Chapter 3
Powers and Spectrum of Class $wF(p, r, q)$ Operators with an Operators Equation

In this chapter we discuss powers of class $wF(p, r, q)$ operators for $1 \geq p > 0$, $1 \geq r > 0$ and $q \geq 1$; and an example is given on powers of class $wF(p, r, q)$ operators. We show that every class $wF(p, r, q)$ operator has SVEP and property $(\beta)$, and Weyl’s theorem holds for $f(T)$ when $f \in H(\sigma(T))$. As a continuation, we consider the equation $K^p = H^{-\frac{\delta}{r}} T^\frac{1}{r} (T^\frac{1}{r} H^\delta + r T^\frac{1}{r}) \frac{p-\delta}{\delta+r} T^\frac{1}{r} H^{-\frac{\delta}{r}}$, where $p > 0, r > 0$ and $p \geq \delta > -r$. As applications, we show that the inclusion relations among class $wA(p, r)$ operators are strict and show a generalization of Aluthge’s result.

Sec (3.1): Powers of Class $wF(p, r, q)$ Operators

Let $H$ be a complex Hilbert space and $B(H)$ be the algebra of all bounded linear operators in $H$, and a capital letter (such as $T$) denote an element of $B(H)$. An operator $T$ is said to be $k$-hyponormal for $k > 0$ if $(T^*T)^k \geq (TT^*)^k$, where $T^*$ is the adjoint operator of $T$. A $k$-hyponormal operator $T$ is called hyponormal if $k = 1$; semi-hyponormal if $k = 1/2$. Hyponormal and semi-hyponormal operators have been studied by many authors, such as [119,171,159,174,135]. It is clear that every $k$-hyponormal operator is $q$-hyponormal for $0 < q \leq k$ by the Löwner-Heinz theorem ($A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $1 \geq \alpha \geq 0$). An invertible operator $T$ is said to be log-hyponormal if $\log T^*T \geq \log TT^*$, see [142,158]. Every invertible $k$-hyponormal operator for $k > 0$ is log-hyponormal since $\log t$ is an operator monotone function. log-hyponormality is sometimes regarded as 0-hyponormal since $(X^k - 1)/k \rightarrow \log X$ as $k \rightarrow 0$ for $X > 0$.

As generalizations of $k$-hyponormal and log-hyponormal operators, many authors introduced many classes of operators, see the following.

Definition (3.1.1)[141,146,148]:
(1) For $p > 0$ and $r > 0$, an operator $T$ belongs to class $A(p, r)$ if
\[
\frac{r}{p+r} |T^*|^r |T|^{2p} |T^*|^r \geq |T^*|^{2r}.
\]
(2) For $p > 0, r \geq 0$ and $q \geq 1$, an operator $T$ belongs to class $F(p, r, q)$ if
\[
\frac{1}{q} |T^*|^q |T|^{2p} |T^*|^q \geq |T^*|^{2(p+r)}.
\]
For each $p > 0$ and $r > 0$, class $A(p, r)$ contains all $p$-hyponormal and log-hyponormal operators. An operator $T$ is a class $A(k)$ operator ([147]) if and only if $T^*$ is a class $A(k, 1)$ operator, $T$ is a class $A(1)$ operator if and only if $T$ is a class $A$. 

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operator ([147]), and $T$ is a class $A(p, r)$ operator if and only if $T$ is a class $F \left( p, r, \frac{p+r}{r} \right)$ operator.

Aluthge-Wang [143] introduced $w$-hyponormal operators defined by $|\bar{T}| \geq |T| \geq |\bar{T}^*|$ where the polar decomposition of $T$ is $T = U|T|$ and $\bar{T} = |T|^{1/2}U|T|^{1/2}$ is called the Aluthge transformation of $T$. As a generalization of $w$-hyponormality, Ito [128] and Yang-Yuan [139,138] introduced the classes $wA(p, r)$ and $wF(p, r, q)$ respectively.

Definition (3.1.2)[141]:

1. For $p > 0, r > 0$, an operator $T$ belongs to class $wA(p, r)$ if
   \[ (|T^*|^r |T|^{2p} |T^*|^r)^{\frac{r}{2p+r}} \geq |T^*|^{2r} \quad \text{and} \quad |T|^{2p} \geq (|T|^p |T^*|^{2r})^{\frac{p}{p+r}}. \]

2. For $p > 0, r \geq 0$, and $q \geq 1$, an operator $T$ belongs to class $wF(p, r, q)$ if
   \[ (|T^*|^r |T|^{2p} |T^*|^r)^{\frac{1}{q}} \geq |T^*|^{\frac{2(p+r)}{q}} \quad \text{and} \quad |T|^2 \geq (|T|^p |T^*|^{2r})^{\frac{1}{q}} \left(1 - \frac{1}{q} \right), \]
   denoting $(1 - q^{-1})^{-1}$ by $q^*$ (when $q > 1$) because $q$ and $(1 - q^{-1})^{-1}$ are a couple of conjugate exponents.

An operator $T$ is a $w$-hyponormal operator if and only if $T$ is a class $wA(\frac{1}{2}, \frac{1}{2})$ operator, $T$ is a class $wA(p, r)$ operator if and only if $T$ is a class $wF(p, r, \frac{p+r}{r})$ operator.

Ito [129] showed that the class $A(p, r)$ coincides with the class $wA(p, r)$ for each $p > 0$ and $r > 0$, class $A$ coincides with class $wA(1, 1)$. For each $p > 0, r \geq 0$ and $q \geq 1$ such that $rq \leq p + r$, [139] showed that class $wF(p, r, q)$ coincides with class $F(p, r, q)$.

Halmos ([171, Problem 209]) gave an example of a hyponormal operator $T$ whose square $T^2$ is not hyponormal. This problem has been studied by many authors, see [169,170,173,175,176]. Aluthge-Wang [169] showed that the operator $T^n$ is $(k/n)$-hyponormal for any positive integer $n$ if $T$ is $k$-hyponormal. In this section, we firstly discuss powers of class $wF(p, r, q)$ operators for $1 \geq p > 0, 1 \geq r > 0$ and $q \geq 1$. Secondly, we shall give an example on powers of class $wF(p, r, q)$ operators.

Theorem (3.1.3)[129,141]: Let $1 \geq p > 0, 1 \geq r > 0$. Then $T^n$ is a class $wA(\frac{p}{n}, \frac{r}{n})$ operator.

Theorem(3.1.4)[172,141]: Let $1 \geq p > 0, 1 \geq r \geq 0, q \geq 1$ and $rq \leq p + r$. If $T$ is an invertible class $F(p, r, q)$ operator, then $T^n$ is a $F(\frac{p}{n}, \frac{r}{n}, q)$ operator.
Theorem (3.1.5)[139,141]: Let $1 \geq p > 0, 1 \geq r \geq 0; q \geq 1$ when $r = 0$ and $\frac{p+r}{r} \geq q \geq 1$ when $r > 0$. If $T$ is a class $wF(p,r,q)$ operator, then $T^n$ is a class $wF\left(\frac{p}{n}, \frac{r}{n}, q\right)$ operator.

Here we generalize them in theorem (3.1.6).

Lemma (3.1.6)[127,141]: Let $\alpha \in \mathbb{R}$ and $X$ be invertible. Then $(X^\alpha X)^{\alpha^{-1}} X$ holds, especially in the case $\alpha \geq 1$, Lemma (3.1.6) holds without invertibility of $X$.

Theorem (3.1.7)[129,141]: Let $A, B \geq 0$. Then for each $p, r \geq 0$, the following assertions hold:

1. \[
\left( B^2 A^p \right)^{\frac{r}{p+r}} B^r \geq B^r \Rightarrow \left( A^2 B^r \right)^{\frac{p}{p+r}} A^p \leq A^p.
\]

2. \[
\left( A^2 B^r \right)^{\frac{p}{p+r}} \leq A^p \text{ and } N(A) \subset N(B) \Rightarrow \left( B^2 A^p \right)^{\frac{r}{p+r}} \geq B^r.
\]

Theorem (3.1.8)[137,141]: Let $T$ be a class $wA$ operator. Then $|T^2|^2 \geq |T|^2$ and $|T|^r \geq |(T^2)^r| \geq \cdots \geq |(T^n)^r| \geq \cdots \geq |(T^n)^r|^2$ held.

Theorem (3.1.9)[139,141]: Let $T$ be a class $wF(p_0, r_0, q_0)$ operator for $p_0 > 0, r_0 \geq 0$ and $q_0 \geq 1$. Then the following assertions hold.

1. If $q \geq q_0$ and $r_0 q \leq p_0 + r_0$, then $T$ is a class $wF(p_0, r_0, q)$ operator.

2. If $q^* \geq q_0^*$, $p_0 q^* \leq p_0 + r_0$ and $N(T) \subset N(T^*)$, then $T$ is a class $wF(p_0, r_0, q)$ operator.

3. If $r q \leq p + r$, then class $wF(p, r, q)$ coincides with class $F(p, r, q)$.

Theorem (3.1.10)[139,141]: Let $T$ be a class $wF(p_0, r_0, q_0)$ operator for $p_0 > 0, r_0 \geq 0$ and $-r_0 < \delta_0 \leq p_0$. Then $T$ is a class $wF(p, r, q)$ operator for $p \geq p_0$ and $r \geq r_0$.

Proposition(3.1.11)[139,141]: Let $A, B \geq 0$; $1 \geq p > 0$, $1 \geq r < 0$; $\frac{p+r}{r} \geq q \geq 1$. Then the following assertions hold.

1. If \[
\left( B^2 A^p \right)^{\frac{r}{q}} \geq B \geq A \text{ and } B \geq C, \text{ then } \left( C^2 A^p \right)^{\frac{r}{q}} \geq C \geq A
\]

2. If \[
\frac{p+r}{r} \geq \left( B^2 C^p \right)^{\frac{r}{q}} \geq A \geq B \text{ and the condition}
\]

\[
(*) \text{If } \lim_{n \to \infty} B^2 x_n = 0 \text{ and } \lim_{n \to \infty} A^2 x_n \text{ exists, then } \lim_{n \to \infty} A^2 x_n = 0
\]

holds for any sequence of vectors $\{x_n\}$, then \[
\lim_{n \to \infty} A^q x_n \leq \left( A^2 C^p \right)^{\frac{r}{q}}
\]

Theorem (3.1.12)[141]: Let $1 \geq p > 0$, $1 \geq r > 0$; $q > \frac{p+r}{r}$. If $T$ is a class $wF(p, r, q)$ operator such that $N(T) \subset N(T^*)$, then $T^n$ is a class $wF\left(\frac{p}{n}, \frac{r}{n}, q\right)$ operator.

In order to prove the theorem, we require the following assertions.
Proof. Put \( \delta = \frac{p+r}{q} - r \), then \(-r < \delta < 0\) by the hypothesis. Moreover, if
\[
(|T^*|^{r}|T|^{2p}|T^*|^r)^{\frac{p+\delta}{p+r}} \geq |T^*|^{2(r+\delta)} \text{ and } |T|^{2(p-\delta)} \geq (|T|^{p}|T^*|^{2r}|T|^p)^{\frac{p-\delta}{p+r}},
\]
then \( T \) is a class \( wA \) operator by Theorem (3.1.10) and Theorem (3.1.7), so that the following hold by taking \( A_n = |T^n|^{\frac{2}{n}} \) and \( B_n = |(T^n)^*|^{\frac{2}{n}} \) in Theorem (3.1.8)
\[
A_n \geq \cdots \geq A_2 \geq A_1 \text{ and } B_1 \geq B_2 \geq \cdots \geq B_n.
\]
(1) Meanwhile, \( A_n \) and \( A_1 \) satisfy the following for any sequence of vectors \( \{x_m\} \), (see \[137\])
\[
\text{if } \lim_{m \to \infty} A_1^{\frac{1}{n}} x_m = 0 \text{ and } \lim_{m \to \infty} A_n^{\frac{1}{n}} x_m \text{ exists, then } \lim_{m \to \infty} A_n^{\frac{1}{n}} x_m = 0.
\]
Then the following holds by Proposition (3.1.11)
\[
(A_n)^{\frac{p+r}{q}} \geq \left( (A_n)^{\frac{p}{2}} (B_1)^r (A_n)^{\frac{p}{2}} \right)^{\frac{1}{q^r}} \geq \left( (A_n)^{\frac{p}{2}} (B_2)^r (A_n)^{\frac{p}{2}} \right)^{\frac{1}{q^r}},
\]
and it follows that
\[
|T^n|^{\frac{2(p-r)}{nq}} \geq (|T^n|^{\frac{p}{n}} |(T^n)^*|^{\frac{2p}{n}} |T^n|^{\frac{p}{n}})^{\frac{1}{q^r}}.
\]
We assert that \( N(T) \subset N(T^*), \) implies \( N(T^n) \subset N((T^n)^*). \)
In fact,
\[
x \in N(T^n) \implies T^{n-1} x \in N(T) \subseteq N(T^*),
\]
\[
\to T^{n-2} x \in N(T^*T) = N(T) \subseteq N(T^*)
\]
\[
\cdots
\]
\[
\implies x \in N(T) \subseteq N(T^*)
\]
\[
\implies x \in N(T^*) \subseteq N((T^n)^*),
\]
thus
\[
\left( |(T^n)^*|^{\frac{r}{n}} |T^n|^{\frac{2p}{n}} |(T^n)^*|^{\frac{r}{n}} \right)^{\frac{1}{q}} \geq |(T^n)^*|^{\frac{2(p+r)}{nq}}
\]
holds by Theorem (3.1.7) and the Löwner-Heinz theorem, so that \( T^n \) is a class \( wF(\frac{p}{n}, \frac{r}{n}, q) \) operator. \( \square \)

**Theorem (3.1.13)[141]**: (Furuta inequality \[124\], in brief FI). If \( A \geq B \geq 0, \) then for each \( r \geq 0, \)
\[
(i) \quad \left( B^r A^p B^r \right)^{\frac{1}{q}} \geq \left( B^r B^p B^r \right)^{\frac{1}{q}}
\]
and
\[
(ii) \quad \left( A^r A^p A^r \right)^{\frac{1}{q}} \geq \left( A^r B^p A^r \right)^{\frac{1}{q}}
\]
hold for \( p \geq 0 \) and \( q \geq 1 \) with \( (1 + r)q \geq p + r. \)
Theorem (3.1.13) yields the Löwner-Heinz inequality by putting \( r = 0 \) in (i) or (ii), of FI. It was shown by Tanahashi [134] that the domain drawn for \( p, q \) and \( r \) in the Figure is the best possible for Theorem (3.1.13).

**Theorem (3.1.14)[141]:** Let \( A \) and \( B \) be positive operators on \( H, U \) and \( D \) be operators on \( \bigoplus_{k=-\infty}^{\infty} H_k \), where \( H_k \cong H \), as follows

\[
U = \begin{pmatrix}
\ddots & 0 \\
\ddots & 1 & 0 \\
1 & 0 & (0) \\
1 & 0 & 1 & 0 \\
\ddots & \ddots & \ddots & \ddots
\end{pmatrix},
\]
where (·) shows the place of the (0, 0) matrix element, and $T = UD$. Then the following assertions hold.

(1) If $T$ is a class $wF(p, r, q)$ operator for $1 \geq p > 0, 1 \geq r \geq 0, q \geq 1$ and $rq \leq p + r$, then $T^n$ is a $wF\left(\frac{p}{n}, \frac{r}{n}, q\right)$ operator.

(2) If $T$ is a class $wF(p, r, q)$ operator such that $N(T) \subset N(T^*)$, $1 \geq p > 0, 1 \geq r \geq 0, q \geq 1$ and $rq > p + r$, then $T^n$ is a $wF\left(\frac{p}{n}, \frac{r}{n}, q\right)$ operator.

**Proof.** By simple calculations, we have

\[
|T|^2 = \begin{pmatrix}
\ddots \\
B & B \\
& (A) \\
& & A \\
& & & \ddots
\end{pmatrix},
\]

and

\[
|T^*|^2 = \begin{pmatrix}
\ddots \\
B & B \\
& (B) \\
& & A \\
& & & \ddots
\end{pmatrix}.
\]
Therefore

\[
\begin{vmatrix}
T^* & T & T^* \\
T & T & T \\
T^* & T & T^* \\
\end{vmatrix} = \begin{pmatrix}
\cdots & B^{p+r} & B^{p+r} \\
B^{p+r} & (B^2 A^p B^2) & A^{p+r} \\
B^{p+r} & A^{p+r} & \cdots \\
\end{pmatrix}
\]

And

\[
\begin{vmatrix}
T^p & T^* & 2T^p \\
T & T & T \\
T^* & T & T^* \\
\end{vmatrix} = \begin{pmatrix}
\cdots & B^{p+r} & B^{p+r} \\
B^{p+r} & (A^2 B^p A^2) & A^{p+r} \\
B^{p+r} & A^{p+r} & \cdots \\
\end{pmatrix}
\]

Thus the following hold for \( n \geq 2 \)

\( T^{n*} T^n \)

\[
\begin{pmatrix}
\cdots & B^n & B^n \\
B^n & B^{n-1} A B^{n-1} \\
\cdots & B^2 A^{n-j} B^2 \\
B^2 A^{n-j} B^2 & B^2 A^{n-1} B^2 \\
\cdots & (A^n) & A^n \\
\end{pmatrix}
\]
And

\[ T^n T^{n^*} = \left\{ \begin{array}{c}
B^n \\
\begin{array}{c}
A^j B^{r-1} A^j \\
A^j B^{r-j} A^j \\
\vdots
\end{array}
\end{array} \right\} \]

Proof of (1). \( T \) is a class \( wF(p, r, q) \) operator is equivalent to the following

\[
\left( B^\frac{1}{q} A^p B^\frac{1}{q} \right)^\frac{p+r}{q} \geq B^\frac{p+r}{q} \quad \text{and} \quad A^\frac{p+r}{q^*} \geq \left( A^\frac{p}{q^*} B^\frac{p}{q^*} \right) A^\frac{p}{q^*},
\]

\( T^n \) belongs to class \( wF\left( \frac{p}{n}, \frac{r}{n}, q \right) \) is equivalent to the following (2) and (3).

\[
\left\{ \begin{array}{c}
(\frac{p}{n} A B^{n-j} A^{\frac{1}{2}})^\frac{1}{q^*} \geq B^\frac{p+r}{q^*} \\
(\frac{p}{n} A B^{n-j} A^{\frac{1}{2}})^\frac{1}{q^*} \geq B^\frac{p+r}{q^*}
\end{array} \right\}
\]

\[
\left\{ \begin{array}{c}
(\frac{p}{n} A B^{n-j} A^{\frac{1}{2}})^\frac{1}{q^*} \geq B^\frac{p+r}{q^*} \\
(\frac{p}{n} A B^{n-j} A^{\frac{1}{2}})^\frac{1}{q^*} \geq B^\frac{p+r}{q^*}
\end{array} \right\}
\]

where \( j = 1, 2, \ldots, n - 1 \).

\[
\left\{ \begin{array}{c}
(\frac{p}{n} A B^{n-j} A^{\frac{1}{2}})^\frac{1}{q^*} \geq B^\frac{p+r}{q^*} \\
(\frac{p}{n} A B^{n-j} A^{\frac{1}{2}})^\frac{1}{q^*} \geq B^\frac{p+r}{q^*}
\end{array} \right\}
\]

where \( j = 1, 2, \ldots, n - 1 \)
We only prove (2) because of Theorem (3.1.7).

**Step 1.** To show
\[
\left( B^\frac{r}{2} \left( B^2 A^{n-j} B \right) \frac{p}{n} B^\frac{r}{2} \right)^{\frac{1}{q}} \geq B^{\frac{p+r}{q}}
\]
for \( j = 1, 2, \ldots, n - 1 \).

In fact, \( T \) is a class \( wF(p, r, q) \) operator for \( 1 \geq p > 0, 1 \geq r \geq 0, q \geq 1 \) and \( rq \leq p + r \) implies \( T \) belongs to class \( wF \left( j, n - j, \frac{n}{\delta + j} \right) \), where \( \delta = \frac{p+r}{q} - r \). By Theorem (3.1.10) and Theorem (3.1.7), thus
\[
\left( B^\frac{j}{2} A^{n-j} B \right)^{\frac{j}{n}} \geq B^{\delta - j} \quad \text{and} \quad A^{n-j-\delta} \geq \left( \frac{n-j}{2} B^{j} A^\frac{n-j}{2} \right)^{\frac{n-j-\delta}{n}}
\]
Therefore the assertion holds by applying (i) of Theorem (3.1.13) to \( \left( B^\frac{j}{2} A^{n-j} B \right)^{\frac{j}{n}} \) and \( B^{\delta+j} \) for \( \left( 1 + \frac{r}{\delta+j} \right) q \geq \frac{p}{\delta+j} + \frac{r}{\delta+j} \).

**Step 2.** To show
\[
\left( \left( A^\frac{j}{2} A^{n-j} A^\frac{j}{2} \right)^{\frac{j}{n}} \right)^r A^\frac{j}{2} \left( A^\frac{j}{2} B^{n-j} A^\frac{j}{2} \right)^{\frac{r}{q}} \geq \left( A^\frac{j}{2} B^{n-j} A^\frac{j}{2} \right)^{\frac{p+r}{nq}}
\]
for \( j = 1, 2, \ldots, n - 1 \).

In fact, similar to Step 1, the following hold
\[
\left( B^\frac{n-j}{2} A^\frac{j}{2} B^{-\frac{j}{n}} \right)^{\frac{j}{n}} \geq B^{\delta+n-j} \quad \text{and} \quad A^{j-\delta} \geq \left( A^\frac{j}{2} B^{n-j} A^\frac{j}{2} \right)^{\frac{1}{n}}
\]
this implies that \( A^j \geq \left( A^\frac{j}{2} B^{n-j} A^\frac{j}{2} \right)^{\frac{j}{n}} \) by Theorem (3.1.7). Therefore the assertion holds by applying (i) of Theorem (3.1.13) to \( A^j \) and \( \left( A^\frac{j}{2} B^{n-j} A^\frac{j}{2} \right)^{\frac{j}{n}} \) for \( \left( 1 + \frac{r}{j} \right) q \geq \frac{p}{j} + \frac{r}{j} \).

Proof of (2). This part is similar to Proof of (1), so we omit it here. \( \square \)

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**Corollary (3.2.15)[232]:** Let \( p = (1 - \epsilon), r = (1 - \epsilon) \) and \( q = (2 + \epsilon) \). If \( T \) is a class \( wF ((1 - \epsilon), (1 - \epsilon), (2 + \epsilon)) \) operator such that \( N(T) \subset N(T^*) \), then \( T^n \) is a class \( wF \left( \frac{(1-\epsilon)}{n}, \frac{(1-\epsilon)}{n}, (2+\epsilon) \right) \) operator.

In order to prove the theorem, we require the following assertions.

**Proof.** Put \( \delta = \frac{-\epsilon (1-\epsilon)}{2+\epsilon} \), then \( \epsilon + 1 < \delta < 0 \) by the hypothesis. Moreover, if
\[
(\|T^*\|^{(1-\epsilon)} \|T\|^{2(1-\epsilon)} \|T^*\|^{(1-\epsilon)}) \geq \|T^*\|^2(1-\epsilon) \|T\|^{2(1-\epsilon)} \text{ and } \|T\|^{2(1-\epsilon)}(2+\epsilon)
\]

then \(T\) is a class \(wA\) operator by Theorem (3.1.10) and Theorem (3.1.7), so that the following hold by taking \(A_n = \|T^n\|^2/\pi\) and \(B_n = \|(T^n)^*\|^2/\pi\) in Theorem (3.1.8)

\[A_n \geq \cdots \geq A_2 \geq A_1 \text{ and } B_1 \geq B_2 \geq \cdots \geq B_n.\]

Meanwhile, \(A_n\) and \(A_1\) satisfy the following for any sequence of vectors \(\{x_m\}\)

(see [137])

If \(\lim_{m \to \infty} A_1^{1/2} x_m = 0\) and \(\lim_{m \to \infty} A_n^{1/2} x_m\) exists, then \(\lim_{m \to \infty} A_n^{1/2} x_m = 0\).

Then the following holds by Proposition (3.1.11)

\[
(A_n)^{2(1-\epsilon)/(2+\epsilon)^*} \geq \left( (A_n)^{(1-\epsilon)/2} (B_1)^{(1-\epsilon)} (A_n)^{(1-\epsilon)/2} \right)^{1/(2+\epsilon)^*}
\]

and it follows that

\[
\|T^n\|^{4(1-\epsilon)/n(2+\epsilon)^*} \geq \left( (\|T^n\|^{(1-\epsilon)/n}) (\|T^n\|^{(1-\epsilon)/n}) \right)^{1/(2+\epsilon)^*}.
\]

We assert that \(N(T) \subseteq N(T^*)\), implies \(N(T^n) \subseteq N((T^n)^*)\).

In fact,

\[
x \in N(T^n) \Rightarrow T^{n-1} x \in N(T) \subseteq N(T^*),
\]

\[
\Rightarrow T^{n-2} x \in N(T^* T) = N(T) \subseteq N(T^*)
\]

\[
\ldots
\]

\[
\Rightarrow x \in N(T) \subseteq N(T^*)
\]

\[
\Rightarrow x \in N((T^n)^*) \subseteq N((T^n)^*),
\]

thus

\[
\left( \|(T^n)^*\|^2 \|T^n\|^{4(1-\epsilon)/n} \|T^n\|^{4(1-\epsilon)/n} \right)^{1/(2+\epsilon)^*} \geq \|(T^n)^*\|^{4(1-\epsilon)/n(2+\epsilon)^*}
\]

holds by Theorem (3.1.7) and the Löwner-Heinz theorem, so that \(T^n\) is a class \(wF\left(\frac{1-\epsilon}{n}, \frac{1-\epsilon}{n}, (2+\epsilon)\right)\) operator.

\[\square\]

Sec(3.2)  Spectrum of Class \(wF(p, r, q)\) Operators

A capital letter (such as \(T\)) means a bounded linear operator on a complex Hilbert space \(\mathcal{H}\). For \(p > 0\), an operator \(T\) is said to be \(p\)-hyponormal if \((T^* T)^p \geq (T^* T)^p\), where \(T^*\) is the adjoint operator of \(T\). An invertible operator \(T\) is said to be log-hyponormal if \(\log(T^* T) \geq \log(T T^*)\). If \(p = 1\), \(T\) is called hyponormal, and if \(p = \frac{1}{2}\), \(T\) is called semi-hyponormal. Log-hyponormality is sometimes regarded as 0-hyponormal since \((X^p - 1)/p \to \log X\) as \(p \to 80\).
0 for $X > 0$. See Martin and Putinar [131] and Xia [135] for basic properties of hyponormal and semi-hyponormal operators. Log-hyponormal operators were introduced by Tanahashi [142], Aluthge and Wang [143], and Fujii et al. [144] independently. Aluthge [145] introduced $p$-hyponormal operators.

As generalizations of $p$-hyponormal and log-hyponormal operators, many authors introduced many classes of operators. Aluthge and Wang [143] introduced $w$-hyponormal operators defined by $|\tilde{T}| \geq |T| \geq |(\tilde{T})^*|$, where the polar decomposition of $T$ is $T = U|T|$ and $\tilde{T} = |T|^{1/2}U|T|^{1/2}$ is called Aluthge transformation of $T$. For $p > 0$ and $r > 0$, Ito [128] introduced class $wA(p, r)$ defined by

$$\left(\langle |T^*|^2 |T^*|^2 \rangle \right)^{2r} \geq \left(\langle |T|^p |T^*|^2 |T|^p \rangle \right)^{p+r} \leq \left(\langle |T|^2 |T^*|^2 |T|^2 \rangle \right)^{2p}.$$ (4)

Note that the two exponents $r/(p + r)$ and $p/(p + r)$ in the formula above satisfy $r/(p + r) + p/(p + r) = 1$, Yang and Yuan [138] introduced class $wF(p, r, q)$.

**Definition (3.2.1) [138,139]:** For $p > 0, r > 0$, and $q \geq 1$, an operator $T$ belongs to class $wF(p, r, q)$ if

$$\left(\langle |T^*|^2 |T^*|^2 \rangle \right)^{\frac{1}{q}} \geq \left(\langle |T^*|^2 |T^*|^2 \rangle \right)^{\frac{p+r}{q}} \geq \left(\langle |T|^p |T^*|^2 |T|^p \rangle \right)^{(1-\frac{2}{q})}.$$ (5)

Denote $(1 - q^{-1})^{-1}$ by $q^*$ when $q > 1$ because $q$ and $(1 - q^{-1})^{-1}$ are a couple of conjugate exponents. It is clear that class $wA(p, r)$ equals class $wF(p, r, (p + r)/r)$. $w$-hyponormality equals $wA(1/2, 1/2)$ [128]. Ito and Yamazaki [129] showed that class $wA(p, r)$ coincides with class $A(p, r)$ (introduced by Fujii et al. [146]) for each $p > 0$ and $r > 0$. Consequently, class $wA(1, 1)$ equals class $A$ (i.e., $|T^2| \geq |T|^2$, introduced by Furuta et al. [147]). Reference [139] showed that class $wF(p, r, q)$ coincides with class $F(p, r, q)$ (introduced by Fujii and Nakamoto [148]) when $rq \leq p + r$.

Recently, there are great developments in the spectral theory of the classes of operators above. We cite [138, 149–157]. In this section, we will discuss several spectral properties of class

$$wF(p, r, q)$$

for $p > 0, r > 0, p + r \leq 1$, and $q \geq 1$.

In this Section, we prove that Riesz idempotent $E_{\lambda}$ of $T$ with respect to each nonzero isolated point spectrum $\lambda$ is selfadjoint and $E_{\lambda}H = \ker(T - \lambda) = \ker(T - \lambda)^*$. also we will show that each class $wF(p, r, q)$ operator has SVEP (single-valued extension property) and Bishop’s property $(\beta)$. and we show that Weyl’s theorem holds for class $wF(p, r, q)$. Now we show that Riesz idempotent. Let $\sigma(T), \sigma_p(T), \sigma_{jp}(T), \sigma_a(T), \sigma_{ja}(T)$, and $\sigma_{r}(T)$ mean the spectrum, point spectrum, joint point spectrum, approximate point spectrum, joint approximate point spectrum, and residual spectrum of an operator $T$, respectively (cf. [138, 158]). $\sigma^{Xia}(T)$ and $\sigma_{iso}(T)$ mean the set $\sigma(T) - \sigma_a(T)$ and the set of isolated
points of $\sigma(T)$, see [158, 135]. If $\lambda \in \sigma_{iso}(T)$, the Riesz idempotent $E_\lambda$ of $T$ with respect $\lambda$ is defined by

$$E_\lambda = \int_{\partial \Delta} (z - T)^{-1} dz,$$

(6)

where $\Delta$ is an open disk which is far from the rest of $\sigma(T)$ and $\partial \Delta$ means its boundary. Stampfli [159] showed that if $T$ is hyponormal, then $E_\lambda$ is selfadjoint and $E_\lambda \mathcal{H} = \ker(T - \lambda) = \ker(T - \lambda)^{\ast}$. The recent developments of this result are shown in [151, 152, 155, 157], and so on.

In this section, it is shown that when $\lambda \neq 0$, this result holds for class $wF(p, r, q)$ with $p + r \leq 1$ and $q \geq 1$. It is always assumed that $\lambda \in \sigma_{iso}(T)$ when the idempotent $E_\lambda$ is considered.

**Theorem (3.2.2)[138, 149]:** Let $\lambda \neq 0$, and let $\{x_n\}$ be a sequence of vectors. Then the following assertions are equivalent.

1. $(T - \lambda)x_n \to 0$ and $(T^{\ast} - \bar{\lambda})x_n \to 0$.
2. $(|T| - |\lambda|)x_n \to 0$ and $(U - e^{i\theta})x_n \to 0$.
3. $(|T|^{\ast} - |\lambda|)x_n \to 0$ and $(U^{\ast} - e^{-i\theta})x_n \to 0$.

**Theorem (3.2.3)[138]:** If $T$ is a class $wF(p, r, q)$ operator for $p + r \leq 1$ and $q \geq 1$, then the following assertions hold.

1. If $Tx = \lambda x$, $\lambda \neq 0$, then $T^{\ast}x = \bar{\lambda}x$.
2. $\sigma_a(T) - \{0\} = \sigma_{ja}(T) - \{0\}$.
3. If $Tx = \lambda x, Ty = \mu y$ and $\lambda \neq \mu$, then $(x, y) = 0$.

**Theorem (3.2.4)[138, 139]:** If $T$ is a class $wF(p, r, q)$ operator, then there exists $\alpha_0 > 0$, which satisfies

$$|T(p, r)|^{2\alpha_0} \geq |T|^{2\alpha_0(p + r)} \geq |T(p, r)^{\ast}|^{2\alpha_0}.$$  (7)

**Lemma (3.2.5)[138]:** If $T$ belongs to class $wF(p, r, q)$ for $p + r \leq 1, \lambda = |\lambda|e^{i\theta} \in \mathbb{C}$, and $\lambda_{p+r} = |\lambda|^{p+r}e^{i\theta}$, then $\ker(T - \lambda) = \ker(T(p, r) - \lambda_{p+r})$.

**Proof.** We only prove $\ker(T - \lambda) \supseteq \ker(T(p, r) - \lambda_{p+r})$ because $\ker(T - \lambda) \subseteq \ker(T(p, r) - \lambda_{p+r})$ is obvious by Theorems (3.2.2)-(3.2.3)

If $\lambda \neq 0$, let $0 \neq x \in \ker(T(p, r) - \lambda_{p+r})$. By Theorem (3.2.4), $(p, r)$ is $\alpha_0$-hyponormal and we have

$$|T(p, r)|x = |\lambda|^{p+r}x = \left|(T(p, r))^{\ast}\right|x,

|T(p, r)|^{2\alpha_0} - |(T(p, r))^{\ast}|^{2\alpha_0} \geq |T(p, r)|^{2\alpha_0} - |T|^{2\alpha_0(p + r)} \geq 0.$$  (8)

Hence $(|T(p, r)|^{2\alpha_0} - |T|^{2\alpha_0(p + r)})x = 0,$

$$\left\|T^{2\alpha_0(p + r)}x - |\lambda|^{2\alpha_0(p + r)}x\right\| \leq \left\|T^{2\alpha_0(p + r)}x - |T(p, r)|^{2\alpha_0}x\right\| + \left\|T(p, r)|^{2\alpha_0}x - |\lambda|^{2\alpha_0(p + r)}x\right\| = 0.$$  (9)

On the other hand, $(T(p, r))^{\ast}x = |\lambda|^{p+r}e^{-i\theta}x$ implies that $|T|^{\ast}U^\ast x = |\lambda|^{p+r}e^{-i\theta}x, T^{\ast} = |\lambda|e^{-i\theta}x$. Therefore,
\[\|(T - \lambda)x\|^2 = \|Tx\|^2 - \lambda(x, Tx) - \bar{\lambda}(Tx, x) + |\lambda|^2\|x\|^2 = \|Tx\|^2 - \lambda(T^*x, x) - \bar{\lambda}(x, T^*x) + |\lambda|^2\|x\|^2 = 0. \quad (10)\]

If \( \lambda = 0 \), let \( 0 \neq x \in \ker(T(p, r)) \), then \( x \in \ker |T| = \ker T \) by Theorem (3.2.4) so that \( \ker(T - \lambda) \supseteq \ker(T(p, r) - \lambda_{p+r}) \). \( \square \)

Lemma (3.2.6)[138,153,160]: If \( A \) is normal, then for every operator \( B, \sigma(AB) = \sigma(BA) \).

Let \( \mathcal{F} \) be the set of all strictly monotone increasing continuous nonnegative functions on \( \mathbb{R}^+ = [0, \infty) \). Let \( \mathcal{F}_0 = \{ \psi \in \mathcal{F} : \psi(0) = 0 \} \). For \( \psi \in \mathcal{F}_0 \), the mapping \( \Phi \) is defined by \( \Phi(\rho e^{i\theta}) = e^{i\theta}\psi(\rho) \) and \( \Phi(T) = U\Phi(|T|)U^{-1} \).

Theorem (3.2.7)[138,161]: If \( \psi \in \mathcal{F}_0 \), then for every operator \( T \), \( \sigma_{ja}(\Phi(T)) = \Phi(\sigma_{ja}(T)) \).

Lemma (3.2.8)[138]: Let \( T \) belong to class \( wF(p, r, q) \) with \( p + r \leq 1, \lambda = |\lambda|e^{i\theta} \in \mathcal{D}, T(t) = U|T|^{1-t+t(p+r)} \), and \( \tau_{t}(\rho e^{i\theta}) = e^{i\theta}\rho^{1+t(p+r-1)} \), where \( t \in [0,1] \). Then
\[
\sigma_{a}(T(t)) = \tau_{t}(\sigma_{a}(T)), \quad \sigma_{r}^{x, a}(T(t)) = \tau_{t}(\sigma_{r}^{x, a}(T)), \quad \sigma(T(t)) = \tau_{t}(\sigma(T)). \quad (11)
\]

Proof. We only need to show that \( \sigma_{a}(T(t)) = \tau_{t}(\sigma_{a}(T)) \) by homotopy property of the spectrum [135, page 19].

Since \( T \) belongs to class \( wF(p, r, q) \) with \( p + r \leq 1, T(t) \) belongs to class \( wF(p/(1 + t(p + r - 1)), r/(1 + t(p + r - 1)), q) \) with \( p/(1 + t(p + r - 1)) + r/(1 + t(p + r - 1)) \leq 1 \). By Theorems (3.2.3)(2) and (3.2.7),
\[
\sigma_{a}(T(t)) - \{0\} = \tau_{t}(\sigma_{a}(T(t)) - \{0\}) = \tau_{t}(\sigma_a(T) - \{0\}). \quad (12)
\]

On the other hand, if \( 0 \notin \sigma_{a}(T) \), then there exists a sequence \( \{x_n\} \) of unit vectors such that \( U|T|x_n \to 0 \). Hence \( |T|x_n = U^{*}U|T|x_n \to 0 \), so that \( |T|^{1/(2m)}x_n \to 0 \) for each positive integer \( m \) by induction. Take a positive integer \( m(t) \) such that \( 1/(2m(t)) \leq 1 + t(p + r - 1) \), then
\[
|T|^{1+t(p+r-1)}x_n = |T|^{1+t(p+r-1)-1/(2m(t))}|T|^{1/(2m(t))}x_n \to 0 \quad (13)
\]

and \( 0 \notin \sigma_{a}(T(t)) \). It is obvious that if \( 0 \notin \sigma_{a}(T(t)) \), then \( 0 \notin \sigma_{a}(T) \) because of \( p + r \leq 1 \). Therefore \( \sigma_{a}(T(t)) = \tau_{t}(\sigma_{a}(T)) \). \( \square \)

Theorem (3.2.9)[138,150]: If \( T \) is \( p \)-hyponormal or log-hyponormal, then \( E_{\lambda} \) is selfadjoint and \( E_{\lambda}\mathcal{H} = \ker(T - \lambda) = \ker(T - \lambda)^{*} \).

Riesz and Sz.-Nagy [162] gave the the formula \( E_{\lambda}\mathcal{H} = \{ x \in \mathcal{H} : \|(T - \lambda)^{n}x\|^{1/n} \to 0 \} \).
Lemma (3.2.10) [138]: For any operator $T$, $|T|^p \ker(T - \lambda) \subseteq |T|^p E_\lambda(p, r) \mathcal{H}$ for $p + r = 1$.

Proof. Let $x \in E_\lambda$, by the formula above we have

$$\| (T(p, r) - \lambda)^n |T|^p x \|^1/n = \| |T|^p (T - \lambda)^n x \|^1/n \to 0.$$  \hspace{1cm} (14)

Hence $|T|^p x \in E_\lambda(p, r) \mathcal{H}$.

Lemma (3.2.11) [138]: If $T$ belongs to class $wF(p, r, q)$ with $p + r \leq 1$, then

$$\ker T = E_0 \mathcal{H} = E_0(p, r) \mathcal{H} = \ker(T(p, r)).$$  \hspace{1cm} (15)

Note that $\lambda_{p+r} \in \sigma_{iso}(T(t))$ if $\lambda \in \sigma_{iso}(T)$ by Lemma (3.2.8), so the notion $E_0(p, r)$ in Lemma (3.2.10) is reasonable.

Proof. Since $T(p, r)$ is $\alpha_0$-hyponormal by Theorem (3.2.4), we only need to prove that $E_0 \mathcal{H} \subseteq E_0(p, r) \mathcal{H}$ for $E_0 \mathcal{H} \supseteq E_0(p, r) \mathcal{H}$ holds by Lemma (3.2.5) and Theorem (3.2.9). We may also assume that $p + r = 1$ by Lemma (3.2.5). It follows from Lemma (3.2.10) that

$$|T|^p E_0(p, r) \mathcal{H} \subseteq |T|^p E_0 \subseteq E_0(p, r) \mathcal{H},$$  \hspace{1cm} (16)

thus $E_0(p, r) \mathcal{H}$ is reduced by $|T|^p$.

Let $x \in E_0 \mathcal{H}$ and $x = x_1 + x_2 \in E_0(p, r) \mathcal{H} \oplus (E_0(p, r) \mathcal{H})^\perp$. Then $|T|^p x \in |T|^p E_0 \mathcal{H} \subseteq E_0(p, r) \mathcal{H}$, $|T|^p x_1 \in E_0(p, r) \mathcal{H}$, $|T|^p x_2 \in (E_0(p, r) \mathcal{H})^\perp$ by (16), and $E_0(p, r) \mathcal{H}$ is reduced by $|T|^p$.

Thus $|T|^p x_2 = |T|^p x - |T|^p x_1 \in E_0(p, r) \mathcal{H}$, $|T|^p x_2 \in E_0(p, r) \mathcal{H} \cap (E_0(p, r) \mathcal{H})^\perp$ so that

$$x_2 \in \ker |T|^p \subseteq \ker(T(p, r)) = E_0(p, r) \mathcal{H}, x \in E_0(p, r) \mathcal{H}.$$

Theorem (3.2.12) [138]: Let $T$ belong to class $wF(p, r, q)$ with $p + r \leq 1, \lambda = |\lambda| e^{i\theta} \in \mathcal{G}$, and $\lambda_{p+r} = |\lambda|^{p+r} e^{i\theta}$, then the following assertions hold.

1. If $\lambda \neq 0$, then $E_\lambda = E_\lambda(p, r)$ and $E_\lambda \mathcal{H} = \ker(T - \lambda) = \ker(T - \lambda)^*$, where $E_\lambda(p, r)$ is the Riesz idempotent of $T(p, r) = |T|^p U |T|^r$ (the generalized Aluthge transformation of $T$) with respect to $\lambda_{p+r}$.

2. If $\lambda = 0$, then $\ker T = E_0 \mathcal{H} = E_0(p, r) \mathcal{H} = \ker(T(p, r))$.

Reference [156] gave an example that the operator $T$ is $w$-hyponormal, $E_0$ is not selfadjoint, and $\ker T \neq \ker T^*$.

An operator $T$ is said to be isoid if $\sigma_{iso}(T) \subseteq \sigma_p(T)$, is said to be reguloid if $(T - \lambda) \mathcal{H}$ is closed for each $\lambda \in \sigma_{iso}(T)$.

Proof. We only need to prove (1) for (2) holds by Lemma (3.2.11). Since $\sigma(T(p, r)) = \sigma(U |T|^p)^{r+p} = \{ e^{i\theta} \rho^{p+r} : e^{i\theta} \rho \in \sigma(T) \}$ by Lemmas (3.2.6) and (3.2.8), $\lambda_{p+r} \in \sigma_{iso}(T(p, r))$. Hence

$$(E_\lambda(p, r) \mathcal{H})^\perp = \ker(E_\lambda(p, r)) = (I - E_\lambda(p, r)) \mathcal{H}$$  \hspace{1cm} (17)

by Theorem (3.2.9), so $\lambda_{p+r} \notin \sigma(T(p, r))_{(E_\lambda(p, r) \mathcal{H})^\perp}$. By Theorem (3.2.3)(1) and Lemma (3.2.5), we have $T = \lambda T \oplus T_{22}$ on $\mathcal{H} = E_\lambda(p, r) \mathcal{H} \oplus (E_\lambda(p, r) \mathcal{H})^\perp$, where $T_{22} = T|_{(\ker(T - \lambda))^\perp}$.
Since $\ker(T - \lambda)$ is reduced by $T, T_{22}$ also belongs to class $wF(p, r, q)$ and $T_{22}(p, r) = T(p, r)|_{(E_\lambda(p, r))^{-1}}$ so that $\lambda \notin \sigma(T_{22})$ because $\lambda_{p+r} \notin \sigma(T_{22}(p, r))$. Hence $T - \lambda = 0 \oplus (T_{22} - \lambda)$ and $\ker(T - \lambda)^* = \ker(T - \lambda) \oplus \ker(T_{22} - \lambda)^* = \ker(T - \lambda)$.

Meanwhile, $E_\lambda = \int_{\partial D}(z - \lambda)^{-1} \oplus (z - T_{22})^{-1} dz = 1 \oplus 0 = E_\lambda(p, r)$. □

**Theorem (3.2.13)**[138]: If $T$ belongs to class $wF(p, r, q)$ with $p + r \leq 1$, then $T$ is isoloid and reguloid.

**Proof.** We only need to prove that $T$ is reguloid for $T$ being isoloid follows from Theorem (3.2.12) easily.

If $\lambda \in \sigma_{iso}(T)$, then $\mathcal{H} = E_\lambda \mathcal{H} + (I - E_\lambda) \mathcal{H}$, where $E_\lambda \mathcal{H}$, and $(I - E_\lambda) \mathcal{H}$ are topologically complemented [163, page 94]. By $T = T|_{E_\lambda \mathcal{H}} + T|_{(I - E_\lambda) \mathcal{H}}$ on $\mathcal{H} = E_\lambda \mathcal{H} + (I - E_\lambda) \mathcal{H}$ and Theorem (3.2.12), we have

$$(T - \lambda) \mathcal{H} = (T|_{(I - E_\lambda) \mathcal{H}} - \lambda)(I - E_\lambda) \mathcal{H}.$$  

Then $\sigma(T|_{(I - E_\lambda) \mathcal{H}}) = \sigma(T) - \{\lambda\}$. □

**Definition (3.2.14)**[138]: An operator $T$ is said to have SVEP at $\lambda \in \mathcal{G}$ if for every open neighborhood $G$ of $\lambda$, the only function $f \in H(G)$ such that $(T - \lambda)f(\mu) = 0$ on $G$ is $0 \in H(G)$, where $H(G)$ means the space of all analytic functions on $G$.

When $T$ have SVEP at each $\lambda \in \mathcal{G}$, say that $T$ has SVEP.

This is a good property for operators. If $T$ has SVEP, then for each $\lambda \in \mathcal{G}, \lambda - T$ is invertible if and only if it is surjective (cf. [164, 153]).

**Definition (3.2.15)**[138]: An operator $T$ is said to have Bishop’s property ($\beta$) at $\lambda \in \mathcal{G}$ if for every open neighborhood $G$ of $\lambda$, the function $f_n \in H(G)$ with $(T - \lambda)f_n(\mu) \to 0$ uniformly on every compact subset of $G$ implies that $f_n(\mu) \to 0$ uniformly on every compact subset of $G$.

When $T$ has Bishop’s property ($\beta$) at each $\lambda \in \mathcal{G}$, simply say that $T$ has property ($\beta$). This is a generalization of SVEP and it is introduced by Bishop [165] in order to develop a general spectral theory for operators on Banach space.

**Lemma (3. 2.16)**[138,153]: Let $G$ be open subset of complex plane $\mathcal{G}$ and let $f_n \in H(G)$ be functions such that $\mu f_n(\mu) \to 0$ uniformly on every compact subset of $G$, then $f_n(\mu) \to 0$ uniformly on every compact subset of $G$.

**Theorem (3.2.17)**[138]: Let $p$ and $r$ be positive numbers. If $p + r = 1$, then $T$ has SVEP if and only If $T(p, r)$ has SVEP, $T$ has property ($\beta$) if and only if $T(p, r)$ has property ($\beta$). In particular, every class $wF(p, r, q)$ operator $T$ with $p + r \leq 1$ has SVEP and property ($\beta$).

This result is a generalization of [153]. Lemma (3.2.16) and the relations between $T$ and its transformation $T(p, r)$ are important:

$$T(p, r)|T|^p = |T|^p U|T|^r \quad |T|^p = |T|^p T,$$

$$U|T|^r T(p, r) = U|T|^r \quad |T|^p U|T|^r = TU|T|^r.$$  

(19)
Proof. We only prove that $T$ has property ($\beta$) if and only if $T(p,r)$ has property ($\beta$) because the assertion that $T$ has SVEP if and only if $T(p,r)$ has SVEP can be proved similarly.

Suppose that $T(p,r)$ has property ($\beta$). Let $G$ be an open neighborhood of $\lambda$ and let $f_n \in H(G)$ be functions such that $(\mu - T) f_n(\mu) \to 0$ uniformly on every compact subset of $G$. By (19), $(T(p,r) - \mu)|T|^p f_n(\mu) = |T|^p(T - \mu) f_n(\mu) \to 0$ uniformly on every compact subset of $G$. Hence $T f_n(\mu) = U|T|^r |T|^p f_n(\mu) \to 0$ uniformly on every compact subset of $G$ for $T(p,r)$ has property ($\beta$), so that $\mu f_n(\mu) \to 0$ uniformly on every compact subset of $G$, and $T$ having property ($\beta$) follows by Lemma (3.2.16).

Suppose that $T$ has property ($\beta$). Let $G$ be an open neighborhood of $\lambda$ and let $f_n \in H(G)$ be functions such that $(\mu - T(p,r)) f_n(\mu) \to 0$ uniformly on every compact subset of $G$. By (19), $(\mu - T)(U|T|^r f_n(\mu)) = U|T|^r(\mu - T(p,r)) f_n(\mu) \to 0$ uniformly on every compact subset of $G$. Hence $T(p,r) f_n(\mu) \to 0$ uniformly on every compact subset of $G$ for $T(p,r)$ has property ($\beta$) so that $\mu f_n(\mu) \to 0$ uniformly on every compact subset of $G$, and $T(p,r)$ having property ($\beta$) follows by Lemma (3.2.16).

For a Fredholm operator $T$, ind $T$ means its (Fredholm) index. A Fredholm operator $T$ is said to be Weyl if ind $T = 0$.

Let $\sigma_e(T), \sigma_w(T)$, and $\pi_{00}(T)$ mean the essential spectrum, Weyl spectrum, and the set of all isolated eigenvalues of finite multiplicity of an operator $T$, respectively (cf. [163, 152]).

According to Coburn [166], we say that Weyl’s theorem holds for an operator $T$ if $\sigma(T) - \sigma_w(T) = \pi_{00}(T)$. Very recently, the theorem was shown to hold for several classes of operators including $w$-hyponormal operators and paranormal operators (cf. [152, 167, 155]).

In this section, we will prove that Weyl’s theorem and Weyl spectrum mapping theorem hold for class $wF(p,r,q)$ operator $T$ with $p + r \leq 1$. We also assume that $p + r = 1$ because of the inclusion relations among class $wF(p,r,q)$ [139].

**Theorem (3.2.18)[138]:** Let $p > 0, r > 0$, and $q \geq 1, s \geq p, t \geq r$. If $T$ is a class $wF(p,r,q)$ operator and $T(s,t)$ is normal, then $\text{ind} T \leq 0$.

This result can be regarded as a good complement of Theorem (3.2.12).

**Proof.** Since $T$ is Fredholm, $|T|^p$ is also Fredholm and ind$(|T|^p T) = 0$. By (19),

\[
\text{ind} T = \text{ind}(|T|^p T) = \text{ind}(T(p,r)|T|^p T) = \text{ind}(T(p,r)).
\]

Hence, $\text{ind} T \leq 0$ for $\text{ind}(T(p,r)) \leq 0$ by Theorem (3.20). □
Theorem (3.2.20)[138]: Let $T$ belong to class $wF(p, r, q)$ with $p + r = 1$ and let $H(\sigma(T))$ be the space of all functions $f$ analytic on some open set $G$ containing $\sigma(T)$, then the following assertions hold.

1. Weyl’s theorem holds for $T$.
2. $\sigma_w(f(T)) = f(\sigma_w(T))$ when $f \in H(\sigma(T))$.
3. Weyl’s theorem holds for $f(T)$ when $f \in H(\sigma(T))$.

This is a generalization of the related assertions of [152].

Proof. 1. Let $\lambda \in \sigma(T) - \sigma_w(T)$, then $T - \lambda$ is Fredholm, $\text{ind}(T - \lambda) = 0$, and $\text{dim} \ker(T - \lambda) > 0$.

If $\lambda$ is an interior point of $\sigma(T)$, there would be an open subset $G \subseteq \sigma(T)$ including $\lambda$ such that $\text{ind}(T - \mu) = \text{ind}(T - \lambda) = 0$ for all $\mu \in G$ [163, page 357]. So $\text{dim} \ker(T - \mu) > 0$ for all $\mu \in G$, this is impossible for $T$ has SVEP by Theorem (3.2.17) [164, Theorem 10]. Thus $\lambda \in \partial \sigma(T) - \sigma_w(T), \lambda \in \sigma_{iso}(T)$ by [163, Theorem 6.8, page 366], and $\lambda \in \pi_{00}(T)$ follows.

Let $\lambda \in \pi_{00}(T)$ then the Riesz idempotent $E_\lambda$ has finite rank by Theorem (3.2.12), and $\lambda \in \sigma(T) - \sigma_w(T)$ follows.

2. We only need to prove that $\sigma_w(f(T)) \supseteq f(\sigma_w(T))$ since $\sigma_w(f(T)) \subseteq f(\sigma_w(T))$ is always true for any operators.

Assume that $f \in H(\sigma(T))$ is not constant. Let $\lambda \not\in \sigma_w(f(T))$ and $f(z) - \lambda = (z - \lambda_1) \cdots (z - \lambda_k)g(z)$, where $\{\lambda_i\}_{i=1}^k$ are the zeros of $f(z) - \lambda$ in $G$ (listed according to multiplicity) and $g(z) \neq 0$ for each $z \in G$. Thus

$$f(T) - \lambda = (T - \lambda_1) \cdots (T - \lambda_k)g(T).$$

(21)

Obviously, $\lambda \not\in f(\sigma_w(T))$ if and only if $\lambda_i \not\in \sigma_w(T)$ for some $i$. Next we prove that $\lambda_i \not\in \sigma_w(T)$ for every $i \in \{1, \ldots, k\}$, thus $\lambda \not\in f(\sigma_w(T))$ and $\sigma_w(f(T)) \supseteq f(\sigma_w(T))$.

In fact, for each $i, T - \lambda_i$ is also Fredholm because $f(T) - \lambda$ is Fredholm. By Theorem (3.2.12) and Lemma (3.2.19), $\text{ind}(T - \lambda_i) \leq 0$ for each $i$. Since $0 = \text{ind}(f(T) - \lambda) = \text{ind}(T - \lambda_1) + \cdots + \text{ind}(T - \lambda_k), \text{ind}(T - \lambda_i) = 0$ and $\lambda_i \not\in \sigma_w(T)$ for each $i$.

3. By Theorem (3.2.13), $T$ is isoloid and it follows from [168] that

$$\sigma(f(T)) = \sigma_w(f(T)) = f(\sigma(T) - \pi_{00}(T)).$$

On the other hand, $f(\sigma(T) - \pi_{00}(T)) = f(\sigma_w(T)) = \sigma_w(f(T))$ by (1)-(2). The proof is complete.

Theorem (3.2.21)[138]: Let $T$ belong to class $wF(p, r, q)$ with $p + r = 1$, then the following assertions hold.

(i) If $m_2(\sigma(T)) = 0$ where $m_2$ means the planar Lebesgue measure, then $T$ is normal.
(ii) If $\sigma_w(T) = 0$, then $T$ is compact and normal.

Theorem (3.2.21)(i) is a generalization of [161] and (ii) is a generalization of [159].
Theorem As generalizations, Bach and Furuta [123] provided a simple proof. Thus, (ii) follows by Theorem (3.2.18).

Next to prove that $T$ is compact, we may assume that $\sigma(T) - \{0\}$ is a countable set for $\sigma(T) - \{0\} \subseteq \sigma_{iso}(T)$. Let $\sigma(T) - \{0\} = \{\lambda_1\}_{\nu=1}^{\infty}$ with $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq 0$ and $\lambda_0 = \lim_{n \to \infty} |\lambda_n|$, then $\lambda_0 = 0$. Since every $E_{\lambda_n}$ has finite rank by Theorems (3.2.12) and (3.2.20), for every $\epsilon > 0$, $\bigoplus_{|\lambda_n| > \epsilon} E_{\lambda_n}$ also has finite rank. Therefore $T$ is compact [163, page 271].

Corollary (3.2.22) [232]: For any operator $\mathcal{T}$, $|\mathcal{T}|^{(1-r)} \ker(T - \lambda) \subseteq |\mathcal{T}|^{(1-r)} E_{\lambda} \mathcal{H} \subseteq |\mathcal{T}|^{(1-r)} (T - \lambda)^n \mathcal{H}$ for $p = 1 - r$.

Proof. Let $x \in E_{\lambda}$, by the formula above we have
\[
\left\| (T((1-r), r) - \lambda)^n |\mathcal{T}|^{(1-r)} x \right\|^{1/n} = \left\| |\mathcal{T}|^{(1-r)} (T - \lambda)^n x \right\|^{1/n} \to 0.
\]
Hence $|\mathcal{T}|^{(1-r)} x \in E_{\lambda}((1-r), r) \mathcal{H}$.

Sec (3.3): The Operator Equation

\[ K^p = H^{\delta} T^2 \left( T^2 H^{\delta+r} T^2 \right)^{p-\delta} T^2 H^{\delta} \text{ and its Applications} \]

A capital letter (such as $T$) means a bounded linear operator on a Hilbert space. $\mathcal{T} \geq 0$ and $\mathcal{T} > 0$ mean a positive operator and an invertible positive operator, respectively.

In [133], Pedersen and Takesaki developed the operator equation $K = \mathcal{T} \mathcal{H} \mathcal{T}$ as a useful tool for the noncommutative Radon–Nikodym theorem. By using Douglas’s majorization theorem [123], Nakamoto [132] provided a simple proof. As generalizations, Bach and Furuta [121,125] gave deep discussion on the equation $K = T (H \mathcal{T} T)^n$.

Theorem (3.3.1) [118,125]: Let $H$ and $K$ be bounded positive operators on a Hilbert space, and assume that $H$ is nonsingular.

(1) The following statements are equivalent for any natural number $n$:

(a) $\alpha H^{\frac{1}{2n}} \geq \left( H^{\frac{1}{2n}} K H^{\frac{1}{2n}} \right)^{\frac{n+1}{n+1}}$ for some $\alpha \geq 0$;

(b) there exists a unique positive operator $T$ such that $\|T\| \leq \alpha$, and

\[ K = T^{\frac{1}{2}} T^2 T^{-\frac{1}{2}} H^{\frac{1}{2}}. \] (23)
(2) If there exists a positive operator $T$ satisfying (23) for some natural number $n$, then, for each natural number $m \leq n$, there exists a positive operator $T_1$ satisfying

$$K = T_1^{\frac{1}{m}} \left( T_1^{\frac{1}{m}} H \frac{1}{m} T_1^{\frac{1}{m}} \right)^m T_1^{\frac{1}{m}}. \quad (24)$$

Lin [130] showed a generalization of Theorem (3.3.1)(1) via Furuta inequality [124] under the restriction $a = 1$.

**Theorem (3.3.2)[118,121]:** Given any natural number $n$ and $m$ with $m < n$, there exist a nonsingular positive operator $H$ and a positive operator $K$ such that Eq. (24) is solvable and (23) is unsolvable.

In this section, as a continuation, we consider the following equation for $p > 0, r > 0$ and $p \geq \delta > -r$

$$K^p = H^\delta T_1^{\frac{1}{m}} (T_1^{\frac{1}{m}} H^{\delta+r} T_1^{\frac{1}{m}})^{\frac{p-\delta}{\delta+r}} T_1^{\frac{1}{m}} H^\delta. \quad (25)$$

Obviously, the special case $p = 1, r = \frac{1}{n}$ and $\delta = 0$ of (25) becomes (23). Theorems (3.3.1)–(3.3.2) are extended to Theorems (3.3.4)–(3.3.5), respectively.

Some applications are obtained. We show that the inclusion relations in the following result are strict. See Theorem (3.3.3) below.

**Theorem (3.3.3)[118,128,129]:** Let $T$ be a class $wA(p, r)$ operator, then $T$ is a class $wA(p_1, r_1)$ operator for $p_1 \geq p > 0$ and $r_1 \geq r > 0$.

A kind of polar decomposition of Aluthge transformation [119] is given. See Theorems (3.3.14)–(3.3.15) below.

**Theorem (3.3.4)[118,123]:** The following assertions are equivalent for any operators $A$ and $B$.

1. $AA^* \leq \lambda B B^*$ for some $\lambda \geq 0$.
2. There exists a $C$ with $A = BC$ and $\|C\| \leq \lambda$.

**Lemma (3.3.5)[118,126,127]:** Let $\alpha \in \mathbb{R}$ and $X$ be invertible. Then

$$(X^*X)^\alpha = X^* (XX^*)^{\alpha-1} X,$$

especially in case $\alpha \geq 1$ the equality holds without invertibility of $X$.

**Theorem (3.3.6)[118,137,139]:** (Furuta type inequality). Let $A, B \geq 0$, $\alpha_0, \beta_0 > 0$, $-\beta_0 < \delta_0 \leq \alpha_0$, $-\beta_0 \leq \delta_0 < \alpha_0$.

(1) If $0 \leq \delta_0 \leq \alpha_0$, then

$$\left( B_2^{\frac{\beta}{2}} A_2^\alpha B_2^\beta \right)^{\frac{\beta_0+\delta_0}{\beta_0+\alpha_0}} \geq B_2^{\beta_0+\delta_0} \Rightarrow \left( B_2^{\frac{\beta}{2}} A_2^\alpha B_2^\beta \right)^{\frac{\beta+\delta_0}{\beta+\alpha}} \geq B_2^{\beta+\delta_0}$$

for any $\alpha \geq \alpha_0$ and $\beta \geq \beta_0$.

(2) If $-\beta_0 \leq \delta_0 \leq 0$ and $N(A) \subset N(B)$, then

$$A_2^{\alpha_0+\delta_0} \geq \left( A_2^{\frac{\alpha_0}{2}} B_2^{\beta_0} A_2^\alpha \right)^{\frac{\alpha_0+\delta_0}{\alpha_0+\beta_0}} \Rightarrow A_2^{\alpha+\delta_0} \geq \left( A_2^{\frac{\alpha}{2}} B_2^{\beta} A_2^\alpha \right)^{\frac{\alpha+\delta_0}{\alpha+\beta_0}}.$$
for any $\alpha \geq \alpha_0$ and $\beta \geq \beta_0$.

Theorem (3.3.6) is important to the proof of (2) of Theorem (3.3.8).

**Lemma (3.3.7)[118,134]:** Let $a, b, d$ and $\theta$ be real numbers and satisfy $a + b > 0$, $ab = d^2$, and $S = \begin{pmatrix} a & de^{-i\theta} \\ d e^{i\theta} & b \end{pmatrix}$. Then

$$S^p = (a + b)^{p-1} S$$

for $p > 0$.

**Theorem (3.3.8)[118]:** Let $H$ and $K$ be bounded positive operators on a Hilbert space, and assume that $H$ is nonsingular.

(1) The following statements are equivalent for any $p > 0, r > 0$ and $p \geq \delta \geq 0$:

(a) $a H^{\delta + r} \geq \left( H^\frac{r}{2} K^p H^\frac{r}{2} \right)^{\frac{\delta + r}{p + r}}$ for some $a \geq 0$;

(b) there exists a unique positive operator $T$ satisfies $\|T\| \leq a$ and (25).

If in additional $H$ is invertible, (1) holds for $p \geq \delta > -r$.

(2) If there exists a positive operator $T$ satisfying (25) for fixed $p > 0, r > 0$ and $p \geq \delta \geq 0$, then, for $p_1 \geq p$ and $r_1 \geq r$, there exists a positive operator $T_1$ satisfying

$$K^{p_1} = H^\frac{r}{2} T_1^2 \left( T_{11}^2 H^{\delta + r_1} T_{12}^2 \right)^{\frac{p_1 - \delta}{\delta + r_1}} T_{11}^{-\frac{1}{2}} H^\frac{r}{2}.$$

(26)

Lin [130] showed case $\delta = \frac{p-nr}{n+1}$ of Theorem(3.3.8)(1) under some restrictions.

**Proof.** The proof is similar to [125].

$(a) \Rightarrow (b)$. By Theorem (3.3.4), there exists a $S$ such that

$$H^\frac{r}{2} K^p H^\frac{r}{2} = H^\frac{r}{2} T_2^2 \left( T_{21}^2 H^{\delta + r_1} T_{22}^2 \right)^{\frac{p_1 - \delta}{\delta + r_1}} T_{22}^{-\frac{1}{2}} H^\frac{r}{2}.$$

Put $T = SS^*$, then $\|T\| \leq a$ and by Lemma (3.3.7),

$$H^\frac{r}{2} K^p H^\frac{r}{2} = H^\frac{r}{2} T_2^2 \left( T_{21}^2 H^{\delta + r_1} T_{22}^2 \right)^{\frac{p_1 - \delta}{\delta + r_1}} T_{22}^{-\frac{1}{2}} H^\frac{r}{2}.$$

So (25) holds for $H$ is singular.

$(b) \Rightarrow (a)$. For $a$ with $\|T\| \leq a$, by Lemma (3.3.7), (25) implies

$$\left( H^\frac{r}{2} K^p H^\frac{r}{2} \right)^{\frac{\delta + r}{p + r}} = \left( H^\frac{r}{2} T_2^2 \left( T_{21}^2 H^{\delta + r_1} T_{22}^2 \right)^{\frac{p_1 - \delta}{\delta + r_1}} T_{22}^{-\frac{1}{2}} H^\frac{r}{2} \right)^{\frac{\delta + r}{p + r}}.$$

$$= H^\frac{r}{2} T H^{\delta + r} \leq a H^{\delta + r}.$$

To show the uniqueness of $T$. Assume that $Z$ also satisfies (25), by (27) we have

$$H^\frac{r}{2} Z H^\frac{r}{2} = \left( H^\frac{r}{2} K^p H^\frac{r}{2} \right)^{\frac{\delta + r}{p + r}} = H^\frac{r}{2} T H^{\delta + r},$$

therefore $Z = T$.

Next to prove (2). By the assumption and (1), (a) holds for some $a > 0$, that is
\[
\left( \frac{p+r}{a^p(p+r)} \right)^{\delta+r} H^{\delta+r} \geq \left( \frac{p+r}{a^p(p+r)} \right)^{\frac{r}{2}} K^{p} \left( \frac{p+r}{a^p(p+r)} \right)^{\frac{r}{2}} H^{\frac{r}{2}} \cdot \left( \frac{p+r}{(p_1+r_1)^{p_1}} \right)^{\delta+r_1}.
\]

So that the following follows from (2) of Theorem (3.3.8):
\[
\left( \frac{p+r}{a^p(p+r)} \right)^{\delta+r_1} H^{\delta+r_1} \geq \left( \frac{p+r}{a^p(p+r)} \right)^{\frac{r_1}{2}} K^{p_1} \left( \frac{p+r}{a^p(p+r)} \right)^{\frac{r_1}{2}} H^{\frac{r_1}{2}} \cdot \left( \frac{p+r}{(p_1+r_1)^{p_1}} \right)^{\delta+r_1}.
\]

Therefore (26) is solvable. □

**Remark (3.3.9)[118]:** For each \( p > 0, r > 0 \) and \( \min\{p, 1\} \geq \delta > -r \), it is clear that the condition (a) is satisfied if \( H \) is invertible or, more generally \( a^{\frac{p+r}{(p_1+r_1)^{p_1}}} H \geq K \) for some \( a \geq 0 \) by (28) and Furuta inequality [124]. In the first case, the solution \( T \) to (25) is given by
\[
T = H^{-\frac{(\delta+r)}{2}} \left( H^{\frac{r}{2}} H^{\frac{r}{2}} \right)^{\frac{\delta+r_1}{(p_1+r_1)}} H^{-\frac{r}{2}}
\]
by (27).

**Theorem (3.3.10)[118]:** Given any positive numbers \( p, r, p_1 \) and \( r_1 \) with \( r_1 > r \), there exist a nonsingular positive operator \( H \) and a positive operator \( K \) such that case \( \delta = 0 \) of Eq. (26) is solvable and case \( \delta = 0 \) of (25) is unsolvable. To give proofs, the following results are needful.

**Proof.** The proof is inspired by [121].

For a natural number \( k \), let \( A_k = \begin{pmatrix} 1 & 0 \\ k & 1 \\ 0 & k^{-1} \end{pmatrix} \) and \( B_k = \begin{pmatrix} 1 & 1+k^2 \\ k^{-1} & k^2 \end{pmatrix} \). Take
\[
H = \bigoplus_{k=1}^{\infty} A_k^{r_1} \quad \text{and} \quad K = \bigoplus_{k=1}^{\infty} K_k^{p_1} \quad \text{where} \quad K_k = \begin{pmatrix} A_k^2 & B_k^{r_1} \\ A_k^{r_1} & B_k \end{pmatrix}.
\]

By Lemma (3.3.9),
\[
K_k = \frac{1}{(1+k^2)k^{2r_1/r_1}} \begin{pmatrix} 1 & k \\ k & k^2 \end{pmatrix}, \quad \text{hence} \quad \left\| K_k^{p_1} \right\| = K^{-2/r_1} \leq 1 \quad \text{and} \quad K \quad \text{is meaningful}.
\]

Next to show that the operators \( H \) and \( K \) satisfy the conditions.

In fact, \( H^{r_1} - \left( H^{\frac{r_1}{2}} K^{p_1} H^{\frac{r_1}{2}} \right)^{(p_1+r_1)} = \bigoplus_{k=1}^{\infty} \left( A_k - B_k \right) \geq 0 \) and this implies case \( \delta = 0 \) of (26) is solvable by (1) of Theorem (3.3.8). Meanwhile, case \( \delta = 0 \) of (25) is unsolvable for \( H \) and \( K \) here. Otherwise, also by (1) of Theorem (3.3.8), \( H \) and \( K \) satisfy (a) for some \( a > 0 \). This implies that
\[
aA_k^{r/r_1} \geq \left( \begin{pmatrix} r & p \\ 2r & p_1 \\ r & 2r_1 \end{pmatrix} \right)^{(p+r)}.
\]

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By Lemma (3.3.7),

\[ a \geq A_k^{-2r_1} \left( A_k^{\frac{r_1}{2}} \frac{1}{1+k^2} A_k^{\frac{r}{2k^2}} \right)^{\frac{p}{p+r}} A_k^{-2r_1} \]

\[ = A_k^{-2r_1} \left( \frac{1}{1+k^2} \right)^{\frac{r_1}{p+r}} \left( \frac{1}{1+k^{2(1-2r)/r_1}} \right)^{\frac{p}{p+r}} \left( \frac{1}{k} \right)^{\frac{1-2r}{k^{2(1-2r)/r_1}}} A_k^{-2r_1} \]

\[ = \left( \frac{1}{1+k^2} \right)^{\frac{p}{p+r}} \left( \frac{1}{1+k^{2(1-2r)/r_1}} \right)^{\frac{p}{p+r}} \left( \frac{1}{k} \right)^{\frac{1-2r}{k^{2(1-2r)/r_1}}} . \] (29)

Therefore,

\[ a \geq \left( \frac{1+k^2}{k^{2(1-2r)/r_1}} \right)^{\frac{p}{p+r}} = \left( \frac{1+k^2}{(k^{2r/r_1}+k^{2(1-2r)/r_1})} \right)^{\frac{p}{p+r}} . \] (30)

So that \( a \geq \infty \) by letting \( k \to \infty \) for \( \max\{2r/r_1, 2(1 - r/r_1)\} < 2 \). This is a contradiction. \( \Box \)

A fact in the proof of Theorem (3.3.10) is useful.

**Theorem (3.3.11)[118]:** Given any positive numbers \( p, r, p_1 \) and \( r_1 \) with \( r_1 > r \), there exist invertible positive operators \( H \) and \( K \) such that

\[ H^{-r_1} \geq \left( K^{\frac{p_1}{2}} K^{\frac{r_1}{2}} \right)^{\frac{r}{p+r}} , aH^{-r} \geq \left( H^{\frac{p}{2}} K^{p} H^{\frac{r}{2}} \right)^{\frac{r}{p+r}} , \]

where \( a \) is an arbitrary positive number.

**Proof.** The operators \( H \) and \( K \) in the proof of Theorem (3.3.10) are suitable. \( \Box \)

We Show Some Applications. For \( q > 0, T \) is called a \( q \)-hyponormal operator if \( (T^* T)^q \geq (T T^*)^q \), where \( T^* \) is the adjoint operator of \( T \). If \( q = 1, T \) is called a hyponormal operator and if \( q = 1/2, T \) is called a semi-hyponormal operator. See Martin and Putinar [131] and Xia [135] for related topics and basic properties of hyponormal operators.

Aluthge [119] introduced Aluthge transformation \( \tilde{T} = |T|^1/2 U |T|^1/2 \) where the polar decomposition of \( T \) is \( T = U |T| \). For each \( p > 0 \) and \( r > 0, \tilde{T}_{p,r} = |T|^p U |T|^r \) is called generalized Aluthge transformation.

As a generalization of \( q \)-hyponormal operators, Ito [128] introduced class \( wA(p, r) \) defined by

\[ (|T^*|^p |T|^{2r} |T^*|^p)^{\frac{p}{p+r}} \geq |T^*|^{2r} \text{ and } (|T|^p |T^*|^{2r} |T|^p)^{\frac{p}{p+r}} \leq |T|^{2p} . \]

See [120, 129, 137, 138] for related topics.

**Lemma (3.3.12)[118]:** For positive operators \( A \) and \( B \) on a Hilbert space \( \mathcal{H} \) define operators \( U \) and \( D \) on \( \bigoplus_{k=-\infty}^{\infty} \mathcal{H}_k \) where \( \mathcal{H}_k \cong \mathcal{H} \) as follows:

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where (0) shows the place of the (0, 0) matrix element, and \( T = UD \). Then the following assertions hold for each \( p > 0, r > 0 \) and \( \beta > 0 \):

1. \( (|T^*|^p |T|^2p |T^*|^r)^\beta \geq |T^*|^{2(p+r)\beta} \) if and only if \( (B^2 A^p B^2)^\beta \geq B^{(p+r)\beta} \).
2. \( |T|^{2(p+r)\beta} \geq (|T|^p |T^*|^{2r} |T|^p)^\beta \) if and only if \( A^{(p+r)\beta} \geq (A^p B^r A^p)^\beta \).

This example appears in [140,141] and is a modification of [122, Theorem 2] and [136, Lemma 1].

**Proof.** By easy calculation,

\[
|T|^2 = \begin{pmatrix} \ldots & B \\ & (A) \\ & \end{pmatrix}, \quad |T^*|^2 = \begin{pmatrix} \ldots & B \\ & (B) \\ & \end{pmatrix},
\]

Therefore

\[
|T^*|^r |T|^2p |T^*|^r = \begin{pmatrix} \ldots & B^{p+r} \\ & (B^2 A^p B^2) \\ & A^{p+r} \\ & \end{pmatrix}.
\]
and

$$|T|^p |T^*|^2 r |T|^p = \begin{pmatrix} \ddots & B^{p+r} \\ (A^p B^r A^p) & \ddots & \ddots \\ & & \ddots & \ddots \\ & & & \ddots & A^{p+r} \end{pmatrix},$$

By comparing the $(0,0)$ elements of the operator matrices above, the assertions hold. □

**Theorem (3.3.13)[118]:** Given any positive numbers $p, r, p_1$ and $r_1$ with $r_1 > r$, there exists an operator $T$ such that $T$ is a class $wA(p_1, r_1)$ operator but not a class $wA(p, r)$ operator. Theorem (3.3.13) implies that the inclusion relations in Theorem (3.3.3) are strict.

**Proof.** By Theorem (3.3.11), there exist invertible positive operators $H$ and $K$ on a Hilbert space $\mathcal{H}$ such that

$$H |_{^{p+r}} \geq \left( H^{\frac{r}{2}} K^{p_1} H^{\frac{r}{2}} \right)^{(p_1 + r_1)}, \quad H |^r r \geq \left( H^{\frac{r}{2}} K^p H^{\frac{r}{2}} \right)^{p+r}.$$

Let $A = H$ and $B = K$, define an operator $T$ on $\bigoplus_{k=-\infty}^{\infty} \mathcal{H}_k$ where $\mathcal{H}_k = \mathcal{H}$ as Lemma (3.3.12). Then $T$ is a class $wA(p_1, r_1)$ operator but not a class $wA(p, r)$ operator by Lemma (3.3.12). □

Aluthge [119] showed a kind of polar decomposition of Aluthge transformation on invertible $q$-hyponormal operators via the equation

$$K = THT.$$

**Theorem (3.3.14)[118, 119]:** Let $T$ be a invertible $q$-hyponormal operator and the polar decomposition of $\widetilde{T}$ be $\overline{\widetilde{T}} = \overline{U}|\overline{T}|$. Then $|\overline{T}| = |T|^{1/2}S^{-1}|T|^{1/2}$ and $\overline{U} = |T|^{1/2}US|T|^{-1/2}$ where $S$ is the solution to the equation $|T| = SU^*|T|US$.

The following assertion say that this result holds for any invertible operator $T$.

**Theorem (3.3.15)[118]:** Let $T$ be an invertible operator and the polar decomposition of $\overline{T}_{p,r}$ be $\overline{T}_{p,r} = \overline{U}_{p,r} |\overline{T}_{p,r}|$. Then $|\overline{T}_{p,r}| = |T|^{1/2}S^{-1}|T|^{1/2}$ and $\overline{U}_{p,r} = |T|^p US|T|^{-r}$ where $S$ is the solution to the equation $|T|^{2r} = SU^*|T|^{2p}US$.

**Proof.** By Remark (3.3.9), the solution $S$ to $|T|^{2r} = SU^*|T|^{2p}US$ exists and $S = H^{-\frac{1}{2}} (H^{\frac{1}{2}} K H^{\frac{1}{2}})^{\frac{1}{2}} H^{-\frac{1}{2}}$ where $H = U^*|T|^{2p}U$ and $= |T|^{2r}$. Hence $S$ is invertible for $T$ is invertible and
\[ |\bar{T}_{p,r}| = (|T|^{-S^{-1}}|T|^{2rS^{-1}}|T|^{-r})^{1/2} = |T|^{-S^{-1}}|T|^{-r}. \]

Moreover, \[ \bar{U}_{p,r} = \bar{T}_{p,r} |\bar{T}_{p,r}|^{-1} = |T|^{pUS}|T|^{-r}. \]

**Corollary 3.3.16 [232]:** Given any positive numbers \( p, r_1 - \epsilon, p_1 \), there exist a nonsingular positive operator \( H \) and a positive operator \( K \) such that case \( \delta = 0 \) of Eq. (26) is solvable and case \( \delta = 0 \) of (25) is unsolvable. To give proofs, the following results are needful.

**Proof.** The proof is inspired by [121].

For a natural number \( k \), let \( A_k = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( B_k = \frac{1}{1+k^2} \begin{pmatrix} k^{-1} \\ k^{-1} \end{pmatrix} \). Take \( H = \bigoplus_{k=1}^{\infty} A_k \) and \( K = \bigoplus_{k=1}^{\infty} K_k \) where \( K_k = A_k^2 B_k r_1 A_k^{-1} \). By Lemma (3.3.9), \( K_k = \frac{1}{(1+k^2)^{k^2p_1/r_1}} \begin{pmatrix} k \\ k \end{pmatrix} \), hence \( \|K_k\| = k^{-2/r_1} \leq 1 \) and \( K \) is meaningful.

Next to show that the operators \( H \) and \( K \) satisfy the conditions.

In fact, \( H^r_1 - \left( \frac{r_1}{H^2 K^2 H^2} \right)^{r_1} = \bigoplus_{k=1}^{\infty} (A_k - B_k) \geq 0 \) and this implies case \( \delta = 0 \) of (26) is solvable by (1) of Theorem (3.3.8). Meanwhile, case \( \delta = 0 \) of (25) is unsolvable for \( H \) and \( K \) here. Otherwise, also by (1) of Theorem (3.3.8), \( H \) and \( K \) satisfy (a) for some \( a > 0 \). This implies that

\[ aA_k^{(r_1-\epsilon)/r_1} \geq \left( A_k^{2r_1/p} K_k^{p_1} A_k^{2r_1/p} \right)^{(r_1-\epsilon)/(p+r_1-\epsilon)}. \]

By Lemma (3.3.7),

\[ a \geq A_k^{2r_1/p} \left\{ \frac{(r_1-\epsilon)}{(1+k^2)^{2p/r_1}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} A_k^{2r_1/p} \right\} \frac{(r_1-\epsilon)}{(p+r_1-\epsilon)} A_k^{2r_1/p}. \]

Therefore,

\[ a \geq \left( \frac{1+k^2}{(k^2(r_1-\epsilon)/r_1 + (1+k^2)^2(1-(r_1-\epsilon)/(r_1))^2)} \right)^{r_1-\epsilon/(p+r_1-\epsilon)} \cdot \left( \frac{1+k^2}{(k^2(r_1-\epsilon)/r_1 + (1+k^2)^2(1-(r_1-\epsilon)/(r_1))^2)} \right)^{r_1-\epsilon/(p+r_1-\epsilon)}. \]

So that \( a \geq \infty \) by letting \( k \rightarrow \infty \) for \( \max\{2(r_1-\epsilon)/r_1, 2(1-(r_1-\epsilon)/(r_1)) \} < 2 \). This is a contradiction. \( \square \)