Chapter 2

Stable Standing Waves and Lyapunov Asymptotic Stability

In this chapter we give sufficient conditions for the stability of the standing waves of least energy for nonlinear Klein-Gordon equations.

Section (2.1): Nonlinear Klein-Gordon Equations:

We give sufficient conditions for the stability of standing waves of the nonlinear Klein-Gordon equation:

$$u_{tt} - \Delta u + u + f(|u|)\arg u = 0, \quad x \in \mathbb{R}^n, \quad n > 2, \tag{1}$$

or equivalently the steady-state solutions of the modulated equation:

$$u_{tt} + 2i\omega u_t - \Delta u + (1 - \omega^2)u + f(|u|)\arg u = 0. \tag{2}$$

We show the stability of the standing waves of lowest energy in the energy norm. They are stable with respect to the lowest energy solution set of

$$-\Delta u + (1 - \omega^2)u + f(|u|)\arg u = 0. \tag{3}$$

The existence of solutions of (3) has already been shown in [42] and [43]. In the generality presented and this problem was solved by Berestycki and Lions in [43]. The condition for stability is very simple. If we define

$$d(\omega) = 1/2\int |\nabla\varphi_\omega|^2\, dx + (1 - \omega^2)/2\int |\varphi_\omega|^2\, dx + \int G(|\varphi_\omega|)dx,$$

where $G' = f$ and φ_ω is a least energy solution of (3), then:

Theorem (2.1.1)[41]: $d(\omega)$ is strictly convex in a neighborhood of ω_0, then φ_{ω_0} is stable.

Equation (1) arises in particle physics. It models the field equation for spin-0 particles [44]. The existence of stable standing waves has, until now, eluded any rigorous proof. Anderson [45] showed by numerical computation that these equations can have stable standing waves. He studied the particular example where $(|u|)\arg u = -|u|^2u + |u|^4u, x \in \mathbb{R}^3$, and showed numerically that there are both stable and unstable standing waves. We have shown in [46] the existence of unstable standing waves for this example when ω is close to 1. Here we show that $d(\omega)$ is strictly convex for some ω and therefore there are stable standing waves. This problem was subsequently considered by Lee [44] and others who arrive at the same conclusion, heuristically, using the principle of least energy.

It can be shown that the condition $d(\omega)$ is convex, is equivalent to the condition that the energy of equation (1) $E(u,v)$ restricted to the charge $Q(u,v)=Q(\varphi_\omega,i\omega\varphi_\omega)$ has a local minimum at $(\varphi_\omega,i\omega\varphi_\omega)$, where the charge $Q(u,v)=\mathrm{Im}\int u\bar{v}dx$. This agrees
with the physical intuition of the problem [44].

The theory of linearized stability does not give a clue to whether there are stable standing waves or not. The spectrum of the linearized problem might lie entirely on the imaginary axis and therefore one cannot deduce the stability of these waves.

It is interesting to compare this result of stability with the instability result of the ground state, i.e., the least energy steady state solution of equation (1). Berestycki and Cazenave [47] showed that for special type of nonlinearities, solutions that are close to the ground state blow up in finite time. In [46] we generalized this result to show instability, but not necessarily blow up, of the ground state for all nonlinearities that we can prove the existence of a ground state for.

Finally, for the Schródinger equation: $iu_t-\Delta u+f(|u|)\arg u=0$. Cazenave and Lions [48] showed the existence of stable standing waves for some nonlinearities. Berestycki and Cazenave [47] showed the existence of unstable standing waves for another type of nonlinearities.

Notation (2.1.2)[41]: We employ here the standard notation

$$H_r^1(\mathbb{R}^n)=\{u,\text{radially symmetric functions on }\mathbb{R}^n$$

$$\|u\|=\left(\int|\nabla u(x)|^2dx+\int|u(x)|^2dx\right)^{1/2}<\infty\},$$

$$L_r^p(\mathbb{R}^n)=\{u,\text{radially symmetric function on }\mathbb{R}^n$$

$$|u|_p=\left(\int|u(x)|^pdx\right)^{1/p}<\infty\}$$

$$C_{0r}^\infty(\mathbb{R}^n)\ =\ \{\text{radially symmetric, infinitely differentiable functions with compact support}\},$$

$$f(s) = o(s) \Leftrightarrow |f(s)/s| \to 0 \quad \text{as} \quad |s| \to 0,$$
$$f(s) = O(s) \Leftrightarrow |f(s)/s| \text{ is bounded as } s \to 0.$$

Consider the nonlinear Klein-Gordon equation

$$u_{tt} - \Delta u + u + f(|u|)\arg u = 0, \quad f(0) = f'(0) = 0, \qquad x \in \mathbb{R}^n, \qquad n > 2. \quad (4)$$

This equation has nontrivial standing waves, $u(x,t) = e^{i\omega t}\varphi(x)$ provided that

$$-\Delta\varphi + (1-\omega^2)\varphi + f(|\varphi|)\arg\varphi = 0 \quad (5)$$

has a nontrivial solution.

Definition (2.1.3)[41]: Let

$$J_\omega(\psi) = 1/2 \int |\Delta\psi|^2 dx + n\left((1-\omega^1)/2 \int |\psi|^2 dx + \int G(|\psi|)dx\right)$$

where $G'(|\psi|) = f(|\psi|)$ and $G(0) = 0$,

$$K_\omega(\psi) \equiv (n-2)/2 \int |\Delta\omega\psi|^2 dx + n\left((1-\omega^1)/2 \int |\psi|^2 dx + \int G(|\psi|dx)\right)$$

$$M_\omega \equiv \{\psi \in H_r^1(\mathbb{R}^n), K_\omega(\psi) = 0, \psi \neq 0\}$$

In order that equation (5) has nontrivial solutions it is sufficient that f and G satisfy [43]:

$$H{:}\begin{cases} H.1 \quad \exists\eta > 0_\ni : G(\eta) < 0 \\ H.2 \quad \lim_{\eta\to\infty} f(\eta)/\eta^l \geqq 0, \quad l < 1 + 4/(n-2). \end{cases}$$

Definition (2.1.4)[41]: Let $\omega^* = \{\inf \omega \geqq 0_\ni : \exists\, \eta(l-\omega^2)\eta^2/2 + G(\eta) < 0\}$. Thus $\omega^* \in [0,1)$. We shall always take $\omega^* < \omega < l$.

Lemma (2.1.5)[41]: For $\omega \in (\omega^*, 1) M_\omega$ is a C^1 hypersurface in $H_r^1(\mathbb{R}^n)$ bounded away from zero.

Proof See [46].

Proposition (2.1.6)[41]: if $\varphi_\omega \in H_r^1(\mathbb{R}^n)$ is a solution of (5), and $\int G(|\varphi_\omega|)dx < \infty$, then

$$K_\omega(\varphi_\omega) = 0$$

Proof. Let $\varphi_\beta(x) = \varphi_\omega(x/\beta)$ then

$$J_\omega(\varphi_\beta) = \beta^{n-2}/2 \int |\nabla\varphi_\omega|^2 dx + \beta^n\left((1-\omega)^2/2 \int |\varphi_\omega|^2 dx + \int G(|\varphi_\omega|)dx\right), \quad (6)$$

since φ_ω is a solution then $\delta J_\omega(\varphi_\omega) = 0 \Longrightarrow d\left(J_\omega(\varphi_\beta)\right)/d\beta|_{\beta=1} = 0$, but

$$d\left(J_\omega(\varphi_\beta)\right)/d\beta|_{\beta=1}$$

$$= (n-2)/2\int |\nabla\varphi_\omega|^2 dx\,| + n(l-\omega^2)/2\int |\varphi_\omega|^2 dx + \int \mathrm{G}(|\varphi_\omega| dx)$$

therefore $K_\omega(\varphi_\omega) = 0$.

Theorem (2.1.7)[41]: Let $\omega^2 \in (\omega^{*2}, 1), n > 2$, then

$$d(\omega) = \inf_{v\in M_\omega} J_\omega(v)$$

is achieved for some $v \neq 0$, and

$$d(\omega)\left\{\inf\frac{1}{n}\int |v|^2 dx\,, K_\omega(v) \leqq 0, v \neq 0\right\}$$

Moreover v satisfies

$$-\Delta v + (l-\omega^2)v + f(|v|)\arg v = 0. \tag{7}$$

Proof. First we show the equivalence of both minimization problems. Consider any function $v \in H_r^1(\mathbb{R}^n)$ such that $K_\omega(v) < 0$. Let $v_\beta(x) = v(x/\beta)$. Then

$$K_\omega(v) = \beta^{n-2}(n-2)/2\int |\nabla v|^2 dx + \beta^n n\left((1-\omega^2)/2\int |v|^2 dx + \int G(v) dx\right) \tag{8}$$

Now for $\beta = 1 K_\omega(v_1) = K_\omega(v) < 0$ and for β close to zero $K_\beta(v_\beta) > 0$. Therefore there exist a $\beta_0 \in (0,1)$ such that $K_\omega(v_{\beta_0}) = 0$ and

$$1/n\int \left|\nabla v_{\beta_0}\right|^2 dx = \beta_0^{n-2}/n\int |\nabla v|^2 dx < 1/n\int |\nabla v| 2 dx$$

Since $J_\omega(v) = 1/n\left(\int |\nabla v|^2 dx + K_\omega(v)\right)$ then

$$d(\omega) = \inf_{v\in M_\omega} J_\omega(v) = \inf\left\{1/n\int |\nabla v|^2 dx + K_\omega(v) = 0, v \neq 0\right\},$$

$$= \inf\left\{1/n\int |\nabla v|^2 dx + K_\omega(v) \leqq 0, v \neq 0\right\}$$

Next, consider any minimizing sequence v_k. Then $(\int |\nabla v_k|^2 dx)$ is bounded. By $H.2$ for every $\varepsilon > 0$ there exist $C_1(\varepsilon) > 0$ such that $G(\eta) > -\varepsilon/2\eta^2 - C_1(\varepsilon)\eta^{l+1}$, where $+4/(n-2)$. Since $K_\omega(v_k) \leqq 0$, then

$$0 \geqq K_\omega(v_k) = (n-2)/2\int |\nabla v_k|^2 dx + n\left((1-\omega^2)/2\int |v_k|^2 dx + \int \mathrm{G}(|v_k|) dx\right)$$

and this implies

$$0 \geqq (n-2)/2 \int |\nabla v_k|^2 dx + n\left((1-\omega^2)/2 \int |v_k|^2 dx + C_1(\varepsilon) \int |v_k|^{l+1} dx\right)$$

Now by Sobolev embedding $H_r^1(\mathbb{R}^n) \hookrightarrow L_r^p(\mathbb{R}^n), 2 < p < 2 + 4/(n-2)$ and since $(\int |\nabla v_k|^2 dx)$ is bounded we get that $\|v_k\|$ is bounded. Therefore there exist a subsequence, also denote it by (v_k) such that

$$v_k \overset{w}{\to} v_0 \in H_r^1(\mathbb{R}^n) \text{ and } v_k \longrightarrow v_0 \in L^p \quad 2 < p < 2 + 4/(n-2),$$

since for radially symmetric $H_r^1(\mathbb{R}^n) \hookrightarrow L_r^p(\mathbb{R}^n)$ is compact for $2 < p < 2 + 4/(n-2)$. By lower semicontinuity of weak limits we have:

$$k_\omega(v_0) = (n-2)/2 \int |\nabla v_0|^2 dx + n\left((1-\omega^2)/2 \int |\nabla v_0|^2 dx + \int G(|\nabla v_0|) dx\right)$$

$$\leqq \lim_{k\to\infty} (n-2)/2 \int |\nabla v_k|^2 dx + n\left((1-\omega^2)/2 \int |\nabla v_k|^2 dx + \int G(|\nabla v_k|) dx\right) = 0$$

And from the above argument the inequalities are equalities and the weak limit is strong. Consequently $v_0 \neq 0$ by Lemma (2.1.5), and

$$d(\omega) = \inf_{v \in M_\omega} J_\omega(v) = J_\omega(v_0).$$

Finally, to show that v_0 satisfies Eq. (7) we have by the Lagrange multiplier method

$$\delta J_\omega(v_0) = \lambda \delta K_\omega(v_0), \tag{9}$$

or

$$-\Delta v_0 + (1-\omega^2) v_0 + f(|v_o|) \arg v_0$$
$$= \lambda(-(n-2)\Delta v_0 + n(1-\omega^2) v_0 + n f(|v_o|) \arg v_0).$$

By Proposition (2.1.6) we have

$$(n-2)/2 \int |\nabla v_0|^2 dx + n\left((1-\omega^2)/2 \int |v_0|^2 dx + \int G(|v_0|) dx\right)$$

$$= \lambda\left[(n-2)^2/2 \int |\nabla v_0|^2 dx + n\left((1-\omega^2)/2 \int |v_0|^2 dx + \int G(|v_0|) dx\right)\right]. \tag{10}$$

But

$$K_\omega(v_0) = (n-2)/2 \int |\nabla v_0|^2 dx + n\left((1-\omega^2)/2 \int |v_0|^2 dx + \int G(|v_0|) dx\right) = 0,$$

therefore

$$0 = \lambda\left[(n-2)^2/2 \int |\nabla v_0|^2 dx - n(n-2)/2 \int |\nabla v_0|^2 dx\right],$$

$$0 = \lambda(n-2)\int |\nabla v_0|^2 dx\,,$$

and this implies that $\lambda = 0$.

Definition (2.1.8)[41]: Let S_ω be the solution set of $d(\omega) = \inf_{v\in M_\omega} J_\omega(v)$.

Corollary (2.1.9)[41]: S_ω is also the solution set of

$$\inf J_\omega(v) = d(\omega), \qquad \omega^2 \in (\omega^{*2}, l)\,, \qquad 1/n\int |\nabla v_0|^2 dx = d(\omega).$$

Proof: Suppose $\exists v$ such that $1/n\int |\nabla v|^2 dx = d(\omega)$ and $J_\omega(v) < d(\omega)$. Then

$$1/nK_\omega(v) = J_\omega(v) - 1/n\int |\nabla v_0|^2 dx < 0.$$

But by Theorem (2.1.7)

$$d(\omega) = \inf\left\{1/n\int |\nabla v|^2 dx\,, K_\omega(v) \leqq 0, v \neq 0\right\}$$

and this contradicts the above assumption. Therefore

$$\inf J_\omega(v) = d(\omega), \qquad 1/n\int |\nabla v|^2 dx = d(\omega).$$

Now to show that the solution set of this problem is S_ω we note that $\forall v$ which is a minimum we have

$$\delta J_\omega(v) = \lambda \Delta v, \qquad \text{or } -(1+\lambda)\Delta v + (1-\omega^2)v + (|v|)\arg v = 0, \qquad (11)$$

and by Proposition (2.1.6)

$$(1+\lambda)(n-2)/2\int |\nabla v|^2 dx + n\left((1-\omega^2)/2\int |v|^2 dx + \int G(|v|)dx\right) = 0$$

$$\Longrightarrow J_\omega(v) = 1/2\int |\nabla v|^2 dx - (1+\lambda)(n-2)/(2n)\int |\nabla v|^2 dx = d(\omega)$$

$$\Longrightarrow J_\omega(v) = 1/n\int |\nabla v|^2 dx - \lambda(n-2)/(2n)\int |\nabla v|^2 dx = d(\omega)$$

$$\Longrightarrow \lambda = 0 \text{ and } \therefore K_\omega(v) = 0 \Longrightarrow v \in S_\omega.$$

Corollary (2.1.10)[41]: Let $v^k \in H_r^1(\mathbb{R}^n)$ be a sequence such that $1/n\int |\nabla v^k|^2 dx > d(\omega)$ and $J_\omega(v^k) \longrightarrow d^1 \leqq d(\omega)$ then v^k has a strongly convergent subsequence $v^k \to \varphi_\omega \in H_r^1(\mathbb{R}^n)$ for some $\varphi_\omega \in S_\omega$ and

$$\left|\int \Big(G(|v^k|)\Big) - G(|\varphi_\omega|)dx\right| \to 0, \qquad d^1 = d(\omega)$$

Proof. Since $\int|\nabla v^k|^2dx$ and $J_\omega(v^k)$ are bounded, v^k is a bounded sequence in $H_r^1(\mathbb{R}^n)$ (see the proof of Theorem (2.1.7). v^k has a weakly convergent subsequence, also denote it by v^k, such that

$$v^k \overset{w}{\to} v_0 \in H_r^1(\mathbb{R}^n), \quad v^k \to v_0 \in L_r^p, \qquad 2 < p < 2 + 4/(n-2)$$

Now $1/n||\nabla v_0|^2dx \leqq \lim_{k\to\infty} 1/n\int|\nabla v^k|^2dx = d(\omega)$ and $K_\omega(v_0) \leqq \lim_{k\to\infty} K_\omega(v^k)$ (by the proof of Theorem (2.1.7)), therefore

$$K_\omega(v_0) \leqq n \lim_{k\to\infty}\left(J_\omega(v^k) - 1/n\int|\nabla v|^2dx\right)$$

or

$$K_\omega(v_0) \leqq d^1 - d(\omega) \leqq 0$$

But from Theorem (2.1.7) we have

$$d(\omega) = \inf\left\{1/n\int|\nabla v|^2dx\,, K_\omega(v) \leqq 0, v \neq 0\right\}$$

Therefore all inequalities are equalities and the weak convergence is strong.

Therefore

$$\int G(|v^k|)dx \to \int G(|\varphi_\omega|)dx \qquad \text{and} \qquad d^1 = d(\omega)$$

Since $1/n\int|\nabla v_0|^2dx = d(\omega)$ and $K_\omega(v_0) = 0 \Longrightarrow v_0 \in S_\omega$ and $\therefore v^k \to v_0 \in H_r^1(\mathbb{R}^n), v_0 \in S_\omega$.

Remark (2.1.11)[41]: This is the only place where radial symmetry is needed. One can generalize the above result to include the space $H_r^1(\mathbb{R}^n)$ by using the notion of "concentrated compactness" introduced by Lions [49]. In this case the sequence $v_k \in H^1(\mathbb{R}^n)$ of Corollary (2.1.10) will have a subsequence vkn such that $v_{k_n}\left(.+y_{k_n}\right) \in H^1(\mathbb{R}^n)$ is relatively compact in $H^1(\mathbb{R}^n)$ for some sequence $\left(y_{k_n}\right)$.

We'll study the behavior of $d(\omega) = 1/n\int|\nabla\varphi_\omega|^2dx$ as a function of the frequency ω.

Lemma (2.1.12)[41]: Let $\omega_1 < \omega_2$ be such that $[\omega_1, \omega_2] \subset (\omega^*, 1)$, then $d(\omega)$ and $\int|\varphi_\omega|^2dx(\varphi_\omega \in S_\omega)$ are uniformly bounded in $\omega \in [\omega_1, \omega_2]$.

Proof. Since K is continuous in $\omega, d(\omega)$ is bounded for $\omega \in [\omega_1, \omega_2]$. Now for $\varphi \in S_\omega, K_\omega(\varphi_\omega) = 0$. By $H.2G(\eta) \geqq -c\eta^{l+1}; l < 1 + 4/(n-2)$, for η large, and $G(0) = G''(0) = 0 \Longrightarrow$ for any $a > 0, \frac{a}{2}\eta^2 + G(\eta) > -C_a\eta^{t_0+1}, l_0 = 1 + 4/(n-2)$,

$$\int \left(\frac{a}{2}|v|^2 + G(|v|)\right) dx \geqq -C_a \int |v|^{t_0+1} dx.$$

Now because $K_\omega(\varphi_\omega) = 0$, and by Sobolev embedding

$$(n-2)/2 \int |\nabla\varphi_\omega|^2 dx + n(1 - \omega^2 - a^2)/2 \int |\varphi_\omega|^2 dx - C_a \left(\int |\varphi_\omega|^2 dx\right)^a \leqq 0$$

for a small.

This implies that $\int |\varphi_\omega|^2 dx$ is uniformly bounded for $\omega \in [\omega_1, \omega_2] \subset (\omega^*, 1)$.

Proposition (2.1.13)[41]: a) $d(\omega)$ is a decreasing function of $\omega \in (\omega^*, 1)$, b) if $\omega_1 < \omega_2$,

i) $d(\omega_2) < d(\omega_1) - (\omega_2^2 - \omega_1^2)/2 \int |\varphi_{\omega_1}|^2 dx + o(\omega_1 - \omega_2)$

ii) $d(\omega_1) < d(\omega_2) - (\omega_2^2 - \omega_1^2)/2 \int |\varphi_{\omega_2}|^2 dx + o(\omega_1 - \omega_2)$

Consequently, $d(\omega)$ is a continuous function of $\omega \in (\omega^*, 1)$.

Proof. a) Let $\omega_1 < \omega_2$, then

$$K_{\omega_2}(\varphi_{\omega_1}) = (n-2)/2 \int |\varphi_{\omega_1}|^2 dx + n\left((1 - \omega_2^2)/2 \int |\varphi_{\omega_1}|^2 dx + \int G(|\varphi_{\omega_1}|) dx\right)$$

or

$$K_{\omega_2}(\varphi_{\omega_1}) = (n-2)/2 \int |\nabla\varphi_{\omega_1}|^2 dx + n((1 - \omega_1^2)/2 \int |\nabla\varphi_{\omega_1}|^2 dx$$

$$+G(|\varphi_{\omega_1}|) dx) - n(\omega_2^2 - \omega_1^2)/2 \int |\nabla\varphi_{\omega_1}| dx,$$

$$K_{\omega_2}(\varphi_{\omega_1}) = K_{\omega_1}(\varphi_{\omega_1}) - n(\omega_2^2 - \omega_1^2)/2 \int |\nabla\varphi_{\omega_1}|^2 dx,$$

but $K_{\omega_2}(\varphi_{\omega_1}) = 0$

$$\Longrightarrow K_{\omega_2}(\varphi_{\omega_1}) = n(\omega_2^2 - \omega_1^2)/2 \int |\nabla\varphi_{\omega_1}|^2 dx < 0$$

Therefore

$$d(\omega_1) = 1/n \int |\nabla\varphi_{\omega_1}|^2 dx > \inf\left\{1/n \int |\nabla v|^2 dx, K_{\omega_2}(v) \leqq 0, v \neq 0\right\},$$

since $K_{\omega_2}(\varphi_{\omega_1}) < 0$,

$$\Longrightarrow d(\omega_1) > d(\omega_2)$$

b) Again let $\omega_1 < \omega_2$ and $\psi_\beta(x) = \varphi_{\omega_1}(x/\beta)$, then

$$K_{\omega_2}(\psi_\beta) = (n-2)\beta^{n-2}/2\int\left|\nabla\varphi_{\omega_1}\right|^2 dx$$

$$+ n\beta^n\left((1-\omega_2^2)/2\int\left|\varphi_{\omega_1}\right|^2 dx + \int G(\left|\varphi_{\omega_1}\right|)dx\right),$$

$$K_{\omega_2}(\psi_\beta) = (n-2)\beta^{n-2}/2\int\left|\nabla\varphi_{\omega_1}\right|^2 dx - \beta^n\big((n-2)/2\int\left|\nabla\varphi_{\omega_1}\right|^2 dx$$

$$+n(\omega_2^2-\omega_1^2)2\int\left|\varphi_{\omega_1}\right|^2 dx\big)$$

Let

$$\Delta_{12} = (\omega_2^2-\omega_1^2)/2\int\left|\varphi_{\omega_1}\right|^2 dx, \tag{12}$$

then $K_{\omega_2}(\psi_\beta) = (n(n-2)d(\omega_1)/2)\beta^{n-2} - n\big((n-2)d(\omega_1)/2 + \Delta_{12}\big)$, and for

$$\beta_2 \equiv 1/\big(1+2\Delta_{12}/(n-2)d(\omega_1)\big)^{1/2}, \qquad K_{\omega_2}(\psi_{\beta_2}) = 0, \tag{13}$$

$$\therefore d(\omega_2) \leqq 1/n\int\left|\nabla\psi_{\beta_2}\right|^2 dx = \beta_2^{n-2}/n\int\left|\nabla\varphi_{\omega_1}\right|^2 dx = \beta_2^{n-2}d(\omega_1) \tag{14}$$

But for $\omega_1-\omega_2$ small, $|\Delta_{12}| < C(\omega_2-\omega_2)$, since $\int|\varphi_\omega|^2 dx$ is bounded. Therefore

$$\beta_2^{n-2} = 1-\Delta_{12}/d(\omega_1) + o(\Delta_{12}), \tag{15}$$

and from equation (14) we get

$$d(\omega_2) \leqq d(\omega_1) - (\omega_2^2-\omega_1^2)/2\int\left|\varphi_{\omega_1}\right|^2 dx + o(\omega_2-\omega_1), \tag{16}$$

or

$$d(\omega_2) \leqq d(\omega_1) - (\omega_2^2-\omega_1^2)/2\int\left|\varphi_{\omega_1}\right|^2 dx + o(\omega_2-\omega_1). \tag{17}$$

To show the second part of b) let $\psi_\gamma(x) = \varphi_{\omega_2}(x/\gamma)$, then

$$K_{\omega_1}(\psi_\gamma) = (n-2)\gamma^{n-2}/2\int\left|\nabla\varphi_{\omega_1}\right|^2 dx$$

$$+ n\gamma^n\left((1-\omega_1^2)/2\int\left|\varphi_{\omega_2}\right|^2 dx + \int G(\left|\varphi_{\omega_2}\right|)dx\right)$$

or

$$K_{\omega_1}(\psi_\gamma) = (n(n-2)d(\omega_2)/2)\gamma^{n-2} - n\big((n-2)d(\omega_2)/2\big)\gamma^n, \tag{18}$$

where

$$\Delta_{12} = (\omega_2^2 - \omega_1^2)/2 \int \left|\varphi_{\omega_2}\right|^2 dx, \tag{19}$$

since $\int \left|\nabla\varphi_{\omega_2}\right|^2 dx = nd(\omega_2)$ and $K_{\omega_2}(\varphi_{\omega_2}) = 0$.

For

$$\gamma_1 \equiv 1/\left(1 - 2\Delta_{2\,1}/(n-2)d(\omega_2)\right)^{1/2},$$
$$K_{\omega_1}(\psi_{\gamma_1}) = 0 \Longrightarrow d(\omega_1) \leqq 1/n \int \left|\nabla\psi_{\gamma_1}\right|^2 dx = \gamma_1^{n-2}/2 \int \left|\nabla\varphi_{\omega_2}\right|^2 dx = \gamma_1^{n-2} d(\omega_2) \tag{20}$$

but for $\omega_1 - \omega_2$ small

$$\gamma_1^{n-2} = 1 + \Delta_{2\,1}/d(\omega_2) + o(\omega_1 - \omega_2) \tag{21}$$

$$\therefore d(\omega_1) < d(\omega_2) + (\omega_2^2 - \omega_1^2)/2 \int \left|\varphi_{\omega_2}\right|^2 dx + o(\omega_1 - \omega_2) \tag{22}$$

The continuity of $d(\omega)$ follows from equation (17) and (22).

Next we'll need this lemma about strictly convex functions.

Lemma (2.1.14)[41]: Suppose $h(\omega)$ is a strictly convex function in a neighborhood of ω_0, then

$\forall \varepsilon > 0\ \exists N(\varepsilon) > 0_{\ni}: |\omega_\varepsilon - \omega_0| = \varepsilon$,

a) $\omega_\varepsilon < \omega_0 < \omega, |\omega - \omega_0| < \varepsilon/2$

$$\left(h(\omega_\varepsilon) - h(\omega)\right)/(\omega_\varepsilon - \omega) \leqq \left(h(\omega_0) - h(\omega)\right)/(\omega_0 - \omega) - 1/N(\varepsilon)$$

b) $\omega < \omega_0 < \omega_\varepsilon, |\omega - \omega_0| < \varepsilon/2$

$$\left(h(\omega_\varepsilon) - h(\omega)\right)/(\omega_\varepsilon - \omega) \geqq \left(h(\omega_0) - h(\omega)\right)/(\omega_0 - \omega) + 1/N(\varepsilon)$$

Proof. The proof is very easy to see geometrically from the picture below.

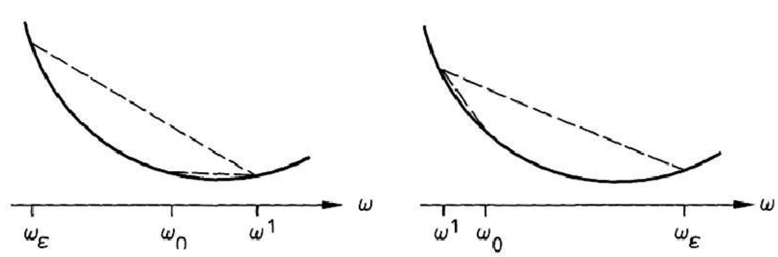

Fig. 2.1

We'll give a proof for the case $\omega_\varepsilon < \omega_0 < \omega$ and the second part follows by an identical argument. Assume that the claim is false. Then there is an $\varepsilon_0 > 0$ and a sequence $\omega_k \ni : |\omega_{\varepsilon_0} - \omega_0| = \varepsilon_0, |\omega_k - \omega_0| < \varepsilon/2$,

$$\big(h(\omega_0) - h(\omega_k)\big)/(\omega_0 - \omega_k) - 1/k < \Big(h\big(\omega_{\varepsilon_0}\big) - h(\omega_k)\Big)/\big(\omega_{\varepsilon_0} - \omega_k\big). \quad (23)$$

Pick ω_1 such that $\omega_{\varepsilon_0} < \omega_1 < \omega_0$, then

$$\big(h(\omega_0) - h(\omega_k)\big)/(\omega_0 - \omega_k) > \big(h(\omega_1) - h(\omega_k)\big)/(\omega_1 - \omega_k). \quad (24)$$

[since $h(\omega)$ is convex]. From equation (23) we get

$$\Big(h\big(\omega_{\varepsilon_0}\big) - h(\omega_k)\Big)/\big(\omega_{\varepsilon_0} - \omega_k\big) > \big(h(\omega_1) - h(\omega_k)\big)/(\omega_1 - \omega_k) - 1/k. \quad (25)$$

Since ω_k is bounded it has a convergent subsequence. Also denote it by ω_k such that $\omega_k \to \omega_2 \geqq \omega_0 > \omega_1 > \omega_{\varepsilon_0}$. Now from Eq. (25) and continuity of $h(\omega)$

$$\Big(h\big(\omega_{\varepsilon_0}\big) - h(\omega_2)\Big)/\big(\omega_{\varepsilon_0} - \omega_2\big) \geqq \big(h(\omega_1) - h(\omega_2)\big)/(\omega_1 - \omega_2) \quad (26)$$

But since $h(\omega)$ is strictly convex

$$\big(h(\omega_1) - h(\omega_2)\big)/(\omega_1 - \omega_2) > \big(h(\omega_1) - h(\omega_2)\big)/\big(\omega_{\varepsilon_0} - \omega_2\big), \quad (27)$$

which contradicts equation (26). Therefore the claim is true.

Theorem (2.1.15)[41]: Suppose that $d(\omega)$ is strictly convex in a neighborhood of ω_0, then for ω close to ω_0 $\exists \eta(\omega) > 0, \eta(\omega_0) = 0$, such that

$$d(\omega) - d(\omega_0) \geqq (\omega_0 - \omega)\omega_0 \int |\varphi_{\omega_0}|^2 dx + \eta(\omega).$$

Proof. Let $\omega < \omega_0, \omega$ close to ω_0. Then from Lemma (2.1.14) and for $\omega < \omega_0 < \omega_1$,

$$\big(d(\omega_1) - d(\omega)\big)/(\omega_1 - \omega) \leqq \big(d(\omega_1) - d(\omega_0)\big)/(\omega_1 + \omega_0) - 1/N(\omega) \quad (28)$$

and from Proposition (2.1.13)

$$\begin{aligned} &\big(d(\omega_1) - d(\omega)\big)/(\omega_1 - \omega_0) \\ &\qquad < -(\omega_1 + \omega_0)/2 \int |\varphi_{\omega_0}|^2 dx + o(\omega_1 - \omega_0)/(\omega_1 - \omega_0) \end{aligned} \quad (29)$$

From equation (28) and (29)

$$\begin{aligned} &\big(d(\omega_1) - d(\omega)\big)/(\omega_1 - \omega) \\ &\qquad \leqq -(\omega_1 + \omega_0)/2 \int |\varphi_{\omega_0}|^2 dx - 1/N(\omega) + o\,(\omega_1 + \omega_0)/(\omega_1 + \omega_0)\ . \end{aligned}$$

Let $\omega_1 \to \omega_0$, then by continuity of $d(\omega)$

$$\left(d(\omega_0) - d(\omega)\right)/(\omega_0 - \omega) \leqq - \ \omega_0 \int \left|\varphi_{\omega_0}\right|^2 dx - 1/N(\omega)$$

or

$$d(\omega) - d(\omega_0) \geqq \omega_0(\omega_0 - \omega) \int \left|\varphi_{\omega_0}\right|^2 dx + (\omega_0 - \omega)/N(\omega). \quad (30)$$

For $\omega > \omega_0$, from Lemma (2.1.14) and $\omega > \omega_0 > \omega_1$, we have

$$\left(d(\omega) - d(\omega_1)\right)/(\omega - \omega_1) \geqq \left(d(\omega_0) - d(\omega_1)\right)/(\omega_0 - \omega_1) + 1/N(\omega), \quad (31)$$

and from Proposition (2.1.13)

$$\begin{aligned}&\left(d(\omega_0) - d(\omega_1)\right)/(\omega_0 - \omega_1)\\ &\qquad \geqq (\omega_0 - \omega_1)/2 \int \left|\varphi_{\omega_0}\right|^2 dx + (\omega_1 - \omega_0)/(\omega_1 - \omega_0). \quad (32)\end{aligned}$$

Again from equation (31) and (32) and letting $\omega_1 \to \omega_0$,

$$\left(d(\omega) - d(\omega_0)\right) \geqq -\omega_0(\omega - \omega_0) \int \left|\varphi_{\omega_0}\right|^2 dx + (\omega - \omega_0)/N(\omega), \quad (33)$$

and this concludes the proof of Theorem (2.1.15).

Now if we consider the Cauchy problem

$$u_{tt} - \Delta u + u + f(|u|) \arg u = 0, \qquad x \in \mathbb{R}^n$$
$$u(0) = u_0 \in H_r^1(\mathbb{R}^n), \qquad u_t(0) = u_t \in L_r^2(\mathbb{R}^n),$$

we don't have strong solutions $\left(u(\cdot) \in C\left([0,T], H_r^1(\mathbb{R}^n)\right), u_t(\cdot) \in C\left([0,T) L_r^2(\mathbb{R}^n)\right)\right)$ for the general nonlinearities we are considering but we always have weak solutions

$$\left(u(\cdot) \in L^\infty\left([0,T), H_r^1(\mathbb{R}^n)\right), u_t(\cdot) \in L^\infty\left([0,T), L_r^2(\mathbb{R}^n)\right)\right),$$

that are weakly continuous in t. Also we don't necessarily have uniqueness, or energy identity, but we always can find a weak solution that satisfies the energy inequality

$$1/2 \int |u_t(t)|^2 dx + J_0(u(t) \leqq 1/2 \int |u_t|^2 dx + J_0(u_0),$$

provided $\int G(|u_0|) dx < \infty$ (see Strauss [50]).

Definition (2.1.16)[41]: Define the metric space X= {completion of $u \in C_{0r}^\infty(\mathbb{R}^n)$ with the metric

$$\varrho(u_1, u_2) = \|u_1 - u_2\| + \left| \int \left(G(|u_1|) - G(|u_1|)\right) dx \right|$$

and define the modulated energy functional of Eq. (2), $E_\omega(u,v) = 1/2\int |v|^2 dx + J_\omega(u), u \in X, v \in L_r^2(\mathbb{R}^n)$.

$$R_\omega^1 \equiv \{u \in X, v \in L_r^2(\mathbb{R}^n); E_\omega(u,v) < d(\omega), K_\omega(u) > 0\} \cup \{(0,0)\}$$
$$= \left\{u \in X, v \in L_r^2(\mathbb{R}^n); E_\omega(u,v) < d(\omega), 1/n \int |\nabla u|^2 dx < d(\omega)\right\},$$
$$R_\omega^2 = \{u \in X, v \in L_r^2(\mathbb{R}^n); E_\omega(u,v) < d(\omega), K_\omega(u) < 0, u \neq 0\}$$
$$= \left\{u \in X, v \in L_r^2(\mathbb{R}^n); E_\omega(u,v) < d(\omega), 1/n \int |\nabla u|^2 dx > d(\omega)\right\}.$$

Lemma (2.1.17)[41]: R_ω^1 and R_ω^2 are invariant regions under the flow of (2) for the solutions that satisfy the energy inequality.

Proof. We'll prove this by contradiction. Let $(u_0, u_1) \in R_\omega^1$ and assume that there exist a t_1 such that $\left(u(t_1), u_1(t_1)\right) \notin R_\omega^1$. By lower semi-continuity of $K_\omega\left(u(t)\right)$ there exist a minimal t_0 such that $\left(u(t_0), u_t(t_0)\right) \notin R_\omega^1$, i.e. $K_\omega\left(u(t_0)\right) \leqq 0$ and $K_\omega\left(u(t)\right) > 0$ for $t \in [0, t_0)$. Now

$$1/n \int |\nabla u(t_0)|^2 dx \leqq \lim_{\substack{t\to t_0 \\ t<t_0}} 1/n \int |\nabla u(t)|^2 dx$$
$$\leqq \lim_{\substack{t\to t_0 \\ t<t_0}} \left(1/n \int |\nabla u(t)|^2 dx + K_\omega u(y)\right),$$

therefore

$$1/n \int |\nabla u(t_0)|^2 dx \leqq \lim_{t\to t_0} J_\omega\left(u(t)\right) \leqq \lim_{t\to t_0} E_\omega\left(u(t), u_t(t)\right) d(\omega)$$

and we also have $K\left(u(t_0)\right) \leqq 0$. This contradicts the definition of

$$d(\omega) = \inf\left\{1/n \int |\nabla u(t)|^2 dx + K_\omega(v) \leqq 0, v \neq 0\right\}$$

Therefore R_ω^1 is invariant under the flow of equation (2). Similarly we can show that R_ω^2 is also invariant.

Lemma (2.1.18)[41]: Let $u(t)$ be a solution of

$$u_{tt} - \Delta u + u + f(|u|) \arg u = 0,$$
$$u(0) = u_0 \in X, \qquad u_t(0) = u_1 \in L_r^2(\mathbb{R}^n),$$

that satisfies the energy inequality. Then for every $K > 0$ there exist $\delta(K)$ such that if

$$\varrho(u_0, \varphi_{\omega_0}) + \left|u_1 - i\omega_0\varphi_{\omega_0}\right|_2 < \delta(K),$$

then $d(\omega_0 + 1/K) \leqq 1/n \int |\nabla u(t)|^2 dx \leqq d(\omega_0 - 1/K) \forall t.$

Proof. Fix $K > 0$ and let $\omega_+ = \omega_0 + 1/K, \omega_- = \omega_0 - 1/K,$ and $u(t) = v_+(t)e^{i\omega+t} = v_-(t)e^{i\omega+t}$. Then

$$v_{\pm tt} + 2i\omega_\pm v_{\pm t} - \Delta v_\pm + (1 - \omega_\pm^2)v_\pm + f(|v_\pm|)\arg v_\pm = 0,$$
$$v_\pm(0) = u_0, \qquad v_{\pm t}(0) = u_1 - i\omega_\pm u_0.$$

The energy inequality of this equation is

$$E_{\omega_\pm}\left(v_\pm(t), v_{\pm t}(t)\right) \leqq 1/2 \int \left|u_1 - i\omega_\pm u_0\right|^2 dx + 1/2 \int |\nabla u_0|^2 dx$$
$$+(1 - \omega_\pm^2)/2 \int |u_0|^2 dx + \int G(|u_0|)dx, \qquad (34)$$

Or

$$1/2 \int \left|v_{\pm t}(t)\right|^2 dx + J_{\omega_\pm}(u(t)) \leqq 1/2 \int \left|u_1 - i\omega_\pm u_0\right|^2 dx + J_{\omega_\pm}(u_0), \qquad (35)$$

but

$$\left|u_1 - i\omega_\pm u_0\right|_2 \leqq \left|u_1 - i\omega_0\varphi_{\omega_0}\right|_2 + \left|\omega_0\varphi_{\omega_0} - \omega_\pm\varphi_{\omega_0}\right|_2 + \left|\omega_\pm\varphi_{\omega_0} - \omega_\pm u_0\right|_2$$
$$\Longrightarrow 1/2 \int \left|u_1 - i\omega_\pm u_0\right|^2 dx \leqq (\omega_0 - \omega_\pm)^2/2 \int \left|\varphi_{\omega_0}\right|^2 dx$$
$$+ C(\omega_\pm, \omega_0)f \mid \int \left|u_1 - i\omega_\pm u_0\right|^2 dx + \int \left|u_0 - \varphi_{\omega_0}\right|^2 dx. \qquad (36)$$

Now since $d(\omega_+) < d(\omega_0) < d(\omega_-)$ and

$$d(\omega_0) \equiv 1/n \int \left|\nabla\varphi_{\omega_0}\right|^2 dx = 1/n \int |\nabla u_0|^2 dx + O(\delta). \qquad (37)$$

If we pick δ small enough we have

$$d(\omega_+) < 1/n \int |\nabla u_0|^2 dx < d(\omega_-), \qquad (38)$$

$$E_{\dot\omega_\pm}(u_0, u_1 - i\omega_\pm u_0) \leqq (\omega_0 - \omega_\pm)^2/2 \int \left|\varphi_{\omega_0}\right|^2 dx + J_{\omega_\pm}(\varphi_{\omega_0}) + O(\delta)$$
$$\leqq (\omega_0 - \omega_\pm)\omega_0 \int \left|\varphi_{\omega_0}\right|^2 dx + J_{\omega_0}(\varphi_{\omega_0}) + O(\delta) \qquad (39)$$

since $(\omega_0^2 - \omega_\pm^2)/2 + (\omega_0 - \omega_\pm)^2/2 = (\omega_0 - \omega_\pm)\omega_0$.

By Theorem (2.1.15) and for δ small

$$(\omega_0 - \omega_\pm)\omega_0 \int \left|\varphi_{\omega_0}\right|^2 dx + d(\omega_0) + O(\delta)$$

$$\leqq (\omega_0 - \omega_\pm)\omega_0 \int \left|\varphi_{\omega_0}\right|^2 dx + d(\omega_0) + \eta(\omega_\pm) \leqq d(\omega_\pm), \quad (40)$$

and therefore from equation (39) we have the energy inequality

$$1/2 \int \left|v_{\pm t}(t)\right|^2 dx + J_{\omega_\pm}\big(u(t)\big) < d(\omega_\pm) \qquad \forall t$$

$$\Rightarrow d(\omega_\pm) < 1/n \int |\nabla u(t)|^2 dx < d(\omega_-) \qquad \forall t \qquad (41)$$

by Lemma (2.1.17)

Theorem (2.1.19)[41]: If $d(\omega)$ is strictly convex at ω_0, then the standing waves of frequency ω_0 are stable in the following sense: for every $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that if $\varrho\big(u_0, \varphi_{\omega_0}\big) + \left|u_1 - i\omega_0\varphi_{\omega_0}\right|_2 < \delta(\varepsilon)$ then

$$\inf_{\psi \in S_{\omega_0}} (\varrho(u(t), \psi) + |u_t(t) - i\omega_0\psi|_2) < \varepsilon \; for \; all \; t.$$

Proof. Assume not. Then $\exists$ sequence $\big(u_{0_k}, u_{1_k}\big), (t^k)$ and an $\varepsilon_0 > 0$ such that

$$\big(u_{0_k}, u_{1_k}\big) \to \big(\varphi_{\omega_0}, i\omega_0\varphi_{\omega_0}\big) \in X \oplus L_r^2$$

and

$$\inf_{\psi \in S_{\omega_0}} \varrho\Big((u_k(t^k), \psi)\Big) + |u_{kt}(t^k) - i\omega_0\psi|_2 > \varepsilon_0.$$

From Lemma (2.1.18), $\exists$ subsequence also denote it by $\Big(u_k(t^k)\Big)$ such that

$$d(\omega_0 + 1/k) \leqq 1/n \int |\nabla u_k(t^k)|^2 dx \leqq d(\omega_0 - 1/k),$$

and $(\int |u_k(t^k)|^2 dx)$ is bounded (by Theorem (2.1.7)). Now as $k \to \infty$

$$1/n \int |u_k(t^k)|^2 dx \to d(\omega_0) \qquad (42)$$

from continuity of $d(\omega)$. From equation (41) we have

$$1/2 \int |v_{+\,kt}(t^k)|^2 dx + J_{\omega_+}\Big(u_k(t^k)\Big) < d(\omega_0 + 1/k),$$

therefore $\exists$ subsequence such that

$$J_{\omega_0}\Big(u_k(t^k)\Big) \to d^1 \leqq d(\omega_0). \qquad (43)$$

By Corollary (2.1.10) equation (42) and (43) imply that $\exists \psi \in S_{\omega_0}$ such that

$$|u_k(t^k) - \psi| \to 0\,,$$

$$J_{\omega_0}\Big(u_k(t^k)\Big) \to d(\omega_0).$$

Again from equation (41) we have

$$\int |v_{+\,kt}(t^k)|^2 dx = |u_{kt}(t^k) - i\omega_+ u_k(t^k)|_2^2 \to 0,$$

and

$$\left|\int \Big(G(|u_k(t^k)|) - G(|\psi|)\Big) dx\right| \to 0,$$

which contradicts the assumption of instability.

We'll present here two examples where we have stable standing waves.

Theorem (2.1.20)[41]: The equation

$$u_{tt} - \Delta u + u - |u|^{p-1}u = 0, \qquad x \in \mathbb{R}^n, \qquad n \geqq 3,$$

has stable standing waves for $1 < p < 1 + 4/n$.

Proof. In order to show the existence of stable standing waves it is sufficient to show that $d(\omega)$ is strictly convex for some interval of ω.

Solution of the equation

$$-\Delta\varphi_\omega + (1 - \omega^2)\varphi_\omega - |\varphi_\omega|^{p-1}\varphi_\omega = 0 \tag{44}$$

has the following scaling property: let $v(x) = (1/\delta)\varphi_\omega(x/\beta)$, then

$$-\delta\beta^2\Delta v + (1 - \omega^2)\delta v - \delta^p|v|^{p-1}v = 0, \tag{45}$$

and for

$$\beta^2 = 1 - \omega^2 = \delta^{p-1}. \tag{46}$$

Equation (44) becomes

$$-\Delta v + v - |v|^{p-1}v = 0. \tag{47}$$

Now

$$\int |\nabla v|^2 dx = \beta^{n-2}/\delta^2 \int |\nabla\varphi_\omega|^2 dx$$

$$\Longrightarrow \int |\nabla\varphi_\omega|^2 dx = (1 - \omega^2)^\alpha \int |\nabla\varphi_0|^2 dx \tag{48}$$

where $\alpha = \big(4 - (n-2)(p-1)\big)/2(p-1)$.

Now it becomes easy to see when $d(\omega) \equiv 1/n \int |\nabla\varphi_\omega|^2 dx$ is strictly convex,

$$d(\omega) = (1-\omega^2)^{\alpha}/n \int |\nabla\varphi_0|^2 dx = (1-\omega^2)^{\alpha} d(0) \tag{49}$$

$$\Rightarrow d''(\omega) = [-2\alpha(1-\omega^2)^{\alpha} + 4\alpha(\alpha-1)\omega^2(1-\omega^2)^{\alpha-2}]d(0)$$

$$\Rightarrow d''(\omega) = -2\alpha\,[-1 + (2\alpha-1)\omega^2](1-\omega^2)^{\alpha-2} d(0), \tag{50}$$

since $1 < p < 1 + 4/n, \alpha > 0, 2\,\alpha - 1 > 0$.

Therefore $d''(\omega) > 0$ implies $-1 + (2\alpha - 1)\omega^2 > 0$,

Moreover $\omega^2 > 1$ for equation (44) to have a solution. Therefore and this set is not empty for $1 < p < 1 + 4/n$.

Remark (2.1.21)[41]: For $1 + 4/n < p < 1 + 4/(n-2)$ we showed that all standing waves obtained by Theorem (2.1.7) are unstable [46].

Another example we'll consider is one which appears in studying spin-0 particles in field theory [44]. The potential, i.e. $G(|w|)$ for this model is of the form $G(|w|) = -|u|^4/4 + |u|^6/6$

Proposition (2.1.22)[41]: The equation

$$-\Delta u + (1-\omega^1)u - |u|^2 u + |u|^4 u = 0, \qquad x \in \mathbb{R}^3 \tag{51}$$

has nontrivial solution φ_ω of lowest energy for $\omega^2 \in (13/16, 1)$. Moreover as ω^2

Proof. For equation (51) to have nontrivial solution it is sufficient to have $1 - \omega^2 > 0$ and $\exists\eta$ such that $(1-\omega^2)\eta^2/2 + G(\eta) < 0$. Now

$$(1-\omega^2)\eta^2/2 - \eta^4/4 + \eta^6/6 < 0$$

for some η if

$$(1/4)^2 - 4(1-\omega^2)/12 > 0$$

$$\Rightarrow \omega^2 > 13/16.$$

We show that $d(\omega) \to \infty$ as $\omega^2 \to 13/16$ by contradiction. Assume that $d(\omega)$ remains bounded then by Theorem (2.1.7) $\|\varphi_\omega\|$ is bounded. This implies that $\exists$ sequence $\omega_k \to 13/16$ and $v \in H_r^1(\mathbb{R}^3)$ such that $\varphi_{\omega_k} \xrightarrow{w} v \in H_r^1(\mathbb{R}^3)$. Again by Theorem (2.1.7)

$$K_{\omega_0}(v) \leqq \underline{\lim}\, K_{\omega_0}(\varphi_{\omega_k}) = \underline{\lim}(\omega_k^2 - \omega_0^2)/2 \int |\varphi_{\omega_k}|^2 dx + K_{\omega_k}(\varphi_{\omega_k}), \tag{52}$$

where $\omega_0^2 = 13/16$. But $K_{\omega_k}(\varphi_{\omega_k}) = 0$, therefore

$$K_{\omega_0}(v) \leqq 0. \tag{53}$$

Now $K_{\omega_0}(u) > 0 \ \ \forall u \in, u \neq 0$ so from (53) we have that $v = 0$. By equation (52) we have that-the convergence is strong. But $d(\omega) = 1/2 \int |\varphi_\omega|^2 dx$ is monotone decreasing function,

$$d(\omega_1) > 0 \ \Longrightarrow 0 = \varliminf_{k\to\infty} d(\omega_k) > d(\omega_1) > d$$

a contradiction. Therefore $d(\omega) \to \infty$ as $\omega \to 13/16$.

Theorem (2.1.23)[41]: The equation

$$u_{tt} - \Delta u + u - |u|^2 u \ + \ |u|^4 u = 0, \qquad x \in \mathbb{R}^3$$

has stable standing waves for ω close to 13/16.

Proof. By Proposition (2.1.22) $d(\omega) \to \infty$ as $\omega \to 13/16$ and by Proposition (2.1.13) $d(\omega)$ is monotone decreasing function of ω. Therefore the graph of $d(\omega)$ looks like

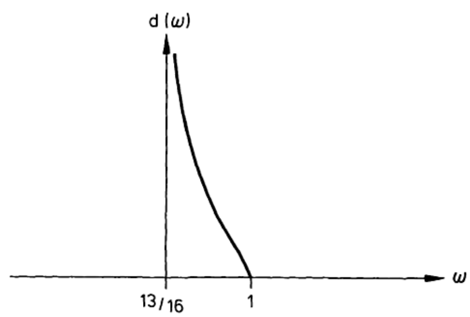

Fig. 2.2

Now it is easy to see that $d(\omega)$ is strictly convex for ω close to 13/16 and by Theorem (2.1.19) these standing waves are orbitally stable.

Remark (2.1.24)[41]: This particular example was studied numerically by Anderson [45] where he showed that for ω^2 close to 13/16 there are stable standing waves and that for ω close to 1 they are unstable and this is precisely what we show in [46].

Section (2.2): Zhukovskij Asymptotic Stability:

Stability of motion in dynamical systems is an old but still active area of studies. At the close of the 19th century, three types of stability were established for motion in continuous dynamical systems, i.e., for solutions of differential equations. The Liapunov stability [52,53-54] and Poincaré stability [53,55] or orbital stability are well-known to the people. But the Zhukovskij stability [53,56] is less known, which was sometimes regarded as a special kind of Poincaré stability [57].

In [53], these types of stability were discussed with stress on the corresponding instability, where these notions of stability were recalled and some examples were given to show the differences and relations between these types of stability. However, geometric meaning of these differences is still unclear, hence it is necessary to go further to make out geometric mechanism of these types of stability. It is found out that in case of asymptotic stability, Liapunov asymptotic stability and Zhukovskij asymptotic stability are of simple and distinct geometric meaning in terms of omega limit set, and the purpose of this section is to demonstrate these observations.

Consider a continuous dynamical system described by autonomous differential equations

$$\frac{dx}{dt} = f(x), \qquad x \in G \subset R^n, \tag{54}$$

where G is a closed bounded domain in R^n, and $f \in C^r(G)(r \geq 1)$.

Denote by $x(t, x_0), 0 \leq t < +\infty$, the solution of (54) with initial point $x_0 = x(0, x_0)$. Let $\mathrm{o}^+(x_0)$ be the forward or positive orbit passing x_0, i.e., $\mathrm{o}^+(x_0) = \{x(t, x_0) | 0 \leq t < +\infty\}$. In addition, let $B_\delta(x)$ be the open ball with center x and radius d. Clearly only bounded orbits are of interest to us, so we shall consider orbits bounded in the domain G.

The well-known Liapunov stability is as follows:

Liapunov stability. A solution $x(t, x_0)$ to system (54) is said to be Liapunov stable if, for each $e > 0$, there exists $\delta(\varepsilon) > 0$, such that for every $y_0 \in B_\delta(x_0)$ the relation

$$\|X(t, x_0) - X(t, y_0)\| < \varepsilon$$

holds for $t \geq 0$.

This stability means that if two orbits are near in the beginning then they remain near together synchronously for all the time $t \geq 0$.

Based on the Liapunov stability, the well-known asymptotic Liapunov stability can be described as the following.

Asymptotic Liapunov stability. A solution $x(t, x_0)$ to system (54) is said to be asymptotically Liapunov stable if, it is Liapunov stable and there exists $\eta > 0$ such that for every $y_0 \in B_\eta(x_0)$, one has the relation

$$\|X(t, x_0) - X(t, y_0)\| \to 0 \quad as \quad t \to \infty$$

Zhukovskij stability. A solution $x(t, x_0)$ to system (54) is said to be Zhukovskij stable if, for any $e > 0$, there exists $\delta(\varepsilon) > 0$, such that for every $y_0 \in B_\delta(x_0)$, one can find two parameterizations τ_1, τ_2 of time t, such that

$$\|x(\tau_1(t), x_0) - x(\tau_1(t), y_0)\| < \varepsilon$$

holds for $t \geq 0$, where τ_1 and τ_2 are homeomorphisms from $[0, +\infty)$ to $[0, +\infty)$ with $\tau_1(0) = \tau_2(0) = 0$.

Asymptotic Zhukovskij stability. A Zhukovskij stable solution $x(t, x_0)$ to system (54) is said to be asymptotically Zhukovskij stable if, for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$, such that for every $y_0 \in B_\delta(x_0)$, one can find two parameterizations τ_1, τ_2 of time t, such that

$$\|x(\tau_1(t), x_0) - x(\tau_2(t), y_0)\| \to 0 \quad as \quad t \to +\infty$$

where τ_1 and τ_2 are homeomorphisms from $[0, +\infty)$ to $[0, +\infty)$ with $\tau_1(0) = \tau_2(0) = 0$.

It is easy to see [53] that (asymptotic) Zhukovskij stability implies (asymptotic) Liapunov stability. However the converse is not true, the reader can convince himself by constructing some examples. Furthermore, it is not difficult to see that these two types of stability are equivalent in case of $x(t, x_0)$ being an equilibrium point.

In this section we shall show that the omega limit set of an asymptotically stable orbit of autonomous system is just an equilibrium point set. This may be a known fact to some people, but for completeness we give a proof here.

Theorem (2.2.1)[51]: If the solution $x(t, x_0)$ to system (54) is asymptotically Liapunov stable, then the omega limit set $\omega(x_0)$ consists of equilibrium points of (54).

Proof. Since the solution $x(t, x_0)$ is asymptotically Liapunov stable, for every value Δt small enough, there holds the following relation:

$$\lim_{t\to\infty}\left(x(t+\Delta t,x_0)-x(t,x_0)\right)=0,$$

or

$$\begin{aligned}
&\lim_{t\to\infty}\left(\frac{d}{dt}x(t,x_0)\Delta t+\mathrm{O}\left(\frac{d^2}{dt^2}x(t,x_0)\Delta t^2\right)\right),\\
&=\lim_{t\to\infty}f\left(x(t,x_0)\right)\Delta t+\mathrm{O}\left(\frac{d}{dt}f(t,x_0)\Delta t^2\right),\\
&=\lim_{t\to\infty}f\left(x(t,x_0)\right)\Delta t+\mathrm{O}\left(\frac{\partial f}{\partial x}\left(x(t,x_0)\right)f\left(x(t,x_0)\right)\Delta t^2\right),\\
&=\lim_{t\to\infty}f\left(x(t,x_0)\right)\left(\Delta t+\mathrm{O}\frac{\partial f}{\partial x}\left(x(t,x_0)\right)\Delta t^2\right)\\
&=0.
\end{aligned}$$

Since $x(t,x_0)$ is contained in the closed bounded domain, of $\frac{\partial f}{\partial x}\left(x(t,x_0)\right)f\left(x(t,x_0)\right)$ is bounded, therefore the above relation holds for arbitrary small Δt. It follows that for Δt small enough,

$$\Delta t+\mathrm{O}\left(\frac{\partial f}{\partial x}\left(x(t,x_0)\right)\right)\Delta t^2\neq 0$$

This means that

$$\lim_{t\to\infty}f\left(x(t,x_0)\right)=0$$

Hence $x(t,x_0)$ must approach an equilibrium point set.

From the computational view point, the above theorem indicates a fact that if one takes an orbit and calculates its Liapunov exponents that are all negative, then one could reasonably expect that this orbit would go to an equilibrium point as t goes to infinity thus showing a way to decide whether iterates of a point can reach a zero point of a map.

It can be expected that the omega limit set of a asymptotically Zhukovskij stable orbit would be of a simpler structure. In fact, the omega limit set should just be a cycle, i.e., a closed orbit. We will demonstrate this observation under a little more conditions.

To facilitate the ensuing arguments, we introduce a notion of uniform asymptotic Zhukovskij stability.

Definition (2.2.2)[51]: A solution $x(t, x_0)$ to system (54) is said to be uniformly asymptotically Zhukovskij stable if, for any $\varepsilon > 0$, there exists $\delta > 0$, which is independent of ε, such that for every $t^0 > 0$, if $y_0 \in B_\delta\big(x(t^0, x_0)\big)$, then one can find two parameterizations τ_1, τ_2 of time t, such that

$$\left\|x\big(\tau_1(t), x(t^0, x_0)\big) - x(\tau_2(t), y_0)\right\| < \varepsilon$$

holds for $t \geq 0$, and

$$\left\|x\big(\tau_1(t), x(t^0, x_0)\big) - x(\tau_2(t), y_0)\right\| \to 0, \qquad as \quad t \to \infty$$

where τ_1 and τ_2 are homeomorphisms from $[0, +\infty)$ to $[0, +\infty)$ with $\tau_1(0) = \tau_2(0) = 0$.

Note that this definition is sharply different from the so-called uniform stability of equilibrium points, for example, see [58].

Theorem (2.2.3)[51]: If a solution $x(t, x_0)$ to system (54) is uniformly asymptotically Zhukovskij stable, then its omega limit set is a periodic orbit.

Proof. Denote by $D_\delta(x)$ the $n-1$ dimensional disc:

$$D_\delta(x) = T_x \cap B_\delta(x),$$

where T_x is the $n-1$ dimensional hyperplane at the point x, and is perpendicular to the vector $f(x)$ of (54) at the regular point x (the point where $f\,(x) \neq\ 0$), $B_\delta(x)$ is the ball defined as before.

Now suppose that the orbit $x(t, x_0)$ to system (54) is uniformly asymptotically Zhukovskij stable. By long tubular flow theorem [59], it is not difficult to see that there exists a $r < \delta$ (δ is the number defined as in Definition (54)) and a Poincaré map from $D_\sigma(x_0)$ into $D_\sigma x(t, x_0)$:

$$\phi_t : D_\sigma(x_0) \to D_\sigma\big(x(t, x_0)\big)$$

for every $t > 0$.

Since $x(t, x_0)$ is uniformly asymptotically Zhukovskij stable, the relation

$$\text{diam } \Big(\phi_t\big(D_\sigma(x)\big)\Big) \to 0 \qquad as \quad t \to \infty$$

holds for every point $x \in x(t, x_0)$, where diam means the diameter of $\phi_t\big(D_\sigma(x)\big)$ in terms of

$$\text{diam}\,(A) = \max_{x,y,\in A} \|x - y\|$$

Let ω be an omega limit point of $x(t, x_0)$, then there exists a sequence $t_j \to \infty$ (as $j \to \infty$) with $x(t_j, x_0) \in D_\sigma(\omega)$ and $x(t_j, x_0) \to \omega$. Therefore there exists $k > 0$ such that

$$\|x(t_k, x_0) - \omega\| < \sigma/8, \qquad D_{\sigma/2}(\omega) \subset D_\sigma\big(x(t_k, x_0)\big).$$

Since

$$\text{diam}\,\Big(\phi_{t_j - t_k}\Big(D_{\sigma/2}(\omega)\Big)\Big) \to 0 \quad as\, j \to \infty,$$

there exists $i > 0$, such that

$$\text{diam}\Big(\phi_{t_j - t_k}\Big(D_{\sigma/2}(\omega)\Big)\Big) < \sigma/8 \qquad \text{for } t \geq i. \tag{55}$$

On the other hand, by the definition of the asymptotic uniform Zhukovskij stability, it is easy to see that there exists a homeomorphism $h : [0, \infty) \to (0, \infty)$ with $\|x(h(t), \omega) - x(t, x(t_k, x_0)\| \to 0$ as $t \to \infty$. Furthermore, one can find $m > k$ such that

$$\|x(t_m, x_0) - \omega\| < \sigma/8$$

and

$$\big\|x(h(t_m - t_k), \omega) - x\big(t_m - t_k, x(t_k, x_0)\big)\big\| < \sigma/8:$$

Note that $x\big(t_m - t_k, x(t_k, x_0)\big) = x(t_m, x_0)\big) \in D_\sigma(\omega)$ so one can choose the homeomorphism h such that $x(h(t_m - t_k), \omega) \in D_\sigma(\omega)$.

It follows that

$$\begin{aligned}
&\|x(h(t_m - t_k), \omega) - \omega\| \\
&\quad \leq \big\|x(h(t_m - t_k), \omega) - x\big(t_m - t_k, x(t_k, x_0)\big)\big\| + \big\|x\big(t_m - t_k, x(t_k, x_0)\big) - \omega\big\| \\
&\quad = \|x(h(t_m - t_k), \omega) - x(t_m, x_0)\| + \|x(t_m, x_0) - \omega\| < \sigma/4.
\end{aligned}$$

Now for every $y \in D^{\sigma/2}(\omega)$, take $t_a = \max\{t_m, t_i\}$, one gets by Theorem (2.2.3)

$$\|x(h(t_a - t_k), y) - x(h(t_a - t_k), \omega)\| < \sigma/8;$$

where h^* is a homeomorphism from $[0, \infty)$ to $[0, \infty)$ such that $x(h^*(t_a - t_k), y) \in D_\sigma(\omega)$.

Now it is obvious that

$$\|x(h^*(t_a - t_k), y) - \omega\|$$
$$\leq \|x(h^*(t_a - t_k), y) - x(h(t_a - t_k), \omega)\| + \|x(h(t_a - t_k), \omega) - \omega\|$$
$$< \sigma/8 + \sigma/4 \ < \sigma/2.$$

This means that the Poincaré map

$$\phi: D_{\sigma/2}(\omega) \to D_{\sigma}(\omega)$$

satisfies

$$\phi\left(D_{\sigma/2}(\omega)\right) \subset D_{\sigma/2}(\omega)$$

It follows from the fixed point theorem that there exists a point x_p such that

$$\phi(x_p) = x_p,$$

and the orbit $x(t, x_p)$ is a periodic orbit.

Because of the uniform asymptotic Zhukovskij stability of $x(t, x_0)$, the orbit $x(t, x_p)$ is an omega limit set of $x(t, x_0)$, again due to the uniform asymptotic Zhukovskij stability of $x(t, x_0)$, this periodic orbit is the unique omega limit set of $x(t, x_0)$.