

Dedication

*To my great father , lovely mother ,
brothers and sisters*

Acknowledgements

Foremost, all my gratitude to the lord of the universe, Allah, the greatest and the most merciful, for giving me the life firstly then the power and patience to complete this work.

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Abstract

We show the role of some Hardy inequalities in the blow-up phenomena of the very weak solution of a linear equation in the sense of Brezis. We find some new Hardy inequalities related to some extended Sobolev spaces, Sobolev–Hardy spaces and Sobolev–Zygmund spaces, or other non-standard weighted spaces. The method is based on the theory of positive solutions and applies to both symmetric and nonsymmetric operators. The constructed Hardy-weight is given by an explicit simple formula involving two distinct positive solutions. We study the multiplicity results of positive solutions for a semi-linear elliptic system involving both concave–convex and critical growth terms. We also study the multiplicity results of positive solutions for a class of quasi-linear elliptic equations involving critical Sobolev exponent.

الخلاصة

أوضحنا دور بعض متباينات هاردي في ظواهر تفجير المتابعة الضعيفة جداً للمعادلة الخطية في حالة بريز . أوجدنا بعض متباينات هاردي الجديدة ذات العلاقة لبعض فضاءات سوبوليف الممدة وفضاءات سوبوليف – هاردي و فضاءات سوبوليف – زيجموند أو أي فضاءات مرجحة غير أساسية . الطريقة مؤسسة على نظرية الحلول الموجبة وتطبق معاً إلى المؤثرات المتماثلة وغير المتماثلة . بناء مرجحة هاردي تعطى بواسطة صيغة بسيطة صريحة تنطوي على حلين موجبين منعزلين . تمت دراسة نتائج تضاعف الحلول الموجبة لشبه نظام الناقصية الخطية المتضمنة معاً التقعرية – التحديبية وحدود النمو الحرج . أيضاً تمت دراسة نتائج تضاعف الحلول الموجبة لعائلة شبه المعادلات الناقصية الخطية المتضمنة أسية سوبوليف الحرجة .

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