Chapter 3

Moment Identities and Random Hermite Polynomials

We recover and extend the sufficient conditions for the invariance of the Wiener measure under random rotations given. As an application we recover, under simple conditions and with short proofs, the anticipative Girsanov identity and quasi-invariance results obtained for quasi-nilpotent shifts on the Wiener space.

Section (3.1): Skorohod Integrals on the Wiener Space:

In [61], sufficient conditions have been found for the Skorohod integral $\delta(Rh)$ to havea Gaussian law when $h \in H = L^2(\mathbb{R}_+, \mathbb{R}^d)$ and R is a random isometry of H, usingan induction argument.

We state a general identity for the moments of Skorohod integrals, which will allow us in particular to recover the result of [61] by a direct proof and to obtain a recurrence relation for the moments of second order Wiener integrals.

We refer to [62] and [63] for the notation recalled in this section. Let $(B_t)_{t\in\mathbb{R}_+}$ denote a standard $\mathbb{R}^d$-valued Brownian motion on the Wiener space $(W,\mu)$with $W = C_0(\mathbb{R}_+, \mathbb{R}^d)$. For any separable Hilbert space X, consider the Malliavin derivative $D$with values in $H = L^2(\mathbb{R}_+, X \otimes \mathbb{R}^d)$, defined by

$$D_t F = \sum_{i=1}^{n} \mathbf{1}_{[0,t_i]}(t)\partial_i f(B_{t_1}, \dots, B_{t_n}), \qquad t \in \mathbb{R}_+,,$$

for F of the form

$$F = f(B_{t_1}, \dots, B_{t_n}), \tag{1}$$

$f \in C_b^\infty(\mathbb{R}^n, X), t_1, \dots, t_n \in \mathbb{R}_+, n \geq 1$. Let $\mathbb{D}_{p,k}(X)$ denote the completion of the space of smooth X-valued random variables under the norm

$$\|u\|_{\mathbb{D}_{p,k}(X)} = \sum_{l=0}^{k} \|D^l u\|_{L^p(W, X\otimes H^{\otimes l})}, \qquad p > 1,$$

where $X \otimes H$ denotes the completed symmetric tensor product of X and H. For all $p, q > 1$ such that $p^{-1} + q^{-1} = 1$ and $k \geq 1$, let

$$\delta : \mathbb{D}_{p,k}(X)(X \otimes H) \to \mathbb{D}_{q,k^{-1}}(X)$$

denote the Skorohod integral operator adjoint of

$$D : \mathbb{D}_{p,k}(X) \to \mathbb{D}_{q,k^{-1}}(X \otimes H),$$

with

$$E[\langle F, \delta(u)\rangle_X] = E\big[\langle DF, u\rangle_{X\otimes H}\big], \qquad F \in \mathbb{D}_{p,k}(X), \;\; u \in \mathbb{D}_{q,k}(X \otimes H).$$

Recall that $\delta(u)$ coincides with the Itô integral of $u \in L^2(W; H)$ with respect to Brownian motion, i.e.

$$\delta(u) = \int_0^\infty u_t dB_t,$$

When u is square-integrable and adapted with respect to the Brownian filtration.

Each element of $X \otimes H$is naturally identified to a linear operator from H to X via

$$(a \otimes b)c = a\langle b, c\rangle, \qquad a \otimes b \in X \otimes H, \qquad c \in H.$$

For $u \in \mathbb{D}_{2,1}(H)$ we identify $Du = (D_t u_s)_{s,t\in\mathbb{R}_+}$ to the random operator $Du : H \to H$ almost surely defined by

$$(Du)v(s) = \int_0^\infty (D_t u_s) v_t dt, \quad s \in \mathbb{R}_+, \qquad v \in L^2(W; H),$$

and define its adjoint D^*u on $H \otimes H$as

$$(D^*u)v(s) = \int_0^\infty \big(D_s^\dagger u_t\big) v_t dt, \quad s \in \mathbb{R}_+, \qquad v \in L^2(W; H),$$

where $D_s^\dagger u_t$ denotes the transpose matrix of $D_t u_s$in $\mathbb{R}^d \otimes \mathbb{R}^d$.

Recall the Skorohod [64] isometry

$$E[\delta(u)^2] = E[\langle u, u\rangle_H] + E[\mathrm{trace}(Du)^2], \qquad u \in \mathbb{D}_{2,1}(H), \tag{2}$$

with

$$\begin{aligned} \mathrm{trace}(Du)^2 &= \langle Du, D^*u\rangle_{H\otimes H} \\ &= \int_0^\infty \int_0^\infty \langle D_s u_t, D_s^\dagger u_t\rangle_{\mathbb{R}^d\otimes\mathbb{R}^d} dsdt, \end{aligned}$$

and the commutation relation

$$D\delta(u) = u + \delta(D^*u), \qquad u \in \mathbb{D}_{2,2}(H). \tag{3}$$

First we state a moment identity for Skorohod integrals, which will be proved.

Corollary (3.1.1)[60]: Let $n \geq 1$and $u \in \mathbb{D}_{n+1,2}(H)$ such that $\langle u, u\rangle_H$ is deterministic and

$$\operatorname{trace}(Du)^{k+1} + \sum_{i=2}^{k} \frac{1}{i} \langle (Du)^{k-i}u, D\operatorname{trace}(Du)^i\rangle_H = 0, \quad a.s., \quad 1 \leq k \leq n. \qquad (4)$$

Then $\delta(u)$ has the same first $n+1$ moments as the centered Gaussian distribution with variance $\langle u, u\rangle_H$.

Proof. The relation $D\langle u, u\rangle = 2(D^*u)u$ shows that

$$\langle (D^{k-1}u)u, u\rangle = \langle (D^*u)^{k-1}u, u\rangle = \frac{1}{2}\langle u, (D^*)^{k-2}D\langle u, u\rangle\rangle = 0, \qquad k \geq 2, \qquad (5)$$

when $\langle u, u\rangle$ is deterministic, $u \in \mathbb{D}_{2,1}(H)$. Hence under Condition (4), Theorem (3.1.3) yields

$$E\left[\left(\delta(u)\right)^{n+1}\right] = n\langle u, u\rangle_H E\left[\left(\delta(u)\right)^{n-1}\right],$$

and by induction

$$E\left[\left(\delta(u)\right)^{2m}\right] = \frac{(2m)!}{2^m m!}\langle u, u\rangle_H^m, \qquad 0 \leq 2m \leq n+1,$$

and $E\left[\left(\delta(u)\right)^{2m+1}\right] = 0, 0 \leq 2m \leq n$, while $E[\delta(u)] = 0$ for all $u \in \mathbb{D}_{2,1}(H)$.

We close this section with some applications.

(i) Random rotations

As a consequence of Corollary (3.1.1) we recover Theorem (3.1.3)-b) of [61], i.e. $\delta(Rh)$ has a centered Gaussian distribution with variance $\langle h, h\rangle_H$ when $u = Rh, h \in H$,and $R$is a random mapping with values in the isometries of H, such that $Rh \in \bigcap_{p>1} \mathbb{D}_{p,2}(H)$ and trace $(DRh)^{k+1} = 0, k \leq 1$. Note that in [61] the condition $Rh \in \bigcap_{p>1,k\geq 2} \mathbb{D}_{p,k}(H)$ is assumed instead of $Rh \in \bigcap_{p>1} \mathbb{D}_{p,2}(H)$.

(ii) Second order Wiener integrals

Let $d = 1$. The second order Wiener integral $I_2(f_2)$ of a symmetric function $f_2 \in H \otimes H = L^2(\mathbb{R}_+^2)$ can be written as $I_2(f_2) = \delta(u)$ with $u_t = \delta\big(f_2(\cdot, t)\big), t \in \mathbb{R}_+$. Its law is infinitely divisible with Lévy measure

$$\nu(dy) = \mathbf{1}_{\{y>0\}} \sum_{k;a_k>0} \frac{1}{2|y|} e^{-\frac{y}{a_k}} dy + \mathbf{1}_{\{y>0\}} \sum_{k;a_k<0} \frac{1}{2|y|} e^{-\frac{y}{a_k}} dy, (6)$$

when f_2 is decomposed as

$$f_2 = \frac{1}{2}\sum_{k=0}^{\infty} a_k h_k \otimes h_k$$

in a complete orthonormal basis $(h_k)_{k\in\mathbb{N}}$ of H. Letting

$$g_2^{(k+1)}(s,t) = \int_{\mathbb{R}^k} f_2(s,t_1)f_2(t_1,t_2)\cdots f_2(t_{k-1},t_k)f_2(t_k,t)\cdots dt_1\cdots dt_k,$$

we have trace $(Du)^{k+1} = \int_{\mathbb{R}^2} g_2^{(k+1)}(s,t)dsdt$, and using the relation

$$\delta(f_1)\delta(g_1) = I_2(f_1\otimes g_1) + \langle f_1, g_1\rangle_H, \qquad f_1, g_1 \in H,$$

we get

$$\langle (Du)^{k-1}u, u\rangle_H = \int_{\mathbb{R}^{k-1}} \delta\big(f_2(\cdot,t_1)\big)f_2(t_1,t_2)\cdots f_2(t_{k-1},t_k)\delta\big(f_2(\cdot,t_k)\big)\cdots dt_1\cdots dt_k,$$

$$= \int_{\mathbb{R}^{k-1}} I_2\big(f_2(\cdot,t_1)\otimes f_2(\cdot,t_k)\big)f_2(t_1,t_2)\cdots f_2(t_{k-1},t_k)dt_1\cdots dt_k$$

$$+ \int_{\mathbb{R}^{k-1}} f_2(t_0,t_1)f_2(t_1,t_2)\cdots f_2(t_{k-1},t_k)f_2(t_k,t_0)dt_0\cdots dt_k$$

$$= I_2\Big(g_2^{(k+1)}\Big) + \mathrm{trace}(Du)^{k+1},$$

hence Theorem (3.1.3) below yields the recurrence relation

$$E\left[\big(I_2(f_2)\big)^{n+1}\right] = \sum_{k=1}^{n}\frac{n!}{(n-k)!}E\left[\big(I_2(f_2)\big)^{n-k}\left(I_2\Big(g_2^{(k+1)}\Big) + 2\,\mathrm{trace}(Du)^{k+1}\right)\right]$$

$$= 2\sum_{k=0}^{n-1}\frac{n!}{k!}\int_{\mathbb{R}^2} g_2^{(n-k+1)}(s,t)dsdtE\left[\big(I_2(f_2)\big)^{k}\right]$$

$$+\sum_{k=0}^{n-1}\sum_{l=1}^{k}\frac{(-1)^{k+1-l}n!}{(k)!\,(k+1)!}\binom{k}{l}l^{k+1}E\left[\big(I_2(f_2)\big)^{k+1}\right]$$

$$+\sum_{k=0}^{n-1}\sum_{l=1}^{k}\frac{(-1)^{k+1-l}n!}{(k)!\,(k+1)!}\binom{k}{l-1}E\left[I_2\left((l-1)f_2 + g_2^{(n-k+1)}\right)^{k+1}\right],$$

for the computation of the moments of second order Wiener integrals, by polarisation of $\big(I_2(f_2)\big)^{n-k}I_2\Big(g_2^{(n-k+1)}\Big)$.

In the sequel, all scalar products will be simply denoted by$\langle\cdot,\cdot\rangle$.

We will need the following lemma.

Lemma (3.1.2)[60]: Let $n \geq 1$ and $u \in \mathbb{D}_{n+1,2}(H)$. Then for all $1 \leq k \leq n$ we have

$$E\left[\left(\delta(u)\right)^{n-k}\langle(Du)^{k-1}u, D\delta(u)\rangle\right] - (n-k)\left[E\left(\delta(u)\right)^{n-k-1}\langle(Du)^{k}u, D\delta(u)\rangle\right]$$

$$= E\left[\left(\delta(u)\right)^{n-k}\left(\langle(Du)^{k-1}u, u\rangle + \operatorname{trace}(Du)^{k+1} + \sum_{i=2}^{k}\frac{1}{i}\langle(Du)^{k-i}u, D\operatorname{trace}(Du)^{i}\rangle\right)\right].$$

Proof. We have $(Du)^{k-1}u \in \mathbb{D}_{(n+1)/k,1}(H), \delta(u) \in \mathbb{D}_{(n+1)/(n-k+1),1}(\mathbb{R})$, and using Relation (8) we obtain

$$E\left[\left(\delta(u)\right)^{n-k}\langle(Du)^{k-1}u, D\delta(u)\rangle\right]$$

$$= E\left[\left(\delta(u)\right)^{n-k}\langle(Du)^{k-1}u, u + \delta(D^{*}u)\rangle\right]$$

$$= E\left[\left(\delta(u)\right)^{n-k}\langle(Du)^{k-1}u, u\rangle\right] + E\left[\left(\delta(u)\right)^{n-k}\langle(Du)^{k-1}u, \delta(Du)\rangle\right]$$

$$= E\left[\left(\delta(u)\right)^{n-k}\langle(Du)^{k-1}u, u\rangle\right] + E\left[\langle D^{*}u, D\left(\left(\delta(u)\right)^{n-k}(Du)^{k-1}u\right)\rangle\right]$$

$$= E\left[\left(\delta(u)\right)^{n-k}\langle(Du)^{k-1}u, u\rangle\right] + E\left[\left(\delta(u)\right)^{n-k}\langle D^{*}u, D((Du)^{k-1}u)\rangle\right]$$

$$+E\left[(D^{*}u, (Du)^{k-1}u) \otimes D\left(\delta(u)\right)^{n-k}\right]$$

$$= E\left[\left(\delta(u)\right)^{n-k}(\langle(Du)^{k-1}u, u\rangle + \langle D^{*}u, D((Du)^{k-1}u)\rangle)\right]$$

$$+(n-k)E\left[\left(\delta(u)\right)^{n-k-1}\langle D^{*}u, ((Du)^{k-1}u) \otimes D\delta(u)\rangle\right]$$

$$= E\left[\left(\delta(u)\right)^{n-k}(\langle(Du)^{k-1}u, u\rangle + \langle D^{*}u, D((Du)^{k-1}u)\rangle)\right]$$

$$+(n-k)E\left[\left(\delta(u)\right)^{n-k-1}\langle(Du)^{k}u, D\delta(u)\rangle\right].$$

Next,

$$\langle D^{*}u, D((Du)^{k-1}u)\rangle = \int_0^\infty \cdots \int_0^\infty \langle D^{\dagger}_{t_{k-1}}u_{t_k}, D_{t_k}\left(D_{t_{k-2}}u_{t_{k-1}} \cdots D_{t_0}u_{t_1}D_{t_k}u_{t_0}\right)\rangle dt_0 \cdots dt_k$$

$$= \int_0^\infty \cdots \int_0^\infty \langle D^{\dagger}_{t_{k-1}}u_{t_k}, D_{t_{k-2}}u_{t_{k-1}} \cdots D_{t_0}u_{t_1}D_{t_k}u_{t_0}\rangle dt_0 \cdots dt_k$$

$$+\int_0^\infty \cdots \int_0^\infty \langle D^\dagger_{t_{k-1}} u_{t_k}, D_{t_k}\big(D_{t_{k-2}} u_{t_{k-1}} \cdots D_{t_0} u_{t_1}\big) u_{t_0} \rangle dt_0 \cdots dt_k$$

$$= \operatorname{trace}(Du)^{k+1} + \sum_{i=0}^{k-2} \int_0^\infty \cdots \int_0^\infty$$

$$\langle D^\dagger_{t_{k-1}} u_{t_k}, D_{t_k} u_{t_{k-1}} \cdots D_{t_{i+1}} u_{t_{i+2}} \big(D_{t_i} D_{t_k} u_{t_{i+1}}\big) D_{t_{i-1}} u_{t_i} \cdots D_{t_0} u_{t_1} u_{t_0} \rangle dt_0 \cdots dt_k$$

$$= \operatorname{trace}(Du)^{k+1} + \sum_{i=0}^{k-2} \frac{1}{k-i} \int_0^\infty \cdots \int_0^\infty$$

$$\langle D_{t_i} \langle D^\dagger_{t_{k-1}} u_{t_k}, D_{t_k} u_{t_{k-1}} \cdots D_{t_{i+1}} u_{t_{i+2} D_{t_k} u_{t_{i+1}}} \rangle, D_{t_{i-1}} u_{t_i} \cdots D_{t_0} u_{t_1} u_{t_0} \rangle \, dt_0 \cdots dt_k$$

$$= \operatorname{trace}(Du)^{k+1} + \sum_{i=0}^{k-2} \frac{1}{k-i} \langle (Du)^i u, D \operatorname{trace}(Du)^{k-i} \rangle.$$

Theorem (3.1.3)[60]: For any $n \geq 1$ and $u \in \mathbb{D}_{n+1,2}(H)$ we have

$$E\left[\big(\delta(u)\big)^{n+1}\right] = \sum_{k=1}^{n} \frac{n!}{(n-k)!}$$

$$E\left[\big(\delta(u)\big)^{n-k} \left(\langle (Du)^{k-1} u, u \rangle_H + \operatorname{trace}(Du)^{k+1} + \sum_{i=2}^{k} \frac{1}{i} \langle (Du)^{k-i} u, D \operatorname{trace}(Du)^i \rangle_H, \right)\right] \quad (7)$$

Where

$$\operatorname{trace}(Du)^{k+1} = \int_0^\infty \cdots \int_0^\infty \langle D^\dagger_{t_{k-1}} u_{t_k}, D_{t_{k-2}} u_{t_{k-1}} \cdots D_{t_0} u_{t_1} D_{t_k} u_{t_0} \rangle_{\mathbb{R}^d \otimes \mathbb{R}^d} dt_0 \cdots dt_k.$$

For $n = 1$ the above identity coincides with the Skorohod isometry (2).

In particular we obtain the following immediate consequence of Theorem (3.1.3). Recall that $\operatorname{trace}(Du)^k = 0, k \geq 1$, when the process u is adapted with respect to the Brownian filtration.

Proof. We decompose

$$E\left[\big(\delta(u)\big)^{n+1}\right] = E\left[\langle u, D\big(\delta(u)\big)^n \rangle\right] = nE\left[\big(\delta(u)\big)^{n-1} \langle u, D\delta(u) \rangle\right]$$

$$= \sum_{k=1}^{n} \frac{n!}{(n-k)!} \left(E\left[\big(\delta(u)\big)^{n-k} \langle (Du)^{k-1} u, D\delta(u) \rangle\right] - (n-k) E\left[\big(\delta(u)\big)^{n-k-1} (Du)^k u, D\delta(u)\right]\right),$$

as a telescoping sum and then apply Lemma (3.1.2), which yields (7).

Finally we state some other consequences of Theorem (3.1.3).

Corollary (3.1.4)[60]: Let $n \geq 1$ and $u \in \mathbb{D}_{n+1,2}(H)$, and assume that

$$\operatorname{trace}(Du)^{k+1} + \sum_{i=2}^{k} \frac{1}{i} \langle (Du)^{k-i}u, D \operatorname{trace}(Du)^i \rangle = 0, \qquad 1 \leq k \leq n. \tag{8}$$

Then we have

$$E\left[\left(\delta(u)\right)^{n+1}\right] = \sum_{k=1}^{n} \frac{n!}{(n-k)!} E\left[\left(\delta(u)\right)^{n-k} \langle (Du)^{k-1}u, u \rangle\right].$$

Corollary (3.1.5)[60]: Let $n \geq 1$and $u \in \mathbb{D}_{n+1,2}(H)$ such that $\langle u, u \rangle$ is deterministic. We have

$$E\left[\left(\delta(u)\right)^{n+1}\right] = n\langle u, u \rangle E\left(\delta(u)\right)^{n-1}$$
$$+ \sum_{k=1}^{n} \frac{n!}{(n-k)!} E\left[\left(\delta(u)\right)^{n-k} \left(\operatorname{trace}(Du)^{k+1} + \sum_{i=2}^{k} \frac{1}{i} \langle (Du)^{k-i}u, D \operatorname{trace}(Du)^i \rangle\right)\right].$$

Section (3.2): Girsanov Identities on the Wiener Space:

It is well known that the Hermite polynomial

$$H_n(x, \mu) = \sum_{0 \leq 2k \leq n} \frac{n! \, (-\mu/2)^k}{k! \, (n-2k)!} x^{n-2k}, \qquad x \in \mathbb{R} \tag{9}$$

With parameter $\mu \in \mathbb{R}$ and generating function

$$e^{tx - t^2\mu/2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x, \mu), \qquad x, t \in \mathbb{R}, \tag{10}$$

Satisfies the identity

$$E[H_n(X, \sigma^2)] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} H_n(X, \sigma^2) e^{-x^2/(2\sigma^2)} dx = 0, \qquad n \geq 1, \tag{11}$$

Where $X \simeq \mathcal{N}(0, \sigma^2)$ is a centered Gaussian random variable with variance $\sigma^2 \geq 0$, since

$$E[H_n(X, \sigma^2)] = \sum_{k=0}^{n} \frac{(2n)! \, (-\sigma^2/2)^k}{k! \, (2n-2k)!} E[X^{2n-2k}]$$

$$= \frac{(2n)!}{n!} \sum_{k=0}^{n} \binom{n}{k} (-\sigma^2/2)^k (\sigma^2/2)^{n-k} = 0$$

From the even Gaussian moment $E[X^{2m}] = (\sigma^2/2)^m (2m)!/m!, m \geq 0$.

The identity (11) holds in particular when X is written as the stochastic integral

$$X = \int_0^\infty f(s) dB_s$$

of a deterministic real-valued function f with respect to the standard Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ and σ^2 is the constant $\sigma^2 = \int_0^\infty |f(s)|^2 ds$.

It is well known, however, that the Gaussianity of X is not required for $E[H_n(X, \sigma^2)]$ to vanish when σ^2 is allowed to be random. Indeed, such an identity also holds in the random adapted case under the form

$$E\left[H_n\left(\int_0^\infty u_t dB_t, \int_0^\infty |u_t|^2 dt\right)\right] = 0, \tag{12}$$

Where $(u_t)_{t\in\mathbb{R}_+}$ is a square-integrable process adapted to the filtration generated by $(B_t)_{t\in\mathbb{R}_+}$, due to the fact that

$$H_n\left(\int_0^\infty u_t dB_t, \int_0^\infty |u_t|^2 dt\right) = n! \int_0^\infty u_{t_n} \int_0^{t_n} u_{t_{n-1}} \int_0^{t_2} u_{t_1} dB_{t_1} \cdots dB_{t_n},$$

is the n-th order iterated multiple stochastic integral of $u_{t_1} \cdots u_{t_n}$ with respect to $(B_t)_{t\in\mathbb{R}_+}$, cf. [66] and [67].

We prove an extension of (12) to the random case, by computing in Theorem (3.2.3) below the expectation

$$E[H_n(\delta(u), \|u\|^2)], \qquad n \geq 1,$$

of the random Hermite polynomial $H_n(\delta(u), \|u\|^2)$, where $\delta(u)$ is the Skorohod integral of a possibly anticipating process $(u_t)_{t\in\mathbb{R}_+}$. In particular we provide conditionson the process $(u_t)_{t\in\mathbb{R}_+}$ for the expectation $E[H_n(\delta(u), \|u\|^2)], n \geq 1$, to vanish. Suchconditions cover the quasi-nilpotence condition of [68] and include the adaptedness of $(u_t)_{t\in\mathbb{R}_+}$, which recovers (12) as a particular case since $\delta(u)$ coincides with the Itô integral when $(u_t)_{t\in\mathbb{R}_+}$ is adapted.

Indeed, it is well known that in the adapted case, (12) and (10) can be used for the proof of the (adapted) Girsanov identity

$$E\left[\exp\left(\int_0^\infty u_t dB_t - \frac{1}{2}\int_0^\infty |u_t|^2 dt\right)\right] = 1,$$

Under the Novikov type condition

$$E\left[\exp\left(\frac{1}{2}\int_0^T |u_t|^2 dt\right)\right] < \infty.$$

Similarly we recover, under simple conditions and with short proofs, the anticipating Girsanov identity obtained in [68] for quasi-nilpotent anticipative shifts of Brownian motion. This also simplifies the proof of classical results on the quasi-invariance of Euclidean motions [70], and on the invariance of random rotations.

The results of this section can be formally summarized by the derivation formula

$$\frac{\partial}{\partial t}E\left[e^{t\delta(u)-\frac{t^2}{2}\|u\|^2}\right] = tE\left[e^{t\delta(u)-t^2\langle u,u\rangle/2}\langle D^*u, D(I_H - tDu)^{-1}u\rangle\right], \qquad (13)$$

For t in a neighborhood of 0, cf. Relation (23) below, where $D$and δ respectively denote the Malliavin gradient and Skorohod integral, showing that

$$E\left[\exp\left(\delta(u) - \frac{1}{2}\|u\|^2\right)\right] = 1,$$

Provided$\langle D^*u, D(I_H - tDu)^{-1}u\rangle = 0$,

For t in a neighborhood of 0, cf. Corollary (3.2.2) below for a formal statement.

We refer to [72] and to Appendix B in [70] for the notation recalled in this section. Let$(B_t)_{t\in\mathbb{R}_+}$denote a standard $\mathbb{R}^d$-valued Brownian motion on the Wiener space (W, P) with $W = C_0(\mathbb{R}_+, \mathbb{R}^d)$. For any separable Hilbert space X, consider the Malliavin derivative D with values in $H = L^2(\mathbb{R}_+, X \otimes \mathbb{R}^d)$, defined by

$$D_t F = \sum_{i=1}^n \mathbf{1}_{[0,t_i]}(t)\partial_i f\left(B_{t_1}, \dots, B_{t_n}\right), \qquad t \in \mathbb{R}_+,$$

For F of the form

$$F = f\left(B_{t_1}, \dots, B_{t_n}\right), \tag{14}$$

$f \in C_b^\infty(\mathbb{R}^n, X), t_1, \dots, t_n \in \mathbb{R}_+, n \geq 1$. Let $\mathbb{D}_{p,k}(X)$ denote the completion of the space of smooth X-valued random variables under the norm

$$\|u\|_{\mathbb{D}_{p,k}(X)} = \sum_{l=0}^{k} \|D^l u\|_{L^p(W, X \otimes H^{\otimes l})}, \qquad p > 1,$$

Where $X \otimes H$ denotes the completed symmetric tensor product of X and H. For all $p, q > 1$ such that $p^{-1} + q^{-1} = 1$ and $k \geq 1$, let

$$\delta : \mathbb{D}_{p,k}(X \otimes H) \to \mathbb{D}_{q,k-1}(X)$$

denote the Skorohod integral operator adjoint of

$$D : \mathbb{D}_{p,k}(X) \to \mathbb{D}_{q,k-1}(X \otimes H),$$

With

$$E[\langle F, \delta(u) \rangle_X] = E\left[\langle DF, u \rangle_{X \otimes H}\right], \qquad F \in \mathbb{D}_{p,k}(X),\ \ u \in \mathbb{D}_{q,k}(X \otimes H).$$

For $u \in \mathbb{D}_{2,1}(H)$ we identify $Du = (D_t u_s)_{s,t \in \mathbb{R}_+}$ to the random operator $Du : H \to H$ almost surely defined by the relation

$$(Du)v(s) = \int_0^\infty (D_t u_s) v_t dt, \qquad s \in \mathbb{R}_+,\ \ v \in L^2(W, H),$$

in which $a \otimes b \in X \otimes H$ is identified to a linear operator $a \otimes b : H \to X$ via

$$(a \otimes b)c = a \langle b, c \rangle_H, \qquad a \otimes b \in X \otimes H, \quad c \in H.$$

The adjoint $D^* u$ of Du on $H \otimes H$ is given by

$$(D^* u)v(s) = \int_0^\infty \left(D_s^\dagger u_t\right) v_t dt, \qquad s \in \mathbb{R}_+,\ \ v \in L^2(W, H),$$

Where $D_s^\dagger u_t$ denotes the transpose matrix of $D_s u_t$ in $\mathbb{R}^d \otimes \mathbb{R}^d$. We will use the commutation relation

$$D\delta(u) = u + \delta(D^* u), \qquad u \in \mathbb{D}_{2,2}(H). \tag{15}$$

Finally, recall that $Du : H \to H$ is a quasi-nilpotent operator if

$$\text{trace}(Du)^k = 0, \qquad k \geq 2, \tag{16}$$

where the trace of $(Du)^k$ is a.s. given for all $k \geq 2$ by

$$\mathrm{trace}(Du)^k = \int_0^\infty \cdots \int_0^\infty \langle D^\dagger_{t_{k-1}} u_{t_k}, D_{t_{k-2}} u_{t_{k-1}} \cdots D_{t_1} u_{t_1} D_{t_k} u_{t_1} \rangle_{\mathbb{R}^d \otimes \mathbb{R}^d} dt_1 \cdots dt_k.$$

In the sequel we will drop the indices in the scalar products and norms in $\mathbb{R}^d \otimes \mathbb{R}^d, H$, and $H \otimes H$, letting in particular $\|u\| = \|u\|_H$.

In Theorem (3.2.1) below we extend Relations (11) and (12) by computing the expectation of the random Hermite polynomial $H_n(\delta(u), \|u\|^2)$ in the Skorohod integral $\delta(u), n \geq 1$. This result will be applied to anticipating Girsanov identities onthe Wiener space. In the sequel, all scalar products in $H$and in $H \otimes H$ will be simply denoted by $\langle \cdot,\cdot \rangle$, with $\|h\|^2 = \langle h, h \rangle_H, h \in H$.

Lemma (3.2.1)[65]: For all $k \geq 0$and $u \in \mathbb{D}_{n+1,2}(H)$ we have

$$\langle D^* u, D((Du)^k u) \rangle = \mathrm{trace}(Du)^{k+2} + \sum_{i=0}^{k-1} \frac{1}{k+1-i} \langle (Du)^i u, D\, \mathrm{trace}(Du)^{k+1-i} \rangle.$$

Proof. From [69] ,we have

$$\begin{aligned}\langle D^* u, D((Du)^k u) \rangle &= \langle D^* u, (Du)^{k+1} \rangle + \langle D^* u, D(Du)^k u \rangle \\ &= \mathrm{trace}(Du)^{k+2} + \sum_{i=0}^{k-1} \frac{1}{k+1-i} \langle (Du)^i u, D\, \mathrm{trace}(Du)^{k+1-i} \rangle.\end{aligned}$$

As a consequence of Lemma (3.2.1), if $Du : H \to H$is a.s. quasi-nilpotent in the senseof (16) then it satisfies (18). This leads to the following corollary of Theorem (3.2.3) below.

Corollary (3.2.2)[65]: Let $u \in \mathbb{D}_{n,2}(H)$ for some $n \geq 1$, such that $Du : H \to H$ is a.s.quasi-nilpotent or satisfies (18). Then we have

$$E[H_n(\delta(u), \|u\|^2)] = 0.$$

Recall that when the process $(u_t)_{t \in \mathbb{R}_+}$is adapted with respect to the Brownian filtration we have trace $(Du)^k = 0, k \geq 2$, cf. [70], and therefore Condition (18) is satisfied. This recovers (12) in the setting of adapted processes since in this case $\delta(u)$ coincides with the Itô integral of $u \in L^2(W, H)$ with respect to Brownian motion, i.e.

$$\delta(u) = \int_0^\infty u_t dB_t. \tag{17}$$

Theorem (3.2.3)[65]: For any $n \geq 0$ and $u \in \mathbb{D}_{n+1,2}(H)$ we have

$$E[H_{n+1}(\delta(u), \|u\|^2)] = \sum_{l=0}^{n-1} \frac{n!}{l!} E\left[\delta(u)^l \sum_{0\leq 2k\leq n-1-l} \frac{(-1)^k}{k!} \frac{\|u\|^{2k}}{2^k} \langle D^* u, D\big((Du)^{n-2k-l-1} u\big)\rangle\right].$$

Clearly it follows from Theorem (3.2.1) that if $u \in \mathbb{D}_{n,2}(H)$ and

$$\langle D^* u, D((Du)^k u)\rangle = 0, \qquad 0 \leq k \leq n-2, \tag{18}$$

then we have

$$E[H_n(\delta(u), \|u\|^2)] = 0, \qquad n \geq 1, \tag{19}$$

which extends Relation (12) to the anticipating case.

Proof. Step 1. We show that for any $n \geq 1$ and $u \in \mathbb{D}_{n+1,2}(H)$ we have

$$E[H_{n+1}(\delta(u), \|u\|^2)] = \sum_{0\leq 2k\leq n-1} (-1)^k \frac{n!}{k!\,2^k(n-2k-1)!} E[\delta(u)^{n-2k-1}\langle u,u\rangle^k \langle u, \delta(D^* u)\rangle]$$
$$+ \sum_{1\leq 2k\leq n} (-1)^k \frac{n!}{k!\,2^k(n-2k)!} E[\delta(u)^{n-2k}\langle u, D\langle u,u\rangle^k\rangle]. \tag{20}$$

For $F \in \mathbb{D}_{2,1}$ and $l, k \geq 1$ we have

$$E[F\delta(u)^{l+1}] = \frac{l+2k+1}{2k} E[F\delta(u)^{l+1}] - \frac{l+1}{2k} E[F\delta(u)^{l+1}]$$
$$= \frac{l+2k+1}{2k} E[F\delta(u)^{l+1}] - \frac{l+1}{2k} E[\langle u, D(\delta(u)^l F)\rangle]$$
$$= \frac{l+2k+1}{2k} E[F\delta(u)^{l+1}] - \frac{(l+1)}{2k} E[F\delta(u)^{l-1}\langle u, D\delta(u)\rangle] - \frac{l+1}{2k} E[\delta(u)^l \langle u, DF\rangle]$$
$$= \frac{l+2k+1}{2k} E[F\delta(u)^{l+1}] - \frac{(l+1)}{2k} E[F\delta(u)^{l-1}\langle u,u\rangle]$$
$$- \frac{l(l+1)}{2k} E[F\delta(u)^{l-1}\langle u, \delta(D^* u)\rangle] - \frac{l+1}{2k} E[\delta(u)^l\langle u, DF\rangle],$$

i.e.

$$E[F\delta(u)^{n-2k+1}] + \frac{(n-2k)(n-2k+1)}{2k} E[F\delta(u)^{n-2k-1}\langle u,u\rangle]$$
$$= \frac{n+1}{2k} E[F\delta(u)^{n-2k+1}] - \frac{(n-2k)(n-2k+1)}{2k} E[F\delta(u)^{n-2k-1}\langle u, \delta(D^* u)\rangle]$$
$$- \frac{n-2k+1}{2k} E[\delta(u)^{n-2k}\langle u, DF\rangle].$$

Hence, taking $F = \langle u,u\rangle^k$, we get

$$E[\delta(u)^{n+1}] = E[\langle u, D\delta(u)^n\rangle]$$

$$= nE[\delta(u)^{n-1}\langle u, D\delta(u)\rangle]$$

$$= nE[\delta(u)^{n-1}\langle u, u\rangle] + nE[\delta(u)^{n-1}\langle u, \delta(D^*u)\rangle]$$

$$= nE[\delta(u)^{n-1}\langle u, \delta(D^*u)\rangle]$$

$$- \sum_{1\le 2k\le n+1} (-1)^k \frac{n!}{(k-1)!\,2^{k-1}(n+1-2k)!}\Bigg(E[\delta(u)^{n-2k+1}\langle u, u\rangle^k]$$

$$+ \frac{(n-2k+1)(n-2k)}{2k} E[\delta(u)^{n-2k-1}\langle u, u\rangle^{k+1}]\Bigg)$$

$$= nE[\delta(u)^{n-1}\langle u, \delta(D^*u)\rangle]$$

$$- \sum_{1\le 2k\le n+1} (-1)^k \frac{n!}{(k-1)!\,2^{k-1}(n+1-2k)!}\Bigg(\frac{n+1}{2k} E[\delta(u)^{n-2k+1}\langle u, u\rangle^k]$$

$$- \frac{(n-2k)(n-2k+1)}{2k} E[\delta(u)^{n-2k-1}\langle u, u\rangle^k \langle u, \delta(D^*u)\rangle]$$

$$- \frac{n-2k+1}{2k} E[\delta(u)^{n-2k}\langle u, D\langle u, u\rangle^k\rangle]\Bigg)$$

$$= - \sum_{1\le 2k\le n+1} (-1)^k \frac{(n+1)!}{k!\,2^k(n+1-2k)!} E[\delta(u)^{n-2k+1}\langle u, u\rangle^k]$$

$$+ \sum_{0\le 2k\le n-1} (-1)^k \frac{n!}{k!\,2^k(n-2k-1)!} E[\delta(u)^{n-2k-1}\langle u, u\rangle^k \langle u, \delta(D^*u)\rangle]$$

$$+ \sum_{1\le 2k\le n} (-1)^k \frac{n!}{k!\,2^k(n-2k)!} E[\delta(u)^{n-2k}\langle u, D\langle u, u\rangle^k\rangle],$$

which yields (20) after using (9).

Step 2.For $F \in \mathbb{D}_{2,1}$and $0 \le i \le l$ we have

$$E\big[F\delta(u)^l\langle (Du)^i u, \delta(D^*u)\rangle\big] - lE\big[F\delta(u)^{l-1}\langle (D^*u)^{i+1}u, \delta(D^*u)\rangle\big]$$

$$= E\big[\langle D^*u, D\big(F\delta(u)^l (Du)^i u\big)\rangle\big] - lE\big[F\delta(u)^{l-1}\langle (D^*u)^{i+1}u, \delta(D^*u)\rangle\big]$$

$$= lE\big[F\delta(u)^{l-1}\langle D^*u, (Du)^i u \otimes D\delta(u)\rangle\big] - lE\big[F\delta(u)^{l-1}\langle (D^*u)^{i+1}u, \delta(D^*u)\rangle\big]$$

$$+E\big[\delta(u)^l\langle D^*u, D\big(F(Du)^i u\big)\rangle\big]$$

$$= lE\big[F\delta(u)^{l-1}\langle D^*u, (Du)^i u \otimes D\delta(u)\rangle\big] + lE\big[F\delta(u)^{l-1}\langle D^*u, (Du)^i u \otimes \delta(D^*u)\rangle\big]$$

$$-lE\big[F\delta(u)^{l-1}\langle (D^*u)^{i+1}u, \delta(D^*u)\rangle\big] + E\big[\delta(u)^l\langle D^*u, D\big(F(Du)^i u\big)\rangle\big]$$

$$= lE\big[F\delta(u)^{l-1}\langle (Du)^{i+1}u, u\rangle\big] + E\big[\delta(u)^l\langle (Du)^{i+1}u, DF\rangle\big]$$

$$+E\left[F\delta(u)^l\langle D^*u, D\big((Du)^i u\big)\rangle\right].$$

Hence, replacing l above with $l-i$, we get

$$E[F\delta(u)^l\langle u, \delta(D^*u)\rangle] = l!\,E[F\langle (Du)^l u, \delta(D^*u)\rangle]$$

$$+\sum_{i=0}^{l-1}\frac{l!}{(l-i)!}E\left[F\delta(u)^{l-i}\langle (Du)^i u, \delta(D^*u)\rangle\right] - (l-i)E\left[F\delta(u)^{l-i-1}\langle (D^*u)^{i+1}u, \delta(D^*u)\rangle\right]$$

$$= l!\,E[F\langle (Du)^l u, \delta(D^*u)\rangle] + \sum_{i=0}^{l-1}\frac{l!}{(l-i-1)!}E\left[F\delta(u)^{l-i-1}\langle (Du)^{i+1}u, u\rangle\right]$$

$$+\sum_{i=0}^{l-1}\frac{l!}{(l-i)!}E\left[\delta(u)^{l-i}\langle (Du)^{i+1}u, DF\rangle\right] + \sum_{i=0}^{l-1}\frac{l!}{(l-i)!}E\left[F\delta(u)^{l-i}\langle D^*u, D\big((Du)^i u\big)\rangle\right]$$

$$= l!\,E[\langle (Du)^{l+1}u, DF\rangle] + \sum_{i=0}^{l-1}\frac{l!}{(l-i-1)!}E\left[F\delta(u)^{l-i-1}\langle (Du)^{i+1}u, u\rangle\right]$$

$$+\sum_{i=0}^{l-1}\frac{l!}{(l-i)!}E\left[\delta(u)^{l-i}\langle (Du)^{i+1}u, DF\rangle\right] + \sum_{i=0}^{l}\frac{l!}{(l-i)!}E\left[F\delta(u)^{l-i}\langle D^*u, D\big((Du)^i u\big)\rangle\right]$$

$$= l!\,E[\langle (Du)^{l+1}u, DF\rangle] + \sum_{i=0}^{l-1}\frac{l!}{(l-i-1)!}E\left[F\delta(u)^{l-i-1}\langle (Du)^{i+1}u, u\rangle\right]$$

$$+\sum_{i=0}^{l}\frac{l!}{(l-i-1)!}E\left[\delta(u)^{l-i+1}\langle (Du)^i u, DF\rangle\right] + \sum_{i=0}^{l}\frac{l!}{(l-i)!}E\left[F\delta(u)^{l-i}\langle D^*u, D\big((Du)^i u\big)\rangle\right],$$

Thus letting $F = \langle u,u\rangle^k$and $l = n-2k-1$ above, and using (20) in Step 1, we get

$$E[H_{n+1}(\delta(u), \|u\|^2)] = \sum_{0\le 2k\le n-1}(-1)^k\frac{n!}{k!\,2^k(n-2k-1)!}E[\delta(u)^{n-2k-1}\langle u,u\rangle^k\langle u, \delta(D^*u)\rangle]$$

$$+\sum_{1\le 2k\le n}(-1)^k\frac{n!}{k!\,2^k(n-2k)!}E[\delta(u)^{n-2k}\langle u, D\langle u,u\rangle^k\rangle]$$

$$= \sum_{0\le 2k\le n}(-1)^k\frac{n!}{k!\,2^k}E[\langle (Du)^{n-2k}u, D\langle u,u\rangle^k\rangle]$$

$$+\sum_{0\le 2k\le n-1}\frac{(-1)^k}{k!\,2^k}\sum_{i=0}^{n-2k-2}\frac{n!}{(n-2(k+1)-i)!}E\left[\langle u,u\rangle^k\delta(u)^{n-2(k+1)-i}\langle (Du)^{i+1}u, u\rangle\right]$$

$$+\sum_{0\le 2k\le n-1}\frac{(-1)^k}{k!\,2^k}\sum_{i=1}^{n-2k-2}\frac{n!}{(n-2k-i)!}E\left[\delta(u)^{n-2k-i}\langle (Du)^i u, D\langle u,u\rangle^k\rangle\right]$$

$$+\sum_{0\le 2k\le n-1}\frac{(-1)^k}{k!\,2^k}\sum_{i=1}^{n-2k-1}\frac{n!}{(n-2k-1-i)!}E\big[\langle u,u\rangle^k\delta(u)^{n-2k-i-1}\langle D^*u,D\big((Du)^iu\big)\rangle\big]$$

$$+\sum_{0\le 2k\le n}(-1)^k\frac{n!}{k!\,2^k}E[\delta(u)^{n-2k}\langle u,D\langle u,u\rangle^k\rangle]$$

$$=\sum_{0\le 2k\le n}(-1)^k\frac{n!}{k!\,2^k}E[\langle (Du)^{n-2k}u,\langle u,u\rangle^k\rangle]$$

$$-\sum_{0\le 2k\le n-1}\frac{(-1)^{k+1}}{(k+1)!\,2^k}\sum_{i=1}^{n-2k-2}\frac{n!}{(n-2(k+1)-i)!}E\big[\delta(u)^{n-2(k+1)}\langle (Du)^iu,D\langle u,u\rangle^{k+1}\rangle\big]$$

$$+\sum_{0\le 2k\le n}\frac{(-1)^k}{k!\,2^k}\sum_{i=1}^{n-2k-1}\frac{n!}{(n-2k-i)!}E\big[\delta(u)^{n-2k-i}\langle (Du)^iu,D\langle u,u\rangle^k\rangle\big]$$

$$+\sum_{0\le 2k\le n-1}\frac{(-1)^k}{k!\,2^k}\sum_{i=1}^{n-2k-1}\frac{n!}{(n-2k-1-i)!}E\big[\langle u,u\rangle^k\delta(u)^{n-2k-i-1}\langle D^*u,D\big((Du)^iu\big)\rangle\big]$$

$$=\sum_{0\le 2k\le n-1}\frac{(-1)^k}{k!\,2^k}\sum_{i=1}^{n-2k-1}\frac{n!}{(n-2k-1-i)!}E[\langle u,u\rangle^k\delta(u)^{n-2k-i-1}\langle D^*u,D((Du)^iu)\rangle],$$

Where we applied the relation

$$\langle u,u\rangle^k\langle (Du)^{i+1}u,u\rangle=\frac{1}{2}\langle u,u\rangle^k\,\langle (Du)^iu,D\langle u,u\rangle\rangle$$
$$=\frac{1}{2(k+1)}\langle (Du)^iu,D\langle u,u\rangle^{k+1}\rangle$$
$$=\frac{1}{2(k+1)}\langle (Du)^iu,D\langle u,u\rangle^{k+1}\rangle,$$

Which follows from $D\langle u,u\rangle=2(D^*u)u$ and the derivation property of the gradient operator D.

The next proposition is an immediate consequence of (19), using the generating function (10). In comparison with Proposition (3.2.1) of [70] we do not require assumptions on the inverse mapping $(I_H-Du)^{-1}$and we show that quasi-nilpotence of Ducan be replaced by the weaker condition (18), while working under a stronger integrability condition. Let $\mathbb{D}_{\infty,2}(H)=\bigcap_{n\ge 1}\mathbb{D}_{n,2}(H)$.

Corollary (3.2.4)[65]: Assume that $u\in\mathbb{D}_{\infty,2}(H)$ with $E\big[e^{|\delta(u)|+\|u\|^2/2}\big]<\infty$, and that $Du:H\to H$ is a.s. quasi-nilpotent, or more generally that (18) holds. Then we have

$$E\left[\exp\left(\delta(u)-\frac{1}{2}\|u\|^2\right)\right]=1. \qquad (21)$$

Proof. From (9) we have the bound

$$|H_n(x,\sigma^2)| \leq \sum_{0\leq 2k\leq n} \frac{(-1)^k}{k!\,2^k}\frac{n!}{(n-2k)!}|x|^{n-2k}(-\sigma^2)^k = H_n(|x|,\sigma^2),$$

hence

$$E\left[\sum_{n=0}^{\infty}\frac{1}{n!}|H_n(\delta(u),\|u\|^2)|\right] \leq E\left[\sum_{n=0}^{\infty}\frac{1}{n!}H_n(|\delta(u)|,-\|u\|^2)\right]$$
$$= E\left[e^{|\delta(u)|+\|u\|^2/2}\right] < \infty.$$

Consequently, by Theorem (3.2.3) and the Fubini theorem we have

$$E\left[\exp\left(\delta(u)-\frac{1}{2}\|u\|^2\right)\right] = 1 + E\left[\sum_{n=0}^{\infty}\frac{1}{(n+1)!}H_{n+1}(\delta(u),\|u\|^2)\right]$$
$$= 1 + \sum_{n=0}^{\infty}\frac{1}{(n+1)!}E[H_{n+1}(\delta(u),\|u\|^2)] = 1.$$

This shows in particular that if $u \in \mathbb{D}_{\infty,2}(H)$ is such that $\|u\|$is deterministic and $Du : H \to H$is a.s. quasi-nilpotent, or more generally (18) holds, then we have

$$E\left[e^{\delta(u)}\right] = e^{\frac{1}{2}\|u\|^2},$$

i.e. $\delta(u)$ has a centered Gaussian distribution with variance $\|u\|^2$, cf. Theorem (3.2.1) of [68] and Corollary(3. 2.2) of [69].

More generally, Corollary (3.2.5) below states an anticipative Girsanov identity (22) that recovers Proposition (3.2.1) of [70] under simpler hypotheses, namely without requirements on the smoothness and integrability of $(I_H - Du)^{-1}$. In the sequel, for$u \in \mathbb{D}_{2,1}(H)$ we let

$$\Lambda_u = \exp\left(\delta(u) - \frac{1}{2}\|u\|^2\right),$$

and we denote by $T_u$the transformation of Wdefined by

$$T_u\omega(t) = \omega(t) + \int_0^t u_s(\omega)ds, \qquad t \in \mathbb{R}_+, \qquad \omega \in W.$$

Corollary (3.2.5)[65]: Assume that $u \in \mathbb{D}_{\infty,2}(H)$ with $E\left[e^{\varepsilon\left(|\delta(u)|+\|u\|^2/2\right)}\right] < \infty$ for some$\varepsilon > 1$, and that $Du : H \to H$is a.s. quasi-nilpotent, or more generally that (18) holds. Then the transformation $Tu : W \to W$ satisfies the Girsanov type identity

$$E[F \circ T_u \Lambda_u] = E[F], \tag{22}$$

for all bounded random variables F.

Proof. For all exponential vectors $\Lambda_f = e^{\int_0^\infty f(t)dB_t - \frac{1}{2}\|f\|^2}, f \in L^2(\mathbb{R}_+)$, we have

$$\begin{aligned}\Lambda_f \circ T_u \Lambda_u &= e^{\int_0^\infty f(t)dB_t - \int_0^\infty f(t)u(t)dt - \frac{1}{2}\|f\|^2} \Lambda_u \\ &= e^{\delta(f+u) - \frac{1}{2}\|f\|^2 - \frac{1}{2}\|u\|^2 - \langle f,u\rangle} \\ &= \Lambda_{u+f},\end{aligned}$$

hence by Corollary (3.2.5) we have

$$E\left[\Lambda_f \circ T_u \Lambda_u\right] = E\left[\Lambda_{u+f}\right] = 1,$$

and we conclude by density of the linear combination of exponential vectors $\Lambda_f, f \in L^2(\mathbb{R}_+)$, in $L^2(W)$.

In particular, if $Tu : W \to W$is invertible, then by Corollary (3.2.5) it is absolutely continuous with respect to the Wiener measure, and

$$\frac{dT_u^* P}{dP} = \Lambda_u.$$

We refer to Corollary (3.2.2) of [70] for sufficient conditions for the invariability of $Tu : W \to W$.

The conditions imposed to obtain the Girsanov identity for Euclidean motions written as the sum of a rotation and a quasi-nilpotent shifts as in Theorem (3.2.2) of [70] can be simplified similarly.

Finally we sketch the proof of the formal identity (13) stated in the introduction, i.e.

$$\frac{\partial}{\partial t} E\left[e^{t\delta(u) - \frac{t^2}{2}\|u\|^2}\right] = tE\left[e^{t\delta(u) - t^2\langle u,u\rangle/2}\langle D^* u, D(I_H - tDu)^{-1} u\rangle\right]. \tag{23}$$

Proof. We have

$$\frac{\partial}{\partial t} E\left[e^{t\delta(u) - \frac{t^2}{2}\|u\|^2}\right] = \sum_{n=0}^{\infty} \frac{t^n}{n!} E[H_{n+1}(\delta(u), \|u\|^2)]$$

$$= \sum_{n=0}^{\infty} t^n E\left[\sum_{l=0}^{n-1} \frac{\delta(u)^l}{l!} \sum_{0\le 2k\le n-1-l} \frac{(-1)^k}{k!}\frac{\|u\|^{2k}}{2^k}\langle D^* u, D((Du)^{n-2k-l-1}u)\rangle\right]$$

$$= tE\left[\sum_{l=0}^{\infty} \frac{t^l \delta(u)^l}{l!} \sum_{n=0}^{\infty} t^n \sum_{0\le 2k\le n} \frac{(-1)^k}{k!}\frac{\|u\|^{2k}}{2^k}\langle D^* u, D((Du)^{n-2k}u)\rangle\right]$$

$$= tE\left[e^{t\delta(u)} \sum_{n=0}^{\infty} t^n \sum_{0\le 2k\le n} \frac{(-1)^k}{k!}\frac{\|u\|^{2k}}{2^k}\langle D^* u, D((Du)^{n-2k}u)\rangle\right]$$

$$= tE\left[e^{t\delta(u)} \sum_{n=0}^{\infty} \frac{(-1)^k}{k!}\frac{\|u\|^{2k}}{2^k} \sum_{n=0}^{\infty} t^n \langle D^* u, D((Du)^{n}u)\rangle\right]$$

$$= tE\left[e^{t\delta(u)-t^2\langle u,u\rangle/2}\langle D^* u, D(I_H - tDu)^{-1}u\rangle\right].$$

In a similar way, from Theorem (3.2.1) of [69] we get

$$\frac{\partial}{\partial t} E\left[e^{t\delta(u)}\right] = \sum_{n=0}^{\infty} \frac{t^n}{n!} E\left[\left(\delta(u)\right)^{n+1}\right]$$

$$= \sum_{n=0}^{\infty} t^n \sum_{k=1}^{n} \frac{1}{(n-k)!} E\left(\langle\left(\delta(u)\right)^{n-k}(Du)^{k-1}u, u\rangle + \langle D^* u, D((Du)^{k-1}u)\rangle\right)$$

$$= t\sum_{n=0}^{\infty} \frac{t^n}{n!} E\left[\left(\delta(u)\right)^{n} \sum_{k=1}^{n} t^k(\langle (Du)^k u, u\rangle + \langle D^* u, D((Du)^k u)\rangle)\right]$$

$$= tE\left[e^{t\delta(u)}(\langle u, (I_H - tDu)^{-1}u\rangle + \langle D^* u, D((I_H - tDu)^{-1}u)\rangle)\right],$$

hence

$$\frac{\partial}{\partial t} E\left[e^{t\delta(u)}\right] = tE\left[e^{t\delta(u)}(\langle u, (I_H - tDu)^{-1}u\rangle + \langle D^* u, D((I_H - tDu)^{-1}u)\rangle)\right].$$