Chapter 1

Amenability and Generalized Notions

In this chapter the Results are given on Banach sequence, Lipschitz algebras and Burling algebras, and on crucial role of approximate identities. We show a result due to N. Grønbæk on characterizing of amenability for Beurling algebras.

Section (1.1): Equivalence with Uniform Notion and Sequence Space

The concept of amenability for a Banach algebra A was introduced by Johmson in 1972, and has proved to be of enormous importance in Banach algebra theory. Several modifications of this notion were introduced. We continue the investigation of these, in particular that of approximate amenability.

Let A be a Banach algebra, and let X be a Banach A-bimodule. A derivation is a linear map $D: A \to X$ such that

$$D(ab) = a \cdot D(a) + D(a) \cdot b \quad (a, b \in A).$$

By a derivation we mean a continuous derivation. For $\mathrm{x} \in \mathrm{X}$, set $\mathrm{ad_x}$: $\mathrm{a} \longmapsto \mathrm{a} \cdot \mathrm{x} - \mathrm{x} \cdot \mathrm{a}, \mathrm{A} \to \mathrm{X}$. Then $\mathrm{ad_x}$ is the inner derivation induced by x. The derivation $\mathrm{D} : \mathrm{A} \to \mathrm{X}$ is approximately inner if there is a net $(\mathrm{x_a})$ in X such that

$$D(a) = \lim_{\alpha}(a \cdot x_\alpha - x_\alpha \cdot a) \quad (a \in A),$$

so that $D = \lim_\alpha \mathrm{ad}_{\mathrm{x_a}}$ in the strong-operator of $\mathcal{B}(A, X)$.

The dual of a Banach space X is denoted by X^*; in the case where X is a Banach A-bimodule, X^* is also a Banach A-bimodule. For the standard dual module definitions.

Definition (1.1.1) [1]:

Let A be a Banach algebra.

(i) A is approximately amenable if, for each Banach A-bimodule X, every derivation $D : A \to X^*$ is approximately inner;

(ii) A is approximately contractible if, for each Banach A-bimodule X, every derivation $D : A \to X$ is approximately inner.

The qualifier sequential prefixed to the above definitions specifies that there is a sequence of inner derivations approximating each given derivation. Similarly, the qualifier weak* prefixed to the definitions of approximate amenability specifies that the convergence in the weak* topology of X^*.

Each amenable Banach algebra is approximately amenable. Some approximately amenable Banach algebras which are not amenable are constructed. Further examples have been shown Ghahramani and Stokke: the Fourier algebra $A(G)$ is approximately amenable for each amenable, discrete group G, but it is known that these algebras are not always amenable.

Throughout, the second dual of a Banach algebra A will always be equipped with the first (or left) Arens product. Thus $(x, y) \longmapsto xy$ is a continuous function of $y \in A^{**}$ for each $x \in A$, and continuous function of $x \in A^{**}$ for each $y \in A^{**}$. Finally, $A^{\#}$ will denote A with identity, denoted by e, adjoined.

Now we can define Goldstine's Theorem [5]: let X be a Banach space, then the image of the closed unit ball $B \subset X$ under the canonical imbedding into the closed unit ball B "of the bidual space X" is weakly *-dense.

Theorem (1.1.2) [1]:

For a Banach algebra A the following are equivalent.

(i) A is approximately contractible;
(ii) A is approximately amenable;
(iii) A is weak*-approximately amenable.

Proof:

It suffices to show that (iii) $\Rightarrow$ (i).

Suppose that (iii) holds. Then $\mathcal{A}^{\#}$ is weak*-approximately amenable. Following the classical argument of B.E.Johnson, there is a net $(M_v) \subset$

$\left(\mathcal{A}^{\#}\,\widehat{\otimes}\,\mathcal{A}^{\#}\right)^{**}$ such that for each $a \in A, a \cdot M_v - M_v \cdot a \to 0$ and $\pi^{**}(M_v) \to e$ in the weak*-topology of $\left(\mathcal{A}^{\#}\,\widehat{\otimes}\,\mathcal{A}^{\#}\right)^{**}$ and A^{**}, respectively.

Now take $\varepsilon > 0$, and finite sets $F \subset \mathcal{A}^{\#}, \Phi \subset (\mathcal{A}^{\#})^{*}$, and $N \subset \left(\mathcal{A}^{\#}\,\widehat{\otimes}\,\mathcal{A}^{\#}\right)^{*}$. Then there is v such that

$$|\langle a \cdot f - f \cdot a, M_v\rangle| = |\langle f, a \cdot M_v - M_v \cdot a\rangle| < \varepsilon$$

and

$$|\langle \phi, \pi^{**}(M_v) - e\rangle| < \varepsilon$$

for all $a \in F, \phi \in \Phi$ and $f \in N$.

By Goldstine's theorem, and the weak*-continuity of π^{**}, there is $m \in \mathcal{A}^{\#}\,\widehat{\otimes}\,\mathcal{A}^{\#}$ such that

$$|\langle f, a \cdot m - m \cdot a\rangle| = |\langle a \cdot f - f \cdot a, m\rangle| < \varepsilon \text{ and } |\langle \phi, \pi(m) - e\rangle| < \varepsilon$$

for all $a \in F, \phi \in \Phi$ and $f \in N$.

Thus there is net $(m_\lambda) \subset \mathcal{A}^{\#}\,\widehat{\otimes}\,\mathcal{A}^{\#}$ such that for every $a \in A, a \cdot m_\lambda - m_\lambda \cdot a \to 0$ and $\pi(m_\lambda) \to e$, weakly in $\mathcal{A}^{\#}\,\widehat{\otimes}\,\mathcal{A}^{\#}$ and $\mathcal{A}^{\#}$, respectively.

Finally, for each finite set $F \subset \mathcal{A}^{\#}$, say $F\{a_1, \dots, a_n\}$,

$$\left(a_1 \cdot m_\lambda - m_\lambda \cdot a_1, \dots, a_n \cdot m_\lambda - m_\lambda \cdot a_n, \pi(m_\lambda)\right) \to (0, \dots, 0, e)$$

weakly in $\left(\mathcal{A}^{\#}\,\widehat{\otimes}\,\mathcal{A}^{\#}\right)^{n} \oplus \mathcal{A}^{\#}$. Thus

$$(0, \dots, 0, e) \in \overline{\mathrm{co}}^{weak}\left\{\left(a_1 \cdot m_\lambda - m_\lambda \cdot a_1, \dots, a_n \cdot m_\lambda - m_\lambda \cdot a_n, \pi(m_\lambda)\right)\right\}.$$

The Hahn-Banach theorem now gives that for each $\varepsilon > 0$, there is $u_{\varepsilon,F} \in \mathrm{co}\{m_\lambda\}$, such that

$$\left\|a \cdot u_{\varepsilon,F} - u_{\varepsilon,F} \cdot a\right\| < \varepsilon \text{ and } \left\|\pi\left(u_{\varepsilon,F}\right) - e\right\| < \varepsilon$$

for $a \in F$. Thus we have (1).

Recall that a Banach algebra A is uniformly approximately amenable if every continuous derivation from A into any dual Banach A-bimodule may be approximated uniformly on the unit ball of A by inner derivations. Clearly any amenable Banach algebra is uniformly approximately amenable. In this section we show that the converse is also true.

Theorem (1.1.3) [1]:

A Banach algebra A is uniformly approximately amenable if and only if it is amenable.

Proof:

Let A be uniformly approximately amenable. Note that A is amenable (uniformly approximately amenable) if and only if its unitization $A^{\#}$ is amenable (respectively uniformly approximately amenable), and so without loss of generality we may assume that A has a unit e. Consider $A \widehat{\oplus} \mathcal{A}^{\mathrm{op}}$ with the product specified by

$$(a \otimes b)(c \otimes d) = ac \otimes db \quad (a, b, c, d \in A).$$

Let $\pi : A \widehat{\oplus} \mathcal{A}^{\mathrm{op}} \to A$ be the product map. To show A is amenable it suffices to show that $\mathcal{K}_0 = \ker(\pi)$ has a bounded right approximate identity, or equivalently, that $\mathcal{K}_0^{**}$ has a right identity.

For $a, b \in A$ and $t \in A \widehat{\oplus} \mathcal{A}^{\mathrm{op}}$, we have

$$(a \otimes b)t = a \cdot t \cdot b. \qquad (1)$$

By the weak* continuity of the actions involved, (1) also for $t \in \left(A \widehat{\otimes} A^{\mathrm{op}}\right)^{**}$. Take $t \in \mathcal{K}_0^{**}$. Then for $s = \sum_j a_j \otimes b_j \in \mathcal{K}_0$, noting that $\sum_j a_j b_j = \pi(s) = 0$, and using (1), we have

$$\begin{aligned} st - s &= \sum_j (a_j \otimes b_j)t - t \sum_j a_j b_j - \sum_j a_j \otimes b_j + e \otimes \sum_j a_j b_j \\ &= \sum_j \left(a_j \cdot t - t \cdot a_j - a_j \otimes e + e \otimes a_j\right) \cdot b_j. \end{aligned}$$

It follows that

$$\|st - s\| \le \sum_j \|a_j\| \|b_j\| \sup_{a \in A_{\mathrm{j}}} \|a \cdot t - t \cdot a - a \otimes e + e \otimes a\|,$$

where A_1 denotes the unit ball of A.

So we have

$$\|st - s\| \leq \|s\| \sup_{a \in A_j} \|a \cdot t - t \cdot a - a \otimes e + e \otimes a\|, \qquad (2)$$

for each $s \in \mathcal{K}_0$. Now take $s \in \mathcal{K}_0^{**}$. Then there is a net $(s_i) \subset \mathcal{K}_0$ such that $\|s_i\| \leq \|s\|$ and $s_i \xrightarrow{wk^*} s$. Thus $s_i t - s \xrightarrow{wk^*} st - s$ and $\|st - s\| \leq \sup_i \|s_i t - s_i\|$. It follows that inequality (2) holds for all $s \in \mathcal{K}_0^{**}$.

Consider the continuous derivation $D : A \to \mathcal{K}_0^{**}$ defined by

$$D(a) = a \otimes e - e \otimes a.$$

From the hypothesis, there is a sequence $(t_n) \subset \mathcal{K}_0^{**}$, and $\varepsilon_n \to 0$ such that

$$\|a \cdot t_n - t_n \cdot a - a \otimes e + e \otimes a\| \leq \varepsilon_n \|a\| \quad (a \in A).$$

Thus, form inequality (2), the multiplication operator $\rho_{t_n} : \mathcal{K}_0^{**} \to \mathcal{K}_0^{**}$ defined by $\rho_{t_n}(s) = st_n$ satisfies $\left\|\rho_{t_n} - id_{\mathcal{K}_0^{**}}\right\| < 1$ for n sufficiently large. Take such n, so that ρ_{t_n} is invertible. By surjectivity, there is $x \in \mathcal{K}_0^{**}$ such that $xt_n = t_n$. Then for each $y \in \mathcal{K}_0^{**}$ we have $(yx - y)t_n = 0$. From the injectivity of ρ_{t_n} this implies $yx = y(y \in \mathcal{K}_0^{**})$. So $\mathcal{K}_0^{**}$ has a right identity, as required.

In contrast to Theorem (1.1.2) the above theorem and indicate that uniform approximate amenability and uniform approximate contractability are not the same.

Corollary (1.1.4) [1]:

If a finite-dimensional Banach algebra is approximately amenable, then it is already amenable.

Proof:

If a Banach algebra A is finite- dimensional and is approximately amenable, then it is uniformly approximately amenable. So the conclusion follows from Theorem (1.1.4).

As usual c_{00} will be the subalgebra of $\mathbb{C}^{\mathbb{N}}$ consisting of sequences having finite support.

Definition (1.1.5) [1]:

A Banach sequence algebra on $\mathbb{N}$ is a Banach algebra A which is a subalgebra of $\mathbb{C}^{\mathbb{N}}$ such that $c_{00} \subset A$.

It is known that a Banach sequence algebra A is approximately amenable whenever it has a bounded approximate identity. Indeed, a simple variant on the argument there shows the following.

Proposition (1.1.6) [1]:

Let A be a commutative semisimple Banach algebra with discrete maximal ideal space, and suppose that A has a bounded approximate identity consisting of elements of compact support. Then A is approximately amenable.

All known approximately amenable algebras have bounded approximate identities, though in general all that can be said is that approximately amenable algebras have one-side, possibly unbounded, approximate identities. Thus it is of interest to know conditions under which an approximately amenable algebra must have a bounded approximate identity. We show the following.

Proposition (1.1.7) [1]:

Either of the following conditions is sufficient for A to be sequentially approximately contractible.

(i) A is a Banach algebra with identity e and there exists $(G_n) \subset A \otimes A$ with $\pi(G_n) = e$ and such that for every $a \in A$,

$$\|a \cdot G_n - G_n \cdot a\| \to 0.$$

(ii) A is a Banach sequence algebra with a bounded sequential approximate identity in c_{00}.

for $n \in \mathbb{N}$, set $E_n = \mathcal{X}[1, n] \in c_{00}, e_n = \mathcal{X}[n]$.

Theorem (1.1.8) [1]:

Let A be a Banach sequence algebra such that (E_{n_k}) is an approximate identity for some increasing sequence $(n_k)_k \geq 0$. Then A is sequentially approximately contractible if and only if A has a bounded sequential approximate identity in c_{00}.

Proof:

Suppose that A is sequentially approximately contractible. We take (E_{n_k}) unbounded otherwise there is nothing to prove. By going to a subsequence if necessary, we may suppose that $P_k = E_{n_{k+1}} - E_{n_k}$ is an unbounded sequence of idempotents. Set $P_0 = E_{n_1}$. Define $T_k : x \longmapsto E_{n_k}x$ for $x \in A$. Then (T_k) converges pointwise to the identity, so by uniform boundedness there is $B > 0$ such that $\|T_k\| \leq B$ for all k. Thus setting $Q_k = T_{k+1} - T_k$, we have $\|Q_k\| \leq 2B$ for each k, yet the implementing elements P_k are unbounded in norm. set $Z_k = P_k/\|P_k\|$.

Now our hypothesis gives sequences $(M_n) \subset A \widehat{\otimes} A$, and $(F_n) \subset A$ such that (F_n) is an approximate identity for A for any $x \in A$,

$$x \cdot M_n - M_n \cdot x - x \otimes F_n + F_n \otimes x \to 0.$$

Indeed, since (E_{n_k}) is an approximate identity for A, it follows that c_{00} is dense in A, so we may assume that $M_n \in c_{00} \otimes c_{00}$ and $F_n \in c_{00}$.

By uniform boundedness, it follows that there is a constant $L \geq 0$ such that

$$\|x \cdot M_n - M_n \cdot x - x \otimes F_n + F_n \otimes x\| \leq L\|x\| \ \ (n \in \mathbb{N}) \qquad (3)$$

Set $x = z_k$ in (3). Then

$$\|Z_k \cdot M_k - M_k \cdot Z_k - Z_k \otimes F_n + F_n \otimes Z_k\| \leq L \ \ (n \in \mathbb{N}). \qquad (4)$$

Write $F_n = \sum_j f_j^{(n)} e_j$, $M_n = \sum_i \left(\sum_j a_{ij}^{(n)} e_j\right) \otimes \left(\sum_\ell b_{i\ell}^{(n)} e_\ell\right)$ where

$$\sum_j \left\|\left(\sum_j a_{ij}^{(n)} e_j\right)\right\| \left\|\sum_\ell b_{i\ell}^{(n)} e_\ell\right\| \leq \|M_n\| + 1.$$

Note that each of the sums here is finite. Now

$$\|P_k\|(Z_k\cdot M_n - M_n\cdot Z_k - Z_k\otimes F_n)$$

$$= P_k\cdot M_n - M_n\cdot P_k - P_k\otimes F_n$$

$$= \sum_j\left(\sum_{j=n_k+1}^{n_k+1} a_{ik}^{(n)} e_j\right)\otimes\left(\sum_j b_{ij}^{(n)} e_j\right) - \sum_i\left(\sum_j a_{ij}^{(n)} e_j\right)\otimes\left(\sum_{\ell=n_k+1}^{n_k+1} b_{i\ell}^{(n)} e_\ell\right) - \sum_{j=n_k+1}^{n_k+1} e_j\otimes\left(\sum_j\sum_{\ell=n_i+1}^{n_i+1} f_\ell^n e_\ell\right).$$

Multiply though on the right by the idempotent P_k, this is a map with bound $2B$. Noting that $Z_kP_k = Z_k$, we have

$$\|P_k\|(Z_k\cdot M_n\cdot P_k - M_n\cdot Z_k\cdot P_k - Z_k\otimes F_n\cdot P_k) = \sum_j\left(\sum_{j=n_k+1}^{n_k+1} a_{ik}^{(n)} e_j\right)\otimes\left(\sum_{\ell=n_k+1}^{n_k+1} b_{i\ell}^{(n)} e_\ell\right) - \sum_i\left(\sum_j a_{ij}^{(n)} e_j\right)\otimes\left(\sum_{\ell=n_k+1}^{n_k+1} b_{i\ell}^{(n)} e_\ell\right) - \sum_{j=n_k+1}^{n_k+1} e_j\otimes\sum_{\ell=n_i+1}^{n_k+1} f_\ell^n e_\ell. \tag{5}$$

Consider the terms on the right-hand side of (5). For each k, $\|P_k\|^{-1}\sum_{j=n_k+1}^{n_k+1} e_j$ has unit norm; and $\sum_{\ell=n_k+1}^{n_k} f_\ell^n e_\ell \to 0$ as $k\to\infty$.

Further,

$$\left\|\sum_i\left(\sum_{j=n_k+1}^{n_k+1} a_{ik}^{(n)} e_j\right) - \sum_i\left(\sum_j a_{ij}^{(n)} e_j\right)\right\| \le (1+B)\left\|\sum_i\left(\sum_j a_{ij}^{(n)} e_j\right)\right\|$$

so the other terms have norm at most

$$\left\|\sum_i\left(\sum_{j=n_k+1}^{n_k+1} a_{ik}^{(n)} e_j\right)-\sum_i\left(\sum_j a_{ij}^{(n)} e_j\right)\right\| \cdot \left\|\left(\sum_{\ell=n_k+1}^{n_k+1} b_{i\ell}^{(n)} e_\ell\right)\right\|$$

$$\leq 2B(1+2B)\sum_i\left\|\sum_j a_{ij}^{(n)} e_j\right\|\left\|\sum_\ell b_{i\ell}^{(n)} e_\ell\right\|$$

$$\leq 2B(1+2B)(\|M_n\|+1).$$

Since $\|P_k\| \to \infty$, it follows that for each n,

$$Z_k \cdot M_n \cdot P_k - M_n \cdot P_k - Z_k \otimes F_n \cdot P_k \to 0 \ \ (k \to \infty).$$

But since from (4),

$$\|Z_k \cdot M_n \cdot P_k - M_n \cdot Z_k \cdot P_k - Z_k \otimes F_n \cdot P_k + F_n \otimes Z_k \cdot P_k\| \leq 2BL$$

for all k, n, we have $\|F_n\| = \lim_k \|F_k \otimes Z_k\| = \lim_k \|F_k \otimes Z_k \cdot P_k\|$ is bounded.

Thus (F_k) is a sequential bounded approximate identity for A contained in c_{00}. The converse is Proposition (1.1.7) (ii).

In particular, consider the Feinstein algebras A_α. Let $\alpha = (\alpha_n)$ be a sequence of strictly positive reals. Define

$$\mathcal{A}_\alpha = \left\{x = (x_n) \in c_0 \colon \|x\| \coloneqq \|x\|_\infty + \sum_{n=1}^{\infty} \alpha_n \,|x_{n+1} - x_n| < \infty\right\}.$$

These algebras have a bounded approximate identity if and only if $\liminf \alpha_n < \infty$, and are amenable if and only if $\sum \alpha_i < \infty$. Moreover, they always have an approximate identity of the form (E_{n_k}).

Corollary (1.1.9) [1]:

The Feinstein algebra A_α is sequentially approximately contractible if and only if $\liminf \alpha_k < \infty$, if and only if it has a bounded approximate identity.

Proof:

If A_α is sequentially approximate contractible, Theorem (1.1.8) shows that A_α has a bounded approximate identity, and so $\liminf \alpha_n < \infty$ as noted above. Conversely, $\liminf \alpha_n < \infty$ implies A_α has a bounded approximate identity, whence A_α is sequentially approximately contractible by Theorem (1.1.8).

Theorem (1.1.8) shows that $\ell^1(\omega)$ under pointwise operations is never sequentially approximately contractible. In fact it is never approximately amenable.

Suppose now that $\alpha_k \equiv 1$ and take a sequence (m_k) such that $m_k > m_{k-1} + 1$, let

$$I = \left\{x \in A_\alpha : x_j = 0 \text{ unless } j \in \{m_k\}\right\}.$$

Then I is a closed ideal in A_α, and I isomorphic to ℓ^1. Under the supposition on (m_k) shows that I complemented in A_α is sequentially approximately contractible, with a bounded approximate identity, yet I is a complemented ideal which is not even approximately amenable. This is in contrast to the situation with amenability.

We remark that taking $I \subset A_\alpha$ to be the ideal " sits " on the even integers, so $Z_1 = 2\mathbb{N} + 1, J$ that on the odd integers so that $Z_1 = 2\mathbb{N}$, then both I and J are isomorphic to ℓ^1, are complemented (but not complementary) ideals in $A_\alpha, I \cap J = \{0\}$, and $I + J$ is dense. This just reflects the fact that one cannot just set terms to zero and expect to remain inside A_α.

Example (1.1.10) [1]:

(Suggested by Garth Dales) Let S be the semigroup $\mathbb{N}$ with product $mn = \min\{m, n\}$, and take $A_\wedge = \ell^1(S)$ with convolution product. The point masses $\{\delta_n : n \in \mathbb{N}\}$ are idempotents with dense span, whence $A_\wedge$ is weekly amenable. However, it is not amenable. We show that $A_\wedge$ is sequentially approximately contractible.

For $a = \sum a_i \delta_i \in A_\wedge$.

$$\delta_n a = \sum_{i=1}^{n} a_i \delta_i + \left(\sum_{i>n} a_i \right) \delta_n \to a$$

as $n \to \infty$, so that (δ_n) is a sequential bounded approximate identity. The Gelfand transform for $A_\wedge$ is the map $\Phi: A_\wedge \to c_0$ given by

$$\Phi(x) = \left(\sum_{i\geq 1} x_i , \sum_{i\geq 2} x_i, \dots \right),$$

which is clearly injective with range containing c_{00}. Thus $A_\wedge$ can be considered as a Banach sequence algebra Proposition (1.1.8) (ii) shows that $A_\wedge$ is sequential approximately contractible, with $G_n = E_n \otimes E_n$ and E_n the required sequences when viewed in $\Phi(A_\wedge)$. Lifting back to $A_\wedge$ gives $F_n = \delta_n \otimes \delta_n$ which satisfies $\pi(F_n) = \delta_n$. However to fit with requires a sequence F_n' satisfying $\pi(F_n') = 2\delta_n$. In fact, setting $\delta_0 = 0$,

$$F_n' = F_n + \sum_{j=1}^{n} (\delta_j - \delta_{j-1}) \otimes (\delta_j - \delta_{j-1})$$

gives an unbounded sequence with the required properties. To see this first note that

$$\pi(F_n') = \delta_n + \sum_{j=1}^{n} (\delta_j - \delta_{j-1})(\delta_j - \delta_{j-1}) = \delta_n + \sum_{j=1}^{n} (\delta_j - \delta_{j-1}) = 2\delta_n.$$

Since

$$\delta_k(\delta_j - \delta_{j-1}) = \begin{cases} \delta_j - \delta_{j-1}, & j \leq k, \\ 0, & k \leq j-1, \end{cases}$$

for $k \leq n$ we have

$$\delta_k \cdot F_n' - F_n' \cdot \delta_k + \delta_n \otimes \delta_k - \delta_k \otimes \delta_n$$

$$= \delta_k \sum_{j=1}^{n} (\delta_j - \delta_{j-1}) \otimes (\delta_j - \delta_{j-1}) - \sum_{j=1}^{n} (\delta_j - \delta_{j-1}) \otimes (\delta_j - \delta_{j-1}) \, \delta_k$$

$$= 0,$$

and for $k > n$,

$$\delta_k \cdot F'_n - F'_n \cdot \delta_k + \delta_k \otimes \delta_k - \delta_k \otimes \delta_n$$
$$= \delta_k \cdot F_n - F_n \cdot \delta_k + \delta_n \otimes \delta_k - \delta_k \otimes \delta_n$$

So for $a \in A_\wedge$,

$$a \cdot F'_n - F'_n \cdot a + \delta_n \otimes a - a \otimes \delta_n$$

$$= \left(\sum_{i>n} a_i \delta_i\right) \cdot F_n - F_n \cdot \left(\sum_{i>n} a_i \delta_i\right) + \delta_n \otimes \left(\sum_{i>n} a_i \delta_i\right) - \left(\sum_{i>n} a_i \delta_i\right) \otimes \delta_n$$
$$= \delta_n \otimes \left(\sum_{i>n} a_i \delta_i\right) - \left(\sum_{i>n} a_i \delta_i\right) \otimes \delta_n$$
$$\to 0 \ (n \to \infty). \tag{6}$$

For the product $mn = \max\{m, n\}$, $A_\vee = \ell^1(S)$ has δ_1 as an identity.

Define the (unbounded) sequence

$$G_n = \delta_n \otimes \delta_n + \sum_{i=2}^{n} (2\delta_i \otimes \delta_i - \delta_i \otimes \delta_{i-1} - \delta_{i-1} \otimes \delta_i) \quad (n \in \mathbb{N})$$

Then $\pi(G_n) = \delta_1$ clear. Further, for $\ell \geq n$,

$$\delta_\ell \cdot G_n - G_n \cdot \delta_\ell$$
$$= \delta_\ell \otimes \delta_1 - \delta_1 \otimes \delta_\ell + 2\sum_{i=2}^{n} (\delta_\ell \otimes \delta_i - \delta_i \otimes \delta_\ell)$$
$$- \sum_{i=2}^{n} (\delta_\ell \otimes \delta_{i-1} + \delta_\ell \otimes \delta_i) + \sum_{i=2}^{n} (\delta_i \otimes \delta_\ell + \delta_{i-1} \otimes \delta_\ell)$$
$$= \delta_n \otimes \delta_\ell - \delta_\ell \otimes \delta_n.$$

And for $\ell < n$.

$$\delta_\ell \cdot G_n - G_n \cdot \delta_\ell$$
$$= \delta_\ell \otimes \delta_1 - \delta_1 \otimes \delta_\ell + 2\sum_{i=2}^{\ell} (\delta_\ell \otimes \delta_i - \delta_i \otimes \delta_\ell)$$
$$+ 2\sum_{i=\ell+1}^{n} (\delta_i \otimes \delta_i - \delta_i \otimes \delta_i) - \sum_{i=2}^{\ell} \delta_\ell \otimes \delta_{i-1} - \sum_{i=\ell+1}^{n} \delta_i \otimes \delta_{i-1}$$
$$- \sum_{i=\ell+2}^{n} \delta_{i-1} \otimes \delta_i - \sum_{i=2}^{\ell+1} \delta_\ell \otimes \delta_i + \sum_{i=2}^{\ell+1} \delta_i \otimes \delta_\ell + \sum_{i=\ell+2}^{n} \delta_i \otimes \delta_{i-1}$$
$$+ \sum_{i=\ell+1}^{n} \delta_{i-1} \otimes \delta_i + \sum_{i=2}^{\ell} \delta_{i-1} \otimes \delta_\ell.$$

Looking at the terms with δ_k as first factor, for various values of k, we have

$$\delta_\ell \otimes \left(\delta_1 + 2\left(\sum_{i=2}^{\ell} \delta_i - \delta_l \right) - \sum_{i=2}^{\ell} \delta_{i-1} - \sum_{i=2}^{\ell+1} \delta_i + \delta_{\ell+1} + \delta_\ell \right) = 0,$$

for $r < l$,

$$\delta_r \otimes (-2\delta_\ell + \delta_l + \delta_l) = 0,$$

and for $r > l$,

$$\delta_r \otimes (-\delta_{r-1} - \delta_{r+1} + \delta_{r-1} + \delta_{r+1}) = 0.$$

Thus $\delta_\ell \cdot G_n - G_n \cdot G_\ell = 0$ for $r < l$.

It follows that for $a = \sum a_i \delta_i \in A_\vee$,

$$a \cdot G_n - G_n \cdot a = \sum_{k \geq n} a_k (\delta_n \otimes \delta_k - \delta_k \otimes \delta_n)$$
$$\to 0 \ \ (n \to \infty). \qquad (7)$$

So $A_\vee$ is sequentially approximately amenable by Proposition (1.1.8) (i).

Section (1.2): Boundedly and Existence Approximate Amenability of Direct Sums

Definition (1.2.1) [1]:

A Banach algebra A is boundedly approximately amenable if for every Banach A-bimodule X, and every continuous derivation $D : A \to X^*$, there is a net , there is a net $(\xi_i) \subset X^*$ such that the net $\left(ad_{\xi_i}\right)$ is norm bounded in $\mathcal{B}(A, X^*)$ and such that

$$D(a) = \lim_{\mathrm{i}} ad_{\xi_i}(a) \quad (a \in A).$$

Replacing X^* with X in the above definition, we then have the notion of boundedly approximately contractible.

Note that it is the net of derivations $\left(ad_{\xi_i}\right)$ that is required to be bounded, not the implementing net (ξ_i). On the other hand, if A is amenable shows that A is boundedly approximately contractible with the implementing net bounded.

A standard argument shows the following.

Proposition (1.2.2) [1]:

A Banach algebra A is boundedly approximately amenable if and only if there is a constant $L_b > 0$ such that for any A-bimodule X, and any continuous derivation $D : A \to X^*$, there is a net $(\xi_i) \subset X^*$ such that

(i) $\sup_i \left\|\mathrm{ad}_{\xi_i}\right\| \le L_b \|D\|$; and
(ii)$D(a) = \lim_i \mathrm{ad}_{\xi_i}(a) \;\; (a \in A)$.

Proof:

The "if" part being trivial, assume that A is boundedly approximately amenable. If there is no such L_b, then for every integer $n \in \mathbb{N}$ there is a module M_n with constant at least n for some norm one derivation D_n from A into M_n^*. Take the module $\ell^1(M_n)$ with dual $\ell^\infty(M_n^*)$. Then the derivation $D = (D_n)$ into the latter has constant at least n, a contradiction.

In terms of the basic characterization of approximate amenability, we have the following.

Theorem (1.2.3) [1]:

Suppose that the Banach algebra A is boundedly approximately amenable. Then there is a net $(M_v) \subset \left(A^{\#} \widehat{\otimes} A^{\#}\right)^{**}$ and a constant $L > 0$ such that for each $a \in A^{\#}, a \cdot M_v - M_v \cdot a \to 0, \pi^{**}(M_v) \to e$, and $\|a \cdot M_v - M_v \cdot a\| \le L\|a\|$. Conversely, if A has this latter property and $\left(\pi^{**}(M_v)\right)$ is bounded, then A is boundedly approximately amenable.

The uniform boundedness principle shows that every sequentially approximately amenable Banach algebra is boundedly approximately amenable.

Proposition (1.2.4) [1]:

Suppose that A is a boundedly approximately amenable Banach algebra. If A is separable as a Banach space, then it is sequentially approximately amenable.

Proof:

Let $\{b_n : n \in \mathbb{N}\}$ be a countable dense subset of A. Let X be a Banach A-bimodule and $D: A \to X^*$ be a continuous derivation. Since A is boundedly approximately amenable, there is a constant $c > 0$ such that for each $n \in \mathbb{N}$ there is $\xi_n \in X^*$ such that

$$\|D(b_k) - (b_k \cdot \xi_n - \xi_n \cdot b_k)\| < \frac{1}{n} \quad (k = 1,2,\dots,n), \quad \text{and}$$

$$\|a \cdot \xi_n - \xi_n \cdot a\| \le c\|a\| \quad (a \in A).$$

This shows that the sequence $(\xi_i) \subset X^*$ satisfies

$$D(b_k) = \lim_{n\to\infty} (b_k \cdot \xi_n - \xi_n \cdot b_k) \quad (k \in \mathbb{N}),$$

and the sequence $\left(\mathrm{ad}_{\xi_n}\right)$ is a bounded net in $B(A, X^*)$. These together with the density of (b_k) in A imply that

$$D(a) = \lim_{n\to\infty} (a \cdot \xi_n - \xi_n \cdot a) \quad (a \in A).$$

Therefore, D is sequentially approximately inner.

Proposition (1.2.5) [1]:

Suppose that A is a boundedly approximately contractible Banach algebra. If A is separable as a Banach space, then it is sequentially approximately contractible.

Example (1.2.6) [1]:

Let $A = c_0(S)$ where S is uncountable. Then A amenable and hence is boundedly approximately contractible, but A cannot be sequentially approximately contractible, for otherwise $c_0(S)$ would have a sequential approximate identity, which is impossible. So, without separability Proposition (1.2.5) is not true.

Example (1.2.7) [1]:

Let ω_0 be the first infinite ordinal, and ω_1 the first uncountable ordinal. For each non-zero ordinal λ, let S_λ be the set λ taken as a semigroup under the product $\wedge$. Consider the resulting algebras $\ell^1(S_\lambda)$.

For $\lambda < \omega_0$ these are finite-dimensional and amenable. We have $\ell^1(S_{\omega_0})$ boundedly approximately amenable as earlier, with $L_b \leq 2$ from Eq. (6).

Indeed, for any ordinal λ the same calculation with S_n replaced $S_{\lambda+n}$ shows that $\ell^1(S_{\lambda+\omega_0})$ is boundedly approximately amenable with $L_b = 2$. Note that (here the factor of 2 is merely a technical device)

$$\ell^1(S_{\omega_1}) = \overline{\cup \{\ell^1(S_{\lambda+2\omega_0}) : \lambda < \omega_1\}}.$$

Further $(S_{\lambda+\omega_0})_{\lambda<\omega_1}$ is an approximate identity for $\ell^1(S_{\omega_0})$ of bound 1 : for $a = \sum a_k \delta_k$, we have

$$S_{\lambda+\omega_0} a = \sum_{k<\lambda+\omega_0} a_k \delta_k + \left(\sum_{\lambda+\omega_0 \leq k < \omega_1} a_k \right) \delta_{\lambda+\omega_0} \to a.$$

Since $S_{\lambda+\omega_0} \in \ell^1(S_{\lambda+2\omega_0})$ and $S_{\lambda+\omega_0} \ell^1(S_{\omega_1}) \subset \ell^1(S_{\lambda+2\omega_0})$ shows that $\ell^1(S_{\omega_1})$ is approximately amenable, and checking the argument shows that $L_b = 2$.

Yet $\ell^1(S_{\omega_1})$ is not sequentially approximately contractible. For if it were then in particular there would be a sequence (u_n) in $\ell^1(S_{\omega_1})$ such that, for every $a \in \ell^1(S_{\omega_1})$,

$$a - u_n a \to 0. \qquad (8)$$

But all the u_n have support in some countable set, and so in an interval $[0, \lambda]$ for some $\lambda < \omega_1$. But then so does $u_n a$ for any a. So (8) fails for $a = \delta_\mu$ for any $\mu > \lambda$.

Give a Banach algebra A with unitization $A^\#$, set $\pi : A^\# \widehat{\otimes} A^{\#\mathrm{op}} \to A^\#$ to the product map, and set $\mathcal{K} = \ker \pi$. One of the standard characterizations of amenability is the existence of a bounded right approximate identity in $\mathcal{K}$. As we now show, boundedly approximate amenability can be characterized in a similar fashion. First a simple lemma.

Lemma (1.2.8) [1]:

A Banach algebra A is boundedly approximately amenable if and only if $A^\#$ is boundedly approximately amenable.

Proof:

Let A be boundedly approximately amenable, X a Banach $A^\#$-bimodule, $D : A^\# \to X^*$ a derivation. By adjusting by an inner derivation of norm at most $4\|D\|$ we may suppose that X is neo-unital, and so $D(e) = 0$.

By assumption, there is $(x_i^*) \subset X^*$ and $M > 0$ such that for $a \in A$:

$$D(a) = \lim_i (a \cdot x_i^* - x_i^* \cdot a),$$

and for all i,

$$\|a \cdot x_i^* - x_i^* \cdot a\| \le M\|a\|.$$

Since $D(e) = 0$ and $e \cdot x^* = x^* \cdot e$ $(x \in X^*)$, it follows that

$$D(a + \alpha e) = \lim_i \big((a + \alpha e) \cdot x_i^* - x_i^* \cdot (a + \alpha e)\big),$$

and

$$\|(a+\alpha e)\cdot x_i^* - x_i^*\cdot(a+\alpha e)\| \le M\|a\| \le \|a+\alpha e\|,$$

so that $A^\#$ is boundedly approximately amenable.

Conversely, let X be an A-bimodule, and $D : A \to X^*$ a derivation. Setting $e\cdot x = x\cdot e = x$ makes X into an $A^\#$-bimodule. Setting $D(e) = 0$ extends D to $A^\#$. Supposing $A^\#$ is boundedly approximately amenable, there is $(x_i^*) \subset X^*$ and $M > 0$ such that for all $a \in A$,

$$D(a) = \lim_i (a\cdot x_i^* - x_i^*\cdot a), \qquad \text{with } \|a\cdot x_i^* - x_i^*\cdot a\| \le M\|a\|,$$

as required.

In the following theorem π still denotes the product map from $A^\# \hat{\otimes} A^{\#\mathrm{op}}$ into $A^\#$ and $\mathcal{K}$ denotes kernel of π.

Theorem (1.2.9) [1]:

A Banach algebra A is boundedly approximately amenable if and only if there is a net $(u_i) \subset \mathcal{K}^{**}$ and $M > 0$ such that:

(i) $k\cdot u_i \to k$ for each $k \in \mathcal{K}$;
(ii) $\|k\cdot u_i\| \le M\|k\|$ for all $k \in \mathcal{K}$ and all i.

Proof:

Suppose that A is boundedly approximately amenable, and let $D : A \to \mathcal{K}^{**}$ be the derivation $D(a) = a\otimes e - e\otimes a$. Then there is a net $(u_i) \subset \mathcal{K}^{**}$ and $M > 0$ such that for all $a \in A$,

$$D(a) = \lim_i (a\cdot u_i - u_i\cdot a),$$
$$\text{with } \|a\cdot u_i - u_i\cdot a\| \le M\|a\| \text{ for all } i.$$

We show that (u_i) has the desired properties.

Let $k = \sum a_n \otimes b_n \in \mathcal{K}$, so that $\sum a_n b_n = 0$. Then

$$k\cdot u_i = \sum_n a_n\cdot u_i\cdot b_n = \sum_n a_n\cdot u_i\cdot b_n - \sum_n u_i\cdot a_n b_n$$
$$= \sum_n (a_n\cdot u_i - u_i\cdot a_n)\cdot b_n,$$

so that

$$\|k \cdot u_i\| \leq \sum_n \|a_n \cdot u_i - u_i \cdot a_n\| \, \|b_n\| \leq M \sum_n \|a_n\| \|b_n\|,$$

and so (ii) is satisfied.

Take $\varepsilon > 0$, and write $k = k_1 + k_2$ where

$$k_1 = \sum_{n=1}^{N} c_n \otimes d_n \in \mathcal{K} \text{ and } \|k_2\| < \varepsilon.$$

This is possible. Then, as above,

$$k_1 \cdot u_i = \sum_{n=1}^{N} c_n \cdot u_i \cdot d_n = \sum_{n=1}^{N} (c_n \cdot u_i - u_i \cdot c_n) \cdot d_n, \tag{9}$$

Since $D(a) = a \otimes e - e \otimes a$ for $a \in A$,

$$k_1 = \sum_{n=1}^{N} c_n \otimes d_n = \sum_{n=1}^{N} (c_n \otimes e - e \otimes c_n) \cdot d_n$$

$$= \sum_{n=1}^{N} D(c_n) \cdot d_n \quad (10)$$

Putting (9) and (10) together,

$$\|k_1 \cdot u_i - k_1\| \leq \sum_{n=1}^{N} \|c_n \cdot u_i - u_i \cdot c_n - D(c_n)\| \, \|d_n\| < \varepsilon,$$

Provided that i is sufficiently large. Since

$$\|k_2 \cdot u_i - k_2\| \leq (M+1)\|k_2\| < (M+1)\varepsilon,$$

we thus have

$$\|k \cdot u_i - k\| \leq (M+2)\varepsilon$$

provided i is sufficiently large. Thus (i) is satisfied.

Now suppose that a net $(u_i) \subset \mathcal{K}^{**}$ as above exists. By Lemma (1.2.8) it suffices to show that $A^{\#}$ is boundedly approximately amenable.

Set $v_i = e \otimes e - u_i \in \left(A^\# \widehat{\otimes} A^{\# \mathrm{op}}\right)^{**}$. Then $\pi^{**}(v_i) = e$, and for $a \in A$,

$$\begin{aligned} a \cdot v_i - v_i \cdot a &= (a \otimes e - e \otimes a) - (a \cdot u_i - u_i \cdot a) \\ &= (a \otimes e - e \otimes a) - (a \otimes e - e \otimes a)u_i \\ &\to 0, \end{aligned} \tag{11}$$

because $a \otimes e - e \otimes a \in \mathcal{K}$. Moreover, there is $m > 0$ such that

$$\|a \cdot v_i - v_i \cdot a\| \le m\|a\| \ \ (a \in A, \text{all } i). \tag{12}$$

Now let X be a unit-linked $A^\#$-bimodule, and let $D : A^\# \to X^*$ be a derivation. Let $\varphi : A^\# \widehat{\otimes} A^\# \to X^*$ be the mapping specified by

$$\varphi(a \otimes b) = a \cdot D(b) \ \ (a, b \in A^\#).$$

Then $\|\varphi\| \le \|D\|$, and for $a \in A^\#, u \in A^\# \widehat{\otimes} A^\#$,

$$\varphi(u \cdot a) = \varphi(u) \cdot a + \pi(u)D(a), \qquad \varphi(a \cdot u) = a \cdot \varphi(u).$$

The natural projection $P : X^{***} \to X^*$ is an $A^\#$-bimodule morphism, $\varphi^{**} : \left(A^\# \widehat{\otimes} A^\#\right)^{**} \to X^{***}$ is weak*-weak* continuous, and the map $\psi = P \circ \varphi^{**} : \left(A^\# \widehat{\otimes} A^\#\right)^{**} \to X^*$ satisfies $\|\psi\| \le \|D\|$. For $a \in A^\#, u \in \left(A^\# \widehat{\otimes} A^\#\right)^{**}$, noting that P is weak* continuous we have

$$\psi(u \cdot a) = \psi(u) \cdot a + \pi^{**}(u) \cdot D(a), \qquad \psi(a \cdot u) = a \cdot \psi(u).$$

In particular, using neo-unitality,

$$\begin{aligned} D(a) &= \pi^{**}(v_i) \cdot D(a) \\ &= \psi(v_i \cdot a) - \psi(v_i) \cdot a \\ &= a \cdot \psi(v_i) - \psi(v_i) \cdot a - \psi(a \cdot v_i - v_i \cdot a). \end{aligned}$$

Thus by (11),

$$D(a) = \lim_i (a \cdot \psi(v_i) - \psi(v_i) \cdot a)$$

whence, by (12),

$$\begin{aligned} \|a \cdot \psi(v_i) - \psi(v_i) \cdot a\| &\le \|D(a)\| + \|\psi\|\|a \cdot v_i - v_i \cdot a\| \\ &\le \|D\|(m+1)\|a\|. \end{aligned}$$

It follows that D is boundedly approximately inner.

The same argument, with appropriate modifications, shows the following.

Theorem (1.2.10) [1]:

The Banach algebra A is boundedly approximately contractible if and only if there is a net $(u_i) \subset \mathcal{K}$ and $M > 0$ such that

(i) $k \cdot u_i \to k$ for each $k \in \mathcal{K}$;
(ii) $\|k \cdot u_i\| \leq M\|k\|$ for all $k \in \mathcal{K}$ and all i.

We improve concerning approximate amenability of the direct sum of Banach algebras as follows. There appears to be a close relation between the existence of two-sided approximate identities in approximately amenable algebras and the approximate amenability of the direct sum of approximately amenable algebras.

Proposition (1.2.11) [1]:

Suppose that A and B are approximately amenable Banach algebras. Suppose that one of A or B has a bounded approximate identity. Then $A \oplus B$ is approximately amenable.

Proof:

Let X be an $(A \oplus B)$-bimodule, and let $D : A \oplus B \to X^*$ be a continuous derivation. Suppose that $(b_\alpha) \subset B$ is a bounded approximate identity for B. Without loss of generality we assume

$$b_\alpha \xrightarrow{wk^*} E \text{ in } B^{**} \text{ and } D(b_\alpha) \xrightarrow{wk^*} \xi \text{ in } X^{***}.$$

Then X^{***} is an $(A \oplus B)^{**} = A^{**} \oplus B^{**}$-bimodule. We can extend the module actions of $A \otimes B$ on X^{***} to actions of $A^\# \oplus B$ on X^{***} by defining

$$e_A \cdot F = F - E \cdot F, \qquad F \cdot e_A = F - F \cdot E, \qquad F \in X^{***},$$

where e_A is the identity for $A^\#$.

Now view D as a derivation from $A \oplus B$ into X^{***}. We extend it to a derivation from $A^\# \oplus B$ into X^{***} by defining $D(e_A) = -\xi$. It is readily seen that after this extension D is still a derivation. For instance, for each $a \in A$,

$$a \cdot D(e_A) + D(a) \cdot e_A = -a \cdot \xi + D(a) - D(a) \cdot E$$
$$= D(a) - \text{weak}^* - \lim_{\alpha} \mathrm{D}(ab_\alpha) = D(a) = D(ae_A).$$

Since $A^\# \oplus B$ is approximately, it is approximately contractible by Theorem (1.1.2). Therefore the extended D is approximately inner. So there exists a net $(F_i) \subset X^{***}$ for which

$$D(a,b) = \lim_i [(a,b) \cdot F_i - F_i \cdot (a,b)], \qquad a \in A, b \in B.$$

Applying the canonical projection from X^{***} to both sides of the above equation, we obtain that the original D is approximately inner. So $A \oplus B$ is approximately amenable.

Proposition (1.2.12) [1]:

Suppose that A and B are approximately amenable Banach algebras. Then, for any neo-unital $(A \oplus B)$-bimodule X, continuous derivations from $A \oplus B$ into X^* are weak* approximately inner.

Proof:

Let $D : A \oplus B \to X^*$ be a continuous derivation. Then D induces (continuous) derivations $D_1 : A \to X^*$ define by $D_1(a) = D(a,0)$, and $D_2 : B \to X^*$ define by $D_2(b) = D(0,b)$. Since A and B are approximately amenable, there are nets $(\xi_i), (\zeta_i) \subset X^*$ such that

$$D_1(a) = \lim_i [(a,0) \cdot \xi_i - \xi_i \cdot (a,0)] \quad (a \in A), \tag{13}$$

$$D_2(b) = \lim_i [(0,b) \cdot \zeta_i - \zeta_i \cdot (0,b)] \quad (b \in B), \tag{14}$$

Let $(l_\alpha^A)(r_\alpha^A)$ respectively be left and right approximate identities of A, and let $(l_\alpha^B)(r_\alpha^B)$ respectively be left and right approximate identities of B. Then we have

$$(a,0) = \lim_\alpha (a,b)(r_\alpha^A, 0) = \lim_\alpha (l_\alpha^A, 0)(a,b) \quad (a \in A),$$

$$(0,b) = \lim_\alpha (a,b)(0, r_\alpha^B) = \lim_\alpha (0, l_\alpha^B)(a,b) \quad (b \in B).$$

These together with equations (13) and (14) imply that there are nets (Φ_v) and (ψ_v) in X^* such that

$$D(a,b) = D_1(a) + D_2(b)$$
$$= \lim_v [(a,b)\cdot \Phi_v - \psi_v \cdot (a,b)] \quad (a \in A, b \in B).$$

Since D is a derivation, (Φ_v) and (ψ_v) in the above equation satisfy

$$(a,b)\cdot(\Phi_v - \psi_v)\cdot(c,d) \xrightarrow{v} 0 \quad (a,c \in A, b,d \in B).$$

So we have

$$D(a,b)(c,d) = \lim_v [(a,b)\cdot \psi_v - \psi_v(a,b)]\cdot(c,d),$$

for all $a,c \in A, b,d \in B$. If X is a neo-unital $(A \oplus B)$-bimodule, this implies that

$$D(a,b) = \text{weak}^* - \lim_v [(a,b)\cdot \psi_v - \psi_v \cdot (a,b)] \quad (a \in A, b \in B).$$

Therefore D is weak* approximately inner.

Proposition (1.2.13) [1]:

If $A \oplus A$ is approximately amenable, then A has a two-sided approximate identity.

Proof:

Make $X = A$ an $A \oplus A$-bimodule by defining module actions as follows.

$$(a,b)\cdot x = ax, \quad x\cdot(a,b) = xb \quad (x \in X, a,b \in A).$$

Then $D(a,b) = a - b$ is derivation from $A \oplus A$ into X. So there exists $(x_i) \subset X$ for which

$$a - b = \lim_i (ax_i - x_i b) \quad (a,b \in A).$$

In particular, we have $\lim_i ax_i = a$ and $\lim_i x_i b = b$ $(a,b \in A)$. So (x_i) is a two-sided approximate identity.

Suppose that A is an approximately amenable Banach algebra. In particular, A has one-sided approximate identity. Consider the topology τ determined by the seminorms $b \longmapsto \|ab\|$ $(a \in A)$.

Proposition (1.2.14) [1]:

Suppose that A is approximately amenable, and that τ is stronger than the weak topology on A. Then A has a two-sided approximate identity.

Proof:

Take $X = A$ as an $(A \oplus A)$-bimodule as above.

Following the argument of proposition (1.2.12), we have that for any derivation $D : A \oplus A \to X$ there is a net (ψ_v) in X such that

$$D(a,b) \cdot (c,d) = \lim_v [(a,b) \cdot \psi_v - \psi_v(a,b)] \cdot (c,d).$$

Applying this to the derivation $D(a,b) = a - b$, we have that for every $c \in A$,

$$(a-b)c = \lim_v (a\psi_v - \psi_v b)c.$$

Hence from the assumption on τ,

$$a - b = \text{weak}^* - \lim_v (a\psi_v - \psi_v b).$$

Thus (ψ_v) is a two-sided weak approximate identity, and standard arguments yield a two-sided approximate identity.

Now we can define Mazure theorem [6]: most will – behaved normed spaces are subspaces of the space of continuous path.

Proposition (1.2.15) [1]:

Suppose that

(i) $\text{span}\{aa^*: a \in A, a^* \in A^*\}$ is dense in A^*; and
(ii) A is boundedly approximately amenable, or
(iii) A is boundedly approximately contractible.

Then A has a two-sided approximate identity.

Proof:

Suppose (i) and (ii) and let D and $X \subset X^{**}$ be as in Proposition (1.2.4). Then there is a net (ξ_ν) in X^{**} such that

$$D(a,b)c = \lim_{\nu}(a \cdot \xi_\nu - \xi_\nu \cdot b)\,\mathrm{c}, \qquad a,b,c \in A,$$

where, moreover, $(a \cdot \xi_\nu - \xi_\nu \cdot b)$ is bounded for each $a,b \in A$. It follows that for $c \in A$ and $c^* \in A^*$,

$$\langle D(a,b), cc^* \rangle = \lim_{\nu}\langle a \cdot \xi_\nu - \xi_\nu \cdot b, cc^* \rangle, \qquad a \in A,$$

and hence for finite sums $c_1c_1^* + \cdots + c_kc_k^*$. But then by boundedness of $(a \cdot \xi_\nu - \xi_\nu \cdot b)$ and hypothesis on A^*,

$$a - b = \text{weak}^*\_\lim_{\mathrm{v}}(a \cdot \xi_\nu \cdot \xi_\nu \cdot b),$$

which suffices.

Supposing (iii) the argument is similar but simpler.

The spanning condition certainly holds if A^* is essential with the usual module operations. It also holds when A is approximately amenable and reflexive as a Banach space. For with (e_i) a right approximate identity for A, we have

$$\langle a^*, a \rangle = \lim_{i}\langle a^*, ae_i \rangle = \lim_{i}\langle e_i a^*, a \rangle,$$

so that $\overline{\mathrm{span}}\{cc^*\}^{\mathrm{weak}} = A^*$, and hence in norm by Mazur's theorem. However, it should be noted that no example of an infinite-dimensional reflexive as a Banach algebra is known. Indeed, it has been conjectured that a reflexive amenable Banach algebra is finite-dimensional.

Proposition (1.2.15) can be strengthened a little.

Proposition (1.2.16) [1]:

Let $M = (\mathrm{span}\{aa^*: a \in A, a^* \in A^*\})^-$. Suppose that A is boundedly approximately amenable and that M is complemented by a closed submodule in A^*. Then A has a two-sides approximate identity.

Proof:

Let N be a complementing closed submodule, such that $A^* = M \oplus N$. By the definition of M, the left action of A annihilates N. Let $D : A \oplus A \to A^{**}$ be given by $D(a,b) = a - b$. Now $A^{**} = M^* \oplus N^*$, let Q be the quotient map of A^{**} onto M^*. Then QD and $(I - Q)D$ are derivations into M^* and N^*, respectively.

Since the right action of A on N^* is trivial, and A has a left approximate identity, $(I - Q)D$ is approximately inner. For QD, the argument of Proposition (1.2.15) gives $(\xi_i) \subset M^*$ with

$$QD(a,b) = \lim_i [(a,b) \cdot \xi_i - \xi_i \cdot (a,b)] \quad (a,b \in A).$$

Thus we have D is weak*-approximately inner, and hence approximately inner. The result follows as in Proposition (1.2.14).

Section (1.3): Lipschitz and Beurling with Discrete Semigroup Algebras

For an infinite compact metric space E and $0 < \alpha < 1$, and $f : E \to \mathbb{C}$, define

$$P_\alpha(f) = \sup\left\{\frac{|f(x) - f(y)|}{d(x,y)^\alpha} : x, y \in E, x \neq y\right\}.$$

Then set

$$\mathrm{Lip}_\alpha(E) = \{f : X \to \mathbb{C}: p_\alpha(f) < \infty\},$$

and

$$\mathrm{lip}_\alpha(E) = \left\{f \in \mathrm{Lip}_\alpha(E) : \frac{|f(x) - f(y)|}{d(x,y)^\alpha} \to 0 \text{ as } d(x,y) \to 0\right\}.$$

On each of these spaces set $\|f\|_\alpha = \|f\|_\infty + P_\alpha(f)$. Then with pointwise multiplication $\mathrm{Lip}_\alpha(E)$ and $\mathrm{lip}_\alpha(E)$ are commutative Banach algebras.

Since $\mathrm{Lip}_\alpha(E)$ fails to be weakly amenable, $0 < \alpha \leq 1$, it cannot be approximately amenable. Of rather more interest is $\mathrm{lip}_\alpha(E)$ where this last statement only hold in general for $1/2 < \alpha < 1$.

Here we make a very modest contribution towards answering the approximate amenability question for these algebras.

With E and α as above, let $A = \mathrm{lip}_\alpha E$, and set

$$X = \{f \in \mathrm{Lip}_\alpha(E \times E): f(x,x) = 0 \ (x \in E)\}.$$

Proposition (1.3.1) [1]:

The derivation $D : A \to X$ given by

$$(Da)(x,y) = a(x) - a(y) \ \ (a \in A, x, y \in E)$$

is non-inner but is sequentially approximately inner.

Proof:

It has been shown that D is non-inner.

For $n \in \mathbb{N}$, set

$$G_n(x,y) = \min\left\{1, n\big(d(x,y)\big)^\alpha\right\} \ \ (x,y \in E).$$

Note that $\|G_n\|_\alpha = 1 + \alpha n^\alpha$. Let $a \in A$, and consider

$$\begin{aligned}(a \cdot G_n - G_n \cdot a - Da)(x,y) \\ = \big(a(x) - a(y)\big)(G_n(x,y) - 1).\end{aligned} \tag{15}$$

We show this converges to 0 in X. Note that uniform convergence to 0 is clear. Assume that the result fails. Without loss of generality, there is $\eta > 0$ such that

$$\|a \cdot G_n - G_n \cdot a - Da\| > \eta \ \ (n \in \mathbb{N}).$$

Thus there exist $x_n, y_n, x_n', y_n' \in E$ such that

$$\begin{aligned}\frac{\left|\big(a(x_n) - a(y_n)\big)(G_n(x,y) - 1) - \big(a(x_n') - a(y_n')\big)(G_n(x_n', y_n') - 1)\,\right|}{[d(x_n, x_n') + d(y_n, y_n')]^\alpha} \\ \geq \eta.\end{aligned} \tag{16}$$

Note that necessarily $\lim_n\big(d(x_n, x_n') + d(y_n, y_n')\big) = 0$, since the numerator in (16) converges uniformly to 0. Write

$$\begin{aligned}a(x_n) - a(y_n) \\ = \big(a(x_n) - a(x_n')\big) + \big(a(x_n') - a(y_n')\big) + \big(a(y_n') - a(y_n)\big).\end{aligned}$$

Since

$$\frac{a(x_n) - a(x_n')}{d(x_n, x_n')^\alpha} \to 0 \ \text{ and } \ \frac{a(y_n) - a(y_n')}{d\big(y_n, y_n'\big)^\alpha} \to 0,$$

we deduce from (16) that

$$\liminf_n \frac{\left|\big(a(x_n) - a(y_n)\big)\big(G_n(x_n, y_n) - G_n(x_n', y_n')\big)\right|}{[d(x_n, x_n') + d(y_n, y_n')]^\alpha} \geq \eta, \tag{17}$$

$$\liminf_n \frac{\left|\big(a(x_n') - a(y_n')\big)\big(G_n(x_n, y_n) - G_n(x_n', y_n')\big)\right|}{[d(x_n, x_n') + d(y_n, y_n')]^\alpha} \geq \eta, \tag{18}$$

Now

$$\begin{aligned}\frac{\left|\big(G_n(x_n, y_n) - G_n(x_n', y_n')\big)\right|}{[d(x_n, x_n') + d(y_n, y_n')]^\alpha} \\ \leq \begin{cases} 0, & \min\{d(x_n, y_n), d(x_n', y_n')\} \geq 1/n\,, \\ 1 + n^\alpha, & \text{otherwise.}\end{cases}\end{aligned}$$

Thus from (17) and (18) it follows that at least one of $d(,x_n y_n) < 1/n$ or $d(x'_n, y'_n) < 1/n$ must hold for infinitely many n. Without loss of generality suppose it is the former. Then (18) gives

$$\eta \leq \liminf_{n} |a(x_n) - a(y_n)|(1 + n^\alpha) \leq \liminf_{n} \frac{|a(x_n) - a(y_n)|}{d(x_n, y_n)^\alpha} \frac{1 + n^\alpha}{n^\alpha} = 0,$$

since $d(x_n, y_n) < 1/n$ for infinitely many n. This contradiction.

In the special case $E = [0,1]$, the same style of argument also shows that for a fixed $y \in [0,1]$, $u_n(x) = \min\{1, \omega(n(x - y))\}$ defines an (unbounded) approximate identity in the maximal ideal $M_y = \{f \in \mathrm{lip}_\alpha[0,1]: f(y) = 0\}$. Thus results are of no help as to the approximate amenability of $\mathrm{lip}_\alpha[0,1]$.

A similar argument, with suitable $G_n \in \mathrm{lip}_\alpha[0,1] \mathbin{\widehat{\otimes}} \mathrm{lip}_\alpha[0,1]$, and more technically involved, shows that for $E = [0,1]$ the derivation above is sequentially approximately inner when considered as mapping into $\mathrm{lip}_\alpha[0,1]^2$.

To show approximate amenability we in effect need to show convergence of (15), for such G_n, in $\mathrm{lip}_\alpha[0,1] \mathbin{\widehat{\otimes}} \mathrm{lip}_\alpha[0,1]$ rather than $\mathrm{lip}_\alpha[0,1]^2$ as above, and the norms involved are not equivalent: $\|z^n \otimes z^n\|_\pi = O(n^{2\alpha}), \|z^n \otimes z^n\|_\pi = O(n^\alpha)$. For any compact metric space E, the natural map $\Phi : \mathrm{lip}_\alpha(E) \mathbin{\widehat{\otimes}} \mathrm{lip}_\alpha(E) \to \mathrm{lip}_\alpha(E^2)$ is a contractive monomorphism, and Hedbeg's theorem can be used to show it has dense range.

Recall that a weight ω on a locally compact group is a continuous function $G \to (0, \infty)$ satisfying

$$\omega(xy) \leq \omega(x)\omega(y) \ \ (x, y \in G).$$

For a weight $\omega, L^1(\omega) = L^1(G, \omega)$ is a Banach algebra under convolution, the Beurling algebra corresponding to ω.

The weight ω is symmetric if $\omega(\mathrm{g}) = \omega(\mathrm{g}^{-1})(\mathrm{g} \in G)$. For any weight ω, its symmetrization is the weight defined by $\Omega(\mathrm{g}) = \omega(\mathrm{g})\omega(\mathrm{g}^{-1})(\mathrm{g} \in G)$.

Throughout Proposition (1.3.2)–Theorem (1.3.5) below we assume that $\omega(e) = 1$.

Proposition (1.3.2) [1]:

Suppose the weight ω is bounded away from 0, and that $L^1(\omega)$ is approximately amenable. Then G is amenable.

Proof:

The hypothesis ensure that $L^1(\omega) \subset L^1(G)$, and hence $UC(G)$ is an $L^1(\omega)$-bimodule. There is an invariant mean on $UC(G)$, so G is amenable.

The precise relation between the behavior of ω and the approximate amenability of $L^1(\omega)$ is unresolved. For example $L^1(\mathbb{R}, e^t) \cong L^1(\mathbb{R})$ is amenable, so boundedness of ω is not necessary. We conjecture that $L^1(\omega)$ will fail to be approximately amenable whenever $\Omega \to \infty$. Indeed, should this not be the case then we have a group G which is amenable by Proposition (1.3.2), with $L^1(\omega)$ approximately amenable but not amenable (see Theorem (1.3.7)). While this remains unresolved, a modified hypothesis yields a weaker result. Some preliminary constructions will be required.

Suppose that G is a locally compact group, ω a continuous weight on G. Define

$$\widehat{\omega}(x) = \liminf_{r\to\infty} \frac{\omega(rx)}{\omega(r)} \quad (x \in G)$$

It is readily seen that $\widehat{\omega}$ is continuous on G and for $x, y \in G$,

$$\begin{aligned} &\omega(x^{-1})^{-1} \le \widehat{\omega}(x) \le \omega(x), \\ &\qquad \widehat{\omega}(xy) \le \widehat{\omega}(x)\omega(y) \wedge \omega(x)\widehat{\omega}(y). \end{aligned} \tag{19}$$

Note that $\widehat{\omega}$ is usually not a weight on G. In fact, $\widehat{\omega}^{-1}$ is a weight since we always have $\widehat{\omega}(xy) \ge \widehat{\omega}(x)\widehat{\omega}(y)$ $(x, y \in G)$.

For $\varphi \in L^1(\widehat{\omega} \times \omega)$, define

$$\pi(\varphi)(x) = \int_G \varphi(\xi, \xi^{-1}, x)\, d\xi \quad (x, \in G).$$

Then $\pi(\varphi) \in L^1(\widehat{\omega})$ with $\|\pi(\varphi)\| \leq \|\varphi\|$. Set π^* to be the adjoint of π^* maps $L^\infty(\widehat{\omega}^{-1})$ into $L^\infty(\widehat{\omega}^{-1} \times \omega^{-1})$.

Lemma (1.3.3) [1]:

Suppose that $\lim_{x\to\infty} \widehat{\omega}(x^{-1})\omega(x) = \infty$. Then $\pi^*|C_0(\widehat{\omega}^{-1})$ maps $C_0(\widehat{\omega}^{-1})$ into $C_0(\widehat{\omega}^{-1} \times \omega^{-1})$.

Proof:

Let $f \in C_0(\widehat{\omega}^{-1})$, and let $\|f\|_{\widehat{\omega}}$ denote its is norm. By definition $\pi^*(f)(x,y) = f(xy)$, and so is certainly continuous on $G \times G$. Take $\varepsilon > 0$, and a compact set $N \subset G$ such that $|f(x)\widehat{\omega}(x)^{-1}| < \varepsilon$ for $x \in G\backslash N$. Set $c = \sup\{\omega(x)\widehat{\omega}(x^{-1}) : x \in N\}$. By hypothesis there is a compact set $K \subset G$ such that

$$\frac{c\|f\|_\omega}{\widehat{\omega}(y^{-1})\omega(y)} < \varepsilon \quad (y \in G\backslash K).$$

Then $A = \{(x,y) : y \in K, xy \in N\}$ is compact in $G \times G$. For $(x,y) \notin A$ and $x, y \notin N$,

$$\left|\frac{\pi^*(f)(xy)}{\widehat{\omega}(x)\omega(y)}\right| \leq \frac{|f(xy)|}{\widehat{\omega}(xy)} < \varepsilon.$$

On the other hand for $(x,y) \notin A$ and $xy \in N$, so that $y \notin K$, (19) gives

$$\begin{aligned}\left|\frac{\pi^*(f)(x,y)}{\widehat{\omega}(x)\omega(y)}\right| &= \frac{|f(xy)|}{\widehat{\omega}(xy)} \frac{\widehat{\omega}(xy)}{\widehat{\omega}(x)\omega(y)} \leq \|f\|_{\widehat{\omega}} \frac{\widehat{\omega}(xy)\omega(y^{-1}x^{-1})}{\widehat{\omega}(y^{-1})\omega(y)} \\ &\leq \frac{c\|f\|_{\widehat{\omega}}}{\widehat{\omega}(y^{-1})\omega(y)} < \varepsilon.\end{aligned}$$

Thus $\pi^*(f) \in C_0(\widehat{\omega}^{-1} \times \omega^{-1})$.

Viewing π as map from $L^1(\omega \times \omega)$, almost the same argument as above yields the following.

Lemma (1.3.4) [1]:

Suppose that $\lim_{x\to\infty} \omega(x^{-1})\omega(x) = \infty$. Then $\pi^*|C_0(\omega^{-1})$ maps $C_0(\omega^{-1})$ into $C_0(\omega^{-1} \times \omega^{-1})$.

When the hypothesis that $\lim_{x\to\infty} \widehat{\omega}(x^{-1})\omega(x) = \infty$ (or the hypothesis that $\lim_{x\to\infty} \omega(x^{-1})\omega(x) = \infty$) holds, set $\tilde{\pi} = \left(\pi^*|C_0(\widehat{\omega}^{-1})\right)^*: M(\widehat{\omega} \times \omega) \to M(\widehat{\omega})$ (or, respectively, $\tilde{\pi} = \left(\pi^*|C_0(\widehat{\omega}^{-1})\right)^*: M(\omega \times \omega) \to M(\omega)$). Then $\tilde{\pi}$ extends π and is weak*-weak* continuous.

Theorem (1.3.5) [1]:

Let ω be a weight function on G.

(i) Suppose that there is a net $(r_\alpha) \subset G$ such that $\lim_\alpha r_\alpha = \infty$ and $\left(\omega(r_\alpha^{-1})\omega(r_\alpha)\right)$ is bounded. Then $L^1(\omega)$ is boundedly approximately contractible if and only if it is amenable;

(ii) Suppose that $\lim_{x\to\infty} \widehat{\omega}(x^{-1})\omega(x) = \infty$. Then $L^1(\omega)$ is not boundedly approximately amenable.

Proof:

We begin by setting up some module machinery. It is routine to check that $C_0(\omega^{-1} \times \omega^{-1})$ is a Banach $L^1(\omega)$-bimodule, and hence a Banach $M(\omega)$-bimodule; the module actions are given by

$$\begin{cases} (\mu \cdot f)(x,y) = \int\limits_G f(x,y\xi)\,d\mu(\xi), \\ (f \cdot \mu)(x,y) = \int\limits_G f(\xi x,y)\,d\mu(\xi), \end{cases} \tag{20}$$

where $x, y \in G, \mu \in M(\omega)$ and $f \in C_0(\omega^{-1} \times \omega^{-1})$. It follows that $M(\omega \times \omega)$ is dual $M(\omega)$-bimodule, with actions

$$\begin{cases} \langle \mu \cdot m, f\rangle = \int\limits_{G^3} f(\xi x,y)\,d\mu(\xi)\,dm(x,y), \\ \langle m \cdot \mu, f\rangle = \int\limits_{G^3} f(x,y\xi)\,d\mu(\xi)\,dm(x,y), \end{cases} \tag{21}$$

where $\mu \in M(\omega \times \omega)$ and $f \in C_0(\omega^{-1} \times \omega^{-1})$.

We also have $C_0(\omega^{-1} \times \omega^{-1})$ is an $M(\omega)$-bimodule with actions given by (20). So $M(\widehat{\omega} \times \omega)$ is a dual $M(\omega)$-bimodule, with module actions given by (21). Moreover, these actions are weak*-weak* continuous in each variable separately.

Finally, the natural dual actions given by

$$\begin{cases} (f \cdot m)(z) = \displaystyle\int_{G\times G} f(x, yz)dm(x, y), \\ (m \cdot f)(z) = \displaystyle\int_{G\times G} f(zx, y)dm(x, y) \end{cases} \tag{22}$$

for $f \in C_0(\omega^{-1} \times \omega^{-1})$ and $m \in M(\widehat{\omega} \times \omega)$ define mapping from $C_0(\widehat{\omega}^{-1} \times \omega^{-1}) \times M(\widehat{\omega} \times \omega)$ into $C_0(\widehat{\omega}^{-1})$.

Note that $M(\omega \times \omega)$ is dual $L^1(\omega)$-bimodule by restricting the operation in (21). Consider the continuous mapping $D : L^1(\omega) \to M(\omega \times \omega)$ given by $D(f) = f \otimes \delta_e - \delta_e \otimes f$. In general, D is a derivation into $\ker \pi$. If $\lim_{x\to\infty} \omega(x^{-1})\omega(x) = \infty$, we can regard D as a derivation into $\ker \tilde{\pi}$ which, by Lemma (1.3.4), is a dual $L^1(\omega)$-bimodule. Now suppose that $L^1(\omega)$ is boundedly approximately contractible or that it is boundedly approximately amenable with $\lim_{x\to\infty} \widehat{\omega}(x^{-1})\omega(x) = \infty$ (which implies that $\lim_{x\to\infty} \omega(x^{-1})\omega(x) = \infty$). Then there is a net (μ_j) ($(\mu_j) \subset \ker \pi$ in the former case and $(\mu_j) \subset \ker \tilde{\pi}$ in the latter case) and $k_0 > 0$ such that for all $\varphi \in L^1(\omega)$),

$$D(\varphi) = \lim_j (\varphi \cdot \mu_j - \mu_j \cdot \varphi), \qquad \text{with } \|\varphi \cdot \mu_j - \mu_j \cdot \varphi\| \le k_0 \|\varphi\|.$$

Set $M_j = \delta_e \otimes \delta_e - \mu_j$ and $k = k_0 + 2$. Then $\pi(M_j) = \delta_e$ (or, respectively, $\tilde{\pi}(M_j) = \delta_e$), and for every $\varphi \in L^1(\omega)$,

$$\varphi \cdot M_j - M_j \cdot \varphi \xrightarrow{j} 0 \text{ and } \|\varphi \cdot M_j - M_j \cdot \varphi\| \le k\|\varphi\| \text{ for all } j. \tag{23}$$

Since the M-bimodule operations are weak*-weak* continuous from (21), it follows from (23) that

$$\|\mu \cdot M_j - M_j \cdot \mu\| \le k\|\mu\| \quad (\mu \in M(\omega)).$$

In particular, $\|\delta_r \cdot M_j - M_j \cdot \delta_r\| \le k\omega(r)$ for each $r \in G$. That is to say,

$$\int_{G\times G} \omega(x)\omega(y)d\left|\delta_r \cdot M_j - M_j \cdot \delta_r\right|(x,y) \le k\omega(r),$$

and so

$$\int_{G\times G} \frac{\omega(rx)\omega(y)}{\omega(r)} d\left|M_j - \delta_{r^{-1}} \cdot M_j \cdot \delta_r\right|(x,y) \le k$$

for $r \in G$ and all j. Then for any compact set $K \subset G \times G$,

$$\begin{aligned}\int_K \frac{\omega(rx)\omega(y)}{\omega(r)} d\left|M_j\right|(x,y) &\le k + \int_K \frac{\omega(rx)\omega(y)}{\omega(r)} d\left|\delta_{r^{-1}} \cdot M_j \cdot \delta_r\right|(x,y) \\ &\le k + \int_{(r,e)K(e,r^{-1})} \frac{\omega(x)\omega(yr)}{\omega(r)} d\left|M_j\right|(x,y) \\ &\le k + \int_{(r,e)K(e,r^{-1})} \omega(x)\omega(y)\, d|M_j(x,y).\end{aligned}$$

But $M_j \in M(\omega \times \omega)$, and so, as $r \to \infty$, the integral on the right-hand side tends to 0.

If $L^1(\omega)$ is boundedly approximately contractible and there is a net $(r_\alpha) \subset G$ such that $r_\alpha \to \infty$ and $\omega(r_\alpha^{-1})\omega(r_\alpha) \le d$ for all α, then we let r tend to ∞ through (r_α). Noting that

$$\frac{\omega(rx)\omega(y)}{\omega(r)} \ge \frac{\omega(x)\omega(y)}{\omega(r^{-1})\omega(r)} \ge \frac{1}{d}\omega(x)\omega(y)$$

when $r = r_\alpha$, we have

$$\frac{1}{d}\left\|M_j\right\| \le k \quad \text{for all } j.$$

Therefore, (M_j) is a bounded net in $M(\omega \times \omega) \subset \left(L^1(\omega) \oplus L^1(\omega)\right)^{**}$, which implies that there is a virtual diagonal for $L^1(\omega)$ is amenable. This together with the remark after Definition (1.2.1) proves the first statement of the theorem.

Now suppose that $L^1(\omega)$ is boundedly approximately amenable and that $\lim_{x\to\infty} \widehat{\omega}(x^{-1})\omega(x) = \infty$. We have

$$\int_K \widehat{\omega}(x)\omega(y)\, d|M_j|(x,y) \le \limsup_{r\to\infty} \int_K \frac{\omega(rx)\omega(y)}{\omega(r)} d||M_j|(x,y) \le k.$$

(In fact, let A be the collection of all compact sets of G with the inclusion as partial order. Then the net $(f_C)_{C\in A}$ with $f_C(x,y) = \inf_{r\in G\setminus C} \frac{\omega(rx)}{\omega(r)} \omega(y)\ \big((x,y) \in K\big)$ is equicontinuous, and so converges to $\widehat{\omega} \times \omega$ in measure on K.) Thus the net (M_j) is bounded in $M(\widehat{\omega} \times \omega)$. By going to a subnet necessary, we may assume that (M_j) converges weak* to some $M \in M(\widehat{\omega} \times \omega)$. Note that weak* continuity of $\tilde{\pi}$ and $\tilde{\pi}(M_j) = \delta_e$ give $\tilde{\pi}(M) = \delta_e$.

Now for each $\varphi \in L^1(\omega), \varphi \cdot M_j - M_j \cdot \varphi \to 0$ in $M(\omega \times \omega)$, and since $\widehat{\omega} \le \omega$, this limit also holds on $M(\widehat{\omega} \times \omega)$. But weak* continuity

$$\varphi \cdot M - M \cdot \varphi = 0 \quad \big(\varphi \in L^1(\omega)\big).$$

By weak* continuity again, we have $\mu \cdot M - M \cdot \mu = 0$ for $\mu \in M(\omega)$, so in particular $M = \delta_{r^{-1}} \cdot M \cdot \delta_r$ for $r \in G$. Thus

$$\|M\|_{\widehat{\omega}\times\omega} = \|\delta_{r^{-1}} \cdot M \cdot \delta_r\| = \int_{G\times G} \widehat{\omega}(r^{-1}x)\omega(yr)\, d|M|(x,y).$$

So for any compact $K \subset G \times G$,

$$\begin{aligned}\infty > \|M\|_{\widehat{\omega}\times\omega} &\ge \int_K \widehat{\omega}(r^{-1}x)\omega(yr)\, d|M|(x,y)\\ &\ge \int_K \frac{\widehat{\omega}(r^{-1})\omega(r)}{\omega(x^{-1})\omega(y^{-1})} d|M|(x,y) \ \ge \frac{\widehat{\omega}(r^{-1})\omega(r)}{C_K} \int_K d|M|(x,y),\end{aligned}$$

where $C_K = \max_{(x,y)\in K^{-1}} \omega(x)\omega(y)$. Letting $r \to \infty$, finiteness of $\|M\|_{\widehat{\omega}\times\omega}$ implies that $\int_K d|M|(x,y) = 0$, and this holding for any compact $K \subset G \times G$ necessitates $M = 0$. But this is a contradiction to $\tilde{\pi}(M) = \delta_e$. Thus the second statement of the theorem is true.

Corollary (1.3.6) [1]:

The Beurling algebras $\ell^1(\mathbb{Z},\omega), \omega(n) = (1 + |n|)^\alpha$ with $\alpha > 0$, are not boundedly approximately amenable and hence are not sequentially approximately amenable.

As noted, approximately amenability for commutative algebras, so, $\ell^1((1+|n|)^\alpha)$ is not approximately amenable for $\alpha \geq 1/2$.

Now we give a new proof for characterization of amenability of Beurling algebras due to N. Gr¢nbæk.

Let Ω be the symmertrization of ω as define in the beginning of this section. The following is essentially.

Theorem (1.3.7) [1]:

Let G be a locally compact group, ω a weight on G with $\omega(e)=1$. Then the following are equivalent:

(i) $L^1(\omega)$ is amenable;
(ii) $L^1(\Omega)$ is amenable;
(iii) G is amenable and Ω is bounded.

The next results together give a new proof of Theorem (1.3.7). In fact we able to dispense with the assumption that $\omega(e)=1$.

Proposition (1.3.8) [1]:

Let ω be a weight function on a locally compact group G, and suppose that $L^1(\omega)$ is amenable. Then Ω is bounded.

Proof:

Let $f \in L^1(\omega)$ have compact support K and be such that $\int_G f(x)\,dx \neq 0$. Certainly $F = f \cdot 1_K \in L^\infty(\omega^{-1})$ since $1_K \in L^\infty(\omega^{-1})$ and $L^\infty(\omega^{-1})$ is a Banach $L^1(\omega)$-bimodule. Then $\pi^*(F) \in L^\infty(\omega^{-1} \times \omega^{-1})$ with

$$\pi^*(F)(x,y) = F(x,y) = \int 1_K(xy\xi) f(\xi)\,d\xi.$$

It follows that $\pi^*(F)(x,y) = 0$ for $xy \notin KK^{-1}$. Set $E = KK^{-1}$, a compact subset of G.

Now suppose that $u \in L^1(\omega \times \omega)^{**}$ is a virtual diagonal for $L^1(\omega)$, so that $u = \delta_g \cdot u \cdot \delta_{g^{-1}}$ $(g \in G)$, and $\pi^{**}(u) \cdot f = f$. Thus

$$\langle \pi^*(F), u\rangle = \langle F, \pi^{**}(u)\rangle = \langle 1_K, \pi^{**}(u)\cdot f\rangle = \langle 1_K, f\rangle \int_K f(x)\,dx$$
$$\neq 0. \tag{24}$$

Define

$$A = \{(x, y): xy \in E\}.$$

Then $\pi^*(F)$ has support contained in A, so $\pi^*(F) = \pi^*(F)1_A$.

Given $\alpha > 0$, define

$$A_\alpha = \{(x, y) \in A: \omega(x)\omega(y) < \alpha\},$$
$$B_\alpha = A\backslash A_\alpha = \{(x, y) \in A: \omega(x)\omega(y) \geq \alpha\}.$$

Clearly $\pi^*(F)1_{A_\alpha}, \pi^*(F)1_{B_\alpha} \in L^\infty(\omega^{-1} \times \omega^{-1})$, and $\pi^*(F) = \pi^*(F)1_{A_\alpha} + \pi^*(F)1_{B_\alpha}$.

Now estimate,

$$\left|\langle \pi^*(F)1_{B_\alpha}, u\rangle\right| \leq \left\|\pi^*(F)1_{B_\alpha}\right\| \cdot \|u\| = \|u\| \sup_{B_\alpha} \left|\frac{\pi^*(F)(x, y)}{\omega(x)\omega(y)}\right|$$
$$= \|u\| \sup_{B_\alpha} \left|\frac{F(xy)}{\omega(xy)} \cdot \frac{\omega(xy)}{\omega(x)\omega(y)}\right| \leq \alpha^{-1}\|u\|\|F\|c_1,$$

where $c_1 = \sup_{t\in E} \omega(t)$. Thus

$$\lim_{\alpha\to\infty} \langle \pi^*(F)1_{B_\alpha}, u\rangle = 0. \tag{25}$$

Further, for any $\mathrm{g} \in G$,

$$\left|\langle \pi^*(F)1_{A_\alpha}, u\rangle\right| = \left|\langle \pi^*(F)1_{A_\alpha} \cdot \delta_{\mathrm{g}} \cdot u \cdot \delta_{\mathrm{g}^{-1}}\rangle\right|$$
$$\leq \|u\|\left\|\delta_{\mathrm{g}^{-1}} \cdot \pi^*(F)1_{A_\alpha} \cdot \delta_{\mathrm{g}}\right\| = \|u\| \sup_{A_\alpha} \left|\frac{\pi^*(F)(x, y)}{\omega(\mathrm{g}^{-1}x)\omega(y\mathrm{g})}\right|$$
$$= \|u\| \sup_{A_\alpha} \left|\frac{F(x, y)}{\omega(x, y)} \cdot \frac{\omega(xy)}{\omega(\mathrm{g}^{-1}x)\omega(y\mathrm{g})}\right|$$
$$\leq \|u\|\|F\| \sup_{A_\alpha} \frac{\omega(xy)\omega(x^{-1})\omega(y^{-1})}{\omega(\mathrm{g}^{-1})\omega(\mathrm{g})}$$
$$\leq \|u\|\|F\| \sup_{A_\alpha} \frac{\omega(xy)\omega^2(y^{-1}x^{-1})\omega(x)\omega(y)}{\omega(\mathrm{g}^{-1})\omega(\mathrm{g})}.$$

Thus

$$\left|\langle \pi^*(F)1_{A_\alpha}, u\rangle\right| \leq \frac{\alpha\|u\|\|F\|c_1c_2^2}{\omega(\mathrm{g}^{-1})\omega(\mathrm{g})}, \tag{26}$$

where $c_2 = \sup_{t\in E^{-1}} \omega(t)$.

Suppose the result is false. Then there is a sequence $(\mathrm{g}_n) \subset G$ such that $\lim_{n\to\infty} \omega(\mathrm{g}_n)\omega(\mathrm{g}_n^{-1}) = \infty$, whence it follows from (26) that for each $\alpha > 0$,

$$\left|\langle \pi^*(F)1_{A_\alpha}, u\rangle\right| = 0. \tag{27}$$

Putting (25) and (27) together, it follows that

$$\langle \pi^*(F), u\rangle = 0.$$

contradicting (24).

The next step we show.

Proposition (1.3.9) [1]:

Let G be a locally compact group, ω a weight on G such that $L^1(\omega)$ is amenable. Then there is a continuous positive character ϕ on G such that

$$\phi(\mathrm{g}) \leq \omega(\mathrm{g}) \quad (\mathrm{g} \in G).$$

Proof:

Let $u \in L^1(\omega\times\omega)^{**}$ be a virtual diagonal for $L^1(\omega)$, so that $\delta_{\mathrm{g}^{-1}} \cdot u \cdot \delta_{\mathrm{g}} = u\ (\mathrm{g} \in G)$ and $\pi^{**}(u)\cdot f = f\left(f \in L^1(\omega)\right)$. For $f \in L^\infty(\omega^{-1}\times\omega^{-1})^+$, define

$$\tilde{u}(f) = \sup\{\mathrm{Re}\langle u, \psi\rangle : 0 \leq |\psi| \leq f, \psi \in L^\infty(\omega^{-1}\times\omega^{-1})\}.$$

Then $\tilde{u} \not\equiv 0$ on $L^\infty(\omega^{-1}\times\omega^{-1})^+$ and $\tilde{u}$ is affine on $L^\infty(\omega^{-1}\times\omega^{-1})^+$, and satisfies $0 \leq \tilde{u}(f) \leq \|u\|\|f\|(L^\infty(\omega^{-1}\times\omega^{-1})^+)$. Thus $\tilde{u}$ can be extended to a bounded linear functional on $L^\infty(\omega^{-1}\times\omega^{-1})$ in the obvious manner. Then $\tilde{u} \neq 0, \langle\tilde{u}, f\rangle \geq 0$ for $f \in L^\infty(\omega^{-1}\times\omega^{-1})^+$, and $\delta_{\mathrm{g}^{-1}}\cdot\tilde{u}\cdot\delta_{\mathrm{g}} = \tilde{u}\ (\mathrm{g}\in G)$.

Now define

$$\widetilde{\omega}(x) = \sup_{g \in G} \omega(g^{-1}xg) \quad (x \in G).$$

Note that $\widetilde{\omega}$ is lower semicontinuous and hence measurable. By Proposition (1.3.8), Ω is bounded, whence $\widetilde{\omega} \in L^\infty(\omega^{-1})$. Further, clearly $\widetilde{\omega}(g^{-1}xg) = \widetilde{\omega}(x)$ $(x, g \in G)$, whence $\widetilde{\omega}(xy) = \widetilde{\omega}(yx)$ $(x, y \in G)$.

Consider $\pi^*(\widetilde{\omega}) \in L^\infty(\omega^{-1} \times \omega^{-1})$. Note that

$$\delta_g \cdot \pi^*(\widetilde{\omega}) \cdot \delta_{g^{-1}} = \pi^*(\widetilde{\omega}) \quad (g \in G).$$

Take $f \in C_c(G)^+$ with $\int f = 1$, let K be the support of f, and set $h = f \cdot 1_K$, where we regard f as an element in $L^1(\omega)$ and 1_K in $L^\infty(\omega^{-1})$. Then h is continuous with support contained in KK^{-1}. Since $\widetilde{\omega}(x) \geq \omega(x) > 0$ for $x \in G$, there is $c > 0$ such that $\widetilde{\omega} \geq ch$, whence $\pi^*(\widetilde{\omega}) \geq c\pi^*(h)$. Thus

$$\langle \tilde{u}, \pi^*(\widetilde{\omega}) \rangle \geq c\langle \tilde{u}, \pi^*(h) \rangle \geq c\mathrm{Re}\langle \tilde{u}, \pi^*(h) \rangle$$

$$= c\mathrm{Re}\langle \pi^{**}(u), h \rangle \geq c\mathrm{Re}\langle f, 1_K \rangle = c > 0.$$

Set $F = \langle \tilde{u}, \pi^*(\widetilde{\omega}) \rangle^{-1} \pi^*(\widetilde{\omega}) \in L^\infty(\omega^{-1} \times \omega^{-1})$, so we have that $\delta_{g^{-1}} \cdot F \cdot \delta_g = F$ $(g \in G)$ and $\langle \tilde{u}, F \rangle = 1$. Now define, for $g \in G$,

$$A_g(x, y) = \frac{1}{2}\left[\log \frac{\omega(gx)\omega(gy^{-1})}{\omega(x)\omega(y^{-1})}\right] F(x, y) \quad (x, y \in G).$$

Then for $g \in G$,

$$\log(g^{-1})\, F \leq A_g \leq \log \omega(g)\, F, \tag{28}$$

so that $A_g \in L^\infty(\omega^{-1} \times \omega^{-1})$. Note that, for $g_1, g_2 \in G$,

$$A_{g_1 g_2} = \delta_{g_2^{-1}} \cdot A_{g_1} \cdot \delta_{g_2} + A_{g_2}. \tag{29}$$

Finally, define

$$\phi(g) = \exp\langle \tilde{u}, A_g \rangle \quad (g \in G).$$

Then (29) gives

$$\phi(g_1 g_2) = \phi(g_1)\phi(g_2) \quad (g_1, g_2 \in G),$$

so that ϕ is a character, and from (28)

$$\phi(\mathrm{g}) \le \exp\langle \tilde{u}, \log \omega(\mathrm{g})\, F\rangle = \omega(\mathrm{g}) \quad (\mathrm{g} \in G)$$

shows ϕ dominated by ω. ϕ bounded (on a neighbourhood of e) shows it is continuous.

Corollary (1.3.10) [1]:

Let G be a locally compact group, ω a weight on G. Then if $L^1(\omega)$ is amenable, G is amenable.

Proof:

By Proposition (1.3.9) there is a continuous positive character $\phi \le \omega$. Then $\Phi : f \longmapsto \phi f$ is continuous monomorphism of $L^1(G,\omega) \to L^1(G)$. Since ϕ is bounded on compact sets. Then a range of Φ contains $C_c(G)$, whence $L^1(G)$ is amenable. It is standard that this equivalent to G being amenable.

Proposition (1.3.11) [1]:

Let G be a locally compact group, ω a weight on G. Then G is amenable and Ω is bounded if and only if $L^1(\Omega)$ is amenable.

Proof:

Supposing G is amenable and Ω is bounded, $L^1(\Omega) \cong L^1(G)$ is amenable. The converse is the symmetric case of Proposition (1.3.8) and Corollary (1.3.10).

The final step is then

Proposition (1.3.12) [1]:

Let G be an amenable locally compact group, ω a weight on G such that Ω is bounded. Then $L^1(\omega)$ is amenable.

A discrete semigroup S is left amenable if the space $\ell^\infty(S)$ admits a functional m such that $m(1) = 1 = \|m\|$ and $m(\ell_x f) = m(f)$ $\big(x \in S, f \in \ell^\infty(S)\big)$. Similarly for right amenable. If S is both left and right amenable, it is amenable. In the case of a group, or even an inverse semigroup, left (or right) amenable implies amenable.

We recall some further standard notions from semigroup theory. Only the left versions will be defined. Let S be a semigroup.

(i) S is regular if for all $s \in S$, there is $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$, it is an inverse semigroup if such s^* exists is unique;

(ii) $T \subseteq S$ is a left ideal group if T is a left ideal in S as well as being a group under the semigroup operation.

Set E_S to be the set of idempotents in S. Note that (1) above both $ss^*, s^*s \in E_S$.

We summarize some known structural implications of amenable of $\ell^1(S)$. In fact a characterization is given.

Theorem (1.3.13) [1]:

Let S be a semigroup with $\ell^1(S)$ amenable. Then:

(i) S is amenable;

(ii) S is regular;

(iii) E_S is finite;

(iv) $\ell^1(S)$ has an identity;

(v) S contains exactly one left ideal group S_0, which is also the only right ideal group, and $S = S_0 z^{-1} = z^{-1} S_0$, for some idempotent z, furthermore S_0 is amenable.

Now suppose that $\ell^1(S)$ is approximately amenable. Example (1.1.10) shows that (iii), (iv) and (v) may fail. On the other hand.

Theorem (1.3.14) [1]:

Let S be a semigroup such that $\ell^1(S)$ is approximately amenable. Then

(i) S is regular;

(ii) S is amenable.

Proof:

The argument is valid as far as showing that for each $v \in S, sS \cap [vv^{-1}] \neq \emptyset$, and that is sufficient to show regularity. Further, the standard argument, applied to an approximate diagonal yield a net $\Lambda_v \subset L^\infty(S)^*$ satisfying $\delta_s \cdot \Lambda_v = \Lambda_v, \Lambda_v \delta_s - \Lambda_v \to 0$ weak* for all $s \in$

S, and $\langle 1, \Lambda_v \rangle \to 1$. The argument at the end of now gives an invariant mean, so that S is amenable.

We give a direct construct of an approximate diagonal for $L^1(\omega)$ to show (iii) ⇒ (i) of Theorem (1.3.7) (that is, Proposition (1.3.12)) without assuming $\omega(e) = 1$. First a simple lemma.

Lemma (1.3.15) [1]:

Let ω be a weight on G (not necessarily satisfying $\omega(e) = 1$). Then the following are equivalent:

(i) Its symmetrization Ω is bounded;

(ii) There is a constant $k > 0$ such that

$$\omega(gh) \geq k\omega(g)\omega(h) \quad (g, h \in G); \qquad (30)$$

(iii) There is a weight $\widetilde{\omega}$ on G, equivalent to ω, with $g \longmapsto \widetilde{\omega}(g)\widetilde{\omega}(g^{-1})$ a constant.

Proof:

(i) ⇒ (ii).

$$\omega(g)\omega(h) \leq \omega(g)\omega(g^{-1})\omega(gh) \leq \Omega(g)\omega(gh) \leq const \cdot \omega(gh).$$

(ii) ⇒ (i). Just take $h = g^{-1}$

(ii) ⇒ (iii). Define

$$\widetilde{\omega}(g) = \left(\frac{\omega(g)}{k\omega(g^{-1})}\right)^{1/2}.$$

Clearly $\widetilde{\omega}(g)\widetilde{\omega}(g^{-1}) = 1/k$. Further,

$$\widetilde{\omega}(g) = \left(\frac{\omega(g^2g^{-1})}{k\omega(g^{-1})}\right)^{1/2} \leq \left(\frac{\omega(g^2)}{k}\right)^{1/2} \leq \frac{\omega(g)}{\sqrt{k}}$$

and

$$\widetilde{\omega}(g) = \left(\frac{\omega(g^2g^{-1})}{k\omega(g^{-1})}\right)^{1/2} \geq \omega(g^2)^{1/2} \geq (k\omega(g)^2)^{1/2} = \sqrt{k}\omega(g).$$

Thus $\widetilde{\omega}$ and ω are equivalent.

Finally,

$$\widetilde{\omega}(gh) = \left(\frac{\omega(gh)}{k\omega(h^{-1}g^{-1})}\right)^{1/2} \leq \left(\frac{\omega(g)\omega(h)}{k^2\omega(h^{-1})\omega(g^{-1})}\right)^{1/2} = \widetilde{\omega}(g)\widetilde{\omega}(h).$$

(iii) ⇒ (ii) is obvious.

Theorem (1.3.16) [1]:

Let ω be a weight on G (not necessarily satisfying $\omega(e) = 1$). Suppose that G is amenable and Ω is bounded. Then $L^1(\omega)$ has a bounded approximate diagonal and hence is amenable.

Proof:

We will use $\|\cdot\|_1$ for the usual L^1-norm $\|\cdot\|_\omega$ the norm in $L^1(\omega) \cdot L_t$ will denote the left translation by t: $(L_t a)(s) = a(t^{-1}s)$. Fix throughout a neighbourhood V of e such that $\omega(g) \leq 2\omega(e)$ for $g \in V$. Let k be the constant given by Lemma (1.3.15) (ii).

Now take $\varepsilon > 0$ and a finite subset $F \subset L^1(\omega)$. Take a compact set K such that

$$\int_{G\setminus K} \omega(t)|f(t)|\, dt < \varepsilon k/\big(8\omega(e)\big) \quad (f \in F).$$

Using Reiter's condition (P_1) there is a $a \in C_{00}(G)^+$ with $\|a\|_1 = 1$ and $\|f\|_\omega \|L_t a - a\|_1 < \varepsilon k/\big(4\omega(e)\big)$ for $t \in K, f \in F$.

Now $f \in L^1(G)$ for each $f \in F$, and so there is a neighbourhood u of e such that for $s \in \operatorname{supp}(a)$, $t \in U, f \in F$,

$$\|L_{sts^{-1}}(f\omega) - f\omega\|_1 < \frac{\varepsilon}{2},$$

$$\|f\|_\omega \left[|\omega(sts^{-1}) - 1| + \frac{|\omega(st^{-1}s^{-1}) - 1|}{\omega(st^{-1}s^{-1})}\right] < \frac{\varepsilon}{2}.$$

Thus we have

$$
\begin{aligned}
\|L_{sts^{-1}}f - f\|_\omega &\le \|L_{sts^{-1}}(f\omega) - f\omega\|_1 + \|L_{sts^{-1}}f(L_{sts^{-1}}\omega - \omega)\|_1 \\
&< \frac{\varepsilon}{2} + \|f(\omega - \ L_{st^{-1}s^{-1}}\omega)\|_1 \\
&\le \frac{\varepsilon}{2}\|f\|_\omega \left\|1 - \frac{L_{st^{-1}s^{-1}}\omega}{\omega}\right\|_\infty \\
&\le \frac{\varepsilon}{2} + \|f\|_\omega \left[|\omega(sts^{-1}) - 1| + \frac{|\omega(st^{-1}s^{-1}) - 1|}{\omega(st^{-1}s^{-1})}\right] \\
&< \varepsilon.
\end{aligned}
$$

Now take $b \in L^1(G)^+$ with $\|b\|_1 = 1$ and $\operatorname{supp}(b) \subset U$. Define $u_{\varepsilon,F} = u$ in $L^1(G \times G)$ by

$$u(s,t) = a(s)b(ts) \vartriangle (s),$$

where $\vartriangle$ is the modular function of G. Since a and b have compact support, $u \in L^1(\omega \times \omega)$ which is, of course, $L^1(\omega) \widehat{\otimes} L^1(\omega)$.

Further, u is bounded independent of ε and F:

$$
\begin{aligned}
\|u\|_{\omega\times\omega} &= \int\limits_{G\times G} \omega(s)a(s)b(ts) \vartriangle (s)\omega(t)\, ds\, dt \\
&= \int\limits_{G\times G} \omega(s)\omega(ts^{-1})a(s)b(t)\, ds\, dt \le \frac{1}{k} \int\limits_{G\times G} \omega(t)a(s)b(t)\, ds\, dt \\
&\le \frac{2\omega(e)}{k}\|a\|_1\|b\|_1 = \frac{2\omega(e)}{k}.
\end{aligned}
$$

Now for $f \in F$,

$$(f \cdot u)(s,t) = \int\limits_G f(v)a(v^{-1}s)b(tv^{-1}s) \vartriangle (v^{-1}s)\, dv,$$

$$(u \cdot f)(s,t) = \int\limits_G a(s)b(tv^{-1}s) \vartriangle (v^{-1}s)f(v)\, dv,$$

so that

$$(f \cdot u - u \cdot f)(s,t) = \int\limits_G \big(a(v^{-1}s) - a(s)\big)b(v^{-1}s) \vartriangle (v^{-1}s)f(v)\, dv.$$

Thus

$$
\begin{aligned}
&\|f\cdot u-u\cdot f\|_{\omega\times\omega}\\
&\qquad\le \int_{G^3}\omega(s)\omega(t)|a(v^{-1}s)-a(s)|b(tv^{-1}s)\\
&\qquad\vartriangle(v^{-1}s)|f(v)|\,dv\,ds\,dt\\
&\qquad\le \int_{G^3}\omega(s)|a(v^{-1}s)-a(s)|b(t)\omega(ts^{-1}v)|f(v)|\,dv\,ds\,dt\\
&\qquad\le \int_{G^3}\frac{\omega(s)\omega(s^{-1}v)}{\omega(v)}\omega(t)|a(v^{-1}s)\\
&\qquad- a(s)|b(t)|f(v)|\omega(v)\,dv\,ds\,dt\\
&\qquad\le \frac{2\omega(s)}{k}\int_{G^3}\|L_v a-a\|_1|f(v)|\omega(v)\,dv\\
&\qquad\le \frac{2\omega}{k}\left(\int_{G\backslash K}+\int_{K}\right)\|L_v a-a\|_1|f(v)|\omega(v)dv\\
&\qquad\le \frac{2\omega(e)}{k}\left(2\int_{G\backslash K}|f(v)|\,\omega(v)dv+\frac{k\varepsilon}{4\omega(e)}\right)<\varepsilon.
\end{aligned}
$$

Further,

$$
\begin{aligned}
\pi(u)*f(t) &= \int_{G\times G} a(s)b(s^{-1}vs)\vartriangle(s)f(v^{-1}t)\,dv\,ds\\
&= \int_{G\times G} a(s)b(v)f(sv^{-1}s^{-1}t)\,dv\,ds,
\end{aligned}
$$

so that

$$
\|\pi(u)*f-f\|_\omega=\int_{G\times G}a(s)b(v)\|L_{svs^{-1}}f-f\|_\omega\,dv\,ds
$$

$$
<\varepsilon\int_{G\times G}a(s)b(v)\,ds\,dv=\varepsilon.
$$

It follows that (u_ε,F) is an approximate diagonal for $L^1(\omega)$ with bound at most $2\omega(e)/k$.

Chapter 2

Banach Algebra and Character Amenability

In this chapter Various necessary and sufficient conditions of a global and a pointwise nature are found for a Banach algebra to posses a φ-mean of norm 1. We also completely determine the size of the set of φ-means for a separable weakly sequentially complete Banach algebra A with no φ-mean in A itself. A number of illustrative examples are discussed.

Section (2.1): ϕ −Means of Norm One

The notion of an amenable Banach algebra was defined and studied in the seminal work of Johnson. One of the fundamental results was that for a locally compact group G, the group algebra $L^1(G)$ is amenable if and only if the group G is amenable. Since then amenability has become a major issue in Banach algebra theory and in harmonic analysis.

We continue our recent investigation of a concept which might be referred to as amenability with respect to a character. Let A be an arbitrary Banach algebra and φ a character of A, that is, a homomorphism from A onto $\mathbb{C}$. We call A φ-amenable if there exists a bounded linear functional m on A^* satisfying $(m, \varphi) = 1$ and $\langle m, f \cdot a\rangle = \varphi(a)\langle m \cdot f\rangle$ for all $a \in A$ and $f \in A^*$. Here $f \cdot a \in A^*$ is defined by $\langle f \cdot a, b\rangle = \langle f, ab\rangle, b \in A$. Any such m is called a φ-mean. This concept considerably generalizes the notion of left amenability for F-algebras which was introduced and studied.

Note that a Banach algebra is called right character amenable if it is φ-amenable for each character φ and has a bounded right approximate identity. Note also that for a locally compact group G (respectively, a discrete semigroup S), the group algebra $L^1(G)$ (respectively, the semigroup algebra $l^1(S)$) is amenable with respect to the trivial character 1 precisely when G is amenable (respectively, S is left amenable). However, $l^1(\mathbb{N})$ is not amenable since it does not have a bounded approximate identity.

We give two characterizations (in terms of cohomology groups and a Hahn-Banach type extension property) of φ-amenability, which are close to results. We mainly focus on φ-means of norm 1. We establish various criteria for their existence. Pointwise conditions, in terms of elements $f \in A^*$ or $a \in \ker \varphi$, the kernel of φ, are given that ensure the existence of φ-means of norm 1.

We concentrate on weakly sequentially complete Banach algebras. We show that if there is no φ-mean in A itself, but there exists a so-called sequential bounded approximate φ-mean, then A admit at least $2^c \varphi$-means, and there are no more if A is separable. We also relate the existence of φ-means to Arens regularity of A. A result of a flavor similar to that of Theorem (2.2.1) is obtained in Theorem (2.2.10). It implies that if A is a separable F-algebra and ϵ denotes the identity of the von Neumann algebra A^*, then there are 2^c ϵ-means of norm 1 with the additional property that $\|m - n\| = 2$ for any two of them.

Finally, we present illustrative examples such as Lipschitz algebras and $L^p(G)$, where G is a compact group.

In this section, the second dual A^{**} of a Banach algebra A will always be equipped with the first Arens product which is defined as follows. For $a, b \in A, f \in A^*$ and $m, n \in A^{**}$, the elements $f \cdot a$ and $m \cdot f$ of A^* and $mn \in A^{**}$ are defined by

$$\langle f \cdot a, b\rangle = \langle f \cdot ab\rangle,\ \ \langle m \cdot f, b\rangle = \langle m, f \cdot b\rangle \text{ and } \langle mn, f\rangle = \langle m, n \cdot f\rangle$$

respectively. With this multiplication, A^{**} is a Banach algebra of A^{**}. Alternatively, the multiplication on A^{**} can be defined by using iterated limits as follows. For $m, n \in A^{**}$, let

$$mn = w^* \lim_{a \to m} \Big(w^* - \lim_{b \to n} ab \Big).$$

In general, the multiplication $(m, n) \to mn$ is not separately continuous with respect to the w^*-topology on A^{**}. But, for fixed $n \in A^{**}$, the mapping $m \to mn$ is w^*-continuous, and also for fixed $a \in A$, the mapping $m \to am$ is w^*-continuous. Moreover, for all $m, n \in A^{**}$ and $\varphi \in \Delta(A)$, the set of all homomorphisms from A onto $\mathbb{C}, \langle mn, \varphi\rangle = \langle m, \varphi\rangle\langle n, \varphi\rangle$. Consequently, each $\varphi \in \Delta(A)$ extends to some element φ^{**} of $\Delta(A^{**})$. The kernel of $\varphi^{**}, \ker \varphi^{**}$, contains $\ker \varphi$ in the same sense that A^{**} naturally contains A. Since each of these ideals has codimension

1, the theory of second polars shows that $\ker\varphi$ is w^*-dense in $\ker\varphi^{**}$ and that $\ker\varphi^{**} = (\ker\varphi)^{**}$.

The Banach algebra A is said to be φ-amenable if there exists $m \in A^{**}$ such that $(m, \varphi) = 1$ and $\langle m, f \cdot a\rangle = \varphi(a)\langle m, f\rangle$ for all $f \in A^*$ and $a \in A$, and any such m is called a φ-mean. The φ-means are nothing but the w^*-cluster points of bounded nets $(u_\gamma)_\gamma$ in A with $\varphi(u_\gamma) = 1$ for all γ and $\|au_\gamma - \varphi(a)u_\gamma\| \to 0$ for all $a \in A$. Consequently, we call such a net $(u_\gamma)_\gamma$ a bounded approximate φ-mean. Given a φ-mean m, the net $(u_\gamma)_\gamma$ can be chosen so that $\|u_\gamma\| \to \|m\|$.

If X is a Banach A-module, then so is the dual X^* with the module actions given by

$$\langle a \cdot f, x\rangle = \langle f, x \cdot a\rangle \quad \text{and} \quad \langle f \cdot a, x\rangle = \langle f, a \cdot x\rangle,$$

$a \in A, x \in X, f \in X^*$. In the following theorem $H^l(A, X^*)$ denotes the first cohomology group of A with coefficients in X^*.

Theorem (2.1.1) [2]:

Let A be a Banach algebra and $\varphi \in \Delta(A)$. Then the following three conditions are equivalent.

(i) A is φ-amenable.

(ii) If X is a Banach A-bimodule such that $a \cdot x = \varphi(a)x$ for all $x \in X$ and $a \in A$, then $H^l(A, X^*) = \{0\}$.

(iii) Give $(\ker\varphi)^{**}$ a second A-bimodule structure by taking the left action to be $a \cdot m = \varphi(a)m$ for $m \in A^{**}$ and taking the right action to be the natural one. Then any continuous derivation $D: A \to (\ker\varphi)^{**}$ is inner.

Proof:

The equivalence of (i) and (ii) has been shown. Trivially, (ii) implies (iii), and therefore we only have to show (iii) $\Leftrightarrow$ (i). Choose any $b \in A$ with $\varphi(b) = 1$. Then $Da = ab - ba, a \in A$, defines a derivation from A into $(\ker\varphi)^{**}$. By (iii), D is inner, so there is $m \in (\ker\varphi)^{**}$ such that $Da = a(-m) - (-m)a$ for all $a \in A$. Then

$$a(b+m) = (b+m)a = \varphi(a)(b+m)$$

for all $a \in A$ and $\langle b+m, \varphi \rangle = \varphi(b) = 1$. So $b+m$ is φ-mean.

The implication (iii) ⇒ (ii) in the above shows that if $H^l(A, X^*) = \{0\}$ for the particular case in which $X = (\ker \varphi)^*$, then all such cohomology groups are zero. We have the following result.

Theorem (2.1.2) [2]:

Let A be a Banach algebra and $\varphi \in \Delta(A)$. Then the following two conditions are equivalent.

(i) A is φ-amenable.

(ii) If X is any Banach A-module and Y is any Banach A-submodule of X and $\mathrm{g} \in Y^*$ is such that the left action of A on g has the form $a \cdot \mathrm{g} = \varphi(a)\mathrm{g}$ for all $a \in A$, then g extends to some $f \in X^*$ such that $a \cdot f = \varphi(a)f$ for all $a \in A$.

Proof:

(i) ⇒ (ii) let $\tilde{\mathrm{g}} \in X^*$ such that $\tilde{\mathrm{g}}$ extends g and $\|\tilde{\mathrm{g}}\| = \|\mathrm{g}\|$. If $a \in A$ satisfies $\varphi(a) = 1$, then $a.\tilde{\mathrm{g}}$ also extends g. Since A is φ-amenable, there exists a net $\left(u_\gamma\right)_\gamma = 1$ in A such that, for all $\gamma \cdot \varphi\left(u_\gamma\right) = 1$ and $\left\|u_\gamma\right\| \leq C$ for some constant $C > 0$ and $\left\|a u_\gamma - \varphi(a) u_\gamma\right\| \to 0$ for all $a \in A$. Then $u_\gamma \cdot \tilde{\mathrm{g}}$ extends g and we may assume that $\left\|u_\gamma \cdot \tilde{\mathrm{g}}\right\| \leq C\|\mathrm{g}\| + 1$ for all γ. After passing to a subnet if necessary, we can also assume that $u_\gamma \cdot \tilde{\mathrm{g}} \to f$ in the w^*-topology for some $f \in X^*$. Clearly, f extends g. Taking w^*-limits, we obtain

$$\begin{aligned} a \cdot f &= \lim_\gamma a \cdot \left(u_\gamma \cdot \tilde{\mathrm{g}}\right) = \lim_\gamma \left(a u_\gamma\right) \cdot \tilde{\mathrm{g}} \\ &= \lim_\gamma \left[\left(a u_\gamma - \varphi(a) u_\gamma\right) \cdot \tilde{\mathrm{g}} + \varphi(a) u_\gamma \cdot \tilde{\mathrm{g}}\right] = \varphi(a) f \end{aligned}$$

for all $a \in A$. So (ii) holds.

(ii) ⇒ (i) Take $X = A^*$ and $Y = \mathbb{C}\varphi$. Let $\varphi^* \in Y^*$ be defined by $\langle \varphi^*, \varphi \rangle = 1$. Then the left action of A on φ^* is given by $a \cdot \varphi^* = \varphi(a)\varphi^*$. By hypothesis, there exists $m \in A^{**}$ such that $m|\gamma = \varphi^*$ and $a \cdot m = \varphi(a)m$ for all $a \in A$. Since $\langle m, \varphi \rangle = \langle \varphi^*, \varphi \rangle = 1, m$ is a φ-mean.

Using w^*-continuity, we easily see that an element $m \in A^{**}$ is a φ-mean for A if and only if for all $n \in A^{**}$ we have $nm = \varphi^{**}(n)m$. It is tempting to introduce a new general concept by saying that, when φ is a complex homomorphism on a complex algebra B, m is a φ-right zero if $nm = \varphi(n)m$ for all $n \in B$ (the term "right zero" in this context comes from the measure algebra on a semigroup with a right zero). However, this is worthwhile, as it would merely be giving a new name to a φ-mean which lies in B. But this viewpoint does reduce the idea of a φ-mean to a purely algebraic one, and sometimes it is easy to prove results in context and then interpret them as applying to Banach algebras. It is trivial to notice that A has a φ-mean if and only if A^{**} has a φ^{**}-mean which lies in A^{**}. The next proposition ant its corollary provide an example of this technique.

Proposition (2.1.3) [2]:

Let B be a complex algebra and $\varphi: B \to \mathbb{C}$ a homomorphism. Let J be an ideal in B with $J \subseteq \ker \varphi$ and let $\tilde{\varphi}: B/J \to \mathbb{C}$ be the homomorphism induced by φ. If J has a right identity and B/J has a $\tilde{\varphi}$-mean in B/J, then B has a φ-mean in B.

Proof:

Let $q: B \to B/J$, so that $\varphi = \tilde{\varphi} \circ q$. Let e be a right identity for J and let $m \in B$ be such that $q(m)$ is a $\tilde{\varphi}$-mean for B/J. Since $q(e) = 0$ we find for all $x \in B$,

$$q(x)q(m - me) = q(x)q(m) = \tilde{\varphi}\big(q(x)\big)\, q(m) = \varphi(x)q(m - me).$$

This shows that $x(m - me) - \varphi(x)(m - me) \in J$. Since e is a right identity for J and $(m - me)e = 0$, we see that in fact $x(m - me) - \varphi(x)(m - me) = 0$, so that $m - me$ is a φ-mean for B.

Corollary (2.1.4) [2]:

Let A be a Banach algebra $\varphi \in \Delta(A)$ and I a closed ideal in A with $I \subseteq \ker \varphi$. Suppose that I has a bounded right approximate identity and that A/I is $\tilde{\varphi}$-amenable, where $\tilde{\varphi} \in \Delta(A/I)$ is the homomorphism induced by φ. Then A is φ-amenable.

Proof:

The statement follows from Proposition (2.1.3) on taking $B = A^{**}$ and $J = I^{**}$. In fact, since I has a bounded right approximate identity, I^{**} has a right identity, and since $B/J = A^{**}/I^{**} = (A/I)^{**}$ and A/I is $\tilde{\varphi}$-amenable and $\tilde{\varphi}^{**} = \widetilde{\varphi^{**}}, B/J$ is $\widetilde{\varphi^{**}}$-amenable. Thus B/J has a $\widetilde{\varphi^{**}}$-mean and the proposition shows that B has a φ^{**}-mean. This says that A is φ-amenable.

Let A be a Banach algebra and $\varphi \in \Delta(A)$. In this section we establish several criteria for A to possess a φ-mean of norm 1. We start by showing that the existence of such a mean is a pointwise property.

Theorem (2.1.5) [2]:

Let A be any Banach algebra and $\varphi \in \Delta(A)$. Suppose that for each $f \in A^*$ there exists $m_f \in A^{**}$ such that $\|m_f\| = \langle m_f, \varphi\rangle = 1$ and $\langle m_f, f \cdot a\rangle = \varphi(a)\langle m_f, f\rangle$ for all $a \in A$. Then A has a φ-mean of norm 1.

Proof:

Define a subsets S of A^{**} by

$$S = \{m \in A^{**}: \|m\| = \langle m, \varphi\rangle = 1\} = \{m \in A^{**}: \|m\| \leq 1, \langle m, \varphi\rangle = 1\}.$$

Then S is w^*-compact and easily seen to be a semigroup for the first Arens product. Let $\mathcal{F}$ denote the collection of all finite subsets F of A^*, and for every $F \in \mathcal{F}$, let

$$S_F = \{m \in S: \langle m, f \cdot a\rangle = \varphi(a)\langle m, f\rangle \text{ for all } f \in F \text{ and } a \in A\}$$

Then S_F is closed in S and $S_{F_1} \supseteq S_{F_2}$ whenever $F_1 \subseteq F_2$. Clearly, every $m \cap \{S_F: F \in \mathcal{F}\}$ is a φ-mean with $\|m\| = 1$. It therefore suffices to show that $S_F \neq \emptyset$ for each $F \in \mathcal{F}$. We achieve this by induction on the number of elements in F.

So suppose that some $m_1 \in S_F$ exists and let g $\in A^*\backslash F$ and set $h = m_1 \cdot$ g $\in A^*$. By hypothesis, there exists $m_2 \in S_{\{h\}}$. Let $m = m_2 m_1 \in A^{**}$. Then $m \in S$ since S is a semigroup. For $f \in F$ and $a, b \in A$, we have

$$\langle m_1 \cdot (f \cdot a), b\rangle = \langle m_1, f \cdot (ab)\rangle = \varphi(a)\langle m_1, f\rangle\varphi(b)$$

Hence $m_1 \cdot \langle f \cdot a \rangle = \varphi(a)\langle m_1, f\rangle\varphi$, and similarly $m_1 \cdot f = \langle m_1, f\rangle\varphi$. It follows that, for $f \in F$ and all $a \in A$

$$\langle m, f \cdot a\rangle = \langle m_2, m_1 \cdot (f \cdot a)\rangle = \varphi(a)\langle m_1, f\rangle\langle m_2, \varphi\rangle$$
$$= \varphi(a)\langle m_2, \langle m_1, f\rangle\varphi\rangle = \varphi(a)\langle m_2, m_1 \cdot f\rangle = \varphi(a)\langle m, f\rangle.$$

Moreover, for all $a \in A$,

$$\langle m, \mathrm{g} \cdot a\rangle = \langle m_2, (m_1 \cdot \mathrm{g}) \cdot a\rangle = \varphi(a)\langle m_2, m_1 \cdot \mathrm{g}\rangle = \varphi(a)\langle m, \mathrm{g}\rangle.$$

So $m \in S_{F\cup\{\mathrm{g}\}}$, and this finishes the proof.

Let A be a Banach algebra and $\varphi \in \Delta(A)$. For $f \in A^*$ and $\epsilon > 0$, let

$$K_{f,\epsilon} = \overline{\{u \cdot f : u \in A, \varphi(u) = 1, \|u\| \leq 1 + \epsilon\}}^{w^*} \subseteq A^*.$$

Clearly, $K_{f,\epsilon}$ is convex and w^*-compact, and so is $K_f = \cap_{\epsilon>0} K_{f,\epsilon}$.

Proposition (2.1.6) [2]:

For $f \in A^*$, the following conditions are equivalent.

(i) There exists $m \in A^{**}$ such that $\|m\| = 1, \langle m, \varphi\rangle = 1$ and $\langle m, f \cdot a\rangle = \varphi(a)\langle m, f\rangle$ for all $a \in A$.
(ii) K_f contains $\lambda\varphi$ for some $\lambda \in \mathbb{C}$.

In fact, $\mathbb{C}\varphi \cap K_f$ equals the set of all $\langle m, f\rangle\varphi$ where m is as in (i).

Proof:

Let m be as in (i), and let $(u_\gamma)_\gamma$ be a net in A such that $\varphi(u_\gamma) = 1$ for all $\gamma \cdot \|u_\gamma\| \to 1$ and $u_\gamma \to m$ in the w^*-topology. Then

$$\langle u_\gamma \cdot f, a\rangle = \langle u_\gamma, f \cdot a\rangle \to \langle m, f \cdot a\rangle = \varphi(a)\langle m, f\rangle$$

for all $a \in A$, and hence $\langle m, f\rangle\varphi \in K_{f,\epsilon}$ for every $\epsilon > 0$.

Conversely, assume that $\lambda\varphi \in K_f$ and let $\epsilon > 0$. There exists a net $(u_{\gamma,\epsilon})_\gamma$ in A such that $\varphi(u_{\gamma,\epsilon}) = 1, \|u_{\gamma,\epsilon}\| \leq 1 + \epsilon$ for all γ and $\lambda\varphi = w^* - \lim_\gamma(u_\gamma \cdot f)$. Let n_ϵ be a w^*-cluster point of the net $(u_{\gamma,\epsilon})_\gamma$ in A^{**}. Then $\|n_\epsilon\| \leq 1 + \epsilon, \langle n_\epsilon, \varphi\rangle = 1$ and $\langle n_\epsilon, f \cdot a\rangle = \lambda\varphi(a)$ for all $a \in A$ since

$$\langle u_{\gamma,\epsilon}, f \cdot a\rangle = \langle u_{\gamma,\epsilon} \cdot f, a\rangle \to \lambda\varphi(a).$$

Let n be a w^*-cluster point of the net $(n_\epsilon)_\epsilon$. Then $\|n\| = 1, \langle n, \varphi\rangle = 1$ and

$$\langle n \cdot f, a\rangle = \langle n, f \cdot a\rangle = \lambda\varphi(a)$$

for all $a \in A$. Finally, let $m = n^2 \in A^{**}$. Then $\langle m, \varphi\rangle = \langle n, \varphi\rangle^2 = 1$ and $\|m\| = 1$. Moreover,

$$\langle m, f\rangle = \langle n, n \cdot f\rangle = \langle n, \lambda\varphi\rangle = \lambda\langle n, \varphi\rangle = \lambda,$$

and hence, for all $a \in A$,

$$\langle m, f \cdot a\rangle = \langle n, (n \cdot f) \cdot a\rangle = \langle n, (\lambda\varphi) \cdot a\rangle = \lambda\varphi(a)\langle n, \varphi\rangle = \varphi(a)\langle m, f\rangle.$$

So m satisfies all the requirements in (i).

Actually, the above proof shows that $\lambda\varphi$belongs to K_f if and only if $\lambda = \langle m, f\rangle$ for some $m \in A^{**}$ as in (i).

As an immediate consequence of Proposition (2.1.6) and Theorem (2.1.5) we obtain

Corollary (2.1.7) [2]:

For a Banach algebra A and $\varphi \in \Delta(A)$, the following are equivalent.

(i) A admits a φ-mean of norm 1.

(ii) For each $f \in A^*, \mathbb{C}\varphi \cap K_f \neq \emptyset$.

The next theorem, which is one of the main results, in particular shows that the existence of a φ-mean of norm 1 is a pointwise property in the sense that it follows from the existence of a certain functional on A^* associated with each of the elements of the ideal ker φ.

Theorem (2.1.8) [2]:

For a Banach algebra A and $\varphi \in \Delta(A)$, the following four conditions are equivalent.

(i) There exists a φ-mean such that $\|m\| = 1$.

(ii) There exists a net $(u_\gamma)_\gamma$ in A such that $\varphi(u_\gamma) = 1$ for all $\gamma, \|u_\gamma\| \to 1$ and $\|au_\gamma\| \to |\varphi(a)|$ for all $a \in A$.

(iii) For each $a \in \ker\varphi$, there exists $m_a \in A^{**}$ with $\|m_a\| \leq 1, \langle m_a, \varphi\rangle = 1$ and $am_a = 0$.

(iv) For each $a \in \ker\varphi$ and $\epsilon > 0$, there exists $u \in A$ such that $\|u\| \leq 1 + \epsilon, \|au\| \leq \epsilon$ and $\varphi(u) = 1$.

Proof:

(ii) $\Rightarrow$ (iv) is clear. Also, (i) $\Rightarrow$ (iii) is simple: if m is a φ-mean, we can choose $m_a = m$ for all $a \in A$. Therefore, in order to establish the theorem it suffices to show the implications (i) $\Rightarrow$ (ii), (iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii) There exists a net $(u_\gamma)_\gamma$ in A with the following properties: $\varphi(u_\gamma) = 1$ for all $\gamma, \|u_\gamma\| \to 1$ and $\|au_\gamma - \varphi(a)u_\gamma\| \to 0$ for all $a \in A$. Thus,

$$\big|\|au_\gamma\| - |\varphi(a)|\big| \leq \big|\|au_\gamma\| - \|\varphi(a)u_\gamma\|\big| + \big|\|\varphi(a)u_\gamma\| - |\varphi(a)|\big|$$
$$\leq \|au_\gamma - \varphi(a)u_\gamma\| + |\varphi(a)| \cdot \big|\|u_\gamma\| - 1\big|$$

(iii) $\Rightarrow$ (iv) Fix $a \in \ker\varphi$ and take any net $(u_\gamma)_\gamma$ in A such that $\|u_\gamma\| \leq 1$ and $u_\gamma \to m_a$ in the w^*-topology. Then $\varphi(u_\gamma) \to 1$. By replacing each u_γ with a scalar multiple of itself and taking a cofinal subnet, we may arrange that $\|u_\gamma\| \leq 1 + \epsilon$ and $\varphi(u_\gamma) = 1$ for all γ. Since $w^* \lim au_\gamma = am_a = 0$ and $au_\gamma \in A$, is in the weak closure of the set $(au_\gamma)_\gamma$ and therefore 0 is in the norm closure of the convex hull of $(au_\gamma)_\gamma$. The set $(u_\gamma)_\gamma$ being contained in the closed hyperplane $\{x \in A: \varphi(x) = 1\}$, we easily reach our conclusion.

(iv) $\Rightarrow$ (i) We claim that for every finite subset of F of A and $\epsilon > 0$, there exists $u_{F,\epsilon}$ such that $\varphi(u_{F,\epsilon}) = 1, \|u_{F,\epsilon}\| \leq 1 + \epsilon$ and

$$\|au_{F,\epsilon} - \varphi(a)u_{F,\epsilon}\| \leq \epsilon$$

for all $a \in F$. Let $F = \{a_1, \dots, a_k\}$, say, and choose $\delta > 0$ such that $(1+\delta)^{k+1} \leq 1 + \epsilon$. By hypothesis, there exists $u_0 \in A$ such that

$\varphi(u_0) = 1$ and $\|u_0\| \leq 1 + \delta$. Since $a_1 u_0 - \varphi(a_1) u_0 \in \ker \varphi$, again by (iv) there exists $u_1 \in A$ such that

$$\varphi(u_1) = 1, \qquad \|u_1\| \leq 1 + \delta \text{ and } \|(a_1 u_0 - \varphi(a_1) u_0) u_1\| \leq \delta.$$

Likewise, $a_2 u_0 u_1 - \varphi(a_2) u_0 u_1 \in \ker \varphi$ and hence there exists $u_2 \in A$ such that

$$\varphi(u_2) = 1, \qquad \|u_2\| \leq 1 + \delta \text{ and } \|(a_2 u_0 u_1 - \varphi(a_2) u_0 u_1) u_2\| \leq \delta.$$

For $j = 1,2$ we have $\|u_j\| \leq 1 + \delta, \varphi(u_j) = 1$ and

$$\|a_j u_0 u_1 u_2 - \varphi(a_j) u_0 u_1 u_2\| \leq \delta(1 + \delta)$$

Proceeding inductively, we see that there exist $1 \leq j \leq k$, such that $\varphi(u_j) = 1, \|u_j\| \leq 1 + \delta$ and for $i = 1, \dots, j$,

$$\|a_i u_0 u_1 \dots\dots u_j - \varphi(a_i) u_0 u_1 \dots\dots u_j\| \leq \delta(1 + \delta)^{j-1} \leq \epsilon.$$

In particular, when $j = k$, setting $u_{F,\epsilon} = \prod_{j=0}^{k} u_j$ gives us $\varphi(u_{F,\epsilon}) = 1$, $\|u_{F,\epsilon}\| \leq 1 + \epsilon$ and $\|a u_{F,\epsilon} - \varphi(a) u_{F,\epsilon}\| \leq \epsilon$ for all$a \in F$. This proves the above claim.

Now, order the pairs $(F, \epsilon), F \subseteq A$ finite, $\epsilon > 0$, in the obvious manner, and let m be a w^*-cluster point of the net $(u_{F,\epsilon})_{F,\epsilon}$ in A^{**}. Then $\|m\| \leq 1$ and $\langle m, \varphi \rangle = 1$ (and hence $\|m\| = 1$) and $am = \varphi(a) m$ for all $a \in A$. So m is the required φ-mean.

Remark (2.1.9) [2]:

Using methods similar to those employed in the proof of Theorem (2.1.8), the following can be shown. Let A be a Banach algebra and $\varphi \in \Delta(A)$. For $C > 0$, the following statements are equivalent.

(i) A has a φ-mean of norm C.

(ii) A contains an approximate φ-mean with norm bound C.

(iii) For each $a \in \ker \varphi$, there exists $m_a \in A^{**}$ with $\|m_a\| = C$, $\langle m_a, \varphi \rangle = 1$ and $a m_a = 0$.

(iv) There exists a net $(u_\gamma)_\gamma$ in A with $\varphi(u_\gamma) = 1$ for all $\gamma = \|u_\gamma\| \to C$ and $a u_\gamma \to 0$ for every $a \in \ker \varphi$.

For a Banach algebra A and $\varphi \in \Delta(A)$, let $N(A, \varphi)$ denote the set of all $f \in A^*$ with the following property: for each $\delta > 0$, there exists a sequence $(a_n)_n$ in A such that $\varphi(a_n) = 1, \|a_n\| \leq 1 + \delta$ for all n and $\|f \cdot a_n\| \to 0$. We now aim at a criterion for a φ-mean of norm 1 involving the set $N(A, \varphi)$ (Theorem (2.1.12) below).

Lemma (2.1.10) [2]:

For a Banach algebra A and $\varphi \in \Delta(A)$, the following hold.

(i) $\varphi \notin N(A, \varphi)$.

(ii) $N(A, \varphi)$ is closed in A^* and closed under scalar multiplication.

(iii) If A is commutative, then $N(A, \varphi)$ is closed under addition.

Proof:

(i) is immediate since $\varphi \cdot a = \varphi$ for all $a \in A$ with $\varphi(a) = 1$.

(ii) Let $f_n \in A^*, n \in \mathbb{N}$ and $f \in A^*$ such that $f_n \to f$. For every n there exists $a_n \in A$ such that $\varphi(a_n) = 1, \|a_n\| \leq 1 + \frac{1}{n}$ and $\|f_n \cdot a_n\| \leq \frac{1}{n}$. Then $\|f \cdot a_n\| \leq \|f - f_n\| \cdot \|a_n\| + \frac{1}{n}$ for all n, whence $f \in N(A, \varphi)$.

(iii) Let $f_1, f_2 \in N(A, \varphi)$ and $\delta > 0$. If $a_j \in A, j = 1,2$, are such that $\varphi(a_j) = 1, \|a_j\| \leq 1 + \delta$ and $\|f_j \cdot a_j\| \leq \delta$, then since A is commutative,

$$\|(f_1 + f_2) \cdot (a_1 a_2)\| \leq \|f_1 \cdot a_1\| \cdot \|a_2\| + \|f_2 \cdot a_2\| \cdot \|a_1\| \leq 2\delta(1 + \delta).$$

It follows that $f_1 + f_2 \in N(A, \varphi)$.

Lemma (2.1.11) [2]:

Suppose that A admits a φ-mean of norm 1. Then $N(A, \varphi)$ is a subspace of A^*.

Proof:

Let $J = \{a \in A: \varphi(a) = 1\}$ and let $\epsilon > 0$. Since A has a φ-mean of norm 1, there exists a net $(u_\gamma)_\gamma$ in A such that $\varphi(u_\gamma) = 1$ and $\|u_\gamma\| \leq 1 + \epsilon$ for all γ and $\|a u_\gamma - u_\gamma\| \to 0$ for every $a \in J$.

Now let $f_1, f_2 \in N(A, \varphi)$. Given $\epsilon > 0$, there exists $a_1, a_2 \in J$ such that $\|f_j \cdot a_j\| \leq \epsilon$ and $\|a_j\| \leq 1 + \epsilon, j = 1,2$. By the first paragraph, there exists $u \in A$ with $\|u\| \leq 1 + \epsilon, \varphi(u) = 1$ and

$$\|a_1 u - a_2 u\| \leq \|a_1 u - u\| + \|u - a_2 u\| < \epsilon.$$

Then

$$\begin{aligned} &\|(f_1 + f_2) \cdot (a_1 u)\| \\ &\leq \|f_1 \cdot (a_1 u)\| + \|f_2 \cdot (a_1 u) - f_2 \cdot (a_2 u)\| + \|f_2 \cdot (a_2 u)\| \\ &\leq \|f_1 \cdot a_1\| \cdot \|u\| + \|f_2\| \cdot \|a_1 u - a_2 u\| + \|f_2 \cdot a_2\| \cdot \|u\| \\ &\leq \epsilon(1 + \epsilon) + \epsilon\|f_2\| + \epsilon(1 + \epsilon) \\ &= \epsilon(2 + 2\epsilon + \|f_2\|). \end{aligned}$$

Since $\varphi(a_1 u) = 1$ and $\|a_1 u\| \leq (1 + \epsilon)^2$ and $\epsilon > 0$ is arbitrary, it follows that $f_1 + f_2 \in N(A, \varphi)$.

Theorem (2.1.12) [2]:

Let A be a Banach algebra and $\varphi \in \Delta(A)$. Then the following two conditions are equivalent.

(i) There exists a φ-mean m with $\|m\| = 1$.

(ii) $N(A, \varphi)$ is a subspace of A^* and $f \cdot a - f \in N(A, \varphi)$ for all $f \in A^*$ and all $a \in A$ with $\varphi(a) = 1$.

Proof:

Let m be a φ-mean of norm 1. By Lemma (2.1.11), $N(A, \varphi)$ is a subspace of A^* and $a \in A$ with $\varphi(a) = 1$. There exists a net $(u_\gamma)_\gamma$ in A such that $\varphi(u_\gamma) = 1, \|u_\gamma\| \to 1$ and $\|a u_\gamma - u_\gamma\| \to 0$ since $\|(f \cdot a - f) \cdot u_\gamma\| \leq \|f\| \cdot \|a u_\gamma - u_\gamma\|$, it follows that $f \cdot a - f \in N(A, \varphi)$.

Conversely, suppose that $N(A, \varphi)$ is a subspace of A^* and that (ii) holds. Since $\varphi \notin N(A, \varphi)$ and $\|\varphi\| = 1$, by the Hahn-Banach theorem there exists $m \in A^{**}$ such that $\|m\| = \langle m, \varphi \rangle = 1$ and $m|_{N(A,\varphi)} = 0$. Then, by (ii), $\langle m, f \cdot a \rangle = \langle m, f \rangle$ for all $f \in A^*$ and all $a \in A$ with $\varphi(a) = 1$ and hence $\langle m, f \cdot a \rangle = \varphi(a)\langle m, f \rangle$ for all $a \in A$.

We shall see in Example (2.2.16) that if $\|m\| > 1$, it can even happen that $N(A, \varphi) = \{0\}$.

The following corollary is an immediate consequence of Lemma (2.1.10) and Theorem (2.1.12).

Corollary (2.1.13) [2]:

If A is a commutative Banach algebra and $\varphi \in \Delta(A)$, then A has a φ-mean of norm 1 if and only if $f \cdot a - f \in N(A, \varphi)$ for all $f \in A^*$ and all $a \in A$ with $\varphi(a) = 1$.

Before proceeding, recall that an F-algebra A is a Banach algebra which is the predual of a von Neumann algebra M such that the identity ϵ of M is a multiplicative linear functional on A. In this case, the ϵ-means of norm 1 are nothing but the topologically left invariant means (TLIM) on A^*. Examples of F-algebra include the group algebra, the Fourier algebra and the Fourier-Stieltjes algebra of a locally compact group. Other examples are the measure algebra of a locally compact semigroup and the predual of a Hopf-von Neumann algebra.

Let A be a Banach algebra and $\varphi \in \Delta(A)$. We say that an element a of A is φ-maximal if it satisfies $\|a\| = \varphi(a) = 1$. Let $P_1(A, \varphi)$ denote the collection of all φ-maximal elements of A. When A is an F-algebra and φ is the identity of the von Neumann algebra A^*, the φ-maximal elements are precisely the positive linear functionals of norm 1 on A^* and hence span A. However, in general $P_1(A, \varphi)$ can be quite small.

Let $X(A, \varphi)$ denote the closed span of $P_1(A, \varphi)$. Then $X(A, \varphi)$ is a closed subalgebra of A.

Were MarKov-Kakutani fixed point theorem [7] said: A commuting family of continuous affine self-mappings of a compact convex subset in locally convex topological vectors space has a common fixed point.

Proposition (2.1.14) [2]:

Let A be a commutative Banach algebra and $\varphi \in \Delta(A) = A$, if $X(A, \varphi) = A$, then A has a φ-mean of norm 1.

Proof:

Let $K = \{m \in A^{**} : \|m\| = \langle m, \varphi \rangle = 1\}$. Then K is a w^*-compact convex subset of A^{**}. For each $a \in P_1(A, \varphi)$, let $T_a : K \to K$ denote the map $m \to am$. Then $a \to T_a$ is a representation of the commutative semigroup $P_1(A, \varphi)$ as $w^* - w^*$-continuous affine mapping from K into K. Therefore, by the Markov-Kakutani fixed point theorem, there exists $m \in K$ with $am = m$ for all $a \in P_1(A, \varphi)$. For all $a \in A$, it then follows that $am = \varphi(a)m$, and hence m is a φ-mean.

Remark (2.1.15) [2]:

Let A be a Banach algebra such that A is a left ideal in A^{**}. Let $\varphi \in \Delta(A)$ and suppose that there exists a φ-mean m. Then there exists a φ-mean in A itself.

To see this, fix $a \in A$ with $\varphi(a) = 1$. If A is a right ideal in A^{**}, then $m = \varphi(a)m = am \in A$. If A is the left ideal in A^{**}, then

$$\langle ma, \varphi \rangle = \langle m, a \cdot \varphi \rangle = \langle m, \varphi \rangle = 1$$

and $b(ma) = \varphi(b)ma$ for all $b \in A$, whence $ma \in A$ is φ-mean.

Section (2.2): Complete Banach Algebras and Invariant Means on F- Algebras

A φ-mean of a Banach algebra A is an element of the second dual of A. There are some aspects of the theory of second duals which are particularly striking for weakly sequentially complete algebras. In this section we offer some results which are relevant to φ-means.

A Banach algebra A is weakly consequentially complete if every sequence $(a_n)_n$ in A which is weakly Cauchy is weakly convergent in A. As is well-known, preduals of von Neumann algebras are weakly sequentially complete. In particular, $L^1(G)$ and $A(G)$, the group algebra and the Fourier algebra of a locally compact group G, are weakly sequentially complete. The w^*-topology on A^{**} induces the weak topology on A, so an easy consequence of the definitions is that if a sequence $(a_n)_n$ in A converges to a w^*-limit $a \in A^{**}$, then in fact $a \in A$. Since bounded subsets in A^{**} are relatively w^*-compact, we see that if $(a_n)_n$ is a bounded sequence in A which has just one w^*-cluster point in A^{**}, then that cluster point is in A.

Theorem (2.2.1) [2]:

Let A be weakly sequentially complete with a sequential bounded approximate φ-mean, but with no φ-mean in A itself. Then A has at least $2^c \varphi$-means. If A is separable, then it has precisely 2^c φ-means.

Proof:

Let $(u_n)_n$ be a sequential bounded approximate φ-mean, and let M denote the set of all w^*-cluster point of $(u_n)_n$ in A^{**}. Each element of M is w^*-compact. We claim that no element of M has a countable neighbourhood based in M. Indeed, suppose that for some $m \in M$, there is a decreasing countable base $(V_k)_k$ of closed neighbourhoods of m in M. Choose w^*-closed neighborhoods $W_k, k \in \mathbb{N}$, of m in A^{**} with $W_k \cap M = V_k$. Then $M \cap (\cap_{k=1}^{\infty} W_k) = \{m\}$, and we can arrange for the sequence $(W_k)_k$ to be decreasing. For each k, select $u_{n_k} \in W_k$. Then every w^*-cluster point of the subsequence $\left(u_{n_k}\right)_k$ lies in each W_k and in M, so must be equal to m. Since A weakly sequentially complete, it follows that $m \in A$, which is impossible by hypothesis.

Thus no point of M has a countable neighbourhoods base. This implies that M has at least 2^c elements. Finally, if$A$ is separable, then A^{**} has a countable w^*-dense subset and hence no more than 2^c elements.

Example (2.2.2) [2]:

(i) If G is a locally compact group, then $\epsilon: f \to \int_G f(x)\,dx$ defines an element of $\Delta(L^1(G))$. In this case, the ϵ-means correspond to the set of topologically left invariant means on $L^\infty(G)$. Suppose that G is amenable, second countable and noncompact. Since then $L^1(G)$ is separable, it follows from Theorem (2.2.1) that L^∞ admits precisely 2^c topologically left invariant means, a fact which is known.

(ii) Let G be a locally compact group and $A(G)$ its Fourier algebra. Then $A(G)^* = VN(G)$, the von Neumann algebra generated by left translation operators on $L^2(G)$. The identity operator 1 on $L^2(G)$ defines an element ϵ of $\Delta(A(G))$ by $\epsilon(u) = \langle I, u\rangle = u(e), u \in A(G)$. Then the set of ϵ-mean coincides with the set of topologically invariant means that studied, if G is second countable, then $L^2(G)$ is separable and hence $A(G)$ is separable and weakly sequentially complete. If, in addition, G, is not discrete, then no ϵ-mean can belong to $A(G)$. Thus the cardinality of the set of topologically invariant means on $VN(G)$ is exactly 2^c.

(iii) Consider the convolution algebra $A = l^1(\mathbb{Z}_+)$. For $z \in \overline{\mathbb{D}}$, the closed unit disc, define $\varphi_z: A \to \mathbb{C}$ by $\varphi_z(a) = \sum_{n=0}^{\infty} a_n z^n, a = (a_n)_n \in A$. Then the map $z \to \varphi_z$ is a homeomorphism between $\overline{\mathbb{D}}$ and $\Delta(A)$. We already know that A is φ_z-amenable if and only if $|z| = 1$. Let $z \in \overline{\mathbb{D}}$ with $|z| = 1$. Since A is weakly sequentially complete and separable, by Theorem (2.2.1) there either exists a φ_z-mean in A itself or there are precisely 2^c φ_z-means. Now, suppose that $u = (u_n)_n \in A$ is a φ_z-mean. Then for all $a \in A$ and $f = (f_n)_n \in l^\infty(\mathbb{Z}_+) = A^*$.

$$\sum_{n=0}^{\infty} f_n\,(a_k u_{n-k}) = \langle f, a * u\rangle = \langle f \cdot a, u\rangle = \varphi_z(a)\langle f, u\rangle$$

$$= \sum_{n=0}^{\infty} a_n z^n \cdot \sum_{n=0}^{\infty} f_n u_n.$$

Taking $f = \delta_k$ and $a = \delta_l, l > k$, we obtain $z^l u_k = 0$. Thus $u = 0$ and hence there are exactly 2^c φ_z-means.

If m_1 and m_2 are two φ-means on A, then $m_1 m_2 = \varphi(m_1)m_2 = m_2$. One of the immediate consequences of Theorem (2.2.1) is therefore that if A satisfies its hypotheses, A^{**} is not commutative, even if A is. There is a formulation of this which makes sense for non-commutative algebras. Define a second multiplication on A^{**} by

$$m \diamond n = w^* - \lim_{b \to n}\left(w^* - \lim_{a \to m} ab\right)$$

(a similar formula to that which determines the multiplication in A^{**}, but with the limits taken in the other order). The product $m \diamond n$ is w^*-continuous in n for fixed $m. A$ is called Arens regular if $m \diamond n = mn$ for all $m, n \in A^{**}$. A condition equivalent to Arens regularity is that mn should be w^*-continuous in n for fixed m. When A is commutative, so that $ba = ab$, we find that in A^{**} we have $m \diamond n = nm$. Thus we have shown that, under the hypotheses of Theorem (2.2.1), a commutative A is not Arens regular. We shall obtain a non-commutative result generalizing this.

We must introduce some additional concepts. We call $m \in A^{**}$ a 2-sided φ-mean if $\langle m, \varphi\rangle = 1$ and for each $f \in A^*$ and $a \in A$ we have not only $\langle m, f \cdot a\rangle = \varphi(a)\langle m, f\rangle$, but also $\langle m, f \cdot a\rangle = \varphi(a)\langle m, f\rangle$. Of course, the latter two conditions are equivalent to $am = \varphi(a)m$ and $ma = \varphi(a)m$ for all $a \in A$, respectively. W^*-continuity then gives $nm = \langle n, \varphi\rangle m$ for all $n \in A^{**}$. However, we cannot conclude that $mn = \langle n, \varphi\rangle m$ unless A is Arens regular. Notice that if A is commutative, every φ-mean is automatically a 2-sided φ-mean.

A bounded net $\left(u_\gamma\right)_\gamma$ in A is called a bounded approximate 2-sided φ-mean if $\varphi\left(u_\gamma\right) = 1$ for all γ and for each $a \in A$,

$$\|au_\gamma - \varphi(a)u_\gamma\| \to 0 \text{ and } \|u_\gamma a - \varphi(a)u_\gamma\| \to 0.$$

Proposition (2.2.3) [2]:

An element m of A^{**} is a 2-sided φ-mean for A if and only if m is a w^*-cluster point of a bounded approximate 2-sided φ-mean.

Proof:

If m is a w^*-cluster point of a bounded approximate 2-sided φ-mean $(u_\gamma)_\gamma$, then for each $a \in A, am$ is a w^*-cluster point of $(u_\gamma)_\gamma$ and this implies that $am = \varphi(a)m$. Similarly, $ma = \varphi(a)m$. Since also $\langle m, \varphi\rangle = \lim_\gamma \varphi(u_\gamma) = 1$, we get that m is a 2-sided φ-mean.

Conversely, let m be a 2-sided φ-mean. Then m is the w^*-limit of some net $(u_\gamma)_\gamma$ in A with $\|v_\gamma\| \to \|m\|$. Then $\varphi(v_\gamma) - 1 = \langle v_\gamma - m, \varphi\rangle \to 0$, and w^*-continuity gives

$$av_\gamma - \varphi(a)v_\gamma \to am - \varphi(a)m = 0 \text{ and } v_\gamma a - \varphi(a)v_\gamma \to ma - \varphi(a)m = 0$$

in the w^*-topology for each $a \in A$. So the nets

$$\left(av_\gamma - \varphi(a)v_\gamma\right)_\gamma \quad and \quad \left(v_\gamma a - \varphi(a)v_\gamma\right)_\gamma$$

in A both converge to 0 weakly for all $a \in A$.

Now take any finite subset $F = \{a_1, \dots, a_k\}$ for A and let

$$C = \left\{\left(\left(a_j v - \varphi(a_j)v\right)_{j=1}^k, \left(va_j - \varphi(a_j)v\right)_{j=1}^k, \varphi(v) - 1\right) : v \in A\right\}.$$

Then in the Banach space $A^{2k} \times \mathbb{C}, 0$ is in the weak closure of C and hence in the norm closure since C is convex. Thus, given $\epsilon > 0$, we can find $V_{F,\epsilon} \in A$ such that $\|V_{F,\epsilon}\| \leq 2\|m\|$, say, $\left|\varphi(V_{F,\epsilon}) - 1\right| < \epsilon$ and for all $a \in F$,

$$\|av_{F,\epsilon} - \varphi(a)v_{F,\epsilon}\| < \epsilon \text{ and } \|v_{F,\epsilon}a - \varphi(a)v_{F,\epsilon}\| < \epsilon.$$

Finally replace $v_{F,\epsilon}$ by a scalar multiple $u_{F,\epsilon} = \lambda_{F,\epsilon}v_{F,\epsilon}$ for which $\varphi(u_{F,\epsilon}) = 1$. Then $|\lambda_{F,\epsilon}| < \frac{1}{1-\epsilon}$ and

$$\left\|au_{F,\epsilon} - \varphi(a)u_{F,\epsilon}\right\| < \frac{\epsilon}{1-\epsilon} \quad \text{and} \quad \left\|u_{F,\epsilon}a - \varphi(a)u_{F,\epsilon}\right\| < \frac{\epsilon}{1-\epsilon}$$

So the net $\left(u_{F,\epsilon}\right)_{F,\epsilon}$ is a bounded approximate 2-sided φ-mean and m is the w^*-limit of $\left(u_{F,\epsilon}\right)_{F,\epsilon}$.

We shall show

Theorem (2.2.4) [2]:

Let A be weakly sequentially complete. Suppose that A has a bounded approximate 2-sided φ-mean, but that there is no 2-sided φ-mean in A itself. Then A is not Arens regular.

In proving Theorem (2.2.4) we will partly follow an idea of Ulger, where he established a parallel result for bounded approximate identities.

Let I be a commutative idempotent semigroup, that is, $i^2 = i$ for all $i \in I$. Define an order on I by $i \leq j$ if $ij = j$. Then I is a directed set with $\max\{i,j\} = ij$.

Proposition (2.2.5) [2]:

Let A be a Banach algebra. Let I be as above and let $h: I \to A$ be a homomorphism into the multiplicative semigroup of A such that $h(I)$ is bounded and $0 \notin h(I)$. If the net $\left(h(i)\right)_i$ has a weak cluster point in A, then $h(I)$ has a maximal element.

Proof:

Let e be a weak cluster point of $\left(h(i)\right)_i$. Take J to be a cofinal subset of I with $w - \lim_{i\in J} h(i) = e$. For $i \leq j$ in J we have $h(i)h(j) = h(j)$. Taking the j-limit gives $h(i)e = e$ and then taking the i-limit gives $e^2 = e$. Since weak and norm closures of convex sets coincide, e is in the norm closure of the convex hull of $\{h(i): i \in J\}$. Thus given $\epsilon > 0$, we can find $j_1, \dots, j_n \in J$ and scalar $\lambda_1, \dots, \lambda_n \geq 0$ with $\sum_{k=1}^{n} \lambda_k = 1$ such that

$$\left\|\sum_{k=1}^{n} \lambda_k h(j_k) - e\right\| \leq \epsilon.$$

For $j \in J$ with $j \geq \max\{j_1, \dots, j_n\}$ we have

$$\left(\sum_{k=1}^{n} \lambda_k h(jk)\right) h(j) = \sum_{k=1}^{n} \lambda_k h(j_k j) = h(j).$$

Because $h(I)$ is commutative, we see that $eh(j) = e$ for all j and therefore

$$\|h(j) - e\| = \left\|\sum_{k=1}^{n} \lambda_k h(j_k) h(j) - eh(j)\right\| \leq \left\|\sum_{k=1}^{n} \lambda_k h(j_k) - e\right\| \cdot \|h(j)\|$$
$$\leq \epsilon \sup_{j\in J} \|h(j)\|.$$

But $h(j) - e$ is an idempotent, so either is zero or satisfies $\|h(j) - e\| \geq 1$. Since $\epsilon > 0$ is arbitrary, it follows that $h(j) = e$. This holds for a cofinal set of j's and consequently e is a maximal element in $h(I)$.

Next we present a general construction which produces subalgebras which have sequential bounded approximate φ-means.

Proposition (2.2.6) [2]:

Let A be a Banach algebra with a bounded approximate 2-sided φ-mean (respectively, a bounded approximateφ-mean) $(\mathrm{u}_\gamma)_\gamma$. Let $\mathrm{X} = \{\mathrm{x}_1, \mathrm{x}_2, \dots\}$ be any countable subset of A. Then there is a closed separable subalgebra A(X) of A which contains X and has a sequential bounded approximate 2-sided φ-mean (respectively, a sequential bounded approximate φ-mean) $(\mathrm{u}_{\gamma_\mathrm{n}})_{\gamma_\mathrm{n}}$ chosen from $(\mathrm{u}_\gamma)_\gamma$.

Proof:

We shall only prove the 2-sided φ-mean case (the other one being easier). If we replace each element of X by any non-zero scalar multiple of itself we do not change $A(X)$, and we may therefore arrange for X to be bounded. Thus let $C > 0$ be such that $\|x_n\|, \|u_\gamma\| \leq \mathrm{C}$ for all n, γ. We choose $u_{\gamma_n}, n \in N$, inductively to satisfy

$$\|x_i u_{\gamma_n} - \varphi(x_i) u_{\gamma_n}\| \leq \frac{1}{n} \quad \text{and} \quad \|u_{\gamma_n} x_i - \varphi(x_i) u_{\gamma_n}\| \leq \frac{1}{n}$$

for $1 \leq i \leq n$, and

$$\|u_{\gamma_i} u_{\gamma_n} - u_{\gamma_n}\| \leq \frac{1}{n} \quad \text{and} \quad \|u_{\gamma_n} u_{\gamma_i} - u_{\gamma_n}\| \leq \frac{1}{n}$$

for $1 \leq i < n$. We take $A(X)$ to be the closed linear span of $X \cup \{u_{\gamma_1}, u_{\gamma_2}, \dots\}$.

Now $(u_{\gamma_n})_{\gamma_n}$ is a bounded approximate 2-sided φ-mean for $A(X)$. This requires a little argument, whereas the corresponding conclusion when dealing with bounded approximate identities is just a simple observation. Take any k elements $a_1, \dots, a_k$ from $X \cup \{u_{\gamma_1}, u_{\gamma_2}, \dots\}$ and let $\epsilon > 0$. Choose N so large that $k\mathrm{C}^{k-1}/N < \epsilon$ and that if $n > N$, then x_n and u_{γ_n} do not belong to $\{a_1, \dots, a_k\}$. Then, for $n > N$, we can estimate the norm

$$\mathrm{C}_n := \left\|a_1 \dots a_k u_{\gamma_n} - \varphi(a_1 \dots a_k) u_{\gamma_n}\right\|$$

as follows:

$$c_n \leq \sum_{j=1}^{k} \left\|a_1 \dots a_j \varphi(a_j + 1) \dots \varphi(a_k) u_{\gamma_n} - a_1 \dots a_{j-1} \varphi(a_j) \dots \varphi(a_k) u_{\gamma_n}\right\|$$

$$\leq \sum_{j=1}^{k} \|a_1\| \dots \|a_{j-1}\| \cdot |\varphi(a_{j+1})| \dots |\varphi(a_k)| \cdot \left\|a_j u_{\gamma_n} - \varphi(a_j) u_{\gamma_n}\right\|$$

$$\leq \sum_{j=1}^{k} \mathrm{C}^{j-1} \mathrm{C}^{k-j} \frac{1}{n} = k\mathrm{C}^{k-1} \frac{1}{n}$$

$$< \epsilon.$$

A parallel calculation deals with $u_{\gamma_n} a_1 \dots a_k$. Similar methods will allow us to treat finite linear combinations of products $a_1 \dots a_k$. We then have a bounded approximate 2-sided φ-mean for the algebra generated algebraically by $X \cup \{u_{\gamma_n} : n \in \mathbb{N}\}$. Standard arguments extend this to the norm closure, that is, A(X).

Corollary (2.2.7) [2]:

Let A be a separable Banach algebra and $\varphi \in \Delta(A)$. If A is φ-amenable, then there exists a sequential bounded approximate φ-mean.

The algebra A(X) constructed in the proof of Proposition (2.2.6) is of course not unique, as it depends on the choice of $(u_{\gamma_n})_n$. We now prove Theorem (2.2.4) in the following form.

Theorem (2.2.8) [2]:

Let A be weakly sequentially complete and Arens regular, and suppose that A has a 2-sided φ-mean m. Then m is unique and contained in A.

Proof:

We first consider the case in which the bounded approximate 2-sided ϕ-mean is sequential, say $(u_n)_n$. If m_1 and m_2 are both 2-sided φ-means, we have $a \cdot m_2 = \varphi(a)m_2$ for all $a \in A$, and choosing a net in A converging w^* to m_1, we get $m_1 m_2 = \langle m_1, \varphi \rangle m_2 = m_2$. In the same way, from $m_1 \cdot a = \varphi(a)m_1$, because A is Arens regular, we get that $m_1 m_2 = m_1$. Thus $m_1 = m_2$, and in particular any two w^*-cluster points of $(u_n)_n$ are equal. Since A is weakly sequentially complete, there exists a cluster point in A itself, which then is the unique 2-sided φ-mean.

Now let A be arbitrary. Take any countable subset X_1 of A and form $A(X_1)$ as in Proposition (2.2.6). Then $A(X_1)$ is weakly sequentially complete and Arens regular and has a sequential bounded approximate 2-sided φ-mean. By the first part of the proof, $A(X_1)$ has a unique 2-sided φ-mean, m_1. If m_1 is a 2-sided φ mean for A, we are finished. Otherwise we can find a countable subset X_2 of A with $m_1 \in X_2$ for which m_1 is not a 2-sided φ-mean. Then $A(X_2)$ contains a 2 sided φ-mean, m_2 say. In particular, $m_1 m_2 = \langle m_1, \varphi \rangle m_2 = m_2$ and similarly $m_2 m_1 = m_2$. Again, if m_2 is a 2-sided φ-mean for A, we are finished. Otherwise, take X_3 with $m_1, m_2 \in X_3$ in order to find m_3, and so on. If this process stops we have found a 2-sided φ-mean in A. If it does not stop, we find a bounded infinite sequence $(m_n)n$ in A with the product $m_k m_l = m_{\max\{k,l\}}$. This is impossible by Proposition (2.2.5).

Let A be an F-algebra. We now use the term topologically left invariant mean (TLIM) rather than ϵ-mean of norm 1. The purpose is to prove the following theorem which was proved for the Fourier algebra of a locally compact group.

Theorem (2.2.9) [2]:

Let A be a separable F-algebra which is ϵ-amenable. Suppose that A* contains a C*-subalgebra B such that B is w^*-dense in A^* and $m(B) = \{0\}$ for every ϵ-mean m. Then there is a linear isometry Θ from $l^1(\mathbb{N})$

into A with the property that each $m \in \Theta^{**}(\beta\mathbb{N}\backslash\mathbb{N})$ is an ϵ-mean. In particular, if $m_1, m_2 \in \Theta^{**}(\beta\mathbb{N}\backslash\mathbb{N})$ are distinct, then $\|m_1 - m_2\| = 2$.

The proof of Theorem (2.2.9) will make substantial use of the following lemma.

Lemma (2.2.10) [2]:

Let A and B be as in Theorem (2.2.9).

(i) If m is a TLIM on A^*, then $\|m - a\| = 2$ for every $a \in P_1(A, \epsilon)$.

(ii) If a net $(u_\gamma)_\gamma$ is an approximate ϵ-mean with $\|u_\gamma\| = 1$ for all γ, then $\lim_\gamma \|u_\gamma - a\| = 2$ for each $a \in P_1(A, \epsilon)$.

Proof:

(i) Since B is w^*-dense in A^*, by the Kaplansky density the unit ball of B is w^*-dense in the unit ball of A^*. Consequently, the map $r: A^{**} \to B^*, m \to m|_B$ is a linear isometry of A^{**} into B^*. Choose a bounded approximate identity $(e_\beta)_\beta$ in B such that $e_\beta \geq 0, \|e_\beta\| \leq 1$ and $e_\beta \leq e_{\beta'}$ if $\beta \leq \beta'$ (such a bounded approximate identity exists in every C^* -algebra). Let $a \in P_1(A, \epsilon)$. Then $\|a\| = \lim_\beta \langle e_\beta, a\rangle$ and hence, given any $\delta > 0$, there exists β such that $\langle e_\beta, a\rangle \geq \|a\| - \delta = 1 - \delta$. Let $g = 2e_\beta - \epsilon \in A^*$. Now, if m is any ϵ-mean, then

$$\langle a - m, g\rangle = \langle a - m, 2e_\beta - \epsilon\rangle = 2\langle a - m, e_\beta\rangle \geq 2(1 - \delta) - 2\langle m, e_\beta\rangle$$
$$= 2 - 2\delta$$

as $\langle m, e_\beta\rangle = 0$. So $\|a - m\| \geq 2 - 2\delta$, and since $\delta > 0$ was arbitrary, $\|a - m\| = 2$.

(ii) If $\|u_\gamma - a\|$ does not converge to 2, then by taking a subnet, we may assume that $\|u_\gamma - a\| \leq 2 - \delta$ for all γ and some $\delta > 0$. Then $\|m - a\| \leq 2 - \delta$ for every w^*-cluster point of $(u_\gamma)_\gamma$. This contradicts (i) since any such m is an ϵ mean.

We now turn to the proof of Theorem (2.2.9). For $a \in A$, let $s(a)$ denote the support of a in A^*, that is, the smallest projection p such that $\langle p, a\rangle = \epsilon(a) = \|a\|$.

If m is a positive linear functional of norm 1 on A^*, then there exists a net $(u_\gamma)_\gamma$ in A such that $u_\gamma \geq 0, \|u_\gamma\| = 1$ (equivalently, $u_\gamma \in P_1(A, \epsilon)$) and $u_\gamma \to m$ in the w^*-topology. Thus $w^*\text{-}\lim(au_\gamma - \epsilon(a)u_\gamma) = 0$ for every $a \in A$.

By an argument similar to the one in the proof of Proposition (2.2.6), we can find a sequence $(u_{y_n})_n$ such that $\|au_{y_n} - \epsilon(a)u_{y_n}\| \to 0$ for all $a \in A$. By Lemma (2.2.10), $\lim_y\|u_{y_n} - a\| = 2$ for all $a \in P_1(A, \epsilon)$. Using Theorem (2.1.8)(iii). We can find a subsequence $(u_{y_n})_j$ of $(u_{y_n})_n$ and sequence $(v_j)_j$ in $P_1(A, \epsilon)$ such that

$$\left\|u_{\gamma_{n_j}} - \mathrm{v}_j\right\| < \frac{1}{2^{j-1}}$$

for all j and $s(v_j)s(v_k) = 0$ if $j \neq k$. Clearly, $(v_j)_j$ is an approximate ϵ-mean.

Let $V = \{v_1, v_2, \ldots\}$. Since V is orthogonal, V is a linearly independent subset of A. Let

$$\Theta: \text{span}\,\{\delta_v : v \in V\} \to \text{span}\, V$$

be defined by

$$\Theta\left(\sum_{j=1}^{n} \lambda_j \delta_{v_j}\right) = \sum_{j=1}^{n} \lambda_j v_j.$$

Clearly, $\|\Theta(\sum_{j=1}^n \lambda_j v_j)\| \leq \sum_{j=1}^n |\lambda_j|$. On the other hand, if $P_j = s(v_j)$, then $(P_j)_j$ is a sequence of pariwise orthogonal projections in A^*. Let M be the w^*-closure of the span of the $P_j, j \in \mathbb{N}$. Then M is a commutative w^*-subalgebra of A^*. For each j, let $\mu_j \in \mathbb{C}$ such that $\mu_j\lambda_j = |\lambda_j|$, and let $q = \sum_{j=1}^n \mu_j P_j \in A^*$. Then $\|q\| = 1$, and

$$\langle\Theta\left(\sum_{j=1}^{n}\lambda_j\delta_{v_j}\right),q\rangle=\sum_{j=1}^{n}|\lambda_j|,$$

and therefore

$$\left\|\Theta\left(\sum_{j=1}^{n}\lambda_j\delta_{v_j}\right)\right\|=\left\|\sum_{j=1}^{n}\lambda_j\delta_{v_j}\right\|.$$

Consequently, Θ extends to a linear isometry, also denoted Θ, from $l^1(V)$ into A.

Finally, since each $\eta\in\beta\mathbb{N}\backslash\mathbb{N}$ is a w^*-cluster point of $\left(\delta_{v_j}\right)_j$, and Θ^{**} is w^*-continuous, it follows that $m=\Theta^{**}(\eta)$ is a w^*-cluster point of the sequence $(\mathrm{v}_j)_j$. So for each w^*-neighbourhood U of m, there exists $n_U\in\mathbb{N}$ such that $v_{n_U}\in U$. Let $\mathcal{U}$ denote the set of all w^*-neighbourhood of m. Then $\left(v_{n_U}\right)_U$ is a subset of the sequence $(v_j)_j$ and $v_{n_U}\to m$ in the w^*-topology. Indeed, otherwise there exists $N\in\mathbb{N}$ such that $n_U\leq N$ for all U, which implies that m is an ϵ-mean in A. However, since $m|_B=0$, this is impossible. Clearly, m is ϵ-mean.

Examples (2.2.11) [2]:

We present two illustrative examples: algebras of Lipschitz functions on compact metric spaces and convolution algebras $L^P(G)$ on a compact group G. In both case, the relevant singletons in $\varDelta(A)$ are open. We therefore start by looking at how openness of $\{\varphi\}$ and φ-amenability are related.

Remark (2.2.12) [2]:

Let A be a Banach algebra and $\varphi\in\varDelta(A)$ and suppose that A is φ-amenable. For every $\psi\in\varDelta(A)$ such that $\psi=\varphi$, there exists $a_\psi\in\ker\psi$ with $\varphi(a_\psi)=1$. So, if m is a φ-mean, then $(m,\varphi)=1$, whereas

$$(m,\psi)=\left(m,\psi\cdot a_\psi\right)=\langle m,\psi\left(a_\psi\right)\psi\rangle=0$$

for all $\psi\neq\varphi$. Hence $\{\varphi\}$ is open in ($\varDelta(A)$, weak).

We can define a Shilov's idempotent [8]: Let A be a commutative Banach algebra and let c be a compact open subset of $\Delta(A)$. Then there exists an idempotent a in A such that $\hat{a}$ equals the characteristic function of C.

Lemma (2.2.13) [2]:

Let A be a semisimple commutative Banach algebra. Let $\varphi \in \Delta(A)$ and suppose that $\{\varphi\}$ is open in $\Delta(A)$. Let a be the unique element of A with $\varphi(a) = 1$ and $\psi(a) = 0$ for all $\psi \in \Delta(A)\backslash\{\varphi\}$.

(i) Then a is a φ-mean for A and it is only one in A^{**}.
(ii) If $\|a\| = 1$, then $N(A, \varphi) = \{f \in A^*: \langle f, a\rangle = 0\}$.

Proof:

The existence of a follows from Shilov's idempotent. However, in the present special situation it is easy to avoid such heavy machinery. To see this, let J be the closed ideal of A defined by

$$J = \{a \in A: \psi(a) = 0 \; for \; all \; \psi \in \Delta(A)\backslash\{\varphi\}\}.$$

Since $\{\varphi\}$ is open in $(\Delta(A), w^*), \Delta(J) = \{\varphi|_J\}$ and hence J is 1-dimensiomal as A is semisimple. Of course, $\ker\varphi + J = A$ and $\ker\varphi \cap J = \{0\}$ since A is semisimple and $\psi(\ker\varphi \cap J) = \{0\}$ for all $\psi \in \Delta(A)$. Thus $A = \ker\varphi \oplus J$ and $A^* = (\ker\varphi)^* \oplus \mathbb{C}_\varphi$.

Let $a \in J$ such that $\varphi(a) = 1$. Then $\psi(a) = 0$ for all $\psi \in \Delta(A)\backslash\{\varphi\}$, and a is the only element of A with these properties since A is semisimple.

(i) For each $x \in A, \varphi(xa) = \varphi(\varphi(x)a)$ and, for $\psi \in \Delta(A)\backslash \{\varphi\}, \psi(xa) = 0 = \psi(\varphi(x)a)$. So $xa = \varphi(x)a$ by semisimplicity and hence a is a φ-mean. Now let $m \in A^{**}$ be any φ-mean for A. Since A commutative, every element of A commutes with every element of A^{**}. Thus
$$m = \varphi(a)m = am = ma = \varphi^{**}(m)a = \langle m, \varphi\rangle a = a.$$

So a is the only φ-mean for A in A^{**}.

(ii) Let $\|a\| = 1$. Since $N(A, \varphi)$ is a proper linear subspace of A^*, by definition of $N(A, \varphi)$ it suffices to show that for any

$f \in A^*, \langle f, a \rangle = 0$ implies $f \cdot a = 0$. Now, every $x \in A$ has a decomposition $x = y + \lambda a$ with $y \in \ker \varphi$ and $\lambda \in \mathbb{C}$. Since $ay \in \ker \varphi \cap J = \{0\}$, for $f \in A^*$,

$$\langle f \cdot a, x \rangle = \langle f, ay \rangle + \lambda \langle f, a \rangle = \lambda \langle f, a \rangle.$$

So $f \cdot a = 0$ whenever $\langle f, a \rangle = 0$.

Since an amenable Banach algebra is φ-amenable for each $\varphi \in \Delta(A)$. If A is commutative and semisimple and the weak and weak* topologies coincide on $\Delta(A)$, then by Lemma (2.2.13), A is φ-amenable if and only if $\{\varphi\}$ is open in $\Delta(A)$. Then condition that the two topologies coincide, however, is quite restrictive. It is for instance satisfied if A is an ideal in A^{**}.

Example (2.2.14) [2]:

Let X be a compact metric space with metric d and let $0 < \alpha \leq 1$. Then $\mathrm{lip}_\alpha X$ is the space of al complex-valued functions u and X such that

$$P_\alpha(u) = \sup\left\{\frac{|u(x) - u(y)|}{d(x, y)^\alpha} : x, y \in X, x \neq y\right\}$$

is finite, and $\mathrm{lip}_\alpha X$ is the subspace of functions satisfying

$$\frac{|u(x) - u(y)|}{d(x, y)^\alpha} \to 0 \text{ as } d(x, y) \to 0.$$

with pointwise multiplication and the norm $\|u\| = \|u\|_\infty + P_\alpha(u), \mathrm{liP}_\alpha X$ is a unital commutative Banach algebra and $\mathrm{lip}_\alpha X$ is a closed subalgebra. These algebras were first studied by Sherbert and later by Bade, Curtis and Dales.

We first treat $\mathrm{lip}_\alpha X$. The map $x \to \varphi_x$, where $\varphi_x(u) = u(x)$ for $u \in \mathrm{liP}_\alpha X$, is a homeomorphism from X onto $\Delta(\mathrm{liP}_\alpha X)$. If x is a non-isolated point of X, then there exist non-zero continuous point derivations at φ_x and hence $\mathrm{liP}_\alpha X$ is not φ_x-amenable. Now, let x be an isolated point of X. Then, by Lemma (2.2.15) (i), there exists a unique φ_x-mean, namely the Dirac function $\delta_x \in \mathrm{liP}_\alpha X$. In view of Section (2.1) where the means are supposed to have norm 1, we point out that

$$\|\delta_x\| = 1 + P_\alpha(\delta_x) = 1 + \sup\left\{\frac{1}{d(x,y)^\alpha} : y \neq x\right\},$$

which, depending on d, can be arbitrarily large.

In light of Lemma (2.2.13) (ii) which shows that $N(A, \varphi)$ is a linear subspace of codimension 1 if there exists a φ-mean of norm 1, it is interesting to note that $N(\mathrm{liP}_\alpha X, \varphi_x) = \{0\}$ for any isolated point x of X. To see this, let $f \in N(\mathrm{liP}_\alpha X, \varphi_x)$. There exists a sequence $(u_n)_n$ in $\mathrm{lip}_\alpha X$ with $u_n(x) = 1$ for all $n, \|u_n\| \to 1$ and $f \cdot u_n \to 0$ in norm. it suffices to show that $u_n \to 1$ in $\mathrm{lip}_\alpha X$ because then $f = f \cdot 1 = \lim_{n\to\infty} f \cdot u_n = 0$. Since

$$1 + P_\alpha(u_n) = |u_n(x)| + P_\alpha(u_n) \leq \|u_n\| \to 1,$$

it follows that $P_\alpha(u_n - 1) = P_\alpha(u_n) \to 0$. Therefore it remains to verify that $u_n \to 1$ uniformly on X. Since X is compact, there exists $C > 0$ such that $d(y,x)^\alpha \leq C$ for all $y \in X$. For $y \neq x$ it follows that

$$|u_n(y) - 1| = |u_n(y) - u_n(x)| \leq C \cdot \frac{|u_n(y) - u_n(x)|}{d(y,x)^\alpha} \leq C_{P_\alpha}(u_n),$$

which tends to zero. So $u_n \to 1$ uniformly on $X \backslash \{x\}$ and hence on all of X.

We now turn to $\mathrm{lip}_\alpha X$. Note that $\mathrm{lip}_1 X$ can be very small since for X a compact interval it consists only of the constant functions. In fact, if $d(x,y) = |x - y|$, then each $u \in \mathrm{lip}_1 X$ is differentiable with $u' = 0$ on X. Thus, let $0 < \alpha < 1$. Then $\mathrm{lip}_1 X$ is dense in $\mathrm{lip}_\alpha X$ and $\Delta(\mathrm{lip}_\alpha X)$ can be identified with X in the same manner as above. However, in contrast to $\mathrm{lip}_1 X$, all continuous point derivations on $\mathrm{lip}_\alpha X$ are zero. Nevertheless, for $x \in X, \mathrm{lip}_\alpha X$ is also φ_x-amenable if and only if x is an isolated point of X. This follows from Theorem (2.2.4) and the remarkable result that $(\mathrm{lip}_\alpha X)^{**}$ is isometrically isomorphic to $\mathrm{lip}_\alpha X$. Indeed, this latter fact implies that the weak* and the weak topologies coincide on $\Delta(\mathrm{lip}_\alpha X)$ since X is homeomorphic to both $\Delta(\mathrm{lip}_\alpha X)$ and $\Delta(\mathrm{lip}_\alpha X)$.

Example (2.2.15) [2]:

Let G be a compact group with normalized Haar measure and consider the convolution algebra $L^P(G), 1 \leq P < \infty$. Let $\hat{G}$ denote the set of all continuous homomorphisms from G into the circle group $\mathbb{T}$, equipped with the topology of uniform convergence. For $\chi \in \hat{G}$, define $\varphi_x : L^P(G) \to \mathbb{C}$ by $\varphi_x(f) = \int_G f(x)\overline{\chi(x)}\, dx$. It is routine to show that map $\chi \to \varphi_x$ is a homomorphism from $\hat{G}$ onto $\Delta\big(L^P(G)\big)$.

Let $\mathrm{q} = \frac{\mathrm{P}}{\mathrm{P}-1}$. Fix $\chi \in \hat{G}$ and define m_χ on $L^{\mathrm{q}}(G) = L^P(G)^*$ by

$$\langle m_\chi, \mathrm{g}\rangle = \int_G \mathrm{g}(x)\chi(x)\, dx, \qquad \mathrm{g} \in L^q(G)$$

Then $\langle m_\chi, \varphi_\chi\rangle = \int_G |\chi(x)|^2\, dx = 1$ and

$$\langle m_\chi, \mathrm{g}\cdot f\rangle = \langle m_\chi, \mathrm{g} * \check{f}\rangle = \int_G\int_G \mathrm{g}(xy)f(y)\chi(x)\, dy\, dx$$

$$= \int_G\int_G \mathrm{g}(x)f(y)\chi(xy^{-1})\, dx\, dy = \varphi_\chi(f)\langle m_\chi, \mathrm{g}\rangle$$

for all $\mathrm{g} \in L^{\mathrm{q}}(G)$ and $f \in L^{\mathrm{P}}(G)$. Thus m_χ is a φ_χ-mean, and we claim that it is the only one. Note that $L^{\mathrm{P}}(G)$ does not have a bounded approximate identity and hence Lemma (2.2.13) (i) does not apply. So let m be a φ-mean and let

$$L = \{\mathrm{g} - \hat{\mathrm{g}}(\bar{\chi})\bar{\chi} : \mathrm{g} \in L^{\mathrm{q}}(G)\}.$$

Then $L = \ker m_\chi$ and, since $\mathrm{g} * \check{\chi} = \breve{\mathrm{g}}(\bar{\chi})\bar{\chi}$, we also have

$$\langle m, \mathrm{g} - \hat{\mathrm{g}}(\bar{\chi})\bar{\chi}\rangle = \langle m, \langle m, \varphi_\chi\rangle \mathrm{g} - \hat{\mathrm{g}}(\bar{\chi})\bar{\chi}\rangle = \langle m, \mathrm{g} * \check{\chi} - \hat{\mathrm{g}}(\bar{\chi})\bar{\chi}\rangle = 0$$

for all $\mathrm{g} \in L^{\mathrm{q}}(G)$. So $m|_l = 0$ and since $\langle m, \varphi_\chi\rangle = \langle m_\chi, \varphi_\chi\rangle$, it follows that $m = m_\chi$.

We now determine $P_1\big(L^P(G), \varphi_\chi\big)$. If $P = 1$, then

$$P_1\big(L^1(G), \varphi_\chi\big) = \big\{f \in L^1(G) : f\bar{\chi} \geq 0, \|f\bar{\chi}\| = 1\big\}.$$

For every $h \in L^1(G)$ and $\chi \in \hat{G}, h\bar{\chi}$ can be written as a linear combination $h\bar{\chi} = \sum_{j=1}^{4} c_j h_j$, where $c_j \in \mathbb{C}, h_j \geq 0$, and $\|h_j\| = 1, 1 \leq j \leq 4$. Hence $h = \sum_{j=1}^{4} c_j h_j\, \chi$ and $h_j\chi \in P_1\big(L^1(G), \varphi_\chi\big)$. So $P_1\big(L^1(G), \varphi_\chi\big)$ spans $L^1(G)$. Alternatively, we could appeal to the fact that $L^1(G)$ is an F-algebra.

We claim that $P_1\big(L^{\mathrm{P}}(G), \varphi_\chi\big) = \{\chi\}$ whenever $P > 1$, so that $P_1\big(L^{\mathrm{P}}(G), \varphi_\chi\big)$ is as small as it can be in this case.

Suppose first that $P \geq 2$. Then $L^{\mathrm{P}}(G) \subseteq L^2(G)$ and hence, for $f \in P_1\big(L^{\mathrm{P}}(G), \varphi_\chi\big)$,

$$1 = \|f\|_P^2 \geq \|f\|_2^2 = \sum_{\eta \in \hat{G}} \big|\hat{f}(\eta)\big|^2 = 1 + \sum_{\eta \neq \chi} \big|\hat{f}(\eta)\big|^2.$$

So $\hat{f}(\eta) = 0$ for $\eta \neq \chi$ and hence $f = \sum_{\eta \in \hat{G}} \hat{f}(\eta)\eta = \chi$ in $L^2(G)$.

Finally, let $1 < p < 2$ and $f \in L^P(G) \subseteq L^1(G)$. Then, by the Hausdorff-Young inequality, $\hat{f} \in l^q(\hat{G})$ and $\|\hat{f}\|_q \leq \|f\|_P$. Thus, if $f \in P_1(L^P(G), \varphi_\chi)$, then

$$1 = \|f\|_P \geq \left(\sum_{\eta \in \hat{G}} \big|\hat{f}(\eta)\big|^{\mathrm{q}}\right)^{1/\mathrm{q}} = \left(1 + \sum_{\eta \neq \chi} \big|\hat{f}(\eta)\big|^{\mathrm{q}}\right)^{1/\mathrm{q}}.$$

Again, $\hat{f}(\eta) = 0$ for $\eta \neq \chi$ and hence, since $L^P(G) \subseteq L^1(G), f \in k\big(\hat{G} \backslash \{\chi\}\big) = \mathbb{C}_\chi$. Then $f = \chi$ since $\hat{f}(\chi) = 1$.

We now determine $\mathrm{N}\big(\mathrm{L}^{\mathrm{P}}(\mathrm{G}), \varphi_\chi\big)$. If G is abelian, it follows easily from Lemma (2.2.13) (ii) that

$$N\big(L^P(G), \varphi_\chi\big) = \big\{f \in L^{\mathrm{q}}(G) \colon \hat{f}(\bar{\chi}) = 0\big\}.$$

We show that the same description of $N\big(L^P(G), \varphi_\chi\big)$ is true when G is an arbitrary compact group.

Observe first that, for $f \in L^q(G)$ and $\eta \in \hat{G}$, we have

$$\widehat{f \cdot \chi}(\eta) = \widehat{f * \bar{\chi}}(\eta) = \int_G \overline{\eta(x)} \left(\int_G f(xy)\chi(y)\, dy \right) dx$$
$$= \int_G \int_G f(x)\overline{\eta(x)}\chi(y)\, dx\, dy = \hat{f}(\eta) \int_G \chi(y)\eta(y)\, dy.$$

The orthogonality relations now imply that $\widehat{f\cdot\chi}=0$ whenever $\hat{f}(\bar{\chi})=0$. Thus $f\cdot\chi=0$, and since $\varphi_{\chi}(\chi)=1$ and $\|\chi\|_{P}=1$, this shows that

$$\{f\in L^{\mathrm{q}}(G):\hat{f}(\bar{\chi})=0\}\subseteq N\big(L^{P}(G),\varphi_{\chi}\big).$$

Conversely, let $f\in N(L^{P}(G),\varphi_{x})$ and let $(\mathrm{g}_n)_n$ be a sequence in $L^{P}(G)$ with $\|f\cdot\mathrm{g}_n\|\to 0$ and $\varphi_{\chi}(\mathrm{g}_n)=1$ for all n. Since

$$\begin{aligned}
\left|\hat{f}(\bar{\chi})\right| &= \left|\int_G f(x)\chi(x)\,dx\cdot\int_G \mathrm{g}_n(y)\overline{\chi(y)}dy\right| \\
&= \left|\int_G\int_G f(xy)\breve{g}_n(y^{-1})\chi(x)\,dydx\right| \\
&\le \left(\int_G\left|\int_G f(xy)\breve{g}_n(y^{-1})\,dy\right|^q dx\right)^{1/q} = \|f\cdot\mathrm{g}_n\|_q,
\end{aligned}$$

which tends to 0. It follows that $\hat{f}(\bar{\chi})=0$ and hence

$$N(L^{P}(G),\varphi_{x})\subset\{f\in L^{P}(G):\hat{f}(\bar{\chi})=0\},$$

as required.

Chapter 3

Approximate and Non Approximate amenability

We give nice condition for c_0 direct-sum of amenable Banach algebras to be approximately amenable, which gives us a reasonably large and varied class. then we examine examples in some details.we show that the two notions of bounded approximate amenability and bounded approximate contractibility are not the same; the direct-sum of two approximately amenable Banach algebras does not have to be approximately amenable; and a 1-condimensional closed ideal in a boundedly approximately amenable Banach algebra need be approximately amenable.

Section (3.1): Approximate Identities

Approximately inner and non-inner derivations arise naturally in the theory of operator algebras in abstract harmonic analysis. The notion of approximate amenability for Banach algebras founded by F. Ghahramani and R.J. Loy in the year 2000 to study the Banach algebras having the property that every continuous derivations from them into a related dual Banach bimodule is approximately inner. Since then various classes of naturally arising approximately amenable and non-amenable Banach algebras have emerged. Such are examples of certain sequence algebras, studied, certain semigroup algebras one studied and certain Fourier algebras studied. So far all of these examples of approximately amenable Banach algebras as well as the synthetic ones (constructed by C_0-direct-sums or projective tensor products) have bounded approximate identities. It is a well-known and significant feature of amenable Banach algebras that they have bounded approximate identities. Several open questions in the theory of approximate amenability have recently been answered, by Choi and Ghahramani. It has been an open question whether approximately amenable Banach algebras must also have bounded approximate identities. In the positive direction, it was shown by Choi. Ghahramani and Zhang that if a boundedly approximately amenable Banach algebra has a multiplier bounded right approximate identity and a multiplier bounded left approximate identity, then it has a bounded approximate identity. In particular, every boundedly approximately contractible Banach algebra has a bounded approximate identity. it is tempting to think that every boundedly approximately amenable Banach algebra must also have a bounded approximate identity. Here we give examples of boundedly approximately amenable Banach algebras which do not have bounded approximate identities.

We will use the abbreviations a.i., l.a.i. and r.a.i. for approximate identity left approximate identity and right approximate identity, respectively. We use the abbreviations b.a., bl..a.i. and b.r.a.i. for bounded such approximate identities, and m.b.a.i, m.b.l.a.i, m,b,r.a.i. for multiplier bounded such approximate identities. All the bounded forms of approximate identity have their associated constants. $L(E,F)$ denotes the Banach space of all continuous linear maps from the Banach space E to the Banach space F, and $K(E,F)$ denotes the closed subspace consisting of the compact operators. $L(E)\big(K(E)\big)$ denotes the Banach algebra $L(E,F)\big(K(E,F)\big)$. If $\mathcal{A}.\mathcal{B}$ are Banach algebras. $\mathcal{A}\,\widehat{\otimes}\,\mathcal{B}$ denotes their projective tensor product, and we use the symbol $\pi: \mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{A}$ to denoted the natural product map with $\pi(a_1 \otimes a_2) = a_1 a_2$.

A Banach algebra $\mathcal{A}$ is approximately contractible if every continuous derivation $d: \mathcal{A} \to E$ from $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule E is approximately inner, that is, it is a limit, in the strong operator topology on $L(\mathcal{A}, E)$, of a suitable net of inner derivations $ad_x(x \in E)$, where $ad_x(a) = a \cdot x - x \cdot \mathcal{A}$ is approximately amenable every continuous derivation $d: \mathcal{A} \to E$ from $\mathcal{A}$ into a and Banach $\mathcal{A}$-binmodule E is approximately inner. $\mathcal{A}$ is boundedly approximately amenable if every continuous derivation from $\mathcal{A}$ into Banach $\mathcal{A}$-bimodule E is the strong limit of a norm-bounded net of inner derivations ad_x (that is, the operators ad_x used in the net are uniformly bounded in $L(\mathcal{A}.E)$. this condition is much weaker than saying that the elements x involved are norm bounded in E –that condition is too strong, implying at once that $\mathcal{A}$ must be amenable).

One can likewise define bounded approximate contractibility, but it turns out that a boundedly approximately contractible Banach algebra must have a b.a.i., so the algebras constructed in the present section do not have this last property. $\mathcal{A}^{\#}$ denotes the unitization of a non-unital Banach algebra $\mathcal{A}$: if $\mathcal{A}$ is already unital, we define $\mathcal{A}^{\#} = \mathcal{A}$.

A Banach algebra is approximately amenable if and only if it is approximately contractible. We shall see that the "bounded" version of this statement is not true: our main construction is of a Banach algebra which is boundedly approximately amenable but which, not having a b.a.i., is not boundedly contractible. We shall show also that the direct-sum of boundedly approximately amenable Banach algebras is not necessarily approximately amenable, and a 1-codimensional closed ideal in a boundedly approximately amenable Banach algebra need not be approximately amenable. We note that in a boundedly approximately contractible Banach algebra a 1-condimensional closed ideal is boundedly approximately contractible.

Most forms of amenability have an equivalent (and sometimes more useful) definition in terms of a suitable diagonal; for those we have defined above, they are as follows. A Banach algebra $\mathcal{A}$ is approximately contractible if there is a net $(d_\alpha)_{\alpha\in A}$ of elements in the Banach $\mathcal{A}$ bimodule $\mathcal{A}^\# \hat{\otimes} \mathcal{A}^\#$ such that $\pi(d_\alpha) = 1$ and the operators $ad_{d\alpha}$ tend to zero in the strong operator topology of $L(\mathcal{A}, \mathcal{A}^\# \hat{\otimes} \mathcal{A}^\#)$. $\mathcal{A}$ is boundedly approximately contractible if, in addition , the net (d_α) can be chosen such that the operators $ad_{d\alpha}$ are uniformly bounded. $\mathcal{A}$ is approximately amenable if is a net $(\Delta_\alpha)_{\alpha\in A}$ of elements in the dual Banach $\mathcal{A}$-bimodule $(\mathcal{A}^\# \hat{\otimes} \mathcal{A}^\#)^{**}$ such that $\pi^{**}(\Delta_\alpha) = 1$ and the operators ad_{Δ_α} tend to zero in the strong operator topology of $L(\mathcal{A}, (\mathcal{A}^\# \hat{\otimes} \mathcal{A}^\#)^{**})$. $\mathcal{A}$ is boundedly approximately amenable if the net (Δ_α) can be chosen such that the operators ad_{Δ_α} uniformly bounded.

Let l^1 denote the well-known space of complex sequences, and l^∞ its dual. It is well known that the Banach algebra $K(l^1)$ is amenable. In this section we renorm $K(l^1)$ with a family of equivalent norms $\|\cdot\|^{[K]}$, in such a way that the b.l.a.i. constant (i.e. the infimum of all M such that the algebra has a b.l.a.i. bounded by M) for $\mathcal{A}^{[K]} = (K(l^1), \|\cdot\|^{[K]})$ is always 1, but the b.r.a.i. constant is precisely $K + 1$. So the C_0-direct-sum $\mathcal{A} = \bigoplus_{K=1}^{\infty} \mathcal{A}^{|K|}$ has a bounded l.a.i. but no bounded r.a.i.

We begin by constructing a bounded right approximate identity for the algebra $K(l^1)$; a simple but not quite trivial task because no sequential such r.a.i. exists.

Let $\mathcal{F}$ denoted the collection of all partitions Π of $\mathbb{N}$ into finitely many non-empty disjoint subset $\left(F_i^{(\Pi)}\right)_{i=1}^{n}$. We define $|\Pi| = n$ and we direct the set $\mathcal{F}$ by saying that $\Pi > \Pi'$ if Π is a refinement of Π', that is, $|\Pi| > |\Pi'|$ and each set $F_i^{(\Pi')}$ is a union of some of the sets $F_j^{(\Pi)}$. With each partition $\Pi \in \mathcal{F}$ with $|\Pi| = n$ we associate the functionals $\left(f_i^{(\Pi)}\right)_{i=1}^{n} \in l^\infty$, where

$$f_i^{(\Pi)}(e_j) = \begin{cases} 1, & if\ j \in F_i^{(\Pi)}; \\ 0, & \text{otherwise}, \end{cases} \qquad (1)$$

and where (e_n) stands for the standard basis of l^1. We write $m_i^{(\Pi)} = \min F_i^{(\Pi)}$ and we note that $f_i^{(\Pi)}\left(e_{m_j^{(\Pi)}}\right) = \delta_{i,j}$. We define the rank-one operators $F_{i,j}^{(\Pi)}$ by $F_{i,j}^{(\Pi)}(x) = e_{m_i^{(\Pi)}} \cdot f_j^{(\Pi)}(x)$, and we define the projection $Q^{(\Pi)} = \sum_{i=1}^{n} F_{i.i}^{(\Pi)}$. We also define the more basic projections $P_n = \sum_{i=1}^{n} E_{i.j}$, where $E_{i.j}(x) = e_i \cdot e_j^*(x)$.

Lemma (3.1.1) [3]:

The sequence $(P_i)_{i=1}^{\infty}$ is a bounded left approximate identity for $K(l^1)$.

Proof:

Let $T \in K(l^1), \varepsilon > 0$ and B be the unit ball of l_1. Let $x_1 \dots x_n$ be an $\varepsilon/2$-net for $T(B)$. Because $P_N x \to x$ for $x \in l^1$, there is an N_0 such that for all $N > N_0$, $\|P_N x_i - x_i\| < \varepsilon/2 , i = 1,2, \dots , n$. Then for $y \in B$, there is an i such that $\|Ty - x_i\| < \mathcal{E}/2$, so

$$\|(I - P_N)Ty\| < \|(I - P_N)x_i\| + \varepsilon/2 < \varepsilon.$$

So $\|T - P_N T\| \leq \varepsilon$, for all $N \geq N_0$ and $(P_i)_{i=1}^{\infty}$ is a bounded left approximate identity.

Lemma (3.1.2) [3]:

For $g \in l^\infty$ we have $Q^{*(\Pi)} g \to g$ as $\Pi \to \mathcal{F}$.

Proof:

Let $\varepsilon > 0$ and write $g_i = g(e_i)$. Suppose $\Pi \in \mathcal{F}$ is sufficiently refined that for each $k = 1, \dots, |\Pi|$ we have

$$\sup\left\{\left|g_i - g_j\right|: i, j \in F_k^{(\Pi)}\right\} \leq \varepsilon.$$

Then for $i \in F_k^{(\Pi)}$ we have

$$\left|g(e_i) - Q^{*(\Pi)}g(e_i)\right| = \left|g_i - g\left(Q^{(\Pi)}e_i\right)\right| = \left|g_i - g_{m_k^{(\Pi)}}\right| \leq \varepsilon.$$

The sets F_k cover $\mathbb{N}$ so $\left\|g - Q^{*(\Pi)}g\right\| \leq \varepsilon$.

Corollary (3.1.3) [3]:

The net $\left(Q^{(\Pi)}\right)_{\Pi\in\mathcal{F}}$ is a bounded right approximate identity for $K\left(l^1\right)$.

Proof:

Given $T \in K\left(l^1\right)$ and $\varepsilon > 0$, we pick n sufficiently large that $\|T - P_nT\| < \varepsilon/3$. The operator $S = P_nT$ is of form $S(x) = \sum_{i=1}^{n} e_i \cdot s_i^*(x)$, for some $s_1^*, \dots, s_n^* \in l^\infty$. From the preceding lemma we can choose $\Pi_0 \in \mathcal{F}$ such that for all $\Pi \geq \Pi_0$ and $i = 1, \dots, n$, we have

$\left|Q^{*(\Pi)}s_i^* - s_i^*\right| < \varepsilon/3n$. Then for any $x \in l^1$,

$$\left\|SQ^{(\Pi)}x - Sx\right\| = \left\|\sum_{i=1}^{n} e_i\left(s_i^* - s_i^*Q^{(\Pi)}\right)x\right\| = \sum_{i=1}^{n}\left|s_i^*(x) - Q^{*(\Pi)}s_i^*(x)\right|$$
$$\leq \|x\|\,\varepsilon/3.$$

Therefore,

$$\left\|TQ^{(\Pi)}x - Tx\right\| \leq \left(\varepsilon/3 + \left\|I - Q^{(\Pi)}\right\| \cdot \|S - T\|\right)\|x\|$$
$$\leq (\varepsilon/3 + 2\varepsilon/3)\|x\| = \varepsilon\|x\|.$$

So, the net $\left(Q^{(\Pi)}\right)$ is a bounded right approximate identity.

Lemma (3.1.4) [3]:

Let $K > 1$. If we renorm $K\left(l^1\right)$ with the equivalent norm

$$\|T\|_K = \|T\| + K \limsup_n \|Te_n\|, \tag{2}$$

then this is an algebra norm, and the left approximate identity P_n has norm 1 in the Banach algebra

$$\mathcal{A}^{[K]} = \left(K(l^1), \|\cdot\|_K\right), \tag{3}$$

but the smallest norm of any bounded right approximate identity in $\mathcal{A}^{[K]}$ is $K + 1$.

Proof:

We have

$$\limsup_n \|TSe_n\| \leq \|T\| \cdot \limsup_n \|Se_n\|.$$

Hence in fact $\|TS\|_K \leq \|T\| \cdot \|S\|_K$, so we have an algebra norm. the P_n have norm 1 because $P_n e_i$ is 0 for all but finitely many i. But let T be the operator such that $T(e_i) = e_1$ for all $i. T \in \mathcal{A}$ and if Q is any operator such that $\|TQ - T\| \leq \varepsilon$ we must have $\|Q\| > 1 - \varepsilon$, but also

$$\limsup \|Qe_n\| \geq \limsup_n \|TQe_n\| / \|T\| > 1 - \varepsilon.$$

also because $\lim \|Te_i\| = 1$. Therefore $\|Q\|_K > (1 + K)(1 - \varepsilon)$, and $1 + K$ is the smallest possible norm for right approximate identity of $\mathcal{A}^{[K]}$. Since $\|\cdot\|^{[K]}$ is at most $K + 1$ times the usual norm on $K(l^1)$, and since the family $\left(Q^{(\Pi)}\right)$ are b.r.a.i. for $K(l^1)$ of norm 1, they are b.r.a.i. for $\mathcal{A}^{[K]}$ of norm exactly $K + 1$, and the b.r.a.i. constant for $\mathcal{A}^{[K]}$ is $K + 1$.

Section (3.2): Non Approximate Amenability

We now give condition for a c_0-direct-sum of amenable Banach algebras. If boundedly approximately amenable.

Corollary (3.2.1) [3]:

The algebra $\mathcal{A} = c_0 - \bigoplus_{K=1}^{\infty} \mathcal{A}^{[K]}$ defined in the previous section is boundedly approximately amenable, but has no b.r.a.i.

Theorem (3.2.3) is proved using the following lemma, which look less general but is in fact enough to give the main result. In the proof of the lemma we use the following result which we think is folklore, as we cannot find a reference for it, so we have sketched a proof. Let E and F be Banach spaces. Then the projective tensor product $E \widehat{\otimes} F^{**}$ has a continuous embedding in $\left(E \widehat{\otimes} F\right)^{**}$. To see this, first we identity the dual space $\left(E \widehat{\otimes} F\right)^{*}$ with $\mathcal{B}(E, F^{*})$. Then we define Θ from $E \widehat{\otimes} F^{**}$ by using duality, as follows:

$$\langle \Theta\left(\sum_{\mathrm{n}=1}^{\infty} e_n \otimes f_n^{**}\right), T\rangle = \sum_{n=1}^{\infty} \langle T(e_n), f_n^{**}\rangle \quad \left(T \in \mathcal{B}(E. F^{*})\right).$$

To see that Θ is injective, it suffices to assume that in that in the equation

$$\Theta\left(\sum_{n=1}^{\infty} e_n \otimes f_n^{**}\right) = 0,$$

The e_n's are linearly independent and use special T 's to conclude that $f_n^{**} = 0$, for all n.

Lemma (3.2.2) [3]:

Let $C \geq 1$ and let $\left(\mathcal{B}^{[K]}\right)_{K=1}^{\infty}$ be a family of amenable Banach algebras. Suppose ε and $\mathcal{F}$ are direct sets, and suppose, for each K, the family $\left(P_m^{[K]}\right)_{m \in \varepsilon}$ is a.b.l.a.i. for $\mathcal{B}^{[K]}$ of norm at most C. Suppose, for each K, ab.r.a.i. $\left(Q_n^{[K]}\right)_{n\in\mathcal{F}}$ for $\mathcal{B}^{[K]}$ is also given, and there is a bounded net $\left(d_{m,n}^{[K]}\right)_{m\in\varepsilon,n\in\mathcal{F}}$ in $\mathcal{B}^{[K]}\widehat{\otimes}\mathcal{B}^{[K]}$ such that

$$\left\|\pi\left(d_{m,n}^{[K]}\right)-P_m^{[K]}+Q_n^{[K]}-Q_n^{[K]}P_m^{[K]}\right\|\to 0, \qquad (4)$$

as $m\to\varepsilon$ and $n\to\mathcal{F}$ (i.e. the set $\varepsilon\times\mathcal{F}$ is given the product order and the limit of the associated net to this direct set is taken); and for $b\in\mathcal{B}^{[K]}$ we have $b\cdot d_{m,n}^{[K]}-d_{m,n}^{[K]}\cdot b\to 0$ as $m\to\mathcal{E}$ and $n\to\mathcal{F}$. Then the c_0-direct-sum $\mathcal{B}=\bigoplus_{K=1}^{\infty}\mathcal{B}^{[K]}$ is boundedly approximately amenable.

Proof of Lemma (3.2.3):

To begin, we need an ultrafilter $\mathcal{U}$ on $\mathcal{E}\times\mathcal{F}$ which refines the order filter on the Cartesian product $\mathcal{E}\times\mathcal{F}$ of our given direct sets. Let us pick such a $\mathcal{U}$, but not just any $\mathcal{U}$. Rather, let us pick an ultrafilter $\mathcal{U}_1$ on $\mathcal{E}$ refining the order filter on $\mathcal{E}$, and an ultrafilter $\mathcal{U}_2$ on $\mathcal{F}$ refining the order filter on $\mathcal{F}$. Let $\mathcal{E}^{\mathcal{F}}$ denote the collection of all functions from $\mathcal{F}$ to $\mathcal{E}$.

For $A\in\mathcal{U}_1, B\in\mathcal{U}_2$ and $h\in\mathcal{E}^{\mathcal{F}}$ we define the subset

$$S(A,B,h) = \{(m,n)\in\mathcal{E}\times\mathcal{F}: m\in A, n\in B \text{ and } m\geq h(n)\}. \qquad (5)$$

These sets are not-empty because B is non-empty, and for each fixed $n\in B$, the collection of $m\in\mathcal{E}$ such that $m\geq h(n)$ meets A because A belongs to the ultrafilter $\mathcal{U}_1$, which refines the order filter on $\mathcal{E}$. Let $\mathcal{G}$ be the collection of all supersets of sets $S(A,B,h)\subset\mathcal{E}\times\mathcal{F}$. Our collection $\mathcal{G}$ is closed under finite intersection and is therefore a filter on $\mathcal{E}\times\mathcal{F}$ (given S_1 and S_2 as in (3.2.2), if we intersection the A sets, interest the B sets, and take a function $h:\mathcal{F}\to\mathcal{E}$ which, at each point $n\in\mathcal{F}$, exceeds the two functions we have been given, then we have an $S\subset S_1\cap S_2$, so $S_1\cap S_2$ being a superset of one of the elementary sets in (5), is in filter). We refine the filter $\mathcal{G}$ to an ultrafilter $\mathcal{U}$. Plainly as $(m,n)\to\mathcal{U}$ we have $m\to\mathcal{U}_1$ and $n\to\mathcal{U}_2$.

We define $P^{[K]}=\lim_{m\to\mathcal{U}_1}P_m^{[K]}\in\mathcal{B}^{[K]^{**}}$ and $Q^{[K]}=\lim_{n\to\mathcal{U}_2}Q_n^{[K]}$: limits being weak -* limits here and for most of this section. We note that for $A\in\mathcal{U}_1, B\in\mathcal{U}_2$ and $f_j\in\mathcal{B}^{[K]^*}$ $(j=1,\ldots,J)$. $\mathcal{U}$ also contains the set

$$S\big(K;A,B,f_1,\dots,f_J,\eta\big) = \big\{(m,n): m \in A, n \in B, \big|\langle Q_n^{[K]}P_m^{[K]} - Q_n^{[K]}P^{[K]}, f_j\rangle\big| < \eta(j = 1,\dots,J)\big\}. \tag{6}$$

(for $P_m^{[K]} \to P^{[K]}$ so for each fixed $n \in \mathcal{F}$ there is an $m_0 = h(n) \in \mathcal{E}$ such that whenever $m \geq m_0$, Eq. (6) holds. Then, $S(A,B,h) \subset S\big(K;A,B,f_1,\dots,f_J,\eta\big)$ so the latter set in the filter $\mathcal{G}$.)

In view of (6), we are sure that $Q_n^{[K]}P_m^{[K]} - Q_n^{[K]}.P^{[K]} \to 0$ in the weak-* topology as $(m,n) \to \mathcal{U}$; since $Q_n^{[K]} \to Q^{[K]}$ we will have $Q_n^{[K]} \cdot P^{[K]} \to Q^{[K]} \square\, P^{[K]}$ (the first Arens product); so

$$Q_n^{[K]}P_m^{[K]} \to Q^{[K]} \square P^{[K]} \tag{7}$$

It is also true that for the element $R^{[K]} = \lim_{(\mathrm{m,n})\to\mathcal{U}} Q_n^{[K]} \otimes P_m^{[K]} \in \big(\mathcal{B}^{[\mathrm{K}]} \mathbin{\widehat{\otimes}} \mathcal{B}^{[\mathrm{K}]}\big)^{**}$ we have

$$\pi^{**}\big(R^{[K]}\big) = Q^{[K]} \square P^{[K]}, \tag{8}$$

and

$$b \cdot R^{[K]} = b \otimes P^{[K]},\ \ R^{[K]} \cdot b = Q^{[K]} \otimes b \tag{9}$$

for each $b \in \mathcal{B}^{[K]}$ (here we regard both $\mathcal{B}^{[K]} \mathbin{\widehat{\otimes}} \mathcal{B}^{[K]^{**}}$ and $\mathcal{B}^{[K]^{**}} \mathbin{\widehat{\otimes}} \mathcal{B}^{[K]}$ as canonically embedded in $\big(\mathcal{B}^{[K]} \mathbin{\widehat{\otimes}} \mathcal{B}^{[K]}\big)^{**}$). Similarly we may define

$$\bar{P}^{[K]} = \lim_{n\to\mathcal{U}_1} \lim_{m\to\mathcal{U}_2} P_m^{[K]} \otimes P_n^{[K]} \tag{10}$$

and we have $\pi^{**}\big(\bar{P}^{[K]}\big) = \lim_n \lim_m P_m^{[K]} P_n^{[K]} = \lim_n P_n^{[K]}$ (because $\big(P_m^{[K]}\big)$ is a.l.a.i.) $=P^{[K]}$ and for $b \in \mathcal{B}^{[K]}$ we have

$$\bar{P}^{[K]}.\, b = \lim_n \lim_m P_m^{[K]} \otimes P_n^{[K]} = P^{[K]} \otimes b. \tag{11}$$

Using the bounded approximate diagonal we have been given, we define $\Delta^{[K]} = \lim_{(m,n)\to\mathcal{U}} d_{m,n}^{[K]}$. It is clear from (4) that

$$\pi^{**}\big(\Delta^{[K]}\big) = P^{[K]} + Q^{[K]} - Q^{[K]} \square P^{[K]}; \tag{12}$$

and for $b \in \mathcal{B}^{[K]}$,

$$b \cdot \Delta^{[K]} = \Delta^{[K]} \cdot b. \tag{13}$$

At this point, we have done all we could do with the individual algebras $\mathcal{B}^{[K]}$; we begin to make suitable definitions involving the algebra $\mathcal{B} = c_0 - \oplus_{K=1}^{[K]} \mathcal{B}^{[K]}$ and its bidual $\mathcal{B}^{**} = l^{\infty} - \oplus_{K=1}^{[K]} \mathcal{B}^{[K]**}$. Let $\mathcal{E}^{[K]} \colon \mathcal{B}^{[K]} \to \mathcal{B}$ denote the natural embedding of $\mathcal{B}^{[K]}$ as a closed ideal of $\mathcal{B}$, and let $\pi^{[K]} : \mathcal{B}^{[K]} \to \mathcal{B}$ be the natural left inverse which picks out the K-th coordinate of an element of $\mathcal{B}$. Write $\rho K = \sum_{r=1}^{K} \mathcal{E}^{[r]} \pi^{[r]}$ for the natural projection onto the first K coordinates of the direct-sum. Let $\bar{\mathcal{E}}^{[K]}$ denote the tensor product $\mathcal{E}^{[K]} \otimes \mathcal{E}^{[K]} \colon \mathcal{B}^{[K]} \widehat{\otimes} \mathcal{B}^{[K]} \to \mathcal{B} \widehat{\otimes} \mathcal{B}$. We define

$$P(K) = \sum_{r=1}^{K} \mathcal{E}^{[K]^{**}} \left(P^{[r]}\right) \in \mathcal{B}^{**} \tag{14}$$

and we let $P(\infty)$ be the weak-* limit of this sequence in the l^{∞}-direct-sum $\mathcal{B}^{**}$ (which exists because the $P^{[r]}$ projection are norm bounded by C independent of r and so the sum resulting from evaluating the terms of $P(K)$ at an element ϕ of the l^1-direct-sum $\mathcal{B}^*$ is Cauchy, being bounded by $C\|\phi\|$). Now $\mathcal{E}^{[K]}(a)^{\mathcal{E}[L]}(b)$ is zero unless $K = L$, in which case it is $\mathcal{E}^{[K]}(ab)$; so for $b \in \mathcal{B}$ with $b_r = \pi^{[r]}(b)$ we have

$$P(K) \cdot b = \sum_{r=1}^{K} \mathcal{E}^{[r]^{**}} \left(P^{[r]} \cdot b_r\right) = \sum_{r=1}^{K} \mathcal{E}^{[r]} \left(b_r\right) = \rho K(b), \tag{15}$$

since $P^{[r]} \cdot x = x$ for $x \in \mathcal{B}^{[r]}$. Therefore

$$P(\infty) \cdot b = b. \tag{16}$$

Likewise, we write

$$Q(K) = \sum_{r=1}^{K} \mathcal{E}^{[r]^{**}} \left(Q^{[r]}\right). \tag{17}$$

Since $x \cdot Q^{[K]} = x$ for $x \in \mathcal{B}^{[K]}$ we have

$$b \cdot Q(K) = \rho K(b) \tag{18}$$

for $b \in \mathcal{B}$. Once again using the fact that $\mathcal{B} \widehat{\otimes} \mathcal{B}^{**}$ is canonically embedded in $\left(\mathcal{B} \widehat{\otimes} \mathcal{B}\right)^{**}$, we define

$$R(K) = \lim_{n\to\mathcal{U}_2}\left(\sum_{r=1}^{K}\mathcal{E}^{[r]}\left(Q_n^{[r]}\right)\right)\otimes P(\infty) \in (\mathcal{B}\otimes\mathcal{B})^{**}, \qquad (19)$$

and since $b_r Q_n^{[r]}$ is norm convergent to b_r as $n \to \mathcal{U}_2$, we have

$$b\cdot R(K) = \left(\sum_{r=1}^{K}\mathcal{E}^{[r]}(b_r)\right)\otimes P(\infty) = \rho K(b)\otimes P(\infty), \quad (20)$$

and by (16),

$$R(K)\cdot b = \lim_{n\to\mathcal{U}_2}\left(\sum_{r=1}^{K}\mathcal{E}^{[r]}\left(Q_n^{[r]}\right)\right)\otimes b = Q(K)\otimes b. \qquad (21)$$

Also

$$\pi^{**}\big(P(K)\big) = \lim_{r=1}\sum_{n\to\mathcal{U}_2}^{K}\mathcal{E}^{[r]^{**}}\left(Q_n^{[r]}P^{[r]}\right)$$

$$= \sum_{r=1}^{K}\mathcal{E}^{[r]^{**}}\left(Q^{[r]}\,\square\,P^{[r]}\right). \qquad (22)$$

We define

$$\bar{P}(K) = \sum_{r=1}^{K}\bar{\mathcal{E}}^{[r]^{**}}\left(\bar{P}^{[r]}\right) \qquad (23)$$

and $\bar{P}(\infty)$ to be any weak-* limit point of the finite sums (such a limit exists because $\left\|\bar{P}^{[K]}\right\| \le C^2$ for all K, and the projective tensor product $\mathcal{B}\mathbin{\widehat{\otimes}}\mathcal{B}$ is the c_0-direct-sum of its " components'' $\mathcal{B}^{[K]}\mathbin{\widehat{\otimes}}\mathcal{B}^{[K]}$ hence its bidual is the l^∞-direct-sum of $\left(\mathcal{B}^{[K]}\mathbin{\widehat{\otimes}}\mathcal{B}^{[K]}\right)^{**}$, and the norm $\|\bar{P}(K)\| = \max\left\{\left\|\bar{P}^{[r]}\right\| : r\le K\right\} \le C^2$ also). Eqs. (10) and (11) then is that

$$\bar{P}(K)\cdot b = \sum_{r=1}^{K}\bar{\mathcal{E}}^{[r]^{**}}\left(P^{[r]}\otimes b_r\right), \qquad (24)$$

and

$$\pi^{**}\big(\bar{P}(K)\big) = \sum_{r=1}^{K} \mathcal{E}^{[r]^{**}}\left(P^{[r]}\right) = P(K). \qquad (25)$$

Then,

$$\pi^{**}\big(\bar{P}(\infty)\big) = \lim_{K} P(K) = P(\infty). \qquad (26)$$

We write

$$\Delta(K) = \sum_{r=1}^{K} \bar{\mathcal{E}}^{[r]^{**}}\left(\Delta^{[r]}\right); \qquad (27)$$

using (13) we find that for each $b \in \mathcal{B}$,

$$b \cdot \Delta(K) = \Delta(K) \cdot b; \qquad (28)$$

And

$$\begin{aligned}\pi^{**}\big(\Delta(K)\big) &= \sum_{r=1}^{K} \bar{\mathcal{E}}^{[r]}\, \pi^{**}\big(\Delta^{[r]}\big) \\ &= \sum_{r=1}^{K} \bar{\mathcal{E}}^{[r]^{**}}\left(P^{[r]} + Q^{[r]} - Q^{[r]} \,\square\, P^{[r]}\right). \qquad (29)\end{aligned}$$

To prove the lemma we need a multiplier bounded approximate diagonal for $\mathcal{B}^{\#}$. We proceed as follows: for each K we define an element $D_K \in \left(\mathcal{B}^{\#} \,\widehat{\otimes}\, \mathcal{B}^{\#}\right)^{**}$ by

$$D_K = 1 \otimes 1 - 1 \otimes P(\infty) - Q(K) \otimes 1 + R(K) + \Delta(K) + \bar{P}(\infty) - \bar{P}(K). \quad (30)$$

We claim that the (D_K) from a multiplier bounded approximate diagonal for $\mathcal{B}^{\#}$ in $\left(\mathcal{B}^{\#} \,\widehat{\otimes}\, \mathcal{B}^{\#}\right)^{**}$, showing that $\mathcal{B}$ is boundedly approximately amenable. For

$$\begin{aligned}\pi^{**}(D_K) = 1 - P(\infty) - Q(K) &+ \sum_{r=1}^{K} \mathcal{E}^{[r]^{**}}\left(Q^{[r]} \,\square P^{[r]}\right) \\ &+ \sum_{r=1}^{K} \mathcal{E}^{[r]^{**}}\left(P^{[r]} + Q^{[r]} - Q^{[r]} \,\square P^{[r]}\right) + P(\infty) - P(K) \\ &= 1 + \sum_{r=1}^{K} \mathcal{E}^{[r]^{**}}\left(P^{[r]} + Q^{[r]}\right) - Q(K) - P(K) = 1.\end{aligned}$$

Furthermore, if $b \in \mathcal{B}$ then

$$\begin{aligned} b \cdot D_K - D_K \cdot b & \\ &= (b \otimes 1 - 1 \otimes b) - (b \otimes P(\infty) - 1 \otimes b) \\ &- (\rho_K(b) \otimes 1 - Q(K) \otimes b) \\ &+ (\rho_K(b) \otimes P(\infty) - Q(K) \otimes b) \\ &+ \left(b \cdot \left(\bar{P}(\infty) - \bar{P}(K)\right) \cdot b\right), \end{aligned}$$

which is a bounded expression since the $Q(K)$ terms disappear:

$$= \left(b - \rho_K(b)\right) \otimes \left(1 - P(\infty)\right) + b \cdot \left(\bar{P}(\infty) - \bar{P}(K)\right) - \left(\bar{P}(\infty) - \bar{P}(K)\right) \cdot b.$$

All the P terms have norm at most C, and the $\bar{P}$ terms have norm at most C^2. Furthermore, the difference $\bar{P}(\infty) - \bar{P}(K)$ is a limit of sums of tensors in the image of $\bar{\mathcal{E}}^{[r]^{**}}$ for $r = K + 1$ to infinity, so

$$b \cdot \left(\bar{P}(\infty) - \bar{P}(K)\right) = \left(b - \rho_K(b)\right) \cdot \left(\bar{P}(\infty) - \bar{P}(K)\right)$$

and

$$\left(\bar{P}(\infty) - \bar{P}(K)\right) \cdot b = \left(\bar{P}(\infty) - \bar{P}(K)\right) \cdot \left(b - \rho_K(b)\right).$$

For every K and every $b \in \mathcal{B}$ we therefore have

$$\|b \cdot D_K - D_K \cdot b\| \leq 6C^2 \|b - \rho_K(b)\| \leq 6C^2 \|b\|.$$

As $K \to \infty$, we have $b \cdot D_K - D_K \cdot b \to 0$ because $b - \rho_K(b) \to 0$. So the sequence of elements D_K is a multiplier bounded approximate diagonal for $\mathcal{B}^{\#}$, which is therefore boundedly approximately amenable.

Theorem (3.2.3) [3]:

Let $C \geq 1$, and let $\left(\mathcal{B}^{[K]}\right)_{K=1}^{\infty}$ be a sequence of amenable Banach algebras. If each $\mathcal{B}^{[K]}$ has b.l.a.i. of norm at most C, then the C_0-direct-sum $\mathcal{B} = \oplus_{K=1}^{\infty} \mathcal{B}^{[K]}$ is boundedly approximately amenable.

Proof of Theorem (3.2.3):

From Lemma (3.2.2) let $\mathcal{E}_0, \mathcal{F}_0$ be directed sets such that for each K we can find a b.l.a.i. $\left(P_m^{[K]}\right)_{m \in \mathcal{E}_0}$ and a b.r.a.i. $\left(Q_n^{[K]}\right)_{n \in \mathcal{F}_0}$ for $\mathcal{B}^{[K]}$; with $\left\|P_m^{[K]}\right\| \leq C$ for all m and K. Let $\mathcal{G}$ be yet another directed set, such that there is a bounded approximate diagonal $\left(d_{\gamma}^{[K]}\right)_{\gamma\in\mathcal{G}} \in \mathcal{B}^{[K]} \otimes \mathcal{B}^{[K]}$ for each K. So, writing

$$u_{\gamma}^{[K]} = \pi\left(d_{\gamma}^{[K]}\right) \tag{31}$$

the net $\left(u_{\gamma}^{[K]}\right)_{\gamma\in\mathcal{G}}$ is a bounded approximate identity for $\mathcal{B}^{[K]}$, and for each $x \in \mathcal{B}^{[K]}$, we have $x \cdot d_{\gamma}^{(K)} - d_{\gamma}^{(K)} \cdot x \to 0$ as $\gamma \to \mathcal{G}$.

Let $\mathcal{E} = \mathcal{E}_0 \times \mathcal{G}$; given the product ordering this is a direct set, and if for $\mathbf{m} = (m, \gamma) \in \mathcal{E}$ we define $P_{\mathbf{m}}^{[K]} = P_{\mathbf{m}}^{[K]}$, the net $\left(P_{\mathbf{m}}^{(K)}\right)_{\mathbf{m}\in\mathcal{E}}$ is a b.l.a.i. for $\mathcal{B}^{[K]}$ of norm at most C.

Let $\mathcal{F} = \mathcal{F}_0 \times \mathbb{N}$; given the product ordering this too is a direct set, and if for $\mathbf{n} = (n, n') \in \mathcal{F}$ we define $Q_{\mathbf{n}}^{[K]} = Q_{\mathbf{n}}^{[K]}$, the net $\left(Q_{\mathbf{n}}^{(K)}\right)_{\mathbf{n}\in\mathcal{F}}$ is a b.r.a.i. for $\mathcal{B}^{[K]}$.

For each $\mathbf{m} = (m, \gamma) \in \mathcal{E}$ and $\mathbf{n} = (n, n') \in \mathcal{F}$, let us pick a $\mathrm{g} \in \mathcal{G}$ such that $\mathrm{g} \geq \gamma$ and Max $\left\{\left\|Q_{\mathbf{n}}^{[K]} u_{\mathrm{g}}^{[K]} - Q_{\mathbf{n}}^{[K]}\right\| \cdot \left\|u_{\mathrm{g}}^{[K]} P_{\mathbf{m}}^{[K]} - P_{\mathbf{m}}^{[K]}\right\|\right\} \leq 1/n'$. We define

$$d_{\mathbf{m},\mathbf{n}}^{[K]} = Q_{\mathbf{n}}^{[K]} \cdot d_{\mathrm{g}}^{[K]} + d_{\mathrm{g}}^{[K]} \cdot P_{\mathbf{m}}^{[K]} - Q_{\mathbf{n}}^{[K]} \cdot d_{\mathrm{g}}^{[K]} \cdot P_{\mathbf{m}}^{[K]}. \tag{32}$$

Then

$$\pi\left(d_{\mathbf{m},\mathbf{n}}^{[K]}\right) = Q_{\mathbf{n}}^{[K]} u_{\mathrm{g}}^{[K]} + u_{\mathrm{g}}^{[K]} P_{\mathbf{m}}^{[K]} - Q_{\mathbf{n}}^{[K]} u_{\mathrm{g}}^{[K]} P_{\mathbf{m}}^{[K]},$$

$$\left\|\pi\left(d_{\mathbf{m},\mathbf{n}}^{[K]}\right) - \left(Q_{\mathbf{n}}^{[K]} + P_{\mathbf{m}}^{[K]} - Q_{\mathbf{n}}^{[K]} P_{\mathbf{m}}^{[K]}\right)\right\| \leq \frac{1}{n'}\left(2 + \left\|P_{\mathbf{m}}^{[K]}\right\|\right) \leq \frac{2 + C}{n'},$$

so

$$\left\|\pi\left(d_{\mathbf{m},\mathbf{n}}^{[K]}\right) - \left(Q_{\mathbf{n}}^{[K]} + P_{\mathbf{m}}^{[K]} - Q_{\mathbf{n}}^{[K]} P_{\mathbf{m}}^{[K]}\right)\right\| \to 0, \text{as } \mathbf{m} \to \mathcal{E} \text{ and } \mathbf{n} \to \mathcal{F}. \tag{33}$$

Also for $x \in \mathcal{B}^{[K]}$, we have

$$
\begin{aligned}
x \cdot d_{\mathbf{m},\mathbf{n}}^{[K]} - d_{\mathbf{m},\mathbf{n}}^{[K]} \cdot x
&= x \cdot d_{\mathrm{g}}^{[K]} - d_{\mathrm{g}}^{[K]} \cdot\ x + \left(x \cdot Q_{\mathbf{n}}^{[K]} - x\right) \cdot\ d_{\mathrm{g}}^{[K]} \cdot \left(1 - P_{\mathbf{m}}^{[K]}\right) \\
&\quad - \left(1 - Q_{\mathbf{n}}^{[K]}\right) \cdot d_{\mathrm{g}}^{[K]} \cdot \left(P_{\mathbf{m}}^{[K]} x - x\right). \qquad (34)
\end{aligned}
$$

As $\mathbf{m} \to \mathcal{E}$ and $\mathbf{n} \to \mathcal{F}$ we have $\mathrm{g} \to \mathcal{G}$ so $\left\| x \cdot d_{\mathrm{g}}^{[K]} - d_{\mathrm{g}}^{[K]} \cdot x \right\| \to 0$; as $\mathbf{m} \to \mathcal{E}$ we have $\left\| P_{\mathbf{m}}^{[K]} x - x \right\| \to 0$; and as $\mathbf{n} \to \mathcal{F}$ we have $\left\| x Q_{\mathbf{n}}^{[K]} - x \right\| \to 0$. Therefore,

$$
\left\| x \cdot d_{\mathbf{m},\mathbf{n}}^{[K]} - d_{\mathbf{m},\mathbf{n}}^{[K]} \cdot x \right\| \to 0, \qquad \text{as } \mathbf{m} \to \mathcal{E} \text{ and } \mathbf{n} \to \mathcal{F}. \qquad (35)
$$

By (33) and (35), the net $\left(d_{\mathbf{m},\mathbf{n}}^{[K]}\right)$ satisfies the requirements of Lemma (3.2.3). Therefore, $\mathcal{B}$ is boundedly approximately amenable.

Corollary (3.2.4) [3]:

Our algebra $\mathcal{A}$ constructed in the preceding as a c_0-direct-sum of the algebras $\left(K\left(l^1\right), \|.\|_K\right)$ has the following properties:

(i) It is boundedly approximately amenable;
(ii) It has no two-side bounded approximate identity.

Hence $\mathcal{A}$ is not boundedly approximately contractible.

Proof:

It only suffices to note that every boundedly approximately contractible Banach algebra has a bounded approximate identity.

It is shown that if a Banach algebra $\mathcal{B}$ is boundedly approximately amenable, has a multiplier bounded right approximate identity, and a multiplier bounded left approximate identity, then it has a bounded approximate identity. The following shows that the existence of such nets in the second dual of the Banach algebra cannot ensure the same conclusion.

Theorem (3.2.5) [3]:

The algebra $\mathcal{A}$ constructed in the preceding section has the following property: $\mathcal{A}^{**}$ has a multiplier-bounded approximate identity for $\mathcal{A}$ with constant 1 (that is, there is a net $(T_\alpha)_{\alpha\in I}$ in $\mathcal{A}^{**}$ such that for all $a \in \mathcal{A}, \alpha \in I$ we have

$$\mathrm{Max}\,\{\|a \cdot T_\alpha \cdot a\|\} \leq \|a\|: \tag{36}$$

and $a \cdot T_\alpha \to a, T_\alpha \cdot a \to a$ as $\alpha \to I$). The m.b.a.i. can be chosen to be sequential.

Proof:

$\mathcal{A}$ is the c_0-direct-sum of the algebras $\mathcal{A}^{[i]}$, each of which has a b.a.i., since it has a b.a,i, and a b.r.a.i., albeit with the bad constant $i+1$. So $\mathcal{A}^{[i]^{**}}$ has an identity $e^{[i]}$ for $\mathcal{A}^{[i]}$, an element such that $e^{[i]} \cdot a = a$ for every $a \in \mathcal{A}^{[i]}$; the m.b.a.i. we want is the sequence

$$E_n = \sum_{i=1}^{n} \mathcal{E}^{[i]^{**}}\left(e^{[i]}\right) \in \mathcal{A}^{**} = l^\infty - \underset{i\,=\,1}{\overset{\infty}{\oplus}} \mathcal{A}^{[i]^{**}}.$$

If $a = (a_i)_{i=1}^{\infty} \in \mathcal{A}$ and $f = (f_i)_{i=1}^{\infty} \in \mathcal{A}^*$ (so f is the l^1-direct-sum of elements $f_i \in \mathcal{A}^{[i]^*}$) then

$$\begin{aligned}\langle a \cdot E_n, f\rangle = \langle E_n, f \cdot a\rangle = \langle E_n, (f_i \cdot a_i)_{i=1}^{\infty}\rangle &= \sum_{i=1}^{n} \langle e^{[i]}, f_i \cdot a_i\rangle \\ &= \sum_{i=1}^{n} \langle a_i, f_i\rangle.\end{aligned}$$

The difference between this and $\langle a, f\rangle$ is at most $\|f\| \cdot \max\{\|a_i\| : i > n\}$, so $a \cdot E_n \to a$ in norm as $n \to \infty$. Likewise $E_n \cdot a \to a$.

The above is more remarkable because $\mathcal{A}$ does not have a m.b.r.a.i.

It was shown that if the Banach algebras $\mathcal{A}$ and $\mathcal{B}$ are approximately amenable and either one has a bounded approximate identity, then the direct-sum $\mathcal{A} \oplus \mathcal{B}$ is approximately amenable. It is tempting to think that the condition on the existence of bounded approximate identity may be dispensed with. However, that is not the case, as the following shows.

Theorem (3.2.6) [3]:

Let $\mathcal{A}^{\mathrm{op}}$ denote the opposite algebra to our algebra $\mathcal{A}$. The algebra $\mathcal{B} = \mathcal{A} \oplus \mathcal{A}^{\mathrm{op}}$ is not approximately amenable.

Note that our proof depend somewhat on special properties of $\mathcal{A}$, but is nonetheless general enough to indicate that it may be difficult to find an approximately amenable Banach algebra which has neither a bounded right approximate identity nor a bounded left approximate identity.

For $x \in l^1$, let us write $\lambda(x) = \sum_{i=1}^{\infty} \langle x, e_i^* \rangle$. Let $T_0 \in K(l^1)$ be the element such that $T_0(x) = e_1 \cdot \langle x, \lambda \rangle$. Evidently $\lim\sup_i \|T_0 e_i\| = 1$ and $\|T_0\|^{[K]} = K + 1$. Let us choose a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Up to scaling, a support functional for T_0 in any of the $\|\cdot\|^{[K]}$ norms is $\phi(T) = \lim_{i \to \mathcal{U}} \langle T e_i, \lambda \rangle$. For if $|\phi(T)| = 1$, then certainly $\lim\sup_i \|T e_i\| \geq 1$ so $\|T\|^{[K]} \geq K + 1$, hence $\|\phi\|^{[K]} \leq \frac{1}{K+1}$: and $\langle \phi, T_0 \rangle = 1 \geq \|\phi\|^{[K]} \cdot \|T_0\|^{[K]}$. So equality must hold, and $\|\phi\|^{[K]} = \frac{1}{K+1}$. Simple calculation shows that $T_0^* \lambda = \lambda$, so we have $\lim_{i \to \mathcal{U}} \langle T_0 S e_i, \lambda \rangle = \lim_{i \to \mathcal{U}} \langle S e_i, \lambda \rangle$ for any $S \in K(l^1)$, that is,

$$\phi(T_0 S) = \phi(S). \tag{37}$$

(This is the special property of $K(l^1)$ that will be used to prove the theorem.)

There is an isometry $E: c_0 \to \mathcal{A}$ sending $\delta = (\delta_i)_{i=1}^{\infty}$ to the sequence $(T_0\delta_1/2, T_0\delta_2/3, T_0\delta_3/3, \dots, T_0\delta_K/(K+1), \dots)$ which is

$$\sum_{K=1}^{\infty} \mathcal{E}^{[K]}\,(\delta_K T_0/(K+1)).$$

Let's write $\phi^{[K]} = \phi \circ \pi^{[K]} \in \mathcal{A}^*$, the linear functional of norm $1/(K+1)$ which applies ϕ to the Kth entry of $a \in \mathcal{A}$. Evidently,

$$\phi^{[K]}\big(E(\delta)\big) = \delta_K/(K+1),$$

and more generally, because of (37) we have

$$\phi^{[K]}(E(\delta) \cdot a) = \delta_K/(K+1) \cdot \phi^{[K]}(a), \tag{38}$$

for any $a \in \mathcal{A}$.

Lemma (3.2.7) [3]:

Let $(M_i)_{i=1}^{\infty}$ be a strictly increasing sequence of positive integers. Suppose the sequence $\delta \in c_0$ is chosen to tend to zero so slowly that $\delta_{2M_n} \geq 2/n$ for all $n \in \mathbb{N}$. Write $\tau = E(\delta)$. Then whenever $a \in \mathcal{A}$ is such that $\|\tau a - \tau\| \leq 1/n^2$ (some $n \in \mathbb{N}$), we have

$$\|a\| \geq M_n, \tag{39}$$

more specifically

$$\left|\phi^{[2M_n]}(a) - 1\right| \leq \frac{1}{2n}. \tag{40}$$

Proof:

Note that (39) follows from (40); for (40) implies

$$\|a\| \geq (2M_n + 1)\left|\phi^{[2M_n]}(a)\right| \geq (2M_n + 1)/2.$$

But

$$\left|\phi^{[2M_n]}(\tau) - \phi^{[2M_n]}(\tau a)\right| \leq \frac{1}{(2M_n + 1)}\|\tau - \tau a\| \leq \frac{1}{n^2(2M_n + 1)},$$

and the left-hand side is

$$\frac{\left|\delta_{2M_n} - \delta_{2M_n}\phi^{[2M_2]}(a)\right|}{2M_n + 1} \geq \left|1 - \phi^{[2M_n]}(a)\right| \cdot \frac{2}{n(2M_n + 1)},$$

so $\left|1 - \phi^{[2M_n]}(a)\right| \leq \frac{1}{2n}$, as required.

Let

$$\Delta = 1 \otimes 1 - 1 \otimes u - v \otimes 1 + d \in \mathcal{B}^{\#} \otimes \mathcal{B}^{\#}, \tag{41}$$

with $d \in \mathcal{B} \widehat{\otimes} \mathcal{B}$. Let P_i $(i = 1,2)$ be the maps which pick the left and right coordinates respectively from the pair $(a_1, a_2) \in \mathcal{A} \oplus \mathcal{A}^{\mathrm{op}} = \mathcal{B}$, and let $p_{12} = P_1 \otimes P_2 \colon \mathcal{B} \widehat{\otimes} \mathcal{B} \to \mathcal{A} \widehat{\otimes} \mathcal{A}^{\mathrm{op}}$. (Obviously there are similar maps P_{11}, P_{21} and P_{22} but it's P_{12} that we 're interested in.) For a proof of Theorem (3.2.6), we claim that provided that the sequence (M_n) increases sufficiently rapidly, it is impossible (regardless of choice of u, v and d) to have

$$\|(\tau,0)\cdot\Delta-\Delta\cdot(\tau,0)\|<1/10 \tag{42}$$

and

$$\|(0,\tau)\cdot\Delta-\Delta\cdot(0,\tau)\|<1/10. \tag{43}$$

To see this, let $\mathcal{X}$ denote the character on $\mathcal{B}^{\#}$ with $\mathcal{X}(\lambda 1+b)=\lambda$ and let $q(x)=x-\mathcal{X}(x)1(x\in\mathcal{B}^{\#})$ and $\bar{q}=q\otimes q$. We write $d_{12}=P_{12}(d), u_i=P_i(u)$ and $v_i=P_i(v)$, and then we apply $P_{12}\bar{q}$ to both sides of (42). Most of the terms disappear, and we get

$$\|\tau\cdot d_{12}-\tau\otimes u_2\|<1/10. \tag{44}$$

We do the same to (43) and we get

$$\|d_{12}\cdot\tau-v_1\otimes\tau\|<1/10. \tag{45}$$

Note that in this last equation $d_{12}\cdot\tau$ refers to the natural right module action of the opposite algebra on $\mathcal{A}\oplus\mathcal{A}^{\mathrm{op}}$, so that $(a_1\otimes a_2)\cdot\tau=a_1\otimes\tau a_2$ for $a_1\in\mathcal{A}, a_2\in\mathcal{A}^{\mathrm{op}}$: where τa_2 denotes the ‘ usual ’ product of elements of $\mathcal{A}$, not the ‘ opposite ’ product.

Lemma (3.2.8) [3]:

Suppose that the sequence M_n increases “sufficiently rapidly”, (41) (42) and (43) hold, and that for some $n\geq 2$ we have

$$\left|\phi^{[2M_n]}(u_2)\right|\in[(1/2+1/n\,,3/2-1/n)] \tag{46}$$

and

$$\left|\phi^{[2M_n]}(v_1)\right|\in[(1/2+1/n\,,3/2-1/n)]. \tag{47}$$

Then we must also have

$$\left|\phi^{[2M_L]}(u_2)\right|\in[(1/2+1/L\,,3/2-1/L)],$$

and

$$\left|\phi^{[2M_L]}(v_1)\right|\in[(1/2+1/L\,,3/2-1/L)],$$

where

$$L=\left\lfloor\sqrt{5(1+2M_n)}\right\rfloor.$$

Proof:

Let P, Q denote the rank I projections onto lin (u_2) and lin (v_1) respectively, with

$$P(x) = u_2 \cdot \frac{\phi^{[2M_n]}(x)}{\phi^{[2M_n]}(u_2)} \tag{48}$$

and

$$Q(x) = v_1 \cdot \frac{\phi^{[2M_n]}(x)}{\phi^{[2M_n]}(v_1)}. \tag{49}$$

We have $\|P\| \leq \|u_2\| \, \|\phi^{[2M_n]}\| / |\phi^{[2M_n]}(u_2)| \leq 2\,\|u_2\|/(1 + 2M_n)$, because $|\phi^{[2M_n]}(u_2)| \geq 1/2$ by (46). Similarly, we have $\|Q\| \leq 2\,\|v_1\|/(1 + 2M_n)$ because of (47). Let's write $(I \otimes P)(d_{12}) = v_1' \otimes u_2$ for some $v_1' \in \mathcal{A}$, and $(Q \otimes I)(d_{12}) = v_1 \otimes u_2'$ for some $u_2' \in \mathcal{A}^{\mathrm{op}}$. Applying $I \otimes P$ to (44) we get

$$\|(\tau v_1' - \tau) \otimes u_2\| \leq \|P\|/10\,; \tag{50}$$

so

$$\|\tau v_1' - \tau\| \leq \frac{1}{5(1 + 2M_n)} \leq 1/L^2\,; \tag{51}$$

so (40) tells us

$$\left|\phi^{[2M_L]}(v_1') - 1\right| < 1/2L. \tag{52}$$

We apply $Q \otimes I$ to (45) and we get

$$\|v_1 \otimes (\tau u_2' - \tau)\| \leq \|Q\|/10\,; \tag{53}$$

so by (40),

$$\|\tau u_2' - \tau\| \leq \frac{1}{5(1+2M_n)} \leq 1/L^2, \tag{54}$$

and hence by (40),

$$\left|\phi^{[2M_L]}(u_2') - 1\right| < 1/2L. \tag{55}$$

Next, let us apply $I \otimes P$ to (45). In view of (38) we have

$$P(b \cdot \tau) = P(\tau b) = \delta_{2M_n} P(b)/(1 + 2M_n) \quad (b \in \mathcal{A}^{\mathrm{op}}).$$

Therefore,

$$(I \otimes P)(d \cdot \tau) = \delta_{2M_n} (I \otimes P)(d)/(1 + 2M_n) \quad \big(d \in \mathcal{A} \widehat{\otimes} \mathcal{A}^{\mathrm{op}}\big).$$

Also

$P(\tau) = u_2 \cdot \phi^{[2M_L]}(\tau)/\phi^{[2M_n]}(u_2) = \delta_{2M_n} u_2/\Big((1 + 2M_n)\phi^{[2M_n]}(u_2)\Big)$
because entry $2M_n$ of τ is $\delta_{2M_n} T_0/(1 + 2M_n)$ and $\phi(T_0) = 1$. So we get

$$\begin{aligned}\left\|v_1' \otimes u_2 - \frac{v_1 \otimes u_2}{\phi^{[2M_n]}(u_2)}\right\| \frac{\delta_2 M_n}{1 + 2M_n} &\leq \|P\|/10 \\ &\leq \frac{\|u_2\|}{5(1 + 2M_n)}\end{aligned} \tag{56}$$

and so

$$\left\|v_1' - \frac{v_1}{\phi^{[2M_n]}(u_2)}\right\| \leq \frac{1}{5\delta_{2M_n}} \leq n/10. \tag{57}$$

This last estimate may not look so strong, but it looks much better if we apply $\phi^{[2M_L]}$ to it and recall that $\left\|\phi^{[2M_L]}\right\| \leq \frac{1}{1+2M_L}$. We get

$$\left|\phi^{[2M_L]}\left(v_1' - \frac{v_1}{\phi^{[2M_n]}(u_2)}\right)\right| \leq \frac{n}{10(1 + 2M_L)}, \tag{58}$$

so

$$\left|\phi^{[2M_L]}(v_1')\phi^{[2M_n]}(u_2) - \phi^{[2M_L]}(v_1)\right| \leq \frac{n\left|\phi^{[2M_n]}(u_2)\right|}{10(1 + 2M_L)} \leq \frac{3n}{20(1 + 2M_L)},$$

since $\left|\phi^{[2M_L]}(u_2)\right| \leq 3/2$. Now $\left|\phi^{[2M_n]}(u_2) - 1\right| \leq \frac{1}{2} - \frac{1}{n}$ and $\left|\phi^{[2M_L]}(v_1') - 1\right| \leq \frac{1}{2L}$, so

$$\left|\phi^{[2M_L]}(v_1')\phi^{[2M_n]}(u_2)\right| \leq \frac{1}{2} - \frac{1}{n} + \frac{1}{2L} + \frac{1}{2L}\left(\frac{1}{2} - \frac{1}{n}\right),$$

and

$$\left|\phi^{[2M_L]}(v_1) - 1\right| \leq \frac{1}{2} - \frac{1}{n} + \frac{1}{2L} + \frac{1}{2L}\left(\frac{1}{2} - \frac{1}{n}\right) + \frac{3n}{20(1 + 2M_L)} \leq \frac{1}{2} - \frac{1}{L}.$$

given a mild growth condition on the sequence (M_n); so

$$\left|\phi^{[2M_L]}(v_1)\right| \in [1/2 + 1/L\,, 3/2 - 1/L].$$

Similarly, if we apply $Q \otimes I$ to (44), we get

$$\left|\phi^{[2M_L]}(u_2)\right| \in [1/2 + 1/L\,, 3/2 - 1/L], \qquad (59)$$

and the proof of the lemma is complete.

Corollary (3.2.9) [3]:

If any Δ exists satisfying (41), (42) and (43), we cannot have $\left|\phi^{[2M_n]}(v_1)\right| \in [1/2 - 1/n\,, 3/2 - 1/n]$ and $\left|\phi^{[2M_n]}(u_2)\right| \in [1/2 + 1/n\,, 3/2 - 1/n]$ for any $n \geq 2$.

For given a mild growth condition we always have $L > n$ in Lemma (3.2.8), by Lemma (3.2.8) we would have $\left|\phi^{[2M_n]}(v_1)\right| \in [1/2 + 1/n\,, 3/2 - 1/n]$ and $\left|\phi^{[2M_n]}(u_2)\right| \in [1/2 + 1/n\,, 3/2 - 1/n]$ for an infinite sequence of values of n. But $\left\|\phi^{[K]}\right\| = 1/(K+1)$ so this is impossible.

But now we can prove Theorem (3.2.6). For if any Δ exists satisfying (42) and (43), we apply $P_1 \cdot (I \otimes \chi)$ to both sides of (42) (where χ is the character), and we get $\|\tau - \tau v_1\| < 1/10$ so by (40), $\left|\phi^{[2M_3]}(v_1) - 1\right| \leq 1/6$. We apply $P_2 \cdot (\chi \otimes I)$ to both sides of (43) and we likewise get $\left|\phi^{[2M_3]}(u_2) - 1\right| \leq 1/6$. So the conditions $\left|\phi^{[2M_n]}(u_2)\right|, \left|\phi^{[2M_n]}(v_1)\right| \in [1/2 + 1/n\,, 3/2 - 1/n]$ would be satisfied with $n = 3$, which by Corollary (3.2.9) is impossible. So no such Δ exists and $\mathcal{A} \oplus \mathcal{A}^{\mathrm{op}}$ is not approximately amenable.

Corollary (3.2.10) [3]:

There is a boundedly approximately amenable Banach algebra that has a l-codimensional closed ideal which is not boundedly approximately amenable.

Proof:

Let $\mathcal{A}$ be our algebra constructed above and let $\mathcal{A}^{\#}$ be the unitization of $\mathcal{A}$. Then from the proof of a Banach result we see that the Banach algebra $\mathcal{B} = \mathcal{A}^{\#} \oplus \mathcal{A}^{\mathrm{op}}$ is boundedly approximately amenable, whereas the l-codimensional ideal $\mathcal{A} \oplus \mathcal{A}^{\mathrm{op}}$ of $\mathcal{B}$ is not boundedly approximately amenable, as seen above.

Chapter 4

Approximate amenability On The Banach Algebra

We use to give examples of Banach spaces X for which the Banach algebra $K(X)$ is approximately amenable but not amenable. Thus we answer a question on existence of such spaces.

Section (4.1): Introduction and Results

The notion of approximate amenability was introduced by R.J. Loy. The first example of an approximately amenable non-amenable Banach algebra, is synthetic. Later, a host of naturally arising example of approximately amenable non-amenable Banach algebras were found amongst: Banach sequence algebras, Fourier algebras and semigroup algebras.

The study of amenability of the Banach algebra $K(X)$ began with the work of B.E. Johnson. Later N. Gronbeak, B.E. Johnson and G.A. Willis made an extensive study of amenability of the Banach algebra $K(X)$, for various Banach space X. A. Blanco made a systematic study of weak amenability of the Banach algebra $A(X)$ of all approximable operator on the Banach space X, for various Banach spaces X. later in 2000 – when approximate amenability was founded – it was natural to ask whether there could be a Banach space X for which $K(X)$ is approximately amenable (but not amenable).

We now recall the definition of approximately amenable Banach algebras. First off, a continuous derivation D from the Banach algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule X is approximately inner, if there exists a net (x_i) of elements of X such that $D(a) = \lim_{\mathrm{i}} a \cdot x_i - x_i \cdot a$, for all $a \in \mathcal{A}$. The Banach algebra $\mathcal{A}$ is approximately amenable if every continuous derivative from $\mathcal{A}$ into the dual Banach bimodule X^* is approximately inner, for all Banach $\mathcal{A}$-binomiales X. As noted in the above definition one can replace X^* by X i.e. approximate amenable and approximate contractibility are the same concepts. We will also be concerned with the concept of pseudo-amenability for Banach algebra. The Banach algebra $\mathcal{A}$ is pseudo-amenable if there is a net (m_i) of elements of $\mathcal{A}\widehat{\otimes}\mathcal{A}$ such that

$$a \cdot m_i - m_i \cdot a \to 0 \quad (a \in \mathcal{A})$$

and

$$\pi(m_i) \cdot a \to a \quad (a \in \mathcal{A}),$$

where $\pi: \mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{A}$ is the so-called product map, specified by $\pi(a \otimes b) = ab$ for all $a, b \in \mathcal{A}$.

Definition (4.1.1) [4]:

Let $b > 0$ be an absolute constant, and X a Banach space. We will say X is "fairly close" to a Hilbert space (with constant b) if the following conditions hold: For every finite sequence $(T_\mu)_{\mu=1}^{m} \subset K(X)$, and every $\epsilon > 0$ we can find a shrinking basis $(x_i)_{i=1}^{\infty}$ for $X^{\mu=1}$ (with co-ordinate functional $(x_i^*) \subset X^*, x_i^*(x_j) = \delta_{i,j}$), and a finite sequence

$$0 = n_0 < n_1 < n_2 < \cdots < n_k = N,$$

with the following properties:

(i) Let $1 \le r \le k$, and $\pi_r = \sum_{i=1+n_{r-1}}^{n_r} x_i \cdot x_i^*$ where $x_i \cdot x_i^*(x) = \langle x_i^* \cdot x \rangle x_i \,, (x \in X)$.

Let

$$\bar{\pi}_r = \sum_{s=1}^{r} \pi_s, \quad (1 \le r \le k).$$

Then $\|\bar{\pi}_r\| \vee \|I - \bar{\pi}_r\| \le b$.

(ii) Let $(e_i)_{i=1}^{\infty}$ denote the unit vector basis of ℓ^2. Let

$$\rho = \sum_{i=1}^{n_1} e_i \cdot x_i^* : X \to \ell^2 \text{ and } \rho' = \sum_{i=1}^{n_1} x_i \cdot e_i^* : \ell^2 \to X,$$

where for $x \in X, e_i \cdot x_i^*(x) = \langle x_i^*, x \rangle e_i$ and for $f \in \ell^2, \langle x_i \cdot e_i^*, f \rangle = \langle e_i^*, f \rangle x_i$ and if we let

$$\sigma = \sum_{i=1+n_1}^{N} e_i \cdot x_i^* \,, \quad \sigma' = \sum_{i=1+n_1}^{N} x_i \cdot e_i^*$$

then $\|\rho\| \vee \|\rho'\| \leq \frac{1}{2}\sqrt{c}$, for a certain $c > 0$, depending on $T_1, \dots, T_m$ and ϵ, $\|\sigma\| \vee \|\sigma'\| \leq \frac{1}{2}\sqrt{b}$, while $k > (b+c)^4 \ /\epsilon$.

(iii) For $\mu = 1 \dots m$ we have $\|T_\mu - \pi_1 T_\mu \pi_1\| < \epsilon$; and for each $j \in [1, n_r], (1 \leq r < k)$ we have

$$\|(I - \bar{\pi}_{r+1}) T_\mu x_j\| < \frac{\epsilon}{n_r \cdot 2^r \cdot (b+c)^2}$$

and

$$\|x_j^* \circ T_\mu (I - \bar{\pi}_{r+1})\| < \frac{\epsilon}{n_r \cdot 2^r \cdot (b+c)^2}$$

The point of this definition is:

Theorem (4.1.2) [4]:

Let X be fairly close to a Hilbert space. Then $K(X)$ is approximate amenable.

Note. We shall show then that certain ℓ^2 -direct sums $X = \oplus_{i=1}^{\infty} \ell_{p_i}^{n_i}$ are fairly close to Hilbert space but $K(X)$ is not amenable. This is because if we split the direct sum into $X_1 = \oplus_{p_i<2} \ell_{p_i}^{n_i}$ and $X_2 = \oplus_{p_i<2} \ell_{p_i}^{n_i}$, then we find that neither X_1 is finitely representable in X_2 nor X_2 is finitely representable in X_1.

This means that $K(X_1 \oplus X_2)$ cannot be amenable. The complete details will be given in Theorem (4.2.8).

Proof:

Given $(T_\mu)_{\mu=1}^{m} \subset K(X)$, with $\|T_\mu\| \leq 1$ say, and $\epsilon > 0$, we seek $a\ \Delta \in K(X) \widehat{\otimes} K(X)$ such that $\|\pi(\Delta) \cdot T_\mu - T_\mu\| < b^2\epsilon$ and $\|T_\mu \cdot \Delta - \Delta \cdot T_\mu\| < (9 + 4b)\epsilon$ $(\mu = 1 \dots m)$ and $\|\pi(\Delta)\| \leq b$. We claim this is enough for our assertion. Perhaps it is best if we prove that first so as to get it out of the way:

Lemma (4.1.3) [4]:

Let $\mathcal{A}$ be a Banach algebra, and $b > 0$. Suppose that for every $T_1, T_2, \dots T_m \in \mathcal{A}$ with $\|T_\mu\| \leq 1$ $(\mu = 1, \dots, m)$, and every $\epsilon > 0$, there is a $\Delta \in \mathcal{A}\widehat{\otimes}\mathcal{A}$ such that $\|\pi(\Delta)\| \leq b, \|\pi(\Delta)\cdot T_\mu - T_\mu\| < b^2\epsilon$ and $\|\Delta \cdot T_\mu - T_\mu \cdot \Delta\| < (9+4b)\epsilon, (\mu = 1 \ldots m)$. Then $\mathcal{A}$ is approximate amenable.

Proof:

Since $\|\pi(\Delta)\| \leq b$, we have approximate amenability of $\mathcal{A}$ as a consequence of (i) ⟺(iii).

So returning to the main proof. We pick a shrinking basis (x_i) and finite sequence $n_0 < n_1 < \cdots < n_k$ as in Definition (4.1.1), for the particular (T_μ) and ϵ. We write $F_{i,j} = x_i \cdot x_j^* \in K(X)$ and $E_{i,j} = e_i \cdot e_j^* \in K(\ell^2)$. We define, for $i \in \mathbb{N}$,

$$r(i) = \begin{cases} r, \text{if } i \in (n_{r-1}, n_r], r \leq k; \\ k+1, \text{if } i > n_k = N, \end{cases} \tag{1}$$

and

$$\lambda(i) = \frac{k+1-r(i)}{k}$$

We then define $\Delta \in K(X)\widehat{\otimes}K(X)$ by

$$\Delta = \frac{1}{N}\sum_{i,j=1}^{N} \lambda(i) F_{i,j} \otimes F_{j,i} \tag{2}$$

Evidently,

$$\pi(\Delta) = \sum_{i=1}^{N} \lambda(i) F_{i,i} = \frac{1}{k}\sum_{r=1}^{k} \bar{\pi}_r,$$

(since both operators S have $S(e_i) = \frac{(k+1-r)}{k} e_i$ if $i \in (n_{r-1}, n_r]$, so $e_i \in \operatorname{lm} \bar{\pi}_r \cap \operatorname{lm} \bar{\pi}_{r+1} \cap \ldots \cap \operatorname{lm} \bar{\pi}_k$, but $e_i \in \ker \bar{\pi}_j$, for $j < r$). Accordingly,

$$\|\pi(\Delta)\| \leq \max\{\|\bar{\pi}_r\|\} \leq b, \tag{3}$$

by (i) of Definition (4.1.1).

Similarly,

$$\|I - \pi(\Delta)\| \le \max\{\|I - \bar{\pi}_r\|\} \le b \tag{4}$$

Furthermore, since $\lambda(i) = 1$ for $i \le n_1$, we have $\pi(\Delta) \cdot \pi_1 = \pi_1$. So

$$\begin{aligned}\left\|\big(I - \pi(\Delta)\big) \cdot T_\mu\right\| &= \left\|\big(I - \pi(\Delta)\big)(I - \pi_1)T_\mu\right\| \\ &\le b \cdot \left\|(I - \pi_1)T_\mu\right\| \end{aligned} \tag{5}$$

by part (iii) of Definition (4.1.1).

Let us now estimate $\|T \cdot \Delta - \Delta \cdot T\|$ for $T \in K(X)$. we have

$$T \cdot \Delta = T \cdot \bar{\pi}_k \cdot \Delta,$$

because $F_{i,j} = \bar{\pi}_k F_{i,j}$, for all $i, j = 1, \dots, N$. Similarly, $\Delta \cdot T = \Delta \cdot \bar{\pi}_k \cdot T$. Now $T(x_i) = \sum_{l=1}^{\infty} T_{l,i} x_l$, where

$$T_{l,i} = \langle x_l^*, T(x_i)\rangle;$$

also

$$x_i^* \circ T = \sum_{l=1}^{\infty} T_{i,l} x_l^*,$$

the latter being a norm-convergent sum in X^* because (x_i) is a shrinking basis, and (x_i^*) the dual basis of X^*. So,

$$\begin{aligned} T \cdot \Delta &= \frac{1}{N} \sum_{i,j=1}^{N} \lambda(i) T \cdot F_{i,j} \otimes F_{j,i} \\ &= \frac{1}{N} \sum_{i,j=1}^{N} \sum_{l=1}^{\infty} \lambda(i) T_{l,i} F_{l,j} \otimes F_{j,i} \\ &= \frac{1}{N} \sum_{j=1}^{N} \sum_{i,l=1}^{\infty} \lambda(i) T_{l,i} F_{l,j} \otimes F_{j.i}, \end{aligned} \tag{6}$$

(since $\lambda(i) = 0$ for $i > N$ anyway). Likewise

$$\begin{aligned}\Delta \cdot T &= \frac{1}{N}\sum_{i,j=1}^{N} \lambda(i) F_{i,j} \otimes F_{j,i} T \\ &= \frac{1}{N}\sum_{j=1}^{N}\sum_{i,l=1}^{\infty} \lambda(i) T_{i,l} F_{i,j} \otimes F_{j,l} \\ &= \frac{1}{N}\sum_{j=1}^{N}\sum_{i,l=1}^{\infty} \lambda(l) T_{l,i} F_{l,j} \otimes F_{j,l}\end{aligned} \tag{7}$$

Accordingly, for any $T \in K(X)$, we have

$$\begin{aligned}T \cdot \Delta - \Delta \cdot T &= \frac{1}{N}\sum_{j=1}^{N}\sum_{i,l=1}^{\infty} \big(\lambda(i) - \lambda(l)\big) T_{l,i} F_{l.j} \otimes F_{j,i} \\ &= \frac{1}{Nk}\sum_{j=1}^{N}\sum_{i,l=1}^{\infty} \big(r(l) - r(i)\big) T_{l,i} F_{l,j} \otimes F_{j,i}\end{aligned} \tag{8}$$

Given our sequence $\left(T_\mu\right)_{\mu=1}^{m}$, let us define $\left(T'_\mu\right)_{\mu=1}^{m}$ by

$$\langle T'_\mu x_i, x_j^* \rangle = \begin{cases} \langle T_\mu x_i, x_j^* \rangle, & \text{if } |r(i) - r(j)| \leq 1, \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

Let us estimate $\|T_\mu - T'_\mu\|$. For $i, j \in \mathbb{N}$, we will have

$$\langle T'_\mu x_i, x_j^* \rangle = \begin{cases} \langle T_\mu x_i, x_j^* \rangle & \begin{array}{c}\text{if } i \in (0, n_1] \text{and } j \in (0, n_2] \\ \text{or } i \in (n_{r-1}, n_r], r \in [2, k), j \in (n_{r-2}, n_{r+1}]; \\ \text{or } i \in (n_{k-1}, n_k] \text{ and } j \in (n_{k-2}, \infty); \\ \text{or } i \in (n_k, \infty] \text{ and } j \in (n_{k-1}, \infty);\end{array} \\ 0 & \text{otherwise.} \end{cases}$$

Hence, if we adopt the convention that $\pi_0 = \bar{\pi}_0 = 0$, we have

$$\begin{aligned}T'_\mu = \sum_{r=1}^{k-1} (\pi_{r-1} + \pi_r + \pi_{r+1}) T_\mu \pi_r + (1 - \bar{\pi}_{k-2}) T_\mu \pi_k \\ + (1 - \bar{\pi}_{k-1}) T_\mu (1 - \bar{\pi}_k)\end{aligned} \tag{10}$$

If we also adopt the convention that $\bar{\pi}_{k+1} = I$, we also have

$$T_\mu - T'_\mu = (I - \bar{\pi}_2) T_\mu \pi_1 \\ + \sum_{r=2}^{k} (\bar{\pi}_{r-2} + 1 - \bar{\pi}_{r+1}) T_\mu \pi_r + \bar{\pi}_{k-1} T_\mu (1 - \bar{\pi}_k) \tag{11}$$

For $j \in (n_{r-1}, n_r]$ we have

$$\left\| (I - \bar{\pi}_{r+1}) T_\mu x_j \right\| < \frac{\epsilon}{n_r 2^r (b+c)^2},$$

by part (iii) of (4.1.1). Hence

$$\left\| (I - \bar{\pi}_{r+1}) T_\mu \pi_r \right\| < \frac{\epsilon}{2^r (b+c)^2}. \tag{12}$$

For $3 \leq r \leq k+1$ and $j \leq n_{r-2}$, we have

$$\left\| x_j^* T_\mu (I - \bar{\pi}_{r-1}) \right\| < \frac{\epsilon}{2^{r-2} n_{r-2} (b+c)^2},$$

by part (iii) of Definition (4.1.1). So

$$\left\| \bar{\pi}_{r-2} T_\mu (I - \bar{\pi}_{r-1}) \right\| < \frac{\epsilon}{2^r (b+c)^2}, \tag{13}$$

and in particular, since $\pi_r = (I - \bar{\pi}_{r-1})_{\pi_r}$, we have

$$\left\| \bar{\pi}_{r-2} T_\mu \pi_r \right\| < \frac{\epsilon \|\pi_r\|}{2^{r-2} (b+c)^2} \leq \frac{2\epsilon b}{2^{r-2} (b+c)^2}. \tag{14}$$

Substituting (12), (14), and (13) into (11), we get

$$\left\| T_\mu - T'_\mu \right\| \leq \frac{\varepsilon}{2(b+c)^2} + \frac{\varepsilon}{4(b+c)^2} + \sum_{r=3}^{k} \frac{\epsilon(1+8b)}{2^r (b+c)^2} + \frac{\epsilon}{2^{k-1} (b+c)^2} \\ \leq \frac{\epsilon}{(b+c)^2} (2 + 2b). \tag{15}$$

Next we estimate $\|\Delta\|$:

$$\Delta = \frac{1}{N} \sum_{i,j=1}^{N} \lambda(i) F_{i,j} \otimes F_{j,i}.$$

With ρ, ρ' and σ, σ' as in part (ii) of Definition (4.1.1), we write $\mathcal{T} = \rho + \sigma$ and $\mathcal{T}' = \rho' + \sigma'$. Then we have $F_{i,j} = \mathcal{T}' E_{i,j\mathcal{T}}$ (when $1 \leq i, j \leq N$) and so, if $\bar{\mathcal{T}}: K(\ell^2)\widehat{\otimes}K(\ell^2) \to K(X)\widehat{\otimes}K(X)$, is the map specified by $\bar{\mathcal{T}}(A\otimes B) = \mathcal{T}'A\mathcal{T}\otimes\mathcal{T}'\,B\mathcal{T}$, then we have $\Delta = \bar{\mathcal{T}}(\Delta_0)$, where $\Delta_0 = \frac{1}{N}\sum_{i,j=1}^{N}\lambda(i)E_{i,j}\otimes E_{j,i}$. Since $\lambda(i) \in [0,1]$, it is straightforward that $\|\Delta_0\| = 1$ in $K(\ell^2)\widehat{\otimes}K(\ell^2)$. So

$$\begin{aligned}\|\Delta\| \le (\|\mathcal{T}\|\cdot\|\mathcal{T}'\|)^2 \le (\|\rho\| + \|\sigma\|)^2(\|\rho'\| + \|\sigma'\|)^2 \le \left(\frac{1}{2}\sqrt{b} + \sqrt{c}\right)^4 \\ \le (b+c)^2, \qquad\qquad (16)\end{aligned}$$

by part (ii) of equation (1). Form (15) and (16) we get

$$\begin{aligned}\left\|\left(T_\mu - T_\mu'\right)\cdot\Delta - \Delta\cdot\left(T_\mu - T_\mu'\right)\right\| \le 2\left\|T_\mu - T_\mu'\right\|\cdot\|\Delta\| \\ \le 2\epsilon(2+2b). \quad (17)\end{aligned}$$

It remains to estimate $\left\|T_\mu'\cdot\Delta - \Delta\cdot T_\mu'\right\|$. For any T, we have $T\cdot\Delta - \Delta\cdot T$ given by (8). But when $T = T_\mu'$ the coefficients $T_{i,j} = T_{\mu,i,j}'$ are zero unless $|r(i) - r(j)| \le 1$ (in which case they are equal to the corresponding coefficients $T_{\mu,i,j}$ of T_μ). Suppressing the index μ, we have

$$T'\cdot\Delta - \Delta\cdot T' = \frac{1}{N\kappa}\sum_{j=1}^{N}\sum_{r=1}^{k}\sum_{\substack{r(l)=r+1\\ r(i)=r}} T_{l,i}F_{l,j}\otimes F_{j,i} - \sum_{\substack{r(l)=r\\ r(i)=r+1}} T_{l,i}F_{l,j}\otimes F_{j,i}$$

For fixed $r, \sum_{j=1}^{N}\sum_{\substack{r(l)=r+1\\ r(i)=r}} T_{l,i}F_{l,i}\otimes F_{j,i} =$

$$\sum_{j=1}^{N}\sum_{r(i)=r}\pi_{r+1}TF_{i,j}\otimes F_{j,i} = \sum_{i,j=1}^{N}\pi_{r+1}T\pi_r F_{i,j}\otimes F_{j,i},$$

where when $r = k$ we define $\pi_{r+1} = \bar{\pi}_{k+1} - \bar{\pi}_k = I - \bar{\pi}_k$. Likewise,

$$\begin{aligned}\sum_{j=1}^{N}\sum_{\substack{r(i)=r+1\\ r(l)=r}} T_{l,i}F_{l,i}\otimes F_{j,i} &= \sum_{j=1}^{N}\sum_{r(i)=r}\pi_r TF_{i,j}\otimes F_{j,i} \\ &= \sum_{j=1}^{N}\sum_{i,j=1}^{N}\pi_r T\pi_{r+1}F_{i,j}\otimes F_{j,i}.\end{aligned}$$

Hence,

$$T' \cdot \Delta - \Delta \cdot T' = \sum_{r=1}^{k} (\pi_r T \pi_{r+1} - \pi_{r+1} T \pi_r) \cdot d, \tag{18}$$

where,

$$d = \frac{1}{Nk} \sum_{i,j=1}^{N} F_{i,j} \otimes F_{j,i}.$$

Now $d = \frac{1}{\kappa} \bar{\mathcal{T}} \left(\frac{1}{N} \sum_{i,j=1}^{N} E_{i,j} \otimes E_{j,i} \right)$, hence

$$\|d\| \leq \frac{(b+c)^2}{k}, \tag{19}$$

by the same argument as for our estimate (16). Now on Hilbert space, the map

$$\theta : S \longmapsto \sum_{r=1}^{k-1} P_r S P_{r+1} - P_{r+1} S P_r,$$

(P_r a family of disjoint orthogonal projections) has norm at most 2. We have

$$\sum_{1}^{k-1} (\pi_r T \pi_{r+1} - \pi_{r+1} T \pi_r)$$
$$= \sum_{1}^{k-1} \{\mathcal{T} \cdot P_r \cdot \mathcal{T}' T_{\mathcal{T}} P_{r+1} \mathcal{T}' - \mathcal{T} P_{r+1} \mathcal{T}' T \mathcal{T} P_r \mathcal{T}'\},$$

for $P_r(e_i) = e_i$ (if $r(i) = r$) or 0 otherwise. So

$$\left\| \sum_{1}^{k-1} \pi_r T \pi_{r+1} - \pi_{r+1} T \pi_r \right\| \leq (\|\mathcal{T}\| \cdot \|\mathcal{T}'\|)^2 \|T\| \leq (b+c)^2 \|T\|,$$

and

$$\left\| \sum_{1}^{k} \pi_r T \pi_{r+1} - \pi_{r+1} T \pi_r \right\| \leq \|T\| \{(b+c)^2 + 2 \cdot \|\pi_k\| \cdot \|I - \bar{\pi}_k\|\}$$
$$\leq \|T\| \cdot \{(b+c)^2 + 4b^2\} \leq 5(b+c)^2$$

Substituting this and (19) in (18) we find

$$\left\|T'_\mu \cdot \Delta - \Delta \cdot T'_\mu\right\| \leq \frac{5(b+c)^2 \cdot (b+c)^2}{k} \leq 5\epsilon,$$

Since $k > (b+c)^4$ by part 2 of Definition (4.1.1). Throwing in (17) we find

$$\left\|T_\mu \cdot \Delta - \Delta \cdot T_\mu\right\| \leq \epsilon(9+4b).$$

So for every $\left(T_\mu\right)_{\mu=1}^{m} \subset K(X)$, with $\left\|T_\mu\right\| \leq 1$, there is a $\Delta \in K(X) \otimes K(X)$ with $\|\pi(\Delta)\| \leq b$ (by (3)), and $\left\|\pi(\Delta) \cdot T_\mu - T_\mu\right\| \leq b^2\epsilon$ (by (5)), and $\left\|T_\mu \cdot \Delta - \Delta \cdot T_\mu\right\| \leq \epsilon(9+4b)$. So the Banach algebra $K(X)$ is approximately amenable.

Section (4.2): Examples

Example (4.2.1) [4]:

Let $a_1 < b_1 < a_2 < b_2 < \cdots$ be a strictly increasing sequence of positive integers, which will be required to satisfy growth conditions. We define $p_i \in [1,3]$ by

$$p_i = \begin{cases} 2 - 1/a_i \text{, if } i \text{ is odd,} \\ 2 + 1/a_i \text{, if } i \text{ is even,} \end{cases}$$

and let the Banach space X be the ℓ^2-direct sum $X = \oplus_{n=1}^{\infty} \ell_{p_n}^{b_n}$, where $\ell_{p_n}^{b_n}$ stands for b_n-dimentional complex ℓ_{p_n}-space. We write $X = X_1 \oplus X_2$ with

$$X_1 = \bigoplus_{n \in 2\mathbb{N}+1} \ell_{p_n}^{b_n}, \quad X_2 = \bigoplus_{n \in 2\mathbb{N}} \ell_{p_n}^{b_n}.$$

We claim that (given growth conditions), X_1 is not finitely representable in X_2, nor is X_2 finitely representable in X_1. This is because X_1, being an ℓ_2-direct sum of ℓ_p^n-spaces with $p \le 2$, has cotype 2, while X_2 (given the growth conditions) does not; whereas X_2, being an ℓ_2-direct sum of ℓ_p^n-spaces with $3 \ge p \ge 2$, has type 2, but X_1 does not. Let us give the full argument:

Lemma (4.2.2) [4]:

The space X_1 has cotype 2, and the space X_2 has type 2.

Proof:

For all $p \in [1,2]$ it is known that the Banach space ℓ_p has cotype 2; furthermore the cotype 2 constant is uniformly bounded (a suitable uniform bound is given, for example. Let C denote such a uniform bound. All the spaces $\ell_{p_n}^{b_n}$(n odd) have cotype 2 constant at most C; therefore, by an elementary and well-known calculation, the cotype 2 constant of the ℓ_2-direct sum $\oplus_{n \in 2\mathbb{N}+1} \ell_{p_n}^{b_n}$ is at most C as well.

Similarly, for $p \in [2,3]$ the type 2 constant of ℓ_p is uniformly bounded, a uniform estimate being given in Veraar; though we could not allow $p \in [2, \infty]$ here, because ℓ_∞ does not have any nontrivial type. But for p on the bounded interval [2,3] (or indeed on [2,N] for fixed N) there is a uniform bound; let's call it T. The spaces $\ell_{p_n}^{b_n}$ (n even) all have type 2 constant at most T. The same elementary calculation then shows that the ℓ_2-direct sum $X_2 = \oplus_{n\in 2\mathbb{N}}\ell_{p_n}^{b_n}$ has type 2 constant at most T.

Lemma (4.2.3) [4]:

Given growth conditions, X_1 does not have type 2, nor does X_2 have cotype 2.

Proof:

By considering the unit vectors $e_i, 1 \le i \le m$, we find that the type 2 constant of ℓ_p^m is at least $m^{\frac{1}{p}-\frac{1}{2}}$ and the cotype 2 constant is at least $m^{\frac{1}{2}-\frac{1}{p}}$. Given an odd n, and $p = p_n = 2 - 1/a_n$, the type 2 constant of ℓ_p^m is at least n provided $m^{\frac{1}{p}-\frac{1}{2}} = m^{\frac{1}{4a_n-2}} > n$, or $m \ge n^{4a_n-2}$. Given an even n, and $p = p_n = 2 + 1/a_n$, the cotype 2 constant of ℓ_p^m is at least n provided $m^{\frac{1}{2}-\frac{1}{p}} = m^{\frac{1}{4a_n+2}} > n$, or $m \ge n^{4a_n+2}$.

So if we impose the growth conditions $b_n > n^{4a_n+2}$ for all $n \in \mathbb{N}$, we find the type 2 constant of $X_1 = \oplus_{n\in 2\mathbb{N}+1}\ell_{p_n}^{b_n}$ is at least n for all $n \in 2\mathbb{N}+1$ and the cotype 2 constant 2 of $X_2 = \oplus_{n\in 2\mathbb{N}}\ell_{p_n}^{b_n}$ is at least n for all $n \in \mathbb{N}$.

Corollary (4.2.4) [4]:

Given growth conditions, X_1 is not finitely representable in X_2, or even in the ℓ_2-direct sum of countably many copies of X_2. The same is true with roles of X_1 and X_2 reserved.

Proof:

The ℓ_2-direct sum of countably many copies of X_2 still has type 2, and is not possible to finitely represent a space not of type 2 in a space which does have type 2. The ℓ_2 direct sum of countably many copies of X_1 still has cotype 2, so X_2 is not finitely representable in it.

We define a Banach-Mazur distance [9]: Is a way to define distance on the set $\mathrm{Q}(n)$ of n-dimensional normed spaces. If X and Y are two finite-dimensional normed space with the same dimension. Let $Gl(X,Y)$ denote the collection of all linear isomorphism $T: X \to Y$. The Banach-Mazur distance between X and Y is defined by

$$\delta(X,Y) = \log(\inf\{\|T\|\|T^{-1}\|: T \in Gl(X,Y)\})$$

Equipped with the metric δ, the space $\mathrm{Q}(n)$ is a compact metric space, called the Banach-Mazur Compactum.

Theorem (4.2.5) [4]:

Given growth conditions, $K(X)$ is not amenable.

Proof:

Evidently $K(X) = \mathcal{F}(X)$ (the closure of the space of finite-rank operators) because X has an obvious Schauder basis. We have X must be approximately primary, i.e. whenever $X \simeq X_1 \oplus X_2$, one of the product maps

$$\pi_1: \mathcal{F}(X, X_1) \widehat{\otimes} \mathcal{F}(X_1, X) \to \mathcal{F}(X),$$

or

$$\pi_2: \mathcal{F}(X, X_2) \widehat{\otimes} \mathcal{F}(X_2, X) \to \mathcal{F}(X)$$

(where $\mathcal{F}(A,B)$ stands for the closure of finite-rank operators from B into A) is surjective, and therefore an open map. In particular, the projections P_n onto the first n elements of the Schauder basis of X must satisfy $P_n = \sum_{k=1}^{\infty} A_k^{(n)} B_k^{(n)}$ with $\sum_{k=1}^{\infty} \left\|A_k^{(n)}\right\| \left\|B_k^{(n)}\right\| \leq C$ (independent of n), and either $B_k^n \in \mathcal{F}(X_1, X), A_k^n \in \mathcal{F}(X, X_1)$, for all k, or $B_k^n \in \mathcal{F}(X_2, X)$, $A_k^n \in \mathcal{F}(X, X_2)$, for all k. Without loss of generality, we may assume that the first of the above two statements holds. Normalizing we can further assume that $\|A_k^n\| = \|B_k^n\|$ for all k, n; so we have $\sum \|A_k^n\|^2 \leq C$. We then have $P_n = A^{(n)} B^{(n)}$, where $A^{(n)} = \oplus_k A_k^n \in \mathcal{F}\left(X, \left(\oplus_{k=1}^{\infty} X_1^{(k)}\right)_2\right)$, $B^{(n)} = \oplus_k B_k^{(n)} \in \mathcal{F}\left(\left(\oplus_{k=1}^{\infty} X_1^{(k)}\right)_2, X\right)$ and $\|A^{(n)}\|, \|B^n\| \leq \sqrt{C}$.

So the Banach-Mazur distance from lm P_n to a subspace of the ℓ_2-direct sum of countably many copies $X_1^{(k)}$ of X_1 (namely the subspace

$B^{(n)}P_n X$) is at most C. Hence X_2 is represented on $\left(\oplus_k X_1^{(k)}\right)_2$ up to C-equivalence; a contradiction, and the proof is complete by symmetry.

Now writing $P_n = 2 + (-1)^n \frac{1}{a_n}$ and $X = \left(\oplus_{n=1}^{\infty} \ell_{p_n}^{b_n}\right)_2$, we claim that if the sequence $a_1 < b_1 < a_2 < b_2 < \cdots$ satisfies growth conditions, then X is approximately amenable. To prove this we shall use Theorem (4.1.2). we will also need the following fairly elementary lemma.

We can define Banach lattice [10]: It is a vector lattice that is at the same time a Banach space with a norm with norm which satisfies the monotonicity condition.

Lemma (4.2.6) [4]:

There is a function $\xi: \mathbb{N}^2 \to \mathbb{N}$ with the following property: Whenever X is a Banach space with 1-unconditional normalized basis $(f_i)_{i=1}^{\infty}$, and whenever $n, m \in \mathbb{N}, y_1, \dots, y_2 \in X$ with $\|y_i\| = 1$, there are vectors $z_1, \dots, z_k \in X, K = \xi(m, n)$, which are disjointedly supported with respect to the basis f_i, and for each $i = 1, \dots, n$ the distance from y_i to the linear span $lin\ (z_1, \dots, z_k)$ is at most $1/m$. In fact, one may take

$$\xi(n, m) = (1 + 4mn)^n. \tag{20}$$

Proof:

Let f_j^* be the support functional for (f_j), with $f_j^*(f_j) = \delta_{i,j}$. For each $j \in \mathbb{N}$ we define a vector $v_j \in \mathbb{C}^n$ by

$$\langle v_j, e_i \rangle = f_j^*(y_i) \quad (i = 1 \dots n) \tag{21}$$

$(y_i)_1^n$ the given vectors in X. We write $E = \{j: v_j \neq 0\}$. The unit ball B_n of $(\mathbb{C}^n, \|\cdot\|_2)$ (the usual Euclidean norm) has for each $\epsilon > 0$ an ϵ-net of size at most $(1 + 2/\epsilon)^n$. we write $\epsilon = 2/mn$ and choose an ϵ-net $(w_i)_{i=1}^{Q}$ for B_n of size $Q \leq (1 + 2/\epsilon)^n = (1 + 4mn)^n$. For each $j \in E$, we pick an $\alpha = \alpha(j) \in [1, Q]$ such that

$$\left\| w_\alpha - \left(v_j / \|v_j\|_2\right) \right\|_2 < \epsilon.$$

Given $\alpha \in [1, Q]$ we write $E_\alpha = \{j \in E: \alpha(j) = \alpha\}$, and in cases when $E_\alpha \neq \emptyset$, we let $I_\alpha \in [1, 1/n]$ be an index such that

$$\left|\langle w_\alpha, e_{I(\alpha)}\rangle\right| = \max x_i\{|\langle w_\alpha, e_i\rangle|\}.$$

Then we define

$$z_\alpha = \sum_{j\in E_\alpha} \langle yI_{(\alpha)}, f_j^*\rangle f_j.$$

The vectors $(z_\alpha)_{\alpha=1}^{Q}$ have disjoint supports E_α. We claim that for each $i = 1 \dots n$, the distance $d(y_i, \mathrm{lin}\{z_\alpha: \alpha = 1 \dots Q\}) < 1/m$. To show this let $A = \{\alpha: z_\alpha \neq 0\}$ and let us decide on an approximating vector $z \in \mathrm{lin}\{z_\alpha\}$, namely

$$z = \sum_{\alpha\in A} \frac{\langle w_\alpha, e_i\rangle}{\langle w_\alpha, eI_\alpha\rangle} \cdot z_\alpha.$$

We claim that $\|z - y_i\| < 1/m$. For if $j \in \mathbb{N}$ is any index such that $f_j^*(y_i) \neq 0$ or $f_j^*(z) \neq 0$, then j belongs to one of the sets $E_\alpha, \alpha = \alpha(j) \in A$, then

$$\langle y_i, f_j^*\rangle = \langle v_i, e_i\rangle, \ \ \frac{\langle y_i, f_j^*\rangle}{\|v_j\|_2} = \frac{\langle v_j, e_i\rangle}{\|v_j\|_2}.$$

So

$$\left|\frac{\langle y_i, f_j^*\rangle}{\|v_j\|_2} - \langle w_\alpha, e_i\rangle\right| \leq \left|\frac{v_j}{\|v_j\|_2} - w_\alpha\right| < \epsilon.$$

Accordingly,

$$\left|\langle y_i, f_j^*\rangle - \langle w_\alpha, e_i\rangle\|v_j\|_2\right| < \epsilon\|v_j\|_2.$$

If $I = I(\alpha)$, we also have

$$\left|\langle y_I, f_j^*\rangle - \langle w_\alpha, e_I\rangle\|v_j\|_2\right| < \epsilon\|v_j\|_2$$

So

$$\left|\langle y_i, f_j^*\rangle - \frac{\langle w_\alpha, e_i\rangle}{\langle w_\alpha, e_I\rangle}\langle y_I, f_j^*\rangle\right| < \epsilon\|v_j\|\left(1 + \left|\frac{\langle w_\alpha, e_i\rangle}{\langle w_\alpha, e_I\rangle}\right|\right) \leq 2\epsilon\|v_j\|_2$$

because $I = I(\alpha)$ is chosen such that $|\langle w_\alpha, e_I\rangle|$ is maximal. So

$$\|y_i - z\| = \left\| y_i - \sum_{\alpha\in A} \frac{\langle w_\alpha, e_i\rangle}{\langle w_\alpha, e_{I(\alpha)}\rangle} \cdot z_\alpha \right\|$$
$$= \left\| y_i - \sum_{\alpha\in A} \frac{\langle w_\alpha, e_i\rangle}{\langle w_\alpha, e_{I_\alpha}\rangle} \cdot \sum_{j\in E_\alpha} \langle y_I, f_j^*\rangle f_j \right\|$$
$$= \left\| \sum_{\alpha\in A} \sum_{j\in E_\alpha} \left(\langle y_i, f_j^*\rangle - \frac{\langle w_\alpha, e_i\rangle}{\langle w_\alpha, eI_{(\alpha)}\rangle} \cdot \langle yI_{(\alpha)}, f_j^*\rangle \right) f_j \right\|$$
$$< \left\| \sum_{\alpha\in A} \sum_{j\in E_\alpha} 2\epsilon \left\| v_j \right\|_2 \cdot f_j \right\|, \tag{22}$$

Because X having 1-unconditional basis (f_j), is a Banach lattice, and $|\gamma_i| \le \delta_i$ implies $\|\sum \gamma_i f_i\| \le \|\sum \delta_i f_i\|$. But $\|v_i\|_2 \le \sum_{i=1}^n |\langle v_i, e_i\rangle| = \sum_{i=1}^n \left|\langle f_j^*, y_i\rangle\right|$ so in the sense of the Banach lattice X, we have $\sum_{\alpha\in A} \sum_{j\in E_\alpha} 2\epsilon \left\|v_j\right\|_2 \cdot f_j \le 2\epsilon \sum_{i=1}^n |y_i|$. Since $\|y_i\| = 1$ by hypothesis equation (22) tell us $\|y_i - z\| \le 2n\epsilon = 1/m$.

Corollary (4.2.7) [4]:

There is a function $\chi: \mathbb{N}^2 \to \mathbb{N}$ with the following property: if a Banach space X has 1-unconditional normalized basis $(f_i)_1^\infty$, if $n, m \in \mathbb{N}$ and $(y_i)_{i=1}^n \in X$ with $\|y_i\| = 1$, then there are vectors $z_1 \dots z_K \in X, K = \chi(n, m)$ disjointly supported, $d(y_i, lin\{z_i\}) \le 1/m$ for all i, and in addition, the support of each y_i is a union of some of the supports of the z_j. In fact, we may take

$$\chi(n, m) = 2^n\big(1 + \xi(n, m)\big), \tag{23}$$

ξ as in Lemma (4.2.6).

Proof:

The ring R of subsets of $\mathbb{N}$ generated by the supports $supp\ (y_i)$ $(i = 1, \dots, n)$ and their complements has less than or equal 2^n atoms (minimal non-empty elements). Given X, n, m and (y_i) first we pick vectors $z_1 \dots z_n$ $(N \le \xi(n, m))$ in accordance with Lemma (4.2.6). We add an extra vector z_{N+1} whose support is $\mathbb{N} \backslash \cup_1^N \text{supp} z_i$, if that set is non-empty. For each atom $E \in R$ we define

$$z_{i,E} = z_i 1_E \quad (i = 1 \dots N+1)$$

that is

$$\langle f_j^*, z_{i,E} \rangle = \begin{cases} \langle f_j^*, z_i \rangle & \text{if } j \in E, \\ 0 & \text{if } j \neq E. \end{cases}$$

The $z_{i,E}$ are disjointly supported, and their linear span contains each z_i, so $d(y_i, \mathrm{lin}\{z_{i,E}\}) \leq 1/m$ for all j. The support of y_i is a union of some of the atoms of R; so it is the union of the supports of the $z_{i,E}$ over $i = 1 \dots N+1$ and appropriate atoms E. The non-zero $z_{i,E}$ can be normalized and there are at most $2^n(1 + \xi(n,m))$ of them. Of course if there are strictly less one can "pad" the sequence out by splitting up some of the $z_{i,E}$ into vectors of smaller support. So one obtains a set of the right size and properties.

Before proceeding to the main proof, we also wish to discuss uniform convexity. Let us impose the modest growth condition $a_1 \geq 2$. Then all the p_n lie in the interval $\left[\frac{3}{2}, \frac{5}{2}\right]$, and all the conjugate indices p' (with $\frac{1}{p} + \frac{1}{p'} = 1$) lie in $\left[\frac{5}{3}, 3\right]$. Now the ℓ_p except $p = 1$ or ∞ are uniformly convex; that is, there is a function $\Delta_p: (0,1] \to (0,1]$ such that whenever $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \epsilon$, we have $\|(x+y)/2\| \leq 1 - \Delta_p(\epsilon)$. For a compact set of values p not including $p = 1$, we can use the same modulus of convexity for all p, e.g. for all $p \in \left[\frac{3}{2}, 3\right]$. Our Banach space $X(X^*)$ are the ℓ_2-direct sum of ℓ_p^n having a common modulus of convexity. Therefore, X and X^* themselves are uniformly convex. Let Δ denote a common modulus of convexity for all such Banach spaces. (Δ is the modulus of convexity for the uncountable ℓ_2-direct sum $\left(\bigoplus_{p \in \left[\frac{3}{2}, 3\right]} \ell_p\right)_2$). We now show the following result.

Theorem (4.2.8) [4]:

If the sequence $a_1 < b_1 < a_2 \dots$ satisfies growth conditions, then our space X is "fairly close" to a Hilbert space (with constant $b = 100$), and therefore $K(X)$ is approximately amenable.

Proof:

Given $\left(T_\mu\right)_{\mu=1}^{m} \in K(X)$ of norm at most 1, and $\epsilon > 0$, we must find a shrinking basis $(x_i)_1^\infty$ with coordinate functional $(x_i^*), 0 = n_0 < n_1 < \cdots n_k = N$, and $c > 0$ such that the conditions of Definition (4.1.1) are satisfied. We may assume that $\epsilon < 1$. We being by choosing n_1 and c, also the finite sequences $(x_i)_{i=1}^{n_1}$ and $(x_i^*)_{i=1}^{n_1}$.

Definition (4.2.9) [4]:

Let $Q_r: \oplus_{n=1}^\infty \ell_{p_n}^{b_n} \to \oplus_{n=1}^r$ be the natural projection onto the first r vectors in the ℓ_2-direct sum. Pick an r_0 large enough that $r_0 > m \vee \frac{1}{\epsilon}$, and

$$\left\|T_\mu - Q_{r_0} T_\mu Q_{r_0}\right\| < \epsilon \quad (\mu = 1 \dots m). \tag{24}$$

We write $B_r = \sum_{s=1}^r b_s$ $(and\ B_0 = 0)$, and let $(f_i)_{i=1+B_{r-1}}^{B_r}$ be the unit vector basis of $l_{p_r}^{b_r}$, so that the entire sequence $(f_i)_{i=1}^\infty$ is the obvious basis of X. Let $(f_i^*)_{i=1}^\infty$ denote the dual basis. Thus, we have

$$Q_r = \sum_{i=1}^{B_r} f_i \cdot f_i^*, \quad (r \in \mathbb{N}). \tag{25}$$

Define $n_1 = B_{r_0},\ x_i = f_i$ and $x_i^* = f_i^*$ for $1 \le i \le n_1$. We also define

$$c = \left[4 \cdot b_{r_0}^{1/(2a_{r_0}-1)}\right] \tag{26}$$

and

$$k = 1 + r_0(b + c)^4. \tag{27}$$

In the notation of Definition (4.1.1), we are committed to $\pi_1 = Q_{r_0}$. Note that the condition $\left\|T_\mu - \pi_1 T_\mu \pi_1\right\| < \epsilon$ of part (iii) of Definition (4.1.1) is satisfied by (24). Note also that $k > (b + c)^4/\epsilon$ because $r_0 > 1/\epsilon$.

Lemma (4.2.10) [4]:

Given growth conditions on the sequences (a_n) and (b_n), the maps $\rho = \sum_{i=1}^{n_1} e_i \cdot x_i^* : X \to \ell_2^{n_1}$ and $\rho' = \sum_1^{n_1} x_i \cdot e_i^* : \ell_2 \to X$ have norm at most $\frac{1}{2}\sqrt{c}$.

Proof:

Let $1 \le r \le r_0$. The natural map $\rho_r = \sum_{i=1+B_{r-1}}^{B_r} e_i \cdot x_i^*$ sends the unit vectors of $\ell_{p_r}^{b_r}$ to some of the unit vectors of ℓ_2, and the map $\rho_r' = \sum_{i=1+B_{r-1}}^{B_r} x_i \cdot e_i^*$ sends unit vectors of ℓ_2 to unit vectors of $\ell_{p_r}^{b_r}$; so $\|\rho_r\| \vee \|\rho_r'\| = b_r^{\left|\frac{1}{p_r}-\frac{1}{2}\right|} = b_r^{1/(4a_r-2)}$ (if r is odd) or $b_r^{1/(4a_r+2)}$ (if r is even).

We can assume, as a growth condition, that the sequence $\left(b_r^{1/(4a_r-2)}\right)_{r\in\mathbb{N}}$ is non-decreasing; so $\|\rho\| = \left\|\sum_{r=1}^{r_0} \rho_r\right\| = \vee_{r=1}^{r_0} \|\rho_r\| \le b_{r_0}^{1/(4a_{r_0}-2)} = \frac{1}{2}\sqrt{c}$, likewise $\|\rho'\| \le \frac{1}{2}\sqrt{c}$ also.

So we now need to choose $n_2 \dots n_k$. In fact we shall also define a sequence $\eta_1 \dots \eta_k$ of small positive reals, as follows:

Definition (4.2.11) [4]:

Given n_1, b, c and k, we define sequences $(\eta_i)_{i=1}^{k}, (\eta_i)_{i=2}^{k}$ recursively as follows: Given $i \in [1, k]$ and the value n_i, we define

$$\eta_i = \frac{1}{5} \cdot \Delta\left(\frac{\epsilon}{2^{i+4} \cdot n_i^2 \cdot (b+c)^2}\right), \tag{28}$$

where Δ is the modulus of convexity as defined above; and η_i, η_i we define $N_i = (2m+1)n_i$ and (when $i < k$)

$$n_{i+1} = \chi\left(N_i, \left\lceil\frac{1}{\eta_i}\right\rceil\right), \tag{29}$$

where $\chi: \mathbb{N}^2 \to \mathbb{N}$ is as in (23) and ξ as in (20).

Note that from (23) we certainly have $n_{i+1} \geq 2^N > n_i$; the sequence (n_i) is strictly increasing as required. We continue by defining some vectors $z_i^{(\gamma)} \in X, z_i^{(\gamma)^*} \in X^*$ $(i = 1 \dots n_\gamma, 1 \leq \gamma \leq k)$ as follows.

Definition (4.2.12) [4]:

We define $z_i^{(1)} = x_i = f_i$ and $z_i^{(1)_*} = f_i^*$ $(i = 1 \dots n_1)$. Given $\gamma \in [1, k)$ and $z_i^{(\gamma)}, z_i^{(\gamma)^*}$ $(i = 1 \dots n_\gamma)$, we define $\Omega_1^{(\gamma)} = \left\{(i, \mu): 1 \leq i \leq n_\gamma, 1 \leq \mu \leq m, T_\mu z_i^{(\gamma)} \neq 0\right\}$, and for $(i, \mu) \in \Omega_1^{(\gamma)}$ we write

$$v_{\mu,i}^{(\gamma)} = \frac{T_\mu z_i^{(\gamma)}}{\|T_\mu z_i^\gamma\|}. \tag{30}$$

We write $\Omega_2^{(\gamma)} = \left\{(i, \mu): 1 \leq i \leq n_\gamma, 1 \leq \mu \leq m, z_i^{(\gamma)^*} \circ T_\mu(1 - \pi_1) \neq 0\right\}$ and for $(i, \mu) \in \Omega_2^{(\gamma)}$ we write $w_{\mu,i}^{(\gamma)} \in X$ for the (unique, because X, X^* are uniformly convex) norm 1 support vectors for the functional $z_i^{(\gamma)^*} \circ T_\mu(1 - \pi_1) \in X^*$. We then write

$$\begin{aligned} S^{(\gamma)} = \left\{z_i^{(\gamma)}\right\} \cup \left\{v_{\mu,i}^{(\gamma)}: (i, \mu) \in \Omega_1^{(\gamma)}\right\} \\ \cup \left\{w_{\mu,i}^{(\gamma)}: (i, \mu) \in \Omega_2^{(\gamma)}\right\}. \end{aligned} \tag{31}$$

There are at most $(2m + 1)_{n_\gamma}$ non-zero elements of $S^{(\gamma)}$. By Corollary (4.2.7) there is a collection of norm-1 vectors $z_i^{(\gamma+1)}$ of size

$$\chi\left((2m + 1)_{n_\gamma}, \left\lceil\frac{1}{\eta_\gamma}\right\rceil\right) = \chi\left(N_\gamma, \left\lceil\frac{1}{\eta_\gamma}\right\rceil\right) = n_{\gamma+1},$$

having disjoint supports, such that for each $s \in S^{(\gamma)}$ we have

$$d\left(s, \operatorname{lin}\left\{z_i^{(\gamma+1)}: 1 \leq i \leq n_{\gamma+1}\right\}\right) \leq \eta_\gamma \tag{32}$$

and the support of $z_j^{(\gamma)}$ is a union of some of the supports of the $z_i^{(\gamma+1)}$ for each j. These are our vectors $\left(z_i^{(\gamma+1)}\right)$. The functionals $\left(z_i^{(\gamma+1)^*}\right)$ are the unique norm 1 support functionals for the vectors $z_i^{(\gamma+1)}$; as always with a uniformly convex space with 1-unconditional basis, the support of $z_i^{(\gamma+1)^*}$ is the same as the support of $z_i^{(\gamma+1)}$.

We know that for each $j = 1 \dots n_1$ and $\gamma = 2 \dots k$, the collection $\left(z_i^{(\gamma)}\right)_{i=1}^{n_\gamma}$ includes a unit vector whose support is the singleton $\{j\}$, which is the support of $z_j^{(1)} = x_j = f_j$. We can rearrange if necessary and assume simply that $z_j^{(\gamma)} = f_j$ (and so $z_j^{(\gamma)^*} = f_j$) for all $\gamma = 1 \dots k, 1 \leq j \leq n_1$. Then for $j \in (n_1, n_k]$, the vector $z_j^{(k)}$ is supported on (n_1, ∞), as is the support functional $z_j^{(k)*}$.

Definition (4.2.13) [4]:

For $r \in [1, k]$ we write $Z^{(r)} = \operatorname{lin}\left\{z_i^{(r)}: 1 \leq i \leq n_r\right\}$; and we define a further set of vectors $\left\{\zeta_i^{(k)}: 1 \leq r \leq k, 1 \leq i \leq n_r\right\} \subset Z^{(k)}$ recursively as follows: $\zeta_i^{(k)} = z_i^{(k)}$ for all i, and for each $r < k$ and $i \in [1, n_r], \zeta_i^{(r)}$ is the unique vector in $\zeta^{(r+1)} = lin\left\{\zeta_j^{(r+1)}: 1 \leq j \leq n_{r+1}\right\}$ which is closets to $z_i^{(r)}$.

As usual, the "closets vectors" in the definition are indeed unique because X is uniformly convex. And as we have discussed, for any r one has $z_i^{(r)} = f_i$ for $1 \leq i \leq n_1$ hence $\zeta_i^{(r)} = f_i$, also when $i \leq n_1$.

Also, for fixed r the $z_i^{(r)}$ are chosen disjointly supported with respect to the standard basis (f_i); and when $r < k$, the support $supp\ z_i^{(r)}$ is a union of some of the supports of the vectors $z_i^{(r+1)}$. We claim that the support $supp\ \zeta_i^{(r)}$ is contained in $supp\ z_i^{(r)}$ for all i and r. When $r = k$, equality holds. Proceeding by reverse induction on r, let us fix $i \in [1, n_r]$ and write $supp\ z_i^{(r)} = \cup_{j \in E}\ supp z_j^{(r+1)}$; and assume that $supp\ \zeta_j^{(r+1)} \subset supp\ z_j^{(r+1)}$ for all $j = 1, \dots, n_{r+1}$. Then for $j \notin E, supp \zeta_j^{(r+1)} \cap supp z_i^{(r)} = \emptyset$, so since (f_i) is a 1-unconditional basis, the unique closets vector to $z_i^{(r)}$ in $\operatorname{lin}\left\{\zeta_j^{(r+1)}\right\}$ is a linear combination of vectors $\left\{\zeta_j^{(r+1)}: j \in E\right\}$ alone. That is, $\zeta_i^{(r)} \in \operatorname{lin}\left\{\zeta_j^{(r+1)}: j \in E\right\}, supp\ \zeta_i^{(r)} \subset$

$\cup_{j\in E}\ supp\zeta_j^{(r+1)} \subset \cup_{j\in E}\ suppz_j^{(r+1)}$ by hypothesis, that is $supp\zeta_i^{(r)} \subset suppz_i^{(r)}$. In fact, we can say a little more:

Lemma (4.2.14) [4]:

The vectors $\zeta_i^{(r)}$ are non-zero, and the distance $\left\|\zeta_i^{(r)} - z_i^{(r)}\right\| \leq 2\eta_r$ for all $r \in [1, \kappa]$ and $i \in [1, n_r]$.

Proof:

We write $\varepsilon_r = \max\left\{\left\|\zeta_i^{(r)} - z_i^{(r)}\right\| : i = 1, \dots, n_r\right\}$. Form (28) we see that $\eta_r \leq 1/5$ for all r, so since $\left\|z_i^{(r)}\right\| = 1$, we only need to prove the second assertion. Since for fixed r, the vectors $z_i^{(r)}$ are disjointly supported, for every $y \in \mathbb{C}^{n_r}$ we have

$$\max|y_i| \leq \left\|\sum_{i=1}^{n_r} y_i z_i^{(r)}\right\| \leq \sum_{i=1}^{n_r} |y_i|. \tag{33}$$

One can define a linear map $\alpha = \alpha^{(r)}: Z^{(r)} \to \zeta^{(r)}$ with

$$\alpha\left(z_i^{(r)}\right) = \zeta_i^{(r)} \tag{34}$$

for each i, and (33) gives us the simple estimate

$$\|\alpha(x) - x\| \leq \|x\| \cdot \sum_{i=1}^{n_r} \left\|z_i^{(r)} - \zeta_i^{(r)}\right\| \leq n_r\varepsilon_r \cdot \|x\|. \tag{35}$$

Now suppose that $r > 1$. By (32), the vectors $z_i^{(r)}$ are chosen so that for each $j \in [1, n_{r-1}]$, the norm distance $d\left(z_j^{(r-1)}, Z^{(r)}\right) \leq \eta_{r-1}$. Fix j and let $z \in Z^{(r)}$ be a vector with $\left\|z - z_j^{(r-1)}\right\| \leq \eta_{r-1}$. Then the vector $\alpha(z)$ lies in $\zeta^{(r)}$, and $\left\|\alpha(z) - z_j^{(r-1)}\right\| \leq \eta_{r-1} + \|\alpha(z) - z\| \leq \eta_{r-1} + n_r\varepsilon_r\|z\| + \eta_{r-1} + n_r\varepsilon_r(1 + \eta_{r-1})$ because $\left\|z_j^{(r-1)}\right\| = 1$ and $\left\|z - z_j^{(r-1)}\right\| \leq \eta_{r-1}$. Since $\zeta_j^{(r-1)}$ is by definition the closets vector in $\zeta^{(r)}$ to

$z_j^{(r-1)}$ we accordingly have $\left\|\zeta_j^{(r-1)} - z_j^{(r-1)}\right\| \leq \eta_{r-1} + n_r\eta_{r-1}(1 + \eta_{r-1})$, and hence,

$$\varepsilon_{r-1} \leq n_r\varepsilon_r(1 + \eta_{r-1}) + \eta_{r-1}. \tag{36}$$

Now for all $r \in [1, k]$, the constant $\eta_r = \frac{1}{5} \cdot \Delta\left(\frac{\varepsilon}{2^{r+4}n_r^2(b+c)^2}\right) \leq \frac{1}{2^{r+4}n_r^2}$ because Δ is a modulus of convexity so $\Delta(h) \leq h$ for all $h \in (0,1]$. Therefore,

$$n_r\eta_r \leq \frac{1}{2^{r+4}n_r} \leq 2^{-r-6}\eta_{r-1}, \qquad \text{if } r > 1; \tag{37}$$

because the constant $n_r = \chi\left(N_{r-1}, \left[\frac{1}{\eta_{r-1}}\right]\right)$ where $\chi(n,m) > \xi(n,m) = (1 + 4nm)^n$ by (23) and (20); so very crudely, we can say $n_r > 4/\eta_{r-1}$. Substituting this in (36) and dividing by η_{r-1}, we have

$$\frac{\varepsilon_{r-1}}{\eta_{r-1}} \leq 2^{-r-6}\frac{\varepsilon_r}{\eta_r}(1 + \eta_r) + 1 \leq 2^{-r-5}\frac{\varepsilon_r}{\eta_r} + 1.$$

So if $\frac{\varepsilon_r}{\eta_r} \leq 2$, certainly $\frac{\varepsilon_{r-1}}{\eta_{r-1}} \leq 2$. But we begin with $\varepsilon_k = 0$; so $\frac{\varepsilon_r}{\eta_r} \leq 2$ for all $r = 1, \dots, k$ by reverse induction.

Corollary (4.2.15) [4]:

For all $r = 1, \dots, k$ and $x \in Z^{(r)}$ we have $\left\|\alpha^{(r)}(x) - x\right\| \leq 2n_r\eta_r \cdot \|x\|/(2^{r+3}n_r)$, and when $r > 1$, we have $\left\|\alpha^{(r)}(x) - x\right\| \leq \|x\| \cdot 2^{-r-5}\eta_{r-1}$.

Proof:

We are now in a position to complete the definition of the sequences $(x_i)_{i=1}^{\infty}, (x_i^*)_{i=1}^{\infty}$.

Definition (4.2.16) [4]:

For $r \in (1, k]$ we define the maps $\beta: X \to \ell_2$ and $\beta': \ell_2 \to X$ by

$$\beta = \sum_{1+n_1}^{n_k} e_i \cdot z_j^{(k)*} \tag{38}$$

and

$$\beta' = \sum_{1+n_1}^{n_k} z_j^{(k)} \cdot e_j^*. \tag{39}$$

We also define a Euclidean seminorm $\|\cdot\|_2$ on X by

$$\|x\|_2 = \|\rho(x) + \beta(x)\|, \tag{40}$$

where $\rho = \sum_{i=1}^{n_1} e_i \cdot x_i^* = \sum_{i=1}^{n_1} z_i^{(k)^*}$ as in Definition (4.1.1). Of course, $\|\cdot\|_2$ is a norm on the finite dimensional subspace $Z^{(k)}$. The subspaces $\xi^{(r)} \subset Z^{(k)}$ are nested, and we have a projection $P = \sum_{i=1}^{n_k} z_j^{(k)} \cdot z_j^{(k)*}$ onto $Z^{(k)}$. We have already defined the sequence $(x_i)_{i=1}^{n_1}$, namely $x_i = f_i$, and it is $\|\cdot\|_2$-orthonormal basis of $\zeta^{(1)} = lin\left\{\zeta_i^{(1)}: 1 \le i \le n_1\right\} = lin\{f_i: 1 \le i \le n_1\}$. We define the sequence $(x_i)_{i=1+n_1}^{n_2}$ to be any $\|\cdot\|_2$-orthonormal basis of the orthonormal complement $\zeta^{(2)} \ominus \zeta^{(1)}$ (noting that this subspace does indeed have dimension exactly $n_2 - n_1$ because the $\zeta_i^{(r)}$ are disjointly supported, and non-zero by Lemma (4.2.14)). The sequence $(x_i)_{i=1+n_2}^{n_3}$ is any orthonormal basis of $\zeta^{(3)} \ominus \zeta^{(2)}$; and so on, unit the sequence $(x_i)_{i=1+n_{k-1}}^{n_k}$ is orthonormal basis of $\zeta^{(k)} \ominus \zeta^{(k-1)}$. Thus we choose $(x_i)_{i=1}^{n_k}$ such that they are an orthonormal basis of the image $PX = Z^{(k)}$.

Now the space X has a Schauder basis, so its closed subspaces of finite codimension all have Schauder bases. The sequence $(x_i)_{i=1+n_k}^{\infty}$ we choose to be an arbitrary Schauder basis of the kernel $\ker P$, so the whole sequence $(x_i)_{i=1}^{\infty}$ is a basis of X. The associated coordinate functionals $(x_i^*)_{i=1}^{\infty} \in X^*$ will satisfy $\sum_{i=1}^{n_k} x_i \cdot x_i^* = P$, so the sequence really does extend the initial sequence $x_i^* = f_i^*$ for $i = 1, \dots, n_1$.

We note that the Schauder basis (x_i) is certainly a shrinking basis because X is reflexive. We claim that our choice of the (x_i) and (x_i^*) satisfies all the other conditions of Definition (4.1.1), hence X is fairly close to a Hilbert space. We now begin to prove this, by getting a decent estimate on the norms of the projections $\bar{\pi}_r$ as in Definition (4.1.1).

Lemma (4.2.17) [4]:

Let $F:\mathbb{N}^2 \to \mathbb{N}$ be any function. Given suitable growth conditions on the underlying sequences $a_1 < b_1 < a_2 < b_2 \ldots$, the following is true: whenever $(z_i)_{i=1}^{\alpha}$ are disjointly supported unit vectors in $lin\{f_j: j > B_r\} \subset X$, and $(z_i^*)_{i=1}^{\alpha}$ are the corresponding support functionals, and $\alpha \le F(r, b_r)$, one has

$$\| \mathcal{T}\| \cdot \| \mathcal{T}'\| \le 1 + \frac{1}{\sqrt{a_{r+1}}} \tag{41}$$

where $\mathcal{T} = \sum_1^{\alpha} e_i \cdot z_i^* : X \to \ell_2$ and $\mathcal{T}' = \sum_1^{\alpha} z_i \cdot e_i^* : \ell_2 \to X$.

Proof:

Consider first the case when all the z_i belong to a single $\ell_{p_s}^{b_s} (s \ge r + 1)$. One has $\|\sum_1^{\alpha} \lambda_i z_i\| = (\sum |\lambda_i|^{p_s})^{1/p_s}$ but $\|\sum_1^{\alpha} \lambda_i e_i\| = (\sum_1^{\alpha} |\lambda_i|^2)^{1/2}$ and routing calculations lead to the conclusion that

$$\| \mathcal{T}\| \vee \| \mathcal{T}'\| \le \alpha^{\left|\frac{1}{p_s} - \frac{1}{2}\right|} \le \alpha^{1/a_s} \quad \left(\text{since } p_s = 2 \pm \frac{1}{a_s}\right) \le \alpha^{1/a_{1+r}}.$$

When the z_i may be supported on several of the $\ell_{p_s}^{b_s}$ we can split z_i into several $z_{i,s} \in \ell_{p_s}^{b_s}$, take the direct sum of the projections $\mathcal{T}_s$ onto

$$\text{lin}\{z_{i,s}: i = 1 \ldots \alpha\},$$

compose it at the ℓ_2-end with a partial isometry such that the images of the sums $\sum_s \mathcal{T}_s z_{i,s}$ (one has $\sum_s \|z_{i,s}\|^2 = 1$ because X is an ℓ_2-direct sum) are the unit vectors e_i, and one obtains the map $\mathcal{T}$. So still, $\| \mathcal{T}\| \le \alpha^{1/a_{r+1}}$ and similarly $\| \mathcal{T}'\| \le \alpha^{1/a_{r+1}}$. Our growth condition is therefore

$$F(r, b_r)^{1/a_{r+1}} = \exp\left(\frac{\log F(r, b_r)}{a_{r+1}}\right) < 1 + \frac{1}{\sqrt{a_{r+1}}} \tag{42}$$

for each r, a perfectly respectable growth condition.

Corollary (4.2.18) [4]:

We can, given growth conditions, be sure that the maps β and β' of (38) and (39) satisfy

$$\|\beta\| \vee \|\beta'\| < 1 + \frac{1}{\sqrt{a_{r_0+1}}} < (1+\eta_k)^{1/2} \tag{43}$$

Proof:

For the first inequality, the preceding lemma tells us that it is only necessary to show that n_k as in Definition (4.2.11) is bounded above by a fix function $F(r_0, b_{r_0})$. Now $n_1 = B_{r_0} \le r_0 b_{r_0}$ because the sequence (b_i) is increasing likewise the constants c, k as defined in (26) and (27) are bounded by suitable functions of r_0 and b_{r_0}. In the same definition, Definition (4.2.9), we chose $r_0 > m$ and $r_0 > 1/\epsilon$, so when we recursively define $n_2 \dots n_k$ and $\eta_1 \dots \eta_k$ by the procedure of Definition (4.2.11), even the last element n_k of the sequence is bounded by a function of r_0 and b_{r_0}. Likewise the small constant η_k has $1/\eta_k$ bounded by a suitable function of r_0 and b_{r_0}; so the second inequality $1 + \frac{1}{\sqrt{a_{r_0+1}}} < (1+\eta_k)^{1/2}$ is just another growth condition.

Corollary (4.2.19) [4]:

With our chosen shrinking basis (x_i), and our chosen sequence $(n_r)_{r=1}^{k}$, the maps σ, σ' as defined in Definition (4.1.1) have norm at most $(1+\eta_k)^{1/2}$. The estimate $\|\sigma\| \vee \|\sigma'\| \le \frac{1}{2}\sqrt{b}$ is satisfied.

Proof:

The map $\sigma = \sum_{i=1+n_1}^{n_k} e_i \cdot x_i^*$ annihilates $\ker P$ and $\mathrm{lm}\pi_1 = \mathrm{lin}\{f_i : 1 \le i \le n_1\}$, and it sends the $\|\cdot\|_2$-orthonormal basis $(x_i)_{i=1+n_1}^{n_k}$ of $\mathrm{lm}P \ominus \mathrm{lm}\pi_1$ to the unit vectors $e_i (i = 1+n_1, \dots, n_k)$. The map $\beta = \sum_{i=1+n_1}^{n_k} e_i \cdot z_i^{(k)^*}$ likewise annihilates $\ker P$ and $\mathrm{lm}\pi_1$, and sends the original $\|\cdot\|_2$-orthonormal basis $z_i^{(k)}$ to the unit vectors $e_i (i = 1 + n_1, \dots, n_k)$. Consequently we have $\sigma = U\beta$ for a suitable unitary operator U on ℓ_2. Similarly we have $\sigma' = \beta' U^*$, so $\|\sigma\| \vee \|\sigma'\| = \|\beta\| \vee \|\beta'\|$ and the result follows from Corollary (4.2.18).

Corollary (4.2.20) [4]:

With our chosen shrinking basis (x_i), and our chosen sequence $(n_r)_{r=1}^{k}$, the maps $\bar{\pi}_r(r = 1, \dots, k)$ as defined in Definition (4.1.1) have norm at most $2 + \eta_k$. The estimate $\|\bar{\pi}_r\| \vee \|I - \bar{\pi}_r\| \leq b$ is satisfied. Furthermore, we have $\|\bar{\pi}_r - \pi_r\| \leq 1 + \eta_k$.

Proof:

The basis constant for X is 1, and $\pi_1 = \sum_{i=1}^{n_1} f_i \cdot f_i^*$ accordingly has norm 1. For $r > 1$, the difference $\bar{\pi}_r - \pi_1 = \sum_{i=1+n_1}^{n_r} x_i \cdot x_i^*$ is equal to the composition $\sigma' q_r \sigma$, where q_r is the orthonormal projection with

$$q(e_i) = \begin{cases} e_i & i \in (n_1, n_r]; \\ 0 & \text{otherwise.} \end{cases} \tag{44}$$

Accordingly $\|\bar{\pi}_r - \pi_1\| \leq \|\sigma\| \cdot \|\sigma'\| \leq 1 + \eta_k$ by our preceding Corollary. The result follows.

We can now establish the rest of the condition of Definition (4.1.1). Form (i) and (ii) of Definition (4.1.1), there is nothing left to prove, so we now establish part (iii). We must show that

$$\left\|(I - \bar{\pi}_{r+1}) T_\mu x_j\right\| < \frac{\epsilon}{n_r \cdot 2_r \cdot (b + c)^2},$$

and

$$\left\|x_j^* \circ T_\mu (I - \bar{\pi}_{r+1})\right\| < \frac{\epsilon}{n_r \cdot 2_r \cdot (b + c)^2},$$

for $j \in [1, n_r], 1 \leq r < k$ and $\mu = 1 \dots m$. Now $I - \bar{\pi}_{r+1}$ is a projection of norm no more than 4 by Corollary (4.2.19), so it is enough to show that

$$\begin{aligned} d\big(T_\mu x_j, \mathrm{lm}\ \bar{\pi}_{r+1}\big) &= d\big(T_\mu x_j, \zeta^{(r+1)}\big) \\ &< \frac{\epsilon}{4n_r \cdot 2^r \cdot (b + c)^2}, \end{aligned} \tag{45}$$

and

$$d\big(x_j^* \circ T_\mu, X^* \circ \bar{\pi}_{r+1}\big) < \frac{\epsilon}{4n_r \cdot 2^r \cdot (b + c)^2}. \tag{46}$$

Let us first establish (45). If $j \leq n_1$ then $x_j = f_j = z_j^{(r)}$, and $d\left(T_\mu z_j^{(r)}, Z^{(r+)}\right) \leq \eta_r$ by (32). If $z \in Z^{(r+1)}$ is the closets vector to $T_\mu z_j^{(r)}$ then $\|z\| \leq 1 + \eta_r$ so $\left\|\alpha^{(r+1)}z - z\right\| \leq (1+\eta_r) \cdot 2^{-r-6}\eta_r$ by Corollary (4.2.15). So $d\left(T_\mu z_j^{(r)}, \zeta^{(r+1)}\right) \leq \left\|T_\mu z_j^{(r)} - \alpha^{(r+1)}z\right\| \leq \eta_r + (1+\eta_r) \cdot 2^{-r-6}\eta_r \leq 2\eta_r$, and (46) is established when we look at the definition of η_r in (28) and remember that $\Delta(h) \leq h$.

If $j > n_1, x_j$ is a sum $\sum_{1+n_1}^{n_r} \lambda_i \zeta_i^{(r)}$ with $\left\|x_j\right\|_2 = 1$. By Corollary (4.2.15), the map $\alpha^{(r)}: Z^{(r)} \to \zeta^{(r)}$ satisfies $\|\alpha(x) - x\| \leq 2n_r\eta_r\|x\| \leq \|x\|/16$ for all $x \in Z^{(r)}$. So the inverse map α^{-1} has $\|\alpha^{-1}\| \leq 16/15$; we have $\alpha^{-1}x_j = \sum_{1+n_1}^{n_r} \lambda_i z_i^{(r)}$, and the $z_i^{(r)}$ are disjointly supported with norm 1, so $\left\|\alpha^{-1}x_j\right\| \geq \max\{|\lambda_i|\}$, so for all $i, |\lambda_i| \leq \left\|\alpha^{-1}x_j\right\| \leq 16\left\|x_j\right\|/15 \leq 16\,\|\sigma'\|/15$ $\left(\text{for } x_j = \sigma' e_j\right) \leq 16\left(1 + 1/\sqrt{a_{1+r_0}}\right)/15$ by Corollary (4.2.19). This estimate is at most $3/2$, given the very modest growth condition $a_1 \geq 7$, so no $|\lambda_i|$ exceeds $3/2$. We have $\left\|x_j - \alpha^{-1}x_j\right\| \leq 2n_r\eta_r\left\|\alpha^{-1}x_j\right\| \leq 3n_r\eta_r$, and so $\left\|T_\mu x_j - T_\mu\alpha^{-1}x_j\right\| \leq 3n_r\eta_r$ also. By (28), and $\Delta(h) \leq h$, we have $\left\|T_\mu x_j - T_\mu\alpha^{-1}x_j\right\| \leq \frac{3}{5} \cdot \frac{\varepsilon}{2^{r+4}n_r(b+c)^2}$. Comparing this with our target (45), we see that it is enough to show

$$d\left(T_\mu\alpha^{-1}x_j, \zeta^{(r+1)}\right) \leq \frac{\varepsilon}{8n_r \cdot 2^r \cdot (b+c)^2}. \tag{47}$$

The vector $T_\mu\alpha^{-1}x_j = \sum_{i=1+n_1}^{n_r} \lambda_i T_\mu z_i^{(r)}$, and $\frac{T_\mu z_i^{(r)}}{\left\|T_\mu z_i^{(r)}\right\|} = v_{\mu,i}^{(r)} \in S^{(r)}$ by Definition (4.2.12). The vectors $z_i^{(r+1)}$ are chosen such that $d\left(v_{\mu,i}^{(r)}, Z^{(r+1)}\right) \leq \eta_r$. The closets vector w to $v_{\mu,i}^{(r)}$ in $Z^{(r+1)}$ has norm at most $1+\eta_r$, and $\left\|\alpha^{(r+1)}w - w\right\| \leq 2^{-r-6}\eta_r \cdot \|w\|$ by Corollary (4.2.15). The vector $\alpha^{(r+1)}w \in \zeta^{(r+1)}$, so $d\left(v_{\mu,i}^{(r)}, \zeta^{(r+1)}\right) \leq \eta_r(1 + 2^{-r-6} \cdot \|w\|) \leq \frac{3}{2}\eta_r$.

Since $\left\|T_\mu z_i^{(r)}\right\| \le 1$, we also have $d\left(T_\mu z_i^{(r)}, \zeta^{(r+1)}\right) \le \frac{3}{2}\eta_r$ also. Now $\alpha^{-1}x_j = \sum_{1+n_1}^{n_r} \lambda_i z_i^{(r)}$ with $|\lambda_i| \le 3/2$, so we have $d\big(T_\mu \alpha^{-1}x_j, \zeta^{(r+1)}\big) \le \frac{9}{4} n_r \eta_r$. Applying (28) again (and $\Delta(h) \le h$), we find $d\big(T_\mu \alpha^{-1}x_j, \zeta^{(r+1)}\big) \le \frac{9}{20} \cdot \frac{\varepsilon}{2^{r+4} n_r (b+c)^2}$. Thus (47), and hence (45) are established.

It remains to establish (46). We first note that

$$x_j^* \circ T_\mu \pi_1 \in \operatorname{lin}\{x_1^* \dots x_{n_1}^*\} = \operatorname{lin}\{x_1^* \circ \bar{\pi}_{r+1} \dots x_{n_1}{}^* \circ \bar{\pi}_{r+1}\},$$

so (46) is exactly equivalent to

$$d\big(x_j^* \circ T_\mu (I - \pi_1), X^* \circ \bar{\pi}_{r+1}\big) < \frac{\epsilon}{4n_r \cdot 2^r \cdot (b+c)^2}. \tag{48}$$

The set S^r includes a supporting vector $w_{\mu,i}^{(r)}$ for the functional $z_i^{(r)*} \circ T_\mu \circ (I - \pi_1)$, and the closets vector w to $w_{\mu,i}^{(r)}$ in $\operatorname{lin}\left\{z_i^{(r+1)}\right\}$ is within distance η_r. That means that we have $\|w\| \le 1 + \eta_r$, and $w' = \alpha^{(r+1)} w$ satisfies $\|w' - w\| \le 2^{-r-6}\eta_r \|w\| \le 2^{-r-5}\eta_r$ by Corollary (4.2.19), so $\left\|w_{\mu,i}^{(r)} - w'\right\| \le 3\eta_r/2$ and $\|w'\| \le 1 + 3\,\eta_r/2$. Accordingly, the real part

$$\begin{aligned} &\Re e \,\langle z_i^{(r)*} \circ T_\mu (I - \pi_1), w' \rangle \\ &\qquad \ge \left\|z_i^{(r)*} \circ T_\mu (I - \pi_1)\right\| \cdot \left(1 - \frac{3}{2}\eta_r\right). \end{aligned} \tag{49}$$

Now $\bar{\pi}_{r+1} w' = w'$, so we also have

$$\Re e \,\langle z_i^{(r)*} \circ T_\mu (I - \pi_1)\bar{\pi}_{r+1}, w' \rangle \ge \left\|z_i^{(r)*} \circ T_\mu (I - \pi_1)\right\| \cdot \left(1 - \tfrac{3}{2}\eta_r\right).$$

We have

$$\|(I - \pi_1)\bar{\pi}_{r+1}\| = \|\bar{\pi}_{r+1} - \pi_1\| \le 1 + \eta_k,$$

so the ratio $\frac{\left\|z_i^{(r)*} \circ T_\mu (I - \pi_1)\bar{\pi}_{r+1}\right\|}{\left\|z_i^{(r)*} \circ T_\mu (I - \pi_1)\right\|} \le 1 + \eta_k$.

Writing $z^* = \left(z_i^{(r)*} \circ T_\mu (I - \pi_1)\right) / \left\|z_i^{(r)} \circ T_\mu (I - \pi_1)\right\|$, and

$$w^* = \left(z_i^{(r)*} \circ T_\mu(I - \pi_1)\bar{\pi}_{r+1}\right) / \left\| z_i^{(r)*} \circ T_\mu(I - \pi_1)\bar{\pi}_{r+1} \right\|,$$

we have $\Re e\langle z^*, w'\rangle \geq 1 - \frac{3}{2}\eta_r$ and

$$\Re e\langle w^*, w'\rangle \geq \left(1 - \tfrac{3}{2}\eta_r\right) / (1 + \eta_k) \geq 1 - \tfrac{5}{2}\eta_r.$$

We also have

$$\|z^*\| = \|w^*\| = 1,$$

And

$$\left\|\frac{z^* + w^*}{2}\right\| \geq \Re e\,\langle \frac{z^* + w^*}{2}, w'\rangle / \|w'\| \geq \frac{(1 - 2\eta_r)}{\|w\prime\|} \geq \frac{(1 - 2\eta_r)}{1 + \frac{3}{2}\eta_r} \geq 1 - 4\eta_r.$$

But $\eta_r = \Delta\left(\frac{\epsilon}{2^{r+4} n_r^2 \cdot (b+c)^2}\right) / 5$, by (28), so by the uniform convexity of X^*,we have $\|z^* - w^*\| < \epsilon / n_r^2 \cdot 2^{r+4} \cdot (b + c)^2$. But $w^* \in X^* \circ \bar{\pi}_{r+1}$ so $d(z^*, X^* \circ \bar{\pi}_{r+1}) < \frac{\epsilon}{2^{r+4} n_r^2 \cdot (b+c)^2}$.

Hence, since

$$\left\| z_i^{(r)*} \right\| = 1, and\ \left\| T_\mu(I - \pi_1) \right\| \leq 1,$$

we also have

$$d\left(z_i^{(r)*} \circ T_\mu(I - \pi_1), X^* \circ \bar{\pi}_{r+1}\right) \leq \frac{\epsilon}{2^{r+4} n_r^2 \cdot (b + c)^2}. \tag{50}$$

For $j \leq n_1, x_j^* = z_j^{(r)*} = f_j^*$ so equation (50) also applies with $z_i^{(r)*}$ replaced by x_j^*, and (46) is established for this j. If $n_1 < j \leq n_r$ let us again write $x_j = \sum_{i=1+n_1}^{n_r} \lambda_j \zeta_j^{(r)}$. The linear function x_j^* annihilates $\ker \bar{\pi}_r$ and lm π_1, and satisfies $x_j^*(x_i) = \delta_{i,j}$ for $n_1 < i \leq n_r$. The (x_i) are a $\|\cdot\|_2$-orthonormal basis of $\zeta^{(r)}$, so, we have

$$\langle x_j^*, y\rangle = \langle (\bar{\pi}_r - \pi_1)y, x_j\rangle = \sum_{i=1+n_1}^{n_r} \lambda_i \,\langle (\bar{\pi}_r - \pi_1)y, \zeta_i^{(r)}\rangle, \tag{51}$$

for all $y \in X$, where $\langle\cdot,\cdot\rangle$ is the inner product associated with $\|\cdot\|_2$.

We write $\zeta_i^{(r)*}$ for the functional in X^* with $\zeta_i^{(r)*}(y) = \langle(\bar{\pi}_r - \pi_1)y, \zeta_i^{(r)}\rangle = \langle\sigma' q_r \sigma y, \zeta_i^{(r)}\rangle$ (where q_r is as in (44)) $= \langle q_r \sigma y, \sigma\zeta_i^{(r)}\rangle$, where the last inner product is in ℓ_2. We will have $\left\|\zeta_i^{(r)*}\right\| \leq \|\sigma\|^2 \cdot \left\|\zeta_i^{(r)}\right\| \leq (1+\eta_k)\left\|\zeta_i^{(r)}\right\|$ (by Corollary (4.2.19)) $\leq (1+\eta_k)(1+2\eta_r)$ (by Lemma (4.2.14)). So the normalized functional $w_i^* = \zeta_i^{(r)^*}/\left\|\zeta_i^{(r)^*}\right\|$ has $w_i^*\left(\zeta_i^{(r)}\right) = \left\|\zeta_i^{(r)}\right\|_2^2/\left\|\zeta_i^{(r)^*}\right\| \geq (1+\eta_k)^{-1}\left\|\zeta_i^{(r)}\right\|_2^2/\left\|\zeta_i^{(r)}\right\|$. The ratio $\left\|\zeta_i^{(r)}\right\|/\left\|\zeta_i^{(r)}\right\|_2$ is equal to $\left\|\beta'\beta\zeta_i^{(r)}\right\|/\left\|\beta\zeta_i^{(r)}\right\|$, and cannot exceed $(1+\eta_k)^{1/2}$ by Corollary (4.2.18); so $\left\|\zeta_i^{(r)}\right\|_2^2/\left\|\zeta_i^{(r)}\right\| \geq (1+\eta_k)^{-1}\left\|\zeta_i^{(r)}\right\|$, and

$$w_i^*\left(\zeta_i^{(r)}/\left\|\zeta_i^{(r)}\right\|\right) \geq (1+\eta_k)^{-2} \geq 1 - 2\eta_r. \tag{52}$$

The norm 1 functional $z_i^{(r)^*}$ has $\Re e z_i^{(r)^*}\left(\zeta_i^{(r)}\right) = 1 + \Re e z_i^{(r)^*}\left(\zeta_i^{(r)} - z_i^{(r)}\right) \geq 1 - 2\eta_r$ by Lemma (4.2.14) again, so $\Re e z_i^{(r)^*}\left(\left\|\zeta_i^{(r)}\right\|\right) \geq (1-2\eta_r)/(1-2\eta_r) \geq 1 - 4\eta_r$. Comparing this with (52), we find that the average $\left(w_i^* + z_i^{(r)^*}\right)/2$ has norm at least $1 - 3\eta_r > 1 - \Delta\left(\frac{\varepsilon}{2^{r+4}n_r^2\cdot(b+c)^2}\right)$ by (28). By the uniform convexity of X^*, we have $\left\|w_i^* - z_i^{(r)^*}\right\| < \frac{\varepsilon}{2^{r+4}n_r^2\cdot(b+c)^2}$, and so $d\left(w_i^* \circ T_\mu(1-\pi_1), X^* \circ \bar{\pi}_{r+1}\right) \leq \frac{\epsilon}{2^{r+4}n_r^2\cdot(b+c)^2}$ by (50), we have $x_j^* = \sum_{i=1+n_1}^{n_r} \lambda_i \zeta_i^{(r)^*} = \sum_{i=1+n_1}^{n_r} \lambda_i \left\|\zeta_i^{(r)}\right\| \cdot w_i^*$, where no $|\lambda_i|$ exceed $3/2$, and no $\left\|\zeta_i^{(r)^*}\right\|$ exceeds $(1+\eta_k)(1+2\eta_r)$; therefore $d\left(x_j^* \circ T_\mu(1-\pi_1), X^* \circ \bar{\pi}_{r+1}\right) \leq \frac{3n_r\varepsilon(1+\eta_k)(1+2\eta_r)}{2^{r+4}n_r^2\cdot(b+c)^2} < \frac{\varepsilon}{2^{r+4}n_r^2\cdot(b+c)^2}$ since no $\eta_i > 1/80$ by (28). Thus (48), and so also (46), are established.

So $K(X)$ is approximately amenable, given growth conditions on the (a_n) and (b_n).

List of Symbols

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