Chapter 1

Operator Valued Besov Spaces and Dyadic Paraproducts with Hankel Operators

We retrieve Peller's characterizations of scalar and vector Hankel operators of Schatten- ven Nuermann class S_p for $1 < p < \infty$. We then emply vector techniques to characterize little Hankel operator of Schatten- ven Nuermann class.

Furthermore, using a bilinear version of our product formula, we obtain characterization for boundedness compactness and Schatten class membership of product of dyadic paraproducts.

Section (1.1): Scalar Dyadic Paraproducts and Besov Spaces

Dyadic paraproducts have been successfully employed in the study of Hankel operators in various settings, we want to look at Schatten class membership of scalar, vector and multivariable dyadic paraproducts and use these to study Schatten class membership of Hankel operators.

Boundedness, compactness and membership of Schatten classes of their paraproducts have been characterized in terms of oscillatory properties of their symbols. Dyadic paraproducts on vector valued spaces (with matrix or more generally operator valued symbols) have also been studied; it has not thus far been possible to characterize the boundedness of paraproducts with operator valued symbols in terms of oscillation properties of the symbol. These difficulties are closely connected with a breakdown of a form of the John-Nireberg Theorem in the operator-valued setting. Here, we want to consider a "P-.John-Nirenberg Theorem', which generalizes easily to the operator setting.

The purpose of the chapter is threefold. First, we show that a "p-John--Nirenberg, Theorem" which can be found can be used to give a comparatively simple. Interpolation free proof of the characterization of Schatten class paraproducts in terms of oscillatory properties of their symbols. Our approach is related to Rochberg. And Semmes' method of nearly weakly orthonormal sequences—indeed, scalar dyadic paraproducts are in some sense the model case for nearly weakly orthonormal sequences. but technically simpler. Using an averaging technique it is possible to retrieve the known characterization of Schatten class Hankel operators at least for $1 < p < \infty$, for a second proof with different methods, Our approach is again interpolation-free and has the advantage that one does not need any nontrivial properties of Besov spaces, for example the atomic decomposition of B_I^I. Secondly, in contrast to the classical John–Nirenberg theorem, the version of "p-John–Nirenberg Theorem" we require extends both to the operator-valued and the multivariable setting, and we thus obtain characterizations for the membership of Schatten classes for vector paraproducts and paraproducts in several variables. Again using averaging, we also obtain known results for vector Hankel operators and new results on "little Hankel" operators.

Finally, using a sesquilinear version of our method, we obtain necessary and sufficient conditions for boundedness, compactness and Schatten class- membership of products of dyadic paraproducts. This part is motivated by the literature about products of Hankel operators, where characterizations of compactness of products of Hankel operators are known, but the corresponding questions about boundedness and Schatten class membership of products of Hankel operators are still open. we give a "p-John–Nirenberg" proof for the characterisation of dyadic paraproducts of Schatten class for $1 \leq P < \infty$. We characterize dyadic paraproducts of Schatten class with operator-valued symbols for $1 \leq P < \infty$. We give an interpolation-free proof of the characterisation of Hankel operators of Schatten class with operator symbols for $1 < P < \infty$ .We use the vector results to characterise little Hankel operators of Schatten class on $H^2(\mathbb{C}^{+n})$ and multivariable dyadic paraproducts of Schatten class. We characterise boundedness, compactness and Schatten class membership of products of paraproducts.

Let $\mathcal{D}$ denote the collection of all dyadic intervals on the real line $\mathbb{R}$, so

$$\mathcal{D} := \{I = I_{n.k} := [2^{-n}k, 2^{-n}(k+1)) : n, k \in \mathbb{Z}\}.$$

Let $\mathcal{D}_n$ denote the collection of intervals in $\mathcal{D}$ of length 2^{-n} . For $I \in \mathcal{D}$. let $\hat{I}$ denote the parent interval of I, let I_+ and I_- denote the left and right halves of I. respectively, $\mathcal{D}(I)$ the collection of dyadic intervals contained in I. $\mathcal{D}(1)'$ the collection of dyadic intervals contained properly in I, and $\mathcal{D}_n(I)$ the intersection of $\mathcal{D}_n$ and $\mathcal{D}(I)$. For $J \in \mathcal{D}'(I)$, we write

$sign(J.I) = 1$ for $J \in \mathcal{D}(I_+)$. $Sign\ (J, I) = -1$ For $J \in \mathcal{D}(I_+)$, We let h_I denote the Haar function corresponding to I, that is

$$h_I = \frac{1}{|\, I \,|^{1/2}} \left(\chi_{I_+} - \chi_{I_-}\right),$$

Where χ_J denotes the characteristic function of an interval J. It is well known that $\{h_I : I \in \mathcal{D}\}$. Forms an orthonormal basis of the Hilbert space $L^2(\mathbb{R})$. Throughout the article, let $C_P$and K_P denote various constants, depending only on p.

For a Hilbert space $\mathcal{H}$, let $\mathcal{g}(\mathcal{H})$ and $\mathcal{H}\ (\mathcal{H})$ denote the collections of bounded operators and compact operators on $\mathcal{H}$, respectively. Any operator $T \in \mathcal{H}(\mathcal{H})$ has a Schmidt decomposition, so there exist orthonormal bases $\{e_n\}$ and $\{\sigma_n\}$ of $\mathcal{H}$ and a sequence $\{\lambda_n\}$ with $\lambda_n \geq 0$ and $\lambda_n \to 0$ such that.

$$Tf = \sum_{n=0}^{\infty} \lambda_n \langle f, e_n \rangle\, \sigma_n \qquad (1).$$

for all $f \in \mathcal{H}$. For $0 < p < \infty$. a compact operator T with such a decomposition belongs to the Schatten-von Neumann p-class, $S_p(\mathcal{H})$. if and only if

$$\|\, T \,\|_{S_p} = \left(\sum_{i=0}^{n} \lambda_n^p\right)^{1/p} < \infty \qquad (2).$$

We shall frequently use the following elementary facts: For $0 < P \leq 2$

$$\|\, T \,\|_{S_p}^p = \inf\left\{\sum_{n\in\mathbb{N}} \|\, Te_n \,\|^p : \{e_n\}_{n\in\mathbb{N}} \text{ orthonormal basis of } \mathcal{H}\right\} \quad (3)$$

For $2 \leq P < \infty$,

$$\|\, T \,\|_{S_p}^p = \sup\{\textstyle\sum_{n\in\mathbb{N}} \|\, Te_n \,\|^p : \{e_n\}_{n\in\mathbb{N}} \text{ orthonormal basis of } \mathcal{H}\ \} \quad (4)$$

For locally integrable function f on $\mathbb{R}$ and $I \in \mathcal{D}$, let $m_I f$ denote the mean value of f on I, i.e

$$m_I f = \frac{1}{|\, I \,|} \int_I f(t)\ dt,$$

And f_I denote the Haar coefficient of f, i.e.

$$f_1 = \langle f, h_1 \rangle = \int_I f(t)\, h_I(t) dt \,.$$

For f locally integrable and $I \in \mathcal{D}$, we write $P_I f = \chi_I(f - m_I f) = \sum_{J \in \mathcal{D}(I)} h_J f_J$. and $P_I' f = \sum_{J \in \mathcal{D}'(I)} h_J f_J$. on $L^2(\mathbb{R})$, P_I and P_I' are the orthogonal projections on $span\{h_J : J \in \mathcal{D}(I)\}$ and $span\{h_J : J \in \mathcal{D}'(1)\}$ respectively.

We shall repeatedly use the following fact: for $I, J \in \mathcal{D}$,

$$m_J(h_I) = \pm \frac{1}{|I|^{1/2}} \; when \; J \in \mathcal{D}(I_{\pm}) \qquad (5)$$

and zero otherwise. For a locally integrable function b, the densely defined dyadic paraproduct with symbol b, π_b is given by.

$$\pi_b f = \sum_{I \in \mathcal{D}} m_I \, f b_I h_I \,.$$

It is easy to see that the adjoint of π_b on $L^2(\mathbb{R})$ is given by

$$\pi_b^* f = \sum_{I \in \mathcal{D}} \frac{\chi_I}{|I|} f_I \bar{b}_I \qquad (6).$$

We want to denote this adjoint operator by $\Lambda_{\bar{b}}$.

Necessary and sufficient condition on the symbol b for π_b to be bounded on $L^2(\mathbb{R})$ or belong to a Schatten class have been obtained. We shall say a locally integrable function b belongs to the dyadic BMO space $BMO^d(\mathbb{R})$ if

$$\|b\|_{BMO^d} = \sup_{I \in \mathcal{D}} \frac{1}{|I|^{1/2}} \|p_I b\| < \infty \,.$$

Note that

$$\frac{1}{|I|^{1/2}} \|p_I b\| = \left(\frac{1}{|\,I\,|} \int_I |b(t) - m_I b|^2 \, dt \right)^{1/2} = \left(\frac{1}{|\,I\,|} \sum_{J \in \mathcal{D}(I)} |b_J|^2 \right)^{1/2} .$$

We say that $b \in BMO^d$ belongs to the dyadic VMO space $VMO^d(\mathbb{R})$ if

$$\lim_{|I| \to 0} \frac{1}{|I|^{1/2}} \|p_I b\| = 0 \qquad (7).$$

$$\lim_{|I|\to 0} \frac{1}{|I|^{1/2}} \|p_I b\| = 0 \qquad (8).$$

$$\lim_{|k|\to\infty} \frac{1}{|I_{n,k}|^{1/2}} \left\|p_{I_{n,k}} b\right\| = 0 \; for\; each\; n \in \mathbb{Z} \qquad (9).$$

Here the limits in (7) and (8) are meant to be uniform limits as $|I| \to 0 \; or \; |I| \to \infty$, respectively. Somewhat loosely, we will write

$$\lim_{|I|\to\infty} \frac{1}{|I|^{1/2}} \|p_I b\| = 0 \qquad (10).$$

if condition (7) – (9) above hold. We understand $I \to \infty \; as \; I$ converging to the point ∞ in the locally compact space $\mathcal{D}$ with the discrete topology.

For $0 < P < \infty$, a locally integrable function b belongs to the dyadic Besov space B_P^d ($\mathbb{R}$), where the Besov space (named after Oleg Vladimirovich Besov) $B_{P,q}^d(\mathbb{R})$ is a complete quasinormed space which is a Banach space when $1 \le p, q \le \infty$. It, as well as the similarly defined Triebel–Lizorkin space, serve to generalize more elementary function spaces and are effective at measuring (in a sense) smoothness properties of functions [5].

Let $\Delta_h f(x) = f(x-h) - f(x)$

and define the modulus of continuity by

$$\omega_p^2(f,t) = \sup_{|h|\le t} \left\|\Delta_h^2 f\right\|_p$$

Let n be a non-negative integer and define: $s = n + \alpha$ with $0 < \alpha \le 1$. The Besov space $B_{P,q}^d(\mathbb{R})$ contains all functions f such that

$$f \in W^{n,p}(\mathbb{R}), \int_0^\infty \left|\frac{\omega_p^2\left(f^{(n)},t\right)}{t^\alpha}\right|^q \frac{dt}{t} < \infty.$$

if

$$\|b\|_{B_p^d} = \left(\sum_{\in \mathcal{D}} \left(\frac{|b_I|}{|I|^{1/2}}\right)^p\right)^{1/p} < \infty.$$

We then have

For a locally integrable function φ, let $\mathcal{Q}_\varphi$ be the so-called "dyadic sweep" or the square of the dyadic square function of φ, that is.

$$\mathcal{Q}_\varphi(t) = \sum_{I \in \mathcal{D}} \frac{\chi_I(t)}{|I|} |\varphi_I|^2, \qquad t \in \mathbb{R} \qquad (11).$$

We need the following elementary property of $\mathcal{Q}_\varphi$

Lemma (1.1.1) [1]: $P_I \mathcal{Q}_\varphi \;=\; P_I \mathcal{Q}_{P_I \varphi}$

Let D_φ be the operator on $L^2(\mathbb{R})$ which is diagonal in the Haar basis and defined by

$$D_\varphi h_I = \; h_I \frac{1}{|I|} \sum_{J \in \mathcal{D}(I)'} |\varphi_J|^2 \;\; (I \in \mathcal{D}).$$

The following identity relates the paraproducts $\pi_\varphi \; and \; \pi_{\mathcal{Q}_\varphi}$

Proposition (1.1.2) [1]:

$$\pi_\varphi^* \pi_\varphi \;=\; \pi_{\mathcal{Q}_\varphi} \;+\; \pi_{\mathcal{Q}_\varphi}^* + D_\varphi.$$

Proof:

It suffices to show that $\langle \pi_\varphi^* \pi_\varphi h_I, h_J \rangle \;= \langle \left(\pi_{\mathcal{Q}_\varphi} + \pi_{\mathcal{Q}_\varphi}^* + \; D_\varphi \right) h_I, h_J \rangle$ for $I, J \in \mathcal{D}$. Note that $\pi_\varphi$is superdiagonal in the Haar basis in the sense that $\pi_\varphi h_I$ has nontrivial Haar coefficient only for $J \subseteq I$ and $\pi_\varphi$is subdiagonal in the Haar basis in the sense that $\pi_\varphi h_I$has nontrivial Haar coefficient only for $J \supsetneq I$.

Furthermore, $supp \; \pi_\varphi h_I \subseteq I$ and $supp(\pi_{\mathcal{Q}_\varphi} + \; \pi_{\mathcal{Q}_\varphi}^* + \; D_\varphi) h_I \subseteq I$ for all $I \in \mathcal{D}$, so we only have to consider the cases.

(i) $\quad I = J$:

$$\langle \left(\pi_{\mathcal{Q}_\varphi} + \pi_{\mathcal{Q}_\varphi}^* + D_\varphi \right) h_I, h_I \rangle = \langle D_\varphi h_I, h_I \rangle = \tfrac{1}{|I|} \textstyle\sum_{J \in \mathcal{D}(I)'} \left|\varphi_J\right|^2 = \langle \pi_\varphi h_I, \pi_\varphi h_I \rangle.$$

(ii) $\quad I \supsetneq J$:

$$\langle \pi_\varphi h_I, \pi_\varphi h_J \rangle = \textstyle\sum_{\mathrm{k} \in \mathcal{D}} |\varphi_{\mathrm{k}}|^2 m_k h_I \, m_k h_J.$$

$$= \frac{sign(J,I)}{|I|^{1/2}|J|^{1/2}}\left(\sum_{k\in\mathcal{D}(J^+)}|\varphi_k|^2 - \sum_{k\in\mathcal{D}(J^-)}|\varphi_k|^2\right)$$

$$= \frac{sign(J,I)}{|I|^{1/2}}\left(Q_\varphi\right)_J = \langle\, \pi_{Q_\varphi} h_I, h_J\rangle = \langle\left(\pi_{Q_\varphi} + \pi^*_{Q_\varphi} + D_\varphi\right) h_I, h_J\rangle.$$

(iii) $I \subsetneq J$:

$$\langle \pi^*_{Q_\varphi} h_I, h_J\rangle = \langle h_I, \pi_{Q_\varphi} h_J\rangle = \langle h_I, \pi^*_\varphi \pi_\varphi h_J\rangle = \langle \pi^*_\varphi \pi_\varphi h_I, h_J\rangle.$$

by (ii).

We now need to temporarily introduce a further scale of function spaces. For $0 < p < \infty$ and $1 \le q < \infty$, we say that $b \in L^2(\mathbb{R})$ belongs to the space $B^d_{p,q}$ if

$$\|b\|_{B^d_{p,q}} = \left(\sum_{I\in\mathcal{D}}\left(\frac{1}{|I|^{1/q}}\|p_I b\|_q\right)^p\right)^{1/p} < \infty \qquad (12)$$

Where $\|\quad\|_q$ denotes the norm in $L^q(\mathbb{R})$. A continuous version of the following "p-John–Nirenberg Theorem" can be found.

Proposition (1.1.3) [1]: Let $0 < p < \infty$. There exists a constant α_p such that for each nonnegative sequence $(a_I)_{I\in\mathcal{D}}$ indexed by the dyadic intervals,

$$\sum_{I\in\mathcal{D}}\left(\frac{1}{|I|}\sum_{J\in\mathcal{D}(I)} a_J\right)^p \le \alpha_p \sum_{I\in\mathcal{D}}\left(\frac{a_I}{|I|}\right)^p .$$

Note that the reverse inequality trivially holds, with constant equal to I. Note also that the $p = \infty$ version of the above statement fails, i.e. there exists no.

Constant C such that

$$\sup_{I\in\mathcal{D}} \frac{1}{|I|}\sum_{J\in\mathcal{D}(I)} a_J \le C \sup_{I\in\mathcal{D}} \frac{a_I}{|I|},$$

Simply take $a_I = |I|$ for all $I \in \mathcal{D}$.

We include the proof of proposition (1.1.3).

Proof: we shall first suppose that $0 < P \leq I$. Then, for all $I \in \mathcal{D}$, we have

$$\left(\frac{1}{|I|}\sum_{J\in\mathcal{D}(I)} a_J\right)^p \leq \frac{1}{|I|^p}\sum_{J\in\mathcal{D}(I)} a_J^p$$

and hence

$$\sum_{I\in\mathcal{D}}\left(\frac{1}{|I|}\sum_{J\in\mathcal{D}} a_J\right)^p \leq \sum_{I\in\mathcal{D}}\frac{1}{|I|^p}\sum_{J\in\mathcal{D}(I)} a_J^p = \sum_{J\in\mathcal{D}}\left(\sum_{k=0}^{\infty}\frac{1}{(2^k|J|)^P}\right) a_J^p.$$

as each dyadic interval J is contained in exactly one dyadic interval of size $2^k|J|$, for $K = 0,1, \ldots.$ summing the infinite geometric series, we see that.

$$\sum_{I\in\mathcal{D}}\left(\frac{1}{|I|}\sum_{J\in\mathcal{D}(I)} a_J\right)^p \leq \frac{2^p}{2^p-1}\sum_{J\in\mathcal{D}}\left(\frac{a_J}{|J|}\right)^p.$$

As required.

We now consider the case $1 < p < \infty$. we see that $I = I_{n,k} \in \mathcal{D}$

$$\left(\frac{1}{|I|}\sum_{J\in\mathcal{D}(I)} a_J\right)^P = \left(2^n\sum_{m=n}^{\infty}\sum_{J\in\mathcal{D}_m(I)} a_J\right)^P$$

$$= \left(\sum_{m=n}^{\infty}(m-n+1)^{-2}(m-n+1)^2 2^{n-m}\sum_{J\in\mathcal{D}_m(I)} 2^m a_J\right)^p$$

$$\leq C_p\sum_{m=n}^{\infty}(m-n+1)^{2p-2}2^{p(n-m)}\left(\sum_{J\in\mathcal{D}_m(I)} 2^m a_J\right)^p.$$

For some constant C_p by Jensen's inequality since $\sum_{m=n}^{\infty}(m-n+1)^{-2} = \pi^2/6$ for all m and $t \mapsto t^p$ is convex Applying Holder's inequality Where $1/P + 1/q = 1$. We see that

$$\left(\sum_{J\in\mathcal{D}_m(I)} 2^m a_J\right)^P \leq \sum_{J\in\mathcal{D}_m(I)}(2^m a_J)^P\left(\sum_{J\in\mathcal{D}_m(I)} I^q\right)^{P/q} = 2^{(m-n)(p-1)}\sum_{J\in\mathcal{D}_m(I)}(2^m a_J)^p.$$

as $|\mathcal{D}_m(I_{n,k})| = 2^{m-n}$, when m ≥ n. consequently, we get

$$\sum_{I\in\mathcal{D}}\left(\frac{1}{|I|}\sum_{J\in\mathcal{D}(I)} a_J\right)^P$$

$$\leq C_p \sum_{n,k\in\mathbb{Z}}\sum_{m=n}^{\infty}(m-n+1)^{2p-2}2^{(n-m)}\sum_{J\in\mathcal{D}_m(I_{n,k})}\left(2^m a_J\right)^p$$

$$= C_p \sum_{m,J\in\mathbb{Z}}\sum_{n=-\infty}^{m}(m-n+1)^{2p-2}2^{(n-m)}\left(2^m a_{I_{m,J}}\right)^p.$$

Changing the order of summation. But, for all $m \in \mathbb{Z}$,

$$\sum_{n=-\infty}^{m}(m-n+1)^{2p-2}2^{(n-m)} = \sum_{I=1}^{\infty} I^{2p-2}\, 2^{-(I+1)} = K_p.$$

Say. Therefore,

$$\sum_{I\in\mathcal{D}}\left(\frac{1}{|I|}\sum_{J\in\mathcal{D}(I)} a_J\right)^P \leq C_p K_p \sum_{J\in\mathcal{D}}\left(\frac{a_J}{|J|}\right)^p.$$

As required.

Corollary (1.1.4) [1]: Let b be a locally integrable function, and let $0 < P < \infty$. Then $b \in B_p^d$ *if and only if* $b \in B_{p,q}^d$ *for* $1 \leq q \leq 2$.

Moreover, $||\mathrm{b}||_{B_p^d}$ is equivalent to the expressions in (12).

Proof: Applying Holder's inequality it is easy to see that $B_{P,2}^d \subseteq B_{P,q}^d \subseteq B_{P,1}^d$. And it also easy to see that $B_{P,1}^d \subseteq B_{P\,.}^d$ all of these embedding are bounded. So it only remains to prove that $B_p^d \subseteq B_{p,2}^d$, and that the embedding is bounded.

By proposition (1.1.3),

$$\|b\|_{B_{p,2}^d} = \sum_{I\in\mathcal{D}}\left(\frac{1}{|I|}\|P_I b\|^2\right)^{p/2} = \sum_{I\in\mathcal{D}}\left(\frac{1}{|I|^{1/2}}\sum_{J\in\mathcal{D}(I)}|b_J|^2\right)^{p/2}$$

$$\leq \alpha_p \sum_{I\in\mathcal{D}}\left(\frac{|b_I|}{|I|^{1/2}}\right)^P = \alpha_p \|b\|_{B_P^d}^P.$$

Before we can deal with the case $q > 2$, we need

Proposition (1.1.5) [1]: Let $0 < \mathrm{p} < \infty$ and $1 \leq q < \infty. Then$

$$\|Q_b\|_{B^d_{p,q}} \leq 2C^2_{2q}\|b\|^2_{B^d_{2P,2q}} \ \left(b \in B^d_{2P,2q}\right).$$

Where C_{2q} is the norm of the dyadic square function on $L^{2q}(\mathbb{R})$ conversely,

$$\|b\|^2_{B^d_{2P,2q}} \leq C^2_{2q} I_P\left(\|Q_b\|_{B^d_{P,q}} + \|b\|^2_{B^d_{2P,2}}\right).$$

Where C_{2q} is the lower bound of the dyadic square function on $L^{2q}(\mathbb{R})$, and I_P is a constant depending only on $P, I_P = 1$ for $p \geq 1$.

Proof: Note that the projections $(P_I)_{I\in\mathcal{D}}$ are uniformly bounded on each $L^2(\mathbb{R})$, $1 \leq q < \infty$, with norms bounded by 2, independent of q, Let $b \in B^d_{2p.2q}$. Then

$$\begin{aligned}
\|Q_b\|_{B^d_{p,q}} &= \left(\sum_{I\in\mathcal{D}} \frac{1}{|I|^{P/q}}\|P_I Q_b\|^P_q\right)^{1/P} = \left(\sum_{I\in\mathcal{D}} \frac{1}{|I|^{P/q}}\left\|P_I Q_{P_I b}\right\|^P_q\right)^{1/P} \\
&\leq 2\left(\sum_{I\in\mathcal{D}} \frac{1}{|I|^{P/q}}\left\|Q_{P_I b}\right\|^P_q\right)^{1/P} \\
&\leq 2C^2_{2q}\left(\sum_{I\in\mathcal{D}} \frac{1}{|I|^{2P/2q}}\|P_I b\|^{2P}_{2q}\right)^{1/P} = 2C^2_{2q}\|b\|^2_{B^d_{2P,2q}}.
\end{aligned}$$

Where the first equality follows from Lemma (1.1.1)

Conversely

$$\begin{aligned}
\left(\sum_{I\in\mathcal{D}} \frac{1}{|I|^{2P/2q}}\|P_I b\|^{2P}_{2q}\right)^{1/P} &\leq C^2_{2q}\left(\sum_{I\in\mathcal{D}} \frac{1}{|I|^{2P/2q}}\left\|Q_{P_I b}\right\|^P_q\right)^{1/P} \\
&= C^2_{2q}\left(\sum_{I\in\mathcal{D}} \frac{1}{|I|^{2P/2q}}\left\|P_I Q_{P_I b} + \chi_I m_I\left(Q_{P_I b}\right)\right\|^P_q\right)^{1/P} \\
&\leq C^2_{2q}\left(\sum_{I\in\mathcal{D}} \frac{1}{|I|^{2P/2q}}\left(\|P_I Q_b\|_q + \frac{1}{|I|^{1-1/q}}\left\|Q_{P_I b}\right\|_I\right)^P\right)^{1/P} \\
&= C^2_{2q}\left(\sum_{I\in\mathcal{D}} \frac{1}{|I|^{2P/2q}}\left(\|P_I Q_b\|_q + \frac{1}{|I|^{1-1/q}}\|P_I b\|^2_2\right)^P\right)^{1/P} \\
&\leq C^2_{2q} I_P\left(\|Q_b\|_{B^d_{P,q}} + \|b\|^2_{B^d_{2P,2}}\right).
\end{aligned}$$

Theorem (1.1.6) [1]: Let $0 < p < \infty$. Then the spaces $B^d_{p,q}, 1 \le q < \infty$, all coincide with the dyadic Besov space B^d_p. The corresponding norms are equivalent.

In the case $p = \infty$, all the spaces $B^d_{\infty,q}$ coincide with BMO^d. as known from the classical John–Nirenberg Theorem.

We shall present a proof of Theorem (1.1.4) here, which only uses the dyadic square function, a bootstrap argument and the following proposition , which covers the case $1 \le q \le 2$.

Proof:

let $2 < P < \infty$. Because of the trivial inclusion $B_{P,q_2} \subseteq B_{P.q_1}$ for $1 \le q_1 \le q_2 < \infty$, we can assune that $q = 2^n$, $n > 1$. We prove by induction over n that for all $n \in \mathbb{N}$. And all $0 < p < \infty$ there exists a constant $K_{P,n}$ such that

$$\|b\|_{B_{p,n}} \le K_{p,n} \|b\|_{B^d_p} \left(b \in B^d_p\right)$$

By Corollary (1.1.6) this is true for $n = 1$ suppose that the statement holds for some $n \in \mathbb{N}$. then by proposition (1.1.5), for each $b \in B^d_P$,

$$\begin{aligned}
\|b\|^2_{B^d_{P,2^{n+1}}} &\le I_P C^2_{2^{n+1}} \left(\|Q_b\|_{B^d_{P/2,2^n}} + \|b\|^2_{B^d_{P,2}}\right) \\
&\le C^2_{2^{n+1}} I_P \left(K_{P/2,n} \|Q_b\|_{B^d_{P/2,1}} + \|b\|^2_{B^d_{P,2}}\right) \\
&\le 2C^2_{2^{n+1}} I_P \left(K_{P/2,n} \|b\|^2_{B^d_{P,2}} + \|b\|^2_{B^d_{P,2}}\right) \\
&\le \|b\|^2_{B^d_P} 2\alpha^2_P C^2_{2^n} I_P \left(K_{P/2,n} + 1\right).
\end{aligned}$$

The theorem follows now with an appropriate choice of $K_{p,n+1.}$

Corollary (1.1.7) [1]: Let 0 < p < ∞ and let $b \in B^d_p$. Then $Q_b \in B^d_{p/2}$, and there exists a constant $C_p > 0$ depending only on p such that $\|Q_b\|_{B^d_{p/2}} \le C_p \|b\|^2_{B^d_P}$.

Proof: Corollary (1.1.4) (or Theorem (1.1.6)) and Proposition (1.1.5).

Now we can give our "p-John-Nirenberg" proof of Theorem (1.1.8) (ii). We will give the full proof only for $p \ge 1$.

Theorem (1.1.8) [1]:

(i) $b \in BMO^d$ if and only if $\pi_b \in \mathcal{g}\big(L^2(\mathbb{R})\big)$;

(ii) for $0 < P < \infty$, $b \in B_p^d$ if and only if $\pi_b \in S_p\big(L^2(\mathbb{R})\big)$;

(iii) $b \in\ VMO^d$ if and only if $\pi_b \in \mathcal{H}(L^2(\mathbb{R}))$.

These results are well known, for the boundedness result and for the result concerning Schatten classes S_P , $1 \leq p < \infty$.

Before we give a "John-Nirenberg type" proof Theorem (1.1.8) (ii).

Proof: Notice that

$$\sum_{I\in\mathcal{D}} \|\pi_b^* h_I\|^P = \sum_{I\in\mathcal{D}} \left\|\bar{b}_I \frac{\chi_I}{|I|}\right\|^P = \sum_{I\in\mathcal{D}} \frac{1}{|I|^{P/2}} |b_I|^P = \|b\|_{B_P^d}^P.$$

For $0 < p < \infty$ by proposition (1.1.3) thus "$\Rightarrow$" follows immediately for $0 < P \leq 2$ from Eq (3).

To prove "$\Rightarrow$" for $2 \leq P < \infty$, note first that for $0 < P < \infty$.

$$\|D_b\|_{S_{P/2}}^{P/2} = \sum_{I\in\mathcal{D}} |\langle \pi_b^* \pi_b h_I, h_I\rangle|^{P/2} = \sum_{I\in\mathcal{D}} \|\pi_b h_I\|^P$$

$$= \sum_{I\in\mathcal{D}} \frac{1}{|I|^{P/2}} \left(\sum_{J\in\mathcal{D}'(I)} \left|b_J\right|^2 \right)^{P/2} \leq K_P \|b\|_{B_P^d}^P$$

By proposition (1.1.3), and that therefore for $2 \leq p \leq 4$

$$\|\pi_b\|_{s_p}^2 = \|\pi_b^* \pi_b\|_{s_{p/2}} \leq 2\left\|\pi_{Q_b}\right\|_{S_{P/2}} + \|D_b\|_{S_{P/2}}$$

$$\leq 2\left\|\pi_{Q_b}\right\|_{S_{P/2}} + K_P\|b\|_{B_P^d}^2 \leq K_P' \left(\|Q_b\|_{B_{P/2}^d} + \|b\|_{B_P^d}^2\right) \leq C_P \|b\|_{B_P^d}^2$$

by Corollary (1.1.7) and the first part of the proof. Inductively, we obtain the result for all p with $2 \leq p < \infty$. To obtain the reverse direction, we define a bounded operator $R: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ of norm 1 by $Rh_I =\ h_I$ for $\in \mathcal{D}$, where Î denotes the parent interval of *I*. Recalling that.

$$\sum_{n=1}^{\infty} |\langle Te_n, \sigma_n\rangle|^P \leq \|T\|_{S_P}^P \qquad (13).$$

For any orthonormal bases $\{e_n\}$, (σ_n), $p \geq 1$ *and* $T \in S_p$, we find that.

$$\|\pi_b\|_{s_p}^P \geq \|\pi_b R\|_{s_p}^P \geq \sum_{I\in\mathcal{D}} |\langle Rh_I, \pi_b^* h_I\rangle|^P = \sum_{I\in\mathcal{D}} \left|\langle h_I, \frac{\chi_I}{|I|}\bar{b}_I\rangle\right|^P$$

$$= \frac{1}{2^{P/2}} \sum_{I\in\mathcal{D}} \left(\frac{|b_I|}{|I|^{1/2}}\right)^P = \frac{1}{2^{P/2}} \|b\|_{B_P^d}^P.$$

$for\ 0 \leq p < \infty$

The implication "⇐" in Theorem (1.1.8) (ii) for $0 < p < 1$ is more difficult to deal with and was first shown by Peng .

Section (1.2): Operators of Besov Spaces and Vectors of Schatten Class with Hankel Operators

Dyadic paraproducts with matrix or operator symbols have been considered .We first introduce some notation for dyadic paraproducts acting on a vector valued Hilbert space, with operator valued symbols.

Let $\mathcal{H}$ denote separable Hilbert space and $L^2(\mathbb{R},\mathcal{H})$ the corresponding vector valued Hilbert space, so

$$L^2(\mathbb{R},\mathcal{H}) := \left\{ g: \mathbb{R} \to \mathcal{H}: \|g\|^2_{L^2(\mathbb{R},\mathcal{H})} = \int_{\mathbb{R}} \|g(t)\|^2_{\mathcal{H}}\, dt \ < \infty \right\}.$$

We may consider $L^2(\mathbb{R},\mathcal{H})$ as the Hilbert space tensor product

$L^2(\mathbb{R}) \otimes \mathcal{H}$ and, for $f \in L^2(\mathbb{R})$ and $x \in \mathcal{H}$, we let $f \otimes x$ denote the element of $L^2(\mathbb{R}.\mathcal{H})$ defined for almost all $t \in \mathbb{R}$ by $f \otimes x(t) = f\ (t)x$.

Let B be a locally SOT-integrable operator valued function on $\mathbb{R}$, so $B(t) \in \mathcal{g}(\mathcal{H})$ for almost all $t \in \mathbb{R}$, and for $I \in \mathcal{D}$ we may formally define the operator $B_I \in \mathcal{g}\ (H)$ given by

$$\langle B_1 x, y\rangle \ = \int_I h_1(t)\, \langle B(t)x, y\rangle dt, x, y \in \ \mathcal{H}.$$

For the definition of SOT integrability, we then define the (dyadic) paraproduct Π_B , acting on elementary tensors in $L^2(\mathbb{R},\mathcal{H})$ by

$$\Pi_B(f \otimes x) = \sum_{I \in \mathcal{D}} m_I f h_I \otimes B_I x,\ f \in L^2(\mathbb{R}), x \in \mathcal{H} \quad (14).$$

and extending by linearity. One would anticipate that the boundedness of such an operator would be characterised by an operator bounded mean oscillation criterion. However, it was shown that the naive generalisation of the scalar BMO condition to the operator case does not imply boundedness of the operator paraproducet.

We shall show, however, that Schatten class membership may be characterised by an operator Besov condition analogous to the scalar condition. These results can also be obtained using Rochberg and Semmes' method of nearly weakly orthonormal sequences , although the vector case does not seem to appear in the literature. We shall first derive an expression for Π_B^*.

Lemma (1.2.1) [1]: if

$$\Lambda_B(f \otimes x) = \sum_{I\in\mathcal{D}} f_I \frac{\chi_I}{|I|} \otimes B_I^* x, f \in L^2(\mathbb{R}), x \in \mathcal{H}.$$

Extending by linearity, then $\Lambda_B = \Pi_B^*$

Proof: This can easily be verified by means of elementary tensors.

We shall follow the same approach as in the scalar case, using dyadic square functions. For an operator valued function B, let Q_B be the "square of the dyadic square function" of B, that is

$$\mathcal{Q}_B(t) = \sum_{I\in\mathcal{D}} B_I^* B_I \frac{\chi_I(t)}{|I|}, t \in \mathbb{R}.$$

Let D_B be the operator on $L^2(\mathbb{R}, \mathcal{H})$ defined on elementary tensors by

$$D_B(f \otimes x) = \sum_{I\in D} \frac{1}{|I|} \sum_{J\in\mathcal{D}(I)'} f_I h_I \otimes B_J^* B_J x.$$

The following identity relates the paraproducts Π_B and $\Pi_{\mathcal{Q}_B}$

Proposition (1.2.2) [1]:

$$\Pi_B^* \Pi_B = \Pi_{\mathcal{Q}_B} + \Pi_{\mathcal{Q}_B}^* + D_B.$$

Proof: It is sufficient to show that

$$\begin{aligned}\langle \pi_B^* \pi_B (h_I \otimes x), h_J \otimes y)\rangle \\ = \langle (\pi_{\mathcal{Q}_B} + \Pi_{\mathcal{Q}_B}^* + D_B)(h_I \otimes x), h_I \otimes y\rangle \qquad (15).\end{aligned}$$

for all $I, J \in \mathcal{D}$ and $x, y \in \mathcal{H}$. This is shown exactly as in Proposition (1.1.2). For $0 < p < \infty$. We shall say that an operator valued function B, lies in the operator valued dyadic Besov space B_P^d , if

$$\|B\|_{B_p^d} = \left(\sum_{I\in\mathcal{D}} \left(\frac{\|B_I\|_{S_P}}{|I|^{1/2}}\right)^P\right)^{1/P} < \infty.$$

We shall show that Schatten class operator valued dyadic paraproducts have symbols which belong to corresponding Besov spaces, thus generalising-the scalar result.

We prove an operator analogue to Corollary (1.1.7). For $P \geq 2$, the Besov norms of B and Q_B are related in the same way as in the scalar case.

Lemma (1.2.3) [1]: if $2 \leq p < \infty$ and $B \in B_P^d$, then $Q_B \in B_{p/2}^d$, with $\|Q_B\|_{B_{P/2}^d} \leq \alpha_P \|B\|_{B_P^d}^2$ for some universal constant α_p.

Proof: Note that $\|\cdot\|_{S_{P/2}}$ is a norm. By the definition of Q_B and (5), we see that.

$$(Q_B)_I = \frac{1}{|I|^{1/2}} \left(\sum_{J \in \mathcal{D}(I_+)} B_J^* B_J - \sum_{J \in \mathcal{D}(I_-)} B_J^* B_J \right).$$

and so

$$\|(Q_B)_I\|_{S_{P/2}} \leq \frac{1}{|I|^{1/2}} \sum_{J \in \mathcal{D}(I)'} \|B_J^* B_J\|_{S_{P/2}} = \frac{1}{|I|^{1/2}} \sum_{J \in \mathcal{D}(I)'} \|B_J\|_{S_P}^2,$$

Which gives

$$\|Q_B\|_{B_{p/2}^d}^{P/2} \leq \sum_{I \in \mathcal{D}} \left(\frac{1}{|I|} \sum_{J \in \mathcal{D}(I)'} \|B_J\|_{S_P}^2 \right)^{P/2} \leq \alpha_P \|B\|_{B_P^d}^P.$$

by Proposition (1.1.3).

The analogous statement for $p = \infty$ is false for infinite-dimensional $\mathcal{H}$. For $I \in \mathcal{D}$, let U_I and V_I be the bounded operators given by

$$U_I : \mathcal{H} \to L^2(\mathbb{R}, \mathcal{H}), U_I x = x_I \otimes x, x \in \mathcal{H},$$

$$V_I : L^2(\mathbb{R}, \mathcal{H}) \to \mathcal{H}, V_I F = \int_I F(t) h_I(t)\, dt, F \in L^2(\mathbb{R}, \mathcal{H}).$$

Then $B_I = V_I \Pi_B U_I$. It follows that, for $0 < p < \infty$, if $\Pi_B \in S_p$, then $B_1 \in S_p$.

Proposition (1.2.4) [1]: If $0 < p \leq 2$ and $B \in B_p^d$ *then* $\Pi_B \in S_p$ and $\|\Pi_B\|_{S_P} \leq \|B\|_{B_P^d}$,

Proof: Again, it will be more convenient to work with adjoints. Let $0 < P < \infty$ and $I \in \mathcal{D}$, Suppose (without. loss of generality, by the discussion above) that B_I^* has Schmidt decomposition,

$$B_I^* x = \sum_{n=0}^{\infty} \lambda_n^I \langle x. e_n^I \rangle \sigma_n^I \, , x \in \mathcal{H}.$$

Where $\{e_n^1\}$ and $\{\sigma_n^1\}$ are orthonormal bases for $\mathcal{H}$ Therefore,

$$\|B_I^*\|_{S_P} = \|B_I\|_{S_P} = \left(\sum_{n=0}^{\infty} (\lambda_n^I)^P \right)^{1/P}.$$

It follows that $\{h_I \otimes e_n^1 : I \in \mathcal{D}, n = 0, 1, \ldots\}$ is an orthonormal basis for $L^2(\mathbb{R}, \mathcal{H})$. It is clear from Lemma (1.2.1) that

$$\Pi_B^* \, (h_I \otimes e_n^1) = \frac{\chi_I}{|I|} \lambda_n^I \otimes \sigma_n^1$$

Consequently,

$$\sum_{n=0}^{\infty} \|\Pi_B^*(h_I \otimes e_n^I)\|^P = \left(\frac{\|B_I\|_{S_P}}{|I|^{1/2}} \right)^P.$$

For each $I \in \mathcal{D}$ and therefore

$$\|\Pi_B\|_{S_P}^P \leq \sum_{I \in \mathcal{D}} \sum_{n=0}^{\infty} \|\Pi_B^*(h_I \otimes e_n^I)\|^P = \|B\|_{B_P^d}^P.$$

by (3).

The rest of this section will be concerned with showing that the statement in Proposition (1.2.4) extends to $p > 2$, and that also the reverse holds. We shall first use Proposition (1.2.4) and Lemma (1.2.3) to show that $B \in B_p^d$ implies $\Pi_B \in S_p$, for $2 < p < \infty$.

Proposition (1.2.5) [1]: If $2 \leq p < \infty$ and $B \in B_p^d$, then

$\Pi_B \in S_p$. Moreover, there exist a constant $C_P > 0$ depending only on p such that $\|\Pi_B\|_{S_P} \leq C_P \|B\|_{B_P^d}$.

Proof: The proof runs along the lines of the proof of Theorem (1.1.8) (ii). We shall suppose that $2^n < P \leq 2^{n+1}$ for $n = 0, 1, ..$ and proceed by induction. The base case ($n = 0$) is covered by Proposition (1.2.4) so suppose that $n \geq 1$. We shall first consider the operator D_B. By definition, $D_B$has block diagonal form $D_B = (E_I)_{I\in\mathcal{D}}$ with respect to the Hilbert space decomposition $L^2(\mathbb{R}, \mathcal{H}) = \oplus_{I\in\mathcal{D}} \mathcal{H}$given by $f \mapsto (h_I)_{I\in\mathcal{D}}$. Here, E_I is defined by $\langle E_I x. y \rangle = \langle \Pi_B^* \, \Pi_B \, h_I \otimes x, h_I \otimes y \rangle$ for $x, y \in \mathcal{H}$ That is, $E_I = \frac{1}{|I|} \sum_{J\in\mathcal{D}(I)'} B_J^* B_J$ Thus

$$\|D_B\|_{S_{P/2}}^{P/2} = \sum_{I\in\mathcal{D}} \|E_I\|_{S_{P/2}}^{P/2} = \sum_{I\in\mathcal{D}} \left\| \frac{1}{|I|} \sum_{I\in\mathcal{D}(I)'} B_J^* B_J \right\|_{S_{P/2}}^{P/2}$$

$$\leq \sum_{I\in\mathcal{D}} \left(\frac{1}{|I|} \sum_{J\in\mathcal{D}(t)'} \|B_J\|_{S_P}^2 \right)^{P/2}$$

$$\leq K_P \sum_{I\in\mathcal{D}} \frac{1}{|I|^{P/2}} \|B_I\|_{S_P}^P = K_P \|B\|_{B_P^d}.$$

by Proposition (1.1.5), since $\|\cdot\|_{S_{P/2}}$ is a norm.

Also, by Lemma (1.2.3), $Q_B \in S_{P/2}$ and $2^{n-1} < p/2 \leq 2^n$.

Hence, by the inductive hypothesis $\Pi_{Q_B} \in S_{P/2}$ and $\|Q_B\|_{B_P^d} \leq C_P \|B\|_{B_p^d}^2$ Consequently, by Proposition (1.2.4), we have

$$\|\Pi_B\|_{S_P}^2 = \|\Pi_B^* \Pi_B\|_{S_{P/2}} \leq 2 \|\Pi_{Q_B}\|_{S_{P/2}} + \|D_B\|_{S_{P/2}}$$

$$\leq 2C_{P/2} \|Q_B\|_{B_{P/2}^d} + \|D_B\|_{S_{P/2}} \leq C_P \|B\|_{B_P^d}^2$$

as required for an appropriate choice of C_P.

Finally, we must show that, for $0 < p < \infty$, $\Pi_B \in S_p$ implies that $B \in B_p^d$ .We will first deal with the case $1 \leq P < \infty$.

Proposition (1.2.6)[1]: If $1 \leq P < \infty$. and $\Pi_B \in S_p$then $B \in B_p^d$

moreover $\|B\|_{B_P^d} \leq C_P \|\Pi_B\|_{S_P}$ for some universal constant C_P .

Proof: Let $1 \leq p < \infty$, and let $T: L^2(\mathbb{R}, \mathcal{H}) \to L^2(\mathbb{R}, \mathcal{H})$ be in the Schatten class S_P. The block diagonal $E = (E_I)_{I\in\mathcal{D}}$ of T, taken with respect to the Hilbert space decomposition $L^2(\mathbb{R}, \mathcal{H}) = \oplus_{I\in\mathcal{D}} \mathcal{H}$, $f \mapsto (h_I)_{I\in\mathcal{D}}$ is then given by

$$Eh_I \otimes e = h_I \otimes E_I e.$$

For $I \in \mathcal{D}$ where $E_I : \mathcal{H} \to \mathcal{H}$ is defined by $\langle E_I e , f\rangle = \langle Te \otimes h_I, f \otimes h_I\rangle$ for $I \in \mathcal{D}, e, f \in \mathcal{H}$. We will use the inequality

$$\|E\|_{S_P}^P = \sum_{I \in \mathcal{D}} \|E_I\|_{S_P}^P \le \|T\|_{S_P}^P \qquad (16).$$

Similarly to the proof of Theorem (1.1.8) (ii), one defines a bounded linear operator $R: L^2(\mathbb{R}, \mathcal{H}) \to L^2(\mathbb{R}, \mathcal{H})$ of norm I by $Rh_I \otimes e = h_{\hat{I}} \otimes e$

For $I \in \mathcal{D}$,$e \in \mathcal{H}$.

Suppose now that $\Pi_B \in S_p$, for each $I \in \mathcal{D}$, let B_I^* have the Schmidt decomposition

$$B_I^* x = \sum_{n=0}^{\infty} \lambda_n^I < x.e_n^I > \sigma_n^I, x \in \mathcal{H}.$$

Where $\{e_n^I\}$ and $\{\sigma_n^I\}$ are orthonormal bases for $\mathcal{H}$.

Applying (16) with $T = \Pi_B R$and using (13), we obtain

$$\begin{aligned}
\|\Pi\|_{S_P}^P \ge \|\Pi_B R\|_{S_P}^P \ge \sum_{I \in \mathcal{D}} \|E_I\|_{S_P}^P \ge \sum_{I \in \mathcal{D}} \sum_{n=1}^{\infty} |< E_I \sigma_n^I, e_n^I >|^P \\
= \sum_{I \in \mathcal{D}} \sum_{n=1}^{\infty} |< \Pi_B R h_I \otimes \sigma_n^I, h_I \otimes e_n^I >|^P \\
= \sum_{I \in \mathcal{D}} \sum_{n=1}^{\infty} \left|< h_{\hat{I}} \otimes \sigma_n^I, \frac{\chi_I}{|I|} B_I^* e_n^I >\right|^P \\
= \sum_{I \in \mathcal{D}} \frac{1}{2^{P/2}} \frac{1}{|I|^{P/2}} \sum_{n=1}^{\infty} |\lambda_n^I|^P \\
= \frac{1}{2^{P/2}} \sum_{I \in \mathcal{D}} \left(\frac{\|B_I\|_{S_P}^P}{|I|^{1/2}} \right)^P = \frac{1}{2^{P/2}} \|B\|_{B_B^d}^P.
\end{aligned}$$

Finally, we shall consider the case $0 < p < 1$. we shall generalize an argument. Note that, for $0 < p < 1$.

$\| \quad \|_{S_P}$ is not a norm. However, if $T = R + S$, then

$$\|\mathrm{T}\|_{S_p}^p \leq \|\mathrm{R}\|_{S_p}^p + \|\mathrm{S}\|_{S_p}^p \qquad (17).$$

We can obtain a partial reverse inequality, in the case that R and S have orthogonal ranges.

Lemma (1.2.7) [1]: Let $0 < p < \infty$ If R and S are operators with orthogonal ranges and $T = R + S$ then

$$\|\mathrm{T}\|_{S_p}^p \leq \tfrac{1}{2}\left(\|\mathrm{R}\|_{S_p}^p + \left\|\mathrm{S}\right\|_{S_p}^p\right).$$

Proof: It follows that $T^*T = R^*R + S^*S,\ as\ R^*S = RS^* = 0$. Therefore, $R^*R \leq T^*T$.

By Douglas' Lemma there exists a contraction Z such that $R = ZT$ and so $\|R\|_{S_p} \leq \|T\|_{S_p}$. similarly, $\|S\|_{S_p} \leq \|T\|_{S_p,}$ and the result follows.

For $m \in \mathbb{Z}$ we define the orthogonal projection Δ_{m} on $L^2\ (\mathbb{R}, \mathcal{H})$ by

$$\Delta_m(f \otimes x) = \sum_{I \in D} \langle f, h_1 \rangle\ h_1 \otimes x.$$

defined here on elementary tensors for $f \in L^2(\mathbb{R})$ and $x \in \mathcal{H}$. We also define, for $m, n \in \mathbb{Z}$,

$$\Pi_B^{n,m} = \Delta_m \Pi_B \Delta_n$$

Lemma (1.2.8) [1]: Let $B \in B_p^d$ If $m \leq n$ then $\Pi_B^{n,m} = 0$. If $m > n$ and $0 < P < 1$ then

$$\|\Pi_B\|_{S_P}^P \leq 2^{(n-m)P/2} \sum_{I \in \mathcal{D}_m} \left(\frac{\|B_I\|_{S_p}}{|I|^{1/2}}\right)^P.$$

Proof: By definition, we see that, for $f \in L^2(\mathbb{R})$ and $x \in \mathcal{H}$

$$\Pi_B^{n,m}(f \otimes x) = \sum_{J \in \mathcal{D}_n} \sum_{I \in \mathcal{D}_m} \langle f, h_J \rangle m_I\left(h_J\right) h_I \otimes B_I x.$$

Therefore, by (5), if $m \leq n$ then $\Pi_B^{n,m} = 0$ and if $m > n$

$$\Pi_B^{n,m}(f \otimes x) = \sum_{J \in \mathcal{D}_n} \sum_{I \in \mathcal{D}_m(J_+)} \frac{\langle f, h_J \rangle}{|J|^{1/2}} h_I \otimes B_I x - \sum_{J \in \mathcal{D}_n} \sum_{I \in \mathcal{D}_m(J_-)} \frac{\langle f, h_J \rangle}{|J|^{1/2}} h_I \otimes B_I x.$$

Let B_I have Schmidt decomposition

$$B_I x = \sum_{I=0}^{\infty} \lambda_I^I < x. e_I^I > \sigma_I^I \,, \qquad (18).$$

So we have

$$\begin{aligned} &\Pi_B^{n,m}(f \otimes x) \\ &= \sum_{J\in\mathcal{D}_n} \sum_{I\in\mathcal{D}_m(J_+)} \sum_{I=0}^{\infty} \frac{\lambda_I^I}{|J|^{1/2}} \langle f \otimes x, h_J \otimes e_I^I \rangle\, h_I \otimes \sigma_I^I \\ &- \sum_{J\in\mathcal{D}_n} \sum_{I\in\mathcal{D}_m(J_-)} \sum_{I=0}^{\infty} \frac{\lambda_I^I}{|J|^{1/2}} \langle f \otimes x, h_J \otimes e_I^I \rangle\, h_I \otimes \sigma_I^I \qquad (19). \end{aligned}$$

Thus $\Pi_B^{n,m}$ has been expressed as the difference of two operators with given Schmidt decompositions, so, by (17),

$$\begin{aligned} \|\Pi_B^{n,m}\|_{S_P}^P &\le \sum_{J\in\mathcal{D}_n} \sum_{I\in\mathcal{D}_m(J_+)} \sum_{I=0}^{\infty} \left(\frac{\lambda_I^I}{|J|^{1/2}}\right)^p + \sum_{J\in\mathcal{D}_n} \sum_{I\in\mathcal{D}_m(J_-)} \sum_{I=0}^{\infty} \left(\frac{\lambda_I^I}{|J|^{1/2}}\right)^p \\ &= \sum_{J\in\mathcal{D}_n} \sum_{I\in\mathcal{D}_m(J)} \left(\frac{\|B_I\|_{S_P}}{|J|^{1/2}}\right)^P = 2^{(n-m)p/2} \sum_{I\in\mathcal{D}_m} \left(\frac{\|B_I\|_{S_P}}{|I|^{1/2}}\right)^P. \end{aligned}$$

Let $B \in B_p^d$ and N be a positive integer (to be determined later) .

For $k = 0, \ldots N-1$, let

$$\Pi_{B,k} = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \Pi_B^{Nn+k,Nn+k+1} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{m} \Pi_B^{Nn+k,Nm+k+1}$$

by Lemma (1.2.8) Let

$$\Pi_{B,k}^{(0)} = \sum_{n=-\infty}^{\infty} \Pi_B^{Nn+k,Nn+k+1}, \Pi_{B,k}^{(1)} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{m-1} \Pi_B^{Nn+k,Nm+k+1}$$

So

$$\Pi_{B,k} \Pi_{B,k}^{(0)} + \Pi_{B,k}^{(1)}$$

Lemma (1.2.9) [1]: For $0 < p \le 1$ and B, N as above

$$\sum_{k=0}^{N-I} \left\| \Pi_{B,k}^{(0)} \right\|_{S_P} \geq C_P \|B\|_{B_P^d}^P.$$

Proof: By (18) and (19), we see that,

$$\Pi_B^{Nn+k,Nn+k+1}(f \otimes x) = \sum_{J \in \mathcal{D}_{Nn+k}} \frac{\langle f, h_J \rangle}{|J|^{1/2}} \left(h_{J_+} \otimes B_{J_+} x - h_{J_-} \otimes B_{J_-} x \right)$$

$$= \sum_{J \in \mathcal{D}_{Nn+k}} \sum_{I=0}^{\infty} \frac{\lambda_I^{J+}}{|J|^{1/2}} \langle f \otimes x, h_J \otimes e_I^{J+} \rangle \, h_{J_+} \otimes \sigma_I^{J+}$$

$$- \sum_{J \in \mathcal{D}_{N_{n+k}}} \sum_{I=0}^{\infty} \frac{\lambda_I^{J-}}{|J|^{1/2}} \langle f \otimes x, h_J \otimes e_I^{J-} \rangle \, h_{J_-} \otimes \sigma_I^{J-}.$$

and so

$$\Pi_{B,k}^{(0)}(f \otimes x)$$

$$= \sum_{n=-\infty}^{\infty} \sum_{J \in \mathcal{D}_{Nn+k}} \sum_{I=0}^{\infty} \frac{\lambda_I^{J+}}{|J|^{1/2}} \langle f \otimes x, h_J \otimes e_I^{J+} \rangle h_{J_+} \otimes \sigma_I^{J+}$$

$$- \sum_{n=-\infty}^{\infty} \sum_{J \in \mathcal{D}_{Nn+k}} \sum_{I=0}^{\infty} \frac{\lambda_I^{J-}}{|J|^{1/2}} \langle f \otimes x, h_J \otimes e_I^{J-} \rangle \, h_{J_-} \otimes \sigma_I^{J-},$$

Since $\langle h_{J_+}, h_{I_-} \rangle = 0$ for all $,J \in \mathcal{D}$, $\Pi_{B,k}^{(0)}$ has been expressed as the difference of two operators with orthogonal ranges and given Schmidt decompositions.

So, by Lemma (1.2.7),

$$\left\| \Pi_{B,k}^{(0)} \right\|_{S_P}^P \geq \frac{1}{2} \left(\sum_{n=-\infty}^{\infty} \sum_{J \in \mathcal{D}_{Nn+k}} \sum_{I=0}^{\infty} \frac{\left(\lambda_I^{J+}\right)^P}{|J|^{P/2}} + \sum_{n=-\infty}^{\infty} \sum_{J \in \mathcal{D}_{Nn+k}} \sum_{I=0}^{\infty} \frac{\left(\lambda_I^{J-}\right)^P}{|J|^{P/2}} \right)$$

$$= \frac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{J \in \mathcal{D}_{Nn+k}} \frac{1}{|J|^{P/2}} \left(\|B_{J_+}\|_{S_P}^P + \|B_{J_-}\|_{S_P}^P \right)$$

$$= \frac{1}{2^{P/2+1}} \sum_{n=-\infty}^{\infty} \sum_{I \in \mathcal{D}_{Nn+k+1}} \left(\frac{\|B_I\|_{S_P}^P}{|I|} \right)^{P/2}$$

Consequently,

$$\sum_{k=0}^{N-1} \left\| \Pi_{B,k}^{(0)} \right\|_{S_P}^{P} \le \frac{1}{2^{P/2+1}} \sum_{I\in\mathcal{D}} \left(\frac{\|B_I\|_{S_P}^{P}}{|I|} \right)^{P} = \frac{1}{2^{P/2+1}} \|B\|_{B_P^d}^{P},$$

In seeking a converse to Proposition (1.2.4) for $0 < p < 1$, we shall first suppose that $B \in B_p^d$: a simple density argument will then give the full result.

Proposition (1.2.10) [1]: Let $0 < p < 1$ .There exists a constant C_P such that if $B \in B_p^d$, then $\|B\|_{B_p^d} \le C_p \|\Pi_B\|_{S_p}$

Proof: Note that

$$\Pi_{B,k} = \left(\sum_{n=-\infty}^{\infty} \Delta_{Nn+k} \right) \Pi_B \left(\sum_{m=-\infty}^{\infty} \Delta_{Nm+k+1} \right),$$

And so $\|\Pi_{B,k}\|_{S_p} \le \|\Pi_B\|_{S_p}$ *as* $\sum_{n=-\infty}^{\infty} \Delta_{Nn+k}$ *and* $\sum_{m=-\infty}^{\infty} \Delta_{Nm+k+1}$ are norm 1 projections. Consequently,

$$N\|\Pi_B\|_{S_P}^P \ge \sum_{k=0}^{N-1} \|\Pi_{B,k}\|_{S_P}^P \ge \sum_{k=0}^{N-1} \left(\left\|\Pi_{B,k}^{(0)}\right\|_{S_P}^P - \left\|\Pi_{B,k}^{(1)}\right\|_{S_P}^P \right)$$

$$\ge C_P \|B\|_{B_P^d}^P - \sum_{k=0}^{N-1} \left\| \Pi_{B,k}^{(1)} \right\|_{S_P}^P \qquad (20).$$

by Lemma (1.2.9). However, for $k = 0, \dots, N - 1$, we have by (17) and Lemma (1.2.8)

$$\begin{aligned}
\left\| \Pi_{B,k}^{(1)} \right\|_{S_P}^P &\le \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{m-1} \|\Pi_B^{Nn+k,Nm+k+1}\|_{S_P}^P \\
&\le \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{m-1} 2^{(Nn-Nm-1)P/2} \sum_{I\in\mathcal{D}_{Nm+k+1}} \left(\frac{\|B_I\|_{S_P}}{|I|^{1/2}} \right)^P \\
&= \sum_{m=-\infty}^{\infty} 2^{-(Nm+1)P/2} \sum_{I\in\mathcal{D}_{Nm+k+1}} \left(\frac{\|B_I\|_{S_P}}{|I|^{1/2}} \right)^P \sum_{n=-\infty}^{m-1} 2^{N_{nP/2}} \\
&= \sum_{m=-\infty}^{\infty} 2^{-(Nm+1)P/2} \sum_{I\in\mathcal{D}_{Nm+k+1}} \left(\frac{\|B_I\|_{S_P}}{|I|^{1/2}} \right)^P \sum_{n=-\infty}^{m-1} \frac{2^{N_{nP/2}}}{2^{N_{P/2}-1}} \\
&= \frac{2^{-P/2}}{2^{N_{P/2}-1}} \sum_{m=-\infty}^{\infty} \sum_{I\in\mathcal{D}_{N_{m+k+1}}} \left(\frac{\|B_I\|_{S_P}}{|I|^{1/2}} \right)^P .
\end{aligned}$$

Therefore,

$$\sum_{k=0}^{N-1} \left\| \Pi_{B,k}^{(1)} \right\|_{S_P}^{P} \leq \frac{2^{-P/2}}{2^{NP/2-1}} \sum_{I\in\mathcal{D}} \left(\frac{\|B_I\|_{S_P}}{|I|^{1/2}} \right)^{P} = \frac{2^{-P/2}}{2^{NP/2-1}} \|B\|_{B_P^d}^{P},$$

So, by (20), we see that, for all N.

$$\|\Pi_B\|_{S_P}^{P} \geq \frac{1}{N}\left(\mathrm{C_P} - \frac{2^{-p/2}}{2^{Np/2}-1} \right) \|B\|_{B_P^d}^{P}.$$

Choose N large enough so that $C_P - \frac{2^{-p/2}}{2^{Np/2}-1} > 0$ to obtain the required result.

Corollary (1.2.11) [1]: Let $0 < p \leq 1$. If $\Pi_B \in S_p$ then $S_p \in B_p^d$. , Moreover, there exists a constant $\mathrm{C_p}$ such that $\|B\|_{B_p^d} \leq C_p \|\Pi_B\|_{S_p}$.

Proof: For any positive integer N, let

$$\mathcal{D}^{(N)} := \{ I_{n,k} \in \mathcal{D} : |n| \leq N, |k| \leq N \} \; and \; B^{(N)}(t) = \sum_{I\in\mathcal{D}^{(N)}} B_I h_I(t).$$

Then $B^{(N)} \in B_p^d$ and so $\|B^{(N)}\|_{B_p^d} \leq C_p \|\Pi_{B^{(N)}}\|_{S_p}$ by Proposition (1.2.10). But,

$$\Pi_{B^{(N)}}(f \otimes x) = \sum_{I\in\mathcal{D}^{(N)}} m_J f h_J \otimes B_J x = P^{(N)} \Pi_B (f \otimes x).$$

Where $P^{(N)}$ is the orthogonal projection on $L^2(\mathbb{R}, \mathcal{H})$ defined by

$$P^{(N)}\left(h_J \otimes x \right) = h_J \otimes x \; if \; J \in \mathcal{D}^{(N)} = 0 \; otherwise \quad (21).$$

Therefore,

$$\|B^{(N)}\|_{B_p^d} \leq C_p \|P^{(N)} \Pi_B\|_{S_p} \leq C_P \|\Pi_B\|_{S_p}$$

for all N. But $\{\|B^{(N)}\|_{B_p^d}\}$ is an increasing sequence and so we see that

$$\|B\|_{B_P^d} = \lim_{N\to\infty} \left\| B^{(N)} \right\|_{B_P^d} \leq C_P \|\Pi_B\|_{S_P}$$

In summary, combining Propositions (1.2.4), (1.2.5), (1.2.11) and Corollary (1.2.6), we obtain the main result.

Theorem (1.2.12) [1]: For $0 < p < \infty$, $\Pi_B \in S_p$ if and only if $B \in B_p^d$. Moreover,

$$C_p \|B\|_{B_p^d} \leq \|\Pi_B\|_{S_p} \leq K_P \|B\|_{B_p^d}$$

Let $I < p < \infty$, and let $B: \mathbb{R} \to \mathcal{g}(\mathcal{H})$ be locally integrable. We say that B is in the operator Besov space $B_P(\mathbb{R})$, if

$$\|B\|_{B_P}^P := \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\|B(x) - B(y)\|_{S_P}^P}{|x - y|^2} \, dxdy < \infty.$$

Theorem (1.2.13) [1]: Let $1 < p < \infty$, and let $B: \mathbb{R} \to \mathcal{g}(\mathcal{H})$ be antianalytic and locally integrable. Then the following are equivalent:

(i) The vector Hankel operator $\Gamma_B: L^2(\mathbb{R}, \mathcal{H}) \to L^2(\mathbb{R}, \mathcal{H})$ is in S_p.

(ii) $B \in B_P(\mathbb{R})$.

We can use our results on vector paraproducts, along with the averaging procedure to obtain an alternative proof of the sufficiency of this condition. One would expect B_p to be continuously included in B_p^d, and we show this here.

Lemma (1.2.14) [1]: Let $I < p < \infty$. Then there exists a constant

$K_P > 0$ such that if $B \in B_P(\mathbb{R})$ then $B \in B_p^d$ and $\|B\|_{B_p^d} \leq K_p \|B\|_{B_p}$.

Proof: It is easily shown that, for any $A \in S_P$ and $J \in \mathcal{D}$.

$$\frac{\|B_J\|_{S_P}}{|J|^{1/2}} \leq \frac{1}{|J|} \int_J \|B(x) - m_J B\|_{S_p} \, dx \leq \frac{2}{|J|} \int_J \|B(x) - A\|_{S_P} \, dx.$$

Letting $A = B(y)$ and then averaging for $y \in J$. we see that

$$\frac{\|B_J\|_{S_P}}{|J|^{1/2}} \leq \frac{2}{|J|^2} \int_J \int_J \|B(x) - B(y)\|_{S_p} \, dxdy.$$

Supposing that $I \in \mathcal{D}_n$ for $n \in \mathbb{Z}$ and using Holder's inequality, we see that

$$\sum_{J\in\mathcal{D}(I)} \frac{\|B_J\|_{S_P}^{P}}{|J|^{P/2}} \leq 2^P \sum_{J\in\mathcal{D}(I)} \frac{1}{|J|^2} \int_J \int_J \|B(x)-B(y)\|_{S_P}^{P}\, dxdy$$

$$= 2^P \sum_{m=n}^{\infty} \sum_{J\in\mathcal{D}_m(I)} \frac{1}{(2^{n-m}|I|)^2} \int_J \int_J \|B(x)-B(y)\|_{S_P}^{P}\, dxdy$$

$$= \frac{2^P}{|I|^2} \int_{\mathbb{R}} \int_{\mathbb{R}} K_I(x,y)\, \|B(x)-B(y)\|_{S_P}^{P} dxdy.$$

where

$$K_I(x,y) = \sum_{m=n}^{\infty} \sum_{J\in\mathcal{D}_m(I)} 2^{2(m-n)} \chi_J(x)\chi_J(y).$$

Clearly, if either $x \notin I$ or $y \notin I$ then $K_P\,(x,y) = 0$. Suppose that $x, y \in I$, with $x \neq y$ and let $J \in \mathcal{D}_m(I)$. Then $\chi_J(x)\chi_J\,(y) = 0$ if

$|x - y| > |J| = 2^{n-m}|I|$. *So,*

$$K_I(x,y) \leq \sum_{m=n}^{n+[\log_2(|I|/|x-y|)]} 2^{2(m-n)} \leq \frac{|I|^2}{3|x-y|^2}.$$

Therefore, for all $n \in \mathbb{Z}$,

$$\sum_{I\in\mathcal{D}_n} \sum_{J\in\mathcal{D}(I)} \frac{\|B_J\|_{S_P}^{P}}{|J|^{P/2}} \leq \frac{2^P}{3} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\|B(x)-B(y)\|_{S_P}^{P}}{|x-y|^2} dxdy.$$

Letting $n \to -\infty$ we see that

$$\|B\|_{B_P^d}^{P} = \sum_{J\in\mathcal{D}} \frac{\|B_J\|_{S_P}^{P}}{|J|^{P/2}} \leq \frac{2^P}{3} \|B\|_{B_P}^{P}.$$

For $\alpha \in \mathbb{R}$ and $r \in \mathbb{R}^+$, let $\mathcal{D}^{\alpha,r}$ denote the translated, dilated dyadic grid given by

$$\mathcal{D}^{\alpha,r} = \{[\alpha + r2^{-n}k, \alpha + r2^{-n}(k+1) \colon n, k \in \mathbb{Z}\}.$$

For $J \in \mathcal{D}^{\alpha,r}$ let $h_J^{\alpha,r}$ denote the corresponding Haar function, normalised in $L^2(\mathbb{R})$. We define the dyadic shift $S^{\alpha,r}$ on $L^2(\mathbb{R})$ by

$$S^{\alpha,r}(f\otimes x)=\sum_{I\in\mathcal{D}^{\alpha,r}}\langle f,h_I^{\alpha,r}\rangle\big(h_{I_+}^{\alpha,r}\otimes x-h_{I_-}^{\alpha,r}\otimes x\big).$$

for an elementary tensor $.f\otimes x\in L^2(\mathbb{R},\mathcal{H})$. Note that.$S^{\alpha,r}$ has norm $\sqrt{2}$ Let $H\colon L^2(R,H)\to L^2(\mathbb{R},\mathcal{H})$ denote the vector Hilbert transform on ℝ.

So

$$H(f\otimes x)=\left(P.v.\int_{\mathbb{R}}\frac{f(\cdot-y)}{y}dy\right)\otimes x.$$

Then, there exists a function $a\in L^2(R)$ and a constant $c_0>0$ such that the operator $T\colon L^2(\mathbb{R},\mathcal{H})\to L^2(\mathbb{R},\mathcal{H})$ given by

$$T(f\otimes x)=c_0H(f\otimes x)+(af)\otimes x\ \ (f\in L^2(\mathbb{R}),x\in\mathcal{H}).$$

is contained in the WOT-closed convex hull of the set $\{S^{\alpha,r}\colon\alpha\in\mathbb{R},r\in\mathbb{R}^+\}$.

We begin by showing, that for $1<p<\infty$ there exists a constant $\tilde{C}_p>0$ such that

$$\|[S^{\alpha,r},B]\|_{S_P}\le\tilde{C}_p\|B\|_{B_P}\ (B\in B_P)\qquad(22).$$

For $\alpha\epsilon\mathbb{R}\ r\epsilon\mathbb{R}^+$, *let* $\Pi_B^{\alpha,r}$ denote the vector paraproduct with respect to the dyadic grid $\mathcal{D}^{\alpha,r}$ given by.

$$\Pi_B^{\alpha,r}f\otimes x=\sum_{I\in\mathcal{D}^{\alpha,r}}m_Ifh_I^{\alpha,r}\otimes B_Ix.$$

And $\Lambda_{B^+}^{\alpha,r}$ its adjoint. Let $R_B^{\alpha,r}$ be the operator defined on elementary tensors by

$$R_B^{\alpha,r}f\otimes x=\sum_{I\in D^{\alpha,r}}f_Ih_I^{\alpha,r}\otimes m_1Bx.$$

Here, m_If , f_I , B_I and m_IB denote Haar coefficients and averages with respect to the grid $\mathcal{D}^{\alpha,r}$.Then it is easily shown that

$$M_B=\Pi_B^{\alpha,r}+\Lambda_B^{\alpha,r}+R_{B,}^{\alpha,r},$$

and

$$\begin{aligned}&\|S^{\alpha,r}M_B - M_B S^{\alpha,r}\|_{S_P}\\ &\qquad\le \|S^{\alpha,r}\Pi_B^{\alpha,r} - \Pi_B^{\alpha,r}S^{\alpha,r}\|_{S_P} + \|S^{\alpha,r}\ \Lambda_B^{\alpha,r} - \Lambda_B^{\alpha,r}\ S^{\alpha,r}\|_{S_P}\\ &\qquad+ \left\|S^{\alpha,r}R_{B,}^{\alpha,r} - R_{B,}^{\alpha,r}S^{\alpha,r}\right\|_{S_P}.\end{aligned}$$

For F a function (scalar, vector or operator valued) defined on R. let

$$U^{\alpha,r}F(t) = r^{-1/2}F((t-\alpha)/r),\ V^{\alpha,r}F(t) = F((t-\alpha)/r),$$

Then $U^{\alpha,r}$ is a unitary map on $L^2(R,\mathcal{H})$. $V^{\alpha,r}$ is an isometry on B_p and

$$U^{\alpha,r}\Pi_B^{\alpha,r}(U^{\alpha,r})^{-1} = \Pi_{V^{\alpha,r}B} \qquad (23).$$

If $B \in B_P$, then

$$\begin{aligned}\|S^{\alpha,r}\Pi_B^{\alpha,r} - \Pi_B^{\alpha,r}S^{\alpha,r}\|_{S_P} &\le 2\|S^{\alpha,r}\|\|\Pi_B^{\alpha,r}\|_{S_P} = 2\sqrt{2}\|\Pi_{V^{\alpha,r}B}\|_{S_P}\\ &\le 2\sqrt{2}C_P\|V^{\alpha,r}B\|_{B_P} = 2\sqrt{2}C_P\|B\|_{B_P}.\end{aligned}$$

by (23), Theorem (1.2.12) and Lemma (1.2.14) It is similarly shown that $\|S^{\alpha,r}\ \Lambda_B^{\alpha,r} - \Lambda_B^{\alpha,r}\ S^{\alpha,r}\|_{S_P} \le 2\sqrt{2}\,C_P\|B\|_{B_P}$.

Finally, it may be seen that

$$\left(S^{\alpha,r}R_{B,}^{\alpha,r} - R_{B,}^{\alpha,r}S^{\alpha,r}\right)(f\otimes x) = \sum_{I\in\mathcal{D}^{\alpha,r}} |I|^{-1/2}f_I\left(h_{I_+} - h_{I_-}\right)\otimes B_I x.$$

If B_I has Schmidt decomposition $B_I = \sum \lambda_n^1 < .\,, e_n^1 > \sigma_n^1$ then

$$\begin{aligned}&\left(S^{\alpha,r}R_{B,}^{\alpha,r} - R_B^{\alpha,r}S^{\alpha,r}\right)(f\otimes x)\\ &\qquad= \sum_{I\in\mathcal{D}^{\alpha,r}}\sum_{n=0}^{\infty}\frac{\sqrt{2}\lambda_n^I}{|I|^{1/2}}\langle f, h_I\rangle\left(\frac{h_{I_+} - h_{I_-}}{\sqrt{2}}\right)\otimes\langle x, e_n^I\rangle\sigma_n^I.\end{aligned}$$

Which is an expression in Schmidt form and so

$$\begin{aligned}\|S^{\alpha,r}R_B^{\alpha,r} - R_B^{\alpha,r}S^{\alpha,r}\|_{S_P}^P &= \sum_{\mathrm{I}\in\mathcal{D}^{\alpha,r}}\sum_{\mathrm{n}=0}^{\infty}\frac{2^{\mathrm{P}/2}(\lambda^{\mathrm{I}})^{\mathrm{P}}}{|\mathrm{I}|^{\mathrm{P}/2}} = 2^{\mathrm{P}/2}\|\mathrm{B}\|_{\mathrm{B}_{\mathrm{P}}^{\mathrm{d}^{\alpha,r}}}\\ &\le \mathrm{K_P}\|\mathrm{B}\|_{\mathrm{B_P}}^{\mathrm{P}}.\end{aligned}$$

by Lemma (1.2.14), where $B_p^{d^{\alpha,r}}$ is the dyadic Besov space defined with respect to $\mathcal{D}^{\alpha,r}$ This shows (22).

Let $B\colon \mathbb{R} \to \mathcal{g}(\mathcal{H})$, $B \in B_P$. and suppose that B is locally bounded with respect to the operator norm on $\mathcal{g}(\mathcal{H})$ .Let $(S_\gamma)_{\gamma\in\Gamma}$ be a net in

$conv\ \{S^{\alpha,r}: \alpha \in \mathbb{R}, r \in \mathbb{R}^+\}$ which converges to the operator *T* introduced above in the weak operator topology. It follows immediately from (22) that $\|[S_\gamma, M_B]\|_{S_p} \leq \tilde{C}_p \|B\|_{S_P}$. To proceed to the WOT- limit, we require the following elementary lemma.

Lemma (1.2.15) [1]: Let $\mathcal{H}$ be a Hilbert space, let $1 \leq p < \infty$, and suppose that $(A_\gamma)_{\gamma\in\Gamma}$. is a bounded net of operators in $S_P \subseteq \mathcal{g}(\mathcal{H})$,

Furthermore, suppose that there exists a dense subspace $\mathcal{A}$ of $\mathcal{H}$ and a sesquilinear form $\mathcal{A} \times \mathcal{A} \to ₵\ (x, y) \mapsto \langle A\, x, y\rangle$, such that $\lim_{\gamma\in\Gamma} \langle A_\gamma x, y\rangle = \langle Ax, y\rangle$ for all $x, y \in \mathcal{A}$. Then A extends to a bounded linear operator on $\mathcal{H}$, $A \in S_P$ and $\|A\|_{S_P} \leq \sup_{\gamma\in\Gamma} \|A_\gamma\|_{S_P}$.

Proof: From

$$|\langle Ax, y\rangle| = \lim_{\gamma\in\Gamma} |\langle A_\gamma x, y\rangle| \leq \sup_{\gamma\in\Gamma} \|A_\gamma\| \|x\| \|y\| \leq \|x\| \|y\| \sup_{\gamma\in\Gamma} \|A_\gamma\|_{S_P}.$$

For all $x, y \in \mathcal{A}$it follows that *A* extends to a bounded sesquilinear form on $\mathcal{H} \times \mathcal{H}$ and therefore defines a bounded linear operator on $\mathcal{H}$, which we also denote by *A*. The estimate for the S_p norm of *A* is easily obtained from the identity

$$\|A\|_{S_p}^p = \sup\left\{\sum_{n=1}^{N} |\langle Ae_n, \sigma_n\rangle|^P : N \in \mathbb{N}, \{e_n\}, \{\sigma_n\}_{n=1}^{N}\ orthonormal\ systems\ in\ \mathcal{H}\right\}.$$

see (13), and the density of $\mathcal{A}$ in $\mathcal{H}$.

We can now finish the proof of the direct implication in the theorem. Let $\mathcal{A} = \{f \in L^2(\mathbb{R}, \mathcal{H}), f\ has\ compact\ support\}$ .Since *B* is locally bounded, the commutator [H, M_B] defines a sesquilinear form on $\mathcal{A} \times \mathcal{A}$, and one has

$$\lim_{\gamma\in\Gamma} \langle [S_\gamma, M_B]x, y\rangle = \langle [T, B]x, y\rangle = \frac{1}{c_0} \langle [H, B]x, y\rangle.$$

by the WOT convergence of $(S_\gamma)_{\gamma\in\Gamma}$ to *T*. Thus by the previous lemma

$$\|[H, M_B]\|_{S_P} \leq \frac{\tilde{C}_P}{C_0} \|B\|_{B_P} \qquad (24).$$

It is not difficult to see that the locally bounded functions are dense in B_p for $p > I$. For a given $B \in B_p$, one can for example choose the sequence given by $B_n(x) = \tilde{B}(x + \frac{i}{n})$, where $\tilde{B}$ denotes the harmonic extension of B to the upper half plane. Then each B_n is locally bounded, and $(B_n)_{n \in \mathbb{N}}$ converges to B in B_p by the Dominated Convergence Theorem, subharmonicity and a vector version of Fatou's Theorem. By density, we obtain (24) for all $B \in B_p$. Using that $P^{\perp}_{H^2([\mathbb{R},\mathcal{H}]}[H, M_B]P_{H^2([\mathbb{R},\mathcal{H}])} = \frac{1}{2i}\Gamma_B$ for antianalytic B finishes the proof.

We note also that a version of Theorem (1.2.13) holds, for $0 < p \leq I$, using appropriate definitions of the operator Besov spaces, for such p,

Section (1.3): Little Hankel Operators and products of Dyadic Paraproducts

In this section, we want to use the vector valued results above to obtain a characterization of Schatten class dyadic paraproducts in several variables and of Schatten class little Hankel operators on certain product domains.

As in the case of vector paraproducts, the method of nearly weakly orthonormal sequences provides an alternative route to obtain the characterization of the symbols of Schatten class paraproducts, although this does appear not explicitly in the literature.

Let $n \in \mathbb{N}$. We write $\mathcal{R} = \mathcal{D}^n$ for the collection of dyadic rectangles in $\mathbb{R}^n$. For $R = I_1 \times \ldots\ldots \times I_n$. Let $h_R(t_1, \ldots., t_n) = h_{I_1}(t_1) \ldots h_{I_n}(t_n)$. The collection $(h_R)_{R\in\mathcal{R}}$ is then the product Haar basis of $L^2(\mathbb{R}^n)$.

For a locally integrable function f on $\mathbb{R}^n$. We denote the Haar coefficient $\langle f, h_R\rangle$ by f_R and the average $\frac{1}{|R|}\int_R f(t_1,\ldots,t_n)\, dt_1 \ldots t_n$ by $m_R f$.

Let $b \in L^2(\mathbb{R}^n)$. The densely defined linear mapping on $L^2(\mathbb{R}^n)$. given by

$$f \mapsto \sum_{R\in\mathcal{R}} h_R b_R m_R f \qquad (25).$$

is the multivariable dyadic paraproduct with symbol b, denoted by π_b.

If we want to make clear that we take the paraproduct in n variables, we write $\pi_b^{(n)}$. For $1 \le i \le n$, let $P_i : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ denote the Riesz projection in the ith variable and $P_i^{\perp}$ denote $I - P_i$. Then $P = P_1 \ldots P_n$is the orthogonal projection from $L^2(\mathbb{R}^n)$ onto the Hardy space $H^2(\mathbb{R}^n)$. We identify functions in the Hardy space $H^2(\mathbb{C}^{+^n})$ on the n-fold product of the upper half planes with their boundary values in $H^2(\mathbb{R}^n)$. in the usual manner, and we write $H^2(\mathbb{R}^n) = H^2(\mathbb{C}^{+^n})$ Let $\overline{H}^2(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \bar{f} \in H^2(\mathbb{R}^n)\}$.

The densely defined linear map on $H^2(\mathbb{R}^n) \longrightarrow \overline{H}^2(\mathbb{R}^n)$ given by

$$f \longmapsto P_{1^\perp} \dots . P_{n^\perp} bf \qquad (26).$$

Is the little Hankel operator with symbol b, denoted by γ_b. Again, we will write $\gamma_b^{(n)}$ if we want to emphasize that the Hankel operator is taken with respect to n variables. The characterizations of bounded multivariable dyadic paraproducts and little Hankel operators in terms of their symbols are by no means a simple extension of the one-dimensional results.

For $n = 2$, boundedness of dyadic paraproducts was characterized in terms of an oscillation property of the symbol over all open sets in $\mathbb{R}^n$ and this gave rise to a characterization of the dual of the Hardy space $H^1(\mathbb{C}^{+^n})$ in terms of oscillation properties. Only recently, it was shown that also the boundedness of little Hankel operators on $H^2(\mathbb{C}^{+^2})$can be characterised in terms of an oscillation property over open sets, in the course of the solution of the long-standing weak factorization problem on $H^1\left(\mathbb{C}^{+^2}\right)$. For $n \geq 3$, no such characterization is known.

Little Hankel operators on the unit ball in $\mathbb{C}^n$, or more generally, on smoothly bounded strictly pseudo convex sets, are much better understood.

The main point of this section is to show that because of the good behaviour of Schatten class vector paraproducts and vector Hankel operators, multivariable paraproducts and little Hankel operators of Schatten class on certain product domains can be characterized quite easily in terms of their symbols.

It is shown that for $b \in H^2\left(\mathbb{C}^{+^2}\right)$, the little Hankel operator γ_b on $H^2(\mathbb{C}^{+^2})$ is of trace class, if and only if $\partial_1^2 \partial_1^2 b$ is integrable on $\mathbb{C}^{+^2}$, that is, b is in the Besov space $B_I(\mathbb{R}^2)$. This appears as a special case of a consideration of tube domains over symmetric cones. It is conjectured that these results extend at least to $1 < p \leq 2$.

We will give here a Besov space characterization of the symbols for $1 < p < \infty$ for little Hankel operators, and for $1 < p < \infty$ for multivariable dyadic paraproducts.

Theorem (1.3.1) [1]: Let $n \in \mathbb{N}$, $1 < p < \infty$, and let $b \in L^2(\mathbb{R}^n)$. Then $\pi_b \in S_P$,, if and only if $(\sum_{R\in\mathcal{R}} \frac{1}{|R|^{p/2}} \ |b_R|^P)^{1/p} < \infty$, and the S_P norm of π_b, is equivalent to this expression.

Proof: We prove this statement by induction over n. For $n = 1$. This is just Theorem (1.1.1). Suppose that the statement is true for some $n \in \mathbb{N}$. given $b \in L^2(\mathbb{R}^{n+1})$, we understand the multivariable paraproduct $\pi_b^{(n+1)}$ as a vector-valued parproduct in one variable, defining $B(t) = \pi_{b(.,\dots,.,t)}^{(n+1)}$ fort $t \in \mathbb{R}$. We write $b_I$for the function on $\mathbb{R}^n$ given by

$$(t_1 \dots, t_n) \to \int_I b(t_1 \dots, t_n, t)\, h_I(t) dt.$$

Then $\pi_b^{(n+1)} = B_I$ and it is easy to see that $\pi_b^{(n+1)}: L^2(\mathbb{R}^{n+1}) \to L^2(\mathbb{R}^{n+1})$ is unitarily equivalent to

$$\Pi_B: L^2(\mathbb{R}, L^2(\mathbb{R}^n)) \to L^2\ (\mathbb{R}, L^2(\mathbb{R}^n))$$

via the natural unitary equivalence $L^2(\mathbb{R}^{n+1}) \to L^2(\mathbb{R}, L^2(\mathbb{R}^n))$

Applying the induction hypothesis and Theorem (1.2.14), we obtain

$$\begin{aligned}\left\|\pi_b^{(n+1)}\right\|_{S_P}^P &= \|\Pi_B\|_{S_P}^P \asymp \sum_{I\in\mathcal{D}} \frac{1}{|I|^{P/2}} \|B_I\|_{S_P}^P = \sum_{I\in\mathcal{D}} \frac{1}{|I|^{P/2}} \left\|\pi_{b_I}^{(n)}\right\|_{S_P}^P \\ &\asymp \sum_{\mathrm{I}\in\mathcal{D}} \frac{1}{|\mathrm{I}|^{\mathrm{P}/2}} \sum_{\mathrm{R}'\in\mathcal{D}^{\mathrm{n}}} \frac{1}{|\mathrm{R}'|^{\mathrm{P}/2}} |(\mathrm{b_I})_{\mathrm{R}'}|^{\mathrm{P}} = \sum_{R\in\mathcal{D}^{n+1}} \frac{1}{|R|^{P/2}} |b_R|^P.\end{aligned}$$

The same method applies for the characterization of the symbols of little Hankel operators on $H^2(\mathrm{c}^{+n})$ of Schatten class S_P $1 < p < \infty$.

We need the following notation. For $i \in \{1, \dots, n\}$,$t \in \mathbb{R}$ and $f: \mathbb{R}^n \longrightarrow \mathbb{C}$, let $\Delta_t^{(i)}$ be the finite difference operator in the ith coordinate given by

$$\left(\Delta_t^{(i)} f\right)(x) = f(x_1, \dots, x_i + t, \dots, x_n) - f(x_1, \dots, x_n)\ x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

For $0 < p < \infty$, we say that $b: \mathbb{R}^n \to \mathbb{C}$ is in $B_P(\mathbb{R}n)$, if

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\left|\left(\Delta_{t_1}^{(1)} \dots \Delta_{t_n}^{(n)} b\right)(x)\right|^P}{\prod_{i=1}^n |t_i|^2} dx_1 \dots dx_n dt_1 \dots dt_n < \infty.$$

We denote the semi norm defined by the pth root of the expression above by $\|b\|_{B_p(\mathbb{R}^n)}$.

Applying the well-known equivalence of the "harmonic analysis" definition and the "complex analysis" definition of analytic Besov class functions coordinate wise for a version on the unit disc), one sees easily that for b analytic in $\mathbb{C}^{+^n}$, the expression on the left is equivalent to

$$\int_{\mathbb{C}^{+n}} \left|\Im_{z_1}\right|^{P-2} \dots \left|\Im_{z_n}\right|^{P-2} \left|\partial_{z_1} \dots \partial_{z_n} b(z)\right|^{P} dz_1 \dots dz_n.$$

Theorem (1.3.2) [1]: Let $I < p < \infty$, and let $b \in \bar{H}^2\,(\mathbb{R}^n)$. Then the .following are equivalent:

(i) $\gamma_b : H^2\,(\mathbb{R}^n) \rightarrow \bar{H}^2\,(\mathbb{R}^n)$ is in S_p;
(ii) $b \in B_P(\mathbb{R}^n)$,

and the $B_P(\mathbb{R}^n)$ norm is equivalent to the S_P norm.

Proof: For n = I, this is just Pellet's characterization of Schatten class Hankel operators in the case $0 < p < \infty$. As before, we use induction over the dimension n. Suppose that the statement above holds for some $n \in \mathbb{N}$.

Let $b \in \bar{H}^2(\mathbb{R}^{n+1})$. We define an operator valued function $B: \mathbb{R} \rightarrow \mathcal{g}(L^2(\mathbb{R}^n))$ by $B(t) = \gamma^{(n)}_{b(.,\dots,.,t)}$ .For each $t \in \mathbb{R}$, $b(\cdot,\dots,\cdot,t)$ is an antianalytic function in n variables, and $b(.,\dots,.,t) \in \bar{H}^2(\mathbb{R}^n)$. for almost every $t \in \mathbb{R}$. It is easy to verify that the vector Hankel operator Γ_B is unitarily equivalent to the little Hankel operator $\gamma_b$via the canonical unitary $L^2(\mathbb{R}, L^2(\mathbb{R}^n)) \rightarrow L^2(\mathbb{R}^{n+1})$. Therefore by Theorem (1.2.13).

$$\begin{aligned}
\left\|\gamma_b^{(n+1)}\right\|_{S_P}^{P} = \|\Gamma_B\|_{S_P}^{P} &\asymp \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\|B(t) - B(s)\|_{S_P}^{P}}{|t-s|^2} dtds \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\|B(s+t) - B(s)\|_{S_P}^{P}}{|t|^2} dtds \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\left\|\gamma^{(n)}_{b(.,\dots,.,s+t)} - \gamma^{(n)}_{b(.,\dots,.,s)}\right\|_{S_P}^{P}}{|t|^2} dtds \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\left\|\gamma^{(n)}_{b(.,\dots,.,s+t)-b(.,\dots,.,s)}\right\|_{S_P}^{P}}{|t|^2} dtds \\
&\asymp \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\|b(.,\dots,.,s+t) - b(.,\dots,.,s)\|_{B_P(\mathbb{R}^{n+1})}^{P}}{|t|^2} dtds = \|b\|_{B_P(\mathbb{R}^{n+1})}
\end{aligned}$$

The same method applies of course for little Hankel operators on domains of the form $D = \mathbb{C}^{+^n} \times \Omega \subseteq \mathbb{C}^{n+m}$ in the case where we have a Besov space type characterization of Schatten class little Hankels on $H^2(\Omega)$, for example, if $\Omega \subseteq \mathbb{C}m$ is a smoothly bounded convex domain of finite type For such domains, we can define the Hardy class $H^2(D) = H^2(\mathbb{C}^{+^n}) \otimes H^2(\Omega) \subseteq L^2(\mathbb{R}^n \times \partial\Omega)$ and, for b$\in \overline{H}^2(D)$, define the little Hankel operator $\gamma_b$on a dense subspace of $H^2(D)$.

Theorem (1.3.3) [1]: Let $D = \mathbb{C}^{+^n} \times \Omega \subset \mathbb{C}^{n+m}$, where Ω is a smoothly bounded convex domain of finite type in $\mathbb{C}^m$. Let $b \in \overline{H}^2(D)$, and let $1 < p < \infty$.

Then the following are equivalent.

$$(i) \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\left\|\Delta_{t_1}^{(1)} \dots \Delta_{t_n}^{(n)} b(x_1, \dots, x_n, \cdot, \dots, \cdot)\right\|_{B_P(\Omega)}^P}{\Pi_{i=1}^n |t_i|^2} dx_1 \dots dx_n dt_1 \dots dt_n < \infty.$$

(ii) $\gamma_b : H^2(D) \to \overline{H}^2(D)$, is in S_P

(For the definitions of $H^2(\Omega)$ and $B_P(\Omega)$, It would be interesting to see whether this method is also useful for domains of the form $U_{z\in\mathbb{C}^+} \{Z\} \times \Omega_z \subset \mathbb{C}^{n+1}$, where Ω_z is a "sufficiently nice" domain in $\mathbb{C}^n$ for each $z \in \mathbb{C}^+$. The case of the light cone, which was studied by Bonami and Peloso, would be an interesting candidate for this approach.

Operator-theoretic properties of the product $\Gamma_f^*\Gamma_g$ of a Hankel operator and the adjoins of a Hankel operators have been studied for a long time, partly motivated by the identity $\Gamma_f^*\Gamma_g = [T_f, T_g)$ where$[T_f, T_g)$ denotes the semi-commutator $T_fT_g - T_{fg}$of the Toeplitz operators T_f and T_g on $H^2(\mathbb{D})$, the Hardy space of the unit disc.

One example for this is the Axler-Chang-Sarason-Volberg Theorem, which characterizes compact products of Hankel operators in terms of certain Douglas algebras .

The study of such products of Hankel operators is in general much more difficult than the study of single Hankel operators. There is still no full characterization of boundedness and Schattcn class membership in terms of oscillation properties of the symbols.It was shown that the natural reproducing kernel condition

$$\lim_{|z|\to 1}\|\Gamma_g k_z\|_2\|\Gamma_f k_z\|_2 = 0 \qquad (27).$$

is equivalent to compactness of the product $\Gamma_f^*\Gamma_g$. Here, $\{k_z\}_{z\in\mathbb{D}}$, denote the normalized reproducing kernels on $H^2(\mathbb{D})$. However, it is an open question whether the reproducing kernel condition

$$\sup_{z\in\mathbb{D}}\|\Gamma_g k_z\|_2\|\Gamma_f k_z\|_2 < \infty \qquad (28).$$

Which can be understood as a "combined" oscillation condition and was shown to be necessary ,implies the boundedness of the product $\Gamma_f^*\Gamma_g$. Slightly stronger sufficient conditions have been found.It is also open for which symbols $g, f \in L^2(\mathbb{T})$ the product of Hankel operators $\Gamma_f^*\Gamma_g$ is in the Schatten-von Neumann class S_p ,although partial results were found and estimates for the singular values of such products have been obtained. In this section, we will again consider dyadic paraproducts as a model case for Hankel operators and study operator-theoretic properties for products $\pi_f^*\pi_g$ of dyadic paraproducts. Asesquilinear version of the dyadic sweep from (II). Given by

$$Q[f,g] = \sum_{I\in\mathcal{D}}\frac{\chi_I}{|I|}f_I\overline{g_I} \quad \left(f,g \in L^2(\mathbb{R})\right) \qquad (29).$$

allows us to address this dyadic analogue in a very simple fashion.

as before we collect some elementary properties of the sesquilinear map Q.

Lemma (1.3.4) [1]: (i) $\|Q[f,g]\|_I \le \|f\|_2\|g\|_2$;

(ii) $P_I Q[f,g] = P_I Q[P_I' f, P_I' g]$

We first need an analogue of Proposition (1.1.2). For $f, g \in L^2(\mathbb{R})$, let $D_{[f,g]}$ be defined on the Haar basis by

$$D_{[f,g]}h_I = \left(\frac{1}{|I|}\sum_{J\in\mathcal{D}'(I)} f_J\overline{g_J}\right)h_I.$$

Lemma (1.3.5) [1]:

$$\pi_g^*\pi_f = \pi_{Q[f,g]} + \Delta_{Q[f,g]} + D_{[f,g]}.$$

Proof: Exactly as in Proposition (1.1.2).

Theorem (1.3.6) [1]: Let $f, g \in L^2(\mathbb{R})$. Then the following are equivalent:

(i) $\pi_f^*\pi_g$ defines a bounded linear operator on $L^2(\mathbb{R}) \to L^2(\mathbb{R})$.

(ii) $Q[f,g] \in BMO^d$, and

$$\sup_{I\in\mathcal{D}} \frac{1}{|I|}\left|\sum_{J\in\mathcal{D}'(I)} f_J\overline{g_J}\right| < \infty \qquad (30).$$

(iii)

$$\sup_{I\in\mathcal{D}} \frac{1}{|I|} \|Q[P_I'f, P_I'g]\|_I < \infty \, .$$

(iv)

$$\sup_{I\in\mathcal{D}} \left\|\pi_g^*\pi_f h_I\right\| < \infty.$$

Proof: (i) ⇨ (iv) obvious.

(iv) ⇨ (ii): Remember from (1.3.5) that $\pi_{[f,g]}$ is the superdiagonal part of $\pi_g^*\pi_f$, $\Delta_{Q[f,g]}$ is the subdiagonal part, and $D_{[f,g]}$ is the diagonal part with respect to the Haar basis. The uniform boundedness of $\| \pi_g^*\pi_f h_I \|$: therefore implies uniform boundedness of $\left\|D_{f,g}\, h_I\right\| = \|< \pi_g^*\pi_f h_I, h_I >\|$ and thereby (30). Furthermore, note that $\frac{1}{|I|}\sum_{I\in\mathcal{D}}\left|(Q[f,g])_J\right|^2 = \left\|\pi_{Q[f,g]}h_I\right\|^2 = \left\|P_I'\pi_g^*\pi_f h_I\right\|^2 \leq \left\|\pi_g^*\pi_f h_I\right\|^2$ for all $I \in \mathcal{D}$ and therefore.

$$Q[f,g] \in BMO^d.$$

(ii)⇨(iii):Note the identity

$$Q[P_I'f, P_I'g] = \chi_I m_I(Q[P_I'f, P_I'g]) + P_I(Q[f,g])$$

Thus

$$\frac{1}{|I|}\|Q[P_I'f, P_I'g]\|_I \leq |m_I(Q[P_I'f, P_I'g])| + \frac{1}{|I|}\|P_I(Q[P_I'f, P_I'g])\|_I$$

$$= \frac{1}{|I|}\left|\sum_{J\in\mathcal{D}'(I)} f_J\bar{g}_J\right| + \frac{1}{|I|}\|P_I Q[f,g]\|_I \qquad (31).$$

(iii) ⇨ (ii): By the uniform boundedness of the projections $(P_I)_{I\in\mathcal{D}}$, on $L^1(\mathbb{R})$, the uniform boundedness of the left-hand side in (31) implies the uniform boundedness of the right-hand side. Therefore

$$\|Q[f,g]\|_{BMO^d} = \sup_{I\in\mathcal{D}} \frac{1}{|I|}\|P_I Q[f,g]\|_I < \infty$$

and also (30) holds.

(ii) ⇨ (i): By Theorem (1.1.8), $\pi_{Q[f,g]}$ and $\Delta_{Q[f,g]}$ arc bounded, and by (30),$D_{[f,g]}$ is bounded. Thus $\pi_g^*\pi_f + \Delta_{Q[f,g]} + \Delta_{f,g}$ is bounded.

Condition (30) looks like a natural sesquilinear analogue to the BMO^d condition. However, a simple example shows that it is not sufficient for the boundedness of $\pi_g^*\pi_f$.

Remark (1.3.7) [1]: There exists functions $f,g \in L^2(\mathbb{R})$ such that (30) holds, but $\pi_g^*\pi_f$ does not define a bounded linear operator on $L^2(\mathbb{R})$.

Proof: Let f,g be defined by the following Haar coefficients. For $k \geq 0$. let $I_k = [0,2^{-k}]$ Let $a > 0,\ 1/2 < a^4 < 1$, and let $f_{I_k} = f_{I_k} = g_{I_k} = a^k\ g_{I_k^-} = a^k$ for each $k \geq 0$.

Let all remaining Haar coefficients of f and g be 0. Then $g,f \in L^2(\mathbb{R})$ and $\sum_{J\in\mathcal{D}'(I)} f_J\bar{g}_J = 0$ for each $I \in \mathcal{D}$, but

$$\sum_{I\subseteq[0,1]} |(Q[f,g])_I|^2 = \sum_{I\subseteq[0,1]} \frac{1}{|I|}\left|\sum_{J\in\mathcal{D}(I^+)} f_J\bar{g}_J - \sum_{J\in\mathcal{D}(I^-)} f_J\bar{g}_J\right|^2$$
$$= \sum_{k=0}^{\infty} \frac{1}{|I_k|}\left|\sum_{J\in\mathcal{D}(I_k^+)} f_J\bar{g}_J - \sum_{J\in\mathcal{D}(I_k^-)} f_J\bar{g}_J\right|^2 = \sum_{k=0}^{\infty} 2^k 4a^{4k} = +\infty.$$

The remark follows now by Theorem (1.2.6) (ii).

A further natural candidate for a "combined" BMO^d condition is given by

$$\sup_{I\in\mathcal{D}} \frac{1}{|I|} \sum_{J\in\mathcal{D}'(I)} |f_J||g_J| \qquad (32).$$

It turns out that this condition leads even to a stronger property. For $\sigma \in \{-I.I\}$ let T_σ, denote the dyadic martingale transforms $L^2(\mathbb{R}) \to L^2(\mathbb{R}), \sum_{I\in\mathcal{D}} h_I f_I \mapsto \sum_{I\in\mathcal{D}} \sigma(I) h_I f_I$. then we have the following result:

Theorem (1.3.8) [1]: Let $f, g \in L^2(\mathbb{R})$.Then the following are equivalent:

(i) For each dyadic martingale transform T_σ, $\pi_g^* T_\sigma \pi_f$ defines a bounded linear operator on $L^2(\mathbb{R})$, and the operators $(\pi_g^* T_\sigma \pi_f)_{\sigma \in \{-1.1\}}\mathcal{D}$ are uniformly bounded. (ii)

$$\sup_{I \in \mathcal{D}} \frac{1}{|I|} \sum_{J \in \mathcal{D}'(I)} |f_J||g_J|$$

$$\sup_{I \in \mathcal{D}, \sigma \in \{-1,1\}^{\mathcal{D}}} \left\| \pi_g^* T_\sigma \pi_f h_I \right\| < \infty,$$

Proof: (i) ⇨ (iii) obvious.

(iii) ⇨ (ii): Let $I \in \mathcal{D}$ Then for each $\sigma \in \{-1,1\}^{\mathcal{D}}$.

$$\langle \pi_g^* T_\sigma \pi_f h_I, h_I \rangle = \frac{1}{|I|} \langle \sum_{J \in \mathcal{D}'(I)} sign(J, I)\sigma(J) h_J f_J, \sum_{J \in \mathcal{D}'(I)} sign(J, I) h_J g_J \rangle$$
$$= \frac{1}{|I|} \sum_{J \in \mathcal{D}'(I)} \sigma(J) \bar{g}_J f_J.$$

Choosing an appropriate sequence $(\sigma(J))_{J \in \mathcal{D}} \in \{-1,1\}^{\mathcal{D}}$, we obtain

$$\frac{1}{|I|} \sum_{J \in \mathcal{D}'(I)} |g_J||f_J| \leq \sqrt{2} \left| \sum_{J \in \mathcal{D}'(I)} \sigma(J) \bar{g}_J f_J \right| \leq \frac{\sqrt{2}}{|I|} \sup_{\sigma \in \{-1,1\}^{\mathcal{D}}} \left| \langle \pi_g^* T_\sigma \pi_f h_I, h_I \rangle \right|$$
$$\leq \frac{\sqrt{2}}{|I|} \sup_{\sigma \in \{-1,1\}^{\mathcal{D}}} \left\| \pi_g^* T_\sigma \pi_f h_I \right\|.$$

(ii) ⇨ (i): Observe that $\pi_g^* T_\sigma \pi_f = \pi_g^* \pi_T \pi_f$ For all $I \in \mathcal{D}$ and all $\sigma \in \{-1,1\}^{\mathcal{D}}$

$$\frac{1}{|I|} \|Q[P_I'(T_\sigma f), P_I' g]\|_I = \frac{1}{|I|} \left\| \sum_{J \in \mathcal{D}'(I)} \frac{\chi_J}{|J|} \sigma(J) \bar{g}_J f_J \right\|_I \leq \frac{1}{|I|} \sum_{J \in \mathcal{D}'(I)} |g_J||f_J|.$$

(1) Follows now from Theorem (1.3.6) .

Unfortunately, when considering products of operators it is not as easy as in the situation to pass from results on paraproducts to results on Hankel operators via averaging. The following remark shows that products of paraproducts and products of Hankel operators behave quite differently.

Remark (1.3.9): A seemingly natural dyadic analogue to Zheng's necessary condition (28) is the following:

$$\sup_{I\in\mathcal{D}}\left\|\pi_f h_I\right\|_2\left\|\pi_g h_I\right\|_2 < \infty \qquad (33).$$

This condition is easily seen to be sufficient for the uniform boundedness of all operator products $\pi_g^* T_\sigma \pi_f$, $\sigma \in, \{-1,1\}^{\mathcal{D}}$, by Theorem (1.3.8) (ii). However, whenever the sets $\{I \in \mathcal{D} : f_I \neq 0\}$ and $\{I \in D : g_I \neq 0\}$ are disjoint, the product $\pi_g^* T_\sigma \pi_f$is 0 for all $\sigma \in, \{-1,1\}$. Thus one sees that (33) is not necessary.

Finally, we want to characterise Schatten class products of paraproducts. First, let us look at the compact case.

Theorem (1.3.10) [1]: Let $f, g \in L^2(\mathbb{R})$. Then the folowing are equivalent:

(i) $\pi_g^* T_f$ defines a compact linear operator on. $L^2(\mathbb{R}) \to L^2(\mathbb{R})$.

(ii) $Q[f,g] \in VMO^d$, and

$$\lim_{I\to\infty} \frac{1}{|I|}\left|\sum_{J\subsetneq I} f_I \bar{g}_I\right| = 0 \qquad (34).$$

(iii)

$$\lim_{I\to\infty} \frac{1}{|I|} \|Q[P_I' f, P_I' g]\|_I = 0.$$

(iv)

$$\lim_{I\to\infty}\left\|\pi_g^* \pi_f h_I\right\| = 0$$

Here, the limits in (ii)- (iv) are meant in the sense of (10), and convergence to 0 is meant to be uniform, as $|I| \to 0$ or $|I| \to \infty$ respectively.

Proof: (i) ⇨ (iv): For $N \in \mathbb{N}$. let $P^{(N)}$ denote the orthogonal projection defined in (21). $(P^{(N)})_{N\in\mathbb{N}}$. Converges to the identity in the strong operator topology, so $\pi_g^* \pi_f P^{(N)} - \pi_g^* \pi_f$ converges to 0 in norm, and we obtain (iv). For the remainder of the proof, one shows (iv) ⇨ (ii) (i) and (ii) ⇔ (iii) along the same lines as in the proof for Theorem (1.3.6), using Theorem (1.1.8) (iii) and Lemma (1.3.5).

Now we look at the Schatten classes S_P, $1 \le P < \infty$. In this case. it turns out that if $\pi_g^* \pi_f \in S_P$then also$\pi_g^* T_\sigma \pi_f \in S_P$ for all $\sigma \in \{-1,1\}^{\mathcal{D}}$, with uniformly hounded S_P norm. We get a natural combined dyadic Besov space condition for the symbols f and g.

Theorem (1.3.11) [1]: Let $f, g \in L^2(\mathbb{R})$, the $1 \le P < \infty$ the following are equivalent:

(i) $\pi_g^* T_\sigma \pi_f \in S_P$ For each $\sigma \in \{-1,1\}^{\mathcal{D}}$,and $(\pi_g^* T_\sigma \pi_f)_{\sigma \in \{-1,1\}^{\mathcal{D}}}$ is bounded in S_P

(ii) $\pi_g^* \pi_f \in S_P$.

(iii) $\mathcal{Q}[f, g] \in B_p^d$, and

$$\sum_{I \in \mathcal{D}} \left| \frac{1}{|I|} \sum_{J \in \mathcal{D}'(I)} f_J \bar{g}_J \right|^P < \infty$$

(iv)

$$\sum_{I \in \mathcal{D}} \frac{1}{|I|^P} |f_I g_I|^P < \infty$$

(v) for all $1 \le q < \infty$.

$$\sum_{I \in \mathcal{D}} \frac{1}{|I|^{P/q}} \|\mathcal{Q}[P_I' f, P_I' g]\|_q^P < \infty$$

Proof: We will show

(ii) ⇨ (iv) ⇨ (v) ⇨ (iii) ⇨ (ii) and (iv) ⇨ (i) ⇨ (ii),

(ii) ⇨ (iv) : For $I \in \mathcal{D}$, Let

$$\psi_I = \frac{1}{|I^{++}|^{1/2}} (\chi_{I^{+++}} - \chi_{I^{+--}})$$

and

$$\psi_I{}' = \frac{1}{|I^{++}|^{1/2}} (\chi_{I^{++-}} - \chi_{I^{--+}}).$$

The sequences $(\psi_I)_{I \in D}$ and $(\psi_I'\ )_{I \in D}$ are not orthonrmal , but it is easy to see that they are the images of the rothonormal Haar basis under bounded linear maps *A,B*. In the notation, $(\psi_I)_{I\in\mathcal{D}}$ and $(\psi_I')_{I\in\mathcal{D}}$ are weakly orthonormal. Therefore,

$$\sum_{I\in\mathcal{D}}\left|\langle\pi_g^*\pi_f\psi_I,\psi_I'\rangle\right|^P=\sum_{I\in\mathcal{D}}\left|\langle B^*\pi_g^*\pi_f Ah_I,h_I\rangle\right|^P\le\|A\|^P\|B\|^P\left\|\pi_g^*\pi_f\right\|_{S_P}^P.$$

Notice that

$$m_I\psi_I=\begin{cases}\frac{1}{2|I^{++}|^{1/2}} & if\ J\subseteq I^{++},\\ -\frac{1}{2|I^{++}|^{1/2}} & if\ J\subseteq I^{+-},\\ \frac{1}{2|I^{++}|^{1/2}} & if\ J\subseteq I^{+++},\\ \frac{1}{2|I^{++}|^{1/2}} & if\ J\subseteq I^{+--},\\ 0 & otherwise,\end{cases}$$

and

$$m_I\psi_I{}'=\begin{cases}\frac{1}{4|I^{++}|^{1/2}} & if\ J=I^{+},\\ -\frac{1}{4|I^{++}|^{1/2}} & if\ J=I^{-},\\ \frac{1}{2|I^{++}|^{1/2}} & if\ J=I^{++},\\ \frac{1}{2|I^{++}|^{1/2}} & if\ J=I^{--},\\ \frac{1}{|I^{++}|^{1/2}} & if\ J\subseteq I^{++}\\ -\frac{1}{|I^{++}|^{1/2}} & if\ J\subseteq I^{--+}\\ 0 & otherwise,\end{cases}$$

Thus $\langle m_J\psi_I,m_J\psi_I'\rangle$ equals $\frac{1}{4|I^{++}|}$ for $J=I^{++}$ and 0 otherwise, giving

$$\sum_{I\in\mathcal{D}}\left|\langle\pi_g^*\pi_f\psi_I,\psi_I'\rangle\right|^P=\frac{1}{4^P}\sum_{I\in\mathcal{D}}\frac{1}{|I^{++}|^P}|f_{I^{-+}}g_{I^{++}}|^P.$$

Adjusting the definitions of ψ_I and ψ_I' we obtain corresponding expressions for

$$\sum_{I\in\mathcal{D}}\frac{1}{|I^{+-}|^p}\left|f_{I^{+-}}g_{I^{+-}}\right|^P,\sum_{I\in\mathcal{D}}\frac{1}{|I^{-+}|^p}\left|f_{I^{-+}}g_{I^{-+}}\right|^P\ and\ \sum_{I\in\mathcal{D}}\frac{1}{|I^{--}|^p}\left|f_{I^{--}}g_{I^{--}}\right|^P$$

Thus (iv) holds.

(iv) ⇨ (v): Let $\phi=\sum_{I\in\mathcal{D}}h_I|f_Ig_I|^{1/2}$. Then

$$\sum_{I\in\mathcal{D}}\frac{1}{|I|^{P/q}}\|Q[P_I'f,P_I'g]\|_q^P\leq\sum_{I\in\mathcal{D}}\frac{1}{|I|^{P/q}}\|Q[P_I'\Phi]\|_q^P$$

$$\leq C_{2q}^{2P}\sum_{I\in\mathcal{D}}\frac{1}{|I|^{P/q}}\|P_I'\Phi\|_{2q}^{2P}=C_{2q}^{2P}\|\phi\|_{B_{2P,2q}^d}^{2P}$$

$$\leq C_{2q}^{2P}K_{2P,2q}\|\phi\|_{B_{2P,2q}^d}^{2P}=C_{2q}^{2P}K_{2P,2q}\sum_{I\in\mathcal{D}}\frac{1}{|I|^P}|f_Ig_I|^P.$$

by Theorem (1.1.6) where C_{2q}denotes the norm of the dyadic square function on L^{2q} and $k_{2P,2q}$denotes the equivalence constant between the $B_{p,q}^d$ and the B_p^d norms from Theorem (1.1.6).

(v) ⇨ (iii): Suppose that (v) holds for some q, $1\leq q<\infty$. Them by Holder's inequality, (v) holds in particular for $q=1$. Note that the projections $(P_I)_{I\in\mathcal{D}}$ are uniformly bounded on $L^1(\mathbb{R})$. We obtaion

$$\sum_{I\in\mathcal{D}}\frac{1}{|I|^P}\|P_IQ[f,g]\|_I^P$$

$$=\sum_{I\in\mathcal{D}}\frac{1}{|I|^P}\|P_IQ[P_I'f,P_I'g]\|_I^P\leq C^P\sum_{I\in\mathcal{D}}\frac{1}{|I|^P}\|Q[P_I'f,P_I'g]\|_I^P.$$

And it follows from Theorem (1.1.4) that $Q[f,g]\in B_p^d$ Furthermore,

$$\sum_{I\in\mathcal{D}}\frac{1}{|I|}\left|\sum_{J\in\mathcal{D}'(I)}f_J\bar{g}_J\right|^P\leq\sum_{I\in\mathcal{D}}\frac{1}{|I|}\|Q[P_I'f,P_I'g]\|_I^P.$$

Thus (iii) holds.

(iii) ⇨ (ii): This follows directly from Lemma (1.3.5).

(iv) ⇨ (i) : Note again that $\pi_g^*T_\sigma\pi_f=\pi_g^*\pi T_\sigma f$, and that condition (iv) is invariant under exchanging f with $T_\sigma f$.

Condition (i) now follows by applying the implication (iv) ⇨(ii) proved above to the symbols $T_\sigma f$ and g.

(i) ⇨ (ii): This is immediate .

Using Theorem (1.2.12) we also obtain a vector version of this result:

Corollary (1.3.12) [1]: Let $\mathcal{H}$ be separable Hilbert space , Let $F, G: \mathbb{R} \to \wp(H)$ be weakly locally integrable, and let $1 \leq p < \infty$. Then the following are equivalent:

(i) $\Pi_G^* \, \Pi_F \in S_p$

(ii)

$$\sum_{I \in \mathcal{D}} \frac{1}{|I|^P} \|G_I^* F_I\|_{S_P}^P < \infty.$$

Proof: The proof (ii) ⇨ (iv) ⇨ (i) ⇨ (ii) in theorem (1.3.11) also works in the vector case. We omit the details here.

In [VI], it was shown that the condition

$$\sup_{k \in \mathbb{N}} \sum_{I \in \mathcal{D}_k} \frac{1}{|I|^P} |f_I g_I|^P < \infty \qquad (35).$$

is necessary for the product of Hankel operators $\Gamma_g^* \Gamma_f$ to be in S_P.

We have seen above that the stronger condition Theorem (1.3.11) (iv) holds whenever $\pi_g^* \pi_f$ is in S_P It would be interesting to know whether Theorem (1.3.11) (iv) holds (at least in some averaze sense) whenever $\Gamma_g^* \Gamma_f$ is in S_P. Conversely, it would be of great interest to know whether a translation and dilation invariant version of condition (1.3.11) (iv) implies that $\pi_g^* \pi_f$ is in S_P. We finish by stating this as a conjecture.

As before, denote by $\mathcal{D}^{\alpha,r}$where and $\alpha \in \mathbb{R}$ $r \in \mathbb{R}^+$ the dyadic grid obtained by dilating the standard dyadic grid $\mathcal{D}$ by r and then translating it by α.

Conjecture (1.3.13) [1]: Let$1 \leq p < \infty$, and let $f, g \in \overline{H}^2(\mathbb{R})$. Suppose that

$$\sup_{\alpha \in \mathbb{R}, r \in \mathbb{R}^+} \sum_{I \in \mathcal{D}^{\alpha,r}} \frac{1}{|I|^P} |f_I g_I|^P < \infty.$$

Then $\Gamma_g^* \Gamma_f \in S_P$

Chapter 2

Continuity in Schatten- von Neumann of p- class of Hankel Operators with Fock Spaces

We investigate Hankel operators $H_{\bar{f}}: A_m^2 \to A_m^{2\perp}$ with anti-holomorphic symbols $\bar{f} = \sum_{k=0}^{\infty} b_k \bar{z}^k \in L^2(\mathbb{C}, |z|^m)$, where A_m^2 are general Fock spaces. We will show that $H_{\bar{f}}$ is not continuous if the corresponding symbol is not a polynomial $\bar{f} = \sum_{k=0}^{\infty} b_k \bar{z}^k$.

Namely in case $2k < m$ the Hankel operators $H_{\bar{z}^k}$ are in the Schatten– von Neumann p-class iff $p > 2m/(m-2k)$; and in case $2k \geq m$ they are not in the Schatten–von Neumann p-class.

Section (2.1): Hankel operators with anti–holomorphic symbols

Hankel operators with the special symbols $\bar{z}^k$, $k \in \mathbb{N}$, have been considered. Here we try to generalize these investigations in order to obtain more insight for general anti-holomorphic symbols

$$\bar{f} = \sum_{k=0}^{\infty} b_k \bar{z}^k \in L^2(\mathbb{C}, |z|^m), m \in \mathbb{N}$$

Where

$$L_m^2 := L^2(\mathbb{C}, |z|^m)$$
$$= \left\{ g\ measurable: \|g\|_m^2: \right.$$
$$\left. = \int_{\mathbb{C}} |g(z)|^2 e^{-|z|^m}\, d\lambda(z) < \infty \right\}.$$

and A_m^2 is the corresponding subspace of entire functions:

$$A_m^2 := A^2(\mathbb{C}, |z|^m)$$
$$= \left\{ g\ entire: \|g\|_m^2 := \int_{\mathbb{C}} |g(z)|^2 e^{-|z|^m}\, d\lambda(z) < \infty \right\}.$$

For convenience we sometimes abbreviate $L^2(\mathbb{C},|z|^m)$by $L^2(|z|^m)$ and $A^2(\mathbb{C},|z|^m)$ by $A^2(|z|^m)$. The subspaces A_m^2 are weighted Bergman spaces with weight function $exp\,\{|z|^m\}$, norm $\|\cdot\|_m$ and associated inner product

$$\langle f,g\rangle_m := \int_{\mathbb{C}} f(z)\bar{g}(z)e^{-|z|^m}\, d\lambda(z),$$

Where $d\lambda$ denotes the Lebesgue measure in $\mathbb{C} \cong \mathbb{R}^2$. the expressions

$$C_{n,m} = \langle z^n, z^n\rangle_m = \int_{\mathbb{C}} |z^n|^2 e^{-|z|^m}\, d\lambda(z) = \|z^n\|_m^2, n, m \in \mathbb{N},$$

are the so-called moments. We denote the spaces, A_m^2 general Fock spaces, where Fock space is the (Hilbert) direct sum of tensor products of copies of a single-particle Hilbert space H

$$\begin{aligned} F_v(H) &= \bigoplus_{n=0}^{\infty} S_v H^{\otimes n} \\ &= \mathbb{C} \oplus H \oplus \big(S_v(H \otimes H)\big) \oplus \big(S_v(H \otimes H \otimes H)\big) \oplus \ldots.. \end{aligned}$$

Here $\mathbb{C}$, a complex scalar, represents the states of no particles, H the state of one particle, $S_v(H \otimes H)$ the states of two identical particles etc.

A typical state in $F_v(H)$ is given by

$$\begin{aligned} |\Psi\rangle_v &= |\Psi_0\rangle_v \oplus |\Psi_1\rangle_v \oplus |\Psi_2\rangle_v \oplus \ldots \\ &= a_0|0\rangle \oplus |\psi_1\rangle \oplus \sum_{ij} a_{ij} \left|\psi_{2\mathrm{i}}, \psi_{2\mathrm{j}}\right\rangle_v \oplus \ldots \end{aligned}$$

Where:

$|0\rangle$ is a vector of length 1, called the vacuum state and $a_0 \in \mathbb{C}$ is a complex coefficient,

$|\psi_1\rangle \in H$ is a state in the single particle Hilbert space,

$$\left|\psi_{2\mathrm{i}}, \psi_{2\mathrm{j}}\right\rangle_v = \frac{1}{2}\Big(|\psi_{2\mathrm{i}}\rangle \otimes |\psi_{2\mathrm{j}}\rangle + (-1)^v |\psi_{2\mathrm{j}}\rangle \otimes |\psi_{2\mathrm{i}}\rangle\Big) \in S_v(H \otimes H),$$

and $a_{ij} = v\, a_{ji} \in \mathbb{C}$ is a complex coefficient etc [6].

and A_2^2 is the classical Fock space. Remember that the Hankel operator with symbol $\bar{f}=\sum_{k=0}^{\infty} b_k\, \bar{z}^k \in L^2(\mathbb{C}, |\, z\, |^m)$ is given by

$$H_{\bar{f}}(h) = (Id - p)(\bar{f}h)$$

$$H_{\bar{f}} = (Id - p)(\bar{f}) : A_m^2 \rightarrow A_m^{2\perp},$$

Where $P : A_m^2 \overset{\perp}{\rightarrow} A_m^2$ denotes the Bergman projection, which has the following integral representation:

$$P(g)(w) = \int_{\mathbb{C}} g(z) K_m(w,z) e^{-|z|^m}\, d\lambda(z), \forall g \in L^2(\mathbb{C}, |\, z\, |^m),$$

Here $K_m(w,z)$is the reproducing kernel, the so called Bergman kernel, which is given by

$$K_m(z,w) = \sum_{k=0}^{\infty} \phi_{k,m}(z) \overline{\phi_{k,m}(w)},$$

Where $\{\phi_{k,m}\}_{k=1}^{\infty}$ is any complete orthonormal system of A_m^2 .

Most results about Hankel operators only deal with essentially bounded symbols ψ. In that case it is well known, that the Hankel operators are bounded with $\|H_{\bar{\psi}}\| \leq \|\psi\|_\infty$.

In the last years further (spectral-) properties of Hankel operators, like compactness, Hilbert-Schmidt or p-Schatten-von Neumann class where a Hilbert–Schmidt operator, named for David Hilbert and Erhard Schmidt, is a bounded operator A on a Hilbert space H with finite Hilbert–Schmidt norm

$$\|A\|_{HS}^2 = Tr|A^*A| := \sum_{i \in I} \|A\, e_i\|^2$$

where $\|\quad\|$ is the norm of H and $\{e_i : i \in I\}$ an orthonormal basis of H for an index set I. Note that the index set need not be countable [7].

There has been some work on Hankel operators on weighted Bergman spaces .For a general introduction in the field of compact operators and for the p-Schatten-von Neumann class. Later on Hankel operators with monomial-symbols $\bar{z}^k$ were studied also on weighted Bergman spaces, especially on generalized Fock spaces It should be also mentioned that there is the following connection between Hankel operators and the $\bar{\partial}$-Neumann problem: the canonical solution operator S to $\bar{\partial}$ is a Hankel operator of the form $S = \left(Id - P\right)\bar{z} = H_{\bar{z}}$; If S is restricted to Bergman spaces, or more generally to holomorphic $(0, q)$ −forms.

Let $H_{\bar{f}}$ be the Hankel operator with general anti-holomorphic symbol $\overline{f}$, i.e. $\bar{f}$ is polynomial or more generally a power series

$$\bar{f} = \sum_{k=0}^{\infty} b_k \bar{z}^k \ \in L^2(|z|^m)$$

Then the following problem arises: if $h \in A^2(|z|^m)$, then it is not clear that $fh \in L^2(|z|^m)$. Even the multiplication with $\bar{z}^n$ is only densely defined as an operator from $L^2(|z|^2)$ to $L^2(|z|^2)$, $\forall n \in \mathbb{N}$.

Example (2.1.1) [2]: $h = \sum_{k=0}^{\infty} a_k \bar{z}^k \ with \ \left|a^k\right|^2 = \frac{1}{k^2 k!}$. A calculation shows

$$\|h\|_2^2 \sim \sum_k \frac{1}{k^2},$$

$$\|\bar{z}^n h\|_2^2 \sim \sum_k \frac{(n+k) \dots (k+1)}{k^2} > \sum_k (n+k) \dots (k+3),$$

So, $h \in L^2(|z|^2)$, but $\bar{z}^n h \notin L^2(|z|^2), \forall n \in \mathbb{N}, n \neq 0$

In this section we investigate continuity of the Hankel operators $H_{\bar{f}}$. In the following we will show that there are no continuous Hankel operators with anti-holomorphic symbols if the corresponding symbol is not a polynomial. Let $\{e_n = \frac{z^n}{c_{n,m}} : n \in \mathbb{N}\}$ be the natural basis of A_m^2 and $C_{n,m}$ the moments corresponding to m. We will suppress the dependence of $C_{n,m}$ on m and will simply write C_n.. In the following we will assume that

$$\bar{f} e_n \in L^2(|z|^m),$$

in order to ensure that $H_{\bar{f}}(e_n)$ can be defined in a suitable way. Clearly $H_{\bar{f}}(e_n) = \bar{f} e_n - P\ (\bar{f} e_n)$.

The following proposition calculates $H_{\bar{f}}(e_n) = \bar{f} e_n - P\ (\bar{f} e_n)$. directly

Proposition (2.1.2) [2]: let $\bar{f} = \sum_k b_k \bar{z}^k \in L^2(|z|^m)$.Then we have

$$H_{\bar{f}}(e_n) = \bar{f}\frac{z^n}{c_n} - P\left(\bar{f}\frac{z^n}{c_n}\right) = \bar{f}\frac{z^n}{c_n} - \sum_{k\le n} b_k \frac{c_n}{c_{n-k}^2} z^{n-k}.$$

Proof: Note that

$$\begin{aligned} P(\bar{f}e_n)(z) &= \frac{1}{c_n} P\left(\textstyle\sum_{k=0}^{\infty} b_k \bar{z}^k z^n\right) \\ &= \frac{1}{c_n}\sum_{l=0}^{\infty} \langle \textstyle\sum_{k=0}^{\infty} b_k \bar{z}^k z^n | e_l\rangle e_l = \frac{1}{c_n}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \frac{b_k}{c_l^2}\langle z^n | z^{l+k}\rangle z^l \\ &= \frac{1}{c_n}\sum_{l=n-k}\sum_{k=n-l} \frac{b_k}{c_l^2} c_n^2 \delta_{n,l+k} z^l = \frac{1}{c_n}\sum_{k\le n} \frac{c_n^2}{c_{n-k}^2} b_k\, z^{n-k}. \end{aligned}$$

Now we calculate the norm of Hankel operators in terms of the moments $c_k$and the coefficients b_k:

$$\begin{aligned} &\langle H_{\bar{f}}\left(\frac{z^n}{c_n}\right) | H_{\bar{f}}\left(\frac{z^m}{c_m}\right)\rangle \\ &\quad = \langle \bar{f}\frac{z^n}{c_n} - \sum_{k\le n} b_k \frac{c_n}{c_{n-k}^2} z^{n-k} | \bar{f}\frac{z^m}{c_m} - \sum_{l\le m} b_l \frac{c_m}{c_{m-l}^2} z^{m-l}\rangle \\ &\quad = \langle \bar{f}\frac{z^n}{c_n} | \bar{f}\frac{z^m}{c_m}\rangle + \sum_{k\le n}\sum_{l\le m} b_k \bar{b}_l c_n c_m \langle \frac{z^{n-k}}{c_{n-k}^2} | \frac{z^{m-1}}{c_{m-l}^2}\rangle \\ &\quad - \sum_{l\le m} \bar{b}_l c_m \langle \bar{f}\frac{z^n}{c_n} | \bar{f}\frac{z^{m-1}}{c_{m-l}^2}\rangle - \sum_{k\le n} b_k c_n \langle \frac{z^{n-k}}{c_{n-k}^2} | \bar{f}\frac{z^m}{c_m}\rangle. \end{aligned}$$

For $\bar{f} = \sum_k^{\infty} b_k \bar{z}^k$ and $n = m$ we have

$$\begin{aligned} \left\| H_{\bar{f}}\left(\frac{z^n}{c_n}\right)\right\|^2 &= \left\|\bar{f}\frac{z^n}{c_n}\right\|^2 + \sum_{k\le n} b_k \bar{b}_k c_n^2 \left\|\frac{z^{n-k}}{c_{n-k}^2}\right\|^2 - \sum_{k\le n} \bar{b}_k c_n \langle \bar{f}\frac{z^n}{c_n} | \frac{z^{n-k}}{c_{n-k}^2}\rangle \\ &- \sum_{k\le n} b_k c_n \langle \frac{z^{n-k}}{c_{n-k}^2} | \bar{f}\frac{z^n}{c_n}\rangle \\ &= \sum_k |b_k|^2 \frac{c_{n+k}^2}{c_n^2} + \sum_{k\le n} |b_k|^2 \frac{c_n^2}{c_{n-k}^2} - 2\sum_{k\le n} |b_k|^2 \frac{c_n^2}{c_{n-k}^2} \\ &= \sum_{k\le n} |b_k|^2 \left[\frac{c_{n+k}^2}{c_n^2} - \frac{c_n^2}{c_{n-k}^2}\right] + \sum_{k>n} |b_k|^2 \frac{c_{n+k}^2}{c_n^2}. \end{aligned}$$

So we can conclude the following characterization .

Proposition (2.1.3) [2]: Let $\overline{f} = \sum_{k}^{\infty} b_k \bar{z}^k \in L^2(|z|^m)$, then the Hankel operator $H_{\bar{f}} \coloneqq \mathrm{A}_{\mathrm{m}}^2 \rightarrow \mathrm{A}_{\mathrm{m}}^{2\perp}$ is bounded if only if there exists a constant C such that

$$\sum_{k\leq n} |b_k|^2 \left[\frac{c_{n+k}^2}{c_n^2} - \frac{c_n^2}{c_{n-k}^2}\right] + \sum_{k>n} |b_k|^2 \frac{c_{n+k}^2}{c_n^2} < C, \forall n \in \mathbb{N}.$$

Note that in case of polynomials $f = \sum_{k}^{N} b_k z^k$ the formula of the norm reduces for $n > N$ to

$$\left\| H_{\bar{f}}\left(\frac{z^n}{c_n}\right)\right\|^2 = \sum_{k}^{N} |b_k|^2 \left[\frac{c_{n+k}^2}{c_n^2} - \frac{c_n^2}{c_{n-k}^2}\right] =: \sum_{k}^{N} |b_k|^2 a_n(k).$$

In this case we have only to investigate a finite sum of $a_n(k)$. in the next section we will give explicit, necessary and sufficient conditions for boundedness and compactness of Hankel operators with polynomial symbols on generalized Fock spaces.

One result will be the following: if k is large enough, i.e., if $2k > m$, then $a_n(k) \rightarrow \infty$ for $n \rightarrow \infty$. Consequently boundedness is only possible for polynomial symbols.

Section (2.2): Hankel Operators with Polynomial Symbols on Fock Spaces and Schatten-von Neumann *P*-class

We recall (Eq. (1)) that in case of polynomials $\overline{f} = \sum_k^N b_k \bar{z}^k$ the formula of the norm simplifies in case $n > N$ to

$$\left\| H_{\bar{f}}\left(\frac{z^n}{c_n}\right)\right\|^2 = \sum_k^N |b_k|^2 \left[\frac{c_{n+k}^2}{c_n^2} - \frac{c_n^2}{c_{n-k}^2}\right].$$

So we have to investigate the asymptotic behaviour of

$$a_n = a_n(k) := \frac{c_{n+k}^2}{c_n^2} - \frac{c_n^2}{c_{n-k}^2} \; for \; n \to \infty,$$

Example (2.2.1) [2]: On the Fock space $A^2(\mathbb{C}, |z|^2)$the moments are given by $c_n^2 \sim n!$ and so we have,

$$a_n = \frac{(n+k)!}{n!} - \frac{n!}{(n-k)!},$$

Consequently for polynomial-symbols $f = \sum_{k=0}^N b_k z^k$ the Hankel operators $H_{\bar{f}}$ are bounded in case $N = 0,1$; but in case $N \leq 2$ they are unbounded. To see this we note that $a_n(0) = 0$ and $a_n(1) = 1$, for all $n \geq 0$. Furthermore, for $k \geq 2$ the coefficient $a_n(k)$ is a polynomial of degree $k - 1$ with leading coefficient equal to

$$\sum_{j=1}^{k} j + \sum_{j=1}^{k-1} j = k^2,$$

There is also a more direct way to see that the operators $H_{\bar{z}^k}$ are not bounded on the Fock space for $k > 1$ .If k_w denotes the normalized reproducing kernel on the Fock space

(defined by $k_w(z) = k_2(z,w)/k_2(w,w)^{1/2} = e^{\bar{w}z - |w|^2/2}$), and τ_w denoted the translation by w (so $\tau_w(z) = z + w$), the formula

$$\left\| H_{\bar{f}} k_w \right\|_2 = \| f o \tau_w - f(w) \|_2$$

is valid for analytic functions f such that $\bar{f} k_w \in L_2^2$, in particular for all polynomials f . Since the functions k_w have unit norm, a necessary condition for boundedness of $H_{\bar{f}}$ on the Fock space is that the norms $\|fo\tau_w - f(w)\|_2$ are uniformly bounded. If $f(z) = z^k$ and $n \geq 2$, then it is easily to see that

$$\|fo\tau_w - f(w)\|_2 = \int_{\mathbb{C}} \left|(z+w)^k - w^k\right|^2 e^{-|z|^2} d\lambda \to \infty ,$$

as $|w| \to \infty$,so that the operator $H_{\bar{z}^k}$ is unbounded for $k > 1$.

Let us recall that on generalized Fock spaces the moments are given by

$$A^2(\mathbb{C}, |z|^m) := \left\{ f \ entire: \|f\|_m^2 := \int_{\mathbb{C}} |f(z)|^2 \exp\{-|z|^m\} \ d\lambda(z) < \infty \right\}.$$

The moments are given by

$$c_k^2 = c_{k,m}^2 = \int_{\mathbb{C}} |z|^{2k} \exp\{-|z|^m\} \, d\lambda(z) = \frac{2\pi}{m} \Gamma\left(\frac{2k+2}{m}\right),$$

Where $\Gamma = \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt , x > 0$, is the Gamma function. We remember Stirling's formula with error term

$$\Gamma(x+1) = x^x e^{-x} \sqrt{2\pi x} \left(1 + \frac{1}{12x} + \frac{1}{288x^2} + \mathcal{O}_3\right),$$

Where $\mathcal{O}_k := \mathcal{O}(1/n^k)$ are the so-called Landau symbols

The following proposition determines the limiting behavior of the sequence $a_n = c_{n+k}^2 / c_n^2 - c_n^2 / c_{n+k}^2$ '

Proposition (2.2.2) [2]:

$$a_n = \frac{c_{n+k}^2}{c_n^2} - \frac{c_n^2}{c_{n-k}^2} \approx C(k,m) n^{\frac{2k}{m}-1},$$

Where

$$C\,(k,m) = \left(\frac{2}{m}\right)^{\frac{2k}{m}} \frac{2k^2}{m},$$

Proof: Using Stirling's formula it is easy to verify that

$$\frac{\Gamma(x+1+P)}{\Gamma(x+1)} \sim (x+\mathrm{P})^{\mathrm{P}} \text{ as } x \to \infty,$$

Thus

$$a_n = \frac{\Gamma\left(\frac{2n+2k+2}{m}\right)}{\Gamma\left(\frac{2\mathrm{n}+2}{\mathrm{m}}\right)} - \frac{\Gamma\left(\frac{2n+2}{m}\right)}{\Gamma\left(\frac{2\mathrm{n}-2\mathrm{k}+2}{\mathrm{m}}\right)}$$

$$\sim \left(\frac{2n+2k+2}{m} - 1\right)^{\frac{2k}{m}} - \left(\frac{2n+2}{m} - 1\right)^{\frac{2k}{m}} \text{ as } n \to \infty.$$

It follows from Eq. (2) that $a_n \to 1$ as $n \to \infty$ if $2k = m$ and $a_n \to \infty$ as $n \to \infty$ if $2k$. also, if $2k < m$, then it is easy to show that Eq. (2) implies that

$$a_k \sim \left(\frac{2k}{m}\right)^2 \left(\frac{2}{m}\right)^{\frac{2k}{m}-1} \frac{2k}{n^m} - 1 \; as \; n \to \infty.$$

So we can conclude the following theorems.

Theorem (2.2.3) [2]: Let $H_{\bar{f}}$ be a Honkel operator with sysmbol $\bar{f} = \sum_{k=0}^{\infty} b_k \bar{z}^k \in L^2(|\mathrm{z}|^{\mathrm{m}})$, which is not a polynomial. Then the hankel operator

$$H_{\bar{f}} = (Id - P)\bar{f}: A^2(\mathbb{C}, |z|^m) \to A^2(\mathbb{C}, |z|^m)^{\perp}.$$

is unbounded.

Proof: We recall that for symbols $\bar{f} = \sum_k^{\infty} b_k \bar{z}^k \in L^2(|\mathrm{z}|^{\mathrm{m}})$, the Hankel operator $H_{\bar{f}} : A_m^2 \to A_m^{2\perp}$ is bounded if and only if there exists a constant C such that

$$\left\| H_{\bar{f}}\left(\frac{z^n}{c_n}\right) \right\|^2 = \sum_{k \le n} |b_k|^2 \, a_n(k) + \sum_{k > n} |b_k|^2 \frac{c_{n+k}^2}{c_n^2} < C, \forall n \in \mathbb{N}$$

In case that the power series symbol $\bar{f}$ is not a polynomial we have n with $2n > m$.

Consequently we have k with $2k > m$. And with proposition (2.2.2) we have $a_n(k) \to \infty$ for all k with $2k > m$. So clearly $H_{\bar{f}}$ is not bounded.

Theorem (2.2.4) [2]: $H_{\bar{f}}$ be Hankel operator operator with polynomial $\bar{f} = \sum_{k=0}^{N} b_k \bar{z}^k$. Then in case $2N \le m$ the Hankel operators

$$H_{\bar{f}} = (Id - P)\bar{f}: A^2(\mathbb{C}, |z|^m) \to A^2(\mathbb{C}, |z|^m)^{\perp}.$$

are bounded; and in case $2N > m$ they are unbounded.

Proof of: We have only to recall that in case $n > N$ we have

$$\left\| H_{\bar{f}}\left(\frac{z^n}{c_n}\right) \right\|^2 = \sum_k^N |b_k|^2 \left[\frac{c_{n+k}^2}{c_n^2} - \frac{c_n^2}{c_{n-k}^2}\right] = \sum_k^N |b_k|^2 a_n(k).$$

and so with Proposition (2.2.2) we get the result.

Now we consider compactness conditions of Hankel operators $H_{\bar{f}}$ with symbol $f = \sum_k^N b_k z^k$ on generalized Fock spaces $A^2(\mathbb{C}, |z|^m)$. we will need the following proposition for the further investigations.

Proposition (2.2.5) [2]: Let $f = \sum_{k=0}^{\infty} a_k z^k$ and $n, r \in \mathbb{N}$ with $n > r$. Then in case $n - r > N$ we get

$$H_{\bar{f}}(z^n) \perp H_{\bar{f}}(z^r).$$

Proof: We have

$$\begin{aligned} \langle H_{\bar{f}}\left(\frac{z^n}{c_n}\right) | H_{\bar{f}}\left(\frac{z^r}{c_r}\right)\rangle \\ &= \langle \bar{f}\frac{z^n}{c_n} | \bar{f}\frac{z^r}{c_r}\rangle + \sum_{k \le n}\sum_{l \le r} b_k \bar{b}_l c_n c_r \langle \frac{z^{n-k}}{c_{n-k}^2} | \frac{z^{r-l}}{c_{r-l}^2}\rangle \\ &- \sum_{l \le r} \bar{b}_l c_r \langle \bar{f}\frac{z^n}{c_n} | \frac{z^{r-1}}{c_{r-l}^2}\rangle - \sum_{k \le n} b_k c_n \langle \frac{z^{n-k}}{c_{n-k}^2} | \bar{f}\frac{z^r}{c_r}\rangle. \end{aligned}$$

Clearly $-N \le k, l, k - I \le N < n - r$ and so by using $\delta_{n-r,k-1,}\delta_{n-r,-1}$ and $\delta_{n-r,k} we\ get\ H_{\bar{f}}(z^n) \perp H_{\bar{f}}(z^r)$.

So we can conclude the following.

Corollary (2.2.6) [2]: Let $f = \sum_{k=0}^{N} a_k z^k$. Then for all $n, r \in \mathbb{N}$ with $n - r > N, r > N$ we get

$$\begin{aligned} \left\| H_{\bar{f}}(e_n) - H_{\bar{f}}(e_r) \right\|^2 &= \left\| H_{\bar{f}}(e_n) \right\|^2 + \left\| H_{\bar{f}}(e_r) \right\|^2 \\ &= \sum_k^N |b_k|^2 \left[\frac{c_{n+k}^2}{c_n^2} - \frac{c_n^2}{c_{n-k}^2}\right] + \sum_k^N |b_k|^2 \left[\frac{c_{r+k}^2}{c_r^2} - \frac{c_r^2}{c_{r-k}^2}\right] \\ &= \sum_k^N |b_k|^2 [a_n(k) - a_r(k)]. \end{aligned}$$

Theorem (2.2.7) [2]: Let $f = \sum_k^N b_k z^k$. Then in case $2N < m$ the Hankel operators.

$$H_{\bar{f}} = (Id - P)\bar{f}: A^2(\mathbb{C}, |z|^m) \to A^2(\mathbb{C}, |z|^m)^{\perp}.$$

are compact and in case $2N = m$ the Hankel operator $H_{\bar{f}}$ fail to be compact.

Proof: Let $2N = m$. With Corollary (2.2.6) we get

$$\left\|H_{\bar{f}}(e_n) - H_{\bar{f}}(e_r)\right\|^2 = \sum_k^N |b_k|^2 [a_n(k) - a_r(k)].$$

and with Proposition (2.2.2) we have for $2k = m = 2r$ that $a_n(k), a_r(k) \to \ const\ (k, m)$. Consequently the Hankel operator $H_{\bar{f}}$ can not be compact.

Compactness is shown for $H_{\bar{f}}: A_m^2 \to A_m^{2\perp}$ in case $2k < m$. so clearly $H_{\bar{f}} = \sum_k^N b_k H_{\bar{z}^k}$ is compact in case $2N < m$.

In this section we will give for the symbol $\bar{z}^k$, $k \in \mathbb{N}$ and for all $p > 0$ a complete characterization of the Schatten-von Neumann p-class membership in terms of k, m and p. Let us start with some definitions.

First we recall that a bounded linear Hilbert space operator $T: H_1 \to H_2$ is called positive, if

$$\langle t\, x, x \rangle \geq 0 \,, \forall x \in H_1.$$

In fact this is equivalent to $T = T^*$ and $\sigma(T) \subseteq [0, x)$.

Definition (2.2.8) [2]: Let H_1, H_2 be Hilbert spaces with a complete orthonormal system $\{e_n\}_{n\in\mathbb{N}}$ of H_1. A positive operator $T : H_1 \to H_2$ is in the trace class if

$$tr(T) := \sum_n^{\infty} \langle Te_n, e_n \rangle < \infty.$$

The definition is independent of the choice of the orthonormal system. Clearly a bounded linear Hilbert space operator T is Hilbert-Schmidt, i.e.

$$\|T\|_{HS}^2 := \sum_{n}^{\infty} \|Te_n\|^2 = \sum_{n}^{\infty} \langle Te_n, Te_n \rangle < \infty,$$

if and only if the operator T^*T is in the trace class. In that case clearly we have

$$\|T\|_{HS}^2 = tr(T^*T).$$

This can be generalized to the definition of the Schatten-von Neumann p-class (or the Schatten ideal S^P):

Definition (2.2.9) [2]: Let $p > 0$ and H_1, H_2 be Hilbert spaces with a complete orthonormal system $\{e_n\}_{n\in\mathbb{N}}$ of H_1. A bounded linear operator $T : H_1 \to H_2$ is in the Schatten-von Neumann p-class if $(T^*T)^{P/2}$ is in the trace class, i.e.,

$$tr\big((T^*T)^{P/2}\big) := \sum_{n}^{\infty} \langle (T^*T)^{P/2} e_n, e_n \rangle < \infty,$$

In that case we define $\|T\|_P^P := tr\big((T^*T)^{P/2}\big)$. so by definition the Schatten-von Neumann 2-class (or the Schatten ideal S^2) is the ideal of Hilbert-Schmidt operators. Now let us turn to our Hankel operators with symbol $\bar{z}^k$, $k \in \mathbb{N}$ and let us repeat some facts. As above we abbreviate $C_{n,m} = C_n$.

Lemma (2.2.10) [2]: Let $k \in \mathbb{N}$ with $n \geq k$ and $e_n(z) = z^n/c_n$. then

$$\langle H_{\bar{z}^k}^* H_{\bar{z}^k} e_n, e_n \rangle = \frac{c_{n+k}^2}{c_n^2} - \frac{c_n^2}{c_{n-k}^2}.$$

Theorem (2.2.11) [2]: Let $p > 0$ and $2k < m$. Then the Hankel operator $H_{\bar{z}^k}: A_m^2 \to A_m^{2\perp}$ is in the Schatten-von Neumann p-class if and only if

$$\sum_{n=k}^{\infty} \left(\frac{c_{n+k}^2}{c_n^2} - \frac{c_n^2}{c_{n-k}^2} \right)^{\frac{P}{2}} < \infty,$$

Proof: In case $2k < m$ with Theorem (2.2.7) the Hankel operator $H_{\bar{z}^k}$ is compact and so the operator $H_{\bar{z}^k}^* H_{\bar{z}^k}$ has the form

$$H_{\bar{z}^k}^* H_{\bar{z}^k}(g) = \sum_{n=0}^{\infty} \lambda_{n,k} \langle g, e_n \rangle e_n.$$

Consequently

$$\langle H_{\bar{z}^k}^* H_{\bar{z}^k} e_n, e_n \rangle = \lambda_{n,k}.$$

and with Lemma (2.2.10),

$$\sum_{n=k}^{\infty} \lambda_{n.k}^{\frac{P}{2}} = \sum_{n=k}^{\infty} \langle H_{\bar{z}^k}^* H_{\bar{z}^k} e_n, e_n \rangle^{\frac{P}{2}} = \sum_{n=k}^{\infty} \left(\frac{c_{n+k}^2}{c_n^2} - \frac{c_n^2}{c_{n-k}^2} \right)^{\frac{P}{2}}.$$

In case of the Fock space ($m = 2$) for all $k \geq 1$ the sequence

$$a_n(k) = \frac{c_{n+k}^2}{c_n^2} - \frac{c_n^2}{c_{n-k}^2} = \frac{(n+k)!}{n!} - \frac{n!}{(n-k)!}.$$

Does not even tend to 0. So we can conclude the following.

Corollary (2.2.12) [2]: Let $p > 0$. On the Fock space the Hankel operators

$$H_{\bar{z}^k} : A^2(\mathbb{C}, |z|^2) \to L^2(\mathbb{C}, |z|^2).$$

are not in the Shatten p-class for all $k \geq 1$.

Theorem (2.2.13) [2]: Let $p > 0$. In case $2k < m$ the Hankel operators

$$H_{\bar{z}^k} = (Id - P)\bar{z}^k : A^2(\mathbb{C}, |z|^m) \to A^2(\mathbb{C}, |z|^m)^{\perp}.$$

are in the Schatten p-class, iff $p > \frac{2m}{m-2k}$. In case $2k \geq m$ the Hankel operators are not in the Schatten p-class.

Proof: Use Eq. (3) to note that

$$\sum_{n \geq k} \left(\frac{c_{n+k}^2}{c_n^2} - \frac{c_n^2}{c_{n-k}^2} \right)^{\frac{P}{2}} \approx \sum_{n=k}^{\infty} \frac{1}{\left(n^{1-\frac{2k}{n}} \right)^{P/2}} < \infty$$

if and only if $(1 - \frac{2k}{m})\, p/2 > 1$, that is $p > 2/(1 - \frac{2k}{m})$.

Chapter 3

Stieltjes Moment Problem and Hankel Operators

We prove that there are nontrivial Hilbert–Schmidt Hankel operators with anti-holomorphic symbols if and only if s is exponentially bounded. In this case, the space of symbols of such operators is shown to be the classical Dirichlet space. We mention that the classical weighted Bergman spaces, the Hardy space and Fock type spaces fall in this setting.

Section (3.1): operators of Hilbert space A^2(s)

We consider Hankel operators and the $\bar{\partial}$-canonical solution operator in a Hilbert space of analytic functions related to a Stieltjes moment sequence. We recall that a sequence $S = (S_d), d \in \mathbb{N}_0$, is.said to be a Stieltjes moment sequence if it has the form

$$s_d = \int_0^{+\infty} t^d d\mu(t),$$

Where μ is a non-negative measure on $[0, +\infty)$, called a representing measure for s. These sequences have been characterized by Sticltjes in terms of some positive definiteness conditions. We denote by $\mathcal{S}$ the set of such sequences. It follows from the above integral representation that each $s \in \mathcal{S}$ is either non-vanishing, that is, $S_d > 0$ for all d, or else $s_d = \delta_{0d}$ for all d. We denote by S^* the set of all non-vanishing elements of S. Fix an element $s = (s_d) \in \mathcal{S}^*$. By Cauchy-Schwarz inequality we see that the sequence.$\frac{s_{d+1}}{s_d}$ is non-decreasing and hence converges as $d \to +\infty$ to the radius of convergence of the entire series

$$F_s(\lambda) := \sum_{d=n-1}^{+\infty} \frac{\lambda^d}{s_{d+1-n}}, \lambda \in \mathbb{C},$$

Set $R_s := \lim_{d\to+\infty} \sqrt{\frac{s_{d+1}}{s_d}} = \sqrt{\lim_{d\to+\infty} s_d d^{\perp}}$.The sequences s for which the radius R_s is finite are called exponentially bounded.

Denote by Ω_s the ball in $\mathbb{C}^n$ centered at the origin with radius R_s with the understanding that $\Omega_s = \mathbb{C}^n$ $when$ $R_s = +\infty$. We denote by $\mathcal{A}^2(s)$ the Hilbert space of those holomorphic functions $f(z) = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha z^\alpha$ on Ω_s that satisfy

$$\sum_{\alpha \in \mathbb{N}_0^n} \frac{\alpha!\, S_{|\alpha|}}{(|\alpha| + n - 1)!} |a_\alpha|^2 < +\infty$$

equipped with the natural inner product

$$\langle f, g \rangle := \sum_{\alpha \in \mathbb{N}_0^n} \frac{\alpha!\, S_{|\alpha|}}{(|\alpha| + n - 1)!} a_\alpha \bar{b}_\alpha.$$

if $f(z) = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha z^\alpha$ and $g(z) = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha z^\alpha$are two elements of $\mathcal{A}^2(s)$.

Now let $\sigma = \sigma_n$ be the rotation invariant probability measure on the unit sphere $\mathbb{S}_n$ in $\mathbb{C}^n$and let μ, be a representing measure of s. We denote by μ_n, the image measure in $\mathbb{C}^n$ of $\mu \otimes \sigma_n$ under the map $(t, \xi) \to \sqrt{t\xi}$ from $[0, +\infty) \times \mathbb{S}_n$ onto$\mathbb{C}^n$. We consider the Hilbert space $L^2(\mu_n)$ of square integrable complex-valued functions in $\mathbb{C}^n$with respect to the measure μ_n. Our first result is the following:

The classical weighted Bergman spaces, weighted Fock spaces and Hardy spaces, where Hardy spaces (or Hardy classes) H^p are certain spaces of holomorphic functions on the unit disk or upper half plane. They were introduced by Frigyes Riesz, who named them after G. H. Hardy. In real analysis Hardy spaces are certain spaces of distributions on the real line, which are (in the sense of distributions) boundary values of the holomorphic functions of the complex Hardy spaces, and are related to the L^pspaces of functional analysis. For $1 \le p \le \infty$ these real Hardy spaces H^p are certain subsets of L^p, while for $p < 1$ the L^p spaces have some undesirable properties, and the Hardy spaces are much better behaved [8], are of the form , $\mathcal{A}^2(s)$; each of these space is associated to an appropriate choice of the sequence s.

To state further results we consider the orthogonal projection P_s, associated to $\mathcal{A}^2(\mu_n)$. It is given for all $g \in L^2(\mu_n)$ by

$$(P_s g)(z) = \int_{\bar{\Omega}_s} K_s(z, w) g(w) d\mu_n(w), z \in \Omega_s.$$

This integral operator can be extended in a natural way to functions g that satisfy $K_s(z,\cdot)g \in L^1(\mu_n)$ for all $z \in \Omega_s$. This extension allows us to define Hankel operators. To do so, denote by $\mathcal{T}(s)$ the class of all $f \in \mathcal{A}^2(s)$ such that $f\varphi K_s(z,\cdot) \in L^1(\mu_n)$ for all holomorphic polynomials φ and $z \in \Omega_s$ and the function

$$H_{\bar{f}}(\varphi)(z) := \int_{\mathbb{C}^n} K_s(z,w)\,\varphi(w)\big[\bar{f}(z) - \bar{f}(w)\big]d\mu_n(w)\ z \in \Omega_s,$$

is the restriction to Ω_s of a function in $L^2(\mu_n)$. This is a densely defined operator from $\mathcal{A}^2(s)$ into $L^2(\mu_n)$ which will be called the Hankel operator $H_{\bar{f}}$ with symbol $\bar{f}$ it can be written in the form

$$H_{\bar{f}}(\varphi) = (I - P_s)\big(\bar{f}\varphi\big).$$

for all holomorphic polynomials φ.

It is not hard to see that the class $\mathcal{T}(s)$ contains all holomorphic polynomials. Finally, if $f \in \mathcal{T}(s)$, we denote by $Spec\,(f\,)$ the set of all multi-indices $k \in \mathbb{N}_0^n$ such that $\frac{\partial^k f}{\partial z^k}(0) \neq 0$.

Our second result is the following :

Theorem (3.1.1) [3]: Suppose that f is a holomorphic polynomial Then

(i) $H_{\bar{f}}$ is bounded if and only if

$$\sup_{d\in\mathbb{N}_0}\left(\frac{s_{d+2|k|}}{s_{d+|k|}} - \frac{s_{d+|k|}}{s_d} + \frac{n-1}{d} - \frac{s_{d+|k|}}{s_d}\right) < +\infty, \qquad (1)$$

For all $k \in Spec(f)$.

(ii) $H_{\bar{f}}$ is compact if and only if

$$\lim_{d\to+\infty}\left(\frac{s_{d+2|k|}}{s_{d+|k|}} - \frac{s_{d+|k|}}{s_d} + \frac{n-1}{d}\frac{s_{d+|k|}}{s_d}\right) = 0, \qquad (2)$$

For all $k \in Spec(f)$.

(iii) If $p > 0$, then $H_{\bar{f}}$ is in the Schatten class $S^P(\mathcal{A}^2(s), L^2(\mu_n)$ if and only if

$$\sum_{d\in\mathbb{N}} d^{n-1}\left(\frac{s_{d+2|k|}}{s_{d+|k|}}-\frac{s_{d+|k|}}{s_d}\right)^{\frac{P}{2}}+(n-1)\,d^{n-1-\frac{P}{2}}\left(\frac{s_{d+|k|}}{s_d}\right)^{\frac{P}{2}}<+\infty,$$

for all $k \in Spec(f)$.

We point out that if the sequence s is exponentially bounded then (1) and (2) hold. The last assertion of Theorem (3.1.2) shows that if $n \geq 2$, and the Schatten class$S^P(\mathcal{A}^2(s), L^2(\mu_n)$ concerns nontrivial Hankel operators with anti-holomorphic symbols, then $p > 2n$. The converse to this statement is not true as shown. In particular, in higher dimensions there are no nontrivial Hilbert-Schmidt Hankel operators with anti-holomorphic symbols. The situation in the one-dimensional case is completely different. More precisely.

The first equality shows the characterization in the latter theorem depends only on the limit $\lim_{d\to\infty} \frac{d_{d+1}}{s_d}$. The above result has been proved by separate methods in the two simple partieular cases of Hardy and Bergman spaces.

Now we shall characterize the boundedness, the compactness and the membership in a Schatten class of S the canonical solution operator of the $\bar{\partial}$ on the space $\mathcal{H}^{(0,1)}(\Omega_s)$ consisting of $(0,1)-$ forms with holomorphic coefficients in $L^2(\mu_n)$ defined by $\bar{\partial}(Sf) = f$ and Sf is orthogonal to holomorphic elements of $L^2(\mu_n)$. The spectral properties of this operator were studied by Haslinger, Haslinger and Helfer and Lovera and Youssfi.

Corollary (3.1.2) [3]: Consider the canonical solution operator S to the $\bar{\partial}$ from $\mathcal{H}^{(0,1)}(\Omega_s)$ to $L^2(\mu_n)$. Then the following are equivalent:

(i) S is bounded on $\mathcal{H}^{(0,1)}(\Omega_s)$.

(ii) For all $j = 1, \dots . n$, the Hankel operator $H_{\bar{z}_j}$ is bounded from $\mathcal{A}^2(s)$ into $L^2(\mu_n)$.

(iii) There is $j = 1, \dots\, n$, such that the Hankel operator $H_{\bar{z}_j}$. is bounded from $\mathcal{A}^2(s)$ into $L^2(\mu_n)$.

(iv) There is a positive constant $C > 0$ such that

$$\left(\frac{s_{d+n}}{s_{d+n-1}}-\frac{s_{d+n-1}}{s_{d+n-2}}\right)+\frac{n-1}{d}\frac{s_{d+n}}{s_{d+n-1}}\leq C$$

for all positive integers d.

Corollary (3.1.3) [3]: Consider the canonical solution operator S to the $\bar{\partial}$ from $\mathcal{H}^{(0,1)}(\Omega_s)$ to $L^2(\mu_n)$ .Then the following are equivalent:

(i) S is compact on $\mathcal{H}^{(0,1)}(\Omega_s)$.

(ii) For all $j = 1, \ldots, n$, the Hankel operator $\mathrm{H}_{\bar{\mathrm{z}}_\mathrm{j}}$ is compact from $\mathcal{A}^2(s)$ into $L^2(\mu_n)$.

(iii) There is $j = 1, \ldots, n$ such that the Hankel operator $\mathrm{H}_{\bar{\mathrm{z}}_\mathrm{j}}$ is compact $\mathcal{A}^2(s)$ into $L^2(\mu_n)$.

(iv) We have

$$\lim_{d\to+\infty}\left(\frac{s_{d+n}}{s_{d+n-1}} - \frac{s_{d+n-1}}{s_{d+n-2}} + \frac{n-1}{d}\frac{s_{d+n}}{s_{d+n-1}}\right) = 0.$$

In each of the two preceding corollaries, the equivalence between the two assertions (i) and (iv) was established in Lovera and Youssfi and later by Haslinger and Lame!.

Corollary (3.1.4) [3]: Consider the canonical solution operator S to the $\bar{\partial}$ from $\mathcal{H}^{(0,1)}(\Omega_s)$ to $L^2(\mu_n)$ and let $p > 0$. Then the following are equivalent:

(i) S is in the Schatten class $\mathcal{S}_P(\mathcal{H}^{(0,1)}(\Omega_s), L^2(\mu_n))$.

(ii) For all $j = 1, \ldots, n$, the Hankel operator $H_{\bar{z}_j}$ is in the Schatten class $\mathcal{S}_P(\mathcal{H}^{(0,1)}(\Omega_s), L^2(\mu_n))$.

(iii) There is $j = 1, \ldots, n$, such that the Hankel operator $H_{\bar{z}_j}$ is in the Schatten class $\mathcal{S}_P(\mathcal{A}^2(s), L^2(\mu_n)$.

(iv) There is a positive constant C such that

$$\sum_{d\in\mathbb{N}} d^{n-1}\left(\frac{s_{d+n}}{s_{d+n-1}} - \frac{s_{d+n-1}}{s_{d+n-2}}\right)^{\frac{P}{2}} + (n-1)\, d^{n-1-\frac{P}{2}}\left(\frac{s_{d+n}}{s_{d+n-1}}\right)^{\frac{P}{2}} \leq C$$

for all positive integers d.

In the latter corollary, the equivalence between the two assertions (i) and (iv) was established in Lovera and Youssfi in the case $p \geq 2$ and later by Haslinger and Lamel in the general case.

To state another result, we let $\mathcal{M}(s)$ be the subspace of $\mathcal{T}(s)$ consisting of those functions f. for which the Hankel operator $H_{\bar{f}}$ is bounded on $\mathcal{A}^2(s)$ .We equip $\mathcal{M}(s)$ with norm

$$\|f\|_{\mathcal{M}(s)} := \|H_{\bar{f}}\| + |f(0)|.$$

The subspace of $\mathcal{M}(s)$consisting of functions f such that $H_{\bar{f}}$ is a compact operator will be denoted by $\mathcal{M}_\infty(s)$. Then it is not hard to see that $\mathcal{M}_\infty(s) is$ a closed subspace of $\mathcal{M}(s)$.

If $p > 0$, we denote by $\mathcal{M}_P(s)$ the subspace of those functions $f \in \mathcal{M}(s)$ such that the Hankel operator $H_{\bar{f}}$ is the Schatten class $\mathcal{S}_P(\mathcal{A}^2(s), L^2(\mu_n))$ We equip $\mathcal{M}_P(s)$ with quasi-norm

$$\|f\|_{\mathcal{M}(s)} := \|H_{\bar{f}}\|_{\mathcal{S}_P} + |f(0)|.$$

Then we have the following "

Theorem (3.1.5) [3]: Let $\mathbb{X} \in \{\mathcal{M}(s), \mathcal{M}_\infty(s), \mathcal{M}_P(s)\}$ and let U be a rotation in $\mathbb{C}^n$. Then the following assertions hold.

(i) If $f \in \mathbb{X}$, then $foU \in \mathbb{X}$ and $\|foU\|_{\mathbb{X}} = \|f\|_{\mathbb{X}}$.
(ii) If $f \in \mathbb{X}$, then $z^k \in \mathbb{X}$ for all $k \in Spec(f)$.
(iii) If the sequence s is either exponentially bounded or satisfies

$$\lim_{d\to+\infty}\left(\frac{s_{d+l}}{s_d^2}\right)^{\frac{1}{d}} = 0 \; for \; all \; l \in \mathbb{N}_0 \quad (3).$$

then the spaces $\mathcal{M}(s), \mathcal{M}_\infty(s)$, and $\mathcal{M}_P(s), p \geq 1$, are Banach spaces and the space $\mathcal{M}_P(s), 0 < p < 1$, is a quasi-Banach space.

We point out that there are examples of Stieltjes moment sequences that do not satisfy (3) as shown by Boas type sequences. There is a sequence of positive real numbers S satisfying $S_0 \geq 1$ and

$S_{n+1} \geq (nS_n)_{n+1}$. It is not hard to see by Theorem (3.1.5) that the spaces $\mathcal{M}(s)$, $\mathrm{M}_\infty(s)$ and $\mathcal{M}_P(s)$ corresponding to such sequences are trivial, namely, they consist only of constant functions.

Another type of Stieltjes moment sequences for which Theorem (3.1.1) applies to show that the corresponding spaces $\mathcal{M}(s), \mathcal{M}_\infty(s)$,and $\mathcal{M}_P(s)$ are trivial are the Stieltjes sequences s that satisfy $S_0 \geq 1$ and

$$s_d^2 \leq \delta_{s_{d+1}s_{d-1}} \, for \; all \; d \geq 1 \quad (4)$$

for some $0 < \delta < 1$. Arbitrary sequences satisfying (4) were studied by Bisgaard and Sasviri and Bisgaard. They were shown to be Stieltjes moment sequences as long as $\sum_{d\geq 1}\delta^{d^2} \leq \frac{1}{4}$.

Let $\mathbb{N}_0^n$ denote the set of all n-tuples with components in the set $\mathbb{N}_0$ of all non-negative integers. If $\alpha = (\alpha_1, .., \alpha_n) \in \mathbb{N}_0^n$, We let $\|\alpha\| := \alpha_1 + \cdots + \alpha_n$ denote the length of α. If $\beta = (\beta_1 \ldots \beta_n) \in \mathbb{N}_0^n$ satisfies $\alpha_j \geq \beta_j$ for all $j = 1, \ldots, n$, then we write $\alpha \geq \beta$. Otherwise, set $\ngeq \beta$.

Finally, if A and B are two quantities, we use the symbol $A \approx B$ whenever $A \leq C_1 B$ and $B \leq C_2 A$, where $C_1$and C_2 are positive constants independent of the varying parameters.

Theorem (3.1.6) [3]: The measure μ_n is supported by the closure of the domain Ω_s.In addition, for each set compact $K \subset \Omega_s$ there exists

$C = C(K) > 0$ Such that

$$\sup_{z\in k}|f(z)| \leq C\|f\|_{L^2(\mu_n)}.$$

for all holomorphic polynomials f in $\mathbb{C}^n$. Furthermore, the space $\mathcal{A}^2(s)$ coincides with the clo-sure of the holomorphic polynomials in $L^2(\mu_n)$ and its reproducing kernel is given by

$$K_s(z,w) = \frac{1}{(n-1)!}F_s^{(n-1)}(\langle z,w\rangle), z, w \in \Omega_s.$$

Proof: We first observe that if a positive real number r satisfies $\mu\,([r,+\infty)) = 0$, then for all non-neutive integers d, we have $s_d \leq r^d \mu\,([0,+\infty))$ and hence $\lim\sup_d s_d^{\frac{1}{d}} \leq r$.

This shows that the radius of converence of the series F_s is smaller that or equal to the infimum of all such real numbers r.

Conversely, suppose that $r > 0$ satisfies $\mu([r,+\infty)) > 0$ then

$$r^d \mu([r,+\infty)) \leq \; s_d$$

For all non-negative integers d. Therefore,

$$r \leq lim_d \inf s_d^{\frac{1}{d}} \leq \ lim_d \ \sup s_d^{\frac{1}{d}}$$

Since

$$sup\{r: \mu([r, +\infty)) \ > \ 0\} = inf\{r: \mu \ ([r, +\infty)) = 0\}$$

we see that $R_s^2 \ = \ lim_{d\to\infty} \ s_d^{\frac{1}{d}}$. Therefore, the measure $\mu_n$is supported the closure $\overline{\Omega}_s$ since both series F_s and $F_s^{(n-1)}$ have the same radius of convergence it follows that for cach $z \in \Omega_s$, the series

$$K_s(z,w) = \frac{1}{(n-1)!} \sum_{0}^{+\infty} \frac{(d+n-1)!}{d!\, s_d} \langle z, w\rangle^d \,, w \in \Omega_s,$$

Converges on $\overline{\Omega}_s$. Moreover, by Fatou's lemma and orthognality of the holomorphic monomials with respect to μ_n we have

$$\int\limits_{\overline{\Omega}_s} |K_s(z,w)|^2 \, d\mu_n(w)$$

$$\leq \left(\frac{1}{(n-1)!}\right)^2 \liminf_{N\to+\infty} \int\limits_{\overline{\Omega}_s} \left| \sum_{d=0}^{N} \frac{(d+n-1)!}{d!\, s_d} \langle z, w\rangle^d \right|^2 \, d\mu_n(w)$$

$$= \left(\frac{1}{(n-1)!}\right)^2 \liminf_{N\to+\infty} \int\limits_{\overline{\Omega}_s} \left| \sum_{d=0}^{N} \frac{(d+n-1)!}{d!\, s_d} \langle z, w\rangle^d \right|^2 \, d\mu_n(w) =$$

$$= K_s(z,z)$$

Hence for any fixed $z \in \Omega_s$, the series $K_s(z,w)$ converges in $L^2(\mu_n)$. In addition , a little computing shows that for any $\alpha \in \mathbb{N}_0^n$, we have

$$\int\limits_{\overline{\Omega}_s} w^\alpha \, K_s(z,w) \, d\mu_n(w)$$

$$= \frac{1}{(n-1)!} \sum_{d=0}^{+\infty} \frac{(d+n-1)!}{d!\, s_d} \int\limits_{\overline{\Omega}_s} w^\alpha \, (z,w)^d \, d\mu_n(w)$$

$$= \frac{1}{(n-1)!} \frac{(|\alpha|+n-1)!}{|\alpha|!\, s_{|\alpha|}} \int\limits_{\overline{\Omega}_s} w^\alpha \, (z,w)^{|\alpha|} \, d\mu_n(w) = z^\alpha.$$

This shows that the kernel $K_s(z,w)$ reproduces holomorphic polynomials. Morcover it satisfies

$$\sup_{z\in K}|f(z)| \le \sup_{z\in K}\sqrt{K_s(z,z)}\|f\|_{L^2(\mu_n)}.$$

For all holomorphic polynomials and each set compact $K\subset\Omega$. The remaining part of the proof follows by standard arguments.

We point out that R_s is always strictly positive.

Lemma (3.1.7) [3]: Suppose that k and l are in $\mathbb{N}_0^n$ then the domain $\mathcal{D}om(H^*_{\bar{z}^k})$ of $H^*_{\bar{z}^k}$ contains all polynomials in w and $\bar{w}$. Moreover, if f is a holomorphic homogeneous polynomial of degree d, then

$$H^*_{\bar{z}^k}H_{\bar{z}^k}f = \frac{s_{d+|l|}}{s_{d+|l|-|k|}}\frac{\Gamma(n+d+|l|-|k|)}{\Gamma(n+d+|l|)}\frac{\partial^{|k|}}{\partial z^k}(z^l f) - \frac{s_d}{s_{d-|k|}}\frac{\Gamma(d+n-|k|)}{\Gamma(d+n)}z^l\frac{\partial^{|k|}}{\partial z^k}f$$

In particular, $H^*_{\bar{z}^k}H_{\bar{z}^k}\ f$ is a holomorphic homogeneous polynomial of degree $d+|l|-|k|$. In particular, for each a in $\mathbb{N}_0^n$ the monomial z^α is an eigenvector for the operator $H^*_{\bar{z}^k}H_{\bar{z}^k}$ and the corresponding eigenvalue $\lambda_\alpha$is given by

$$\lambda_\alpha = \frac{s_{|\alpha|+|k|}}{s_{|\alpha|}}\frac{\Gamma(n+|\alpha|)}{\Gamma(n+|\alpha|+|k|)}\frac{(\alpha+k)!}{\alpha!} - \frac{s_{|\alpha|}}{s_{|\alpha|-|k|}}\frac{\Gamma(|\alpha|+n-|k|)}{\Gamma(|\alpha|+n)}\frac{\alpha!}{(\alpha-k)!}$$

if $\alpha \ge k$ and

$$\lambda_\alpha = \frac{s_{|\alpha|+|k|}}{s_{|\alpha|}}\frac{\Gamma(n+|\alpha|)}{\Gamma(n+|\alpha|+|k|)}\frac{(\alpha+k)!}{\alpha!}$$

Otherwise.

For simplicity reasons, we introduce some notations. We set

$$f_n(t_1,\dots,t_n) := -|k|^2t^k + \sum_{j=1}^{n}k_j^2\frac{t^k}{t_j}, t\in\mathbb{R}^n \tag{5}$$

With that understanding that $K_j^2\frac{t^k}{t_j} = 0$as long as $k_j = 0$

And $\frac{t_j^{k_j}}{t_j} = t_j^{k_{j-1}} \; for \; k_j \geq 1$. we also let

$$t(\alpha) := \left(\frac{1+\alpha_1}{|\alpha|+n}, \dots, \frac{1+\alpha_n}{|\alpha|+n}\right), \qquad \alpha \in \mathbb{N}_0^n.$$

Lemma (3.1.8) [3]: The function f_n given by (5) Satisfies $f_n(t_1, \dots, t_n) \geq 0$ for all non-negative real numbers $t_1, \dots, t_n$ that satisfy $t_1 + \cdots + t_n = 1$. In particular , $f_n(t(\alpha)) \geq 0 \; for \; all \; \alpha \in \mathbb{N}_0^n$

Proof: Setting $r_j = \frac{k_j}{|k|}$, the lemma follows from the inequality

$$\sum_{j=1}^{n} \frac{r_j^2}{t_j} \geq 1 ,$$

Which holds for all $t_1, \dots, t_n, \dots, \in [0, +\infty)$ that satisfy :

$$t_1 + \cdots + t_n = r_1 + \cdots + r_n = 1 ,$$

This inequality, in turn, can be proved by induction on n.

Lemma (3.1.9) [3]: Suppose that α and k are in $\mathbb{N}_0^n$. If $n = 1$, set $\gamma_{\alpha,k} := 0$ and if $n > 1$, set

$$\gamma_{\alpha,k} := \frac{1}{n-1}\left(\frac{\Gamma(n+|\alpha|)}{\Gamma(n+|\alpha|+|k|)}\frac{(\alpha+k)!}{\alpha!} - \frac{\Gamma(|\alpha|+n-|k|)}{\Gamma(|\alpha|+n)}\frac{\alpha!}{(\alpha-k)!}\right),$$

Then $\gamma_{\alpha,k} \geq 0$, for all $\alpha \in \mathbb{N}_0^n$ that satisfy $\alpha \geq k$. In addition, if $n \geq 2$, then

$$\gamma_{\alpha,k} = \frac{1}{n-1}\frac{1}{d+n}\left(f_n(t(\alpha)) + O\left(\frac{1}{d}\right)\right),$$

for all $k, \alpha \in \mathbb{N}_0^n$, satisfying $\alpha \geq k$, where $d := |\alpha|$.

Proof: We consider the particular case of the constant Stieltjes moment sequence $s_d = 1, d \in \mathbb{N}_0$ represented by the Dirac measure $\mu = \delta_1$. if $\alpha \in \mathbb{N}_0^n$, then $(n-1)_{\gamma_{\alpha,k}}$, is the eigenvalue of $H_{\bar{z}^k}^* H_{\bar{z}^k}$ corresponding to the eigenvector z^α. Applying the previous lemma we sec that $(n-1)_{\gamma_{\alpha,k}} \geq 0$ and hence the first part of the lemma follows. Next, we prove the second part of lemma. From the property of the Gamma function.

$$\frac{\Gamma(x+y)}{\Gamma(x+z)} = x^{y-z}\left(1+\frac{(y-z)(y+z-1)}{2x}+O\left(\frac{1}{x^2}\right)\right) \; as\; x \to +\infty\,,$$

where y and z are real numbers, we get

$$\frac{\Gamma(d+n)}{\Gamma(d+n+|k|)} = (d+n)^{-|k|}\left(1-\frac{|k|(|k|-1)}{2(d+n)}+O\left(\frac{1}{d^2}\right)\right) \; as\; d \to +\infty,$$

$$\frac{\Gamma(d+n-|k|)}{\Gamma(d+n)} = (d+n)^{-|k|}\left(1+\frac{|k|(|k|+1)}{2(d+n)}+O\left(\frac{1}{d^2}\right)\right) \; as\; d \to +\infty,$$

By the proof of a Lemma, we have, when $\alpha \geq k$,

$$\frac{(\alpha+k)!}{\alpha!} = \prod_{j=1}^{n}\left(1+\alpha_j\right)^{k_j} + \sum_{j=1}^{n}\frac{k_j\left(k_j-1\right)}{2}\left(1+\alpha_j\right)^{k_j-1}\prod_{l\neq j}(1+\alpha_l)^{k_l} + q(\alpha) = (d+n)^{|k|}\left[t^k + \frac{h(t)-g(t)}{d+n} + \frac{q(\alpha)}{(d+n)^{|k|}}\right].$$

Where

$$h(t) \coloneqq \sum_{j=1}^{n}\frac{k_j^2 t^k}{2t_j},\;\; g(t) \coloneqq \sum_{j=1}^{n}\frac{k_j\; t^k}{2t_j}.$$

Using a similar argument, we also have

$$\frac{\alpha!}{(\alpha-k)!} = \prod_{j=1}^{n}\left(1+\alpha_j\right)^{k_j} - \sum_{j=1}^{n}\frac{k_j\left(k_j-1\right)}{2}\left(1+\alpha_j\right)^{k_j-1}\prod_{l\neq j}(1+\alpha_l)^{k_l} + r(\alpha) == (d+n)^{|k|}\left[t^k - \frac{h(t)+g(t)}{d+n} + \frac{r(\alpha)}{(d+n)^{|k|}}\right]$$

Where q and r are polynomials of degree at most $|k| - 2$.

$$\frac{\Gamma(d+n)}{\Gamma(d+n+|k|)}\frac{(\alpha+k)!}{\alpha!}-\frac{\Gamma(d+n-|k|)}{\Gamma(d+n)}\frac{\alpha!}{(\alpha-k)!}$$
$$=(d+n)^{-|k|}\frac{(\alpha+k)!}{\alpha!}\left(1-\frac{|k|(|k|-1)}{2(d+n)}+O\left(\frac{1}{d^2}\right)\right)$$
$$-(d+n)^{-|k|}\frac{\alpha!}{(\alpha-k)!}\left(1+\frac{|k|(|k|+1)}{2(d+n)}+O\left(\frac{1}{d^2}\right)\right)$$
$$=\left(1-\frac{|k|(|k|-1)}{2(d+n)}+O\left(\frac{1}{d^2}\right)\right)\left[t^k+\frac{h(t)-g(t)}{d+n}+O\left(\frac{1}{d^2}\right)\right]$$
$$-\left(1+\frac{|k|(|k|+1)}{2(d+n)}+O\left(\frac{1}{d^2}\right)\right)\left[t^k-\frac{h(t)+g(t)}{d+n}+O\left(\frac{1}{d^2}\right)\right]$$
$$=\frac{1}{d+n}\left(-|k|^2t^k+2h(t)+O\left(\frac{1}{d^2}\right)\right)$$

The lemma now follows since $f_n(t)=-|k|^2t^k+\ 2h(t)$.

Lemma (3.1.10) [3]: If $\alpha\in\mathbb{N}_0^n$, then the eigenvalue λ_α of the operator $H^*_{\bar{z}^k}H_{\bar{z}^k}$ satisfies

$$\lambda_\alpha=\left(\frac{s_{|\alpha|+|k|}}{s_{|\alpha|}}-\frac{s_{|\alpha|}}{s_{|\alpha|-|k|}}\right)\left(\left(t(\alpha)\right)^k+O\left(\frac{1}{d}\right)\right)+\frac{n-1}{d+n}\frac{s_{|\alpha|}}{s_{|\alpha|-|k|}}\left(f_n(t(\alpha))+O\left(\frac{1}{d}\right)\right).$$

If $\alpha\ \geq k\ and$

$$\lambda_\alpha=\frac{s_{|\alpha|+|k|}}{s_{|\alpha|}}O\left(\frac{1}{d}\right),$$

Otherwise.

Proof: By Lemma (3.1.7) and the definition of $\gamma_{\alpha,k}$, we have

$$\lambda_\alpha=\left(\frac{s_{|\alpha|+|k|}}{s_{|\alpha|}}-\frac{s_{|\alpha|}}{s_{|\alpha|-|k|}}\right)\frac{\Gamma(d+n)}{\Gamma(d+n+|k|)}\frac{(\alpha+k)!}{\alpha!}+(n-1)\gamma_{n,k}\frac{s_{|\alpha|}}{s_{|\alpha|-|k|}}$$

By the estimates in the proof of Lemma (3.1.10) we deduce that

$$\frac{\Gamma(d+n)}{\Gamma(d+n+|k|)}\frac{(\alpha+k)!}{\alpha!}=\left(t^{t(\alpha)}+O\left(\frac{1}{d}\right)\right).$$

The latter equation, combined with Lemma (3.1.9), completes the proof of the first part of the lemma. To prove the remaining part of the lemma, suppose that for some $j_0=1\ldots.,n$ we have that $k_{j_0}\geq 1$ and $\alpha_{j_0}<k_{j_0}$.Then by Lemma (3.1.7) we have

$$\lambda_\alpha = \frac{s_{|\alpha|+|k|}}{s_{|\alpha|}} \frac{\Gamma(d+n)}{\Gamma(d+n+|k|)} \frac{(\alpha+k)!}{\alpha!}$$

Set

$\alpha' = (\alpha_1, \dots \alpha_{j_{0-1}}, 0, \alpha_{j_{0+1}}, \dots \dots, \alpha_n$ *and* $k' = (k_1, \dots, k_{j_{0-1}}, 0, k_{j_{0+1}}, \dots, k_n)$

Arguing like in a Lemma we get

$$\frac{(\alpha+k)!}{\alpha!} \le (2k_{J_0})! \left[\prod_{j=1, j\neq j_0}^{n} (1+\alpha_j)^{k_j} \right.$$
$$\left. + \sum_{j=1, j\neq j_0}^{n} \frac{k_j(k_j-1)}{2} (1+\alpha_j)^{k_j-1} \prod_{i\neq j, j_0} (1+\alpha_j)^{k_j} \right] + q(\alpha').$$

Where $q(\alpha')$ is a polynomial of degree at most $|k'|-2$ this inequality, combined with the estimate

$$\frac{\Gamma(d+n)}{\Gamma(d+n+|k|)} = O\left(\frac{1}{(d+n)^{|k|}}\right).$$

gives the second part of the lemma.

Theorem (3.1.11) [3]: *Fix* $k \in \mathbb{N}_0^n$ and consider the Hankel operator $H_{\bar{z}^k}$ from the dense subspace of $\mathcal{A}^2(s)$ consisting of holomorphic polynomials into $L^2(\mu_n)$. Then:

(i) $H_{\bar{z}^k}$ is bounded if and only if

$$\sup_{d\in\mathbb{N}_0} \left(\frac{s_{d+2|k|}}{s_{d+|k|}} - \frac{s_{d+|k|}}{s_d} + \frac{n-1}{d} \frac{s_{d+|k|}}{s_d} \right) < +\infty \tag{6}$$

(ii) $H_{\bar{z}^k}$ Compact if and only if

$$\lim_{d\to+\infty} \left(\frac{s_{d+2|k|}}{s_{d+|k|}} - \frac{s_{d+|k|}}{s_d} + \frac{n-1}{d} \frac{s_{d+|k|}}{s_d} \right) = 0 \tag{7}$$

Proof: We consider the sequence $(\lambda_\alpha)_\alpha$ of eigenvalues of the $H_{\bar{z}^k}^* H_{\bar{z}^k}$. Let $\sum_n$ be the simplex consisting of those $t = (t_1, \dots, t_n) \in \mathbb{R}^n$ such that

$t_j \ge 0$ and $t_1 + \dots + t_n = 1$. Since the set

$$\bigcup_{d\in\mathbb{N}_0}\left\{\left(\frac{\alpha_1+1}{d+n},\dots\dots,\frac{\alpha_n+1}{d+n}\right),|\alpha|=d\right\}$$

is dense in $\sum_n$, it follows that

$$\sup_{|\alpha|=d} f_n\left(\frac{\alpha_1+1}{d+n},\dots\dots,\frac{\alpha_n+1}{d+n}\right)\approx\sup_{t\in\Sigma_n} f_n(t)$$

and

$$\sup_{|\alpha|=d} t(\alpha)^k\approx\sup_{t\in\Sigma_n} t^k.$$

As d tends to $+\infty$, these estimates, combined with Lemma (3.1.10), implies that $(\lambda_\alpha)_\alpha$,is bounded if and only if (6) holds and $\lim_{|\alpha|\to+\infty}\lambda_\alpha=0$ if and only if (7) holds. The theorem now follows since $H_{\bar{z}^k}$ is bounded if and only if $H^*_{\bar{z}^k}H_{\bar{z}^k}$ is bounded and compactness of $H_{\bar{z}^k}$is equivalent to that of $H^*_{\bar{z}^k}H_{\bar{z}^k}$.

Next, let $p>0$. we shall study the membership of the operator $H_{\bar{z}^k}$ in a Schatten class $\mathcal{S}_P$.

Recall that $H_{\bar{z}^k}$ is in $\mathcal{S}_P$if and only if $H^*_{\bar{z}^k}H_{\bar{z}^k}$ is in $\mathcal{S}_{\frac{p}{2}}$, that is to say the series $\sum\lambda_\alpha^{\frac{p}{2}}$ is convergent.

Let d be an integer. We shall estimate the sum$S_d=\sum_{|\alpha|=d}\lambda_\alpha^P$ when $d\to+\infty$. The calculations above lead to study the cases $\alpha\geq k$ and its opposite separately. Let $\mathcal{B}_d:=\{\alpha\in\mathbb{N}_0^n,|\alpha|=d\}$. We partition $\mathcal{B}_d=\mathcal{B}'_d\cup\mathcal{B}''_d$, where $\mathcal{B}'_d=\{\alpha\in\mathcal{B}_d,\alpha\geq k\}$ and $\mathcal{B}''_d=\mathcal{B}_d\setminus\mathcal{B}'_d$ .Thus S_d can be written in the form $S_d=S'_d+S''_d$ where $S'_d=\sum_{\alpha\in\mathcal{B}'_d}\lambda_\alpha^P$ and $S''_d=\sum_{\alpha\in\mathcal{B}''_d}\lambda_\alpha^P$. We shall use the following lemmas .

Lemma (3.1.12) [3]: If $n\geq 2$, then we have the estimates

$$\#\mathcal{B}_d\approx\#\mathcal{B}'_d\approx\frac{d^{n-1}}{(n-1)!}\,and\,\#\mathcal{B}''_d\approx d^{n-2}\,as\,d\to+\infty.$$

Lemma (3.1.13) [3]: Suppose that $n\geq 2$ and g is a continuous on $\mathbb{R}^{n-1}$. Consider the open set $D:=\{(t_1,\dots,t_{n-1})\in\mathbb{R}_+^{n-1},\sum_{j=1}^{n-1}t_j<1\}$. For a multi-index $\beta=(\beta_1,\dots,\beta_{n-1})\ in\ \mathbb{N}_0^{n-1}$, set

$$C_{\beta,d} := \left(\frac{\beta_1 + 1}{d} \dots \dots \frac{\beta_{n-1} + 1}{d}\right),$$

$$\mathbb{J}_d := \left\{\beta \in \mathbb{N}_0^{n-1} : \prod_{j=1}^{n-1} \left[\frac{\beta_j}{d}, \frac{\beta_j + 1}{d}\right] \subset D\right\}.$$

The $\lim_{d\to+\infty} \frac{1}{d^{n-1}} \sum_{\beta\in\mathbb{J}_d} g(c_{\beta.d}) = \int_D g(t)\, dt.$

The above results enable us to estimate S_d when $d = |a| \to +\infty$.

Lemma (3.1.14) [3]: If $p > 0$, then

$$S_d \approx d^{n-1}\left(\frac{s_{d+|k|}}{s_d} - \frac{s_d}{s_{d-|k|}}\right)^P + (n-1)d^{n-1-P}\left(\frac{s_d}{s_{d-|k|}}\right)^P.$$

Proof: Recall that $S_d = S_d' + S_d''$ where $S_d' = \sum_{\alpha\in\mathcal{B}_d'} \lambda_\alpha^p$ and $S_d'' = \sum_{\alpha\in\mathcal{B}_d''} \lambda_\alpha^p$. First we shall estimate S_d' By Lemma (3.1.10), we know that this sum has the following expansion when $d = |a| \to +\infty$

$$S_d' \approx \left(\frac{s_{d+|k|}}{s_d} - \frac{s_d}{s_{d-|k|}}\right)^P \sum_{\alpha\in\mathcal{B}_d'} \left(\left(t(\alpha)\right)^k + O\left(\frac{1}{d}\right)\right)^P$$
$$+ \left(\frac{n-1}{d+n}\frac{s_d}{s_{d-|k|}}\right)^P \sum_{\alpha\in\mathcal{B}_d'} \left(f_n\left(t(\alpha)\right) + O\left(\frac{1}{d}\right)\right)^P.$$

Using the properties of the function $x \to x^P$ and Lemma (3.1.14) we see that there exists a constant $M > 0$, such that

$$\sum_{\alpha\in\mathcal{B}_d'} \left(\left(t(\alpha)\right)^k + O\left(\frac{1}{d}\right)\right)^P$$
$$\approx d^{n-1} \int_D t_1^{Pk_1} \dots t_{n-1}^{Pk_{n-1}} \left(1 - \sum_{j=1}^{n} t_j\right)^{Pk_n} dt \sum_{\alpha\in\mathcal{B}_d'} \left(f_n\left(t(\alpha)\right) + O\left(\frac{1}{d}\right)\right)^P$$
$$\approx d^{n-1} \int_D \left(f_n\left(t_1, \dots, t_{n-1}, 1 - \sum_{j=1}^{n} t_j\right)\right)^P dt$$

Therefore,

$$S_d' \approx d^{n-1}\left(\frac{s_{d+|k|}}{s_{|\alpha|}} - \frac{s_d}{s_{d-|k|}}\right)^P + d^{n-1}\left(\frac{n-1}{d+n}\frac{s_d}{s_{d-|k|}}\right)^P$$
$$\approx d^{n-1}\left(\frac{s_{d+|k|}}{s_{|\alpha|}} - \frac{s_d}{s_{d-|k|}}\right)^P + (n-1)d^{n-1-P}\left(\frac{s_d}{s_{d-|k|}}\right)^P.$$

as $d \to +\infty$.

To estimate S_d'' we observe that if $n = 1$, then

$$S_d'' \le S_{|k|}''$$

On the other hand, if $n \geq 2$, by Lemma (3.1.10) we see that for $\alpha \in \mathcal{B}_d''$ we have

$$\lambda_\alpha^P = (n-1)\left(\frac{s_{d+|k|}}{s_d}\right)^P O(d^{-P}).$$

Since $\#\mathcal{B}_d'' \approx d^{n-2}$ we see that $S_d'' = O\left(\frac{S_{d+|k|}'}{d^P}\right)$ the lemma follows from the relation $S_d = S_d' + S_d''$.

We then characterize the Schatten class membership of $H_{\bar{z}^k}$.

Theorem (3.1.15) [3]: Let $k \in \mathbb{N}_0^n$. Then the Hankel operator $H_{\bar{z}^k}$, is in the Schattn $\mathcal{S}^P(\mathcal{A}^2(s), L^2(\mu_n))$ if and only if

$$d^{n-1}\left(\frac{s_{d+2|k|}}{s_{d+|k|}} - \frac{s_{d+|k|}}{s_d}\right)^{\frac{P}{2}} + (n-1)d^{n-1-\frac{P}{2}}\left(\frac{s_{d+|k|}}{s_d}\right)^{\frac{P}{2}} < +\infty, \qquad (7)$$

Proof: We use that the operator $H_{\bar{z}^k}$ is in the Schatten class $\mathcal{S}^P(\mathcal{A}^2(s), L^2(\mu_n))$ if and only if $H_{\bar{z}^k}^* H_{\bar{z}^k}$ is in $\mathcal{S}^{\frac{P}{2}}(\mathcal{A}^2(s))$. Therefore, the theorem follows from Lemma (3.1.15).

Lemma (3.1.16) [3]: If U is a unitary transformation in $\mathbb{C}^n$, the operator $U f := f \, O \, U$ is a unitary isometry, form $L^2(\mu_n)$ onto itself and from $\mathcal{A}^2(s)$ onto itself. Moreover the following assertions hold.

(i) If $f \in \mathcal{M}(s)$, then $Uf \in \mathcal{M}(s)$ and $\|Uf\|_{\mathcal{M}(s)} = \|f\|_{\mathcal{M}(s)}$.
(ii) If $f \in \mathcal{M}_\infty(s)$, then $Uf \in \mathcal{M}_\infty(s)$.
(iii) If $f \in \mathcal{M}_P(s)$, then $Uf \in \mathcal{M}_P(s)$ and

$$\|Uf\|_{\mathcal{M}_P(s)} = \|f\|_{\mathcal{M}_P(s)}$$

Proof: Let U be a unitary transformation in $\mathbb{C}^n$ and denote U^* its adjoint, which is also its inverse. It is clear that the operator U is a unitary isometry from $L^2(\mu_n)$onto itself and from $\mathcal{A}^2(s)$onto itself. Let f be in $\mathcal{M}(s)$. If g is a holomorphic polynomial, then by a change of variable we see that

$$H_{\overline{Uf}}(g)(z) = \int_{\mathbb{C}^n} K_s(Uz,w)g(U^*w)\left[\overline{Uf}(z) - \bar{f}(w)\right]d\mu_m(w)$$

$$= \int_{\mathbb{C}^n} K_s(Uz,w)(U^*g)(w)\left[\bar{f}(Uz) - \bar{f}(w)\right]d\mu_m(w)$$

$$= H_{\bar{f}}(U^*g)(Uz) = \left(UH_{\bar{f}}U^*\right)(g)(z).$$

Therefore,

$$H_{\overline{Uf}} = U\, H_{\bar{f}}\, U^* \tag{8}$$

and thus $\|H_{\overline{Uf}}\| = \|\, H_{\bar{f}}\|$, showing that

$$\|Uf\|_{\mathcal{M}_P(s)} = \|f\|_{\mathcal{M}_P(s)}$$

This proves part (i) of the lemma. The proof of parts (ii) and (iii) of the lemma are similar.

Let $\mathbb{T}^n := \{\zeta = (\,\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n \colon |\zeta_j| = 1, j = 1, \dots n\}$ and for $\zeta = (\,\zeta_1, \dots, \zeta_n) \in \mathbb{T}^n$ let U_ζ be the unitary linear transformation in $\mathbb{C}^n$ defined by $U_\zeta(z) = (\zeta_1 z_1, \dots . \zeta_n z_n)$,for all z = $(z_1\,, \dots\; z_n) \in \mathbb{C}^n$.

We have the following.

Lemma (3.1.17) [3]: If $f \in \mathcal{T}(s)$ and $g \in \mathcal{A}^2(s)$, the mappings $\zeta \to U_\zeta g$ and $\zeta \mapsto H_{\overline{U}_\zeta f}(g)$ are continuous from $\mathbb{T}^n$ to $L^2(\mu_n)$.

Proof: Let $g \in \mathcal{A}^2(\mu_n)$ and write $g(z) = \sum_{\alpha \in \mathbb{N}_0^n} a_k z^a$ If $\zeta, \eta \in \mathbb{T}^n$, then

$$\left\|(U_\xi - U_\eta)g\right\|_{L^2(\mu_n)}^2 = \left\|goU_\xi - goU_\eta\right\|_{L^2(\mu_n)}^2$$

$$= \sum_{\alpha \in \mathbb{N}_0^n} |a_\alpha|^2 \left\|(U_\xi z)^\alpha - (U_\eta z)^\alpha\right\|_{L^2(\mu_n)}^2$$

$$= \sum_{\alpha \in \mathbb{N}_0^n} |a_\alpha|^2 C_\alpha |\xi^\alpha - \eta^\alpha|^2.$$

Where

$$C_\alpha = \int_{\mathbb{C}^n} |z^\alpha|^2 \, d\mu_n(z), \alpha \in \mathbb{N}_0^n.$$

Since

$$\sum_{\alpha \in \mathbb{N}_0^n} |a_\alpha|^2 C_\alpha < +\infty \quad and \ |\xi^\alpha - \eta^\alpha|^2 \le 4\,,$$

The dominated convergence theorem leads to

$$\lim_{\xi \to \eta} \left\| \left(U_\xi - U_\eta\right) g \right\|_{L^2(\mu_n)} = 0,$$

Showing that the mapping $\zeta \to U_\zeta g$ is continuous from $\mathbb{T}^n$ to $L^2(\mu_n)$. this, combined with the fact that U_ζ is unitary and the equalities

$$\begin{aligned} H_{\overline{U_\xi f}} - H_{\overline{U_\eta f}} &= U_\xi H_{\bar f} U_{\bar\xi} - U_\eta H_{\bar f} U_{\bar\eta} \\ &= U_\xi H_{\bar f} U_{\bar\xi} - U_\xi H_{\bar f} U_{\bar\eta} + U_\xi H_{\bar f} U_{\bar\eta} - U_\eta H_{\bar f} U_{\bar\eta} \\ &= U_\eta H_{\bar f}\left(U_{\bar\xi} - U_{\bar\eta}\right) + \left(U_\xi - U_\eta\right) H_{\bar f} U_{\bar\eta}\,, \end{aligned}$$

shows that the mapping $\zeta \longmapsto H_{\overline{Uf}}\,(g)$ is also continuous from $\mathbb{T}^n$ to $L^2(\mu_n)$.

Lemma (3.1.18) [3]: Assume that $f \in \mathcal{T}(s)$.

(i) monomial $z^k is$ in $\mathcal{M}(s)$.

(ii) if $f \in \mathcal{M}_\infty(s)$, then for any multi-index $k \in Spec(f)$, the monomial $z^k is$ in $\mathcal{M}_\infty(s)$.

(iii) if $p > 0$ and $f \in \mathcal{M}_P(s)$, then for any multi-index $k \in Spec(f)$, the monomial $z^k is$ in $\mathcal{M}_P(s)$.

Proof: To prove (i), suppose that $f \in \mathcal{M}(s)$ and write $f(z) = \sum_{k \in \mathbb{N}_0^n} a_k z^k$. By the Cauchy formula we have

$$a_k z^k = \int_{\mathbb{T}^n} f\left(U_\xi z\right) \bar\xi^k \, dm_n(\xi),$$

where $dm_n(\xi)$ is the normalized Lebesgue measure on $\mathbb{T}.^n$ if g is a holomorphic polynomial and $h \in L^2(\mu_n)$, an application of Fubini's theorem leads to

$$\int_{\mathbb{T}^n} \langle H_{\overline{U_\xi f}}(g), h\rangle \,\bar{\xi}^k \,dm_n(\xi) = \langle H_{\overline{a_k z^k}}(g), h\rangle \tag{9}$$

By Lemmas (3.1.18), (3.1.19) we see that

$$\left\| H_{\overline{a_k z^k}}(g) \right\|_{L^2(\mu_n)} \leq \int_{\mathbb{T}^n} \left\| H_{\overline{U_\xi f}}(g) \right\| dm_n(\xi) \tag{10}$$

Since $\| H_{\overline{Uf}}\,(g) \leq \| H_{\bar{f}} \| \| g \|_{L^2(\mu_n)}$ for all ζ in $\mathbb{T}^n$, it follows that $H_{\overline{a_k \zeta^k}}$ is bounded and $a_k z^k$ is $\mathcal{M}(s)$.Therefore, $z^k \in \mathcal{M}(s)$ as long as $\frac{\partial^k f}{\partial_{\zeta^k}}(0) \neq 0$. This proves part (i) of the lemma. Suppose now that $f \in \mathcal{M}_\infty(s)$. And let (g_q) be a sequence in $\mathcal{A}^2(s)$.which converges weakly to 0.

$$\lim_{q\to+\infty} \left\| H_{\overline{U_\xi f}}(g_q) \right\|_{L^2(\mu_n)} = 0,\ for\ all\ \xi \in \mathbb{T}^n\ ,$$

so that by (10) and the dominated convergence theorem we see that

$$\lim_{q\to+\infty} \left\| H_{\overline{a_k z^k}}(g_q) \right\|_{L^2(\mu_n)} = 0.$$

and hence $z^k \in \mathcal{M}_\infty(s)$. whenever $\frac{\partial^k f}{\partial_{\zeta^k}}\,(0) \neq 0$. Therefore part (ii) of the lemma holds. To establish dzk the remaining part of the lemma, we recall that if T is a compact operator from $\mathcal{A}^2(s)$ to $L^2(\mu_n)$ then its singular numbers $\mathcal{V}_q(T)$, $q \in \mathbb{N}_0$, are given by

$$\mathcal{V}_q(T) \coloneqq \inf_{A\in\mathcal{R}_q} \|T - A\|\ ,$$

where R_q is the space of all operators from $\mathcal{A}^2(s)$ to $L^2(\mu_n)$ with finite rank at most q. Assume that $f \in \mathcal{M}_P(s)$ .Then the sequence $\mathcal{V}_q(H_{\bar{f}})_q$ is in l^P. Moreover, there are an orthonormal system $(u_q)_q$ in $\mathcal{A}^2(s)$ and an orthonormal system $(\mathcal{V}_q)_q$ in $L^2(\mu_n)$ such that

$$H_{\bar{f}} = \sum_{q=0}^{+\infty} \mathcal{V}_q(H_{\bar{f}})\langle\cdot, u_q\rangle \mathcal{V}_q\ ,$$

Where the series converges in the operator norm. If q is a positive integer, consider the operator with rank at most q given by.

$$A_q := \sum_{j=0}^{q-1} \mathcal{V}_j(H_{\bar{f}})\langle \cdot, u_{kj}\rangle \mathcal{V}_{kj} ,$$

Where for each integer $j = 0, \dots , q-1$ and $z \in \mathbb{C}^n$ the functions u_{kj} and v_{kj} are defined by

$$u_{kj}(z) := \int_{\mathbb{T}^n} (U_\xi u_j)(z)\bar{\xi}^k \, dm_n(\xi) \ and \ \mathcal{V}_{kj}(z) := \int_{\mathbb{T}^n} (U_\xi h_j)(z)\bar{\xi}^k \, dm_n(\xi)$$

The dominated convergence theorem, combined with (9) and (8), yields

$$\langle \left(H_{\overline{a_k z^k}} - A_q\right)(g), h\rangle = \int_{\mathbb{T}^n} \sum_{j=q}^{+\infty} \mathcal{V}_j(H_{\bar{f}})\langle U_{\bar{\xi}} g, u_j\rangle \langle \mathcal{V}_j U_{\bar{\xi}} h\rangle \bar{\xi}^k \, dm_n(\xi) .$$

Due to the facts that the sequence $\left(\mathcal{V}_j(H_{\bar{f}})\right)_j$ is non-increasing and the systems $(u_j)_j$ and $(v_j)_j$are orthonormal it follows that

$$\left|\sum_{j=q}^{+\infty} \mathcal{V}_q(H_{\bar{f}})\langle U_{\bar{\xi}} g, u_j\rangle \langle \mathcal{V}_j U_{\bar{\xi}} h\rangle\right| \le \mathcal{V}_q(H_{\bar{f}})\|g\|_{\mathcal{A}^2(s)}\|h\|_{L^2(\mu_n)}$$

for all $\zeta \in \mathbb{T}^n$, $g \in \mathcal{A}^2(s)$ and $h \in L^2(\mu_n)$. Hence

$$\left\|H_{\overline{a_k z^k}} - A_q\right\| \le \mathcal{V}_q(H_{\bar{f}}) .$$

This implies that

$$\mathcal{V}_q\left(H_{\overline{a_k z^k}}\right) \le \mathcal{V}_q(H_{\bar{f}}).$$

Showing that $a_k z^k \in \mathcal{M}_P(s)$, consequently, $z^k \in \mathcal{M}_P(s)$ if and only if $\frac{\partial^k f}{\partial_{z^k}}(0)$, the proof of the lemma is now complete.

Lemma (3.1.19) [3]: Suppose that $R_s = +\infty$ and the sequence s satisfies (3). Then the function $w \to g(w)\, K_s(z,w)$ is in $L^2(\mu_n)$ for all holomorphic polynomials g and $z \in \mathbb{C}^n$.

Proof: We first observe that

$$K_s(z,w)=\frac{1}{(n-1)!}\sum_{0}^{+\infty}\frac{(d+n-1)!}{d!_{s_d}}\langle z,w\rangle^d, z\in\mathbb{C}^n, w\in\mathbb{C}^n\ ,$$

Therefore, for any $\alpha\in\mathbb{N}_0^n$ and $z\in\mathbb{C}^n$.

$$\int_{\mathbb{C}^n}|w^\alpha K_s(z,w)|^2dm_n(w)$$

$$=\left(\frac{1}{(n-1)!}\right)^2\sum_{0}^{+\infty}\left(\frac{(d+n-1)!}{d!_{s_d}}\right)^2\int_{\mathbb{C}^n}|w^\alpha\langle z,w\rangle^d|^2dm_n(w)$$

$$\le\left(\frac{1}{(n-1)!}\right)^2\sum_{0}^{+\infty}\left(\frac{(d+n-1)!}{d!_{s_d}}\right)^2|z|^{2d}\int_0^R t^{|\alpha|+d}\,d\mu(t)$$

$$=\left(\frac{1}{(n-1)!}\right)^2\sum_{0}^{+\infty}\left(\frac{(d+n-1)!}{d!_{s_d}}\right)^2 s_{|\alpha|+d|z|^{2d}}$$

Now assumption (3) ensures that the latter series converges for all $z\in\mathbb{C}^n$.

Lemma (3.1.20) [3]: Assume that s satisfies (2). Then the spaces $\mathcal{M}(s)$ and $\mathcal{M}^2(s)$, p $\ge$ 1 are Banach spaces and $\mathcal{M}_P\ (s)$, $0<p<1$, i.s a quasi-Banach space.

Proof: We prove the lemma for $\mathcal{M}(s)$. Let $(f_q)_{q\in\mathbb{N}_0}$ be a Cauchy sequence in $\mathcal{M}(s)$. without loss of generality we may assume that

$f_q(0)=0$ For all n. the sequence $(H_{\bar{f}_q})_{q\in\mathbb{N}_0}$ is a Cauchy sequence of bounded operators on $\mathcal{A}^2(s)$.Therefore, there is an operator T in $\mathcal{A}^2(s)$ such that $(H_{\bar{f}_q})_{q\in\mathbb{N}_0}$ converges to T in the norm operator. Let $f:=\overline{T(1)}$ be the conjugate of the image $T(1)$ of the constant function 1 under T since $H_{\bar{f}_q}(1)=\bar{f}_q$, it follows that

$$\|f_q-f\|_{L^2(\mu_n)}=\|\bar{f}_q-T(1)\|_{L^2(\mu_n)}=\left\|H_{\bar{f}_q}(1)-T(1)\right\|_{L^2(\mu_n)}\le\left\|H_{\bar{f}_q}-T\right\|\|1\|_{L^2(\mu_n)}$$

Showing that

$$\lim_{q\to\infty}\|f_q-f\|_{L^2(\mu_n)}=0 \qquad (11)$$

Thus $f\in\mathcal{A}^2(s)$, we shall show that the Hankel operator $H_{\bar{f}}$ with symbol $\bar{f}$ is bounded. We shall prove that $f\in\mathcal{T}(s)$ and $H_{\bar{f}}$ coincides with T on holomorphic polynomials. Let g be a holomorphic polynomial. We first observe by Lemma (3.1.19) that for all $z \in \mathbb{C}^n$. we have :

$$\left|P_s\left((\bar{f}-\bar{f}_q)g\right)(z)\right| \leq \|f_q - f\|_{L^2(\mu_n)} \|gK_s(z,\cdot)\|_{L^2(\mu_n)}$$

So that (11) we see that $\lim_{q\to+\infty} p_s((\bar{f} - \bar{f}_q)(g)(z) = 0$. Since again by (11) we have that $\lim_{q\to+\infty}((\bar{f} - \bar{f}_q)(g)(z) = 0$ .it follows that

$$\lim_{q\to+\infty}(H_{\overline{f_a}} - H_{\bar{f}})(g)(z) = 0.$$

This proves that $Tg = H_{\bar{f}}(g)$ and hence $f \in \mathcal{T}(s)$ and $T = H_{\bar{f}}$. Therefore $\mathcal{M}(s)$ is a Banach space. The proof of that $\mathcal{M}_P(s)$ is a Banach space for $p \geq 1$, and a quasi-Banach space for $0 < p < 1$ is similar.

Theorem (3.1.21) [3]: Suppose that $n = 1$ and f is a non constant holomorphic function in $f \in \mathcal{T}(s)$. Then $H_{\bar{f}}$ is in. the Hilbert-Schmidt class $S^P(\mathcal{A}^2(s), L^2(\mu_n)$if and only if s is exponentially bounded and f is in the classical Dirichlet space $\mathcal{D}(\Omega_s)$. In addition, the trace $Tr(H_{\bar{f}}^* H_{\bar{f}})$ of $H_{\bar{f}}^* H_{\bar{f}}$ is given by

$$Tr(H_{\bar{f}}^* H_{\bar{f}}) = \frac{1}{\pi}\int_{\Omega_s} |f'(z)|^2 dA(z) = \int_{\Omega_s} |f(z) - f(w)|^2 |K_s(z,w)|^2 dA(z) dA(w).$$

Where dA(z) is the Lebesgue measure in $\mathbb{C}$.

Proof: Suppose that $n = 1$ and f is as in the hypothesis of Theorem (3.1.21). A straight-forward calculation appealing to Lemma (3.1.7) shows that for all non-negative integers j, k we have :

$$Tr(H_{\bar{z}^k}^* H_{\bar{z}^j}) = 0 \text{ as long as } j \neq k$$

and

$$Tr(H_{\bar{z}^k}^* H_{\bar{z}^k}) = \sum_{d=k}^{+\infty}\left(\frac{s_{d+2k}}{s_{d+|k|}} - \frac{s_{d+k}}{s_d}\right) + \sum_{d=0}^{k-1}\frac{s_{d+k}}{s_d} = kR_S^2.$$

Writing $f = \sum_{k\in\mathbb{N}} a_k z^k$ yields

$$Tr(H_{\bar{f}}^* H_{\bar{f}}) = R_S^2 \sum_{k\in\mathbb{N}} k|a_k|^2 = \frac{1}{\pi}\int_{\Omega_s} |f'(z)|^2\, dA(z).$$

This proves the first equality of the theorem. Next we prove the second equality. Writing$K_s(z,w) \sum_{k=0}^{\infty} f_k\,(z)\bar{f}_k(w)$, where (f_k) is an orthonormal basis of $\mathcal{A}^2(s)$, we observe by a standard argument that for any positive operator T on $\mathcal{A}^2(s)$.we have

$$Tr(T) = \sum_{k=0}^{\infty} \langle Tf_k, f_k\rangle \mathcal{A}^2(s) = \int_{\Omega_s} \langle \mathrm{TK_s}(\cdot,\mathrm{z}), \mathrm{K_s}(\cdot,\mathrm{z})\rangle_{\mathcal{A}^2(s)}\, dA(z).$$

Applying this equality to $T = H_{\bar{f}}^*\, H_{\bar{f}}$ and using the reproducing property of the kernel K_s implies that

$$Tr\left(H_{\bar{f}}^* H_{\bar{f}}\right) = \int_{\Omega_s} |f(z) - f(w)|^2 |K(z,w)|^2\, dA(z)$$

and hence completes the proof of the theorem.

Chapter 4

Membership of Hankel Operators in a Class of Lorentz Ideals

We will show that the Lorentz ideal C_p^- is the collection of operators A satisfying thecondition $\|A\|_p^- = \sum_{j=1}^{\infty} j^{-\frac{p-1}{p}} s_j(A) < \infty$. Now we consider Hankel operators $H_f\colon H^2(S) \to L^2(S, d\sigma) \oplus H^2(S)$, where $H^2(S)$ is the Hardy space on the unit sphere S in $\mathbb{C}^n$. Hence we characterize the membership $H_f \in C_p^-, 2n < p < \infty$.

Section (4.1): Symmetric Guge Functions with Decomposition and Modified Kernel

The study of Hankel operators has a long and rich history. We are particularly interested in one kind of Hankel operators: those on the Hardy space of the unit sphere. Let us begin by describing our basic setting.

Let S be the unit sphere$\{z\colon |z| = 1\}$ in C^n. In this chapter, the complex dimension n is always assumed to be greater than or equal to 2. Let $d\sigma$ be the standard spherical measure on S.

That is, $d\sigma$ is the positive, regular Borel measure on S with σ(S) = 1 that is invariant under the orthogonal group $O(2n)$, i.e., the group of isometries on $\mathbb{C}^n \cong R^{2n}$, which fix 0.

Recall that the Hardy space $H^2(s)$ is the norm closure in $L^2\,(S, d\sigma)$ of the collection of polynomials in the complex variables $z_1, \ldots, z_n$. As usual, we let P denote the orthogonal projection from $L^2(S, d\sigma)$ onto $H^2(s)$. The main object of study is, the Hankel operator $H_f : H^2(S) \longrightarrow L^2\,(S, d\sigma) \ominus H^2(s)$, is defined by the formula

$$H_f = (1 - P)M_f\,|H^2(S).$$

We consider symbol functions $f \in L^2\,(S, d\sigma)$. Recall that the problems of boundedness and compactness of H_f were settled .Later, we characterized the membership of H_f in the Schatten class $C_{p,}\ 2n < p < 1$. Moreover, it was shown that the membership $H_f \in C_{2n}$ implies $H_f = 0$. More recently, we characterized the membership of H_f in the ideal C_p^+,2n $< p < 1$.

We turn our attention to the membership of H_f in the Lorentz ideal C_p^-.

Before going any further, it is necessary to recall the definition of these operator ideals.

Given an operator A, we write $s_1(A), \ldots, s_j(A)$ ... for its s-numbers For each $1 < p < \infty$, the formula

$$\|A\|_P^- = \sum_{i=1}^{\infty} \frac{s_j(A)}{j^{(p-1)/p}}$$

defines a symmetric norm for operators .On any separable Hilbert space $\mathcal{H}$, the set $C_p^- = \{A \in \beta(\mathcal{H}) : \|A\|_p^- < \infty\}$ is a norm ideal .

Closely associated with the Lorentz ideals C_p^- are the ideals C_p^+ , which are defined as follows: each $1 \leq p < \infty$, the formula

$$\|A\|_p^+ = \sup_{j\geq 1} \frac{s_1(A) + S_2(A) + \cdots + S_j(A)}{1^{-1/p} + 2^{-1/p} + \cdots + j^{-1/p}}$$

also defines a symmetric norm for operators .On any separable Hilbert space $\mathcal{H}$, we have the norm ideal

$$C_p^+ = \{A \in \beta(\mathcal{H}) : \|A\|_p^+ < \infty\}$$

As we mentioned, the C_p^+ 's were the ideals of interest. These ideals will play an important supporting role.

Compared with the more familiar Schatten class

$$C_p = \{A \in \beta(\mathcal{H}) : \|A\|_P < \infty\},$$

Where

$$\|A\|_p = \{tr((A^*A)^{P/2})\}^{1/2}$$

for all $1 < p < \infty$ we have the relation $c^+_{\mathrm{p}'} \subset c^-_p \subset c_p \subset c^+_p$.
with all the inclusions being proper. This explains the + and $-$ in the notation: C^-_p is slightly smaller than C_p, whereas C^+_p, is slightly larger than C_p.

Since the membership problem $H_f \in C^+_p$, $2n < p < \infty$, was settled, the obvious next step is to determine the membership $H_f \in C^-_p$, $2n < p < \infty$, But this next step,however natural it is, turns out to be quite a challenge. We have a sizable collection of techniques from previous investigations but these techniques alone are not sufficient for the membership problem $H_f \in C^-_p$. The reason for that is that the norm $\|.\|^-_p$ is much harder to work with than $\|.\|^+_p$.
But, with considerable effort, we have finally developed the necessary additional techniques. Combining these additional techniques with techniques from previous investigations, we are able to characterize the membership $H_f \in C^-_p$, $2n < p < \infty$.

It is well known that, if $p, q \in (1, \infty)$ are such that $p^{-1} + q^{-1} = 1$, then C^+_q is the dual of C^-_p .This duality was quite useful, sometimes even crucial, in the investigations of many problems in the past. Instead, we must exploit a diferent kind of relation between the families

$$\{C^-_p : 2 < p < \infty\} \; and \; \{C^+_p : 2 < p < \infty\}$$

To state the result, it is necessary to recall the notion of symmetric gauge functions. Let $\hat{c}$ be the linear space of sequences$\{a_j\}_{j\in\mathrm{N}}$, where $a_j \in R$ and for every sequence the set $\{j \in N : a_j \neq 0\}$ is finite. A symmetric gauge function (also called symmetric norming function) is a map $\Phi: \hat{c} \to [0, \infty)$ that has the following properties:

(a) Φ is a norm on $\hat{c}$.

(b) $\Phi(\{1,0, \dots ,0, \dots\}) = 1$.

(c) $\Phi(\{a_j\}_{j\in\mathrm{N}}) = \Phi(\{|a_{\pi(j)}|\}_{j\in\mathrm{N}})$ for every bijection $\pi: N \to N$.

Each symmetric gauge function Φ gives rise to the symmetric norm

$$\|A\|_\Phi = \sup_{j\geq 1} \Phi(\{s_1(A), \dots, s_j(A), 0, \dots ,0, \dots\})$$

for operators. On any separable Hilbert space H, the set of operators

$C_{\Phi} = \{A \in \beta(\mathcal{H}): \|A\|_{\Phi} < \infty\}$. is a norm ideal .
If X is an unbounded operator, then its s-numbers are not
defined. But it will be convenient to adopt the convention that $\|X\|_{\Phi} = 1$ whenever X is an unbounded operator.

In particular, associated with the ideal $C_p^-$is the symmetric gauge function Φ_p^- , which is defined as follows. Let 1 < p < ∞. For each $\{a_j\}_{j\in N} \in \hat{c}$, define

$$\Phi_p^-\left(\{a_j\}_{j\in N}\right) = \sum_{j=1}^{\infty} \frac{|a_\pi(j)|}{j^{p-1/p}},$$

where $\pi : N \to N$ is any bijection such that $|a_\pi(j)| \geq |a_\pi(j+1)|$ for every $j \in N$, which exists because $a_j = 0$ for all but afinite number of j's. Then we have $C_p^- = C_{\Phi_p^-}$ Similarly, for each $1 \geq p < \infty$ we define the symmetric gauge function

$$\Phi_p^-\left(\{a_j\}_{j\in N}\right) = \sup_{j\geq 1} \frac{|a_\pi(1)| + |a_\pi(2)| + \cdots + |a_\pi(j)|}{1^{-1/p} + 2^{-1/p} + \cdots + j^{-1/p}} , \{a_j\}_{j\in N} \in \hat{c}$$

Where, again, $\pi : N \to N$ is any bijection such that$|a_\pi(j)| \geq |a_\pi(j+1)|$ for every $j \in N$.

Then $C_p^+ = C_{\Phi_p^+}$. Theorem we state the following if Φ is a symmetric gauge function and if $0 < \|H_f\|_{\Phi} < 1$ for some $f \in L^2(S, d\sigma)$, then $C_{\Phi} \supset C_{2n}^+$ .we need to extend the domains of definition of symmetric gauge functions beyond the space $\hat{c}$. Let Φ be any symmetric gauge function. Suppose that $\{b_j\}_{j\in N}$ is an arbitrary sequence of real numbers, i.e. suppose that the set $\{j \in \mathrm{N}, b_j \neq 0\}$ is not necessarily finite. Then we define

$$\Phi\left(\{b_j\}_{j\in N}\right) = \sup_{j\geq 1} \Phi(\{b_{1,} \ldots, b_j, o, \ldots, 0, \ldots\}) \qquad (1).$$

Thus for every bounded operator A we can simply write
$\|A\|_{\Phi} = \Phi(\{s_1(A), \ldots, s_j(A), \ldots\})$.

We also need to deal with sequences indexed by sets other than N. If W is a countable,infinite set, then we define

$$\Phi(\{b_\alpha\}_{\alpha\in W}) = \Phi\left(\{b_{\pi(j)}\}_{j\in \mathrm{N}}\right).$$

where $\pi: N \to W$ is any bijection . From the definition of symmetric gauge functions we see that the value of $\Phi(\{b_\alpha\}_{\alpha\in W})$ is independent of the choice of the bijection π.

For a finite index set F = $\{x_1, \dots, x_\ell\}$, we define

$$\Phi(\{b_x\}_{x\in F}) = \Phi(\{b_{x_1}, \dots, b_{x_\ell}, 0, \dots, 0, \dots\})$$

Let us write B for the open unit ball $\{z : |z| < 1\}$ in C^n.

Let β be the Bergman metric on B. That is,

$$\beta(z,\omega) = \frac{1}{2}\log\frac{1+|\varphi_z(\omega)|}{1-|\varphi_z(\omega)|} \quad z, w \in B$$

where φ_z is the Mobius transform of B .For each $z \in B$. and each $a > 0$,we define the corresponding -ball $D(z,a) = \{\omega \in B : \beta(z,\omega) < a\}$ $z, w \in B$.

(i) Let a be a positive number.

A subset Γof B is said to be a-separated if $D(z,a) \cap D(\omega,a) = 0$ for all distinct elements z, ω in Γ.

(ii) Let $0 < a < b < \infty$. A subset Γ of B is said to be an a, b-lattice if it is a-separated and has the property $\bigcup_{z\in\Gamma} D(z,b) = B$.

Recall that the normalized reproducing kernel for the Hardy space $H^2(S)$ is given by the formula

$$K_z(\omega) = \frac{(1-|z|^2)^{n/2}}{1-\langle\omega,z\rangle^n}, |z| < 1, |\omega| \le 1$$

For $f \in L^2(S,d\sigma)$ and $z \in B$, we define

$$Var(f;\, z) = \|(f - \langle fk_z, k_z\rangle)k_z\|^2$$

We think of $Var(f;z)$ as the "variance" of f with respect to the probability measure $|k_z|^2 d\sigma$ on S.

We know from previous investigations that the scalar quantity $Var(f;z)$ plays an extremely important role in the study of Hankel operators.

One can formulate a rather broad conjecture about the membership of Hankel operators H_f in a norm ideal C_Φ.

Suppose that Φ is a symmetric gauge function satisfying the condition $C_\Phi \supset C_{2n}^+$, which is necessary for C_Φ to contain any $H_f \neq 0$.Then the general conjecture is that a Hankel operator H_f belongs to C_Φ if and only if

$$\Phi\left(\left\{Var^{\frac{1}{2}}(f - Pf;\, z)\right\}_{z\in\Gamma}\right) < \infty,$$

for some a; b-lattice Γ in B with $b \ge 2a$. But the challenge is to prove this conjectured result for specific symmetric gauge functions, where success depends in no small measure on the “user-friendliness” of the Φ in question, the solution of this problem for the symmetric gauge functions Φ_p^+, $2n < p < \infty$, represented the limit of what could be done with the techniques available then. Now, newly developed techniques allow us to finally solve this problem for the symmetric gauge functions Φ_{p}^- , $2n < p < \infty$

Theorem (4.1.2) [4]: Let $2n < p < \infty$ be given. Let $0 < a < b < \infty$ be positive numbers such that $b \geq 2a$. Then there exist constants $0 < c \leq C < \infty$ which depend only on the given p, a, b and the complex dimension n such that the inequality

$$c\Phi_{\mathrm{p}}^-\left(\left\{Var^{\frac{1}{2}}(f - Pf;\, z)\right\}_{z\in\Gamma}\right) \leq \left\|\mathrm{H}_f\right\|_p^-$$
$$\leq c\,\Phi_{\mathrm{p}}^-\left(\left\{Var^{\frac{1}{2}}(f - Pf;\, z)\right\}_{z\in\Gamma}\right).$$

holds for every $f \in L^2(S, d\sigma)$ and every a; b-lattice Γ in B.

Next let us explain some of the dificulties involved in the proof of Theorem (4.1.2). Recall that, an extremely important role was played by the inequality

$$c\left(\Phi_p^+(\{\alpha_k\}_{k\in\mathrm{N}})\right)^r \leq \Phi_p^+\left(\{\alpha_k^r\}_{k\in\mathrm{N}}\right) \leq c\left(\Phi_p^+(\{\alpha_k\}_{k\in\mathrm{N}})\right)^r \qquad (2).$$

where $1 < r < \infty$, $1 < \rho < \infty$ and $p = \rho r$. For the lack of a better term, one might call (2) the power-transformation property of the family of symmetric gauge functions $\Phi_{p\,,}^+ 1 < p < \infty$.

This power-transformation property is needed because, e.g., at certain point in our estimates, what we can prove are inequalities of the form

$$\Phi\left(\left\{\left\|A_{z,t}\right\|^2\right\}_{z\in F}\right) = \Phi\left(\left\{\langle A^*A\psi_{z,t}, \psi_{z,t}\rangle\right\}_{z\in F}\right) \leq C\|A^*A\|_\Phi \qquad (3)$$

but what we need to prove are inequalities of the form

$$\Phi\left(\left\{\left\|A\psi_{z,t}\right\|\right\}_{z\in F}\right) \leq C\|A\|_\Phi \qquad (4)$$

The power-transformation property is precisely what allows us to deduce (4) from (3).

But, the first stumbling block is that there is no analogue of this power transformation property for the family of symmetric gauge functions Φ_p^- , $1 < p < \infty$. Thus our only hope is to somehow “make (2) work for the Φ_p^--problem” so to speak.

Thanks to a rather complicated relation between Φ_p^- and

$$\Phi_{r'}^{+}, \Phi_{r}^{+}, 1 < r' < p < r < \infty,$$

this idea actually works.

Another major difficulty is the proof of a "reverse Hölder's inequality" of the form

$$\Phi\left(\{J_t(g;k,j)\}_{(k,j)\in I}\right) \le C\Phi\left(\{J(g;k,j)\}_{(k,j)\in I}\right) \qquad (5)$$

Here $t \ge 1$ and J_t "has the exponent t inside the integral", making (5) a reverse Holder 's inequality. The proof of this inequality in the case of Φ_p^+ again depended on the power-transformation property. But for the proof of this inequality in the case of Φ_p^- , even the above-mentioned relation between Φ_p^- and $\Phi_{r'}^+, \Phi_r^+$ does not help. Instead, we must take an entirely new approach. We exploit a property of Φ_p^- called (DQK). Condition (DQK) was introduced for a completely different purpose, but it turns out to be exactly what is needed to prove (5). We are able to show that (5) actually holds for every symmetric gauge function that satisfies condition (DQK).

We begin by establishing the all too important relation between Φ_P^- and $\Phi_{r'}^+, \Phi_r^+$. The proof of Theorem (4.1.2) depends on a crucial relation between the symmetric gauge functions Φ_P^- and $\Phi_{r'}^+, \Phi_r^+$,where $1 < r' < p < r < \infty$. Our task in this section is to establish this relation. Let us introduce the following notation. For every sequence of non-negative numbers $a = \{a_1, \ldots, a_j, \ldots\}$. and every $s > 0$, we denote $N(a;s) = card\{j \in \boldsymbol{N}: a_j > s\}$.

Lemma (4.1.3) [4]: Let $1 < p < \infty$. Then for every sequence of non-negative numbers $a = \{a_1, \ldots, a_j, \ldots\}$. we have

$$\int_0^\infty \{N(a;s)\}^{1/P} \le \Phi_P^-(a) \le P \int_0^\infty \{N(a;s)\}^{1/P} \qquad (6)$$

Proof: Given any $1 < p < \infty$. it is trivial that

$$k^{1/P} \le \sum_{j=1}^{k} \frac{1}{j^{(P-1)/P}} \le 1 + \int_1^k \frac{1}{x^{(P-1)/P}}\, dx \le P k^{1/P} \qquad (7)$$

for every $k \in \boldsymbol{N}$. For the given p, define the measure μ_P^- on $\boldsymbol{N}$ by the formula

$$\mu_P^-(E) = \sum_{j\in E} \frac{1}{j^{(P-1)/P}}, E \subset \boldsymbol{N}.$$

By the monotone convergence theorem and (1), it suffices to consider the case where the sequence $a = \{a_1, \dots, a_j, \dots\}$. has only a finite number of nonzero terms. For such a sequence, rearranging the terms if necessary, we may assume that it is non-increasing, i.e., $a_1 \geq a_2 \geq \cdots \geq a_j$ ..... For such an $a = \{a_1, \dots, a_j, \dots\}$. we have

$$\Phi_P^-(a) = \sum_{j=1}^{\infty} \frac{a_j}{j^{(P-1)/P}} = \int_0^{\infty} \mu_P^-\left(\left(j \in \boldsymbol{N}: a_j > s\right)\right) ds \qquad (8)$$

Where the second = follows from Fubini's theorem. Suppose that $a_1 > 0$, for otherwise (6) holds trivially. Since the sequence $a = \{a_1, \dots, a_j, \dots\}$. is non-increasing, for each $0 < s < a_1$, we have $a_j > s$ if $1 \leq j \leq N(a; s)$ and $a_j \leq s$ if $j > N(a; s)$. Thus for every $0 < s < a_1$ we have

$$\mu_P^-\left(\left(j \in \boldsymbol{N}: a_j > s\right)\right) = \mu_P^-(\{1, \dots, N(a; s)\}) = \sum_{j=1}^{N(a;s)} \frac{1}{j^{(P-1)/P}}.$$

Combining this with (7), we obtain

$$\{N(a; s)\}^{1/P} \leq \mu_P^-\left(\left(j \in \boldsymbol{N}: a_j > s\right)\right) \leq P\{N(a; s)\}^{1/P} \qquad (9)$$

for $0 < s < a_1$. On the other hand, it is obvious that if $s \geq a_1$, then

$$\mu_P^-\left(\left(j \in \boldsymbol{N}: a_j > s\right)\right) = \mu_P^-(0) = 0 = \{N(a; s)\}^{1/P} \qquad (10)$$

Obviously, (6) follows from the combination of (8), (9) and (10).

Proposition (4.1.4) [4]: For every sequence of non-negative numbers $a = \{a_1, \dots, a_j, \dots\}$. and every $s > 0$, define the sequence $a^{\vee}(s) = \{a_1^{\vee}(s), \dots, a_j^{\vee}(s), \dots\}$. where

$$a_j^{\wedge}(s) = \begin{cases} 0 & if\ a_j > s \\ a_j & if\ a_j \leq s \end{cases}, j \in \boldsymbol{N}$$

Then given any $1 < p < r < \infty$, there exists a constant $0 < C_{2.2} < \infty$ such that

$$\int_0^{\infty} \left(\frac{1}{s} \Phi_r^+(a^{\vee}(s))\right)^{r/p} ds \leq C_{2.2} \Phi_P^-(a) \qquad (11)$$

for every sequence of non-negative numbers $a = \{a_1, \dots, a_j, \dots\}$.

Proof: Let $1 < p < r < \infty$ be given. By the monotone convergence theorem and (1), it suffices to consider the case where $a = \{a_1, \dots, a_j, \dots\}$. has only a finite number of nonzero terms. For each $i \in \boldsymbol{Z}$, define

$$v(i) = card\{j \in \boldsymbol{N}: 2^{-i} < a_j \leq 2^{-i+1}\} \qquad (12)$$

Suppose that $2^{-i} < s \leq 2^{-i+1}$ for some $i \in \boldsymbol{Z}$. For such an s, by the definition of Φ_r^+ , there is a subset $\varepsilon(s)$ of N with $card(\varepsilon(s)) = k(s) \in \boldsymbol{N}$ such that

$$\Phi_r^+(a^\vee(s)) = \frac{\sum_{j\in\varepsilon(s)} a_j^\vee(s)}{1^{-1/r} + \cdots + (k(s))^{-1/r}} \leq \frac{\sum_{j\in\varepsilon(s)} a_j^\vee(s)}{(k(s))^{-1/r}}$$

Define $j \in \varepsilon(s): 2^{-i-m} < a_j^\vee(s) = \sum_{m=0}^{\infty} \sum_{j\in E_{S,M}} a_j^\vee (s)$.

If j, i and m are such that $a_j^\vee(s) > 2^{-i-m}$, then $a_j^\vee(s) = a_j$.

Therefore

$$Card\left(E_{s,m}\right) \leq \{v(i+m), k(s)\}$$

Hence for each $m \geq 0$ we have

$$\frac{1}{(k(s))^{1-(1/r)}} \sum_{j\in E_{s,m}} a_j^\vee(s) \leq \frac{2}{2^{i+m}} . \frac{Card\left(E_{s,m}\right)}{(k(s))^{1-(1/r)}} \leq \frac{2}{2^{i+m}}\{v(i+m)\}^{1/r} .$$

Combining this with the above, we conclude that if $2^{-i} < s \leq 2^{-i+1}$, then

$$\Phi_r^+(a^\vee(s)) \leq \frac{1}{(k(s))^{1-(1/r)}} \sum_{m=0}^{\infty} \sum_{j\in E_{S,M}} a_j^\vee (s) \ \leq 2 \sum_{m=0}^{\infty} \frac{1}{2^{i+m}}\{v(i+m)\}^{1/r}$$

Consequently, we have

$$\frac{1}{s}\Phi_r^+(a^\vee(s)) \leq 2 \sum_{m=0}^{\infty} \frac{1}{2^m}\{v(i+m)\}^{1/r} \ for\ every\ s \in (2^{-i}, 2^{-i+1}] \qquad (13)$$

Since $r/p > 1$, we have $r/p = (1+\epsilon)/(1-\epsilon)$ for some $0 < \epsilon < 1$. That is, $(r/p)/(1-\epsilon) = 1+\epsilon$. Factoring 2^{-m} in the form $2^{-m} = 2^{-\epsilon m}.2^{-(1-\epsilon)m}$ a simple application of Holder's inequality to (13) gives us

$$\left(\frac{1}{s}\Phi_r^+(a^\vee(s))\right)^{r/p} \leq C \sum_{m=0}^{\infty} \frac{1}{2^{(1+\epsilon)m}}\{v(i+m)\}^{1/P}$$

for $s \in (2^{-i}, 2^{-i+1}]$. Therefore

$$\int_0^\infty \left(\frac{1}{s}\Phi_r^+(a^\vee(s))\right)^{r/p} ds = \sum_{i=-\infty}^{\infty}\int_{2^{-i}}^{2^{-i+1}} \left(\frac{1}{s}\Phi_r^+(a^\vee(s))\right)^{r/p} ds$$
$$\le C\sum_{i=-\infty}^{\infty}\frac{1}{2^i}\sum_{m=0}^{\infty}\frac{1}{2^{(1+\epsilon)m}}\{v(i+m)\}^{1/P}$$
$$= C\sum_{i=-\infty}^{\infty}\sum_{m=0}^{\infty}\frac{1}{2^{\epsilon m}}\cdot\frac{1}{2^{i+m}}\{v(i+m)\}^{1/P} =$$
$$= C\sum_{k=-\infty}^{\infty}\frac{1}{2^k}\{v(k)\}^{1/P}\sum_{m=0}^{\infty}\frac{1}{2^{\epsilon m}}$$
$$= C_1\sum_{k=-\infty}^{\infty}\frac{1}{2^k}\{v(k)\}^{1/P} \qquad (14)$$

By (12), we have $v(k) \le N(a;s)$ for every $s \in (2^{-k-1}, 2^{-k}]$. Thus

$$\sum_{k=-\infty}^{\infty}\frac{1}{2^k}\{v(k)\}^{1/P} = 2\sum_{k=-\infty}^{\infty}\frac{1}{2^{k+1}}\{v(k)\}^{1/P} \le 2\sum_{k=-\infty}^{\infty}\int_{2^{-k-1}}^{2^{-k}}\{N(a;s)\}^{1/P} ds$$
$$= \int_0^\infty \{N(a;s)\}^{1/P} ds \le \Phi_P^-(a) \qquad (15)$$

where the last $\le$ is an application of Lemma (4.1.3). Obviously, the proposition follows from the combination of (14) and (15).

Proposition (4.1.5) [4]: For every sequence of non-negative numbers $a = \{a_1, \dots, a_j, \dots\}$ and every $s > 0$, define the sequence $a^\wedge(s) = \{a_1^\wedge(s), \dots, a_j^\wedge(s), \dots\}$, where

$$a_j^\wedge(s) = \begin{cases} a_j & if\ a_j > s \\ 0 & if\ a_j \le s \end{cases}, j \in \boldsymbol{N}$$

Then given any $1 < r' < p < \infty$, there exists a constant $0 < C_{2.3} < \infty$ such that

$$\int_0^\infty \left(\frac{1}{s}\Phi_{r'}^+(a^\wedge(s))\right)^{r'/P} ds \le C_{2.3}\Phi_P^-(a)$$

for every sequence of non-negative numbers $a = \{a_1, \dots, a_j, \dots\}$.

Proof: Let $1 < r' < p < \infty$ be given. Again, by the monotone convergence theorem and (1), it suffices to consider the case where $a = \{a_1, \dots, a_j, \dots\}$, has only afinite number of nonzero terms. For each $i \in \boldsymbol{Z}$, let $v(i)$ be given by (12). Suppose that $2^{-i} < s \le 2^{-i+1}$ for some $i \in \boldsymbol{Z}$.

By the definition of $\Phi_{r'}^{+}$, there is a subset $\mathcal{F}(s)$ of N with $card(\mathcal{F}(s)) = k'(s) \in \boldsymbol{N}$ such that

$$\Phi_{r'}^{+}(a^{\wedge}(s)) = \frac{\sum_{j\in\mathcal{F}(s)} a_j^{\wedge}(s)}{1^{-1/r'} + \cdots + (k'(s))^{-1/r'}} \leq \frac{\sum_{j\in\mathcal{F}(s)} a_j^{\wedge}(s)}{(k'(s))^{-1/r'}}$$

Define $F_{s,m} = \{j \in \mathcal{F}(s): 2^{-i+m} < a_j^{\wedge}(s) \leq 2^{-i+m+1}\}$. for each $m \in Z_+$. By definition, if $a_j^{\wedge}(s) > 0$, then $a_j^{\wedge}(s) > s$. Since $s > 2^{-i}$, we have

$$\sum_{j\in\mathcal{F}(s)} a_j^{\wedge}(s) = \sum_{m=0}^{\infty} \sum_{j\in\mathcal{F}_{s,m}} a_j^{\wedge}(s)$$

We have $card(F_{s,m}) \leq \sup\{v(i-m), k'(s)\}$ for every $m \geq 0$. Therefore

$$\frac{1}{s}\Phi_{r'}^{+}(a^{\wedge}(s)) \leq \frac{1}{(k'(s))^{-1/r'}} \sum_{m=0}^{\infty} \sum_{j\in\mathcal{F}_{s,m}} a_j^{\wedge}(s) \leq 2\sum_{m=0}^{\infty} 2^{-i+m}\{v(i-m)\}^{1/r'}$$

Consequently,

$$\frac{1}{s}\Phi_{r'}^{+}(a^{\wedge}(s)) \leq 2\sum_{m=0}^{\infty} 2^{m}\{v(i-m)\}^{1/r'}$$

Since $0 < r'/P < 1$, it follows that

$$\left(\frac{1}{s}\Phi_{r'}^{+}(a^{\wedge}(s))\right)^{r'/P} \leq 2\sum_{m=0}^{\infty} 2^{mr'/P}\{v(i-m)\}^{1/P}$$

for $2^{-i} < s \leq 2^{-i+1}$. Thus

$$\int_0^{\infty} \left(\frac{1}{s}\Phi_{r'}^{+}(a^{\wedge}(s))\right)^{r'/P} ds = \sum_{i=-\infty}^{\infty} \int_{2^{-i}}^{2^{-i+1}} \left(\frac{1}{s}\Phi_{r'}^{+}(a^{\wedge}(s))\right)^{r'/P} ds$$

$$\leq 2\sum_{i=-\infty}^{\infty} 2^{-i} \sum_{m=0}^{\infty} 2^{mr'/P}\{v(i-m)\}^{1/P}$$

$$= 2\sum_{i=-\infty}^{\infty} \sum_{m=0}^{\infty} \frac{1}{2^{(1-(r'/P))m}} \cdot \frac{1}{2^{i-m}}\{v(i-m)\}^{1/P}$$

$$= 2\sum_{k=-\infty}^{\infty} \frac{1}{2^k}\{v(k)\}^{1/P} \sum_{m=0}^{\infty} \frac{1}{2^{(1-(r'/P))m}}.$$

Recalling (15), the proof is now complete.

Although Theorem (4.1.2) is about membership in the ideal C_P^-, the fact that we need Propositions (4.1.4) and (4.1.5) clearly indicates that symmetric gauge functions Φ_P^+ , $1 < p < \infty$, will be an important part of our analysis. We end this section with some facts about these symmetric gauge functions, which will be needed later on.

Lemma (4.1.6)[4]: Suppose that $1 < p < \infty$. Let $\alpha = \{\alpha_1, \dots, \alpha_k, \dots.\}$ be a non-increasing sequence of non-negative numbers. Define

$$F_P(\alpha) = \sup_{k\geq 1} k^{1/P}\, \alpha_k$$

Then

$$\frac{P-1}{P} F_P(\alpha) \leq \Phi_P^+(a) \leq F_P(\alpha)$$

Lemma (4.1.7)[4]: Let $1 < r < \infty$, $1 < p < \infty$. and $p = \rho r$. Then for every sequence $\alpha = \{\alpha_1, \dots, \alpha_k, \dots.\}$ of non-negative numbers we have

$$\frac{\rho - 1}{\rho}\left(\Phi_P^+(\{\alpha_k\}_{k\in N})\right)^r \leq \Phi_\rho^+(\{\alpha_k^r\}_{k\in N}) \leq \left(\frac{P}{P-1}\Phi_P^+(\{\alpha_k\}_{k\in N})\right)^r.$$

If Φ_P denotes the symmetric gauge function for the Schatten class C_P, $1 < p < \infty$, then, of course, for every sequence of non-negative numbers $a = \{a_1, \dots., a_k, \dots\}$ we have the following well-known inequality of weak-type:

$$N(a; s) \leq \left(\frac{\Phi_P(a)}{s}\right)^P \qquad (16).$$

for $s > 0$. But for the purpose of this section, (16) is not good enough; we need an improved version of it. More specifically, we need to replace the $\Phi_P(a)$ above by $\Phi_P^+(a)$.

Lemma (4.1.8)[4]: Suppose that $1 < p < \infty$. Then for every sequence of non-negative numbers $a = \{a_1, \dots., a_k, \dots\}$ and every $s > 0$ we have

$$N(a; s) \leq \left(\frac{P}{P-1}\right)^P \left(\frac{1}{s}\Phi_P^+(a)\right)^P \qquad (17)$$

Proof: Given an $s > 0$, set $M = \{j \in \boldsymbol{N}: a_j > s\}$. If $card(M) = \infty$, then $\Phi_P^+(a) = \infty$,and therefore (17) holds in this case. Obviously, (17) also holds in the case $M = 0$, Suppose that $card(M) = m \in N$. Then there is a bijection $\pi: \{1, \dots, m\} \to M$ such that

$$a_{\pi(1)} \geq \cdots \geq a_{\pi(m)}$$

Since $a_{\pi(m)} > s$, by Lemma (4.1.6) we have

$$sm^{1/P} < a_{\pi(m)} m^{1/P} \leq \sup_{1\leq k\leq m} a_{\pi(m)} k^{1/P}$$
$$\leq \frac{P}{P-1}\Phi_P^+(\{a_{\pi(1)}, \dots, a_{\pi(m)}\}) \leq \frac{P}{P-1}\Phi_P^+(a).$$

Solving for $m(= N(a; s))$, we find that $m \leq \{P/(P-1)\}^P (\Phi_P^+(a)/s)^P$

Although (17) is only a slight improvement of (16), we will see that this improvement makes quite a difference. In fact, (16) is the reason why Propositions (4.1.4) and (4.1.5) are useful for our purpose.

It is well known that the formula

$$d(\zeta,\xi) = |1 - \langle \zeta,\xi\rangle|^{1/2}, \zeta,\xi \in S \tag{18}$$

defines a metric on S .Throughout, we denote

$$B(\zeta,r) = \{x \in S: |1 - \langle \zeta,\xi\rangle|^{1/2} < r\}$$

for $\zeta \in S$ and $r > 0$. There is a constant $2^{-n} < A_0 < \infty$ such that

$$2^{-n} r^{2n} \le \sigma(B(\zeta,r)) \le A_0 r^{2n} \tag{19}$$

for all $\zeta \in S$ and $0 < r \le \sqrt{2}$. Note that the upper bound actually holds when $r > \sqrt{2}$.

Next we need to recall the spherical decomposition. For each integer $k \ge 0$, let $\{u_{k,j}, \dots, u_{k,m(k)}\}$be a subset of S which is maximal with respect to the property

$$B(u_{k,j}, 2^{-k-1}) \cap B(u_{k,j'}, 2^{-k-1}) = 0 \; for\ all\ 1 \le j < j' \le m(k) \tag{20}$$

The maximality of $\{u_{k,1}, \dots, u_{k,m(k)}\}$implies that

$$\bigcup_{j=1}^{m(k)} B(u_{k,j}, 2^{-k}) = S \tag{21}$$

For each pair of $k \ge 0$ and $1 \le j \le m(k)$, define

$$T_{k,j} = \{ru: 1 - 2^{-2k} \le r^2 < 1 - 2^{-2(k+1)}, u \in B(u_{k,j}, 2^{-k})\} \tag{22}$$

We define the index set

$$I = \{(k,j): k \ge 0, 1 \le j \le m(k)\}$$

Recall that for each pair of $0 < t < \infty$ and $z \in \boldsymbol{B}$, we define

$$\psi_{z,t}(\zeta) = \frac{(1 - |z|^2)^{(n/2)+t}}{(1 - \langle \zeta, z\rangle)^{n+t}}$$

$|\zeta| \le 1$. In terms of the normalized reproducing kernel k_z and the Schur multiplier

$$m_z(\zeta) = \frac{1 - |z|}{1 - \langle \zeta, z\rangle} \tag{23}$$

We have the relation

$$\psi_{z,t} = (1 + |z|)^t m_z^t k_z \, .$$

We think of z, t as a modified kernel function, i.e., a modified version of k_z.

Definition (4.1.9) [4]: (a) A partial sampling set is afinite subset F of the open unit ball B with the property that $card(F \cap T_{k,j}) \leq 1$ for every $(k,j) \in I$.

(b) For any partial sampling set F and any $t > 0$, denote

$$R_F^{(t)} = \sum_{z \in F} \psi_{z,t} \otimes \psi_{z,t}$$

The next proposition shows the benefit of modifying k_z:

Proposition (4.1.10)[4]: For each $t > 0$, there is a constant $C_{3.2}(t)$ such that the inequality

$$\Phi\left(\left\{\langle B\psi_{z,t}, \psi_{z,t}\rangle\right\}_{z \in F}\right) \leq C_{3.2}(t)\|B\|_\Phi$$

holds for every partial sampling set F, every symmetric gauge function Φ, and every non-negative self-adjoint operator B on the Hardy space $H^2(S)$.

Proof: Let Φ be any symmetric gauge function. Then it has the following property: For non-negative numbers $a_1 \geq \cdots \geq a_v \geq 0$ and $b_1 \geq \cdots \geq b_v \geq 0$ in descending order, if $a_1 + \cdots + a_j \leq b_1 + \cdots + b_j$ for every $1 \leq j \leq v$.

then

$$\Phi(\{a_1, \dots, a_v, 0, .., 0 \dots\}) \leq \Phi(\{b_1, \dots, b_v, 0, .., 0 \dots\})$$

Let $t > 0$ be given,there is a constant $C_{3.2}(t)$ such that

$$\left\|R_F^{(t)}\right\| \leq C_{3.2}(t)$$

for every partial sampling set F.

Let B be a non-negative self-adjoint operator, and suppose that F is a partial sampling set with $card(F) = m$. Then we can enumerate the elements in F as $z_1, \dots, z_m$ in such a way that

$$\langle B\psi_{z_1,t}, \psi_{z_1,t}\rangle \geq \cdots \geq \langle B\psi_{z_M,t}, \psi_{z_M,t}\rangle$$

For each $1 \leq k \leq m$, define the subset $F_k = \{z_1, \dots, z_k\}$ of F. Then each F_k is also a partial sampling set, and we have $\left\|R_{F_k}^{(t)}\right\| \leq \left\|R_F^{(t)}\right\| \leq C_{3.2}(t)$ for every $1 \leq k \leq m$. In terms of s-numbers, this implies that $s_j\left(BR_{F_k}^{(t)}\right) \leq C_{3.2}(t)s_j(B)$.

for every $j \geq 1$ .Write $\|\cdot\|_1$ for the norm of the trace class. Since rank$\left(R_{F_k}^{(t)}\right) \leq k$, we have

$$\langle B\psi_{z_1,t},\psi_{z_1,t}\rangle+\cdots+\langle B\psi_{z_k,t},\psi_{z_k,t}\rangle = tr\big(BR_{F_k}^{(t)}\big) \le \big\|BR_{F_k}^{(t)}\big\|_1$$
$$= s_1\big(BR_{F_k}^{(t)}\big)+\cdots+s_k\big(BR_{F_k}^{(t)}\big) \le C_{3.2}(t)\{s_1(B)+\cdots+s_k(B)\}$$

Since this holds for every $1 \le k \le m$ .by the property of Φ that we mentioned in the previous paragraph, we have

$$\Phi\left(\left\{\langle B\psi_{z,t},\psi_{z,t}\rangle\right\}_{z\in F}\right) \le C_{3.2}(t)\Phi\left(\left\{s_j(B)\right\}_{j\in \boldsymbol{N}}\right) = C_{3.2}(t)\|B\|_{\Phi}$$

proving the proposition.

Proposition (4.1.11) [4]: Given any pair of $t > 0$ and $2 < p < \infty$, there exists a constant $C_{3.3}(t,p)$ such that the inequality

$$\Phi_P^{+}\left(\left\{\|A\psi_{z,t}\|\right\}_{z\in F}\right) \le C_{3.3}(t,p)\|A\|_{\rho}^{+}$$

holds for every bounded operator $A: H^2(S) \to L^2(S, d\sigma)$ and every partial sampling set F.

Proof: Let $t > 0$ and $2 < p < \infty$ be given. Set $\rho = p/2$. Then $\rho > 1$ and $p = 2\rho$. Let $C = \{\rho/(\rho-1)\}^{1/2}$. Let $A: H^2(S) \to L^2(S, d\sigma)$ be any bounded operator and let F be any partial sampling set. Applying Lemma (4.1.7) with $r = 2$, we have

$$\Phi_P^{-}\left(\left\{\|A\psi_{z,t}\|\right\}_{z\in F}\right) \le C\left(\Phi_P^{-}\left(\left\{\|A\psi_{z,t}\|^2\right\}_{z\in F}\right)\right)^{1/2}$$
$$= C\left(\Phi_P^{-}\left(\left\{\langle A^*A\psi_{z,t},\psi_{z,t}\rangle\right\}_{z\in F}\right)\right)^{1/2} \tag{24}$$

On the other hand, Proposition (4.1.10) gives us

$$\Phi_P^{-}\left(\left\{\langle A^*A\psi_{z,t},\psi_{z,t}\rangle\right\}_{z\in F}\right) \le C_{3.2}(t)\|A^*A\|_{\rho}^{+} \tag{25}$$

Again applying Lemma (4.1.7) with $r = 2$, we have

$$\|A^*A\|_{\rho}^{+} = \left\|(A^*A)^{2/2}\right\|_{\rho}^{+} \le \left\{\tfrac{P}{P-1}\left\|(A^*A)^{1/2}\right\|_{\rho}^{+}\right\}^2 = \left\{\tfrac{P}{P-1}\|A\|_{\rho}^{+}\right\}^2 \tag{26}$$

Thus if we set $C_{3.3}(t,p) = C\{C_{3.2}(t,p)\}^{1/2}\{P/(P-1)\}$, then the proposition follows from the combination of (24), (25) and (26).

Proposition (4.1.12)[4]: Given any pair of $t > 0$ and $2 < p < \infty$, there exists a constant $C_{3.4}(t,p)$ such that the inequality

$$\Phi_P^{-}\left(\left\{\|A\psi_{z,t}\|\right\}_{z\in F}\right) \le C_{3.4}(t,p)\|A\|_{\rho}^{-} \tag{27}$$

holds for every bounded operator $A: H^2(S) \to L^2(S, d\sigma)$ and every partial sampling set F.

Proof: Let $t > 0$ and $2 < p < \infty$ be given. We pick an r' such that $2 < r' < p$. To prove (27), we only need to consider compact $A: H^2(S) \to L^2(S, d\sigma)$, for otherwise the inequality holds for the trivial reason that its right-hand side is infinity. But for a compact A, we have the representation

$$A = \sum_{j=1}^{\infty} a_j x_j \otimes y_j$$

where $\{x_j : j \in \boldsymbol{N}\}$ and $\{y_j : j \in \boldsymbol{N}\}$ are orthonormal sets in $L^2(S, d\sigma)$ and $H^2(S)$ respectively, and $a_j \geq 0$ for every $j \in \boldsymbol{N}$. For every $s > 0$, define the operators

$$A_s = \sum_{a_j > s} a_j x_j \otimes y_j \quad and \; B_s = \sum_{a_j \leq s} a_j x_j \otimes y_j$$

It follows from Proposition (4.1.5) that

$$\int_0^\infty \left(\frac{1}{s} \|A_s\|_{r'}^{+} \right)^{r'/P} ds \leq C_{2.3} \|A\|_{\rho}^{-} \tag{28}$$

On the other hand, it is obvious that $\|B_s\| \leq s$. Since $\|\psi_{z,t}\| \leq 2^t$, we have

$$\|B_s \psi_{z,t}\| \leq 2^t s \tag{29}$$

for all $z \in B$ and $s > 0$.

Let a partial sampling set F be given. With somewhat abuse of notation, let us write

$$N(F; \lambda) = card\{z \in F : \|A_{\psi_{z,t}}\| > \lambda\}$$

for $\lambda > 0$. By Lemma (4.1.3), we have

$$\Phi_P^{-} \left(\{\|A\psi_{z,t}\|\}_{z \in F} \right) \leq P \int_0^\infty \{N(F; \lambda)\}^{1/P} \, d\lambda$$

$$(1 + 2^t) P \int_0^\infty \{N(F; (1 + 2^t)s)\}^{1/P} \, ds \tag{30}$$

where the last step is the substitution $\lambda = (1 + 2^t)s$ Define

$$N(s) = card\{z \in F : \|A_s \psi_{z,t}\| > s\}$$

for $s > 0$. Since $A = A_s + B_s$, we have $\|A\psi_{z,t}\| \leq \|A_s \psi_{z,t}\|$ for all $s > 0$ and $z \in F$. Therefore (29) implies that for every $s > 0$,

$$N(F; (1 + 2^t)s) \leq N(s)$$

Applying Lemma (4.1.8) and Proposition (4.1.11), we have

$$N(s) \leq \left(\frac{r'}{r'-1}\right)^{r'} \left(\frac{1}{2}\Phi^{+}_{r'}\{\|A_s\psi_{z,t}\|\}_{z\in F}\right)^{r'}$$
$$\leq \left(\frac{r'}{r'-1}\right)^{r'} \left(\frac{1}{s}C_{3.3}\|A_s\|^{+}_{r'}\right)^{r'}.$$
Thus if we set $C = \{r'C_{3.3}(t,r')/(r'-1)\}^{r'/P}$ then
$$\{N(F;(1+2^t)s)\}^{1/P} \leq \{N(s)\}^{1/P} \leq \left(\frac{1}{s}\|A_s\|^{+}_{r'}\right)^{r'/P}$$
for every $s > 0$. Substituting this in (30) and recalling (28), we obtain
$$\Phi^{-}_{P}\left(\{\|A\psi_{z,t}\|\}_{z\in F}\right) \leq (1+2^t)PC\int_0^\infty \left(\frac{1}{s}\|A_s\|^{+}_{r'}\right)^{r'/P} ds$$
$$\leq (1+2^t)PCC_{2.3}\|A\|^{-}_{P}$$
This completes the proof of the proposition.

Definition (4.1.13) [4]: A partial sampling map is a map φ from a set X into B which has the property that $card\{x \in X: \varphi(x) \in T_{k,j}\} \leq 1$ for every $(k,j) \in I$

Lemma (4.1.14) [4]: There exists a natural number $M_{3.6}$ determined by the complex dimension n such that the following is true: Let L be a subset of I and suppose that $z: L \to \boldsymbol{B}$ map satisfying the condition $z(k,j) \in T_{k,j}$for every $(k,j) \in L$ Then there is a partition $L = E_1 \cup \ldots . \cup E_{M_{3.6}}$. such that for every $1 \leq v \leq M_{3.6}$, the map $z: E_v \to \boldsymbol{B}$ is a partial sampling map.

Proof: By (22), we have $T_{k,j} \cap T_{k',j} = 0$; for all $k \neq k'$ in Z_+ and $1 \leq j \leq m(k), 1 \leq j \leq m(k')$,. By (19), (20) and (22), there is an $M \in N$ determined by the complex dimension n such that the inequality
$$card\{i: 1 \leq j \leq m(k), T_{k,j} \cap T_{k,i} \neq 0\} \leq M \tag{31}$$
holds for every $(k,j) \in I$. Let us show that $M_{3.6} = M^2$ suffices for our purpose.

Let $L \subset I$, and suppose that $z: L \to \boldsymbol{B}$ is a map such that $z(k,j) \in T_{k,j}$ for every$(k,j) \in L$. Then by (31), for every $(k,j) \in I$ we have
$$\sum_{T_{k,j}\cap T_{k,i}\neq 0} card\{\ell \in \{1,\ldots,m(k)\}: z(k,\ell) \in T_{k,i}\} \leq M^2 = M_{3.6} \tag{32}$$
We pick a subset E_1 of L that is maximal with respect to the condition that the restricted map $z: E_1 \to \boldsymbol{B}$ be a partial sampling map.

Suppose that $m \geq 1$ and that we have defined pairwise disjoint subsets $E_1, \ldots, E_m$ of L.
We then define E_{m+1} to be a subset of $L \backslash (E_1 \cup \ldots \cup E_m)$ that is maximal with respect to the condition that the restricted map $z: E_{m+1} \rightarrow \boldsymbol{B}$ be a partial sampling map. Then the proof will be complete once we show that $E_{M_{3.6+1}} = 0$ Assume the contrary, i.e., assume that there were some $(k^*, j^*) \in E_{M_{3.6+1}}$. We will show that this leads to a contradiction.

First of all, we have

$$z(k^*, j^*) \in T_{k^*, j^*} \tag{33}$$

By the maximality of the sets $E_1, \ldots., E_{M_{3.6}}$, for each $1 \leq v \leq M_{3.6}$, the map z fails to satisfy Definition (4.1.13) on the set $E_v \cup \{(k^*, j^*)\}$. Since z is partial sampling on E_v, this means that for each $1 \leq v \leq M_{3.6}$ there is a $(k_v, \ell_v) \in E_v$ such that

$$\{z(k_v, \ell_v), z(k^*, j^*)\} \subset T_{k'_v, i_v}.$$

for some $(k'_v, i_v) \in I$. By (33), this implies $k'_v = k^* = k_v$ and $T_{k^*, i_v} \cap T_{k^*, j^*} \neq 0$; for every $1 \leq v \leq M_{3.6}$. Thus z maps the set $\{(k^*, j^*), (k^*, \ell_1), \ldots, (k^*, \ell_{M_{3.6}})\}$ into $\bigcup_{T_{k^*, j^*} \cap T_{k^*, i} \neq 0} T_{k^*, i}$.
Since the set $\{(k^*, j^*), (k^*, \ell_1), \ldots, (k^*, \ell_{M_{3.6}})\}$ contains $M_{3.6} + 1 = M_2 + 1$ elements, this contradicts (32). This completes the proof of the lemma.
In addition to the index set I, let us also define $I_m = \{(k, j) \in I: k \leq m\}$ for each $m \in Z_+$. The following is the main goal of this section:
Proposition (4.1.15) [4]: Let $2 < p < \infty$ and $0 < t < 1$ Suppose that $w_{k,j} \in T_{k,j}$ for every $(k, j) \in I$. Then the inequality

$$\Phi_{\overline{P}}^{-}\left(\left\{\left\|A\psi_{w_{k,j},t}\right\|\right\}_{(k,j)\in I_m}\right) \leq C_{3.4}(t, P) M_{3.6} \|A\|_{\overline{P}}^{-} \tag{34}$$

holds for every bounded operator $A: H^2(S) \rightarrow L^2(S, d\sigma)$ and every $m \geq 1$, where $C_{3.4}(t, p)$and $M_{3.6}$ are the constants provided by Proposition (4.1.12) and Lemma (4.1.14) respectively.
Proof: First of all, a symmetric gauge function Φ has the following obvious property: If X is any countable set and if $X = X_1 \cup \ldots \cup X_N$, then for every map $\varphi: X \rightarrow [0, \infty)$ we have

$$\Phi(\{\varphi(x)\}_{x\in X}) \leq \Phi(\{\varphi(x)\}_{x\in X_1}) + \cdots + \Phi(\{\varphi(x)\}_{x\in X_N}) \tag{35}$$

Let $m \geq 1$ be given and consider the map $(k, j) \mapsto w_{k,j}$ from I_m into B.

Since $w_{k,j} \in T_{k,j}$ for every (k, j), by Lemma (4.1.14) there is a partition $I_m = E_1 \cup \ldots \cup E_{M_{3.6}}$.

such that for every $1 \leq i \leq M_{3.6}$, the map $(k,j) \mapsto w_{k,j}$is partial sampling on E_i. By Definition (4.1.13), this means that the map $(k,j) \mapsto w_{k,j}$is injective on E_i and $\{w_{k,j}:(k,j) \in E_i\}$ is a partial sampling set as defined in Definition (4.1.9). Hence Proposition (4.1.12) gives us

$$\Phi_P^- \left(\left\{ \left\| A\psi_{w_{k,j},t} \right\| \right\}_{(k,j)\in E_i} \right) \leq C_{3.4}(t,P)\|A\|_P^-$$

for every bounded operator $A: H^2(S) \to L^2(S, d\sigma)$ and every $1 \leq i \leq M_{3.6}$. By (35), we also have

$$\Phi_P^- \left(\left\{ \left\| A\psi_{w_{k,j},t} \right\| \right\}_{(k,j)\in I_m} \right) \leq \sum_{i=1}^{M_{3.6}} \Phi_P^- \left(\left\{ \left\| A\psi_{w_{k,j},t} \right\| \right\}_{(k,j)\in E_i} \right)$$

Obviously, the proposition follows from the above two inequalities.

Section (4.2): Radial Contractions and Local Inequality with Lower Bound and Small Factor

For each $\ell \in N$ we define the radial contraction

$$\rho\ell(z) = \begin{cases} \left(1 - 4^{\ell}(1 - |z|^2)\right)^{1/2} (z/|z|) & if\ 4^{\ell}(1 - |z|^2) < 1 \\ 0 & if\ 4^{\ell}(1 - |z|^2) \geq 1 \end{cases} \tag{36}$$

$z \in B$. One can better understand these $\rho\ell$ in terms of the following relations: we have

$$\begin{cases} \rho\ell(z)/|\rho\ell(z)| = z/|z| \quad and \\ 1 - |\rho\ell(z)|^2 = 4^{\ell}(1 - |z|^2) \end{cases} \tag{37}$$

if $4^{\ell}(1 - |z|^2) < 1$. Recall that the Schur multiplier m_z is given by (23).

A key ingredient in the proof of the lower bound in Theorem (4.1.2) is the following local inequality for Hankel operators:

Theorem (4.2.1) [4]: Given any $0 < \delta \leq 1/2$, there exists a constant $0 < C(\delta) < 1$ which depends only on δ and the complex dimension n such that the inequality

$$Var^{1/2}(f - Pf; z) \leq C(\delta) \sum_{\ell=1}^{\infty} \frac{1}{2^{(1-\delta)\ell}} \left\| M_{m_{\rho\ell(z)}} H_f k_{\rho\ell(z)} \right\|$$

holds for all $f \in L^2(S, d\sigma)$, and $z \in B$.

Next we again turn to the symmetric gauge function Φ_P^-.

Lemma (4.2.2)[4]: Let $1 < p < \infty$. Let X, Y be countable sets and let $N \in \boldsymbol{N}$.

Suppose that $T: X \to Y$ is a map that is at most N-to-1. That is, $card\{x \in X: T(x) = y\} \leq N$ for every $y \in Y$. Then for every set of real numbers $\{a_y\}_{y \in Y}$ we have

$$\Phi_P^-\left(\{a_{T(x)}\}_{x \in X}\right) \leq \max\{P, 2\} N^{1/P} \Phi_P^-\left(\{a_y\}_{y \in Y}\right)$$

We will now bring the radial contractions $\rho\ell$ defined by (36) into our estimates. Recall that the index set I was defined in Section 3 and that for each $m \in Z_+$, we write $I_m = \{(k;\ j) \in I: k \leq m\}$.

Lemma (4.2.3) [4]: There exists a constant $C_{4.3}$ which depends only on the complex dimension n such that the following holds true: Let $h: \boldsymbol{B} \to [0, \infty)$ be a map such that $\sup_{w \in T_{k,j}} h(w) < \infty$ for every $(k, j) \in I$. For each $(k, j) \in I$, let $w_{k,j} \in T_{k,j}$ be such that

$$h(w_{k,j}) \geq \frac{1}{2} \sup_{w \in T_{k,j}} h(w) \qquad (38)$$

Suppose that $z_{k,j} \in T_{k,j}$ for every $(k,j) \in I$. Then the inequality

$$\Phi_P^- \left(\left\{h\left(\rho\ell(z_{k,j})\right)\right\}_{(k,j)\in I_m}\right) \leq \max\{P,2\}\, C_{4.3} 2^{2n\ell/P} \Phi_P^- \left(\left\{h(w_{k,j})\right\}_{(k,j)\in I_m}\right)$$

holds for all $m, \ell \in N$ and $1 < p < \infty$.

Proof: First of all, by (20) and (19), there exists a natural number C_1 such that for all integers $0 \leq k' \leq k$ and $1 \leq i \leq m(k')$, we have

$$card\{j \in \{1, \dots, m(k)\}: B(u_{k,j}, 2^{-k}) \cap B(u_{k',i} 2^{-k'}) \neq 0\} \leq C_1 2^{2n(k-k')} \quad (39)$$

Let $h, w_{k,j}$ and $z_{k,j}$, $(k,j) \in I$, be as in the statement of the lemma. Let $\ell \in N$. By (37) and (22), we have

$$\rho\ell\left(\bigcup_{j=1}^{m(k)} T_{k,j}\right) \subset \bigcup_{i=1}^{m(k-\ell)} T_{k-\ell,i} \ \ if\ k > l \qquad (40)$$

Consider any $1 < p < \infty$ and $m \in N$. First let us consider the case where $m > l$. Then $I_m = I_\ell \cup I_{m,\ell}$ where

$$I_{m,\ell} = \{(k,j) \in I: \ell \leq k \leq m\}$$

By (40), for each $(k,j) \in I_{m,\ell}$, there is an $\eta(k,j) \in \{1, \dots, m(k-\ell)\}$such that $\rho\ell(z_{k,j}) \in T_{k-\ell,\eta(k,j)}$. We now define a map $\varphi: I_{m,\ell} \to I_m$ by the formula

$$\varphi(k,j) = (k-\ell, \eta(k,j)), \qquad (k,j) \in I_{m,\ell}$$

This map ensures that $\rho\ell(z_{k,j}) \in T_{\varphi(k,j)}$, $(k,j) \in I_{m,\ell}$ By (38), we have

$$h\left(\rho\ell(z_{k,j})\right) \leq 2h(w_{\varphi(k,j)}) for\ every\ (k,j) \in I_{m,\ell}$$

Consequently,

$$\Phi_P^- \left(\left\{h(\rho\ell(z_{k,j}))\right\}_{(k,j)\in I_{m,\ell}}\right) \leq 2\Phi_P^- \left(\left\{h(w_{\varphi(k,j)})\right\}_{(k,j)\in I_{m,\ell}}\right) \qquad (41)$$

By (36), if $(k,j) \in I_{m,\ell}$ then

$$\frac{\rho\ell(z_{k,j})}{|\rho\ell(z_{k,j})|} = \frac{z_{k,j}}{|z_{k,j}|}$$

Since $z_{k,j} \in T_{k,j}$and $\rho\ell(z_{k,j}) \in T_{\varphi(k,j)} = T_{k-\ell,\eta(k,j)}$, by (22), the above identity implies

$$B(u_{k,j}, 2^{-k}) \cap B(u_{k-\ell,\eta(k,j)}, 2^{-k+\ell}) \neq 0$$

Combining this with (39), we see that for each $i \in \{1, \dots, m(k-\ell)\}$
$card\{j \in \{1, \dots, m(k)\}: \eta(k,j) = i\} \leq C_1 2^{2n\ell}$.

In other words, the map $\varphi: I_{m,\ell} \to I_m$ is at most $C_1 2^{2n\ell}$-to-1. By Lemma (4.2.2), this means

$$\Phi_P^- \left(\left\{h\left(w_{\varphi(k,j)}\right)\right\}_{(k,j)\in I_{m,\ell}}\right)$$
$$\leq \max\{P,2\}\, C_1^{1/P} 2^{2n\ell/P} \Phi_P^- \left(\left\{h\left(w_{k,j}\right)\right\}_{(k,j)\in I_m}\right)$$

Since $C_1^{1/P} \leq C_1$, if we combine the above with (41), we obtain

$$\Phi_P^- \left(\left\{h\left(\rho\ell\left(z_{k,j}\right)\right)\right\}_{(k,j)\in I_{m,\ell}}\right)$$
$$\leq \max\{P,2\}\, 2C_1 2^{2n\ell/P} \Phi_P^- \left(\left\{h\left(w_{k,j}\right)\right\}_{(k,j)\in I_m}\right) \qquad (42)$$

Next we consider the set I_ℓ.

Note that by (20) and (19), there is a natural number C_2 such that

$$m(k) \leq C_2 2^{2nk}\ for\ every\ k \geq 0 \qquad (43)$$

By (36), we have

$$\rho\ell\left(\bigcup_{j=1}^{m(k)} T_{k,j}\right) \subset \bigcup_{i=1}^{m(0)} T_{0,j}\ if\ 0 \leq k \leq \ell.$$

Therefore there is a map $\varphi: I_\ell \to I_0$ such that

$$\rho\ell\left(z_{k,j}\right) \in T_{\psi(k,j)}\ for\ every\ (k,j) \in I_\ell$$

Combining this relation with (38), we have

$$\Phi_P^- \left(\left\{h\left(\rho\ell\left(z_{k,j}\right)\right)\right\}_{(k,j)\in I_\ell}\right) \leq 2\Phi_P^- \left(\left\{h\left(w_{\psi(k,j)}\right)\right\}_{(k,j)\in I_\ell}\right)$$

By (43), $card(I_\ell) \leq C_2 \sum_{k=0}^{\ell} 2^{2nk} \leq 2C_2 2^{2n\ell}$ Therefore the map $\psi: I_\ell \to I_0$ is at most $2C_2 2^{2n\ell}$-to-1. Applying Lemma (4.2.2) again, we obtain

$$\Phi_P^- \left(\left\{h\left(w_{\psi(k,j)}\right)\right\}_{(k,j)\in I_\ell}\right) \leq \max\{P,2\}\, 2C_2 2^{2n\ell/P} \Phi_P^- \left(\left\{h\left(w_{k,j}\right)\right\}_{(k,j)\in I_0}\right)$$

Therefore

$$\Phi_P^- \left(\left\{h\left(\rho\ell(z_{k,j})\right)\right\}_{(k,j)\in I_\ell}\right)$$
$$\leq \max\{P,2\}\, 2C_2 2^{2n\ell/P} \Phi_P^- \left(\left\{h\left(w_{k,j}\right)\right\}_{(k,j)\in I_0}\right) \qquad (44)$$

Combining this with (42), we see that in the case $m > l$ we have

$\Phi_P^-\left(\{h(\rho\ell(z_{k,j}))\}_{(k,j)\in I_m}\right)$

$$\leq \Phi_P^-\left(\{h(\rho\ell(z_{k,j}))\}_{(k,j)\in I_{m,\ell}}\right)$$
$$+\Phi_P^-\left(\{h(\rho\ell(z_{k,j}))\}_{(k,j)\in I_\ell}\right)$$
$$\leq \max\{P,2\}\,(2C_1+4C_2)2^{2n\ell/P}\Phi_P^-\left(\{h(w_{k,j})\}_{(k,j)\in I_m}\right).$$

On the other hand, if $m \leq \ell$,then (44) gives us

$\Phi_P^-\left(\{h(\rho\ell(z_{k,j}))\}_{(k,j)\in I_m}\right) \leq \Phi_P^-\left(\{h(\rho\ell(z_{k,j}))\}_{(k,j)\in I_\ell}\right)$

$$\leq \max\{P,2\}\,4C_2 2^{2n\ell/P}\Phi_P^-\left(\{h(w_{k,j})\}_{(k,j)\in I_0}\right)$$
$$\leq \max\{P,2\}\,4C_2 2^{2n\ell/P}\Phi_P^-\left(\{h(w_{k,j})\}_{(k,j)\in I_m}\right)$$

This completes the proof of the lemma.

Proposition (4.2.4) [4]: Given any $2n < p < \infty$, there exists a constant $C_{4.4}(p)$ which depends only on p and the complex dimension n such that the following estimate holds: Let $f \in L^2(S, d\sigma)$. For each $(k,j) \in I$, let $w_{k,j} \in T_{k,j}$be such that

$$\left\|M_{m_{w_{k,j}}}H_f k_{w_{k,j}}\right\| \geq \frac{1}{2}\sup_{w\in T_{k,j}}\left\|M_{m_w}H_f k_w\right\| \tag{45}$$

Let $z_{k,j} \in T_{k,j}$, $(k,j) \in I$. Then for every $m \in N$ we have

$\Phi_P^-\left(\{Var^{1/2}(f-Pf;z_{k,j})\}_{(k,j)\in I_m}\right)$

$$\leq C_{4.4}(P)\Phi_P^-\left(\left\{\left\|M_{m_{w_{k,j}}}H_f k_{w_{k,j}}\right\|\right\}_{(k,j)\in I_m}\right) \tag{46}$$

Proof: Since $p > 2n$, there is a $0 < \delta \leq 1/2$, such that if we set $\epsilon = 1-\delta-(2n/P)$.
then $\epsilon > 0$. Let $f \in L^2(S, d\sigma)$and let $w_{k,j}$ and $z_{k,j}$be as in the statement of the proposition. By Theorem (4.2.1), we have

$$Var^{1/2}(f-Pf;z_{k,j}) \leq C(\delta)\sum_{\ell=1}^{\infty}\frac{1}{2^{(1-\delta)\ell}}\left\|M_{m_{\rho\ell(z_{k,j})}}H_f k_{\rho\ell(z_{k,j})}\right\|$$

for every$(k,j) \in I$. Since Φ_P^- is a norm on $\hat{c}$, it follows that

$\Phi_P^-\left(\{Var^{1/2}(f-Pf;z_{k,j})\}_{(k,j)\in I_m}\right)$

$$\leq C(\delta)\sum_{\ell=1}^{\infty}\frac{1}{2^{(1-\delta)\ell}}\Phi_P^-\left(\left\{\left\|M_{m_{\rho\ell(z_{k,j})}}H_f k_{\rho\ell(z_{k,j})}\right\|\right\}_{(k,j)\in I_m}\right) \tag{47}$$

for every $m \in N$. Next, we define

$$h(w) = \left\|M_{m_w} H_f k_w\right\|, w \in B$$

Then (45) tells us that this map $h: B \to [0, \infty)$ and the points $w_{k,j}$, $(k,j) \in I$, satisfy condition (48). This allows us to apply Lemma (4.2.3) to obtain

$$\sum_{\ell=1}^{\infty} \frac{1}{2^{(1-\delta)\ell}} \Phi_P^{-} \left(\left\{ \left\| M_{m_{\rho\ell(z_{k,j})}} H_f k_{\rho\ell(z_{k,j})} \right\| \right\}_{(k,j)\in I_m} \right)$$

$$\leq P C_{4.3} \sum_{\ell=1}^{\infty} \frac{2^{2n\ell/P}}{2^{(1-\delta)\ell}} \Phi_P^{-} \left(\left\{ h(w_{k,j}) \right\}_{(k,j)\in I_m} \right)$$

$$= P C_{4.3} \sum_{\ell=1}^{\infty} \frac{1}{2^{\epsilon\ell}} \Phi_P^{-} \left(\left\{ \left\| M_{m_{w_{k,j}}} H_f k_{w_{k,j}} \right\| \right\}_{(k,j)\in I_m} \right)$$

Combining this with (47), we see that the proposition holds for the constant $C_{4.4} = PC(\delta) C_{4.3} \sum_{\ell=1}^{\infty} 2^{\epsilon\ell}$.

Propositions (4.1.15) and (4.2.4) represent the two main steps in the proof of the lower bound in Theorem (4.1.2). The remaining step in the proof of the lower bound is to bridge the gap between the right-hand side of (46) and the left-hand side of (34), which only involves existing ideas. Nonetheless, we repeat all the necessary details here for completeness.

Lemma (4.2.5) [4]: There is a constant $0 < C_{5.1} < \infty$ such that

$$\left\|M_{m_z} H_f k_z\right\| \leq \left\|H_f \psi_{z,t}\right\| + C_{5.1} Var^{1/2}(f - Pf; z)$$

for all $f \in L^2(S, d\sigma), z \in B$ and $0 < t \leq 1$.

Proposition (4.2.6) [4]: Given any $2n < p < \infty$, there is a constant $C_{5.2}(p)$ such that the following holds true: Let $f \in L^2(S, d\sigma)$. For each (k,j), let $z_{k,j} \in T_{k,j}$ satisfy the condition

$$Var^{1/2}(f - Pf; z_{k,j}) \geq \frac{1}{2} \sup_{z\in T_{k,j}} Var^{1/2}(f - Pf; z) \qquad (48)$$

Then

$$\Phi_P^{-} \left(\left\{ Var^{1/2}(f - Pf; z_{k,j}) \right\}_{(k,j)\in I} \right) \leq C_{5.2}(P) \left\|H_f\right\|_P^{-}$$

Proof: Let $f \in L^2(S, d\sigma)$be given. For each $(k,j) \in I$ we pick a $w_{k,j} \in T_{k,j}$. such that

$$\left\|M_{m_{w_{k,j}}} H_f k_{w_{k,j}}\right\| \geq \frac{1}{2} \sup_{w\in T_{k,j}} \left\|M_{m_w} H_f k_w\right\|$$

Then by Proposition (4.2.4) we have

$$\Phi_P^-\left(\left\{Var^{1/P}(f-Pf;z_{k,j})\right\}_{(k,j)\in I_m}\right)$$
$$\leq C_{4.4}(P)\Phi_P^-\left(\left\{\left\|M_{m_{w_{k,j}}}H_f k_{w_{k,j}}\right\|\right\}_{(k,j)\in I_m}\right).$$

for every $m\in N$. Applying Lemma (4.2.5) to each $\left\|M_{m_{w_{k,j}}}H_f k_{w_{k,j}}\right\|$, for $0<t\leq 1$ we have

$$\Phi_P^-\left(\left\{Var^{1/P}(f-Pf;z_{k,j})\right\}_{(k,j)\in I_m}\right)$$
$$\leq C_{4.4}(P)\Phi_P^-\left(\left\{\left\|H_f\psi_{w_{k,j,t}}\right\|\right\}_{(k,j)\in I_m}\right)$$
$$+C_{4.4}(P)C_{5.1}t\Phi_P^-\left(\left\{Var^{1/P}(f-Pf;w_{k,j})\right\}_{(k,j)\in I_m}\right).$$

Since $w_{k,j}\in T_{k,j}$. it follows from (48) that $Var(f-Pf;w_{k,j})\leq 2Var(f-Pf;z_{k,j})$. Hence

$$\Phi_P^-\left(\left\{Var^{1/P}(f-Pf;z_{k,j})\right\}_{(k,j)\in I_m}\right)$$
$$\leq C_{4.4}(P)\Phi_P^-\left(\left\{\left\|H_f\psi_{w_{k,j,t}}\right\|\right\}_{(k,j)\in I_m}\right)$$
$$+2C_{4.4}(P)C_{5.1}t\Phi_P^-\left(\left\{Var^{1/P}(f-Pf;z_{k,j})\right\}_{(k,j)\in I_m}\right)$$

Now, for the given $2n<p<\infty$, we pick $0<t\leq 1$ such that $2C_{4.4}(P)C_{5.1}t\leq 1/2$. This fixes the value of t in terms of p, and from the above inequality we obtain

$$\Phi_P^-\left(\left\{Var^{1/P}(f-Pf;z_{k,j})\right\}_{(k,j)\in I_m}\right)$$
$$\leq C_{4.4}(P)\Phi_P^-\left(\left\{\left\|H_f\psi_{w_{k,j,t}}\right\|\right\}_{(k,j)\in I_m}\right)$$
$$+(1/2)\Phi_P^-\left(\left\{Var^{1/P}(f-Pf;z_{k,j})\right\}_{(k,j)\in I_m}\right).$$

Since I_m is a finite set, the quantity $\Phi_P^-\left(\left\{Var^{1/P}(f-Pf;z_{k,j})\right\}_{(k,j)\in I_m}\right)$ is finite. Therefore after the obvious cancellation the above inequality becomes

$$\Phi_P^-\left(\left\{Var^{1/2}(f-Pf;z_{k,j})\right\}_{(k,j)\in I_m}\right)$$
$$\leq 2C_{4.4}(P)\Phi_P^-\left(\left\{\left\|H_f\psi_{w_{k,j,t}}\right\|\right\}_{(k,j)\in I_m}\right)$$

Assuming $\left\|H_f\right\|_P^-<\infty$, an application of Proposition (4.1.15) to the right-hand side gives us

$$\Phi_P^-\left(\left\{Var^{1/2}(f-Pf;z_{k,j})\right\}_{(k,j)\in I_m}\right)\leq 2C_{4.4}(P)C_{3.4}(t,P)M_{3.6}\left\|H_f\right\|_P^-$$

Since this holds for every $m \in N$, by (1) we have

$$\Phi_P^- \left(\{Var^{1/2}(f - Pf; z_{k,j})\}_{(k,j)\in I}\right) \leq 2C_{4.4}(P)C_{3.4}(t,P)M_{3.6}\|H_f\|_P^-$$

Thus the proposition holds for the constant $C_{5.2}(p) = 2C_{4.4}(P)C_{3.4}(t,P)M_{3.6}$

Lemma (4.2.7) [4]: Given any $0 < a < \infty$, there exists a natural number K which depends only on a and the complex dimension n such that the following holds true: Suppose that Γ is an a-separated subset of B. Then there exist pairwise disjoint subsets $\Gamma_1, \dots, \Gamma_k$ of Γ such that $\bigcup_{i=1}^{k} \Gamma_i = \Gamma$ and such that $card(\Gamma_i \cap T_{k,j}) \leq 1$ for all $i \in \{1, \dots, K\}$ and $(k,j) \in I$.

With the above preparation, we now have proof of the lower bound in Theorem (4.1.2) Let $2n < p < 1$ and $a > 0$ be given. We need to find a $0 < C_1 < \infty$ that depends only on p, a and n such that the inequality

$$\Phi_P^- \left(\{Var^{1/P}(f - Pf; z)\}_{z\in\Gamma}\right) \leq C_1\|H_f\|_P^- \qquad (49)$$

holds for every $f \in L^2(S, d\sigma)$and every a-separated Γ set in B.

Let an a-separated set Γ in B be given. Then Lemma (4.2.7) provides the partition

$$\Gamma = \Gamma_1 \cup \dots . \cup \Gamma_k \qquad (50)$$

where K depends only on a and n, such that

$$card(\Gamma_i \cap T_{k,j}) < 1 \; for\ all\ i \in \{1, \dots, K\}\ and\ (k,j) \in I \qquad (51)$$

Let $f \in L^2(S, d\sigma)$. For each $(k,j) \in I\ pick\ z_{k,j} \in T_{k,j}$ such that (48) holds. Combining (48) with (51), we see that

$$\Phi_P^- \left(\{Var^{1/P}(f - Pf; z)\}_{z\in\Gamma_i}\right) \leq 2\Phi_P^- \left(\{Var^{1/P}(f - Pf; z_{k,j})\}_{(k,j)\in I}\right)$$

for every $i \in \{1, \dots, K\}$. Proposition (4.2.6) tells us that

$$\Phi_P^- \left(\{Var^{1/P}(f - Pf; z_{k,j})\}_{(k,j)\in I}\right) \leq C_{5.2}(P)\|H_f\|_P^-$$

Therefore

$$\Phi_P^- \left(\{Var^{1/P}(f - Pf; z)\}_{z\in\Gamma_i}\right) \leq 2C_{5.2}(P)\|H_f\|_P^-$$

$i \in \{1, \dots, K\}$.. By (50) and (35) we have

$$\Phi_P^- \left(\{Var^{1/P}(f - Pf; z)\}_{z\in\Gamma}\right) \leq \sum_{i=1}^{k} \Phi_P^- \left(\{Var^{1/P}(f - Pf; z)\}_{z\in\Gamma_i}\right)$$

By the above two inequalities, (49) holds for the constant $C_1 = 2KC_{5.2}(P)$.

We now turn to the upper bound in Theorem (4.1.2). One of the main ingredients in the proof of the upper bound is a reverse HÖlder's inequality. But whereas for the symmetric gauge function Φ_P^+ , here the inequality must cover Φ_P^-, which makes its proof a much more difficult task. We will see that the key to the proof of the reverse Holder's inequality is a certain cancellation, and what enables this cancellation to take place is a certain "small factor". Here we must take an approach that is to obtain the requisite "small factor".

For any $a = \{a_j\}_{j \in \boldsymbol{N}}$ and $N \in \boldsymbol{N}$, define the sequence $a^{[N]} = \{a_j^N\}_{j \in \boldsymbol{N}}$ by the formula

$$a_j^N = a_i \text{ if } (i-1)N + 1 \leq j \leq iN \text{ , } i \in \boldsymbol{N} \qquad (52)$$

In other words, $a^{[N]}$ is obtained from a by repeating each term N times. Alternately, we can think of $a^{[N]} as\ a \oplus \ldots \oplus a,$ the "direct sum" of N copies of a.

Definition (4.2.8) [4]: A symmetric gauge function Φ is said to satisfy condition (DQK) if there exist constants $0 < \theta < 1$ and $0 < \alpha < \infty$ such that

$$\Phi\left(a^{[N]}\right) \geq \alpha N^{\theta} \Phi(a)$$

for every $a \in \hat{c}$ and every $N \in \boldsymbol{N}$.

The relevance of Definition (4.2.8) to what we do in this chapter is the following:

Lemma (4.2.9) [4]: For each $1 < p < \infty$, the symmetric gauge function Φ_P^- satisfies condition (DQK). More precisely, we have $\Phi_P^-\left(a^{[N]}\right) \geq N^{1/P} \Phi_P^-(a)$ for all $a \in \hat{c}$ and $N \in \boldsymbol{N}$.

Proof: Let $1 < p < \infty$. It suffices to consider $a = \{a_j\}_{j \in \boldsymbol{N}}$ where the terms are nonnegative and in descending order, i.e.,

$$a_1 \geq a_2 \geq \cdots \geq a_j \geq \cdots$$

Then by (52) and the definition of Φ_P^-, for every $N \in \boldsymbol{N}$ we have

$$\Phi_P^-\left(a^{[N]}\right) = \sum_{i=1}^{\infty} a_i \sum_{j=1}^{N} \frac{1}{\left((i-1)N + j\right)^{(P-1)/P}} \geq \sum_{i=1}^{\infty} \frac{a_i N}{(iN)^{(P-1)/P}}$$
$$= N^{1/P} \Phi_P^-(a)$$

as promised.

The proof of the reverse Holder's inequality for Φ_P^- will be based on condition (DQK).

But for the proof itself it will be more convenient to work with (DQK), rather than with the specific Φ_P^- .

Recall that for each $k \geq 0$, we introduced $\{u_{k,1}, \dots . u_{k,m}(k)\}$, which is a subset of S that is maximal with respect to (20). For each $(k,j) \in I$, we now define

$$A_{k,j} = B(u_{k,j}, 2^{-k+1}), B_{k,j} = B(u_{k,j}, 2^{-k+2}) \qquad (53)$$

and

$$C_{k,j} = B(u_{k,j}, 2^{-k+3}).$$

Definition (4.2.10) [4]: For each $i \in Z_+$ and each $(k,j) \in I$, we set $E_i(k,j) = \{(k+i,j') \in I: A_{k+i,j} \cap B_{k,j} \neq 0\}$.

Definition (4.2.11) [4]: Suppose that $g \in L^2(S, d\sigma)$.

(a) For each $1 \leq t < \infty$ and each $(k,j) \in I$, define

$$J_t(g;k,j) = (\frac{1}{\sigma(B_{k,j})}\int_{B_{k,j}} |g - gB_{k,j}|\, d\sigma)^{1/t}$$

(b) For each $k \in Z_+$, define the function Rkg on S by the formula

$$(R_{k,g})(\zeta) = \frac{1}{\sigma(B(\zeta, 2^{-k-2}))}\int_{B(\zeta,2^{-k-2})} g d\sigma\,, \qquad \zeta \in S$$

(c) For $1 \leq t < \infty$, $i \in Z_+$ and $(k,j) \in I$, define

$$G_{t,i}(g;k,j) = (\frac{1}{\sigma(B_{k,j})}\int_{B_{k,j}} |g - R_{k+i}g|\, d\sigma)^{1/t}$$

and

$$H_{t,i}(g;k,j) = (\frac{1}{\sigma(B_{k,j})}\int_{B_{k,j}} |R_{k+i}g - gB_{k,j}|\, d\sigma)^{1/t}$$

(d) For each $(k,j) \in I$, define

$$J(g;k,j) = \frac{1}{\sigma(C_{k,j})}\int_{C_{k,j}} |g - gC_{k,j}|\, d\sigma.$$

Lemma (4.2.12) [4]: There is a constant $C_{6.5}$ such that

$$G_{t,i}^t(g;k,j) \leq 2^{t-1}C_{6.5}\sum_{(k+i,j')\in E_i(k,j)} \frac{\sigma(B_{k+i,j'})}{\sigma(B_{k,j})} J_t^t(g;k+v,j')$$

for all $g \in L^2(S, d\sigma)$, $1 \leq t < \infty$, $i \in Z_+$ and $(k,j) \in I$.

Proof: By (19) and (53), there is a constant C_1 such that

$$\frac{\sigma(B_{k+i,j'})}{\sigma(B(\zeta, 2^{-k-i-2}))} \leq c_1.$$

for all k; $i \in Z_+, j' \in \{1, \dots, m(k + i)\}$ and $\zeta \in S$. Let $g \in L^2(S, d\sigma)$, $1 \le t < \infty$, 1, $i \in Z_+$ and $(k,j) \in I$. Then by Definition (4.2.10) and (21) we have

$$\int_{B_{k,j}} |g - R_{k+i}g|^t \, d\sigma \le \sum_{(k+i,j') \in E_i(k,j)} \int_{A_{k+v,j'}} |g - R_{k+i}g|^t \, d\sigma$$

$$\le 2^{t-1} \sum_{(k+i,j') \in E_i(k,j)} \int_{B_{k+i,j'}} \left|g - gB_{k+i,j'}\right|^t d\sigma$$

$$+ 2^{t-1} \sum_{(k+i,j') \in E_i(k,j)} \int_{A_{k+i,j'}} \left|gB_{k+i,j'} - R_{k+i}g\right|^t d\sigma \qquad (54)$$

For each $\zeta \in A_{k+v,j'}$ we have $B(\zeta, 2^{-k-i-2}) \subset B_{k+v,j'}$. Therefore

$$\left|gB_{k+v,j'} - R_{k+i}g(\zeta)\right|^t \le \frac{1}{\sigma B(\zeta, 2^{-k-i-2})} \int_{B(\zeta, 2^{-k-i-2})} \left|gB_{k+v,j'} - g\right|^t d\sigma$$

$$\le C_1 J_t^t(g; k + v, j')$$

for every $\zeta \in A_{k+v,j'}$. Hence

$$\int_{A_{k+v,j'}} \left|gB_{k+v,j'} - R_{k+i}g\right|^t d\sigma \le C_1 \sigma\big(A_{k+v,j'}\big) J_t^t(g; k + v, j').$$

Substituting this in (54), we see that if we set $C_{6.5} = 1 + C_1$, then the lemma holds.

Lemma (4.2.13) [4]: Suppose that X and Y are countable sets and that N is a natural number. Suppose that $T: X \to Y$ is a map that is at most N-to-That is, for every $y \in Y$, $card\{x \in X: T(x) = y\} \le N$. Then for every set of real numbers $\{b_y\}_{y \in Y}$,and every symmetric gauge function Φ, we have

$$\Phi\left(\{b_{T(x)}\}_{x \in X}\right) \le N\Phi\left(\{b_y\}_{y \in Y}\right)$$

The next lemma is the most crucial step in the proof of our reverse Holder's inequality: extraction of the requisite “small factor”.

Lemma (4.2.14) [4]: Let Φ be a symmetric gauge function satisfying condition (DQK). Let $1 \le t < \infty$ and $\epsilon > 0$ also be given. Then there exists a natural number $v \in N$ which depends only on Φ, t, ϵ and the complex dimension n such that

$$\Phi\left(\{J_t\ (g; k+v, \eta(k,j))\}_{(k,j)\in I_m}\right) \leq \epsilon\Phi\left(\{J_t\ (g; k,j)\}_{(k,j)\in I_{m+v}}\right)$$

for all $g \in L^2(S, d\sigma)$. and $m \in N$.

Proof: We begin by fixing a number of necessary constants. First of all, by (20) and (19), there is a natural number $M_1 \in N$ such that

$$card\{j' \in \{1, ..., m(k)\}: C_{k,j'} \cap C_{k,j} \neq 0\} \leq M_1 \tag{55}$$

for every $(k,j) \in I$. Let $m \in N$. By a standard maximality argument, there is a partition

$$I_m = I_1 \cup ....\cup I_{M_1} \tag{56}$$

of the truncated index set I_m such that for each pair of $q \in \{1, ..., M_1\}$ and $k \in Z_+$, if

$$(k,j); (k,j') \in I_q \ and\ j \neq j' \tag{57}$$

then $C_{k,j} \cap C_{k,j'} = 0$.

Again by (20) and (19), there are constants $0 < c_1 \leq C_1 < \infty$ such that

$$c_1 2^{-2ni} \leq \frac{\sigma\left(B(\zeta, 2^{-i}r)\right)}{\sigma(B(\xi, r))} \leq c_1 2^{-2ni} \tag{58}$$

holds for all $\zeta, \xi \in S, 0 < r \leq\ 8$ and $i \in Z_+$. In particular, we have $\sigma(B_{k,j}) \leq C_2\sigma\left(A_{k,j}\right)$and $\sigma(C_{k,j}) \leq C_2\sigma\left(B_{k,j}\right)$ for every $(k,j) \in I$, where $C_2 = (2^{2n}/c_1)$. Note that for every $i \in Z_+$, if $(k+i, j') \in E_i(k,j)$, then $A_{k+ij'} \subset C_{k,j}$. Combining these facts with (55), we see that if we set $C_3 = M_1 C_2^2$, then

$$\sum_{(k+v,j')\in E_i(k,j)} \frac{\sigma\left(B_{k+v,j'}\right)}{\sigma\left(B_{k,j}\right)} \leq C_3 \tag{59}$$

for all $i \in Z_+$,and $(k,j) \in I$.

Suppose that Φ is a symmetric gauge function satisfying condition (DQK). Then Definition (4.2.8) implies that there exist constants $0 < \theta < 1$ and $0 < C_4 < \infty$ such that

$$\Phi(a) \leq C_4 N^{-\theta}\Phi\left(a^{[N]}\right) for\ all\ a\ \in \hat{c}\ and\ N \in \boldsymbol{N} \tag{60}$$

Let $1 \leq t < \infty$ be given. We write $C_5 = 2^{t-1}C_{6.5}$, where $C_{6.5}$ is the constant provided by Lemma (4.2.12) Let $\epsilon > 0$ also be given. We pick an $N_0 \in \boldsymbol{N}$ such that

$$(4C_3)^{1/t} C_4 N^{-\theta} \leq \frac{\epsilon}{2M_1 C_5^{1/t}} \tag{61}$$

Finally, with No so chosen, we pick a $v \in\ N$ such that

$$(4N_0C_1C_5 2^{-2nv})^{1/t}M_1 \le \epsilon/2 \qquad (62)$$

What remains is to show that the lemma holds for this v.

Let $g \in L^2(S, d\sigma)$ be given. It suffice to consider the case where $J_t(g; k, j) < \infty$ for every $(k, j) \in I_{m+v}$ For each$(k, j) \in I_m$ Lemma (4.2.12) gives us

$$G_{t,v}^t(g; k, j) \le C_5 \sum_{(k+v,j')\in E_v(k,j)} \frac{\sigma(B_{k+v,j'})}{\sigma(B_{k,j})} J_t^t(g; k+v, j')$$

$$= C_5 \sum_{(k+v,j')\in \tilde{E}_v(k,j)} \frac{\sigma(B_{k+v,j'})}{\sigma(B_{k,j})} J_t^t(g; k+v, j') \qquad (63)$$

Where $\tilde{E}_v(k,j) = \{(k+v, j') \in E_v(k,j) : J_t^t(g; k+v, j') > 0\}$. Now, for every $(k, j) \in I_m$, we have the decomposition

$$\tilde{E}_v(k,j) = \bigcup_{\ell=-\infty}^{\infty} X_\ell(k,j) \, .$$

where $X_\ell(k,j)$, is the collection of $(k+v, j') \in E_v(k,j)$, satisfying the condition

$$2^{\ell-1} < J_t^t(g; k+v, j') \le 2^\ell \qquad (64)$$

$\ell \in Z$. For each $(k, j) \in I_m$, define the sets

$$Z^{(1)}(k,j) = \{\ell \in Z : 1 \le card(X_\ell(k,j)) \le N_0\}$$

and

$$Z^{(2)}(k,j) = \{\ell \in Z : card(X_\ell(k,j)) \le N_0\}$$

It follows from (63) that

$$G_{t,v}^t(g; k, j) \le C_5\{T^{(1)}(k,j) + T^{(2)}(k,j)\} \qquad (65)$$

Where, for $i = 1,2$,

$$T^{(i)}(k,j) = \sum_{\ell\in z^{(i)}(k,j)} \sum_{(k+v,j')\in X_\ell(k,j)} \frac{\sigma(B_{k+v,j'})}{\sigma(B_{k,j})} J_t^t(g; k+v, j')$$

Let us first consider $T^{(1)}(k,j)$. Suppose that $(k,j) \in I_m$is such that $z^{(i)}(k,j) \ne 0$.

Since $E_v(k,j)$ is afinite set, the set $z^{(1)}(k,j)$is also finite and, consequently, has a largest element $\mu(k,j)$. Thus there is an $\eta(k,j) \in \{1, \dots, m(k+v)\}$ such that $(k+v, \mu(k,j)) \in X_{\mu(k,j)}(k,j)$. By (64), we have

$$2^{\mu(k,j)} \le 2J_t^t(g; k+v, \eta(k,j))$$

By (58), $\sigma(B_{k+v,j'}/\sigma(B_{k,j}) \le C_1 2^{-2nv}$. Since $card(X_\ell(k,j)) \le N_0$ for every $\ell \in z^{(1)}(k,j)$and since $\mu(k,j)$ is the largest element in $z^{(1)}(k,j)$, we have

$$T^{(1)}(k,j) \leq \sum_{\ell=-\infty}^{\mu(k,j)} N_0 C_1 2^{-2nv} 2^{\ell} = 2C_1 2^{-2nv} 2^{\mu(k,j)}$$
$$\leq 4N_0 C_1 2^{-2nv} J_t^t(g; k+v, \eta(k,j))$$

If $(k,j) \in I_m$ is such that $z^{(1)}(k,j) = 0$ then $T^{(1)}(k,j) = 0$. Thus we conclude that for every $(k,j) \in I_m$, there is an $\eta(k,j) \in \{1, \dots, m(k+v)\}$ such that $(k+v, \mu(k,j)) \in E_v(k,j)$ and such that

$$T^{(1)}(k,j) \leq 4N_0 C_1 2^{-2nv} J_t^t\big(g; k+v, \eta(k,j)\big) \quad (66)$$

Now define the map $\varphi: I_m \to I_{m+v}$. by the formula $\varphi(k,j) = (k+v, \eta(k,j))$.

$(k,j) \in I_m$. If $k \in Z_+$ and $j_1, j_2 \in \{1, \dots, m(k)\}$ are such that $\eta(k,j_1) = \eta(k,j_1)$, then, by the definition of η we have $E_v(k,j_1) \cap E_v(k,j_2) \neq 0$ By (53), if $A_{k+i,j'} \cap B_{k,j} \neq 0$,then $A_{k+i,j'} \subset C_{k,j}$. Hence the condition $E_v(k,j_1) \cap E_v(k,j_2) \neq 0$, implies $C_{k,j_1} \cap C_{k,j_2} \neq 0$. By (55), the map $\varphi: I_m \to I_{m+v}$is at most M_1-to-1. Thus Lemma (4.2.13) gives us

$$\Phi\left(\left\{J_t\ (g; k+v, \eta(k,j))\right\}_{(k,j)\in I_m}\right) = \Phi\left(\left\{J_t\big(g; \varphi(k,j)\big)\right\}_{(k,j)\in I_m}\right)$$
$$\leq \mathrm{M}_1 \Phi\left(\left\{J_t(g; k,j)\right\}_{(k,j)\in I_{m+v}}\right)$$

By (66), we have

$$\left(T^{(1)}(k,j)\right)^{1/t} \leq (4N_0 C_1 2^{-2nv})^{1/t} J_t(g; k+v, \eta(k,j))$$

for every $(k,j) \in I_m$The combination of these two inequalities gives us

$$\Phi\left(\left\{\left(T^{(1)}(k,j)\right)^{1/t}\right\}_{(k,j)\in I_m}\right)$$
$$\leq (4N_0 C_1 2^{-2nv})^{1/t} \mathrm{M}_1 \Phi\big(\{J_t(g; k,j)\}_{(k,j)\in I_{m+v}}\big) \quad (67)$$

It follows from (65) that

$$G_{t,v}(g; k,j) \leq C_5^{1/t} \left\{\left(T^{(1)}(k,j)\right)^{1/t} + \left(T^{(2)}(k,j)\right)^{1/t}\right\}$$

Hence

$$\Phi\left(\left\{G_{t,v}(g; k,j)\right\}_{(k,j)\in I_m}\right)$$
$$\leq C_5^{1/t} \Phi\left(\left\{\left(T^{(1)}(k,j)\right)^{1/t}\right\}_{(k,j)\in I_m}\right)$$
$$+ C_5^{1/t} \Phi\left(\left\{\left(T^{(2)}(k,j)\right)^{1/t}\right\}_{(k,j)\in I_m}\right)$$
$$\leq (\epsilon/2)\Phi\big(\{J_t(g; k,j)\}_{(k,j)\in I_{m+v}}\big) + C_5^{1/t} \Phi\left(\left\{\left(T^{(2)}(k,j)\right)^{1/t}\right\}_{(k,j)\in I_m}\right).$$

where the second $\leq$follows from (67) and (62). Thus the proof of the lemma is reduced to the proof of the inequality

$$C_5^{1/t}\Phi\left(\left\{\left(T^{(2)}(k,j)\right)^{1/t}\right\}_{(k,j)\in I_m}\right) \le (\epsilon/2)\Phi\left(\{J_t(g;k,j)\}_{(k,j)\in I_{m+v}}\right)$$

By (56) and (35), this inequality will follow if we can show that

$$\Phi\left(\left\{\left(T^{(2)}(k,j)\right)^{1/t}\right\}_{(k,j)\in I_q}\right) \le \frac{\epsilon}{2M_1C_5^{1/t}}\Phi\left(\{J_t(g;k,j)\}_{(k,j)\in I_{m+v}}\right) \quad (68)$$

for every $q \in \{1, \dots, M_1\}$.

To prove (68), consider any $q \in \{1, \dots, M_1\}$ and define $\tilde{I}_q = \{(k,j) \in I_q: z^{(2)}(k,j) \neq 0\}$. Again, each $z^{(2)}(k,j)$.is a finite set because $card(E_v(k,j)) < \infty$. Thus for each $(k,j) \in \tilde{I}_q$, $z^{(2)}(k,j)$. has a largest element $\lambda(k,j)$That is,

$$card(X_{\lambda(k,j)}(k,j) > N_0 \quad (69)$$

and $\ell \notin z^{(2)}(k,j)\ if\ \ell > \lambda(k,j)$. For each $(k,j) \in \tilde{I}_q$, pick an $h(k,j) \in \{1, \dots, m(k+v)\}$ such that $(k+v, h(k,j)) \le X_{\lambda(k,j)}(k,j)$.

Since $X_{\lambda(k,j)}$ is the largest element in $z^{(2)}(k,j)$, by (64) we have

$$J_t^t(g;k+v,j') \le 2J_t^t(g;k+v,(k,j))\ for\ every\ (k+v,j') \in \bigcup_{\ell\in z^{(2)}(k,j)} X_\ell(k,j)$$

Combining this with the definition of $T^{(2)}(k,j)$ and with (59), we obtain $T^{(2)}(k,j) \le 2C_3J_t^t(g;k+v,(k,j))$.

Thus $\left(T^{(2)}(k,j)\right)^{1/t} \le (2C_3)^{1/t}J_t\ \left(g;k+v,(k,j)\right)$ for every $(k,j) \in \tilde{I}_q$.

Consequently,

$$\Phi\left(\left\{\left(T^{(2)}(k,j)\right)^{1/t}\right\}_{(k,j)\in I_q}\right) = \Phi\left(\left\{\left(T^{(2)}(k,j)\right)^{1/t}\right\}_{(k,j)\in \tilde{I}_q}\right) \le (2C_3)^{1/t}\Phi\left(\left\{J_t\ (g;k+v,h(k,j))\right\}_{(k,j)\in\tilde{I}_q}\right) \quad (70)$$

Recall that the condition $A_{k+i,j'} \cap B_{k,j} \neq 0$; implies $A_{k+i,j'} \subset C_{k,j}$. Combining this fact with (57), we have $E_v(k_1,j_1) \cap\ E_v(k_2,j_2) = 0$; for all $(k_1,j_1) \neq (k_2,j_2)$ in $\tilde{I}_q$Therefore

$$X_{\lambda(k_1,j_1)}(k_1,j_1) \cap X_{\lambda(k_2,j_2)}(k_2,j_2) = 0\ for\ all\ (k_1,j_1) \neq (k_2,j_2) \quad (71)$$

in $\tilde{I}_q$.

Note that (64) also gives us

$$J_t\ (g; k+v,(k,j)) \leq 2^{1/t} J_t\ (g; k+v, j') \tag{72}$$
for every $(k+v, j') \in X_{\lambda(k,j)}(k,j)$.

If $(k,j) \in \tilde{I}_q$, then, of course, $X_{\lambda(k,j)}(k,j) \subset I_{m+v}$. Thus, if we re-enumerate the numbers $\{J_t\ (g; k+v, h(k,j))\}_{(k,j)\in\tilde{I}_q}$ in the form $b = \{b_1, \dots, b_i\}$, then it follows from the combination of (72), (71) and (69) that $\Phi\left(b^{[N_0]}\right) \leq 2^{1/t}\Phi\left(\{J_t(g;k,j)\}_{(k,j)\in I_{m+v}}\right)$.

Applying (60), we now have

$$\Phi\left(\{J_t\ (g; k+v, h(k,j))\}_{(k,j)\in\tilde{I}_q}\right) = \Phi(b) \leq C_4 N_0^{-\theta}\Phi\left(b^{[N_0]}\right)$$
$$\leq 2^{1/t} C_4 N_0^{-\theta}\Phi\left(\{J_t(g;k,j)\}_{(k,j)\in I_{m+v}}\right)$$

Combining this with (70) and (61), we have

$$\Phi\left(\left\{\left(T^{(2)}(k,j)\right)^{1/t}\right\}_{(k,j)\in I_q}\right) \leq (4C_3)^{1/t} C_4 N_0^{-\theta}\Phi\left(\{J_t(g;k,j)\}_{(k,j)\in I_{m+v}}\right)$$
$$\leq \frac{\epsilon}{2M_1 C_5^{1/t}}\Phi\left(\{J_t(g;k,j)\}_{(k,j)\in I_{m+v}}\right). \tag{73}$$

This proves (68) and completes the proof of the lemma.

Lemma (4.2.15) [4]: There exists a constant $C_{6.8}$ which depends only on the complex dimension n such that the inequality

$$H_{t,i}(g; k, j) \leq C_{6.8} 2^{2ni} J(g; k, j).$$

holds for all $g \in L^2(S, d\sigma)$, $(k,j) \in I.\ i \in Z_+$ and $1 \leq t < \infty$.

Proof: Let $g \in L^2(S, d\sigma)$ and $(k,j) \in I$. If $\zeta \in B_{k,j}$ and $i \in Z_+$ then $B(\zeta, 2^{-k-i-2}) \subset C_{k,j}$, and consequently

$$|(R_{k+i}g)(\zeta) - gC_{k+i}| \leq \frac{1}{B(\zeta, 2^{-k-i-2})} \int_{B(\zeta, 2^{-k-i-2})} |g - gC_{k+i}|\, d\sigma$$
$$\leq \frac{\sigma(C_{k+i})}{B(\zeta, 2^{-k-i-2})} \cdot \frac{1}{\sigma(C_{k+i})} \int_{\sigma(C_{k+i})} |g - gC_{k+i}|\, d\sigma$$
$$\leq (2^{10n}/C_1) 2^{2ni} J(g; k, j).$$

where the third $\leq$ follows from (58). On the other hand,

$$|gC_{k+i} - gB_{k+i}| \frac{1}{\sigma(B_{k+i})} \int_{B_{k+i}} |gC_{k+i} - g|\, d\sigma \leq \frac{\sigma(C_{k+i})}{\sigma(B_{k+i})} \cdot \frac{1}{\sigma(C_{k+i})} \int_{C_{k+i}} |gC_{k+i} - g|\, d\sigma$$
$$\leq (2^{2n}/C_1) 2^{2ni} J(g; k, j).$$

where the last $\leq$ again follows from (58). Write $C_{6.8} = (2^{10n}/C_1) + (2^{2n}/C_1)$. Then the above two inequalities together give us

$$|(R_{k+i}g)(\zeta) - gC_{k+i}| \le C_{6.8}2^{2ni}J(g;k,j).$$
for every $\zeta \in B_{k,j}$. Recalling Definition (4.2.11) (c), the lemma follows.

Definition (4.2.16) [4]:

(a) For each $(k,j) \in I$, we set $E(k,j) = \{(k',j') \in I : k' \le k, d(u_{k',j'}, u_{k,j}) < 2^{-k+5}\}$ and $G(k,j) = \{(k',j') \in I : k' \le k, A_{k',j'} \cap B_{k+i} \ne 0\}$.

(b) For $g \in L^2(S, d\sigma)$ and $(k,j) \in I$. we set $M(g;k,j) = sup\{J(g;k',j') : (k',j') \in E(k,j)\}$.

Proposition (4.2.17) [4]: Let $1 \le t < \infty$. Then there exists a constant $C_{6.10} = C_{6.10}(t,n)$ such that the inequality
$$J_t(g;k,j) \le C_{6.10}M(g;\ k;\ j).$$
holds for all $g \in L^2(S, d\sigma)$ and $(k,j) \in I$,

Obviously, Proposition (4.2.17) follows from a more structured version of the well-known John-Nirenberg theorem, a version that incorporates our particular decomposition scheme (20), (21) and (53). As such, the proof of Proposition (4.2.17) is relegated to the Appendix [4].

Proposition (4.2.18) [4]: Let $1 \le t < \infty$. There exists a constant $C_{6.11} = C_{6.11}(t,n)$ such that if Φ is any symmetric gauge function, $g \in L^2(S, d\sigma)$and $\ell \in Z_+$, then
$$\Phi\left(\{J_t(g;\ell,i)\}_{i=1}^{m(\ell)}\right) \le C_{6.11}\Phi(\{J(g;k,j)\}_{(k,j)\in I}).$$

Proof: By (20) and (19), there is a natural number L such that the inequality
$$card\{j' \in \{1, \dots, m(k)\} : d(u_{k,j'}, u_{k,j}) < 2^{-k+6}\} \le L \qquad (74)$$
holds for every $(k,j) \le I$. Let $1 \le t < 1$ be given Let $g \in L^2(S, d\sigma)$ and symmetric gauge function Φ also be given To prove (73), it suffices to consider the case where $\Phi(\{J(g;k,j)\}_{(k,j)\in I}) < \infty$. Note that this implies
$$\sup_{(k,j)\le I} J(g;k,j) < \infty.$$

Let $\ell \in Z_+$. Then for each $i \in \{1, \dots, m(k)\}$ there is an $h(i) \in E(\ell, i)$ such that
$$J(g;h(i)) \ge \frac{1}{2}M(g;\ell,i).$$
Applying Proposition (4.2.17), we have
$$J_t(g;\ell,i) \le C_{6.10}M(g;\ell,i) \le 2C_{6.10}J(g;h(i)).$$
$i \in \{1, \dots, m(k)\}$. Consequently,

$$\Phi\left(\{J(g;\ell,i)\}_{i=1}^{m(\ell)}\ \right) \le 2C_{6.10}\Phi\left(\{J(g;h(i))\}_{i=1}^{m(\ell)}\ \right) \qquad (75)$$

If $i,i' \in \{1,\dots,m(k)\}$ are such that $h(i)=h(i')$, then $E(\ell,i) \cap E(\ell,i') \neq 0$ which means that there is some (k_0,j_0) such that $d(u_{\ell,i}\,,u_{k_0,j_0}) < 2^{-\ell+5}$ and $d(u_{\ell,i'}\,,u_{k_0,j_0}) < 2^{-\ell+5}$.

Hence if $h(i)=h(i')$, then $d(u_{\ell,i}\,,u_{\ell,i'}) < 2^{-\ell+6}$. Thus, by (74), the map $h:\{1,\dots,m(\ell)\} \to I$. is at most L-to-1. Therefore it follows from Lemma (4.2.13) that

$$\Phi\left(\{J(g;h(i))\}_{i=1}^{m(\ell)}\ \right) \le L\Phi\left(\{J(g;k,j)\}_{(k,j)\in I}\right).$$

Combining this with (75), we see that the proposition holds for the constant $C_{6.11} = 2LC_{6.10}$.

After the extensive preparation above, here is our reverse HÖlder's inequality:

Proposition (4.2.19) [4]: Let Φ be a symmetric gauge function satisfying condition (DQK), and let $1 \le t < \infty$. Then there exists a constant $C_{6.12}$ which depends only on Φ ,t and the complex dimension n such that

$$\Phi\left(\{J_t(g;k,j)\}_{(k,j)\in I}\right) \le C_{6.12}\Phi\left(\{J(g;k,j)\}_{(k,j)\in I}\right) \qquad (76)$$

for every $g \in L^2(S,d\sigma)$

Proof: Given $\Phi$and t as in the statement of the proposition, Lemma (4.2.14) provides a $v \in N$ such that

$$\Phi\left(\{G_{t,v}(g;k,j)\}_{(k,j)\in I_m}\right) \le \frac{1}{2}\Phi\left(\{J_t(g;k,j)\}_{(k,j)\in I_{m+v}}\right) \qquad (77)$$

for all $g \in L^2(S,d\sigma)$ and $m \in N$. By Lemma (4.2.15), we also have

$$\Phi\left(\{H_{t,v}(g;k,j)\}_{(k,j)\in I_m}\right) \le C_{6.8}2^{2nv}\Phi\left(\{J(g;k,j)\}_{(k,j)\in I_m}\right) \qquad (78)$$

for all $g \in L^2(S,d\sigma)$and $m \in$ To prove (76), we only need to consider $g \in L^2(S,d\sigma)$ satisfying the condition $\Phi\left(\{J(g;k,j)\}_{(k,j)\in I}\right) < \infty$. By Proposition (4.2.17), this implies $J_t(g;k,j) < \infty$ for every $(k,j) \in I$.

Since $I_{m+v} = I_m \cup \{I_{m+v}\backslash I_m\}$, by (35) we have

$$\begin{aligned}
&\Phi\left(\{J_t(g;k,j)\}_{(k,j)\in I_{m+v}}\right)\\
&\qquad \le \Phi\left(\{J_t(g;k,j)\}_{(k,j)\in I_m}\right) + \Phi\left(\{J_t(g;k,j)\}_{(k,j)\in I_{m+v}\backslash I_m}\right)\\
&\qquad \le \Phi\left(\{J_t(g;k,j)\}_{(k,j)\in I_m}\right) + \sum_{\ell=m+1}^{m+v}\Phi\left(\{J_t(g;\ell,i)\}_{i=1}^{m(\ell)}\right).
\end{aligned}$$

Applying Proposition (4.2.18), we obtain

$$\Phi\left(\{J_t(g;k,j)\}_{(k,j)\in I_{m+v}}\right)$$
$$\leq \Phi\left(\{J_t(g;k,j)\}_{(k,j)\in I_m}\right) + vC_{6.11}\Phi\left(\{J(g;k,j)\}_{(k,j)\in I}\right)$$

Substituting this in (77), we have

$$\Phi\left(\left\{G_{t,v}(g;k,j)\right\}_{(k,j)\in I_m}\right)$$
$$\leq \frac{1}{2}\Phi\left(\{J_t(g;k,j)\}_{(k,j)\in I_m}\right) + vC_{6.11}\Phi\left(\{J(g;k,j)\}_{(k,j)\in I}\right).$$

By Definition (4.2.11), $J_t(g;k,j) \leq G_{t,v}(g;k,j) + H_{t,v}(g;k,j)$. Thus, combining the above inequality with (78), we find that

$$\Phi\left(\{J_t(g;k,j)\}_{(k,j)\in I_m}\right)$$
$$\leq \Phi\left(\left\{G_{t,v}(g;k,j)\right\}_{(k,j)\in I_m}\right) + \Phi\left(\left\{H_{t,v}(g;k,j)\right\}_{(k,j)\in I_m}\right)$$
$$\leq \frac{1}{2}\Phi\left(\{J_t(g;k,j)\}_{(k,j)\in I_m}\right)$$
$$+ (vC_{6.11} + C_{6.8}2^{2nv})\Phi\left(\{J(g;k,j)\}_{(k,j)\in I_m}\right)$$

Thus the obvious cancellation in the above leads to

$$\Phi\left(\{J_t(g;k,j)\}_{(k,j)\in I_m}\right) \leq 2(vC_{6.11} + C_{6.8}2^{2nv})\Phi\left(\{J(g;k,j)\}_{(k,j)\in I_m}\right)$$

Since $m \in N$ is arbitrary, recalling (1), we conclude that the proposition holds for the constant $C_{6.12} = 2(vC_{6.11} + C_{6.8}2^{2nv})$.

Section (4.3): Upper Bound

We now turn to the estimate of $\|P, M_g\|_P^-$. As it happens, this estimate involves a new and quite elaborate interpolation scheme. In other words, this is not the standard kind of interpolation [3]. Our estimate of $\|P, M_g\|_P^-$will be realized through an interpolation between the norms $\|\cdot\|_{r'}^+$ and $\|\cdot\|_r^+$ where $r' < p < r$. What complicates the matter is that estimates of $\|\cdot\|_{r'}^+$ and $\|\cdot\|_r^+$ are themselves obtained through interpolation between Schatten classes. Thus the estimate of $\|[P, M_g]\|_P^-$is really a two-stage interpolation.

For each operator A we introduce the distribution function

$$N_A(s) = card\{j \in N: s_j(A) > s\},$$

$s > 0$, where $s_1(A), s_2(A), \dots, s_j(A), \dots$ are the s-numbers of A. Also recall that we have the inequality

$$N_{A+B}(s) \leq N_A(s/2) + N_B(s/2)$$

We define the measure

$$d\mu(x, y) = \frac{d\sigma(x) d\sigma(y)}{|1 - \langle x, y\rangle|^{2n}}$$

on $S \times S$. For each $1 < p < \infty$, let $L^P_{sym}(S \times S, d\mu)$ be the collection of functions F on $S \times S$ which are L^P with respect to $d\mu$ and which satisfy the condition

$$|F(x, y)| = |F(y, x)|, \qquad (x, y) \in S \times S.$$

For each $F \in L^P_{sym}(S \times S, d\mu)$, define T_F to be the integral operator on $L^2(S, d\sigma)$with the kernel function

$$K_F(x, y) = \frac{F(x, y)}{(1 - \langle x, y\rangle)^n}.$$

For these operators we have the following weak-type inequality:

Proposition (4.3.1) [4]: Given any $2 < p < \infty$, there is a constant $C_{7.1} = C_{7.1}(p, n)$ such that

$$N_{TF}(t) \leq \frac{C_{7.1}}{t^P} \int \int \frac{|F(x, y)|^P}{|1 - \langle x, y\rangle|^{2n}} d\sigma(x) d\sigma(y).$$

For all $F \in L^P_{sym}(S \times S, d\mu)$ and $t > 0$.

Definition (4.3.3) [4]:

(a) A subset Y of $S \times S$ is said to be symmetric if for every $(x, y) \in S \times S$, we have $(x, y) \in Y$ if and only if $(x, y) \in Y$.

(b) $S \times S$, we let $C(g, Y\ )$
denote the integral operator on $L^2(S, d\sigma)$with the kernel function

$$\chi Y(x, y) = \frac{g(y) - g(x)}{(1 - \langle x, y\rangle)^n}.$$

Definition (4.3.4) [4]: (a) For each $k \in Z_+$, let $E_k = \{(x, y) \in S \times S : 2^{-k} \leq d(x, y) < 2^{-k+1}\}$. (b) For each $(k, j) \in I$, we set $D_{k,j} = B_{k,j} \times B_{k,j}$, where $B_{k,j}$is defined in (53).
(c) For each $(k, j) \in I$, we set $R_{k,j} = D_{k,j} \setminus E_k$.

We are now ready to carry out the out two-stage interpolation for $\|P, M_g\|_P^-$. The first interpolation is a more refined version:

Proposition (4.3.4) [4]: Let $2 < p < t < \infty$. Then there is a constant $C_{7.4} = C_{7.4}(p, t, n)$.
Such that the following estimate holds: Suppose that G is a subset of I and that Y is a measurable, symmetric subset of $S \times S$ satisfying the condition

$$Y \subset \bigcup_{(k,j)\in G} R_{k,j}\ .$$

Then

$$\|C(g; Y)\|_P^+ \leq C_{7.4}\ \Phi_P^+\big(\{J_t(g; k, j)\}_{(k,j)\in G}\big)\ for\ every\ g \in L^2(S, d\sigma)$$

Proof: Let $2 < p < t < \infty$. By (19), it is elementary that there is a constant C such that

$$2^{4nk} \int \int_{D_{k,j}} |g(x) - g(y)|^t d\sigma(x) d\sigma(y) \leq C J_t^t(g; k, j).$$

for all $g \in L^2(S, d\sigma)$ and $(k, j) \in I$. Let G and Y be as in the statement of the proposition.

To prove the proposition, it suffices to consider $g \in L^2(S, d\sigma)$. satisfying the condition $C_{7.4}\ \Phi_P^+\big(\{J_t(g; k, j)\}_{(k,j)\in G}\big) < \infty$

Let us estimate $N_{C(g;Y)}(s), s > 0$. For this, we will decompose the integral operator $C(g; Y)$in the form $C(g; Y) = A_s + B_s$ and take advantage of the inequality

$$N_{C(g;Y)}(s) \leq N_{A_s}(s/2) + N_{B_s}(s/2)$$

We will then estimate $N_{A_s}(s/2)$by Proposition (4.3.2) and estimate $N_{B_s}(s/2)$ by using the Hilbert-Schmidt norm $\|B_s\|_2$. But first we need to define A_s and B_s. Let us write

$$R = 2^{1/P}\frac{P}{P-1}\,\Phi_P^+\big(\{J_t(g;k,j)\}_{(k,j)\in G}\big) \tag{79}$$

Set $\mathcal{N} = N$ in the case $card(G) = \infty$ and set $\mathcal{N} = \{1, \dots, m\}$ in the case $card(G) = m < \infty$. By Lemma (4.1.6), there is a bijection $\pi: \mathcal{N} \to G$ such that

$$J_t(g;\pi(i)) \le \frac{R}{i^{1/P}} \, for\ every\ i \in \mathcal{N} \tag{80}$$

Let $G(s) = \{\pi(i): 1 \le i < (R/s)^P\}$. We define

$$W(s) = \bigcup_{(k,j)\in G(s)} \big(Y \cap R_{k,j}\big) \ and\ F(s) = Y \backslash W(s).$$

Now we let A_s and B_s be the integral operators on $L^2(S, d\sigma)$ with the kernel functions

$$\chi F(s)(x,y)\frac{g(y)-g(x)}{(1-\langle x,y\rangle)^n} \ \ and\ \chi W(s)(x,y)\frac{g(y)-g(x)}{(1-\langle x,y\rangle)^n}.$$

respectively. We first estimate $N_{A_s}(s/2)$.

Since $Y \subset \bigcup_{(k,j)\in G} R_{k,j}$ by assumption, we have $F(s) \subset \bigcup_{(k,j)\in G\backslash G(s)} R_{k,j}$ Hence

$$\int\int_{F(s)} \frac{|g(y)-g(x)|^t}{|1-\langle x,y\rangle|^{2n}} d\sigma(x)d\sigma(y)$$

$$\le \sum_{(k,j)\in G\backslash G(s)} \int\int_{R_{k,j}} \frac{|g(y)-g(x)|^t}{|1-\langle x,y\rangle|^{2n}} d\sigma(x)d\sigma(y)$$

$$\le \sum_{(k,j)\in G\backslash G(s)} 2^{4nk}\int\int_{D_{k,j}} |g(y)-g(x)|^t d\sigma(x)d\sigma(y)$$

$$\le C \sum_{(k,j)\in G\backslash G(s)} J_t^t(g;k,j) = C \sum_{i\ge (R/s)^P} J_t^t\big(g;\pi(i)\big)$$

$$\le C \sum_{i\ge (R/s)^P} \big(R/i^{1/P}\big)^t \ \ \big(by(80)\big) \le R^t.C_1\left(\max\left\{1,\frac{R}{2}\right\}\right)^{P(1-(t/P))}$$

$$= C_1 R^t\left(\max\left\{1,\frac{R}{2}\right\}\right)^{P-t}.$$

where the last $\le$ is the reason why we must require $t > p$. Since the set $F(s)$ is symmetric, we can apply Proposition (4.3.2) to obtain

$$N_{A_s}\left(\frac{s}{2}\right) \le C_{7.1}(s/2)^t \int \int\limits_{F(s)} \frac{|g(y)-g(x)|^t}{|1-\langle x,y\rangle|^{2n}} d\sigma(x)d\sigma(y)$$

$$\le C_{7.1}(s/2)^t . C_1 R^t \left(\max\left\{1, \frac{R}{2}\right\}\right)^{P-t} \le 2^t C_1 C_{7.1} R^P s^{-P} \qquad (81)$$

where the last $\le$ also uses the assumption $t > p$.To estimate $N_{B_s}(s/2)$, note that

$$\|B_s\|_2^2 = \int \int\limits_{W(s)} \frac{|g(y)-g(x)|^t}{|1-\langle x,y\rangle|^{2n}} d\sigma(x)d\sigma(y)$$

$$\le \sum_{(k,j)\in G(s)} \int \int\limits_{R_{k,j}} \frac{|g(y)-g(x)|^t}{|1-\langle x,y\rangle|^{2n}} d\sigma(x)d\sigma(y)$$

$$\le \sum_{(k,j)\in G(s)} 2^{4nk} \int \int\limits_{D_{k,j}} |g(y)-g(x)|^t d\sigma(x)d\sigma(y)$$

$$\le C_2 \sum_{(k,j)\in G(s)} J_t^2(g;k,j).$$

where the last follows from (19) and HÖlder's inequality. Recalling (80), we have

$$\|B_s\|_2^2 \le C_2 \sum_{\pi(i)\in G(s)} J_t^2(g;\pi(i)) \le C_2 \sum_{1\le i<(R/s)^P} \left(R/i^{1/P}\right)^t$$

$$\le C_3 R^2 . (R/s)^{P(1-(t/P))} = C_3 R^2 s^{-P+2}$$

Therefore

$$N_{B_s}(s/2) \le (2/s)^2 \|B_s\|_2^2 \le 4C_3 R^P s^{-P}.$$

Combining this with (81) and recalling (79), we have

$$N_{C(g;Y)}(s) \le \{2^t C_1 C_{7.1} + 4C_3\} R^P s^{-P}$$

$$= C_4 \left\{\Phi_P^+\left(\{J_t(g;k,j)\}_{(k,j)\in G}\right)\right\}^P s^{-P}.$$

If $v \in N$ and $a_v > 0$ are such that $N_T(a_v) < v$, then $s_v(T) \le a_v$. Hence it follows from the above inequality that the s-numbers of $C(g;Y)$ satisfy the condition

$$s_v\left(C(g;Y)\right) \le (2C_4)^{1/P} \Phi_P^+\left(\{J_t(g;k,j)\}_{(k,j)\in G}\right) v^{-1/P}$$

for every $v \in N$. Therefore

$$\|C(g;Y)\|_P^+ \le (2C_4)^{1/P} \Phi_P^+\left(\{J_t(g;k,j)\}_{(k,j)\in G}\right)$$

This completes the proof of the proposition.

The second stage of our interpolation requires the estimates obtained.

Proposition (4.3.5) [4]: Let $2 < p < \infty$. Then there is a constant $C_{7.5} = C_{7.5}(p, n)$ such that $\|P, M_g\|_P^- \le C_{7.5}\Phi_P^-(\{J(g; k, j)\}_{(k,j)\in I})$ for every $g \in L^2(S, d\sigma)$.

Proof. Given $2 < p < \infty$, we pick a t such that $p < t < \infty$. Lemma (4.2.8) tells us that the symmetric gauge function Φ_P^- satisfies condition (DQK). Thus, by Proposition (4.2.19),

$$\Phi_P^-(\{J_t(g; k, j)\}_{(k,j)\in I}) \le C_{6.12}\Phi_P^-(\{J \quad (g; k, j)\}_{(k,j)\in I})$$

for every $g \in L^2(S, d\sigma)$. Hence it suffices to show that there is a constant C such that

$$\|P, M_g\|_P^- \le C\Phi_P^-(\{J_t(g; k, j)\}_{(k,j)\in I}) \tag{82}$$

for every $g \in L^2(S, d\sigma)$.

To prove (82), we pick r and r' such that $2 < r' < p < r < t$. Given $g \in L^2(S, d\sigma)$,let us estimate $N_{[P;M_g]}(s), s > 0$. The idea is to decompose $[P; M_g]$ in the form $C(g; X_s) + C(g; Y_s)$ and take advantage of the inequality

$$N_{[P;M_g]}(s) \le N_{C(g;X_s)}(s/2) + N_{C(g;Y_s)}(s/2) \tag{83}$$

The sets X_s and Y_s are chosen as follows. Let Δ denote the diagonal $\{(x;\ x): x \in S\}$ in $S \times S$. Then, of course, $(\sigma \times \sigma)(\Delta) = 0$. For each $s > 0$ we set $E(s) = \{(k;\ j) \in I: J_t(g; k, j) \le s\}$.

We then define $X_s = \bigcup_{(k,j)\in E(s)} R_{k,j}$ *and* $Y_s = (S \times S)\backslash(X_s \cup \Delta)$. Since $2 < r < t$, it follows from Proposition (4.3.4) that

$$\|C(g; X_s)\|_r^+ \le C_{7.4}(r, t)\Phi_r^+(\{J_t(g; k, j)\}_{(k,j)\in E(s)})$$

By Lemma (4.2.8), we have

$$N_{C(g;X_s)}(s/2) \le \left(\frac{r}{r-1}\right)^r \left(\frac{2}{s}\|C(g; X_s)\|_r^+\right)^r$$

Setting $C_1 = \{2C_{7.4}(r, t)r/(r-1)\}^r$, from the above two inequalities we obtain

$$N_{C(g;X_s)}(s/2) \le C_1\left(\frac{1}{s}\Phi_r^+(J_t\{g; k, j\}_{(k,j)\in E(s)})\right)^r \tag{84}$$

By (21) and (53), we have $\bigcup_{j=1}^{m(k)} D_{k,j} \subset E_k$ for every $k \in Z_+$. Also, it is obvious that $\bigcup_{k=0}^{\infty} E_k = (s \times s)\backslash\Delta$. Consequently, $\bigcup_{(k,j)\in I} R_{k,j} = (s \times s)\backslash\Delta$.

Therefore $Y_s \subset \bigcup_{(k,j)\in I\setminus E(s)} R_{k,j}$.

Since $2 < r' < t$, it follows from Proposition (4.3.5) that

$$\|C(g;Y_s)\|_{r'}^{+} \leq C_{7.4}(r',t)\Phi_{r'}^{+}\left(\{J_t(g;k,j)\}_{(k,j)\in E(s)}\right)$$

Then another application of Lemma (4.2.8) gives us

$$N_{C(g;Y_s)}(s/2) \leq C_2\left(\frac{1}{s}\Phi_{r'}^{+}\left(\{J_t(g;k,j)\}_{(k,j)\in E(s)}\right)\right)^{r'} \tag{85}$$

where $C_2 = \{2C_{7.4}(r',t)r'/(r'-1)\}^{r'}$. Note that $(a+b)^{1/P} \leq a^{1/P} + b^{1/P}$ for all $a,b \in [0,\infty)$.Thus if we write $C_3 = (max\{C_1,C_2\})^{1/P}$, then it follows from (83), (84) and (85) that

$$\begin{aligned}\left\{N_{[P,M_g]}(s)\right\}^{1/P}&\\ &\leq C_3\left(\frac{1}{s}\Phi_{r}^{+}\left(\{J_t(g;k,j)\}_{(k,j)\in E(s)}\right)\right)^{r/P}\\ &+ C_3\left(\frac{1}{s}\Phi_{r'}^{+}\left(\{J_t(g;k,j)\}_{(k,j)\in E(s)}\right)\right)^{r'/P}\end{aligned} \tag{86}$$

Since $2 < p < r$, it follows from Proposition (4.1.4) that

$$\int_0^\infty \left(\frac{1}{s}\Phi_{r}^{+}\left(\{J_t(g;k,j)\}_{(k,j)\in E(s)}\right)\right)^{r/P} ds \leq C_{2.2}\Phi_{P}^{-}\left(\{J_t(g;k,j)\}_{(k,j)\in I}\right)$$

Similarly, since $2 < r' < P$, Proposition (4.1.5) tells us that

$$\int_0^\infty \left(\frac{1}{s}\Phi_{r'}^{+}\left(\{J_t(g;k,j)\}_{(k,j)\in I\setminus E(s)}\right)\right)^{r'/P} ds \leq C_{2.3}\Phi_{P}^{-}\left(\{J_t(g;k,j)\}_{(k,j)\in I}\right)$$

Combining the above two inequalities with (86), we obtain

$$\int_0^\infty \left\{N_{[P,M_g]}(s)\right\}^{1/P} ds \leq C_3(C_{2.2}+C_{2.3})\Phi_{P}^{-}\left(\{J_t(g;k,j)\}_{(k,j)\in I}\right)$$

Now an application of Lemma (4.1.3) gives us

$$\begin{aligned}\left\|P,M_g\right\|_{P}^{-} &\leq P\int_0^\infty \left\{N_{[P,M_g]}(s)\right\}^{1/P} ds\\ &\leq PC_3(C_{2.2}+C_{2.3})\Phi_{P}^{-}\left(\{J_t(g;k,j)\}_{(k,j)\in I}\right)\end{aligned}$$

That is, (82) holds for the constant $C = PC_3(C_{2.2}+C_{2.3})$ This completes the proof.

Proposition (4.3.5) is the essential part of the proof of the upper bound in Theorem (4.1.2).

What remains in the proof of the upper bound is to bring $Var^{1/2}(g;\ z)$ and Bergman lattice into the picture. But this last step has been taken care of previously:

Proposition (4.3.6) [4]: Given any positive number $0 < b < \infty$ there is a constant $C_{7.6}$ which depends only on b and n such that if Γ is a countable subset of B with the property that $\cup_{z\in\Gamma} D\,(z, b) = B$, then

$$\Phi(\{J\ \ (g; k, j)\}_{(k,j)\in I}) \le C_{7.6}\Phi\left(\left\{Var^{1/2}(g;\ z)\right\}_{z\in\Gamma}\right)$$

for every $g \in L^2(S, d\sigma)$ and every symmetric gauge function Φ.

Proof of the upper bound in Theorem (4.1.2) Given an $f \in L^2(S, d\sigma)$, write $g = f - Pf$.

Then $H_f = H_g$. Let $2 < p < \infty$ and $b > 0$. Let Γ be a countable subset of B such that $\cup_{z\in\Gamma} D\,(z, b) = B$. Applying Propositions (4.3.5) and (4.3.6), we have

$$\begin{aligned}\|H_f\|_P^- = \|H_g\|_P^- \le \|P, M_g\|_P^- &\le C_{7.6}\Phi_P^-(\{J\ \ (g; k, j)\}_{(k,j)\in I})\\ &\le C_{7.5}C_{7.6}\Phi_{\mathrm{P}}^-\left(\left\{Var^{1/2}(g;\ z)\right\}_{z\in\Gamma}\right)\\ &= C_{7.5}C_{7.6}\Phi_{\mathrm{P}}^-\left(\left\{Var^{1/2}(f - Pf;\ z)\right\}_{z\in\Gamma}\right).\end{aligned}$$

This completes the proof of Theorem (4.1.2).

Lemma (4.3.7) [4]: There exists a constant $C_{8.1}$ such that $\left(\widetilde{M}_{k+3}f\right)(x) \le C_{8.1}(M_k f)(x)$ for all $f \in L^1(S, d\sigma)$, $x \in S$, and $k \in Z_+$.

Proof: By (58), there is a constant $C_{8.1}$such that

$$\frac{\sigma(A_{k,i})}{\sigma(B_{k+3,i})} \le C_{8.1}$$

for all $k \in Z_+$, $j \in \{1, \dots, m(k)\}$ and $j \in \{1, \dots, m(k+3)\}$. Let $f \in L^1(S, d\sigma), x \in S$, and $k \in Z_+$. be given. By (21), there is a $j^* \in \{1, \dots, m(k)\}$ such that $x \in B(u_{k,j^*}, 2^{-k})$.

By (53), we have $A_{k,j^*} \supset B_{k+3,i}$. Again by (53), if $i \in \{1, \dots, m(k+3)\}$. is such that $x \in B_{k+3,i}$, then $B(x, 2^{-k}) \supset B_{k+3,i}$. Thus $A_{k,j^*} \supset B_{k+3,i}$ for every $i \in \{1, \dots, m(k+3)\}$ such that $x \in B_{k+3,i}$Therefore if $x \in B_{k+3,i}$ then

$$\frac{1}{\sigma(B_{k+3,i})}\int_{B_{k+3,i}} |f|d\sigma \le \frac{\sigma(A_{k,j^*})}{\sigma(B_{k+3,i})}\frac{1}{\sigma(A_{k,j^*})}\int_{A_{k,j^*}} |f|d\sigma \le C_{8.1}(M_k f)(x)$$

Combining this with the definition of $\left(\widetilde{M}_{k+3}f\right)(x)$, the lemma follows.

Lemma (4.3.8) [4]: There exist constants $C_{8.2}$and $C_{8.3}$ such that the following estimates hold:
Suppose $f \in L^1(S, d\sigma)$, $(k, j) \in I$ and $r > 0$ satisfy the condition

$$\frac{1}{\sigma(C_{k,i})} \int_{C_{k,i}} |f| d\sigma \leq r \tag{87}$$

Then there exists a subset G of $G(k, j)$ such that

$$(a) \quad |f(x)| \leq C_{8.2^r} \; for \; \sigma - a.e. x \in B_{k,i} \setminus \left\{ \bigcup_{(k,i) \in G} B_{k,i} \right\}$$

$$(b) \quad \sum_{(k,i) \in G} \sigma(B_{k,i}) \leq \frac{M_1}{r} \int_{C_{k,i}} |f| d\sigma$$

where M_1 is the natural number in (55);
(c) for every$(k, i) \in G$. we have

$$\frac{1}{\sigma(B_{k,i})} \int_{B_{k,i}} |f| d\sigma \leq C_{8.2^r}.$$

Proof: By (19), there is a constant $0 < C_{8.2} < \infty$ such that

$$\frac{\sigma\left(B(\zeta, 2^{-k+5})\right)}{\sigma(B(\zeta, 2^{-k}))} \leq C_{8.2} \; .$$

for all $\zeta, \xi \in S$ and $k \in Z_+$Suppose that (87) holds. Then define

$$B = \left\{ x \in B_{k,i} : \limsup_{k \to \infty} (M_k f)(x) > C_{8.2^r} \right\}.$$

It follows from (19) that if x is a Lebesgue point for $|f|$then

$$\lim_{k \to \infty} (M_k f)(x) = |f(x)|.$$

Hence $|f(x)| \leq C_{8.2^r}$ for $\sigma - a.e. x \in B_{k,j} \setminus B$ Consequently it suffices to find a subset G of $G(k, j)$ such that

$$\bigcup_{(k,i) \in G} B_{k,i} \supset B, \tag{88}$$

and such that estimates (b) and (c) hold. To find such a G, we first recall that if $k \geq k$ and if $A_{k,i} \cap B_{k,j} \neq 0$ then$A_{k,i} \subset C_{k,j}$.

Let $x \in B_{k,j}$. For any $1 \leq v \leq 3$, if $i \in \{1, \dots, m(k + v)\}$ is such that $x \in A_{k+v,i}$, then

$$\frac{1}{\sigma(A_{k+v,i})}\int_{A_{k+v,i}}|f|d\sigma \leq \frac{\sigma(C_{k,j})}{\sigma(A_{k+v,i})}\cdot\frac{1}{\sigma(C_{k,j})}\int_{C_{k,j}}|f|d\sigma \leq C_{8.2^r}.$$

This shows that $(M_{k+v}f)(x) \leq C_{8.2^r}$ for all $x \in B_{k,j}$and $v = 1,2,3$. Thus for each $x \in B$, there is a natural number $k(x) > k + 3$ such that

$$\left(M_{k(x)}f\right)(x) \leq C_{8.2^r} and \left(M_{k(x)-3}f\right)(x) \leq C_{8.2^r}.$$

Set $C_{8.3} = C_{8.1}C_{8.2}$. By Lemma (4.3.7), the second inequality above implies

$$\left(M_{k(x)}^{\sim}f\right)(x) \leq\ C_{8.3^r} \tag{89}$$

For each $x \in B$, there is an $i(x) \in \{1, \dots, m(k+v)\}$ such that $x \in\ A_{k(x),i(x)}$ and

$$\frac{1}{\sigma(A_{k(x),i(x)})}\int_{A_{k(x),i(x)}}|f|d\sigma = \left(M_{k(x)}f\right)(x) \leq C_{8.2^r} \tag{90}$$

Let $\mathcal{G} = \{(k(x),i(x)): x \in B\}$.

Then, of course, $\mathcal{G} \subset G(k,j)$ and $\cup_{(k,i)\in\mathcal{G}} A_{k,i} \supset B$. Our desired set G will be a subset of $\mathcal{G}$, defined as follows. Recall that $k(x) \geq k + 4$ for every $x \in B$. We define

$$C_{k+4} = \{(k(x),i(x)): x \in B\ and\ k(x) = k+4\}.$$

Inductively, suppose that $\ell \geq 4$ and that we have defined G_{k+q} for every $4 \leq q \leq \ell$. Then we define

$$C_{k+\ell+1} = \left\{(k(x),i(x)): x \in B\ , k(x) = k+\ell+1\ and\ A_{k(x),i(x)} \cap \left\{\bigcup_{q=4}^{\ell}\ \bigcup_{(k,i)\in G_{k+q}} A_{k,i} = 0\right\}\right\}$$

This defines $C_{k+\ell}$ for every $\ell \geq 4$. Let $G = \cup_{\ell=4}^{\infty} C_{k+\ell}$.

Let us verify that G has the desired properties. First of all, by the above inductive process, if $x \in B$ is such that $(k(x),i(x)) \notin G$, then there is a $(k,i) \in G$ with $k < k(x)$ such that $A_{k(x),i(x)} \cap A_{k,i} \neq 0$. Since $k < k(x)$, this implies $A_{k(x),i(x)} \subset B_{k,i}$. Hence (88) holds.

To verify (b), for each $\ell \geq 4$,we define

$$\Delta_\ell = \bigcup_{(k+\ell,i)\in G_{k+\ell}} A_{k+\ell,i}$$

It follows from (55) that

$$\sum_{(k+\ell,i)\in G_{k+\ell}} \chi A_{k+\ell,i} \leq M_1 \chi \Delta_\ell$$

for every $\ell \geq 4$. By (90), we have

$$C_{8.2}\sigma\left(A_{k+\ell,i}\right) < \frac{1}{r} \int_{A_{k+\ell,i}} |f| d\sigma$$

for every $(k+\ell, i) \in G_{k+\ell,i}$. Combining the above two inequalities, we have

$$\sum_{(k+\ell,i)\in G_{k+\ell}} C_{8.2}\sigma\left(A_{k+\ell,i}\right) < \frac{1}{r} \sum_{(k+\ell,i)\in G_{k+\ell}} \int_{A_{k+\ell,i}} |f| d\sigma \leq \frac{M_1}{r} \int_{\Delta_\ell} |f| d\sigma$$

Since $C_{8.2}\sigma\left(A_{k+\ell,i}\right) \geq \sigma\left(B_{k+\ell,i}\right)$ for every $(k+\ell, i) \in G_{k+\ell,i}$we obtain

$$\sum_{(k+\ell,i)\in G_{k+\ell}} \sigma\left(B_{k+\ell,i}\right) \leq \frac{M_1}{r} \int_{\Delta_\ell} |f| d\sigma$$

$\ell \geq 4$ If $(k+\ell, i) \in G_{k+\ell,i}$, then $A_{k+\ell,i} \cap B \neq 0$. Hence $\Delta_\ell \subset G_{k+\ell,i}$for every $\ell \geq 4$ The definition of the $G_{k+\ell,i}$'Sensures that$\Delta_\ell \cap \Delta_{\ell'} = 0$ for all $4 \leq \ell < \ell'$. Therefore

$$\sum_{(k,i)\in G} \sigma\left(B_{k,i}\right) < \sum_{\ell=4}^{\infty} \sum_{(k+\ell,i)\in G_{k+\ell}} \sigma\left(B_{k+\ell,i}\right) \leq \frac{M_1}{r} \sum_{\ell=4}^{\infty} \int_{\Delta_\ell} |f| d\sigma \leq \frac{M_1}{r} \int_{C_{k,j}} |f| d\sigma$$

proving (b). Finally, (c) follows simply from (89). Indeed for each $x \in B$, we have

$$\frac{1}{\sigma\left(B_{k(x),i(x)}\right)} \int_{B_{k(x),i(x)}} |f| d\sigma \leq \left(\widetilde{M}_{k(x)} f\right)(x) \leq C_{8.3^r}$$

This completes the proof.

Proposition (4.3.9) [4]: There exists a constant $C_{8.4}$ such that if $g \in L^2(S, d\sigma)$ and $(k, j) \in I$ satisfy the condition $0 < M(g; k, j) < \infty$ and if $s > 0$, then

$$\frac{\sigma\left(\left\{x \in B_{k,j} : \left|g(x) - gB_{k,j}\right| > s\right\}\right)}{\sigma\left(B_{k,j}\right)} \leq 2exp\left(\frac{-s}{C_{8.4}M(g; k, j)}\right) \qquad (91)$$

Proof: By (19), there is a constant C_1 such that $\sigma(C_{k,i}) \leq C_1(B_{k,i})$. for every $(k,i) \in I$. It is easy to see that $|\varphi B_{k,i} - \varphi C_{k,i}| \leq C_1 J(\varphi; k,i)$, for all $\varphi \in L^2(S, d\sigma)$and $(k,i) \in I$. By the homogeneity of (91), it suffices to consider the case where $g \in L^2(S, d\sigma)$ and $(k,i) \in I$ .satisfy the condition $M(\varphi; k,i) = 1$. Note that

$$\frac{1}{\sigma(C_{k,i})} \int_{C_{k,i}} |g - gB_{k,i}|\, d\sigma \leq \frac{1}{\sigma(C_{k,i})} \int_{C_{k,i}} |g - gC_{k,i}|\, d\sigma + |gC_{k,i} - gB_{k,i}|$$
$$\leq 1 + C_1$$

Now we apply Lemma (4.3.8) to the pair of $f = |g - gB_{k,i}|$ and (k,j), and to the number

$$r = 2C_1 M_1(1 + C_1) \tag{92}$$

where M_1 is the natural number that appears in (55). By Lemma (4.3.8), there is a subset $G^{(1)} of\ G(k,j)$ such that

$$|g(x) - gB_{k,i}| \leq C_{8.2^r}\ for\ \sigma - a.e.x \in B_{k,j} \backslash \left\{ \bigcup_{(k,i)\in G^{(1)}} B_{k,i} \right\},$$

$$\sum_{(k,i)\in G^{(1)}} \sigma(B_{k,i}) \leq \frac{M_1}{r} \int_{C_{k,i}} |g - gB_{k,i}|\, d\sigma \leq \frac{M_1}{r}(1 + C_1)\sigma(C_{k,i})$$
$$\leq \frac{1}{2}\sigma(B_{k,i})$$

And

$$\frac{1}{\sigma(B_{k,i})} \int_{B_{k,i}} |g - gB_{k,i}|\, d\sigma \leq C_{8.2^r}\ for\ every\ (k,i) \in G^{(1)}$$

This last inequality implies that

$$|gB_{k,i} - gB_{k,j}| \leq C_{8.2^r}\ for\ every\ (k,i) \in G^{(1)}$$

Also, since $G^{(1)} \subset G(k,j)$, for every $(k,i) \in G^{(1)}$ we have $k \geq k + 1$ and $d(u_{k,i}, u_{k,j}) < 2.2^{-k+2} = 2^{-1}.2^{-k+4}$.

Inductively, suppose that $\ell < 1$ and that we have a subset $G^{(\ell)}$ of $\{(k,i) \in I: k \geq k + \ell\}$. such that

$$|g(x) - gB_{k,i}| \leq C_{8.2^r} + (\ell + 1)C_{8.3^r}\ for\ \sigma - a.e.x$$
$$\in B_{k,j} \backslash \left\{ \cup_{(k,i)\in G^{(\ell)}} B_{k,i} \right\} \tag{93}$$

$$\sum_{(k,i)\in G^{(1)}} \sigma(B_{k,i}) \leq \frac{1}{2^\ell}\sigma(B_{k,i}) \tag{94}$$

and

$|gB_{k,i} - gB_{k,j}| \leq C_{8.2^r}$ $andd(u_{k,i}, u_{k,j})$

$$< (2^{-1} + \cdots + 2^{-\ell}).2^{-k+4} \qquad (95)$$

for every $(k,i) \in G^{(\ell)}$. This last condition means $G^{(\ell)} \subset E(k,j)$ (see Defnition (4.2.16), which together with the condition $M(g;\, k,j) = 1$ implies

$$\frac{1}{\sigma(C_{k,i})}\int_{C_{k,i}} |g - gB_{k,i}|\, d\sigma \leq \frac{1}{\sigma(C_{k,i})}\int_{C_{k,i}} |g - gC_{k,i}|\, d\sigma + |gC_{k,i} - gB_{k,i}| \leq 1 + C_1$$

for every $(k,i) \in G^{(\ell)}$. Thus the above argument can be repeated. That is, we apply Lemma (4.3.8) to each triple of $(k,i) \in G^{(\ell)}, f_{k,i} = |g - gB_{k,i}|$ and the same r given by (92).This gives us a subset $G_{k,i}^{(\ell+1)}$ of $G(k,i)$ for each $(k,i) \in G^{(\ell)}$.

We set

$$G^{(\ell+1)} = \bigcup_{(k,i)\in G^{(\ell)}} G_{k,i}^{(\ell)}.$$

By Lemma (4.3.8) (a) and (95):

$$|g(x) - gB_{k,i}| \leq C_{8.2^r} + \ell C_{8.3^r}$$

For

$$\sigma - a.e.x \in \left\{\cup_{(k,i)\in G^{(\ell)}} B_{k,i} \backslash \cup_{(k,i)\in G^{(\ell+1)}} B_{k,i}\right\}.$$

Combining this with (93), we have

$|g(x) - gB_{k,i}| \leq C_{8.2^r} + \ell C_{8.3^r}$ $for\ \sigma - a.e.x$

$$\in \left\{\cup_{(k,i)\in G^{(\ell)}} B_{k,i} \backslash \cup_{(k,i)\in G^{(\ell+1)}} B_{k,i}\right\}.$$

Also,

$$\sum_{(k,i)\in G^{(\ell+1)}} \sigma(B_{k,i}) \leq \sum_{(k,i)\in G^{(\ell)}} \frac{1}{2}\sigma(B_{k,i}) \leq \frac{1}{2^{\ell+1}}\sigma(B_{k,i}),$$

And

$$|gB_{k,i} - gB_{k,j}| \leq (\ell+1)C_{8.3^r}\ for\ every\ (k,i) \in G^{(\ell+1)}.$$

Furthermore, if $(k,i) \in G^{(\ell+1)}$then there is a $(k',i') \in G^{(\ell)}$.such that $(k,i) \in G(k',i')$. Since $k' \geq k + \ell$ this implies $d(u_{k,i}, u_{k',j'}) < 2^{-1}.2^{-k'+4} \leq 2^{-\ell-1}.2^{-k+4}$. By the triangle inequality,
$d(u_{k,i}, u_{k,j}) < (2^{-1} + \cdots + 2^{-\ell} + 2^{-\ell-1}).2^{-k+4}$ $for\ every\ (k,i) \in G^{(\ell+1)}$.
This completes the inductive selection of the sets $G^{(1)}, G^{(2)}, \ldots, G^{(\ell)}, \ldots$.

To complete the proof of the proposition, let us write $C = max\{C_{8.2}, C_{8.3}\}r$, where, as we recall, r is fixed in (92). Suppose that $s \geq C$. Then there is an $\ell \in N$ such that

$$\ell C \leq s < (\ell + 1)C.$$

By (93) and (94), we have

$$\frac{\sigma\left(\left\{x \in B_{k,j}: \left|g(x) - gB_{k,j}\right| > s\right\}\right)}{\sigma\left(B_{k,j}\right)} \leq \frac{1}{2^{\ell}} = 2e^{-(\ell+1)\log 2}$$

$$\leq 2exp\left(-\frac{\log 2}{C}s\right)$$

On the other hand, if $0 < s < C$, then

$$2exp\left(-\frac{\log 2}{C}s\right) \geq 2exp\left(-\frac{\log 2}{C}C\right) = 1$$

$$\geq \frac{\sigma\left(\left\{x \in B_{k,j}: \left|g(x) - gB_{k,j}\right| > s\right\}\right)}{\sigma\left(B_{k,j}\right)}.$$

Hence the proposition holds for the constant $C_{8.4} = C/\log 2$,

Proof of Proposition (4.2.17). For any $1 \leq t < \infty, g \in L^2(S, d\sigma)$. and $(k,j) \in I$,, we have

$$J_t^t(g; k, j) = \frac{1}{\sigma\left(B_{k,j}\right)} \int_{B_{k,j}} \left|g - gB_{k,j}\right|^t d\sigma$$

$$= t\int_0^{\infty} s^{t-1} \frac{\sigma\left(\left\{x \in B_{k,j}: \left|g(x) - gB_{k,j}\right| > s\right\}\right)}{\sigma\left(B_{k,j}\right)} ds.$$

Applying Proposition (4.3.9) to the fraction in the last integral and making the obvious substitution, we obtain

$$J_t^t(g; k, j) \leq 2t\left(C_{8.4}M(g; k, j)\right)^t \int_0^{\infty} u^{t-1}e^{-u}du\,.$$

Thus Proposition (4.2.17) holds for the constant

$$C_{6.10} = (2t)^{1/t}C_{8.4}\left(\int_0^{\infty} u^{t-1}e^{-u}du\right)^{1/t}$$

This completes the proof.

List of symbols

symbol		page
$\| \quad \|$	norm	3
L^2	Hilbert Space	3
inf	infimum	3
Sup	Supremum	3
BMO	Bounded mean Oscillation	4
VMO	Vanishing Meam Oscillation	4
B_P^d	Dyadic Besov Space	5
Supp	Support	6
$\langle , \rangle$	Inner product	12
$\otimes$	Tensor product	14
$\oplus$	Direct Sum	18
A^2	FOCK Space	51
exp	exponential	52
tr	Trace class	55
H^P	Hardy Space	59
Max	maximum	128

References:

[1] Sandra Pott_ and Martin P. Smith:

Paraproducts and Hankel operators of Schatten class via p-John–Nirenberg Theorem, Journal of Functional Analysis 217 (2004) 38–78.

[2] Wolfgang Knirsch, Georg Schneider:

Continuity and Schatten–von Neumann p-class membership of Hankel operators with anti-holomorphic symbols on (generalized) Fock spaces, J. Math. Anal. Appl. 320 (2006) 403–414.

[3] Hélène Bommier-Hato, El Hassan Youssfi:

Hankel operators and the Stieltjes moment problem, Journal of Functional Analysis 258 (2010) 978–998.

[4] Quanlei Fang and Jingbo Xia:

On the Membership of Hankel Operators in a Class of Lorentz Ideals, Journal of Functional Analysis , Volume 267, issue 4, 2014, 1137- 1187.

[5] Besov, O. V. :

On a certain family of functional spaces. Embedding and extension theorems, *Dokl. Akad. Nauk SSSR* 126 (1959), 1163–1165.

[6] Bargmann, V. :

On a Hilbert space of analytic functions and associated integral transform I. *Comm. Pure Math.1961. Appl.* 14: 187–214.

[7] Voitsekhovskii, M.I.:

"H/h047350", in Hazewinkel, Michiel, *Encyclopedia of Mathematics*, Springer, 2001, ISBN 978-1-55608-010-4.

[8] Cima, Joseph A.; Ross, William T.:

The Backward Shift on the Hardy Space, American Mathematical Society, 2001, ISBN 0-8218-2083-4.