Chapter 1

One –Dimensional Finite Approximation

1.1 Variational Formulation of Two-Point Boundary –Value Problems

In this chapter we want to discuss a basis of finite element approximation and weaker variational statement of the problem. Moreover, to provide some symmetry in the formulation, we generally prefer to choose a formulation in which the trial function u and test function v have a same degree of smoothness that is we would like to have a variational statement in which the highest order of derivatives of u that appears is the same that of v. For more details see [1,2,3,4,5] To demonstrate how this to done, consider a boundary –value problem defined by the system

$$\left.\begin{aligned} -\frac{d}{dx}\left[k(x)\frac{du(x)}{dx}\right] + c(x)\frac{du(x)}{dx} + b(x)u(x) &= f(x) \\ for\ points\ x \in \Omega_i\ , i = 1,2,3,4 & \\ \left[\!\left[k(x)\frac{du(x)}{dx}\right]\!\right] = 0 \qquad & at\ x = x_1 \\ -\left[\!\left[k(x)\frac{du(x)}{dx}\right]\!\right] = \hat{f} \qquad & at\ x = x_2 \\ \left[\!\left[k(x)\frac{du(x)}{dx}\right]\!\right] = 0 \qquad & at\ x = x_3 \\ \alpha_0\frac{du(0)}{dx} + \beta_0 u(0) = \gamma_0 \quad , \quad \alpha_l\frac{du(l)}{dx} + \beta_l u(l) &= \gamma_l \end{aligned}\right\} \tag{1.1}$$

Where the domain is divided into four smooth subdomains Ω_i by the interface points $x_0 = 0,\ x_1, x_2\ , x_3\ and\ x_4 = l$. when

$$[\![k(x_i)\acute{u}(x_i)]\!] = \lim_{x \to x_i^+} k(x)\acute{u}(x) - \lim_{x \to x_i^-} k(x)\acute{u}(x)$$

Now we can arrive quite easily at a variational statement of this problem first note that the solution u is quite smooth inside each of the subdomains Ω_i. Indeed, by virtue of the fact that the differential

equation in (1.1) must hold in these subdomains, u is at least twice differentiable there, we construct the residual error function r.

$$r(x) = -[k(x)\acute{u}(x)\acute{]}+c(x)\acute{u}(x) +b(x)u(x)\text{-}f(x) \ , \ \ x \in \Omega_i \ , i = 1,2,3,4$$

Multiply r by sufficiently smooth test function v defiend over the entire interval $0 \le x \le l$, and integrate the first term in the product rv by parts over each subdomains . The result over subdomain Ω_i is of the form:

$$\int_{\Omega_i} rv\,dx = -k\acute{u}v|_{x_{i-1}}^{x_i} + \int_{\Omega_i}(k\acute{u}\acute{v} + c\acute{u}v + buv)\,dx - \int_{\Omega_i} fv\,dx \qquad (1.2)$$

$$i = 1,2,3,4$$

Now , since u is the solution of our problem ,

$$\int_{\Omega_i} rv\,dx = 0 \quad and \quad \sum_{i=1}^{4}\int_{\Omega_i} rv\,dx = 0 \qquad (1.3)$$

Thus substituting (1.2) into (1.3) yields:

$$\int_0^l (k\,\acute{u}\acute{v} + c\acute{u}v + buv)\,dx + k(0)\acute{u}(0)v(0) + [\![k(x_1)\acute{u}(x_1)]\!]v(x_1)$$

$$+[\![k(x_2)\acute{u}(x_2)]\!]v(x_2) + [\![k(x_3)\acute{u}(x_3)]\!]v(x_3) - k(l)\acute{u}(l)v(l)$$

$$= \int_0^l \tilde{f}v\,dx \qquad (1.4)$$

For all smooth test functions v.

And the function $\tilde{f}$ appearing on the right-hand side of (1.4) is understood to be the "smooth part" or integrable part of the source f. In view of (1.1),

$$[\![k(x_1)\acute{u}(x_1)]\!] = 0 \quad , \quad -[\![k(x_2)\acute{u}(x_2)]\!] = \hat{f}$$

$$[\![k(x_3)\acute{u}(x_3)]\!] = 0 \quad , \quad \acute{u}(0) = \frac{\gamma_0 - u(0)\beta_0}{\alpha_0} \quad , \quad \acute{u}(l) = \frac{\gamma_l - u(l)\beta_l}{\alpha_l}$$

Thus (1.4) reduces to

$$\int_0^l (k\acute{u}\acute{v} + c\acute{u}v + buv)\ dx$$
$$= \int_0^l \tilde{f}\, v\ \ dx + \hat{f}v(x_2) - \frac{k(0)}{\alpha_0}\,[\gamma_0 - u(0)\beta_0]v(0)$$
$$+ \frac{k(l)}{\alpha_l}\,[\gamma_l - u(l)\beta_l]v(l) \qquad (1.5)$$

for all admissible test function v .

A variational statement of the two-point boundary-value problem (1.1) now takes on the following form : find a function u such that (1.5) holds for all test functions v in asuitable class of admissible function.

This is a rather remarkable result . We have managed to transform the entire system of differential equations , jump conditions , and boundary conditions in (1.1) in to a single equation in which all of the features of the solution and the discontinuous data are intrinsically present.

The variational problem (1.5) characterizes the solution as a function defined over the entire interval $0 \leq x \leq l$, rather than piecewise as in (1.1) . Nevertheless, it is clear that any solution of (1.1) is automatically a solution of (1.5), we henceforth view the variational statement(1.5) as given variation-nal problem . It will always include the classical problem as special case whenever the solution is sufficiently smooth .

Problem (1.5) is still incompletely defined . Our study established that the specification of the appropriate space of admissible functions lies at the heart of our analysis. In addition , the character of the boundary terms in (1.5) deserves further comment .

We now list several fundamentally important observations which lead to a more concrete definition of variational statement of our problem:

1- By integrating $(k\acute{u})v$ once by part, we have produced an integral involving products of the first derivatives of trial functions u and test function v. Thus, if we wish to identify a class of admissible functions on which smoothness assumptions are barely strong enough to make this integral well defined, it is sufficient to make u and v to be members of a class of functions, denoted H^1, whose derivatives of order 1 and less are square-integrable over Ω. In other words a test function v will belong to H^1 if

$$\int_0^l [(\acute{v})^2 + v^2]\, dx < +\infty \qquad (1.6)$$

With these conventions, it is clear that the variational statement of problem (1.1) is as follows : Find a function $u \in H^1$ such that (1.5) holds for all test function

$$v \in H^1 \qquad (1.7)$$

We also encounter frequency the subclass of functions in H^1 that vanish at $x = 0, x = l$ we denote this class by H_0^1, that is $v = v(x)$ is member of H_0^1 if :

(a) v satisfies (1.6)

(b) $v(0) = 0$, $v(l) = 0$

In particular, consider the case in which we have essential boundary conditions of form

$$u(0) = \frac{\gamma_0}{\beta_0} \qquad \text{and} \qquad u(l) = \frac{\gamma_l}{\beta_l} \qquad (1.8)$$

Instead of the natural boundary conditions in (1.1), then the variational boundary-value problem becomes:

Find a function u in H^1 satisfying (1.8) such that

$$\int_0^l (k\acute{u}\acute{v} + c\acute{u}v + buv)\, dx = \int_0^l \tilde{f} v\, dx + \hat{f} v(x_2) \qquad (1.9)$$

For all $v \in H_0^1$

2- Boundary conditions for a problem such as (1.1) cannot be arbitrarily constructed ;they must be in some sense ,compatible with the governing differential equation of the problem . For example the specification of an arbitrary set of boundary conditions to define a specific solution of a differential equation may lead to an "ill-posed" problem in which the solution does not exist at all exists but not uniquely defined or "well-behaved". Another useful feature of the variational formulation is that whenever the boundary conditions can be incorporated naturally into an integration by-parts formula , as was the case in our derivation of (1.5), they are automatically compatible with the differential equation Thus , there are intrinsic features of the variational statement of a boundary-value problem that serve to characterize well-posed problems .

3- Recall that in our discussion of physical origins of two-point problems , boundary conditions fell into two categories:
essential boundary conditions , in which the value of the solution u is specified , and the natural boundary conditions , in which $\acute{u}$ or a combination of u and $\acute{u}$ is specified . Exactly the same classification arises naturally in variational formulations such as (1.5). Suppose that u and v are in class H^1. Then derivatives of u and v and higher may not exist . If v is barely smooth enough to be in H^1, it is impo- ssible to be impose conditions on derivatives of v of order 1 or higher. From this in observation, it follows that boundary conditions enter variational boundary-value problems of the type in (1.5) in two distinct ways : the essential boundary conditions , which involve the specification on of values of the solution, enter the

problem the definition of the of the space of admissible functions , whereas the natural boundary conditions which involve the specification of derivatives of the solution, dictate the actual form of the variational equation .
Inparticular, we see from the form of problem (1.5) that natural boundary conditions appear on the right-hand side of the variational equality (1.5).

To fix ideas, consider as examples

(i) $-u''(x) + u(x) = f(x)$, $0 < x < l$

$u(0) = 0$, $u(l) = 0$

(ii) $-u''(x) + u(x) = f(x)$, $0 < x < l$

$\acute{u}(0) = \gamma_0$, $\acute{u}(l) = \gamma_l$

According (1.5) and (1.9) , the variational statements of these problems are :

(v-i) find u in H_0^1 such that

$$\int_0^1 (\acute{u}\acute{v} + uv)\, dx = \int_0^1 fv\, dx \quad \text{for all} \quad v \in H_0^1$$

Where in H_0^1 is class of functions v satisfying (1.6) and vanishing at the boundaries : $v(0) = 0 = v(l)$.

(v-ii) find $u \in H^1$ such that

$$\int_0^1 (\acute{u}\acute{v} + uv)\, dx = \int_0^1 fv\, dx - \gamma_0 v(0) + \gamma_l v(l) \quad , \quad for\ all \quad v \in H^1$$

Clearly , the essential boundary conditions (v-i) enter the variational problem through the definition of the class H_0^1 of admissible test functions , whereas the natural boundary conditions in (v-ii) enter as data on the right-hand side of the equation.

(4) As a final remark , we may also define as the energy norm for problem (1.5) when $b(x) > 0$,

$$\|v\|_E = \left[\int_0^l (k\acute{v}^2 + bv^2)\, dx\right]^{1/2} \tag{1.10}$$

Frequently , we employ the equivalent $H^1 -$ norm

$$\|v\|_1 = \left[\int_0^l (\acute{v}^2 + v^2)\, dx\right]^{1/2} \tag{1.11}$$

This norm provides a natural measure of error in approximations to problem like (1.5) .

Galerkin Approximations:

The Galerkin approximation of second-order boundary-value problems follows exactly the same lines as those discussed for the model problem . We identify a finite set of basis functions $\{\emptyset_1, \emptyset_2, \dots, \emptyset_N\}$ in H^1 that define a finite-dimensional subspace of test functions H^h in H^1. We then seek a function $u_h \in H^h$ of the form :

$$u_h(x) = \sum_{j=1}^N \alpha_j \emptyset_j(x) \tag{1.12}$$

Which satisfies the variational problem on H^h. For problem (1.5) , this procedure leads to the discrete problem

$$\int_0^l (k\acute{u}_h\acute{v}_h + c\acute{u}_h v_h + bu_h v_h)\, dx = \int_0^l \tilde{f} v_h\, dx + \hat{f} v_h(x_2)$$

$$-\frac{k(0)}{\alpha_0}[\gamma_0 - u_h(0)\beta_0]v_h(0) + \frac{k(l)}{\alpha_l}[\gamma_l - u_l(l)\beta_l]v_h(l) \tag{1.13}$$

For all $v_h \in H^h$

Or , equivalently,

$$\sum_{j=1}^N K_{ij}\alpha_j = F_i \quad , \; i = 1,2, \dots, N \tag{1.14}$$

Where the stiffness matrix K_{ij} is now of the form

$$K_{ij} = \int_0^1 \left(k\acute{\emptyset}_i\acute{\emptyset}_j + c\acute{\emptyset}_i\emptyset_j + b\emptyset_i\emptyset_j\right) - \frac{k(0)\beta_0}{\alpha_0}\emptyset_i(0)\emptyset_j(0)$$

$$+ \frac{k(l)\beta_l}{\alpha_l}\emptyset_i(l)\emptyset_j(l) \qquad (1.15)$$

And the components of the load vector are:

$$F_i = \int_0^l \tilde{f}\emptyset_i\, dx + \hat{f}\emptyset_i(x_2) - \frac{k(0)}{\alpha_0}\gamma_0\emptyset_i(0) + \frac{k(l)}{\alpha_l}\gamma_l\emptyset_i(l) \qquad (1.16)$$

With $1 \le i, j \le N$

Upon solving (1.14) for the coefficients α_j our Galerkin approximation of the problem is obtained immediately from (1.12) as before , this method of approximation becomes very powerful whenever we have a systematic technique for constructing the basis functions $\emptyset_i$. Such a systematic is provided , by finite element techniques , we note that the stiffness matrix k_{ij} is not symmetric whenever the coefficient c in (1.15) is not identically zero.

Minimization of Energy Functionals:

Our reference to certain weak forms of boundary-value problems as " variation " statements arises from the fact that , whenever the operators involved possess a certain symmetry to be identified below , a weak form of the problem can be obtained which is precisely that arising in standard problems in the calculus of variations . In such cases , the variational boundary-value problem represents a characterization of the function u which minimizes "or maximizes" the energy of the problem since one may find this interpretation useful from time to time , we summarize here some of these variational concepts.

Let us consider once again a class H_0^1 of functions v defined on interval $0 < x < l$ and vanishing at the endpoints . Suppose that J denotes a real valued function defined on H_0^1 given by

$$J(v) = \frac{1}{2}\int_0^l (k\acute{v}^2 + bv^2 - 2fv)\, dx \tag{1.17}$$

Where k , b , and f are given functions of x , with k and b satisfying $0 < k_0 \leq k(x) < \infty$ and $b(x) \geq 0$ for all x , k_0 being a positive constant . We may generally regard J as energy of certain physical system , note that J is "a function of functions" (i.e , the domain of J is the class H_0^1 of admissible functions) and that the values of J are real numbers . Any function with these properties is called a functional.

Now a classical minimization problem in the calculus of variations is to seek a particular function $u \in H_0^1$ at which J assumes it is smallest value over the whole class H_0^1 . In other words , "u is the minimize of J over H_0^1" means that

$$J(u) \leq J(v) \quad , \; for\ all \quad v \in H_0^1 \tag{1.18}$$

The minimization problem now reduces to one of characterizing the minimizing function u . Toward this end , we consider an arbitrary function $\eta \in H_0^1$ of the form $\eta = u + \epsilon v$, v arbitrary, where ϵ is a positive number, then η can be made as close to u as possible by choosing ϵ small enough . The "perturbation" ϵv in u is called a variation in u and is often written δu . The value of energy at η is

$$\begin{aligned} J(\eta) &= J(u + \epsilon v) \\ &= J(u) + \epsilon \delta J(u; v) + \epsilon^2 \delta^2 J(v) \end{aligned} \tag{1.19}$$

Where by a direct calculation using (1.17),

$$\delta J(u; v) = \int_0^1 (k\acute{u}\acute{v} + buv - fv)\, dx \tag{1.20}$$

and

$$\delta^2 J(v) = \frac{1}{2}\int_0^l (k\acute{v}^2 + bv^2)\, dx \tag{1.21}$$

The quantity $\delta J(u; v)$ is called the first variation in J at u and $\delta^2 J(v)$ is called the second variation in J at u . It is clear that $\delta J(u; v)$ can also be calculated using the formula:

$$\delta J(u;v) = lim_{\epsilon\to 0}\frac{1}{\epsilon}[J(u+\epsilon v) - J(u)]$$

$$= \frac{\partial}{\partial\epsilon}J(u+\epsilon v)|_{\epsilon=0} \qquad (1.22)$$

Now since $\delta^2 J(v)$ is , because of our assumptions on k and b, always greater than or equal to zero, and since u is minimize of J,

$$J(u) \le J(u+\epsilon v) = J(u) + \epsilon\delta J(u;v) + \epsilon^2\delta^2 J(v)$$

Thus,

$$\delta J(u;v) + \epsilon\delta^2 J(v) \ge 0$$

or taking the limit as ϵ goes to zero,

$$\delta J(u;v) \ge 0 \qquad \text{for all} \quad v \in H_0^1$$

But this inequality must also hold if v replaced by $-v$, so that if u is the actual minimize of J, we must have

$$\delta J(u;v) = 0 \qquad \text{for all} \quad v \in H_0^1 \qquad (1.23)$$

In other words , the minimize of J is characterized as the solution of the variational boundary-value problem

$$\int_0^l (k\acute{u}\acute{v} + buv)\,dx = \int_0^l fv\,dx \qquad \text{for all } v \in H_0^1 \qquad (1.24)$$

We recognize (1.24) as a variational statement of the classical boundary-value problem

$$\left.\begin{aligned} -[k(x)\acute{u}(x)\acute{]} + b(x)u(x) = f(x), \quad 0 < x < l \\ u(0) = 0 \quad , \quad u(l) = 0 \end{aligned}\right\} \qquad (1.25)$$

These concepts drawn from variation calculus are the basis for our use of the term "variational boundary-value problem" when we refer to the weak statement (1.24) of (1.25) . We continue to refer to problems of the form (1.24) as variational problems even in those cases in which they cannot be derived from a problem of minimizing some energy functional.

It is also clear that the problem of approximating the solution u of (1.24) can be approached as one of seeking a minimize of J in some finite-dimensional subspace H^h of H_0^1 . Then

If $u_h(x) = \sum_{i=1}^{N} u_i \emptyset_i(x)$,

$$J(u_h) = \frac{1}{2}\int_0^l [k(\acute{u}_h)^2 + bu_h^2 - 2fu_h]\, dx$$

$$= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} u_j \left[\int_0^l (k\acute{\emptyset}_i\acute{\emptyset}_j + b\emptyset_i\emptyset_j - 2f\emptyset_i)\, dx\right] u_i \qquad (1.26)$$

$$= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} u_i\, K_{ij} u_j - \sum_{i=1}^{N} F_i u_i$$

Where k_{ij} and F_i are components of the stiffness matrix and the load vector, respectively:

$$K_{ij} = \int_0^l \left(k\acute{\emptyset}_i\acute{\emptyset}_j + b\emptyset_i\emptyset_j\right) dx \quad ; \quad F_i = \int_0^l f\emptyset_i\, dx \qquad (1.27)$$

$J(u_h)$ is minimize by choosing the coefficients u_i , so that

$$\frac{\partial J(u_h)}{\partial u_i} = 0 \quad ; \quad i = 1,2, \dots, N \qquad (1.28)$$

Thus , once again , we arrive at the system of equations

$$\sum_{i=1}^{N} K_{ij} u_j = F_i \ ; \qquad i = 1,2, \dots, N \qquad (1.29)$$

The approximation scheme outlined above is called the Ritz method We see that the Ritz method can be used as a basis for constructing finite element approximations of variational boundary-value problems whenever the problem is equivalent to finding a function u which makes the first variation of an energy functional J vanish for all admissible variations v . We conclude with a final observation of some importance .

Note that the governing differential equation in linear second-order problems can be written compactly in the operator form

$$Au = f \tag{1.30}$$

Where A is the differential operator for the problem . If u and v are arbitrary smooth functions vanishing at $x = 0$ and $x = l$, the operator A is said to be formally self-adjoint whenever

$$\int_0^l vAu\, dx = \int_0^l uAv\, dx \tag{1.31}$$

It can be shown that an energy functional J of the type in (1.1) exists for a given boundary-value problem only when the operator A for the problem is self-adjoint . For self-adjoint problems and therefore, for all problems derivable from an energy functional in the manner outlined above , the stiffness matrix (1.27) resulting from a Ritz approximation will always be symmetric . clearly , when Ritz's method is applicable , it leads to the same system of equations as Galerkin's method .The operator was not self-adjoint. For this reason , it is clear that Galerkin's method is applicable to a wider class of problems than is Ritz's method .

Finite Element Interpolation:

While we have presented the finite element method as a technique for systematically applying Galerkin's method to the approximate solution of boundary-value problems, a brief reflection reveals that the underlying ideas also provide a basis for methods of interpolation.

Indeed , the finite element concepts can be used to construct curve-fitting schemes where in any given function g can be approximated by a system of piecewise polynomials , the values of g which coincide with those of g at prescribed nodal points in the domain of g . When viewed in this way , a variety of choices of element shape functions come to mind which are merely bases of well-known methods of interpolating smooth functions .

Suppose that we are given a function g defined on interval $0 \leq x \leq l$ and the g is smooth enough to be continuously differentiated k times and

that its derivative of order $k+1$ is bounded (finite) on this interval . We wish to construct a finite element approximation (an interpolant) g_h of g that coincides with g at the nodal points , and we wish to estimate the accuracy of such approximations . We begin by partitioning the interval into a collection of finite elements. Then comes the problem of showing just how general shape functions can be constructed. We describe next a technique for generating polynomial shape functions of any degree k (i.e , each shape function Ψ_i^e will contain monomials in x up to x^k , k being a positive integer) . The technique leads to the Lagrange families of finite elements , the name "Lagrange" being borrowed from the notion of Lagrange interpolation , from which these element families are derived .

A Lagrange finite element employing polynomials of degree k is constructed as follows:

1- We consider a typical finite element Ω_e isolated from the mesh and we establish a local coordinate system ξ , with origin now at the center of the element , scaled so that $\xi = -1$ at the left endpoint and $\xi = 1$ at the right endpoint , this is achieved by the simple linear stretching transform for the general element Ω_e

$$\xi = \frac{2x-(x_i+x_{i+1})}{x_{i+1}-x_i} \qquad (1.32)$$

So that points x such that $x_i \le x \le x_{i+1}$ are transformed to points ξ such that $-1 \le \xi \le 1$. We perform our element calculations on this reference or "master" element $\widehat{\Omega}$ and denote the shape functions on the master element by $\widehat{\Psi}_i(\xi)$.

2- For shape functions of degree k , we identify $k+1$ nodes (including the endpoints) which divide the element into k equal segments let ξ_i , $i = 1,2,\dots,k+1$, denote the ξ-coordinates of each node . For each node ξ_i , we form the product of k linear functions $(\xi - \xi_j)$, $j = 1,2,\dots,k+1$, $j \neq i$. Note that this product is zero at all nodes except i .

These functions are of the form:

Node 1*:* $(\xi - \xi_2)(\xi - \xi_3) \ \dots \ (\xi - \xi_{k+1})$

Node 2*:* $(\xi - \xi_1)(\xi - \xi_3)(\xi - \xi_4) \ \dots \ (\xi - \xi_{k+1})$

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Node *i :* $(\xi - \xi_1)\dots(\xi - \xi_{i-1})(\xi - \xi_{i+1}) \ \dots \ (\xi - \xi_{k+1})$

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Node *k+1:* $(\xi - \xi_1)(\xi - \xi_2) \ \dots \ (\xi - \xi_k)$

For each node *i*. We evaluate the corresponding product in step 2 at $\xi = \xi_i$ and divide the product functions by this value . This normalizes the polynomials so that $\widehat{\Psi}_i(\xi_i) = 1$ and produces the correct shape function $\widehat{\Psi}_i(\xi)$ corresponding to each node I . For example ,

$$\widehat{\Psi}_1(\xi) = \frac{(\xi - \xi_2)(\xi - \xi_3) \dots (\xi - \xi_{k+1})}{(\xi_1 - \xi_2)(\xi_1 - \xi_3) \dots (\xi_1 - \xi_{k+1})}$$

$$\widehat{\Psi}_2(\xi) = \frac{(\xi - \xi_1)(\xi - \xi_3) \dots (\xi - \xi_{k+1})}{(\xi_2 - \xi_1)(\xi_2 - \xi_3) \dots (\xi_2 - \xi_{k+1})}$$

or , in general ,

$$\widehat{\Psi}_i(\xi) = \frac{(\xi-\xi_1)(\xi-\xi_2)\dots(\xi-\xi_{i-1})(\xi-\xi_{i+1})\dots(\xi-\xi_{k+1})}{(\xi_i-\xi_1)(\xi_i-\xi_2)(\xi_i-\xi_{i-1})(\xi_i-\xi_{i+1})\dots(\xi_i-\xi_{k+1})} \qquad (1.33)$$

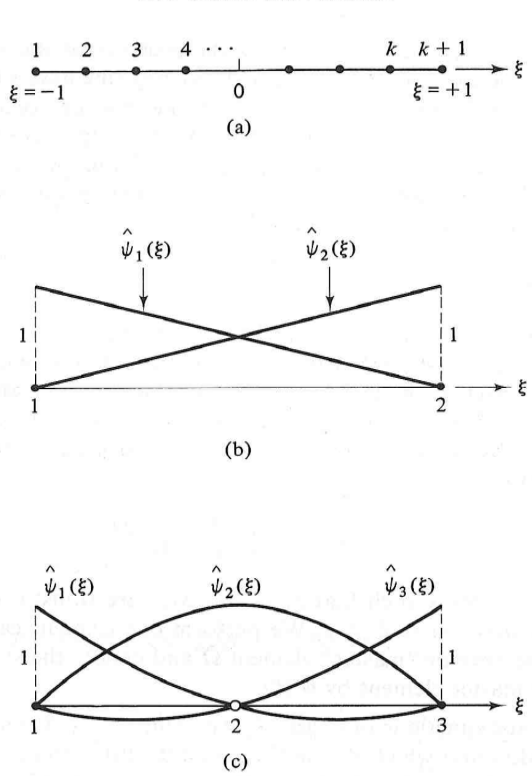

Figure 1.1 (a) Amaster element $\hat{\Omega}$ with k=1 nodes ; (b) linear shape functions corresponding to k=1 and (c) an element with three nodes and piecewise quadratic shape functions (k=2)

These functions have the property that

$$\widehat{\Psi}_i(\xi_j) = \begin{cases} 1 & if\ i = j \\ 0 & if\ i \neq j \end{cases} \tag{1.34}$$

Which implies that $\widehat{\Psi}_i(\xi)$ is linearly independent . These *k+1* functions define a basis for the set of all polynomials of degree k and we say that the basis $\widehat{\Psi}_i$ is complete .

This implies that any polynomial of degree k or less can be represented uniquely in terms of the Lagrange polynomial basis. This property carries over to the global basis functions $\emptyset_i$, every polynomial of degree $\leq k$ can be expressed in a unique way as a linear combination of the basis functions $\emptyset_i$ generated using the Lagrange shape functions in (1.33) .

Note that for $k = 1$ (linear shape function) , we have two nodes and shape functions are :

$$\left.\begin{aligned} \widehat{\Psi}_1(\xi) &= \frac{\xi - \xi_2}{\xi_1 - \xi_2} = \frac{1}{2}(1 - \xi) \\ \widehat{\Psi}_2(\xi) &= \frac{\xi - \xi_1}{\xi_2 - \xi_1} = \frac{1}{2}(1 + \xi) \end{aligned}\right\} \tag{1.35}$$

If we introduce the change of coordinates $\xi = 2\bar{x}/h - 1$, where $\bar{x} = x - x_i$, then $\Psi_1^e(x) = 1 - \bar{x}/h$ and $\Psi_2^e(x) = \bar{x}/h$. Upon connecting elements together to form the finite element mesh , the element functions match up to produce piecewise-linear basis functions $\emptyset_i$.

For $k = 2$ (quadratic shape functions) , we have three nodes and the shape functions

$$\widehat{\Psi}_1(\xi) = \tfrac{1}{2}\xi(\xi - 1) \quad , \widehat{\Psi}_2(\xi) = 1 - \xi^2 \quad , \widehat{\Psi}_3(\xi) = \tfrac{1}{2}\xi(\xi + 1) \tag{1.36}$$

The corresponding global basis functions $\emptyset_i$.

An estimate of the error for piecewise-linear Lagrange interpolation can be obtained using Taylor series .

let $E = \mathrm{g} - \mathrm{g}_h$ be the interpolation error function and consider an arbitrary element Ω_e with points $x_i \le x \le x_{i+1}$ in the mesh . We assume that g has bounded second derivatives . Now , on Ω_e , $E = \mathrm{g} - \mathrm{g}_h$ can be expanded in a local Taylor series about any interior point $\bar{x}$ *:*

$$E(x) = E(\bar{x}) + \acute{E}(\bar{x})(x - \bar{x}) + \frac{1}{2}\acute{\acute{E}}(\xi)(x - \bar{x})^2 \tag{1.37}$$

Where ξ is a point between $\bar{x}$ and x .

Since g_h is the interpolant of g , the error E is zero at the endpoints x_i, x_{i+1}.

We next select $\bar{x}$ to be that point at which $|E|$ is maximum . At this point , $\acute{E}(\bar{x}) = 0$, so that (1.37) reduces to

$$E(x) = E(\bar{x}) + \frac{1}{2}\acute{E}(\xi)(x - \bar{x})^2 \tag{1.38}$$

For $x_i \leq x \leq x_{i+1}$.

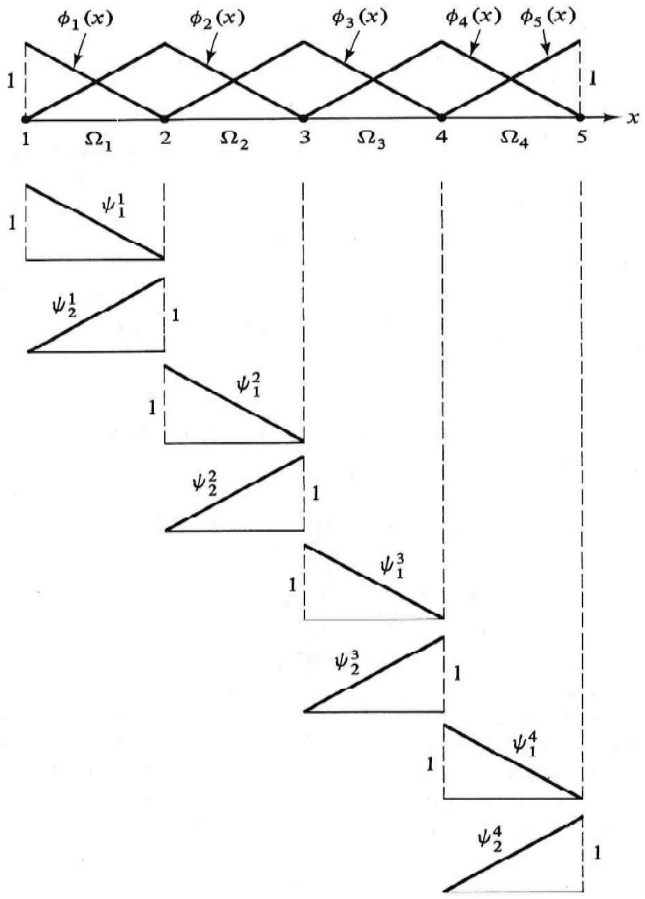

Figure 1.2 piecewise –linear basis functions $\emptyset_i$ for a 4-element mesh generated by linear shape functions ,Ψ_1^e, Ψ_2^e defined over each element

Next , We set $x = x_i$ or x_{i+1} whichever is closer to $\bar{x}$ (say x_i) then

$$E(\bar{x}) = -\frac{1}{2}\acute{E}(\xi)(x_i - \bar{x})^2 \tag{1.39}$$

hence

$$|E(\bar{x})| = \frac{1}{2}\left|\acute{E}(\xi)\right|(x_i - \bar{x})^2 \tag{1.40}$$

Since $x_{i+1} - x_i = h$, then $|x_i - \bar{x}| \leq h/2$ in (1.37) and we have the error bound

$$|E(\bar{x})| \leq \frac{h^2}{8}\left|\acute{E}(\xi)\right| \tag{1.41}$$

Finally, $E = \mathrm{g} - \mathrm{g}_h$ implies $\acute{E} = \acute{\mathrm{g}} - \acute{\mathrm{g}}_h = \acute{\mathrm{g}}$ within Ω_e . Introducing this result in (1.41) and maximizing over all the elements , we obtain the final estimate

$$\underset{0\leq x\leq l}{max}|E(x)| \leq \frac{h^2}{8}\underset{0\leq x\leq l}{max}\left|\acute{g}(x)\right| \tag{1.42}$$

Since $\acute{g}$ is bounded on the domain , $\acute{\mathrm{g}} \leq C \leq \infty$ in this inequality , C being a constant.

A similar procedure can be used to derive error bounds for Lagrange elements of higher degree . For a finite element employing complete polynomials of degree , the error bound assumes the form

$$\|E\|_\infty = \underset{0\leq x\leq l}{max}|E(x)| \leq Ch^{k+1} \tag{1.43}$$

C being a constant independent of h . This estimate indicates that the finite element interpolant g_h of g will converge to g (in the $\|\bullet\|_\infty$ −norm) at a rate of $k + 1$ as h approaches zero .

Since , in general , the local Taylor's series expansion of g will contain polynomial terms of all degrees up through degree k , it is important that the interpolant (and , hence , also the shape functions in each element) be able to represent each of these terms , if for example , the shape functions contain independent terms proportional to x^0 (constant) , x^2 , x^3 , ... , x^k but none proportional to x^1, then the error will , in general , be only proportional to h instead of h^{k+1} as indicated in (1.43) . If constants are missing from the shape functions , then the representation need not converge at all .Thus , the requirement that the set of shape functions contain complete polynomial is of considerable importance.

As a simple example of the finite element interpolation , consider an interpolation of the function $g(x) = sin\,\pi x$ on interval $0 \leq x \leq 1$ by two quadratic elements , the nodes are at $x = 0.0, 0.25, 0.50, 0.75$, and 1.0 , and the value of g at these nodes 0.0 ,0.707 ,1.0 ,0.707 , and 0.0, so that the finite element interpolant is

$$g_h(x) = 0.707\emptyset_2(x) + \emptyset_3(x) + 0.707\emptyset_4(x) \qquad (1.44)$$

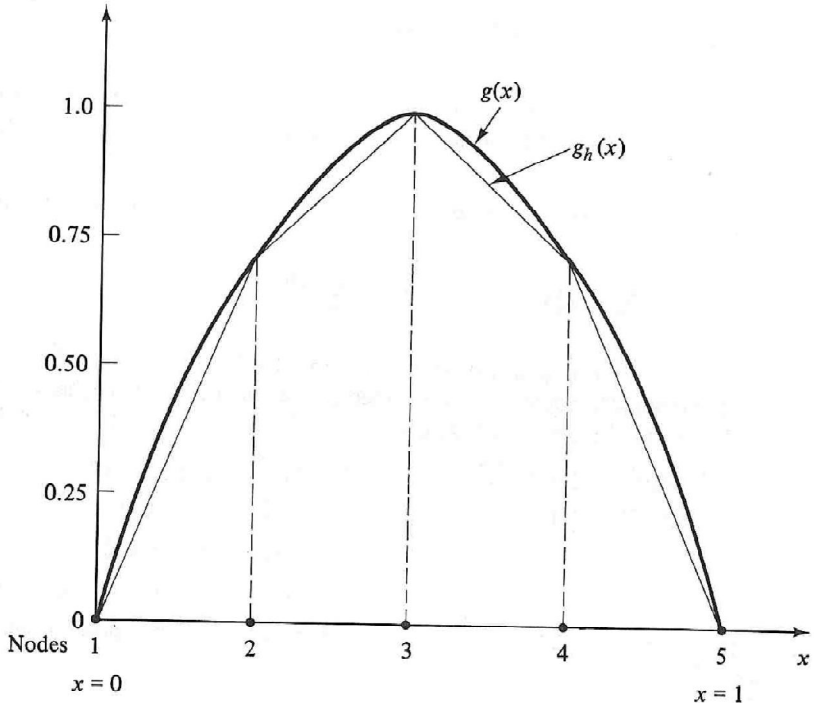

Figure 1.3 Interpolation of g(x)=sin πx using two quadratic elements

To obtain an estimate of the interpolation error , note that $\max_{0\leq x\leq l}\left|\acute{g}(x)\right| = \pi^2$, so that $|g(x) - g_h(x)| \leq ch^3$, where $c = \pi^2/48$.

A final comment of considerable importance should be made . In arriving at the error bound (1.43) , we assumed that the given function g is so smooth that it has continuous derivatives of order $\leq k$. Suppose that it does not . Assume, to the contrary, that g has continuous derivatives of only order s where $0 \leq s \leq k$. Then, no matter how large the degree k of

our interpolation g_h , only its first s terms may be effective in approximating g . then instead of (1.43) , we have

$$\max_{0\le x\le l}|g(x)-g_h(x)|\le ch^3 \tag{1.45}$$

And the accuracy of our approximation, being independent of k , cannot be increased by increasing the degree of the polynomials defining Ψ_i^e. We can improve the accuracy by reducing h as long as $s>0$.

1.2 Finite Element Approximation

At this point in our study , we have accumulated sufficient information to complete a detailed finite element analysis of second –order two-point boundary-value problems . In this section , we describe all of steps through which (1.5) and Galerkin approximation (1.15) can be used as a basis for the analysis of quite two-point boundary-value problems by the finite element method . The procedure is outlined as follows .

Partitioning Ω and Selection of Shape Functions

We begin by partitioning Ω into a number of finite elements Ω_e of length h_e (thus $\sum_e h_e = l$) , we will , for definiteness , assume that the domain of the exact solution of our problem is composed of the four natural smooth subdomains . Suppose that the concentrated source terms in f is located at the coordinate $x=\bar{x}\,(\bar{x}=x_2)$. Then the flux $\sigma=-k\acute{u}$ will experience a jump $[\![\sigma]\!]=\hat{f}$ at $\bar{x}$. However , our element shape functions will always have the property that their derivatives are continuous within each element and , therefore , they cannot accommodate a jump such as this . For this reason , we will always construct our mesh so that a nodal point is located at all points of discontinuity of the data . Then terms such as $\hat{f}v_h(\bar{x})$ representing prescribed jumps will never enter the local equations which characterize the approximation over individual elements . These terms enter the analysis when the contributions of individual elements are summed .

We now focus our attention on typical element Ω_e and consider possible choices of shape functions Ψ_i^e . We have at our disposal any member of

Lagrange families of shape functions . We could , for instance , use linear, quadratic , or cubic shape functions or , for the matter, shape functions consisting of polynomials of any degree k .

As k increases , the bandwidth of the resulting stiffness matrix increases and thus , in general , so does the computational effort required in solving the final system of equations . For this reason , it is rare that shape functions containing polynomials of degree higher than $k = 2$, or $k = 3$ are used in applications . We shall generally use linear or quadratic shape functions in the analysis of one-dimensional problems.

Calculation of Element Matrices and Equations

Having selected an appropriate set of shape functions , we now come to a crucial step in the analysis, the calculation of local approximations of the problem over each element . To see how this is done , note that in the actual problem, for any smooth subdomain Ω_i between points s_1 and s_2, we have, for all admissible v .

$$\int_{s_1}^{s_2}(k\acute{u}\acute{v} + c\acute{u}v + buv)\,dx = \int_{s_1}^{s_2}\tilde{f}v\,dx + \sigma(s_1)v(s_1) - \sigma(s_2)v(s_2) \qquad (1.46)$$

Where $\sigma(s_i)$ is the flux at points s_i , *i=1, 2* . Again note that the fluxes $\sigma(s_i)$ appear as given natural boundary data in the right-hand side of this equation. Now let us consider a typical element Ω_e in the finite element mesh with endpoints $s_1^e \, and \, s_2^e$. Using (1.46), We formulate a variatinal statement of our problem for this single element independent of whatever boundary conditions might be actually imposed at $x = 0$ and $x = l$. Thus over each element we have a variational boundary-value problem of the form :

$$\int_{s_1^e}^{s_2^e}(ku_h^{e'}v_h^{e'} + cu_h^{e'}v_h^e + bu_h^e v_h^e)\,dx$$

$$= \int_{s_1^e}^{s_2^e}\tilde{f}v_h^e\,dx + \sigma(s_1^e)v_h^e(s_1^e) - \sigma(s_2^e)v_h^e(s_2^e) \qquad (1.47)$$

Where u_h^e and v_h^e represent restrictions of u_h and v_h to Ω_e . It is important to realize that the $\sigma(s_i^e)$ are the true values of the flux at (s_i^e) and not their approximations ; as explained earlier, the quantities $\sigma(s_i^e)$ correspond to natural boundary conditions at the endpoints of Ω_e . Note also that , unlike (1.16) , no point-source terms such as $\hat{f}v_h^e(\bar{x})$ appear in (1.47) because of our decision to locate end nodal points of elements at these points .

We remark that we have chosen to use a global *x*-coordinate system in (1.47) only in order to clarify how the contributions from each element are actually summed together in generating the final stiffness and load matrices. In actual computations, these contributions are generally calculated for a master (reference) element in terms of normalized local coordinate ξ in (1.32) and are then transformed to the appropriate coordinates for each element in the mesh . Note that u_h^e in (1.47) is of the form

$$u_h^e(x) = \sum_{j=1}^{N_e} u_j^e \Psi_j^e(x) \tag{1.48}$$

Where N_e is the number of nodes in Ω_e , Ψ_j^e are the shape functions for this element , and u_j^e is the value of u_h^e at the node x_j^e of the element

$$u_j^e = u_h^e\,(x_j^e) \quad , \quad j=1,2,\ldots, N_e \tag{1.49}$$

Upon substituting (1.48) into (1.47) and taking $v_h^e = \Psi_i^e$ we arrive at the system of linear equations of the form

$$\sum_{j=1}^{N_e} k_{ij}^e\, u_j^e = f_i^e + \sigma(s_1^e)\Psi_i^e(s_1^e) - \sigma(s_2^e)\Psi_i^e(s_2^e) \tag{1.50}$$

In (1.50) , k_{ij}^e are the entries in the element stiffness matrix and f_i^e are the components of the element load vector for Ω_e.

$$k_{ij}^e = \int_{s_1^e}^{s_2^e} \left(k\Psi_i^{\acute{e}}\Psi_j^{\acute{e}} + c\Psi_i^{\acute{e}}\Psi_j^e + b\Psi_i^e\Psi_j^e \right) dx \,,$$

$$i,j = 1,2,\dots,N_e \qquad (1.51)$$

$$f_i^e = \int_{s_1^e}^{s_2^e} \tilde{f}\Psi_i^e \, dx$$

In actual finite element calculations, the integrals in (1.51) are rarely evaluated in closed form . Instead, the entries k_{ij}^e are generally computed using numerical integration rules, which are of sufficient accuracy. Also, it is common practice to calculate f_i^e using the interpolant of f rather than f itself for example , if $\tilde{f}$ is the continuous part of f (excluding point sources) and if

$$f_h^e(x) = \sum_{i=1}^{N_e} \tilde{f}\,(x_i^e)\Psi_i^e(x)$$

Then, instead of the formula in (1.51), we use

$$f_i^e = \int_{s_1^e}^{s_2^e} f_h^e\Psi_i^e \, dx \qquad (1.52)$$

In this way we can define the data f in our approximation by specifying its values at the nodal points.

Element Assembly

Having calculated the matrices and equations describing our approximation over each finite element , the next step in our analysis is to assemble the equations describing the approximation on the entire mesh by adding up the contributions to these equations furnished by each element.

To fix ideas , consider the special case in which linear shape functions of the form in (1.35) are used . Each element then has two nodes , and therefore there are two equations per element of the following form:

$$\left.\begin{aligned} k_{11}^e u_1^e + k_{12}^e u_2^e &= f_1^e + \sigma(s_1^e) \\ k_{21}^e u_1^e + k_{22}^e u_2^e &= f_2^e + \sigma(s_2^e) \end{aligned}\right\} \qquad (1.53)$$

Here the subscripts 1 and 2 are labels of the endpoint nodes on a typical element and $\sigma(s_1^e)$ and $\sigma(s_2^e)$ represent the actual values of the flux $\sigma = -k\acute{u}$ at the nodes . Of course , these subscripts are to be relabeled upon assembling the elements so as to coincide with appropriate node numbers *1,2,3,...N* in the final mesh . For example , if the element is to fit between nodes 6 and 7 in a mesh , u_1^e in (1.53) is actually u_6 , u_2^e is u_7, $\sigma(s_1^e)$ is the of $-k\acute{u}$ as the node at x_6 is approached from the right , and $\sigma(s_2^e)$ is $-k\acute{u}$ as x_7 approached from the left .

We now assemble the equations describing the entire collection of elements comprising our mesh by sweeping through all elements , one at a time , and using (1.53) to calculate the contributions of each of them . Consider for example , a mesh containing *N-1* elements and N nodes , numbered consecutively 1,2,...,N . This means that there results N equations in N degrees of freedom describing the assembled system of elements , and we must allocate space in the computer for a system of this size, thus, we anticipate calculating an $N \times N$ stiffness matrix $K = [K_{ij}]$ and an $N \times 1$ load vector $F = \{f_i\}$, $i, j = 1,2, \dots, N$.

We initiate the assembly process by setting $K_{ij} = 0$ and $F_i = 0$ for element Ω_1, between nodes 1 and 2 , (1.53) yields the equations

$$k_{11}^1 u_1 + k_{12}^1 u_2 = f_1^1 + \sigma(0)$$
$$k_{21}^1 u_1 + k_{22}^1 u_2 = f_2^1 + \sigma(x_1^-)$$

Where $\sigma(0) = \sigma(0^+)$ is the actual flux at node 1 as this node is approached from the right and $\sigma(x_1^-)$ is the flux at the node at $x_1$at this node is approached from the left.

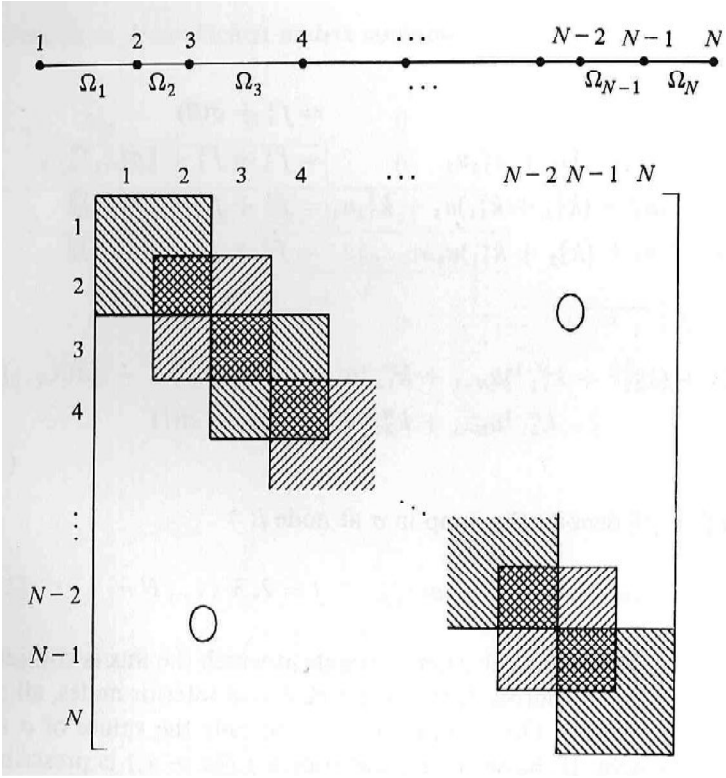

Figure 1.4 A finite -element mesh with N nodes and N-1 elements and the assembled stiffness matrix with the shaded blocks of entries representing the contributions of each element ; the symbols 0 represent the fact that outside the diagonal blocks all entries are zero

These equations are added into the first and second rows of the $N \times N$ system describing the entire mesh .

We next go to element Ω_2 since it lies between nodes 2 and 3, its contributions , calculating using (1.53), are added to the equations in rows 2 and 3 . Since two elements and three nodes have been (activated) we now have the three equations

$$k_{11}^1 u_1 + k_{12}^1 u_2 = f_1^1 + \sigma(0)$$
$$k_{21}^1 u_1 + (k_{22}^1 + k_{11}^2) u_2 + k_{12}^2 u_3 = f_2^1 + f_1^2 - \sigma(x_1^-) + \sigma(x_1^+)$$
$$k_{21}^2 u_2 + k_{22}^2 u_3 = f_2^2 - \sigma(x_2^-)$$

Continuing this process through the entire system of N elements , we arrive at the system :

$$\left.\begin{aligned}
k_{11}^1 u_1 + k_{12}^1 u_2 &= f_1^1 + \sigma(0) \\
k_{21}^1 u_1 + (k_{22}^1 + k_{11}^2) u_2 + k_{12}^2 u_3 &= f_2^1 + f_1^2 + [\![\sigma(x_1)]\!] \\
k_{21}^2 u_2 + (k_{22}^2 + k_{11}^3) u_3 + k_{12}^3 u_4 &= f_2^2 + f_1^3 + [\![\sigma(x_2)]\!] \\
k_{21}^3 u_3 + (k_{22}^3 + k_{11}^4) u_4 + \cdots &= f_2^3 + f_1^4 + [\![\sigma(x_3)]\!] \\
&\vdots \\
k_{21}^{N-2} u_{N-2} + (k_{22}^{N-2} + k_{11}^{N-1}) u_{N-1} + k_{12}^{N-1} u_N &= f_2^{N-2} + f_1^{N-1} + [\![\sigma(x_{N-1})]\!] \\
k_{21}^{N-1} u_{N-1} + k_{22}^{N-1} u_N &= f_2^{N-1} - \sigma(l)
\end{aligned}\right\} \tag{1.54}$$

Wherein $[\![\sigma(x_j)]\!]$ denotes the jump in σ at nodes j:

$$[\![\sigma(x_j)]\!] = \sigma(x_j^+) - \sigma(x_j^-) \ , \qquad j = 2,3, \dots, N-1 \tag{1.55}$$

Recall that $[\![\sigma]\!] = 0$ at the interior points at which the flux is continuous .

If there are no point sources in the data f located at interior nodes , all of the interior jump terms in (1.54) must be zero and only the values of σ at the boundaries remain. However, a point source $\hat{f}_j \delta(x - x_j)$ is prescribed at any interior node x_j, then we must set

$$[\![\sigma(x_j)]\!] = \hat{f}_j$$

In (1.54)

Let us assume that the point source $\hat{f}\delta(x - \bar{x})$ is located at node *3:*$\bar{x} = x_3$, then the linear system of equations for the entire mesh assumes the form :

$$\begin{bmatrix} \tilde{K}_{11} & K_{12} & 0 & 0 & \cdots & 0 & 0 \\ K_{21} & K_{22} & K_{23} & 0 & \cdots & 0 & 0 \\ 0 & K_{32} & K_{33} & K_{34} & \cdots & 0 & 0 \\ 0 & 0 & K_{43} & K_{44} & \cdots & 0 & 0 \\ . & . & . & . & \cdots & . & . \\ 0 & 0 & 0 & 0 & \cdots & K_{N-1,N-1} & K_{N-1,N} \\ 0 & 0 & 0 & 0 & \cdots & K_{N,N-1} & \tilde{K}_{NN} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ . \\ . \\ . \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} \tilde{F}_1 \\ F_2 \\ F_3 \\ F_4 \\ . \\ . \\ . \\ F_{N-1} \\ \tilde{F}_N \end{bmatrix} \tag{1.56}$$

Where the $N \times N$ coefficient matrix contains the entries

$$\begin{bmatrix} k_{11}^1 & k_{12}^1 & 0 & 0 & \cdots & 0 & 0 \\ k_{21}^1 & k_{22}^1 + k_{11}^2 & k_{12}^2 & 0 & \cdots & 0 & 0 \\ 0 & k_{21}^2 & k_{22}^2 + k_{11}^3 & k_{12}^3 & \cdots & 0 & 0 \\ 0 & 0 & k_{21}^3 & k_{22}^3 + k_{11}^4 & \cdots & 0 & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & \cdots & k_{22}^{N-2} + k_{11}^{N-1} & k_{12}^{N-1} \\ 0 & 0 & 0 & 0 & \cdots & k_{21}^{N-1} & k_{22}^{N-1} \end{bmatrix} \tag{1.57}$$

And

$$\begin{bmatrix} \tilde{F}_1 \\ F_2 \\ F_3 \\ . \\ . \\ . \\ F_{N-1} \\ \tilde{F}_N \end{bmatrix} = \begin{bmatrix} f_1^1 + \sigma(0) \\ f_2^1 + f_1^2 \\ f_2^2 + f_1^3 + \hat{f} \\ . \\ . \\ . \\ f_2^{N-2} + f_1^{N-1} \\ f_2^{N-1} - \sigma(l) \end{bmatrix} \tag{1.58}$$

The stiffness coefficients $\tilde{k}_{11}$ and $\tilde{k}_{NN}$ and the load components $\tilde{F}_1$ and $\tilde{F}_N$ are not yet of the general form described in (1.15) and (1.16) because they do not contain the boundary contributions . We discus modifications in these terms for various choices of boundary conditions subsequently .

Notes that the stiffness matrix in (1.56) is sparse (there are many zeros) and that the location of the element matrices , indicated in the bordered blocks in(1.56), Overlap in rows and columns corresponding to shared nodes . Note also that if the coefficient c in (1.46) is not identically zero , the matrix will be unsymmetric .

1.3 Boundary Conditions

An extremely important feature of the development up to this point is that no boundary conditions have , as yet been applied . Thus , (1.56) is applicable to a wide range of the boundary conditions . Consider , for instance the following cases :

General Natural Boundary Conditions:

These correspond to the general case in which a linear combination of u and $\acute{u}$ are prescribed :

$$\alpha_0 \acute{u}(0) + \beta_0 u(0) = \gamma_0 \quad , \qquad \alpha_l \acute{u}(l) + \beta_l u(l) = \gamma_l \tag{1.59}$$

In our approximation of this case, we set

$$\acute{u}_h(0) = \frac{\gamma_0 - \beta_0 u_h(0)}{\alpha_0} \quad \text{and} \quad \acute{u}_h(l) = \frac{\gamma_l - \beta_l u_h(l)}{\alpha_l} \tag{1.60}$$

Where, of course, $u_h(0) = u_1$ and $u_h(l) = u_N$.

Then (1.56) reduces to

$$\begin{bmatrix} \tilde{K}_{11} - \frac{k(0)\beta_0}{\alpha_0} & K_{12} & 0 & \cdots & 0 & 0 \\ K_{21} & K_{22} & K_{23} & \cdots & & \\ 0 & K_{32} & K_{33} & \cdots & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & K_{N-1,N-1} & K_{N-1,N} \\ 0 & 0 & 0 & \cdots & K_{N,N-1} & \tilde{K}_{NN} + \frac{k(l)\beta_l}{\alpha_l} \end{bmatrix} \times \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \cdot \\ \cdot \\ \cdot \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} f_1^1 - \frac{k(0)\gamma_0}{\alpha_0} \\ F_2 \\ F_3 \\ \cdot \\ \cdot \\ \cdot \\ F_{N-1} \\ f_2^{N-1} + \frac{k(l)\gamma_l}{\alpha_l} \end{bmatrix} \tag{1.61}$$

If the final $N \times N$ stiffness matrix in (1.61) is invertible , we can solve (1.60) for the unknown nodal values $u_1, u_2, \ldots, u_N$. Other features of the approximation, such as the approximate flux $\sigma_h = -k\acute{u}_h$, can then be easily evaluated .

Dirichlet Boundary Conditions :

Boundary conditions of the type

$$u(0) = \frac{\gamma_0}{\beta_0} \quad , \quad u(l) = \frac{\gamma_l}{\beta_l} \tag{1.62}$$

Follow as a special case of (1.59) when $\alpha_0 = \alpha_l = 0$. Essential boundary conditions of this form are usually called Dirichlet boundary conditions of and the corresponding boundary-value problem is referred to as a Dirichlet problem for the function u .

In this case $u_h(0) = u_1 = \gamma_0/\beta_0$ and $u_h(l) = u_N = \gamma_l/\beta_l$, so that only N-2 unknown nodal values $u_2, u_3 , \dots , u_{N-1}$ (1.57) reduces to the $(N-2) \times (N-2)$ system :

$$\begin{bmatrix} \mathrm{K}_{22} & \mathrm{k}_{23} & 0 & \dots & 0 & 0 \\ \mathrm{k}_{32} & \mathrm{k}_{33} & \mathrm{k}_{34} & \dots & 0 & 0 \\ 0 & \mathrm{k}_{43} & \mathrm{k}_{44} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \mathrm{k}_{N-1,N-2} & \mathrm{k}_{N-1,N-1} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ u_4 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

$$= \begin{bmatrix} F_2 - \dfrac{k_{21}\gamma_0}{\beta_0} \\ F_3 \\ F_4 \\ \vdots \\ F_{N-1} - \dfrac{k_{N-1,N}\gamma_l}{\beta_l} \end{bmatrix} \tag{1.63}$$

And the two auxiliary equations corresponding to nodes 1 and N ,

$$\left.\begin{aligned} \tilde{k}_{11}\left(\frac{\gamma_0}{\beta_0}\right) + k_{12}u_2 &= f_1^1 + \sigma(0) \\ k_{N,N-1}u_{N-1} + \tilde{k}_{NN}\left(\frac{\gamma_l}{\beta_l}\right) &= f_2^{N-1} - \sigma(l) \end{aligned}\right\} \tag{1.64}$$

The reduced stiffness matrix in (1.63) is nonsingular, so that (1.63) can be solved for the unknown nodal values $u_2 , u_3, \dots , u_{N-1}$.

Of course (1.64) also provides useful information . Once u_2 and u_{N-1} are known, the approximations of the endpoint fluxes can be computed using (1.64).

Neumann Boundary Conditions :

When only the derivative of u is specified at each end , (1.59) reduces to

$$\acute{u}(0) = \frac{\gamma_0}{\alpha_0} \quad , \quad \acute{u}(l) = \frac{\gamma_l}{\alpha_l} \tag{1.65}$$

Whenever $\beta_0 = \beta_l = 0$. Natural boundary conditions of this type are called Neumann boundary conditions and the corresponding boundary-value problem is referred to as a Neumann problem for the function u .

Neumann problems frequently require some special considerations for certain forms of the governing differential equation . In particular , consider the case in which the coefficients $c = c(x)$ and $b = b(x)$ are identically zero so that the boundary –value problem becomes one of solving the differential equation

$$-(k(x)\acute{u}(x)\acute{)} = f(x) \tag{1.66}$$

On smooth subdomains , subject to the end conditions (1.65) in this case , the solution u is determined only to within an arbitrary constant c_0 ; that is , if $u = u(x)$ satisfies (1.65) and (1.66), then $u + c_0$ is also a solution . Because of the analogy of (1.65) with equations describing mechanical systems , the constant c_0 is sometimes referred to as a rigid motion , and this rigid motion must be specified if we are to obtain a unique solution to our problem . Moreover , the finite element approximation(1.56) of this Neumann problem will also contain an arbitrary rigid motion . Since solutions to (1.56) will then be nonunique , the stiffness matrix K in (1.57) will necessarily be singular .

The presence of a rigid motion in the solution to a Neumann problem leads to another consideration of fundamental importance : the data $f, \gamma_0, \alpha_0, \gamma_l$ and α_l cannot be specified arbitrarily , they must be compatible in a sense we will now make clear .

Since the variational form of this Neumann problem (with $c, b \equiv 0$) is to find $u \in H^1$ such that

$$\int_0^l k\acute{u}\acute{v}\, dx = \int_0^l \tilde{f} v\, dx + \hat{f} v(\bar{x})$$

$$-k(0)\left(\frac{\gamma_0}{\alpha_0}\right) v(0) + k(l)\left(\frac{\gamma_l}{\alpha_l}\right) v(l) \qquad (1.67)$$

For all $v \in H^1$

And since $u = c_0$ = constant is a solution, this equation must also hold for $u = c_0$ and the choice $v = 1$. Hence, the data must be such that

$$\int_0^l \tilde{f}\, dx + \hat{f} - \frac{k(0)\gamma_0}{\alpha_0} + \frac{k(l)\gamma_l}{\alpha_l} = 0 \qquad (1.68)$$

The compatibility condition (1.68) is a necessary condition for the existence of a solution to (1.67). We remark that from a physical viewpoint (1.68) is a global conservation law; it reflects the requirement that the flux σ be conserved over the entire body Ω.

For the discrete problem corresponding to (1.67), this compatibility condition assumes the form (see (1.58))

$$\tilde{F}_1 + \sum_{i=2}^{N-1} F_i + \tilde{F}_N = 0 \qquad (1.69)$$

To eliminate the rigid motion, we can specify the value u_j of u_h and any node j equal to an arbitrary constant c_0. For instance, setting $u_1 = c_0$ in (1.61) (with it understood that $c = b = 0$) yields the reduced system

$$\begin{bmatrix} \mathrm{K}_{22} & \mathrm{k}_{23} & 0 & \dots & 0 & 0 \\ \mathrm{k}_{32} & \mathrm{k}_{33} & \mathrm{k}_{34} & \dots & 0 & 0 \\ 0 & \mathrm{k}_{43} & \mathrm{k}_{44} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \mathrm{k}_{\mathrm{N,N-1}} & \tilde{\mathrm{k}}_{\mathrm{NN}} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ u_4 \\ \vdots \\ u_N \end{bmatrix} = \begin{bmatrix} F_2 - K_{21} c_0 \\ F_3 \\ F_4 \\ \vdots \\ f_2^{N-1} + \frac{k(l)\gamma_l}{\alpha_l} \end{bmatrix}$$

(1.70)

and the equation

$$\tilde{k}_{11}c_0 + \mathrm{K}_{12}u_2 = f_1^1 - \frac{k(0)\gamma_0}{\alpha_0} \tag{1.71}$$

Equation (1.70) is uniquely solvable for u_2 , u_3 , … , u_N in terms of c_0. frequently , we simply set c_0=0 . Notice that we can solve for u_2 using either (1.70) or (1.71) . it is remarkable fact that the condition (1.69) guarantees that these two systems will be compatible , the value of u_2 obtained from (1.70) will be exactly the same as that of (1.71) whenever (1.69) holds.

Mixed Boundary Conditions:

When an essential boundary condition is applied at one boundary point and a natural boundary condition at the other, a mixed boundary-value problem for the function u is obtained . For example , one mixed problem is characterized by the end conditions

$$u(0) = \frac{\gamma_0}{\beta_0} \quad , \quad \acute{u}(l) = \frac{\gamma_l}{\beta_l} \tag{1.72}$$

and another by

$$\alpha_0\acute{u}(0) + \beta_0 u(0) = \gamma_0 \quad , \quad u(l) = \frac{\gamma_l}{\beta_l} \tag{1.73}$$

Since at least one of these conditions specifies the value of u at an endpoint , the solution will contain no rigid motions . Consider the case (1.72) . Then $u_1 = \gamma_0/\beta_0$, $\sigma(l) = -k(l)\,\gamma_l/\alpha_l$ and (1.56) reduces to the system

$$\begin{bmatrix} K_{22} & K_{23} & \cdots & 0 & 0 \\ K_{32} & K_{33} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & K_{N,N-1} & \tilde{K}_{NN} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ \cdot \\ \cdot \\ u_N \end{bmatrix} = \begin{bmatrix} F_2 - \frac{K_{21}\gamma_0}{\beta_0} \\ F_3 \\ \cdot \\ \cdot \\ f_2^{N-1} + \frac{k(l)\gamma_l}{\alpha_l} \end{bmatrix} \tag{1.74}$$

and the equation

$$\tilde{k}_{11}\left(\frac{\gamma_0}{\beta_0}\right) + \mathrm{K}_{12}u_2 = f_1^1 - \sigma(0) \tag{1.75}$$

We solve (1.74) for $u_2, u_3, \ldots, u_N$ and use (1.75) to obtain an approximation $\sigma_h(0)$ of $\sigma(0)$, if desired. Conditions (1.73) are handled in a similar fashion.

This completes our description of the finite element analysis of two-point boundary-value problems of the type in (1.5). What remains to be done is to implement the procedure described above by developing a finite element computer program, which is not port of this thesis.

Chapter 2

Two-Dimensional Problems

2.1 Introduction :

The principle ingredients of the finite element method for constructing approximate solutions of problems are :

1. The formulation of the problem in a variational framework in which the appropriate space H of admissible functions is identified .
2. The construction of a finite element mesh and piecewise-polynomial basis functions defined on the mesh,which generate a finite-dimensional subspace of H .
3. The construction of an approximation of the variational boundary-value problem on a finite element subspace H^h of H. This entails the calculation of element matrices and the generation of a sparse system of linear algebraic equations in the values of the approximate solution at nodal points in the mesh .
4. The solution of the algebraic system .
5. The examination of the characteristics of the approximate solution and if possible , an estimation of the inherent approximation error .

These steps from the basis of most finite element methods for not only one dimensional problems , but , more importantly , for boundary-value problems in two and three dimensions.

Our objective here is to develop, in a logical manner, the natural extension of the earlier developments to two dimensions . The governing equation for the boundary-value problem now becomes a partial differential equation which is to be satisfied by the solution inside some two-dimensional domain Ω boundary data are thus specified on the curve defining the boundary of Ω .

Instead of the line elements , the finite elements now assume simple two-dimensional shapes , such as triangles and quadrilaterals , and these elements fit together to make up the discretization (the mesh) approximating the domain Ω of the solution u . The inherent ability of such elements to represent domains with very irregular geometries underlies the practical value of the method for the approximate solution of difficult boundary-value problems in numerous research and industrial applications .

2.2 Two-Dimensional Boundary-Value Problems

The development of boundary-value problems describing physical phenomena in two dimensions follows closely the one-dimension treatment , differing only in aspects dictated by the higher dimensionality . In this section we sketch the development of linear, elliptic , self-adjoint second-order boundary-value problems , based on classical conservation principles .

Some Preliminaries

The domain $\bar{\Omega}$ of our problem is composed of two parts , the interior Ω and the boundary $\partial\Omega$. We consider only bounded domains (i.e , no part of Ω extends to infinity in any direction) ; with reasonably smooth boundaries . In general , the boundary can be defined by the parametric equations $x = x(s)$ and $y = y(s)$, where s is the arc length along $\partial\Omega$ measured from some arbitrary reference point . When referring to the value of a function , say g , which is defined at points on the boundary , we will write $g(s) \equiv g(x(s), y(s))$ for s in $\partial\Omega$.

The primary dependant variable in our problem is the state variable $u = u(x, y)$. As an essential condition , we require that u be a smooth function in Ω .

The degree of smoothness we require depends on the data of the problem , including the functions $x(s)$, $y(s)$ that define $\partial\Omega$. We shall assume throughout this section that all functions are smooth enough for the operations we perform to be valid .

Our physical statement of the problem will contain expressions involving the rate change with respect to position in Ω of the scalar field u .

In general , the value of u at a point (x, y)changes at different rates as we move from (x, y)in the different directions . The rate of change of u at a point (x, y) is defined by a vector-valued function , denoted ∇u , called the gradient of u . If i and j denote unit vectors directed along the x- and y-axis , respectively then the components of ∇u at the point (x, y) with respect to these basis vectors are $\partial u(x, y)/\partial x$ and $\partial u(x, y)/\partial y$, so that

$$\nabla u(x, y) = \frac{\partial u(x,y)}{\partial x}\,\mathrm{i} + \frac{\partial u(x,y)}{\partial y}\,\mathrm{j} \tag{2.1}$$

Note that (2.1) can be interpreted as the construction of a vector field ∇u by operating on u with the vector differential operator

$$\nabla = \frac{\partial}{\partial x}\,\mathrm{i} + \frac{\partial}{\partial y}\,\mathrm{j} \tag{2.2}$$

The gradient $\nabla u(x, y)$ determines the total rate of change of u at (x, y) in any direction . In particular , let t has components $\cos\theta$ and $\sin\theta$ so that $t = \cos\theta\,\mathrm{i} + \sin\theta\,\mathrm{j}$. The rate at which u changes as one moves from a point (x, y) in the direction of t is called the directional derivative of u with respect to t and is written $du(x, y)/dt$. The directional derivative is calculated according to

$$\frac{du(x,y)}{dt} = \nabla u(x, y) \cdot t = \frac{\partial u(x,y)}{\partial x}\cos\theta + \frac{\partial u(x,y)}{\partial y}\sin\theta \tag{2.3}$$

The second quantity of physical interest in our boundary-value problems is the flux σ . The flux , like the gradient of u , is a vector-valued function or vector field , a flux field is represented schematically as vectors that very in magnitude and direction with in $\bar{\Omega}$. The flux vector $\sigma(s)$ at the point s on the boundary is shown in Fig.2.1b .

The flux crossing the boundary at s is given by the component of $\sigma(s)$ in the direction of a unit outward normal $\mathrm{n}(s)$ to $\partial\Omega$:

$$\sigma_n(s) = \sigma(s) \cdot \mathrm{n}(s) \tag{2.4}$$

The component in the direction of the unit tangent $\tau(s)$ is $\sigma(s).\tau(s)$, as indicated in the figure

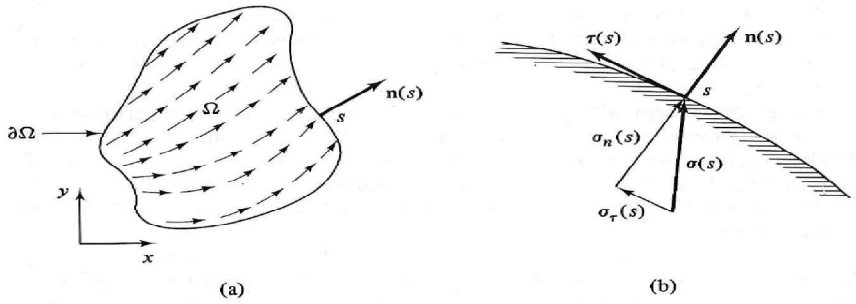

Figure 2.1 : (a) representation of the vector –valued function $\sigma = \sigma(x, y)$; (b) resolution of the flux $\sigma(x)$ at appoint s on $\partial\Omega$ into normal flux $\sigma_n(s)$ and the tangent flux $\sigma_r(s)$ components.

Consider an arbitrary subregion of Ω , say ω , containing the point P_0 whose coordinates are (x_0, y_0) . Figure 2.2a shows the distribution of normal flux $\sigma_n(s)$ across the boundary $\partial\omega$. The net (total) flux crossing the boundary is :

$$\sum_{\omega} \equiv \int_{\partial\omega} \sigma_n(s)\, ds \tag{2.5}$$

If we divide $\sum_{\omega} by$ the area A_{ω} of the subregion , we can view the result as the average value of the amount of σ flowing into ω per unit area . The limit of this ratio as ω decreases in size , always containing the point P_0 , is called the divergence of the flux at P_0 and is often abbreviated div $\sigma(x_0, y_0)$. Taking as ω the square subregion containing P_0, we calculate $\sum_{\omega} = \Delta\sigma_x \Delta y + \Delta\sigma_y \Delta x$. then , using the mean-value theorem to expand component σ_x and $\sigma_y$of σ about point P_0 , we divide by the area $\Delta x \Delta y$ and calculate the limit as Δx , Δy go to zero . In this way , we obtain the formula for the divergence of the vector field σ at the point (x, y)

$$div\ \sigma(x, y) = \frac{\partial \sigma_x(x, y)}{\partial x} + \frac{\partial \sigma_y(x, y)}{\partial y} \tag{2.6}$$

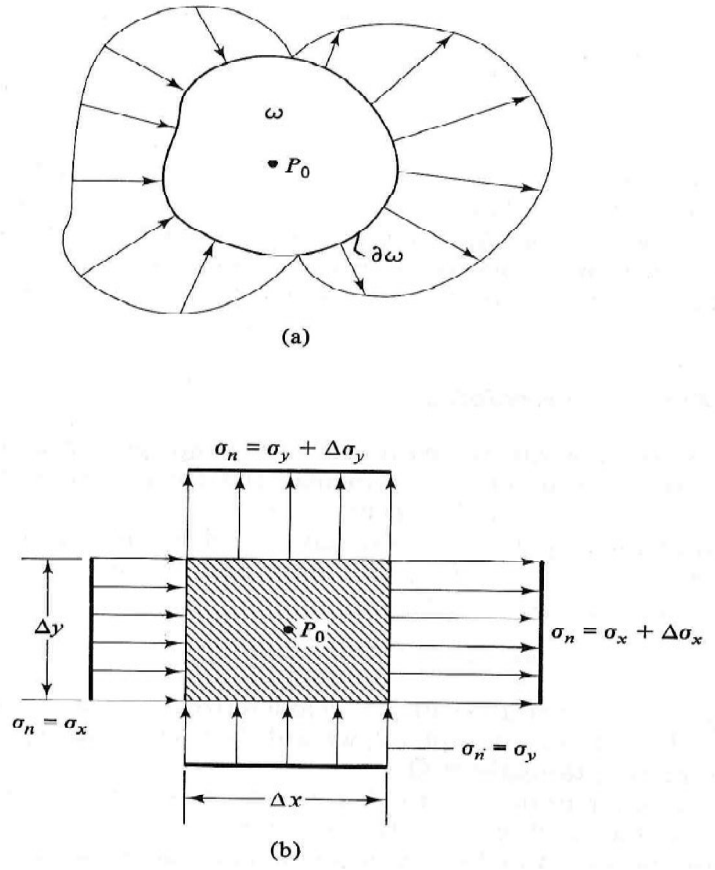

Figure 2.2: (a) distribution of normal flux on boundary $\partial\omega$ of subregion ω ;(b) magnitude of normal flux on boundary of square region

Note that by using the vector operator ∇ defined in (2.2) , the divergence of σ can also be written

$$div\ \sigma = \nabla \cdot \sigma \tag{2.7}$$

Recall the definition of $\nabla.\sigma$ as the density of the net flux per unit area at a point . It follows that the total flux $\sum$ into the region Ω can be written

$$\sum = \int_{\Omega} \nabla \cdot \sigma\, dx\, dy \tag{2.8}$$

Provided that Ω and σ are smooth enough . It then follows from (2.4) , (2.5) , and (2.8) that

$$\int_{\Omega} \nabla \cdot \sigma\, dx\, dy = \int_{\partial\Omega} \sigma \cdot n\, ds \tag{2.9}$$

The relation between the area integral and the boundary integral in (2.9) is an important tool in applied mathematics and is referred to as the Gauss divergence theorem . Although we have stated (2.9) in terms of a special vector field σ defined on a two-dimensional domain , the result is generally the same for any vector or tensor field in any number of dimensions .

Physical Principles

The physical situations we wish to highlight are governed by a linear constitutive law and the conservation principle . The linear constitutive equation in our physical problem establishes that , at each point in the body , the flux is proportional to the gradient of the state variable . The factor of proportionality is denoted k and is referred to as the material modulus (coefficient or property) , thus

$$\sigma(x, y) = -k(x, y)\nabla u(x, y) \tag{2.10}$$

Clearly , different materials will be characterized by different material moduli , $k = k(x, y)$,we will always assume that $|k(x, y)| > k_0$ =constant > 0 throughout $\bar{\Omega}$.

The conservation principle (or balance law)states that within any portion of domain , the net flux across the boundary of that part must equal the total quantity produced by internal source . The mathematical implications of the balance law takes different forms for different parts of the domain.

Let us next examine the implications of the conservation principle consider, for example, a portion ω of material surrounding a point P_0 inside Ω (fig 2.3) in which the material properties are smooth . The balance law defined earlier establishes that the net flux across the boundary $\partial\omega$ given by (2.5) , must be balanced by the total quantity supplied by sources within ω . If f denotes the source per unit area , then we must have

$$\int_{\partial\omega} \sigma \cdot n \, ds = \int_{\omega} f \, dx \, dy \tag{2.11}$$

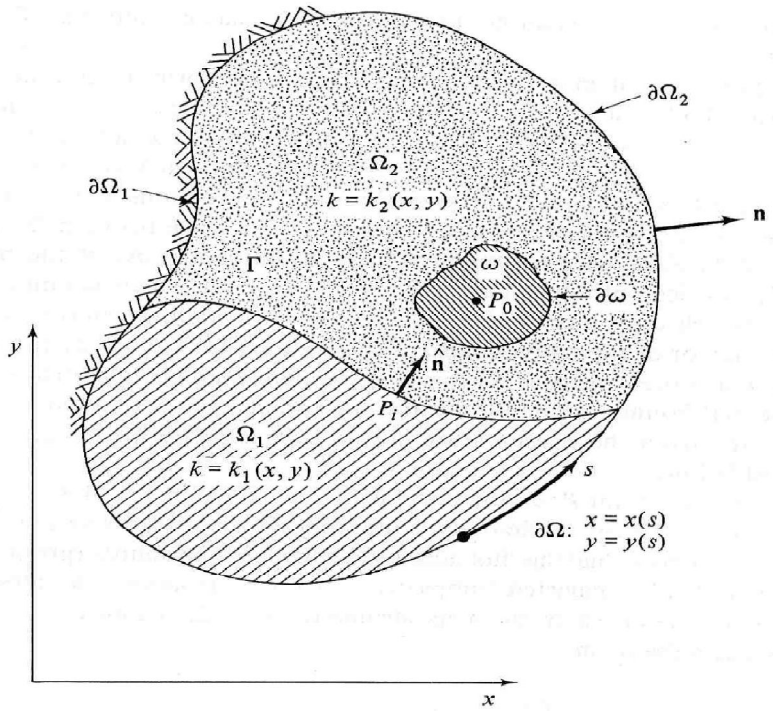

Figure 2.3 Domain of a two-dimensional boundary-value problem

Using the divergence theorem (2.9) to transform the surface integral in (2.11) to an area integral , we have

$$\int_{\omega}(\nabla \cdot \sigma - f)\, dx\ dy = 0 \tag{2.12}$$

For all subregion ω in Ω . Since ω is an arbitrary region in which $\nabla \cdot \sigma$ and f are smooth , the integrand in (2.12) must be zero at all point interior to ω . Thus , with in such "smooth " regions the local form of the balance law is

$$\nabla \cdot \sigma(x, y) = f(x, y) \tag{2.13}$$

For completeness , we suppose that , in addition to f , there may exist internal sources with an intensity proportional to u .

Letting the proportionality factor be $-b(x, y)$, (2.13) becomes

$$\nabla \cdot \sigma(x, y) + b(x, y)u(x, y) = f(x, y) \tag{2.14}$$

Examples of sources proportional to u include temperature-dependent exothermic chemical reactions and distributed elastic supports of elastic membranes.

The conservation principle takes on a different form at interfaces and boundaries . To fix ideas, let us consider the particular case in which the body $\bar{\Omega}$ is composed of two distinct materials, one occupying a subregion Ω_1 and the other subregion Ω_2.Within each of these regions, the modulus k is assumed to given by smooth functions (constant) k_1 and k_2 . The curve defining the interface between Ω_1 and Ω_2 is denoted Γ . Similarly , we suppose that the boundary $\partial\Omega$ of the body is naturally divided into two parts $\partial\Omega_1$ and $\partial\Omega_2$, on which conditions are imposed which characterize the effect of the surroundding exterior medium on the behavior of the body . On $\partial\Omega_1$, we suppose that the values of the state variable u are prescribed as $u(s) = \hat{u}(s)$, so that it is on this portion of $\partial\Omega$ that essential boundary conditions are prescribed. Natural boundary conditions arising from the conservation law will be specified on $\partial\Omega_2$, as will be explained below .

Consider the point P_i on the interface Γ, as shown in figures 2.3 and 2.4 shows a material strip of the body containing P_i . This strip is assumed to be sufficiently narrow that the flux across its ends and the source(proportional to the area) can be neglected compared to the net flux across the sides of the strip. As the thickness of the strip shrinks to zero , the balance law for this strip assumes the form

$$\sum = \int_{s_1}^{s_2} (-\sigma^{(-)} \cdot n + \sigma^{(+)} \cdot n)\, ds = 0$$

Where s_1 and s_2 are the endpoints of the strip. Because the region of integration is arbitrary , the local balance law at points on the interface reduces to conditions on the jump in $\sigma \cdot n = \sigma_n$ across Γ :

$$[\![\sigma_n\,(s)]\!] = \sigma_n^{(+)}(s) - \sigma_n^{(-)}(s) = 0 \qquad , s \in \Gamma \tag{2.15}$$

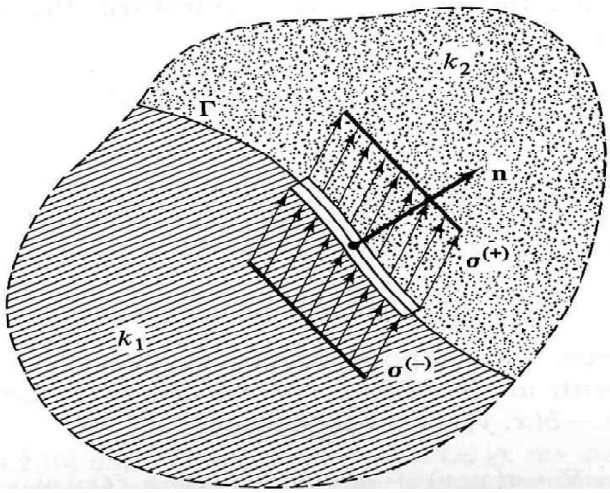

Figure 2.4 A strip of the domain containing a portion of the interface Γ in the neighborhood of p_i

We now turn to the boundary conditions on $\partial\Omega_2$. A region containing a typical boundary point P_b shown in fig 2.5

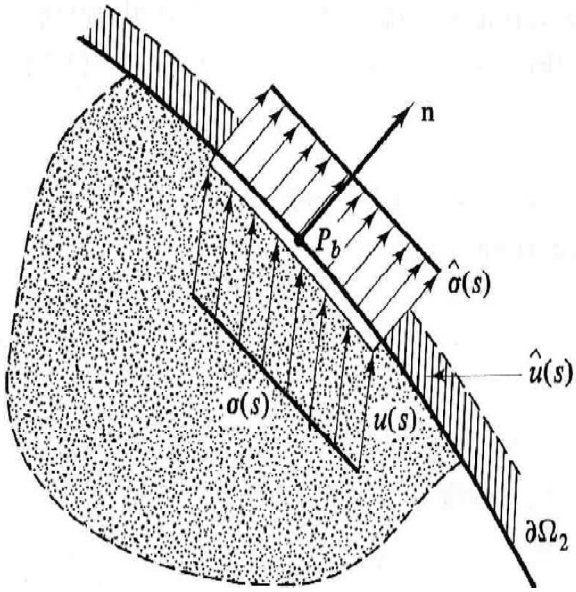

Figure 2.5 A strip of the domain containing a portion of the boundary $\partial\Omega_2$ in the neighborhood of point p_b

The value of the normal flux $\hat{\sigma}(s)$ through the surrounding material immediatly adjacent to the boundary is assumed to be proportional to the difference between the value $u(s)$ on the boundary and the given value $\hat{u}(s)$ in the exterior medium . Thus , if $p(s)$ is the factor of proportionality at s ,

$$\hat{\sigma}(s) \equiv p(s)[u(s) - \hat{u}(s)]$$

Balancing the flux in the strip containing P_b gives

$$\sigma_n(s) \equiv \sigma(s) \cdot n(s) = \hat{\sigma}(s) \tag{2.16}$$

Hence ,

$$\sigma_n(s) = p(s)[u(s) - \hat{u}(s)] \qquad , s \in \partial\Omega_2 \tag{2.17}$$

The physical situation often dictates other forms of the boundary conditions , which can be viewed as special cases of (2.16) or (2.17) . For example , we recover the essential boundary condition on u from (2.17) by taking the limit as $p(s) \to \infty$ while $\sigma_n(s)$ remains bounded . This procedure corresponds to a two-dimensional version of the penalty method . Boundary conditions (2.16) and (2.17) , which are consequences of the conservation principle , are called natural boundary conditions . Boundary conditions that prescribe the state variable are essential boundary conditions.

2.3 Statement of the Boundary-Value Problem

The final mathematical statement of our boundary-value problem is obtained by eliminating σ and $\sigma_n$from(2.14) through (2.17) using the constitutive equation (2.10). The data defining the problem consist of the following :

1. The boundaries $\partial\Omega_1$, $\partial\Omega_2$, and the interface Γ defined by the parametric equations

$$x = x(s) \qquad , \quad y = y(s) \quad s \in \partial\Omega \quad \text{or } s \in \Gamma$$

2. The source distribution $f = f(x, y)$ in Ω_i $, i = 1,2.$
3. The material coefficients $k_i = k_i(x, y)$ and $b_i = b_i(x, y)$ for $(x, y) \in \Omega_i$, $i = 1,2$.
4. The prescribed value $\hat{u}(x)$ for $s \in \partial\Omega_1$.

5. The value of boundary coefficient $p(s)$and $\hat{u}(x)$on $\partial\Omega_2$ or the prescribed value $\hat{\sigma}(s)$ for $\in \partial\Omega_2$.

Given the following data , the problem is to find the function $u = u(x,y)$ that satisfies :

1. The governing partial differential equation at points interior to the smooth subdomains Ω_1 and Ω_2 ,

$$-\nabla \cdot \big(k(x,y)\nabla u(x,y)\big) + b(x,y)u(x,y) = f(x,y) \qquad (2.18)$$

For $(x,y) \in \Omega_i, i = 1,2$

2. The jump condition at points on the interface Γ ,

$$[\![k\nabla u \cdot n]\!] = 0 \quad , \quad s \in \Gamma \qquad (2.19)$$

3. The essential boundary condition on $\partial\Omega_1$,

$$u(s) = \hat{u}(s) \quad , \qquad s \in \partial\Omega_1 \qquad (2.20)$$

4. The natural boundary condition on $\partial\Omega_2$,

$$\left.\begin{array}{ll} -k(s)\dfrac{\partial u(s)}{\partial n} = p(s)[u(s) - \hat{u}(s)] \,, & s \in \partial\Omega_2 \\ \textit{or} & \\ -k(s)\dfrac{\partial u(s)}{\partial n} = \hat{\sigma}(s) \qquad\qquad , & s \in \partial\Omega_2 \end{array}\right\} \qquad (2.21)$$

The special case of problem (2.18) , in which $b \equiv 0$ and only natural boundary conditions of the form $-k(s)[\partial u(s)/\partial n] = \hat{\sigma}(s)$ are specified and $\partial\Omega_2 = \partial\Omega$, requires two qualifying remarks . First , we note that (2.18) Then determines the solution u only to within an additive constant . The second point is that , in order for a solution u to exist at all , the data f and $\hat{\sigma}$ must

satisfy a compatibility relation . This relation is simply a statement of the conservation principle for the entire domain Ω and requires that f and $\hat{\sigma}$ be such that

$$\int_{\Omega} f \, dx \, dy = \int_{\partial\Omega} \hat{\sigma} \, ds \tag{2.22}$$

The development and statement of the boundary-value problem have been made assuming that all functions were as smooth as necessary for the operations indicated to be valid . The only consideration made for lack of smoothness in any of the data was the treatment of the material interface Γ across which the coefficients k , b and the source f may be discontinuous .

2.4 Variational Boundary-Value Problems

The construction of our variational formulation of the boundary-value problem (2.18) begins , as usual , by defining the residual r :

$$r(x,y) = -\nabla \cdot [k(x,y)\nabla u(x,y)] + b(x,y)u(x,y) - f(x,y)$$

To "test" the residual over arbitrary subregions , we multiply r by a sufficiently smooth test function v , integrate over each domain in which rv is smooth , and set the resulting weighted average of r equal to zero . For problem whose domain is shown in fig 2.3 , we must integrate separately over Ω_1 and Ω_2 since the second derivatives of u are not integrable along the interface Γ this procedure gives

$$\int_{\Omega_1} [-\nabla \cdot (k\nabla u) + bu - f]v \, dx \, dy + \int_{\Omega_2} [-\nabla \cdot (k\nabla u) + bu - f]v \, dx \, dy = 0 \tag{2.23}$$

A two-dimensional "integration-by-parts formula" is needed to reduce the first term in each of these integrals to terms containing only first derivatives .

By product rule for differentiation , we find that :

$$\left.\begin{aligned} &\nabla\cdot(vk\nabla u) = k\nabla u\cdot\nabla v + v\nabla\cdot(k\nabla u) \\ or\quad &\\ &v\nabla\cdot(k\nabla u) = \nabla\cdot(vk\nabla u) - k\nabla u\cdot\nabla v \end{aligned}\right\} \qquad (2.24)$$

Substitution (2.24) in to (2.23) yields

$$\begin{aligned} \int_{\Omega_1}(k\nabla u\cdot\nabla v + buv - fv)dxdy + \int_{\Omega_2}(k\nabla u\cdot\nabla v + buv & \\ -fv)dxdy - \int_{\Omega_1}\nabla\cdot(vk\nabla u)dxdy & \qquad (2.25) \\ -\int_{\Omega_2}\nabla\cdot(vk\nabla u)dxdy = 0 & \end{aligned}$$

The last two integrals in (2.25) can be transformed into boundary integrals using the divergence theorem (2.9) with $(vk\nabla u)$ used in place of σ. We obtain

$$\begin{aligned} -\int_{\Omega_1}\nabla\cdot(vk\nabla u)\,dx\,dy - \int_{\Omega_2}\nabla\cdot(vk\nabla u)\,dx\,dy & \\ = -\int_{\partial(\Omega_1)}k\frac{\partial u}{\partial n}v\,ds - \int_{\partial(\Omega_2)}k\frac{\partial u}{\partial n}v\,ds & \end{aligned} \qquad (2.26)$$

Where $\partial(\Omega_1)$ and $\partial(\Omega_2)$ are boundaries of subregions Ω_1 and $\Omega_2$the direction of integration is counterclockwise in each of $\Omega_1$and Ω_2, and $\partial u/\partial n = \nabla u\cdot n$. We must be careful to identify the functions in (2.26) with the domains on which they are define. In fig 2.6 which shows Ω_1 and Ω_2 separated for clearly , the boundary of each domain is divided into two parts-the parts of $\partial(\Omega_i)$ that do not coincide with the interface Γ are denoted $\partial(\Omega_i) - \Gamma$, $i = 1,2$. We decompose each of the boundary integrals in (2.26) into two corresponding parts, obtaining

$$\begin{aligned} -\int_{\partial(\Omega_1)-\Gamma}k\frac{\partial u}{\partial n}v\,ds - \int_{\partial(\Omega_2)-\Gamma}k\frac{\partial u}{\partial n}v\,ds & \\ +\int_{\Gamma}(-k\frac{\partial u}{\partial n})_1 v\,ds + \int_{\Gamma}(-k\frac{\partial u}{\partial n})_2 v\,ds & \end{aligned} \qquad (2.27)$$

Where in the notation

$$\left(-k\frac{\partial u}{\partial n}\right)_i \text{ indicates that } \left(-k\frac{\partial u}{\partial n}\right)$$

Is to be evaluated on region i. Noting that the outward normal n_1 to region Ω_1 is the negative of n_2 at each point on Γ, we rewrite the sum of the last two integrals in (2.27) as

$$\int_\Gamma \left[-k^{(+)}\frac{\partial u^{(+)}}{\partial n} + k^{(-)}\frac{\partial u^{(-)}}{\partial n}\right] v\, ds \tag{2.28}$$

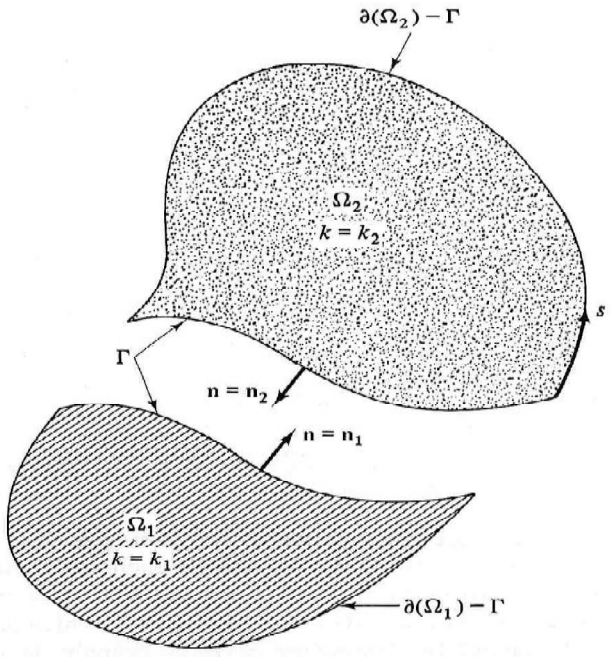

Figure 2.6 Decomposition of regions of integration for boundary integrals around $\partial(\Omega_1)$ and $\partial(\Omega_2)$

Where we have used the notation introduced in the preceding section . We recognize that the integrand (2.28) is exactly $v[\![\sigma_n(s)]\!]$, which , according to (2.19) , is zero . Hence , the integral in (2.28) vanishes .

Returning to (2.27) , we note that the first two integrals can be combined into a single integral over the entire boundary $\partial\Omega$. We also note that the integrals of the first two integrals in (2.25) contain at most first derivatives of u and v , so, if u and v are smooth enough, these integrals can be combined into a single integral over the entire domain Ω . The result is

$$\int_{\Omega}(k\nabla u\cdot\nabla v+buv-fv)dxdy-\int_{\partial\Omega}k\frac{\partial u}{\partial n}v\,ds=0 \qquad (2.29)$$

Substitution of natural boundary condition in (2.21) , for example , into the boundary integral in(2.29) gives the variational equation

$$\int_{\Omega}(k\nabla u\cdot\nabla v+buv-fv)dxdy+\int_{\partial\Omega}p(u-\hat{u})vds=0 \qquad (2.30)$$

Which must hold for all admissible test functions v . Since we require that $u=\hat{u}$ on $\partial\Omega_1$

$$\int_{\partial\Omega}p(u-\hat{u}\,)vds=\int_{\partial\Omega_2}puv\,ds-\int_{\partial\Omega_2}\gamma v\,ds$$

Where we have denoted $\gamma=p\hat{u}$, Hence , our problem becomes one of finding a function u such that $u=\hat{u}$ on $\partial\Omega_1$ and

$$\int_{\Omega}(k\nabla u\cdot\nabla v+buv)dxdy+\int_{\partial\Omega_2}puv\,ds=\int_{\Omega}fv\,dxdy+\int_{\partial\Omega_2}\gamma v\,ds \qquad (2.31)$$

for all admissible test functions v .

As in earlier discussions , we will regard (2.31) as the given boundary-value problem . Then , if (2.31) holds for all smooth test functions v and if the data and the solution u are sufficiently smooth , a solution of (2.31) will also be a solution of (2.18) . Conversely , any solution of (2.18) will always automatically satisfy (2.31).

There remains the important issue of specifying the appropriate class of admissible functions for problem (2.31) . We observe that the area integrals in (2.31) are well defined whenever u and v and their first partial derivatives are smooth enough to be square-integrable over Ω . Thus , we require that all admissible functions v be such that

$$\int_{\Omega} \left[\left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 + v^2 \right] dxdy < \infty \qquad (2.32)$$

Adopting a minor modification in the notation used in previous chapter , we refer to the class of functions satisfying (2.32) as $H^1(\Omega)$ the superscript "1" reflecting the fact that first derivatives are square-integrable and (Ω) indicating the domain over which these functions are defined .

As was the case with one-dimensional problems , note that the natural boundary conditions (the conditions on $\partial\Omega_2$) enter (2.31) in the statement of the problem itself . These conditions appear in the term $\int_{\partial\Omega_2} puv \, ds$ and $\int_{\partial\Omega_2} \gamma v \, ds$.

The essential boundary conditions enter the problem in the definition of the classes of admissible functions . We choose as test functions v in $H^1(\Omega)$ such that $v = 0$ on $\partial\Omega_1$. The solution u must be a function in $H^1(\Omega)$ such that $u = \hat{u}$ on $\partial\Omega_1$.

Our variational boundary-value problem can now be stated concisely in the following form:

Find a function $u \in H^1(\Omega)$ such that $u = \hat{u}$
on $\partial\Omega_1$ and (4.9) holds for all $v \in H^1(\Omega)$ (2.33)
such that $v = 0$ on $\partial\Omega_1$.

Chapter 3

Finite Element Interpolation

3.1 Discretization

This stage of our analysis represents a direct but nontrivial extension of the ideas discussed earlier for one –dimensional problems . Having a variational statement of our model problem, we proceed to construct a finite element mesh representing Ω . In the one-dimensional problem , this amounted to partitioning an interval into line elements connected at nodal points at their ends . For two-dimensional problems , the discretization of Ω is less straightforward .

The basic idea is to continue to represent approximate solutions u_h and test functions v_h by polynomials defined piecewise over geometrically simple subdomins of subregion Ω_h , with Ω_h now in the *x,y*-plane . Our first concern is to choose a discretization that will be general to model irregular domains but consist of element simple enough to minimize computational effort . In fig 3.1 simple triangles and or quadrilaterals can be used for this purpose . If $\partial\Omega$ is curved , as in the figure , there will always be some discretization error , since the finite element mesh Ω_h , constructed as collection of triangular or quadrilateral elements , will not perfectly coincide with the given domains Ω . However , as the mesh is refined , Ω_h can approximate Ω with increasing accuracy .

Another reason for considering elements of simple shapes such as traingles is that there is a natural correspondence between the number and location of nodal points in an element and the number of terms used in the local polynomial approximation . Recall that the piecewise-linear approximations in one dimension , the restriction of a test function to an element Ω_e was a linear function of the form

$$v_h^e(x) = \alpha_1 + \alpha_2 x$$

α_1 and α_2 being constants . Since the element has two nodes , the constants α_1 and α_2 are uniquely determined by specifying the value v_h^e at the endpoints of the element . Having done this , a continuous function v_h is produced by demanding that functions v_h^e and v_h^{e+1} in adjacent elements have same value at their common node.

An analogous situation exists in two dimensions . A linear function in two dimensions of form

$$v_h(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y$$

With three constants : α_1 , α_2 , and α_3 . Thus , three independent values of v_h must be specified to determine these constants , which means that the element should have three nodes , suggesting a triangle with nodes at the vertices . Moreover , if two adjacent triangles in the mesh share one side (and , hence , share two nodes) , a function continuous across the interface of these elements will be produced by demanding that the linear functions on the each element have the same values at the common nodes .

Similarly , a bilinear function

$$v_h(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$

Has four constants and might qualify as a shape function for an element with four nodes (a rectangle) . Likewise the quadratic

$$v_h(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2$$

Having six parameters , could be used to construct an element with six nodes (e . g , a triangle with a node at each vertex and at the midpoint of each side), and so on .

We now furnish some details as to how such two-dimensional finite element representations can be constructed , it is formative to consider the finite element concept as a device for interpolating a given function $g = g(x, y)$ defined on Ω .

As before , our aim is to construct an interpolant g_h of g of the form

$$g_h(x,y) = \sum_{j=1}^{N} g_j \emptyset_j(x,y) \quad , \quad (x,y) \in \Omega_h \tag{3.1}$$

Where $\emptyset_1(x,y), \emptyset_2(x,y), \ldots, \emptyset_N(x,y)$ are basis functions defined over Ω_h satisfying

$$\emptyset_i(x_j,y_j) = \begin{cases} 1 & if\ i = j \\ 0 & if\ i \neq j \end{cases} \tag{3.2}$$

where (x_j,y_j) are coordinates of nodal points in the finite element mesh . When (3.2) holds , we have

$$g_h(x_j,y_j) = g_j \quad , j = 1,2,\ldots,N \tag{3.3}$$

So by setting g_j=g(x_j,y_j) , g_h will coincide with (and , therefore , interpolate) the given function g at the nodes . We must deal with two basic requirements:

1. The construction of the local shape functions Ψ_i^e defined over each element Ω_e in the mesh , must be such that when patched together , they produce basis functions satisfy (3.2).
2. In anticipation of solving our model problem , the resulting basis functions $\emptyset_i$ must be square-integrable and have square-integrable first partial derivatives ; that is , they must satisfy

$$\int_{\Omega_h} \left[\left(\frac{\partial \emptyset_i}{\partial x}\right)^2 + \left(\frac{\partial \emptyset_i}{\partial y}\right)^2 + \emptyset_i^2 \right] dxdy < \infty \ , \ i = 1,2,\ldots,N \tag{3.4}$$

This requirement is satisfied by constructing the functions $\emptyset_i$ to be continuous across interelement boundaries .

3.2 Piecewise-Linear Interpolation on Triangles

Since the linear function ,

$$v_h(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y$$

Determines a plane surface , the use of linear interpolation on a triangle will result in the approximation of a given smooth function v by a planar function of the type shown in fig 3.1 . Suppose that Ω_h consists of a collection of E triangular elements and that we consider such a linear interpolation over a typical finite element Ω_e . Then the restriction of v_h to Ω_e will be of the form

$$v_h^e(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y \qquad \text{for } (x,y) \in \Omega_e \tag{3.5}$$

We determine the three constants from the conditions

$$v_1 = v_h^e(x_1,y_1) = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1$$

$$v_2 = v_h^e(x_2,y_2) = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2$$

$$v_3 = v_h^e(x_3,y_3) = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3$$

Where (x_i, y_i) , $i = 1,2,3$ are coordinates of the three vertices of the triangle . solving this system for α_1 , α_2, and α_3 we find

$$\alpha_1 = \frac{1}{2A_e}[v_1(x_2y_3 - x_3y_2) + v_2(x_3y_1 - x_1y_3) + v_3(x_1y_2 - x_2y_1)]$$

$$\alpha_2 = \frac{1}{2A_e}[v_1(y_2 - y_3) + v_2(y_3 - y_1) + v_3(y_1 - y_2)]$$

$$\alpha_3 = \frac{1}{2A_e}[v_1(x_3 - x_2) + v_2(x_1 - x_3) + v_3(x_2 - x_1)]$$

where A_e is the area of element Ω_e . Thus , eliminating α_1, α_2, and α_3 from (3.5) yields

$$v_h^e(x,y) = v_1\Psi_1^e(x,y) + v_2\Psi_2^e(x,y) + v_3\Psi_3^e(x,y) \tag{3.6}$$

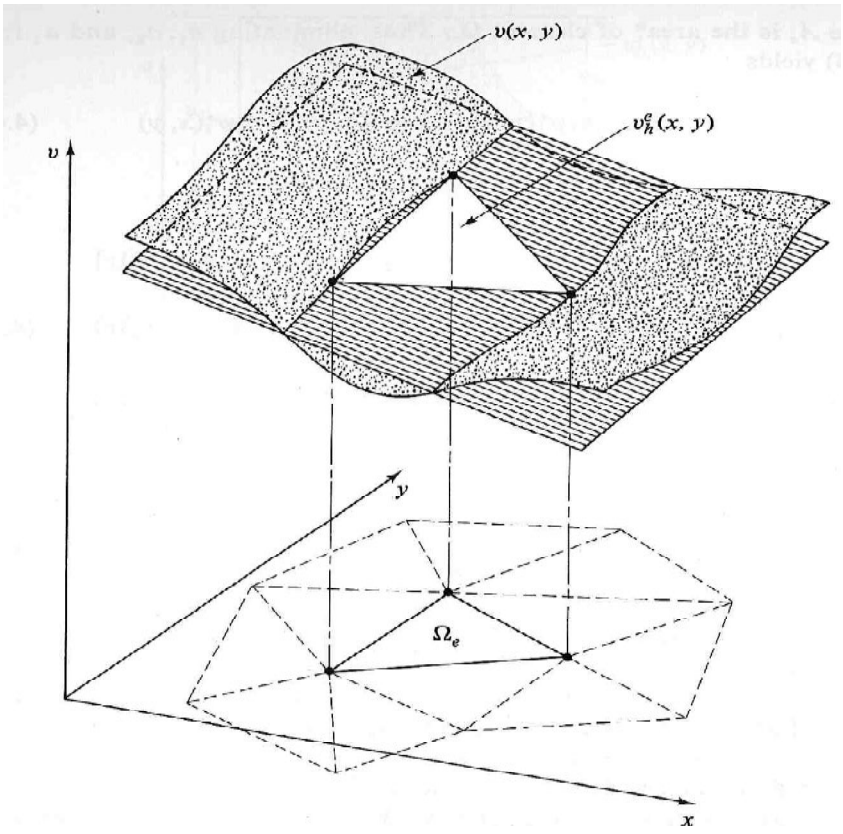

Figure 3.1 Illustration of the idea that the three values of v_h^e at the vertices of a triangular element Ω_e determine a plane which intersects surface v=v(x,y) at three points

Where $\Psi_i^e(x, y)$ are the element shape functions,

$$\left.\begin{aligned}
\Psi_1^e(x,y) &= \frac{1}{2A_e}[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y] \\
\Psi_2^e(x,y) &= \frac{1}{2A_e}[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y] \\
\Psi_3^e(x,y) &= \frac{1}{2A_e}[(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y]
\end{aligned}\right\} \qquad (3.7)$$

Notice that

$$\Psi_i^e(x_j, y_j) = \begin{cases} 1 & if\ i = j \\ 0 & if\ i \neq j \end{cases} \qquad i, j = 1,2,3 \qquad (3.8)$$

Now let us determine the type of "global" basis functions $\emptyset_i(x, y)$ that these shape functions produce . The basis functions $\emptyset_i(x, y)$ (*i=1,2,...,N*) are constructed in the same manner , the shape functions Ψ_i^e corresponding to adjacent elements in the mesh are simply patched together , to produced a "pyramid" function $\emptyset_i$ at each nodal point in the mesh . Clearly , each $\emptyset_i$ is piecewise-linear , it is assumes a value of unity at node (x_i, y_i) , and it is zero at all other nodes (x_j, y_j) , $j \neq i$ and therefore , satisfies (1.2) . For a boundary node , we have an analogous situation in that $\emptyset_i$ assumes the form of a portion of a "pyramid" . Of equal importance , the functions produced in this way are continuous across interelement boundaries and , therefore , over Ω_h ; their first partial derivatives are step functions and , hence , are square-integrable . Thus , such basis functions would be appropriate choices for constructing finite element approximations of the model problems .

Other Triangle Elements

Other triangle elements involving higher-degree polynomial in x and y can be easily constructed . Let us firstly display the terms appearing in polynomials of various degree in two variables in the following tabular form :

1	degree 0
$x \ \ y$	degree 1
$x^2 \ \ xy \ \ y^2$	degree 2
$x^3 \ \ x^2y \ \ xy^2 \ \ y^3$	degree 3
$x^4 \ \ x^3y \ \ x^2y^2 \ \ xy^3 \ \ y^4$	degree 4
.	
.	

This triangular array is called Pascal's triangle . Note that a complete polynomial of degree k in x and y will have exactly $\frac{1}{2}(k+1)(k+2)$ terms . Thus , a polynomial of degree k can be uniquely determined by specifying its value at $\frac{1}{2}(k+1)(k+2)$ points in the plane . Moreover , the location of entries in Pascal's triangle suggests a symmetric location of nodal points in triangular elements that will produce exactly the right number of nodes to define a polynomial interpolant of any degree . For instance , the six terms in a quadratic polynomial will be determined by specifying the value of v_h^e at six nodal points in a triangle , one at each vertex and one at a midpoint of each side-precisely the location of entries in the triangle formed by the quadratic in Pascal's triangle . Similarly, a complete cubic, having 10 terms, leads to a triangular element with 10 nodes . The location of nodes is , again, determined by Pascal's triangle : one at each vertex , two on each side dividing each side into three equal lengths, and one at the centroid . Similarly, a complete quartic leads to 15 nodes , and so on . The family of finite elements generated in this manner is illustrated in Fig 3.2a .

Another important feature of these elements is that they produce , for polynomials of degree > 0 , basis functions that are continuous over the domain and, therefore , have square-integrable first partial derivatives Consider, for example, two adjacent six node triangles Ω_e and Ω_{e+1}in the mesh . The local interpolants v_h^e and v_h^{e+1} are quadratic polynomials that must coincide at the three nodal points common to each element . However, the specification of three values of a quadratic in one dimension uniquely determines that quadratic . Hence , v_h^e and v_h^{e+1} will coincide everywhere on the common boundary of the two elements , and v_h will therefore , be continuous across this boundary . The idea is illustrated in Fig 3.2b . Similarly , for cubic elements , values at four nodes are matched on common boundaries . Since a one –dimensional cubic is determined by specifying four independent values , a continuous piecewise cubic function is formed by patched together .

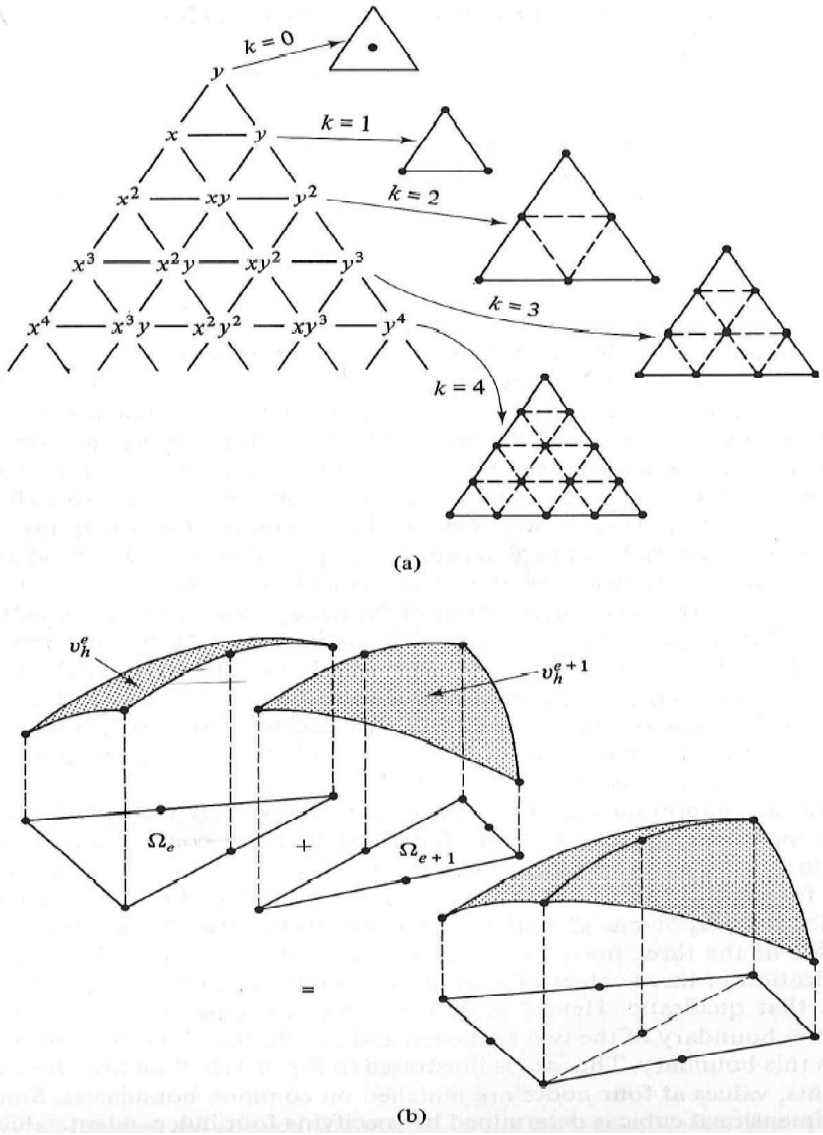

Figure 3.2 (a) use of Pascal's triangle to generate various triangular elements over which complete polynomial shape of any degree k are defined ; (b) illustrations for the case k=2 , that basis functions produced by such elements are continuous across interelement boundaries

3.3 Rectangular Elements

By taking the "product" of sets of polynomials in x with sets of polynomials in y, shape functions for a variety of rectangular elements can

be obtained , we show these ideas can be used to construct general quadriclateral elements .

For examples , a linear polynomial in x is characterized by a linear combination of monomials $(1, x)$. The tensor product with monomials $(1, y)$ produced the matrix of four functions

$$\begin{bmatrix}1\\x\end{bmatrix}[1 \;\; y] = \begin{bmatrix}1 & y\\x & xy\end{bmatrix} \tag{3.9}$$

And a linear combination of entries in this matrix produces a local bilinear polynomial of the form

$$v_h^e(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy \tag{3.10}$$

There are four constants in (3.10) and four elements in the tensor product (3.9) Thus , if we visualize a rectangular element with four nodes , one at each corner, the function v_h^e in (3.10) is completely and uniquely determined by specifying its values at these four nodal points . Moreover , along the sides x =constant and y =constant , v_h^e is linear in x *or* y . Thus , if two such rectangular elements Ω_e and Ω_{e+1} have a common side in mesh , a function that is continuous across their common interelement boundary will be produced by demanding that v_h^e and v_h^{e+1} assume the same values at nodes common to each element . Hence , shape functions obtained using (3.10) will produce basis functions $\emptyset_i$ which have square-integrable first derivatives over Ω_h .

By considering tensor products of polynomials of higher degree , element shape functions can be constructed which contain polynomials of any desired degree and which lead to basis functions that are continuous throughout Ω_h . For example , the product of two quadratics yields a matrix with nine entries :

$$\begin{bmatrix}1\\x\\x^2\end{bmatrix}[1 \quad y \quad y^2] = \begin{bmatrix}1 & y & y^2\\x & xy & xy^2\\x^2 & x^2y & x^2y^2\end{bmatrix} \tag{3.11}$$

A biquadratic local interpolant v_h^e is obtained by forming a linear combination of all nine terms in this matrix . By constructing a rectangular element with nine nodes , one node located in the element at the point corresponding to the location of each entry in the foregoing matrix , an element is produced which leads to piecewise-biquadratic basis functions continuous on all Ω_h. Similarly , a tensor product of cubics leads to an element with 16 nodes and bicubic shape functions ; and so on . Various rectangular elements produced by tensor products of polynomials are illustrated in Fig 3.3

Interpolation Error

Suppose that a smooth function g is given . Further , assume that we wish to interpolate g by a finite element representation g_h which contains complete polynomials of degree k . As in the one-dimensional , if partial

	1	y	y^2	y^3	y^4	$\cdots$
1	1	y	y^2	y^3	y^4	$\cdots$
x	x	xy	xy^2	xy^3	xy^4	$\cdots$
x^2	x^2	x^2y	x^2y^2	x^2y^3	x^2y^4	$\cdots$
x^3	x^3	x^3y	x^3y^2	x^3y^3	x^3y^4	$\cdots$
x^4	x^4	x^4y	x^4y^2	x^4y^3	x^4y^4	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

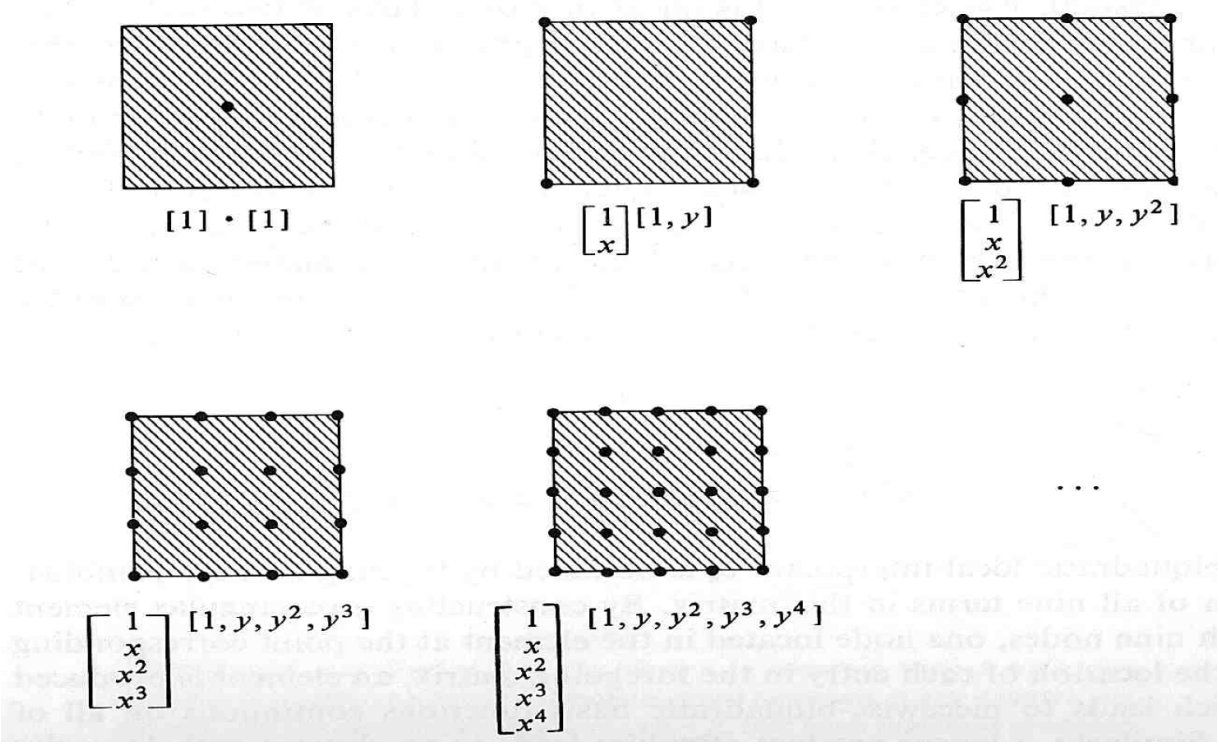

Figure 3.3 matrix containing terms of a tensor product of polynomials and various rectangular elements obtained using such tensor products .

derivatives of g of order $k+1$ are bounded in Ω_e the interpolation error satisfies

$$\begin{aligned} \|\mathrm{g}-\mathrm{g_h}\|_{\infty,\Omega_e} &= \max_{(x,y)\in\Omega_e} |\mathrm{g}(x,y)-\mathrm{g}_h(x,y)| \\ &\leq Ch_e^{k+1} \quad , \end{aligned} \tag{3.12}$$

where C is a positive constant and h_e is the "diameter" of Ω_e ; that is , h_e is the largest distance between any two points in Ω_e . As in the one-dimensional case note that this estimate holds only if all terms in a complete polynomial of degree k appear in g_h . Similarly ,

$$\left\|\frac{\partial \mathrm{g}}{\partial x}-\frac{\partial \mathrm{g}_h}{\partial x}\right\|_{\infty,\Omega_e} \leq C_1 h_e^{k+1} \quad \text{and} \quad \left\|\frac{\partial \mathrm{g}}{\partial y}-\frac{\partial \mathrm{g}_h}{\partial y}\right\|_{\infty,\Omega_e} \leq C_2 h_e^{k+1} \tag{3.13}$$

The H^1-norm in two dimensions is defined by

$$\|g\|_1^2 = \int_\Omega \left[g^2 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 \right] dx\, dy \qquad (3.14)$$

Suppose that $\Omega_h = \Omega$ and h is the maximum diameter of all elements in the mesh . It can also be shown that for reasonable meshes and refinement ,

$$\|g - g_h\|_1 \leq C_3 h_e^k \qquad (3.15)$$

for h sufficiently small . This estimate holds only if g_h contains a complete polynomial of degree k . For example , if g_h is piecewise-linear on triangles $k = 1$ and we say that the H^1-interpolation error is $O(h)$. Similarly if g_h is piecewise-bilinear , so that $g_h^e = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$, then (3.15) holds with $k = 1$ even though g_h^e contains a quadratic term xy . The key is that g_h^e contains a complete polynomial of degree $k = 1$ but not $k = 2$; the terms x^2 and y^2 are missing . Similarly , if g_h^e is a tensor product of quadratics , it will only contain complete polynomials of degree $k = 2$, even though cubic and quadratic terms appear in such shape functions . These extra terms furnish enough nodal points for the element to provide for the generation of continuous basis functions , but they do not contribute to asymptotic rate of convergence of the interpolation error .

3.4 Finite Element Approximations:

Approximation of Two-Dimensional Boundary-Value Problems

Let us now return to the problem described previously sections . In particular , consider the general variational boundary-value problem (2.33) . let $H^1(\Omega)$ denote the class of functions satisfying (2.32) and defined over the whole domain Ω . Our problem is then to find a function u in $H^1(\Omega)$ such that $u = \hat{u}$ on $\partial\Omega_1$ and such that

$$\int_{\Omega}\left[k\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)+buv\right]dxdy+\int_{\partial\Omega_2}puv\,ds = \int_{\Omega}fv\,dx\,dy+\int_{\partial\Omega_2}\gamma v\,ds \tag{3.16}$$

For all $v \in H^1(\Omega)$ such that $v = 0$ on $\partial\Omega_1$ and where $\gamma = p\hat{u}$.

The approximation of (3.16) follows the pattern now familiar . We replace Ω by a domain Ω_h that consist of a collection of E finite elements and N nodal points and we define an N-dimensional subspace H^h of $H^1(\Omega_h)$ by constructing an appropriate set of global basis functions $\emptyset_i$, $i = 1,2,\dots,N$. Since the shape functions are continuous within each element , they can not modal a jump in material properties there . Hence , we always choose the location of nodes and element boundaries to coincide with interfaces at which jumps in the modulus k occur , as indicated in fig 3.4 . A typical test function in H^h will be of the form

$$v_h(x,y) = \sum_{j=1}^{N} v_j \emptyset_j(x,y) \tag{3.17}$$

Where $v_j = v_h(x_j, y_j)$. In general , the Dirichlet data $\hat{u}$ given on $\partial\Omega_1$ is approximated by its interpolant $\hat{u}_h(s) = \sum \hat{u}_j \emptyset_j\,(x(s), y(s))$ the sum being taken over all nodes on the approximation $\partial\Omega_{1h}$ of $\partial\Omega_1$.

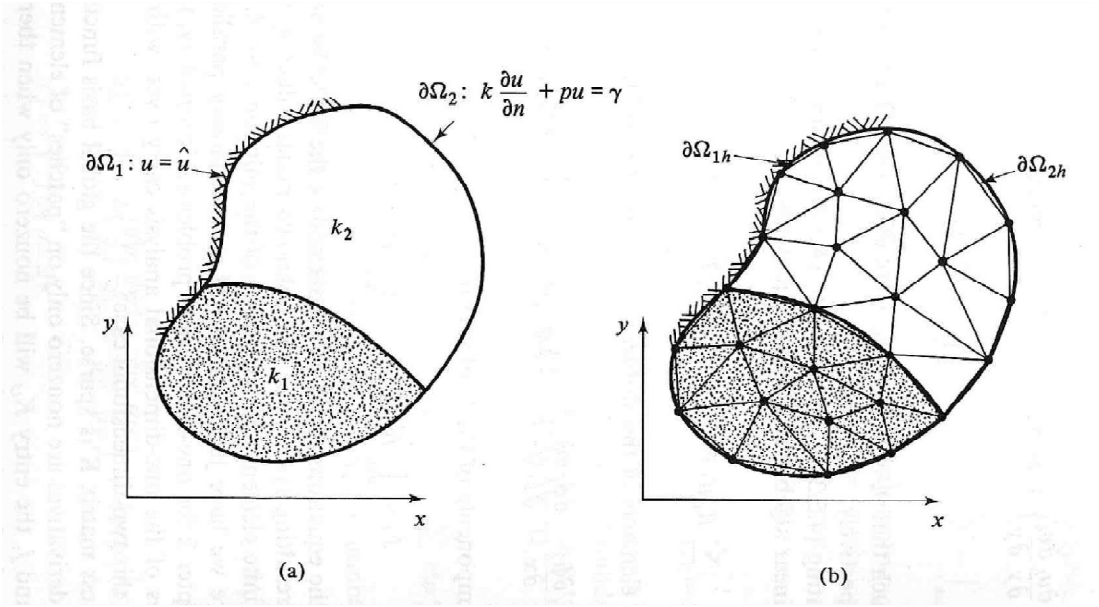

Figure 3.4 (a) the domain Ω for the modal problem and (b) its finite element discretization Ω_h

Our approximation (3.16) then consists of seeking a function u_h in H^h,

$$u_h(x,y) = \sum_{j=1}^{N} u_j \emptyset_j(x,y) \tag{3.18}$$

Such that $u_j = \hat{u}_j$ at the nodes on $\partial\Omega_{1h}$ and

$$\int_{\Omega_h} \left[k\left(\frac{\partial u_h}{\partial x}\frac{\partial v_h}{\partial x} + \frac{\partial u_h}{\partial y}\frac{\partial v_h}{\partial y}\right) + bu_h v_h \right] dxdy + \int_{\partial\Omega_{2h}} pu_h v_h \, ds = \int_{\Omega_h} f v_h \, dx \, dy + \int_{\partial\Omega_{2h}} \gamma v_h \, ds \tag{3.19}$$

For all $v_h \in H^h$, such that $v_h = 0$ on $\partial\Omega_{1h}$. Here $\partial\Omega_{1h}$ and $\partial\Omega_{2h}$ approximate $\partial\Omega_1$ and $\partial\Omega_2$, respectively .

Upon substituting (3.17) and (3.12) into (3.19) and simplifying terms , we arrive at the linear algebraic system of equations

$$\sum_{j=1}^{N} K_{ij} u_j = F_i \quad , \quad i = 1,2, \dots, N \tag{3.20}$$

Where K_{ij} are the elements of stiffness matrix for the problem ,

$$K_{ij} = \int_{\Omega_h} \left[k \left(\frac{\partial \emptyset_i}{\partial x} \frac{\partial \emptyset_j}{\partial x} + \frac{\partial \emptyset_i}{\partial y} \frac{\partial \emptyset_j}{\partial y} \right) + b \emptyset_i \emptyset_j \right] dxdy + \int_{\partial \Omega_{2h}} p \emptyset_i \emptyset_j \, ds \tag{3.21}$$

and F_i are the components of the load vector,

$$F_i = \int_{\Omega_h} f \emptyset_i \, dx \, dy + \int_{\partial \Omega_{2h}} \gamma \emptyset_i \, ds \tag{3.22}$$

We next modify the equations (3.20) to accommodate the Dirichlet data and finally solve the resulting system for the unknown nodal values u_j there by determining the finite element approximation of the solution u to (3.16) .

The procedure we have just outlined , of course , closely parallels that for one-dimensional problems . Fortunately , most of the other features of the one-dimensional analysis carry over , with minor modifications , to this two-dimensional case :

1. The stiffness matrix K is sparse . Since the global basis functions $\emptyset_i$ and $\emptyset_j$ and their derivatives are nonzero only on " patches " of elements containing nodes i , j , the entry K_{ij} will be nonzero only when there is an element containing both node i and node j .
2. In the present case , K is symmetric (owing to the fact that the operator is self-adjoint) . Note also that if a judicious numbering of nodes is used , K will be banded ; that is the nonzero elements in K will form a band containing the main diagonal of the matrix . The fact that K is sparse , symmetric matrix can be thoroughly exploited in designing efficient algorithms for solving linear systems of the form (3.20).
3. Each of the integrals in (3.21) and (3.22) can be calculated as the sum of contributions furnished by each element in the mesh . However , the interpretation of such a procedure is interesting and deserves some elaboration . Let Ω_e denote a typical finite element in the mesh . On Ω_e, the exact solution u of our boundary-value problem satisfies

$$\int_{\Omega_e}(k\nabla u\cdot\nabla v+buv)dxdy=\int_{\Omega_e}fv\,dx\,dy-\int_{\partial\Omega_e}\sigma_n v\,ds$$

For every admissible v, where σ_n is the normal component of flux at the element boundary . Next , let u_h^e and v_h^e denote the restrictions of the approximations u_h and v_h to Ω_e then the local approximation of the variational boundary-value problem over Ω_e assumes the form

$$\int_{\Omega_e}(k\nabla u_h^e\cdot\nabla v_h^e+bu_h^e v_h^e)dxdy=\int_{\Omega_e}fv_h^e\,dx\,dy-\int_{\partial\Omega_e}\sigma_n v_h^e\,ds \qquad (3.23)$$

Here σ_n is the actual (exact) flux across $\partial\Omega_e$ and , although not given as data in the original problem , appears as data in natural boundary conditions on $\partial\Omega_e$. Since v_h=0 on $\partial\Omega_{1h}$, there will be no contribution to the last integral of (3.23) from elements with sides coincident with $\partial\Omega_{1h}$. Since u_h^e and v_h^e are of the form

$$u_h^e(x,y)=\sum_{j=1}^{N_e}u_j^e\Psi_j^e(x,y)\quad,\quad v_h^e(x,y)=\sum_{j=1}^{N_e}v_j^e\Psi_j^e(x,y)$$

Ψ_j^e being the local shape functions for Ω_e and $N_e$the number of nodes in Ω_e , (3.23) leads to the linear system

$$\sum_{j=1}^{N_e}k_{ij}^e u_j^e=f_i^e-\sigma_i^e\ ,\ i=1,2,\dots,N_e \qquad (3.24)$$

Where

$$k_{ij}^e=\int_{\Omega_e}\left[k\left(\frac{\partial\Psi_i^e}{\partial x}\frac{\partial\Psi_j^e}{\partial x}+\frac{\partial\Psi_i^e}{\partial y}\frac{\partial\Psi_j^e}{\partial y}\right)+b\Psi_i^e\Psi_j^e\right]dxdy \qquad (3.25)$$

$$f_i^e=\int_{\Omega_e}f\Psi_i^e\,dxdy \qquad (3.26)$$

$$\sigma_i^e=\int_{\partial\Omega_e}\sigma_n\Psi_i^e\,ds \qquad (3.27)$$

Here k_{ij}^e and f_i^e are the components of the element stiffness matrix and load vector , respectively , for element Ω_e and σ_e is an element flux vector, obtained

by assigning to node i of Ω_e a weighted average $\int_{\partial\Omega_e}\sigma_n\Psi_i^e\,ds$ of the actual flux σ_n across $\partial\Omega_e$.

Formally, the global system of equations (3.20) is obtained by summing (3.24) over all E elements in the mesh . We expand the element matrices in (3.25), (3.26), and (3.27) to $N\times N$ and $N\times 1$ order matrices corresponding to the order of global matrices in (3.21) and (3.22) . For example , k^ewill become an $N\times N$ matrix k^ewill zeros everywhere except those rows and columns corresponding to nodes within element Ω_e and f^e and σ^e will be expanded to $N\times 1$ vectors F^e and $\sum^e$ with nonzero entries only in those rows corresponding to nodes in Ω_e. Then the first terms in the global matrices in (3.21) and (3.22) are obtained as the sums

$$\left.\begin{aligned}&\textstyle\sum_{e=1}^{E}\int_{\Omega_e}\left[\left(\frac{\partial\emptyset_i}{\partial x}\frac{\partial\emptyset_j}{\partial x}+\frac{\partial\emptyset_i}{\partial y}\frac{\partial\emptyset_j}{\partial y}\right)+b\emptyset_i\emptyset_j\right]dx\,dy=\sum_{e=1}^{E}k_{ij}^e\\&\textstyle\sum_{e=1}^{E}\int_{\Omega_e}f\,\emptyset_i\,dxdy=\sum_{e=1}^{E}F_i^e\quad,\quad i,j=1,2,\dots,N\end{aligned}\right\}\tag{3.28}$$

And

$$\textstyle\sum_{e=1}^{E}(k_{ij}^e u_j-F_i^e+\sum_i^e)=0\quad,\qquad i=1,2,\dots,N\tag{3.29}$$

Notice that the contributions to K_{ij} and F_i from boundary conditions (recall (3.21) and (3.22)) must enter the problem through the terms $\sum_i^e$. Continuing we node that the sum of the contour integrals can be written in the form

$$\textstyle\sum_{e=1}^{E}\sum_i^e=S_i^{(0)}+S_i^{(1)}+S_i^{(2)}\quad,\qquad i=1,2,\dots,N\tag{3.30}$$

Where

$$\begin{aligned}S_i^{(0)}&=\textstyle\sum_{e=1}^{E}\int_{\partial\Omega_e-\partial\Omega_h}\sigma_n\emptyset_i\,ds\\S_i^{(1)}&=\textstyle\int_{\partial\Omega_{1h}}\sigma_n\emptyset_i\,ds\\S_i^{(2)}&=\textstyle\int_{\partial\Omega_{2h}}\sigma_n\emptyset_i\,ds\end{aligned}\tag{3.31}$$

Here $\partial\Omega_e - \partial\Omega_h$ is the portion of the boundary $\partial\Omega_e$ of Ω_e not on $\partial\Omega_h$ (i.e, the part of $\partial\Omega_e$ that consists of interelement boundaries) . We interpret the quantities in (3.31) as follows .

$S_i^{(0)}$:Since only involves terms on $\partial\Omega_e - \partial\Omega_h$, this vector is defined only at interior nodes i . To interpret $S_i^{(0)}$, consider an interior patch of four elements having node 1 in common such as that indicated in Fig 3.5 . Using (3.23) and (3.29) , we easily verify that , for this node , $S_1^{(0)}$ has the form

$$S_1^{(0)} = \sum_{e=1}^{4} \int_{\partial\Omega_e} \sigma_n \emptyset_1 \, ds$$

$$= \int_{\Gamma_1} [\![\sigma_n]\!] \emptyset_1 \, ds + \int_{\Gamma_2} [\![\sigma_n]\!] \emptyset_1 \, ds + \int_{\Gamma_3} [\![\sigma_n]\!] \emptyset_1 \, ds + \int_{\Gamma_4} [\![\sigma_n]\!] \emptyset_1 \, ds$$

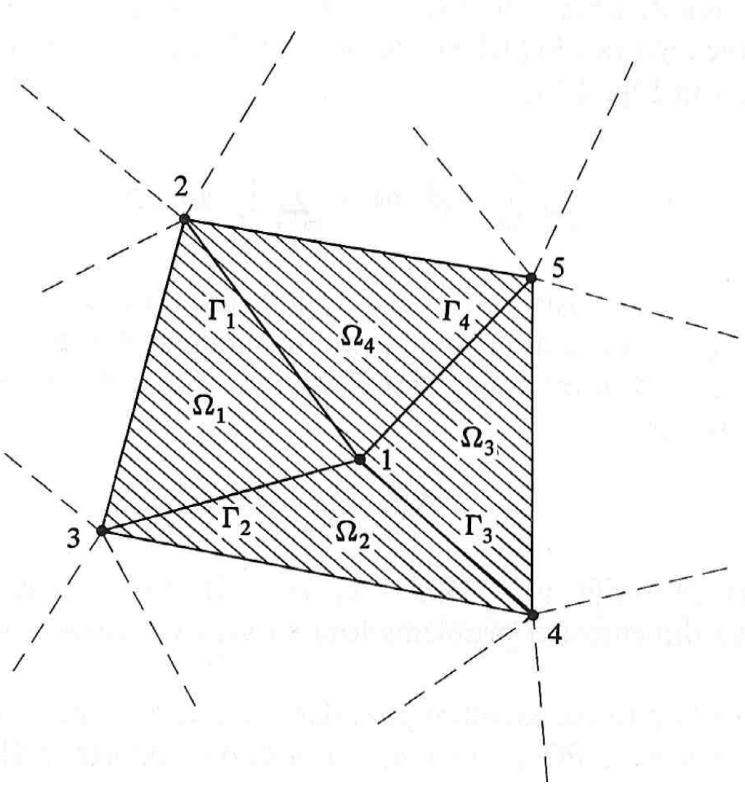

Figure 3.5 An interior patch of four elements sharing node 1

From the conservation law, $[\![\sigma_n]\!] = 0$ across an interface where no point or line sources are applied . Thus , if f is smooth in the patch shown in Fig. 3.5, we have

$$S_1^{(0)} = 0 \tag{3.32}$$

Naturally , there is no need to evaluate these zero contributions (3.29) and (3.30) , so that they may be excluded from element calculations .

An exception to (3.32) occurs when the source function f contains a line source or concentrated point source . Then $[\![\sigma_n]\!]$ equals the intensity of the line source and is no longer zero . In the case of a point source , our present variational formulation is not strictly applicable . However , we can include the effects of point sources in the finite element analysis if we proceed as follows . Suppose that f has the form

$$f(x, y) = \tilde{f}(x, y) + \hat{f}\delta(x - x_i, y - y_i) \tag{3.33}$$

Where $\tilde{f}$ is smooth (integrable) part of f and $\hat{f}\delta(x - x_i, y - y_i)$ denotes a point source of intensity $\hat{f}$ at a point $(x_i, y_i) \in \Omega_h$. As in our study of one-dimensional problems , that the mesh Ω_h is always constructed so that nodal points are located at points where point sources act . Then only the smooth part $\tilde{f}$ of f appears in the integrals in (3.23) and (3.26) . In this case , note that for the interior node in Fig . 3.5,

$$S_1^{(0)} = \sum_{e=1}^{4} \int_{\partial\Omega_e} \sigma_n \emptyset_1 \, ds = \sum_{m=1}^{4} \int_{\Gamma_m} [\![\sigma_n]\!] \emptyset_1 \, ds$$

The presence of the basis function $\emptyset_1$ indicates that $S_1^{(0)}$ represents the weighted average of these jumps at the interior node 1 . We choose to balance nonzero jumps in flux by concentrated sources $\hat{f}$, we set

$$S_1^{(0)} = \hat{f}$$

Whenever $f(x,y) = \tilde{f}(x,y) + \hat{f}\delta(x - x_i, y - y_i)$. We remark that the point sources in two-dimension problems leads to very irregular (singular) solutions u .

$S_i^{(1)}$: According to the essential boundary conditions , the values of u_h prescribed at nodes on $\partial\Omega_{1h}$. Since σ_n is not known on $\partial\Omega_{1h}$, $S_i^{(1)}$ cannot be prescribed there . However , once all of the nodal displacements $u_1, u_2, \dots, u_N$ have been determined , an approximation of $S_i^{(1)}$ can be calculated directly from (4.14) , if desired .

$S_i^{(2)}$: On $\partial\Omega_{2h}$, the natural boundary condition is specified . There we set

$$\sigma_n(s) = p(s)u_h(s) - \gamma(s)$$

So that , approximately ,

$$\begin{aligned} S_i^{(2)} &\approx \int_{\partial\Omega_{2h}} \left[p \sum_{j=1}^{N} u_j \emptyset_j - \gamma\right] \emptyset_i ds \\ &= \sum_{j=1}^{N} p_{ij} u_j - \gamma_i \end{aligned} \tag{3.34}$$

where

$$\gamma_i = \int_{\partial\Omega_{2h}} \gamma \emptyset_i \, ds = \sum_{e=1}^{E} \int_{\partial\Omega_{2h}^e} \gamma \emptyset_i \, ds = \sum_{e=1}^{E} \gamma_i^e \tag{3.35}$$

and

$$p_{ij} = \int_{\partial\Omega_{2h}} p \emptyset_i \emptyset_j \, ds = \sum_{e=1}^{E} \int_{\partial\Omega_{2h}^e} p \emptyset_i \emptyset_j \, ds = \sum_{e=1}^{E} p_{ij}^e \tag{3.36}$$

Here $\partial\Omega_{2h}^e$ is aportion of $\partial\Omega_e$ intersecting Ω_{2h} .

Returning now to (3.29) , we arrive at the system of equations ,

$$\sum_{j=1}^{N} K_{ij} \, u_j = F_i - S_i^{(1)} \quad , \quad i = 1,2, \dots, N \tag{3.37}$$

where

$$K_{ij} = \sum_{e=1}^{E} (K_{ij}^e + p_{ij}^e) \tag{3.38}$$

$$F_i = \sum_{e=1}^{E}(F_i^e + \gamma_i^e) \tag{3.39}$$

We now impose boundary conditions on $\partial\Omega_{1h}$ and proceeds to solve the resulting system of equations for the unknown nodal values .

An Example

We shall briefly outline the analysis of a simple example problem . Consider the formulation of a finite element approximation of problem

$$\left.\begin{array}{ll} -\Delta u(x,y) = f(x,y) & in\ \Omega \\ u = 0 & on\ \Gamma_{41} \\ \frac{\partial u}{\partial n} = 0 & on \Gamma_{12}, \Gamma_{25}, \Gamma_{67}, and\ \ \Gamma_{74} \\ \frac{\partial u}{\partial n} + \beta u = \gamma & on\ \Gamma_{56} \end{array}\right\} \tag{3.40}$$

Where Ω is the polygonal domain shown in Fig .3.7a and $\Gamma_{41}, \Gamma_{12}, \ldots, \Gamma_{74}$ are the segments of the boundary . In this case , $\partial\Omega_1 = \Gamma_{41}$ and $\partial\Omega_2$ consists of the segments $\Gamma_{12}, \Gamma_{25}, \Gamma_{56}$, Γ_{67} and Γ_{74} . Our analysis of this problem proceeds as follows:

1. We partition the domain into six triangular elements , as in Fig .3.6b , over which linear approximations u_h of the solution u of (3.40) are defined . The six elements and seven nodes are numbered as shown . Note that $\partial\Omega_1 = \partial\Omega_{1h}$ and $\Omega_2 = \partial\Omega_{2h}$.

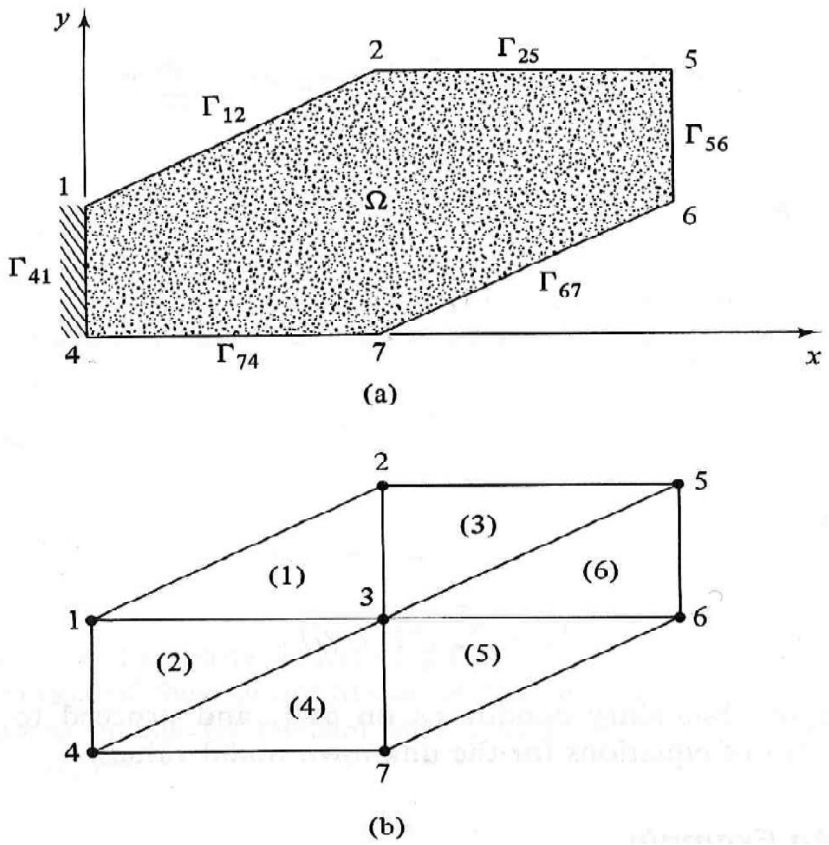

Figure 3.6 (a) The polygonal domain Ω in problem (4.25)and (b) a finite-element model of this domain

2. Next ,we use (3.25) and (3.26) to compute the element matrices K^e and F^e , e=1,2…,6 , and expand these to 7 × 7 matrices :

   Element 1

$$\mathbf{K}^1 = \begin{bmatrix} k_{11}^1 & k_{12}^1 & k_{13}^1 & 0 & 0 & 0 & 0 \\ k_{21}^1 & k_{22}^1 & k_{23}^1 & 0 & 0 & 0 & 0 \\ k_{31}^1 & k_{32}^1 & k_{33}^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{F}^1 = \begin{bmatrix} f_1^1 \\ f_2^1 \\ f_3^1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Element 2

$$\mathbf{K}^2 = \begin{bmatrix} k_{11}^2 & 0 & k_{12}^2 & k_{13}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ k_{21}^2 & 0 & k_{22}^2 & k_{23}^2 & 0 & 0 & 0 \\ k_{31}^2 & 0 & k_{32}^2 & k_{33}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{F}^2 = \begin{bmatrix} f_1^2 \\ 0 \\ f_2^2 \\ f_3^2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Element 6

$$\mathbf{K}^6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{11}^6 & 0 & k_{12}^6 & k_{13}^6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{21}^6 & 0 & k_{22}^6 & k_{23}^6 & 0 \\ 0 & 0 & k_{31}^6 & 0 & k_{32}^6 & k_{33}^6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{F}^6 = \begin{bmatrix} 0 \\ 0 \\ f_1^6 \\ 0 \\ f_2^6 \\ f_3^6 \\ 0 \end{bmatrix}$$

3. As the contributions from each element are calculated , strating from element 1 and continuing through element 6, they are added to their appropriate locations in the global stiffness and load matrices . At this stage , we have the system :

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & 0 & 0 & 0 \\ K_{21} & K_{22} & K_{23} & 0 & K_{25} & 0 & 0 \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} & K_{37} \\ K_{41} & 0 & K_{43} & K_{44} & 0 & 0 & K_{47} \\ 0 & K_{52} & K_{53} & 0 & \tilde{K}_{55} & \tilde{K}_{56} & 0 \\ 0 & 0 & K_{63} & 0 & \tilde{K}_{65} & \tilde{K}_{66} & K_{67} \\ 0 & 0 & K_{73} & K_{74} & 0 & K_{76} & K_{77} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ \tilde{F}_5 \\ \tilde{F}_6 \\ F_7 \end{bmatrix} - \begin{bmatrix} \Sigma_1 \\ 0 \\ 0 \\ \Sigma_4 \\ \Sigma_5 \\ \Sigma_6 \\ 0 \end{bmatrix} \quad (3.41)$$

Where $F_1 = f^{\,1}_{\,1} + f^{\,2}_{\,1}$, $F_2 = f^{\,1}_{\,2} + f^{\,3}_{\,2}$, etc , and the $\sum_i$ are defined using (3.30) . the entries market with $\sim$ will be modfied in the final system of equations upon the application of the natural boundary conditions on Γ_{56} .

4. Since nonhomogeneous conditions are applied only on the segment connecting node 5 and 6, the matrices γ and P defined in (3.35) and (3.36) are of the form

$$\boldsymbol{\gamma} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \gamma_5 \\ \gamma_6 \\ 0 \end{bmatrix}, \qquad \mathbf{P} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & P_{55} & P_{56} & 0 \\ 0 & 0 & 0 & 0 & P_{65} & P_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.42}$$

Thus (3.41) becomes

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & 0 & 0 & 0 \\ K_{21} & K_{22} & K_{23} & 0 & K_{25} & 0 & 0 \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} & K_{37} \\ K_{41} & 0 & K_{43} & K_{44} & 0 & 0 & K_{47} \\ 0 & K_{52} & K_{53} & 0 & K_{55} & K_{56} & 0 \\ 0 & 0 & K_{63} & 0 & K_{65} & K_{66} & K_{67} \\ 0 & 0 & K_{73} & K_{74} & 0 & K_{76} & K_{77} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} F_1 - \Sigma_1 \\ F_2 \\ F_3 \\ F_4 - \Sigma_4 \\ F_5 \\ F_6 \\ F_7 \end{bmatrix} \tag{3.43}$$

Wherein

$K_{55} = \widetilde{K}_{55} + P_{55}$, $K_{56} = \widetilde{K}_{56} + P_{56}$, $F_5 = \widetilde{F}_5 + \gamma_5$ etc.

5. We now impose the essential conditions u_1=u_4=0 on Γ_{41} . In this way , we obtain the invetible system of five equations and five unknown

$$\begin{bmatrix} K_{22} & K_{23} & K_{25} & 0 & 0 \\ K_{32} & K_{33} & K_{35} & K_{36} & K_{37} \\ K_{52} & K_{53} & K_{55} & K_{56} & 0 \\ 0 & K_{63} & K_{65} & K_{66} & K_{67} \\ 0 & K_{73} & 0 & K_{76} & K_{77} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} F_2 \\ F_3 \\ F_5 \\ F_6 \\ F_7 \end{bmatrix} \tag{3.44}$$

Which we solve for the nodal values u_2, u_3, u_5, u_6 and u_7 .The remaining pair of equations can then be used to calculate the approximate fluxes $\sum_1$ and $\sum_4$ at 1 and 4 :

$$\left.\begin{aligned} -\sum_1 &= k_{12}u_2 + k_{13}u_3 + k_{14}u_4 - F_1 \\ -\sum_4 &= k_{43}u_3 + k_{47}u_7 - F_4 \end{aligned}\right\} \tag{3.45}$$

Other features of the solution can now be evaluated since , by (3.18) , u_h is now completely determined .

Solution let :

$$\mathrm{a}(\Psi_i, \Psi_j) = \begin{cases} \frac{2}{h} & i = j \\ \frac{-1}{h} & |i-j| = 1 \\ 0 & otherwise \end{cases}$$

then element 1

$$= \frac{1}{h}\begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 2

$$= \frac{1}{h}\begin{bmatrix} 2 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\vdots$$

Element 6

$$= \frac{1}{h}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 2 & 0 \end{bmatrix}$$

As the contributions from each element are calculated , starting from element 1 and continuing throught element 6, they are added to their appropriate locations in the global stiffness and load matrices .

After imposing the boundary conditions , we solve the resulting system for the unknown nodal values u_j by writing a program code using matlab or any programming languages , which is not part of this thesis .

References

[1] E.B. Becker, G.F. Carey & J.T. Oden, (1981), Finite Elements An Introduction Volume 1, Prentice-Hall , USA.

[2] J. Chaskalovic,(2008), Finite Element Methods For Engineering Sciences, Springer, Berlin .

[3] M.S. Gockenbach, (2006), Understanding And Implementing The Finite Element Method, SIAM, USA .

[4] C. Johnson, (1987), Numerical Solution of Partial Differential Equations by The Finite Element Methods, Cambridge University Press, Cambridge.

[5] O.C. Zienkiewicz, R.L. Taylor & J.Z. Zhu, (2005), The Finite Element Method: Its Basis And Fundamentals, Elsevier Butter Worth- Heinemann, Oxford.