) : f \in F^2(H_n)\otimes F\right\} \qquad (70)$$

and

$$\mathrm{H}_1 = \{\Theta_2 f \oplus X_\Theta^*(\Delta_2 f \oplus g) : f \in F^2(H_n)\otimes F, g \in \overline{\Delta_1(F^2(H_n)\otimes \varepsilon)}\}$$

. $$\ominus \left\{\Theta\varphi \oplus \Delta_\Theta \varphi : \varphi \in F^2(H_n)\otimes \varepsilon\right\}, \qquad (71)$$

respectively, where $\Delta_j := (I - \Theta_j^*\Theta_j)^{1/2}, j = 1,2$ Moreover, under the same identification, the row contraction $\tilde{T}$ is unitarily equivalent to the row contraction **T** := [**T**$_1$, . . . ,**T**$_n$] defined on the Hilbert space

$$\mathrm{H} := \left[(F^2(H_n)\otimes \varepsilon_*) \oplus \overline{\Delta_\Theta(F^2(H_n)\otimes \varepsilon)}\right] \ominus \left\{\Theta g \oplus \Delta_\Theta g \oplus : g \in F^2(H_n)\otimes \varepsilon\right\},$$

and

$$T_i^*(f \oplus \Delta_\Theta g) := (S_i^* \otimes I\varepsilon_*) f \oplus C_i^*(\Delta_\Theta g), \quad i = 1,\dots,n,$$

where each operator C_i is defined by

$$C_i(\Delta_\Theta g) := \Delta_\Theta(S_i \otimes I_\varepsilon) g, \qquad g \in F^2(H_n)\otimes \varepsilon,$$

and $S_1, \ldots, S_n$ are the left creation operators on $F^2(H_n)$.

Since the factorization $\Theta = \Theta_2\Theta_1$ is regular, X_Θ is a unitary operator which identifies the subspace $\overline{\Delta_\Theta(F^2(H_n)\otimes\varepsilon)}$ with $\overline{\Delta_2(F^2(H_n)\otimes F)}\oplus$ $\overline{\Delta_1(F^2(H_n)\otimes\varepsilon)}$ and the operator C_i with $\begin{pmatrix} F_i & 0 \\ 0 & E_i \end{pmatrix}$, for each i = 1, . . . , n. Under this identification the Hilbert spaces **H**, **H$_1$**, and **H$_2$** are identified with H, H_1, and H_2, respectively, and the row contraction **T** is unitarily equivalent to the row contraction T.

Part III. We prove the converse of the theorem. Due to the above identification, it is enough to assume that the factorization $\Theta = \Theta_2\Theta_1$ is regular and the subspaces **H**$_1$ and **H**$_2$ are defined as above by relations (71) and (70), respectively. Since X_Θ is a unitary operator and using Lemma (5.2.3), we have

Hence, we obtain

$$\mathrm{H}_1 = G_2 \ominus \left\{\Theta\varphi + \Delta_\Theta\varphi : \varphi \in F^2(H_n)\otimes\varepsilon\right\}.$$

On the other hand, we have

$$\left[\left(F^2(H_n)\otimes\varepsilon_*\right)\oplus\overline{\Delta_\Theta(F^2(H_n)\otimes\varepsilon)}\right]\ominus G_2$$

$$=\left[\left(F^2(H_n)\otimes\varepsilon_*\right)\oplus X_\Theta^*\left(\overline{\Delta_2F^2(H_n)\otimes F}\right)\oplus\overline{\Delta_1(F^2(H_n)\otimes\varepsilon)}\right]\ominus G_2$$

$$=\left[\left(F^2(H_n)\otimes\varepsilon_*\right)\oplus X_\Theta^*\left(\overline{\Delta_2F^2(H_n)\otimes F}\right)\oplus\{0\}\right]\ominus\left\{\Theta_2f \oplus X_\Theta^*(\Delta_2f\oplus\{0\}) : f\in F^2(H_n)\otimes F\right\}.$$

Consequently,

$$H_2 = \left[\left(F^2(H_n)\otimes\varepsilon_*\right)\oplus\overline{\Delta_\Theta(F^2(H_n)\otimes\varepsilon)}\right]\ominus G_2 .$$

Hence, and taking into account the definition of H_1, we deduce that **H** = $H_1\oplus H_2$.

It remains to prove that the subspace **H**$_2$ is invariant under the operators $T_1^*,\ldots,T_n^*$. If f∈ $F^2(H_n)\otimes\varepsilon_*$ and g∈ $\overline{\Delta_2(F^2(H_n)\otimes F)}$, then the vector x:=f $\oplus X_\Theta^*(g\oplus 0)$ is in **H**$_2$ if and only if

$$\Theta_2^* f + \Delta_2 g = 0. \tag{72}$$

Indeed, using relation (70), one can prove that the condition

$$\left\langle f\oplus X_\Theta^*(g\oplus 0), \Theta_2\varphi\oplus X_\Theta^*(\Delta_2\varphi\oplus 0)\right\rangle = 0 \quad for\,any\,\varphi\in F^2(H_n)\otimes F$$

is equivalent to (72). Since

$$T_i^* x = T_i^*(f \oplus X_\Theta^*(g \oplus 0)) = (S_i^* \otimes I_{\varepsilon_*})f \oplus C_i^* X_\Theta^*(g \oplus 0)$$

for each i = 1, . . . , n, to prove that $T_i^* x \in \mathbf{H}_2$, it is enough to show that

$$\left\langle (S_i^* \otimes I_{\varepsilon_*})f \oplus C_i^*(X_\Theta^*(g \oplus 0)), \Theta_2\varphi \oplus X_\Theta^*(\Delta_2\varphi \oplus 0)\right\rangle = 0$$

for any $\varphi \in F^2(H_n) \otimes F$ Since Θ is a multi-analytic operator, the latter condition is equivalent to

$$(S_i^* \otimes I_F)\Theta_2^* f + \Delta_2 P_1 X_\Theta C_i^* X_\Theta^*(g \oplus 0) = 0, \tag{73}$$

where P_1 is the orthogonal projection of the direct sum $\overline{\Delta_2(F^2(H_n) \otimes F} \oplus \overline{\Delta_1(F^2(H_n) \otimes \varepsilon}$ onto $\overline{\Delta_2(F^2(H_n) \otimes F}$ Using Lemma (5.2.3) and the definition of the operators C_i, E_i , and F_i, we deduce that

$$\begin{aligned}\Delta_2 P_1 X_\Theta C_i^* X_\Theta^*(g \oplus 0) &= \Delta_2 P_1 X_\Theta X_\Theta^* \begin{pmatrix} F_i^* & 0 \\ 0 & E_i^* \end{pmatrix}(g \oplus 0) \\ &= \Delta_2 F_i^* g = (S_i^* \otimes I_F)\Delta_2 g.\end{aligned}$$

Hence, and using relation (72), we have

$$\left(S_i^* \otimes I_F\right)\Theta_2^* f + \Delta_2 P_1 X_\Theta C_i^* X_\Theta^*(g \oplus 0) = \left(S_i^* \otimes I_F\right)\left(\Theta_2^* f + \Delta_2 g\right) = 0,$$

which proves relation (73). This shows that $T_i^* H_2 \subset H_2$ for any i=1,…,n. Consequently,the subspace $\mathbf{H_1} = \mathbf{H} \ominus \mathbf{H_2}$is invariant under the operators $T_1,\dots,T_n$. This completes the proof of the theorem.

Now we can reformulate Theorem (5.2.4) in terms of the functional model of a c.n.c. row contraction provided by Theorem (5.2.2). This version will be useful later on.

Theorem (5.2.5)[142]: Let $\Theta : F^2(H_n) \otimes \varepsilon \to F^2(H_n) \otimes \varepsilon_*$ be a purely contractive multi-analytic operator such that

$$\overline{\Delta_\Theta\left(F^2(H_n) \otimes \varepsilon\right)} = \overline{\Delta_\Theta\ [(F^2(H_n) \otimes \varepsilon) \ominus \varepsilon]}$$

and let $\mathbf{T} := [\mathbf{T}_1, . . . ,\mathbf{T}_n]$ be defined on the Hilbert space

$$\mathrm{H} := \left[\left(F^2(H_n) \otimes \varepsilon_*\right) \oplus \overline{\Delta_2\left(F^2(H_n) \otimes \varepsilon\right)}\right] \ominus \left\{\Theta g \oplus \Delta_\Theta g : g \in F^2(H_n) \otimes \varepsilon\right\},$$

and

$$T_i^*(f \oplus \Delta_\Theta g) := (S_i^* \otimes I_{\varepsilon*})f \oplus C_i^*(\Delta_\Theta g), \quad i = 1,\dots,n,$$

where each operator C_i is defined by

$$C_i(\Delta_\Theta g) := \Delta_\Theta(S_i \oplus I_\varepsilon)g, \quad g \in F^2(H_n) \otimes \varepsilon,$$

and $S_1, \dots, S_n$ are the left creation operators on $F^2(H_n)$.

If $\mathbf{H}_1 \subseteq \mathbf{H}$ is an invariant subspace under each operator $\mathbf{T}_i$, i = 1, . . . , n, then there is a regular factorization

$$\Theta = \Theta_2\Theta_1$$

where $\Theta_1 : F^2(H_n) \otimes \varepsilon \to F^2(H_n) \otimes F$ and $\Theta_2 : F^2(H_n) \otimes F \to F^2(H_n) \otimes \varepsilon_*$ are contractive multi-analytic operators such that, if X_Θ is the operator defined by (33), then the subspaces $\mathbf{H}_1$and $\mathbf{H}_2 := \mathbf{H} \ominus \mathbf{H}_1$ have the representations:

$$\mathrm{H}_1 = \left\{\Theta_2 f \oplus X_\Theta^*(\Delta_2 f \oplus g) : f \in F^2(H_n) \otimes F, g \in \overline{\Delta_1(F^2(H_n) \otimes \varepsilon}\right\}$$

$$\ominus \left\{\Theta\varphi \oplus \Delta_\Theta\varphi : \varphi \in F^2(H_n) \otimes \varepsilon\right\}$$

and

$$\mathrm{H}_2 = \left[\left(F^2(H_n) \otimes \varepsilon_*\right) \oplus X_\Theta^*\left(\overline{\Delta_2(F^2(H_n) \otimes F)}\right) \oplus \{0\}\right]$$

$$\ominus \left\{\Theta_2 f \oplus X_\Theta^*(\Delta_2 f \oplus 0) : f \in F^2(H_n) \otimes F\right\}$$

Conversely, every regular factorization $\Theta = \Theta_2\Theta_1$ generates via the above formulas the subspaces $\mathbf{H}_1$ and $\mathbf{H}_2$ with the following properties:

(i) $\mathbf{H}_1$ is an invariant subspace under each operator $\mathbf{T}_i$, i = 1, . . . , n;

(ii) $\mathbf{H}_2 = \mathbf{H} \ominus \mathbf{H}_1$.

In what follows we need the following factorization result for contractive multi analytic operators [168].

Lemma (5.2.6)[142]: Let $\Theta \in R_n^\infty \overline{\otimes} B(\varepsilon, G)$ be a contractive multi-analytic operator. Then Θ admits a unique decomposition $\Theta = \psi \oplus \Lambda$ with the following properties:

(i) $\Psi \in R_n^\infty \overline{\otimes} B(\varepsilon_0, G_0)$ is purely contractive,.e $\left\|P_{G_0}\psi\, h\right\| < \|h\|$ for any $h \in \varepsilon_0, h \neq 0$

(ii) $\Lambda = \mathrm{I} \otimes U \in R_n^\infty \overline{\otimes} B(\varepsilon_u, G_u)$ is a unitary operator;

(iii) $\varepsilon = \varepsilon_0 \oplus \varepsilon_u \quad and \quad G = G_0 \oplus G_u .$

Moreover, the purely contractive part of an outer or inner multi-analytic operator is also outer or inner, respectively.

The next result is an addition to Theorem (5.2.2)

Proposition (5.2.7)[142]: Let $\Theta : F^2(H_n) \otimes \varepsilon \to F^2(H_n) \otimes \varepsilon_*$ be a contractive multi-analytic operator such that

$$\overline{\Delta_\Theta\left(F^2(H_n) \otimes \varepsilon\right)} = \overline{\Delta_\Theta\ [(F^2(H_n) \otimes \varepsilon) \ominus \varepsilon]}$$

and let T := [T$_1$, . . . ,T$_n$] be the functional model associated with Θ, as in Theorem (5.2.2).

(i) The characteristic function of **T** := [**T**$_1$, . . . ,**T**$_n$] coincides with the purely contractive part of Θ.

(ii) The space **H** defined in Theorem (5.2.2) is different from {0} if and only if there is no unitary operator $U \in B(\varepsilon, \varepsilon_*)$ such that $\Theta = I \otimes U$.

Proof. According to Lemma (5.2.6), the multi-analytic operator Θ admits the decomposition $\Theta = \Phi \oplus A \, with \, \psi \in R_n^\infty \overline{\otimes} B(\varepsilon_0, \varepsilon_{*0})$ purely contractive and A = $I \otimes U \in R_n^\infty \overline{\otimes} B(\varepsilon_u, \varepsilon_{*u}), \, where U \in B(\varepsilon_u, \varepsilon_{*u})$ is a unitary operator, $\varepsilon = \varepsilon_0 \oplus \varepsilon_u \; and \; \varepsilon_* = \varepsilon_{*0} \oplus \varepsilon_{*u}$. Notic that

$$F^2(H_n) \otimes \varepsilon_* = (F^2(H_n) \otimes \varepsilon_{*u}) \oplus (F^2(H_n) \otimes \varepsilon_{*0}) \;\; and$$
$$F^2(H_n) \otimes \varepsilon = (F^2(H_n) \otimes \varepsilon_u) \oplus (F^2(H_n) \otimes \varepsilon_0).$$

On the other hand, we have

$$\{\Theta g \oplus \Delta_\Theta g : g \in F^2(H_n) \otimes \varepsilon\} = (F^2(H_n) \otimes \varepsilon_{*u}) \oplus \{\Phi\varphi \oplus \Delta_\Phi \varphi : \varphi \in F^2(H_n) \otimes \varepsilon_0\}.$$

Now, using the definition of the Hilbert space **H**, one can identify **H** with

$$H_0 := \left[\left(F^2(H_n) \otimes \varepsilon_{*0}\right) \oplus \overline{\Delta_\Phi(F^2(H_n) \otimes \varepsilon_0)}\right] \ominus \left\{\Phi\varphi \oplus \Delta_\Phi\varphi : \varphi \in F^2(H_n) \otimes \varepsilon_0\right\}.$$

Due to this identification, the row contraction **T** := [**T**$_1$, . . . ,**T**$_n$] is unitarily equivalent to $\mathbf{T}^0 := [\mathbf{T}_1^0, ..., \mathbf{T}_n^0]$, which is defined on H_0 in the same manner as **T** is defined on **H**. Since $\Delta_\Theta = \Delta_\Phi \oplus 0$, it is easy to see that

$$\overline{\Delta_\Phi\left(F^2(H_n) \otimes \varepsilon\right)} = \overline{\Delta_\Phi\ [(F^2(H_n) \otimes \varepsilon) \ominus \varepsilon]}$$

According to the second part of Theorem (5.2.2) the characteristic function of $\mathbf{T}^0$ coincides with the multi-analytic operator Φ which coincides with the characteristic function of $\mathbf{T}$.

We prove now part (ii). If $\Theta = I \otimes U$ for some unitary operator $U \in B(\varepsilon, \varepsilon_*)$, then $\Delta_\Theta = 0$ and

$$\mathbf{H} = [F^2(H_n) \otimes \varepsilon_*] \ominus \Theta(F^2(H_n) \otimes \varepsilon) = \{0\}.$$

If Θ is not a unitary multi-analytic operator, then, according to Lemma (5.2.6) it has a non-trivial purely contractive part. By part (i), Theorems

$$\dim D_* = \dim \varepsilon_{*0}, \qquad \dim D = \dim \varepsilon_0,$$

where ε and ε_{*0} are not both equal to $\{0\}$. Since $D_* \subset H$ and $D \subset H^{(n)},$ we deduce that $H \neq \{0\}$.This completes the proof.

The following result is an important addition to Theorem (5.2.5) (and hence also to Theorem (5.2.4).

Theorem (5.2.8)[142]: Under the conditions of Theorem (5.2.8), let $\mathbf{H} = \mathbf{H}_1 \oplus \mathbf{H}_2$ be the decomposition corresponding to the regular factorization $\Theta = \Theta_2\Theta_1$**,** and let

$$T_i = \begin{pmatrix} A_i & * \\ 0 & B_i \end{pmatrix}, \qquad i = 1, \dots, n$$

be the corresponding triangulation of $\mathbf{T} := [\mathbf{T}_1, \dots, \mathbf{T}_n]$. Then the characteristic functions of the row contractions $\mathbf{A} := [\mathbf{A}_1, \dots, \mathbf{A}_n]$ and $\mathbf{B} := [\mathbf{B}_1, \dots, \mathbf{B}_n]$ coincide with the purely contractive parts of the multi-analytic operators Θ_1 and Θ_2, respectively.

Moreover, the invariant subspace $\mathbf{H}_1$ under the operators $\mathbf{T}_1, \dots, \mathbf{T}_n$ is non-trivial if and only if the regular factorization $\Theta = \Theta_2\Theta_1$ is non-trivial, i.e., each factor is not a unitary constant.

Proof. Define the operator U from the Hilbert space

$$(F^2(H_n) \otimes \varepsilon_*) \oplus X_\Theta^* \overline{\left(\Delta_2(F^2(H_n) \otimes F\right) \oplus \{0\}})$$

to

$$(F^2(H_n) \otimes \varepsilon_*) \oplus \overline{\left(\Delta_2(F^2(H_n) \otimes F\right)}$$

by setting

$$U(f \oplus X^*(g \oplus 0)) := f \oplus g$$

for any f $\in F^2(H_n) \otimes \varepsilon_*$ and g $\in \overline{\left(\Delta_2(F^2(H_n) \otimes F\right)}$. Since X_Θ is unitary, so is U. Using the definition of $\mathbf{H_2}$ (see relation (70)), we deduce that U$\mathbf{H}_2 = \hat{H}_2$, where

$$\hat{H}_2 := [(F^2(H_n) \otimes \varepsilon_*) \oplus \overline{\Delta_2(F^2(H_n) \otimes F)}] \ominus \left\{\Theta_2\varphi \oplus \Delta_2\varphi : \varphi \in F^2(H_n) \otimes F\right\}. \quad (74)$$

Set $\Gamma_i^* := UB_i^*U^*, i = 1,...,n$ and denote by P_1 the orthogonal projection of the direct sum $\overline{\Delta_2(F^2(H_n) \otimes F)} \oplus \overline{\Delta_1(F^2(H_n) \otimes \varepsilon)}$ *onto* $\overline{\Delta_2(F^2(H_n) \otimes F)}$. Using Lemma (5.2.3), we deduce that

$$P_1 X_\Theta C_i^* X_\Theta^* (g \oplus 0) = P_1 \begin{pmatrix} F_i^* & 0 \\ 0 & E_i^* \end{pmatrix} \begin{pmatrix} g \\ 0 \end{pmatrix} = F_i^* g$$

for any g $\in \overline{\Delta_2(F^2(H_n) \otimes F)}$ and i = 1, . . . , n. Hence and using the definitions for the row contraction [$\mathbf{T}_1$, . . . ,$\mathbf{T}_n$] and the unitary operator U, we have

$$\begin{aligned}
\Gamma_i^*(f \oplus g) &= UT_i^*(f \oplus X_\Theta^*(g \oplus 0)) \\
&= U[(S_i^* \otimes I_{\varepsilon_*})f \oplus C_i^* X_\Theta^*(g \oplus 0)] \\
&= (S_i^* \otimes I_{\varepsilon_*})f \oplus P_1 X_\Theta C_i^* X_\Theta^*(g \oplus 0) \\
&= (S_i^* \otimes I_{\varepsilon_*})f \oplus F_i^* g
\end{aligned}$$

for any f $\in F^2(H_n) \otimes \varepsilon_*$ and g $\in \overline{\Delta_2(F^2(H_n) \otimes F)}$ such that f $\oplus$g $\in\mathbf{H_2}$, and i=1,…,n .

Since

$$\overline{\Delta_\Theta\left(F^2(H_n) \otimes \varepsilon\right)} = \overline{\Delta_\Theta\ [(F^2(H_n) \otimes \varepsilon) \ominus \varepsilon]},$$

one can use again Lemma (5.2.3) to deduce that

$$\overline{\Delta_2(F^2(H_n) \otimes F)} = \overline{\Delta_2(F^2(H_n) \otimes F)}.$$

Now, due to Proposition (5.2.7), we infer that the characteristic function of the row contraction $[\Gamma_1, \ldots, \Gamma_n], \Gamma_i \in B(\hat{H}_2)$ (and hence also [$\mathbf{B}_1$, . . . ,$\mathbf{B}_n$]), coincides with the purely contractive part of the multi-analytic operator Θ_2.

Taking into account the definition of the subspace H_1 (see relation (71)) and the fact that $\Theta = \Theta_2\Theta_1$, one can see that, for each f$\in F^2(H_n) \otimes F$ and $g \in \overline{\Delta_1 F^2(H_n) \otimes \varepsilon}$, the vector $\Theta_2 f \oplus X_\Theta^*(\Delta_2 f \oplus g)$ is in $\mathbf{H_1}$ if and only if

$$\left\langle \Theta_2 f \oplus X_\Theta^*(\Delta_2 f \oplus g), \Theta_2\Theta_1\varphi \oplus X_\Theta^*(\Delta_2\Theta_1\varphi \oplus \Delta_1\varphi)\right\rangle = 0$$

for any $\varphi \in F^2(H_n)\otimes\varepsilon$. The latter equation is equivalent to

$$\Theta_1^*\Theta_2^*\Theta_2 f + \Theta_1^*\Delta_2^2 f + \Delta_1 g = 0$$

Since $\Delta_2^2 = I - \Theta_2^*\Theta_2$ the above equation is equivalent to

$$\Theta_1^* f + \Delta_1 g = 0 \tag{75}$$

If x := $\Theta_2 f \oplus X_\Theta^*(\Delta_2 f \oplus g) \in H_1$, then we have

$$T_i^* x = (S_i^* \otimes I\varepsilon_*)\Theta_2 f \oplus C_i^* X_\Theta^*(\Delta_2 f \oplus g)$$

for each i = 1, . . . , n. Since Θ_2 is a multi-analytic operator and

$$f = \sum_{j=1}^{n}(S_j S_j^* \otimes I_F) f + f(0),$$

where f (0) := $P_{1\otimes F} f$, we deduce that

$$\begin{aligned} T_i^* x &= \left[\Theta_2(S_i^* \otimes I_F) f + (S_i^* \otimes I\varepsilon_*)\Theta_2 f(0)\right] \oplus C_i^* X_\Theta^*(\Delta_2 f \oplus g) \\ &= u + v, \end{aligned}$$

where

$$u := \Theta_2(S_i^* \otimes I_F) f \oplus \left[X_\Theta^*(\Delta_2(S_i^* \otimes I_F) f \oplus E_i^* g)\right]$$

and

$$v := (S_i^* \otimes I_{\varepsilon_*})\Theta_2 f(0) \oplus \left[C_i^* X_\Theta^*(\Delta_2 f \oplus g) - X_\Theta^*(\Delta_2(S_i^* \otimes I_F) f \oplus E_i^* g)\right].$$

Now notice that u $\in$ **H**$_1$. Indeed, using the above characterization of the elements of **H**$_1$, it is enough to show that

$$\Theta_1^*(S_i^* \otimes I_F) f + \Delta_1 E_i^* g = 0 \qquad i = 1,\dots,n \tag{76}$$

Using relation (75) and the definition of E_i, we have

$$\Theta_1^*(S_i^* \otimes I_F) f + \Delta_1 E_i^* g = (S_i^* \otimes I_\varepsilon)(\Theta_1^* f + \Delta_1 g) = 0$$

which proves (76) and therefore u $\in$ **H**$_1$.

Now we prove that v $\in$ **H**$_2$. First, notice that due to Lemma (5.2.3), we have

$$C_i^* X_\Theta^*(0 \oplus g) = X_\Theta^*(0 \oplus E_i^* g), \quad g \in \overline{\Delta_1(F^2(H_n)\otimes\varepsilon)},,$$

and therefore

$$v = (S_i^* \otimes I\varepsilon_*)\Theta_2 f(0) \oplus \left[C_i^* X_\Theta^*(\Delta_2 f \oplus 0) - X_\Theta^*(\Delta_2(S_i^* \otimes I_F) f \oplus 0)\right] \tag{77}$$

Using again Lemma (5.2.3) and the definition of F_i , we infer that

$$C_i^* X_\Theta^*(\Delta_2 f \oplus 0) = C_i^* X_\Theta^* \left(\Delta_2 \left(\sum_{j=1}^n S_j S_j^* \otimes I_F \right) f(0) \oplus 0 \right) + C_i^* X_\Theta^* (\Delta_2 f(0) \oplus 0)$$

$$= X_\Theta^* \left(F_i^* \Delta_2 \left(\sum_{j=1}^n S_j S_j^* \otimes I_F \right) f \oplus 0 \right) + C_i^* X_\Theta^* (\Delta_2 f(0) \oplus 0)$$

$$= X_\Theta^* (\Delta_2 (S_i^* \otimes I_F) f \oplus 0) + C_i^* X_\Theta^* (\Delta_2 f(0) \oplus 0)$$

$$= X_\Theta^* (\Delta_2 (S_i^* \otimes I_F) f \oplus 0) + X_\Theta^* (F_i^* \Delta_2 f(0) \oplus 0).$$

Consequently, relation (77) implies

$$v = (S_i^* \otimes I_{\varepsilon *}) \Theta_2 f(0) \oplus X_\Theta^* (F_i^* \Delta_2 f(0) \oplus 0)$$

Due to the definition of the subspace $\mathbf{H_2}$, to prove that v $\in$ $\mathbf{H}_2$, it is enough to show that

$$\Theta_2^* (S_i^* \otimes I_{\varepsilon *}) \Theta_2 f(0) + \Delta_2 F_i^* \Delta_2 f(0) = 0$$

for each i = 1, . . . , n. Since

$$\Delta_2 F_i^* = (S_i^* \otimes I_F) \Delta_2, \qquad i = 1, ..., n$$

and Θ_2 is multi-analytic, we have

$$\Theta_2^* (S_i^* \otimes I_{\varepsilon *}) \Theta_2 f(0) + \Delta_2 F_i^* \Delta_2 f(0) = (S_i^* \otimes I_F)(\Theta_2^* \Theta_2 + \Delta_2^2) f(0)$$

$$= (S_i^* \otimes I_F) f(0) = 0.$$

Hence, v $\in$ $\mathbf{H}_2$. Now, using the fact that $T_i^* x = u + v$ and the definitions for u and v, we deduce that the operator $A_i^* := P_{H_1} T_i^* |_{H_1}$ satisfies the equation

$$A_i^* (\Theta_2 f \oplus X_\Theta^* (\Delta_2 f \oplus g)) = \Theta_2 (S_i^* \otimes I_F) f \oplus \left[X_\Theta^* (\Delta_2 (S_i^* \otimes I_F) f \oplus E_i^* g) \right] \tag{78}$$

for any $\Theta_2 f \oplus X_\Theta^* (\Delta_2 f \oplus g)$ ∈ H_1 and i = 1, . . . , n.

$$\left\{ \Theta_2 f \oplus X_\Theta^* (\Delta_2 f \oplus g) : f \in F^2(H_n) \otimes F, g \in \overline{\Delta_1 (F^2(H_n) \otimes \varepsilon)} \right\}$$

to the direct sum $(F^2(H_n) \otimes F) \oplus \overline{\Delta_1 (F^2(H_n) \otimes \varepsilon)}$ by setting

$$\Omega (\Theta_2 f \oplus X_\Theta^* (\Delta_2 f \oplus g)) := f \oplus g. \tag{79}$$

Since

$$\left\| \Theta_2 f \oplus X_\Theta^* (\Delta_2 f \oplus g) \right\|^2 = \left\| \Theta_2 f \right\|^2 + \left\| X_\Theta^* (\Delta_2 f \oplus g) \right\|^2$$

$$= \left\langle \Theta_2^* \Theta_2 f, f \right\rangle + \left\| \Delta_2 f \right\|^2 + \left\| g \right\|^2$$

$$= \left\| f \oplus g \right\|^2,$$

it is clear that Ω is a unitary operator. Notice also that

$$\Omega\ (\Theta\varphi \oplus \Delta_\Theta\varphi) = \Omega\ (\Theta_2\Theta_1\varphi \oplus X^*_\Theta(\Delta_2\Theta_1\varphi \oplus \Delta_1\varphi)$$
$$= \Theta_1\varphi \oplus \Delta_1\varphi$$

for any $\varphi \in F^2(H_n) \otimes \varepsilon$. Consequently, $\Omega H_1 = \hat{H}_1$, where

$$\hat{H}_1 := \left[(F^2(H_n) \otimes F) \oplus \overline{\Delta_1(F^2(H_n) \otimes \varepsilon)}\right] \ominus \{\Theta_1\varphi \oplus \Delta_1\varphi : \varphi \in F^2(H_n) \otimes \varepsilon\}. \quad (80)$$

Setting $A_i := \Omega A_i \Omega^*$ relation (63) implies

$$A_i^*(f \oplus g) = (S_i^* \otimes I_F)f \oplus E_i^* g,\quad f \oplus g \in H_1,$$

for any i = 1, . . . , n. Once again, Lemma (5.2.3) implies

$$\overline{\Delta_1\left(F^2(H_n) \otimes \varepsilon\right)} = \overline{\Delta_1\ [(F^2(H_n) \otimes \varepsilon) \ominus \varepsilon]}$$

Now, using Proposition (5.2.7), we infer that the characteristic function of the row contraction $[A_1, . . . , A_n]$, $A_i \in B(\hat{H}_1)$ (and hence also $[\mathbf{A}_1, . . . ,\mathbf{A}_n]$), coincides with the purely contractive part of the multi-analytic operator Θ_1. Due to the relations (74), (80), and Proposition (5.2.7),the subspaces $\hat{H}_1$ and $\hat{H}_2$ (and hence also $\mathbf{H_1}$ and $\mathbf{H_2}$) are different from {0} if and only if both multi-analytic operators Θ_1 and Θ_2 are not unitary constant, i.e., the factorization $\Theta = \Theta_2\Theta_1$ is non-trivial. This completes the proof.

Now, combining Theorems (5.2.4) and (5.2.8), we can deduce the following result.

Theorem (5.2.9)[142]: Let $T:=[T_1, \ldots, T_n]$ be a completely non-coisometric row contraction on a separable Hilbert space H. Then, there is a non-trivial invariant subspace under each operator $T_1,\ldots, T_n$ if and only if the characteristic function Θ_T has a non-trivial regular factorization.

Concerning the uniqueness in Theorem (5.2.5) (and also Theorem (5.2.4)), we can prove the following result, which shows the extent to which a joint invariant subspace determines the corresponding regular factorization of the characteristic function.

Theorem (5.2.10)[142]: Under the conditions of Theorem (5.2.5) let

$$\Theta = \Theta_2\Theta_1 \quad \text{and} \quad \Theta = \Theta_2'\Theta_1'$$

be two regular factorizations of the purely contractive multi-analytic operator Θ, and let ε ,F, ε_* and $\varepsilon, F', \varepsilon_*$ be the corresponding Hilbert spaces. Let **H**$_1$⊂ **H** and **H′**$_1$ ⊂ **H** be the invariant subspaces under each operator **T$_i$** , i = 1, . . . , n, corresponding to the above factorizations. If **H**⊂**H′**$_1$, then there is a multi-analytic operator $\Psi : F^2(H_n)\otimes F \to F^2(H_n)\otimes F'$ such that

$$\Theta_1' = \psi\,\Theta_1$$

Moreover, if **H** = **H′**$_1$

$$\Theta_1' = (I\otimes\psi_0)\Theta_1$$

for some unitary operator $\psi_0 \in B(F,F')$ and, consequently, the multi-analytic operators Θ and Θ_1' coincide.

Proof. We associate with the factorization Θ = $\Theta_2\Theta_1$ the subspace

$$M := \left\{ \Theta_2 f \oplus X_\Theta^*(\Delta_2 f\oplus g) : f\in F^2(H_n)\otimes F, g\in \overline{\Delta_1(F^2(H_n)\otimes\varepsilon)} \right\}.$$

Similarly, we define the subspace M' associated with the factorization Θ = $\Theta_2'\Theta_1'$ **H**$_1$ ⊆ **H′**$_1$ relation (71) and its analogue for **H′**$_1$ imply M $\subseteq M'$ Consequently, for each $f\in F^2(H_n)\otimes F$, there exist $f'\in F^2(H_n)\otimes F'$ and $g' \in \overline{\Delta_1'(F^2(H_n)\otimes\varepsilon)}$ such that

$$(\Theta_2 f \oplus X_\Theta^*(\Delta_2 f\oplus 0) = \Theta_2' f' \oplus X_\Theta'^*(\Delta_2 f'\oplus g'). \qquad (81)$$

Hence and using the definition of the unitary operators X_Θ and X_Θ', we have

$$\|f\|^2 = \|\Theta_2 f\oplus X_\Theta^*(\Delta_2 f\oplus g)\|^2 = \|\Theta_2' f'\oplus X_\Theta'^* \ (\Delta_2 f'\oplus g')\|^2 = \|f'\|^2 + \|g'\|^2.$$

Therefore, it makes sense to define the contraction $Q: F^2(H_n)\otimes F \to F^2(H_n)\otimes F'$ and R : $F^2(H_n)\otimes F \to \overline{\Delta_1'(F^2(H_n)\otimes\varepsilon)}$ by setting Qf := f′ and Rf := g′,respectively. Now, we show that Q is a multi-analytic $Q(S_i\otimes I_F) = (S_i\otimes I_{F'})Q$, i = 1, . . . , n.

Let f_1, . . . , f_n be arbitrary elements in $F^2(H_n)\otimes\varepsilon$. Taking into account the definitions for C_i and X_Θ, and the fact that

$$(S_j^*\otimes I_F)\Delta_2^2(S_i\otimes I_F) = \delta_{ij}\Delta_2^2, \ \ i,j=1,\dots,n$$

we deduce that

$$\left\langle C_i X_\Theta^*(\Delta_2 f \oplus 0), \Delta_\Theta(\sum_{j=1}^n (S_j \otimes I_\varepsilon) f_j)\right\rangle$$

$$= \langle (\Delta_2 f \oplus 0), X_\Theta \Delta_\Theta f_i\rangle = \langle (\Delta_2 f \oplus 0), \Delta_2 \Theta_1 f_i \oplus \Delta_1 f_i\rangle = \langle \Delta_2^2 f, \Theta_1 f_i\rangle$$

and

$$\left\langle X_\Theta^*(\Delta_2 (S_i \otimes I_F) f \oplus 0), \Delta_\Theta(\sum_{j=1}^n (S_j \otimes I_\varepsilon) f_j)\right\rangle$$

$$= \left\langle \Delta_2 (S_i \otimes I_F) f \oplus 0, \Delta_2 \Theta_1 (\sum_{j=1}^n (S_j \otimes I_\varepsilon) f_j) \oplus \Delta_1 (\sum_{j=1}^n (S_j \otimes I_\varepsilon) f_j)\right\rangle$$

$$= \left\langle \Delta_2 (S_i \otimes I_F) f, \Delta_2 \Theta_1 (\sum_{j=1}^n (S_j \otimes I_\varepsilon) f_j)\right\rangle$$

$$= \sum_{j=1}^n \left\langle (S_j^* \otimes I_F) \Delta_2^2 (S_i \otimes I_F) f, \Theta_1 f_j\right\rangle$$

$$= \langle \Delta_2^2 f, \Theta_1 f_i\rangle$$

Hence, and taking into account that

$$\overline{\Delta_\Theta\left(F^2(H_n) \otimes \varepsilon\right)} = \overline{\Delta_\Theta\ [(F^2(H_n) \otimes \varepsilon) \ominus \varepsilon]}$$

we deduce that

$$C_i X_\Theta^*(\Delta_2 f \oplus 0) = X_\Theta^*(\Delta_2 (S_i \otimes I_F) f \oplus 0) \ \ for\, any\, f \in F^2(H_n) \otimes F. \quad (82)$$

Similar calculations show that

$$C_i X_\Theta^*(0 \oplus \Delta_1 \varphi) = X_\Theta^*(0 \oplus \Delta_1 (S_i \otimes I_\varepsilon) \varphi) \quad (83)$$

for any $\phi \in F^2(H_n) \otimes \varepsilon$ and i = 1, … , n. Moreover, similar relations to (82) and (68) hold with X'_Θ, Δ'_1, and Δ'_2 instead of X_Θ, Δ_1 and Δ_2, respectively. Since

$$C_i X_\Theta'^*(0 \oplus \Delta'_1 \varphi) = X_\Theta'^*(0 \oplus \Delta'_1 (S_i \otimes I_\varepsilon) \varphi) \quad (84)$$

for any $\varphi \in F^2(H_n) \otimes \varepsilon$ and i = 1, . . . , n, by taking appropriate limits, we deduce that

$$C_i X_\Theta'^*(\{0\} \oplus \overline{\Delta'_1(F^2(H_n) \otimes \varepsilon)}) \subseteq X_\Theta'^*(\{0\} \oplus \overline{\Delta'_1(F^2(H_n) \otimes \varepsilon)}).$$

Consequently, for each $g' \in \overline{\Delta'_1(F^2(H_n) \otimes \varepsilon)}$ there exists $g'' \in \overline{\Delta'_1(F^2(H_n) \otimes \varepsilon)}$ such that

$$C_i X_\Theta'^*(0 \oplus g') = X_\Theta'^*(0 \oplus g''). \quad (85)$$

Now, notice that using relations (82), (81), (84), and (85), we obtain

$$\begin{aligned}\Theta_2(S_i\otimes I_F)f\oplus X_\Theta^*(\Delta_2(S_i\otimes I_F)f\oplus 0) &= (S_i\otimes I\varepsilon_*\oplus C_i)(\Theta_2 f\oplus X_\Theta^*(\Delta_2 f\oplus 0))\\ &= (S_i\otimes I\varepsilon_*\oplus C_i)(\Theta_2' f'\oplus X_\Theta'^*(\Delta_2' f'\oplus g'))\\ &= \Theta_2'(S_i\otimes I_{F'})f'\oplus X_\Theta'^*(\Delta_2'(S_i\otimes I_{F'})f'\oplus g'')\end{aligned}$$

for any f $\in$ F $F^2(H_n)\otimes F$. Hence and using the definition of Q, we deduce that

$$Q(S_i\otimes I_F)f=(S_i\otimes I_{F'})f'=(S_i\otimes I_{F'})Qf,\quad f\in F^2(H_n)\otimes F,$$

which proves that Q is a multi-analytic operator.

Since M$\subset M'$, we have

$$\bigcap_{k=0}^{\infty}\bigoplus_{|\alpha|=k}\left[(S_\alpha\otimes I_{\varepsilon_*})\oplus C_\alpha\right]M\subseteq\bigcap_{k=0}^{\infty}\bigoplus_{|\alpha|=k}\left[(S_\alpha\otimes I_{\varepsilon_*})\oplus C_\alpha\right]M'. \tag{86}$$

Using Lemma (5.2.3), definition (79) of the unitary operator Ω, and relations (82), (83), one can prove that

$$\left[(S_i\otimes I_{\varepsilon_*})\oplus C_i\right]\Omega^*=\Omega^*\left[(S_i\otimes I_F)\oplus E_i\right].$$

Indeed, we have

$$\begin{aligned}&\left[(S_i\otimes I_{\varepsilon_*})\oplus C_i\right]\Omega^*(f\oplus\Delta_1\phi)\\ &=\Theta_2(S_i\otimes I_F)f\oplus C_iX_\Theta^*(\Delta_2 f\oplus\Delta_1\varphi)\\ &=\Theta_2(S_i\otimes I_F)f\oplus X_\Theta^*(\Delta_2(S_i\otimes I_F)f\oplus\Delta_1(S_i\otimes I_\varepsilon)\varphi)\\ &=\Omega^*[(S_i\otimes I_F)f\oplus\Delta_1(S_i\otimes I_\varepsilon)\varphi]\\ &=\Omega^*[(S_i\otimes I_F)\oplus E_i](f\otimes\Delta_1\varphi)\end{aligned}$$

for any $f\in F^2(H_n)\otimes F$ *and* $\varphi\in F^2(H_n)\otimes\varepsilon.$

Now, due to the fact that $[S_1\otimes I_F,\dots,S_n\otimes I_F]$ is a multi-shift and $[E_1,\dots,E_n]$ is a Cuntz row isometry, the noncommutative Wold decomposition implies

$$\begin{aligned}&\bigcap_{k=0}^{\infty}\bigoplus_{|\alpha|=k}\left[(S_\alpha\otimes I_{\varepsilon_*})\oplus C_\alpha\right]M\\ &=\Omega^*\left\{\bigcap_{k=0}^{\infty}\left[\bigoplus_{|\alpha|=k}(S_\alpha\otimes I_F)(F^2(H_n)\otimes F)\right]\oplus\bigcap_{k=0}^{\infty}\left[\bigoplus_{|\alpha|=k}E_\alpha\overline{\Delta_1(F^2(H_n)\otimes\varepsilon)}\right]\right\}\\ &=\Omega^*(\{0\}\oplus\overline{\Delta_1(F^2(H_n)\otimes\varepsilon)})\\ &=\{0\oplus X_\Theta^*(0\oplus g):g\in\overline{\Delta_1(F^2(H_n)\otimes\varepsilon)}\}.\end{aligned}$$

A similar relation can be obtain for the set on the right-hand side of the inclusion (86). Hence and using relation (86), we obtain

$$\{0\oplus X_\Theta^*(0\oplus g): g\in\overline{\Delta_1(F^2(H_n)\otimes\varepsilon)}\}\subseteq\{0\oplus X_\Theta'^*(0\oplus g'): g'\in\overline{\Delta_1'(F^2(H_n)\otimes\varepsilon)}\}.$$

Consequently, for each g $\in \overline{\Delta_1(F^2(H_n)\otimes\varepsilon)}$ there exists $g'\in\overline{\Delta_1'(F^2(H_n)\otimes\varepsilon)}$ such that

$$X_\Theta^*(0\oplus g)=X_\Theta'^*(0\oplus g'). \tag{87}$$

Since X_Θ *and* X_Θ' are unitary operators, we can define the isometry

$$V:\overline{\Delta_1(F^2(H_n)\otimes\varepsilon)}\to\overline{\Delta_1'(F^2(H_n)\otimes\varepsilon)}$$

by setting $Vg:=g'$. *For each* $\varphi\in F^2(H_n)\otimes\varepsilon$, *we have*

$$\Theta\varphi\oplus\Delta_\Theta\varphi=\Theta_2'\Theta_1'\varphi\oplus X_\Theta'^*(\Delta_2'\Theta_1'\varphi\oplus\Delta_1'\varphi). \tag{73}$$

On the other hand, using the operators Q,R,V and relation (81), we deduce that

$$\begin{aligned}\Theta\varphi\oplus\Delta_\Theta\varphi&=\Theta_2\Theta_1\varphi\oplus X_\Theta^*(\Delta_2\Theta_1\varphi\oplus\Delta_1\varphi)\\&=\left[\Theta_2\Theta_1\varphi\oplus X_\Theta^*(\Delta_2\Theta_1\varphi\oplus 0)\right]+\left[0\oplus X_\Theta^*(0\oplus\Delta_1\varphi)\right]\\&=\left[\Theta_2'Q\Theta_1\varphi\oplus X_\Theta'^*(\Delta_2'Q\Theta_1\varphi\oplus R\Theta_1\varphi)\right]+\left[0\oplus X_\Theta'^*(0\oplus V\Delta_1\varphi)\right]\\&=\Theta_2'Q\Theta_1\varphi\oplus X_\Theta'^*(\Delta_2'Q\,\Theta_1\varphi\, y),\end{aligned}$$

where y := $R\Theta_1\varphi+V\Delta_1\varphi$ *is in* $\overline{\Delta_1'(F^2(H_n)\otimes\varepsilon)}$. Using the latter relation and (88), we obtain

$$\Theta_2'\Theta_1'\varphi=\Theta_2'Q\Theta_1\varphi\quad and\quad \Delta_2'\Theta_1'\varphi=\Delta_2'Q\Theta_1\varphi.$$

Since the mapping $\Theta_2'f'\oplus\Delta_2'f'\mapsto f'$ is isometric, we deduce that

$$\Theta_1'\varphi=Q\Theta_1\varphi,\qquad \varphi\in F^2(H_n)\otimes\varepsilon, \tag{89}$$

which proves the first part of the theorem.

Now assume that $\mathbf{H}_1$ = $\mathbf{H}'_1$ A closer look at the above proof reveals that $Q(F^2(H_n)\otimes F)=F^2(H_n)\otimes F'$ and V is a unitary operator. Taking into account relations (87) and (81), we obtain

$$\begin{aligned}\Theta_2 f\oplus X_\Theta^*(\Delta_2 f\oplus 0)&=\left[\Theta_2'f'\oplus X_\Theta'^*(\Delta_2'f'\oplus 0)\right]+\left[0\oplus X_\Theta'^*(0\oplus g')\right]\\&=\left[\Theta_2'f'\oplus X_\Theta'^*(\Delta_2'f'\oplus 0)\right]+\left[0\oplus X_\Theta'^*(0\oplus V^*g')\right].\end{aligned}$$

Hence, we get

$$\Theta_2 f \oplus X_\Theta^*(\Delta_2 f \oplus (-V^* g')) = \Theta_2' f' \oplus X_\Theta'^*(\Delta_2' f' \oplus 0).$$

Taking the norms, we have

$$\|f\|^2 + \|g'\|^2 = \|f'\|^2.$$

Combining this with $\|f\|^2 + \|f'\|^2 = \|g'\|^2$., we obtain $\|f\| = \|f'\|$., which shows that is a unitary multi-analytic operator. Due to [166], this implies $Q = I \otimes \Psi_0$, for some unitary operator $\Psi_0 \in B(F, F')$ Using relation (89), we complete the proof.

We prove the existence of a unique triangulation of type

$$\begin{pmatrix} C_{.0} & 0 \\ * & C_{.1} \end{pmatrix} \tag{90}$$

for any row contraction T := [T1, . . . , Tn], and prove the existence of joint invariant subspaces for certain classes of row contractions.

We need a few definitions. A row contraction $T := [T_1, \ldots, T_n], T_i \in B(H)$, is of class $C_{\cdot 1}$ if

$$\lim_{k\to\infty} \sum_{|\alpha|=k} \|T_\alpha^* h\|^2 \neq 0 \quad \text{for any } h \in H, h \neq 0.$$

We say that a row contraction $T := [T_1, \ldots, T_n]$, $T_i \in B(H)$, has a triangulation of type (90) if there is an orthogonal decomposition

H= $H_0 \oplus H_1$ with respect to which

$$T_i = \begin{pmatrix} A_i & 0 \\ * & B_i \end{pmatrix}, \quad i = 1, \ldots, n$$

and the entries have the following properties:

(i) $T_i^* H_0 \subset H_0$ *for any* $i = 1, \ldots, n$;

(ii) $A := [A_1, \ldots, A_n]$ *is of class* $C_{.0}$;

(iii) $B := [B_1, \ldots, B_n]$ *is of class* $C_{.1}$

The type of the entry denoted by * is not specified.

Theorem (5.2.11)[142]: Every row contraction $T := [T_1, \ldots, T_n], T_i \in B(H)$, has a triangulation of type

$$\begin{pmatrix} C_{.0} & 0 \\ * & C_{.1} \end{pmatrix}$$

Moreover, this triangulation is uniquely determined.

Proof. First, notice that the subspace

$$H_0 := \left\{ h \in H : \lim_{k\to\infty} \sum_{|\alpha|=k} \|T_\alpha^* h\|^2 = 0 \right\}$$

is invariant under each operator T_i^*, i= 1, . . . , n. The decomposition H $= H_0 \oplus H_1$, where $H_1 := H \ominus H_0$, i yields the triangulation

$$T_i^* = \begin{pmatrix} A_i^* & * \\ 0 & B_i^* \end{pmatrix}, \quad i = 1,\dots,n,$$

where $A_i^* := T_i^* |_{H_0}$ *and* $B_i^* := P_{H_1} T_i^* |_{H_1}$ *for each* $i = 1,\dots,n$. Since

$$\lim_{k\to\infty} \sum_{|\alpha|=k} \|A_\alpha^* h\|^2 = \lim_{k\to\infty} \sum_{|\alpha|=k} \|T_\alpha^* h\|^2 = 0, \quad h \in H_0,$$

the row contraction A:= [A_1, . . . , A_n] is of class $C_{.0}$. Now, we need to show that

$$\lim_{k\to\infty} \sum_{|\alpha|=k} \|B_\alpha^* h\|^2 \neq 0 \quad \textit{for all } h \in H_1,\ h \neq 0.$$

Let V := [V_1, . . . , V_n], $V_i \in$ B(K), be the minimal isometric dilation of the row contraction T :=[T_1, . . . , T_n] .For every m = 1, . . . , the isometries V_α,|α| = m, have orthogonal ranges. Therefore, we have

$$\begin{aligned} \left\| \sum_{|\alpha|=m} V_\alpha \left(\sum_{|\beta|=k} V_\beta T_\beta^* \right) P_{H_0} T_\alpha^* h \right\|^2 &= \sum_{|\alpha|=m} \left\| \left(\sum_{|\beta|=k} V_\beta T_\beta^* \right) P_{H_0} T_\alpha^* h \right\|^2 \\ &= \sum_{|\alpha|=m} \sum_{|\beta|=k} \|T_\beta^* P_{H_0} T_\alpha^* h\|^2 \end{aligned}$$

For any h $\in$H since P_{H_0} T^*_α h$\in H_0$w e have

$$\lim_{k\to\infty} \sum_{|\beta|=k} \|T_\beta^* P_{H_0} T_\alpha^* h\|^2 = 0. \tag{91}$$

According to [134], we have

$$P_R h = \lim_{k\to\infty} \sum_{|\alpha|=k} V_\alpha T_\alpha^* h \quad \textit{for any } h \in H \tag{92}$$

Where P_R is the orthogonal projection of the minimal isometric dilation space K on the subspace R in the Wold decomposition $K = R \oplus M_V(L_*)$. Now, using relations (91) and (92), we obtain

$$
\begin{aligned}
P_R h &= \lim_{k\to\infty}\sum_{|\alpha|=m}\sum_{|\beta|=k} V_\alpha V_\beta T_\alpha^* T_\beta^* \\
&= \lim_{k\to\infty}\sum_{|\alpha|=m} V_\alpha\left(\sum_{|\beta|=k} V_\beta T_\beta^*\right) P_{H_0} T_\alpha^* h + \lim_{k\to\infty}\sum_{|\alpha|=m} V_\alpha\left(\sum_{|\beta|=k} V_\beta T_\beta^*\right) P_{H_1} T_\alpha^* h \\
&= \sum_{|\alpha|=m} V_\alpha P_R P_{H_1} T_\alpha^* h.
\end{aligned}
$$

Hence, we deduce that

$$
\begin{aligned}
\|P_R h\|^2 &= \left\|\sum_{|\alpha|=m} V_\alpha P_R P_{H_1} T_\alpha^* h\right\|^2 = \sum_{|\alpha|=m}\left\|P_R P_{H_1} T_\alpha^* h\right\|^2 \\
&\le \sum_{|\alpha|=m}\left\|P_{H_1} T_\alpha^* h\right\|^2 = \sum_{|\alpha|=m}\left\|B_\alpha^* h\right\|^2
\end{aligned}
$$

for any h $\in$ H. Let h $\in H_1$,h $\neq 0$,and assume that $\lim_{m\to\infty}\sum_{|\alpha|=m}\|B_\alpha^* h\|^2 = 0$.

The above relation shows that $P_R h=0$ and, due to (91), we deduce that $h \in H_0$, which is a contradiction.

Now, we prove the uniqueness. Assume that there is another decomposition **H**$=M_0 \oplus M_1$ which yields the triangulation

$$
T_i = \begin{pmatrix} C_i & 0 \\ * & D_i \end{pmatrix}, \quad i = 1,\ldots,n
$$

of type $\begin{pmatrix} C_{.0} & 0 \\ * & C_{.1} \end{pmatrix}$, where $C_i^* := T_i^*|_{M_0}$ and $D_i^* := P_{M_1} T_i^*|_{M_1}$ for each i = 1, . . ., n. To prove uniqueness, it is enough to show that $H_0 = M_0$. Notice that if $h \in M_0$, then, due to the fact that the row contraction $[C_1, \ldots, C_n]$ is of class $C_{\cdot 0}$, we have

$$
\lim_{m\to\infty}\sum_{|\alpha|=m}\|T_\alpha^* h\|^2 = \lim_{m\to\infty}\sum_{|\alpha|=m}\|C_\alpha^* h\|^2 = 0
$$

Hence, $h \in H_0$, which proves that $H_0 \subseteq M_0$. Assume now that $h \in H_0 \ominus M_0$. Since $h \in M_1$, we have

$$
\lim_{m\to\infty}\sum_{|\alpha|=m}\|D_\alpha^* h\|^2 = \lim_{m\to\infty}\sum_{|\alpha|=m}\|P_{M_1} T_\alpha^* h\|^2 \le \lim_{m\to\infty}\sum_{|\alpha|=m}\|T_\alpha^* h\|^2 = 0
$$

Consequently, since the row contraction $[D_1, \ldots, D_n]$ is of class $C_{\cdot 1}$, we must have $h = 0$. Hence,we deduce that $H_0 \ominus M_0 = \{0\}$, which shows that $M_0 = H_0$. This completes the proof.

Corollary (5.2.12)[142]: If T := [T_1, … , T_n] is a row contraction such $T \notin C_{\cdot 0}$ and T $\notin C_{\cdot 1}$, then there is a non-trivial joint invariant subspace under $T_1, \ldots, T_n$.

Any row contraction admits a triangulation of type

$$\begin{pmatrix} C_c & 0 \\ * & C_{cnc} \end{pmatrix}$$

where C_c (respectively C_{cnc}) denotes the class of coisometric (respectively c.n.c.) row contractions. Notice that $C_c \subset C_{\cdot 1}$. Combining this result with the triangulation of Theorem (5.2.11), we obtain another triangulation for row contractions, that is,

$$\begin{pmatrix} C_{.0} & 0 & 0 \\ * & C_c & 0 \\ * & * & C_{cnc} \bigcap C_{.1} \end{pmatrix}$$

Corollary (5.2.13)[142]: If T := [T_1, . . . , T_n], $T_i \in B(H)$, is a row contraction such

$$T_1 T_1^* + \ldots + T_n T_n^* \neq \mathrm{I}$$

and there is a non-zero vector $h \in H$ such that $\sum_{|\alpha|=k} \left\| T_\alpha^* h \right\|^2 = \left\| h \right\|^2$ for any k = 1, 2,…, then there is a non-trivial invariant subspace under the operators $T_1, \ldots, T_n$.

We recall from [163] that if

$$T_1 T_1^* + \ldots + T_n T_n^* = \mathrm{I}$$

then a subspace M is invariant under $T_1, \ldots, T_n$ if and only if

$$T_1 P_M T_1^* + \ldots + T_n P_M T_n^* \leq P_M$$

where P_M is the orthogonal projection on M. We also mention that the case when $T \in C_{\cdot 0}$ is treated in the next corollary.

Lemma (5.2.14)[142]: Let $\Theta : F^2(H_n) \otimes \varepsilon \to F^2(H_n) \otimes \varepsilon_*$ be a contractive multi-analytic operator and assume that it has the factorization

$$\Theta = \Theta_2 \Theta_1$$

where Θ_1: $F^2(H_n) \otimes \varepsilon \to F^2(H_n) \otimes F$ and Θ_2: $F^2(H_n) \otimes F \longrightarrow F^2(H_n) \otimes \varepsilon_*$ are contractive multi-analytic operators.

(i) If Θ_2 is inner, then the factorization $\Theta = \Theta_2\Theta_1$ is regular.

(ii) If Θ is inner, then the factorization $\Theta = \Theta_2\Theta_1$ is regular if and only if Θ_1 and Θ_2 are inner multi-analytic operators.

(iii) If rank $\Delta_\Theta < \infty$, then

$$rank\ \Delta_\Theta = rank\ \Delta_{\Theta_2} + rank\ \Delta_{\Theta_1}$$

if and only if the factorization $\Theta = \Theta_2\Theta_1$ is regular.

Corollary (5.2.15)[142]: If T := [T_1, . . . , T_n] is a row contraction of class $C_{\cdot 0}$, then the non-trivial joint invariant subspaces under $T_1,\ldots, T_n$ are parametrized by the non-trivial inner factorizations of the characteristic function Θ_T of T (i.e., $\Theta_T = \Theta_2\Theta_1$ with Θ_1 and Θ_2 inner multi-analytic operators). Moreover, the subspaces $\mathbf{H}_1$ and $\mathbf{H}_2$ in Theorem (5.2.4) become

$$\mathrm{H}_1 = \{\Theta_2 f : f \in F^2(H_n)\otimes F\} \ominus \{\Theta_T f : f \in F^2(H_n)\otimes D\} \quad \text{and} \quad \mathrm{H}_2 = \{F^2(H_n)\otimes D_*\}$$
$$\ominus \{\Theta_2 f : f \in F^2(H_n)\otimes F\}$$

where D and D_* are the defect spaces of T .

Now, we consider some examples that explicitly illustrate the correspondence between joint invariant subspaces and factorizations of the characteristic function.

Example (5.2.16)[142]: Let $\Theta := 1/\sqrt{2}(R_1^2R_2 + R_1R_2^2)$, where R_1, R_2 are the right creation operators on $F^2(H_2)$ the full Fock space with 2 generators. Since $R_i^*R_j = \delta_{ij}I, i, j = 1,2$ we have $\Theta^*\Theta = I$ .On the other hand, is a purely contractive inner multi-analytic operator. Define the Hilbert space

$$H := F^2(H_2) \ominus [F^2(H_2)\otimes(e_2\otimes e_1^2 + e_2^2\otimes e_1)]$$

and the row contraction T := [T_1,T_2], where $T_i := P_H S_i I_H$ and S_1, S_2 are the left creation operators on $F^2(H_2)$. According to Theorem (5.2.2), the characteristic function of T coincides with the multi-analytic operator Θ .

We consider now some regular factorizations of Θ_T and write down the corresponding joint invariant subspaces for T_1, T_2. First, notice that

$$\Theta_T = R_1\left(1/\sqrt{2}R_1R_2 + 1/\sqrt{2}R_2^2\right)$$

and the multi-analytic operators $\Theta_1 := 1/\sqrt{2}R_1R_2 + 1/\sqrt{2}R_2^2$ and $\Theta_2 := R_1$ are isometries on $F^2(H_2)$. Therefore, due to Lemma (5.2.14), the factorization $\Theta_T = \Theta_2\Theta_1$ is regular. Taking into account Corollary (5.2.15), we deduce that the joint invariant subspace under T_1, T_2 corresponding to the above factorization is

$$M := \left[F^2(H_2)\otimes e_1\right] \ominus \left[F^2(H_2)\otimes(e_2\otimes e_1^2 + e_2^2\otimes e_1)\right]$$

Another regular factorization of Θ_T is

$$\Theta_T = \left(1/\sqrt{2}R_1^2 + 1/\sqrt{2}R_1R_2\right)R_2.$$

As above, one can see that this is a regular factorization and the corresponding joint invariant subspace for T_1, T_2 is

$$N := \left[F^2(H_2)\otimes(e_1^2 + e_2\otimes e_1)\right] \ominus \left[F^2(H_2)\otimes(e_2\otimes e_1^2 + e_2^2\otimes e_1)\right].$$

Let us consider a class of examples when the regular factorizations have factors which are not multi-analytic operators with scalar coefficients.

Example (5.2.17)[142]: Let $\Theta \in B(F^2(H_n))$ be an inner multi-analytic operator with $\Theta(0) = 0$.Due to the structure of multi-analytic operators,we have $\Theta = R_1\phi_1 + \cdots + R_n\phi_n$ for some multi-analytic operators $\phi_1, \ldots, \phi_n \in B(F^2(H_n))$. Since $R_i^*R_j = \delta_{ij}I, i, j = 1,\ldots,n$ it is clear that Θ is inner if and only if

$$\varphi_1^*\varphi_1 + \ldots + \varphi_n^*\varphi_n = I. \tag{93}$$

In this case, Θ is purely contractive and we have the factorization $\Theta = \Theta_2\Theta_1$, where

$$\Theta_1 := \begin{bmatrix}\varphi_1\\ \vdots\\ \varphi_n\end{bmatrix} \; and \; \Theta_2 := [R_1,\ldots,R_n]$$

are inner multi-analytic operators. Clearly, the factorization $\Theta = \Theta_2\Theta_1$ is regular. Define the Hilbert space $H := F^2(H_n) \ominus \Theta F^2(H_n)$ and the row contraction $T := [T_1,\ldots,T_n]$, where $T_i := P_H S_i|_H$ *and* $S_1,\ldots,S_n$ are the left creation operators on the full Fock space $F^2(H_n)$. According to Theorem (5.2.2), the characteristic function of T coincides with the multi-analytic operator Θ.

The joint invariant subspace under $T_1, \ldots, T_n$ corresponding to the regular factorization $\Theta_T = \Theta_2\Theta_1$ is

$$M = \left[F^2(H_n)\otimes e_1 + \ldots + F^2(H_n)\otimes e_n\right] \ominus \Theta F^2(H_n)$$

As examples of $\phi_1, \ldots, \phi_n$ satisfying relation (63), one can take $\varphi_i = 1/\sqrt{n}\,V_i, i = 1,\ldots,n$, where V_i is any isometry in R_n^∞ (e.g., any product R_α, $\alpha \in \mathbf{F}_n^+$)

We remark that if $\Psi \in B(F^2(H_n))$ is an inner multi-analytic operator with Fourier representation $\psi = \sum_{|\alpha|\geq m} a_\alpha R_\alpha, m = 1,2,\ldots$,then it admits the regular factorization

$$\psi = \left[R_\beta : |\beta| = m\right]\begin{bmatrix} \Phi_{(\beta)} \\ \vdots \\ |\beta| = m \end{bmatrix},$$

Where $\Phi_{(\beta)} \in B(F^2(H_n))$ are multi-analytic operators such that $\sum_{|\beta|=m} \Phi^*_{(\beta)}\Phi_{(\beta)} = I$. Now,one can write Example (5.2.17) in this more general setting. For examples of inner multi-analytic operators we refer to [169,170].

We recall [165] that any multi-analytic operator admits an essentially unique inner–outer factorization.

Theorem (5.2.18)[142]: Let T:= [T_1,…, T_n] be a completely non-coisometric row contraction. The inner–outer factorization of the characteristic function Θ_T induces (cf. Theorem (5.2.8) the triangulation of type

$$\begin{pmatrix} C_{.0} & 0 \\ * & C_{.1} \end{pmatrix}$$

for the row contraction T .

In particular, if the inner–outer factorization of the characteristic function is non-trivial, then there is a non-trivial joint invariant subspace under the operators T_1,...,T_n.

Proof. Suppose that the multi-analytic operatorΘ: $F^2(H_n)\otimes\varepsilon \to F^2(H_n)\otimes\varepsilon_*$ coincides with the characteristic function of the c.n.c. row contraction$T := [T_1, \ldots, T_n]$. $Let\ \Theta = \Theta_i\Theta_0$ be the canonical inner–outer factorization of Θ. Since Θ_i is inner, Lemma (5.2.14) implies that the factorizationis regular. Therefore, according to Theorem (5.2.4) (see also Theorem (5.2.5)) and Theorem (5.2.8), the above factorization yields a triangulation

$$T_i = \begin{pmatrix} B_i & 0 \\ * & A_i \end{pmatrix}, \quad i = 1,\dots,n$$

of $\mathbf{T} := [\mathbf{T_1}, \dots, \mathbf{T_n}]$, the functional model of T , such that the characteristic functions of $\mathbf{B}:=[\mathbf{B}_1, \dots, \mathbf{B}_n]$ and $\mathbf{A} := [\mathbf{A}_1, \dots, \mathbf{A}_n]$ coincide with the purely contractive parts of Θ_i and Θ_0, respectively. Due to Lemma (5.2.6), the purely contractive part of an outer or inner multi-analytic operator is also outer or inner, respectively. We recall from [132] that a c.n.c. row contraction is of class $C_{\cdot 0}$ (respectively $C_{\cdot 1}$) if and only if the corresponding characteristic function is inner (respectively outer) multi-analytic operator. Finally, using the last part of Theorem (5.2.8), we can complete the proof.

We obtain criteria for joint similarity of n-tuples of operators to Cuntz row isometries. In particular, we prove that a completely non-coisometric row contraction $T:= [T_1, \dots, T_n]$ is jointly similar to a Cuntz row isometry if and only if the characteristic function of T is an invertible multi-analytic operator. This is a multivariable version of a result of Sz.-Nagy and Foias [161], concerning the similarity to unitary operators.

Extending some results obtained by Sz.-Nagy [161], Sz.-Nagy, Foias [134], and the author [152,163] we provide necessary and sufficient conditions for a power bounded n-tuple of operators on a Hilbert space to be jointly similar to a Cuntz row isometry.

We need the following well-known result (see, e.g., [134]).

Lemma (5.2.19)[142]: Let M , N, X and Y be subspaces of a Hilbert space H such that

$$H = M \oplus N = X \oplus Y$$

if

$$P_M X = M \ \ and \ \|P_M x\| \geq c \ \| x \|, \quad x \in X,$$

for some constant c >0, then

$$P_N Y = N \ and \ \| P_N y \| \geq c \| y \|, \quad y \in Y.$$

We recall a few facts concerning the geometric structure of the minimal isometric dilation of a row contraction. Let $T := [T_1,\dots, T_n]$,

$T_i \in B(H)$, be a row contraction and let $V := [V_1, \ldots, V_n]$ be its minimal isometric dilation on a Hilbert space $K \supseteq H$. In [153], we proved that $K = R \oplus M_V(L_*)$ and

$$P_R h = \lim_{k\to\infty} \sum_{|\alpha|=k} V_\alpha T_\alpha^* h, \quad h \in H \ , \tag{94}$$

where P_R is the orthogonal projection of K onto R. Moreover, if T is a one-to-one row contraction, then

$$\overline{P_R H} = R \, . \tag{95}$$

The next result provides necessary and sufficient conditions for a c.n.c. row contraction to be jointly similar to a Cuntz row isometry, in terms of the corresponding characteristic function.

Theorem (5.2.20)[142]: Let $T := [T_1, \ldots, T_n]$, $T_i \in B(H)$, be a completely non-coisometric row contraction.Then T is jointly similar to a Cuntz row isometry $W := [W_1, \ldots, W_n]$, $W_i \in B(\mathcal{W})$, i.e.,

(i) $W_1 W_1^* + \ldots + W_n W_n^* = I_W$;

(ii) $ST_i = W_i S$, $i = 1, \ldots, n$, for invertible operator $S : H \to \mathcal{W}$,

if and only if the characteristic function Θ_T is an invertible multi-analytic operator.

In this case,

$$\left\|\Theta_T^{-1}\right\| = \min\left\{ \|X\| \left\|X^{-1}\right\| : \left[X^{-1} T_1 X, \ldots, X^{-1} T_n X \right] \ is\ a\ Cuntz\ row\ isometry \right\}.$$

Proof. Suppose that the row contraction $T := [T_1, \ldots, T_n]$ is jointly similar to a Cuntz row isometry $W := [W_1, \ldots, W_n]$, $W_i \in B(\mathcal{W})$, i.e.,

$$W_1 W_1^* + \ldots + W_n W_n^* = I_W$$

and $T_i = S^{-1} W_i S, i = 1, .., n$, for some invertible operator $S : H \to \mathcal{W}$. Since $ST_\alpha = W_\alpha S$ and $T_\alpha^* S^* = S^* W_\alpha^*$ *for any* $\alpha \in \mathbb{F}_n^+$, *we have*

$$S\left(\sum_{|\alpha|=k} T_\alpha T_\alpha^* \right) S^* = \sum_{|\alpha|=k} W_\alpha S S^* W_\alpha^* \geq \frac{1}{\left\|S^{*-1} S^{-1}\right\|} \sum_{|\alpha|=k} W_\alpha W_\alpha^* = \frac{1}{\left\|S^{-1}\right\|^2} I$$

for any $k = 1, 2, \ldots$. Therefore,

$$\sum_{|\alpha|=k} \left\langle T_\alpha T_\alpha^* h, h \right\rangle \geq \left\|S^{*-1} h\right\|^2 \frac{1}{\left\|S^{-1}\right\|^2} \geq \frac{1}{\left\|S^*\right\|^2 \left\|S^{-1}\right\|^2} \|h\|^2 ,$$

which, due to relation (94), implies

$$\|P_R h\| \geq \frac{1}{\|S\|\|S^{-1}\|}\|h\|, \quad h \in H . \tag{96}$$

Notice that the operator $[T_1, \dots, T_n]$ is one-to-one. Indeed, the relation

$$S^{-1}W_1Sh_1 + \dots + S^{-1}W_nSh_n = 0, \quad h_i \in H, \ i=1,\dots,n,$$

implies

$$W_1Sh_1 + \dots + W_nSh_n = 0$$

Since W_i are isometries with orthogonal ranges, we have

$$W_iSh_i = 0, \quad i = 1, \dots, n,$$

whence $h_i = 0$, $i = 1, \dots, n$. Therefore $[T_1, \dots, T_n]$ is one-to-one. According to (95), we have $\overline{P_R H} = R$. Due to relation (96), the subspace $P_R H$ is closed. Therefore, $P_R H = R$ and the operator

$$X := P_R|H : H \to R$$

is invertible. According to (94), we have

$$V_i^* P_R h = \lim_{k\to\infty} \sum_{|\alpha|=k} V_i^* V_\alpha T_\alpha^* h = \lim_{k\to\infty} \sum_{|\alpha|=k-1} V_\beta T_\beta^* V_\alpha T_i^* h = P_R T_i^* h$$

for any $h \in H$ and $i = 1, \dots, n$. Consequently, we have

$$T_i X^* = X^* W_i, \quad i = 1,\dots,n,$$

where $W_i := V_i|_R$, $i = 1, \dots, n$. Due to the noncommutative Wold decomposition applied to the row isometry $[V_1, \dots, V_n]$, the subspace R is reducing under each isometry V_i, $i = 1, \dots$, and $[W_1, \dots, W_n]$ is a Cuntz row isometry.

Now, due to the geometric structure of the minimal isometric dilation of T, we have (see relation (47))

$$K = R \oplus M_V(L_*) = H \oplus M_V(L)$$

Since $P_R H = R$, we can use relation (96) and Lemma (5.2.19) to deduce that

$$P_{M_V(L_*)} M_V(L) = M_V(L_*) \ \ and \ \ \|P_{M_V(L_*)} x\| \geq \frac{1}{\|S\|\|S^{-1}\|}\|x\|, \quad x \in M_V(L).$$

Therefore, the operator

$$Q := P_{M_V(L_*)} \mid M_V(L) : M_V(L) \to M_V(L_*)$$

is an invertible contraction with $\|Q^{-1}\| \le \|S\|\|S^{-1}\|$. Since Q is unitarily equivalent to the characteristic function Θ_T of T , we deduce that Θ_T is an invertible multi-analytic operator and $\|\Theta_T{}^{-1}\| \le \|S\|\|S^{-1}\|$.

Conversely, assume that the characteristic function Θ_T (and hence Q) is an invertible contraction and $\|\Theta_T{}^{-1}\| \le 1/c$ for some constant c >0. Applying again Lemma (5.2.19), we deduce that

$$P_R H = R \quad and \quad \|P_R H\| \ge c\|h\|, \quad h \in H.$$

This shows that the operator X := $P_R|_H$: H→R is invertible and $\|X^{-1}\| \le 1/c$. As in the first part of the proof, we have $X^*(V_i \mid R) = T_i X^*$ for any i = 1, . . . , n. This proves the similarity to a Cuntz row isometry. Notice also that, since $\|X\| \le 1$, we have

$$\|X^{*-1}\|\|X^*\| = \|X^{-1}\|\|X\| \le \frac{1}{c}.$$

To prove the last art of the theorem, let c > 0 be such that $\|\Theta_T{}^{-1}\| = \frac{1}{c}$ The converse of this theorem implies the existence of on invertible operator X such that $[X^{-1}T_1X,..., X^{-1}T_nX]$ is a Cuntz row isometry and

$$\|X\|\|X^{-1}\| \le \frac{1}{c} = \|\Theta_T^{-1}\|.$$

On the other hand, using the first part of the proof, we have

$$\|\Theta_T^{-1}\| \le \|X\|\|X^{-1}\|.$$

Therefore, $\|\Theta_T^{-1}\| = \|X\|\|X^{-1}\|$ and the proof is complete.

Corollary (5.2.21)[142]: If T := [T_1, . . . , T_n], $T_i \in B(H)$, is a completely non-coisometric row contraction jointly similar to a Cuntz row isometry, then T is jointly similar to the Cuntz part in the Wold decomposition of the minimal isometric dilation of T . Moreover, in this case, T is similar to the model row contraction C := [C_1,..., C_n],where for each i = 1, … , n,

$$C_i : \overline{\Delta_{\Theta_T}(F^2(H_n) \otimes D)} \to \overline{\Delta_{\Theta_T}(F^2(H_n) \otimes D)}$$

is defined by

$$C_i(\Delta_{\Theta_T} f) := \Delta_{\Theta_T}(S_i \otimes I_D)f,\; f \in F^2(H_n)\otimes D,$$

and $\Delta_{\Theta_T} := (I - \Theta_T^*\Theta_T)^{1/2}$ where Θ_T is the characteristic function of T .

Proof. The first part of the theorem follows from the proof of Theorem (5.2.20). Now, using the model theory for c.n.c. row contractions (see Theorems (5.2.1) and (5.2.2), one can complete the proof.

Now we consider the case when T := [T_1, . . . , T_n] is an arbitrary row contraction.

Theorem (5.2.22)[142]: Let T := [T_1, . . . , T_n], $T_i \in$ B(H), be a row contraction. Then T is jointly similar to a Cuntz row isometry W := [W_1, . . . , W_n], $W_i \in \mathcal{W}$, if and only if T is one-to-one and the operator

$$P := \left(SOT - \lim_{k\to\infty} \sum_{|\alpha|=k} T_\alpha T_\alpha^* \right)^{1/2} \tag{97}$$

is invertible.

Moreover, if this is the case, then the row contraction T := [T_1, . . . , T_n] is jointly similar to the Cuntz part R := [R_1, . . . , R_n] in the Wold decomposition of the minimal isometric dilation of T .

Proof. Assume T is a similar to W, i.e., there exists an invertible operator S :H $\to \mathcal{W}$ such that $T_i = S^{-1}W_iS, i = 1,\dots,n$ As in the proof of Theorem (5.2.20), one can show that the operator [T_1, . . . , T_n] is one-to-one. According to (95), we have $\overline{P_R H}$ = R. On the other hand, due to relation (94), we deduce that

$$\|P_R h\|^2 = \lim_{k\to\infty} \sum_{|\alpha|=k} \|T_\alpha^* h\|^2 = \|Ph\|^2, \qquad h \in H, \tag{98}$$

where operator P is well defined by (97), due to the fact that $\{\sum_{|\alpha|=k} T_\alpha T_\alpha^*\}_{k=1}^\infty$ is a decreasing sequence of positive operators. Notice that, since $\{W_\alpha\}_{|\alpha|=k}$ are isometries with orthogonal ranges, we have

$$\begin{aligned}\sum_{|\alpha|=k} \|T_\alpha^* h\|^2 &\ge \|S^{-1}\|^{-2} \sum_{|\alpha|=k} \|W_\alpha^* S^{*-1} h\|^2 = \|S^{-1}\|^{-2} \|S^{*-1}h\|^2 \\ &\ge \left(\|S^{-1}\|^2 \|S\|^2 \right)^{-1} \|h\|^2\end{aligned}$$

for any h $\in$ H. Therefore

$$\|P_R h\|^2 = \|Ph\|^2 \geq \left(\|S^{-1}\|^2 \|S\|^2\right)^{-1} \|h\|^2$$

for any h $\in$H. Hence, it follows that the operators P and $P_R|_H$ are one-to-one and have closed ranges. Since $\overline{P_R H}$ =R, it is clear that the operator X:H→R is invertible.

According to relation (94), we have

$$V_i^* P_R h = \lim_{k\to\infty} \sum_{|\alpha|=k-1} V_\beta T_\beta^* T_i^* h = P_R T_i^* h$$

for any h $\in$H and i = 1, . . . , n. Consequently, we deduce that

$$XT_i^* = R_i^* X, \qquad i = 1,\dots,n, \tag{99}$$

where X := $P_R|_H$ and R_i := $V_i|_R$, i = 1, . . . , n. Therefore, T := [T_1, . . . , T_n] is jointly similar to R := [R_1, . . . , R_n].

Conversely, assume that the row contraction [T_1, . . . , T_n] is one-to-one and the operator P is invertible. Then relation (98) implies $P_R|_H$ is one-to-one and has closed range. On the other hand, by (95), we have $\overline{P_R H} = R$. Therefore, the operator X := $P_R|_H$:H→R is invertible and, due to relation (69), the row contraction [T_1, . . . , T_n] is jointly similar to the Cuntz row isometry [$V_1|_R$, . . . , $V_n|_R$]. The proof is complete.

We recall [163] that an n-tuple [T_1, . . . , T_n], of operators T_i $\in$B(H), is power bounded if there is a constant M >0 such that

$$\sum_{|\alpha|=k} \|T_\alpha^* h\|^2 \leq M^2 \|h\|^2, \quad h \in H,$$

for any k = 1, 2, ...

Theorem (5.2.23)[142]: Let [T_1, . . . , T_n] be a one-to-one power bounded n-tuple of operators on a Hilbert space H such that, for any non-zero element h $\in$ H, $\sum_{|\alpha|=k} \|T_\alpha^* h\|^2$ does not converge to 0 as k→∞. Then there exists a Cuntz row isometry [W_1, . . . , W_n], W_i $\in$B(H), such that

$$T_i X = XW_i, \qquad i = 1, \dots, n,$$

for some one-to-one operator X $\in$ B(H) with range dense in H.

Proof. For each h $\in$H, h $\neq$0, denote

$$c(h) := \inf_{k=1,2,\dots} \left(\sum_{|\alpha|=k} \|T_\alpha^* h\|^2 \right)^{1/2}.$$

Since [T_1,…, T_n] is a power bounded n-tuple of operators, there is a constant M >0 such that

$$\sum_{|\alpha|=k} \left\| T_\alpha^* h \right\|^2 \le M^2 \|h\|^2, \quad h \in H, \tag{100}$$

for any k = 1, 2, ... If c(h) = 0 and ε >0, then there is k_0 such that

$$\left(\sum_{|\alpha|=k_0} \left\| T_\alpha^* h \right\|^2 \right)^{1/2} \le \frac{\varepsilon}{M}.$$

Hence and using (100), we deduce that

$$\sum_{|\alpha|=m+k_0} \left\| T_\alpha^* h \right\|^2 = \sum_{|\beta|=k_0} \left\langle T_\beta \left(\sum_{|\gamma|=m} T_\gamma T_\gamma^* \right) T_\beta^* h, h \right\rangle$$

$$\le M^2 \sum_{|\beta|=k_0} \left\langle T_\beta T_\beta^* h, h \right\rangle \le \varepsilon^2$$

for any $m \ge 0$. Consequently, $\lim_{k\to\infty} \sum_{|\alpha|=k} \left\| T_\alpha^* h \right\|^2 = 0$, which contradicts the hypothesis. Therefore, we must have c(h) $\neq 0$ for any h $\in$ H, h $\neq 0$.

Now, for each h, $h' \in$ H, we define

$$[h, h'] := \lim_{k\to\infty} \sum_{|\alpha|=k} \left\langle T_\alpha^* h, T_\alpha^* h' \right\rangle$$

where LIM is a Banach limit. Due to the properties of the Banach limit, $[\cdot,\cdot]$ is a bilinear form on H and we deduce that

$$\left[h, h \right] := \lim_{k\to\infty} \sum_{|\alpha|=k} \left\| T_\alpha^* h \right\|^2 \ge c(h)^2 > 0 \quad if \;\; h \in H \quad h \neq 0$$

and [h,h] $\le M^2 \|h\|^2$. Moreover, we have

$$[h, h] = \sum_{i=1}^{n} [T_i^* h, T_i^* h], \qquad h \in H.$$

Due to a well-known theorem on bounded Hermitian forms, there exists a self-adjoint operator P $\in$ B(H) such that

$$[h, h'] = \left\langle Ph, h' \right\rangle \quad for\, any\; h, h' \in H,$$

and, due to the above considerations, we have

$$0 < \left\langle Ph, h \right\rangle < M^2 \|h\|^2, \quad h \in H, h \neq 0 \;. \tag{101}$$

Now, we show that $P = \sum_{i=1}^{n} T_i P T_i^*$ Indeed, we have

$$\langle Ph,h\rangle = \lim_{k\to\infty}\sum_{|\alpha|=k+1}\|T_\alpha^* h\|^2 = \lim_{k\to\infty}\sum_{i=1}^{n}\sum_{|\alpha|=k}\|T_\alpha^* T_i^* h\|^2$$
$$= \sum_{i=1}^{n}[T_i^* h, T_i^* h] = \sum_{i=1}^{n}[PT_i^* h, T_i^* h]$$
$$= \sum_{i=1}^{n}\left\langle \sum_{i=1}^{n} T_i P T_i^* h, h\right\rangle$$

for any h $\in$ H, which proves our assertion. Notice that relation (101) shows that the operator X:= $P^{1/2}$ is one-to-one and has range dense in H. Since $\sum_{i=1}^{n}\|XT_i^* h\|^2 = \|Xh\|^2$ for any h $\in$H, it is clear that

$$\sum_{i=1}^{n}\|XT_i^* X^{-1}x\|^2 = \|x\|^2$$

for any x in the domain on X^{-1}. Hence and due to the fact that the domain on X^{-1} is dense in H, the operators $V_i^* := XT_i^* X^{-1}, i = 1,\dots,n,$ can be extended by continuity on H. Using the same notation for the corresponding extensions, we have

$$\sum_{i=1}^{n}\|V_i^* h\|^2 = \|h\|^2,\ h\in H,$$

and $V_i^* X = XT_i^*, i = 1,\dots,n$ .This shows that [V_1, . . . , V_n] is a co-isometry from $H^{(n)}$ to H such that

$$T_iX = XV_i, \qquad i= 1, \dots, n.$$

Assume now that $h_i \in$ H and $\sum_{i=1}^{n} V_i h_i = 0$. Then $\sum_{i=1}^{n} T_i X h_i = 0$. Since [$T_1$,…, T_n] and X are one-to-one operators, we must have $h_i = 0$ for each
i = 1,…, n. Consequently, [V_1,…, V_n] is a one-to-one co-isometry, and therefore a unitary operator from $H^{(n)}$ to H. This implies that V_1,…, V_n are isometries on H with $V_1V_1^* + \dots + V_nV_n^* = I_H$ $=I_H$. The proof is complete.

As a consequence of Theorem (5.2.23), we deduce the following criterion for joint similarity of a power bounded n-tuple of operators to a Cuntz row isometry.

Corollary (5.2.24)[142]: Let [T_1,…, T_n] be a one-to-one power bounded
n-tuple of operators on a Hilbert space H. Then [T_1,…, T_n] is jointly similar to a Cuntz row isometry if and only if there exists a constant c >0 such that

$$\sum_{|\alpha|=k}\|T_\alpha^* h\|^2 \ge c\|h\|^2,\quad h\in H \quad , \tag{102}$$

for any k = 1, 2, ...

Proof. The direct implication can be extracted from the proof of Theorem (5.2.20). Conversely, if condition (102) holds, then, using the proof of Theorem (5.2.23), we have

$$c(h) \geq \sqrt{c}\|h\|, \qquad h \in H, h \neq 0.$$

Moreover, the positive operator $P \in B(H)$ has the properties

$$T_i P^{1/2} = P^{1/2} V_i, \qquad i = 1, \ldots, n,$$

where $[V_1, \ldots, V_n]$ is a Cuntz isometry, and

$$\langle Ph, h \rangle \geq c\|h\|^2, \qquad h \in H, \quad h \neq 0$$

Since the latter inequality shows that $P^{1/2}$ is an invertible operator, the result follows.

Chapter 6

Minimal-Volume Projections with Sufficient Enlargements of Minimal-Volume

A symmetric with respect to 0 bounded closed convex set A in a finite dimensional normed space X is called a sufficient enlargement for X (or of $B(X)$) if for arbitrary isometric embedding of X into a Banach space Y there exists a projection $P: Y \to X$ such that $P(B(Y))$ is a subset of A (by $B(X)$ we denote the unit ball). In particular the author investigate sufficient enlargements whose support functions are in some directions close to those of the unit ball of the space, sufficient enlargements of minimal volume, sufficient enlargements for euclidean spaces. We devoted to a description of the shape of such images of the cube. The shape is characterized in terms of zonotopes spanned by scalar multiples of rows of totally unimodular matrices. The main results of the chapter: (1) Each minimal-volume sufficient enlargement is linearly equivalent to a zonotope spanned by multiples of columns of a totally unimodular matrix. (2) If a finite-dimensional normed linear space has a minimal-volume sufficient enlargement which is not a parallelepiped, then it contains a two-dimensional subspace whose unit ball is linearly equivalent to a regular hexagon.

Section (6.1): Normed Linear Spaces and Sufficient Enlargements:

Definition (6.1.1)[171]: A in a finite dimensional normed space X is called a sufficient enlargement for X (or of B(X)) if for arbitrary isometric embedding $X \subset Y$ (Y is a Banach space) there exists a projection $P : Y \to X$ such that $P(B(Y)) \subset A$. A minimal sufficient enlargement is defined to be a sufficient enlargement no proper subset of which is a sufficient enlargement.

The notion of sufficient enlargement is implicit in B.Grünbaum's in [82], it was explicilty introduced by the present author in [76].

The notion of sufficient enlargement is of interest because it is a natural geometric notion, it characterizes possible shadows of symmetric convex body onto a subspace, whose intersection with the body is given.

The main purpose of the present section is to continue investigation of sufficient enlargements started in [76]. We investigate sufficient enlargements whose support functions are in some directions close to those of the unit ball of the space, we have devoted to sufficient enlargements for euclidean spaces. We have refer to [172] and [78] for background on Banach space theory and to [83] for background on the theory of convex bodies.

Let X and Y be finite dimensional normed spaces and T:X→Y be a linear operator. An l_∞ − factorization of T is a pair of operators $u_1 : X \to l_\infty$ and $u_2 : l_\infty \to Y$ satisfying $T=u_2u_1$. The l_∞ −factorable norm of T is defined to be the inf $\|u_1\|\|u_2\|$, where the if is taken over all. l_∞ −factorizations.

An absolute projection constant of a finite dimensional normed linear space X is defined to be the smallest positive real number $\lambda(X)$ such that for every isometric embedding $X\subset Y$ there exists a continuous linear projection $P:Y \to X$ with $\|P\| \le \lambda(X)$.

We shall use the following observations.

Proposition (6.1.2)[171]: [76] Let A be a ball in a finite dimensional normed linear space X. The space X normed by the gauge functional of A will be denoted by X_A. The ball A is a sufficient enlargement for X if and only if the L_∞- factorable norm of the natural identity mapping from X to X_A is ≤ 1.

Proposition (6.1.3)[171]: [82] A symmetric with respect to 0 parallelepiped containing B(X) is a sufficient enlargement for X .

Proposition (6.1.4)[171]: [82] Convex combination of sufficient enlargements for X is a sufficient enlargement for X .

Theorem (6.1.5)[171]: Let X be an n − dimensional normed space. Let $\{f_i\}_{i=1}^n \subset S(X^*)$ be a basis of X^* and let vectors $x_i \in S(X)$ be such that $fi(x_i) = 1$ and for some $c_2 > 0$ and each $f \in B(X^*)$ there exists at most one element i in the set $\{1, . . . , n\}$ for which $|f(x_i)| \ge 1 - c_2$.

Let A be a sufficient enlargement for X such that for some $c_1 \geq 0$ it is contained in the parallelepiped $\{x : |f_i(x)| \leq 1 + c_1,\ i \in \{1, \ldots, n\}\}$

Let $c_3 = 1 - \frac{2-c_2}{c_2} c_1$. Suppose $c_3 > 0$. Then A contains the parallelepiped

$$Q := \{x : |f_i(x)| \leq c_3,\ i \in \{1, \ldots, n\}\}.$$

Proof. Let $\{f_i\}_{i=n+1}^{\infty} \subset S(X^*)$ be such that $(\forall x \in X)\ (\|x\| = \sup\{|f_i(x)| : i \in \mathbf{N}\})$. Then the operator $E : X \to l_\infty$ defined by $Ex := \{f_i\}_{i=1}^{\infty}$ is an isometric embedding. Let $P: l_\infty \to E(X)$ be a projection for which

$P(B(l_\infty)) \subset E(A)$.

The condition of the theorem imply that there exists a partition of $\mathbf{N}$ into subsets $F_1, \ldots, F_n$ such that for $i \in F_j$ we have $f_i(x_k) < 1 - c_2$ for

$k \neq j$.

Let us show that $P(B(l_\infty))$ contains $E(Q)$. Observe that the first n coordinate functionals on l_∞ are norm-preserving extensions of functional $f_i E^{-1} : E(X) \to \mathbf{R}$.

Therefore in order to prove that $A \supset Q$ it is sufficient to prove that for every collection $\{\theta_i\}_{i=1}^{n}$, $\theta_1 = \pm 1$ there exists a vector

$z_\theta \in B(l_\infty)$ and real numbers $b_1, \ldots, b_n \geq c_3$ such that

$$Pz_\theta = (\theta_1 b_1, \theta_2 b_2, \ldots, \theta_n b_n, b_{n+1}, b_{n+2}, \ldots)$$

for some $b_{n+1}, b_{n+2}, \cdots \in \mathbf{R}$.

We introduce z_θ as the sequence $\{d_k\}_{k=1,}^{\infty}$ where $d_k = \theta_j f_k(x_j)$ if $k \in F_j$. In particular, $d_1 = \theta_1, \ldots, d_n = \theta_n$. Let us show that $P z_\theta$ satisfies the requirement above. Let

$$Pz_\theta = (\alpha_1, \ldots, \alpha_n, \alpha_{n+1}, \ldots).$$

Suppose that for some $m \in \{1, \ldots, n\}$ we have $\alpha_m \notin [\theta_m c_3, \theta_m \infty)$. Let us consider the family of vectors

$$y\delta = (1+\delta)\theta_m E(x_m) - \delta z_\theta \ (\delta > 0).$$

When $\delta > 0$ is small enough, then $y_\delta \in B(l_\infty)$. More precisely, by the conditions of the theorem it happens at least when $(1-c_2)(1+\delta)+\delta \le 1$, that is, when $\delta \le \frac{c_2}{2-c_2}$.

On the other hand the m−th coordinate of $P\, y_\delta$ is equal to

$$(1+\delta)\theta_m - \delta\alpha_m = \theta_m + \delta(\theta_m - \alpha_m).$$

So for $0 \le \delta \le c_2/(2-c_2)$ we have $|\theta_m + \delta(\theta_m - \alpha_m)| \le 1 + c_1$. Hence

$$1+\frac{c_2}{2-c_2}(1-c_3) < 1+c_1 \text{ or } c_3 > 1-\frac{2-c_2}{c_2}c_1.$$

This contradicts the condition on c_3.

Corollary (6.1.6)[171]: Let X be an n-dimensional normed space and Q be a parallelepiped circumscribed about B(X). Suppose there exist points $\{x_i\}_{i=1}^n$ on faces of Q (one point on the union of each pair of symmetric faces) such that $x_i \in B(X)$ and for every pair (x_i, x_j), $x_i \neq x_j$ and every $f \in B(X^*)$ at least one of the numbers $|f(x_i)|$ is less than 1. Then Q is a minimal sufficient enlargement for X .

Proof. By Proposition (6.1.3) only minimality requires a proof. Let $\{f_i\}_{i=1}^n \subset B(X^*)$ be such that $Q = \{x : |f_i(x)| \le 1,\ i \in \{1, . . . , n\}\}$.

By compactness of $B(X^*)$ there exists $c_2 > 0$ satisfying the condition of Theorem (6.1.5). Let $A \subset Q$ be a sufficient enlargement for X. Applying Theorem (6.1.5) with $c_1 = 0$ we get $A \supset Q$. Hence the sufficient enlargement Q is minimal.

The next result shows that the condition of the Corollary is not necessary for Q to be a minimal sufficient enlargement.

Theorem (6.1.7)[171]: There exist a two-dimensional normed linear space X and functionals $f_1, f_2 \in B(X^*)$ such that the following conditions are satisfied:

(i) There exists precisely one point $x_1 \in B(X)$ such that $f_1(x_1)=1$ and precisely one point $x_2 \in B(X)$ such that $f_2(x_2) = 1$.

(ii) The parallelogram $C= \{x:|f_1(x)|\le 1,|f_2(x)|\le 1\}$ is a minimal sufficient enlargement.

(iii) There exist a linear functional $f_3 \in B(X^*)$ such that $|f_3(x_1)| = |f_3(x_2)| = 1$.

Proof. Consider the space whose unit ball is the euclidean disc intersected with the strip

$$\{(a_1, a_2) : |a_1 - a_2| \leq 1\}.$$

Let $x_1 = (1, 0)$, $x_2 = (0, 1)$ and let f_1 and f_2 be the coordinate functionals. It is clear that Condition (i) of the theorem is satisfied.

In our case $C = \{(a_1, a_2) : |a_1| \leq 1, |a_2| \leq 1\}$.

It is clear that the functional $f_3(a_1, a_2) = a_1 - a_2$ satisfies Condition (iii) of the theorem.

It remains to show, that C is a minimal sufficient enlargement.

Let $\{f_i\}_{i=4}^{\infty} \subset S(X^*)$ be such that $(\forall x \in X)$ $(\|x\| = \sup\{|f_i(x)| : i \in \mathbf{N}\})$. Then the operator $E : X \to l_\infty$ defined by $Ex := \{f_i(x)\}_{i=1}^{\infty}$ is an isometric embedding.

Now, if we suppose that C is not a minimal sufficient enlargement, then there exists a projection $P : l_\infty \to E(X)$, such that the closure of its image is a proper part of E(C). We show that this. gives us a contradiction.

Consider the vectors

$$x_1(\varepsilon) := (\cos\varepsilon, \sin\varepsilon), \quad x_2(\varepsilon) := (\sin\varepsilon, \cos\varepsilon) \in B(X), \quad 0 < \varepsilon < \pi/4.$$

It is clear that for $0 < \varepsilon < \pi/4$ the following is true (the reader is advised to draw the picture): for each $f \in B(X^*)$ either

$$|f(x_1(\varepsilon))| \leq 1 - \tan\varepsilon \quad \text{or} \quad |f(x_2(\varepsilon))| \leq 1 - \tan\varepsilon.$$

Therefore there exists a partition $\mathbf{N} = A_1(\varepsilon) \cup A_2(\varepsilon)$ such that $|f_i(x_1(\varepsilon))| \leq 1 - \tan\varepsilon$ for $i \in A_2(\varepsilon)$ and $|f_i(x_2(\varepsilon))| \leq 1 - \tan\varepsilon$ for $i \in A_1(\varepsilon)$.

Now for $\theta = (\theta_1, \theta_2)$, where $\theta_1 = \pm 1$, $\theta_2 = \pm 1$, we define $z_\theta(\varepsilon) \in l_\infty$ as the vector, whose i–th coordinates coincide with the coordinates of $\theta_1 Ex_1(\varepsilon)$ for $i \in A_1(\varepsilon)$ and with the coordinates of $\theta_2 Ex_2(\varepsilon)$ for $i \in A_2(\varepsilon)$.

It is clear that $z \in B(l_\infty)$. Let

$$Pz_\theta(\varepsilon) = (\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots) \in l_\infty.$$

Let us show that

$$\theta_1\alpha_1 \geq \cos\varepsilon - 2(1-\cos\varepsilon)/\varepsilon, \qquad (1)$$

$$\theta_2\alpha_2 \geq \cos\varepsilon - 2(1-\cos\varepsilon)/\varepsilon. \qquad (2)$$

Because $\varepsilon > 0$ and $\theta = (\theta_1, \theta_2) = (\pm 1, \pm 1)$ are arbitrary(1) and (2) imply $P\,(B(l_\infty)) \supset E(C)$, so we get a contradiction.

Suppose that either (1) or (2) is not satisfied. Without loss of generality, we may assume that (1) is not satisfied.

Consider the family of vectors

$$y_\delta = (1+\delta)\theta_1 E(x_1(\varepsilon)) - \delta z_\theta(\varepsilon) \in l_\infty \ (\delta > 0).$$

From the definition of $z_\theta(\varepsilon)$ it is easy to derive that

$$\|y_\delta\|_\infty \leq \max\{1, (1+\delta)(1-\tan\varepsilon)+\delta\}.$$

Hence if δ is such that $2\delta/(1+\delta) \leq \tan\varepsilon$, then $\|y_\delta\|_\infty \leq 1$. In particular, $\left\|y_{\frac{\varepsilon}{2}}\right\|_\infty \leq 1$. Since $P\,(B(l_\infty)) \subset E(C)$, then the modulus of the first coordinate of $Py_{\varepsilon/2} \in l_\infty$ is ≤ 1. On the other hand, we have

$$Py_{\varepsilon/2} = (1+\varepsilon/2)\theta_1 E(x_1(\varepsilon)) - (\varepsilon/2) P\, z_\theta(\varepsilon).$$

Hence the first coordinate of $Py_{\varepsilon/2}$ is

$$(1+\varepsilon/2)\theta_1 \cos\varepsilon - (\varepsilon/2)\alpha_1.$$

We have

$$|(1+\varepsilon/2)\theta_1\cos\varepsilon - (\varepsilon/2)\alpha_1| = |(1+\varepsilon/2)\cos\varepsilon - (\varepsilon/2)\theta_1\alpha_1| >$$

$$(1+\varepsilon/2)\cos\varepsilon - (\varepsilon/2)(\cos\varepsilon - 2(1-\cos\varepsilon)/\varepsilon) = 1.$$

This contradiction implies that (1) and (2) are valid. Theorem (6.1.7) is proved.

By a prism in $\mathbf{R^n}$ we mean the Minkowski sum of a set A lying in an $(n-1)$–dimensional hyperplane and a line segment that is not parallel to the hyperplane. The set A is called a basis of the prism.

It turns out that if a sufficient enlargement A for X is such that its boundary intersects S(X) in a smooth point, then A should contain a prism, which is also a sufficient enlargement, so the investigation of such enlargement can be in certain sense reduced to investigation of $(n-1)$–dimensional sufficient enlargement.

Theorem (6.1.8)[171]: Let X be an $n-$dimensional normed space and let $x_1 \in S(X)$ be a smooth point and $h \in S(X^*)$ be its supporting functional. Let $\{x_i\}_{i=2}^{n} \subset S(X)$ be such that $\{x_i\}_{i=1}^{n}$ is a basis in X and $h(x_i) = 0$ for $i \in \{2, \dots, n\}$. Suppose that A is the a sufficient enlargement for X, which is contained in the set $\{ x \in X: |h(x) \le 1 \}$. Then there exists a symmetric with respect to 0 prism M with basis parallel to $\operatorname{lin}\{x_2, \dots, x_n\}$ such that

(i) $M \subset A$;

(ii) M is a sufficient enlargement for X.

Proof. We consider the natural isometric embedding E of X into $C(S(X^*))$: every vector is mapped onto its restriction (as a function on X^*) to $S(X^*)$. We introduce the following notation: $C = C(S(X^*))$ and

$B_C = B(C(S(X^*)))$.

Since A is a sufficient enlargement for X, then there exists a projection $P: C \to \operatorname{lin}\{Ex_i\}_{i=1}^{n}$, such that

$$P(B_C) \subset E(A) \tag{3}$$

Projection P can be represented as $P(f) = \sum_{i=1}^{n} \mu_i(f) Ex_i$, where μ_i are measures on $S(X^*)$.

Inclusion (3) implies that $\|\mu_1\| \le 1$. Since P is a projection we have $\mu_j(Ex_i) = \delta_{i,j}$ $(i, j = 1, \dots, n)$. In particular, $\mu_1(Ex_1) = 1$. Because x_1 is a smooth point, the function $|Ex_1| \in C$ attains its maximum only at h and $-h$. Hence μ_1 can be represented as $\mu_1 = b_{1,1}\delta_h + b_{2,1}\delta_{-h}$, where δ_h and δ_{-h} are Dirac measures, $b_{1,1} \ge 0$, $b_{2,1} \le 0$ and $b_{1,1} - b_{2,1} = 1$.

Now, for $i = 2, \dots, n$ we find representations

$$\mu_i = b_{1,i}\delta_h + b_{2,i}\delta_{-h} + \nu_i,$$

where ν_i don't have atoms in h and $-h$. To unify the notation we set $\nu_1 = 0$.

We introduce new measures

$$\omega_i := (b_{1,i} - b_{2,i})\delta_h + \nu_i.$$

It is clear that $\omega_j(Ex_i) = \delta_{i,j}$ (i, j =1, . . . , n). Hence $Q(f) := \sum_{i=1}^{n} \omega_i(f) Ex_i$ is also a projection onto lin$\{Ex_i\}_{i=1}^{n}$

Let us show that

$$Q(B_C) \subset cl(P(B_C)) \tag{4}$$

Let $f \in B_C$. Since ν_i don't have atoms in $\pm h$, then for every $\varepsilon > 0$ there exists a function $g \in B_C$ such that $g(-h) = -f(h)$, $g(h) = f(h)$ and $|\nu_i(f) - \nu_i(g)| < \varepsilon$ for all $i \in \{1, . . . , n\}$. This implies that

$$\forall i \in \{1, . . . , n\}\ |\omega_i(f) - \mu_i(g)| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary (4) follows. Hence $Q(B_C) \subset E(A)$. Now we shall show that $M := E^{-1}(cl(Q(B_C)))$ is the required prism.

The fact that M is a sufficient enlargement follows by a standard argument from the fact that C is an L_∞-space (see [172, 78,173]).

It remains to show that E(M) is a prism with basis parallel to lin $\{Ex_2, . . . , Ex_n\}$.

We have

$$E(M) = cl\{f(h)Ex_1 + \sum_{i=2}^{n}(b_{1,i} - b_{2,i}) f(h) Ex_i + \sum_{i=2}^{n} v_i(f) Ex_i : f \in B_c).$$

It is clear that the closures of the sets

$$\Gamma_\alpha := \{\sum_{i=2}^{n} v_i(f) Ex_i : f \in B_c, f(h) = \alpha\}$$

don't depend on α. So M is a prism of required form. The theorem is proved.

Definition (6.1.9)[171]: A sufficient enlargement A for l_2^n is said to be small if

$$\int_{O(n)} T(A) d\mu(T) = \lambda(l_2^n) B(l_2^n),$$

where μ is the normalized Haar measure on the orthogonal group O(n) and $\lambda\left(l_2^n\right)$ is the absolute projection constant.

The following result supplies us with a wide and interesting class of small sufficient enlargements.

Theorem (6.1.10)[171]: Let G be a finite subgroup of O(n) such that each linear operator on $\mathbf{R}^n$ co mmuting with all elements of G is a scalar multiple of the identity. Then for every $y \in S(l_2^n)$ the Minkowski sum of segments

$$A = \frac{n}{|G\}} \sum_{g \in G} [-g(y), g(y)]$$

is a small sufficient enlargement for l_2^n.

Proof. First we prove

$$(\forall x \in \mathbf{R}^n)\ (x = \frac{n}{|G|} \sum_{g \in G} \langle x, g(y) \rangle g(y)). \tag{5}$$

Let us introduce a linear operator $T : l_2^n \rightarrow l_2^n$ by the equality

$$Tx = \sum_{g \in G} \langle x, g(y) \rangle g(y) \tag{6}$$

Let us show that hT= Th for each $h \in G$. In fact

$$hT(x) = \sum_{g \in G} \langle x, g(y) \rangle hg(y) = \sum_{g \in G} \langle h(x), gh(y) \rangle hg(y) =$$
$$\sum_{g \in G} \langle h(x), g(y) \rangle g(y) = T\, h(x).$$

Hence $T = \lambda I$ for some $\lambda \in \mathbf{R}$.

The equality of traces in (6) shows that $\lambda_n = |G|$. Hence $\lambda = \frac{|G|}{n}$.The assertion (5) follows.

Now, (5) implies that the identity operator on l_2^n admits factorization $I = T_2T_1$, where $T_1 : l_2^n \rightarrow l_\infty^G$ and $T_2 : l_\infty^G \rightarrow l_2^n$ are defined as follows

$$T_1(x) = \{\langle x, g(y) \rangle\}_{g \in G} \ and\ T_2(\{a_g\}_{g \in G}) = \frac{n}{|G|} \sum_{g \in G} a_g g(y).$$

It is clear that $\|T_1\| = 1$ and $A = T_2(B(l_\infty^G))$, therefore A is a sufficient enlargement (see Proposition (6.1.2).

The enlargement A is small by the following observation. A calculation of B. Grünbaum [82] shows that

$$\forall z \in l_2^n \int_{O(n)} T([-z, z]) d\mu(T) = \frac{\|z\| \lambda(l_2^n)}{n} B(l_2^n) \tag{7}$$

Therefore

$$\int_{O(n)} T(A)du(T) = \frac{n}{|G|}\sum_{g\in G}\frac{\|g(y)\|\lambda(l_2^n)}{n}B(l_2^n) = \lambda(l_2^n)B(l_2^n).$$

Theorem (6.1.11)[171]: Let A be a sufficient enlargement for $l_2^{n+m} = l_2^n \oplus l_2^m$ and suppose that the images A_1 and A_2 of A by the orthogonal projections onto l_2^n and l_2^m are small sufficient enlargements for l_2^n and l_2^n. Then $A = A_1 + A_2$ (Minkowski sum).

Proof: We claim: if A_1 and A_2 are small sufficient enlargements for ln and lm, then $A_1 + A_2 \subset l_2^{n+m}$ is a small sufficient enlargement.

At the moment we do not need the fact that $A_1 + A_2$ is a sufficient enlargement, but because the proof is simple, we sketch it. By Proposition (6.1.2) the fact that A_1 is a sufficient enlargement for ln means that the L_∞–factorable norm of the identical embedding of l_2^n nto $\mathbf{R}^n$ normed by the gauge functional of A_1 is not greater than 1, the analogous assertion is valid for l_2^m and A_2. Now, it is easy to see that the L_∞–factorable norm of the identical embedding of $l_2^n \oplus_2 l_2^m$ into $\mathbf{R}^{n+m}$ normed by the gauge functional of

$A_1 + A_2$ is ≤ 1.

The fact that the sufficient enlargement $A_1 + A_2$ is small can be proved in the following way:

$$\int_{O(n+m)} T(A_1 + A_2)d\mu(T) =$$

$$\int_{O(n+m)} T\left(\int_{O(n)} T_1(A_1)d\mu_1(T_1) + \int_{O(m)} T_2(A_2)d\mu_2(T_2)\right)d\mu(T) =$$

(here μ_1 and μ_2 are normalized Haar measures on O(n) and O(m) respectively)

$$\int_{O(n+m)} T(\lambda(l_2^n)B(l_2^n) + \lambda(l_2^m)B(l_2^m))d\mu(T) =$$

$$\int_{O(n+m)} T\left(\int_{O(n)} T_1(Q_1)d\mu_1(T_1) + \int_{O(m)} T_2(Q_2)d\mu_2(T_2)\right)d\mu(T) =$$

(here Q_1 and Q_2 are cubes cicumscribed about $B(l_2^n)$ and $B(l_2^m)$) respectively)

$$\int_{O(n+m)} T(Q_1+Q_2)d\mu(T)=\lambda(l_2^{n+m})B(l_2^{n+m})$$

(by B.Grünbaum's result [82]).

Let X be a finite dimensional normed linear space. Denote by M the set of all sufficient enlargements of minimal volume for X. Results of [87](Theorem 6.1.12) imply the following result.

Theorem (6.1.12)[171]: The set M contains a parallelepiped. Easy examples (e.g. two dimensional space whose ball is regular hexagon) show that M may contain balls which are not parallelepipeds. But it turns out that for Euclidean spaces M contains only parallelepipeds.

Theorem (6.1.13) [171]: If A is a sufficient enlargement of minimal volume for l_2^n, then A is a cube circumscribed about B(l_2^n).

Proof. Let A be a sufficient enlargement for l_2^n and volA = 2^n.We may assume without loss of generality (see Proposition (6.1.2) that A is a zonoid. Therefore (see [83), its support function can be represented in the form

$$h(A,x)=\int_{S^{n-1}}|\langle x,v\rangle|\,d\rho(v)\ \ for\ x\in \mathbf{R}^n$$

with some even measure ρ on S^{n-1}.

We denote by D the set of all smooth points on the boundary of A. It is known (see [83]) that the complement of D in the surface of A has zero surface measure. Let T:D → S^{n-1} be the spherical image map (see [83]), that is: T (d) is the unique outer unit normal vector of A at d. Let μ be the measure on S^{n-1} defined by

$$\mu(\Omega) = m_{n-1}(T^{-1})(\Omega)),$$

where m_{n-1} is the surface area measure on the boundary of A.

It is clear that

$$volA=\frac{1}{n}\int_{S^{n-1}}h(A,x)d\mu(x)=\frac{1}{n}\int_{S^{n-1}}\int_{S^{n-1}}\left|\langle x,v\rangle\right|d\rho(v)d\mu(x).$$

The (n−1)−dimensional volume of the orthogonal projection of A onto the hyperplane orthogonal to $\omega \in S^{n-1}$ can be computed as

$$\alpha(\omega) = \frac{1}{2} \int_{S^{n-1}} |\langle x, \omega \rangle| d\mu(x).$$

We proceed by induction on the dimension. The case n = 1 is trivial. Suppose that we have proved the result for n − 1. Now, let A be a sufficient enlargement for $l_2^n 2$ and vol A = 2^n.

By Fubini theorem

$$2^n = volA = \frac{1}{n} \int_{S^{n-1}} 2\alpha(\omega) d\rho(\omega).$$

Since A is a sufficient enlargement, it is easy to derive from (7) that var(ρ) ≥ n.

It is clear that an orthogonal projection of A onto an (n−1)−dimensional subspace is a sufficient enlargement for l_2^{n-1}. It is clear also that every parallelepiped containing $B(l_2^{n-1})$ has volume $\geq 2^{n-1}$. Therefore by Theorem (6.1.12) $\alpha(\omega) \geq 2^{n-1}$. It follows that almost everywhere (in the sense of ρ) $\alpha(\omega) = 2^{n-1}$.

By induction hypothesis orthogonal projections in directions w for which $\alpha(\omega) = 2^{n-1}$ are cubes. Let us choose one such direction, say ω_1, and let us denote by $\omega_2, \omega_3, \ldots, \omega_n$ an orthonormal basis in the subspace orthogonal to ω_1 such that the orthogonal projection of A onto lin$\{\omega_2, \ldots, \omega_n\}$ is

$$[-\omega_2, \omega_2] + \cdots + [-\omega_n, \omega_n].$$

In particular

$$A \subset \{x : |\langle x, \omega_2 \rangle| \leq 1\}.$$

By Theorem (6.1.8) . A contains a prism M with the basis parallel to

$$\text{lin } \{\omega_1, \omega_3, \omega_4, \ldots, \omega_n\}$$

such that M is a sufficient en largement for l_2^n. Since A is a sufficient enlargement of minimal volume then M =A. Let N = A∩ lin$\{\omega_1, \omega_3, \omega_4, \ldots, \omega_n\}$

It is easy to see that N is a sufficient enlargement for $l_2^{n-1}$and $vol_n A = 2 vol_{n-1} N$. Hence $vol_{n-1} N = 2^{n-1}$. By induction hypothesis N is a cube. Hence A is also a cube.

Section (6.2): Cubes and Totally Unimodular Matrices:

Let $K^m \subset \mathbf{R}^m$ be defined by $K^m = \{(x_1,..., x_m) : |x| \leq 1$ for every $i \in \{1,..., m\}\}$. We refer to K^m as an m-cube. Let L be a linear subspace in $\mathbf{R}^m$ and $P : \mathbf{R}^m \to L$ be a linear projection onto L. The set $P(K^m)$ will be called a projection of K^m in L. Using a compactness argument it can be proved that for every $m \in \mathbf{N}$ and for every subspace

$L \subset R^m$ there exists a linear rojection that minimizes the volume of $P(K^m)$. In such a case the set $P(K^m)$ will be called a minimal-volume projection of K^m in L.

Volumes of projections of convex sets and related optimization problems is one of the natural objects of study in convex geometry. Many problems of this type have been already studied, see [175–181,182,183,89], and references therein.

Usually only orthogonal projections are considered and the standard optimization problem is to find a subspace such that the volume of the orthogonal projection onto it is minimal or maximal.

We consider a different problem. It arises in the study of Projections in normed linear spaces, see [171]. The problem is to characterize the shape of minimal-volume projections of cubes. Some steps in this direction were made in [89], where some classes of minimal-volume projections of K^m were found and the normed linear spaces corresponding to them were studied.

We say that subsets A and B of linear spaces X and Y , respectively, are linearly equivalent if there exists a linear isomorphism T between the subspace spanned by A in X and the subspace spanned by B in Y such that $T(A) = B$.

We give a complete description of the set of minimal-volume projections of K^m up to linear equivalence. To present the description we need some definitions.

A real matrix A with entries 0, 1, and −1 is called totally unimodular if determinants of all submatrices of A are equal to −1, 0 or 1. See [184,95] for survey of results on totally unimodular matrices and their applications.

A Minkowski sum of (finitely many) line segments in $\mathbf{R}^n$ is called a zonotope (see [185,83,94,92] for basic facts on zonotopes). We shall consider zonotopes that are sums of line segments of the form $[-x, x]$. Let $a_1,\ldots, a_m$ be some collection of vectors in $\mathbf{R}^n$.The Minkowski sum

$$\sum_{i=1}^{m}[-a_i, a_i]$$

will be called the zonotope spanned by $a_1,\ldots, a_m$.

Observe that any projection of the m-cube is a zonotope spanned by m vectors. The main result of this section he following

We denote by $\{e_i\}_{i=1}^{m}$ the standard basis in $\mathbf{R}^m$. The proof of the theorem is based on the following observation:

Lemma (6.2.1)[171]: (Minimality condition). Let $S : \mathbf{R}^m \to L$ be a linear projection onto. Le t$\{x_1,\ldots, x_l\}$ be an orthonormal basis in L and let$\{q_1,\ldots,q_{m-l}\}$ be an orthonormal basis in the kernel of S. The set $S(K^m)$ is a minimal-volume projection of K^m in L if and only if

$$|\det[x_1,\ldots, x_l, q_1,\ldots, q_{m-l}]| = |\det[x_1,\ldots, x_l, e_{i(1)},\ldots, e_{i(m-l)}]|$$

$$\times \sum_{\{j(1),\ldots,j(l)\}\subset(1,\ldots m\}} \left|\det[q_1,\ldots q_{m-l}, e_{j(1)} \ldots e_{j(l)}]\right|$$

where $\{i(1), \ldots, i(m-l)\}$ are chosen to maximize

$$|\det[x_1,\ldots, x_l, e_{i(1)},\ldots, e_{i(m-l)}]|.$$

Lemma (6.2.2)[171]: (Image shape lemma). Let $P : \mathbf{R}^m \to \mathbf{R}^m$ be a linear projection. Let $q_1,\ldots, q_{m-l}$ be an orthonormal basis in its kernel ker P. Let $\tilde{q}_1,\ldots, \tilde{q}_l$ be such that $\tilde{q}_1,\ldots, \tilde{q}_l, q_1,\ldots, q_{m-l}$ is an orthonormal basis in $\mathbf{R}^m$. Then $P(K^m)$ is linearly equivalent to the zonotope spanned by rows of the matrix $\tilde{Q} = [\tilde{q}_1,\ldots,\tilde{q}_l]$.

Proof. It is enough to observe that:

Images of K^m under two linear projections with the same kernel are linearly equivalent. Hence $P(K^m)$ is linearly equivalent to the image of the orthogonal projection with the kernel $\ker P$.

The matrix $\tilde{Q}\,\tilde{Q}^T$, where by $\tilde{Q}^T$ we denote the transpose of $\tilde{Q}$, is the matrix of the orthogonal projection with the kernel ker P .

Theorem (6.2.3)[171]: An l-dimensional zonotope Z is linearly equivalent to a minimal- volume projection of K^m if and only if it is linearly equivalent to the zonotope spanned by multiples of rows of a totally unimodular m × r matrix of rank l.

Proof. The lemmata imply that in order to prove the "if" part it is enough to show that for every totally unimodular $m\times r$ matrix A of rank l and for every diagonal $m\times m$ matrix D with positive entries on the diagonal there exists an orthonormal sequence $\tilde{q}_1,\ldots,\tilde{q}_l$ such that

(i) The zonotope spanned by rows of $[\tilde{q}_1,\ldots,\tilde{q}_l]$ is linearly equivalent to the zonotope spanned by rows of DA.

(ii) If $q_1,\ldots,q_{m-l}$ are such that $\tilde{q}_1,\ldots,\tilde{q}_l, q_1,\ldots,q_{m-l}$ is an orthonormal basis in $\mathbf{R}^m$, then there exists an orthonormal sequence $x_1,\ldots,x_l$ such that $[x_1,\ldots,x_l]$ and $[q_1,\ldots,q_{m-l}]$ satisfy the minimality condition of Lemma (6.2.1).

We rearrange columns of A in order to get a matrix whose first l columns are linearly independent. It is clear that the zonotope spanned by rows of $D\times$ (the obtained matrix) is linearly equivalent to the zonotope spanned by rows of DA.

Hence without loss of generality we may assume that the first l columns of A are linearly independent, where l is the rank of A. Also it is clear that if the first l columns $a_1,\ldots,a_l$ of A are linearly independent, then the zonotope spanned by rows of $[a_1,\ldots,a_l]$is linearly equivalent to the zonotope spanned by rows of A. So without loss of generality we may assume that A is an m× l matrix of rank l.

Using the Gram–Schmidt orthonormalization process we get that there exists an invertible $l\times l$ matrix C_1 such that columns of AC_1 form an rthonormal set. This set will play the role of $x_1,\ldots,x_l$ in the construction (see (ii)).

Using the Gram– Schmidt orthonormalization process again we get that there exists an invertible $l\times l$ matrix C_2 such that columns of DAC_2 form an orthonormal set. This set will play the role of $\tilde{q}_1,\ldots,\tilde{q}_l$ in our construction (see (i)).

The condition (i) is satisfied because the matrix C_2 is invertible.

Let $q_1,\ldots,q_{m-l}\in\mathbf{R}^m$ be such that $\tilde{q}_1,\ldots,\tilde{q}_l, q_1,\ldots,q_{m-l}$ form an orthonormal basis in $\mathbf{R}^m$. It remains to show that (ii) is satisfied.

Let $M = \binom{m}{l}$ We denote by u_i (i = 1,...,M) the $l \times l$ minors of $[x_1,..., x_l]$ (ordered in some way). We denote by ω_i (i = 1,...,M) the $l \times l$ minors of $[\tilde{q}_1,..., \tilde{q}_l]$ ordered in the same way as the u_i . We denote by $v_i = (i = 1,..., \binom{m}{m-l} = M)$ their complementary $(m-l) \times (m-l)$ minors of $[q_1,..., q_{m-l}]$. Using the word complementary we mean that all minors are considered as minors of the matrix $[\tilde{q}_1,..., \tilde{q}_l,..., q_{m-l}]$, see [93].

By the Laplacian expansion (see [93]),

$$\det[x_1,...,x_l, q_1,..., q_{m-l}] = \sum_{i=1}^{M} \theta_i u_i v_i$$

and

$$\det[\tilde{q}_1,...,\tilde{q}_l, q_1,..., q_{m-l}] = \sum_{i=1}^{M} \theta_i \omega_i v_i \tag{8}$$

for proper signs θ_i.

Since the matrix $[\tilde{q}_1,...,\tilde{q}_l, q_1,..., q_{m-l}]$ is orthogonal, then

$$\det [\tilde{q}_1,...,\tilde{q}_l, q_1,..., q_{m-l}] = \pm 1. \tag{9}$$

We need one result on compound matrices. We refer to [93] for necessary definitions and background. The result that we need is

A compound matrix of an orthogonal matrix is orthogonal (see [93]).

This result implies, in particular, that the Euclidean norms of the vectors $\{\omega_i\}_{i=1}^{M}$ and $\{v_i\}_{i=1}^{M}$ in $\mathbf{R}^M$ are equal to 1.

From (8) and (9) we get that either

(i) $\omega_i = \theta_i v_i$ for every i

or

(ii) $\omega_i = -\theta_i v_i$ for every i.

Without loss of generality we assume that $\omega_i = \theta_i v_i$ for all i (we replace q_1 by $-q_1$ if it is not the case).

Observe that

$$[x_1,..., x_l] = D^{-1} [\tilde{q}_1,..., \tilde{q}_l]\, C_2^{-1}\, C_1.$$

Hence $u_i = \beta_i \omega_i \det C_2^{-1} \det C_1$, where β_i are some positive numbers determined by the diagonal entries of D^{-1}. Denote $\det C_2^{-1} \det C_1$ by α. We get

$$u_i = \beta_i \omega_i \alpha. \tag{10}$$

On the other hand $[x_1, \ldots, x_l] = AC_1$ and A is totally unimodular. Therefore u_i is equal to $\det C_1$, 0 or $-\det C_1$ for every i. Let $\Omega = \{i : u_i \neq 0\}$, then $|u_i|$ is the same for all $i \in \Omega$.

The minimality condition of Lemma (6.2.1) (that we need to verify) can be written as

$$\left|\sum_{i=1}^{M} \theta_i u_i v_i\right| = \max_i |u_i| \sum_{i=1}^{M} |v_i|. \tag{11}$$

We have

$$\left|\sum_{i=1}^{M} \theta_i u_i v_i\right| = \left|\sum_{i\in\Omega} \theta_i u_i v_i\right| =.$$

(we use (a), (10), and $\beta_i > 0$)

$$\sum_{i\in\Omega} \beta_i \omega_i^2 |\alpha| = \sum_{i\in\Omega} \left|\beta_i \omega_i^2 \alpha\right| |\omega_i| = .$$

(we use (a), (10), and the fact that $|ui|$ is constant when $i\in\Omega$)

$$\sum_{i\in\Omega} |u_i| |v_i| = \max_{i\in\Omega} |u_i| \sum_{i\in\Omega} |v_i|.$$

It remains to observe that from (a) and (10) $u_i = 0$ if and only if $v_i = 0$. Hence

$$\max_{i\in\Omega} |u_i| \sum_{i\in\Omega} |v_i| = \max_i |u_i| \sum_{i=1}^{M} |v_i|.$$

Hence (11) is proved and the proof of the "if" part of Theorem (6.2.3) is finished.

Proof of the "only if": Let a linear projection $P: \mathbf{R}^m \to \mathbf{R}^m$ be such that $P(K^m)$ is a minimal-volume projection of K^m. Let $\{q_1, \ldots, q_{m-l}\}$ be an orthonormal basis in ker P. Let $\{x_1, \ldots, x_l\}$ be an orthonormal basis in the image of P, and let $\{\tilde{q}_1, \ldots, \tilde{q}_l\}$ be such that $\{\tilde{q}_1, \ldots, \tilde{q}_l, q_1, \ldots, q_{m-l}\}$ is an orthonormal basis in $\mathbf{R}^m$. According to Lemma (6.2.2) it is enough to show that the zonotope spanned by rows of $\tilde{Q} = [\tilde{q}_1, \ldots, \tilde{q}_l]$ is linearly equivalent to the zonotope spanned by multiples of rows of some totally unimodular $m\times l$ matrix. It is clear that it is enough to show that $\widetilde{Q} = DAC$, where D is a diagonal $m\times m$ matrix, A is a totally unimodular $m\times l$ matrix, and C is an invertible $l \times l$ matrix.

We let $M = \binom{m}{l}$ and introduce the numbers u_i , v_i , and ω_i $(i = 1, \ldots, M)$ in the same way as in the first part of the proof. Since $P\ (K^m)$ is a minimal-volume projection, then the minimality condition from Lemma (6.2.1) is satisfied, that is

$$\left|\sum_{i=1}^{M} \theta_i u_i v_i\right| = \max_i \left|u_i\right| \sum_{i=1}^{M} \left|v_i\right|. \tag{12}$$

Also, as in the first part of the proof, either

(i) $\omega_i = \theta_i v_i$ for every i

or

(ii) $\omega_i = -\theta_i v_i$ for every i.

Let $\Gamma = \{i : v_i \neq 0\} = \{i : \omega_i \neq 0\}$. The equality (12) is satisfied if and only if the following three conditions are satisfied:

(iii) the numbers $\{u_i\}_{i\in\Gamma}$ have the same absolute value, let us denote it by μ;

(iv) the numbers $\{u_i\, v_i\, \theta_i\}_{i\in\Gamma}$ have the same sign;

(v) $|u_i| \leq \mu$ if $i \notin \Gamma$.

By (i) and (ii) the condition (iii) is equivalent to

(iv') the numbers $\{u_i\, \omega_i\}_{i\in\Gamma}$ have the same sign.

Our approach to finding matrices D and C mentioned above is the following. Let $X = [x_1, \ldots x_l]$.

First we find invertible $l \times l$ matrices C_1 and C_2, and a permutation $m\times m$ matrix R such that the first l rows of $Q^* = R\widetilde{Q}\, C_1$ and $X^* = RXC_2$ are identity $l \times l$ matrices, and conditions similar to (iii), (iv`), and (iv) are satisfied.

The second step is to show that replacing some of the entries of X^* by zeros we get a totally unimodular matrix $\widetilde{A}$ satisfying $Q^* = \widetilde{D}\widetilde{A}\widetilde{S}$, where $\widetilde{D}$ is a diagonal $m\times m$ matrix and $\widetilde{S}$ is a diagonal $l \times l$ matrix. Hence

$$\widetilde{Q} = R^{-1} Q^* C_1^{-1} = R^{-1}\, \widetilde{D}\widetilde{A}\widetilde{S}\, C_1^{-1} = DAC,$$

where $D = R^{-1}\widetilde{D}\,R$, $A = R^{-1}\ \widetilde{A}$, $C = \widetilde{S}\,C_l^{-1}$.

The first step. The condition (iv') implies that either $u_i = \mu$ sign ω_i for all $i \in \Gamma$ or $u_i = -\mu$ sign ω_i for all $i \in \Gamma$. Therefore there exists i such that $u_i \neq 0$ and $\omega_i \neq 0$. Therefore we can multiply both X and $\widetilde{Q}$ by invertible $l \times l$ matrices from the right, and by the same permutation m×m matrix from the left (observe that ultiplication by such permutation matrix is equivalent to simultaneous permutation of rows of X and $\widetilde{Q}$) to get matrices Q * and X* satisfying the conditions:

(i) The first l rows in each of them form an $l \times l$ identity matrix.

(ii) Absolute values of $l \times l$ minors of X* are at most 1.

(iii) If some $l \times l$ minor ω of Q * is nonzero, then the corresponding $l \times l$ minor (the minor with the same rows) in X^* is equal to sign ω.

Let $e_1 . . , e_l$ be the rows of the identity matrix of order l. Let $x_i{}^*$ be rows of X^*, and let q_i^* be rows of Q *.

We show that the conditions (i) and (iii)) imply that if q_{ij}^* is a nonzero entry of Q Q *,then x_{ij}^* = sign q_{ij}^* , where by x_{ij}^* we denote the corresponding entry of X^*.

To prove this statement we apply (iii) to the minors corresponding to the submatrices with rows

$$e_1, . . . ,\ e_{j-1},\ e_{j+1}, . . . ,\ e_l\ ,\ x_i{}^*$$

and

$$e_1, . . . ,\ e_{j-1},\ e_{j+1}, . . . , e_l\,, q_i{}^*$$

in X^* and Q *, respectively.

In a similar way we get

(ii+) Absolute values of all minors of X^* are at most 1.

and

(iii+) some minor (of any order) of Q * is equal to $\omega \neq 0$, then the corresponding minor in X^* is sign ω.

For each şub matrix Q_s^* of Q * (in particular for Q * itself) we introduce a graph $G(Q_s^*)$ whose vertices are nonzero entries of Q_s^*; two vertices are adjacent in $G(Q_s^*)$ if and only if the corresponding (nonzero) entries are either in the same row or in the same column of Q_s^*. Edges joining two entries in one row will be called horizontal, edges joining two entries in one column will be called vertical.

A sub matrix Q_s^* will be called *connected* if the following two conditions are satisfied:

(I) Each column and each row of Q_s^* contains a nonzero entry.

(II) The graph $G(Q_s^*)$ is connected.

A sub matrix Q_s^* of Q^* is called a connected component of Q^* if it is a maximal connected submatrix of Q^*.

It is clear that there are two types of zero entries of Q^*: some of them are entries of some connected components of Q^* and some are not.

Lemma (6.2.4)[171]: *If* $q_{ij}^* = 0$ and q_{ij}^* is an entry of some connected component of Q^*, then $x_{ij}^* = 0$.

Proof. We shall prove this statement for each connected submatrix using the induction on the number of columns of a submatrix. For connected submatrices Q_s^* of Q^* with one column there is nothing to prove: all entries of Q_s^* should be nonzero by (I) in the definition of a connected submatrix.

Consider a connected submatrix Q_s^* with two columns. There should be a row, let it be the row number k, such that both entries of Q_s^* in that row are nonzero. Consider the 2 × 2 submatrix of Q_s^* formed by rows number k and i.

Since q_{ij}^*=0 and Q_s^* is connected, then the 2 × 2 submatrix has exactly 3 nonzero entries. Hence its determinant is nonzero. Using (iii$^+$) we get that the determinant of the corresponding submatrix in X^* is ±1.

On the other hand, since in Q^* this submatrix has exactly 3 nonzero entries, then the corresponding submatrix in X^* has at least three entries equal to ±1.

Therefore its determinant can be ±1 if and only if the remaining entry is 0, that is $x_{ij}^{*} = 0$.

Suppose that we have already proved the result for connected submatrices with k columns *($k \geq 2$)*. Let us prove it for a connected submatrix with $k + 1$ columns.

Assume the contrary. Let Q_s^{*} be a minimal connected submatrix with $k + 1$ columns that violates the condition, that is it contain a zero entry q_{ij}^{*} such that $x_{ij}^{*} \neq 0$. Such q_{ij}^{*} will be called a *violator*. The word *minimal* here means that after removal of any row we get either a disconnected submatrix or a submatrix without violators. By x_s^{*} we denote the corresponding submatrix in X^{*}.

So let q_{ij}^{*} be a violator. Let q_{it}^{*} and q_{rj}^{*} be nonzero entries in Q_s^{*}. Such nonzero entries exist by the part (I) of the definition of a connected matrix. Let P be a shortest path in G(Q_s^{*}) joining q_{it}^{*} and q_{rj}^{*}. It is clear that in a shortest path vertical and horizontal edges are alternating and that a shortest path contains at most 2 vertices in each row of Q_s^{*} and atmost 2 vertices in each column of Q_s^{*}. Using another choice of q_{it}^{*} and q_{rj}^{*} if necessary we may assume that the first edge is vertical and the last edge is horizontal.

Let us consider the minimal submatrix V of Q^{*} containing q_{ij}^{*} and all entries of the path. The submatrix V is connected and is a submatrix of Q_s^{*}. Since V contains a violator, it implies that $V = Q_s^{*}$. Hence the path has vertices in each column of Q_s^{*} and in each row of Q_s^{*}. It is easy to see that it implies that Q_s^{*} is of size (k + 1) ×(k + 1) and that columns and rows of Q_s^{*} can be renumbered in such a way that for the obtained matrix $T = \{t_{ij}\}_{i,j=1}^{k+1}$ the path (presented by listing its vertices) is

$$t_{1,1},\ t_{2,1},\ t_{2,2},\ t_{3,2},\ t_{3,3}, \ldots,\ t_{k,k},\ t_{k+1,k},\ t_{k+1,k+1},$$

and q_{ij}^{*} corresponds to $t_{1,k+1}$.

It is clear that all other entries of T (and, hence, Q_s^{*}) are zeros, because otherwise there is a shorter path. (We skip an elementary proof of this step. It can be obtained by sketching pictures corresponding to the situation $t_{i,j} \neq 0$, $i \neq j$, $i \neq j+1$ for the cases $i<j$ and $i>j$. Observe that we need to use the condition $k+1 \geq 3$). Therefore det $T \neq 0$ and det $Q_s^* \neq 0$.

Let W be the matrix obtained from x_s^* by the same renumbering that was used to get T from Q_s^*. Observe that by the minimality and by the inductive hypothesis q_{ij}^* is the only violator in Q_s^*. Therefore the only nonzero entries in W are

$$\omega_{1,1}, \omega_{2,1}, \omega_{2,2}, \omega_{3,2}, \omega_{3,3}, \ldots, \omega_{k,k}, \omega_{k+1,k}, \omega_{k+1,k+1}, \text{ and } \omega_{1,k+1}.$$

By (iii$^+$) the absolute values of all of these entries, except, possibly, $\omega_{1,k+1}$ are equal to 1. By (ii$^+$) $|\omega_{1,k+1}| \leq 1$. Hence $|\det \omega| \neq 1$ and $|\det X_s^*| \neq 1$. We get a contradiction with the condition (iii+).

We replace all entries in X^* that correspond to those zero entries of Q^* that do not belong to any connected component of Q^* by zeros and denote the obtained matrix by $\tilde{A}$.

Let us show that the matrix $\tilde{A}$ is totally unimodular, that is all of its minors are equal to 0, 1, or -1.

Connected components of $\tilde{A}$ are defined in the same way as for Q^*. Observe that by Lemma (6.2.4) and the definition of $\tilde{A}$, the graphs $G(\tilde{A})$ and $G(Q^*)$ are the same.

First consider a minor of $\tilde{A}$ corresponding to a submatrix of a connected component of $\tilde{A}$. By the definition of $\tilde{A}$ it follows that the minor is a minor of X^* also. By Lemma (6.2.4) and (iii$^+$) it follows that all entries of the minor are 0 or ± 1. Hence the minor is an integer. Since it is a minor of X^*, by (ii$^+$) the absolute value of this integer is at most 1. Hence the integer should be equal to 0, 1, or -1.

Observe that the definition of a connected component implies that two different connected components cannot have entries in the same row or in the same column. By the definition of $\tilde{A}$ all entries of $\tilde{A}$ that are not in any of the connected components are equal to 0. Hence each minor of A is either 0 or is a product of minors corresponding to square submatrices of some connected components. Hence

$\widetilde{A}$ is totally unimodular.

The discussion above implies also that each minor of Q^* is either 0 or is a product of minors corresponding to square submatrices of some components. Therefore $\widetilde{A}$ and Q^* satisfy the condition:

$\widetilde{(iii)}$ If some minor of Q^* is equal to $\omega \neq 0$, then the corresponding minor in $\widetilde{A}$ is equal to sign ω.

Note: We have not proved that, if some minor of Q^* is zero, then the corresponding minor of $\widetilde{A}$ is also zero.

Lemma (6.2.5)[171]: There exist a diagonal $l \times l$ matrix $\widetilde{S}$ and a diagonal $m \times m$ matrix $\widetilde{D}$ with positive entries on the diagonals such that $Q^* = \widetilde{D}\widetilde{A}\widetilde{S}$.

Proof. Assume the contrary. Let Q_s^* be a minimal submatrix of Q^* such that it cannot be multiplied by diagonal matrices with positive diagonals from both sides in order to get the corresponding submatrix $\widetilde{A}_s$ of $\widetilde{A}$. Saying *minimal* we mean that each submatrix of Q_s^* can be multiplied by the diagonal matrices in such a way that we get the corresponding submatrix of $\widetilde{A}$.

It is clear that the minimality condition implies that each row and each column of Q_s^* (and $\widetilde{A}_s$) contains at least two nonzero entries.

Simultaneously renumbering rows and columns of Q_s^* and $\widetilde{A}_s$ we get two matrices, say $Y = \{y_{i,j}\}_{i=1,j=1}^{u\ \ v}$ *and* $Z = \{Z_{i,j}\}_{i=1,j=1}^{u\ \ v}$, satisfying the following conditions.

(I) $\exists \{d_i\}_{i=1}^{u}, (d_i > 0) \exists \{s_j\}_{j=1}^{v-1}, (s_j > 0)$ such that $y_{i.j} = d_i z_{i'j} s_j$ for all $i = 1, \ldots, u$ and $j = 1, \ldots, v-1$. Such $\{d_i\}_{i=1}^{u}$ and $\{s_j\}_{j=1}^{v-1}$ are not unique, but we fix some choice of them at this time.

(II) $\forall\, s_v \in \mathbf{R}$, $s_v > 0$, $\exists i \in 1, \ldots, u$ such that $y_{i,v} \neq d_i z_{i,v} s_v$.

By Lemma (6.2.4) and (iii^+) the definition of $\widetilde{A}$ implies

$$z_{i,j} = \operatorname{sign} y_{i,j}. \tag{13}$$

Hence we get from (I) and (II) that there exist pairs (i_1, i_2) of integers in $\{1, \ldots, u\}$

such that

$$\left|d_{i_1}^{-1} y_{i_1,v}\right| \neq \left|d_{i_2}^{-1} y_{i_2,v}\right|. \tag{14}$$

We call such pairs of integers *incompatible*.

Let us remove the last column from Y and consider connected omponents of the obtained matrix Y_s.

An incompatible pair (i_1, i_2) will be called *connected in* Y_s if there exists a path in the graph $G(Y_s)$ joining an entry in the i_1th row of Y_s with an entry in the i_2 th row of Y_s. Otherwise the pair (i_1, i_2) will be called disconnected in Y_s.

Let us show that if all incompatible pairs are disconnected in Y_s, then we can find positive numbers $\{\tilde{d}_i\}_{i=1}^{u}$ *and* $\{\tilde{s}_j\}_{j=1}^{v}$ such that

$$y_{i,j} = \tilde{d}_i z_{i,j} \tilde{s}_j \quad \forall i \in \{1,..u\} \quad \forall j \in \{1,...,v\}, \tag{15}$$

contrary to the assumption.

In fact, different connected components cannot have nonzero entries in the same row or in the same column. Therefore there exist partitions $\{V_C\}$ and $\{H_C\}$ of the sets $\{1,\ldots,v-1\}$ and $\{1,\ldots,u\}$, respectively, where C runs over the set of all components of Y_s, V_C is the set of numbers of all columns intersecting the component C, H_C is the set of numbers of all rows intersecting C. The observation above (about at least two nonzero entries in each row and column of Q_s^*) implies that $\cup_C V_C = \{1,\ldots, v-1\}$ and $\cup_C H_C = \{1,\ldots, u\}$.

If all in compatible pairs are disconnected in Y_s, then the nonzero values of $\left|d_i^{-1} y_{i,v}\right|$ are the same for all $i \in H_C$, where C is any component of $G(Y_s)$.

If there exist nonzero values of the form $\left|d_i^{-1} y_{i,v}\right|$ $(i \in H_C)$, we let $r(C)$ be their common value. If all numbers $\left|d_i^{-1} y_{i,v}\right|$ $(i \in H_C)$ are equal to 0, we let $r(C) = 0$.

Let

$$\tilde{d}_i = \begin{cases} d_i & \text{if } i \in H_c \text{ and } r(C) = 0, \\ r(C).d_i & \text{if } i \in H_c \text{ and } r(C) \neq 0 \end{cases}$$

and

$$\tilde{s}_i = \begin{cases} s_j & \text{if } j \in V_C \text{ and } r(C) = 0, \\ s_{j/r(C)} & \text{if } j \in V_C \text{ and } r(C) \neq 0, \\ 1 & \text{if } j = v. \end{cases}$$

Straightforward verification shows that (15) is satisfied.

Hence the assumption that Q_s^* is a minimal submatrix of Q^* such that there are no diagonal matrices satisfying the condition described at the begining of the lemma implies that there exist incompatible pairs (i_1, i_2) that are connected in Y_s. For each such pair we choose a shortest path among all paths in $G(Y_s)$ joining a nonzero entry in the i_1th row of Y_s and a nonzero entry in the i_2th row of Y_s. We minimize the length of the path over all incompatible pair(s), connected in Y_s.

So let (i_1, i_2) be an incompatible, connected in Y_s pair and P be a path in $G(Y_s)$ joining a nonzero entry in the i_1th row of Y_s and a nonzero entry in the i_2th row of Y_s and such that any other path joining two nonzero entries from rows corresponding to incompatible pairs is at least of the same length as P. It is clear that vertical and horizontal edges are alternating in P, and the first and the last edges are vertical.

Let ω be the minimal submatrix of Y containing $y_{i_1,v}, y_{i_2,v}$ and all entries corresponding to vertices of P. We renumber columns and rows of ω in such a way that in the obtained matrix (we shall keep the notation ω for it) the path P corresponds to

$$\omega_{1,1},\ \omega_{2,1},\ \omega_{2,2},\ \omega_{3,2},\ \omega_{3,3},\ \dots,\ \omega_{m-1,m-1},\ \omega_{m,m-1},$$

the entry $y_{i_1,v}$ corresponds to $\omega_{1,m}$, and the entry $y_{i_2,v}$ corresponds to $\omega_{m,m}$.

We renumber $\{d_i\}$ in the corresponding way and get $\{t_i\}_{i=1}^m$.

The minimality property of P implies that the only nonzero entries of ω are

$$\omega_{1,1},\ \omega_{2,1},\ \omega_{2,2},\ \omega_{3,2},\ \omega_{3,3},\dots,\ \omega_{m-1,m-1},\ \omega_{m,m-1},\ \omega_{m,m}, \text{ and } \omega_{1,m}.$$

(The existence of other nonzero entries would imply the existence of a shorter path of the same type. It is easy to verify this for all possible cases. Observe that in the case when additional nonzero entries are in the last column we need to consider another incompatible pair.)

Let us show that det $\omega \neq 0$. Assume the contrary, that is det $\omega = 0$.

The condition (14) corresponds to

$$\left|t_1^{-1}\omega_{1,m}\right| \neq \left|t_m^{-1}\omega_{m,m}\right|. \tag{16}$$

On the other hand

$$\det\omega = \prod_{i=1}^{m}\omega_{i,i} + (-1)^{m-1}\left(\prod_{i=1}^{m-1}\omega_{i+1.i}\right).\omega_{1,m}.$$

Hence det $\omega = 0$ implies that

$$\prod_{i=1}^{m}\left|\omega_{i,i}\right| = \left(\prod_{i=1}^{m-1}\left|\omega_{i+1,i}\right|\right)\cdot\left|\omega_{1,m}\right|. \tag{17}$$

The conditions (I) and (13) imply that

$$\frac{\left|\omega_{i,i}\right|}{\left|\omega_{i+1,i}\right|} = \frac{t_i}{t_i+1}$$

Hence (17) implies

$$\frac{t_1}{t_m} = \frac{\left|\omega_{1.m}\right|}{\left|\omega_{m.m}\right|},$$

We get a contradiction to (16). Hence det $\omega \neq 0$.

On the other hand, consider the submatrix U of Z corresponding to ω. Let us renumber entries of U in the same way as we did it for ω. Then the condition (13) implies that the only nonzero entries of U are

$$u_{1,1},\ u_{2,1},\ u_{2,2},\ u_{3,2},\ u_{3,3},\ \ldots, u_{m-1,m-1}, u_{m,m-1}, u_{m,m}, \text{ and } u_{1,m}.$$

Hence

$$\det U = \prod_{i=1}^{m} u_{i,,i} + (-1)^{m-1} = \left(\prod_{i=1}^{m-1} u_{i+1,i}\right)u_{1,m}.$$

Since all nonzero entries of U are equal to ±1, and U is totally unimodular (as a matrix obtained by renumbering of columns and rows of a submatrix of a totally unimodular matrix), then det $U = 0$.

Since renumbering of rows and columns can change the signs of determinants only the equalities det U=0 and det $\omega \neq 0$ contradict the condition $(\widetilde{iii})$. This contradiction proves the lemma and the “only if” part of the theorem.

Section (6.3): Finite-dimensional Normed Linear Spaces:

We devoted to a generalization of the main results of [187], where similar results were proved in the dimension two. We refer to [187, 188] for more background and motivation.

All linear spaces considered will be over the real's. By a space we mean a normed linear space, unless it is explicitly mentioned otherwise. We denote by B_X (S_X) the unit ball (sphere) of a space X. We say that subsets A and B of finite-dimensional linear spaces X and Y, respectively, are linearly equivalent if there exists a linear isomorphism T between the subspace spanned by A in X and the subspace spanned by B in Y such that T(A)=B. By a symmetric set K in a linear space we mean a set such that $x\in K$ implies $-x\in K$.

Our terminology and notation of Banach space theory follows [189]. By B_p^n, $1 \leq p \leq \infty$, $n\in\mathbf{N}$ we de note the closed unit ball of ℓ_p^n Our terminology and notation of convex geometry follows [83].

We use the term ball for a symmetric, bounded, closed, convex set with interior points in a finite-dimensional linear space.

Definition (6.3.1)[186]: (See [76]) .A ball A in a finite-dimensional normed space X is called a sufficient enlargement (SE) for X (or of B_X) if, for an arbitrary isometric embedding of X into a Banach space Y, there exists a projection $P: Y\to X$ such that $P(B_Y)\subset A$. A sufficient enlargement A for X is called a minimal-volume sufficient enlargement (MVSE) if vol A $\leq$ vol D for each SE D for X.

It can be proved, using a standard compactness argument and Lemma (6.3.10) below, that minimal-volume sufficient enlargements exist for every finite-dimensional space.

Recall that a real matrix A with entries −1, 0, and 1 is called totally unimodular if all minors (that is, determinants of square submatrices) of A are equal to −1, 0, or 1. See [184] and [190] for a survey of results on totally unimodular matrices and their applications.

A Minkowski sum of finitely many line segments in a linear space is called a zonotope (see [191,185,192,83,94] for basic facts on zonotopes). We consider zonotopes that are sums of line segments of the form $I(x) = \{\lambda x : -1 \le \lambda \le 1\}$. For a $d \times m$ totally unimodular matrix with columns τ_i ($i = 1, \dots, m$) and real numbers a_i we consider the zonotope Z in $\mathbf{R}^d$ given by

$$Z = \sum_{i=1}^{m} I(a_i \tau_i).$$

The set of all zonotopes that are linearly equivalent to zonotopes obtained in this way over all possible choices of m, of a rank d totally unimodular $d \times m$ matrix, and of positive numbers a_i ($i = 1, \dots, m$) will be denoted by T_d. Observe that each element of T_d is d-dimensional in the sense that it spans a d-dimensional subspace. It is easy to describe all $2 \times m$ totally unimodular matrices and to show that T_2 is the union of the set of all symmetric hexagons and the set of all symmetric parallelograms.

The class T_d of zonotopes has been characterized in several different ways, see [193, 194, 195, 196, 174, 197]. We shall use a characterization of T_d in terms of lattice tiles. Recall that a compact set $K \subset \mathbf{R}^d$ is called a lattice tile if there exists a basis $\{x_i\}_{i=1}^{d}$ in $\mathbf{R}^d$ such that

$$\mathbf{R}^d = \bigcup_{m_1,\dots,m_d \in \mathbf{Z}} \left(\left(\sum_{i=1}^{d} m_i x_i \right) + K \right),$$

and the interiors of the sets $(\sum_{i=1}^{d} m_i x_i) + K$ are disjoint. The set

$$A = \left\{ \sum_{i=1}^{d} m_i x_i : m_1, \dots, m_d \in \mathbf{Z} \right\}$$

is called a lattice. The absolute value of the determinant of the matrix whose columns are the coordinates of $\{x_i\}_{i=1}^{d}$ is called the determinant of A and is denoted d(A), see [198] .

Theorem (6.3.2)[186]: [194,196] A d-dimensional zonotope is a lattice tile if and only if it is in T_d.

It is worth mentioning that lattice tiles in $\mathbf{R}^d$ do not have to be zonotopes, see [199, 200, 201], and [202].

The main result of [174] can be restated in the following way. (A finitedimensional normed space is called polyhedral if its unit ball is a polytope.)

Theorem (6.3.3)[186]: A ball Z is linearly equivalent to an MVSE for some d-dimensional polyhedral space X if and only if $Z \in T_d$.

In [187] it was shown that for d = 2 the statement of Theorem (6.3.3) is valid without the restriction of polyhedrality of X. The main purpose is to prove the same for each $d \in \mathbf{N}$. It is clear that it is enough to prove

Lemma (6.3.4)[186]: (See [187,90]). The set of all sufficient enlargements for a finite-dimensional normed space X is closed with respect to the Hausdorff metric.

Theorem (6.3.5)[186]: Each MVSE for a d-dimensional space is in T_d.

Using Theorem (6.3.4) we show that spaces having non-parallelepipedal MVSE cannot be strictly convex or smooth. More precisely, we prove

Proof. (We assume that Lemmas (6.3.6) and (6.3.7) have been proved.) Let X be a d-dimensional space and let A be an MVSE for X.

Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence satisfying $\psi_d > \varepsilon_n > 0$ and $\varepsilon_n \downarrow 0$ Let $\{Y_n\}_{n=1}^{\infty}$ be a sequence of polyhedral spaces satisfying

$$\frac{1}{1+\varepsilon_n} B_X \subset B_{Y_n} \subset B_X \tag{19}$$

Then A is an SE for Y_n. Let B_n be an MVSE for Y_n. Then $(1 + \varepsilon_n)B_n$ is an SE for X. Since A is a minimal-volume SE for X, we have

$$\mathrm{vol}A \leq \mathrm{vol}\,((1 + \varepsilon_n)B_n) = (1 + \varepsilon_n)^d \,\mathrm{vol}\, B_n.$$

By Lemma (6.3.7) for every $n \in \mathbf{N}$ there exists an SE $\tilde{A}_n$ *for* y_n satisfying

$$\tilde{A}_n \subset A$$

and

$$d(\tilde{A}_n, T_n) \leq t_d(\varepsilon_n) \tag{20}$$

for some $T_n \in T_d$.

The condition (19) implies that $(1 + \varepsilon_n)\,\tilde{A}_n$ is an SE for X.

The sequence $\{(1+\varepsilon_n)\,\tilde{A}_n\}_{n=1}^{\infty}$ is bounded (all of its terms are contained in $(1 + \varepsilon_1)A$). By the Blaschke selection theorem [83] the sequence $\{(1 + \varepsilon_n\,\tilde{A}_n\}_{n=1}^{\infty}$

contains a subsequence convergent with respect to the Hausdorff metric. We denote its limit by D, and assume that the sequence $\{(1 + \varepsilon_n)\, \tilde{A}_n\}_{n=1}^{\infty}$ itself converges to D.

Observe that each $\tilde{A}_n$ contains $(1/(1 + \varepsilon_1))B_X$ and is contained in A. By (20) we may assume without loss of generality that T_n are balls in X satisfying

$$\frac{1}{1+\varepsilon_1} B_X \subset \tilde{A}_n \subset T_n \subset t_d(\varepsilon_n)\tilde{A}_n \subset t_d(\varepsilon_n)A \tag{21}$$

It is clear that D is the Hausdorff limit of $\{\tilde{A}_n\}_{n=1}^{\infty}$. From (21) we get that D is the Hausdorff limit of $\{T_n\}_{n=1}^{\infty}$. By Lemma (6.3.6) we get $D \in \mathcal{T}_d$.

By Lemma (6.3.4) the set D is an SE for X. Since $(1 + \varepsilon_n)\, \tilde{A}_n \subset (1 + \varepsilon_n)A$, and $(1 + \varepsilon_n)A$ is Hausdorff convergent to A, we have $D \subset A$. On the other hand, A is an MVSE for $\mathbf{X}$, hence $D = A$ and $A \in \mathcal{T}_d$.

Lemma (6.3.6)[186]: Let $T_n \subset \mathbf{R}^d$, $n \in \mathbf{N}$ be such that $T_n \in \mathcal{T}_d$, and $\{T_n\}_{n=1}^{\infty}$ converges with respect to the Hausdorff metric to a d-dimensional set T. Then $T \in \mathcal{T}_d$.

Proof. By Theorem (6.3.2) the sets T_n are lattice 2tiles. Let $\{A_n\}_{n=1}^{\infty}$ be lattices corresponding to these lattice tiles. Since volume is continuous with respect to the Hausdorff metric (see [83]), the supremum $\sup_n \mathrm{vol}(T_n)$ is finite. Since T_n is a lattice tile with respect to A_n, the determinant of A_n satisfies $d(A_n) = \mathrm{vol}(T_n)$. (Although I have not found this result in the stated form, it is well known. It can be proved, for example, using the argument from [198].) Hence $\sup_n d(A_n) < \infty$. Since T is d-dimensional, there exists $r > 0$ such that $rB_2^d \subset T$. Choosing a smaller $r>0$, if necessary, we may assume that $rB_2^d \subset T_n$ for each n. Therefore the lattices $\{A_n\}_{n=1}^{\infty}$ satisfy the conditions of the selection theorem of Mahler (see, for example, [198], where the reader can also find the standard definition of convergence for lattices).Hence the sequence $\{A_n\}_{n=1}^{\infty}$ contains a subsequence which converges to some lattice A . It is easy to verify that T tiles $\mathbf{R}^d$ with respect to A.

On the other hand, the number of possible distinct columns of a totally unimodular matrix with columns from $\mathbf{R}^d$ is bounded from above by 3^d, because each entry is 0, 1, or −1. (Actually a much better exact bound is known, see [190].) Using this we can show that T is a zonotope by a straightforward argument. Also we can use the argument from [83] and the observation that a convergent sequence of measures on the sphere of ℓ_2^d, each of whom has a finite support of cardinality $\leq 3^d$, converges to a measure supported on $\leq 3^d$ points. Thus, T is a zonotope and a lattice tile. Applying Theorem (6.3.2) again, we get $T \in \mathcal{T}_{\mathbf{d}}$.

Lemma (6.3.7)[186]: (Main lemma) For each $d \in \mathbf{N}$ there exist $\psi_d > 0$ and a function $t_d : (0, \psi_d) \to (1, \infty)$ satisfying the conditions:

(i) $\lim_{\varepsilon \downarrow 0} t_{\mathbf{d}}(\varepsilon) = 1$;

(ii) If Y is a d-dimensional polyhedral space, B is an MVSE for Y , and A is an SE for Y satisfying

$$\operatorname{Vol} A \leq (1+\varepsilon)^{d} \operatorname{vol} B \tag{18}$$

for some $0 < \varepsilon < \psi_d$, then A contains a ball $\tilde{A}$ satisfying the conditions:

(i) $d(\tilde{A}, T) \leq t_{\mathbf{d}}(\varepsilon)$ for some $T \in \mathcal{T}_{\mathbf{d}}$, where by $d(\tilde{A}, T)$ we denote the Banach– Mazur distance;

(ii) $\tilde{A}$ is an SE for Y .

Proof. In our argument the dimension d is fixed. Many of the parameters considered below depend on d, although we do not reflect this dependence in our notation.

Since Y is polyhedral, we can consider Y as a subspace of ℓ_∞^m .

Let P: $\ell_\infty^m \to Y$

be a linear projection satisfying $P(B_\infty^m) \subset A$ (such a projection exists because A is an SE). Let $\tilde{A} = P(B_\infty^m)$. It is easy to see that $\tilde{A}$ is an SE for Y . It remains to show that $\tilde{A}$ is close to some $T \in \mathcal{T}_{\mathbf{d}}$ with respect to the Banach–Mazur distance.

We consider the standard inner product on ℓ_∞^m . (The unit vector basis is an orthonormal basis with respect to this inner product.)

Let $\{q_1,\ldots,q_{m-d}\}$ be an orthonormal basis in ker P. Let $\{y_1,\ldots, y_{\mathbf{d}}\}$ be an orthonormal basis in Y. Let $\tilde{q}_1,\ldots,\tilde{q}_d$ be such that $\{\tilde{q}_1,\ldots,\tilde{q}_d, q_1,\ldots,q_{m-d}\}$ is an orthonormal basis in ℓ_∞^m.

Lemma (6.3.8)[186]: (Image Shape Lemma) Let P and $\tilde{q}_1,\ldots,\tilde{q}_d$ be as above. Denote by $\tilde{Q}$ $[\tilde{q}_1,\ldots,\tilde{q}_d]$ the matrix whose columns are $\tilde{q}_1,\ldots,\tilde{q}_d$. Let $z_1, \ldots, z_m$ be the columns of the transpose matrix $\tilde{Q}^T$. Then $P(B_\infty^m)$ is linearly equivalent to the zonotope $\sum_{i=1}^{m} I(z_i) \subset \mathbf{R}^d$.

Proof: It is enough to observe that:

(i) Images of B_∞^m under two linear projections with the same kernel are linearly equivalent. Hence, $P(B_\infty^m)$ is linearly equivalent to the image of the orthogonal projection with the kernel $\ker P$.

(ii) The matrix $\tilde{Q}\tilde{Q}^T$ is the matrix of the orthogonal projection with the kernel ker P. By Lemma (6.3.8) we may replace $\tilde{A}$ by

$$Z = \sum_{i=1}^{m} I(z_i) \tag{22}$$

in the estimate (i) of Lemma (6.3.7).

Let $M = \binom{m}{d}$ We denote by u_i ($i = 1, \ldots, M$) the $d \times d$ minors of $[y_1, \ldots, y_d]$ (ordered in some way). We denote by w_i ($i = 1, \ldots, M$) the $d \times d$ minors of $[\tilde{q}_1,\ldots,\tilde{q}_d]$ ordered in he same way as the u. We denote by v_i $(i = 1,\ldots,\binom{m}{m-d} = M)$ their complementary $(m - d) \times (m - d)$ minors of $[\tilde{q}_1,\ldots,\tilde{q}_{m-d}]$. Using the word complementary we mean that all minors are considered as minors of the matrix $[\tilde{q}_1,\ldots,\tilde{q}_d, \tilde{q}_1,\ldots,\tilde{q}_{m-d}]$, see [93].

By the Laplacian expansion (see [93])

$$\det\left[y_1,\ldots y_d, q_1,\ldots,q_{m-d}\right] = \sum_{i=1}^{M} \theta_i u_i v_i$$

and

$$\det\left[\tilde{q}_1,\ldots\tilde{q}_d, q_1,\ldots,q_{m-d}\right] = \sum_{i=1}^{M} \theta_i w_i v_i \tag{23}$$

for proper signs θ_i.

Since the matrix $[\tilde{q}_1,\ldots,\tilde{q}_d, q_1,\ldots,q_{m-d}]$ is orthogonal, we have

$$\det[q_1,\ldots,q_d, q_1,\ldots,q_{m-d}] = \pm 1. \tag{24}$$

We need the following result on compound matrices. (We refer to [93] for necessary definitions and background.)

A compound matrix of an orthogonal matrix is orthogonal (see [93]).

This result implies, in particular, that the Euclidean norms of the vectors $[w_i]_{i=1}^M$ and $\{v\}_{i=1}^M$ in $\mathbf{R^M}$ are equal to 1.

From (23) and (24) we get that either

(i) $w_i = \theta_i v_i$ for every i or

(ii) $w_i = -\theta_i v_i$ for every i.

Without loss of generality, we assume that $w_i = \theta_i v_i$ for all i (we replace q_1 by $-q_1$ if **it** is not the case).

We compute the volume of $\widetilde{A}$ and B with the normalization that comes from the Euclidean structure introduced above. It is well known (see [89]) and is easy to verify that with this normalization

$$vol\,\widetilde{A} = \frac{2^d}{\left|\sum_{i=1}^M \theta_i u_i v_i\right|} \sum_{i=1}^M |v_i|$$

and

$$vol\,B = \frac{2^d}{\max_i |u_i|}$$

for each MVSE B for Y.

Remark (6.3.9)[186]: After the publication of [89] I learned that the formula for the volume of a zonotope used in [89] can be found in [203].

Since vol $\widetilde{A} \le$ volA, the inequality(18) implies that

$$\max_i |u_i| \sum_{i=1}^M |v| \le (1+\varepsilon)^d \left|\sum_{i=1}^M \theta_i u_i v_i\right| \tag{25}$$

By (a) the inequality (25) can be rewritten as

$$\max_i |u_i| \sum_{i=1}^M |w_i| \le (1+\varepsilon)^d \left|\sum_{i=1}^M u_i w_i\right| \tag{26}$$

We need the following two observations:

(i) $2^d \sum_{i=1}^{M} |w_i|$ is the volume of Z in $\mathbf{R}^d$.

(ii) The vector $\{u_i\}_{i=1}^{M}$ is what is called the Grassmann coordinates, or the Plucker coordinates of the subspace Y $\subset \mathrm{R}^{\mathbf{m}}$, see [204] and [205]. Recall that Y is spanned by the columns of the matrix $[y_1, \ldots, y_d]$. **It** is easy to see that if we choose another basis in Y , the Grassman (Plucker) coordinates will be multiplied by a constant.

We denote by Z_ε ($\varepsilon > 0$) the set of all d-dimensional zonotopes in $\mathbf{R}^d$ satisfying the condition (26) with an equality. More precisely, we define Z_ε as the set of those d-dimensional zonotopes Z in $\mathbf{R}^d$ for which

(i) There exists m $\in$ **N** and a rank d matrix $\tilde{Q}$ of size m $\times$ d such that, $Z = \sum_{i=1}^{m} I(z_i)$ where $z_i \in \mathbf{R}^d$, $i = 1, \ldots, m$, are rows of $\tilde{Q}$.

(ii) There exists a rank d matrix Y of size m $\times$ d such that, if we denote the d $\times$ d minors of $\tilde{Q}$ by $\{w_i\}_{i=1}^{\infty}$ where $M = \binom{m}{d}$ and the d $\times$ d minors of Y , ordered in the same way as the w_i, by $\{u_i\}_{i=1}^{\infty}$ then

$$\max_i |u_i| \sum_{i=1}^{M} |w_i| = (1+\varepsilon)^d \left| \sum_{i=1}^{M} u_i w_i \right| \tag{27}$$

and there is no Y for which

$$\max_i |u_i| \sum_{i=1}^{M} |w_i| < (1+\varepsilon)^d \left| \sum_{i=1}^{M} u_i w_i \right|$$

Many objects introduced below depend on Z and ε, although sometimes we do not reflect this dependence in our notation.

Let $Z \in Z_\varepsilon$. We shall change the system of coordinates in $\mathbf{R}^d$ twice. First we introduce in $\mathbf{R}^d$ a new system of coordinates such that the unit (Euclidean) ball B_2^d of $\mathbf{R}^d$ is the maximal volume ellipsoid in Z. now on we consider the vectors $z_\mathbf{i}$ introduced in Lemma (6.3.8) as vectors in $\mathbf{R}^d$ and not as d-tuples of real numbers.

It is easy to see that the support function of Z is given by

$$h_Z(x) = \sum_{i=1}^{m} |\langle x, z_i \rangle|.$$

It is more convenient for us to write this formula in a different way. We consider the set

$$\left\{ \frac{z_1}{\| z_1 \|}, \ldots, \frac{z_m}{\| z_m \|}, -\frac{z_1}{\| z_1 \|}, \ldots, -\frac{z_m}{\| z_m \|} \right\}. \tag{28}$$

If the vectors in (28) are pairwise distinct, we let μ to be the atomic measure on the unit (Euclidean) sphere S whose atoms are given by $\mu(z_i/\|z_i\|) = \mu(-z_i/\|z_i\|) = \|z_i\|/2$. **It** is easy to see that

$$h_z(x) = \int_S |\langle x, z \rangle| \, d\mu(z). \tag{29}$$

The defining formula for μ should be adjusted in the natural way if some of the vectors in (28) are equal.

Conversely, if μ is a nonnegative measure on S supported on a finite set, then (29) is a support function of some zonotope (see [83] for more information on this matter).

Dealing with subsets of S we use the following terminology and notation. Let $x_0 \in S$, $r>0$. The set $\Delta(x_0, r) := \{x \in S : \|x - x_0\| < r \text{ or } \|x + x_0\| < r\}$, where $\|.\|$ is the ℓ_2-norm, is called a cap. If $0 < r < \sqrt{2}$, then $\Delta(x_0, r)$ consists of two connected components. In such a case both x_0 and $-x_0$ will be considered as centers of $\Delta(x_0, r)$.

We are going to show that if $\varepsilon > 0$ is small, then the inequality (26) implies that all but a very small part of the measure μ is supported on a union of small caps centered at a set of vectors which are multiples of a set of vectors satisfying the condition: if we write their coordinates with respect to a suitably chosen basis, we get a totally unimodular matrix. Having such a set, it is easy to find $T \in \mathcal{T}_d$ which is close to Z with respect to the Banach–Mazur distance, see Lemma (6.3.29).

For any two numbers $\omega, \delta > 0$ we introduce the set

$$\Omega(\omega, \delta) := \{x \in S: \mu(\Delta(x, \omega)) \geq \delta\}$$

(recall that by S we denote the unit sphere of ℓ_2^d). In what follows $c_1(d), c_2(d), \ldots,$

$C_1(d)$, $C_2(d)$, . . . denote quantities depending on the dimension d only. Since d is fixed throughout our argument, we regard them as constants.

First we find conditions on ω and δ under which the set $\Omega(\omega, \delta)$ contains a normalized basis $\{e_i\}_{i=1}^{d}$ whose distance to an orthonormal basis can be estimated in terms of d only.

Lemma (6.3.10)[186]: There exist $0<c_1(d)$, $C_1(d)$, $C_2(d)<\infty$, such that for $\omega \le \frac{1}{6d}$ $\delta \le c_1(d)\omega^{d-1}$ there is a normalized basis $\{e_i\}_{i=1}^{d}$ in the space $\mathbf{R}^d$ satisfying the conditions:

(i) $\mu(\Delta(e_i, \omega)) \ge \delta$.

(ii) If $\{o_i\}_{i=1}^{d}$ is an orthonormal basis in $\mathbf{R}^d$, then the operator $N:\mathbf{R}^d \to \mathbf{R}^d$ given by $N o_i = e_i$ satisfies $\|N\| \le C_1(d)$ and $\|N^{-1}\| \le C_2(d)$, where the norms are the operator norms of N, N^{-1} considered as operators from ℓ_2^d into ℓ_2^d.

Proof. We need an estimate for $\mu(S)$. Observe that if K_1 and K_2 are two symmetric zonotopes and $K_1 \subset K_2$, then $\mu_1(S) \le \mu_2(S)$ for the corresponding measures μ_1 and μ_2 (defined as even measures satisfying (29) with $Z = K_1$ and $Z = K_2$, respectively) .To prove this statement we integrate the equality (29) with respect to x over the Haar measure on S.

Now we use the assumption that B_2^d is the maximal volume ellipsoid in Z. Let is the maximal volume ellipsoid in Z. Let $\sum_{i=1}^{n}\gamma_i x_i \otimes x_i$ be the F. John representation of the identity operator corresponding to Z (see [42]). Then

$$Z \subset \{x : |\langle x, x_i \rangle| \le 1 \ \forall i \in \{1, . . . , n\}\} .$$

Since $x = \sum_{i=1}^{n}\langle x, x_i \rangle \gamma_i x_i$ for each $x \in \mathbf{R}^d$, we have $Z \subset \sum_{i=1}^{n}\left[-\gamma_i x_i, \gamma_i x_i\right]$. Since $\sum_{i=1}^{n}\gamma_i = d$, this implies $\mu(S) \le d$.

Using the well-known computation, which goes back to B. Grünbaum ([206], see, also, [207]) one can find estimates for $\mu(S)$ from below, which imply $\mu(S) \ge \sqrt{d}$ For our purposes the trivial estimate $\mu(S) \ge 1$ is sufficient (this estimate follows immediately from $Z \supset B_2^d$, because this inclusion implies $h_Z(x) \ge \|x\|$).

We denote the normalized Haar measure on S by η. It is well known that there exist $c_2(d) > 0$ such that

$$\eta(\Delta(x, r)) \geq c_2(d)r^{d-1} \qquad \forall r \in (0, 1)\ \forall x \in S. \tag{30}$$

Using a standard averaging argument and $\mu(S) \geq 1$, we get that there exists $e_1 \in S$ such that

$$\mu(\Delta(e_1, \omega)) \geq c_2(d)\omega^{d-1}.$$

Consider the closed $(\frac{1}{3d}+\omega)$ -neighborhood (in the ℓ_2^d metric) of the line L_1 spanned by e_1. Let Δ_1 be the intersection of this neighborhood with S. Our purpose is to estimate $\mu(S\backslash\Delta_1)$ from below. Let $x \in S$ be orthogonal o e_1. Then

$$1 \leq h_Z(x) \leq 1 \cdot \mu(S\backslash\Delta_1) + (\frac{1}{3d}+\omega).d,$$

where the left-hand side inequality follows from the fact that Z contains B_2^d. Therefore $\mu(S\backslash\Delta_1) \geq 1-(\frac{1}{3d}+\omega)\,d$.

We erase all measure μ contained in Δ_1, use a standard averaging argument again, and find a vector e_2 such that

$$\mu(\Delta(e_2, \omega)\backslash\Delta_1) \geq c_2(d)\ \omega^{d-1}\left((1-\frac{1}{3d}+\omega)d\right).$$

Since $\mu(\Delta(e_2, \omega)\backslash\Delta_1) > 0$, the vector e_2 is not in the $\frac{1}{3d}$ -neighborhood of L_1.

Let Δ_2 be the intersection of S with the closed$(\frac{1}{3d}+\omega)$ - neighborhood of L_2 = lin$\{e_1, e_2\}$ (that is, L_2 is the linear span of $\{e_1, e_2\}$). Let $x \in S$ be orthogonal to L_2. Then

$$1 \leq h_Z(x) \leq 1.\mu\ (S\backslash\Delta_2)+(\frac{1}{3d}+\omega).d,$$

where the left-hand side inequality follows from the fact that Z contains B_2^d. Therefore

$$\mu(S\backslash\Delta_2) \geq 1-(\frac{1}{3d}+\omega)\,d.$$

Using the standard averaging argument in the same way as in the previous step we find a vector e_3 such that

$$\mu(\Delta(e_3,\omega)\backslash\Delta_2) \geq c_2(d)\ \omega^{d-1}\left(1-\left(\frac{1}{3d}+\omega\right)d\right).$$

Since $\mu(\Delta(e_3,\ \omega)\backslash\Delta_2) > 0$, the vector e_3 is not in the $\frac{1}{3d}$-neighborhood of L_2.

We continue in an obvious way. As a result we construct a normalized basis $\{e_1,\ldots, e_d\}$ satisfying the conditions.

(i) $\mu(\Delta(e_i,\omega)) \geq c_2(d)\ \omega^{d-1}(1-\frac{1}{3d}+\omega)d)$.

(ii) $\operatorname{dist}(ei\ ,\ \operatorname{lin}\{e_j\}_{j=1}^{i-1} \geq \frac{1}{3d}, i=2,\ldots,d$, where $\operatorname{dist}(\cdot,\ \cdot)$ denotes the distance from a vector to a subspace.

If $\omega < \frac{1}{6d}$ the inequality (i) implies

$$\mu(\Delta(e_i,\omega)) \geq \frac{1}{2}c_2(d)\omega^{d-1},$$

and we get the estimate (i) of Lemma (6.3.10) with $c_1(d) = c_2(d)/2$.

To estimate $\|N\|$ and $\|N^{-1}\|$, we let $\{o_i\}_{i=1}^{d}$ be the basis obtained from $\{e_i\}$ are using the Gram–Schmidt orthonormalization process. Let $N : \mathbf{R}^d \to \mathbf{R}^d$ be defined by $N\,o_i = e_i$.The estimate $\|N\| \leq C_1(d)$ with $C_1(d) = \sqrt{d}$ follows because the vectors $\{e_i\}_{i=1}^{d}$ are normalized and the vectors $\{o_i\}_{i=1}^{d}$ form an orthonormal set .

To estimate $\|N^{-1}\|$ we observe that the matrix of N with respect to the basis $\{o_i\}$ is of the form

$$N = \begin{pmatrix} N_{11} & N_{12} & \cdots & N_{1d} \\ 0 & N_{22} & \cdots & N_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N_{dd} \end{pmatrix}$$

and that the inequality (ii) implies $N_{ii} \geq \frac{1}{3d}$. We have

$$T=\begin{pmatrix} N_{11} & 0 & \cdots & 0 \\ 0 & N_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N_{dd} \end{pmatrix}.\begin{pmatrix} 1 & \frac{N_{12}}{N_{11}} & \cdots & \frac{N_{1d}}{N_{11}} \\ 0 & 1 & \cdots & \frac{N_{2d}}{N_{22}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}=D(I+U),$$

where $\mathbf{I}$ is the identity matrix,

$$D=\begin{pmatrix} N_{11} & 0 & \cdots & 0 \\ 0 & N_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N_{dd} \end{pmatrix} \; and \; U=.\begin{pmatrix} 0 & \frac{N_{12}}{N_{11}} & \cdots & \frac{N_{1.d-1}}{N_{11}} & \frac{N_{1d}}{N_{11}} \\ 0 & 0 & \cdots & \frac{N_{2.d-1}}{N_{22}} & \frac{N_{2d}}{N_{22}} \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & \frac{N_{d-1.d}}{N_{d-1.d-1}} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Therefore

$$\mathrm{N}^{-1}=(\mathbf{I}+\mathrm{U})^{-1}\ \mathrm{D}^{-1}=(\mathbf{I}-\mathrm{U}+\mathrm{U}^{2}-\cdots+(-1)^{d-1}\mathrm{U}^{d-1})\mathrm{D}^{-1}, \qquad (31)$$

the identity $(\mathbf{I}+\mathrm{U})^{-1}=(\mathbf{I}-\mathrm{U}+\mathrm{U}^{2}-\cdots+(-1)^{d-1}\mathrm{U}^{d-1})$ follows from the obvious equality $\mathrm{U}^{d}=0$. The definition of U and $N_{ii}\geq\frac{1}{3d}$. imply that columns of U are vectors with Euclidean norm at most 3d, hence $\|\mathrm{U}\|\leq 3d^{\frac{3}{2}}$. Therefore the identity Therefore the identity (31) implies the following estimate for $\|\mathrm{N}^{-1}\|$:

$$\left\|N^{-1}\right\|\leq\frac{\|U\|^{d}-1}{\|U\|-1}.\left\|D^{-1}\right\|\leq\frac{3^{d}d^{\frac{3d}{2}}-1}{3d^{\frac{3}{2}}-1}.3d.$$

Denoting the right-hand side of this inequality by $\mathrm{C}_2(\mathrm{d})$ we get the desired estimate.

Lemma (6.3.11)[39]: Let $c_2(d)$ be the constant from (30), then

$\mu(S\backslash((\Omega(\omega,\delta))_{\omega}))\leq\frac{\delta}{c_2(d)\omega^{d-1}}.$

Proof. Assume the contrary, that is, $\mu(S\backslash((\Omega(\omega,\delta))_{\omega}))>\frac{\delta}{c_2(d)\omega^{d-1}}$. Then, using a standard averaging argument as in Lemma (6.3.10), we find a point x such that

$$\mu(\Delta(x,\omega)\backslash((\Omega(\omega,\delta))_{\omega}))\geq c_2(d)\,\omega^{d-1}\cdot.\frac{\delta}{c_2(d)\omega^{d-1}}=\delta.$$

By the definition of $\Omega(\omega,\delta)$ this implies $x\in\Omega(\omega,\delta)$. On the other hand, since the set $\Delta(x, \omega)\backslash((\Omega(\omega,\delta))_{\omega})$ is non-empty, it follows that

$x \notin \Omega(\omega, \delta)$. We get a contradiction.

For each $Z \in Z\varepsilon$ we apply Lemma (6.3.10) with $\omega = \omega(\varepsilon) = \varepsilon^{4k}$ and $\delta = \delta(\varepsilon) = \varepsilon^{4dk}$, where $0 < k < 1$ is a number satisfying the conditions

$$k < \frac{1}{6+4d^2} \quad and \quad k < \frac{1}{2d+4d^2}, \tag{32}$$

we choose and fix such number k for the rest of the proof. It is clear that there is $\Xi_0 = \Xi_0(d,k) > 0$ such that the conditions $\omega(\varepsilon) \le \frac{1}{6d}$ and

$\delta(\varepsilon) \le c_1(d)(\omega(\varepsilon))^{d-1}$ are satisfied for all $\varepsilon \in \overset{d}{(}0, \Xi_0)$, where $c_1(d)$ is the constant from Lemma (6.3.10). In the rest of the argument we consider $\varepsilon \in (0, \Xi_0)$ only. Let $\{e_i\}_{i=1}^{d}$ be one of the bases satisfying the conditions of Lemma (6.3.10) with the described choice of ω and δ. Now we change the system of coordinates in $\mathbf{R}^d \supset Z$ the second time. The new system of coordinates is such that $\{e_i\}_{i=1}^{d}$ is its unit vector basis. We shall modify the objects introduced so far (Ω, μ, etc.) and denote their versions corresponding to the new system of coordinates by $\breve{\Omega}, \breve{\mu}$, etc. All these objects depend on Z, ε, and the choice of $\{e_i\}_{i=1}^{d}$.

We denote by Š the Euclidean unit sphere in the new system of coordinates. We denote by N : S → Š the natural normalization mapping, that is, $N(z) = z/\|z\|$, where $\|z\|$ is the Euclidean norm of z with respect to the new system of coordinates. The estimates for $\|N\|$ and $\|N^{-1}\|$ from Lemma (6.3.10) imply that the Lipschitz constants of the mapping N and its inverse N^{-1} : Š → S can be estimated in terms of d only.

We introduce a measure $\breve{\mu}$ on Š as an atomic measure supported on a finite set and such that $\breve{\mu}(N(z)) = \mu(z)\,\|z\|$ for each $z \in S$, where $\|z\|$ is the norm of z in the new system of coordinates. Using the definition of the zonotope Z it is easy to check that the function

$$\breve{h}_Z(x) = \int_{\breve{S}} \left|\langle x, \check{z}\rangle \right| d\breve{\mu}(\check{z}),$$

where $\langle .,. \rangle$ is the inner product in the new coordinate system, is the support function of Z in the new system of coordinates.

We define $\breve{\Omega} = \breve{\Omega}(\omega, \delta)$ as $N(\boldsymbol{\Omega}(\omega, \delta))$. It is clear that $e_i \in \breve{\Omega}$. Everywhere below we mean coordinates in the new system of coordinates (when we refer to $\|\cdot\|$, Δ, etc).

The observation that N and N^{-1} are Lipschitz, with Lipschitz constants estimated in terms of d only, implies the following statements:

(i) There exist $C_3(d)$, $C_4(d) < \infty$ such that

$$\breve{\mu}(\breve{S} \setminus ((\breve{\Omega}(\omega,\delta))_{c_3(d)\omega(\varepsilon)})) \leq C_4(d)\frac{\delta}{\omega^{d-1}} \tag{33}$$

(we use Lemma (6.3.11).

(ii) There exist $c_3(d) > 0$ and $C_5(d) < \infty$ such that

$$\breve{\mu}\,(\Delta(x, C_5(d)\omega)) \geq c_3(d)\delta \quad \forall x \in \breve{\Omega}(\omega, \delta) \tag{34}$$

(we use the definitions of $\boldsymbol{\Omega}(\omega, \delta)$ and $\breve{\Omega}(\omega, \delta)$).

(iii)There exists a constant $C_6(d)$ depending on d only, such that

$$\mathrm{vol}(Z) \leq C_6(d). \tag{35}$$

Let $\breve{Q}$ be the transpose of the matrix whose columns are the coordinates of z_i in the new system of coordinates. We denote by $\breve{w}_i (i = 1, \ldots, M)$ the $d \times d$ minors of $\breve{Q}$ ordered in the same way as the w_i. The vector $\{\breve{w}_i\}_{i=1}^M$ is a scalar multiple of $\{w_i\}_{i=1}^M$. Therefore (27) implies

$$\max_i |u_i| \sum_{i=1}^{M} |\breve{w}_i| = (1+\varepsilon)^d \left| \sum_{i=1}^{M} u_i \breve{w}_i \right|. \tag{36}$$

The volume of Z in the new system of coordinates is $2^d \sum_{i=1}^{M} |\breve{w}_i|$.

To show that if $\varepsilon > 0$ is small, then the inequality (36) implies that all but a very small part of the measure $\breve{\mu}$ is supported "around" multiples of vectors represented by a totally unimodular matrix in some basis, we need the following lemma. It shows that the inequality (36) implies that the measure $\breve{\mu}$ cannot have non-trivial "masses" near (d + 2)-tuples of vectors satisfying certain condition.

Lemma (6.3.12)[186]: Let $\chi(\varepsilon), \sigma(\varepsilon)$, and $\pi(\varepsilon)$ be functions satisfying the following conditions:

(i) $\lim_{\varepsilon\downarrow 0}\chi(\varepsilon) = \lim_{\varepsilon\downarrow 0}\sigma(\varepsilon) = \lim_{\varepsilon\downarrow 0}\pi(\varepsilon) = 0$;

(ii) $\varepsilon = o((\chi(\varepsilon))^2(\sigma(\varepsilon))^d)$ as $\varepsilon \downarrow 0$;

(iii) $\pi(\varepsilon) = o(\chi(\varepsilon))$ as $\varepsilon \downarrow 0$;

(iv) There is a subset $\Phi_0 \subset (0, \Xi_0)$ such that the closure of Φ_0 contains 0, and for each $\varepsilon \in \Phi_0$ there exist $Z \in Z_\varepsilon$ and points $x_1, \ldots, x_{d-2}, p_1, p_2, p_3, p_4$ in the corresponding $\breve{S}$, such that

$$\breve{\mu}(\Delta(z, \pi(\varepsilon))) \geq \sigma(\varepsilon) \quad \forall z \in \{x_1, \ldots, x_{d-2}, p_1, p_2, p_3, p_4\}. \tag{37}$$

Let u_0 be the set of pairs (ε, Z) in which $\varepsilon \in \Phi_0$ and Z satisfies the condition from (iv). Let $\Phi_1 \subset \Phi_0$ be the set of those $\varepsilon \in \Phi_0$ for which there exists $(\varepsilon, Z) \in u_0$ such that the corresponding points $x_1, \ldots, x_{d-2}, p_1, p_2, p_3, p_4$ satisfy the condition .

$$|\det(H_{\alpha,\beta})| \geq \chi(\varepsilon) \tag{38}$$

for all matrices $H_{\alpha,\beta}$ whose columns are the coordinates of $\{x_1, \ldots, x_{d-2}, p_\alpha, p_\beta\}$, $\alpha, \beta \in \{1, 2, 3, 4\}$, $\alpha \neq \beta$, with respect to an orthonormal basis $\{e_i\}_{i=1}^d$ in $\mathbf{R}^d$. Then there exists $\Xi_1 > 0$ such that $\Phi_1 \cap (0, \Xi_1) = \emptyset$

Proof. We assume the contrary, that is, we assume that 0 belongs to the closure of Φ_1. For each $\varepsilon \in \Phi_1$ we choose $Z \in Z\varepsilon$ such that $(\varepsilon, Z) \in u_0$ and the condition (37) is satisfied. We show that for sufficiently small $\varepsilon > 0$ this leads to a contradiction.

We consider the following perturbation of the matrix $H_{\alpha,\beta}$: each column vector z in it is replaced by a vector from $\Delta(z, \pi(\varepsilon))$. We denote the obtained perturbation of the matrix $H_{\alpha,\beta}$ by $H^p_{\alpha,\beta}$. We claim that

$$|\det(H^p_{\alpha,\beta})| \geq \chi(\varepsilon) - d\,.\,\pi(\varepsilon). \tag{39}$$

To prove this claim we need the following lemma, which we state in a bit more general form than is needed now, because we shall need it later.

Lemma (6.3.13)[186]: Let $x_1, \ldots, x_d, z \in \ell_2^d$ be such that $\max_{2 \le i \le d} \|x_i\| \le m$ and $\|z - x_1\| \le I$. Then

$$|\det[z, x_2, \ldots, x_d] - \det[x_1, x_2, \ldots, x_d]| \le I.m^{d-1}.$$

This lemma follows immediately from the volumetric interpretation of determinants.

To get the inequality (39) we apply Lemma (6.3.13) d times with $m=1$ and $I=\pi(\varepsilon)$.

Since $Z \in Z_\varepsilon$, it can be represented in the form $Z = \sum_i I(z_i)$. First we complete our proof in a special case when the following condition is satisfied:

All vectors z_i whose normalizations $z_i/\|z_i\|$ belong to the sets $\Delta(z, \pi(\varepsilon))$, $z \in \{x_1, \ldots, x_{d-2}, p_1, p_2, p_3, p_4\}$, have the same norm τ and there are equal amounts of such vectors in each of the sets $\Delta(z, \pi(\varepsilon))$, $z \in \{x_1, \ldots, x_{d-2}, p_1, p_2, p_3, p_4\}$, we denote the common value of the amounts by F.

The inequality (37) implies

$$F.\tau \ge \sigma(\varepsilon).$$

We denote by A the set of all numbers $i \in \{1, \ldots, M\}$ satisfying the condition: the normalizations of columns of the minor $\breve{w}_i$ form a matrix of the form $H^P_{\alpha,\beta}$, for some $\alpha, \beta \in \{1, 2, 3, 4\}$.

We need an estimate for $\sum_{i\in A} |\breve{w}_i|$. The inequality (39) implies $|\breve{w}_i| \ge \tau^d(\chi(\varepsilon) - d \cdot \pi(\varepsilon))$ for each $i \in A$

On the other hand, the cardinality $|A|$ of A is $6F^d$. In fact there are F^{d-2} choose two of the sets $\Delta(p_j, \pi(\varepsilon))$, $j = 1, 2, 3, 4$, and there are F^2 ways to choose one vector $z_i/\|z_i\|$ in each of them. Therefore $|A| = 6F^d$ and

$$\sum_{i\in A} |\tilde{w}_i| \ge 6F^d \tau^d(\chi(\varepsilon) - d \cdot \pi(\varepsilon)) \ge 6(\sigma(\varepsilon))^d(\chi(\varepsilon) - d \cdot \pi(\varepsilon)). \quad (40)$$

We assume for simplicity that $\max_i |u_i| = 1$ (if it is not the case, some of the sums below should be multiplied by maxi $|u_i|$). The u_i are defined above the equality (27). Then the condition (36) can be rewritten as

$$(1+\varepsilon)^d \left|\sum_{i=1}^{M} u_i \breve{w}_i\right| \geq \sum_{i\in A}\left|\breve{w}_i\right| + \sum_{i\notin A}\left|\breve{w}_i\right|. \tag{41}$$

On the other hand,

$$(1+\varepsilon)^d \left|\sum_{i=1}^{M} u_i \breve{w}_i\right| \leq (1+\varepsilon)^d \left|\sum_{i\in A} u_i \breve{w}_i\right| + (1+\varepsilon)^d \sum_{i\notin A}\left|\breve{w}_i\right|. \tag{42}$$

From (41) and (42) we get

$$(1+\varepsilon)^d \left|\sum_{i=1} u_i \breve{w}_i\right| \geq \sum_{i\in A} |\breve{w}_i| - ((1++\varepsilon)^d - 1) \sum_{i\notin A}\left|\breve{w}_i\right|. \tag{43}$$

As is well known, $2^d \sum_{i=1}^{M}\left|\breve{w}_i\right|$ is the volume of Z, hence $\sum_{i=1}^{M}\left|\breve{w}_i\right| \leq 2^{-d} C_6(d)$.

Using this observation and the inequalities (40) and (43) we get

$$\left|\sum_{i\in A} u_i \breve{w}_i\right| \geq \left(\frac{1}{(1+\varepsilon)^d} - \frac{((1+\varepsilon)^d - 1)C_6(d)2^{-d}}{6(\sigma(\varepsilon))^d\,(\chi(\varepsilon) - d.\pi(\varepsilon))}\right) \sum_{i\in A}\left|\breve{w}_i\right|.$$

(We use the fact that $\chi(\varepsilon) - d \cdot \pi(\varepsilon) > 0$ if $\varepsilon > 0$ is small enough.) The conditions (19) and (20) imply that there exists $\psi > 0$ such that

$$\left(\frac{1}{(1+\varepsilon)^d} - \frac{((1+\varepsilon)^d - 1)C_6(d)2^{-d}}{6(\sigma(\varepsilon))^d\,(\chi(\varepsilon) - d.\pi(\varepsilon))}\right) > (1 - 0.04(\chi(\varepsilon) - d.\pi))) \tag{44}$$

is satisfied if $\varepsilon \in (0, \psi)$. The right-hand side is chosen in the form needed below.

Let $\psi > 0$ be such that the statement above is true. Then for $\varepsilon \in (0, \psi)$ we have

$$\left|\sum_{i\in A} u_i \breve{w}_i\right| \geq (1 - 0.04(\chi(\varepsilon) - d.\pi(\varepsilon))) \sum_{i\in A}\left|\breve{w}_i\right|. \tag{45}$$

Recall that u_i are $d \times d$ minors of some matrix $[y_1, \ldots, y_d]$. We need the Plucker relations, see [204] or [205]. The result that we need can be stated in the following way: if $\gamma_1, \ldots, \gamma_{d-2}, \kappa_1, \kappa_2, \kappa_3, \kappa_4$ are indices of $d + 2$ rows of $[y_1, \ldots, y_d]$, then

$$t_{1,2}t_{3,4} - t_{1,4}t_{3,2} + t_{2,4}t_{3,1} = 0, \tag{46}$$

where $t_{\alpha,\beta}$ is the determinant of the $d \times d$ matrix whose rows are the rows of $[y_1, \ldots, y_d]$ with the indices $\gamma_1, \ldots, \gamma_{d-2}$, κ_α, and κ_β. Note that (46) can be verified by a straightforward computation (which is very simple if we make a suitable change of coordinates before the computation).

Now we Show that (45) cannot be satisfied. Let ψ be a set Consist in g of d +2 vectors $z_{k_1}, z_{k_2}, z_{k_3}, z_{k_4}, z_{\gamma_1}, \ldots z_{\gamma_{d-2}}$, formed in the following way. We choose vectors, $(z_{k_i}/\| z_{k_i}\|) \in \Delta(p_i, \pi(\varepsilon))$, i = 1, 2, 3, 4, and choose vectors $(z_{\gamma_i}/\| z_{\gamma_i}\|) \in \Delta(x_i, \pi(\varepsilon))$, i = 1, . . . , d − 2. To each such selection there corresponds a set of 6 minors $\breve{w}_i$ of the form $\tau^d \det(H^P_{\alpha,\beta})$, we denote this set of six minors by $\{\breve{w}\}_{i \in M(\Psi)}$

One of the immediate consequences of the Plücker relation (46) is that for any such (d + 2)-tuple ψ

$$|u_i| \le \frac{1}{\sqrt{2}} \quad \text{for some } i \in M(\Psi). \tag{47}$$

(Here we use the assumption that $\max_i |u_i| = 1$.)

For each ψ we choose one such $i \in M(\psi)$ and denote it by $s(\psi)$. The estimate (39) and the condition (I) imply that

$$\tau^d \ge |\breve{w}_i| \ge \tau^d(\chi(\varepsilon) - d \cdot \pi(\varepsilon)) \tag{48}$$

for every $i \in A$.

Hence for every (d + 2)-tuple ψ of the described type we have

$$\left| \sum_{i \in M(\psi)} u_i \breve{w}_i \right| \le \sum_{i \in M(\psi \setminus (s(\psi))} \left|\breve{w}_i\right| + \frac{1}{\sqrt{2}}\left|\breve{w}_{s(\psi)}\right| \le \sum_{i \in M(\psi)} |\breve{w}_i| - \frac{\sqrt{2}-1}{\sqrt{2}} |\breve{w}_{s(\psi)}|$$

$$= \sum_{i \in M(\psi)} |\breve{w}_i| \left(1 - \frac{(\sqrt{2}-1|\breve{w}_{s(\psi)}|}{\sqrt{2}\sum_{i \in M(\psi)} |\breve{w}_i|} \right)$$

$$\le \sum_{i \in M(\psi)} |\breve{w}_i| \left(1 - \frac{(\sqrt{2}-1)\tau^d (\chi(\varepsilon) - d.\pi(\varepsilon))}{\sqrt{2}.6\tau^d} \right)$$

$$< \sum_{i \in M(\psi)} |\breve{w}_i| (1 - 0.04)(\chi(\varepsilon) - d.\pi(\varepsilon))).$$

Thus

$$\left| \sum_{i \in M(\psi)} u_i \breve{w}_i \right| < \sum_{i \in M(\psi)} \left|\breve{w}_i\right| (1 - 0.04)(\chi(\varepsilon) - d.\pi(\varepsilon))). \tag{49}$$

Recall that F is the number of vectors z_i corresponding to each of the sets $\Delta(z, \pi(\varepsilon))$, $z \in \{x_1, \ldots, x_{d-2}, p_1, p_2, p_3, p_4\}$. Simple counting shows that for an arbitrary collection $\{Y_i\}_{i \in A}$ of numbers we have

$$\sum_{\psi} \sum_{i \in M(\psi)} \gamma_i = F^2 \sum_{i \in A} \gamma_i .$$

Using (49) we get that

$$F^2 \left| \sum_{i \in A} u_i \breve{w}_i \right| = \left| \sum_{\psi} \sum_{i \in M(\psi)} u_i \breve{w}_i \right| \leq \sum_{\psi} \left| \sum_{i \in M(\psi)} u_i \breve{w}_i \right|$$

$$< \sum_{\psi} \sum_{i \in M(\psi)} \left| \breve{w}_i \right| (1 - 0.04(\chi(\varepsilon) - d.\pi(\varepsilon)))$$

$$= F^2 \sum_{i = A} |\breve{w}_i| (1 - 0.04(\chi(\varepsilon) - d.\pi(\varepsilon))).$$

If $\varepsilon \in (0, \psi)$, we get a contradiction with (45).

To see that the general case can be reduced to the case (I) we need the following observation:

Let $\tau_1, \tau_2 > 0$ be such that $\tau_1 + \tau_2 = 1$. We replace the row with the coordinates of z_j in $\breve{Q}$ by two rows, one of them is the row of coordinates of $\tau_1 z_j$ and the other is the row of coordinates of $\tau_2 z_j$. The zonotope generated by the rows of the obtained matrix coincides with Z. In the matrix $[y_1, \ldots, y_d]$ we replace the jth row by two copies of it. It is easy to see that if we replace the sequences $\{\{u_i\}_{i=1}^{M}$ and $\{\breve{w}_i\}_{i=1}^{M}$ by sequences of $d \times d$ minors of these new matrices, the condition (36) is still satisfied.

We can repeat this 'cutting' of vectors z_j into 'pieces' with (36) still being valid.

Therefore, we may assume the following: among z_j corresponding to each of the sets $\Delta(z, \pi(\varepsilon))$, $z \in \{x_1, \ldots, x_{d-2}, p_1, p_2, p_3, p_4\}$ there exists a subset $\Phi(z, \pi(\varepsilon))$ consisting of vectors having the same length τ, and such that the sum of norms of vectors from $\Phi(z, \pi(\varepsilon))$ is $\geq \frac{\sigma(\varepsilon)}{2}$, moreover, we may assume that the numbers of such vectors in the subsets $\Phi(z, \pi(\varepsilon))$ are the same for all $z \in \{x_1, \ldots, x_{d-2}, p_1, p_2, p_3, p_4\}$.

Lemma (6.3.12) in this case can be proved using the same argument as before, but with A being the set of those minors $\check{w}_i$ for which rows are from $\Phi(z, \pi(\varepsilon))$. Everything starting with the inequality (40) can be shown in the same way as before; only some constants will be changed (because we need to replace $\sigma(\varepsilon)$ by $\frac{\sigma(\varepsilon)}{2}$).

Let $\rho(\varepsilon) = \varepsilon^k$, $\nu(\varepsilon) = \varepsilon^{3k}$. For a vector s we denote its coordinates with respect to $\{e_i\}_{i=1}^d$ by $\{s_i\}_{i=1}^d$

Lemma (6.3.14)[186]: If

$$k < \frac{1}{6+4d^2} \tag{50}$$

then there exists $\Xi_2 > 0$ such that for $\varepsilon \in (0, \Xi_2)$, $s, t \in \check{\Omega}(\omega(\varepsilon), \delta(\varepsilon))$, and $\alpha, \beta \in \{1, \ldots, d\}$, the inequality

$$\min\{|s_\alpha|, |s_\beta|, |t_\alpha|, |t_\beta|\} \geq \rho(\varepsilon), \tag{51}$$

implie

$$\left| \det \begin{pmatrix} s_\alpha & t_\alpha \\ s_\beta & t_\beta \end{pmatrix} \right| < \nu(\varepsilon) \tag{52}$$

Proof: Assume the contrary, that is, there exists a subset $\Phi_2 \subset (0, 1)$, having 0 in its closure and such that for each $\varepsilon \in \Phi_2$ there exist $Z \in Z_\varepsilon$, $s, t \in \check{\Omega}(\omega(\varepsilon), \delta(\varepsilon))$ and α, β satisfying the condition (51), and such that

$$\left| \det \begin{pmatrix} s_\alpha & t_\alpha \\ s_\beta & t_\beta \end{pmatrix} \right| \geq \nu(\varepsilon) \tag{52}$$

We apply Lemma (6.3.12) with $\{x_1, \ldots, x_{d-2}\} = \{e_i\}_{i \neq \alpha, \beta}$, $\{p_1, p_2, p_3, p_4\} = \{e_\alpha, e_\beta, s, t\}$. Using a straightforward determinant computation we see that the condition (38) is satisfied with $\chi(\varepsilon) = \min\{1, \rho(\varepsilon), \nu(\varepsilon)\} = \varepsilon^{3k}$ (we consider $\varepsilon < 1$).

The inequality (34) implies that the condition (iv) of Lemma (6.3.12) is satisfied with $\pi(\varepsilon) = C_5(d)\omega(\varepsilon) = C_5(d)\varepsilon^{4k}$ and $\sigma(\varepsilon) = c_3(d)\delta(\varepsilon) = c_3(d)\varepsilon^{4dk}$. It is clear that the conditions (ii) and (iii) of Lemma (6.3.12) are satisfied. To get (19) we use the condition (50). Applying Lemma (6.3.12), we get the existence of the desired Ξ_2.

For each vector from $\check{\Omega}(\omega(\varepsilon), \delta(\varepsilon))$ we define its top set as the set of indices of coordinates whose absolute values $\geq \rho(\varepsilon)$.

The collection of all possible top sets is a subset of the set of all subsets of $\{1,\ldots,d\}$, hence its cardinality is at most 2^d. We create a collection $\Theta(\omega(\varepsilon), \delta(\varepsilon)) \subset \breve{\Omega}(\omega(\varepsilon), \delta(\varepsilon))$ in the following way: for each subset of $\{1, \ldots, d\}$ which is a top set for at least one vector from $\breve{\Omega}(\omega(\varepsilon), \delta(\varepsilon))$, we choose one of such vectors; the set $\Theta(\omega(\varepsilon), \delta(\varepsilon))$ is the set of all vectors selected in this way.

In our next lemma we show that each vector from $\breve{\Omega}(\omega(\varepsilon), \delta(\varepsilon))$ can be reasonably well approximated by a vector from $\Theta(\omega(\varepsilon), \delta(\varepsilon))$. Therefore (as we shall see later), to prove Lemma (6.3.7) it is sufficient to find a "totally unimodular" set approximating $\Theta(\omega(\varepsilon), \delta(\varepsilon))$.

Lemma (6.3.15)[186]: Let $\rho(\varepsilon)$ and $\nu(\varepsilon)$ be as above and let k and Ξ_2 be numbers satisfying the conditions of Lemma (6.3.14). Let $\varepsilon \in (0, \boldsymbol{\Xi_2})$, $Z \in Z_\varepsilon$, and let $s, t \in \breve{\Omega}(\omega(\varepsilon), \delta(\varepsilon))$ be two vectors with the same top set Σ. Then

$$\min\{\|t+s\|, \|t-s\|\} \leq \sqrt{2\frac{\nu(\varepsilon)}{(\rho(\varepsilon))^2} + 4d\,\rho(\varepsilon)^2}. \tag{53}$$

Proof. Observe that if $\rho(\varepsilon) = \varepsilon^k > \frac{1}{\sqrt{d}}$, the statement of the lemma is trivial. Therefore we may assume that $\rho(\varepsilon) \leq \frac{1}{\sqrt{d}}$. In such a case Σ contains at least one element.

First we show that the signs of different components of s and t "agree" on Σ in the sense that either they are the same everywhere on Σ, or they are the opposite everywhere on Σ. In fact, assume the contrary, and let $\alpha, \beta \in \Sigma$ be indices for which the signs "disagree". Then, as is easy to check,

$$\left|\det\begin{pmatrix} s_\alpha & t_\alpha \\ s_\beta & t_\beta \end{pmatrix}\right| = |s_{\boldsymbol{\alpha}}||t_{\boldsymbol{\beta}}| + |s_{\boldsymbol{\beta}}||t_{\boldsymbol{\alpha}}| \geq 2(\rho(\varepsilon))^2 > \nu(\varepsilon),$$

and we get a contradiction. We consider the case when the signs of $t_{\boldsymbol{\alpha}}$ and s_α are the same for each $\alpha \in \Sigma$, the other case can be treated similarly (we can just consider $-s$ instead of s).

We may assume without loss of generality that $|t_\alpha| \geq |s_\alpha|$ for some $\alpha \in \Sigma$. We show that in this case

$$\left|t_{\beta}\right| \geq \left(1-\frac{\nu(\varepsilon)}{(\rho(\varepsilon))^{2}}\right)\left|s_{\beta}\right|$$

for all $\beta \in \Sigma$. In fact, if $|t_{\beta}| < \left(1-\frac{\nu(\varepsilon)}{(\rho(\varepsilon))^{2}}\right) |s_{\beta}|$ for some $\beta \in \Sigma$, then

$$\nu(\varepsilon) > \left|\det\begin{pmatrix} s_{\alpha} & t_{\alpha} \\ s_{\beta} & t_{\beta}\end{pmatrix}\right| \geq |t_{\alpha}|\,|s_{\beta}| - |s_{\alpha}|\,|t_{\beta}| \geq |s_{\alpha}|\,|s_{\beta}|\frac{\nu(\varepsilon)}{(\rho(\varepsilon))^{2}} \geq \nu(\varepsilon),$$

a contradiction.

We have

$$\|t-s\|^{2} = \|t\|^{2} + \|s\|^{2} - 2(t, s) \leq 2 - 2\sum_{\alpha\in\Sigma}(1-\frac{\nu(\varepsilon)}{(\rho(\varepsilon))^{2}}s_{\alpha}^{2} + 2\sum_{\alpha\notin\Sigma}\rho(\varepsilon)^{2}$$

$$\leq 2\frac{\nu(\varepsilon)}{(\rho(\varepsilon))^{2}} + 4\sum_{\alpha\notin\Sigma}\rho(\varepsilon)^{2} \leq 2\frac{\nu(\varepsilon)}{(\rho(\varepsilon))^{2}} + 4d\rho(\varepsilon)^{2}.$$

Let $\Theta(\omega(\varepsilon),\delta(\varepsilon))=\{b_j\}_{j=1}^{J}$, where $J\leq 2^{d}$. We may and shall assume that $\{e_i(\varepsilon)\}_{i=1}^{d} \subset \Theta(\omega(\varepsilon), \delta(\varepsilon))$ (see Lemma (6.3.10) We denote $d \cdot 2^{d}$ by n and introduce $d \cdot n$ functions: $\varphi_1(\varepsilon), \dots, \varphi_{d\cdot n}(\varepsilon)$, such that

$$\varphi_1(\varepsilon) \geq \cdots \geq \varphi_{d\cdot n}(\varepsilon) = \rho(\varepsilon) = \varepsilon^{k}, \tag{54}$$

$$\varphi_{\alpha}(\varepsilon) = (\varphi_{\alpha+1}(\varepsilon))^{\frac{1}{d+1}}. \tag{55}$$

We consider the matrix X whose columns are $\{b_j\}_{j=1}^{J}$ We order the absolute values of entries of this matrix in non-increasing order and denote them by $a_1 \geq a_2 \geq \cdots \geq a_{d\cdot J}$. Let j_0 be the least index for which

$$\varphi_{d\cdot j_0}(\varepsilon) > a_{j_0}. \tag{56}$$

The existence of j_0 follows from $\{e_i(\varepsilon)\}_{i=1}^{d} \subset \Theta(\omega(\varepsilon), \delta(\varepsilon))$. The definition of j_0 implies that $a_j \geq \varphi_{d\cdot j}(\varepsilon)$ for $j < j_0$, hence $a_j \geq \varphi_{d\cdot(j_0-1)}(\varepsilon)$ for $j \leq j_0-1$.

We replace all entries of the matrix X except $a_1, \dots, a_{j_0-1}$ by zeros and denote the obtained matrix by $G = (G_{ij})$, $i = 1, \dots, d$, $j = 1, \dots, J$, and its columns by $\{g_j\}_{j=1}^{J}$. It is clear that

$$\|g_j - b_j\| \le d \cdot \varphi_{d_{j_0}}(\varepsilon). \tag{57}$$

We form a bipartite graph G on the vertex set $\{\bar{1}, \ldots, \bar{d}\} \cup \{1, \ldots, J\}$, where we use bars in $\bar{1}, \ldots, \bar{d}$ because these vertices are considered as different from the vertices 1, . . . ,d, which are in the set $\{1, \ldots, J\}$. The edges of G are defined in the following way: the vertices $\bar{i}$ and j are adjacent if and only if $G_{ij} \neq 0$. So there is a one-to-one correspondence between edges of G and non-zero entries of G. We choose and fix a maximal forest F in G (We use the standard terminology, see, e. g. [90].)

For each non-zero entry of G we define its level in the following way:

(i) The level of entries corresponding to edges of *F* is 1.

(ii) For a non-zero entry of G which does not correspond to an edge in *F* we consider the cycle in G formed by the corresponding edge and edges of *F*. We define the level of the entry as the half of the length of the cycle (recall that the graph G is bipartite, hence all cycles are even).

Observation (6.3.16)[186]: One of the classes of the bipartition has d vertices. Hence no cycle can have more than 2d edges, and the level of each vertex is at most d.

To each entry G_{ij} of level f we assign a square submatrix G(ij) of G all other entries in which are of levels at most f−1. We do this in the following way. To entries corresponding to edges of F we assign the 1 × 1 matrices containing these entries. For an entry G_{ij} which does not correspond to an edge in *F* we consider the corresponding edge e in G and the cycle C formed by e and edges of *F*. Then we consider the entries in G corresponding to edges of C and the minimal submatrix in G containing all of these entries. Now we consider all edges in G corresponding to non-zero entries of this submatrix. We choose and fix in this set of edges a minimum-length cycle *M* containing e. We define G(ij) as the minimal submatrix of G containing all entries corresponding to edges of *M*. It is easy to verify that:

(i) G(ij) is a square submatrix of G.

(ii) Non-zero entries of G(ij) are in one-to-one correspondence with entries of *M*.

(iii) The expansion of the determinant of G(ij) according to the definition contains exactly two non-zero terms.

(iv) All non-zero entries of G(ij) except Gij have level $\leq f-1$.

Lemma (6.3.17)[186]: Let $k < 1/(2d+4d^2)$. If $\varepsilon > 0$ is small enough, then there exists a d×J matrix $\tilde{G}$ such that:

(i) If some entry of G is zero, the corresponding entry of $\tilde{G}$ is also zero.

(ii) The entries of level 1 of $\tilde{G}$ are the same as for G;

(iii) All other non-zero entries of $\tilde{G}$ are perturbations of entries of G satisfying the following conditions:

(I) If G_{ij} is of level f, then $|G_{ij} - \tilde{G}_{ij}| < \varphi_{d \cdot j0-f+1}(\varepsilon)$.

(II) For each non-zero entry G_{ij} of level ≥ 2 of G the determinant of the submatrix $\tilde{G}$(ij) of $\tilde{G}$ corresponding to G(ij) is zero.

Proof. Let G_{ij} be an entry of level f. Since, as it was observed above, all entries of G(ij) have level $\leq f-1$, we can prove the lemma by induction as follows.

(i) We let $\tilde{G}_{ij} = G_{ij}$ for all G_{ij} of level one.

(ii) Let $f \geq 2$.

We assume that for all entries G_{ij} of levels $\ell(G_{ij})$ satisfying $2 \leq \ell(G_{ij}) \leq f-1$ we have found perturbations $\tilde{G}_{ij}$ satisfying

$$|G_{ij} - \tilde{G}_{ij}| \leq \varphi_{j_0 - l(G_{ij})+1}(\varepsilon),$$

such that $\det(\tilde{G}(ij)) = 0$. (Note that this assumption is vacuous if $f = 2$.)

Inductive step: Let Gij be an entry of level f. If $\varepsilon > 0$ is small enough we can find a number $\tilde{G}_{ij}$ such that $|\tilde{G}_{ij} - G_{ij}| \leq \varphi d_{j_0 - f+1}(\varepsilon)$ and $\det(\tilde{G}(ij))=0$. Observe that by the induction hypothesis and the observation that all other entries of G(ij) have levels $\leq f-1$, all other entries of $\tilde{G}$ (ij) have already been defined.

So let G_{ij} be an entry of level f, and G(ij) be the corresponding square submatrix. Renumbering rows and columns of the matrix G we may assume that the matrix G(ij) looks like the one sketched below for some $h \leq f$.

$$G(ij)=\begin{pmatrix} a_1 & 0 & \dots & 0 & G_{ij} \\ b_1 & a_2 & \dots & 0 & 0 \\ . & . & & . & . \\ . & . & \ddots & . & . \\ . & . & & . & . \\ 0 & 0 & \dots & a_{h-1} & 0 \\ 0 & 0 & \dots & b_{h-1} & a_h \end{pmatrix}$$

Therefore the matrix G (possibly, after renumbering of columns and rows) has the form

$$\begin{pmatrix} a_1 & 0 & \cdots & 0 & G_{ij} & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots \\ b_1 & a_2 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & \cdots & a_{h-1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \cdots & b_{h-1} & a_h & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ * & * & \cdots & * & * & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ * & * & \cdots & * & * & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\ * & * & \cdots & * & * & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots \\ * & * & \cdots & * & * & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots \end{pmatrix} \tag{58}$$

We have assumed that we have already found entries $\{\tilde{a}_n\}_{n=1}^{h}$ *and* $\{\tilde{b}_n\}_{n=1}^{h-1}$ of $\tilde{G}$ which are perturbations of $\{a_n\}_{n=1}^{h}$ *and* $\{b_n\}_{n=1}^{h-1}$. The entries 1 shown (58) are the only non-zero entries in their columns, therefore the corresponding edges of G should be in F. Let us denote the perturbation of G_{ij} we are looking for by $\tilde{G}_{ij}$. The condition (II) of Lemma (6.3.17) can be written as

$$\prod_{n=1}^{h}\tilde{a}_n+(-1)^{h-1}\prod_{n=1}^{h-1}\tilde{b}_n.\tilde{G}_{ij}=0. \tag{59}$$

So it suffices to show that the number $\tilde{G}_{ij}$, found as a solution of (59) satisfies $|\tilde{G}_{ij}-G_{ij}|<\varphi_{d.j0-f+1}(\varepsilon)$. To show this we assume the contrary. Since there are finitely many possibilities for j_0 and f, the converse can be described as existence of j_0 and f, such that there is a subset $\Phi_3\subset(0,1)$, whose closure contains 0, satisfying the condition:

For each $\varepsilon\in\Phi_3$ there is $Z\in Z_\varepsilon$ such that after proceeding with all steps of the construction we get: all the conditions above are satisfied, but

$$\left|\prod_{n=1}^{h}\tilde{a}_n+(-1)^{h-1}\prod_{n=1}^{h-1}\tilde{b}_n.G_{ij}\right|>\varphi_{d.j0-f+1}(\varepsilon)\prod_{n=1}^{h-1}\left|\tilde{b}_n\right| \tag{60}$$

We need to get from here an estimate for |det(G(ij))| from below. To get it we observe that the inequality (60) is an estimate from below of the determinant of the matrix

$$G'(ij)=\begin{pmatrix}\tilde{a}_1 & 0 & \cdots & 0 & G_{ij}\\ \tilde{b}_1 & \tilde{a}_2 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & \tilde{a}_{h-1} & 0\\ 0 & 0 & \cdots & \tilde{b}_{h-1} & \tilde{a}_h\end{pmatrix}$$

To get from here an estimate for det(G(ij)) from below we observe the following: The ℓ_2-norm of each column of G_{ij} is ≤ 1, the l_2-distance between a column of G_{ij} and the corresponding column of G(ij) is at most $2\varphi_{dj0-f+2}(\varepsilon)$. Hence the ℓ_2-norm of each column of $G(ij)$ is $\le 1+2\varphi_{dj0-f+2}(\varepsilon)$. Applying Lemma (6.3.13) h times we get

$$|\det(G(ij))|\ge|\det(G'(ij))|-h\cdot 2\varphi_{dj0-f+2}(\varepsilon)(1+2\varphi_{dj0-f+2}(\varepsilon))^{h-1}$$

The induction hypothesis implies

$$|\tilde{b}_l|\ge\varphi_{d(j0-1)}(\varepsilon)-\varphi_{dj0-f+2}(\varepsilon),$$

we get

$$\begin{aligned}|\det(G(ij))|\ge{}&\varphi_{dj0-f+1}(\varepsilon)\cdot(\varphi_{d(j0-1)}(\varepsilon)-\varphi_{dj0-f+2}(\varepsilon))^{h-1}\\ &-h\cdot 2\varphi_{dj0-f+2}(\varepsilon)(1+2\varphi_{dj0-f+2}(\varepsilon))^{h-1}.\end{aligned} \tag{61}$$

Let us keep the notation $\{g_j\}_{j=1}^{J}$ for columns of the matrix (58). We consider the following six d×d minors of this matrix: the corresponding submatrices contain the columns $\{g_2,\ldots,g_{h-1},g_{h+1},\ldots,g_d\}$, and two out of the four columns $\{g_1,g_h,g_{d+1},g_{d+2}\}$. Observe that $g_{h+1}=e_{h+1},\ldots,g_d=e_d, g_{d+1}=e_1, g_{d+2}=e_2$.

The absolute values of the minors are equal to

$$|\det G(ij)|\ \cdot\left|\prod_{n=2}^{h}a_n\right|,\ \left|\prod_{n=1}^{h-1}b_n\right|,|a_1|\cdot\left|\prod_{n=2}^{h-1}b_n\right|,\ \left|\prod_{n=2}^{h}b_n\right|,\ \left|\prod_{n=2}^{h-1}b_n\right| \tag{62}$$

The first number in (62) was estimated in (61). All other numbers are at least

$(\varphi_{d(j0-1)}(\varepsilon))^{h-1}$, it is clear that this number exceeds the number from (61).

We are going to use Lemma (6.3.12) with $\{x_1, \ldots, x_{d-2}\} = \{N(g_2), \ldots, N(g_{h-1}), N(g_{h+1}), \ldots, N(g_d)\}$ and $\{p_1, p_2, p_3, p_4\} = \{N(g_1), N(g_h), N(g_{d+1}), N(g_{d+2})\}$. (Recall that $N(z) = z/\|z\|$.) Our definitions imply that $\|b_j\| = 1$ and $\|g_j\| \le 1$, because g_j is obtained from b_j by replacing some of the coordinates by zeros. Hence the inequality (61) and the remark above on the numbers (62) imply that the condition (38) is satisfied with

$$\chi(\varepsilon) = \varphi_{dj0-f+1}(\varepsilon) \,.\, (\varphi_{d(j0-1)}(\varepsilon) - \varphi_{dj0-f+2}(\varepsilon))^{h-1}$$

$$- h \,.\, 2\varphi_{dj0-f+2}(\varepsilon)(1 + 2\varphi_{dj0-f+2}(\varepsilon))^{h-1}. \tag{63}$$

The inequality (57), the inclusion $b_j \in \check{\Omega}(\omega(\varepsilon), \delta(\varepsilon))$ and (34) imply that the condition (37) is satisfied with $\pi(\varepsilon) = 2d \,.\, \varphi_{dj0}(\varepsilon) + C_5(d)\omega(\varepsilon)$ and $\sigma(\varepsilon) = c3(d)\delta(\varepsilon)$. So it remains to show that the condition (55) implies that the conditions (ii) and (iii) of Lemma (6.3.12) are satisfied.

By (55), (63), the inequality $2 \le h \le f \le d$, and the trivial observation that all functions $\varphi_\alpha(\varepsilon)$ do not exceed 1 for $0 \le \varepsilon \le 1$, we have

$$(\varphi_{dj0-f+1}(\varepsilon))^{d} = O(\chi(\varepsilon)). \tag{64}$$

Now we verify the condition (iii) of Lemma (6.3.12). The part (II) can be verified as follows. The conditions (54) and (38), together with $f \ge 2$ and $\omega(\varepsilon) = \varepsilon^{4k}$, imply that $\pi(\varepsilon) = O(\varphi_{dj0}(\varepsilon)) = o((\varphi_{dj0-f+1}(\varepsilon))^{d}) = o(\chi(\varepsilon))$.

To verify the condition (ii) of Lemma (6.3.12) it suffices to observe that (64) and (54) imply $(\rho(\varepsilon))^{d} = O(\chi(\varepsilon))$. Hence (ii) is satisfied if $2dk + 4d^2k < 1$. This inequality is among the conditions of Lemma (6.3.17). Hence we can apply Lemma (6.3.12) and get the conclusion of Lemma (6.3.17).

Now let $\tilde{G}$ be an approximation of G by a matrix satisfying the conditions of Lemma (6.3.17). We use the same maximal forest F in G as above. It is easy to show (and the corresponding result is well known in the theory of matroids, see, for example, [208]) that multiplying columns and rows of $\tilde{G}$ by positive numbers we can make entries corresponding to edges of F to be equal to ±1. Denote the obtained matrix by $\hat{G}$.

Lemma (6.3.18)[186]: If $\tilde{G}$ satisfies the conditions of Lemma (6.3.17)., then $\hat{G}$ is a matrix with entries −1, 0, and 1.

Proof. Assume the contrary, that is, there are entries is a matrix with entries $\hat{G}_{ij}$ which are not in the set $\{-1,0,1\}$. Let $\hat{G}_{ij}$ be one of such entries satisfying the additional condition: the level $\ell(G_{ij})$ is the minimal possible among all entries $\hat{G}_{ij}$ which are not in $\{-1, 0, 1\}$. Denote by $\hat{G}$ (ij) the submatrix of $\hat{G}$ which corresponds to G(ij).

Then, by observations preceding Lemma (6.3.17), the expansion of $\det\hat{G}$ (ij) contains two non-zero terms: one of them is 1 or −1, the other is $\hat{G}_{ij}$ or $-\hat{G}_{ij}$. Our assumptions imply that $\det\hat{G}$ (ij) ≠ 0. This contradicts det $\tilde{G}$(ij)= 0, because $\hat{G}$ is obtained from $\tilde{G}$ using multiplications of columns and rows by numbers.

In Lemma (6.3.19) we show that for functions $\varphi_\alpha(\varepsilon)$ chosen as above, the matrix $\hat{G}$ should be totally unimodular for sufficiently small ε. In Lemma (6.3.22) we show how to estimate the Banach–Mazur distance between Z and T_d in the case when $\hat{G}$ is totally unimodular.

Lemma (6.3.19)[186]: If ε > 0 is small enough, the matrix $\hat{G}$ is totally unimodular.

Observation (6.3.20)[186]: Each d×d minor of $\tilde{G}$ is a product of the corresponding minor of $\hat{G}$ and a number ζ satisfying $(\varphi_{d(j0-1)}(\varepsilon)/2)^{\mathbf{d}} \le \zeta \le 1$.

Proof. Consider a square submatrix $\tilde{S}$ in $\tilde{G}$ and the corresponding submatrix $\hat{S}$ in $\hat{G}$. If the corresponding minor is zero, there is nothing to prove. If it is non-zero, we reorder columns and rows of $\tilde{S}$ in such a way that all entries on the diagonal become non-zero, and do the same reordering with $\hat{S}$. Let τ_i , c_j > 0 be such that after multiplying rows of $\hat{S}$ by τ_i and columns of the resulting matrix by c_j we get $\hat{S}$. Then

$$\det(\tilde{S}) = \det(\hat{S})\prod_i \tau_i \prod_j c_j .$$

On the other hand, $\tau_i c_i \ge \varphi_{d(j0-1)}(\varepsilon)/2$, because the diagonal entry of $\hat{S}$ is ±1, and the absolute value of the diagonal entry of $\widetilde{S}$ is $\geq \varphi_{d(j0-1)}(\varepsilon)/2$. The conclusion follows.

Lemma (6.3.21)[186]: Let D be a d×J matrix with entries −1, 0, and 1, containing a d × d identity submatrix. If D is not totally unimodular, then it contains d+2 columns $\{\hat{x}_1 ..., \hat{x}_{d-2}, \hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4\}$ such that far all six choices of two vectors from the set $\{\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4\}$ minors obtained by joining them to $\{\hat{x}_1, ..., \hat{x}_{d-2}\}$ are non-zero.

Proof. Our argument follows [209] (see, also, [190]), where a similar statement is attributed to R. Gomory.

Suppose that D is not totally unimodular, then it has a square submatrix S with |det(S)|≥ 2. Let S be of size h × h. Reordering columns and rows of D (if necessary), we may assume that D is of the form:

$$D = \begin{pmatrix} S & 0 & I_h & * \\ * & I_{d-h} & 0 & * \end{pmatrix},$$

where I_h and I_{d-h} are identity matrices of sizes h × h and (d − h) × (d − h), respectively, 0 denote matrices with zero entries of the corresponding dimensions, and * denote matrices of the corresponding dimensions with unspecified entries.

We consider all matrices which can be obtained from D by a sequence of the following operations:

(i) Addition or subtraction a row to or from another row,

(ii) Multiplication of a column by −1,

provided that after each such operation we get a matrix with entries −1, 0, and 1.

Among all matrices obtained from D in such a way we select a matrix $\hat{D}$ which satisfies the following conditions:

(i)Has all unit vectors among its columns;

(ii) Has the maximal possible number ξ of unit vectors among the first d columns.

Observe that ξ < d because the operations listed above preserve the absolute value of the determinant and at the beginning the absolute value of the determinant formed by the first d columns was ≥ 2. Let d_r be one of the first d columns of $\hat{D}$ which is not a unit vector. Let $\{i_1, . . . , i_t\}$ be indices of its non-zero coordinates.

Then at least one of the unit vectors $e_{i_1}, \ldots, e_{i_t}$ is not among the first d columns of $\hat{D}$ (the first d columns of $\hat{D}$ are linearly independent). Assume that e_{i_1} is not among the first d columns of $\hat{D}$. We can try to transform $\hat{D}$ adding subtracting the row number i_1 to from rows number $i_2, \ldots, i_t$ (and multiplying the column number r by (-1), if necessary) into a new matrix $\tilde{D}$ which satisfies the following conditions:

(i) Has among the first d columns all the unit vectors it had before;

(ii) Has e_{i_1} as its column number r;

(iii) Has all the unit vectors among its columns.

It is not difficult to verify that the only possible obstacle is that there exists another column d_t in $\hat{D}$, such that for some $s \in \{2, \ldots, t\}$

$$\left|\det = \begin{pmatrix} D_{i_1 r} & D_{i_1 t} \\ D_{i_s r} & D_{i_s t} \end{pmatrix}\right| = 2, \tag{65}$$

where by D_{ij} we denote entries of $\hat{D}$. By the maximality assumption, a submatrix satisfying (65) exists. It is easy to see that letting

$\{\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4\} = \{d_r, d_s, e_{i_1}, e_{i_s}\}$ and $\{\hat{x}_1 \ldots, \ldots, \hat{x}_{d-2}\} = \{e_1, \ldots, e_d\} \setminus \{e_{i_1}, e_{i_s}\}$,

we get a set of columns of $\hat{D}$ satisfying the required condition.

Since the operations listed above preserve the absolute values of d×d minors, the corresponding columns of D form the desired set.

We continue our proof of Lemma (6.3.19). Assume the contrary. Since there are finitely many possible values of j_0, there is j_0 and a subset $\Phi_4 \subset (0, 1)$, whose closure contains 0, satisfying the condition:

For each $\varepsilon \in \Phi_4$ there is $Z \in Z_\varepsilon$ such that following the construction, we get the preselected value of j_0, and the obtained matrix $\hat{G}$ is not totally unimodular.

Since the entries of $\hat{G}$ are integers, the absolute values of the minors are at least one. We are going to show that the corresponding minors of G are also 'sufficiently large', and get a contradiction using Lemma (6.3.12).

By the observation above the corresponding minors of $\tilde{G}$ are at least $(\varphi_{d(j_0-1)}(\varepsilon)/2)^d$. The Euclidean norm of a column in $\tilde{G}$ is at most $1 + d\varphi_{d(j_0-1)+1}(\varepsilon)$.

Applying Lemma (6.3.13) d times we get that the corresponding minor of G are at least

$$(\varphi_{d(j0-1)}(\varepsilon)/2)^{d\varepsilon} - d^2\, \varphi_{d(j0-1)+1}(\varepsilon) \cdot (1 + d\varphi_{d(j0-1)+1}(\varepsilon))^{d-1}.$$

We are going to (use Lemma 6.3.12) for $x_1, \ldots, x_{d-2}, p_1, p_2, p_3, p_4$ defined in the following way. Let $\check{x}_1, \ldots, \check{x}_{d-2}, \check{p}_1, \check{p}_2, \check{p}_3, \check{p}_4$ be the columns of G corresponding to the columns $\hat{x}_1, \ldots, \hat{x}_{d-2}, \hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4$ of $\hat{G}$ and $x_1, \ldots, x_{d-2}, p_1, p_2, p_3, p_4$ be their normalizations (that is, $x_1 = \check{x}_1/\|\check{x}_1\|$, etc). Since norms of columns of G are ≤ 1, the condition (38) of Lemma (6.3.12) is satisfied with

$$\chi(\varepsilon) = (\varphi_{d(j0-1)}(\varepsilon)/2)^d - d^2\varphi_{d(j0-1)+1}(\varepsilon) \,.\, (1 + d\varphi_{d(j0-1)+1}(\varepsilon))^{d-1}.$$

Now we recall that columns $\{g_j\}$ of G satisfy (57) for some vectors $b_j \in \check{\Omega}(\omega(\varepsilon), \delta(\varepsilon))$. Hence the distance from $x_1, \ldots, x_{d-2}, p_1, p_2, p_3, p_4$ to the corresponding vectors b_j is $\leq 2d\varphi_{dj0}(\varepsilon)$. By (34) the condition (37) is satisfied with

$$\pi(\varepsilon) = 2d\varphi_{dj0}(\varepsilon) + C_5(d)\omega(\varepsilon)$$

and

$$\sigma(\varepsilon) = c_3(d)\delta(\varepsilon).$$

The fact that the conditions (ii) and (iii) of Lemma (6.3.12) are satisfied is verified in the same way as at the end of Lemma (6.3.17), the only difference is that instead of (64) we have $(\phi_{d(j0-1)}(\varepsilon))^d = O(\chi(\varepsilon))$. This does not affect the rest of the argument. Therefore, under the same condition on k as in Lemma (6.3.23), we get, by Lemma (6.3.12), that $\hat{G}$ should be totally unimodular if $\varepsilon > 0$ is small enough.

Lemma (6.3.22)[186]: If $\hat{G}$ is totally unimodular, then there exists a zonotope $T \in \mathcal{T}_d$ such that

$$d(Z, T) \leq t_d(\varepsilon),$$

where $t_d(\varepsilon)$ is a function satisfying $\lim_{\varepsilon \downarrow 0} t_d(\varepsilon) = 1$.

Proof. Observe that the matrix $\tilde{G}$ can be obtained from $\hat{G}$ using multiplications of rows and columns by positive numbers. Hence, re-scaling the basis $\{e_i\}$, if necessary, we get: columns of $\tilde{G}$ with respect to the re-scaled basis are of the form $a_i\tau_i$, where τ_i are columns of a totally unimodular matrix.

We are going to approximate the measure $\breve{\mu}$ by a measure $\hat{\mu}$ supported on vectors which are normalized columns of $\tilde{G}$. Recall that $\breve{\mu}$ is supported on a finite subset of $\breve{S}$.

The approximation is constructed in the following way. We erase the measure $\breve{\mu}$ supported outside $(\breve{\Omega}(\omega(\varepsilon), \delta(\varepsilon)))_{C3(d)\omega(\varepsilon)}$. The total mass of the measure erased in this way is small by (33). As for the measure supported on B:= $(\breve{\Omega}(\omega(\varepsilon), \delta(\varepsilon)))_{C3(d)\omega(\varepsilon)}$, we approximate each atom of it by the atom of the same mass supported on the nearest normalized column of $\tilde{G}$. We denote the nearest to $z \in \text{supp}\ \breve{\mu}$ normalized column of $\tilde{G}$ by A(z). If there are several such columns, we choose one of them.

Now we estimate the distance from a point of $(\breve{\Omega}(\omega(\varepsilon), \delta(\varepsilon)))_{C3(d)\omega(\varepsilon)}$ to the nearest normalized column of $\tilde{G}$. The distance from this point to $\breve{\Omega}(\omega(\varepsilon), \delta(\varepsilon))$ is $C_3(d)\omega(\varepsilon)$, the distance from a point from $\breve{\Omega}(\omega(\varepsilon), \delta(\varepsilon))$ to the point from $\Theta(\omega(\varepsilon), \delta(\varepsilon))$ with the same top set (or its opposite), by Lemma (6.3.2), can be estimated from above by $\sqrt{2\frac{v(\varepsilon)}{(\rho(\varepsilon))^2}+4d\rho(\varepsilon)^2}$ the distance from a point in $\Theta(\omega(\varepsilon), \delta(\varepsilon))$ to the corresponding column of G is estimated in (57), it is $\leq d\,.\varphi_{dj0}(\varepsilon)$, so it is $\leq d\,.\varphi_1(\varepsilon)$, and the distance from a column of G to the corresponding column of $\tilde{G}$ is $\leq d.\varphi_{d(j0-1)+1}(\varepsilon) \leq d.\varphi_1(\varepsilon)$. Since we have to normalize this vector, the total distance from a point of $(\breve{\Omega}(\omega(\varepsilon), \delta(\varepsilon)))_{C3(d)\omega(\varepsilon)}$ to the nearest normalized column of $\tilde{G}$ can be estimated from above by

$$C_3(d)\omega(\varepsilon)+\sqrt{2\frac{v(\varepsilon)}{(\rho(\varepsilon))^2}+4d\rho(\varepsilon)^2}+4d.\varphi_1(\varepsilon).$$

It is clear that this function, let us denote it by $\zeta(\varepsilon)$, tends to 0 as $\varepsilon \downarrow 0$, recall that $\rho(\varepsilon) = e^k$, $\nu(\varepsilon) = \varepsilon^{3k}$, $\omega(\varepsilon) = \varepsilon^{4k}$, $\phi_1(\varepsilon) = \varepsilon^{\left(\frac{1}{d+1}\right)^{d_{n-1}}}$. The obtained measure corresponds to a zonotope from T_d. Let us denote this zonotope by T.

Since the dual norms to the gauge functions of Z and T are their support functions, we get the estimate

$$d(T,Z) \le \sup_{u \in \breve{S}} \frac{\breve{h}_Z(u)}{\breve{h}_T(u)} \cdot \sup_{u \in \breve{S}} \frac{\breve{h}_T(u)}{\breve{h}_Z(u)}.$$

So it is enough to show that

$$C_1(d,\varepsilon) \le \frac{\breve{h}_T(u)}{\breve{h}_Z(u)} \le C_2(d,\varepsilon), \tag{66}$$

where $\lim_{\varepsilon\downarrow 0} C_1(d, \varepsilon) = \lim_{\varepsilon\downarrow 0} C_2(d, \varepsilon) = 1$.

Observe that Lemma (6.3.10) implies that there exists a constant $0 < C_7(d) < \infty$ such that

$$C_7(d) \le \breve{h}_Z(u), \quad \forall u \in \breve{S}. \tag{67}$$

We have

$$\begin{aligned}
\breve{h}_Z(u) &= \int_{\breve{S}} |\langle u,z \rangle| d\breve{\mu}(z) \\
&\le \int_{\breve{S}\setminus B} |\langle u,z \rangle| d\breve{\mu}(z) + \int_{\breve{S}} |\langle u,z \rangle| d\hat{\mu}(z) + \sum_{z \in \operatorname{supp} \breve{\mu} \cap B} (|\langle u,z \rangle - \langle (u, A(z) \rangle|) \breve{\mu}(z) \\
&\le C_4(d) \frac{\delta(\varepsilon)}{\omega^{d-1}(\varepsilon)} + \breve{h}_T(u) + \zeta(\varepsilon)\breve{\mu}(\breve{S}), \qquad \forall u \in \breve{S}.
\end{aligned}$$

In a similar way we get

$$\begin{aligned}
\breve{h}_T(u) &= \int_{\breve{S}} |\langle u,z \rangle| d\hat{\mu}(z) \le \int_{B} |\langle u,z \rangle| d\breve{\mu}(z) + \sum_{z \in \operatorname{supp} \breve{\mu} \cap B} (|\langle u,z \rangle - \langle u, A(z) \rangle|) \breve{\mu}(z) \\
&\le \breve{h}_Z(u) + \zeta(\varepsilon)\breve{\mu}(\breve{S}), \qquad \forall u \in S.
\end{aligned}$$

Using (67) we get

$$1 - \frac{C_4(d)\dfrac{\delta(\varepsilon)}{\omega^{d-1}(\varepsilon)}}{C_7(d)} - \frac{\zeta(\varepsilon)\breve{\mu}(\breve{S})}{C_7(d)} \le \frac{\breve{h}_T(u)}{\breve{h}_Z(u)} \le 1 + \frac{\zeta(\varepsilon)\breve{\mu}(\breve{S})}{C_7(d)}.$$

It is an estimate of the form (66).

It is clear that Lemma (6.3.22) completes our proof of Lemma (6.3.7).

Theorem (6.3.23)[186]: Let X be a finite-dimensional normed linear space having an MVSE that is not a parallelepiped. Then X contains a two-dimensional subspace whose unit ball is linearly equivalent to the regular hexagon.

Proof. We start by proving Theorem (6.3.5) for polyhedral X. In this case we can consider X as a subspace of ℓ_∞^m for some m$\in$**N**. Since X has an MVSE which is not a parallelepiped, there exists a linear projection P: $\ell_\infty^m \to X$ such that P (B_∞^m) has the minimal possible volume, but P (B_∞^m) is not a parallelepiped. Let d = dim X, let $\{q_1, \ldots, q_{m-d}\}$ be an orthonormal basis in ker P and let $\{\tilde{q}_1, \ldots, \tilde{q}_d\}$ be an orthonormal basis in the orthogonal complement of ker P . As it was shown in Lemma (6.3.8), P(B_∞^m) is linearly equivalent to the zonotope spanned by rows of $\tilde{Q}$ = $[\tilde{q}_1, \ldots, \tilde{q}_d]$. By the assumption this zonotope is not a parallelepiped. It is easy to see that this assumption is equivalent to: there exists a minimal linearly dependent collection of rows of $\tilde{Q}$ containing ≥3 rows. This condition implies that we can reorder the coordinates in ℓ_∞^m and multiply the matrix $\tilde{Q}$ from the right by an invertible d × d matrix C_1 in such a way that $\tilde{Q}\, C_1$ has a submatrix of the form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_d \end{pmatrix},$$

where $a_1 \neq 0$ and $a_2 \neq 0$. Let χ be a matrix whose columns form a basis of X. The argument of [205] implies that χ can be multiplied from the right by an invertible d × d matrix C_2 in such a way that χC_2 is of the form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ sign\, a_1 & sign\, a_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}$$

where at the top there is an d × d identity matrix, and all minors of the matrix χC_2 have absolute values ≤ 1.

Changing signs of the first two columns, if necessary, we get that the subspace X $\subset \ell_\infty^m$ is spanned by columns of the matrix

$$\begin{pmatrix} \pm 1 & 0 & 0 & \cdots & 0 \\ 0 & \pm 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 1 & * & \cdots & * \\ b_1 & c_1 & * & \cdots & * \\ b_2 & c_2 & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{m-l-1} & c_{m-l-1} & * & \cdots & * \end{pmatrix}$$

The condition on the minors implies that $|b_i| \le 1$, $|c_i| \le 1$, and $|b_i - c_i| \le 1$ for each i. Therefore the subspace, spanned in ℓ_∞^m by the first two columns of the matrix (68) is isometric to $\mathbf{R}^2$ with the norm

$$\|(\alpha, \beta)\| = \max(|\alpha|, |\beta|, |\alpha + \beta|).$$

It is easy to see that the unit ball of this space is linearly equivalent to a regular hexagon. Thus, Theorem (6.3.23) is proved in the case when X is polyhedral.

Proving the result for general, not necessarily polyhedral, space, we shall denote the space by Y. We use Theorem (6.3.5). Actually we need only the following corollary of it:

Each MVSE is a polyhedron.

Lemma (6.3.24)[186]: Let Y be a finite dimensional space and let A be a polyhedral MVSE for Y . Then there exists another norm on Y such that the obtained normed space X satisfies the conditions:

(i) X is polyhedral;

(ii) $B_X \supset B_Y$;

(iii) A is an MVSE for X.

So we consider the space Y as being embedded into a polyhedral space X with the embedding satisfying the conditions of Lemma (6.3.24).

By the first part of the proof the space X satisfies the conditions of Theorem (6.3.23) and we may assume that X is a subspace ℓ_∞^m in the way described in the first part of the proof. So X is spanned by columns - let us denote them by $e_1, \ldots, e_d$ - of the matrix (68) in ℓ_∞^m . It is easy to see that to finish the proof it is enough to show that the vectors $e_1, e_2, e_1 - e_2$ are in B_Y .

It turns out each of these points is the center of a facet of a minimum-volume paral-lelepiped containing B_X . In fact, let $\{f_i\}_{i=1}^m$ be the unit vector basis of ℓ_∞^m . Let P_1 and P_2 be the projections onto Y with the kernels $\mathrm{lin}\{f_{d+1}, \ldots, f_m\}$ and $\mathrm{lin}\{f_1, f_{d+2}, \ldots, f_m\}$, respectively (recall that Y, as a linear space, coincides with X). The analysis from [89] shows that $P_1(B_\infty^m)$ and $P_2(B_\infty^m)$ have the minimal possible volume among all linear projections of B_∞^m into X. It is easy to see that $P_1(B_\infty^m)$ and $P_2(B_\infty^m)$ are parallelepipeds.

We show that e_1, e_2 are centers of facets of $P_1(B_\infty^m)$, and that $e_1 - e_2$ is the center of a facet of $P_2(B_\infty^m)$. In fact, the centers of facets of $P_1(B_\infty^m)$ coincide with $P_1(f_1), \ldots, P_1(f_d)$, and it is easy to check that $P_1(fi) = e_i$ for $i = 1, \ldots, d$. As for P_2, we observe that $e_1 - e_2 \in \mathrm{lin}\{f_1, f_2, f_{d+2}, \ldots, f_m\}$, and the coefficient near f_2 in the expansion of $e_1 - e_2$ is ± 1. Therefore $P_2(f_2) = \pm(e_1 - e_2)$.

Since the projections P_1 and P_2 satisfy the minimality condition from [174] (see, also [89]), the parallelepipeds $P_1(B_\infty^m)$and $P_2(B_\infty^m)$ are MVSE for X. Hence, by the conditions of Lemma (6.3.24), they are MVSE for Y also. Hence, they are minimum-volume parallelepipeds containing B_Y. On the other hand, it is known, see [210], that centers of facets of minimal-volume parallelepipeds containing B_Y should belong to B_Y , we get $e_1, e_2, e_1 - e_2 \in B_Y$.

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