Chapter 1

Linear Elliptic Equations and New Hardy Inequalities

We apply results then provide two applications of inequalities. We improve recent results by showing that the blow-up phenomena of the gradient can also occur in Hardy spaces. The Hardy inequalities for Sobolev–Zygmund spaces are obtained via an integral formula estimating the oscillation in a ball of radius $r$of a general function$u$in the usual Sobolev space. We shall give a pointwise estimate for the solution $u$of linear equation $-\Delta u = -\mathrm{div}(F)$for a bounded function F, using the distance function δ.[1]

Section (1.1): An Equivalence Relation Between The Growth of The Data and The Growth of The Solution

The main motivation of this chapter comes from the study of the blow-up of the gradient of very weak solution of linear equation. We recall that

$$\begin{cases} \text{there is a unqiue solution } u \in L^1(\Omega) \text{satisfying} \\ -\int_\Omega u\Delta\varphi dx = \int_\Omega f\varphi dx, \forall\varphi \in C^2(\overline{\Omega}), \varphi = 0 \text{ on } \partial\Omega, \\ \text{where } f \in L^1(\Omega;\delta) \text{with } \delta(x) \text{is the distance of } x \text{ to the boundary of } \Omega. \end{cases} \quad (1)$$

In particular, there exists a $c > 0$: $\int_\Omega |\Delta u|\,\delta dx \le c\int_\Omega |f|\delta dx$. We recall below (see Proposition (1.1.6)) some known results in that sense,

One of the main features of this chapter is to emphasize the role of the Hardy inequalities in the study of the Brezis' problem. But at the same time, we give an alternative proof of the blow-up in $L(\log L)$ and we generalize this blow-up result by proving that it happens also in some weighted spaces or Hardy spaces. The techniques to obtain such results rely partly on some new Hardy inequalities and the following equivalence, for u solution of(1), $f \ge 0$,

$$\frac{u}{\delta} \in L^1(\Omega) \text{if and only if} \int_\Omega f\delta(1 + |\log\delta|)dx \; is \text{ finite.} \quad (2)$$

Here, Ω is a bounded open set of class $C^{2,1}$. It is well known that the following Hardy inequality$\left|\frac{v}{\delta}\right|_{L^p(\Omega)} \leq c_p|\nabla v|_{L^p(\Omega)}$ holds for $v \in W_0^{1,p}(\Omega), 1 < p \leq +\infty$ but not for $p = 1$. Nevertheless, we know that the following Hardy inequality holds true:

$$\int_\Omega \frac{|v|}{\delta} dx \leq c \int_{\Omega_*} |\nabla v|_{**}(t) dt = c|\nabla v|_{L(\log L)} \text{whenever } v \in W_0^1 L(\log L) \quad (3)$$

with$|\nabla v|_{**}(t) = \frac{1}{t}\int_0^t |\nabla v|_*(\sigma) d\sigma$ for $t > 0, |\nabla v|_*$ the decreasing monotone rearrangement of$|\nabla v|$, and $W_0^1 L(\log L) = \left\{\varphi \in W_0^{1,1}(\Omega): \int_{\Omega_*} |\nabla\varphi|_{**}(t)dt < +\infty\right\}, \Omega_* = (0, |\Omega|)$ =means $|\Omega|$ (see next paragraph for more details). Therefore, we recover from (2) and (3),that is if $f \notin L^1\big(\Omega; \delta(1 + |\log\delta|)\big), f \geq 0$, then the solution u of(1) satisfies

$$\int_{\Omega_*} |\nabla u|_{**}(t) dt = +\infty. \quad (4)$$

Some of our results, in this chapter will generalize the Hardy inequality (3). Namely we shall prove in Theorem (1.2.1).

If Ω is an open bounded Lipschitzian set of $\mathbb{R}^N$, then there exists a constant $c_\Omega > 0$ such that

$$\int_\Omega \frac{|v(x)|}{\delta x} dx \leq c_\Omega \int_\Omega |\nabla v|(1 + |\log\delta|) dx \equiv c_\Omega |v|,$$
$$\forall v \in W_0^1(\Omega; 1 + |\log \delta|).$$

Again from relation (2) and Theorem (1.2.1), we recover the result shown that

$$\int_\Omega |\nabla u||\log\delta|\, dx = +\infty, \quad (5)$$

whenever u is solution of (1) with $f \notin L^1(\Omega; \delta(1 + |\log\delta|)), f \geq 0$.

A more general Hardy inequality including (3) can be obtained using the Hardy space insteadof $L(\log L)$. More precisely, if we denote by $\mathcal{H}^1(\Omega)$ the Hardy space, we can associate theSobolev space

$$W_0^1\mathcal{H}^1(\Omega)\text{the closure of}C_c^\infty(\Omega)\text{in }\left\{\varphi \in W_0^{1,1}(\Omega): \frac{\partial\varphi}{\partial x_i} \in \mathcal{H}^1(\Omega), i = 1, \ldots, N\right\}:$$

$$W_0^1\mathcal{H}^1(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|}.$$

For nonnegative functions we shall define

$$W_{0_+}^1\mathcal{H}^1(\Omega) = \overline{C_{c_+}^\infty(\Omega)}^{\|\cdot\|}.$$

We shall prove here in Theorem (1.2.16) (Hardy inequality in Hardy space):

Assume that Ω is an open bounded set of class C^2. Then there exists a constant $c > 0$ suchthat

$$\forall\psi \in W_{0_+}^1\mathcal{H}^1(\Omega), \int_\Omega \frac{\psi}{\delta}(x)dx \le c\sum_{i=1}^N \left|\frac{\partial\psi}{\partial x_i}\right|_{\mathcal{H}^1}.$$

Note that if $\frac{\partial\varphi}{\partial x_i} \in L(\log L), \varphi \in W_0^{1,1}(\Omega)$, then $\frac{\partial\varphi}{\partial x_i} \in \mathcal{H}^1(\Omega)$.

We will show that the blow-up result given in relation (4) is also true in $\mathcal{H}^1(\Omega)$, that is to say:

if $f \notin L^1(\Omega; \delta(1 + \log\delta)), f \in L_+^1(\Omega; \delta)$, then the very weak solution of (1) satisfies

$$\sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|_{\mathcal{H}^1(\Omega)} = +\infty. \qquad (6)$$

Since the dual of $L(\log L)$ is L_{exp}, the above results lead us to reconsider the study of Hardyinequalities for Sobolev spaces in the “borderline” cases

$$W_0^1L_{exp}^\alpha(\Omega) = \left\{v \in W_0^{1,1}(\Omega): |\nabla v| \in L_{exp}^\alpha(\Omega)\right\}, \text{ where } \alpha > 0$$

$$L_{exp}^{\alpha}(\Omega) = \left\{ v \in L^1(\Omega) : \exists \lambda(v) = \lambda > 0 \text{ so that} \int_{\Omega} e^{\lambda |v(x)|^{\frac{1}{\alpha}}} dx < +\infty \right\}.$$

Indeed, we will show that three exists$c_\Omega > 0, \forall \varphi \in W_0^1 L_{exp}^{\alpha}(\Omega)$, one has

$$|\varphi(x)| \leq c_\Omega \delta(x)(1 + |\log \delta(x)|)^{\alpha} |\nabla \varphi|_{L_{exp}^{\alpha}(\Omega)}. \qquad (7)$$

The idea the proof relies on the observation that there is a link between the oscillation of $\varphi$and (7) namely, we will show that there exists a constant $\gamma\Omega > 0$ such that$\forall \varphi \in W_0^1 L_{exp}^{\alpha}(\Omega), \forall B(x;r)$,ball of radius $r > 0$ centered at x ,contained in Ω , one has

$$\underset{B(x,r)}{\operatorname{osc}} \varphi \leq \gamma \Omega (1 + |\log r|)^{\alpha} r |\nabla \varphi|_{L_{exp}^{\alpha}(\Omega)}. \qquad (8)$$

The proof of (8) relies on an integral formula related to the relative rearrangement of$|\nabla \varphi|$with respect to φ.

Let us mention that we will also prove a similar property for the solution $u \in H_0^1(\Omega)$, of$(\mathcal{L}_F)$

$$-\Delta u = -div\ (F), F \in L^{\infty}(\Omega)^N.$$

In particular,we willshow here in Theorem (1.2.12).

If Ω is an open bounded set of class C^2,then the unique solution u of $(\mathcal{L}_F)$ satisfies, there exists a constant $c_\Omega > 0$:

$$|u(x)| \leq c_\Omega |F|_\infty \delta(x)(1 + |\log \delta(x)|), \qquad \forall x \in \Omega.$$

For a Lebesgue measurable set E of Ω we denote by $|E|$ its measure.

The decreasing rearrangement of a measurable function $u: \Omega \to \mathbb{R}$is given by

$$u_*: \Omega_* =]0, |\Omega|[\to \mathbb{R}, \quad u_*(s) = \inf\{t \in \mathbb{R}: |u > t| \leq s\},$$

$$u_*(0) = \operatorname*{ess\,sup}_{\Omega} u, \quad u_*(|\Omega|) = \operatorname*{ess\,sup}_{\Omega} u.$$

We shall use the following Lorentz spaces, for $1 < p < +\infty, 1 \leq q \leq +\infty$

$$L^{p,q}(\Omega) = \left\{v: \Omega \to \mathbb{R} \text{ measureable} |v|^q_{L^{p,q}} = \int_0^{|\Omega|} \left[t^{\frac{1}{p}} |v|_{**}(t)\right]^q \frac{dt}{t} < +\infty\right\},$$

For $q < +\infty$:

$$L^{p,\infty}(\Omega) = \left\{v: \Omega \to \mathbb{R} \text{ measureable} |v|_{L^{p,\infty}} = \sup_{t \le |\Omega|} t^{\frac{1}{p}} |v|_{**}(t) < +\infty\right\},$$

χ_E is the characteristic function of a set $E \subset \Omega$ and $|v|_{**}(t) = \frac{1}{t}\int_0^t |v|_*(s)\, ds$ for $t \in \Omega_* =]0, |\Omega|[$.

We denote by $\partial_i = \frac{\partial}{\partial x_i}, \partial_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$. We define the following sets

$$W^1 L^{p,q}(\Omega) = W^1\left(\Omega, |\cdot|_{p,q}\right) = \{v \in W^{1,1}(\Omega): |\nabla v| \in L^{p,q}(\Omega)\}.$$

We shall denote by c various constants depending only on the data.

The notation $\approx$ stands for equivalence of nonnegative quantities, that is

$$\Lambda_1 \approx \Lambda_2 \quad \Leftrightarrow \quad \exists c_1 > 0, c_2 > 0 \ \text{ such that} c_1\Lambda_1 \le \Lambda_1 \le c_2 \Lambda_2.$$

$B(x; r)$ will denote the ball of $\mathbb{R}^N$ centered at x of radius $r > 0$.

$$C_c^m(\Omega) = \{\varphi \in C^m(\Omega): \varphi \text{ has compact support}\}$$
$$C_{c_+}^m(\Omega) = \{\varphi \in C_c^m(\Omega), \varphi \ge 0\}.$$

For $\alpha > 0$, we set

$$L^\alpha_{exp}(\Omega) = \left\{v: \Omega \to \mathbb{R} \text{ measurable: } \|v\|_\alpha = \sup_{t \le |\Omega|} \frac{|v|_*(t)}{\left(1 + \log\frac{|\Omega|}{t}\right)^\alpha} < +\infty\right\},$$

$$L(\log \mathrm{L}) = \left\{v: \Omega \to \mathbb{R} \text{ measurable: } |v|_{L(\log L)} = \int_0^{|\Omega|} |v|_{**}(t) dt < +\infty\right\}.$$

We note that $L^1_{exp}(\Omega) = L_{exp}(\Omega)$ and $L(\log L)$ are associate each other.

To define the Hardy spaces for Ω, we first recall the definition of $\mathcal{H}^1(\mathbb{R}^N)$ And BMO$(\mathbb{R}^N)$.

We first begin with the definition of$\mathcal{H}^1(\mathbb{R}^N)$. we consider$\theta \in C_c^\infty(\mathbb{R}^N)$such that $\theta(x) = 0$ for$|x| \geq 1$, $\int_{\mathbb{R}^N} \theta dx = 1$, $|\nabla\theta|_\infty \leq 1$. We set $\theta_t(x) = t^{-N}\theta\left(\frac{x}{t}\right)$ for $t > 0$.

For $f \in L^1(\mathbb{R}^N)$ we define

$$M_\theta f(x) =$$

and

$$\mathcal{H}^1(\mathbb{R}^N) = \{f \in L^1(\mathbb{R}^N): M_\theta f \in L^1(\mathbb{R}^N)\}.$$

Property (1.1.1) [1]:

(i) If $f \in L^q(\mathbb{R}^N)$, $q > 1$ or in $L(\log L)$having compact support and $\int_{\mathbb{R}^N} f(x)\, dx = 0$, then $f \in \mathcal{H}^1(\mathbb{R}^N)$. Moreover, we have a constant $c > 0$ such that $M_\theta f(x) \leq cMf(x)$for $a.e.x$, Mf, is the usual Hardy–Littelwood maximal operator.

(ii) There exists a constant $c > 0$ such that, $\forall f \in \mathcal{H}^1(\mathbb{R}^N)$

$$|f|_{L^1(\mathbb{R}^N)} \leq c|M_\theta f|_{L^1(\mathbb{R}^N)}.$$

(iii) $\mathcal{H}^1(\mathbb{R}^N)$endowed with the norm $|f|_{\mathcal{H}^1} = |M_\theta f\,|_{L^1}$ is a Banach space.

(iv) The set $\left\{\varphi \in C_c^\infty(\mathbb{R}^N): \int_{\mathbb{R}^N} f(x)dx = 0\right\}$ is dense in $\mathcal{H}^1(\mathbb{R}^N)$.

We shall use the dual of $\mathcal{H}^1(\mathbb{R}^N)$called $\mathrm{BMO}(\mathbb{R}^N)$(set of bounded mean oscillation functions)is defined as follows:

Definition (1.1.2) [1]: (BMO$(\mathbb{R}^N)$)

For any cube $Q$of $\mathbb{R}^N$and $f \in L^1_{loc}(\mathbb{R}^N)$, we denote by $|Q|$itsmeasure and $f_Q = \frac{1}{|Q|}\int_Q\ f(x)dx$the average of $f$over Q. We will say that $f \in\ \mathrm{BMO}(\mathbb{R}^N)$if

$$\sup_{Q\subset\mathbb{R}^N}\frac{1}{|Q|}\int_Q |f(x)-fQ|dx < +\infty.$$

A function $f \in \mathrm{BMO}(\mathbb{R}^N)$is said to be in $\log \mathrm{VMO}(\mathbb{R}^N)$(vanishing mean oscillation with logarithmrate) if it is bounded and

$$\sup_{Q\subset\mathbb{R}^N}\frac{|\log|Q||}{|Q|}\int_Q |f(x)-fQ|\,dx < +\infty.$$

Next we want to introduce the notion of $\mathcal{H}^1(\Omega)$as follows:

Definition (1.1.3) [1]:($\mathcal{H}^1(\Omega)$)

Let Ωbe a bounded open set and $\Delta_0(\Omega) = \left\{\varphi \in C_c^\infty(\Omega): \int_\Omega \varphi(x)\,dx = 0\right\}$. According to the above property, we can identify $\Delta_0(\Omega)$as a subspace of$\mathcal{H}^1(\mathbb{R}^N)$. We set for $\varphi \in \Delta_0(\Omega)$

$$|\varphi|_{\mathcal{H}^1(\Omega)} = |\varphi|_{\mathcal{H}^1(\mathbb{R}^N)}.$$

Then we define

$\mathcal{H}^1(\Omega) = \overline{\Delta_0(\Omega)}^{|\cdot|_{\mathcal{H}^1(\mathbb{R}^N)}}$, the closure of $\Delta_0(\Omega)$with respect to the above norm.

For $\varphi \in C_c^\infty(\Omega)$ we have $\frac{\partial\varphi}{\partial x_i} \in \mathcal{H}^1(\Omega)$ for $i = 1, \dots, N$. We denote by

$$\|\nabla\varphi\|_{\mathcal{H}^1(\Omega)} = \sum_{i=1}^{N}\left|\frac{\partial\varphi}{\partial x_i}\right|_{\mathcal{H}^1(\Omega)}.$$

and

$$W^1\mathcal{H}^1(\Omega) = \left\{\varphi \in L^1(\Omega): \text{for } i = 1, \dots, N, \frac{\partial\varphi}{\partial x_i} \in \mathcal{H}^1(\Omega)\right\}.$$

We endow it by the following norm

$$|||\varphi||| = \|\nabla\varphi\|_{\mathcal{H}^1(\Omega)} + |\varphi|_{L^1(\Omega)} \text{and} C_c^\infty(\Omega) \subset W^1\mathcal{H}^1(\Omega).$$

We define

$$W_0^1\mathcal{H}^1(\Omega) = \overline{C_c^\infty(\Omega)}^{|||\cdot|||}, W_{0_+}^1\mathcal{H}^1(\Omega) = \overline{C_{c_+}^\infty(\Omega)}^{|||\cdot|||}.$$

Many properties can be developed for these spaces, we only state few of them which are necessary for our chapter.

Proposition (1.1.4) [1]:

(i) The embedding of $W_0^1\mathcal{H}^1(\Omega)$ is continuous.

(ii) The Poincaré —Sobolev inequality holds true on $W_0^1\mathcal{H}^1(\Omega)$ that is there exists a constant $c > 0$ such that for all $\psi \in W_0^1\mathcal{H}^1(\Omega)$

$$|\psi|_{L^1(\Omega)} \leq c\|\nabla\psi\|_{\mathcal{H}^1(\Omega)}.$$

Proof:

Let $\psi \in W_0^1\mathcal{H}^1(\Omega)$. Then we have a sequence $\psi_n \in C_c^\infty(\Omega)$

$$|\psi_n - \psi|_{L^1} + \|\nabla(\psi_n - \psi)\|_{\mathcal{H}^1} \xrightarrow[n\to+\infty]{} 0.$$

But we have (see Property (1.1.1))

$$\left|\frac{\partial}{\partial x_i}(\psi_n - \psi_m)\right|_{L^1} \leq c\left|M_\theta\left(\frac{\partial}{\partial x_i}(\psi_n - \psi_m)\right)\right|_{L^1},$$

which shows that

$$|\nabla(\psi_m - \psi)|_{L^1} \leq c\|\nabla(\psi_n - \psi)\|_{\mathcal{H}^1}.$$

In particular$\psi_m \to \psi$ in $W_0^1(\Omega)$,and there exists $c > 0, \forall\psi \in W_0^1\mathcal{H}^1(\Omega)$.

$$|\psi|_{L^1(\Omega)} \leq c\|\nabla\psi\|_{\mathcal{H}^1(\Omega)},$$

and

$$|||\cdot|||\text{is equivalent to}\|\nabla.\|_{\mathcal{H}^1}.$$

The following lemma due to Hajlasz implies relation (3).

Lemma (1.1.5) [1]:

Let Ω be an open and bounded subset of $\mathbb{R}^N$. Supposethat there exists a constant $b > 0$ such that

$$|B(x,r) \cap \Omega^c| \geq b|B(x,r)| \text{for every } x \in \partial\Omega, \text{and } r > 0 \qquad (9)$$

(for instance if $\partial\Omega$ is C^1). Then there exists a constant $c > 0$ depending only on N and b suchthat the inequality

$$|v(x)| \leq c\delta(x)M(|\nabla v|)(x) \qquad (10)$$

holds for all $v \in C_c^\infty(\Omega)$ and all $x \in \Omega$.

Here, $M(|\nabla u|)$ is the maximal function of $|\nabla u|$.

Indeed, to obtain relation (3), we have a constant $c_\Omega > 0$

$$c_\Omega M(|\nabla v|)_*(s) \leq |\nabla u|_{**}(s), \qquad \forall s > 0, \qquad (11)$$

and from relations (10) and (11), we deduce relation (3), knowing from Donaldson Trudinger'sresult that $C_c^\infty(\Omega)$ is dense in the closed set $W_0^1 L(\log L)$.

We recall also the following results obtained previously for the very weak solution of (1):

Proposition (1.1.6) [1]:

Let Ω be an open bounded set of $C^{2,1}$in $\mathbb{R}^N$, $f \in L^1(\Omega,\delta)$. Then there exists a constant $c > 0$ such that for any solution u of (1), one has:

(i) $|\nabla u|_{L^{1+\frac{1}{N},\infty}(\Omega,\delta)} \leq c|f|_{L^1(\Omega;\delta)}, |u|_{L^{N'},\infty(\Omega)} \leq c|f|_{L^1(\Omega;\delta)}, N \geq 2.$

(ii) If $f \geq 0$, then $u \geq 0$.

(iii) If $f \in L^1(\Omega;\ \delta(1 + |\log\delta|))$, then

$$u \in W_0^{1,1}(\Omega) \quad \text{and } |\nabla u|_{L^1(\Omega)} \leq c|f|_{L^1(\Omega;\delta(1+|\log\delta|))}.$$

(iv) If Ω is a ball, f is radial or $N = 1$ and $f \in L^1(]a,b[= \Omega,\delta)$, then

$$u \in W_0^{1,1}(\Omega) \quad \text{and} \quad |\nabla u|_{L^1(\Omega)} \leq c|f|_{L^1(\Omega;\delta)}.$$

Here $\delta(x)$ is the distance of $x \in \Omega$ to be boundary $\partial\Omega$.

Now we shall prove the equivalence relation (2) for the very weak solution (1) which shallmotive the next paragraph concerning Hardy inequalities.

Let us consider u the very weak solution of (1). We then have

Theorem (1.1.7) [1]:

Let Ω be an open bounded set of $C^{2,1}$in $\mathbb{R}^N$. There exists a constant $c > 0$ depending only on Ω such that for all $f \in L^1(\Omega, \delta(1 + |\log \delta|)), f \geq 0$, and every weak solution$u$ of (1), one has

$$\int_\Omega \frac{u}{\delta} dx \leq c \int_\Omega f(1 + |\log \delta|)\delta dx ,$$

and

$$\int_\Omega f\delta |\log \delta| dx \leq c\left[\int_\Omega f\delta\, dx + \int_\Omega \frac{u}{\delta} dx\right].$$

Proof:

Since the set $\{\varphi \in C^2(\bar{\Omega}): \varphi = 0 \text{ on } \partial\Omega\}$ is dense in $X_p(\Omega) = W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ for any $p > 1$, we can easily replace the set of test functions by $X_p(\Omega), p > N$ (fixed).

Consider $\varphi_1 \in X_p(\Omega)$, the first eigenfunction of the operator $-\Delta$ say $-\Delta\varphi_1 = \lambda_1\varphi_1, \lambda_1 > 0$.

We know that there exist $c_0 > 0, c_1 > 0$:

$$c_0\delta \leq \varphi_1 \leq c_1\delta. \tag{12}$$

For $0 < \varepsilon < \frac{1}{2}$let $\varphi = \varphi_1 \log(\varphi_1 + \varepsilon)$. Then $\varphi \in C^2(\bar{\Omega}), \varphi = 0$ on $\partial\Omega$. Therefore, one can useit as a test function

$$-\int_{\Omega} f\varphi_1 \log(\varphi_1 + \varepsilon)\, dx$$

$$= \int_{\Omega} u\Delta(\varphi_1 \log(\varphi_1 + \varepsilon))dx. \qquad (13)$$

Developing the Laplacian term, one has

$$\Delta(\varphi_1 \log(\varphi_1 + \varepsilon)) = \Delta\varphi_1 \left(\log(\varphi_1 + \varepsilon) + \frac{\varphi_1}{\varphi_1 + \varepsilon}\right) + |\nabla\varphi_1|^2 \left[\frac{1}{\varphi_1 + \varepsilon} + \frac{\varepsilon}{(\varphi_1 + \varepsilon)^2}\right]. \quad (14)$$

Since $-\Delta\varphi_1 = \lambda_1\varphi_1$ and $c_6 = \mathrm{Max}_{\Omega}|\nabla\varphi_1|^2 < +\infty$, we derive from relations (13) and (14).

$$\left|\int_{\Omega} f\varphi_1 \log(\varphi_1 + \varepsilon)\, dx\right| \leq +\lambda_1 \left|\int_{\Omega} u\varphi_1 \log(\varphi_1 + \varepsilon)\, dx\right| + \lambda_1 \int_{\Omega} \frac{\varphi_1^2 u}{\varphi_1 + \varepsilon} dx + c_6 \int_{\Omega} u \left[\frac{1}{\varphi_1 + \varepsilon} + \frac{\varepsilon}{(\varphi_1 + \varepsilon)^2}\right] dx. \qquad (15)$$

We estimate each term of relation (15) as follows

$$c_6 \int_{\Omega} u \left[\frac{1}{\varphi_1 + \varepsilon} + \frac{\varepsilon}{(\varphi_1 + \varepsilon)^2}\right] dx \leq c_7 \int_{\Omega} \frac{u}{\delta} dx, \qquad (16)$$

$$\lambda_1 \int_{\Omega} \frac{\varphi_1^2 u}{\varphi_1 + \varepsilon} dx \leq c_8 \int_{\Omega} \frac{u}{\delta} dx, \qquad (17)$$

$$\lambda_1 \int_\Omega u|\varphi_1 \log(\varphi_1 + \varepsilon)|\, dx \le c_{10} \int_\Omega \frac{u}{\delta}\, dx. \qquad (18)$$

Therefore, we have

$$\left| \int_\Omega f\varphi_1 \log(\varphi_1 + \varepsilon)\, dx \right| \le c_{11} \int_\Omega \frac{u}{\delta}\, dx. \qquad (19)$$

Next, we write

$$\int_\Omega f\varphi_1 |\log(\varphi_1 + \varepsilon)|\, dx$$

$$= -\int_\Omega f\varphi_1 \log(\varphi_1 + \varepsilon)\, dx$$

$$+ 2 \int_{\{\varphi_1 + \varepsilon > 1\}} f\varphi_1 \log(\varphi_1 + \varepsilon)\, dx. \qquad (20)$$

For $0 < \varepsilon < \frac{1}{2}$, we have

$$2 \int_{\{\psi_1 + \varepsilon > 1\}} f\, \varphi_1 \log(\varphi_1 + \varepsilon)\, dx$$

$$\le c_{12} \int_\Omega f\delta\, dx. \qquad (21)$$

So, from (19) to (21), we have

$$\int_\Omega f\varphi_1 |\log(\varphi_1 + \varepsilon)|\, dx \le c_{13} \left[\int_\Omega \frac{u}{\delta}\, dx + \int_\Omega f\delta\, dx \right]. \qquad (22)$$

From which we derive the second statement, letting $\varepsilon \to 0$. While for the first statement, we userelations (13) and (14) to have

$$\int_{\Omega} \frac{u|\nabla\varphi_1|^2}{\varphi_1 + \varepsilon}\left(1 + \frac{\varepsilon}{\varphi_1 + \varepsilon}\right) dx$$

$$= \lambda_1 \int_{\Omega} u\varphi_1 \left(\log(\varphi_1 + \varepsilon) + \frac{\varphi_1}{\varphi_1 + \varepsilon}\right) dx$$

$$- \int_{\Omega} f\varphi_1 log(\varphi_1 + \ \varepsilon)\, dx. \qquad (23)$$

From which we derive

$$\int_{\Omega} \frac{u|\nabla\varphi_1|^2}{\varphi_1 + \varepsilon} dx \le c_{14} \int_{\Omega} u dx + c_{14} \int_{\Omega} f\ \varphi_1 |\log(\varphi_1 + \varepsilon)|\, dx. \qquad (24)$$

Using Proposition (1.1.4) and relation (12)

$$\int_{\Omega} \frac{u|\nabla\varphi_1|^2}{\varphi_1 + \varepsilon} dx \ \le c_{15} \int_{\Omega} f(1 + |\log\delta|)\delta\ \ dx. \qquad (25)$$

We can define Hopf's maximum principle: (Let $u = u(x), x = (x_1, \dots, x_n)$, be a c^2 function which satisfies the differential inequality

$$Lu = \sum_{i,j} a_{ij}(x)\, \partial^2_{x_i x_j} u + \sum_{i} b_i(x)\, \partial_{x_i} u \ge 0$$

in a domain Ω. Suppose the (symmetric) matrix $[a_{ij}] = [a_{ij}(x)]$ is locally uniformly positive in Ω (that is, for any given compact subset Ω′ of Ω, the quadratic from

$$\sum_{ij} a_{ij}(x)\, n_i n_j$$

is positive and uniformly bounded from 0 for all x in Ω' and all vectors n in $\mathbb{R}^n$ with $|n| = 1$), and the coefficients $a_{ij}, b_i = b_i(x)$ are locally bounded in Ω.

If u takes a maximum value M in Ω, then $u \equiv M$ in Ω.) [5]

By the Hopf maximum, we deduce that there exists a neighbourhood of boundary denoted by $\Omega_0 \subset \Omega$ such that $\min_{x\in\bar{\Omega}_0} |\nabla\varphi_1|^2(x) > 0$.

Thus,

$$\int_{\Omega_0} \frac{u}{\delta} dx \le c_{16} \int_{\Omega} f(1 + |\log\delta|)\delta \, dx, \tag{26}$$

using relation (25). While on $\Omega\backslash\bar{\Omega}_0$, $\inf_{\Omega\backslash\bar{\Omega}_0} \delta(x) > 0$ and then

$$\int_{\Omega\backslash\bar{\Omega}_0} \frac{u}{\delta} dx \le c_{17} \int_{\Omega} u\, dx \le c_{18} \int_{\Omega} f\delta\, dx. \tag{27}$$

From the two last relations, we have

$$\int_{\Omega} \frac{u}{\delta} dx \le c_{19} \int_{\Omega} f(1 + |\log\delta|)\delta\, dx. \tag{28}$$

As a corollary of the above theorem, we can deduce the same equivalence replacing the operator $-\Delta$ by a general operator, say let us consider

$$L_\varphi = -\sum_{i,j=1}^{N} \partial_i\big(a_{ij}(x)\partial_j\varphi\big) + \sum_{i=1}^{N} b^i(x)\partial_i\varphi + c_0(x)\varphi,$$

under the same assumption say $a_{ij} \in C^{0,1}(\bar{\Omega}), b^i \in C^{0,1}(\bar{\Omega}), c_0 \in L^\infty(\Omega), c_0 \ge 0, \exists\nu_0 > 0$ such that $\forall\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$

$$\sum_{i,j=1}^{N} a_{i,j}(x)\xi_i\xi_j \ge \nu_0|\xi|^2,$$

$c_0(x) - \frac{1}{2}\sum_{i=1}^{N} \partial_i b^i(x) \geq 0$ a.e. in Ω. We denote by L^*its adjoint.

Corollary (1.1.8) [1]:

Let $f \geq 0, f \in L^1(\Omega; \delta)$ and w be the very weak solution of

$$Lw = f \text{ in } \Omega, \qquad w = 0 \text{ on } \partial\Omega$$

in the sense of

$$\int_\Omega wL^*\varphi \, dx = \int_\Omega f\varphi \, dx, \tag{29}$$

$\forall \varphi \in X_p(\Omega) = W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega), p > N.$

If $f \in L^1(\Omega;\ \delta(1 + |\log \delta|))$, then

$$\int_\Omega \frac{w}{\delta} dx \leq c \int_\Omega f(1 + |\log \delta|)\delta \, dx,$$

and

$$\int_\Omega f\delta|\log \delta| \, dx \leq c\left(\int_\Omega f\varphi \ dx + \int_\Omega \frac{w}{\delta} dx\right).$$

Proof:

Let us denote by $G_{-\Delta}$the Green function associated to $-\Delta$ with the Dirichlet boundarycondition and by G_L the one associated to L with the Dirichlet boundary condition.

One has $G_{L^*}(y,x) = G_L(x,y), \forall (x,y) \in \Omega, x \neq y$. According to Stampacchia's result one has two constants

$$\begin{gathered} c_{20} > 0, \qquad c_{21} > 0, \forall (x,y) \in \Omega \times \Omega \setminus \{(x,x), x \in \Omega\}, \\ c_{20}G_{-\Delta}(x,y) \leq G_L(x,y) \leq c_{21}G_{-\Delta}(x,y). \end{gathered} \tag{30}$$

Therefore, we have $c_{20}u(x) \leq w(x)c_{21}u(x)$, a.e. $x \in \Omega$, where

$$u(x) = \int_{\Omega} G_{-\Delta}(x,y)f(y)dy \text{ is the very weak solution of (1)} \tag{31}$$

and

$$w(x) = \int_{\Omega} G_L(x,y)f(y)dy \text{ is the very weak solution of (29).}$$

Therefore, from relation (31), we derive

$$c_{20}\int_{\Omega}\frac{u}{\delta}dx \le \int_{\Omega}\frac{w}{\delta}dx \le c_{21}\int_{\Omega}\frac{u}{\delta}dx. \tag{32}$$

From relation (32) and Theorem (1.1.7) we get the result.

A first consequence of the above Theorem (1.1.7) and its Corollary (1.1.8) is that we recover the blowup result.

Theorem (1.1.9) [1] :

Let u be the very weak solution of (1) with $f \ge 0, f \in L^1(\Omega,\delta)$. If $f \notin L^1(\Omega;\delta(1+|\log\delta|))$, then

$$\int_{\Omega_*} |\nabla u|_{**}(t)\,dt = +\infty.$$

Proof:

If $\int_{\Omega_*}|\nabla u|_{**}(t)\,dt < +\infty$, then $u \in W^{1,1}(\Omega)$. Applying a density argument, u satisfies: $\forall\varphi \in C^2(\bar{\Omega})$ with $\varphi = 0$ on $\partial\Omega$

$$\int_{\Omega} \nabla u \cdot \nabla\varphi\,dx = \int_{\Omega} f\varphi\,dx$$

Integrating by parts this last relation leads to $\int_{\partial\Omega} \gamma_0 u \frac{\partial\varphi}{\partial n} d\Gamma = 0$. Since $f \geq 0$ then $\gamma_0 u \geq 0$,choosing φ such that $\frac{\partial\varphi}{\partial n} > 0$, we deduce that

$$\gamma_0 u = 0 \text{ on } \partial\Omega, \quad u \in W_0^{1,1}(\Omega),$$

where γ_0 denotes the trace operator and $\frac{\partial\varphi}{\partial n}$ is the normal trace of φ.

Therefore, one has

$$u \in W_0^1 L(\log L) = \left\{ \varphi \in W_0^{1,1}(\Omega), \int_{\Omega_*} |\nabla\phi|_{**}(t)\, dt < +\infty \right\}.$$

By using the result of Donaldson Trudinger, we know that the closure of $C_c^\infty(\Omega)$ with respectto the norm $\|\varphi\| = \int_{\Omega_*} |\nabla\phi|_{**}(t) dt = |\nabla\varphi|_{L(\log L)}$ is $W_0^1(L(\log L))$. Therefore, using relation (3)(consequence of Lemma (1.1.5)), we have

$$0 \leq \int_\Omega \frac{u}{\delta} dx \leq c_\Omega |\nabla u|_{L(\log L)} < +\infty.$$

But Theorem (1.1.7) implies $\int_\Omega \frac{u}{\delta} dx = +\infty$ with is a contradiction.

We have shown also that if $f \notin L^1(\Omega, \delta(1 + |\log\delta|)), f \geq 0$, then the very weak solution u of (1) satisfies $\int_\Omega |\nabla u||\log\delta| \, dx = +\infty$. We can recover such result using the sameargument as in Theorem (1.1.9) provided that we show the Hardy inequality given in Theorem (1.2.1). Asfar as we know such inequality has not been proved yet. Moreover, we can show that Theorem (1.2.1) yields relation (3).

Section (1.2): Non-Standard SobolevSpaces and Hardy Inequalities

We shall consider the following norm on $C_c^{0,1}(\Omega)$ (set of Lipschitz functions having compactsupport)

$$\|v\| = \int_\Omega |\nabla v|(1 + |\log \delta|)(x)\ dx$$

and we define the following Sobolev space $W_0^1(\Omega; 1 + |\log \delta|) = \overline{W_c^{1,1}}^{\|\cdot\|}$ as the closure of$W_c^{1,1}(\Omega) = \{v \in W^{1,1}(\Omega) \text{ with compact support}\}$, with respect to $\|\cdot\|$.

We note that $\|v\|$is equivalent to$\int_\Omega |\nabla v||\log \delta|\,dx$ and for $p > 1$

$$W_0^{1,p}(\Omega) \subset_> W_0^1(\Omega; L \log L) \subset_> W_0^1(\Omega; 1 + |\log \delta|),$$

where $\subset_>$ stands for continuous embedding.

One has

Theorem (1.2.1) [1]:

Assume that Ω is an open bounded Lipschitzian set of $\mathbb{R}^N$. Then there exists a constant $c_\Omega > 0$ such that

$$\int_\Omega \frac{|v(x)|}{\delta(x)}dx \le c_\Omega \int_\Omega |\nabla v|(1 + |\log \delta|)\,dx \equiv c_\Omega\|v\|, \ \ \forall v \in W_0^1(\Omega; 1 + |\log \delta|).$$

Proof:

Since the boundary $\partial\Omega$ Lipschitzian, we can decompose Ω as $\Omega = \Omega_0 \cup (\cup_{1\le i\le m} \Omega_i)$, with $\operatorname{dist}(\Omega_0, \partial\Omega) > 0$ and $(\bar{\Omega}_i)$ are a covering ofthe boundary.

Furthermore, there exist an open set $\mathcal{O}_i \subset \mathbb{R}^{N-1}$, a number $0 < \beta < 1$, a system of coordinates$(x_{i1}, \dots, x_{iN})$ for $i = 1, \dots, m$ and a family of Lipschitz functions $a_i: \mathcal{O}_i \to \mathbb{R}$ such that for anypoint $x \in \partial\Omega \cap \partial\Omega_i$ can be written as $x = (x_i', a_i(x_i'))$ and

$$\Omega_i = \{x = (x_i', x_{iN}) \text{with} x_i' \in \mathcal{O}_i,\ \ a_i(x_i') < x_{iN} < a_i(x_i') + \beta\}.$$

For $x \in \Omega_i$, the distance $\delta(x)$ to the boundary is equivalent to $x_{iN} - a_i(x_i')$.Let $\upsilon \in W_c^{1,1}(\Omega)$, for $i = 1, \dots, m, \upsilon \geq 0$, one has

$$\int_{\Omega_i} \frac{\upsilon(x)}{\delta(x)} dx \leq c_i \int_{\mathcal{O}_i} dx_i' \int_{a_i(x_i')}^{a_i(x_i')+\beta} \upsilon(x) \frac{\partial}{\partial x_{iN}} \log\big(x_{iN} - a_i(x_i')\big)\, dx_{iN}. \qquad (33)$$

Integrating by part this last relation, we have

$$\int_{a_i(x_i')}^{a_i(x_i')+\beta} \upsilon(x) \frac{\partial}{\partial x_{iN}} \log\big(x_{iN} - a_i(x_i')\big)\, dx_{iN}$$

$$= -\int_{a_i(x_i')}^{a_i(x_i')+\beta} \frac{\partial \upsilon}{\partial x_{iN}}(x) \log\big(x_{iN} - a_i(x_i')\big)\, dx_{iN} + \upsilon(x_i', a_i(x_i') + \beta) \log \beta$$

$$\leq -\int_{a_i(x_i')}^{a_i(x_i')+\beta} \frac{\partial \upsilon}{\partial x_{iN}}(x) \log\big(x_{iN} - a_i(x_i')\big)\, dx_{iN} \qquad (34)$$

(since $\upsilon \geq 0, \log \beta < 0$).

From relations (33) and (34), we derive

$$\int_{\Omega_i} \frac{\upsilon(x)}{\delta(x)} dx \leq c_{i1} \int_{\mathcal{O}_i} \int_{a_i(x_i')}^{a_i(x_i')+\beta} \left|\frac{\partial \upsilon}{\partial x_{iN}}\right| (x)(1 + |\log \delta(x)|) dx. \qquad (35)$$

From relation (35), we derive

$$\int_{\Omega_i} \frac{\upsilon(x)}{\delta(x)} dx \leq c_\Omega \int_{\Omega} |\nabla \upsilon|(x)\, (1 + |\log \delta(x)|) dx. \qquad (36)$$

For a signed $\upsilon \in W_c^{1,1}(\Omega)$, one has $|\upsilon| \in W_c^{1,1}(\Omega)$ and then relation (36) holds true since

$$|\nabla v(x)| = |\nabla|v(x)|| \text{a.e.}$$

Since $W_c^{1,1}(\Omega)$ is dense in $W_0^1(\Omega, 1 + |\log\delta|)$ we derive the result.

Corollary (1.2.2) [1]:

This is a Corollary of Theorem (1.2.1). Under the same assumption as for Theorem (1.2.1), there exists a constant $c_\Omega > 0$ such that

$$\int_\Omega \frac{|v(x)|}{\delta(x)} dx \leq c_\Omega \int_{\Omega_*} |\nabla v|_{**}(t) dt$$

$$\forall v \in W_0^1(\Omega; L(\log L)).$$

Proof:

If $v \in W_0^1(\Omega; L(\log L|))$, we have

$$\int_\Omega |\nabla v|(1 + |\log\delta|)\, dx \leq \|\nabla v\|_{L(\log L|)} \|1 + |\log\delta|\|_{L_{exp(\Omega)}},$$

and $1 + |\log\delta| \in L_{exp}(\Omega)$, we deduce the result.

One of the purpose of this paragraph is to show the relations (7) and (8). To do this, we recallthe following results and definition:

Definition (1.2.3) [1]:

Let $u \in L^1(\Omega), v \in L^1(\Omega)$. Then the quotient$\frac{(u+\lambda v)_* - u_*}{\lambda}$converges as $\lambda \to 0$ to a function denoted by v_{*u} in L^1-weak (that is for $\sigma\ (L^1; L^\infty)$-topology). Moreover, if $L(\Omega, \rho)$ is a Banach function space, with the norm ρ being rearrangement invariant, forsimplicity, we write

$$\rho(v_*) = \rho(v),$$

then

$$\rho(v_{*u}) \leq \rho(v) \text{whenever } v \in L(\Omega; \rho) \tag{37}$$

υ_{*u} is called the relative rearrangement of υ with respect to u.

Theorem (1.2.4) [1]:

Let $u \in W^{1,1}(\Omega), \bar{u} = u|_{B(x,r)}$, the restriction of u to $B(x,r) \subset \Omega$. Then

$$\underset{B(x,r)}{\operatorname{osc}} u \leq \frac{\alpha_N^{1-\frac{1}{N}}}{\alpha_{N-1}} \int_0^{\alpha_N r^N} s^{\frac{1}{N}-1}(|\nabla\bar{u}|_{*\bar{u}})_*(s)\, ds,$$

where α_m denotes the measure of the unit ball in $\mathbb{R}^m$, $|\nabla\bar{u}|_{*\bar{u}}$ is the relative rearrangement of$|\nabla\bar{u}|$ with respect to $\bar{u}$,

$$\underset{B(x,r)}{\operatorname{osc}} u = \underset{B(x,r)}{\operatorname{ess\,sup}}\, u - \underset{B(x,r)}{\operatorname{ess\,inf}}\, u.$$

We recall the following Zygmund spaces for $\alpha > 0$

$$L^{\alpha}_{exp}(\Omega) = \left\{\upsilon : \Omega \to \mathbb{R} \text{ measurable} \|\upsilon\|_{L^{\alpha}_{exp}(\Omega)} = \sup_{0<t\leq|\Omega|} \frac{|\upsilon|_*(t)}{\left(1 + \log\frac{|\Omega|}{t}\right)^{\alpha}} < +\infty\right\},$$

$$W^1L^{\alpha}_{exp}(\Omega) = \left\{\upsilon \in W^{1,1}(\Omega) : |\nabla\upsilon| \in L^{\alpha}_{exp}(\Omega)\right\},$$

$$W^1_0L^{\alpha}_{exp}(\Omega) = W^{1,1}_0(\Omega) \cap W^1L^{\alpha}_{exp}(\Omega).$$

Theorem (1.2.5) [1]:

Let Ω be an open bounded set of $\mathbb{R}^N, r > 0, x \in \Omega$ such that $B(x;r) \subset \Omega, u \in W^1L^{\alpha}_{exp}(\Omega)$. Then

$$\underset{B(x,r)}{\operatorname{osc}} u \leq \frac{\alpha_N^{1-\frac{1}{N}}}{\alpha_{N-1}} e^{\frac{1}{N}} N^{\alpha+1} |\Omega|^{\frac{1}{N}} \Gamma\big(\alpha + 1; \omega_N(r)\big) \|\nabla u\|_{L^{\alpha}_{exp}(\Omega)}$$

where $\Gamma(a;x) = \int_x^{+\infty} e^{-t} t^{a-1}\, dt$,

$$\omega_N(r) = \frac{1}{N}\left(1 - \log\left(\frac{\alpha_N}{|\Omega|} r^N\right)\right).$$

Proof:

One has

$$\int_0^{\alpha_N r^N} t^{\frac{1}{N}}(|\nabla\bar{u}|_{*\bar{u}})(t)\,\frac{dt}{t} \leq \int_0^{\alpha_N r^N} t^{\frac{1}{N}}|\nabla u|_*(t)\,\frac{dt}{t} \qquad (38)$$

We then have from Theorem (1.2.4)

$$\underset{B(x,r)}{\operatorname{osc}} u \leq \frac{\alpha_N^{1-\frac{1}{N}}}{\alpha_{N-1}}\|\nabla u\|_{L^\alpha_{exp}} \int_0^{\alpha_N r^N} t^{\frac{1}{N}}\left(1+\log\frac{|\Omega|}{t}\right)^\alpha \frac{dt}{t}.$$

By usual change of variables one has

$$\int_0^{\alpha_N r^N} t^{\frac{1}{N}}\left(1+\log\frac{|\Omega|}{t}\right)^\alpha \frac{dt}{t} = e^{\frac{1}{N}}N^{\alpha+1}|\Omega|^{\frac{1}{N}} \int_{\omega_N(r)}^{+\infty} e^{-t}t^\alpha dt,$$

with $\omega_N(r)$ as in the theorem. This ends the proof.

Corollary (1.2.6) [1]:

Under the same assumptions as for Theorem (1.2.5), one has for all$u \in W^1L^\alpha_{exp}(\Omega), B(x,r) \subset \Omega$

$$\underset{B(x,r)}{\operatorname{osc}} u \leq c_N(\alpha;|\Omega|)r(1+|\log r|)^\alpha\|\nabla u\|_{L^\alpha_{exp}(\Omega)}.$$

Proof:

By the asymptotic expansion of $\Gamma(a;x)$

$$\Gamma\,(a;\ x) \underset{x\to+\infty}{\sim} e^{-x}x^{a-1}.$$

Therefore,

$$\Gamma(\alpha+1;\omega_N(r)) \underset{r\to 0}{\sim} e^{-\omega_N(r)}\omega_N(r)^\alpha. \qquad (39)$$

Using Theorem (1.2.6), with relation (39) we deduce the result.

Corollary (1.2.7) [1]:

This is another Corollary of Theorem (1.2.5). Under the same assumption as for Theorem (1.2.5), one has for $u \in W_0^1 L_{exp}^{\alpha}(\Omega)$ and for all $x \in \bar{\Omega}$

$$|u(x)| \leq c_N(\Omega)\delta(x)(1 + |\log \delta\,(x)|)^{\alpha}\|\nabla u\|_{L_{exp}^{\alpha}},$$

where $c_N(\Omega)$ is constant depending only on N and Ω.

Proof:

Let $\bar{\Omega}$ be an open bounded set such that $\forall x \in \Omega, B(x; \mathrm{diam}(\Omega)) \subset \tilde{\Omega}$, where $\mathrm{diam}(\Omega)$is the diameter of Ω. For $u \in W_0^1 L_{exp}^{\alpha}(\Omega)$, we consider $\tilde{u}$ its extension by zero, we have $\tilde{u} \in W_0^1 L_{exp}^{\alpha}(\tilde{\Omega})$ and according to Corollary (1.2.6) of Theorem (1.2.5), $\forall x \in \Omega$

$$\operatorname*{osc}_{B(x,r)} \tilde{u} \leq c_N\left(\alpha, \left|\tilde{\Omega}\right|\right) r(1 + |\log r|)^{\alpha}\|\nabla \tilde{u}\|_{L_{exp}^{\alpha}}.$$

Since $B(x, \delta(x)) \subset \tilde{\Omega}, \delta(x) = \mathrm{dist}(x;\ \partial\Omega)$ for $x \in \Omega$, we deduce

$$\operatorname*{osc}_{B(x,\delta(x))} \tilde{u} \leq c_N(\Omega)\delta(x)(1 + |\log \delta(x)|)^{\alpha}\|\nabla \tilde{u}\|_{L_{exp}^{\alpha}}.$$

Thus

$$|u(x)| \leq \operatorname*{osc}_{B(x,\delta(x))} \tilde{u} \leq c_N(\Omega)\delta(x)(1 + |\log \delta(x)|)^{\alpha}\|\nabla u\|_{L_{exp}^{\alpha}}.$$

We shall provethat this behavior similar to Corollaries (1.2.6) and (1.2.7) can be obtained for

$$(\mathcal{L}_F)\begin{cases} -\Delta u = -\mathrm{div}(F), \\ u \in H_0^1(\Omega), \\ F \in L^{\infty}(\Omega)^N \end{cases}$$

under the assumption that Ω is of class C^2.

Now we can define the Lax–Milgram theorem (Let H be a Hilbert space and V a normed space. Let $B: H \times V \to R$ be a continuous, bilinear function. Then the following are equivalent:

(i) (coercivity) for some constant $c > 0$,

$$\inf_{\|v\|_v=1} \sup_{\|h\|_H\leq 1} |B(h,v)| \geq c;$$

(ii) (existence of a "weak inverse") for each continuous linear functional $f \in V^*$, there is an element $h \in H$ such that $B(h,v) = \langle f,v\rangle$ for all $v \in V$.) [6]

Note that from Lax–Milgram theorem the problem$(\mathcal{L}_F)$ has a unique solution.

To obtain the local behavior, we shall give a direct proof of the following theorem:

Theorem (1.2.8) [1]:

Let u be the unique solution of $(\mathcal{L}_F)$. Then there exists a constantc_Ω depending only on N and Ω such that for all $x \in \Omega$, all $r > 0$ with $B(x,5r) \subset \Omega$

$$\operatorname*{osc}_{B(x;r)} u \leq c_\Omega |F|_\infty r(1 + |\log r|).$$

Proof:

We first assume that $F \in C_c^\infty(\Omega)^N$. Let $x \in \Omega, r > 0$ such that $B(x;5r) \subset \Omega$ (we settemporarily $B(5r) = B(x;5r)$).

Let $x_j \in B(x;r), j = 1{,}2$ such that $u(x_1) = \sup_{\bar{B}(x;r)} u, u(x_2) = \inf_{\bar{B}(x;r)} u$. Using the Greenrepresentation of the solution u, one has

$$u(x_1) - u(x_2)$$
$$= \int_\Omega \left[\nabla_y G(x_1,y) - \nabla_y G(x_2,y)\right] \cdot F(y)dy. \qquad (40)$$

Thus, we derive from relation (40) that

$$0 \leq u(x_1) - u(x_2) \leq |F|_\infty (I_1 + I_2), \qquad (41)$$

where $I_j = \int_\Omega |\nabla_y G(x_j;y) - \nabla_y G(x;y)|\, dy, j = 1{,}2.$

We spilt each I_j as follows

$$I_j = I_{jr} + I_{jr}^c, \qquad I_{jr} = \int\limits_{B(5r)} \left|\nabla_y G(x_j; y) - \nabla_y G(x, y)\right| dy. \tag{42}$$

By the mean value theorem, one has

$$I_{jr}^c = \int\limits_{\Omega\backslash B(5r)} \left|\nabla_y G(x_j; y) - \nabla_y G(x; y)\right| dy$$

$$\leq |x - x_j| \int\limits_{\Omega\backslash B(5r)} \int\limits_0^1 \left|\nabla_{xy}^2 G\big(x + t(x_j - x); y\big)\right| dt dy. \tag{43}$$

Let us set $x_j(t) = x + t(x_j - x)$. Then it is known that

$$\left|\nabla_{xy}^2 G\big(x_j\,(t); y\big)\right| \leq c_{0N}(\Omega)\left|x_j(t) - y\right|^{-N}. \tag{44}$$

For $y \in \Omega\backslash B(5r)$, we have

$$\left|x_j(t) - y\right| \geq |x - y| - \left|x - x_j(t)\right|, \tag{45}$$

and

$$\left|x - x_j(t)\right| \leq \left|x_j - x\right| \leq r \leq \frac{1}{5}|x - y|.$$

Therefore, one has

$$\left|x_j(t) - y\right| \geq |x - y| - \frac{1}{5}|x - y| = \frac{4}{5}|x - y|. \tag{46}$$

From (43) to (46), then we have

$$I_{jr}^c \leq c_{rN}(\Omega)\left|x - x_j\right| \int\limits_{\Omega\backslash B(5r)} |x - y|^{-N} dy. \tag{47}$$

Since

$$\int_{\Omega\backslash B(5r)} |x-y|^N dy \leq c_{2N} \int_{5r}^{\mathrm{diam}(\Omega)} t^{-N} t^{N-1} dt \leq c_{3N}(1+|\log r|). \qquad (48)$$

From (47) and (48) we deduce

$$I_{jr}^C \leq c_{4N} r(1+|\log r|). \qquad (49)$$

While for the term I_{jr}, we bound it as follows

$$I_{jr} \leq \int_{B(5r)} \left|\nabla_y G(x,y)\right| dy + \int_{B(5r)} \left|\nabla_y G(x_j;y)\right| dy. \qquad (50)$$

Since$B(5r) \subset B(xj;5r+|x-xj|) \doteq Bj$, and $|\nabla_y G(x,y)| c_{5N}|x-y|^{1-N}$, we obtain fromrelation (50)

$$\begin{aligned} I_{jr} &\leq c_{5N} \int_{B(5r)} |x-y|^{1-N} dy + c_{5N} \int_{B_j} |x_j - y|^{1-N} dy \\ &\leq c_{6N} \int_0^{5r} t^{1-N} t^{N-1} dt + c_{6N} \int_0^{5r+|x_j - x|} dt \,, I_{jr} \leq c_{7N} r \\ &\leq c_{7N} r(1+|\log r|). \end{aligned} \qquad (51)$$

Combining relations (41), (42), (49) and (51), we derive

$$0 \leq u(x_1) - u(x_2) = \underset{B(x;r)}{\mathrm{osc}}\, u \leq c_{8N}(\Omega)|F|_\infty r(1+|\log r|). \qquad (52)$$

If $F \in L^\infty(\Omega)^N$, we consider $F_n \in C_c^\infty(\Omega)^N$ such that $|F_n|_\infty \leq |F|_\infty$ and $F_n(x) \xrightarrow[x\to+\infty]{} F(x)$a.e. Then the solution $u_n \in H_0^1(\Omega)$ if $-\Delta u_n = -\mathrm{div}(F_n)$ is such that $|\nabla u_n|_{L^p} \leq c_p |F_n|_{L^p}$ for any $p < +\infty$.

Therefore u_n converges uniformly to u solution of $(\mathcal{L}_F)$. Since relation (52) holds for u_n, wecan pass easily to the limit to conclude.

Arguing as in the preceding proof, the boundary growth of the solution can be obtained provided that $\partial\Omega$ satisfies for instance the regularity property given in relation (9) which is the caseif $\partial\Omega \in C^2$. Then we have

Theorem (1.2.9) [1]:

If Ω is a bounded open set of class C^2then there exists aconstant $c_\Omega > 0$ such that $\forall \bar{x} \in \partial\Omega$, one has $r(\bar{x}) > 0$ such that $\forall r < r(\bar{x}), \forall x \in B(\bar{x}; r) \cap \Omega$.

$$|u(x)| \leq c_\Omega |F|_\infty r(1 + |\log r|),$$

where u is the solution of $(\mathcal{L}_F)$.

The proof of this Theorem (1.2.9) as for the next one is similar to the one given for Theorem (1.1.8), wesketch only the proof of Theorem (1.2.10).

The above proof leads to the following behavior of u with respect to the distance function.

Theorem (1.2.10) [1]:

Assume that Ω is an open bounded set of class C^2. Then the unique solution u of$(\mathcal{L}_F)$ satisfies, there exists a constant $c_\Omega > 0$:

$$|u(x)| \leq c_\Omega |F|_\infty \delta(x)(1 + |\log \delta(x)|), \qquad \forall x \in \Omega.$$

Proof:

Let $x \in \Omega$ and $\bar{x} \in \partial\Omega$ such that $|x - \bar{x}| = \delta(x)$. Thus, the segment $]\bar{x}, x]$ is containedin Ω. We consider $r = \frac{\delta(x)}{2}, K(2r) = B(x, \delta(x)) \cap \Omega$. Then for $F \in C_c^\infty(\Omega)^N$

$$u(x) = \int_{\Omega\setminus K(2r)} [\nabla_y G(x; y) - \nabla_y G(\bar{x}, y)] F(y)\, dy$$

$$+ \int_{K(2r)} [\nabla_y G(x; y) - \nabla_y G(\bar{x}; y)]\, F(y) dy$$

$$\leq |F|_\infty (I_3 + I_4)$$

we have

$$|I_3| \leq c_{0N}|x-\bar{x}| \int_{\Omega \setminus K(2r)} |x(t)-y|^{-N} \text{ with } \; x(t) = x + t(\bar{x}-x), \qquad (53)$$

and $|x(t)-y| \geq \frac{1}{2}|x-y|$ arguing as in the proof of Theorem (1.2.8). Thus, relation (53) implies

$$|I_3| \leq c_{0N}\delta(x) \int_{\Omega \setminus K(2r)} |x-y|^{-N} dy \leq c_{0N}\delta(x) \int_{\delta(x)}^{\operatorname{diam}(\Omega)} \frac{dt}{t}$$
$$\leq c'_{0N}\delta(x)(1 + |\log \delta(x)|). \qquad (54)$$

While for I_4, one has

$$|I_4| \leq c \int_{B(x;\delta(x))} |x-y|^{1-N} dy + c \int_{B(\bar{x};\delta(x)+|x-\bar{x}|)} |\bar{x}-y|^{1-N} dy \leq c\delta(x). \qquad (55)$$

Thus, from (53) to (55),

$$|u(x)| \leq c_{\Omega}|F|_{\infty}\delta(x)(1 + |\log \delta(x)|). \qquad (56)$$

Thanks to the above theorem, we can weaken the regularity assumption for the domain Ω,supposed to be of class $C^{2,1}$, to obtain the existence of a very weak solution when $f \in L^1(\Omega; \delta(1 + |\log \delta|))$.

Corollary (1.2.11) [1]:

Let $f \in L^1(\Omega; \delta(1 + |\log \delta|))$ with Ω being an open bounded setof class C^2. Then there exists a unique solution $u \in W_0^{1,1}(\Omega)$ such that

$$\int_{\Omega} \nabla u \nabla \varphi dx = \int_{\Omega} f \varphi dx,$$

$\forall \varphi \in C^2(\bar{\Omega})$ with $\varphi = 0$on$\delta\Omega$.

Proof:

Let $f_k = \min(|f|; k)\,\mathrm{sign}(f)$ and $u_k \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u_k \nabla \varphi dx = \int_\Omega f_k \varphi dx, \qquad \forall \varphi \in H_0^1(\Omega). \tag{57}$$

Setting

$$F_k(x) = \begin{cases} \dfrac{\nabla u_k}{|\nabla u_k|} & \text{if } \nabla u_k \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

from Lax–Milgram theorem and regularity result we have a unique φ_k satisfying

$$\varphi_k \in \bigcap_{p<+\infty} W_0^{1,p}(\Omega), \int_\Omega \nabla \varphi_k \nabla \varphi dx = \int_\Omega F_k \nabla \varphi dx, \qquad \forall \varphi \in H_0^1(\Omega). \tag{58}$$

Then, taking φ_k as the test function in (57), we have

$$\int_\Omega f_k \varphi_k dx = \int_\Omega \nabla u_k \nabla \varphi_k dx = \int_\Omega |\nabla u_k| dx. \tag{59}$$

Using Theorem (1.2.10), we have

$$\begin{aligned} \int_\Omega |\nabla u_k| dx &\leq c_\Omega \left(\int_\Omega f_k \delta (1 + |\log \delta|) dx \right) |F_k|_\infty \\ &\leq c_\Omega \int_\Omega |f_k| \delta (1 \\ &+ |\log \delta|) dx. \end{aligned} \tag{60}$$

By linearity, we conclude that (u_k) is a Cauchy sequence in $W_0^{1,1}(\Omega)$. Thus, we have a solution $u$and it is the very weak solution (1).

As we stipulate in the introduction, we can improve Hardy inequality (3) by replacing$L(\log L)$ by $\mathcal{H}^1(\Omega)$.

Lemma (1.2.12) [1]:

Assume that Ω is a bounded open set of class C^2. Then

$$\log \delta \in \mathrm{BMO}(\mathbb{R}^N).$$

Proof:

The function δ belongs to the Muckhenouptclass A_p for all $p > 2$. Moreover, there exists a constant $c(\Omega, p)$ such that

$$\left(\frac{1}{|Q|}\int_Q \delta(x)dx\right)\left(\frac{1}{|Q|}\int_Q \delta(x)^{-\frac{1}{p-1}}dx\right)^{p-1} \leq c(\Omega, p) \text{ for any cube } Q \text{ in} \mathbb{R}^N. \quad (61)$$

But it is well known that the above estimate implies log δ belongs to the BMO set say

$$\frac{1}{|Q|}\int_Q \left|\log \delta - (\log \delta)_Q\right| dx \leq \log c(\Omega, p), \quad (62)$$

where $(\log \delta)_Q = \frac{1}{|Q|}\int_Q (\log \delta) dx$.

Corollary (1.2.13) [1]:

This is a Corollary of Lemma (1.2.12). Let φ_1 be the first eigenvalue of $-\Delta$ in Ω with Dirichlet boundarycondition, and define $w: \mathbb{R}^N \to \mathbb{R}$ by

$$w(x) = \begin{cases} \varphi_1(x) & \text{if } x \in \Omega, \\ \delta(x) & \text{if } x \notin \Omega. \end{cases}$$

Then $\log w \in \mathrm{BMO}(\mathbb{R}^N)$.

Proof:

We have two constants $c_0 > 0, c_1 > 0$ such that $c_0\delta(x) \leq \varphi_1(x) \leq c_1\delta(x)$. Therefore, wehave

$$\min(1; c_0)\delta(x) \leq w(x) \max(1; c_1)\delta(x). \quad (63)$$

From relation (61) one has $w \in A_p$ for all $p > 2$ and then $\log w \in \mathrm{BMO}(\mathbb{R}^N)$.

We shall use the following result on multiplication in $\mathrm{BMO}(\mathbb{R}^N)$.

Theorem (1.2.14) [1]:

Let $a \in \log \mathrm{VMO}(\mathbb{R}^N)$ (Vanishing Mean Oscillation withlogarithm rate) with a being constant outside of a ball $B(O, r_0)$. Then the multiplier operation$f \to af$ maps $\mathrm{BMO}(\mathbb{R}^N)$ into itself and is bounded.

As a consequence of the above results, we have:

Lemma (1.2.15) [1]:

There exists a constant $c(\Omega) > 0$ such that $\forall \psi \in C_c^\infty(\Omega)$ with$\int_\Omega \psi(x)dx = 0$, for$i = 1, \dots, N$, for all $k > 0$

(i) $|\int_\Omega \frac{\partial \varphi_1}{\partial x_i} \log \varphi_1 \, \psi dx| \leq c(\Omega)|\psi|_{\mathcal{H}^1(\Omega)}$, and

(ii)$|\int_\Omega \frac{\partial \varphi_1}{\partial x_i} T_k(\log \delta)\psi dx| \leq c(\Omega)|\psi|_{\mathcal{H}^1(\Omega)}$ where

$$T_k(\sigma) = \begin{cases} \sigma & \text{if} |\sigma| \leq k, \\ k \operatorname{sign}(\sigma) & \text{otherwise.} \end{cases}$$

Proof:

Since$\frac{\partial \varphi_1}{\partial x_i} \in C^1(\bar{\Omega})$, an extension theorem for Lipschitzian functions, allowsus to extend it to a function a in $C_b^{0,1}(\mathbb{R}^N)$ with $|a|_{C_b^{0,1}} = |\frac{\partial \varphi_1}{\partial x_i}|_{C^1(\bar{\Omega})}$. More precisely, for $x \in \mathbb{R}^N$

$$a(x) = \overline{\frac{\partial \varphi_1}{\partial x_i}}(x) = \operatorname{Max}\left\{\sup\left\{\frac{\partial \varphi_1}{\partial x_i}(y) - \left|\frac{\partial \varphi_1}{\partial x_i}\right|_{C^1(\bar{\Omega})} |x-y|; y \in \bar{\Omega}\right\}; \inf_{y \in \Omega} \frac{\partial \varphi_1}{\partial x_i}(y)\right\}$$

$$\in C_b^{0,1}(\mathbb{R}^N)$$

and

$$|a(x) - a(y)| \leq \left|\frac{\partial \varphi_1}{\partial x_i}\right|_{C^1(\bar{\Omega})} |x-y|, \qquad \forall x, y \in \mathbb{R}^N.$$

Therefore $a \in \log \mathrm{VMO}(\mathbb{R}^N)$. Let w be as in Corollary (1.2.13) of Lemma (1.2.12) i.e.

$$w(x) = \begin{cases} \varphi_1(x) & \text{if } x \in \Omega, \\ \delta(x) & \text{if } x \notin \Omega. \end{cases}$$

Then $\log w \in \mathrm{BMO}(\mathbb{R}^N)$ and $a \log w \in \mathrm{BMO}(\mathbb{R}^N)$.

Now, consider $\psi \in C_c^\infty(\Omega)$ with $\int_\Omega \psi(x)dx = 0$, then $\psi \in \mathcal{H}^1(\mathbb{R}^N)$. Therefore by the duality $\mathcal{H}^1(\mathbb{R}^N) - BMO(\mathbb{R}^N)$ one has

$$\left| \int_{\mathbb{R}^N} a(x) \log w(x) \psi(x) dx \right| \leq c |a \log w|_{\mathrm{BMO}} |\psi|_{\mathcal{H}^1}. \tag{64}$$

Since $\psi = 0$ on $\mathbb{R} \backslash \Omega$, we have

$$\int_\Omega \frac{\partial \varphi_1}{\partial x_i} \log \varphi_1 \psi \, dx = \int_{\mathbb{R}^N} a(x) \log w(x) \psi(x) dx. \tag{65}$$

From relations (64) and (65) we derive statement (i).

While for the second statement, we know from a property of BMO functions that $T_k(\log \delta) \in \mathrm{BMO}(\mathbb{R}^N)$ since $\log \delta \in \mathrm{BMO}$, therefore $aT_k(\log \delta) \in \mathrm{BMO}(\mathbb{R}^N)$. Moreover, we know that

$$|T_k(\log \delta)|_{\mathrm{BMO}} \leq 2 |\log \delta|_{\mathrm{BMO}}, \tag{66}$$

therefore, we have for $\psi \in C_c^\infty(\Omega)$ with $\int_\Omega \psi(x)\, dx = 0$,

$$\left| \int_{\mathbb{R}^N} a(x) T_k(\log \delta(x)) \psi(x) dx \right| \leq c |\psi|_{\mathcal{H}^1}. \tag{67}$$

Since

$$\int_\Omega \frac{\partial \varphi_1}{\partial x_i} T_k(\log \delta) \psi dx = \int_{\mathbb{R}^N} a T_k(\log \delta) \psi dx. \tag{68}$$

We derive the second statement.

As an application of Lemma (1.2.15), we shall prove:

Theorem (1.2.16) [1]:

Let $W^1_{0_+}\mathcal{H}^1(\Omega) = \overline{C^\infty_{c_+}(\Omega)}^{|||\cdot|||}$. Assume that Ω isan open bounded set of class C^2. Then there exists a constant $c > 0$ such that

$$\forall\psi \in W^1_{0_+}\mathcal{H}^1(\Omega), \int\limits_{\Omega} \frac{\psi}{\delta}(x)dx \leq c\sum_{i=1}^{N}\left|\frac{\partial\psi}{\partial x_i}\right|_{\mathcal{H}^1}$$

Proof:

Let $\psi \in W^1_{0_+}\mathcal{H}^1(\Omega)$. There exists a sequence $\psi_n \in C^\infty_c(\Omega), \psi_n \geq 0$ suchthat

$$\frac{\partial\psi_n}{\partial x_i} \to \frac{\partial\psi}{\partial x_i} \text{in} \mathcal{H}^1(\Omega) \text{ for } i = 1, \ldots, N$$

(due to the definition of $\mathcal{H}^1(\Omega)$). We set $v(x) = -\varphi_1 \log\varphi_1 + \varphi_1$ where φ_1 is the first eigenfunction of $-\Delta$ with the Dirichlet boundary condition. Thus $\nabla v(x) = -\nabla\varphi_1(\log\varphi_1)$ and$-\Delta v = \lambda_1\varphi_1(\log\varphi_1) - \frac{|\nabla\varphi_1|^2}{\varphi_1}$in Ω. Therefore, $v$satisfies the following equation in Ω

$$(\text{BVP})_0 \begin{cases} -\Delta v - \lambda_1 v = -\lambda_1\varphi_1 + \dfrac{|\nabla\varphi_1|^2}{\varphi_1} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Using ψ_n as a test function, we derive

$$\int\limits_{\Omega} \psi_n \frac{|\nabla\varphi_1|^2}{\varphi_1} dx = \sum_{i=0}^{N}\int\limits_{\Omega} \frac{\partial v}{\partial x_i}\frac{\psi_n}{\partial x_i}dx - \lambda_1\int\limits_{\Omega}(v - \varphi_1)\psi_n dx, \tag{69}$$

we have from Lemma (1.2.15),

$$\sum_{i=1}^{N}\int\limits_{\Omega} \frac{\partial v}{\partial x_i}\frac{\psi_n}{\partial x_i}dx = -\sum_{i=1}^{N}\int\limits_{\Omega}\frac{\partial\varphi_1}{\partial x_i}(\log\varphi_1)\frac{\partial\psi_n}{\partial x_i}dx \leq c\sum_{i=1}^{N}\left|\frac{\partial\psi_n}{\partial x_i}\right|_{\mathcal{H}^1}. \tag{70}$$

By the Sobolev–Poincaré inequality, we have

$$\lambda_1 \int_\Omega (\upsilon - \varphi_1)\psi_n \le c \int_\Omega |\psi_n| \le c \int_\Omega |\nabla\psi_n| \le c \sum_{i=1}^{N} \left|\frac{\partial \psi_n}{\partial x_i}\right|_{\mathcal{H}^1}. \tag{71}$$

From relation (69) to (71), we deduce letting $n \to +\infty$

$$\int_\Omega \psi \frac{|\nabla\varphi_1|^2}{\varphi_1} dx \le c \sum_{i=1}^{N} \left|\frac{\partial \psi}{\partial x_i}\right|_{\mathcal{H}^1}. \tag{72}$$

There exists a neighborhood of the boundary denoted Ω_0 such $\mathrm{Min}_{\overline{\Omega}_0}|\nabla\varphi_1|^2 > 0$ and$\mathrm{Min}_{\Omega\setminus\Omega_0}\delta(x) > 0$. Then

$$\int_{\Omega_0} \frac{\psi}{\delta} dx \le c \int_\Omega \psi \frac{|\nabla\varphi_1|^2}{\varphi_1} dx \le c \sum_{i=1}^{N} \left|\frac{\partial \psi}{\partial x_i}\right|_{\mathcal{H}^1}, \tag{73}$$

and

$$\int_{\Omega\setminus\Omega_0} \frac{\psi}{\delta} dx \le c \int_\Omega \psi dx \le c|\nabla\psi|_{L^1} \le c \sum_{i=1}^{N} \left|\frac{\partial \psi}{\partial x_i}\right|_{\mathcal{H}^1}. \tag{74}$$

Adding these two last equations we derive the result.

Theorem (1.2.17) [1]:

Let $f \in L^1(\Omega, \delta)\setminus L^1(\Omega; \delta(1 + |\log\delta|)), f \ge 0$. Thenthe unique very weak solution u of (1) satisfies

$$\sum_{i=1}^{N} \left|\frac{\partial u}{\partial x_i}\right|_{\mathcal{H}^1} = +\infty.$$

Proof:

Assume that$\sum_{i=1}^{N} \left|\frac{\partial u}{\partial x_i}\right|_{\mathcal{H}^1} < +\infty$, we have $u \in W_0^{1,1}(\Omega)$ (since $\mathcal{H}^1 \subset L^1$) and

$$\int_{\Omega} \nabla u \nabla \psi dx = \int_{\Omega} f \psi dx, \qquad \forall \psi \in W^{1,\infty}(\Omega) \cap H_0^1(\Omega).$$

We choose as a test function $\psi_k = \varphi_1 T_k(\log \delta) \in W^{1,\infty}(\Omega) \cap H_0^1(\Omega)$ since

$$\frac{\partial \psi_k}{\partial x_i} = \frac{\partial \varphi_1}{\partial x_i} T_k(\log \delta) + \frac{\varphi_1}{\delta} T_k'(\log \delta) \frac{\partial \delta}{\partial x_i} \in L^{\infty}(\Omega),$$

$$-\int_{\Omega} f \varphi_1 Tk(\log \delta) dx = -\sum_{i=1}^{N} \int_{\Omega} \frac{\partial u}{\partial x_i} \frac{\partial \psi_k}{\partial x_i} dx$$

$$= \sum_{i=1}^{N} - \int_{\Omega} \frac{\partial \varphi_1}{\partial x_i} T_k(\log \delta) \frac{\partial u}{\partial x_i} dx$$

$$+ \sum_{i=1}^{N} - \int_{\Omega} \frac{\varphi_1}{\delta} T_k'(\log \delta) \frac{\partial \delta}{\partial x_i} \frac{\partial u}{\partial x_i} dx. \qquad (75)$$

Applying Lemma (1.2.15), since $\frac{\partial u}{\partial x_i} \in \mathcal{H}^1(\Omega)$ (then there exists $\varphi_n \in C_c^{\infty}(\Omega)$ with $\int_{\Omega} \varphi_n(x) dx = 0$ such that $\varphi_n \to \frac{\partial u}{\partial x_i}$ in $\mathcal{H}^1$). We have

$$\int_{\Omega} \frac{\partial \varphi_1}{\partial x_i} T_k(\log \delta) \frac{\partial u}{\partial x_i} dx \leq c(\Omega) \left| \frac{\partial u}{\partial x_i} \right|_{\mathcal{H}^1}. \qquad (76)$$

Then from relations (75) and (76), we have

$$-\int_{\Omega} f\, \varphi_1 T_k(\log \delta) dx$$

$$\leq c \left[\sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right| dx + \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|_{\mathcal{H}^1} \right]. \qquad (77)$$

We know that $\int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right| dx \leq c \left| \frac{\partial u}{\partial x_i} \right|_{\mathcal{H}^1}$, therefore relation (77) implies

$$-\int_{\Omega} f\varphi_1 T_k(\log\delta)dx \le c\sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|_{\mathcal{H}^1}. \qquad (78)$$

Since

$$\int_{\Omega} f\varphi_1 |Tk(\log\delta)|dx = -\int_{\Omega} f\varphi_1 T_k(\log\delta)dx + 2\int_{\{\delta\ge 1\}} f\varphi_1 T_k(\log\delta)dx$$

$$\le c\sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|_{\mathcal{H}^1} + c\int_{\Omega} f\delta dx \text{ since } |\log\delta| \le M \text{ whenever } \delta \ge 1. \qquad (79)$$

Letting $k \to +\infty$ in relation (79), we deduce

$$\int_{\Omega} f\varphi_1 |\log\delta| dx \le c\sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|_{\mathcal{H}^1} + c\int_{\Omega} f\delta dx, \qquad (80)$$

which contradicts the fact that $\int_{\Omega} f\delta|\log\delta|dx = +\infty$.

Chapter 2

Second-Order Elliptic Operator and Optimal Hardy Weight

For a general subcritical second-order elliptic operator Pin a domain $\Omega \in \mathbb{R}^n$, we construct Hardyweight W which is optimal in the following sense. The operator $P - \lambda W$ is subcritical in Ω for all $\lambda < 1$, null-critical in Ω for $\lambda = 1$, and supercritical near any neighborhood of infinity in Ω for any $\lambda > 1$. Moreover, if P is symmetric and $W > 0$, then the spectrum and the essential spectrum of $W^{-1}P$ are equal to $[1, \infty)$, and the corresponding Agmon metric is complete. [2]

Section (2.1): Construction of Hardy-Weights

Let P be a symmetric and nonnegative second-order linear elliptic operatorwith real coefficients which is defined on a domain $\Omega \subset \mathbb{R}^n$ or on anoncompact manifold Ω, and let q be the associated quadratic form definedon $C_0^\infty(\Omega)$. A Hardy-type inequality with a weight$W \geq 0$ has the form

$$q(\varphi) \geq \lambda \int_\Omega W(x)|\varphi(x)|^2 dx \text{ for all } \varphi \in C_0^\infty(\Omega), \tag{1}$$

where $\lambda > 0$ is a constant. Such an inequality aims to quantify the positivity of P: for instance, if (1) holds with $W \equiv 1$ it means that the bottom of thespectrum of the corresponding operator is positive. A nonnegative operator$P$ is called critical in Ω if the inequality $P \geq 0$ cannot be improved, meaning that (1) holds true if and only if $W \equiv 0$. On the other hand, when (1)holds with a nontrivial W, then the operator is subcritical inΩ. Given asubcritical operator P in Ω, there is a huge set of weights W satisfying the inequality (1); We will call these weights, Hardy-weights. A naturalquestion is to find “large”Hardy-weights.

We can define Agmonmetric (Let $E \in \mathbb{C}$. The Agmon metric for $h_0(\varepsilon)$ at the energy E is defined by

$$g^E = \min(T, \inf\{|Im\ \varepsilon||\varepsilon \in S_T, h_0(\varepsilon) = E|\}))\ [7].$$

The search for Hardy-type inequalities with “as large as possible” weightfunction W was proposed by Agmon, and we feel that it deservesthe name Agmon’s problem. Agmon raised this problem in connection withhis theory of exponential decay for solutions of second-order elliptic equations.Given a Hardy-type inequality (1), there is an associated Agmonmetric; if this Riemannian metric turns out to be complete, then Agmon'stheory gives the exponential decay at infinity (with respect to the Agmonmetric) of solutions of the equation $Pu = f$.

Before proceeding, we recall a classical Hardy-type inequality, in order tomotivate the concept of "large" Hardy-weights:

Example (2.1.1)[2]:

For$\Omega = \mathbb{R}^n \backslash \{0\}, n \geq 3$, the following Hardy-type inequality for $P = -\Delta$ holds

$$\int_{\mathbb{R}^n \backslash \{0\}} |\nabla\varphi|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n \backslash \{0\}} \frac{|\varphi(x)|^2}{|x|^2} \qquad \forall\varphi \in C_0^\infty(\mathbb{R}^n \backslash \{0\}). \qquad (2)$$

In Example (2.1.1), the Hardy-weight W decays to zero at infinity and blowsup at zero, and furthermore its behavior is borderline for the Hardy-typeinequalities under consideration. Perhaps the easiest way to illustrate this is the following: for any $\varepsilon \in \mathbb{R}$, define a smooth positive weight W_ε which isequal to

$$W_\varepsilon \coloneqq |x|^{-2+\varepsilon}$$

outside the unit ball. If $\varepsilon < 0$, then W_ε is a short-range potential, while if$\varepsilon \geq 0$, then W_ε is long-range. More precisely, if $\varepsilon < 0$, then for any constant$C > 0$ there exists $R > 0$ such that

$$\int_{\mathbb{R}^n} |\nabla\varphi|^2 dx \geq C \int_{\mathbb{R}^n} W_\varepsilon(x)|\varphi|^2 dx \qquad \forall\varphi \in C_0^\infty(\{|x| > R\}), \qquad (3)$$

and the operator $W_\varepsilon^{-1}P$has a discrete positive spectrum. In particular, thecorresponding Rayleigh-Ritz variational problem admits a minimizer. On the other hand, for any $\varepsilon > 0$, there are no constants $C > 0$ and $R > 0$such that (3) holds true, and the bottom of the (essential) spectrum of theoperator $W_\varepsilon^{-1}P$equals 0. Therefore, W_0, which agrees with $|x|^{-2}$ outsidethe unit ball, is the only long-range potential in the family $\{W_\varepsilon\}_{\varepsilon\in\mathbb{R}}$ suchthat the Hardy-type inequality (1) holds. Moreover, $\lambda = C_H := (n-2)^2/4$ is the best constant for (1) not only in the punctured space, but in a fixedneighborhood of either zero or infinity. On the other hand, the correspondingRayleigh-Ritz variational problem does not admit a minimizer.

This indicates that the weight $C_H|x|^{-2}$ is a "large" Hardy-weight for $P = -\Delta$ on $\mathbb{R}^n\setminus\{0\}$.

Agmon's theory gives the following (almost optimal) a priori decay estimatesfor nongrowing solutions u of the Poisson equation in $\mathbb{R}^n$: for every$\varepsilon > 0$, there is a constant $C = C(f, \varepsilon)$ such that for every x outside the unitball,

$$|u(x)| \leq C|x|^{2-n-\varepsilon}.$$

We thus might expect that the construction of good Hardy-weights will leadto valuable spectral information about P.

In this chapter we use a general albeit simple construction of Hardy-weightswhich allows one to recover practically all classical Hardy inequalities in aunified way. We use this construction to study Agmon's problem. In particular,in some important cases we find an optimal Hardy weight; Thisincludes the case of a general nonselfadjoint operator P defined on a punctureddomain.

Our construction relies on two observations, which are both well known.First, usingAgmon-Allegretto-Piepenbrink (AAP) theory, we will see that there is a correspondence betweenpositive supersolutions of P and nonnegative Hardy-weights. Explicitly, toevery positive supersolution$v$ of P, we associate the weight $W := Pv/v$,which satisfies (1) with $\lambda = 1$. The second step (that we call the supersolutionconstruction) is a way of producing positive supersolutions of P −hence Hardy-weights. The construction is the following: let v_0 and v_1 betwo linearly independent positive (super)solutions of the equation $Pu = 0$in Ω. Then for $0 \leq \alpha \leq 1$, the function

$$v_\alpha \coloneqq v_0^{1-\alpha} v_1^{\alpha}$$

is a positive supersolution of the equation $Pu = 0$ in Ω, thus yielding aHardy-weight $W_\alpha \coloneqq Pv_\alpha/v_\alpha$. We will find that all these weights are proportional,

$$W_\alpha = 4\alpha(1-\alpha)W(v_0, v_1), \qquad W(v_0, v_1) = \frac{1}{4}\left|\nabla \log\left(\frac{v_0}{v_1}\right)\right|_A^2,$$

and the prefactor $4\alpha(1-\alpha)$ achieves its maximum 1 at $\alpha = 1/2$. In particular, if the equation $Pu = 0$ admits two linearly independent positive(super)solutions in Ω, then P is subcritical in Ω. Moreover, with the freedomof choosing v_0 and v_1, this construction allows us in fact, to recoverin a unified way all the classical Hardy inequalities. It is also a very easymethod for producing new examples.

We show that with a careful choice of v_0 and v_1, the preceding construction gives rise to Hardy-weights $W(v_0, v_1)$ which deserve the title of optimal weights.We first give a temporary definition of optimal weights.

Definition (2.1.2) [2]:

Consider a symmetric subcritical operator in Ω, and let Wbe a nonzero nonnegative weight satisfying the Hardy inequality

$$q(\varphi) \geq \lambda \int_{\Omega} W(x)|\varphi(x)|^2 dx \quad \text{for all} \quad \varphi \in C_0^{\infty}(\Omega), \qquad (4)$$

with $\lambda > 0$. We denote by $\lambda_0 = \lambda_0(P, W, \Omega)$the best constant satisfying (4); λ_0 is called the generalized principal eigenvalue. The weight $\lambda_0 W$ issaid to be an optimal Hardy-weight for the operator P in Ωif the followingproperties hold:

(i) The operator $P - \lambda_0 W$ is critical in Ω; that is, the inequality

$$q(\varphi) \geq \int_{\Omega} V(x)\varphi^2(x) dx \qquad \forall \varphi \in C_0^{\infty}(\Omega)$$

is not valid for any $V \gneqq \lambda_0 W$.

(ii) The constant λ_0 is also the best constant for (4) with test functionssupported in the exterior of any fixed compact set in Ω.

(iii) The operator $P - \lambda_0 W$ is null-critical in Ω; that is, the correspondingRayleigh-Ritz variational problem

$$\inf_{\varphi \in \mathcal{D}_P^{1,2}(\Omega)} \left\{ \frac{q(\varphi)}{\int_{\Omega} W(x)|\varphi(x)|^2 dx} \right\} \qquad (5)$$

admits no minimizer. Here $\mathcal{D}_P^{1,2}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ withrespect to the norm $u \mapsto \sqrt{q(u)}$.

Properties (ii) and (iii) indicates in a way that W is “long range”. Notethat contrary to the “short range” case, the validity of (i) in the case ofa “long range” potential is quite delicate. Indeed, it is known that $P - \lambda_0 W$ is always critical when property (ii) does not hold. On the other hand, in the “long range” case $P - \lambda_0 W$ is in generalsubcritical.

In order to motivate the definition, let us mention that the weight $C_H|x|^{-2}$of Example (2.1.1) is an optimal Hardy-weight.

Motivated byExample (2.1.1), we study in detail the case of a general (nonsymmetric) subcriticaloperator P in the punctured domain$\Omega^\star := \Omega\backslash\{0\}$: Theorem (2.1.16) states that if one chooses two positive solutions v_0, and v_1 appropriately in$\Omega^\star$, then for $\alpha = 1/2$, the corresponding weight $W(v_0, v_1)$ constructed by thesupersolution construction is an optimal Hardy-weight in $\Omega^\star$. The followingtheorem states the result for symmetric operators.

Theorem (2.1.3) [2]:

Let P be a symmetric subcritical operator in Ω, and let$G(x) := G_P^\Omega(x, 0)$ be its minimal positive Green’s function with a pole at$0 \in \Omega$. Let u be a positive solution of the equation $Pu = 0$ in Ωsatisfying

$$\lim_{x\to\infty} \frac{G(x)}{u(x)} = 0, \qquad (6)$$

where ∞ is the ideal point in the one-point compactification of Ω. Consider the supersolution$v := \sqrt{Gu}$. Then

$$W := \frac{Pv}{v} = \frac{1}{4}\left|\nabla \log\left(\frac{G}{u}\right)\right|_A^2$$

is an optimal Hardy-weight with respect to P and the punctured domain$\Omega^\star = \Omega\backslash\{0\}$. If furthermore $W > 0$, then the spectrum and the essentialspectrum of the Friedrichs extension of the operator $-W^{-1}\Delta$ on $L^2(\Omega, Wdx)$are equal to $[\lambda_0, \infty)$ and the corresponding Agmon metric is complete.

We assumethat $0 \in \Omega$ and denote $\Omega^\star \coloneqq \Omega \setminus \{0\}$. In addition, we fix a reference point$x_1 \in \Omega, x_1 \neq 0$. When there is no danger of confusion we will omit indices.In particular, for a matrix $A(x) = [a^{ij}(x)]$and a vector field $b(x)$ we denote

$$(A(x)\xi)^i = \sum_{j=1}^{n} a^{ij}(x)\,\xi_j, \quad b(x) \cdot \xi = \sum_{j=1}^{n} b^j(x)\xi_j\,, \text{ where } \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n.$$

Moreover, for $x \in \Omega$we introduce a norm on $\mathbb{R}^n$ associated to a positivedefinite symmetric matrix $A(x)$,

$$|\xi|_A^2 \coloneqq \xi \cdot A\xi.$$

We write $\Omega_1 \Subset \Omega_2$ if $\Omega_2$is open, $\overline{\Omega_1}$is compact and $\overline{\Omega_1} \subset \Omega_2$.

Before embarking to the general setting and proofs, we give a short proofof Theorem (2.1.3) for the case of the classical Hardy inequality (2). This willilluminate the main ideas and steps of the proof in the general case.

Example (2.1.4) [2]:

Let $P = -\Delta$ be the Laplace operatoron $\Omega^\star \coloneqq \mathbb{R}^n \setminus \{0\}$, where $n \geq 3$, and denote by $G(x) := |x|^{2-n}$ thecorresponding positive minimal Green function with a pole at zero (up to amultiplicative constant).

Consider the positive superharmonic function in $\Omega^\star$.

$$v(x) \coloneqq \sqrt{G(x)\mathbf{1}} = G(x)^{1/2} = |x|^{(2-n)/2}.$$

We obtain the Hardy-weight $Pv/v = C_H|x|^{-2}$, and by the (AAP) theory weget the classical Hardy inequality (2).

To prove that we indeed obtain an optimal Hardy-weight, we analyze theoscillatory properties of the corresponding radial equation

$$-u'' - \frac{n-1}{r}u' - \eta \frac{C_H}{r^2} u = 0 \qquad r \in (0, \infty), \tag{7}$$

where $\eta \in \mathbb{R}^n$. Note that (7) is Euler's equation. Consequently, for $\eta \neq 1$two linearly independent solutions of (7) are given by

$$u \pm (r) = r^{(2-n)/2}\left(r^{(2-n)/2}\right)^{\pm\sqrt{1-\eta}}, \tag{8}$$

while for $\eta = 1$ two linearly independent solutions of (7) are expressed by

$$u_+(r) = r^{(2-n)/2}, \qquad u_-(r) = r^{(2-n)/2}\log(r^{2-n}). \tag{9}$$

The difference in the structure of the solutions for $\eta < 1, \eta = 1$ and $\eta > 1$cannot be over-stressed.

For $\eta < 1$ both solutions are positive, and therefore, the operator$P - \eta C_H|x|^{-2}$is subcritical in $\Omega^\star$.

On the other hand, for $\eta = 1$ only $u_+(r) = r^{(2-n)/2}$ is positive, andmoreover, it is dominated by $|u_-|$ near both ends $r = 0$ and $r = \infty$. ByProposition (2.2.1) we infer that u_+ is a ground state and the operator $P - W$is critical in $\Omega^\star$, where $W := -\Delta(u_+)/u_+ = C_H|x|^{-2}$ is the correspondingHardy-weight.

Furthermore, an elementary calculation shows that for $\eta = 1$ we have thatthe ground state u_+ is not in $L^2(\Omega^\star, W\,dx)$, which shows the null-criticalityof the Hardy operator $-\Delta - C_H|x|^{-2}$ in $\Omega^\star$.

Finally, for $\eta > 1$ the solution of (7) given by

$$\Re e\{u_+(r)\} = r^{(2-n)/2}\cos\left[\frac{\sqrt{\eta-1}}{2}\log(r^{2-n})\right] \tag{10}$$

oscillates near zero and near infinity, and therefore, the best possible constantfor the validity of the Hardy inequality in any neighborhood of eitherthe origin or infinity is also C_H. In particular, the bottom of the spectrumand the bottom of the essential spectrum of the corresponding weightedLaplacian (with weight $W = C_H^{-1}|x|^2$) is equal 1.

The entire (essential) spectrum of the operator $\tilde{P} := C_H^{-1}|x|^2(-\Delta)$is obtainedby an explicit spectral representation of the operator $\tilde{P}$ restricted tothe radial functions, using the Mellin transform. Denote by $L^2_{\mathrm{rad}}(\Omega^\star, Wdx)$the subspace of radially symmetric functions in$L^2\,(\Omega^\star, Wdx)$. Recall thatthe Mellin transform $\mathcal{M}: L^2(0,\infty) \to L^2(\mathbb{R})$ is the unitary operator defined by

$$\mathcal{M}f(\xi) := \frac{1}{\sqrt{2\pi}}\int_0^\infty f(r)r^{\mathrm{i}\xi-\frac{1}{2}}dr.$$

In fact, the composition of the unitary operator

$$L^2\left((0,\infty), r^{n-1}\frac{C_H}{r^2}dr\right) \to L^2(0,\infty); \quad f(r) \to \frac{\sqrt{|n-2|}}{2}f\left(r^{1/(n-2)}\right),$$

and the Mellin transform, gives a unitary operator

$$\mathfrak{U}: L^2_{\mathrm{rad}}(\Omega^\star, Wdx) \cong L^2\left((0,\infty), r^{n-1}\frac{C_H}{r^2}dr\right) \to L^2(\mathbb{R}),$$

which is a spectral representation for $\tilde{P}$ restricted to radial functions. Inthis representation, $\tilde{P}$ is just the multiplication by $(1 + 4\xi^2)$. Indeed, thisfollows from the fact that due to (7) and (8) (with $\xi = \sqrt{\eta - 1}/2$), wehave

$$\left(C_H^{-1}|x|^2(-\Delta) - (4\xi^2 + 1)\right)(r^{n-2})^{\mathrm{i}\xi-\frac{1}{2}} = 0. \qquad (11)$$

The proof (Theorem (2.1.14)) in the general case is basedon similar considerations and calculations. Loosely speaking, to obtain thegeneral result, we just replace $r^{(2-n)}$ in equations (8), (9), (10), and(11) by G/u

We review the theory of positive solutions and formulateour main result for nonsymmetric operators defined on punctureddomains.

Let $\Omega \subset \mathbb{R}^n, n \geq 2$be a domain (or more generally, a smooth noncompactmanifold Ω of dimension n). We assume that ν is a positive measure on Ω, satisfying $d\nu = f$ vol with f a positive function; vol being the volume formof Ω(which is just the Lebesgue measure in the case of a domain of $\mathbb{R}^n$).Consider a second-order elliptic operator P with real coefficients which (inany coordinate system $(U; x_1, \ldots, x_n)$) is either of the form

$$Pu = -a^{ij}(x)\partial_i\partial_j u + b(x) \cdot \nabla u + c(x)u, \qquad (12)$$

or in the divergence form

$$Pu = -\mathrm{div}\left[\left(A(x)\nabla u + u\tilde{b}(x)\right)\right] + b(x) \cdot \nabla u + c(x)u, \qquad (13)$$

Here, the minus divergence is the formal adjoint of the gradient with respectto the measure ν. We assume that for every $x \in \Omega$the matrix $A(x) := [a^{ij}(x)]$is symmetric and that the real quadratic form

$$\xi \cdot A(x)\xi := \sum_{i,j=1}^{n} \xi_i a^{ij}(x)\xi_j \qquad \xi \in \mathbb{R}^n \tag{14}$$

is positive definite. Moreover, throughout the chapter it is assumed that$P$ is locally uniformly elliptic, and the coefficients of P are locally sufficientlyregular in Ω. All our results hold for example when P is of the form(13), and A, f are locally Hölder continuous,$b, \tilde{b} \in L^p_{\mathrm{loc}}(\Omega; \mathbb{R}^n, dx)$, and$c \in L^{p/2}_{\mathrm{loc}}(\Omega; \mathbb{R}, dx)$for some $p > n$. However it would be apparent fromthe proofs that any conditions that guarantee standard elliptic theory aresufficient.

The formal adjoint P^* of the operator P is defined on its natural space$L^2(\Omega, \mathrm{d}\nu)$. When P is in divergence form (13) and $b = \tilde{b}$, the operator

$$Pu = -\mathrm{div}[(A\nabla u + ub)] + b \cdot \nabla u + cu,$$

is symmetric in the space $L^2(\Omega, \mathrm{d}\nu)$. Throughout the chapter, we call thissetting the symmetric case. We note that if P is symmetric and b is smoothenough, then P is in fact a Schrödinger-type operator of the form

$$Pu = -\mathrm{div}(A\nabla u) + (c - \mathrm{div} b)u.$$

Definition (2.1.5) [2]:

Denote by $C_P(\Omega)$ the cone of all positive solutions of the elliptic equation $Pu = 0$ in Ω. The operator P is said to be nonnegative inΩ, and write $P \geq 0$ in Ω, if $C_P(\Omega) \neq \emptyset$. We say that P satisfies the positiveLiouville theorem in Ω if $\dim C_P(\Omega) = 1$.

For a nonzero (real valued) function W, let

$$\lambda_0 = \lambda_0(P, W, \Omega) := \sup\{\lambda \in \mathbb{R} | P - \lambda W \geq 0 \text{ in} \Omega\}$$

be the generalized principal eigenvalue of the operator P with respect to thepotential W in Ω. We also denote

$$\lambda_\infty := \lambda_\infty(P, W, \Omega) := \sup\{\lambda \in \mathbb{R} | \exists K \subset\subset \Omega \text{ s.t. } P - \lambda W \geq 0 \text{ in} \Omega \backslash \mathrm{K}\}.$$

Clearly, $\lambda_0 \leq \lambda_\infty$. Moreover, $P - \lambda_0 W \geq 0$ in Ω. If P is a symmetricoperator, then in light of the Agmon-Allegretto-Piepenbrink (AAP) theorem , λ_0 and λ_∞ have the following spectral interpretation:

Proposition (2.1.6) [2]:

Assume that the operator P is a symmetric in $L^2(\Omega, \mathrm{d}\nu)$, and $W > 0$. Suppose also that $\lambda_0(P, W, \Omega) > -\infty$. Define

$$\tilde{P} := W^{-1} P.$$

Then $\tilde{P}$ is symmetric on $L^2(\Omega, W\mathrm{d}\nu)$, has the same quadratic form as P,and λ_0 (resp. λ_∞) is the infimum of the spectrum (resp. essential spectrum)of the Friedrichs extension of $\tilde{P}$.

Denote by q the quadratic form associated to P, and assume that $P \geq 0$ inΩ. Then the following Hardy-type inequality holds true with the best constant$\lambda_0 = \lambda_0(P, W, \Omega) \geq 0$:

$$q(\varphi) \geq \lambda_0 \int_\Omega W \varphi^2 d\nu \qquad \forall \varphi \in C_0^\infty(\Omega). \qquad (15)$$

Next, we introduce the definition of (sub)criticality:

Definition (2.1.7) [2]:

Assume that $P \geq 0$ in Ω. The operator P is said to besubcritical in Ωif there exists a nonzero nonnegative continuous function Wsuch that $\lambda_0(P, W, \Omega) > 0$,otherwise, P is critical in Ω. So, in the criticalcase, $\lambda_0(P, W, \Omega) = 0$ for any nonnegative nonzero continuous function W.

If $P \ngeq 0$ in Ω, then P is said to be supercritical in Ω.

The (sub)criticality of P in Ω has an equivalent characterization in termsof the structure of the cone of positive solutions $\mathcal{C}_P(\Omega)$. This characterizationis based on the notion of positive solution of minimal growth, andit is a key to our theorems and proofs. We recall the definition.

Definition (2.1.8) [2]:

(i) Let $\Subset \Omega$, and let u be a positive solution of the equation $Pw = 0$ in $\Omega\setminus\mathrm{K}$. We say that u is a positive solution of minimalgrowth in a neighborhood of infinity in Ω if for any $K \Subset K' \Subset \Omega$with smoothboundary and any (regular) positive supersolution$v \in C\big((\Omega\setminus K') \cup \partial K'\big)$of the equation $Pw = 0$ in $\Omega\setminus\mathrm{K}'$satisfying $u \le v$on $\partial K'$, we have $u \le v$ in$\Omega\setminus\mathrm{K}'$.

(ii) Let $x_1 \in \Omega$. A positive solution of the equation

$$Pu = 0 \qquad \text{in}\,\Omega\setminus\{x_1\}$$

of minimal growth in a neighborhood of infinity in $\Omega$is called a positiveminimal Green function, if the singularity at x_1 is not removable. Theappropriately normalized Green's function is denoted by $G_P^{\Omega}(x, x_1)$.

The aforementioned characterization of a subcritical operator is given inthe following proposition.

Proposition (2.1.9) [2]:

Suppose that $P \ge 0$ in Ω. The operator P is subcriticalin Ω if and only if it admits a positive minimal Green function $G_P^{\Omega}(x, x_1)$in Ω. Moreover, in the critical case, the equation $Pu = 0$ admits a unique(up to multiplicative constant) positive global solution in Ω, which is calledAgmon's ground state (or in short a ground state).

The operator P is subcritical (resp. critical) in Ω if and only if its formaladjoint $P^{\star}$. is subcritical (resp. critical) in Ω.

We note that a ground state of a critical operator P in Ω is a positiveglobal solution of the equation $Pu = 0$ in Ωthat has minimal growth in aneighborhood of infinity in Ω.

Let P be subcritical in Ω and $W \gneqq 0$. Clearly $\lambda_0 \coloneqq \lambda_0(P, W, \Omega) \ge 0$,but λ_0 might be either 0 or positive. Moreover, the operator $P - \lambda W$ issubcritical in $\Omega$for $0 \le \lambda < \lambda_0$, but $P - \lambda_0 W$ might be either subcritical orcritical in Ω. The case of a perturbation by a compactly supported potentialis well understood. In particular, we have:

Proposition (2.1.10) [2]:

Let P be a subcritical operator in Ω and $W \geq 0$ a nonzerobounded compactly supported weight in Ω (or more generally, W is a semismallperturbation potential of the operator P in Ω). Then $\lambda_0(P, W, \Omega) > 0$.Moreover, the operator $P - \lambda W$ is critical in Ω for $\lambda = \lambda_0$, and subcriticalfor $0 \leq \lambda < \lambda_0$.

Next, we define null-criticality.

Definition (2.1.11) [2]:

We say that the operator $P - W$ is null-critical (resp.positive critical) in Ω with respect to the measure $Wd\nu$ if $P - W$ is criticalin Ω, and $\varphi_0\varphi_0^\star \notin L^1(\Omega, W\mathrm{d}\nu)$(resp. $\varphi_0\varphi_0^\star \in L^1(\Omega, W\mathrm{d}\nu)$, where φ_0, and$\varphi_0^\star$are the corresponding ground states of $P - W$ and $P^\star - W$ in Ω.

Positive criticality is closely related to the large time behavior of the heat kernel. Moreover, if P is symmetric, it is equivalent to theexistence of a minimizer for the corresponding variational problem. Indeed,let q be the quadratic form associated to a subcritical operator P in Ω.Consider the space $\mathcal{D}_P^{1,2}(\Omega)$, the completion of $C_0^\infty(\Omega)$ with respect to thenorm $u \mapsto \sqrt{q(u)}$. Since P is subcritical, we know that $\mathcal{D}_P^{1,2}(\Omega) \hookrightarrow W_{\mathrm{loc}}^{1,2}(\Omega)$ and $\lambda_0(P, W, \Omega)$is characterized by the Rayleigh-Ritz variationalproblem:

$$\lambda_0 = \inf_{u \in \mathcal{D}_P^{1,2}(\Omega)\setminus\{0\}} \frac{q(u)}{\int_\Omega u^2 W \mathrm{d}\nu}. \qquad (16)$$

We have

Lemma (2.1.12) [2]:

Assume that P is symmetric and $W > 0$ in Ω. Then $P - W$is positive-critical in Ω if and only if the infimum in the variational problem(16) is attained, and the infimum is equal 1. Furthermore, if it is the case,then the corresponding ground state φ_0 satisfies $\varphi_0 \in \mathcal{D}_P^{1,2}(\Omega)$, and realizesthe infimum uniquely (up to a multiplicative constant).

Finally, we define precisely what we mean by saying that W is “as largeas possible” weight function.

Definition (2.1.13) [2]:

Let P be a subcritical operator in Ω. A nonzero nonnegativefunction W is said to be an optimal Hardy-weight with respect to P andthe domain Ω if $P - W$ is null-critical in Ω, and for any $\lambda > 1$, the operator$P - \lambda W$ is supercritical in any neighborhood of infinity in Ω.

Let $V \in C_0^\infty(\mathbb{R}^n)$ be a potential such that the operator $-\Delta + V(x)$ iscritical in $\mathbb{R}^n$. Consider the operator $P := -\Delta + 1 + V(x)$, and the potential $W(x) := 1$. Then

$$\lambda_0(P, W, \mathbb{R}^n) = \lambda_\infty(P, W, \mathbb{R}^n) = 1$$

On the other hand, the operator $P - W$ is null-critical in $\mathbb{R}^n$ for $n \leq 4$, and positive-critical if $n > 4$.

The following theorem provides the existence of an optimal Hardy-weight.

Theorem (2.1.14) [2]:

Let P be a subcritical operator in Ω, and let $G(x) := G_P^\Omega(x, 0)$ be its minimal positive Green function with a pole at$0 \in \Omega$. Let u be a positive solution of the equation $Pu = 0$ in Ω satisfying

$$\lim_{x \to \infty} \frac{G(x)}{u(x)} = 0, \qquad (17)$$

where ∞ is the ideal point in the one-point compactification of Ω. Considerthe positive supersolution

$$v := \sqrt{Gu}$$

of the operator P in $\Omega^\star$. Then for the associated Hardy-weight

$$W := \frac{Pv}{v} = \frac{1}{4}\left|\nabla \log\left(\frac{G}{u}\right)\right|_A^2 \tag{18}$$

we have $\lambda_0(P, W, \Omega^\star) = 1$, and W is an optimal Hardy-weight with respectto P and the punctured domain $\Omega^\star$.

Assume further that P is a symmetric operator and W is positive in $\Omega^\star$,then the spectrum and the essential spectrum of the Friedrichs extension of the operator $W^{-1}P$ on $L^2(\Omega^\star, W\mathrm{d}\nu)$ is equal to $[1, \infty)$, and the correspondingAgmon metric

$$\mathrm{d}s^2 := W(x)\sum_{i,j=1}^{n} a_{ij}(x)\mathrm{d}x^i\mathrm{d}x^j\,, \quad \text{where}[a_{ij}] := [a^{ij}]^{-1}$$

is complete.

Remark (2.1.15) [2]:

(i) If P is a symmetric operator, or more generally if$G_P^\Omega(x, y) \asymp G_P^\Omega(y, x)$, then a global positive solution u satisfying (17) always exists.

(ii) If u_0, u_1 are two positive solutions of $Pu = 0$ near infinity in Ω suchthat

$$\lim_{x\to\infty} \frac{u_0(x)}{u_1(x)} = 0, \tag{19}$$

then u_0 is a positive solution of minimal growth in a neighborhood of infinityin Ω(see Proposition (2.2.1)). Therefore, in Theorem (2.1.14) we must take $u_0 = G$(the Green function) as a solution satisfying (19).

(iii) By the uniqueness of the ground state, it follows that $v = \sqrt{Gu}$ is the ground state of $P - W$ in $\Omega^\star$.

As a consequence of the criticality of $P - W$, we get the following positiveLiouville theorem:

Corollary (2.1.16) [2]:

Under the assumptions of Theorem (2.1.14), suppose that $\tilde{v}$is a positive supersolution of the equation $(P - W)w = 0$ in $\Omega^\star$. Then $\tilde{v}$ isactually a solution of the above equation, and is equal (up to a multiplicativeconstant) to $\sqrt{Gu}$.

We prove Theorem (2.1.14) in four steps, see theorems (2.2.2), (2.2.5), (2.2.10), and (2.2.14).

We recall a standard procedure to eliminatethe zero-order term of the operator P. Denote by $\mathcal{V}$ the space $C^{2,\alpha}_{\mathrm{loc}}(\Omega)$(resp. $W^{1,2}_{\mathrm{loc}}(\Omega)$) if P is of the form (12) (resp. (13)). Let $h \in \mathcal{V}$ be apositive continuous function and define a map

$$T_h : \mathcal{V} \to \mathcal{V}, \quad v \to \frac{v}{h}. \tag{20}$$

The operator $P_h \coloneqq T_h \circ P \circ T_h^{-1}$given more explicitly by

$$P_h u = \frac{P(hu)}{h} \tag{21}$$

is called the h-transform of P.

Fix $\varphi \in C_P(\Omega)$. Then the corresponding h-transform is called a groundstate transform. Clearly,

$$P_\varphi \mathbf{1} = 0.$$

Moreover, we have

Proposition (2.1.17) [2]:

Let $\varphi \in C_P(\Omega)$, and let P_φ bethe corresponding ground state transform. Then

$$\lambda_0(P_\varphi, W, \Omega) = \lambda_0(P, W, \Omega), \qquad \lambda_\infty(P_\varphi, W, \Omega) = \lambda_\infty(P, W, \Omega).$$

Moreover, P_φ. is subcritical in Ω if and only if P is subcritical in Ω.

The map $T_\varphi|_{\mathcal{V}\cap L^2(\Omega,\mathrm{d}\nu)}$extends to an isometry between $L^2(\Omega, \mathrm{d}\nu)$ and$L^2(\Omega, \varphi^2 \mathrm{d}\nu)$. In the symmetric case this implies that P and P_φ are unitaryequivalent.

Proof:

The map T_φ respects the structure of positive solutions,

$$T_\varphi C_P(\Omega) = C_{P_\varphi}(\Omega),$$

and preserves support of functions, namely $\operatorname{supp} v = \operatorname{supp} T_\varphi v$. The claimabout λ_0 and λ_∞ then follows from their definitions and Proposition (2.1.9).The last two claims about the isometry are standard. When P is symmetricit provides independent proof of the spectral claims of the proposition.

We note that in the subcritical case, the corresponding Green's functionsatisfies

$$G_{P_\varphi}^{\Omega}(x, y) = \frac{1}{\varphi(x)} G_P^{\Omega}(x, y)\varphi(y).$$

On the other hand, in the critical case (i) is the ground state of the equation$P_\varphi u = 0$ in Ω. In addition, if the operator P is symmetric, then

$$P_\varphi u = -\frac{1}{\varphi^2} \operatorname{div}(\varphi^2 A(x)\nabla u), \tag{22}$$

and P_φ is manifestly symmetric in $L^2(\Omega, \varphi^2 \mathrm{d}\nu)$.

Calculations are genuinely simplified after a ground state transform. Indeed, if $P\mathbf{1} = 0$, then

$$P(uv) = uP(v) - 2A\nabla u \cdot \nabla v + vP(u), \tag{23}$$

$$P\big(f(v)\big) = f'^{(v)} P(v) - f''(v)|\nabla v|_A^2, \tag{24}$$

holds for all functions $u, v \in \mathcal{V}$ and $f \in C^2(\mathbb{R})$.

The construction of the optimal Hardy-weight using the supersolutionmethod is based on the following simple observation

Lemma (2.1.18) [2]:

Let v_j be two positive solutions(resp. supersolutions) of the equation $Pu = 0, j = 0,1$, in a domain Ω, andlet $v \coloneqq v_1/v_0$. Then for any $0 \leq \alpha \leq 1$ the function

$$v_\alpha(x) \coloneqq \big(v_1(x)\big)^\alpha \big(v_0(x)\big)^{1-\alpha} = v^\alpha(x) v_0(x) \tag{25}$$

is a positive solution (resp. supersolution) of the equation

$$[P - 4\alpha(1-\alpha)W(x)]u = 0 \quad \text{in}\Omega, \tag{26}$$

where W is the Hardy-weight given by

$$W(x) := \frac{|\nabla v|_A^2}{4v^2} \geq 0. \tag{27}$$

In fact, v_j are linearly independent if and only if $W \neq 0$.

Optimizing (26) in α, we find for $\alpha = 1/2$:

Corollary (2.1.19) [2]:

The function $\sqrt{v_0 v_1}$ is a positive (super)solution of the equation

$$[P - W(x)]u = 0 \quad \text{in}\Omega.$$

In particular, $P - W \geq 0$ in Ω.

We call the above procedure the supersolution construction, and the correspondingpotential W is called a Hardy-weight. When v_j are positivesolutions it is often useful to apply the ground state transform with respectto v0. This h-transform maps the pair of solutions (v_0, v_1) of P to a pair of solutions $(\mathbf{1}, v_1/v_0)$ of the equation $P_{v_0}u = 0$. For example, (26) is thenobtained by applying (23) and (24) with $= P_{v_0}$, and $f(t) = t^\alpha$. Notethat the Hardy-weight W is unchanged under this ground state transform.

Remark (2.1.20) [2]:

Lemma (2.1.18) has a straightforward generalization to the casewhen v_j are positive (super)solutions of $(P - V_j)v_j = 0, j = 0,1$. In that case v_α is a (super)solution of the equation

$$[P + (1-\alpha)V_0 + \alpha V_1 - 4\alpha(1-\alpha)W]u = 0. \tag{28}$$

Example (2.1.21) [2]:

Suppose that $P = -\Delta$, and assume that Ω is a smoothbounded convex domain. Consider the function $v_0(x) := \delta(x) := \mathrm{dist}(x, \partial\Omega)$which due to the convexity is a positive superharmonic function in Ω, and let$v_1 := 1$. Then the associated weight $W(x) = \delta(x)^{-2}/4$ is the correspondingHardy-weight, and we get the well known Hardy inequality

$$\int_{\Omega} |\nabla\phi|^2 dx \geq \frac{1}{4}\int_{\Omega} \frac{|\phi|^2}{\delta(x)^2} dx \qquad \forall \phi \in C_0^{\infty}(\Omega). \tag{29}$$

It is known that the operator $-\Delta - W$ is subcritical in Ω, but

$$\lambda_0(-\Delta, \delta(x)^{-2}, \Omega) = \lambda_\infty(-\Delta, \delta(x)^{-2}, \Omega) = 1/4. \tag{30}$$

That is, $1/4$ is the best constant in the above inequality in a strong sense.In fact, (30) can be deduced from Theorem (2.2.4) (see Example (2.3.17)). Notealso, that if one takes instead the superharmonic function $v_0(x) = \delta(x)^{\beta}$with $0 < \beta < 1$, then one obtains the Hardy inequality without the bestconstant.

The supersolution construction can be generalized to the case of finitelymany positive supersolutions.

Proposition (2.1.22) [2]:

Suppose that $P \geq 0$ in Ω, and let $u_1, \ldots, u_N$ be positive (super)solutions of $Pv = 0$ in Ω. Let $\alpha_1, \ldots, \alpha_N$ be nonnegative numberssuch that$\sum_{i=1}^{N} \alpha_i = 1$.

Then

$$u \coloneqq \prod_{j=1}^{N} u_j^{\alpha_j} \tag{31}$$

is a positive supersolution of the equation $Pv = 0$ in Ω. Moreover, u is apositive (super)solution of $(P - W)_v = 0$ in Ω, where

$$W \coloneqq \sum_{i<j} \alpha_i \alpha_j \left|\nabla \log\left(\frac{u_i}{u_j}\right)\right|_A^2 .$$

Proof:

Consider the function u defined by (31). We compute that

$$Pu - \sum_{i=1}^{N} \alpha_i \frac{Pu_i}{u_i} u = \left(\sum_{i=1}^{N} \alpha_i (1-\alpha_i) \left| \frac{\nabla u_i}{u_i} \right|_A^2 - 2 \sum_{i<j} \alpha_i \alpha_j \langle A \frac{\nabla u_i}{u_i}, \frac{\nabla u_j}{u_j} \rangle \right) u$$

$$= \left(\sum_{i=1}^{N} \alpha_i (1-\alpha_i) \left| \frac{\nabla u_i}{u_i} \right|_A^2 + \sum_{i<j} \alpha_i \alpha_j \left[\left| \frac{\nabla u_i}{u_i} - \frac{\nabla u_j}{u_j} \right|_A^2 - \left| \frac{\nabla u_i}{u_i} \right|_A^2 - \left| \frac{\nabla u_j}{u_j} \right|_A^2 \right] \right) u$$

$$= \left(\sum_{i=1}^{N} \alpha_i \left(1 - \sum_{j=1}^{N} \alpha_j \right) \left| \frac{\nabla u_i}{u_i} \right|_A^2 + W \right) u = Wu,$$

since by hypothesis$\sum_{i=1}^{N} \alpha_i = 1$.

The supersolution construction given in Proposition (2.1.22) will be used when we study the case of a subcritical operator which is definedon a manifold with N ends, with $N \geq 2$.

Let us focus again on the case of two ends. Let W be the Hardy-weightgiven in Lemma (2.1.18) by (27). The set of solutions of the equation

$$(P - \lambda W)u = 0 \quad \text{in} \Omega$$

for $\lambda \in \mathbb{R}$ plays a crucial role throughout the article. Indeed, under theassumptions of Lemma (2.1.18), for $\lambda < 1$ the equation $P - \lambda W$ admits twopositive (super)solutions

$$v_{\alpha_\pm}(x) = \left(v_1(x)\right)^{\alpha_\pm} \left(v_0(x)\right)^{1-\alpha_\pm}, \text{where } \alpha_\pm := \frac{1 \pm \sqrt{1-\lambda}}{2}. \qquad (32)$$

At the maximum $\lambda = 1$ the construction gives a positive (super)solution $v_1/2$ of $(P - W)u = 0$. We obtain a second solution for $\lambda = 1$ by differentiating(26) with respect to the parameter α and substituting $\alpha = \frac{1}{2}$,

$$\partial_\alpha \{[P - 4\alpha(1-\alpha)W(x)]v_\alpha\}|_{\alpha=\frac{1}{2}} = (P - W)\left[\sqrt{v_0 v_1} \log\left(\frac{v_0}{v_1}\right)\right] = 0.$$

To avoid justification of the differentiating with respect to α, we give anindependent proof of this formula.

Lemma (2.1.23) [2]:

Assume that P is a subcritical operator in Ω. Let v_j be twolinearly independent positive solutions of the equation $Pu = 0$ in Ω, where$j = 0,1$. Let W be the associated Hardy-weight given by (27). Then theequation

$$(P - W)u = 0 \quad \text{in}\Omega, \tag{33}$$

admits a solution $w := \sqrt{v_0 v_1} \log\left(\frac{v_0}{v_1}\right)$.

Proof:

In light of the ground state transform with respect to the functionv_0, we may assume that $v_0 = 1$, and let us denote $v := v_1$. So, $P\mathbf{1} = Pv = 0$and, by the construction of W, $(P - W)v^{1/2} = 0$ in Ω. Then using (23)and (24) we obtain

$$P\left(v^{1/2} \log v\right) = P\left(v^{1/2}\right) \log v - 2A\nabla v^{1/2} \cdot \nabla \log v + v^{1/2} P(\log v)$$

$$= P\left(v^{1/2}\right) \log v + v^{1/2} \frac{1}{v} P(v)$$

$$= W v^{1/2} \log v.$$

Section (2.2): Null-Criticality and The Essential Spectrum

In the present section we show the first assertion of the main theorem (Theorem (2.1.14)). Namely, we show that under assumption (17), the operator $P - W$ is critical in $\Omega^\star$. We start with a preliminary result.

Proposition (2.2.1) [2]:

Let P be a second-order elliptic operator in and let u_0, u_1 be two positive solutions of $Pu = 0$ near infinity in such that

$$\lim_{x \to \infty} \frac{u_0(x)}{u_1(x)} = 0.$$

Then u_0 is a positive solution of minimal growth in a neighborhood of infinity in Ω.

Proof:

Let K be a smooth compact set in such that u_0 and u_1 are positive and continuous in $(\Omega\backslash K)\cup\partial K$, and are solutions of $Pu = 0$ in $\Omega\backslash K$. Let $\{\Omega_k\}$be an exhaustion of Ω, such that $K\subset\Omega_0$, and let w_k be the solution of the following Dirichlet problem:

$$\begin{cases} Pw_k = 0 & \text{in}\Omega_{\mathrm{k}}\backslash\mathrm{K}, \\ w_k(x) = u_0 & \text{on}\,\partial K, \\ w_k(x) = 0 & \text{on}\,\partial\Omega_k. \end{cases} \tag{34}$$

Then by the generalized maximum principle, $\{w_k\}_{k\in\mathbb{N}}$ is an increasing sequence of nonnegative functions, satisfying $w_k\leq u_0$, and therefore, converging to a positive solution w of $Pu = 0$ in $\Omega\backslash K$, that clearly has minimal growth at infinity in Ω. Thus, it is enough to show that $u_0 = w$ in $\Omega\backslash K$. We obviously have $w\leq u_0$. On the other hand, by hypothesis, if $\varepsilon > 0$, there is k_ε such that $u_0\leq\varepsilon u_1$ on $\partial\Omega_k$k, for every $k\geq k_\varepsilon$. By the generalized maximum principle, this implies that $u_0\leq w_k + \varepsilon u_1$ in $\Omega_{\mathrm{k}}\backslash K$ and it follows $u\leq w + \varepsilon u_1$ in $\Omega\backslash K$. By letting $\varepsilon\to 0$, we conclude that $u_0\leq w$. Thus, $u_0 = w$ in $\Omega\backslash K$.

We are ready to show the criticality statement of Theorem (2.1.14).

Theorem (2.2.2) [2]:

Under the hypotheses of Theorem (2.1.14), the operator $P - W$is critical in $\Omega^\star := \Omega\backslash\{0\}$, and has a ground state $\sqrt{Gu}$.

Proof:

By Corollary (2.1.19) and Lemma (2.1.23), the equation $(P - W)u = 0$ admits two solutions

$$u_0 = \sqrt{Gu}\text{and}u_1 = -\sqrt{Gu}\log\left(\frac{G}{u}\right).$$

By assumption (17), these solutions are positive near infinity and

$$\lim_{x\to\infty}\frac{u_0(x)}{u_1(x)}=0.$$

Proposition (2.2.1) then implies that u_0 is a positive solution of the equation $(P-W)u=0$ of minimal growth in a neighborhood of infinity in Ω. By the same argument and using the positive solution $-u_1$ in a neighborhood of zero, we conclude that u_0 has minimal growth in a neighborhood of zero. The second part of Lemma (2.3.6) implies now that u_0 has minimal growth at infinity in $\Omega^\star$. Therefore, u_0 is a ground state of $P-W$ in $\Omega^\star$, so, $P-W$is critical in $\Omega^\star$.

Alternative proof:

(i) Let $\alpha\in(0,1/2)$ and consider $v_\alpha:=G^\alpha u^{1-\alpha}$. Then v_α and $v_{(1-\alpha)}$ are positive solutions of $P-4\alpha(1-\alpha)W$ that satisfies

$$\frac{v_\alpha}{v_{(1-\alpha)}}=\left(\frac{G}{u}\right)^{2\alpha-1}.$$

Therefore, assumption (17) and the singularity of Green's function at 0imply

$$\lim_{x\to\infty}\frac{v_{(1-\alpha)}(x)}{v_\alpha(x)}=0\ \text{ and}\lim_{x\to 0}\frac{v_\alpha(x)}{v_{(1-\alpha)}(x)}=0.$$

Consequently, applying Proposition (2.2.1), we deduce that v_α has minimal growth at zero, and $v_{(1-\alpha)}$ has minimal growth at infinity (both for the operator $P-4\alpha(1-\alpha)W$). This implies that $\sqrt{G}u=\lim_{\alpha\to 1/2}v_\alpha=\lim_{\alpha\to 1/2}v_{(1-\alpha)}$ has minimal growth at zero and at infinity for $P-W$, as we explain now.

Indeed, let v be a positive supersolution for $P-W$ in a neighborhood of zero, that we assume for simplicity to be $B(0{,}1)\setminus\{0\}$. Then v is a positive supersolution of $P-4\alpha(1-\alpha)W$ in $B(0{,}1)\setminus\{0\}$ for $0\le\alpha\le 1$. Since on $\partial B(0{,}1)$, v and v_α are bounded above and below by positive constant that does not depend on α, we deduce that there is a constant C independent of αsuch that

$$v_\alpha\le Cv.$$

Letting $\alpha\to 1/2$, we deduce that

$$\sqrt{Gu} \leq Cv,$$

hence $\sqrt{Gu}$ has minimal growth at zero. The proof at infinity repeats the same argument with the solution $v_{(1-\alpha)}$.

Alternative proof:

(ii) Here we explain how to prove the criticality of $P - W$, using once more the log solution, but without the use of the notion of minimal growth. By performing a ground state transform with respect to u, we can assume that $u = 1$.

We need to prove that the operator $Q := P - W$ is a critical operator in Ω^*. Notice that the supersolution construction gives that $Q(G^{1/2}) = 0$on Ω^*, where G is the Green function for P with a pole 0. Let us perform a ground state transform for Q with respect to its positive solution $G^{1/2}$. We get a second-order elliptic operator .$\tilde{Q} := Q_{G^{1/2}}$. By Lemma (2.1.17), the operator $\tilde{Q}$ is critical in Ω^* if and only if Q is critical in Ω^*. By Lemma (2.1.23) we have,

$$\tilde{Q}(\log(G)) = 0 \quad \text{in} \Omega^\star.$$

So, in $\Omega^\star$, we have two solutions of the equation $\tilde{Q}u = 0$, namely 1 and $w := \log(G)$. Note that

$$\lim_{x\to\infty} w(x) = -\infty, \quad \lim_{x\to 0} w(x) = \infty,$$

where the first limit is due to our assumption (17).

We claim that this implies that $\tilde{Q}$ is critical in $\Omega^\star$.

Assume on the contrary that $\tilde{Q}$ is subcritical in $\Omega^\star$, and let $\tilde{G}(x) = G_{\tilde{Q}}^{\Omega^\star}(x, x_1)$ be the corresponding Green function with a pole at $x_1 \in \Omega^\star$. Let K be a compact annular domain around 0 containing x_1 such that $G(x) = M$ on the inner boundary and $G(x) = M^{-1}$ in the outer boundary, where $M > 1$ is a large positive number. So, $\Omega = K_0 \cup K \cup K_\infty$ where $K_0$is a neighborhood of 0, and K_∞ is a neighborhood of ∞.

By the minimality of $\tilde{G}$ and the fact that $\tilde{Q}\mathbf{1} = 0$, we have

$$\inf_{x\in\Omega^*} \tilde{G}(x) = 0.$$

Therefore, either $\liminf_{x\to 0} \tilde{G}(x) = 0$ or l$\liminf_{x\to\infty} \tilde{G}(x) = 0$. Suppose first that $\liminf_{x\to 0} \tilde{G}(x) = 0$, and let

$$D_k^0 := \{x \in K_0 | \, M < G(x) < k\}.$$

$D_k^0$is a union of open, relatively compact, connected sets in $\Omega^\star$, whose boundaries are contained in $\{x : G(x) = M\} \cup \{x : G(x) = k\}$. Furthermore, the sequence $\{D_k^0\}_{k\in\mathbb{N}}$ is increasing and is an exhaustion of $K_0 \backslash \{0\}$. Let v_k be the solution of the Dirichlet problem

$$\begin{cases} \tilde{Q}u = 0 & \text{in} D_k^0, \\ u(x) = 1 & \text{on } \partial D_k^0 \cap \{x : G(x) = M\}, \\ u(x) = 0 & \text{on } \partial D_k^0 \cap \{x : G(x) = k\}, \end{cases} \qquad (35)$$

Let $C > 0$ such that $\tilde{G} \geq C^{-1}$ on $\{x : G(x) = M\}$. Then by the maximum principle $0 \leq v_k \leq C\tilde{G}$. For k big enough, the set $\partial D_k^0 \cap \{x : G(x) = M\}$is independent of k, and by the maximum principle v_k is a bounded nondecreasing sequence, converging to a positive function v_0 which solves the equation $\tilde{Q}u = 0$ in $K_0 \backslash \{0\}$, and satisfies $v_0 \leq C\tilde{G}$ in $K_0 \backslash \{0\}$. On the other hand, we have an explicit formula for v_k:

$$v_k(x) = \frac{\log k - w(x)}{\log k - \log M}.$$

Hence $v_0 = 1$, and consequently $\tilde{G} \geq C^{-1}$ in $K_0 \backslash \{0\}$ which contradicts our assumption.

A similar argument shows that $\liminf_{x\to\infty} \tilde{G}(x) = 0$ cannot happen. Hence, we obtain a contradiction to our assumption that $\tilde{Q}$ is subcritical in $\Omega^\star$.

We present three proofs of Theorem (2.2.2). The shortest one uses the log solution for $P - W$, as well as the notion of minimal growth and is as follows:

Now, we prove that for any $\lambda > 1$ the equation $(P - \lambda W)u = 0$ does not admit any positive solution neither in any neighborhood of infinity in Ω, nor in any punctured neighborhood of 0.

We first state the following lemma.

Lemma (2.2.3) [2]:

Let v_j be two positive solutions of the equation $Pu = 0$, $j = 0,1$, in a domain Ω, and let $v := v_1/v_0$. Then for any $\lambda \in \mathbb{R}$ and $\alpha \in \mathbb{C}$ satisfying $\lambda = 4\alpha(1-\alpha)$, the function

$$v_\alpha(x) := \big(v_1(x)\big)^{\alpha}\big(v_0(x)\big)^{1-\alpha} = v^{\alpha}(x)v_0(x) \qquad (36)$$

is a solution of the equation

$$[P - \lambda W(x)]u = 0 \quad \text{in}\Omega, \qquad (37)$$

where

$$W(x) := \frac{|\nabla v|_A^2}{4v^2} \geq 0. \qquad (38)$$

We have the following theorem.

Theorem (2.2.4) [2]:

Under the assumptions of Theorem (2.1.14) we have

$$\lambda_\infty(P, W, \Omega) = \lambda_\infty(P, W, \Omega^\star) = 1.$$

More precisely, for any $\lambda > 1$ the equation $(P - \lambda W)u = 0$ does not admit any positive solution neither in any neighborhood of infinity in Ω, nor in any punctured neighborhood of 0.

Proof:

To simplify the notations we assume that $u = \mathbf{1}$ in the assumptions of Theorem (2.1.14) (in particular, $P\mathbf{1} = 0$ in Ω). The general case then follows by ground state transform (see Proposition (2.1.17)).

Fix $\lambda > 1$ and K a compact subset of Ω containing 0. We need to show that the operator $P - \lambda W$ cannot be nonnegative on $K^c := \Omega \setminus K$.

By Lemma (2.2.3), we have

$$(P - \lambda W)G^{\alpha} = 0 \quad \text{in} K^c,$$

where α is a complex number satisfying $4\alpha(1-\alpha) = \lambda$. Inverting the relation, we get that

$$(P - \lambda W)G^{\frac{1}{2}+\mathrm{i}\xi} = 0,$$

where

$$\xi := \frac{\sqrt{\lambda - 1}}{2}.$$

By taking the real part

$$\varphi := \Re\mathrm{e}\left(G^{\frac{1}{2}+\mathrm{i}\xi}\right)\cos(\xi \log(G)),$$

we obtain an oscillatory solution of the equation

$$(P - \lambda W)u = 0 \quad \text{in} K^c.$$

We claim that the existence of such an oscillatory solution φ implies that $P - \lambda W$ is supercritical in K^c (i.e. $P - \lambda \ngeq 0$ in K^c).

Indeed, since $\lim_{x\to\infty} G(x) = 0$, we can find a connected component U of the open, relatively compact set $\{x : 0 < a < G(x) < b\}$ contained in K^c, where a and b are chosen so that

$$\cos(\xi \log a) = \cos(\xi \log b) = 0,$$

and such that φ has a constant sign on U, for example $\varphi > 0$ on U. Then since φ vanishes on the boundary of U and is positive on U, it has a local maximum point in U. If the generalized maximum principle for $P - \lambda W$ would hold, we would deduce that φ is zero on U, which is a contradiction. Therefore, the generalized maximum principle for $P - \lambda W$ does not hold in K^c, and hence $P - \lambda W \ngeq 0$ in K^c. Since K is an arbitrary compact set containing 0, it follows that $P - \lambda W$ cannot admit a positive (super)solution in any neighborhood of infinity in Ω.

Similarly, one shows that for any $\lambda > 1$, the generalized maximum principle for $P - \lambda W$ does not hold in any punctured neighborhood of the origin.

The next result demonstrates that the asymptotic behavior of the constructed optimal Hardy-weight near 0 is exactly like the classical Hardy potential near the origin. Without loss of generality we may assume that the matrix $A = [a^{ij}]$at 0 is equal to the identity matrix.

Theorem (2.2.5) [2]:

Assume that $n \geq \mathbf{3}$, the coefficients of $\boldsymbol{P}$ are smooth enough near 0, and $a^{ij}(0) = \delta_{ij}$. Suppose further that the assumptions of Theorem (2.1.14) holds true. Then

$$\lim_{x\to 0}|x|^2 W(x) = C_H = \left(\frac{n-2}{2}\right)^2.$$

Proof:

It is well known that near the origin we have $G(x) \sim |x|^{n-2}$. Moreover, we know also the asymptotic near 0 of $|\nabla G(x)|$. Hence, an elementary calculation shows that

$$\lim_{x\to 0}\frac{|x|^2\left|\nabla\left(\frac{G}{u}\right)\right|^2}{4\left|\left(\frac{G}{u}\right)\right|^2} = C_H.$$

The next result demonstrates that if P is symmetric, Theorem (2.2.4) implies that the decay of the weight W near infinity is "optimal" in the following sense.

Corollary (2.2.6) [2]:

Suppose that P satisfies the assumptions of Theorem (2.1.14), and assume further that P is a symmetric operator. Then for every $\lambda > 1$and every locally regular potential $\widetilde{W}$ such that $\widetilde{W} = W$ outside a compact neighborhood of 0, the (Friedrichs extension of the) operator $P - \lambda\widetilde{W}$ has an infinite negative spectrum, in the sense that

$$\sup\{\dim(F) : F \subset \mathcal{D}_P^{1,2}(q), q|F < 0\} = \infty,$$

where q is the quadratic form associated to $P - \lambda\widetilde{W}$.

In particular, if

$$\lim_{x \to \infty} W(x) = 0,$$

then (the Friedrichs extension of) $P - \lambda\widetilde{W}$ has an infinite number of negative eigenvalues accumulating at zero.

Proof:

In view of $\lambda_\infty(P, W, \Omega) = \lambda_\infty(P, \widetilde{W}, \Omega)$, the first part follows from Theorem (2.2.4). In fact, $\lambda_\infty(P, \widetilde{W}, \Omega) = 1$implies that for every $\lambda > 1$, the operator $P - \lambda\widetilde{W}$ cannot be nonnegative in any neighborhood of infinity, that is $\lambda_\infty(P - \lambda\widetilde{W}, \mathbf{1}, \Omega) \leq 0$. For the sake of completeness, that the operator $P - \lambda\widetilde{W}$ has a finite negative spectrum if and only if $\lambda_\infty(P - \lambda\widetilde{W}, \mathbf{1}, \Omega) > 0$.

The negative spectrum is the union of the negative essential spectrum and the negative eigenvalues of finite multiplicities. However, under the condition $\lim_{x \to \infty} W(x) = 0$, the essential spectrum of the Friedrichs extension of $P - \lambda\widetilde{W}$ is contained in $[0, \infty)$.

Under the hypotheses of Theorem (2.1.14), we know (by Theorem (2.2.2)) that the operator $P - W$ is critical in $\Omega^\star$. Let φ_0 be the ground state of $P - W$, and φ_0^* be the ground state of $P^\star - W$ (which is also a critical operator in $\Omega^\star$). We study integrability properties of these ground states. In particular, if P is symmetric, we study whether the corresponding ground state belongs to $L^2(\Omega^\star, W\mathrm{d}v)$. Note that since φ_0 is continuous its integrability is determined by its behavior at infinity and zero.

Definition (2.2.7) [2]:

Assume that $P - W$ is critical in $\Omega^\star$, and let φ_0 and $\varphi_0^\star$ be the ground states of $P - W$, and $P^\star - W$, respectively. We say that $P - W$is null-critical at infinity if

$$\int_{\Omega \setminus K} \varphi_0(x)\varphi_0^\star(x) W \,\mathrm{d}v = \infty,$$

for (any) compact set K containing zero. Similarly, we define null-criticality at zero.

We have:

Theorem (2.2.8) [2]:

Under the assumptions of Theorem (2.1.14), the operator $P - W$ is null-critical at infinity and at zero.

Proof:

Recall that the explicit form of φ_0 is known. On the other hand, in contrast to the symmetric case, the explicit form of $\varphi_0$is unknown in the nonsymmetric case. Consequently, the proof is much subtler. Therefore, to illustrate the idea of the proof in the general case, we first present the proof in the symmetric case.

So, let us first assume that P is a symmetric operator. We assume as before that $P\mathbf{1} = 0$, the general case then follows by the ground state transform. Recall that for $\xi \geq 0$, the function

$$\varphi_\xi := G^{1/2} \cos(\xi \log(G))$$

solves the equation

$$(P - (4\xi^2 + 1)W)u = 0.$$

In particular $\varphi_0 = G^{1/2}$ is the ground state.

Define a set

$$\Omega_\xi := \left\{x : -\frac{\pi}{2\xi} < \log G(x) < 0\right\} \tag{39}$$

and consider the solutions $\varphi_\xi, \varphi_{3\xi}$. These solutions as formal eigenfunctions of a mixed value boundary problem on Ω_ξ lead to the following orthogonality relation

$$\int_{\Omega_\xi} \varphi_\xi \varphi_{3\xi} W \, \mathrm{d}v = 0. \tag{40}$$

Let us prove (40) in detail. Assume first that $\Omega_\xi$is regular enough, then we have the following Green formula for P:

$$\int_{\Omega_\xi} \left(P[\varphi_\xi]\varphi_{3\xi} - \varphi_\xi P[\varphi_{3\xi}]\right) \mathrm{d}v = \int_{\partial\Omega_\xi} \langle A\nabla[\varphi_\xi]\varphi_{3\xi} - A\nabla[\varphi_{3\xi}]\varphi_\xi, \vec{\sigma}\rangle \, d\sigma, \quad (41)$$

where $\mathrm{d}\sigma$ is the induced measure on $\partial\Omega_\xi$ and $\vec{\sigma}$ is the outward unit normal vector field on $\partial\Omega_\xi$. By construction, the functions φ_ξ, φ_ξ vanish on the set $\log G = -\pi/(2\xi)$. On the other hand, on the part of the boundary contained in $\{\log G = 0\}$ we have

$$\varphi_\zeta = 1 \quad \text{and} \nabla\varphi_\zeta = \nabla\varphi_0 \qquad (42)$$

for all ζ. It follows that the right hand side of the Green formula (41) vanishes. This establishes (40) since the left hand sides of (40) and (41) are nonzero multiple of each other.

For a nonregularΩ_ξ the claim follows by approximation of $\Omega_\xi$by regular domains.

Now, assume that $\varphi_0 \in L^2(\Omega \backslash K, W\mathrm{d}v)$ and note that

$$|\varphi_\xi| \leq \varphi_0$$

for all $\xi \geq 0$. Letting $\xi \to 0$ in (40), we conclude by the dominated convergence theorem that

$$\int_{\{G<1\}} \varphi_0^2 W \, \mathrm{d}v = 0,$$

which is a contradiction since $\varphi_0 > 0$ and $W \gneqq 0$ on $\{G < 1\}$. The proof of the null-criticality near zero is analogous.

The general case: The proof follows the same idea as above, but since an explicit formula for the ground state $\varphi_0^\star$ of the adjoint operator $P^\star - W$ is not available, we construct instead an approximating sequence for φ_0^*.

Consider the domain $\Omega_\xi$defined by (39), and let $\varphi_\xi^\star$ be the solution of the Dirichlet problem

$$\begin{cases} (P^\star - W)u = 0 & \text{in}\Omega_\xi, \\ u(x) = \varphi_0^\star & \text{on}\{\log G = 0\}, \\ u(x) = 0 & \text{on}\{\log G = -\pi/(2\xi)\}. \end{cases} \tag{43}$$

Since $P^\star - W$ is subcritical in Ω_ξ, the generalized maximum principle implies that $\varphi_\xi^\star$ is positive, $\varphi_\xi^\star \le \varphi_0^*$ on Ω_ξ, and the sequence $\{\varphi_\xi^*\}$ is increasing with respect to ξ.

Therefore, as $\xi \searrow 0$, we have $\varphi_\xi^\star \to \varphi^\star \le \varphi_0^\star$ locally uniformly in $\Omega^\star \setminus K$, where $K = \{G > 1\}$ is a neighborhood of zero, and φ^* is a nonnegative solution of the equation $(P^\star - W)u = 0$ in $\Omega \setminus K$. Since $\varphi_0^\star$ is a ground state of $P^\star - W$ in $\Omega^\star$, it has minimal growth at infinity of Ω, and hence $\varphi_0^\star \le \varphi^\star$. Thus, $\varphi^\star = \varphi_0^\star$, and we obtain

$$\lim_{\xi \searrow 0} \varphi_\xi^\star = \varphi_0^\star.$$

We use Green's formula for the operator $Q := P - W$:

$$\int_{\Omega_\xi} Q[u]\varphi_\xi^\star \,\mathrm{d}v = \int_{\Omega_\xi} \left(Q[u]\varphi_\xi^\star - uQ^\star\left[\varphi_\xi^\star\right]\right) \mathrm{d}v = B.T., \tag{44}$$

where u is either φ_ξ or $\varphi_{3\xi}$, and $B.T.$ is the corresponding boundary term. We claim that $B.T.$ is independent of the choice of either φ_ξ or $\varphi_{3\xi}$. Indeed, the claim readily follows from (42),(43), and the explicit form

$$B.T. = \int_{\{G=1\}} \langle A\nabla[\varphi_0]\varphi_0^\star - A\nabla\left[\varphi_\xi^\star\right]\varphi_0 + \mathbf{b}\varphi_0\varphi_0^\star - \tilde{\mathbf{b}}\varphi_0\varphi_0^\star, \vec{\sigma}\rangle \, d\sigma. \tag{45}$$

We have

$$\int_{\Omega_\xi} 4\xi^2 \varphi_\xi \varphi_\xi^\star W \,\mathrm{d}v = \int_{\Omega_\xi} Q\left[\varphi_\xi\right]\varphi_\xi^\star \,\mathrm{d}v = B.T.$$

$$= \int_{\Omega_\xi} Q\left[\varphi_{3\xi}\right]\varphi_\xi^\star \,\mathrm{d}v = \int_{\Omega_\xi} 4(3\xi)^2 \varphi_{3\xi}\varphi_\xi^\star W \,\mathrm{d}v.$$

Hence,

$$\int_{\Omega_\xi} \varphi_\xi \varphi_\xi^\star W \, \mathrm{d}v = 9 \int_{\Omega_\xi} \varphi_{3\xi} \varphi_\xi^\star W \, \mathrm{d}v.$$

Assuming that $\varphi_0 \varphi_0^\star W$ is v-integrable in$\Omega \backslash K$, we can pass to the limit $\xi \to 0$ and obtain the contradiction $1 = 9$. The case of a nonregular domain Ω_ξ can again be treated by approximations. The proof of null-criticality near zero is analogous.

Corollary (2.2.9) [2]:

Assume further that P is subcritical in Ω, symmetric in $L^2(\Omega, \mathrm{d}v)$, and $P\mathbf{1} = 0$. Then

$$\frac{|\nabla G|_A^2}{G} \tag{46}$$

is not v-integrable neither near 0 nor near infinity in Ω.

We assume that P is a subcritical symmetric operator defined on Ω. We continue our study of the supersolution construction with the pair (u, G), where $G(x) = G_P^\Omega(x, 0)$and u satisfy (17). Moreover, we assume that the corresponding (optimal) Hardy-weight W is strictly positive in $\Omega^\star$.

Recall that for any $\lambda > 1$, the function

$$\varphi_\xi := \varphi(\xi, x) = u \left(\frac{G}{u}\right)^{1/2} \exp(\mathrm{i}\xi \log(G/u)) \tag{47}$$

with $\xi = \pm\sqrt{\lambda - 1}/2$ solves the equation

$$(P - \lambda W)u = (P - (1 + 4\xi^2)W)u = 0 \quad \text{in}\Omega^\star.$$

So, for any $\lambda > 1$ the equation $(P - \lambda W)u = 0$ admits (at least two) "nongrowing" generalized eigenfunctions. Therefore, Šnol's principle (or Blochtype property) suggests that the spectrum σ and the essential spectrum σ_{ess} of $W^{-1}P$ in $L^2(\Omega^\star, W\mathrm{d}v)$ is equal to $[1, \infty)$. In fact, for such an operator P, we find an invariant subspace "spanned" by the functions φ_ξ on which P has a canonical form with purely absolutely continuous spectrum that is equal to $[1, \infty)$.

Define $\mathcal{U}_{\text{rad}}(\Omega^\star)$to be the space of measurable functions that are proportional to u on the level sets of G/u, and denote by $L^2_{\text{rad}}(\Omega^\star, W\mathrm{d}v)$ the space $L^2(\Omega^\star, W\mathrm{d}v) \cap \mathcal{U}_{\text{rad}}(\Omega^\star)$. Explicitly, $v \in \mathcal{U}_{\text{rad}}(\Omega^\star)$if and only if $v = uf(G/u)$ for some measurable function $f : (0, \infty) \to \mathbb{C}$.

Lemma (2.2.10) [2]:

Under the normalization $u(0) = 1$, the map

$$L^2_{\text{rad}}(\Omega^\star, W\mathrm{d}v) \to L^2\left((0,\infty), \frac{1}{4t^2}\mathrm{d}t\right),$$

$$v = uf(G/u) \mapsto f(t), \tag{48}$$

is an isometry.

Proof:

Assume first that P has smooth coefficients. Then by Sard's lemma, almost every point $t \in \mathbb{R}_+$ is a regular value of the function G/u, and hence for such points t, the set $\{G/u = t\}$ is a smooth $(n-1)$-dimensional submanifold. Note also that the function $W|\nabla(G/u)|^{-1}$ is smooth in $\Omega^\star$(see the computation below).

On the other hand, by Green's formula, for any smooth neighborhood $\widetilde{\Omega}$of 0, we have

$$\int_{\partial\widetilde{\Omega}} \langle uA\nabla G - GA\nabla u, \vec{\sigma}\rangle \,\mathrm{d}\sigma = \gamma, \tag{49}$$

where $\gamma = u(0) = 1$.

Consequently, the coarea formula and (49) imply that for any two functions in $L^2_{\text{rad}}(\Omega^\star, W\mathrm{d}v)$we have

$$\int_{\Omega^\star} uf\left(\frac{G}{u}\right)ug^*\left(\frac{G}{u}\right)W\,\mathrm{d}v = \int_{\Omega^\star} uf\left(\frac{G}{u}\right)ug^*\left(\frac{G}{u}\right)\frac{W}{|\nabla(G/u)|_A}\mathrm{d}v$$

$$= \int_0^\infty \mathrm{d}t \int_{\{G/u=t\}} f(t)g^*(t)u^2\frac{1}{4t^2}\langle A\nabla\left(\frac{G}{u}\right), \vec{\sigma}\rangle\,\mathrm{d}\sigma$$

$$= \int_0^\infty \mathrm{d}t\, f(t) g^*(t) \frac{1}{4t^2} \int_{\{G/u=t\}} \langle (uA\nabla G - GA\nabla u), \vec{\sigma} \rangle \,\mathrm{d}\sigma$$

$$= \int_0^\infty f(t) g^*(t) \frac{1}{4t^2} \mathrm{d}t, \tag{50}$$

where in passing from the second line to the third line of (50) we used the coarea formula, and that $\nabla(G/u)$ is parallel (in the metric $|\cdot|_A$) to the normal vector $\vec{\sigma}$ of the level set $\{G/u = t\}$, and therefore,

$$\frac{W}{|\nabla(G/u)|_A} = \frac{1}{4(G/u)^2} |\nabla(G/u)|_A = \frac{\langle A\nabla(G/u), \vec{\sigma} \rangle}{4t^2}.$$

Hence, in the smooth case we have the isometry

$$\int_{\Omega^\star} uf\left(\frac{G}{u}\right) ug^*\left(\frac{G}{u}\right) W \,\mathrm{d}v = \int_0^\infty f(t) g^*(t) \frac{1}{4t^2} \mathrm{d}t. \tag{51}$$

The regular case is obtained by a standard approximation argument (note that one may assume that $u = \mathbf{1}$).

In the sequel of the present section, we assume that the positive solution u is normalized so that $u(0) = 1$.

Before proceeding with the study of the essential spectrum we note that the proof of Lemma (2.2.10) implies the following corollary, which allows us to estimate in average the potential W, and provides (in the symmetric case) an alternative proof of the null-criticality of the operator $P - W$ near 0 and ∞.

Corollary (2.2.11) [2]:

Suppose that the hypotheses of Theorem (2.1.3) are satisfied, and that $u(0) = 1$. Then for any $0 < a < b$ and $\xi \in \mathbb{R}$ we have

$$\int_{\left\{a \le \frac{G}{u} \le b\right\}} uGW \,\mathrm{d}v = \int_{\left\{a \le \frac{G}{u} \le b\right\}} |\varphi_\xi|^2 W \,\mathrm{d}v = \frac{1}{4}(\log b - \log a). \tag{52}$$

Proof:

As in (50), we use the coarea formula on the domain $\left\{a \le \frac{G}{u} \le b\right\}$ (instead of the domain $\Omega^\star$) with the functions $f(x) = x$ and $g(x) = \mathbf{1}$, to obtain

$$\int_{\left\{a \le \frac{G}{u} \le b\right\}} uGW \, \mathrm{d}v = \frac{1}{4}\int_a^b t^{-1} \, \mathrm{d}t = \frac{1}{4}(\log b - \log a).$$

Theorem (2.2.12) [2]:

Suppose that the hypotheses of Theorem (2.1.3) are satisfied, and $W > 0$ in $\Omega^\star$. Then the spectrum σ and the essential spectrum σ_{ess} of (the Friedrichs extension of) $\tilde{P} := W^{-1}P$ acting on $L^2(\Omega^\star, W\mathrm{d}v)$satisfy

$$\sigma(\tilde{P}, \Omega^\star) = \sigma_{\mathrm{ess}}(\tilde{P}, \Omega^\star) = [1, \infty).$$

In fact, the spectrum of $\tilde{P}$ restricted to $L^2_{\mathrm{rad}}(\Omega^\star, W\mathrm{d}v)$ is purely absolutely continuous with respect to the Lebesgue measure.

Moreover, for any neighborhood $U \subset \Omega^\star$of 0 or infinity of Ω, the (essential) spectrum of the Friedrichs extension of the operator $\tilde{P}$ on $L^2(U, W\mathrm{d}v)$satisfies

$$\sigma(\tilde{P}, U) = \sigma_{\mathrm{ess}}(\tilde{P}, U) = [1, \infty).$$

Proof:

Using formulas (23) and (24) we find that

$$\frac{1}{W} P\big(uf(G/u)\big) = -4uf''(G/u)\left(\frac{G}{u}\right)^2. \qquad (53)$$

This proves that $L^2_{\mathrm{rad}}(\Omega^\star, W\mathrm{d}v)$is an invariant subspace of $\tilde{P}$, and the operator restricted to this subspace is unitarily equivalent to the symmetric operator

$$D : L^2\left((0, \infty), \frac{1}{4t^2}\mathrm{d}t\right) \to L^2\left((0, \infty), \frac{1}{4t^2}\mathrm{d}t\right)$$

defined by

$$(Df)(t) := -4t^2 f''(t). \tag{54}$$

The spectral representation of D, in terms of the Mellin transform (with $n = 1$), has been derived in Section (2.1) (see in particular, (11)). More explicitly, it is the composition of the Mellin transform with the isometry from $L^2\left((0,\infty), \frac{1}{4t^2}\mathrm{d}t\right)$to $L^2\big((0,\infty), \mathrm{d}t\big)$, which is given by

$$f(t) \mapsto \frac{1}{2} f\left(\frac{1}{t}\right). \tag{55}$$

It follows

$$\sigma\big(D, (0,\infty)\big) = \sigma_{\mathrm{ac}}\big(D, (0,\infty)\big) = [1, \infty). \tag{56}$$

Recall that by Theorems (2.2.2) and (2.2.5) we have

$$\sigma\big(\tilde{P}, \Omega^\star\big) = \sigma_{\mathrm{ess}}\big(\tilde{P}, \Omega^\star\big) \subset [1, \infty).$$

Therefore, (57) implies that

$$\sigma\big(\tilde{P}, \Omega^\star\big) = \sigma_{\mathrm{ess}}\big(\tilde{P}, \Omega^\star\big) = [1, \infty).$$

It remains to explain why we can localize the spectral result at a neighborhood $U \subset \Omega^\star$ of either 0 or infinity of Ω.

It is not difficult to check using the above results that $\tilde{P}$ on $L^2_{\mathrm{rad}}(\Omega^\star, W\mathrm{d}v)$is unitarily equivalent to the operator

$$\widetilde{D}f = -4(t^2 f')' \text{defined on} L^2\big((0,\infty), \mathrm{d}t\big).$$

Moreover, a neighborhood of 0 (resp. of ∞) in $\Omega^\star$ corresponds to a neighborhood of 0 (resp. of ∞) in $(0, \infty)$.

Therefore, it is enough to prove that the essential spectrum of $\widetilde{D}$ restricted to a neighborhood of 0 or ∞ in $(0,\infty)$ is $[1,\infty)$. First, we know that the essential spectrum is preserved under compactly supported perturbation, and this implies that $\sigma_{\mathrm{ess}}\left(\widetilde{D}, (0,\infty)\right)$ is equal to the union of $\sigma_{\mathrm{ess}}\big(\widetilde{D}, U_0\big)$ and $\sigma_{\mathrm{ess}}\big(\widetilde{D}, U_\infty\big)$, where

U_0 (resp. U_∞) is any neighborhood of 0 (resp. ∞) in $(0, \infty)$. Let U_0 be a neighborhood of 0, and define U_∞ to be the neighborhood of ∞ obtained from U_0 by the transformation $t \mapsto \frac{1}{t}$. Consider the following isometry T between $L^2(U_0, \mathrm{d}t)$ and $L^2(U_\infty, \mathrm{d}t)$ given by

$$Tf(t) = \frac{1}{t} f\left(\frac{1}{t}\right).$$

A computation shows that

$$T\widetilde{D} = \widetilde{D}T,$$

and this implies that the essential spectrum of $\tilde{P}$ restricted to U_0 is equal to the essential spectrum of $\tilde{P}$ restricted to U_∞. Since the union of these two essential spectra is $[1, \infty)$, we get that each one is equal to $[1, \infty)$.

Collecting the transformations (48),(54), and (55), we obtain a spectral representation of $\tilde{P} = W^{-1}P$ restricted to $L^2_{\mathrm{rad}}(\Omega^\star, W\mathrm{d}v)$.

We provide below a more detailed and explicit construction of the above transform $\mathcal{F}$ using methods related to classical Fourier transform. This also gives independent proof of Theorem (2.2.12).

Alternative proof:

The idea is to find a spectral representation of $\tilde{P}$ restricted to $L^2_{\mathrm{rad}}(\Omega^\star, W\mathrm{d}v)$, that is a unitary operator

$$U : L^2_{\mathrm{rad}}(\Omega^\star, W\mathrm{d}v) \longmapsto L^2(\mathbb{R})$$

such that $U\tilde{P}U^{-1}$ is the multiplication by a real function with values in $[1, \infty)$. Since the ground state transform is unitary, we may assume that $u = 1$. For the sake of brevity, we will denote $\mathcal{F}f(\xi)$ by $\hat{f}(\xi)$. We thus have to prove that for every $f \in C_0^\infty(\Omega^\star)$which is constant on the level sets of G, the following two identities hold

$$\int_{\Omega^\star} |f|^2 W \, \mathrm{d}v = \int_{\mathbb{R}} \left|\hat{f}\right|^2 W \, \mathrm{d}\xi \quad \text{(Plancherel – type formula)} \qquad (57)$$

and

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{\mathbb{R}} \hat{f}(\xi) \varphi(-\xi, x) \, \mathrm{d}\xi \quad \forall x \in \Omega^\star \text{(the inversion formula).} \qquad (58)$$

For a fixed $r > 0$, we define $\Omega(r)$ to be the open, relatively compact set

$$\Omega(r) := \{-r\pi < \log(G) < r\pi\},$$

and for any $k \in \mathbb{Z}$, we denote

$$\varphi_k^r(x) := \varphi\left(\frac{k}{r}, x\right) = \sqrt{G} \exp\left(\mathrm{i} \frac{k}{r} \log(G)\right) \quad x \in \Omega(r).$$

Consider the "torus"$\mathbf{T}_r$ to be the closure of $\Omega(r)$ divided by the equivalence relation

$$x \equiv y \Leftrightarrow \log G(x) = \log G(y) \bmod (2\pi r).$$

The set of complex valued continues functions $C(\mathbf{T}_r; \mathbb{C})$ can be identified to the set of complex valued continuous functions on the closure of $\Omega(r)$, each of which is constant on the level sets of G, and its value on the set $\{\log G = -\pi r\}$ is equal to its value on the set $\{\log G = \pi r\}$. In particular, for every $k \in \mathbb{Z}$, we have $\exp\left(i \frac{k}{r} \log G\right) \in C(\mathbf{T}_r; \mathbb{C})$. We also define the space $L^2_{\text{rad}}(\mathbf{T}_r; \mathbb{C})$, with the induced measure from Ω_r. We want to decompose the elements of $L^2_{\text{rad}}(\Omega(r); \mathbb{C})$ "Fourier series" with respect to the family $\{\varphi_k^r\}_{k \in \mathbb{Z}}$. First, we check the orthonormality.

Corollary (2.2.13) [2]:

The operator $\mathcal{F}$ given by

$$\mathcal{F}f(\xi) := \sqrt{\frac{2}{\pi}} \int_{\Omega^\star} f(x) \varphi(\xi, x) W(x) \, \mathrm{d}v(x) \quad \xi \in \mathbb{R}, \qquad (59)$$

(where $\varphi(\xi, x)$ is defined by (47)) is a well defined unitary operator from $L^2_{\mathrm{rad}}(\Omega^\star, W\mathrm{d}v)$ onto $L^2(\mathbb{R}, \mathrm{d}\xi)$, whose inverse is given by

$$\mathcal{F}^{-1}g(x) = \sqrt{\frac{2}{\pi}}\int_{\mathbb{R}} g(\xi)\varphi(-\xi, x)\,\mathrm{d}\xi.$$

Furthermore,

$$\mathcal{F}\frac{1}{W}P\mathcal{F}^{-1}f(\xi) = (1 + 4\xi^2)f(\xi).$$

Lemma (2.2.14) [2]:

For any $r > 0$ it holds

$$\frac{2}{\pi r}\int_{\Omega(r)} \varphi_k^r\overline{\varphi_l^r}W\,\mathrm{d}v = \delta_{k,l} \quad \forall k, l \in \mathbb{Z}.$$

Proof:

Notice that $\overline{\varphi_l^r} = \varphi_{-l}^r$. If $k \neq l$ and $k \neq -l$, then $\varphi_k^r$and $\varphi_l^r$are generalized eigenfunctions of P with different associated eigenvalues, and to prove their orthogonality we need to establish the identity

$$\int_{\Omega(r)} (P[\varphi_k^r]\varphi_l^r - \varphi_k^r P[\varphi_k^r])\,\mathrm{d}v = 0.$$

To this end, we have to check that the boundary term in the corresponding Green formula is zero. This boundary term is given by

$$B.T. := \int_{\partial\Omega(r)} \langle A\nabla[\varphi_k^r]\varphi_l^r - A\nabla[\varphi_l^r]\varphi_k^r, \vec{\sigma}\rangle\,\mathrm{d}\sigma.$$

We compute

$$\nabla\varphi(\xi,\cdot) = \exp(\mathrm{i}\xi\log(G))\left(\nabla G^{1/2}\right) + \mathrm{i}\xi G^{1/2}\exp(\mathrm{i}\xi\log(G))\,\nabla G.$$

Since $\exp\left(\mathrm{i}\frac{k}{r}\log(G)\right)$ and $\exp\left(\mathrm{i}\frac{l}{r}\log(G)\right)$are constant (equal to $(-1)^k$ and$(-1)^l$ respectively) on $\partial\Omega(r)$, we have

$$B.T. = i(-1)^{k+1}\frac{(k-l)}{r}\int_{\partial\Omega(r)}\langle A\nabla[G],\vec{\sigma}\rangle\,\mathrm{d}\sigma.$$

On the other hand, applying the Green formula on the pair $(1, G)$, we obtain

$$\int_{\Omega(r)} P[G]\mathbf{1}\,\mathrm{d}v - \int_{\Omega(r)} GP[\mathbf{1}]\,\mathrm{d}v = \int_{\partial\Omega(r)}\langle A\nabla[G]\mathbf{1} - A\nabla[\mathbf{1}]G,\vec{\sigma}\rangle\,\mathrm{d}\sigma,$$

and recalling that we assumed that $P\mathbf{1} = 0$, and that also $PG = 0$ on $\Omega(r)$, we get

$$\int_{\partial\Omega(r)}\langle A\nabla[G],\vec{\sigma}\rangle\,\mathrm{d}\sigma = 0,$$

and thus $B.T. = 0$.

If $k \in \mathbb{Z}$ and $l = -k \neq 0$, then $\varphi_k^r\overline{\varphi_{-k}^r} = \varphi_{2k}^r\varphi_0$and the orthogonality of φ_{2k} and φ_0 have been already established. On the other hand, for $k \in \mathbb{Z}$, and $l = k$, we have

$$\int_{\Omega(r)}|\varphi_k^r|^2 W\,\mathrm{d}v = \int_{\Omega(r)} GW\,\mathrm{d}v,$$

and the integral is equal to $\pi r/2$ according to Corollary (2.2.11).

Continuation of the alternative proof of Theorem (2.2.12): Since $\mathbf{T}_r$ is compact, the Stone-Weierstrass theorem implies that the vector space generated by the sequence $\{\exp(\mathrm{i}k/r\log(G))\}_{k\in\mathbb{Z}}$is dense in $C(\mathbf{T}_r;\mathbb{C})$ (in the topology of uniform convergence). Therefore, the orthonormal series $\left\{(\pi r/2)^{-1/2}\varphi_k^r\right\}_{k\in\mathbb{Z}}$is complete in $L^2_{\mathrm{rad}}(\Omega(r);\mathbb{C})$. Consequently, by Parseval's equalities, the following discrete analogues of (57) and (58) are available for every $f \in L^2_{\mathrm{rad}}(\Omega(r);\mathbb{C})$:

$$\int_{\mathbf{T}_r}|f|^2 W\,\mathrm{d}v = \frac{1}{r}\sum_{k\in\mathbb{Z}}\left|\hat{f}\left(\frac{k}{r}\right)\right|^2 \tag{60}$$

and

$$f(x) = \frac{1}{r}\sqrt{\frac{2}{\pi}}\sum_{k\in\mathbb{Z}} \hat{f}\left(\frac{k}{r}\right)^2 \varphi\left(-\frac{k}{r}, x\right) \tag{61}$$

Fix now $f \in C_0^\infty(\Omega^\star) \cap L^2_{\mathrm{rad}}(\Omega^\star, W\mathrm{d}v)$, and choose$r > 0$ such that the support of f is included in $\Omega(r)$ (this is possible since the fact that G tends to 0 at infinity implies that $\{\Omega(r)\}_{r>0}$is an exhaustion of $\Omega^\star$). Let us apply (60) and (61) to the function $g := \exp(\mathrm{i}\alpha \log(G))\, f$, for $\alpha \in (0, 1/r)$: we get

$$\int_{\mathbf{T}_r} |f|^2 W\, \mathrm{d}v = \frac{1}{r}\sum_{k\in\mathbb{Z}} \left|\hat{f}\left(\frac{k}{r} + \alpha\right)\right|^2,$$

and

$$f = \frac{1}{r}\sqrt{\frac{2}{\pi}}\sum_{k\in\mathbb{Z}} \hat{f}\left(\frac{k}{r} + \alpha\right) \varphi\left(-\frac{k}{r} - \alpha, \cdot\right).$$

We integrate these two equalities with respect to $\alpha \in (0, 1/r)$: recalling that f has support in $\Omega(r)$, we obtain

$$\int_{\Omega^\star} |f|^2 W\, \mathrm{d}v = \int_{\mathbb{R}} \left|\hat{f}\right|^2 \mathrm{d}\xi,$$

and

$$f(x) = \sqrt{\frac{2}{\pi}}\int_{\mathbb{R}} \hat{f}(\xi)\varphi(-\xi, x)\, \mathrm{d}\xi.$$

This is exactly (57) and (58).

We conclude this section with the following conjecture that arises naturally from our study.

Section (2.3): The Induced AgmonMetric and Rellich-Type Inequalities with Boundary Singularities and Applications

In the this section we prove that the Agmon metric corresponding to optimal Hardy-weight W in $\Omega^\star$ is complete. The completeness of $\Omega^\star$ in this metric implies sharp decay estimates for solutions of the equations the $Pu = f$ in $\Omega^\star$ (below).

Lemma (2.3.1) [3]:

Suppose that the assumptions of Theorem (2.1.14) are satisfied and let W be the corresponding optimal Hardy-weight. Assume further that W is strictly positive. Then $\Omega^\star$ is complete in the Agmon (Riemannian) metric

$$\mathrm{d}s^2 : W(x) \sum_{i,j=1}^{n} a_{ij}(x)\, \mathrm{d}x^i \mathrm{d}x^j, \qquad \text{where} [a_{ij}] := [a^{ij}]^{-1}. \tag{62}$$

Proof:

Let γ be a curve in $\Omega^\star$ such that $\gamma(t) \to \infty$ in $\Omega^\star$when $t \to T$. Here, T is finite or infinite. We have to show that the length $L(\gamma)$ of γ for the metric $\mathrm{d}s^2$ is infinite. Denoting $v := \frac{G}{u}$, we compute

$$L(\gamma) = \int_0^T \sqrt{W(\gamma(s))}\, |\gamma'|_{A^{-1}} \mathrm{d}s = \frac{1}{2}\int_0^T |\nabla \log v|_A(\gamma(s))|\gamma'|_{A^{-1}}\, \mathrm{d}s. \tag{63}$$

Define ∇_A to be the gradient with respect to the metric $|\cdot|_A$. For a function f and a vector $v \in T_x\Omega$, by definition of the gradient, we have the following identity

$$\mathrm{d}f_x(v) = \langle \nabla f, v\rangle = \langle A\nabla_A f, v\rangle,$$

which shows that $\nabla_A = A^{-1}\nabla$. From this, we see that

$$|\nabla_{A^{-1}} f|_{A^{-1}}^2 = \langle A^{-1}\nabla_{A^{-1}} f, \nabla_{A^{-1}} f\rangle = \langle A\nabla f, \nabla f\rangle = |\nabla f|_A^2.$$

Using this last identity, we get

$$L(\gamma) = \frac{1}{2}\int_0^T |\nabla_{A^{-1}} \log v|_{A^{-1}}(\gamma(s))|\gamma'(s)|_{A^{-1}}\, \mathrm{d}s \geq \frac{1}{2}\int_0^T \left|\frac{\mathrm{d}}{\mathrm{d}s} \log v\,(\gamma(s))\right| \mathrm{d}s$$

$$\geq \frac{1}{2}\left|\int_0^T \frac{\mathrm{d}}{\mathrm{d}s}\big(\log v\,(\gamma(s))\big)\, \mathrm{d}s\right| = \frac{1}{2}\lim_{t\to T}|\log v(\gamma(t)) - \log v(\gamma(0))|.$$

Since $\gamma(t) \to \infty$ in $\Omega^\star$ as $t \to T$, and $\lim_{x\to 0}|\log v(x)| = \infty$, we deduce that $L(\gamma) = \infty$.

Let P be a Schrödinger operator of the form

$$Pu = -\sum_{i,j=1}^{n} \partial_i\left(a^{ij}(x)\partial_j u\right) + c(x)u \tag{64}$$

defined on a domain $\Omega \subset \mathbb{R}^n$. A theorem of Agmon states that under certain conditions on P, solutions u of the equation $Pu = f$ in Ω that do not grow too fast, in fact, decay rapidly. The main condition which is required for the validity of the theorem is given by

$$(P\phi, \phi) \geq \int_\Omega \lambda(x)|\phi|^2\, \mathrm{d}x \quad \forall \phi \in C_0^\infty(\Omega), \tag{65}$$

where λ is a nonnegative weight function. The decay is then given in terms of a function h satisfying

$$|\nabla h(x)|_A^2 < \lambda(x) \text{a.e. } \Omega \tag{66}$$

Any Hardy-weight W given by (27) provides us with a natural candidate for λ and h. Assume that our Hardy-weight W obtained by the supersolution construction with a pair (v_0, v_1) is strictly positive a.e, in Ω, and set

$$\lambda := W, \quad h := \frac{\mu}{2}\log\left(\frac{v_0}{v_1}\right),$$

where $0 < \mu < 1$. Then λ and h clearly satisfy (66). Suppose also that a solution u of $Pu = f$ in Ω satisfies the growth condition. By Lemma (2.3.1) the induced Riemannian metric

$$\mathrm{d}s^2 := W(x)\sum_{i,j=1}^{n} a_{ij}(x)\,\mathrm{d}x^i \mathrm{d}x^j, \ \text{where}\left[a_{ij}\right] := \left[a^{ij}\right]^{-1} \tag{67}$$

is complete. Therefore, the following Rellich-type inequality holds true

$$(1-\mu^2)^2\int_\Omega |u|^2 W(x)\left(\frac{v_0}{v_1}\right)^\mu \mathrm{d}x \le \int_\Omega \frac{|Pu|^2}{W(x)}\left(\frac{v_0}{v_1}\right)^\mu \mathrm{d}x. \tag{68}$$

Assume that for some $0 < \mu < 1$ we have

$$\int_\Omega \frac{|Pu|^2}{W(x)}\left(\frac{v_0}{v_1}\right)^\mu \mathrm{d}x < \infty.$$

Then letting $\mu \to 0$ (using the monotone and dominated convergence theorems) we obtain the following Rellich-type inequality:

$$\int_\Omega |u|^2 W(x)\,\mathrm{d}x \le \int_\Omega \frac{|Pu|^2}{W(x)}\mathrm{d}x. \tag{69}$$

Indeed, without loss of generality assume that $P\mathbf{1} = 0$. Then using (23) and (24) it follows that for any two smooth enough functions u and v with $u \in C_0^\infty(\Omega)$ we have

$$(Pu, uv^2) = (P(uv), uv) + \frac{1}{2}(u^2, Pv^2) - (Pv, u^2 v). \tag{70}$$

Now let w be a positive solution of the equation $Pw = 0$ in Ω, and let $0 \le \mu \le 1$. We use (70) with the pair $u \in C_0^\infty(\Omega)$ and $v = w^{\mu/2}$, recalling that

$$Pw^\mu = 4\mu(1-\mu)Ww^\mu, \quad \text{where } W := \frac{Pw^{1/2}}{w^{1/2}}.$$

It follows that

$$(Pu, uw^\mu) = \left(P\left(w^{\mu/2}\right), w^{\mu/2}u\right) + [2\mu(1-\mu) - \mu(2-\mu)](u^2, Ww^\mu)$$

$$\ge (1-\mu^2)\int_\Omega u^2 W w^\mu\,\mathrm{d}x, \tag{71}$$

where we used the Hardy inequality $P - W \ge 0$ to derive the second line. Assume now that $W > 0$ in Ω, then the Cauchy-Schwarz inequality implies the following Rellich-type inequality

$$(1-\mu^2)^2 \int_\Omega u^2 W w^\mu \, \mathrm{d}x \leq \int_\Omega (Pu)^2 \frac{1}{W} w^\mu \, \mathrm{d}x \quad \forall u \in C_0^\infty(\Omega).$$

Therefore, for a general symmetric, subcritical operator P, and a positive Hardy-weight W obtained by the supersolution construction with a pair $(v_0, v_1)$of two positive solutions, we obtain for $0 \leq \mu \leq 1$ that

$$(1-\mu^2)^2 \int_\Omega |u|^2 W(x) \left(\frac{v_0}{v_1}\right)^\mu \mathrm{d}x \leq \int_\Omega \frac{|Pu|^2}{W(x)} \left(\frac{v_0}{v_1}\right)^\mu \mathrm{d}x \;\; \forall u \in C_0^\infty(\Omega). \qquad (72)$$

Moreover, using an approximation argument, it follows that if $P - W \geq 0$is critical in Ω, and $0 \leq \mu < 1$, then $(1-\mu^2)^2$is the best constant for the inequality (72).

We summarize these results in the following corollary.

Corollary (2.3.2) [3]:

Assume that P is a symmetric subcritical operator in Ω, and let $W > 0$ be a Hardy-weight obtained by the supersolution construction with a pair (v_0, v_1) of two positive solutions v_0 and v_1 of the equation $Pu = 0$ in Ω. Fix $0 \leq \lambda \leq 1$. Then

(i) For fixed $0 \leq \mu < 1$ and all $u \in C_0^\infty(\Omega)$ the following Rellich-type inequality holds true

$$\lambda(1-\mu^2)^2 \int_\Omega |u|^2 W(x) \left(\frac{v_0}{v_1}\right)^\mu \mathrm{d}x \leq \int_\Omega \frac{|Pu|^2}{W(x)} \left(\frac{v_0}{v_1}\right)^\mu \mathrm{d}x. \qquad (73)$$

(ii) For any $0 \leq \alpha \leq 1$ and all $u \in C_0^\infty(\Omega)$ the following Hardy-Rellichtype inequality holds true

$$\lambda \int_\Omega |u|^2 W(x) \, \mathrm{d}x \leq \alpha \int_\Omega uP[u] \, dx + (1-\alpha) \int_\Omega \frac{|Pu|^2}{W(x)} \mathrm{d}x \qquad (74)$$

(iii) If $P - W$ is critical in Ω, then $\lambda = 1$ is the best constant in inequalities (73) and (74).

Example(2.3.3) [3]:

Consider the Poisson equation in the punctured space $\Omega^\star = \mathbb{R}^n \backslash \{0\}, n \geq 3$ with the optimal Hardy-weight

$$W(x) := \left(\frac{n-2}{2}\right)^2 |x|^{-2}.$$

The corresponding induced Riemannian metric is given by

$$\mathrm{d}s^2 := W(x) \sum_{i=1}^{n} \left(\mathrm{d}x^i\right)^2.$$

By Lemma (2.3.1), $\Omega^\star$ is complete in the above Agmon metric. By (68),(69), and (72), for any $0 \leq \mu < 1$ the following Rellich-type inequality (with the best constant) holds true

$$\left(\frac{n-2}{2}\right)^4 (1-\mu^2)^2 \int_{\Omega^\star} \frac{|u(x)|^2}{|x|^{2+(n-2)\mu}} \mathrm{d}x \leq \int_{\Omega^\star} |\Delta u|^2 |x|^{2-(n-2)\mu} \,\mathrm{d}x \;\; \forall u \in C_0^\infty(\Omega) \quad (75)$$

In fact, it is known that $\left(\frac{n-2}{2}\right)^4 (1-\mu)^2$is indeed the best constant for the above inequality. Note also that the choice $\mu = 2/(n-2)$ recovers the classical Rellich inequality:

$$\frac{n^2(n-4)^2}{16} \int_{\Omega^\star} \frac{|u(x)|^2}{|x|^4} \mathrm{d}x \leq \int_{\Omega^\star} |\Delta u|^2 \,\mathrm{d}x, \qquad \forall u \in C_0^\infty(\Omega^\star).$$

Now we explain how our results can be extended to the case of boundary singularities, where the singularities of the Hardy-weight are located at $\partial\Omega \cup \{\infty\}$and not at an isolated interior point of Ω as above. So, we apply the supersolution construction with two global positivesolutions u_0, u_1 of the equation $Pu = 0$ in Ω that have singularities "at the boundary", instead of at an interior point, and get an optimal Hardy-weight W in the entire domain Ω. To understand the setting, we begin by presenting an example.

Example (2.3.4) [3]:

Let $P = -\Delta$, and consider the cone

$$\Omega := \{x \in \mathbb{R}^n \mid r > 0, \omega \in \sum\},$$

where $\sum$ is a Lipschitz domain in the unit sphere $S^{n-1} \subset \mathbb{R}^n, n \geq 2$, and (r, ω) denotes the spherical coordinates of x. Let θ be the principal eigenfunctionof the (Dirichlet) Laplace-Beltrami operator on with eigenvalue$\lambda_0 = \lambda_0(\sum)$, and set

$$\alpha_j := \frac{2 - n + (-1)^j\sqrt{(2-n)^2 + 4\lambda_0}}{2}.$$

Then for $j = 0$ (resp. $j = 1$) the positive harmonic function $u_j(r, w) := r^{\alpha_j}\theta(\omega)$ is the (unique) Martin kernel at ∞ (resp. 0).

Applying the supersolution construction with the pair (u_0, u_1), we obtainthe Hardy-weight

$$W(x) := \frac{(n-2)^2 + 4\lambda_0}{4|x|^2}.$$

Consequently, the corresponding Hardy-type inequality reads as

$$\int_\Omega |\nabla\phi|^2 \, \mathrm{d}x \geq \frac{(n-2)^2 + 4\lambda_0}{4|x|^2}\int_\Omega \frac{|\phi|^2}{|x|^2}\mathrm{d}x \quad \forall \phi \in C_0^\infty(\Omega). \tag{76}$$

It follows from Theorem (2.3.4) that W is an optimal Hardy-weight, and thatthe spectrum and the essential spectrum of $W^{-1}(-\Delta)$ is $[1, \infty)$. Note that(76) and the global optimality of the constant is known.

We assume that theMartin boundary $\delta\Omega$ of $\Omega$and P is equal to the minimal Martin boundary and consists of $\partial\Omega \cup \{\xi_0, \xi_1\}$, where $\partial\Omega \backslash \{\xi_0, \xi_1\}$is assumed to be a regular manifold of dimension $n - 1$ without boundary (in fact, it is enough to assume that $\partial\Omega\backslash\{\xi_0, \xi_1\}$ is Lipschitz and satisfies the interior sphere condition). Note that it might be that one or two of the Martin points ξ_0, ξ_1 belong to $\partial\Omega$.

We denote by $\widehat{\Omega}$ the Martin compactification of Ω. Hence,

$$\widehat{\Omega} := \overline{\Omega} \cup \{\xi_0, \xi_1\}.$$

We assume that there exists a bounded domain $D \subset \Omega$ such that ξ_0 and $\xi_1$belongs to two different connected components of $\widehat{\Omega}\backslash\overline{D}$that are neighborhoods of ξ_0 and ξ_1.

We need the following definition of minimal growth at a portion of the boundary $\delta\Omega$:

Definition (2.3.5) [3]:

Let $\omega \subset \delta\Omega$be a closed set, and let u be a positive solution of $Pu = 0$ in a neighborhood $\Omega_1 \subset \Omega$of ω. We say that u has minimal growth at ω if for every positive supersolution v of the equation $Pu = 0$ in a relative neighborhood of ω, we have

$$u \leq Cv$$

in a neighborhood $\Omega_2 \subset \Omega_1$ of ω.

We need two lemmas. The first one concerns minimal growth:

Lemma (2.3.6) [3]:

Assume that the coefficients of P are locally regular up to a Lipschitz portion Γ of $\partial\Omega$. Let W be a nonnegative potential which is $L^{\infty}_{\mathrm{loc}}$up to Γ, such that $P - W \geq 0$ in Ω.

(i) Let $\omega \subset \Gamma$ be the closure of a nonempty open set, and let u be a positive solution of $P - W$ in a relative neighborhood of ω. The following are equivalent:

a) $u$has minimal growth for $P - W$ at ω.

b) u vanishes continuously on ω.

(ii) Let $\omega = \omega_1 \cup \omega_2$, where ω_1 and ω_2 are closed sets in $\delta\Omega$, and let u be a positive solution of $P - W$ in a neighborhood of ω. If u has minimal growth for $P - W$ at ω_1 and at ω_2, then u has minimal growth for $P - W$ at ω.

Proof:

(i) First, we extend P (resp. W) in a neighborhood U of ω in $\mathbb{R}^n$such that the corresponding extension $\hat{P}$ (resp. $\widehat{W}$) has Hölder continuous coefficients (resp. the extension is L^{∞}). If U is small enough, then the extended operator $\hat{P}-\widehat{W}$ is nonnegative in U, and we can find a positive solution θ of the equation $(\hat{P}-\widehat{W})u = 0$ in U. By elliptic regularity, $\theta \in C_{\text{loc}}^{1,\alpha}(U)$, and therefore $\tilde{P} := \theta^{-1}(\hat{P}-\widehat{W})\theta$ has Hölder continuous coefficients in $\Omega \cap U$. By performing a ground state transform with respect to Ω, we see that it is enough to prove the lemma for $\tilde{P}$ instead of $P-W$; so we will assume that u is a solution of $\tilde{P}$ instead. The fact that (ia) implies (ib) now follows from.

For the proof that (ib) implies (ia) we may assume that ω is bounded. Let $\mathcal{O} \subset \Omega$be a neighborhood of ω on which u is a positive solution of the equation $\tilde{P}u = 0$ that vanishes continuously on ω. Let $\{\Omega_k\}_{k\in\mathbb{N}}$ be an exhaustion of Ω such that $\mathcal{O}_k := \mathcal{O} \cap \Omega_k$ is regular. Let $w := \lim_{k\to\infty} w_k$, where wk solves the Dirichlet problem

$$\begin{cases} \tilde{P}w_k = 0 & \text{in} \mathcal{O}_k, \\ w_k(x) = u & \text{on } \partial\mathcal{O} \cap \partial\mathcal{O}_k, \\ w_k(x) = 0 & \text{on } \partial\Omega_k \cap \partial\mathcal{O}_k. \end{cases} \tag{77}$$

Then w has minimal growth at ω (this follows from the local boundary Harnack principle. For every $\varepsilon > 0$, we can find k_0 big enough such that $u < \varepsilon$ on $\partial\Omega_k \cap \partial\mathcal{O}_k$for every $k \geq k_0$. Then, since $\tilde{P}\mathbf{1} = 0, u + \varepsilon$ is a solution of $\tilde{P}$, and by the maximum principle $u < w_k + \varepsilon$. Letting $k \to \infty$ and then $\varepsilon \to 0$, we obtain $u \leq w$, which concludes the first part of the lemma.

Part (ii) follows directly from the definition of minimal growth.

We now turn to the second lemma concerning the regularity of the supersolution construction and the corresponding Hardy-weight on a portion of the boundary where the solutions u_0 and u_1 vanish.

Lemma (2.3.7) [3]:

Let Σ be an open subset of $\partial\Omega$. Assume that Ω is equipped with a Riemannian metric $\mathfrak{g}$, regular up to Σ. Let u_0 and u_1 be two positive functions defined in a neighborhood $\Omega' \subset \Omega$of Σ that are C^2 up toΣ and vanish continuously on Σ. Suppose that the gradients of u_0 and u_1 restricted to vanish nowhere. Then

$$W := \frac{1}{4}\left|\nabla \log\left(\frac{u_0}{u_1}\right)\right|^2$$

has a continuous extension up to (here the gradient and its norm are computed with respect to $\mathfrak{g}$ and not to the Euclidean metric). If, in addition, u_0/u_1has a continuous extension to Σ, then u_0/u_1 is in fact C^1 up to Σ.

Proof:

Let us denote by $\vec{\sigma}$ the unit exterior normal to Σ. Since u_0 and $u_1$vanishes on Σ, the gradient of u_0 and u_1 are collinear to $\vec{\sigma}$ on Σ. Next, we claim that near we have for $i = 0{,}1$,

$$\frac{|\nabla u_i|_A}{u_i} = \frac{1}{\delta} + g_i, \tag{78}$$

where δ is the distance to $\partial\Omega$ with respect to the metric given by $\mathfrak{g}$, and g_i is continuous up to Σ. Indeed, for x_1 be a point of Σ, let γ_{x1} be the unit speed geodesic starting at x_1, with $\gamma'(0) = -\vec{\sigma}$ the interior normal. Let $r \geq 0$ be the coordinate on γ (so that $r = \delta$ in restriction to γ_{x1}, for r small enough), then the restricting u_i (resp. $|\nabla u_i|$) to γ provides us with a function $f_i(r)$(resp. $g_i(r)$). Notice that f_i is C^2, g_i is C^1 and $f_i'(0) = g_i(0) = |\nabla u_i| \neq 0$(this comes from the fact that ∇u_i is collinear to $\vec{\sigma}$, since ui vanishes on Σ). A Taylor expansion in r gives (dropping the subscript i)

$$\frac{g(r)}{f(r)} = \frac{g(0) + g'(0) + o(r)}{rf'(0) + \frac{r^2}{2}f''(0) + o(r^2)} = \frac{1}{r} + \left(\frac{g'(0)}{f'(0)} - \frac{f''(0)}{2f'(0)}\right) + o(r),$$

hence (78) follows. From the same kind of consideration, we get in a neighborhood of Σ,

$$\frac{\nabla u_i(x)}{u_i(x)} = \frac{1}{\delta}\gamma'_{\exp^{-1}(x)}(x) + X_i,$$

where X_i is a continuous vector field defined in a neighborhood ofΣ and $\exp^{-1}$ is the mapping sending a point x to the unique point on $x_1 \in \Sigma$ such that $x \in \gamma_{x_1}$. The lemma follows at once, by noticing that

$$\nabla\left(\frac{u_0}{u_1}\right) = \frac{u_0}{u_1}\left(\frac{\nabla u_0}{u_0} - \frac{\nabla u_1}{u_1}\right) = \frac{u_0}{u_1}(X_0 - X_1),$$

and that

$$W = \frac{\left|\nabla\left(\frac{u_0}{u_1}\right)\right|^2}{4\left|\frac{u_0}{u_1}\right|^2}.$$

We also need the following analogue of Proposition (2.2.1) for a domain with boundary:

Proposition (2.3.8) [3]:

Let P be a second-order nonnegative elliptic operator on Ω either of the form (12) or (13) with coefficients that are locally regular up to $\partial\Omega\backslash\{\xi\}$, where $\xi \in \partial\Omega$. If u and v are two positive solutions of the equation $Pw = 0$ in a relative neighborhood of ξ, which satisfy

$$\lim_{\substack{x\to\xi \\ x\in\Omega}} \frac{u(x)}{v(x)} = 0,$$

and both vanish on a punctured neighborhood of ξ in $\delta\Omega$, then u has minimal growth at ξ.

Proof:

The proof is almost exactly the same as the proof of Proposition (2.2.1). This time, we take a sequence of bounded sets $\{\Omega_k := B_1\backslash B_k\}$, where $\{B_k\}$is a decreasing sequence of relative neighborhoods in $\widehat{\Omega}$ of ξ converging to ξsuch that $\partial\Omega_k$is piecewise smooth. With this definition, $\{\partial\Omega_k\}$ exhausts a punctured neighborhood $\xi \in \partial\Omega$(cf. the proof of Proposition (2.2.1)). Let $w := \lim_{k\to\infty} w_k$, where w_k is the solution of the Dirichlet problem

$$\begin{cases} Pw_k = 0 & \text{in}\,\Omega_k, \\ w_k(x) = u & \text{on}\ \partial B_1 \backslash \partial\Omega, \\ w_k(x) = 0 & \text{on}(\partial\Omega_k \cap \partial\Omega) \cup \partial B_k. \end{cases} \tag{79}$$

It follows (using the boundary Harnack principle and arguments) that w has minimal growth at ξ. The end of the proof follows exactly the lines of the proof of Proposition (2.2.1).

We now establish the following.

Theorem (2.3.9) [3]:

Assume that P is subcritical in Ω. Suppose that the corresponding Martin boundary $\delta\Omega$ is equal to the minimal Martin boundary and is equal to $\partial\Omega \cup \{\xi_0, \xi_1\}$, where $\partial\Omega \backslash \{\xi_0, \xi_1\}$ is assumed to be a regular manifold of dimension $n-1$ without boundary, and the coefficients of $\boldsymbol{P}$ are locally regular up to $\partial\Omega \backslash \{\xi_0, \xi_1\}$.

Denote by $\widehat{\Omega}$ the Martin compactification of Ω, and assume that there exists a bounded domain $D \subset \Omega$ such that ξ_0 and ξ_1 belongs to two different connected components D_0 and D_1 of $\widehat{\Omega} \backslash \overline{D}$ such that each D_j is aneighborhood in $\widehat{\Omega}$of ξ_j , where $j = 0,1$.

Let u_0 and u_1 be the minimal Martin functions at ξ_0 and ξ_1 respectively. Consider the supersolution $v := \sqrt{u_0 u_1}$, and assume that

$$\lim_{\substack{x \to \zeta_0 \\ x \in \Omega}} \frac{u_1(x)}{u_0(x)} = \lim_{\substack{x \to \zeta_1 \\ x \in \Omega}} \frac{u_0(x)}{u_1(x)} = 0. \tag{80}$$

Then the associated Hardy-weight $W := P\,v/v$ is optimal in Ω. Moreover, if P is symmetric and W does not vanish on $\widehat{\Omega} \backslash \{\xi_0, \xi_1\}$, then the (essential) spectrum of the operator $W^{-1}P$ acting on $L^2(\Omega, W\mathrm{d}v)$ is $[1, \infty)$.

Proof:

We know that u_i vanishes continuously on $\partial\Omega\backslash\{\xi_0,\xi_1\}$. Also, by Hopf's boundary point lemma, we know that the gradient of u_i does not vanish on $\partial\Omega$. Define a metric $\mathfrak{g}$ on Ω, regular up to $\partial\Omega$, by

$$\mathfrak{g}(\cdot,\cdot) := \langle A^{-1}\cdot,\cdot\rangle.$$

We have $\nabla_{\mathfrak{g}} = A\nabla$, and therefore,

$$W := \frac{\left|\nabla\mathfrak{g}\left(\frac{u_0}{u_1}\right)\right|_{\mathfrak{g}}^2}{4\left|\frac{u_0}{u_1}\right|^2} = \frac{\left|\nabla\left(\frac{u_0}{u_1}\right)\right|_A^2}{4\left|\frac{u_0}{u_1}\right|^2}.$$

Now, we can apply Lemma (2.3.7) with $\Sigma = \partial\Omega\backslash\{\xi_0,\xi_1\}$, to get that W is continuous up to the boundary $\partial\Omega\backslash\{\xi_0,\xi_1\}$. Also, we know that u_0/u_1 has a continuous positive extension up to $\partial\Omega\backslash\{\xi_0,\xi_1\}$. Hence, the log solution

$$\sqrt{u_0u_1}\log\left(\frac{u_0}{u_1}\right),$$

as well as the oscillating solutions

$$\sqrt{u_0u_1}\cos\left(\xi\log\left(\frac{u_0}{u_1}\right)\right),$$

vanish continuously on $\partial\Omega\backslash\{\xi_0,\xi_1\}$. By elliptic regularity up to the boundary, since W is continuous up to $\partial\Omega\backslash\{\xi_0,\xi_1\}$, all these solutions are in fact $C^{1,\alpha}$ up to $\partial\Omega$, for some $\alpha\in(0,1)$.

Consequently, (80) and Proposition (2.3.8) imply that $\sqrt{u_0u_1}$ has minimal growth at ξ_0 and ξ_1. It also vanishes continuously on $\partial\Omega\backslash\{\xi_0,\xi_1\}$, and therefore has minimal growth on $\partial\Omega\backslash\{\xi_0,\xi_1\}$ by Lemma (2.3.6). Therefore, again by Lemma (2.3.6), it has minimal growth on $\delta\Omega$, i.e. at infinity in Ω, and the criticality of $P-W$ follows.

The optimality of the constant 1 near ξ_0 and ξ_1 follows from the existence of the oscillating solutions. Such a solution contradicts the generalized maximum principle near ξ_0 and ξ_1 for the operator $P-\lambda W$ with the corresponding $\lambda > 1$ (as in Theorem (2.2.4)).

Concerning the null-criticality, the proof follows the same lines as in the proof of Theorem (2.2.8); here again we use the vanishing of the oscillating solutions on $\partial\Omega\backslash\{\xi_0,\xi_1\}$. This implies that the boundary of Ω will not cause trouble in the various integrations by part. The same remark also applies to the proof concerning the entire spectrum in the symmetric case.

The following example deals with an important class of operators with boundary singularities which satisfy the assumptions of Theorem (2.3.9), and in particular (80).

Example (2.3.10) [2]:

Consider a Fuchsian linear subcritical elliptic operator of the form (12) defined on the cone $\Omega := \{x \in \mathbb{R}^n \mid r > 0, \omega \in \Sigma\}$, where Σ is a Lipschitz domain in the unit sphere S^{n-1} in $\mathbb{R}^n, n \geq 2$, and (r, ω) denotes the spherical coordinates of x. We assume that the coefficients of P are up to the boundary locally Hölder continuous except at the origin. The operator P has Fuchsian singularities both at 0 and ∞ means that there exists a positive constant M such that near 0 and ∞ we have

$$M^{-1}\sum_{i=1}^{n}\xi_i^2 \leq \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \leq M\sum_{i=1}^{n}\xi_i^2 \quad \xi \in \mathbb{R}^n,$$

and

$$|x|\sum_{i=1}^{n}|b_i(x)| + |x^2||c(x)| \leq M.$$

It is known that the Martin boundary of Ω for P is equal to the minimal Martin boundary, and is the union of the Euclidean boundary and ∞. For $j = 0$ (resp. $j = 1$), denote by u_j the minimal Martin function with pole 0 (resp. ∞). u_j vanish on $\partial\Omega\backslash\{0\}$, and

$$\lim_{\substack{x\to 0\\ x\in\Omega}}\frac{u_1(x)}{u_0(x)} = \lim_{\substack{x\to\infty\\ x\in\Omega}}\frac{u_0(x)}{u_1(x)} = 0. \tag{81}$$

Applying Theorem (2.3.9), we conclude that if W is the weight obtained by the supersolution construction applied to u_0 and u_1, then W is an optimal Hardy-weight. Moreover, in the symmetric case the spectrum of $W^{-1}P$ is equal to $[1, \infty)$. In particular, the Hardy-weight of Example (2.3.4) is optimal. The same conclusions hold true for a bit more general domains.

The following example deals with the case where one of the conditions of (80) is not satisfied.

Example (2.3.11) [2]:

Let $P = -\Delta$ and $\Omega = \mathbb{R}^n_+, n > 1$. Let $v_0(x) := C_n\, x_n/|x|^n$ be the Poisson kernel at the origin, and $v_1 := \mathbf{1}$. We note that in contrast to the pair (v_0, x_n), the pair $(v_0, \mathbf{1})$ does not satisfy one of the assumptions in (80). An elementary computation shows that

$$W(x) := \frac{1}{4}\left(\frac{1}{|x_n|^2} + \frac{n(n-2)}{|x|^2}\right),$$

which is obviously greater than the corresponding well known Hardy potential $1/(2|x_n|)^2$, and we get the following Hardy inequality

$$\int_{\mathbb{R}^n_+} |\nabla\phi|^2 dx \geq \int_{\mathbb{R}^n_+} W(x)|\phi|^2 dx \qquad \forall\phi \in C_0^\infty(\mathbb{R}^n_+).$$

GeorgiosPsaradakis kindly informed us recently that indeed the above inequalitycan be improved, and in fact, the following improved Hardy inequalityholds true

$$\int_{\mathbb{R}^n_+} |\nabla\phi|^2 dx \geq \frac{1}{4}\int_{\mathbb{R}^n_+} \left(\frac{1}{|x_n|^2} + \frac{(n-1)^2}{|x|^2}\right)|\phi|^2 dx \qquad \forall\phi \in C_0^\infty(\mathbb{R}^n_+). \qquad (82)$$

We show below, that this inequality is in fact optimal.

We note that, for every $0 \leq \mu \leq \frac{1}{4}$one can considerthe Hardy inequality

$$\int_{\mathbb{R}^n_+} |\nabla\phi|^2 dx \geq \int_{\mathbb{R}^n_+} \left(\frac{\mu}{|x_n|^2} + \frac{\beta(\mu)}{|x|^2}\right)|\phi|^2 dx \qquad \forall\phi \in C_0^\infty(\mathbb{R}^n_+), \qquad (83)$$

wherefor a fixed $\mu, \beta(\mu) \coloneqq 1 - n - \sqrt{1-4\mu}$ is the best constant. Moreover, inequality (83) cannot be improved by a Sobolev term.

Claim: The Hardy inequality (83) is optimal. In particular, the operator$-\Delta - \frac{\mu}{|x_n|^2} - \frac{\beta(\mu)^2}{4|x|^2}$is critical in $\mathbb{R}^n_+$ with the ground state $\psi(x) \coloneqq x_n^{\alpha_+}|x|^{\beta(\mu)/2}$. Furthermore, no Sobolev improvement of (83) is possible.

Indeed, for $\mu \leq 1/4$, consider the subcritical operator

$$P_\mu \coloneqq -\Delta - \frac{\mu}{|x_n|^2}$$

in $\Omega = \mathbb{R}^n_+$. Let α_+ be the largest root of the equation $\alpha(1-\alpha) = \mu$, andlet

$$\beta(\mu) \coloneqq 1 - n - \sqrt{1-4\mu}$$

be the nonzero root of the equation

$$\beta\left(\beta + n - 1 + \sqrt{1-4\mu}\right) = 0.$$

Then

$$w_0(x) \coloneqq x_n^{\alpha_+}, w_1(x) \coloneqq x_n^{\alpha_+}|x|^{\beta(\mu)}$$

are two positive solutions of the equation $P_\mu u = 0$ in Ω. Moreover, w_1 hasminimal growth on $\partial\Omega$, and w_0 has minimal growth on $\partial\Omega \cup \{\infty\}\backslash\{0\}$. Inparticular,

$$\lim_{x\to 0}\frac{w_0(x)}{w_1(x)} = \lim_{x\to\infty}\frac{w_1(x)}{w_0(x)} = 0.$$

Although the potential $|x_n|^{-2}$ is not smooth on $\partial\Omega\backslash\{0\}$, it can be easilychecked that the proof of Theorem (2.3.9) applies also to the case of theoperator P_μ in Ω with the pair of the positive solutions w_0 and w_1. This yields that inequality (83) is optimal. The criticality of $-\Delta - \frac{\mu}{|x_n|^2} - \frac{\beta(\mu)^2}{4|x|^2}$implies that no Sobolev improvement is possible.

The criticality result for Hardy-weights obtained by a particular supersolutionconstruction (Theorem (2.2.2)) can be extended to the case where wehave a finite number of ends in Ω, instead of just two ends (e.g., one isolatedsingularity and ∞).

Definition (2.3.12) [2]:

Let M be a noncompact manifold. We say that M has$N$-ends $E_1, \ldots, E_N$, if each E_i is a smooth non-compact connected manifoldwith boundary such that

$$M = \bigcup_{i=0}^{N} E_i, \qquad \text{and} \bigcap_{i=1}^{N} E_i = \emptyset,$$

where E_0 is a relatively compact, open set of M. We denote the ideal "infinity" point of each E_i by x_i (that is, x_i is the ideal limit point when$x \to \infty$ in $M \cap E_1 \backslash \overline{(\partial E_1 \cap M)}$).

Lemma (2.3.13) [2]:

Let P be a symmetric operator on a manifold M with ends$E_1, \ldots, E_N$. For $i = 1{,}2$, let $P_i = P + W_i$ be a nonnegative operator in M,where W_i is a potential, and let $\phi_i$be a positive solution of $P_i u = 0$ in E_1.Assume further that

$$\varphi_2 \leq C\varphi_1 \text{in} E_1,$$

and that φ_1 has a minimal growth at x_1 with respect to P_1. Then φ_2 has aminimal growth at x_1 with respect to P_2.

Proof:

We first modify φ_i so that it has minimal growth for P_i on E_1 (seenas a manifold with boundary). To this purpose, let us consider U a compact,smooth, open set which is a neighborhood of ∂E_1 in E_1. Let ψ be a positive solution of $P_1 u = 0$ in U, with minimal growth at $\partial E_1 \cap M$. Now considera positive function $\tilde{\varphi}_1(x)$ (resp. $\tilde{\varphi}_2(x)$) which is equal to $\psi(x)$ on a neighborhoodof ∂E_1, and to $\varphi_1(x)$ (resp. $\varphi_2(x)$) near x_1. Let $\widetilde{W}_i$ be a potential such that $(P + \widetilde{W}_i)\tilde{\varphi}_i = 0$ in E_1. By the (AAP) theorem, $\tilde{P}_i := P + \widetilde{W}_i$is nonnegative. Also, by construction, $\tilde{\varphi}_1$ has minimal growth (globally)in E_1, considered as a subdomain of M, and therefore $\tilde{P}_1$ is critical in E_1.Furthermore, we still have (with a different constant C)

$$\tilde{\varphi}_2 \leq C\tilde{\varphi}_1 \text{in} E_1.$$

Now, $\tilde{P}_2$ is critical in E_1,and $\tilde{\varphi}_2$ is its ground state. Therefore, $\tilde{\varphi}_2$ has minimal growth (globally) onE_1. Since $\tilde{\varphi}_2(x) = \varphi_2(x)$ near x_1, the lemma is proved.

We now formulate the result that claims thatin the case of finitely many ends the supersolution construction produces an$N-1$-parameter family of critical Hardy-weights.

Theorem (2.3.14) [2]:

Suppose that P is a symmetric subcritical operator in amanifold M with ends $E_1, \ldots, E_N, N \geq 2$. Assume that for each $1 \leq i \leq N$there exists a function u_i which is a positive solution of the equation $Pu = 0$in M of minimal growth near each end $x_j, j \neq i$, and satisfying

$$\lim_{x \to x_i} \frac{u_j(x)}{u_i(x)} = 0 \qquad \forall j \neq i.$$

Consider the supersolution construction

$$v \coloneqq \prod_{j=1}^{N} u_j^{\alpha_j},$$

where $0 < \alpha_j \leq 1/2$ for all $1 \leq j \leq N$, and$\sum_{j=1}^{N} \alpha_j = 1$.

Then the corresponding Hardy-weight $W := Pv/v$ is critical with respectto P and M.

Proof:

Note that by the definition of minimal growth, for each i, and every$k, j \neq i$ we have

$$u_j(x) \asymp u_k(x) \text{as} \quad x \to x_i.$$

Denote $\hat{u}_i \coloneqq \prod_{j \neq i} u_j^{\alpha_j}$. Fix $1 \leq i \leq N$ and $k \neq i$. Then near x_i the following inequality holds

$$\upsilon = u_i^{\alpha_i} \hat{u}_i^{1-\alpha_i} \leq C u_i^{\alpha_i} u_k^{1-\alpha_i} = C(u_i u_k)^{1/2} \left(\frac{u_k}{u_i}\right)^{1/2-\alpha_i} \leq (u_i u_k)^{1/2}.$$

Recall that it follows from the proof of Theorem (2.2.2) (or Theorem (2.3.9)) that$(u_i u_k)^{1/2}$ has minimal growth at x_i with respect the symmetric operator$P - W_{i,k}$, where $W_{i,k}$ is the Hardy-weight corresponding to the pair (u_i, u_k).

Hence, by Lemma (2.3.13), υ has minimal growth at x_i. Since this is true forall $1 \leq i \leq N$, it follows that the operator $P - W$ is critical in M.

Now we present some further examples, and discuss some additionalapplications and extensions. First, we present a straightforwardexample of an optimal Hardy-weight.

Example (2.3.15) [2]:

Consider the Laplace operator $P = -\Delta$ on the unit disk$\Omega = B(0,1) \subset \mathbb{R}^2$. Take $\upsilon_0(x) \coloneqq -\frac{1}{2\pi} \log|x|$, the Green function of theunit ball with a pole at the origin, and let $\upsilon_1 \coloneqq \mathbf{1}$. Then the correspondingoptimal Hardy-weight is given by $W(x) = (4|x| \log|x|)^{-2}$defined on$B(0,1)^\star \coloneqq B(0,1) \setminus \{0\}$. We obtain the classical Leray inequality withthe best constant

$$\int_{B(0,1)^\star} |\nabla\phi|^2 dx \geq \frac{1}{4} \int_{B(0,1)^\star} \frac{|\phi|^2}{(|x| \log|x|)^2} dx \quad \forall \phi \in C_0^\infty(B(0,1)^\star),$$

In particular, the operator $-\Delta - W$ is null-critical in $B(0,1)^\star$,and $\lambda_0(-\Delta, W, B(0,1)^\star) = \lambda_\infty(-\Delta, W, B(0,1)^\star) = 1$.

Analogously, in higher dimension $n \geq 3$, let $\upsilon_0(x) \coloneqq C_n(|x|^{2-n} - 1)$ bethe Green function of the unit ball $B(0,1)$ with a pole at the origin, and let$\upsilon_1 \coloneqq \mathbf{1}$. Then

$$W(x) = \frac{(n-2)^2}{4\big(|x|(1 - |x|^{n-2})\big)^2},$$

and the following optimal inequality holds true

$$\int_{B(0,1)^\star} |\nabla\phi|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_{B(0,1)^\star} \frac{|\phi|^2}{\left(|x|(1-|x|^{n-2})\right)^2} dx \quad \forall\phi \in C_0^\infty(B(0,1)^\star).$$

In particular, the operator $\Delta - W$ is null-critical in $B(0,1)^\star$, furthermore, we have

$$\lambda_0(-\Delta, W, B(0,1)^\star) = \lambda_\infty(-\Delta, W, B(0,1)^\star) = 1.$$

Example (2.3.16) [2]:

The aim of the present example is to give an alternativeproof that $1/4$ is the best constant in the classical Hardy inequality (29)for a smooth convex bounded domain Ω (see the discussion in Example (2.1.21)).

If we use the supersolution construction with $P = -\Delta, u_0 = G$ (the Green function), and $u_1 = \mathbf{1}$, we get an optimal Hardy-weight $W := \frac{1}{4}\left|\frac{\nabla G}{G}\right|^2$.Recall that Gvanishes on $\partial\Omega$ (in fact, $G(x) \asymp \delta(x)$ near the boundary). ByHopf's lemma, $\partial G / \partial\vec{\sigma}$ does not vanish on $\partial\Omega$, where $\vec{\sigma}$ is the outer normalvector to $\partial\Omega$. Hence, by the proof of Lemma (2.3.7), we have

$$W(x) \sim \frac{1}{4\delta(x)^2} \text{ as } \; x \to \partial\Omega. \tag{84}$$

Since we know that $\lambda_\infty(P, W, \Omega) = 1$, we deduce that $1/4$ is indeed the bestconstant in the classical Hardy inequality (29). It is also easy to deduce from the fact that $P - W$ is null-critical that the classical Hardy inequality (29)has no minimizer (this also follows from the subcriticality of $-\Delta - \delta(x)^2/4$).We do not know if the asymptotic of W given by (86) remains true if Ω hasa rougher boundary. On the other hand, the spectrum and the essential spectrum of $-4\delta(x)^2\Delta$on $L^2\left(\Omega, \left(4\delta(x)\right)^{-2} d\mathcal{V}\right)$ is equal to $[1, \infty)$.

In the next two examples we apply the supersolution construction topositive solutions with boundary singularities.

Example (2.3.17) [2]:

Consider the operator $Pu \coloneqq -u''$ on $\mathbb{R}_+$, and apply thesupersolution construction with the positive solutions $u_0(x) = x, u_1(x) = \mathbf{1}$.By Theorem (2.3.9), we readily get the classical Hardy inequality on $\mathbb{R}_+$ withthe optimal Hardy-weight $W(x) := 1/(4x^2)$. We note that the corresponding transform (59) is just the classical Mellin transform.

Example (2.3.18) [2]:

Consider the operator $Pu \coloneqq -u'' + u$ defined on $\mathbb{R}$, with$u_0(x) = e^x, u_1(x) = e^{-x}$. Applying Theorem (2.3.9) we obtain the optimal Hardy-weight $W := \mathbf{1}$, and we get the trivial inequality $P - W = -\mathrm{d}^2/dx^2 \geq 0$ in $\mathbb{R}$. The corresponding transform (59) is just the classical Fouriertransform on $\mathbb{R}$.

The supersolutionconstruction provides bounds for solutions near infinity in termsof the Green function G and a global solution u. In particular, we have

Lemma (2.3.19) [2]:

Let P be a subcritical operator in Ω, and let W be a Hardyweightin Ω associated to a pair (v_1, v_2), where v_1, v_2 are positive solutions of the equation $Pu = 0$ in Ω. Let v be a positive supersolution of the equation

$$(P - V)v = 0$$

of minimal growth at infinity with respect to $P - V$ in Ω.

Suppose further that

$$V(x) \leq 4\alpha(1-\alpha)W(x) \text{in} \Omega'$$

holds true for some $1/2 \leq \alpha \leq 1$, and some neighborhood $\Omega'$of infinity in Ω.Then for any $1/2 \leq \beta \leq \alpha$ there exists a constant C such that the inequality

$$v(x) \leq C v_1^{1-\beta}(x) v_2^{\beta}(x)$$

holds true in a neighborhood of infinity of Ω.

Example (2.3.20) [2]:

Let Ω be a smooth bounded convex domain and u a positivesolution of the equation

$$(P - V)u = 0,$$

of minimal growth in a neighborhood of infinity in Ω. Suppose further thatfor some $1/2 \leq \alpha \leq 1$, the inequality $V(x) \leq \alpha(1 - \alpha)\delta^{-2}(x)$ holds true ina neighborhood Ω' of infinity of Ω. Then

$$u(x) \leq C\delta(x)^{\alpha} \text{in} \Omega'.$$

In order to apply Lemma (2.3.19) for the pair (u, G), where G is the Greenfunction and u is a global positive solution satisfying (17), one needs toknow the behavior of the optimal Hardy-weight W near infinity, and tocompare pointwise$V$ and W, if V is a (non-optimal) Hardy potential. Infull generality, it seems hopeless to get an asymptotic of W at infinity, since∇G might vanish on a nonempty set with an accumulation point at infinityin Ω.

However, in the symmetric case we have an asymptotic of W in averageat infinity, as follows from Corollary (2.2.11), which, if u is normalized so that$u(0) = 1$, gives that

$$\int_{\left\{a \leq \frac{G}{u} \leq b\right\}} uGW \, d\nu = \frac{1}{4}(\log b - \log a).$$

Moreover, in average, we can compare W and any Hardy-weight V nearinfinity.

Proposition (2.3.21) [2]:

Suppose that P is symmetric and the hypotheses of Theorem (2.1.3) are satisfied (with P, u, G and W as in the theorem). Let Vbe a nonnegative potential such that $P - V \geq 0$ in $\Omega^{\star}$. Then for every$1 < a < b < \infty$(or $-\infty < a < b < -1$),we have

$$\int_{\left\{a\le\log\frac{G}{u}\le b\right\}} uGV\,d\nu \le \int_{\left\{a-1\le\log\frac{G}{u}\le b+1\right\}} uGW\,d\nu = \frac{5}{4}[\log(b+1)-\log(a-1)].$$

Proof:

By performing a ground state transform, wve may assume that $u = 1$.We start with the following inequality

$$\int_{\Omega^\star} V\,v^2\mathrm{d}\nu \le \int_{\Omega^\star} v\,P[v]\mathrm{d}\upsilon \qquad \forall v \in C_0^\infty(\Omega^\star),$$

which holds true by our assumption.

Fix $1 < a < b < \infty$, and let ψ be a smooth nonnegative cut-off functionsupported in $\{a-1 \le \log G \le b+1\}$, such that $\psi = 1$ on $\{a \le \log G \le b\}$.Set $v \coloneqq G^{1/2}\psi$, and recall that $(P-W)(G)^{1/2} = 0$. Therefore, by (23) we have

$$\int_{\{a-1\le\log G\le b+1\}} vP[v]\,d\nu = \int_{\{a-1\le\log G\le b+1\}} \left[GW\psi^2 - \frac{1}{2}\langle\nabla G, \nabla\psi^2\rangle_A + G\psi P[\psi]\right] d\nu.$$

Now, integrate by part the last term to get

$$\int_{\{a\le\log G\le b\}} GV\,d\nu \le \int_{\{a-1\le\log G\le b+1\}} GW\psi^2\,d\nu + \int_{\{a-1\le\log G\le b+1\}} G|\nabla\psi|_A^2\,d\nu \quad (85)$$

Consider the function ψ defined by

$$\psi(x) \coloneqq \begin{cases} 1 & x \in \{a \le \log G \le b\}, \\ b+1-\log G(x) & x \in \{b \le \log G \le b+1\}, \\ \log G(x) - a + 1 & x \in \{a-1 \le \log G \le a\}, \\ 0 & \text{elswhere}. \end{cases}$$

Now, take a sequence $\{\psi_k\} \subset C_0^\infty(\Omega^\star)$ of smooth function $0 \le \psi_k \le \psi$ whichconverges in $W^{1,2}$ to ψ. Since ψ is in $W_0^{1,2}(\Omega^\star)$, we can find such a sequence$\{\psi_k\}$. Applying (87) to ψ_k and passing to the limit as $k \to \infty$ gives

$$\int_{\{a\le\log G\le b\}} GV\psi\,d\nu \le \int_{\{a-1\le\log G\le b+1\}} GW\psi^2\,d\nu + \int_{\{a-1\le\log G\le b+1\}} G|\nabla\psi|_A^2\,d\nu.$$

We use finally the fact that ψ is supported in $\{a-1 \leq \log G \leq b+1\}$, that$0 \leq \psi \leq 1$ and that $|\nabla\psi|_A^2 \leq 4W$to get the result.

It is natural to formulate the following conjecture about the pointwiseasymptotic of the optimal Hardy-weight W.

The main result (Theorem (2.1.14)) providesus with an optimal Hardy-weight W defined in the punctured domain$\Omega^\star$ rather in Ω. This drawback can be easily relaxed using the followingregularization procedure. Let $\widetilde{W} \leq W$ be a (locally) regular nonnegativepotential in Ω such that $\widetilde{W} = W$ outside a punctured neighborhood of the origin. Clearly, $P - \widetilde{W}$ is subcritical in Ω. Let $V \in C_0^\infty(\Omega)$ be a smoothnonzero nonnegative function such that $P - \widetilde{W} - V$ is critical in Ω(see,Lemma (2.1.10)). Then the potential $\widetilde{W} \coloneqq \widetilde{W} - V$ is critical in Ω, null-criticalat infinity of Ω, and $\lambda_\infty\big(P, \widetilde{W}, \Omega\big) = 1$. Moreover, in the symmetric case, byTheorem (2.2.12), the corresponding spectrum and essential spectrum of $\widetilde{W}^{-1}P$is equal $[1, \infty)$. So, $\widetilde{W}$ is an optimal Hardy-weight for P in Ω.

We briefly discuss some extensionsof the previous results to the case of p-Laplacian type equations. We assume that $p \neq 2$. The celebrated p-Laplacian is the quasilinear ellipticoperator

$$\Delta_p(u) \coloneqq \mathrm{div}(|\nabla u|^{p-2}\nabla u).$$

Let $V \in L^\infty_{\mathrm{loc}}(\Omega)$ be a given function (potential), we consider the functional

$$Q_{\mathrm{V}}(\phi) \coloneqq \int_\Omega (|\nabla\varphi|^p + V|\varphi|^p)\mathrm{d}x \ \varphi \in C_0^\infty(\Omega) \tag{86}$$

and the associated differential operator

$$Q_V'(u) \coloneqq -\Delta_p(u) + V|u|^{p-2}u. \tag{87}$$

The notions of criticality and subcriticality of Q_V have been studied in this context. In particular,the Agmon-Allegretto-Piepenbrink theorem extends to this case. Due to the nonlinearity of the operator, if the potential V isnonzero it is likely that our supersolution construction will not yield in generalan optimal weight, as we can see from the following result in the radiallysymmetric case.

Theorem (2.3.22) [2]:

Assume that the functional

$$Q\mathrm{v}(\phi) := \int_{\Omega} (|\nabla\phi|^{p} + V(|x|)|\phi|^{p})\mathrm{d}x \quad \phi \in C_0^{\infty}(\Omega), \qquad (88)$$

is subcritical in a radially symmetric domain $\Omega \subset \mathbb{R}^n$,where the potential$V$ is radially symmetric. Suppose further that either $1 < p \leq 2$ and $V \leq 0$, or $p \geq 2$ and $V \geq 0$. Let v_0, v_1 be two linearly independent positive radiallysymmetric supersolutions of the equation $Q_V'(u) = 0$ in $\Omega^\star := \Omega \backslash \{0\}$.

Define the function

$$v_\alpha(|x|) := \big(v_1(|x|)\big)^{\alpha}\big(v_0(|x|)\big)^{1-\alpha} \quad x \in \Omega^\star,$$

where $0 \leq \alpha \leq 1$, and let

$$W_\alpha(t) := \alpha(1-\alpha)(p-1)\left|\left[\log\left(\frac{v_0(t)}{v_1(t)}\right)\right]'\right|^{2}\left|\left[\log\big(v_\alpha(t)\big)\right]'\right|^{p-2}.$$

Then v_α is a positive supersolution of the equation

$$Q_V' - W_\alpha(u) = 0 \quad \text{in}\Omega^\star, \qquad (89)$$

and the following improved inequality holds

$$Q\mathrm{v}(\phi) \geq \int W_\alpha|\phi|^{p}\mathrm{d}x \quad \forall\phi \in C_0^{\infty}(\Omega^\star).$$

Moreover, if $p \neq 2$ and V is not identically zero, then for every $\alpha \in (0,1)$ the functional $Q_V - W_\alpha$ is subcritical in $\Omega^\star$.

Notice that in the case where $p \neq 2$, the supersolution constructionyields a weight W_α which is not easy to optimize with respect to α. Onthe other hand, for the case of the p -Laplacian itself in a general domain$\Omega \subset \mathbb{R}^n$, we can take $u_0 = \mathbf{1}$, and thus optimize W_α:

Proposition (2.3.23) [2]:

Assume that υ is a positive supersolution (resp. solution) of the equation $-\Delta_p(u) = 0$ in Ω. Then for $\alpha \in (0,1)$, υ^α is a positivesupersolution (resp. solution) of the equation $Q'_{W_\alpha}(u) = 0$ in Ω, where

$$W_\alpha \coloneqq \alpha^{p-2}\alpha(1-\alpha)(p-1)\left|\frac{\nabla \upsilon}{\upsilon}\right|^p.$$

In particular, for the optimal value $\alpha = \frac{p-1}{p}$, the following logarithmic Caccioppoliinequality holds:

$$\left(\frac{p-1}{p}\right)^p \int_\Omega \left|\frac{\nabla \upsilon}{\upsilon}\right|^p |\varphi|^p \mathrm{d}x \leq \int_\Omega |\nabla \varphi|^p \mathrm{d}x \qquad \varphi \in C_0^\infty(\Omega), \tag{90}$$

where υ is any positive p-superharmonic function in Ω.

Chapter 3

Multiple Solutions for Semi-Linear Elliptic Systems with Sign-Changing Weight

With the help of the Nehari manifold and the Lusternik–Schnirelmann category, we investigate how the coefficient $h(x)$of the critical nonlinearity affects the number of positive solutions of that problem and get a relationship between the number of positive solutions and the topology of the global maximum set of h. [3]

This chapter is concerned with the multiplicity of positive solutions to the following elliptic system:

$$(E_{f,g})\begin{cases} -\Delta u = f(x)|u|^{q-2}u + \dfrac{\alpha}{\alpha+\beta}h(x)|u|^{\alpha-2}|v|^{\beta}, & \text{in } \Omega, \\ -\Delta v = g(x)|v|^{q-2}v + \dfrac{\beta}{\alpha+\beta}h(x)|u|^{\alpha}|v|^{\beta-2}v, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$with smooth boundary, $\alpha, \beta > 1$ satisfy $\alpha + \beta = 2^* = \frac{2N}{N-2}$ $(N \geq 3)$and $1 < q < 2$. Moreover, we assume that f, g and h satisfy the following conditions.

(H_1) $f, g \in C(\overline{\Omega})$.

(H_2) There exist a non-empty closed set $M = \{x \in \overline{\Omega}; h(x) = \max_{x\in\overline{\Omega}} h(x) = 1\}$ and a positive number $\rho > 2$ when$N \geq 6$, $\rho > \frac{N-2}{2}$ when $3 \leq N \leq 5$ such that $h(z) - h(x) = O(|x-z|^{\rho})$ as $x \to z$ and uniformly in $z \in M$.

(H_3) $f(x), g(x) > 0$for $x \in M$.

Remark (3.1) [3]:

Let $M_r = \{x \in \mathbb{R}^N; \operatorname{dist}(x, M) < r\}$ for $r > 0$. Then by (H_1)–(H_3), there exist $C_0, r_0 > 0$ such that

$$f(x), g(x), h(x) > 0 \quad \text{for all} \quad x \in M_{r_0} \subset \Omega$$

and

$$h(z) - h(x) \leq C_0|x - z|^\rho \text{for all } \; x \in B_{r_0}(z)$$

uniformly in $z \in M$, where $B_{r_0}(z) = \{x \in \mathbb{R}^N;\ |x - z| < r_0\}$.

For $f \equiv \lambda, g \equiv \mu$, we have that $(E_{f.g})$ permits at least two positive solutions when the pair of parameters (λ, μ), belongs to a certain subset of $\mathbb{R}^2$. In further studies involving sign-changing weight functions, two positive solutions were obtained for the subcritical case $2 < \alpha + \beta < 2^*$and for the critical case $\alpha + \beta = 2^*$. The tool of them is thedecomposition of the Nehari manifold.

For $2 < q < 2^*$, if $N > 4, 0 \in \Omega, f, g$ and h satisfy the following conditions.

(A_1) f, g and h are positive continuous functions in $\overline{\Omega}$.

(A_2) There exist k points $a^1, a^2, \ldots, a^k$in Ω such that

$$h(a^i) = \max_{x \in \overline{\Omega}} h(x) = 1 \text{for } \; 1 \leq i \leq k,$$

and for some $\rho > N, h(x) - h(a^i) = O(|x - a^i|^\rho)$ as $x \to a^i$and uniformly in i.

(A_3) Choose $\rho_0 > 0$ such that

$$\overline{B_{\rho 0}(a^i)} \bigcap \overline{B_{\rho 0}(a^i)} = \emptyset \quad \text{for} \quad i \neq j \quad \text{and} \quad 1 \leq i, j \leq k,$$

and$\bigcup_{i=1}^k \overline{B_{\rho 0}(a^i)} \subset \Omega$, where$\overline{B_{\rho 0}(a^i)} = \{x \in \mathbb{R}^N;\ |x - z| \leq \rho_0\}$.

$(E_{f.g})$admits at least k positive solutions when f and g are small enough.

We aim to investigate how the coefficient $h(x)$ of the critical nonlinearity affects the number ofpositive solutions of $(E_{f.g})$ when $1 < q < 2$ in this work. We try to consider the relationship between the number ofpositive solutions and the topology of the global maximum set of h by the idea of category. Furthermore, we will study $(E_{f.g})$ under the conditions (H_1)–(H_3), i.e., we do not need to assumef, g, h are positive solutions and $0 \in \Omega$. We have the following.

Remark (3.2) [3]:

Suppose (A_1)–(A_3) hold. By Theorem (3.20), we obtain that$\left(E_{f.\mathrm{g}}\right)$ has at least $k+1$ positive solutions when$\|f\|_{Lq^*}$and $\|\mathrm{g}\|_{Lq^*}$are small enough.

We give some notations and preliminary results. Then, we discusssome concentration behavior and prove Theorem (3.20).

We propose to study $\left(E_{f.\mathrm{g}}\right)$ in the framework of the Sobolev space $H = H_0^1(\Omega)\times H_0^1(\Omega)$ using the standard norm

$$\|(u,v)\|_H = \left(\int_\Omega |\nabla u|^2 + |\nabla v|^2 \mathrm{d}x\right)^{\frac{1}{2}}.$$

Denote

$$S_{\alpha,\beta} := \inf_{(u,v)\in H\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 + |\nabla v|^2 \mathrm{d}x}{\left(\int_\Omega |u|^\alpha + |v|^\beta \mathrm{d}x\right)^{\frac{2}{\alpha+\beta}}}.$$

wededuce that

$$S_{\alpha,\beta} = \left(\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right)S.$$

where S is the best Sobolev constant, that is

$$S := \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\,\mathrm{d}x}{\left(\int_\Omega |u|^{2^*}\,\mathrm{d}x\right)^{\frac{2}{2^*}}}.$$

It is well known that S is independent of Ω, and for each $\varepsilon > 0$,

$$u_\varepsilon(x) = \frac{[N(N-2)\varepsilon^2]^{(N-2)/4}}{(\varepsilon^2 + |x|^2)^{(N-2)/2}} \qquad (1)$$

is a positive solution of critical problem

$$-\Delta u = |u|^{2^*-2}u \quad \text{in}\mathbb{R}^N$$

with $\int_{\mathbb{R}^N}|\nabla v_\varepsilon|^2\,\mathrm{d}x = \int_{\mathbb{R}^N}|v_\varepsilon|^{2^*}\,\mathrm{d}x = \frac{1}{N}S^{N/2}$. Actually, S is never attained on a domain $\Omega \neq \mathbb{R}^N$.

Positive solutions to $(E_{f,\mathrm{g}})$ will be obtained as critical points of the corresponding energy functional $I_{f,\mathrm{g}} : H \to \mathbb{R}$ given by

$$I_{f,\mathrm{g}}(u,v) = \frac{1}{2}\|(u,v)\|_H^2 - \frac{1}{q}\int_\Omega \left(fu_+^q + \mathrm{g}v_+^q\right)\mathrm{d}x - \frac{1}{\alpha+\beta}\int_\Omega hu_+^\alpha v_+^\beta\,\mathrm{d}x,$$

where $u_+ = \max\{u,0\}$ and $v_+ = \max\{v,0\}$. From the assumption, it is easy to prove that $I_{f,\mathrm{g}}$ is well defined in H and $I_{f,\mathrm{g}} \in C^2(H,\mathbb{R})$.

As $I_{f,\mathrm{g}}$ is not bounded below on H, we consider the behaviors of $I_{f,\mathrm{g}}$ on the Nehari manifold

$$N_{f,\mathrm{g}} = \left\{(u,v) \in H\backslash\{0\};\ I'_{f,\mathrm{g}}(u,v)(u,v) = 0\right\}.$$

Clearly, $(u,v) \in N_{f,\mathrm{g}}$ if and only if

$$\int_\Omega |\nabla u|^2 + |\nabla v|^2\,\mathrm{d}x - \int_\Omega \left(fu_+^q + \mathrm{g}v_+^q\right)\mathrm{d}x - \int_\Omega hu_+^\alpha v_+^\beta\,\mathrm{d}x = 0.$$

On the Nehari manifold $N_{f,\mathrm{g}}$, from the Sobolev embedding theorem and the Young inequality,

$$I_{f,\mathrm{g}}(u,v) = \left(\frac{1}{2}-\frac{1}{2^*}\right)\int_\Omega |\nabla u|^2 + |\nabla v|^2\,\mathrm{d}x$$

$$-\left(\frac{1}{q}-\frac{1}{2^*}\right)\int_\Omega \left(fu_+^q + \mathrm{g}v_+^q\right)\mathrm{d}x$$

$$\geq \left(\frac{1}{2}-\frac{1}{2^*}\right)\|(u,v)\|_H^2 - \left(\frac{1}{q}-\frac{1}{2^*}\right)\left(\|f_+\|_{L^{q^*}} + \|\mathrm{g}_+\|_{L^{q^*}}\right)C\|(u,v)\|_H^q \tag{2}$$

$$\geq -\left(\|f_+\|_{L^{q^*}} + \|\mathrm{g}_+\|_{L^{q^*}}\right)^{2/(2-q)}C, \tag{3}$$

where C denotes positive constants (possibly different) independent of $(u, v) \in H$. Let

$$\psi_{f,g}(u, v) := I'_{f,g}(u, v)(u, v)$$

$$= \int_\Omega |\nabla u|^2 + |\nabla v|^2 \, dx - \int_\Omega \left(f u_+^q + g v_+^q\right) dx - \int_\Omega h u_+^\alpha v_+^\beta \, dx.$$

Then for $(u, v) \in N_{f,g}$,

$$\psi'_{f,g}(u, v)(u, v) = (2 - q)\|(u, v)\|_H^2 - (2^* - q) \int_\Omega h u_+^\alpha v_+^\beta \, dx \tag{4}$$

$$= (2 - 2^*)\|(u, v)\|_H^2 + (2^* - q) \int_\Omega \left(f u_+^q + g v_+^q\right) dx. \tag{5}$$

We split $N_{f,g}$ into three parts:

$N_{f,g}^+ = \{(u, v) \in N_{f,g};\ \psi'_{f,g}(u, v)(u, v) > 0\}$;

$N_{f,g}^0 = \{(u, v) \in N_{f,g};\ \psi'_{f,g}(u, v)(u, v) = 0\}$;

$N_{f,g}^- = \{(u, v) \in N_{f,g};\ \psi'_{f,g}(u, v)(u, v) < 0\}$.

In the sequel, we shall use Λ_* to denote different small parameters. Then we have the following results.

Lemma (3.3) [3]:

Suppose that (u_0, v_0) is a local minimum for $I_{f,g}$on $N_{f,g}$. Then, if $(u_0, v_0) \notin N_{f,g'}^0$, (u_0, v_0) is a critical point of $I_{f,g}$.

Lemma (3.4) [3]:

There exists $\Lambda_* > 0$ such that if $\|f_+\|_{Lq^*} + \|g_+\|_{Lq^*} \in (0, \Lambda_*), N^0_{f,g} = \emptyset$. By Lemma (3.4), for $\|f_+\|_{Lq^*} + \|g_+\|_{Lq^*} \in (0, \Lambda_*)$, we write $N_{f,g} = N^+_{f,g} \cup N^-_{f,g}$and define

$$\theta^+_{f,g} = \inf_{(u,v)\in N^+_{f,g}} I_{f,g}(u,v)\,; \qquad \theta^-_{f,g} = \inf_{(u,v)\in N^-_{f,g}} I_{f,g}(u,v).$$

For each $(u,v) \in H$ with$\int_\Omega \; h u_+^\alpha v_+^\beta \, \mathrm{d}x > 0$, set

$$t_{\max} = \left(\frac{(2-q)\|(u,v)\|_H^2}{(2^*-q)\int_\Omega \; h u_+^\alpha v_+^\beta \, \mathrm{d}x} \right)^{\frac{1}{\alpha+\beta-2}} > 0.$$

Then

Lemma (3.5) [3]:

For each $(u,v) \in H$ with$\int_\Omega \; h u_+^\alpha v_+^\beta \, \mathrm{d}x > 0$, we have the following.

(i) If $\int_\Omega \left(f u_+^q + g v_+^q\right) \mathrm{d}x \leq 0$, there is a unique $t^- > t_{\max}$such that $(t^-u, t^-v) \in N^-_{f,g}$and $I_{f,g}(t^-u, t^-v) = \sup_{t\geq 0} I_{f,g}(tu, tv)$.

(ii) If $\int_\Omega \left(f u_+^q + g v_+^q\right) \mathrm{d}x > 0$, there are unique $0 < t^+ < t_{\max} < t^-$such that $(t^+u, t^+v) \in N^+_{f,g}, (t^-u, t^-v) \in N^-_{f,g}$and $I_{f,g}(t^+u, t^+v) = \inf_{0\leq t\leq t_{\max}} I_{f,g}(tu, tv)\,; I_{f,g}(t^-u, t^-v) = \sup_{t\geq 0} I_{f,g}(tu, tv)$.

Lemma (3.6) [3]:

If $\|f_+\|_{Lq^*} + \|g_+\|_{Lq^*} \in (0, \Lambda_*)$, then

(i) $\theta^+_{f,g} < 0$;

(ii) $\theta^-_{f,g} \geq \rho_0$for some $\rho_0 > 0$.

Remark (3.7) [3]:

From Lemmas (3.5) and (3.6), it is easy to know if $(u,v) \in N^-_{f,g}$,

$$\int_{\Omega} hu_{+}^{\alpha}v_{+}^{\beta}\,\mathrm{d}x > 0.$$

Next we establish that $I_{f,\mathrm{g}}$satisfies the $(\mathrm{PS})_c$-condition for $c \in \left(-\infty, \theta_{f,\mathrm{g}}^{+} + \frac{1}{N}S_{\alpha,\beta}^{N/2}\right)$.

Lemma (3.8) [3]:

For $\|f_{+}\|_{Lq^*} + \|\mathrm{g}_{+}\|_{Lq^*} \in (0, \Lambda_*), I_{f,\mathrm{g}}$satisfies the $(\mathrm{PS})_c$-condition for $c \in \left(-\infty, \theta_{f,\mathrm{g}}^{+} + \frac{1}{N}S_{\alpha,\beta}^{N/2}\right)$.

Proof:

Let $\{(u_k, v_k)\} \subset H$ be a $(\mathrm{PS})_c$-sequence for $I_{f,\mathrm{g}}$and $c \in \left(-\infty, \theta_{f,\mathrm{g}}^{+} + \frac{1}{N}S_{\alpha,\beta}^{N/2}\right)$. After a standard argument,we know that $\{(u_k, v_k)\}$ is bounded in H. Thus, there exist a subsequence still denoted by $\{(u_k, v_k)\}$ and $(u, v) \in H$ such that$(u_k, v_k) \rightharpoonup (u, v)$ weakly in H. By the compactness of Sobolev embedding, we get

- $\int_{\Omega} \left(f(u_k)_{+}^{q} + \mathrm{g}(v_k)_{+}^{q}\right)\mathrm{d}x = \int_{\Omega} \left(fu_{+}^{q} + \mathrm{g}v_{+}^{q}\right)\mathrm{d}x + o(1);$
- $\|(u_k - u, v_k - v)\|_H^2 = \|(u_k, v_k)\|_H^2 - \|(u, v)\|_H^2 + o(1);$
- $\int_{\Omega} h(u_k - u)_{+}^{\alpha}(v_k - v)_{+}^{\beta}\,\mathrm{d}x = \int_{\Omega} h(u_k)_{+}^{\alpha}(v_k)_{+}^{\beta}\,\mathrm{d}x - \int_{\Omega} hu_{+}^{\alpha}v_{+}^{\beta}\,\mathrm{d}x + o(1).$

Moreover, we can obtain $I'_{f,\mathrm{g}}(u, v) = 0$ in H^{-1}(the dual space of H). Since $I'_{f,\mathrm{g}}(u_k, v_k) = c + o(1)$and $I'_{f,\mathrm{g}}(u_k, v_k) = o(1)$ in H^{-1}, we deduce that

$$\frac{1}{2}\|(u_k - u, v_k - v)\|_H^2 - \frac{1}{2^*}\int_{\Omega} h(u_k - u)_{+}^{\alpha}(v_k - v)_{+}^{\beta}\,\mathrm{d}x = c - I_{f,\mathrm{g}}(u, v) + o(1) \quad (6)$$

and

$$o(1) = I'_{f,\mathrm{g}}(u_k, v_k)(u_k - u, v_k - v) = \left(I'_{f,\mathrm{g}}(u_k, v_k) - I'_{f,\mathrm{g}}(u, v)\right)(u_k - u, v_k - v)$$

$$= \|(u_k - u, v_k - v)\|_H^2 - \int_{\Omega} h(u_k - u)_{+}^{\alpha}(v_k - v)_{+}^{\beta}\,\mathrm{d}x + o(1).$$

Now we may assume that

$$\|(u_k - u, v_k - v)\|_H^2 \to l \text{ and} \int_\Omega h(u_k - u)_+^\alpha (v_k - v)_+^\beta \,\mathrm{d}x \to l \text{ as } k \to \infty,$$

for some $l \in [0, +\infty)$.

Suppose $l \neq 0$ and notice the fact $h \leq 1$, using the Sobolev embedding theorem and passing to the limit as $k \to \infty$, wehave

$$l \geq S_{\alpha,\beta} l^{\frac{2}{2^*}},$$

that is,

$$l \geq S_{\alpha,\beta}^{N/2}. \tag{7}$$

Then by (6), (7) and $(u, v) \in N_{f,g} \cup \{0\}$,

$$c = I_{f,g}(u, v) + \frac{1}{N} \geq \theta_{f,g}^+ \frac{1}{N} S_{\alpha,\beta}^{N/2},$$

which contradicts the definition of c. Hence $l = 0$, i.e., $(u_k, v_k) \to (u, v)$ strongly in H.

Then we obtain the existence of a local minimizer for $I_{f,g}$on $N_{f,g}^+$.

Lemma (3.9) [3]:

For $\|f_+\|_{Lq^*} + \|g_+\|_{Lq^*} \in (0, \Lambda_*)$, the functional $I_{f,g}$has a minimizer $\left(u_{f,g}^+, v_{f,g}^+\right) \in N_{f,g}^+$and it satisfies:

(i) $I_{f,g}\left(u_{f,g}^+, v_{f,g}^+\right) = \theta_{f,g}^+$;

(ii) $\left(u_{f,g}^+, v_{f,g}^+\right)$is a positive solution of $\left(E_{f,g}\right)$;

(iii) $I_{f,g}\left(u_{f,g}^+, v_{f,g}^+\right) \to 0$as $\|f_+\|_{Lq^*}, \|g_+\|_{Lq^*} \to 0$.

(iv) $\left\|\left(u_{f,g}^+, v_{f,g}^+\right)\right\|_H \to 0$as $\|f_+\|_{Lq^*}, \|g_+\|_{Lq^*} \to 0$.

Proof:

(i)–(ii) are consequences . Moreover, by (3) and Lemma (3.6),

$$0 > I_{f,g}\left(u^+_{f,g}, v^+_{f,g}\right) \geq -\left(\|f_+\|_{Lq^*} + \|g_+\|_{Lq^*}\right)^{2/(2-q)} C.$$

We obtain $I_{f,g}\left(u^+_{f,g}, v^+_{f,g}\right) \to 0$ as $\|f_+\|_{Lq^*}, \|g_+\|_{Lq^*} \to 0$.

Now we show (iv). By $\left(u^+_{f,g}, v^+_{f,g}\right) \in N^+_{f,g}$ and (5),

$$\left\|\left(u^+_{f,g}, v^+_{f,g}\right)\right\|^2_H \leq \frac{2^* - q}{2^* - 2} \int_\Omega \left(f_+\left(u^+_{f,g}\right)^q + g_+\left(u^+_{f,g}\right)^q\right) dx$$

$$\leq C\left(\|f_+\|_{Lq^*} + \|g_+\|_{Lq^*}\right)\left\|\left(u^+_{f,g}, v^+_{f,g}\right)\right\|^q_H. \qquad (8)$$

Since $I_{f,g}$ is coercive and bounded below on $N_{f,g}$, $\left(u^+_{f,g}, v^+_{f,g}\right)$ is bounded in H and so that by (8) we know

$$\left\|\left(u^+_{f,g}, v^+_{f,g}\right)\right\|^{2-q}_H \leq C\left(\|f_+\|_{Lq^*} + \|g_+\|_{Lq^*}\right).$$

Then

$$\left\|\left(u^+_{f,g}, v^+_{f,g}\right)\right\|_H \to 0 \quad \text{as} \|f_+\|_{Lq^*} + \|g_+\|_{Lq^*} \to 0$$

Now, we will recall and prove some lemmas which are crucial in the proof of the main theorem. For $b > 0$, we define

$$J^b_\infty(u, v) = \frac{1}{2}\|(u, v)\|^2_H - \frac{b}{2^*}\int_\Omega h u^\alpha_+ v^\beta_+ \, dx$$

and

$$N^b_\infty(u, v) = \{(u, v) \in H\backslash\{0\};\ (J^b_\infty)'(u, v)(u, v) = 0\}.$$

Then we have the following.

Lemma (3.10) [3]:

For each $(u, v) \in N^-_{f,g}$, we have the following.

(i) There is a unique $t^b_{(u,v)}$ such that $\left(t^b_{(u,v)}u, t^b_{(u,v)}v\right) \in N^b_\infty$ and

$$\max_{t\geq 0} J_\infty^b(tu,tv) = J_\infty^b\left(t_{(u,v)}^b u, t_{(u,v)}^b v\right) = \frac{1}{N} b^{\frac{2-N}{2}} \left(\frac{\|(u,v)\|_H^{2^*}}{\int_\Omega \ h u_+^\alpha v_+^\beta \, \mathrm{d}x}\right)^{\frac{N-2}{2}}.$$

(ii) For $\mu \in (0,1)$, there is a unique $t_{(u,v)}^1$ such that $\left(t_{(u,v)}^1 u, t_{(u,v)}^1 v\right) \in N_\infty^1$. Moreover,

$$J_\infty^1\left(t_{(u,v)}^1 u, t_{(u,v)}^1 v\right) \leq (1-\mu)^{-\frac{N}{2}} \left(I_{f,\mathrm{g}}(u,v) + \frac{2-q}{2q} \mu^{\frac{q}{q-2}} C\left(\|f_+\|_{Lq^*} + \|\mathrm{g}_+\|_{Lq^*}\right)^{\frac{2}{2-q}}\right).$$

Proof:

(i) For each $u \in N_{f,\mathrm{g}}^-$, let

$$\bar{h}(t) = J_\infty^b(tu,tv) = \frac{1}{2} t^2 \|(u,v)\|_H^2 - \frac{b}{2^*} t^{2^*} \int_\Omega \ h u_+^\alpha v_+^\beta \, \mathrm{d}x.$$

Then since Remark (3.7), we have $\bar{h}(t) \to -\infty$ as $t \to \infty$,

$$\bar{h}'(t) = t\|(u,v)\|_H^2 - b t^{2^*-1} \int_\Omega \ h u_+^\alpha v_+^\beta \, \mathrm{d}x$$

and

$$\bar{h}''(t) = t\|(u,v)\|_H^2 - b(2^*-1) t^{2^*-2} \int_\Omega \ h u_+^\alpha v_+^\beta \, \mathrm{d}x.$$

Set

$$t_{(u,v)}^b = \left(\frac{\|(u,v)\|_H^{2^*}}{\int_\Omega \ b h u_+^\alpha v_+^\beta \, \mathrm{d}x}\right)^{\frac{1}{2^*-2}} > 0.$$

Then $h'\left(t_{(u,v)}^b\right) = 0, t_{(u,v)}^b u \in N_\infty^b$ and $h''\left(t_{(u,v)}^b\right) = (2-2^*)\|(u,v)\|_H^2 < 0$. Hence there is a unique $t_{(u,v)}^b$ such that $\left(t_{(u,v)}^b u, t_{(u,v)}^b v\right) \in N_\infty^b$ and

$$\max_{t\geq 0} J_\infty^b(tu,tv) = J_\infty^b\left(t_{(u,v)}^b u, t_{(u,v)}^b v\right) = \frac{1}{N} b^{\frac{2-N}{2}} \left(\frac{\|(u,v)\|_H^{2^*}}{\int_\Omega hu_+^\alpha v_+^\beta \,\mathrm{d}x}\right)^{\frac{N-2}{2}}.$$

(ii) For $\mu \in (0,1)$, we have

$$\int_\Omega f_+\left(t_{(u,v)}^b u\right)_+^q + \mathrm{g}_+\left(t_{(u,v)}^b v\right)_+^q \mathrm{d}x \leq \left(\|f_+\|_{q^*} + \|\mathrm{g}_+\|_{q^*}\right) C \left\|\left(t_{(u,v)}^b u, t_{(u,v)}^b v\right)\right\|_H^q$$

$$\leq \frac{2-q}{2}\left(\left(\|f_+\|_{q^*} + \|\mathrm{g}_+\|_{q^*}\right) C \mu^{-\frac{q}{2}}\right)^{\frac{2}{2-q}} + \frac{q}{2}\left(\mu^{\frac{q}{2}} \left\|\left(t_{(u,v)}^b u, t_{(u,v)}^b v\right)\right\|_H^q\right)^{\frac{2}{q}}$$

$$= \frac{2-q}{2} \mu^{\frac{q}{q-2}} C\left(\|f_+\|_{q^*} + \|\mathrm{g}_+\|_{q^*}\right)^{\frac{2}{2-q}} + \frac{q\mu}{2}\left\|\left(t_{(u,v)}^b u, t_{(u,v)}^b v\right)\right\|_H^2.$$

Then let $b = \frac{1}{1-\mu}$ and by part (i),

$$I_{f,\mathrm{g}}(u,v) = \max_{t\geq 0} I_{f,\mathrm{g}}(tu,tv) \geq I_{f,\mathrm{g}}\left(t_{(u,v)}^{\frac{1}{1-\mu}} u, t_{(u,v)}^{\frac{1}{1-\mu}} v\right)$$

$$\geq \frac{1-\mu}{2}\left\|\left(t_{(u,v)}^{\frac{1}{1-\mu}} u, t_{(u,v)}^{\frac{1}{1-\mu}} v\right)\right\|_H^2 - \frac{1}{2^*}\left(t_{(u,v)}^{\frac{1}{1-\mu}}\right)^{2^*} \int_\Omega hu_+^\alpha v_+^\beta \,\mathrm{d}x$$
$$- \frac{2-q}{2q} \mu^{\frac{q}{q-2}} C\left(\|f_+\|_{q^*} + \|\mathrm{g}_+\|_{q^*}\right)^{\frac{2}{2-q}}$$

$$= (1-\mu) J_\infty^{\frac{1}{1-\mu}}\left(t_{(u,v)}^{\frac{1}{1-\mu}} u, t_{(u,v)}^{\frac{1}{1-\mu}} v\right) - \frac{2-q}{2q} \mu^{\frac{q}{q-2}} C\left(\|f_+\|_{q^*} + \|\mathrm{g}_+\|_{q^*}\right)^{\frac{2}{2-q}}$$

$$= (1-\mu)^{\frac{N}{2}} \frac{1}{N}\left(\frac{\|(u,v)\|_H^{2^*}}{\int_\Omega hu_+^\alpha v_+^\beta \,\mathrm{d}x}\right)^{\frac{N-2}{2}} - \frac{2-q}{2q} \mu^{\frac{q}{q-2}} C\left(\|f_+\|_{q^*} + \|\mathrm{g}_+\|_{q^*}\right)^{\frac{2}{2-q}}$$

$$= (1-\mu)^{\frac{N}{2}} J_\infty^1\left(t_{(u,v)}^1 u, t_{(u,v)}^1 v\right) - \frac{2-q}{2q} \mu^{\frac{q}{q-2}} C\left(\|f_+\|_{q^*} + \|\mathrm{g}_+\|_{q^*}\right)^{\frac{2}{2-q}}.$$

This completes the proof.

Following the same method as and Remark (3.1), let $\eta(x) \in C_0^\infty(\mathbb{R}^N)$be a radially symmetric function with $0 \le \eta \le 1$, $|\nabla\eta| \le C$, and

$$\eta(x) = \begin{cases} 1, & \text{if} |x| \le \frac{r_0}{2}, \\ 0, & \text{if} |x| \ge r_0. \end{cases}$$

For any $z \in M$, we define

$$\omega_{\varepsilon,z}(x) = \eta(x-z)v_\varepsilon(x-z)$$

where $v_\varepsilon(x)$ is given by (1). We know

$$\int_\Omega |\nabla\omega_{\varepsilon,z}|^2 \, \mathrm{d}x = S^{\frac{N}{2}} + O(\varepsilon^{N-2}) \text{and} \int_\Omega |\omega_{\varepsilon,z}|^{2^*} \, \mathrm{d}x = S^{\frac{N}{2}} + O(\varepsilon^N). \qquad (9)$$

We have the following.

Lemma (3.11) [3]:

$$\int_\Omega h|\omega_{\varepsilon,z}|^{2^*} \, \mathrm{d}x = \begin{cases} S^{N/2} + o(\varepsilon^2), & \text{if } N \ge 6, \\ S^{N/2} + o\left(\varepsilon^{\frac{N-2}{2}}\right), & \text{if } 3 \le N \le 5. \end{cases}$$

Then we have the following results.

Lemma (3.12) [3]:

There exist $\varepsilon_0 > 0$ small enough such that for $\varepsilon \in (0, \varepsilon_0)$, we have $\sigma(\varepsilon_0) > 0$ and

$$\sup_{t\ge0} I_{f,\mathrm{g}}\left(u_{f,\mathrm{g}}^+ + t\sqrt{\alpha}\omega_{\varepsilon,z}, v_{f,\mathrm{g}}^+ + t\sqrt{\beta}\omega_{\varepsilon,z}\right) < \theta_{f,\mathrm{g}}^+ + \frac{1}{N} S_{\alpha,\beta}^{N/2} - \sigma(\varepsilon_0) \text{ uniformly in } z \in M$$

Furthermore, there exists $t_z^- > 0$such that

$$\left(u_{f,\mathrm{g}}^+ + t_z^-\sqrt{\alpha}\omega_{\varepsilon,z}, v_{f,\mathrm{g}}^+ + t_z^-\sqrt{\beta}\omega_{\varepsilon,z}\right) \in N_{f,\mathrm{g}}^- \text{for all } \; z \in M.$$

Lemma (3.13) [3]:

We have

$$\inf_{(u,v)\in N_{\infty}^{1}} J_{\infty}^{1}(u,v) = \inf_{(u,v)\in N^{\infty}} J^{\infty}(u,v) = \frac{1}{N} S_{\alpha,\beta}^{N/2},$$

where $J^{\infty}(u,v) = \frac{1}{2}\|(u,v)\|_{H}^{2} - \frac{1}{2^{*}}\int_{\Omega} u_{+}^{\alpha} + v_{+}^{\beta}\, \mathrm{d}x$ and $N^{\infty} = \{(u,v) \in H\backslash\{0\};\ (J^{\infty})'(u,v)(u,v) = 0\}$.

Proof:

We have

$$\inf_{(u,v)\in N^{\infty}} J^{\infty}(u,v) = \frac{1}{N} S_{\alpha,\beta}^{N/2}.$$

Thus it suffices to show that $\inf_{(u,v)\in N_{\infty}^{1}} J_{\infty}^{1}(u,v) = \frac{1}{N} S_{\alpha,\beta}^{N/2}$. Since

$$\max_{t\geq 0}\left(\frac{a}{2}t^{2} - \frac{b}{2^{*}}t^{2^{*}}\right) = \frac{1}{N}\left(\frac{a}{b^{2/2^{*}}}\right)^{N/2} \text{ for any } a > 0 \text{ and } b > 0,$$

by (9) and Lemma (3.11) we deduce that

$$\sup_{t\geq 0} J_{\infty}^{1}\left(t\sqrt{\alpha}\omega_{\varepsilon,z}, t\sqrt{\beta}\omega_{\varepsilon,z}\right) = \frac{1}{N}\left(\frac{(\alpha+\beta)\int_{\Omega} \left|\nabla\omega_{\varepsilon,z}\right|^{2} \mathrm{d}x}{\left(\alpha^{\frac{\alpha}{2}}\beta^{\frac{\beta}{2}}\int_{\Omega} h\left|\omega_{\varepsilon,z}\right|^{2^{*}} \mathrm{d}x\right)^{2/2^{*}}}\right)^{N/2}$$

$$= \frac{1}{N} S_{\alpha,\beta}^{N/2} + O(\varepsilon^{N-2}).$$

Then we obtain

$$\inf_{(u,v)\in N_{\infty}^{1}} J_{\infty}^{1}(u,v) \leq \frac{1}{N} S_{\alpha,\beta}^{N/2}, \text{ as } \varepsilon \to 0^{+}.$$

Since $h \leq 1$, for each $(u,v) \in H\backslash\{0\}$, we have

$$\sup_{t\geq 0} J^{\infty}(tu,tv) \leq \sup_{t\geq 0} J_{\infty}^{1}(tu,tv).$$

Hence

$$\frac{1}{N}S_{\alpha,\beta}^{N/2} = \inf_{(u,v)\in N^{\infty}} J^{\infty}(u,v) = \inf_{(u,v)\in H\setminus\{0\}} \sup_{t\geqq 0} J^{\infty}(tu,tv)$$

$$\leq \inf_{(u,v)\in\setminus\{0\}} \sup_{t\geq 0} J_{\infty}^{1}(tu,tv) = \inf_{(u,v)\in N_{\infty}^{1}} J_{\infty}^{1}(u,v) \leq \frac{1}{N}S_{\alpha,\beta}^{N/2}.$$

This completes the proof.

We use the idea of category to get positive solutions of $E_{f,g}$ in H and give the proof of Theorem (3.20).Initially, we give the following two lemmas related to the category.

Proposition (3.14) [3]:

Let R be a $C^{1,1}$complete Riemannian manifold (modeled on a Hilbert space) and assume $F \in C^1(R,\mathbb{R})$boundedfrom below. Let $-\infty < \inf_R F < a < b < +\infty$. Suppose that h satisfies the (PS)-condition on the sublevel $\{u \in R; F(u) \leq b\}$and that a is not a critical level for F. Then

$$\#\,\{u \in F^a;\ \nabla F(u) = 0\} \geq \operatorname{cat}_{F^a}(F^a),$$

where $h^a \equiv \{u \in H; h(u) \leq a\}$.

Proposition (3.15) [3]:

Let $Q, \Omega^+$and Ω^-be closed sets with $\Omega^- \subset \Omega^+$. Let $\phi : Q \to \Omega^+, \varphi : \Omega^- \to Q$ be two continuous maps such that $\phi \circ \varphi$ is homotopically equivalent to the embedding $j : \Omega^- \to \Omega^+$. Then $\operatorname{cat}_Q(Q) \geq \operatorname{cat}_{\Omega^+}(\Omega^-)$.

The proof of Theorem (3.20) is based on Propositions (3.14) and (3.15). To argue further, we need to introduce the following lemma.

Lemma (3.16) [3]:

Let $\{(u_k,v_k)\} \subset H$ be a nonnegative function sequence with$\int_\Omega (u_k)_+^{\alpha}(v_k)_+^{\beta}\,dx = 1$ and $\|(u_k,v_k)\|_H^2 \to S_{\alpha,\beta}$. Thenthere exists a sequence $\{(x_k,\varepsilon_k)\} \in \mathbb{R}^N \times \mathbb{R}^+$such that

$$\omega_k(x) = \left(\omega_k^1(x),\omega_k^2(x)\right) := \varepsilon_k^{\frac{N-2}{2}}\left(u_k(\varepsilon_k x + x_k), v_k(\varepsilon_k x + x_k)\right)$$

contains a convergent subsequence denoted again by $\{\omega_k\}$ such that $\omega_k \to \omega = (\omega^1, \omega^2)$ strongly in $\mathfrak{D}^{1,2}(\mathbb{R}^N) \times \mathfrak{D}^{1,2}(\mathbb{R}^N)$ with $\omega^1(x) > 0$ and $\omega^2(x) > 0$ in $\mathbb{R}^N$. Moreover, we have $\varepsilon_k \to 0$ and $x_k \to x_0 \in \overline{\Omega}$ as $k \to \infty$.

Next we define the continuous map $\Phi : H \backslash G \to \mathbb{R}^N$ by

$$\Phi(u, v) := \frac{\int_\Omega x\left(u - u_{f,g}^+\right)_+^\alpha \left(v - v_{f,g}^+\right)_+^\beta \mathrm{d}x}{\int_\Omega \left(u - u_{f,g}^+\right)_+^\alpha \left(v - v_{f,g}^+\right)_+^\beta \mathrm{d}x},$$

where $G = \left\{(u, v) \in H;\ \int_\Omega \left(u - u_{f,g}^+\right)_+^\alpha \left(v - v_{f,g}^+\right)_+^\beta \mathrm{d}x = 0\right\}$. Then we have the following.

Lemma (3.17) [3]:

For each $0 < \delta < r_0$, there exist $\Lambda_\delta, \delta_0 > 0$ such that if

$(u, v) \in N_\infty^1, N_\infty^1(u, v) < \frac{1}{N} S_{\alpha,\beta}^{N/2} + \delta_0$ and $\|f_+\|_{Lq^*} + \|g_+\|_{Lq^*} < \Lambda_\delta$, then $\Phi(u, v) \in M_\delta$.

Proof:

Suppose the contrary. Then there exists a sequence $\{(u_k, v_k)\} \subset N_\infty^1$ such that $J_\infty^1(u_k, v_k) = \frac{1}{N} S_{\alpha,\beta}^{N/2} + o(1), \|f_+\|_{Lq^*} + \|g_+\|_{Lq^*} = o(1)$, and

$$\Phi(u_k, v_k) \notin M_\delta \text{for all } k.$$

It is easy to show that $\{(u_k, v_k)\}$ is bounded in H and there is a sequence $\{t_k^\infty\} \subset \mathbb{R}^+$ such that $(t_k^\infty u_k, t_k^\infty v_k) \in N^\infty$ and

$$\frac{1}{N} S_{\alpha,\beta}^{N/2} \le J^\infty(t_k^\infty u_k, t_k^\infty v_k) \le J_\infty^1(t_k^\infty u_k, t_k^\infty v_k) \le J_\infty^1(u_k, v_k) = \frac{1}{N} S_{\alpha,\beta}^{N/2} + o(1).$$

We obtain $t_k^\infty = 1 + o(1)$ as $k \to \infty$ and

$$\lim_{k\to\infty} J^\infty(u_k, v_k) = \lim_{k\to\infty} \frac{1}{N} \|(u_k, v_k)\|_H^2 = \lim_{k\to\infty} \frac{1}{N} \int_\Omega (u_k)_+^\alpha + (v_k)_+^\beta \mathrm{d}x$$

$$= \lim_{k\to\infty} \frac{1}{N} \int_{\Omega} h(u_k)_+^{\alpha}(v_k)_+^{\beta}\,\mathrm{d}x$$

$$= \frac{1}{N} S_{\alpha,\beta}^{N/2} + o(1). \qquad (10)$$

Define

$$u_k = \left(\frac{(u_k)_+}{\left(\int_{\Omega} (u_k)_+^{\alpha}(v_k)_+^{\beta}\,\mathrm{d}x\right)^{1/(\alpha+\beta)}}, \frac{(v_k)_+}{\left(\int_{\Omega} (u_k)_+^{\alpha}(v_k)_+^{\beta}\,\mathrm{d}x\right)^{1/(\alpha+\beta)}} \right).$$

We see that $\int_{\Omega} (U_k^1)_+^{\alpha}(U_k^2)_+^{\beta}\,\mathrm{d}x = 1$. It follows from (10) and the definition of $S_{\alpha,\beta}$ that

$$\lim_{k\to\infty} \left\| (U_k^1, U_k^2) \right\|_H^2 = S_{\alpha,\beta}.$$

By Lemma (3.18), there is a sequence $\{(x_k, \varepsilon_k)\} \in \mathbb{R}^N \times \mathbb{R}^+$ such that $\varepsilon_k \to 0, x_k \to x_0 \in \overline{\Omega}$ and $\omega_k(x) = \varepsilon_k^{\frac{N-2}{2}} \left(U_k^1(\varepsilon_k x + x_k), U_k^2(\varepsilon_k x + x_k) \right) \to (\omega_1, \omega_2)$ strongly in $\mathfrak{D}^{1,2}(\mathbb{R}^N) \times \mathfrak{D}^{1,2}(\mathbb{R}^N)$ with $\omega_1 > 0$ and $\omega_2 > 0$ in $\mathbb{R}^N$ as $k \to \infty$. Then by (10),

$$1 = o(1) + \int_{\Omega} h(U_k^1)_+^{\alpha}(U_k^2)_+^{\beta}\,\mathrm{d}x$$

$$= \varepsilon_k^{-N} \int_{\Omega} h \left(\omega_k^1 \left(\frac{x - x_k}{\varepsilon_k} \right) \right)_+^{\alpha} \left(\omega_k^2 \left(\frac{x - x_k}{\varepsilon_k} \right) \right)_+^{\beta} \mathrm{d}x + o(1) = h(x_0),$$

as $k \to \infty$, which implies $x_0 \in M$. By the Lebesgue dominated convergence theorem again, we have

$$\Phi(u_k, v_k) = \frac{\int_{\Omega} x\left(u_k - u_{f_k,g_k}^+\right)_+^{\alpha}\left(v_k - v_{f_k,g_k}^+\right)_+^{\beta}\,\mathrm{d}x}{\int_{\Omega} \left(u_k - u_{f_k,g_k}^+\right)_+^{\alpha}\left(v_k - v_{f_k,g_k}^+\right)_+^{\beta}\,\mathrm{d}x}$$

$$= \frac{\int_\Omega \; x(u_k)_+^\alpha (v_k)_+^\beta \, \mathrm{d}x}{\int_\Omega \; (u_k)_+^\alpha (v_k)_+^\beta \, \mathrm{d}x} = +o(1), \quad \text{as} \|(f_k)\|_{Lq^*} + \|(\mathrm{g}_k)\|_{Lq^*} \to 0$$

$$= \frac{\varepsilon_k^{-N} \int_\Omega \; x \left(\omega_k^1 \left(\frac{x - x_k}{\varepsilon_k} \right) \right)_+^\alpha \left(\omega_k^2 \left(\frac{x - x_k}{\varepsilon_k} \right) \right)_+^\beta \mathrm{d}x}{\varepsilon_k^{-N} \int_\Omega \; \left(\omega_k^1 \left(\frac{x - x_k}{\varepsilon_k} \right) \right)_+^\alpha \left(\omega_k^2 \left(\frac{x - x_k}{\varepsilon_k} \right) \right)_+^\beta \mathrm{d}x} + o(1),$$

$$\to x_0 \in M \quad \text{as} \quad k \to \infty,$$

which is a contradiction.

Lemma (3.18) [3]:

There exists $\Lambda_\delta > 0$ small enough such that if $\|f_+\|_{Lq^*} + \|\mathrm{g}_+\|_{Lq^*} < \Lambda_\delta$ and $(u,v) \in N_{f,\mathrm{g}}^-$ with $I_{f,\mathrm{g}}(u,v) < \frac{1}{N} S_{\alpha,\beta}^{N/2} + \frac{\delta_0}{2}$ ($\delta_0$is given in Lemma (3.17)), then $\Phi(u,v) \in M_\delta$.

Proof:

By Lemma (3.10), for $\mu \in (0,1)$, there is a unique $t_{(u,v)}^1$such that $\left(t_{(u,v)}^1 u, t_{(u,v)}^1 v\right) \in N_\infty^1$and

$$J_\infty^1\left(t_{(u,v)}^1 u, t_{(u,v)}^1 v\right) \le (1-\mu)^{-\frac{N}{2}} \left(I_{f,\mathrm{g}}(u,v) + \frac{2-q}{2q} \mu^{\frac{q}{q-2}} c\left(\|f_+\|_{Lq^*} + \|\mathrm{g}_+\|_{Lq^*}\right)^{\frac{2}{2-q}} \right).$$

Thus there exists $\Lambda_\delta > 0$ small enough such that if $\|f_+\|_{Lq^*} + \|\mathrm{g}_+\|_{Lq^*} < \Lambda_\delta$ and $I_{f,\mathrm{g}}(u,v) < \frac{1}{N} S_{\alpha,\beta}^{N/2} + \frac{\delta_0}{2}$,

$$J_\infty^1\left(t_{(u,v)}^1 u, t_{(u,v)}^1 v\right) \le \frac{1}{N} S_{\alpha,\beta}^{N/2} + \delta_0.$$

By Lemma (3.17) and $\left\|\left(u^{+}_{f,\mathrm{g}}, v^{+}_{f,\mathrm{g}}\right)\right\|_{H} \to 0$as $\|(f_k)\|_{Lq^*} + \|(\mathrm{g}_k)\|_{Lq^*} \to 0$, we complete the proof.

Now we denote $c_{f,\mathrm{g}} := \theta^{+}_{f,\mathrm{g}} + \frac{1}{N} S^{N/2}_{\alpha,\beta} - \sigma(\varepsilon_0)$and consider the filtration of the manifold of $N^{-}_{f,\mathrm{g}}$as follows:

$$N^{-}_{f,\mathrm{g}}\left(c_{f,\mathrm{g}}\right) ;= \left\{(u,v) \in N^{-}_{f,\mathrm{g}};\ I_{f,\mathrm{g}} \leq c_{f,\mathrm{g}}\right\}.$$

Then $\mathrm{cat}_{M_\delta}(M)$critical points of $I_{f,\mathrm{g}}$will be obtained from $N^{-}_{f,\mathrm{g}}\left(c_{f,\mathrm{g}}\right)$in the following.

Lemma (3.19) [3]:

Let $\delta, \Lambda_\delta > 0$ be as in Lemmas (3.17) and (3.18). Then for $\|f_+\|_{Lq^*} + \|\mathrm{g}_+\|_{Lq^*} < \Lambda_\delta$, $I_{f,\mathrm{g}}$has at least $\mathrm{cat}_{M_\delta}(M)$criticalpoints in $N^{-}_{f,\mathrm{g}}\left(c_{f,\mathrm{g}}\right)$.

Proof:

For $z \in M$, by Lemma (3.12), we can define

$$\Gamma(z) = \left(u^{+}_{f,\mathrm{g}} + t^{-}_{z}\sqrt{\alpha}\,\omega_{\varepsilon,z}, v^{+}_{f,\mathrm{g}} + t^{-}_{z}\sqrt{\beta}\,\omega_{\varepsilon,z}\right) \in N^{-}_{f,\mathrm{g}}\left(c_{f,\mathrm{g}}\right).$$

Furthermore, $I_{f,\mathrm{g}}$satisfies the (PS)-condition on $N^{-}_{f,\mathrm{g}}\left(c_{f,\mathrm{g}}\right)$. Moreover, it follows from Lemma (3.18) that $\Phi\left(N^{-}_{f,\mathrm{g}}\left(c_{f,\mathrm{g}}\right)\right) \subset M_\delta$for$\|f_+\|_{Lq^*} + \|\mathrm{g}_+\|_{Lq^*} < \Lambda_\delta$. Define $\xi : [0,1] \times M \to M_\delta$ by

$$\xi(\theta, z) = \Phi\left(u^{+}_{f,\mathrm{g}} + t^{-}_{z}\sqrt{\alpha}\,\omega_{(1-\theta)\varepsilon,z}, v^{+}_{f,\mathrm{g}} + t^{-}_{z}\sqrt{\beta}\,\omega_{(1-\theta)\varepsilon,z}\right) \in N^{-}_{f,\mathrm{g}}\left(c_{f,\mathrm{g}}\right).$$

Then straightforward calculations provide that $\xi(0,z) = \Phi \circ F(z)$and $\lim_{\theta\to 1} -\xi(\theta, z) = z$. Hence $\Phi \circ F$ is homotopic tothe inclusion $: M \to M_\delta$. By Propositions (3.14) and (3.15), $I_{f,\mathrm{g}}$has at least $\mathrm{cat}_{M_\delta}(M)$critical points in $N^{-}_{f,\mathrm{g}}\left(c_{f,\mathrm{g}}\right)$.

Theorem (3.20) [3]:

Assume (H_1)–(H_3) hold. Then for each $\delta < r_0$, there exists $\Lambda_\delta > 0$ such that if$\|f_+\|_{Lq^*} + \|\mathrm{g}_+\|_{Lq^*} < \Lambda_\delta\left(E_{f,\mathrm{g}}\right)$has at least $\mathrm{cat}_{M_\delta}(M) + 1$ distinct positive solutions, where $f_{+} = \max\{f, 0\}$, $g_{+} = \max\{g, 0\}$, $q^{*} = \frac{2^{*}}{2^{*}-q}$and cat means theLusternik–Schnirelmann category.

Proof:

Note Lemmas (3.9) and (3.19), and applying $N_{f,g}^{+} \cap N_{f,g}^{-} = \emptyset$and the strong maximum principle, weobtain the conclusion of Theorem (3.20).

Chapter 4

Multiple Solutions for aClass ofQuasi-Linear EllipticEquations InvolvingCritical SobolevExponent

With the help of Neharimanifold and a mini-max principle, we prove that problem admits at least two or three positive solutions under different conditions. [4]

Thischapteris concerned with the multiplicity of solutions to the following nonlinear p-Laplacian equation:

$$
(E_{f,g})\begin{cases} -\Delta_p u = g(x)|u|^{p^*-2}u + f(x), & x \in \Omega \\ u > 0, & x \in \Omega \\ u = 0, & x \in \partial\Omega, \end{cases}
$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u), 1 < p < N, p^* = \frac{Np}{N-p}$, $\Omega$is an open bounded domain in $\mathbb{R}^N$with smooth boundary and f, g are two real functions on $\bar{\Omega}$.Moreover, we assume that thedomain Ω satisfies

(H)$B_\rho(0) \cap \Omega = \emptyset$ and $B_{\frac{1}{\rho}}(0)\backslash\overline{B_\rho(0)} \subset \Omega$ for $\rho > 0$is sufficiently small, where $B_r(0) = \{x \in \mathbb{R}^N; |x| < r\}$.

Under the assumption$f(x) \not\equiv 0$ and $g(x) \not\equiv 1$, $(E_{f,g})$ can be regarded as a perturbationproblem of the following equation:

$$
(\bar{E})\begin{cases} -\Delta_p u = |u|^{p^*-2}u, & x \in \Omega \\ u > 0, & x \in \Omega \\ u = 0, & x \in \partial\Omega, \end{cases}
$$

For the case $p = 2$, it is well known the existence of solutions of $(\bar{E})$ is affected by the shapeof the domain Ω. This has been the focus of a great deal of research by several authors. Inparticular, the first striking result is due to that if Ω is star-shaped with respect to some point, $(\bar{E})$ has no solution. However, if Ω is an annulus, $(\bar{E})$ admits a solution. For a non-contractible domain Ω,$(\bar{E})$ has a solution.

When $p = 2, f(x) > 0$ and $g \equiv 1, (\tilde{E})$ admits at least two solutions. The idea is to divide Nehari manifold into differentparts and apply the Ekelandvariational principle. In a similar way, the existence of four solutions for $(E_{f,1})$ in a non-contractible domain for $\|f\|_{L^2(\Omega)}$issufficiently small and $0 \leq f \not\equiv 0$.

For the case $p \neq 2. (\hat{E}_{f,g})$has at least two solutions if $\lambda > 0$ is sufficiently small and f, g satisfy some integrability in $\overline{\Omega}$.

In this work we aim to obtain a better information on the number of solutions of $(E_{f,g})$ byusing the Nehari manifold and a mini-max principle.

Set $X := W_0^{1,p}(\Omega)$and X^*denotes the usual dual space of X. The following assumptionsare used in this chapter:

(f) $0 \leq f \not\equiv 0$ and $f(x) \in X^*$.

(g_1) $g \in C(\overline{\Omega})$ and $g_+ = \max\{g, 0\} \not\equiv 0$ in Ω.

(g_2) $g(x) \geq 0$.

(g_3) $g(x) \equiv 1$.

The main results of this chapter are concluded in the following theorems. In the first tworesults, we consider Ω as a general bounded domain, and for the third result, we assume thatthe domain is non-contractible.

Theorem (4.1) [4]:

Suppose (f) and (g_1) hold. Then there exists $\Lambda_1 > 0$ such that $(E_{f,g})$ has at least one solution if $0 < \|f\|_{X^*} < \Lambda_1$.

Proof:

Suppose (f) and (g_1) hold. From Lemma (4.10), it follows that there exists $\Lambda_1 = \min\{\lambda_1, \lambda_2\}$ such that $(E_{f,g})$ has at least one solution if $0 < \|f\|_{X^*} < \Lambda_1$.

Theorem (4.2) [4]:

Suppose (f), (g_1) and (g_1) hold. Then there exists $\Lambda_2 > 0$ such that $(E_{f,g})$has at least two solutions if $0 < \|f\|_{X^*} < \Lambda_2$.

Proof:

Since (f), (g_1) and (g_2) hold and suppose $0 < \|f\|_{X^*} < \Lambda_2 = \min\{\lambda_1, \lambda_2\}$, we from Lemma (4.10) get the first solution $u_{f,g}^+ \in N_{f,g}^+$and from Lemma (4.11) get the second solution $u_{f,g}^- \in N_{f,g}^-$. Moreover, $N_{f,g}^+ \cap N_{f,g}^- = \emptyset$, this implies that $u_{f,g}^+$and $u_{f,g}^-$ are distinct.

Theorem (4.3) [4]:

Suppose (H), (f) and (g_3) hold. Then there exists $\Lambda_3 > 0$ such that $(E_{f,g})$has at least three solutions if $0 < \|f\|_{X^*} < \Lambda_3$.

Proof:

We complete the proof by Lemmas (4.10), (4.11), (4.22), (4.23) and the fact

$$-\infty < \alpha_{f,1}^+ < 0 < \alpha_{f,1}^- < \frac{1}{2N}\left(c_0^{\frac{N}{p}} + S^{\frac{N}{p}}\right) < \gamma_{f,1} < \frac{2}{N} S^{\frac{N}{p}},$$

if $0 < \|f\|_{X^*} < \Lambda_3$.

Associated with $(E_{f,g})$, we consider the energy functional $J_{f,g}$ for each $u \in X$,

$$J_{f,g}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \, dx - \frac{1}{p^*}\int_\Omega g u_+^{p^*} \, dx - \int_\Omega f u \, dx,$$

where $u_\pm = \pm \max\{\pm u, 0\}$. From the assumption, it is easy to prove that $J_{f,g}$ is well definedin X and $J_{f,g} \in C^1(X, \mathbb{R})$. Furthermore, let $u \in X$ be a critical point of $J_{f,g}$, then

$$\int_\Omega |\nabla u_-|^p \, dx = \int_\Omega f u_- \, dx \leq 0,$$

since $f(x) \geq 0$. This implies that

$$u_{-} = 0 \ \text{ a.e. in}\Omega.$$

Thus $u \geq 0$.

Now we define Harnack inequality (Let u be a non negative weak super solution of the equation

$$\operatorname{div} A(x, u, \nabla u) + B(x, u, \nabla u) = 0$$

in Ω. Let B_r be a ball such that $B_{3r} \Subset \Omega$ and let M be a constant such that $u \leq M$ in B_{3r}. Then there exists c depending on n, M, a_0, b_0, P and the weight v such that

$$\omega^{-1}(B_{2r}) \int_{B_{2r}} u\omega \, dx \leq c\{\min_{B_r} u + h(r)\}$$

where

$$h(r) = \left[\varphi\left(\left(\frac{e}{\omega}\right)^{\frac{p}{p-1}}; 3r\right) + \varphi\left(\frac{g}{\omega}; 3r\right)\right]^{\frac{1}{p}} + \left[\varphi\left(\frac{p}{\omega}; 3r\right)\right]^{\frac{1}{p-1}})$$ [8]. By Harnack inequality, we obtain that $u > 0$ in Ω.

For convenience, we will denote positive constant (possibly different) as c from then on.

Throughout the chapter by $|\cdot|_r$ we denote the L^r-norm. On the space X we consider thenorm

$$\|u\| = \left(\int_{\Omega} |\nabla u|^p \, dx\right)^{\frac{1}{p}}.$$

Set also

$$\mathcal{D}^{1,p}(\mathbb{R}^N) := \left\{u \in L^{p^*}(\mathbb{R}^N);\ \frac{\partial u}{\partial x_i} \in L^p(\mathbb{R}^N) \text{for } i = 1, \dots, N\right\}$$

equipped with the norm

$$\|u\|_{\mathcal{D}^{1,p}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |\nabla u|^p \, dx\right)^{\frac{1}{p}}.$$

We then define the Palais-Smale (simply by (PS)) sequences, (PS)-values, and (PS)-conditionsin X for $J_{f,g}$ as follows.

Definition (4.4) [4]:

(i) For $\beta \in \mathbb{R}$, a sequence $\{u_k\}$ is a $(\mathrm{PS})_\beta$ -sequence in X for $J_{f,g}$ if $J_{f,g}(u_k) = \beta + o(1)$ and $J'_{f,g}(u_k) = o(1)$strongly in X^* as $k \to \infty$;

(ii) $\beta \in \mathbb{R}$ is a (PS)-value in X for $J_{f,g}$ if there exists a $(\mathrm{PS})_\beta$-sequence in X for $J_{f,g}$;

(iii) $J_{f,g}$ satisfies the $(\mathrm{PS})_\beta$-condition in X if every $(\mathrm{PS})_\beta$-sequence in X for $J_{f,g}$ contains a convergent subsequence.

Lemma (4.5) [4]:

Let $\{u_k\}$ be a (PS)-sequence of $J_{f,1}\big(J_{f,g}\text{for } g \equiv 1\big)$ in X. Then there existsa number $n \in \mathbb{N}$, sequences $\{\varepsilon_k^j\}, \{x_k^j\}, 1 \le j \le n$of radii $\varepsilon_k^j \to 0$ (as $k \to \infty$), and points$x_k^j \in \Omega$, a solution $u_0 \in X \subset \mathcal{D}^{1,p}(\mathbb{R}^N)$ to $(E_{f,1})\big((E_{f,g})\text{for } g \equiv 1\big)$, and nontrivial solutions$u^j \in \mathcal{D}^{1,p}(\mathbb{R}^N), 1 \le j \le n$to the "limiting problem" of $(E_{f,1})$, such that a subsequence of $\{u_k\}$, still denoted by $\{u_k\}$, satisfies

$$\left\| u_k - u_0 - \sum_{j=1}^{n} u_k^j \right\|_{\mathcal{D}^{1,p}(\mathbb{R}^N)} \to 0 \text{ as } k \to \infty,$$

here u_k^jdenotes the scaled function

$$u_k^j(x) = \left(\varepsilon_k^j\right)^{-\frac{N-p}{p}} u^j\left(\frac{x - x_k^j}{\varepsilon_k^j}\right), 1 \le j \le n.$$

Moreover,

$$J_{f,g}(u_k) \to J_{f,1}(u_0) + \sum_{j=1}^{n} J^{\infty}(u^j) \text{ as } k \to \infty,$$

where $J^{\infty}(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p \, dx - \frac{1}{p^*}\int_{\mathbb{R}^N} u_+^{p^*} \, dx$.

As $J_{f,g}$ is not bounded below on X, we consider the behaviors of $J_{f,g}$ on the Nehari manifold

$$N_{f,g} = \{u \in X\backslash\{0\};\ u_+ \not\equiv 0 \text{ and} \langle J'_{f,g}(u), u\rangle = 0\},$$

where $\langle\, ,\rangle$ denotes the usual duality between X and X^*. Clearly, for each $u \in X$ with $u_+ \not\equiv 0, u \in N_{f,g}$ if and only if

$$\int_{\Omega} |\nabla u|^p \, dx - \int_{\Omega} g u_+^{p^*} \, dx - \int_{\Omega} f u \, dx = 0.$$

Thus, on the Nehari manifold $N_{f,g}$, we have

$$\begin{aligned} J_{f,g}(u) &= \frac{1}{p}\int_{\Omega} |\nabla u|^p \, dx - \frac{1}{p^*}\int_{\Omega} g u_+^{p^*} \, dx - \int_{\Omega} f u \, dx \\ &= \left(\frac{1}{p} - \frac{1}{p^*}\right)\int_{\Omega} |\nabla u|^p \, dx - \left(1 - \frac{1}{p^*}\right)\int_{\Omega} f u \, dx \\ &\geq \left(\frac{1}{p} - \frac{1}{p^*}\right)\|u\|^p - \left(1 - \frac{1}{p^*}\right)\|f\|_{X^*}\|u\|. \end{aligned} \tag{1}$$

Hence $J_{f,g}$ is coercive and bounded below on $N_{f,g}$.

We now define

$$\psi_{f,g}(u) := \langle J'_{f,g}(u), u\rangle = \int_{\Omega} |\nabla u|^p \, dx - \int_{\Omega} g u_+^{p^*} \, dx - \int_{\Omega} f u \, dx.$$

Then for $u \in N_{f,g}$,

$$\langle \psi_{f,g}(u), u\rangle = (p-1)\int_{\Omega} |\nabla u|^p \, dx - (p^* - 1)\int_{\Omega} g u_+^{p^*} \, dx \tag{2}$$

$$= (p - p^*) \int_\Omega |\nabla u|^p \, dx + (p^* - 1) \int_\Omega f u \, dx. \qquad (3)$$

We split $N_{f,g}$ into three parts:

$$N_{f,g}^{+} = \{u \in N_{f,g};\ \langle \psi'_{f,g}(u), u \rangle > 0\};$$

$$N_{f,g}^{0} = \{u \in N_{f,g};\ \langle \psi'_{f,g}(u), u \rangle = 0\};$$

$$N_{f,g}^{-} = \{u \in N_{f,g};\ \langle \psi'_{f,g}(u), u \rangle < 0\}.$$

Then we have the following results.

Lemma (4.6) [4]:

There exists $\lambda_1 > 0$ such that if $0 < \|f\|_{X^*} < \lambda_1$, we have $N_{f,g}^{0} = \emptyset$.

Proof:

Suppose otherwise, that is $N_{f,g}^{0} \neq \emptyset$ for all $f \in X^*$. Then for $\in N_{f,g}^{0}$, we from(2),(3), and the Sobolev imbedding theorem obtain that there is a positive constant $c$independent of u such that

$$\int_\Omega |\nabla u|^p \, dx \leq c \left(\int_\Omega |\nabla u|^p \, dx \right)^{\frac{p^*}{p}} \text{and} \int_\Omega |\nabla u|^p \, dx \leq \|f\|_{X^*} \left(\int_\Omega |\nabla u|^p \, dx \right)^{\frac{1}{p}}$$

or

$$\int_\Omega |\nabla u|^p \, dx \geq c^{-\frac{p}{p^*-p}} \text{and} \int_\Omega |\nabla u|^p \, dx \leq (\|f\|_{X^*})^{\frac{p}{p-1}}.$$

If $\|f\|_{X^*}$ is sufficiently small, this is impossible. Thus we can conclude that there exists $\lambda_1 > 0$ such that if $0 < \|f\|_{X^*} < \lambda_1$, we have $N_{f,g}^{0} \neq \emptyset$.

For each $u \in X$ with$\int_\Omega g u_+^{p^*} \, dx > 0$, we set

$$t_{\max} = \left(\frac{(p-1)\int_\Omega |\nabla u|^p \, dx}{(p^*-1)\int_\Omega \; g u_+^{p^*} \, dx}\right)^{\frac{1}{p^*-p}} > 0.$$

Lemma (4.7) [4]:

Suppose that $\|f\|_{X^*} \in (0, \lambda_1)$ and $u \in X$ is a function satisfying with $\int_\Omega \; g u_+^{p^*} \, dx > 0$.

(i) If $\int_\Omega \; f u \, dx \leq 0$, then there exists a unique $t_{f,g}^- > t_{\max}$ such that $t_{f,g}^- u \in N_{f,g}^-$ and

$$J_{f,g}\left(t_{f,g}^- u\right) = \sup_{t \geq 0} J_{f,g}(tu).$$

(ii) If $\int_\Omega \; f u \, dx > 0$, then there exists a unique $t_{f,g}^\pm$ such that $0 < t_{f,g}^+ < t_{\max} < t_{f,g}^-$, $t_{f,g}^+ u \in N_{f,g}^+$ and $t_{f,g}^- \in N_{f,g}^-$. Moreover,

$$J_{f,g}\left(t_{f,g}^+ u\right) = \inf_{0 \leq t \leq t_{\max}} J_{f,g}(tu), \left(t_{f,g}^- u\right) = \sup_{t \geq t_{f,g}^+} J_{f,g}(tu).$$

We remark that it follows Lemma (4.5), for $0 < \|f\|_{X^*} < \lambda_1$, we write $N_{f,g} = N_{f,g}^+ \cup N_{f,g}^-$. Furthermore, by Lemma (4.6), it follows that $N_{f,g}^+$ and $N_{f,g}^-$ are nonempty, and by (1), we maydefine

$$\alpha_{f,g}^+ = \inf_{u \in N_{f,g}^+} J_{f,g}(u)\,; \;\; \alpha_{f,g}^- = \inf_{u \in N_{f,g}^-} J_{f,g}(u).$$

Then we have the following result.

Lemma (4.8) [4]:

(i) $\alpha_{f,g}^+ < 0$.

(ii) There exist $\lambda_2, d_0 > 0$ such that $\alpha_{f,g}^- \geq d_0$ if $0 < \|f\|_{X^*} < \lambda_2$.

Proof:

(i) Given $u \in N_{f,g}^+$, from (3) we obtain

$$J_{f,g}(u) = \left(\frac{1}{p}-\frac{1}{p^*}\right)\int_\Omega |\nabla u|^p\,dx - \left(1-\frac{1}{p^*}\right)\int_\Omega fu\,dx$$

$$\leq \left[\left(\frac{1}{p}-\frac{1}{p^*}\right) - \left(1-\frac{1}{p^*}\right)\frac{p^*-p}{p^*-1}\right]\int_\Omega |\nabla u|^p\,dx$$

$$= \frac{p^*-p}{p^*}\left(\frac{1}{p}-1\right)\int_\Omega |\nabla u|^p\,dx < 0.$$

This yields $\alpha_{f,g}^+ < 0$.

(ii) For $u \in N_{f,g}^-$, by (2) and the Sobolev embedding theorem, we get

$$(p-1)\int_\Omega |\nabla u|^p\,dx < (p^*-1)\int_\Omega g u_+^{p^*}\,dx$$

$$\leq (p^*-1)|g|_\infty S^{-\frac{p^*}{p}}\left(\int_\Omega |\nabla u|^p\,dx\right)^{\frac{p^*}{p}},$$

where S is the best constant of the embedding of $X \hookrightarrow L^{p^*}(\Omega)$. Thus there exists $c > 0$ suchthat

$$\int_\Omega |\nabla u|^p\,dx \geq c.$$

Moreover,

$$J_{f,g}(u) = \left(\frac{1}{p}-\frac{1}{p^*}\right)\int_\Omega |\nabla u|^p\,dx - \left(1-\frac{1}{p^*}\right)\int_\Omega fu\,dx$$

$$\geq \left(\frac{1}{p}-\frac{1}{p^*}\right)\int_\Omega |\nabla u|^p\,dx - \left(1-\frac{1}{p^*}\right)\|f\|_{X^*}\|u\|$$

$$= \|u\|\left[\left(\frac{1}{p}-\frac{1}{p^*}\right)\|u\|^{p-1} - \left(1-\frac{1}{p^*}\right)\|f\|_{X^*}\right].$$

Hence, there exist $\lambda_2, d_0 > 0$ such that $\alpha_{f,g}^- \geq d_0$ if $0 \leq \|f\|_{X^*} < \lambda_2$.

Lemma (4.9) [4]:

(i) If $0 < \|f\|_{X^*} < \lambda_1$, then $J_{f,g}$ has a $(\mathrm{PS})_{\alpha^+_{f,g}}$-sequence $\{u_k\} \subset N^+_{f,g}$.

(ii) If $0 < \|f\|_{X^*} < \min\{\lambda_1, \lambda_2\}$, then $J_{f,g}$ has a $(\mathrm{PS})_{\alpha^-_{f,g}}$-sequence $\{u_k\} \subset N^-_{f,g}$.

Now, we establish the existence of a local minimum for $J_{f,g}$ on $N_{f,g}$.

Lemma (4.10) [4]:

Assume (f) and (g_1) hold. If $0 < \|f\|_{X^*} < \min\{\lambda_1, \lambda_2\}$, the functional $J_{f,g}$has a minimizer $u^+_{f,g} \in N^+_{f,g}$and it satisfies

(i) $J_{f,g}\left(u^+_{f,g}\right) = \alpha^+_{f,g}$;

(ii) $u^+_{f,g}$ is solution of $\left(E_{f,g}\right)$;

(iii) $\left\|u^+_{f,g}\right\| \to 0$ as $\|f\|_{X^*} \to 0$;

(iv) $J_{f,g}\left(u^+_{f,g}\right) \to 0$ as $\|f\|_{X^*} \to 0$.

Proof:

By (1) and Lemma (4.9) (i), there exists a minimizing sequence $\{u_k\} \subset N^+_{f,g}$suchthat

$$J_{f,g}(u_k) = \alpha^+_{f,g} + o(1) \text{and} J'_{f,g}(u_k) = o(1) \text{in} X^*.$$

After a standard argument, we get that $\{u_k\}$ is bounded in X. Passing to asubsequence, there exists $u^+_{f,g}$ such that as $k \to \infty$,

(a) $u_k \rightharpoonup u^+_{f,g}$weakly in X,

(b) $u_k \rightharpoonup u^+_{f,g}$weakly in $L^{p^*}(\Omega)$,

(c) $u_k \to u^+_{f,g}$strongly in $L^r(\Omega)$for all $1 \leq r < p^*$.

Moreover, we can obtain $J'_{f,g}\left(u^+_{f,g}\right) = 0$ in X^* and $u^+_{f,g} \in N_{f,g}$is a nontrivial solution of$\left(E_{f,g}\right)$, since $f \not\equiv 0$.

Next, we prove that $u_k \to u^+_{f,g}$ strongly in X and $J_{f,g}(u^+_{f,g}) = \alpha^+_{f,g}$. From Lemma (4.7), (1), and fact $u_k, u^+_{f,g} \in N_{f,g}$, it follows that

$$\alpha^+_{f,g} \le J_{f,g}(u^+_{f,g}) = \frac{1}{N}\|u^+_{f,g}\|^p - \left(\frac{p^*-1}{p^*}\right)\int_\Omega f u^+_{f,g}\,dx$$

$$\le \liminf_{k\to\infty}\left(\frac{1}{N}\|u_k\|^p - \left(\frac{p^*-1}{p^*}\right)\int_\Omega f u_k\,dx\right)$$

$$= \liminf_{k\to\infty} J_{f,g}(u_k) = \alpha^+_{f,g},$$

which implies that $J_{f,g}(u^+_{f,g}) = \alpha^+_{f,g}$and $\lim_{k\to\infty}\|u_k\|^p = \|u^+_{f,g}\|^p$. Standard argument shows that$u_k \to u^+_{f,g}$strongly in X. Moreover, since $N^0_{f,g} = \emptyset$, we obtain $u^+_{f,g} \in N^+_{f,g}$.

Finally, by (3) and$u^+_{f,g} \in N^+_{f,g}$, we obtain

$$\|u^+_{f,g}\|^{p-1} < \frac{p^*-1}{p^*-p}\|f\|_{X^*},$$

which implies that $\|u^+_{f,g}\| \to 0$ as $\|f\|_{X^*} \to 0$, and so $J_{f,g}(u^+_{f,g}) \to 0$ as $\|f\|_{X^*} \to 0$.

Lemma (4.11) [4]:

Assume $(f), (g_1)$ and (g_2) hold. If $0 < \|f\|_{X^*} < \min\{\lambda_1, \lambda_2\}$, then thereexists$u^-_{f,g} \in N^-_{f,g}$and it satisfies

(i) $J_{f,g}(u^-_{f,g}) = \alpha^-_{f,g}$;

(ii) $u^-_{f,g}$is a solution of $(E_{f,g})$.

Proof:

By (1) and Lemma (4.9) (ii), there exists a minimizing sequence $\{u_k\} \subset N^-_{f,g}$suchthat such that

$$J_{f,g}(u_k) = \alpha^-_{f,g} + o(1) \text{and} J'_{f,g}(u_k) = o(1) \text{in} X^*.$$

After a standard argument, we get that $\{u_k\}$ is bounded in X. Passing to asubsequence, there exists $u_{f,g}^-$ such that as $k \to \infty$,

(a) $u_k \rightharpoonup u_{f,g}^-$weakly in X,
(b) $u_k \rightharpoonup u_{f,g}^-$weakly in $L^{p^*}(\Omega)$,
(c) $u_k \to u_{f,g}^-$strongly in $L^r(\Omega)$for all $1 \leq r < p^*$.

Moreover, we can obtain $J'_{f,g}\left(u_{f,g}^-\right) = 0$ in X^* and $u_{f,g}^- \in N_{f,g}$is a nontrivial solution of$\left(E_{f,g}\right)$, since $f \not\equiv 0$.

Next, we prove that $u_{f,g}^- \in N_{f,g}^-$. On the contrary, if $u_{f,g}^- \in N_{f,g}^+$, then by $N_{f,g}^- \cup \{0\}$isclosed in X, we have

$$\left\|u_{f,g}^-\right\| < \lim_{k\to\infty} \inf\|u_k\|.$$

From (g_1), (g_2) and $u_{f,g}^- > 0$ in Ω, we have

$$\int_\Omega g\left(u_{f,g}^-\right)^{p^*} dx > 0.$$

Thus, by Lemma (4.6), there exists a unique $t_{f,g}^- u_{f,g}^- \in N_{f,g}^-$. If $u \in N_{f,g}$, then it is easy to seethat

$$J_{f,g}(u) = \frac{1}{N}\|u\|^p - \frac{p^*-1}{p^*}\int_\Omega fu\, dx.$$

So we can deduce that

$$\alpha_{f,g}^- \leq J_{f,g}\left(t_{f,g}^-, u_{f,g}^-\right) < \lim_{k\to\infty} J_{f,g}\left(t_{f,g}^- u_k\right) \leq \lim_{k\to\infty} J_{f,g}(u_k) = \alpha_{f,g}^-.$$

This is a contradiction. Hence $u_{f,g}^- \in N_{f,g}^-$.

Then by the same argument as that in Lemma (4.10), we get that $u_k \to u_{f,g}^-$strongly in $X$and $J_{f,g}\left(u_{f,g}^-\right) = \alpha_{f,g}^- > 0$.

Now, we suppose (H), (f) and (g_3) hold throughout.

Denote $V := \left\{u \in X; |u|_{p^*} = 1\right\}$,

$$\hat{J}_{f,1}(u) = \max_{t\geq 0} J_{f,1}(tu) : V \to \mathbb{R}$$

and

$$\hat{J}_{0,1}(u) = \max_{t\geq 0} J_{0,1}(tu) : V \to \mathbb{R}.$$

Lemma (4.12) [4]:

For each $u \in N_{f,1}^{-}$ and $b > 0$, we have

(i) There is a unique $t_{0,b}^{-}$ such that $t_{0,b}^{-}u \in N_{0,b}$ and

$$\max_{t\geq 0} J_{0,b}(tu) = J_{0,b}(t_{0,b}^{-}u) = \frac{1}{N} b^{\frac{p-N}{p}} \left(\frac{\|u\|^{p^*}}{\int_\Omega u_+^{p^*}\, dx} \right)^{\frac{N-p}{p}}.$$

(ii) For $\mu \in (0,1)$, there is a unique $t_{0,1}^{-}$ such that $t_{0,1}^{-}u \in N_{0,1}$. Moreover,

$$J_{f,1}(u) \geq (1-\mu)^{\frac{N}{p}} J_{0,1}(t_{0,1}^{-}u) - \frac{1}{q\mu^{\frac{q}{p}}} \|f\|_{X^*}^{q}$$

and

$$J_{f,1}(u) \leq (1+\mu)^{\frac{N}{p}} J_{0,1}(t_{0,1}^{-}u) + \frac{1}{q\mu^{\frac{q}{p}}} \|f\|_{X^*}^{q},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

(iii) $\alpha_{f,1}^{-} \to \frac{1}{N} S^{\frac{N}{p}}$ as $\|f\|_{X^*} \to 0$.

Proof:

(i) For each $u \in N_{f,1}^{-}$, let

$$h(t) = J_{0,b}(tu) = \frac{1}{p} t^p \|u\|^p - \frac{1}{p^*} t^{p^*} \int_\Omega b u_+^{p^*}\, dx.$$

Then $h(t) \to -\infty$ as $t \to \infty$,

$$h'(t) = t^{p-1}\|u\|^p - t^{p^*-1}\int_\Omega bu_+^{p^*}\,dx$$

and

$$h''(t) = (p-1)t^{p-2}\|u\|^p(p^*-1)t^{p^*-2}\int_\Omega bu_+^{p^*}\,dx.$$

Let

$$t_{0,b}^- = \left(\frac{\|u\|^p}{\int_\Omega bu_+^{p^*}\,dx}\right)^{\frac{1}{p^*-p}} > 0.$$

Then $h'(t_{0,b}^-) = 0, t_{0,b}^- u \in N_{0,b}$ and $(t_{0,b}^-)^{2-p} h''(t_{0,b}^-) = (p-p^*)\|u\|^p < 0$. Hence there is a unique $t_{0,b}^-$ such that $t_{0,b}^- u \in N_{0,b}$ and

$$\max_{t\geq 0} J_{0,b}(tu) = J_{0,b}(t_{0,b}^- u) = \frac{1}{N} b^{\frac{p-N}{p}}\left(\frac{\|u\|^{p^*}}{\int_\Omega u_+^{p^*}\,dx}\right)^{\frac{N-p}{p}}.$$

(ii) For $\mu \in (0,1)$, we have

$$\left|\int_\Omega fu\,dx\right| \leq \|f\|_{X^*}\|u\| \leq \frac{\mu}{p}\|u\|^p + \frac{1}{q\mu^{\frac{q}{p}}}\|f\|_{X^*}^q,$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Then by part (i),

$$J_{f,1}(u) = \max_{t\geq 0} J_{f,1}(tu) \geq J_{f,1}\left(t_{0,\frac{1}{1-\mu}}^- u\right)$$

$$\geq \frac{1-\mu}{p}\left\|t_{0,\frac{1}{1-\mu}}^- u\right\|^p - \frac{1}{p^*}\int_\Omega \left(t_{0,\frac{1}{1-\mu}}^- u\right)_+^{p^*} dx - \frac{1}{q\mu^{\frac{q}{p}}}\|f\|_{X^*}^q$$

$$= (1-\mu)J_{0,\frac{1}{1-\mu}}\left(t^{-}_{0,\frac{1}{1-\mu}}u\right) - \frac{1}{q\mu^{\frac{q}{p}}}\|f\|^{q}_{X^*}$$

$$= (1-\mu)^{\frac{N}{p}}\frac{1}{N}\left(\frac{\|u\|^{p^*}}{\int_{\Omega} u_{+}^{p^*}\,dx}\right)^{\frac{N-p}{p}} - \frac{1}{q\mu^{\frac{q}{p}}}\|f\|^{q}_{X^*}$$

$$= (1-\mu)^{\frac{N}{p}}J_{0,1}\big(t^{-}_{0,1}u\big) - \frac{1}{q\mu^{\frac{q}{p}}}\|f\|^{q}_{X^*}.$$

Similarly, we obtain

$$J_{f,1}(u) \le \max_{t\ge 0}\left\{\frac{1+\mu}{p}\|tu\|^{p} - \frac{1}{p^*}\int_{\Omega}(tu)_{+}^{p^*}\,dx\right\} + \frac{1}{q\mu^{\frac{q}{p}}}\|f\|^{q}_{X^*}$$

$$= (1+\mu)J_{0,\frac{1}{1+\mu}}\left(t^{-}_{0,\frac{1}{1+\mu}}u\right) + \frac{1}{q\mu^{\frac{q}{p}}}\|f\|^{q}_{X^*}$$

$$= (1+\mu)^{\frac{N}{p}}J_{0,1}\big(t^{-}_{0,1}u\big) + \frac{1}{q\mu^{\frac{q}{p}}}\|f\|^{q}_{X^*}.$$

(iii) It follows from part (ii) and the fact that $\alpha_{0,1} = \frac{1}{N}S^{\frac{N}{p}}$, where $\alpha_{0,1} = \inf_{u\in N_{0,1}} J_{0,1}(u)$ and $N_{0,1} = \{u \in X\setminus\{0\};\ u_{+} \not\equiv 0 \text{ and} \langle J'_{0,1}(u), u\rangle = 0\}$.

Lemma (4.13) [4]:

Suppose $\{u_k\}$ is a $(\mathrm{PS})_\beta$-sequence of $J_{f,1}$ with

$$\frac{1}{N}S^{\frac{N}{p}} < \beta < \frac{2}{N}S^{\frac{N}{p}} - \varepsilon_0$$

for some $\varepsilon_0 > 0$. Moreover, we let $u_k \rightharpoonup u_0 \notin N^{+}_{f,1}$ weakly in X. Then there exists $\lambda_0 > 0$ such that for $0 < \|f\|_{X^*} < \lambda_0$, $\{u_k\}$ has a convergent subsequence.

Proof:

By Definition (4.4), we have

$$\frac{1}{N}S^{\frac{N}{p}} < \beta = J_{f,1}(u_k) + o(1) = J_{f,1}(u_0) + \sum_{j=1}^{\iota} J^{\infty}(u^j) < \frac{2}{N}S^{\frac{N}{p}} - \varepsilon_0,$$

where $\iota \in \mathbb{N}$.

We suppose $0 < \|f\|_{X^*} < \min\{\lambda_1, \lambda_2\}$. Then, it follows from Lemma (4.5) that $u_0 \in N_{f,1}^{-}$and $J_{f,1}(u_0) \geq \alpha_{f,1}^{-} > 0$. However, by Lemma (4.12) (iii), we know that there exists $0 < \lambda_0 < \min\{\lambda_1, \lambda_2\}$such that for $0 < \|f\|_{X^*} < \lambda_0$,

$$J_{f,1}(u_0) + \sum_{j=1}^{\iota} J^{\infty}(u^j) \geq J_{f,1}(u_0) + \iota\frac{1}{N}S^{\frac{N}{p}} \geq \alpha_{f,1}^{-} + \frac{1}{N}S^{\frac{N}{p}} > \frac{2}{N}S^{\frac{N}{p}} - \varepsilon_0,$$

when $\iota \geq 1$. This contradicts to our assumption. Therefore $\iota = 0$ and by Definition (4.4), wecomplete the proof.

It is well known that the best Sobolev constant

$$S = \inf\left\{\int_{\mathbb{R}^N} |\nabla u|^p \, dx; u \in \mathcal{D}^{1,p}(\mathbb{R}^N), |u|_{p^*} = 1\right\}$$

is attained by the functions

$$u_\delta(x) = \left(\delta^{\frac{p}{p-1}} N\left(\frac{N-p}{p-1}\right)^{p-1}\right)^{\frac{N-p}{p^2}} \left(\delta^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}}\right)^{\frac{p-N}{p}}$$

for any $\delta > 0$. Moreover, the functions $y_\delta(x)$ are the only positive radial solutions of

$$-\Delta_p u = |u|^{p^*-2}u$$

in $\mathbb{R}^N$. Hence,

$$S\left(\int_{\mathbb{R}^N} |u_\delta|^{p^*} dx\right)^{\frac{p}{p^*}} = \int_{\mathbb{R}^N} \left|\nabla_{u_\delta}\right|^p dx = \int_{\mathbb{R}^N} |u_\delta|^{p^*} dx = S^{\frac{N}{p}}.$$

Let $\varphi_\rho \in C_0^\infty(\mathbb{R}^N)$ be a radially symmetric function such that

$$\varphi_\rho(x) = \begin{cases} 0, & 0 \le |x| \le \dfrac{3\rho}{2}, \\ 1, & 2\rho \le |x| \le \dfrac{1}{2\rho}, \\ 0, & |x| \ge \dfrac{3}{4\rho} \end{cases}$$

and

$$u_\delta^e(x) = \left(\delta^{\frac{p}{p-1}} N \left(\frac{N-p}{p-1}\right)^{p-1}\right)^{\frac{N-p}{p^2}} \left(\delta^{\frac{p}{p-1}} + |x - (1-\delta)e|^{\frac{p}{p-1}}\right)^{\frac{p-N}{p}},$$

where $e \in \mathbb{S} = \{x \in \mathbb{R}^N;\ |x| = 1\}$ and $0 < \delta < 1$. Set

$$\omega_\rho^\delta(x) = \varphi_\rho(x) u_\delta^e(x) \text{and} \vartheta_\rho^{\delta.e}(x) = \frac{\omega_\rho^{\delta,e}(x)}{\left|\omega_\rho^{\delta,e}(x)\right|_{p^*}} \in X.$$

Lemma (4.14) [4]:

$\left\|\vartheta_\rho^{\delta,e}(x)\right\|^p \to S$ as $\delta \to 0$uniformly in $e \in \mathbb{S}$.

Proof:

It is sufficient to show

$$\left|\omega_\rho^{\delta,e}\right|_{p^*}^{p^*} \to S^{\frac{N}{p}}, \quad \left\|\omega_\rho^{\delta,e}\right\|^p \to S^{\frac{N}{p}} (4)$$

uniformly in $e \in \mathbb{S}$ as $\delta \to 0$. In order to show (4), we estimate

$$|u_\delta^e(x)|_{p^*}^{p^*} - \left|\omega_\rho^{\delta,e}(x)\right|_{p^*}^{p^*} = \int_{\mathbb{R}^N} \left(1 - \varphi_\rho(x)^{p^*}\right) |u_\delta^e(x)|^{p^*} dx$$

$$\le c \int_{B_{2\rho}} \frac{\delta^{\frac{N}{p-1}}}{\left(\delta^{\frac{p}{p-1}} + |x-(1-\delta)e|^{\frac{p}{p-1}}\right)^N} dx + c \int_{\mathbb{R}^N \setminus B_{\frac{1}{2\rho}}} \frac{\delta^{\frac{N}{p-1}}}{\left(\delta^{\frac{p}{p-1}} + |x-(1-\delta)e|^{\frac{p}{p-1}}\right)^N} dx$$

$$= o(1)$$

as $\delta \to 0$ for all $e \in \mathbb{S}$. Similarly,

$$\int_{\mathbb{R}^N} \left|\nabla \omega_\rho^{\delta,e}(x)\right|^p dx - \int_{\mathbb{R}^N} |\nabla u_\delta^e(x)|^p dx$$

$$= \int_\Omega \left|\nabla \varphi_\rho(x) u_\delta^e(x) + \varphi_\rho(x) \nabla u_\delta^e(x)\right|^p dx - \int_{\mathbb{R}^N} |\nabla u_\delta^e(x)|^p dx$$

$$\leq \int_\Omega \left|\varphi_\rho(x)^p - 1\right| |\nabla u_\delta^e(x)|^p dx + c \int_{\mathbb{R}^N} \left|\nabla \varphi_\rho(x) u_\delta^e(x)\right|^p dx$$
$$+ c \int_{\mathbb{R}^N} \left|\varphi_\rho(x) \nabla u_\delta^e(x)\right|^{p-1} \left|\nabla \varphi_\rho(x) u_\delta^e(x)\right| dx$$

$$\leq \int_{\left(\mathbb{R}^N \setminus B_{\frac{1}{2\rho}}\right) \cup B_{2\rho}} |\nabla u_\delta^e(x)|^p dx + c\rho^p \int_{B_{\frac{3}{4\rho}} \setminus B_{\frac{1}{2\rho}}} |u_\delta^e(x)|^p dx + c\rho^{-p} \int_{B_{2\rho}} |u_\delta^e(x)|^p dx$$
$$+ c \left(\int_{\left(\mathbb{R}^N \setminus B_{\frac{1}{2\rho}}\right) \cup B_{2\rho}} |\nabla u_\delta^e(x)|^p dx \right)^{\frac{p}{p-1}}$$
$$+ c \left(\rho^p \int_{B_{\frac{3}{4\rho}} \setminus B_{\frac{1}{2\rho}}} |u_\delta^e(x)|^p dx + \rho^{-p} \int_{B_{2\rho}} |u_\delta^e(x)|^p dx \right)^{\frac{1}{p}} \to 0$$

as $\delta \to 0$ uniformly in $e \in \mathbb{S}$. Note the fact

$$\int_{\mathbb{R}^N} |\nabla u_\delta^e(x)|^p dx = \int_{\mathbb{R}^N} |u_\delta^e(x)|^{p^*} dx = S^{\frac{N}{p}},$$

we complete the proof.

Lemma (4.15) [4]:

There is a $\rho_0 > 0$ such that for $0 < \rho < \rho_0$,

$$\sup_{0<\delta<1, e\in\mathbb{S}} \left\|\vartheta_\rho^{\delta,e}(x)\right\|^p < 2^{\frac{p}{N}} S.$$

Proof:

The assertion can be verified similarly as Lemma (4.14). Let us define

$$\Phi : V \to \mathbb{R}^N, \Phi(u) = \int_{\mathbb{R}^N} x|u|^{p^*}\, dx,$$

where the function u is extended to $\mathbb{R}^N$by setting $u = 0$ outside Ω. Set

$$A_0 = \{u \in V; \Phi(u) = 0\}.$$

Lemma (4.16) [4]:

Let $c_0 = \inf_{u \in A_0} \|u\|^p$, then $S < c_0$.

Proof:

Apparently, $c_0 \geq S$. To show $S < c_0$, we argue by contradiction. Suppose that$c_0 = S$, then there is a sequence $\{v_k\} \subset X$ such that

$$|v_k|_{p^*} = 1, \Phi(v_k) = 0 \text{ and } \|\nabla v_k\|^p \to S \text{ as } k \to \infty.$$

So the sequence $u_k = S^{\frac{N-p}{p^2}} v_k$satisfies

$$J_{0,1}(v_k) \to \frac{1}{N} S^{\frac{N}{p}} \text{and} J'_{0,1}(u_k) \to 0 \text{ as } k \to \infty.$$

There exist $\{\varepsilon_k\}, \{x_k\} \in \Omega, \varepsilon_k \to 0$ as $k \to \infty$, for the functions$v_k = S^{-\frac{N-p}{p^2}} u_{\varepsilon_k, x_k}(x)$, where

$$u_{\varepsilon_k, x_k}(x) = \left(\varepsilon_{k^{\frac{p}{p-1}}} \left(\frac{N-p}{p-1}\right)^{p-1}\right)^{\frac{N-p}{p^2}} \left(\varepsilon_{k^{\frac{p}{p-1}}} + |x - x_k|^{\frac{p}{p-1}}\right)^{\frac{p-N}{p}},$$

we have

$$\Phi(v_k) = x_k.$$

From (H), we know $x_k \neq 0$ and this contradicts to our assumption that $\Phi(v_k) = 0$. Hence$S < c_0$.

Lemma (4.17) [4]:

There holds $\lim_{\delta\to 0} \Phi\left(\vartheta_\rho^{\delta,e}(x)\right) = e$.

Proof:

Since

$$\int_{\mathbb{R}^N} (x-e)\left|\omega_\rho^{\delta,e}(x)\right|^{p^*} dx$$

$$= \int_{\mathbb{R}^N} (x-e)|u_\delta^e(x)|^{p^*} dx + \int_{\mathbb{R}^N} (x-e)\left(\varphi_\rho(x)^{p^*} - 1\right)|u_\delta^e(x)|^{p^*} dx$$

$$= \left((1-\delta)e - e\right) + \int_{\mathbb{R}^N} (x-e)\left(\varphi_\rho(x)^{p^*} - 1\right)|u_\delta^e(x)|^{p^*} dx \to 0$$

as $\delta \to 0$. Thus

$$\Phi\left(\vartheta_\rho^{\delta}(x)\right) - e = \frac{\int_{\mathbb{R}^N} (x-e)\left|\omega_\rho^{\delta,e}(x)\right|^{p^*} dx}{\int_{\mathbb{R}^N} \left|\omega_\rho^{\delta,e}(x)\right|^{p^*} dx} \to 0$$

as $\delta \to 0$.

To pursue further, we need the following result.

Lemma (4.18) [4]:

Let Kbe a compact metric space, $K_0 \subset K$ be a closed set, X be a Banach space and $\chi \in C(K_0, X)$. Let us define the complete metric space M by

$$M = \{k \in C(K, X); k(s) = \chi(s) \text{if } s \in K_0\}$$

with the usual distance. Let $\varphi \in C^1(X, \mathbb{R})$and let us define

$$\overline{c} = \inf_{k\in M} \sup_{s\in K} \varphi\big(k(s)\big), \quad \hat{c} = \sup_{\chi(K_0)} \varphi.$$

If $\overline{c} > \hat{c}$, then for each $\varepsilon > 0$ and each $k \in M$ satisfying

$$\sup_{s\in K} \varphi\big(k(s)\big) \leq \overline{c} + \varepsilon,$$

there exists $v \in X$ such that

$$\overline{c} - \varepsilon \leq \varphi(v) \leq \sup_{s\in K} \varphi\big(k(s)\big), \quad \mathrm{dist}\big(v, k(K)\big) \leq \varepsilon^{\frac{1}{2}}, \quad \|\varphi'(v)\| \leq \varepsilon^{\frac{1}{2}}.$$

Let $r_0 = \frac{1}{2} - \delta_0$ and

$$\overline{B}_{r_0} = \left\{\left(\frac{1}{2} - \delta\right) e \in \mathbb{R}^N;\ \left|\left(\frac{1}{2} - \delta\right)\right| \leq r_0, 0 < \delta \leq \frac{1}{2}\right\},$$

where $\delta_0 > 0$ is small enough. Then we set

$$F = \left\{h \in C\big(\overline{B}_{r_0}, V\big); h|_{\partial\overline{B}_{r_0}} = \vartheta_\rho^{\delta,e}(x)\right\}$$

and

$$c_1 = \inf_{h\in F} \sup_{\left(\frac{1}{2}-\delta\right)e\in\overline{B}_{r_0}} \left\|h\left(\left(\frac{1}{2} - \delta\right) e\right)\right\|^p. \qquad (5)$$

Lemma (4.19) [4]:

For $h \in F$, we have $h\big(\overline{B}_{r_0}\big) \cap A_0 \neq \emptyset$.

Proof:

It is equivalent to show that for any $h \in F$, there exist $\left(\frac{1}{2} - \tilde{\delta}\right) \tilde{e} \in \overline{B}_{r_0}, \tilde{e} \in \mathbb{S}$ suchthat

$$\Phi\left(h\left(\left(\frac{1}{2} - \tilde{\delta}\right) \tilde{e}\right)\right) = 0.$$

Set $\theta\left(\left(\frac{1}{2}-\delta\right)e\right) \equiv \Phi\left(h\left(\left(\frac{1}{2}-\delta\right)e\right)\right)$, we claim that

$$d\left(\theta, B_{r_0}, 0\right) = d\left(I, B_{r_0}, 0\right) \neq 0.$$

In fact, if $\left(\frac{1}{2}-\delta\right)e \in \partial B_{r_0}$, we have

$$h\left(\left(\frac{1}{2}-\delta\right)e\right)(x) = \vartheta_\rho^{\delta,e}(x), \Phi\left(h\left(\left(\frac{1}{2}-\delta\right)e\right)(x)\right) = \Phi\left(\vartheta_\rho^{\delta,e}(x)\right) = e + o(1)$$

as $\delta \to 0$. Then we consider the homotopy

$$G\left(t, \left(\frac{1}{2}-\delta\right)e\right) = (1-t)\theta\left(\left(\frac{1}{2}-\delta\right)e\right) + tI, 0 \leq t \leq 1$$

If $\left(\frac{1}{2}-\delta\right)e \in \partial B_{r_0}$,

$$G\left(t, \left(\frac{1}{2}-\delta\right)e\right) = (1-t)\left(e + o(1)\right) + t\left(\frac{1}{2}-\delta_0\right)e \neq 0$$

as $\delta \to 0$. So the claim is proved and there exist $\left(\frac{1}{2}-\tilde{\delta}\right)\tilde{e} \in \overline{B}_{r_0}$, $\tilde{e} \in \mathbb{S}$such that

$$\Phi\left(h\left(\left(\frac{1}{2}-\tilde{\delta}\right)\tilde{e}\right)\right) = 0.$$

We complete the proof.

By Lemmas (4.16) and (4.19), we obtain

$$S < c_0 \leq c_1.$$

On the other hand, it follows from Lemma (4.15) and (5) that for $0 < \rho < \rho_0$,

$$c_1 \leq \sup_{\left(\frac{1}{2}-\delta\right)e \in \overline{B}_{r_0}} \left\|\vartheta_\rho^{\delta,e}(x)\right\|^p \leq \sup_{0<\delta<1, e \in \mathbb{S}} \left\|\vartheta_\rho^{\delta,e}(x)\right\|^p < 2^{\frac{p}{N}} S.$$

Thus

$$S < c_0 \leq c_1 < 2^{\frac{p}{N}} S.$$

Then we define

$$\gamma_{f,1} = \inf_{h\in F} \sup_{\left(\frac{1}{2}-\delta\right)e\in\overline{B}_{r_0}} \hat{J}_{f,1}\left(h\left(\left(\frac{1}{2}-\delta\right)e\right)\right),$$

$$\gamma_{0,1} = \inf_{h\in F} \sup_{\left(\frac{1}{2}-\delta\right)e\in\overline{B}_{r_0}} \hat{J}_{0,1}\left(h\left(\left(\frac{1}{2}-\delta\right)e\right)\right)$$

Lemma (4.20) [4]:

We have

$$\frac{1}{N}S^{\frac{N}{p}} < \frac{1}{N}c_0^{\frac{N}{p}} \le \gamma_{0,1} < \frac{2}{N}S^{\frac{N}{p}}.$$

Proof:

Since

$$\hat{J}_{0,1}(u) = \max_{t\ge 0, u\in V} J_{0,1}(tu) = \frac{1}{N}\left(\int_{\Omega} |\nabla u|^p\, dx\right)^{\frac{p^*}{p^*-p}}$$

and

$$\inf_{h\in F} \sup_{\left(\frac{1}{2}-\delta\right)e\in\overline{B}_{r_0}} \hat{J}_{0,1}\left(h\left(\left(\frac{1}{2}-\delta\right)e\right)\right) = \frac{1}{N}\inf_{h\in F} \sup_{\left(\frac{1}{2}-\delta\right)e\in\overline{B}_{r_0}} \left\|h\left(\left(\frac{1}{2}-\delta\right)e\right)\right\|^N,$$

we obtain the assertion from (5) and the fact $S < c_0 \le c_1 < 2^{\frac{p}{N}}S$.

Lemma (4.21) [4]:

$$\hat{J}_{f,1}\left(\vartheta_{\rho}^{\delta,e}(x)\right) = \frac{1}{N}S^{\frac{N}{p}} + o(1)$$

as $\delta \to 0$.

Proof:

It is easy to obtain $\vartheta_\rho^{\delta,e}(x) \rightharpoonup 0$ weakly in Xas $\delta \to 0$. Solving

$$\frac{\mathrm{d}J_{f,1}\left(t\vartheta_\rho^{\delta,e}(x)\right)}{\mathrm{d}t} = t^{p-1}\frac{\int_\Omega \left|\nabla\omega_\rho^{\delta,e}(x)\right|^p dx}{\left|\omega_\rho^{\delta,e}(x)\right|_{p^*}^{p^*}} - t^{p^*-1} - \frac{\int_\Omega f\omega_\rho^{\delta,e}(x)\,dx}{\left|\omega_\rho^{\delta,e}(x)\right|_{p^*}} = 0,$$

we see that $\int_\Omega f\omega_\rho^{\delta,e}(x)\,dx \to 0$ and

$$t^-\left(\vartheta_\rho^{\delta,e}\right) = \left|\omega_\rho^{\delta,e}\right|_{p^*} + o(1)\text{as } \delta \to 0.$$

As a result,

$$\hat{J}_{f,1}\left(\vartheta_\rho^{\delta,e}(x)\right) = J_{f,1}\left(t^-\left(\vartheta_\rho^{\delta,e}\right)\vartheta_\rho^{\delta,e}\right) = \frac{1}{N}S^{\frac{N}{p}} + o(1).$$

Lemma (4.22) [4]:

There exists $\Lambda_3 > 0$ such that for $0 < \|f\|_{X^*} < \Lambda_3$, we have

$$\frac{1}{N}S^{\frac{N}{p}} < \gamma_{f,1} < \frac{2}{N}S^{\frac{N}{p}} - \varepsilon_0$$

for some $\varepsilon_0 > 0$.

Proof:

For $\in V$, by Lemma (4.12) we obtain

$$(1-\mu)^{\frac{N}{p}}\hat{J}_{0,1}(u) - \frac{1}{qu^{\frac{q}{p}}}\|f\|_{X^*}^q \le \hat{J}_{f,1}(u) \le (1+\mu)^{\frac{N}{p}}\hat{J}_{0,1}(u) + \frac{1}{qu^{\frac{q}{p}}}\|f\|_{X^*}^q.$$

Thus

$$\hat{J}_{0,1}(u) + \left((1-\mu)^{\frac{N}{p}} - 1\right)\hat{J}_{0,1}(u) - \frac{1}{qu^{\frac{q}{p}}}\|f\|_{X^*}^q$$

$$\leq \hat{J}_{f,1}(u) \leq \hat{J}_{0,1}(u) + \left((1+\mu)^{\frac{N}{p}} - 1\right)\hat{J}_{0,1}(u) + \frac{1}{qu^{\frac{q}{p}}}\|f\|_{X^*}^{q}.$$

Hence, for any $\varepsilon > 0$, there exist $\mu(\varepsilon), \Lambda(\varepsilon) > 0$ such that if $0 < \|f\|_{X^*} < \Lambda(\varepsilon)$, we have

$$\gamma_{0,1} - \varepsilon < \gamma_{f,1} < \gamma_{0,1} + \varepsilon.$$

Fix a small $0 < \varepsilon_0 < \frac{\frac{2}{N}S^{\frac{N}{p}} - \gamma_{0,1}}{2}$. Since

$$\frac{1}{N}S^{\frac{N}{p}} < \frac{1}{N}c_0^{\frac{N}{p}} \leq \gamma_{0,1} < \frac{2}{N}S^{\frac{N}{p}},$$

there exists $0 < \Lambda_3 < \min\{\lambda_0, \lambda_1, \lambda_2\}$such that if $0 < \|f\|_{X^*} < \Lambda_3$, we get

$$\frac{1}{N}S^{\frac{N}{p}} < \frac{1}{2N}\left(c_0^{\frac{N}{p}} + S^{\frac{N}{p}}\right) < \gamma_{f,1} < \frac{2}{N}S^{\frac{N}{p}} - \varepsilon_0.$$

By Lemma (4.21), $\hat{J}_{f,1}\left(\vartheta_\rho^{\delta,e}(x)\right) = \frac{1}{N}S^{\frac{N}{p}} + o(1)$ as $\delta \to 0$. Therefore

$$\gamma_{f,1} > \hat{J}_{f,1}\left(\vartheta_\rho^{\delta,e}(x)\right) = \frac{1}{N}S^{\frac{N}{p}} + o(1)$$

for δsmall enough and the conclusion is obtained.

Lemma (4.23) [4]:

Suppose $0 < \|f\|_{X^*} < \Lambda_3$. Then there exists a solution $u$of $\left(E_{f,1}\right)$ suchthat $J_{f,1}(u) = \gamma_{f,1}$.

Proof:

By Lemma (4.18), there exists a $(\mathrm{PS})_{\gamma_{f,1}}$ sequence $\{u_k\} \subset X$for $\hat{J}_{f,1}$. Furthermore,

$$u_k = h_k\left(\left(\frac{1}{2} - \delta_k\right)e_k\right),$$

where $\left(\frac{1}{2}-\delta_k\right)e_k \in \overline{B}_{r_0}$and $h_k \in F$. Since $\{u_k\}$is bounded in X, we may assume that $u_k \rightharpoonup u_0$weakly in Xas $k \to \infty$. By Lemmas (4.13) and (4.22), we only need to prove that if $u_0 \in N_{f,1}^{+}$, $u_k \to u_0$ strongly in X. Indeed, suppose otherwise, we have

$$\|u_0\| < \lim_{n\to\infty} \inf \|u_k\|.$$

Since $\left(\frac{1}{2}-\delta_k\right)e_k \in \overline{B}_{r_0}$, we may obtain $0 < \delta_0 \leq \frac{1}{2}$and $e_0 \in \mathbb{S}$ such that

$$\left(\frac{1}{2}-\delta_k\right)e_k \to \left(\frac{1}{2}-\delta_0\right)e_0 \in \overline{B}_{r_0}, \text{as } k \to \infty.$$

Moreover, it is easy to see that $F$is convex and closed. Therefore,

$$u_k = h_k\left(\left(\frac{1}{2}-\delta_k\right)e_k\right) \to h_0\left(\left(\frac{1}{2}-\delta_0\right)e_0\right) = u_0 \in V,$$

where $h_0 \in F$. Hence, there exists a unique $t_0 > 0$ such that

$$\gamma_{f,1} \leq \hat{J}_{f,1}(u_0) = J_{f,1}(t_0u_0) = \frac{1}{N}\|t_0u_0\|^p - t_0\left(\frac{p^*-1}{p^*}\right)\int_\Omega fu_0\,dx$$

$$< \lim_{n\to\infty} \inf\left(\frac{1}{N}\|t_0u_k\|^p - t_0\left(\frac{p^*-1}{p^*}\right)\int_\Omega fu_k\,dx\right)$$

$$\leq \lim_{n\to\infty} \hat{J}_{f,1}(u_k) = \gamma_{f,1}.$$

This is absurd. Thus we obtain $u_k \to u_0$ strongly in X.

List of Symbols

Symbol		Page
L^1	Lebesgue measure	1
$W_0^{1,p}$	Sobolev space	2
L^p	Lebesgue space	2
H^1	Hardy space	2
Osc	oscillation	4
inf	infimum	4
ess	essential	4
$L^{p,q}$	The Lebesgue measure	4
sup	supremum	5
BMO	bounded mean oscillation	5
a.e	almost everywhere	6
loc	locally	6
VMO	vanising mean oscillation	6
min	minimum	13
L^∞	Lebesgue space	14
sign	signature	27
max	maximum	29
BVP	bounded value problem	32
Im	imaginary	35
AAP	Agmon-Allegretto-Piepenbrink	37
L^2	Hilbert space	39
Re	real	41
rad	radial	42
supp	support	50
dim	dimension	61
mod	modulas	72
cat	category	114
PS	Palais-Smale	123

References

[1] Jean-Michel Rakotoson : New Hardy inequalities and behaviour of linear elliptic equations, Journal of Functional Analysis 263 (2012) 2893–2920.

[2] Baptist Devyver, Martin Fraas and Yehuda Pinchover :Optimal Hardy Weight for second-order elliptic operator, Journal of Functional Analysis vol. 266, issue-7, 1/4/2014 4422-4489.

[3] Haining Fan :Multiple positive solutions for semi-linear elliptic systems with sign-changing weight,J. Math. Anal. Appl. 409 (2014) 399–408 .

[4]Haining Fan and Xiaochun LIU :Multiple positive solutions for a class of quasi-linear elliptic equations involving critical Sobolev exponent,ActaMathematicaScientia 2014,34B(4):1111–1126 .

[5] P.Pucci, J.Serrin, The Maximum principle, Birkhäuser, 2000. Progress in Nonlinear Differential Equations and their Applications (73) (via Vanvitelli 1) .

[6] M. Chipot, Elements of Nonlinear Angalysis, Birkhäuser Advanced Texts Basler Lehrbücher, Institutfür Mathematic 2000 Mathematics subject classification 35-01; 35Q72,35J60, 35k 55 .

[7] S. Agmon, " Lecture on Exponential Decay of solutions of Second-Order Elliptic Equations: Bounds on Eigen functions of N-body Schrödinger operators", Mathmatical Notes, 2, Princeton University Press, Princeton, 1982.

[8] Trudinger NS. On Harnack type inequalities and their application to quasilinear elliptic equations. Comm. Pure Appl Math, 1967, 20:721-747.