Sudan University of Science and Technology

College of Graduate Studies

Categories and Chain Complexes in Homological Algebra

الأصناف ومجمعات السلاسل في الجبر الهمولوجي

A thesis Submitted in fulfillment of the requirements for the degree of Philosophy in mathematics

By

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Dedication

To my family for their
cooperation and encouragement

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Abstract

The thesis exposes the basic language of categories and functions.

We construct the projective, inductive limits, kernel, cokernel, product, co product. Complexes in additive categories and complexes. in abelian categories.

The study asked when dealing with abelian category c, we assume that c is full Abelian.

The thesis prove the Yoned lemma, Five lemma, Horseshoe lemma and Snake lemma an then it give rise to an exact sequence, and introduce the long and short exact sequence.

We consider three Abelian categories c, c', c'' an additive bi functor F: cxc' → c'' and we assume that F is left exact with respect to each of its argument, and the study assume that each injective object I∈C the functor F (1,.): c' → c'' is exact.

The study shows important theorm and proving it if R is ring R = e {x_1, ..., x_n }, the Kozul complex KZ (R) is an object with effective homology.

We prove the cone reduction theorem ∈

(if p = (f,g,h): $C_* \Rightarrow D_*$, and p' = f',f',h'):

$C'_* \Rightarrow D'_*$, be two reduction and Ø: $C_* \longleftarrow C'_*$ a chain complex morphism, then these data define a canonical reduction

P'' = (f'', g'', h'' : cone (Ø) $\Rightarrow$ cone (f' Ø g').

The study gives a deep concepts and nation of completely multi – positive linear maps between C *- algebra and shows they are completely multi positive

We gives interpretation and explain how whiteheal theorem is important to homological algebra. .

The study construct the localization of category when satisfies its suitable conditions and the localization functors.

The thesis is splitting on De Rham co-homology in the module category and structures on categories of complexes in abelian categories.

The thesis applies triangulated categories to study the problem B = D (R), the unbounded derived category of chain complexes, and how to relate between categories and chain complexes.

الخـلاصــة

توضح هذه الأطروحة اللغة الأساسية الأصناف ووظائفها.

اهتمت الدراسة بتركيب أو بناء الحدود المشروعية الاستنتاجية للنواة الإضافية والنـواة الثانويـة والضـرب الـداخلي والضـرب الخـارجي فـي الأصـناف الجمعيـة التركيبية(أو المعقدة) الإضافية والتركيبات في الأصناف الأبيلية.

يطرح البحث السؤال في كيفية التعامل مع الصنف C حيث تفترض أن C فئة صنف أبيلي كامل.

تناولت الدراسة تمهيدية يونـدليما (Yoned lemma)وتمهيدية فايف (Five lemma)وتومهيدية الهورشي (Horsehose lemma)وتمهيدية سناك (Snak lemma)وبراهينها ومن ثم أعطت النتائج للسلسلة التامة التي تعمل على تقديم السلاسل ذو المدى الطويل والقصير وكيف يمكن تطبيقها والاستفادة من ذلك.

تهدف الدراسة إلى بناء ثلاث أصناف أبيلية Cً, C.C وا إلى دوال إضافية أخرى مثل الدالة F وقد تركت لوحدها فيما يتعلق بكل حجياتها وتفترض الدراسة أن كل مدخل موضوعي ينتمي إلى الوظيفة الخاصة به (I∈C) وهي:

$F(I, .) \rightrightarrows$ وتؤدي تماماً إلى $\acute{C} \longrightarrow C.$

وتوضح الدراسة أهمية النظرية التي تنص على أنه إذا كانت R تمثل حلقة وان $R = (x_1 \ldots\ldots\ldots x_n)$ وان تركيب كوزل KZ(R) Kozul هو شيء ذو تجانس أو تماثل فعال ينص برهان النظرية على ان إذا كانت.

$$P = (f,h,g)\ C_* \rightrightarrows D_*$$

$\acute{P} = (\acute{f}, \acute{g}, \acute{h})$ تودي إلى أن $C'_* \rightrightarrows D'_*$

يختصر إلى المجموعة الخالية C َ ⟵ C: ∅

تعطي الدراسة مفاهيم عميقة عن الرواسم الخطية الكلية الموجبية بين *C الجبر. وتوضح أنها متعددة الموجبية كلياً .

وأهتمت الدراسة بنظرية وايتهل (WHITEHEAL) وأعطينا تفسيرات ووضحنا كيف أن مبرهنة وايتهل مهمة في الجبر الهمولوجي.

تهتم الدراسة إلى بعض التناقضات الخاصة بالتجانس المصاحب الذي قام به ديرهام (De Rham) في الأصناف الأبيلية.

تطبق الأطروحة الأصناف المثلثية لـ(R) B=D التي تخص الأصناف غير المحدودة في السلسلة التركيبية(المعقدة) وكيف تقارن أو تخلق علاقة بين الأصناف وسلاسل الأصناف المركبة (المعقدة).

The Contents

Chapter One

The Language of Categories

The aim of the study is to introduce the language of categories and to present the basic notions of homological algebra first from an elementary point of view, with the notion of derived functors, next with a more sophisticated approach, with the introduction of triangulated and derived categories.

After having introduced the basic concepts of categories theory and particular those of projective and inductive limits, we treat with some details additive and abelian categories and construct the derived. The thesis show the important concepts of triangulated and derived categories.

This thesis is of five Chapter.

In chapter one we expose the basic language of categories and functors. A key point is Yoneda Lemma, which asserts that a category c may be embedded in category ĉ of contra variant functors on c with values in the category set of sets.

This naturally leads to the concept of representable functor. Many examples are treated, in particular in relation with the categories set of sets and Mod (A) of A-modules, for a (non necessarily commutative) ring A.

In chapter two we construct the projective and inductive limits, as a particular case, the kernels and co-kernels, product and coproduces.

We introduce the notions filtrant category and co final functors, and we study with some care filtrant inductive limits in the category set of sets.

Chapter three deals with additive categories and study the category of complexes in sub categories in particular, we introduce the shifted complex, the mapping cone of morphism, the homotopy of complexes and the simple complex associated with a double complex, with application to bifunctors. We also briefly study the simplicial category and explain how to associate complexes to simplicial objects.

Chapter four deals with abelian category and develop chapter three, and gives the relations between additive categories and abelian categories C. The chapter also study the injective resolution in constructive the derived functors of left exact functor.

Finally chapter four studies Kozul complexes and shows the important of derived category of K-modules with examples and applications.

In chapter five we study the homotopy category k (c) of derived category c is \derived functors.

H^o: k (c) → c is co homological and the derived category D (c) of c is obtained by localizing k (c) with respect to the family of quasi-homomorphism. The chapter constructs the localization of category with respect to a family of morphism, satisfying suitable conditions and we construct the localization of functors, and Localization of categories appears in particular in the constructing of derived categories.

We introduce triangulated categories, triangulated functors, and we give some results. We also study triangulated categories and functors, and we explain here this construction with some examples.

The word homology was first used in topological context by Hennery Poincare in 1895. Who used it to think about manifolds which were the boundaries of higher-dimensional manifolds it was Emmy Noether in 1920 who began thinking of homology in terms of groups and who developed algebraic techniques such as the idea of Modules over a ring. These are both absolutely crucial in gradients in the Modern theory of homological algebra, yet for the next twenty years homology theory was to remain confined to the realer of topology.

In 1942 came the first more forward towards homological algebra as we know it today, with the arrival of a paper by Samuel Eilenbery and Saunders Mac-lame.

In it we find Hom and Ext defined for the very first time, and a long with it the notions of a functor and natural isomorphism. These were needed to provide a precise language for talking about the properties of Hom (A,B); in particular the fact that it varies naturally, conveniently in A and conversantly in B. Only three years later this language was expanded to include category and natural equivalence. However this terminology was not widely accepted by the mathematical community until the appearance of Chartan and Eilenberg's book in 1956.

Chartan and Eilenberg's book was truly a revolution in the subject, and in fact it was here that the term "Homological algebra" was first coinet. The book used derived functor in a systematic way which united all the previous homology theories. Which in the past ten years had arisen in group theory? Lie algebras and algebraic geometry. The sheer list of terms that were first defined in the book may give an idea of how much of this project is due to the existence of that one book! They defined what it means for an object to be projective and injective resolutions. It is here that we find the first motion of Hom being left exact and the first occurrence of Ext as the right derived functors of Hom.

Until 1970, Chartan and Eilenberg's book was the bible on Homological Algebra, and the subject started be coming standard course material at many universities. Other books gradually started appearing, such as the Hilteon and Stammback book which much of this project is based around. Nowadays homological algebra is a fundamental tool in mathematics, where it has helped to write the foundations of algebraic geometry, to prove the wail conjectures, and to in rent powerful new methods such as algebraic K-Theory.[1].

Many examples are treated, in particular in relation with the categories set of sets and commutative ring A.

In this chapter we introduce some basic notions of category theory which are constant use in various fields of mathematics, without spending too much time on this language. After giving the main definitions on categories and functors, we prove the yoneda lemma, theorem and some propositions and linear maps and modules.

We also introduce the notions of representable functors and adjoint functors.

We start by recalling some basic notions on sets and on modules over a ring, and which shall some important examples.

Section (1.1) Sets and Maps:

The aim of this section is to fix some notations and to recall some elementary constructions on sets.

If $f: X \longrightarrow Y$ is a map from set X to set Y, we shall say that f is an isomorphism and write $f: X \xrightarrow{\sim} Y$, if there exists an isomorphism $f: X \xrightarrow{\sim} Y$, we say that x and y are isomorphic and write $x \simeq y$.

We shall denote by Hom_{set} (X,Y), or simply Hom (x,y), the set of all maps form x to y. if g: Y $\longrightarrow$ Z is another map, we can define the composition g of : X $\longrightarrow$ Z. Hence, we get two maps:

go : Hom (X,Y) $\longrightarrow$ Hom (X,Z),

of : Hom (Y,Z) $\longrightarrow$ Hom (Y,Z),

Notice that if X = {x} and Y={y} are two sets with one element each, then there exists a unique isomorphism X $\xrightarrow{\sim}$ Y, of courses, if x and y are finite sets with the same cardinal π >1, Xx and X and still isomorphic, but the isomorphism is no more unique.

In the sequel we shall denote by Ø the empty set and by {pt} a set with one element. Note that for any set x, there is a unique map Ø $\longrightarrow$ X and a unique map X $\longrightarrow$ {Pt}.

If $\{x_i\}_{i\in I}$ is a family of sets indexed by a set I. The product of the $X_{i's}$, denoted $\pi_{i\in I}\, x_i$, or simply $\pi_{i\in I}\, x_i$, is defined as

(1.1) $\pi_i\, x_i = \{x_i\}_{i\in I}$, $x_i \in x_i$ for all $i\in I$

If I: {1.2} one uses the notation X_1 x X_2. If $x_i = X$

for all i∈I, one uses the notation x', note that

(1.2) Hom (I,X) $\simeq$ X',

For a set y, there is natural isomorphism

(1.3) Hom $(y, \pi_i\, x_i) \simeq \pi_i$ Hom (y, x_i)

For three sets I, x, y, there are natural isomorphism

(1.4) $\text{Hom}(I \times X, y) \simeq \text{Hom}[I, \text{Hom}(X,Y)]$

$\simeq \text{Hom}(X,Y)^{-}$

If $\{x_i\}_{i \in I}$ is a family of sets indexed by asset I, one may also consider their disjoint union, also called their co-product. The co-product of the $X_{i's}$ is denoted $U_{i \in I} X_i$ or simply $U_i X_i$ If

$I = \{1.2\}$ one uses the notation $X_1 \cup X_2$

If $X_i = X$ for all $i \in I$,one uses the notation $X^{(1)}$. Note that

(1.5) $X \times I \simeq X^{(1)}$

Consider two sets x and y and two maps f,g from x to y, we write for short f,g: $X \rightrightarrows Y$.

The kernel (or equalize) of (f,g), denoted Ker (f,g), is defined as

(1.6) $\text{Ker}(f,g) = \{ x \in X; f(x) = g(x) \}$.

Note that for a set Z, one has

(1.7) $\text{Hom}(Z, \text{Ker}(f,g) \simeq \text{Ker}(\text{Hom}(Z,X) \rightrightarrows \text{Hom}(Z,Y)$.

Let us recall a few elementary definitions.

* A relation R on a set X is a subset of X x X. One writes xRy if $(x,y) \in R$.

* The opposite relation R^{op} is defined by $(y)x \in R^{op}$ y if and only if yRx.

* A relation R is reflexive if it contains the diagonal, that is, xRx for all x∈X.

* A relation R is symmetric if xRy implies yRx.

* A relation R is anti-symmetric if xRy implies yRx implies x=y,

* A relation R is a transitive if xRy and yRz implies xRz.

* A relation R is an equivalence relation if it is reflexive, symmetric and transitive.

* A relation R is a pre-order if it is reflexive and transitive. If moreover it is anti-symmetric, then one says that R is an order on x. A pre-order is often denoted ≤. A set endowed with a pre-order is called apo set.
* Let (I, ≤) be apo set. One says that (I,≤) is filtrant (one also says "directed") if I is non empty and for any i.j ∈I there exists j∈J with i ≤ j.
* Assume (I, ≤) is a filtrate apo set and let JcI be a sub set. One says that J is co-final to I if for any i∈I there exists j∈J with i ≤ j.

If R is a relation on a set X, there is a smaller equivalence relation which contains R.

(Take the intersection of all subsets of X x X which contain R and which are equivalence relations).

Let R be an equivalence relation on set X. A subset S of X is saturated if xRy and ∈RΨ implies yRx. One then defines a new set X/R and a canonical map f: X →X/R as follows: the elements of X/R are the saturated subsets of X and the map f associates to x∈X the unique saturated set S such that x∈S.[6,]

Chain Maps:

Definition (1.1.1): Let R be a commutative ring and let M, and N. be R-comp-lexes. A chain map $F_{.}: M_{.} \rightarrow N_{.}$ is sequence $\{f_i: M_i \rightarrow N_i\}_{i \in Z}$ making the next "ladder-diagram" commute.

$$
\begin{array}{ccccccccc}
M. & \cdots & \xrightarrow{\delta M_{i+1}} & M_i & \xrightarrow{\delta M_i} & M_{i-1} & \xrightarrow{\delta M_{i-1}} & \cdots \\
\downarrow F. & & & \downarrow F_i & & \downarrow F_{i-1} & & \\
N. & \cdots & \xrightarrow{\delta N_{i+1}} & N_i & \xrightarrow{\delta N_i} & N_{i-1} & \xrightarrow{\delta N_{i-1}} & \cdots
\end{array}
$$

Chain maps are also called "morphism of R-complexes". An isomorphism from $\underset{.}{M}$ to $\underset{.}{N}$ is chain map $\underset{.}{F}: \underset{.}{M} \rightarrow \underset{.}{N}$. such that each map $F_i: M_i \rightarrow N_i$ is an isomorphism.

Example (1.1.1): Here is a chain map over the ring R = Z/12Z.

$$
\begin{array}{ccccccccc}
M. & \cdots \xrightarrow{6} & Z/12Z & \xrightarrow{4} & Z/12Z & \xrightarrow{6} & Z/12Z & \xrightarrow{4} & \cdots \\
\downarrow F. & & \downarrow 2 & & \downarrow 3 & & \downarrow 2 & & \\
N. & \cdots \xrightarrow{4} & Z/12Z & \xrightarrow{6} & Z/12Z & \xrightarrow{4} & Z/12Z & \xrightarrow{6} & \cdots
\end{array}
$$

The next result states that a chain map induces maps on homology. [15].

Proposition (1.1.2): Let R be a commutative ring and let F.: M. → N. be a chain map.

(a) For each i, we have F_i [(Ker (δM_i)] C Ker (δN_i).

(b) For each i, we have F_i [(Im (δM_{i+1})] C Ker (δN_{i+1}).

(c) For each, i the map H_i (F.): H_i (M) → H_i (N.)

Given by H_i (F.) $(\overline{m}) = \overline{F_i\,(m)}$ is a well-defined R− module homomorphism.

Proof:

(a) and (b): chase diagram in Definition (1.1.1).

(b) The map H_i (F.) is well-defined by parts (a) and

(c) It is straight forward to show that it is R−linear.

Definition (1.1.2): Let R be a commutative ring, let U∈R be a multiplicatively closed sub set, and let M. be R−complex. The localized complex $U^{-1}M.$ is the sequence.

$$U^{-1}M. = \cdots \xrightarrow{U^{-1}\delta M_{i-1}} U^{-1}M_i \xrightarrow{U^{-1}\delta M_i} U^{-1}M_i \xrightarrow{U^{-1}\delta M_{i-1}} U^{-1}M_{i-1} \cdots$$

There is an isomorphism of $U^{-1}R$−complex $U^{-1}M. \cong (U^{-1}R) \otimes RM.$

Let F.: M. → N. be a chain map of R−complexes

Define $U^{-1}F.:\ U^{-1}M. \to U^{-1}N$

To be the sequence of maps $\{U^{-1}F_i: U^{-1}M_i \to U^{-1}N_i\}$. [15]

Remark (1.1.3):[15] Let R be a commutative ring, let U∈R be a multiplicatively closed subset, and let M. be an R−complexes. The sequence $U^{-1}M.$ is a $U^{-1}R$−complexes. The natural maps $M_i \to U^{-1}M_i$ from a chain map $M. \to U^{-1}M_i$ if F.: M. → N. is a chain map of R−complexes, then the sequence

$$U^{-1}F.\ :\ U^{-1}M. \to U^{-1}N.$$

Is a chain map of $U^{-1}R$–complexes that makes the following diagram commute?

$$\begin{array}{ccc} M. & \xrightarrow{F.} & N. \\ \downarrow & & \downarrow \\ U^{-1}M. & \xrightarrow{U^{-1}F.} & U^{-1}N \end{array}$$

Where the unlabeled vertical maps are the natural ones.

The natural isomorphism's is $(U^{-1}R)\otimes_R M_i \rightarrow U^{-1}M_i$

Form an isomorphism of $U^{-1}R$–complexes $(U^{-1}R)\otimes_R M_i \xrightarrow{\cong} U^{-1}M.$
making the next diagram commute

$$\begin{array}{ccc} (U^{-1}R)\otimes_R M. & \xrightarrow{(U^{-1}R)\otimes_R F.} & (U^{-1}R)\otimes_R N. \\ \cong\downarrow & & \downarrow\cong \\ U^{-1}M. & \xrightarrow{U^{-1}F.} & U^{-1}N \end{array}$$

Mapping Cones:

In this section, we discuss the mapping cone of chain map, which gives another important short exact sequence of chain maps.[15].

Definition (1.1.3): Let R be commutative ring, and let X. be an R−complex. The suspension or shift of X. is the sequence ∑X. defined as $(\sum X)_i = X_{i-1}$ and $\delta \sum X_i = -\delta x_{i-1}$ [15]

Remark (1.1.4): Let R be a commutative ring, and let X. be an R−complex. Diagrammatically, we see that ∑X. is essentially obtained by shifting X. one degree to the left.

$$X. = \cdots \xrightarrow{\delta X_{i+1}} X_i \xrightarrow{\delta X_i} X_{i-1} \xrightarrow{\delta X_{i-1}} \cdots$$

$$\sum X. = \cdots \xrightarrow{-\delta X_i} X_{i-1} \xrightarrow{-\delta X_{i-1}} X_{i-2} \xrightarrow{\delta X_{i-2}} \cdots$$

It follows readily that ∑X. is an R−complex and that there is an isomorphism $H_n(\sum X.) = H_{n-1}(X.)$ for each n. [15]

Definition(1.1.4): (15) Let R be commutative ring, and let f.: X.→Y. be a chain map. The mapping cone of f. is the sequence cone (f.) defined as follows

$$\text{Cone (f.)} = \cdots \longrightarrow \begin{matrix} Y_i \\ \otimes \\ X_{i-1} \end{matrix} \xrightarrow{\begin{pmatrix} \delta^y_i & f_{i-1} \\ 0 & -\delta^x_{i-1} \end{pmatrix}} \begin{matrix} Y_{i-1} \\ \otimes \\ X_{i-2} \end{matrix} \xrightarrow{\begin{pmatrix} \delta^y_i & f_{i-2} \\ 0 & -\delta^x_{i-2} \end{pmatrix}} \begin{matrix} Y_{i-2} \\ \otimes \\ X_{i-3} \end{matrix} \longrightarrow \cdots$$

In other words, we have

$\text{Cone}(f)_i = y_i \otimes x_{i-1}$

$\delta_i\text{cone}(f) = y_i \otimes x_{i-1} \rightarrow y_{i-1} \otimes x_{i-2}$

$$\delta_i \text{cone}(f) \begin{bmatrix} y_i \\ x_{i-1} \end{bmatrix} = \begin{bmatrix} \delta y i f i_{i-1} \\ o \delta x_{i-1} \end{bmatrix} \begin{bmatrix} y_i \\ x_{i-1} \end{bmatrix} = \begin{bmatrix} \delta_i^y (y_i) + f_{i-1}(x_{i-1}) \\ -\delta x_{i-1}\ (x_{i-1}) \end{bmatrix}$$

$$= \begin{bmatrix} \delta_i^y (y_i) + f_{i-1}(x_{i-1}) \\ -\delta x_{i-1}\ (x_{i-1}) \end{bmatrix}$$

Proposition (1.1.5): Let R be cumulative ring and let f.: X. → Y. be a chain map. The sequence cone (f.) is an R−complex

Proof:

It is straight forward to show that each map

$\delta_i^{\text{cone}(f)}$ is an R−module homomorphism. Since X. and Y.

are R−complexes, we have $\delta x_{i-2} \delta x_{i-1} = 0$ and

$\delta_{i-1}^y\ \delta_i^y = 0$ for each i. since f. is a chain map, we have

$\delta_{i-1}^y f_{i-1} = f_{i-2} \delta x_{i-1}$ for each i. These facts give the last equality in the following computation;

$$\delta_{i-1}^{\text{cone}(f)}\ \delta_{(f)}^{\text{cone}} = \begin{bmatrix} \delta_{i-1}^y & f_{i-2} \\ 0 & -\delta x_{i-2} \end{bmatrix} \begin{bmatrix} \delta_i^y & f_{i-1} \\ 0 & -\delta x_{i-1} \end{bmatrix}$$

$$= \begin{bmatrix} \delta\delta\delta_{i-1}\ {}_i^y\ {}_{i-1}^y & f_{i-1}\ \ f_{i-2}\ \delta x_{i-1} \\ 0 & \delta x_{i-2} \delta x_{i-1} \end{bmatrix} = \begin{bmatrix} o & o \\ o & o \end{bmatrix}$$

This shows that $\delta_{i-1}^{\text{cone}(f)} \delta_i^{\text{cone}(f)} = 0$ and hence the desired result. [15]

Section (1.2) Modules and linear maps:

Let M and N be two A-modules. An A-linear map f: M → N is also called a morphism of A-Modules. One denotes by Hom_A (M,N) the set of A-linear maps f: M → N. This is clearly a K-Module. In fact one defines the action of K on Hom_A (M,N) by setting; [λf (m)] = λ [f(m)]. Hence (λf) (am) = λf (am) = λaf (m) = aλf (m) = a (λf (m), and λf∈ Hom_A (M,N).

There is a natural isomorphism Hom_A (A,M) ≃ M; to u∈ Hom_A (A,M) one associates u (I) and to m∈M one associates the linear map A → M, a → am. More generally, if I is an ideal of A then Hom_A (A/I,M) ≃ {m∈M; Im = 0}

Note that if A is a k-algebra and L∈Mod (k), M∈ Mod (A), the k-module Hom_K (L,M) is naturally endowed with as structure of a left A-module.

If N is a right A-module, then Hom_K (N,L) becomes a left A-module. [6, 71. 72].

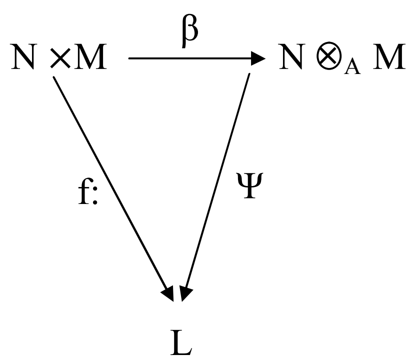
Example (1.2.6): Let W_n (k) denote as above the Wey algebra. Consider the left W_n (k)–linear map W_n (k) → k [x_1……,x_n].wn (k) ∈ p p (I) ∈ (k) {x_1…, x_n} This map is clearly surjective and its kernel is the left ideal generated by (δ_1……,δ_n). Hence, one has the isomorphism of left W_n (k) – modules;

$$W_n (k) / \sum w_n (k) \delta_i \xrightarrow{\sim} [x_1 \ldots\ldots, x_n].$$ [6, 71. 72].

Proposition (1.2.7): The map β is (A,K) – bilinear and for any k-module L and any (A,K) – bilinear map f: N X M → L, the map f factorizes uniquely through a k-linear map.

$$\Psi : N \otimes_A M \rightarrow L.$$

The proposition is visualized by the diagram



Consider an A liner map f: M → L. It defines a linear map id_N x f: N_X

M → N×L, hence a (A,K) – bilinear map N×M → $N \otimes_A L$,

and finally a k-linear map

$$Id_N \otimes f : N \otimes_A M \rightarrow N \otimes_A L.$$

One constructs similarly $g \otimes id_M$ associated to g: N → L.

There is natural isomorphism

$$A \otimes_A M \simeq M \text{ and } N \otimes_A A \simeq N.$$

Denote by BIL (N×M, L) the k-module of (A,K) –bilinear maps from N×M to L. One has the isomorphism

$$\begin{aligned} \text{BIL } (N\times M, L) &\simeq \text{Hom}_K (N \otimes_A M,L) \\ &\simeq \text{Hom}_A [\, M, \text{Hom}_K (M,L)\,] \\ &\simeq \text{Hom}_A [\, N, \text{Hom}_K (M,L)\,] \end{aligned}$$

For $L \in$ Mod (k) and $M \in$ Mod (A), the k-module $L \otimes_A M$ is naturally endowed with a structure of a left a-module. For M, N∈ Mod (A) and L∈ Mod (k), we have the isomorphism

$$Hom_A (L\otimes_K N, M) \simeq Hom_A [N, Hom_K (L,M)]$$

$$\simeq Hom_K [L, Hom_A (N,M)]$$

If A is commutative, there is an isomorphism:

$N\otimes_A M \simeq M\otimes_A N$ given by $n \otimes m \rightarrow m\otimes n$.

Moreover, the tensor product is associative, that is, if L, M, N are A-modules, there are natural isomorphism $L\otimes_A (M\otimes_A N)\simeq (L\otimes_A M) \otimes_A N$.

One simply writes $L\otimes_A M\otimes_A N$. [6, 71. 72].

Definition (1.2.1): A linear map from V to w is a function T: v →w with the following properties.

(i) T (μ + ν) = T (μ) + T (ν) for all μ,ν∈V

(ii) T (a ν) = a T (ν) for all ν∈V and a∈F

The set of all linear maps from V to W is denoted by Ψ (v, w)

Definition (1.2.2):

If T,S∈Ψ (ν,w) we define the product of S and T to be (ST) (ν) = S [T (ν)] for ν∈V as the product of S and T. [4]

Definition (1.2.3) [4] For T∈ (ν,w), the null space of T, is the subset of ν consisting of the vector that T mps to 0: null (T) ={ν∈V" Tv=0}

Definition (1.2.4): A linear map T: ν → w is called injective whenever μv ∈V. and Tu = Tv, we have u = v [4]

Proposition (1.2.8): Let T∈Ψ (ν,w), then T is injective if and only if null T. = {0}.

Proof:

suppose T is injective, since T (0) = 0, O∈ null T and so {0}∈null T. let v ∈ null T. Then T (0) = 0 = T (v) yields v = 0 because T is injective. Thus null T∈{0}. Therefore, null T = {0}. Then T(u-v) = conversely, assume null T = {0}. And let u, v∈V. If T u = Tv, and Tu – Tv = 0 which shows u-p ∈null T. Thus u = v and there fore, T is injective

Which shows u− ν ∈ mull T. Thus u = r and therefore, T is injective. [4.42. 70].

Definition (1.2.5): A linear map T: ν → w is called surjective if its range equals w.[4. 42.70].

Proposition (1.2.9): If T∈Ψ(ν,w), then range T is a subspace of w
proof:

by definition, range T = {Tν/ν∈V}∈w, and let $w_{1.}w_{2.}$ ∈ w range T.

$w_2 = T\nu_2$. By linearity of T, $w_1 + w_2 = T\nu_1 + T\nu_2 = T(\nu_1 + \nu_2)$ which shows $w_1 + w_2$∈ range T. For all w_1 , w_2∈ rang T.Let we range T and let K be a scalar. Then there exists ν∈V such that w = Tν. By linearity of T.

Kw = KTν = T (kν) which shows Kw∈ range T of all we range T and for all scalars K. Therefore, range T is a subspace ow. [4.42.70].

Definition (1.2.6): Let T T∈Ψ (ν,w) and let b_1 = {ν_1………ν_n} be a basis for ν and b_1= {w_1………w_n} be a base for w. Then the matrix of T with aspect to the bases b_1 and b_2 is

$$\begin{bmatrix} a_{i1} & \dots\dots\dots\dots & a_{in} \\ \vdots & & \\ \vdots & & \\ a_{mi} & \dots\dots\dots.. & a_{mn} \end{bmatrix}$$

where the a_i;∈F are determined by $T\nu_k = a_{1k}, w_1 + \dots\dots\ a_{mk}w_m$ for each k = 1. ………., n

Definition (1.2.7): Let b = {ν_1………ν_n} be basis for ν and let ν∈V.

We define the matrix of ν, denoted by M(ν), to be the n-by$_{-1}$ matrix [b_1,………b_n} denoted by

by $\nu = b_1\nu_1 + \dots\dots + b_n\nu_m$. [4.3. 42. 70].

Proposition (1.2.10): If T∈Ψ (ν,w), then m (Tν) = m(T) m(ν) for all ν∈V

Proof:

Let (ν_1………,ν_n) be basis of ν and (w_1………,w_n) be basis of w. If μ∈V, then there exists b_1……..…b_n∈F such that

$\nu = b_1\nu_1 + \dots\dots b_n\nu_n$ so that M(ν) =

$$\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

For each k, $1 \leq k \leq n$ we write $T\nu_k =$

$a_{1k}, w_1 + \ldots\ldots. a_{mk} w_m$ and so by definition of the matrix of a linear map T:

$$M = \begin{bmatrix} a_{i1} & \ldots\ldots\ldots\ldots & a_{in} \\ \vdots & & \\ \vdots & & \\ a_{mi} & \ldots\ldots\ldots.. & a_{mn} \end{bmatrix}$$

By linearity of T: $T\nu = b_1\, T\nu_1 + \ldots\ldots\ldots\ldots\ldots + b_n\, T\nu_n$

$$= b_1 \left(\sum_{j=1}^{m} a_{j1}\, w_j\right) + \ldots\ldots + b_n \left(\sum_{j=1}^{m} a_{jn} w_j\right)$$

$$= w_1 (a_{j1}\, b_a + \ldots\ldots.. + a_{1n}\, b_n) + \ldots. + w_m (a_{m1} b_1 + \ldots\ldots.. + a_{mn}\, b_n)$$

$$m(T\nu) = \begin{pmatrix} a_{j1} b_1 + \ldots\ldots\ldots a_{1n} b_1 \\ \vdots \\ \vdots \\ a_{m1} b_1 + \ldots\ldots.. a_{mn} b_1 \end{pmatrix} = m(T)\, m(\nu)$$

Where the last equality holds by definition of matrix multiplication. [4. 42. 70]

Definition (1.2.8): A linear map $T \in \Psi$ (ν,w) is called inventible if there exists a linear map $S \in L$ (w,ν) such that ST equals the identity map on V and TS equal the identity map on W.

Definition (1.2.9): Given $T \in \Psi(\nu,w)$. A linear map $S \in L(w,\nu)$ satisfying

ST = I and TS = I is called an increase of T. [4.1.2].

Proposition (1.2.11): A linear map is invertible if and only if it is injective and surjective.

Proof:

Suppose $T\in\Psi(\nu,w)$ in inventible with inverse T.

Let $u,\nu \in V$. If $Tu = T\nu$ then

$u = (T^{-1}) u) = T^{-1} (Tu) = T^{-1} (T\nu) = T^{-1}T(\nu) = \nu$

and so T is injective, if $w\in W$, then $\nu = T^{-1}w\in W$

with $T\nu = T^{-1}(T^{-1}w) = w$

Shows T is subjective.Assume T is injective and subjective. For each $w \in W$ assign $T(\nu) = w$, such $S(w) = \nu$ exists because T is subjective and is unique since T is injective. Then $T(\nu) = w$ shows $ST(\nu) = S(w) = \nu$ so that ST is the identity on V. Also, $TS(W) = T(SW) = T\nu = w$ shows Ts is the identity on w. thus S and T are linear since.

(i) If $w_{1.}w_2\in W$, then there exists a unique ν_1 and ν_2 such that $T\nu_1 = w_1$ and $T\nu_2 = w_{2.}$ $S(w_1) = \nu_{1.}$ and $S(w_2) = \nu_{2.}$ by linearity of T, $S(w_1) + S(w_2)$.

(ii) If $w\in W$ and $k\in F$ then there exists a unique $\nu\in V$, such that $T\nu = w$ and $S(w) = \nu$. By linearity of T, $S(Kw) = S(Kt\nu) = S(Kt\nu) = S(Tk\nu) = K\nu = Ks(\nu)$.

Therefore S is linear and is the inverse of T. [4.42].

Example(1.2.12): Show that every linear map from a one-dimensional vector space to it self is multiple location by some scalar. More precisely, prove that if dim V=1 and T∈L (ν,ν), then there exists a∈F such that Tν = aν for all ν∈V.

Solution:

Let {w} be a basis of ν, and let ν∈V.

Then there exists c∈F such that ν∈ cν.

Applying Tyields,

Tν = T (cw) = cTw = c (aw) = (ca) w = a (cw) = aν

Where Tw – aw since Tw ∈ν and {w} is a basis of ν [4].

Example (1.2.13) (Linear Extension): Suppose that V is finite-dimensional. Prove that any linear map on a subspace of ν can be extended to a linear map on ν.

In other words, show that if u is a subspace of ν and s∈Ψ (u,w), then there exists T∈Ψ (ν,w) such that Tu = su for all u∈U.

Solution:

Let $(u_{1.}\dots\dots\dots u_n)$ be basis of ν and extend this basis of u to a basis of ν, say $(u_{1.}\dots\dots\dots u_n, \nu_{1.}\dots\dots\dots \nu_n)$.

Define T as the linear extension of S, as follows T (u_i) for $1 \leq i \leq n$ and T $(\nu_j) = \nu_j$ for for $1 \leq j \leq m$.

Then for all u∈U,

$$T (u) = T (a_1u_1 + \dots\dots\dots + a_nu_n)$$

$= a_1Tu_1 + \ldots\ldots\ldots + a_nTu_n$

$= a_1Su_1 + \ldots\ldots\ldots + a_nSu_n$

$= S\ (a_1u_1 + \ldots\ldots\ldots + a_nu_n)$

$= S\ (u)$

Where $a_{1.}\ldots\ldots a_n \in F$. By definition of T, $T \in \Psi\ (\nu,w)$. [4.4.70]

Example (1.2.14): Suppose that T is a linear map form ν to F. Prove that if $u \in V$ is not in null T, then ν = null $T \otimes \{au/a \in F\}$

Solution:

Let $U = \{au/a \in F\}$. The following argument show

V = null T+U, and null $T \cap U = \{0\}$, respectively

(i) Let $\nu \in V$ with $T\nu = b$. Since $Tu \neq 0$ there is $au_1 \in U$ such that $Tu_1 = b$. Then we can write $\nu = u + (\nu - u_1)$, and $\nu - u_1 \in$ null T. This gives V = null T+U.

(ii) Let $\nu \in$ null $T \cap U$, there exists $a \in F$ such that ν = all and so $T\ (\nu) = aTu = 0$ since $T\nu \in$ null T.
Thus $a = o$ and so $\nu = o$ meaning null $T \cap U \subseteq \{o\}$.

Since $o \in$ null$T \cap U$, if follows null $T \cap U$ = [o]. [4.42. 70].

Example (1.2.15) [4]: Suppose that $T \in l\ (u,w)$ is injective and $(\nu_{1.}\ldots\ldots\ldots\nu_n)$ is linearly independent in ν, prove that $(T\nu_{1.}\ldots\ldots\ldots T\nu_n)$ is linearly independent in w.

Solution:

Suppose $a_1T\nu_1 + \ldots\ldots\ldots a_nT\nu_n = o$ in w where $a_{1.}\ldots\ldots\ldots,a_n \in F$.

Then by linearity of T, T $(a_1v_1+\ldots\ldots\ldots a_nv_n) = o$.

Since T (o) and T is injective,

$a_1v_1+\ldots\ldots\ldots a_nv_n = o$ since $(v_1\ldots\ldots v_n)$ is linearly independent $a_1 = \ldots\ldots\ldots\ldots = a_n = o$

Therefore, $(Tv_1\ldots\ldots\ldots,Tv_n)$ is linearly independent.

Example (1.2.16): Prove that if $s_1\ldots\ldots s_n$ are injective linear maps such that $s_1\ldots s_n$ makes sense, then $s_1\ldots s_n$is injective

Solution:

Suppose v and w are any vectors and $(s_1\ldots s_n)$ v = $(s_2\ldots\ldots s_n)$ w. Then by definition of composition, S_1 $(s_3\ldots\ldots s_n)$ v = S_1 $(s_2\ldots\ldots s_n)$ w, and since S_1 is injective $S_2(s_2\ldots\ldots s_n)$ v = S_2 $(s_3\ldots\ldots s_n)$ w. Since $s_2\ldots\ldots s_n$ are all injective v=w as desired showing $s_1\ldots\ldots s_n$ is injective. [4.42. 70].

Example (1.2.17): Suppose that v and w are finite-dimensional and T∈Ψ (v,w). Prove that there exists a subjective linear map from v on tow if and only if dim w $\leq$ dim v.

Solution:

Suppose T is a linear map of v onto w, then

Dim v = dim null T + dim range T.

Since dim null ≥ 0, dim v$\geq$ dim range T = dim w, since T is onto. Conversely, assume m = dim w $\leq$ dim v = n with bases of w.

Say ($w_{1.}$………,w_n) and of ν say ($\nu_{1.}$………ν_n). define T to be the linear extension of:

$$\begin{cases} T(\nu_i) = w_i \text{ if } 1 \le i \le m \\ T(\nu_i) = o_i \text{ if } i > m \end{cases}$$

Then T is surjective since: if w∈W then there exists

a_i∈F such that w = a_1w_1+……….a_mw_m = a_1 T (ν_1) +…….+

a_mT (ν_m) = T ($a_1\nu_1$+………. $a_m\nu_m$) showing every element in w is in range T. [4. 42. 70].

Example (1.2.18): Suppose that ν is finite-dimensional and if an only if there T∈Ψ (w,ν). Prove that T is Surjective exists S∈ L (w,ν) such that TS is the identity map on w.

Solution:

We will present two proofs

(i) Suppose s∈Ψ (w,ν)and Ts – Iw.

Let w∈W. Then ν = S (w) ∈V is such that T (ν) = Ts (w) = w and therefore T is surjective. Since ν is a finite-dimensional vectors space and T is linear map, w is finite-dimensional. Let ($T\nu_{1.}$….,$T\nu_n$) abasis for V. Since T is surjective, ($T\nu_1$… $T\nu_n$) spans w, also since T is surjective dim w ≤ dim ν = n.

Any spanning set reduced to a basis say ($T\nu_{1.}$.…..,$T\nu_n$) is a basis of w where m ≤ n. Define S as the linear extension of S ($T\nu_i$) = ν_i for each $1 \leq i \leq m$. Then, for all w∈W, $s(w) = s(a_1Tv_1+........+a_m sTv_m)$ $= a_1v_1 +.......+ a_mv_m = w$ for scalars $a_1........,a_m$∈F, and so Ts = Iw.

(ii) Suppose T is surjective. There exists a subspace U of V such that U∩ null T = {0} and rang T = {Tu: u∈V}. Define T_1: u → w by T_1u = Tu for u∈U. Notice T_1 is injective and surjective and SOT_1 has an inverse. Deine $s = T_1^{-1}$ we have $Tsw = T_1T_1^{-1}$ w = w for all w∈W. [4.1. 70].

Example (1.2.19): Prove that every linear map from mat (n;i;f) to mat (m;1.f) is given by matrix multiplication. In other words, prove that if T∈l (mat (n,1.F), mat (m,1.F) then there exists an m-by-n matrix A such that TB = AB for every B∈ mat (n,1.F).

Solution:

Let $(e_1.......,e_n)$ be a basis for mat (n,1.F) and let $(v_1........,v_m)$ be a basis for mat (m,1.F) for each k, there exists $a_1k,........, a_{mk}$∈F such that $Tek = a_1kv_1 +..........+ a_mkv_m$. Define the mxn matrix A as follows

$$A = [\, Te_1 \dots\dots\dots\dots Te_n]$$

If B∈ mat (n,1.F) there exists $b_1.........,b_n$∈F such that

$B = b_1e_1+............+ b_ne_n$; and thus

$TB = T(b_1e_1+............+ b_ne_n) = b_1Te_1+............+ b_nTe_n = BA$ as desired, notice the word "the" follows since $(v_1.........,v_n)$ is a basis. In other words one bases have been chosen, the matrix A is unique. [4. 2. 70].

Theorem (1.2.20): Let A and B be c-algebra. Then the linear map T given by.

(i) T is an isomorphism from M_n[B (A,B)] onto B [M_n(A,), B];

(ii) T maps P_k^n [A,B] into P_k [M_n(A,)B] and

T^{-1} maps P_{kn} [M_n(A,), B] into P_k^n [A,B] for each k = 1.2…

(iii) T is an isomorphism from P_∞^n [A,B] into P_∞ [M_n(A,) B],

Proof: It is clear that the linear map T is one-to-one.

(i) Let {E_{ij} | i.j = 1……n} be the standard matrix untis in M_n.

Then a ⊗ E_{ij} is the nxn matrix in M_n (A) with a at the (I,j) component and zeros elsewhere. For $\in$B [M_n (A), B], define the linear maps $Ø_{ij}$: A ⊗ B by $Ø_{ij}$ (a) = Ø ($a^n_{i,j=1}E_{ij}$) for a ∈A and 1 ≤ I,j ≤ n. Then we have

$T[(Ø_{ij}) \otimes^n_{i,j=1}] [(a_{ij})] = \sum_{i,j=1}^{n} Ø_{ij} (a_{ij}) = Ø [(a_{ij})]$, and so that the linear map T is onto.

(ii) Let T $[Ø_{ij}]^n_{i,j=1}$ be a k-multi-postive linear map from A to B.

For a while, we will use the notatian Ø instead of the linerar map [$Ø_{ij}$] from M_n (A) into M_n (B).

We define the linear map T : M_n(B)→ B by

$$T[(b_{ij}) = \sum_{i,j=1}^{n} b_{ij}, \quad [b_{ij}] \in M^n_{i,j=1}(B).$$

Then we have $T([\emptyset_{ij}]_{i,j=1}^{n})$ is a k-positive linear map of $M_n(A)$ into B. In order to show that $T^{-1}(P_{kn}[M_n(A), B]) \leq P_k^n[A,B]$ for each positive integer k, let $[\emptyset_{ij}]_{i,j=1}^{n} = T^{-1}(\emptyset)$ for any $\emptyset \in B[M_n(A), B]$. First, suppose that Ø is an n-positive linear maps of $M_n(A)$ into B. Define the linear maps $\emptyset_n : M_n(A) \rightarrow M_n[M_n(A)]$ by

$$\Psi_n[a_{ij}] = [a_{ij} \otimes E_{ij}]_{i,j=1}^{n}, \ [a_{ij}]_{i,j=1}^{n} \in M_n(A)$$

Then Ψ_n is completely positive, and we have $[(\emptyset \otimes In) o\Psi n]$

$$[(a_{ij})_{i,j=1}^{n}] = [(\emptyset \otimes In) [(a_{ij} \otimes E_{ij})_{i,j=1}^{n}]$$

$$= [\emptyset(a_{ij} \otimes E_{ij})]_{i,j=1}^{n} = \emptyset[(a_{ij})],$$

for each $[a_{ij}]_{i,j=1}^{n} \in M_n(A)$. Since $\emptyset \otimes In$ and Ψ_n are positive linear maps, $[\emptyset_{ij}]_{i,j=1}^{n} = (\emptyset \otimes I_n) o\Psi n$ is a multi-positive linear maps from A into to B. From the relation

$$[\emptyset_{ij}]_{i,j=1}^{n} \otimes I_k = [(\emptyset \otimes I_n) o\Psi n] \otimes I_k = (\emptyset \otimes I_{nk}) o (\Psi_n \otimes I_k),$$

We see that if $\emptyset \in P_{kn}[M_n(A), B]$ then $[\emptyset_{ij}] \in P_k^n[A,B]$.

(iii) Let map $(\emptyset_{ij})_{ij=1}^{n}$ pe a completely multi – positive linear maps from A in to B. Then $\emptyset \in P_\infty[M_n(A) M_n(B)]$. Since $T(\emptyset_{ij})_{i,j=1}^{n} = T o\emptyset$ and T is completely positive. The linear maps $T([\emptyset_{ij}])$ is completely positive. To show that $T(P_\infty^n [A,B]) P_\infty[M_n(A,)B]$, assume that $\emptyset \in P_\infty[M_n(A,)B]$. Since $([\emptyset_{ij}] \otimes = (\emptyset I_{nk}) o\Psi_n$ and the linear maps lpn is completely positive, the linear maps $[Ø_{ij}]$ is completely multi-positive, which completes the proof.Define the linear map S: M_n (A,B), → B (A_1 M_n (B) byS ($[Ø_{ij}]_{i,j=1}^{n}$) (a) = $[Ø_{ij}(a)]_{i,j=1}^{n}$ ∈ M_n [B(A,B) and a∈ A for each$[Ø_{ij}]$ [5].

Definition (1.2.10): Let $[Ø_{ij}]_{i,j=1}^{n}$ be an nxn matrix of linear map from a c*−algebra A in to a c*−algebra B. Then $[Ø_{ij}]_{i,j=1}^{n}$ may be considered as a linear map from M_n (B)by

(1.1) $[Ø_{ij}] : [a_{ij}] \rightarrow [\, Ø_{ij}(a_{ij})\,]_{i,j=1}^{n}$, $[a_{ij}] \in M_n$ (A)

We say that $[Ø_{ij}]_{i,j=1}^{n}$ is a multi-positive (respectively, k-multi positive or completely multi-positive) linear map from A into B if the linear map $[Ø_{ij}]$ in (1.1) is positive.

We denote by P_k^n [A,B] (respectively, P_k^n [A,B] the cone of all k-multi-positive linear maps. If n = 1. then P_k^n [A.B] coincide with P_k [A,B] and P_∞ [A,B], [5].

Proposition (1.2.21): Let $[Ø_{ij}]_{i,j=1}^{n}$ be multi-positive linear map from a quintal c^*−algebra A into a c^*−algebra B. Then we have:

(i) $Ø_{ij}(a^*) = Ø_{ij}(a^*)$ for each a∈A and i,j = 1. ……. n;

(ii) $[\, Ø_{ij}(a_i^* a^* a a_i)\,]_{i,j=1}^{n} \leq \| a \|^2 [\, Ø_{ij}(a_i^* a_i)\,]_{i,j=1}^{n}$ for each a_1………,a_n , a∈A.

Proof:

Let B (A,B) denote the space of all bounded linear maps from A into B. We define the linear map T: M_n [B (A), B] by

$$T\ ([\emptyset_{ij}])\ ([a_{ij}])\ =\ \sum_{i,j=1}^{n}\ \emptyset_{ij}\ (a_{ij})$$

for $[\emptyset_{ij}] \in M_n$ [B (A,B)] and $[a_{ij}] \in M_n$ (A). [5].

Theorem (1. 2.22): Let A and B be c^*−algebra. Then the linear map S given by (satisfies the following).

(i) S is an isomorphism from M_n[B (A,B)] onto B [A,M_n(B)];

(ii) S maps P_k^n [A,B] into P_k [A,M_n(B) and S^{-1} maps P_{kn} [A, M_n(B)] into P_k^n [A,B] for each k = 1.2. ……

(iii) S is an isomorphism from P_∞^n [A,B] onto P_∞ [A, M_n(B)].

Proof:

Clearly, S is one-to-one. Let $\psi \in$ B [A,M_n(B).

We denote by Ψ_{ij} (a) the (i,j) component of Ψ (a) $\in M_n$(B) for each $a \in A$ and i,j = 1………,n. Then

$[\ \psi_{ij}\ ]_{i,j=1}^{n} \in M_n$[B (A,b)] and

$$S\ (\ [\Psi_{ij}]\ )\ (a)\ =\ [\Psi_{ij}\ (a)\ ]_{i,j=1}^{n} =\ \Psi\ (a),\ a \in A$$

Therefore, it follows that S is onto.

(i) Let $[\Psi_{ij}]_{i,j=1}^{n}\ \in P_k^n$ [A,B] for each k = 1.2………. Define the linear map $\theta : A \rightarrow M_n(B)$ by

$$\theta (a) = \sum_{i,j=1}^{n} a \otimes E , a \in A$$

Then S ([Ψ_{ij}]) = $\Psi o \theta$, where Ψ denotes the linear map from $M_n(A)$ into $M_n(B)$. Since $\theta \in P_\infty$ [A, M_n (A)] and $\Psi \in P_k$ [M_n (A), M_n (B)], we see that S ([Ψ_{ij}]) is a k–positive linear map of A into M_n (B).

We shall show that S^{-1} (P_{kn} [A, M_n (B)] $\subset P_k^n$ [A,B] for each positive integer k. First assume that $\Psi \in P_n$ [A,M_n (B)].

(ii) Let $([\Psi_{ij}]_{i,j=1}^{n} = S^{-1} (\Psi)$. We define the linear map

$T_n : M_n [M_n (B) \rightarrow M_n(B)$ by

$$T_n (\sum_{i,j=1}^{n} x_{ij} \otimes E_{ij}) = \sum_{i,j=1}^{n} x_{ij} \otimes E_{ij} , \ x_{ij} \in M_n (B)$$

Where x_{ij} the (i,j) component of x_{ij}. Then T_n is completely positive. For each $[x_{ij}]_{i,j=1}^{n} \in M_n$ (A) we have

$$T_n o [\Psi \otimes I_n)] ([a_{ij}]_{i,j=1}^{n}) = T_n \sum_{i,j=1}^{n} [\Psi_{k1} (a_{ij})]_{i,j=1}^{n} \otimes E_{ij})]$$

$$= \sum_{i,j=1}^{n} \Psi_{ij}(a_{ij}) \otimes E_{ij} = \Psi ([a_{ij}]).$$

Thus, it follows that $\Psi = T_n$ o ($\Psi \otimes$ In). Since $\Psi \otimes$ In and $= T_n$ are positive linear, the linear map $[\Phi_{ij}]_{i,j=1}^{n}$ is a multi -positive linear map from A into B. By the equality

$$[\Psi_{ij}]_{i,j=1}^{n} \otimes I_k = (T_n \text{ o } (\Psi \otimes I_n) \otimes I_k = (T_n \otimes I_k) \text{ o } (\Psi \otimes I_{nk}),$$

We obtain that $[\Psi_{ij}] \in P_k^n$ [A,B] when ever $\Psi \in$ Pkn [A,Mn (B)], Now it only remains to establish the property.

(iii) Let $[\Psi_{ij}]_{i,j=1}^{n}$ be completely multi-positive linear map from A into B. Then $\Psi \in P_\infty [M_n (A), M_n (B)]$.

Since $[\Psi_{ij}]_{i,j=1}^{n} = \Psi \text{ o } \theta$ and θ is completely positive,

We have s ($[\Psi_{ij}] \in P_\infty [A, M_n (B)]$.If $\Psi \in P_\infty[A,M_n(B)]$,

then we got ($[\Psi_{ij}]$ $[A,B] \in P_k^n$ since $\Psi = T_n \text{ o } (\Psi I_n)$ and T_n

is completely positive. This completely, the proof. [5].

Corollary (1.2.23) [5]**:** The map V:B $[M_n (A), (B)] \rightarrow$ B $[A, M_n (B)]$ given by

$V =$ So $T^{-1}$is an isomorphism preserving the complete positivity.

Modules:

Definition (1.2.11):

A left module over the ring R (or left R−module) is an abelian group A together with a notion of multiplication by elements of R', which satisfies the following four axioms for all $a_1 a_{1.} a_2 \in A$ and $r_1 r_{1.} r_2 \in R$

(1) $(r_{1.} + r_2)\, a = r_{1.} a + r_2\, a$

(2) $(r_1 r_2) a = r_{1.}[r_2 (a)]$

(3) Ira = a

(4) $r (a_1 + a_2) = ra_1 + ra_2$

So a module is generalization of a vector space, with the coefficients of the elements being taken from a ring rather than a filed. A right module is defined similarly, only multiplication would then be from the right.

It is worth noting for future reference that an abelian group is the same thing as aZ-modules for n > 0 we, define na to be a + …+ a (n, times), if n = 0 we put na = 0, and if n < 0 we say

Na = − (−na) [1.25. 35].

Example (1.2.23): Let R be a commutative ring. The ring R is an R−is an R−Module.

More generally, the set.

$$R^n = \left\{ \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} \middle| \; r_1, \ldots\ldots, r_n \in R \right\}$$

is an R−module. For I = 1. …….., n we set

$$e_i = \begin{pmatrix} S_{1,j} \\ \vdots \\ S_{n,j} \end{pmatrix}$$

The with standard basis vector. Here, $S_{i,j}$ is the kronecken delta. The set $\{e_1, \ldots\ldots., e_n\}$ is a basis for R^n.

These modules satisfy our first universal mapping property which defines them up to isomorphism.

The proof of part (b) highlights the importance of the universal mapping property. [15].

Proposition (1.2.24): Let R be a commutative ring, and let n be positive integer. Let Mbe R−module, and let $m_1, \ldots\ldots., m_n \in M$

(a) There exists a unique R−module homomorphism

$f : R^n \rightarrow m$ such that $f(e_i) = m_i$ for each $i = 1..., n$

(b) Assume that M satisfies the following for every R−module P and for every sequence $P_1, \ldots\ldots., P_n \in P$, There exists a unique R−module homomorphism

$f : M \rightarrow P$ such that $f(m_i) = P_i$ for each $i = 1\ldots\ldots., n$.

Then $M \simeq R^n$.

Proof. (a) for the existence, let $f : R^n \rightarrow M$ be Given by

$$\sum_i r_i e_i \rightarrow \sum_i r_i e_i \rightarrow$$

The fact that $\{e_1, \ldots\ldots., e_n\}$ is a basis for R^n shows that

f is well-defined, it is straight forward to show that

f is an R−module homomorphism such that $f(e_i) = m_i$

for each $i = 1, \ldots, n$.

For the uniqueness, assume that $g: R^n \to M$ is an

R−module homomorphism such that $g(e_i) = m_i$

for each $i = 1, ..., n$. Since g is linear, we have

$$g\left(\sum_i r_i e_i\right) = \sum_i r_i\, g(e_i) = \sum_i r_i m_i = f\left(\sum_i r_i e_i\right).$$

Since $\{ e_1 ....e_n \}$ generates R^n, this shows $g = f$

(b) By assumption, there exists an R−module homomorphism

$f : M \to R^n$ such that $f(m_i) = e_i$ for each $i = 1,...,n$.

We claim that $gf = Im$ and $fg = I R^n$. (once this is shown, we will have $m \simeq R^n$ viaf.) The map $gf : M \to M$ is an R−module homomorphism such that

$$gf(m_i) = g[\, f(m_i)\,] = g(e_i) = m_i \quad \text{for } i = 1, ...,n.$$

Hence, the uniqueness' condition in our assumption implies

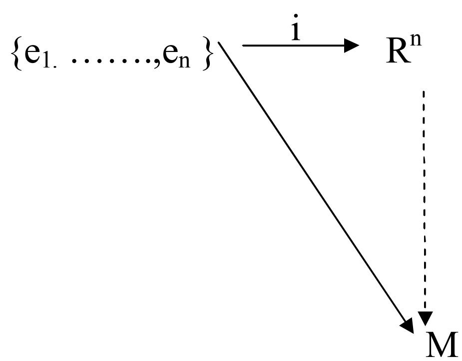
If = Im. The equality $fg = I R^n$ is rarified similar using to uniqueness from part (a).

Here is a useful restatement of proposition in terms of commutative diagrams. [15].

Remark (1.2.25): Let R be a commutative ring. Let $j : \{e_1 ....,e_n\} \to R^n$

denote the inclusion (of sets). For every function (map of Sets)

fo : $\{e_{1.} \ldots\ldots., e_n\} \rightarrow m$ there exists a unique R−module homomorphism $f : R^n \rightarrow m$ making following diagram commute



Hence some notation from linear algebra. [15].

Remark (1.2.26): Let R be a commutative ring integer nm $k \geq 1$ and let $h : R^k \rightarrow R^n$ be an R−module homomorphism. We can represent h by an n x k matrix with entries in R as follows. Write elements of R^k and R^n as column vectors with entries in R.

Let $e_{1.} \ldots\ldots., e_k \in R^k$ be standard basis. For $j = 1. \ldots\ldots\ldots., k$ write

$$h(e_j) = \begin{pmatrix} e_{1,j} \\ \vdots \\ a_{1,j} \\ \vdots \\ a_{n,j} \end{pmatrix}$$

Then is represented by the n x k matrix

$$\{h\} = (a_{ij}) = \begin{pmatrix} a_{1,1} \ldots a_{1,j} \ldots a_{1,k} \\ \vdots \\ a_{ij} \ldots a_{i,j} \ldots a_{i,k} \\ \vdots \\ a_{n,1} \ldots a_{n,j} \ldots a_{n,k} \end{pmatrix}$$

In the following since: For reach vector

$$\begin{pmatrix} r_1 \\ \vdots \\ r_k \end{pmatrix} \in R^k$$

We have $$h\begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} = h\left(\sum_j r_1 e_j\right) = \sum_j r_j h(e_j) = \sum_j r_j \begin{pmatrix} a_{1,j} \\ \vdots \\ a_{nj} \end{pmatrix} =$$

$$\begin{pmatrix} a_{1,j} \ldots a_{1,k} \\ \vdots \\ a_{n,,1} \ldots a_{n,k} \end{pmatrix} \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}$$

In particular, the image of h is generated by the columns of the matrix (a_{ij}). [15]

Section (1.3) Categories and Functions:

Definition (1.3.12):

Let Ψ be a class of objects A, B, C, ……. Together with a set of morphism Ψ (A,B), for each A,B ∈Ψ, and a law of composition.

Ψ (A,B XΨ (B,C) →Ψ (A,C)

(f,g) → g of

Then Ψ is a category if it satisfies the following axioms;

(1) The sets Ψ $(A_{1.}B_1)$, Ψ $(A_{2.}B_2)$ are disjoint unless $A_1 = A_{2.}$ $B_1 = B_2$.

(2) (Associative law of composition) given f ∈Ψ (A,B), g ∈Ψ (B,C), h ∈Ψ (C, D), then h o (g of) = hog) of

(3) (Existence of identities) to each object A∈Ψ there is a morphism I_A∈Ψ (A,A) such that, for any f ∈Ψ (A,B), g ∈Ψ (C,A), fI_A= f I_{Ag}= g. [13.25. 35].

Terminology:

* If f $\in\Psi$ (A,B) then we think of f as a function from A (the domain) to B (the codomain) or range), and write $f : A \to B$. (However, note that temorphism of a category may not always be functions in the usual sense).

* The morphism I_A, is uniquely determined by Axiom (3) and is called the identity morphism of A.

* A morphism $f : A \to B$ is called an is morphism if there exists a morphism $g : B \to A$ such that

$gof = I_A$, $fog = I_B$. In this case we write $g = f^{-1}$ [1.25. 35].

Examples (1.3.27): The following are examples of categories;

(1) Set; the objects are sets and the morphism are function.
(2) Mod_R: The objects are R−modules and the morphism are R−module homomorphisms.
(3) Grp : The objects are groups and the morphism are group homomorphisms.
(4) Top : The objects are topological spaces and the morphism are continuous maps.
(5) Vec: The objects are vector spaces and the morphism is linear transformations.
(6) Abgrp : The objects are a Belgian groups and the morphism are group homomorphism.
(7) Ho Top: The objects are topological space and the morphism are homogony classes of continuous functions

(note here that the morphism are not function).

(8) Diff : The objects are differentiable manifolds and the morphism are smooth maps. [1.25. 35].

A category C consists of:

(i) A set ob (c) whose elements are called the objects ofc.

(ii) For each x,y ∈ ob (c), a set Hom_c (x,y) whose elements are called the morphism from X to y.

(iii) For any X,Y,Z ∈ob (c), a map, called the composition, Hom_c (X,Y)x Hom_c (Y,Z) → Hom_c (X,Z).

(a) o is associative,

(b) for each X∈ ob (c), there exists id_x∈ Hom (X,X) such that for all f ∈ Hom_c (Y,X) and g∈ Hom_c (Y,X), $foid_x = f$, $id_x og = g$.

Note that id_x∈ Hom (X,X) is characterized by the condition in (b) [6,71.72].

Notation (1.3.28): One often writes X∈C instead o X∈ ob (c) and

f : X → Y (or else f : Y → X) instead of f ∈ Hom_c (X,Y). one call X the source and y the target of f.

A morphism f : X → Y is an isomorphism if there exists

g: X → Y such that fog = id_y and gof = id_x

In such a case, one writes $f : X \xrightarrow{\sim} Y$ or simply

$X \simeq Y$. of course g is unique, and one also denotes it by f^{-1}

A morphism f : X → Y is a homomorphism (resp, an epimorphism) if or any morphism g_1 and $g_{1=2}$.

$fog_1 = fog_2$ (resp, g_1 of = g_2 of) implies $g_1 = g_2$.

One sometimes writes f : X→Y else X →Y (resp, f :X →Y) to dente a monomorphism (resp, an epimorphism).

Two morphisms f and g are patrolled if they have the same sources and targets, visualized by f,g : X$\rightrightarrows$Y. one introduces the opposite category C^{op}.

ob (C^{op}) = ob (c), Hom_{cop} (X,Y) = Hom (Y,X), the identity morphism and the composition of morphism being the obvious ones.

A category ć is a sub category of c, denoted ć ⊂ ć,

If : ob (ć) ⊂ ob (c), $Hom_{ć}$ (X,Y) ⊂ Hom_c (X,Y) of any x,y ∈ć, the composition o in ć is induced by the composition in c and identity morphism in ć is a full sub category if for all X,Y∈ć, $Hom_{ć}$ (X,Y) = Hom_c (X,Y).

A category is discrete if the only morphism are the identity morphisms. Note that a set is naturally identities with a discrete category.

A category c is finite if the family of all morphism in c (hence, in particular, the family of objects) is a finite sel.

A category c is a groupoid if all morphism are is omorphism's.[6.71. 72].

Definition (1.3.14):

(i) an object P∈c is called initial if for all X∈c,Hom_c (p,x) ≃ {pt}. One often denotes by $Ø_c$ an initial object in c.

(ii) One says that p is terminal if p is initial in C^{op}, i.e. for all X∈c,Hom_c (x,p) ≃ {pt}. One often denotes by p^t_c a terminal object in c.

(iii) One says that p is a zeno-object if it is both initial and terminal. In such a case, one often denotes it by o. If c has a zero object, for any objects X, Y∈c, the morphism obtained as the composition X → o →Y is still denote by o: X → Y.

Note that initial (resp. terminal) objects are unique up to unique isomorphism. [6, 71. 72. 93. 94].

Example (1.3.29):

(i) In the category set, Ø is initial and {pt} is terminal.

(ii) The zero module o is a zero-object in mod (A).

(iii) The category associated with the ordered set (z ≤) has neither initial nor terminal object. [6, 71. 72].

Definition (1.3.15): Let c and ć be two categories, A functor F: c → ć consists of a map F: ob (c) → ob (ć) and for all X, Y∈c, of a map still denoted by F: Hom_c (X,Y) → $Hom_{ć}$ [F (X), F (Y) such that

F (id_x) = $id_{F(x)}$ F (fog) = F (f) o F (g).

A contravariant functor from c to ć is a functor from c^{op} to ć. In other words, it satisfies F (gof) = F (f) o F (g). If one wishes to put the emphasis the fact that a functor is not contravariant, one says it is covariant.

One denotes by op: $c \rightarrow c^{op}$ the contravariant functor, associated with $id_{cop.}$ [1. 25. 35].

Example (1.3.30): Let A be k-algebra

(i) Let M∈ Mod (A). The functor.
Hom_A (M,.): Mod (A)→Mod (K)
Associates Hom_A (M,K) to the A−module K and to an alinear map g: K → L it associates.

$$Hom_A(m,g): Hom_A(M,K) \xrightarrow{go} Hom_A(M,L)$$

$$(M \xrightarrow{h} K): \rightarrow (M \xrightarrow{h} K \xrightarrow{g} L).$$

Clearly Hom_A (M,.) is a functor from the category Mod (A) of A−module to the category Mod (K) of K−modules.

(ii) Similarly, for N∈Mod (A), the contravariant functor

$$Hom_A(.,N) : Mod(A) \rightarrow Mod(K)$$

Associate Hom_A (K,N) to the A−module K and to an a linear map g: k → L it associates. $Hom_A(g,N): Hom_A(L, N) \xrightarrow{og} Hom_A(K, N)$ $(L \xrightarrow{h} N) \rightarrow (K \xrightarrow{g} L \xrightarrow{h} N)$.

Clearly the functor Hom_A (M,.) commutes with products that is, $Hom_A(M, \Pi_i N_i) \simeq Hom_A(M, N_i)$ and the functor Hom_A (.,N) commutes with direct sums, that is,

Hom_A ($\otimes_i$ M_i, N) $\simeq \Pi_i$ Hom_A (M_i, N)

(iii) Let N be right A−module. Then N $\otimes_A$. : Mod (A) → Mod (K) is a functor. Clearly, the functor N $\otimes_A$. Commutes with direct sums, that is,

$$N \otimes_A (\otimes_i M_i) \simeq \otimes (N \otimes_A M_i),$$

and similarly for the functor. $\otimes_A$M. [6, 71. 72].

Definition (1.3.16): Let F : C → Ć be a functor

(i) One says that F is faithful (resp, full, resp, fully faithful) if for X,Y ∈c Hom_c (X,Y) → $Hom_{ć}$ [f (X), F(Y)] is injective (resp, surjective, resp, bijective).

(ii) One says that F is essentially sujctive if for each Y∈Ć there exists X∈c and an isomorphism F (X) ≃ Y.

(iii) One says that F is conservative if any morphism f: X →Y in c is an isomorphism as soon as F (f) is an isomorphism. [6.71. 72. 93. 94].

Example (1.3.31):

(i) Let C be a category and let X∈C. Then Hom_c (X, .) is a functor from C to set and Hom_c (.,X) is a functor from C^{op} to set.

(ii) The forgetful functor for: Mod (A) → set associates to an A−module M, and to a linear map f the map f. The functor for is faithful and conservative but not fully faithful.

(iii) The forgetful factor for: Top → set [diffident similarly as in (ii) is faithful. It is neither fully faithful nor conservative.

(iv) The forgetful function for set →Re is faithful and conservative.

One defines the product of two categories c and ć by: ob (c x ć) = ob (c) x ob (ć).

Hom_{cxc} [(X,X'), (Y,Y')] = Hom_c (X,Y) x $Hom_{\in}$ (X,Y').

A bi functor F: cxć→ć" is a functor on the product category. This means that for X∈c and X∈c and X'∈ć, F(X,.): ć→ "ć, and F (.,X¯): c →c" are ć, the diagram below commutes.

$$\begin{array}{ccc} F\,(X,X') & \xrightarrow{F\,(x,g)} & F\,(X,X') \\ {\scriptstyle F\,(y,x')}\downarrow & & \downarrow{\scriptstyle F\,(f,y')} \\ F\,(y,x') & \xrightarrow{F(y,g)} & F\,(y,y') \end{array}$$

In fact, (f,g) = $(id_{y,g})$ o $(f,id_{x'})$ = $(f,id_{y'})$ o $(id_{x,g})$. [6].

Example (1.3.32):

(i) Hom_c (.,.): C^{op} x c → set is a bi functor.

(ii) If A is a k−algebra, Hom_A (.,.): Mod $(A)^{op}$ x

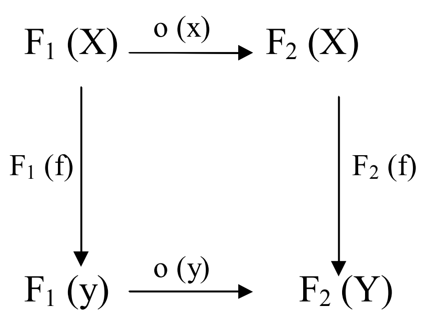
Mod (A) → Mod (K) are bifunctor. [6.71.72].

Definition (1.3.17): Let $F_1.F_2$ are two functors from c to ć. A morphism of functor:

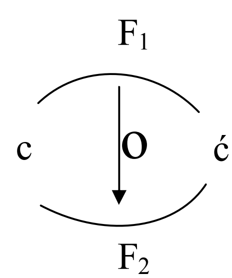
θ $F_1 \rightarrow F_2$ is the data for all X∈c of a morphism

θ (x): F_1 (X)→ F_2 (X) such that for all

F : x → y, the diagram below commutes;



A morphism of functor is visualized by a diagram



Hence, by considering the family of functors from c to ć and the morphism of such functors, we get a new category.

Notation (1.3.33): We denote by f ct (c,ć) the category of functors from c to ć. One may also use the shorter notation ($ć^c$) [6,71.72].

Theorem (1.3.34): The functor F: c →ć is an equivalence of categories if and only if F is fully faithful and essentially surjective.

Proof:

If two categories are equivalent, all results and concepts in one of them have their counter parts in the other one. This is why this notion of equivalence of categories plays an important role in Mathematics. [6.71.72].

Example: (1.3.35) Our most important example of a functor is Hom (A,B), the set of all R−module homorphism from A to B. This is easily seen to be an abelian group under addition. For any $A \in Mod_R$ we can define a covariant functor Hom (A,−): $Mod_R \rightarrow$ AbGrp in the following way.

(i) To every R−module B Hom (A,−) assigns the abelian group Hom (A,B).

(ii) To every R−module homomorphism $f : x \rightarrow y$ is assigns the morphism Hom (A, f) = f* : Hom (A,X) → Hom (A,Y) given by f* (g) = fog.

It can be easily shown that this definition does indeed satisfy the two conditions given in the definition of functor.

Similarly, we can construct the contravariant functor Hom (−,B) in the abvious way. [1.25 35].

Example (1.3.36): One of the motivating examples of a functor is provided by the fundamental group.

To be careful, we must define the category Top* to be the category of pointed topological space (X,x).

Where X is a topological space and x $\in$ X is a chosen base point. A morphism f: (X,X) → (Y,Y) is a continuous map f: X → Y such that f (X) = Y.

The fundamental group Π_1 ($X_{1.}$x) is the set of homotopy classes (equivalence classes under continuous deformation) of paths starting and ending at x, and every

f : (X,x) → (Y,y) induces a homomorphism of groups

f* : Π_1 (X,x)→Π_1 (Y, y) thus Π_1 is a functor from the category Top* into the category Grp. [1.25. 35].

Example (1.3.37): Let Ψ be the category set and let Q be the category Grp, with x a set and y a group. Let G: Grp → set be the forgetful functor, which associates with each group its underlying set. Then G has a left adjoint given by the free functor F: set → Grp which associates to every set the free group generated by words of that set. Then there is a natural equivalence which associates to any function x → Gy the corresponding homomorphism FX → Y. [1.25. 35].

Definition (1.3.18): Let c be a category. One defines the categories

$\hat{c}$= Fct (C^{op}, set), $\check{c}$= Fct (C^{op}, set^{op}) and the functors

$h_c : c \to \hat{c}$, X →Hom_c(.,x)

$k_c : c \to \check{c}$, X →Hom_c(x.,) since there is a natural equivalence of categories

$$\check{c} \simeq c^{op,\ \wedge,op}$$

We shall concentrate our study on $C^{\wedge}$. [6,71.72].

Proposition (1.3.38): (The Yoneda Lemma)

For $A\in \hat{c}$ and $X\in c$, there is an isomorphism

$Hom_{c^\wedge}$ (h_c (x),A) $\simeq$ A (x), functoral with respect to X and A.

Proof:

One constructs the morphism

Ψ : $Hom_{c^\wedge}$ (h_c (x),A) $\rightarrow$ A (x) by the chain of morphism

$Hom_{c^\wedge}$ (h_c (x),A) $\rightarrow Hom_{set}$ (Hom_c (X ,x), A (x) $\rightarrow$ A (k), where the last map is associated with id_x.

To construct Ψ: A (x) $\rightarrow$ $Hom_{c^\wedge}$ (h_c $(x)_c$, A), it is enough to associate with $S\in A$ (x) and $Y\in c$ a map from Hom_c (y,x) to A (y).

It is defined by the chain of maps Hom_c (y,x) $\rightarrow$

Hom_{set} [A (x), A (y)] $\rightarrow$ A (y) where the last map is associated with $S\in A$ (x).

One checks that ψ and ψ are inverse to each other [6.71.72].

Corollary (1.3.39): The functor h_c is fully faithful.

Proof:

For X and Y in c, one has $Hom_{c^\wedge}$ [h_c (X), h_c (Y)] $\simeq$ h_c (Y) (X) = Hom_c (X,Y).

One calls h_c the Yoneda embedding.

Hence, one may consider c as a full sub category of $c^{\wedge}$. In particular, for X∈c, h_c (x) determines X up to unique isomorphism, that is, an isomorphism $h_c(X) \simeq h_c(Y)$ determines a unique isomorphism $X \simeq T$, [6.71.72].

Corollary (1.3.40): Let c be a category and let f: x → y be a morphism in c.

(i) Assume that for any Z∈c, the map Hom_c (Z,X) of Hom_c(Z,Y)is bijective. Then f is an isomorphism.

(ii) Assume that for any Z∈c, the Hom_c (Z, y) $\xrightarrow{\text{of}}$ Hom_c (X,Z) is bijective. Then f is an isomorphism.

Proof:

(i) By the hypothesis, h_c (f) : h_c (X) → h_c (Y) is an isomorphism in $c^{\wedge}$. Since h_c is fully faithful, this implies that f is an isomorphism.

(ii) Follows by replacing c with C^{op}. [6.71.93.94].

Section (1.4) Representable functors, adjoint functors:

Definition (1.4.18):

(i) One says that a functor F from C^{op} to set is representable if there exists X∈c such that F (y) $\simeq Hom_c$ (Y,X) functorially in Y∈c. In other words. F $\simeq$ h_c (x) in $c^{\wedge}$. such an object X is called a representative of F.

(ii) Similarly, a functor G : c → set is representable if there exists X∈c such that G (y) $\simeq$$Hom_c$ (X,Y) functorially in Y∈c.

It is important to notice that the isomorphism above determine x up to unique isomorphism.

Representable functors provides a categories language to deal with universal problems. Let us illustrate this by an example. [6.71.72].

Example (1.4.41): Let A be a k−algebra. Let N be a right A−module, M a left A−module and La K−module. Denote by B (NxM,L) the set of A,K bilinear maps from N×M to L. Then the functor F: L → B (N×M,L) is representable by $N \otimes_A M$ [6.71.72].

Definition (1.4.19): Let F: c → ć and G: ć → c to be two functors. One says that (F,G) is a pair of adjoint functors or that F is a left adjoint to G, or that G is a right adjoint to F if there exists an isomorphism of bifunctors:

$$Hom_{ć} [F (.),) \simeq Hom_c [., G (.)]$$

If G is an adjoint to F, then G is unique up to isomorphism. In fact, G (y) is a representable of the functor x →Hom_c [F (x), y].

The isomorphism is given isomorphism.

$Hom_{ć}$ [Fo∈ (.), .] ≃ Hom_{c} [G (.) G (.)],

$Hom_{ć}$ [F (.) F (.)] ≃ Hom_{c} [., GoF (.)],

In particular, we have morphism x → GoF (x), functorial in X∈C, and morphism GoF (y), → y functorial in y∈Ć. In other words, we have morphism of functors.

FoG → $id_{ć}$, id_{c}→ GoF, [72.6.7].

Example (1.4.42):

(i) Let X∈ set using the bijection, we get that the functor Hom_{set} (x,.) : set → set is right adjoint to the functor .xX,

(ii) Let A be a K−algebra and let L Mod (k). using the first isomorphism in (i), we get that the functor Hom_{k} (L,.): Mod (A) to Mod (A) is right adjoint to the functor . $\otimes_{k}$L.

(iii) Let A ba K−algebra. Using the isomorphisms in (ii) with N = Am we get that the functor for: Mo (A) → Mod (K) which, to an A−module, associates the under lying K – module is right adjoint to the functor A : Mod (K) → Mod (A) (extension of scalars). [6.71.72. 93].

Section (1.5) Free Modules:

Definition (1.5. 20): F∈ R−module if free ↔ F is isomorphic to a sum of copies of R. That is $F \cong \sum_{i \in J} F_j$ where $F_j \cong R$.

$F_j \cong R \leftrightarrow F_j = Rx_j$ for some $x_j \in F_j$ with $rx_j = o$ if and only if r = o.

If so then $X := \{ x_j : j \in J \}$ is called an R−basis of F. Every x∈F has a unique expression $X = \sum x_i r_j$ with $r_i \in R$ and alost all $r_i = o$. [3].

Definiton (1.5.21): An R−module M is called a free Modul if M a dmit a basis, In other words M is free if there exists a subset of M such that M is generated by S, and s is a linearly independent set we regard (o) as a free Module whose basis is the empty set [2].

Theorem (1.5.43): Let M be fee R−module with a basis $(e_1, \ldots\ldots, e_n)$ then $M = R^n$

Proof:

Define a mapping $\theta : M \rightarrow R^n$ by $\varnothing\ (\sum_{i=1}^{n} r_i e_j) = \sum_{i=1}^{n} r_i f_j$ where if = (0, …1.0,…0)∈ R^n because $\sum_{i=1}^{n} r_i e_i = \sum_{i=1}^{n} r_i e_i$ implies by the linear independence of ei's Ri = ri¨ for all i, Ø is well defended.

Ifm = $\sum_{i=1}^{n} r_i e_i$, m'' =$\sum_{i=1}^{n} r_i e_i$ and Ø (m + M) = Ø (m + Ø) (m) and Ø (rm) = rØ (m). Further, if Ø (m) o, then $r_i f_j = o$, This implies (r,………,m) = o, an hence each $r_i = o$. Proving that Ø is (1-1). It is clear that Ø is onto, and hence, Ø is an isomorphism [2].

Lemma (1.5.44): Every M∈R−mod is ahomomrphis image of a free module.

Proof:

Take F free, with basis indexed by all elements of M,

say $X = \{ x_m : m \in M \}$ (such F exists….).

Define $f : X \to M$ (a map on a set) by

$f\ (xm) := m$. this is already onto.

Take $f : F \to M$ be the R−homomorphism extending f.

A free resolution of M is an exact sequence.

$$\ldots\ldots\ldots \to F_n \to F_{n-1} \to F_o \to M \to o$$

In which each F_i is a free R−module every M has a free resolution. [3].

Proposition (1.5.45): Suppose $B \xrightarrow{B} C \to o$ is exact. If F is free and $\alpha : F \to C$ is any R−module homomorphism then there exists V : $F \to B$ with $\alpha = Bv$. (v not unique in general].

Proof:

Let $X = \{X_i\}$ j∈J be anR−basis for F.

For each j there is some $b_j \in B$ such that $B\ (b_j) = \alpha\ (x_i)$.

Then there is a map (on sets) $\varnothing : X \to B$ such that $\varnothing\ (x_j) = b_j$ [by axiom of choice].

There is an R−module homomorphism.

γ. F → B with $\gamma (x_j) = Ø (x_j)$ for all j

$\alpha = \beta\gamma$: check on basis x. [3].

Theorem (1.5.46): For any set x, the sub-module $R^{(X)}$ of the function module $R^{(X)}$ which is spanned by the free R–module on the set {Ex/x∈X}.

Proof:

First we given another description of the sub–module $R^{(X)}$. Define the "support" of any function f : X → R to be the subset.

Supp (f) {x:x∈X} and f (x) ≠ o.

of x, the point wise definitions of the module operations in the function module $R^{(X)}$ show that; supp (f + g)⊂ (supp f) U (supp), supp (kf) ⊂ supp (f).

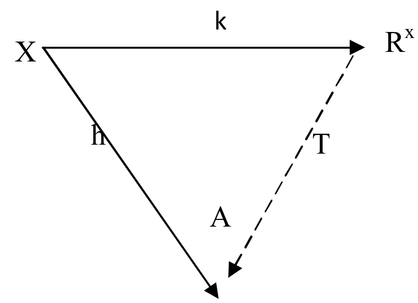
Therefore the set of all those functions f: x → R which have finite support is closed under sum and under all scalar multiples, so sub-module of $R^{(x)}$.

Now each function Ex has support the one-element set {x}, so Ex∈D if f is any function with finite support, say $\{x_1, \ldots\ldots, x_n\}$ then f is determined by its values $f(x_i) Ex_i + \ldots\ldots\ldots. f(x_n) Ex_n$ (Both functions displayed are equal at all x_i and hence at all x).

Therefore the sub-module$R^{(x)}$ spanned by all the element Ex is identical to the sub-module D of all functions of finite support.

The assignment X → EX gives a function k : X → $R^{(x)}$.

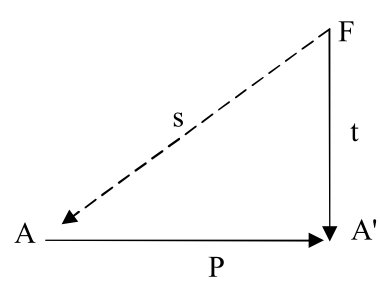
To show that R^x = D is free on {Ex/x∈X} we must show that to each function h on x to an R−module A, there is exactly one R−linear function T: $R^x \rightarrow$ A with Tok = h, as displayed in the diagram.



Now to k = h states that t (Ex) = h (x) for all x, so any such linear map t must have t (f) = f (x_1) (hx_1) + ………+ { f (x_n) (hx_1)] for each function of finite support, as displayed above. This shows that t is unique if t exists; conversely one may verify as before that the function t: $R^x \rightarrow$ A defined by this formula is indeed R−linear, hence R^x is free [2].

Theorem (1.5.47): If P : A → A' is an epimorphism of R−modules, each morphism t : F → A' with domain a free R−module F can be written as a composite t = pos for some morphism S : F → A of R−module.

Proof:



F = R^x is free on some set x and the horizontal map p is an epimorphism; we wish to find a linear maps which makes the diagram commute, ad if pos = t (we also say that t "factors through" p or that t "lifts" to s) for each x∈X, t (Ex) is an element of A', since the epimorphism p is subjective as a function, we can choose to each x∈X some elements Ex∈A with p (Ex) = t (Ex). Since F is free, there is a linear map S: F → A with S (Ex) = Ex for each x. Then (pos) (Ex) = t (Ex) for each free generator Ex, so the composite pos must be t, as desired [2.3].

Section (1.6) Grothendieck's theorem:

Let R be a commutive ring and let us consider the scheme.

$$[\ P'\ R = \text{Proj}\ R \mid x_0, x_1 \mid, Q\ \},$$

We see that the category of quasi-coherent sheaves over the projective line can be considered in terms of certain representation of the quiver . →. ←.

For if we take the basis affine open sets D + (x_0) D+ (x_1) YD+ (x_0) D+ (x_1)∩ D ± (x_0 x_1) covering the projective line. W have the inclusions.

$$D^{+}(x_0) \leftarrow D^{+}\ (x_0\ x_1) \rightarrow D^{+}(x_1)$$

So applying the structure sheaf Q associated to p' (R) we get.

$$Q\ [D^{+}(x_0) \rightarrow Q\ (D^{+}(x_0\ x_1) \leftarrow Ø\ (D^{+}(x_1)\ ],$$

but now, $Q\ [D^{+}(x_0)\ ] = R \mid x_0, x_1 \mid (x_0)\ Q\ (D^{+}x_0\ x_1)\ ] =$

$R \mid x_0, x_1 \mid (x_0\ x_1)$ and $Q\ [D^{+}(x_1)\ ] = R \mid x_0, x_1 \mid (x_1)$

So we may identify

$R \mid x_0, x_1 \mid (x_0)$, $R \mid x_0, x_1 \mid (x_{0\,,} X_1)$, and $R \mid x_0, x_1 \mid (x_0)$

With the rings $R \mid x_1/x_0 \mid$, $R \mid x_1/x_0, x_0/x_1 \mid$, $R \mid x_0/x_1 \mid$ respectively. So if we call $x= x_1/x_0$ we follow that the schem [P'(R),Q] can be seen as a representation of the quiver . →. ←., given by $R\ \{x\} \rightarrow R\ \{s, x^{-1}\} \leftarrow R\ \{x^{-1}\}$

Hence, a sheaf of quasi-coherent modules F on P' (R) is a sheaf of Q−modules, that is, are presentation of the form

$$M \xrightarrow{\ f\ } P \quad \xleftarrow{\ \partial\ } \quad N,$$

With $M \in R \mid x \mid$−mod and $N \in R \mid x^{-1} \mid$−mod and $p \in R \mid x, x^{-1} \mid$ −mod, and with f a $R \mid x \mid$ −linear map and g a $R \mid x^{-1} \mid$ −linear,

Satisfying the quasi-coherence property, that is $F \mid \text{spec} - R \mid x \mid \cong M$, $F \mid \text{space } R \mid x^{-1} \mid \cong N$ and $F \mid \text{space } R \mid, x^{-1} \mid \cong P$. since M and N are also quasi-coherent, it follows that

$M \mid \text{spec } R \mid n, x^{-1} \mid \cong P\tilde{N} \mid \text{spec } R \mid, \bar{x}^{-1} \mid$,

So $p = \{(\text{spec } R \mid x, x^{1\text{-}} \mid, \bar{p} \cong \acute{M}\ (\text{spec } R \mid x, x^{-1} \mid = S^{-1}m$

$P = [\ (\text{spec } R \mid, x^{-1} \mid , p\ ] \cong N\ (\text{spec } R \mid x, x^{-1} \mid = T^{-1}$ m being $S = \{\ 1.\ x, x^{-2.} \ldots \}$, $T = \{\ 1.\ x^{-1.}\ x^{-2}\ ,,,,,\}$ and the isomorphism are just $S^{-1}f$ and $T^{-1}S$.

Considering the category in this way we are able to give a short and elementary proof of Grothendieck's theorem.

We present some well-knows results concerning quasi-coherent sheaves over p' (R) that are easy to prove in terms of our representation. We shall use these later in proving Grothendieck's theorem.

Some of the results presented now are included.

We begin by classifying all representation of the form

$$R \mid x \mid \xrightarrow{f} R \mid x, x^{-1} \mid \xleftarrow{S} R \mid x^{-1} \mid. \ [12.83.\ 87].$$

Proposition (1.6.48):

Each representation of the form $R \mid x \mid \xrightarrow{f} R \mid x, x^{-1} \mid \xleftarrow{s}$ $R \mid x^{-1} \mid$ is isomorphic to some $R \mid x \mid \rightarrow R \mid x, x^{-1} \mid \xleftarrow{x^n} R \mid x^{-1} \mid$, with $n \in Z$.

Proof. We may define a pai of adjoint functor (D,H) between the categories of R {x}. Modules and Q_{co} [P' (R)] defined by D (L) $= \xrightarrow{i} S^{-1} L \xleftarrow{id} S^{-1} L$ is a right adjoint of H:

Qco [p' (R)] → R {x} –mod, given by H (m → p ← B) = M.

Then, by using this, we have

$$\begin{array}{ccccc}
R \mid x \mid & \xrightarrow{f} & R \mid x, x^{-1} \mid & \leftarrow & R \mid x^{-1} \mid \\
\downarrow id & & \downarrow h^{-1} & & \downarrow h^{-1} og \\
R \mid x \mid & \xrightarrow{f} & R \mid x, x^{-1} \mid & \xleftarrow{g} & R \mid, x^{-1} \mid
\end{array}$$

Where $h = (s^{-1} f)^{-1.}$ and from this it follows

$$\begin{array}{ccccc}
R \mid x \mid & \longrightarrow & R \mid x, x^{-1} \mid & \xleftarrow{d} & R \mid x^{-1} \mid \\
\downarrow id & & \downarrow h & & \downarrow id \\
R \mid x \mid & \rightarrow & R \mid x, x^{-1} \mid & \rightarrow & R \mid x, x^{-1} \mid
\end{array}$$

(where $d=h^{-1} og=(s^{-1}f)of$). Then, since columns are isomorphism we deduce that

$(R \mid x \mid \xrightarrow{f} R \mid x, x^{-1}\mid \xleftarrow{s} R \mid x^{-1}\mid) \cong (R \mid x \mid \rightarrow R \mid x, x^{-1}\mid \xleftarrow{d}$

$R \mid x^{-1}\mid)$

(notice $R \mid x\mid \rightarrow R \mid x, x^{-1}\mid \xleftarrow{d} R \mid x^{-1}\mid$ in Q_{co} [p' (R)] because T^{-1} $d = S^{-1}$ f o T^{-1} g is an isomorphism). But if T^{-1} d:

$R \mid x, x^{-1}\mid \rightarrow R \mid x, x^{-1}\mid$ is an isomorphism, T^{-1} d (1) must be a unit of $R \mid x, x^{-1} \mid$, so $d = u,x^n$, with u∈Z; and n∈Z; in fact we can supposed $= x_n$ because $R \mid x \mid \rightarrow R \mid x, x^{-1}\mid \xleftarrow{x^n} R \mid x^{-1} \mid$

and $R \mid x \mid \rightarrow R\mid x,x^{-1}\mid \xleftarrow{x^n} R \mid x^{-1} \mid$ are obriously isomorphic, finally, we see that x^n and x^m give isomorphic representation if, and only if, n = m, if

$R \mid x \mid \rightarrow R \mid x, x^{-1}\mid \xleftarrow{x^n} R \mid x^{-1}\mid$ and $R \mid x \mid \rightarrow R \mid x, x^{-1}\mid \xleftarrow{x^n}$

$R \mid x^{-1} \mid$ are isomorphic, we have a diagram

$$\begin{array}{ccccc} R \mid x \mid & \longrightarrow & R \mid x, x^{-1} \mid & \xleftarrow{x^n} & R \mid x^{-1} \mid \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ R \mid x \mid & \rightarrow & R \mid x, x^{-1} \mid & \xleftarrow{x^n} & \alpha R \mid x, x^{-1}\mid \end{array}$$

With commutative squares, But it is dear that α = z,B = k, n and γ = Z, for some 0 ≠ k, z, Z∈R, | ∈Z, and then, by the commutatively of the first square, it follows k, x^1 = Z, sok = z and I = 0, and from the second square, z x^n = z x^m, so n = m,

In terms of quasi – coherent shears, are presentation $R \mid x \mid \rightarrow$

$R|x, x^{-1}| \xleftarrow{x^n} R|x^1|$, with n∈Z, corresponds to the (unique) line bundles of degree n over p, which is denoted by Q (n). So this justifies the following definition.

Definition (1.6.22): A representation $R | x | \rightarrow R | x, x^{-1}| \xleftarrow{x^n} R |x^{-1}|$, n∈Z is denoted by Q (n). [12.83. 87].

Proposition (1.6.49):

$$Q(n) \otimes Q(m) \cong Q(n+m).$$

Proof:

This is obvious because, in general, if A,B are R−modules,

$T_A^{-1} T_R^{-1} T_B^{-1} T^{-1}(A \cong B)$, for any multiplicatively closed set T, and this isomorphism is precisely $a/t \otimes b/t' \rightarrow (a \otimes b)/tt'$

(notice that $S^{-1} R | x | = R | x, x^{-1}|$.

Another well-known result which is easy to prove under our notation is the following.

Proposition (1.6.50): Let m,n∈Z be two integers. Then Hom (Q (m), Q (n) is trivial if m > n and is equal to the space of polynomials of degree n-m whenever $m \leq n$.

Proof:

Let (f. g, h) be a morphism between Q (m) and Q (n), so f is an R | x | −morphism, g is an $R | x^{-1.} x |$ −morphism and h is an $R | x^{-1} |$ −morphism. Then we must have $x^{m-n} g(I) = h(I) \in R | x^{-1} |$, hence $m-n \leq 0$ and g (I) = f (I) is a polynomial of degree less than or equal to n−m which determines uniquely the morphism Q (m) → Q (n). [12.83. 87}.

Corollary (1.6.51): The space of 0−cohomologies of Q(n) is trivial if N < 0 and is the space of polynomials of degree less than or equal to n wherever $n \geq 0$.

Proof:

This is obvious, by noticing that H^o [Q(n) = Hom [Q (o), Q(n)], and applying proposition [1.6.50].

It is very well know the proposition that vector bundles over the projective line, $p^{1.}$ and direct sums of line bundles in an essentially unique way (Grothendieck's theorem.

The representation of Q co [$p^1(k)$] which correspond to vectors bundles are M → P ← N, with M, N finitely generated and free (for example $k \mid x \mid \to k \mid x^{-1.} x \mid \quad k \mid x^{-1} \mid$ $\overset{x}{\leftarrow}$). In this secton, we are going to prove this theorem, in terms of representation of the quiver . → . ←. [12.83. 87].

Theorem (1.6.52) (Grothendieck): Each representation of Qco [p' (k)] of the form M → P ← N, with M,N finitely generated and free, is direct sum of

$$Q(j_i) \equiv k \mid x \mid \to k \mid x^{-1.} x \mid \overset{x^h}{\underset{x^h}{\leftleftarrows}} k \mid x^{-1} \mid ,$$

$j_i \in Z$ i = 1. ………., n with $j_1 \leq j_2 \leq \ \dots \leq j_n$ moreover the integers $(j_1, \dots, j_n)$ are uniquely determined.

Proof:

First of all note we can suppose $M \xleftarrow{x^h} P \xleftarrow{s} N$, with

$M = k \mid x \mid^n$, $P = K \mid x^{-1.} x \mid^n$, $N = K \mid x^{-1} \mid^n$, is of the form $M \rightarrow P \xleftarrow{h} N$,

By using the right adjoint functor .

Let p be the nxn matric associated to h,p = (P_{ij}), $P_{ij} \in K \mid x^{-1.} x \mid$. We known the $k \mid x \mid$ −linear map h has a unique extension to a $k \mid x^{-1.} x \mid$ −isomorphism between $k \mid x^{-1.} x \mid^n$, so det (p) is a unit of $k \mid x^{-1.} x \mid$,

that is, det (p) = $ux^{1.}$ $I \in Z$, $0 \neq u \in R$, in fact, we can suppose det (P) = x^i , $I \in Z$ changing abase of N corresponds to our column operations on p, so we can assume p is a diagonal matrix,

This proves that each of our representation is a direct sum as desired.

To get uniqueness we follow an argument given by Grauert and Remmert in theorem.

Let us suppose we have two de compositions.

$$Q(j_1) \otimes \ldots\ldots\ldots \otimes (j_n) \qquad Q(k_1) \otimes \ldots\ldots\ldots \otimes Q(k_n)$$

With $j_1 \leq \ldots\ldots\ldots\ldots \leq j_n$ and $k_1 \leq \ldots\ldots\ldots\ldots \leq k_n$. Let I be the first index or which $j_i \neq k_i$ and suppose $j_i < k_i$. By proposition (1.6.49) we have

$$Q(j_i - j_1) \otimes \ldots\ldots\ldots \otimes Q \otimes Q(j_i - j_{i+1}) \otimes \ldots\ldots \otimes Q(j_i - j_1)$$

$$\cong Q(j_i - k_1) \otimes \ldots\ldots\ldots \otimes Q(j_i - k_i) \otimes Q(j_i - j_{i+1}) \otimes \ldots\ldots \otimes Q(k_n)$$

Then the number of Q (t)'s with $t \geq o$ is different in both sides, which leads to a contradiction, by corollary (1.6.51), with the dimension of o−cohomologies in both sides. [12.83. 87].

Remark (1.6.53): Straight forward modifications of the proof of theorem (1.6.52) allow to prove the analogous result for, a non commutative case, that is for the decomposition of a "non commutative" vector bundle of the form

$$k \mid x;\sigma \mid \xrightarrow{f} k \ |x, x^{-1.} x; \sigma \mid \xleftarrow{s} k \mid x^{-1.}\sigma| ,$$

Where $\sigma : k \rightarrow k$ is an auto orphism. [12.83 → 87].

Chapter Two

Limits

The aim of this chapter is to construct the projective and inductive Limits and, as a particular case, the kernels and cockerels, products and coproducts. We introduce the notions filtrant category and cofinal functors, and study will some care filtrant inductive Limits in the category set of sets. Finally, we define right or left exact functors and give some examples.

And how we describe and continue the study of categories.

We also analyze some related notions, in particular those of cofinal categories, filtrant categories and exact functors.

Special attention will be paid to filtrant inductive Limits in the categories set and Mod (A).

Section (2.1) Products and co-products:

Let c be a category and consider a family $\{x_i\}_{i\in I}$ of objects of c indexed by a (small) set I, consider the two functors.

$$(2.1)\ C^{op} \to \text{set},\ Y \to \Pi_i\ \text{Hom}_c\ (x_1\ x_i),$$

$$(2.2)\quad c \to \text{set},\ Y \to \Pi_i\ \text{Hom}_c\ (x_i,\ y).\ [6].$$

In groups, rings and modules we have the notions of direct product and direct sum. Given a family of sets $\{A_i\}_{i\in I}$ we build A: = $\Pi_{i\in I}A_i$, which has as elements families $(a_i)_{i\in I}$ of elements $a_i \in A_i$.

For each k∈I we have projections P_k: A → A_k

Defined by $P_k(a_i)_{i\in I} = a_k$. Then important property a direct product has is that whenever we have a family of maps we can always lift the if in to a single map$\{f_i$: S → A so that p_i of = f_i for each I∈I. This is a notion we can easily generalize to an arbitrary category. [1.25. 35].

Definition (2.1.1.): Let $\{x_i\}_{i\in I}$ be a family of objects of the category Ψ. Then a product (A; p_i) of the objects A_i is an object A, together with morphism $P_i : A \to A_i$, called projections, with the universal property; given any objects S and morphism $f_i : S \to A_i$, there exists a unique morphism $f = \{f_i\}: S \to A$ with $P_i f = f_i$.

There is no guarantee that the product will all ways exist in Ψ, but if it does then the universal property guarantees it is essentially unique, as the next proposition shows. [1.25. 35].

Proposition (2.1.1): If $(A; p_i)$ and $(B;q_i)$ are products of the objects A_i, then A and B are canonically isomorphic (i.e. the isomorphism between them is unique).

Proof:

Using definition (2.1.1), first choose S = B and $f_i = q_i$ to get a unique $f:B \to A$ with $p_i f = q_i$. Then we put s = A and $f_i = p_i$ to get a unique

$h: A \to B$ with $q_i h = p_i$. This gives us

$p_i f h = q_i h = p_i$ and $q_i hf = p_i f = q_i$.

But $p_i IdA = p_i$ and $q_i IdB = q_i$ so be by the uniqueness given in the universal property, we must have fh = IdA and hf = Id_B. This A and B are isomorphic. The natural next step is to dualise the notion of product. [1.25. 35].

Definition (2.1.2): Let $\{M_i\}_{i\in I}$ be a family of objects of the category Ψ. Then a co- product (m, q_i) of the objects m_i is object m, together with morphisms $q_i : M_i \to M$, called injections, with the universal property; given any objects and morphisms $f_i: M_i \to S$ there exist a unique morphism $f_i = (f_i): M \to S$ with $f_{qi} = f_i$ [1.25. 35].

Notation (2.1.2): When talking about products, we often write $A = \Pi A_i$. For coproducts we write $M = \Pi M_i$. [1.25. 35].

Example (2.1.3):

(i) In the category Mod_R of (left)R−module the product is the direct product and the coproduct is the direct sum. In this case we write

instead of Π. The injections $q_i : M_i \to \otimes_{i\in j} M_j$ are defined by $q_i(m_i)=\otimes(n_j)_{i\in J}$ with $n_i = m_i$ and $n_j = o$ for $j \neq i$.
It is worth noting that, for a finite family of modules, the product and coproduct are the same.

(ii) In the category set, the product is the usual Cartesian product and the coproduct is the disjoint union. [1.25. 35].

Definition (2.1.3):

(i) Assume that the functor (2.1) is representable. In this case one denotes by $\Pi_i X_i$ a representative and calls this object the product of the X_i's.
In case I has two elements, say I = {1.2}, one simply denotes this object by $X_1 x X_2$.

(ii) Assume that the functor (2.2) is representable. In this case one denotes by $\Pi_i X_i$ a representative and calls this object the product of the X_i's. In case I has two elements, say I = {1.2}, one simply denotes this object by $X_1 U X_2$.

(iii) If for any family of objects $\{x_i\}_{i\in I}$, the product (resp, coproduct) exits, one says that the category c admits products (resp, coproducts) indexed by I.

(iv) If $X_i = X$ for all $_{i\in I}$, one writes.

$$X^f : = \Pi_i x_i, X^{(1)} = \Pi_i x_i,$$

Note that the coproduct in c is the product in C^{op}. By this definition, the product or the coproduct exists if and only if one has the isomorphisms, functorial with respect to y∈C:

(2.3) $Hom_c (y_i \Pi_i X_i) \cong \Pi Hom_c (y_i, x_i)$, [6].

(2.4) $Hom_c (\Pi_i X_i, y) \cong \Pi Hom_c (x_i, y)$, [6].

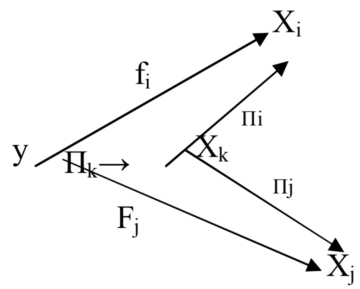
Assume that $\Pi_i X_i$ exists. By choosing $y = \Pi_i x_i$ in (2.3), we get the $\Pi_i, \Pi_j X_j, \to X_i$

Similarly, assume that $\Pi_i X_i$, exists. By choosing $y = \Pi_i X_i$ in (2.4), we get the morphisms.

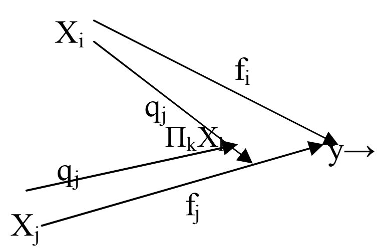
$$\varepsilon_i : X_i \to \Pi_j X_j,$$

The isomorphism (2.3) may be translated as follows. Given an object y and a family of morphisms

$f_i : Y \to X_i$, this family factorizes uniquely through $\Pi_i X_i$. This is visualized by the diagram



The isomorphism (2.4) may be translated as follows. Given an object y and a family of morphisms $f_i: X_i \to Y$, this family factorizes uniquely through $\Pi_i X_i$. This is visualized by the diagram [6].



Example (2.1.4):

(i) The category set admits products (that is, products indexed by small sets) and the two definitions (I,) and that given in definition (2.1.3) coincide.

(ii) The category set admits coproducts indexed by small sets, namely, the disjoint union.

(iii) Let A be a ring. The category Mod (A) admits products, as defined in (ii), the category Mod (A) also admits co-produces, which are the direct sums defined in (ii) and are denoted $\otimes$.

(iv) Let X be a set and denoted byXthe category of subset of X, (the set X is ordered by inclusion, hence defines a category). for $S_{1.}$ $S_2 \in X$, their product in the category X is their intersection and their coproduct is their union. [6.71.72].

Remark (2.1.5): The forgetful functor for: Mod (A) → set commutes with product but does not commute with coproducts. That is the reason why the coproduct in the category Mod (A) is called and denoted differently. [6.71.72].

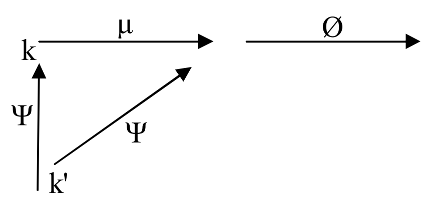
Section (2.2) Kernels and co kernels:

Note: When we talk about kernels and co kernels we will always assume that the category in question has a zero object (and hence zero morphisms) other wise the definition would make no sense. [1.25. 35].

Definition (2.2.4):

(i) The kernel of a morphism Ø : A → B in a category Ψ is a morphism μ : K → A such that (i) Ø μ = o,

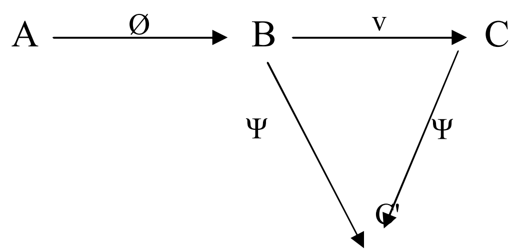
(ii) If ΦΨ = o, then Ψ = μ Ψfor some unique Ψ.



This is a good example of the philosophy of category theory; instead of thinking of the kernel as the space k, we think of the kernel as the morphism μ instead. Now, the canonical definition of the co kernel of map Ø : A → B is CokerØ = B/ImQ. However, we may convert this into a definition about morphisms instead, as we did before; [1.25. 35].

Definition (2.2.5): The cokernel of a morphismØ : A → B in a category Ψ is a morphism V : B → C such that (i) VØ = o,

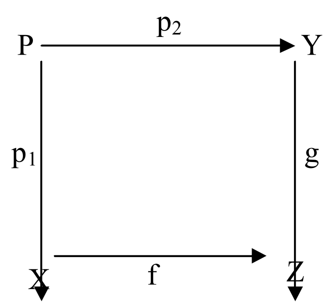
(ii) If $\Psi \emptyset = o$, then Ψ V for some unique Ψ



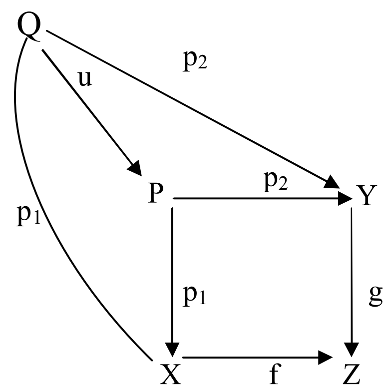
As the realer may already have guessed, the co kernel is simply the dual of the kernel! It is the same diagram, only the arrows have been reversed. In the category of R−modules we can interpret the cokerenl as been a measure of how surjestrve the map is, in the sense that a map is surjective if and only if its co kernel is zero.

This is dual to the notion of the kernel measuring the injectivty of a map. (Also notice the kernels are monomorphisms and co kernels are epimorphisms). [1.25. 35].

Definition (2.2.6): Given f: $X \to Z$ and g: $y \to Z$ in Ψ, a pull-back of f and g consists of an object p and a pair of morphism p: $P \to X$ and P_2: $P \to Y$ such that the following diagram commutes;



Moreover, the pull-back must have the following universal property; given q_1: $Q \to X$ and q_2: $Q \to Y$ with $fq_1 := fq_2$. there exists a unique U: $Q \to P$ with $q_1 := q_1 u$, $q_2 = q_2 u$;



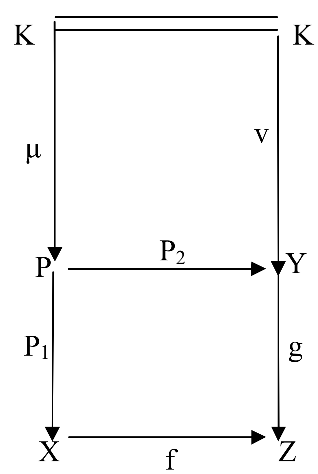
The dual notion of a pull-back is called a push-out.

We now give a nice theorem which combines the idea of pull-backs with the new definition of kernel; [1.25. 35].

Theorem (2.2.6): Consider a pull-back diagram as shown in definition (2.2.6) for a general category Ψ. Then.

(i) If (k, μ) is the kernel of p_i, (k,p_2μ) is the kernel of g.
(ii) If (k, v) is the kernel of g then v can be factored as V = p_2 μ where (k, μ) is the kernel of p_i,

Proof: First note that not every morphism has a kernel, so the statement is not trivial.



(i) Let v = p_2 μ. Then gv = gp_2 μ = f p_2 μ=0, so we need only show that if gT = 0 then T=VTO for some unique To. So suppose T: A→Y is such a rival for V. Then, since f o = o, then pull-back property shows that there exists o:A→P such that 0= p_2 μσ, T=pσ. The first of these two properties implies that σ = μ To for some unique To: K→A by the universal property of the kernel μ. Substituting this into the second property gives us T =μT= p_2μTo = vTo, as required. Thus v is the kernel of g.

(ii) We have gv = 0, so by the same pull-back argument as in (i) we know there exists μ: K→P with the property p_1 μ = o, v = p_2 μ. It remains to show that μ is the kernel of $p_{2.}$ μ so for some T: ep. Then we suppose P_1T= o, P_1T=0= P_1μ and P_2T=VT 0 =P_2μ T0 and by the uniqueness of the pull-back ($p_{1.}$ p_2) we deduce that T = μ To.[1.95. 35].

Definition (2.27): Let c be a category and consider two parallel arrows f, g: $X_0 \rightrightarrows X_1$ in c consider the two functors

(2.5) C^{op} → set, Y → ker [$\mathrm{Hom}_c(Y,X_0) \rightrightarrows \mathrm{Hom}_c(Y,X_1)$].

(2.6) c→ set, Y → ker [$\mathrm{Hom}_c(X_{1.}Y) \rightrightarrows \mathrm{Hom}_c(X_0,Y)$].

(i) Assume that the functor in eq (2.5) is reprentable. In this case one denotes by kernel (f,g) a representative and calls this object a kernel (one also says a equalizer) of (f,g).
(ii) Assume that the functor in (2.6) is representable.

In this case one denotes by Coker (f,g) a representative and calls this object a co-kernel (one also says a co-equalizer) of (f,g).

(iii) A sequence Z → X_0 $\underset{\rightarrow}{X_0}$ (resp, $X_0 \rightrightarrows X_1$→ Z) is exact if Z is isomorphic to the kernel (resp. co-kernel) of $X_0 \rightrightarrows X_1$.
(iv) Assume that the category C admits a zero-object 0. Let f: X→Y be amorphism in c.

A kernel (resp. a co-kernel) of f, if it exists, is a kernel (resp. a co-kernel) of f, o: $X \rightrightarrows Y$. it is denoted Ker (f) [resp, Coker (f)].

Note that the co-kernel in c is the kernel in c^{op}. By this definition, the kernel or the cockerel of f, g: X_0X_1 existed if and only if one has the isomorphism functorial in Y∈c;

(2.7) Hom_c (y, Ker (f,g)$\simeq$ Ker [Hom_c (Y, X_0) $\rightrightarrows$ $Hom_c(Y,X_1)$],

(2.8) Hom_c[Cocker (f,g), y] $\simeq$ Ker [Hom_c (X_1.Y) $\rightrightarrows$ $Hom_c(X_0,Y)$].

Assume that Ker (f,g) exists, By choosing y = Ker (f,g) in (2.7), we get the morphism.

$$h: Ker (X_0 \rightrightarrows X_1) \to X_0$$

Note that h is a mono-morphism. Indeed, consider a pair of parallel arrows a, b; Y$\rightrightarrows$X such that aok = bok = w. Then wof = aokof = aokog = bokog = wog. Hence w factors uniquely through K, and this implies a = b. Similarly, assume that Coker (f,g) exist.

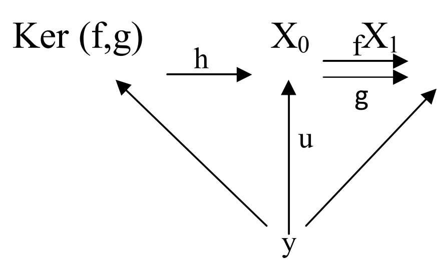
By choosing y – Coker (f,g) in (2.8), we get the morphism

$$K: X_1 \to Coker (X_0 \rightrightarrows X_1)$$

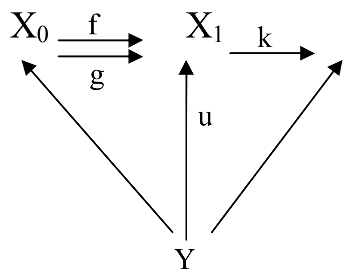
Note that K is an epimorphism.

The isomorphism (2.7) may be translated as follows.

Given an object y and a morphism u: Y $\to X_0$ such that fou = gou, the morphism u factors uniquely through Ker(f,g). This is visualized by the diagram.



The isomorphism (2.8) may be translated as follows. Given an object y and amorphism u: $X3_1 \to Y$ such that uof = uog, the morphism u factors uniquely through Coker (f,g). This is visualized by diagram. Coker (f,g). [6.71.72]

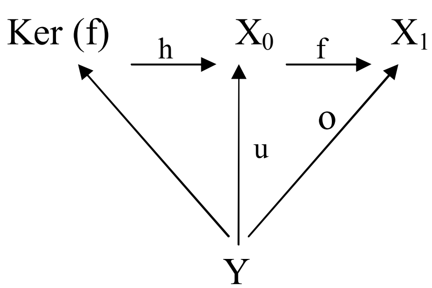


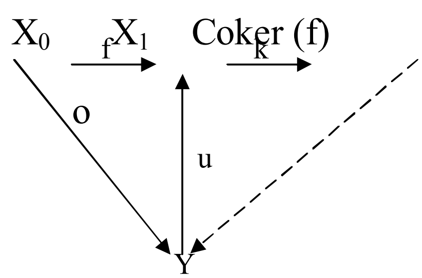
Example (2.2.7): [6.71.72]

(i) The category set admits kernels and the two definitions that given in definition (2.2.7) coincide.

(ii) The category set admits co kernels. If f,g: $Z_0 \rightrightarrows Z_1$ are two maps, the co kernel of (f.g) is the quotient set Z_1/R where R is the equivalence relation generated by the relation x vy if there exists $z \in Z_0$ with f (z) = x and g (z) – y.

(iii) Let A be a ring. The category Mod (A) admits a zero object. Hence, the kernel or the co kernel of morphism f means the kernel or the co kernel of (f,o). As already mentioned, the kernel of linear map f: M → N is the Z A−module f^{-1} (o) and the co kernel is the quotient module M/imf. The kernel and co kernel are visualized by the diagrams.





Section (2.3) Limits:[6.71.72].

Let us generalize and unify the preceding constructions. In the sequel, I will denote a (small) category. Let c be a category. A functor α: I→ c (resp β: I^{op} →c) is sometimes called an inductive (resp, projective) system in C indexed by I.

For example, if (i. $\leq$) is a pre-ordered set, I the associated category, an inductive system indexed by I is the data of a family $(x_i)_{i\in I}$ of objects of c and for all $i \leq j$, a morphism $X_i \rightarrow X_j$ with the natural compatibility conditions.

Definition (2.3.8):

(i) Assume that the functor X $\overset{\lim}{\longleftarrow}$ $Hom_c(X,\beta)$ is represent-able. We denote by $\overset{\lim}{\longleftarrow}$ β its representative and say that the functor β admits a projective Limit in c.

In other words, we have the isomorphism, functorial in X∈C.

(2.10) $$Hom_c(X, \overset{\lim}{\longleftarrow} \beta) \cong \overset{\lim}{\longleftarrow} Hom_c(X, \beta).$$ [6].

(ii) Assume that the functor X $\rightarrow\overset{\lim}{\longleftarrow}$ $Hom_c(\alpha, X)$ is represent-able. We denote by $\overset{\lim}{\longleftarrow}$ α its representative and say that the functor α admits an inductive Limit.

(2.11) $$Hom_c(\overset{\lim}{\longrightarrow}\alpha, X) \cong \overset{\lim}{\longleftarrow} Hom_c(\alpha, X).$$ [6].

When C = set this definition of $\overset{\lim}{\longleftarrow}$ β coincides with the former one, in view of lemma (2.3.8).

Notice that both projective and inductive Limits are defined using projective limits in set;

Assume that $\overset{\lim}{\longleftarrow}$ β exists in c. one gets;

$$Hom_c(\overset{\lim}{\longleftarrow} \beta, \beta) \simeq Hom_c(\overset{\lim}{\longleftarrow} \beta, \overset{\lim}{\longleftarrow} \beta$$ and the identity of $\overset{\lim}{\longleftarrow}$ β defines a family of morphism.

$$P_i : \overset{\lim}{\longleftarrow} \beta \rightarrow \beta(i).$$

Consider a family of morphisms $\{ f_i : X \rightarrow \beta(i) \}_{i\in I}$

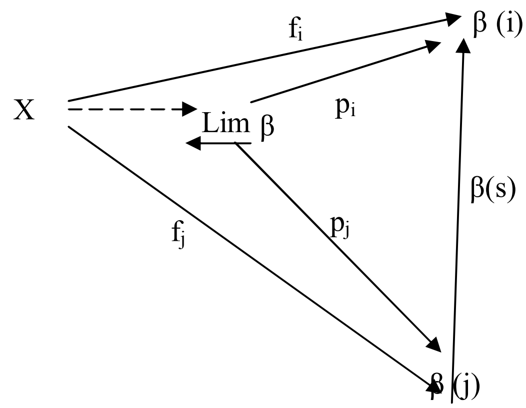
In c satisfying the compatibility conditions

(2.12) $f_i = f_i$ of (s) for all $S \in Hom_c(i, j)$. [6].

This family of morphism is nothing but an element of $\overset{\lim}{\longleftarrow}$ Hom[X, β (i)], hence by (2.10), an element of Hom (X, $\overset{\lim}{\longleftarrow}$ β, X). Therefore, $\overset{\lim}{\longleftarrow}$ β is characterized by the "universal property";

(2.13) For all X∈C and all family of morphism $\{ f_i : X \to P \{i\}_{i\in I}$In c satisfying (2.12), all morphism f_i' factorize uniquely through $\overset{\lim}{\longleftarrow}$ β. [6].

This is visualized by the diagram



Similar, assume that $\underrightarrow{\text{Lim}}$ α exists in c, one gets;

$$\overset{\lim}{\longleftarrow} \text{Hom}_c (\alpha, \overset{\lim}{\longrightarrow} \alpha) \simeq \text{Hom}_c (\overset{\lim}{\longrightarrow} \alpha, \overset{\lim}{\longrightarrow} \alpha)$$

And the identity of $\overset{\lim}{\longrightarrow}$ α defines a family of morphisms

$$P_i : \alpha (i) \to \overset{\lim}{\longrightarrow} \alpha$$

Consider a family of morphism $\{f_i: \alpha (i) \to X \}_{i\in I_n}$ c satisfying the compatibility conditions.

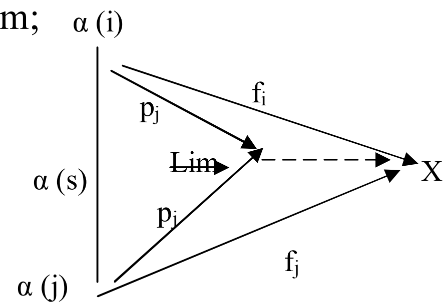
(2.14) $f_i = f_j$ of (s) for all S∈ Hom_I (i,j) [6]

This family of morphism is nothing but an element of

Hom (α (i), X), hence by (2.11), an element of Hom ($\overset{\lim}{\longrightarrow}$ α, X), Therefore, $\overset{\lim}{\longrightarrow} \alpha$ is characterized by the "universal property".

$$(2.15)\begin{cases} \text{For all } X \in C \text{ and all family of morphism } \{ fi : \alpha(i) \\ \to X\}i \in I \text{In c satisfying (2.14), all morphism } fi's \text{ factorize} \\ \text{uniquely through } \alpha. \end{cases}$$

This is visualized by the diagram;



[6]

Projective Limits in set.

Assume first that c is the category set and let us consider projective system. One sets.

(2.9) $\overset{\lim}{\longleftarrow}\beta = \{ \{x_i\} \}_{i\in T}I\beta$ (i); β (s) $(x_i) = x_i$ for all

S∈Hom_c (i, j). [6]

The next result is obvious.

Lemma (2.3.8): Let β : $I^{op} \to$ set be a functor and let X∈ set.

There is a natural isomorphism .

proof:

Hom_{set} (X, $\overset{\lim}{\longleftarrow}\beta$) $\cong \overset{\lim}{\longleftarrow} Hom_{set}$ (X, β),

where Hom_{set} (X, β) denotes the functor : $I^{op} \to$ set, i →

Hom_{set} [X, β (i)] [6.71.72].

Projective and inductive Limits:

Consider now two functors β: $I^{op} \to$ c and

$\alpha : I \to c$. For $X \in C$, we get functors from I^{op} to set;

$Hom_c(X, \beta): I^{op} \ni i \to Hom_c[\, X, \beta\,(i)\,] \in$ set,

$Hom_c(\alpha, X) : I^{op} \ni i \to Hom_c\,(\alpha, X) \in$ set. [6].

We will discuss only inductive Limits, since the notion of projective Limits is dual. Let c be a category, I be a preordered set and $A = (A_i, U_{i,j})$, be an inductive system over I with values in c ($u_{i,j}$ is morphism $A_j, \to A_i$, defined for $i \geq j$). We call (generalized) inductive Limits of a system consisting of $A \in C$ and a family (u_i) of morhism u_i: $A_i, \to A$, satisfying the following conditions: (a) for $I \leq j$, we have $u_i = u_j\, u_{ji}$; (b) for every $B \in C$ and every family (u_i) of morphism u_i: $A_i \to B$, such that $u_i =$ uui for all $i \in I$.

If [A, (u_i)] is an inductive Limit of $A = (A_i, U_{ij})$, and if [B, (u_i)] is an inductive Limit of a second inductive system, $B = (B_i, U_{ij})$ and finally if $w = (w_i)$ is morphism from A to B, then there exists a unique morphism w: $A \to B$ such that for all $i \in I$, $w\, u_i = u_i w_i$.

In particular, two inductive Limits of the same inductive system are canonically isomorphic (in an obvious way), so it is natural to choose, for every inductive system that admits an inductive Limit, a fixed inductive Limit (for example, using Hilbert's T symbol) which we will denote by $\overset{\lim}{\longrightarrow}$ A or $\underset{i \in I}{\underset{\to}{\lim}}$ A_i and which we will call the inductive Limit of the given inductive system.

If I and c are such that $\overset{\lim}{\longrightarrow}$ A exists for every system A over I with values in c, it follows form the preceding that $\overset{\lim}{\longrightarrow}$ A is a covariant functor defined over the category of inductive system on I with values in c, [11].

Proposition (2.3.9): Let c be an abelian category satisfying Axiom AB (existence of arbitrary direct sums) and let I be an increasing filtered preordered set. Then for every inductive system A over I with values in c, the

A exists, and it is a right exact additive functor on A. If c satisfies Axion AB, this functor is even exact, and then the kernel of the canonical morphism $u_i : A_i \to \overset{\lim}{\longrightarrow} A$ is the sup of kernel of the morphism $U_{ji} : A_i \to A_j$ for $j \geq I$ (in particular, if the U_{ji} are injective, so are the U_i).

To construct an inductive Limits of $A = (A_i U_{ij})$ we consider $KS = A_i$ and for every pair $I \leq j$, $T = I \leq A_i$. If u_i: $A_i \to S$ and U_{ij}: $A_i \to T$ are the inclusion in to those coproducts, there are two maps d,e: $T \to S$ defined as the unique maps for which $d\, w_{ij} = u_i$ and $e\, w_i = u_j\, U_{ij}\, S$, for all $i \leq j$.

Then $\overset{\lim}{\longrightarrow} A$ is the co equalizer of d and e. We see[11].

Example (2.3.10): Let X be a set and let x be the category. Let $\beta: I^{op} \to x$ and $\alpha: I \to X$ b two functors. Then $\overset{\lim}{\longleftarrow} \beta \cong \cap_I \beta(i)$, $\overset{\lim}{\longrightarrow} \alpha \cong U_i\, \alpha(i)$.

Example (2.3.11):

(i) When the category I is discrete, projective and inductive limits indexed by I are nothing but products and co- products indexed by I.

(ii) Consider the category I with two objects and two parallel morphisms other than, densities, visualized by $\overset{\lim}{\longrightarrow}$ A functor $\alpha: I \to c$ is characterized by two parallel arrows in c; $\rightrightarrows$

$$(2.16) \qquad f, g : X_0 \rightrightarrows X_1$$

In the sequel we shall identify such a functor with the diagram (2.15). Then the kernel (resp, co kernel)of (f, g) is nothing but the projective (resp, inductive) Limit of the functor α.

(iii) If I is the empty category and $\alpha: I \to c$ is a functor, then $\overset{\lim}{\longleftarrow} \alpha$ exists in c if and only if c has a terminal object and in this case

$$\alpha \cong \overset{\lim}{\longleftarrow} P_c^t$$

Similarly, $\overset{\lim}{\longrightarrow}$ α exists in c if and only if c has an initial object $Ø_c$, and in this case $\overset{\lim}{\longrightarrow}$ $\alpha \cong Ø_c$.

(iv) if I admits a terminal object, say i. and if $\beta: I^{op} \to C$ and $\alpha: I \to c$ are functors, then

$$\overset{\lim}{\longleftarrow} \beta \cong \beta\,(i_0) \overset{\lim}{\longrightarrow} \alpha \cong x\,(i_0).$$

This follows immediately of (2.15) and (9,13).

If every functor from I^{op} to c admits a projective Limit, one says that c admits projective Limits indexed by I. If this property holds for all categories I (resp. finite categories I), one says that C admits projective (resp. finite projective) Limits, and similarly with inductive Limits. [6.71.72].

Remark (2.3.11): Assume that c admits projective (resp, inductive) Limits indexed by I. then $\overset{\lim}{\longleftarrow}$: Fct ($I^{op}$, c) → c [resp, $\overset{\lim}{\longrightarrow}$: Fct (I,c) → c] is functor.

Projective Limits as kernels and products.

We have seen that products and kernels (resp. co products and co kernels) are particular cases of projective (resp. inductive) Limits. One can show that conversely, projective Limits can be obtained as kernels of products and inductive Limits can be obtained as co kernel of co products.

Recall that for a category I, Mor (i) denote the set of morphism in I.

There are two natural maps (source and target) from Mor (i) to ob (I);

$$\sigma : \text{Mor}\,(I) \to \text{ob}\,(I),\ (S;\ i \to j) \to i,$$

$$T : \text{Mor}\,(I) \to \text{ob}\,(I),\ (S;\ i \to j) \to j,$$

Let c be a category which admits projective Limits and let $\beta: T^{op} \to c$ be a functor. For S; i → j. we get two morphism in c.

$$B(i) \times \beta(j) \underset{\beta(s)}{\overset{Id\,\beta(i)}{\rightrightarrows}} \beta(i)$$

From which we deduce to morphism in c:

B [σ (s)]. These morphisms define the two morphism in c.
$Л_i \in I\ \beta(i) \rightrightarrows$

(2.17) $\Pi_{i\in I}\,\beta(i) \underset{b}{\overset{a}{\rightrightarrows}} \Pi_{S\in Mor(I)}\,\beta[\sigma(s)\}$. [6].

Similarly, assume that c admits inductive Limits and let α: → c be a functor. By reversing the arrow, one gets the two morphism in c;

(2.18) $\Pi_{S\in Mor(I)}\alpha[\sigma(s)\} \underset{b}{\overset{a}{\rightrightarrows}} \Pi_{i\in I}\alpha(i)$. [6].

Proposition (2.3.12):

(i) $\overset{\lim}{\leftarrow}\beta$ is the kernel of (a, b) in (2.17).

(ii) $\overset{\lim}{\rightarrow}\ \alpha$ is the co kernel of (a, b) in (2.18).

Sketch of proof. By the definition of projective and inductive Limits we are reduced to check (i) when c = Set and in this case this is obvious.

In particular, a category c admits finite projective Limits if and only if it satisfies;

(i) c admits a terminal object.
(ii) For any X, Y∈ ob (c), the product X x Y exists in c,
(iii) For any parallel arrows in c, f, g: $X \rightrightarrows Y$, the kernel exists in c.

There is a similar result for finite inductive Limits, replacing a terminal object by an initial object, products by co products and kernels by co kernels.

Example (2.3.13): The category set admits project give and inductive Limits, as well as the category Mod (A) for a ring A.

Indeed, both categories admit products, co products kernels and co kernels. [6. 71.72].

Section (2.4) Properties of Limits:

Double Limits:

For two categories I and c, recall the notation c' = Fct (I,c) and for a third category J, recall the equivalence Fct (Ix J,c) ~ Fct [I, Fct (J.c)]. Consider bi functor B: T^{op} X ⟶J^{op}C. It defines a functor

(2.19) $\underleftarrow{\text{Lim}}\ \beta \simeq \underleftarrow{\text{Lim}}\ \underleftarrow{\text{Lim}}\ \beta_J \sim \underleftarrow{\text{Lim}}\ \underleftarrow{\text{Lim}}\ \beta_I$. [6].

Similarly, if α: I x J → c is a bi functor, it defines a functor α_J: I → c^J and one has the isomorphisms.

(2.20) $\underrightarrow{\text{Lim}}\ \alpha \simeq \underrightarrow{\text{Lim}}\ (\underrightarrow{\text{Lim}}\ \alpha_J) \simeq \underrightarrow{\text{Lim}}\ (\alpha_i) \underrightarrow{\text{Lim}}$

In other words:

(2.21) $\underleftarrow{\text{Lim}}_{i,j}\ \beta (i, j) \simeq \underleftarrow{\text{Lim}}_{j}\ \underleftarrow{\text{Lim}}_{i}\ [\beta (i,j)] \simeq \underleftarrow{\text{Lim}}_{i}\ \underleftarrow{\text{Lim}}_{j}\ [\beta (i, j)]$,

(2.22) $\underrightarrow{\text{Lim}}_{i\,j}\ \alpha (i, j) \simeq \underrightarrow{\text{Lim}}_{j}\ [\underrightarrow{\text{Lim}}_{i}\ [\alpha (i,j)] \simeq \underrightarrow{\text{Lim}}_{i}\ \underrightarrow{\text{Lim}}_{j}\ [\alpha (i, j)]$.

Consider a functor β: I^{op}→ Fct (J^{op}, c). It defines a functor

β: I^{op} x J^{op}→c, hence for each j∈J, a functor β (j):

I^{op}→ c. Assuming that c admits projective Limits indexed by I, one checks easily that j → $\underrightarrow{\text{Lim}}$ β (j) is a functor, that is, an object of Fct (J^{op}, c), and is a projective Limit of β, There is a similar result for inductive Limits.[6].

Proposition (2.4.14):(6. 71. 72) Let I be a category and assume that c admits projective Limits indexed by I. then for any category J, the category $C^{J^{op}}$ admits projective Limits indexed by I. Moreover, if β: I^{op}→$C^{J^{op}}$is a functor, then, then $\underleftarrow{\text{Lim}}$ β∈C^J is given by

$$(\underleftarrow{\text{Lim}}\ \beta)(j) = \underleftarrow{\text{Lim}}\ [\beta (j)],\ j \in J$$

Similarly, assume that c admits inductive Limits indexed by I. Then for any category J, the category c^J admits inductive Limits indexed by I. Moreover, if α: I → c^J is a functor, then $\underrightarrow{\text{Lim}}\,\alpha \in c^J$ is given by

$$(\underrightarrow{\text{Lim}}\,\alpha)(j) \simeq \underrightarrow{\text{Lim}}\,[\alpha(j)],\ j \in J.$$

Corollary (2.4.15): (6. 71. 72)

Let c be a category. Then the categories ĉ and č admit inductive and projective Limits. [6.71.72].

Section (2.5) Composition of Limits:

Let I,C and ć be categories and let α: I → c,

β: I^{op}→ c and F: c → ć be functors. When c and ć admit projective or inductive Limits indexed by I, there are natural morphisms.

(2.23) $F(\underleftarrow{\text{Lim}}\ \beta) \rightarrow \underleftarrow{\text{Lim}}\ (F\circ\beta)$.

(2.24) $\underrightarrow{\text{Lim}}\ F(F\circ\alpha) \rightarrow F(\underrightarrow{\text{Lim}}\,\alpha)$.

This follows immediately of (2.15) and (2.13). [6].

Definition (2.5.9): Let I be a category and let F: c → ć be functor.

(i) Assume that c and ć admit projective Limits indexed by I. One says that F commutes with such Limits if (2.23) is an isomorphism.

(ii) Similar, assume that c and ć admit indexed by I. One says that F commutes with such Limits if (2.24) is an isomorphism. [6. 71. 93. 94].

Examples (2.5.16):

(i) Let c be a category which admits projective Limits indexed by I and let X∈c. By (2.10), the functor Hom_c (X,.): C → set commutes with projective Limits indexed by I.

Similarly, if c admits inductive Limits indexed by I, then functor $Hom_c \rightarrow (.,X)$: $C^{op} \rightarrow$ set commutes with projective Limits indexed by I, by (2.11).

(ii) Let I and J be two categories and assume that c admits projective (resp, inductive) Limits indexed by IxJ. Then the functor $\underleftarrow{Lim}$: Fct (J^{op}, c) → c [resp. $\underrightarrow{Lim}$ L Fact (J,C) → c] commutes with projective (resp, inductive) limits indexed by I. This follows from the isomorphisms (2.19) and (2.20).

(iii) Let kbe a field, C= ć = mode (k), and let X∈C. Then the functor Hom_k (X,.) does not commute with inductive limit I f^k X is infinite dimensional. [6.71].

Proposition (2.5.17): Let F: c → ć be a functor and let I be a category.

(i) Assume that c and ć admit projective limits indexed I and F admits a left adjoint G: ć → c. Then F commutes with projective limits indexed by I, that is,

$$F [\underleftarrow{Lim}_{i} \beta (i) \simeq \underleftarrow{Lim}_{i} F [\beta (i)],$$

(ii) Similarly, if c and ć admit inductive limits indexed by I and F admits a right adjoint, then F commutes with such limits.

Proof:

It is enough to prove the first assertion, to check that (2.23) is an isomorphism.

Let Y∈Ć. One has the chain of isomorphisms

$$Hom_c (Y,F [\underleftarrow{Lim}_{i} \beta (i)) \simeq Hom_c [G (y), \underleftarrow{Lim}_{i} \beta (i)]$$

$$\simeq \underleftarrow{Lim}_{i} Hom_c [G (y), \beta (i)]$$

$$\simeq \underleftarrow{Lim}_{i} Hom_c [Y,F (\beta (i)]$$

$$\simeq Hom_{Ćn} [y, \underleftarrow{Lim}_{i} F (\beta (i)]$$

Section (2.6) Filtrant inductive limits.

Since it admits co products and co kernels, the category set admits inductive limits. We shall construct them more explicitly.

Let α: I → set be a functor and consider the relation on [$U_{i \in I}\alpha(i)$].

(2.25) $\begin{cases} \alpha(i) \in xRy \in \alpha(i) \text{ if there exists } K \in I,\ s: I \rightarrow k \\ \text{And } t: j \rightarrow k \text{ with } \alpha(s)(x) = \alpha(t)(y). \end{cases}$

The relation R is reflexive and symmetric but is not transitive in general. [6].

Proposition (2.6.17): With the notations above, denote by ~ the equivalence relation generated by R. Then

$$\underset{\longrightarrow}{\text{Lim}}\ \alpha \simeq [\ U_{i \in I}\alpha(i)\ ] / \sim$$

Proof:

Let S∈ set. By the definition of the projective limit in set we get:

$\underset{\longleftarrow}{\text{Lim}}$ Hom (α, s) $\simeq$ {{ p (I,x) $]_{i \in I, x \in \alpha(i)}$; p (i,x) ∈S, p (i,x) =

p (i,y) if there exists s: i → j with α(s) (x) = y}.

The result follows:

In the category set one uses the notation U better than Π. For a ring A, the category Mod (A) admits co products and co kernels. Hence the category Mod (A) admits inductive limits. One shall be aware that the functor for: Mod (A)→ set does not commute with inductive limits. For example, if I is empty and

α: I → Mod (A) is a functor, then α(I) = {0} and for ({0}) is not an initial object in set. [6.71.72].

Definition (2.6.10): A category I is called filtrate if it satisfies the conditions (i) –(iii) below.

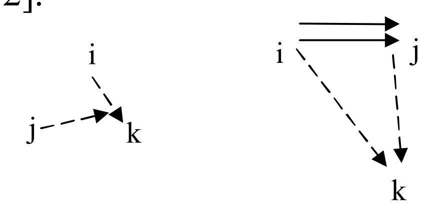
(i) I is non empty,

(ii) For any i and j in I, there exists K∈I and a morphism i →K, j → k.

(iii) For any parallel morphisms f, g; : I $\rightrightarrows$ J, there exists a morphism h: j → k such that hof = hog.

One says that I is co-filtrant if I^{op} is filtrant .

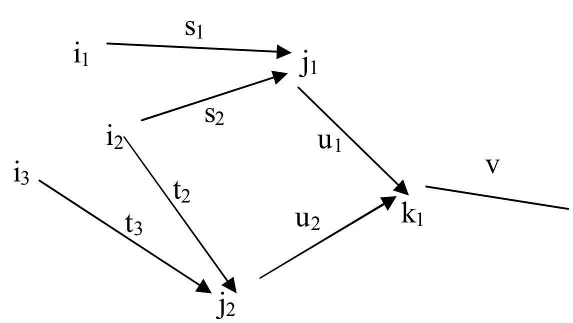
The conditions (ii) – (iii) of being filtrant are visualized by the diagrams; of course, if (I, ≤) is a non- empty directed ordered set, then the associated category I is filtrant [6.71.72].



Proposition (2.6.18): Let α: I → set be a functor, with I filtrant. The relation R given in (2.25) on $II_i\alpha$ (i) is an equivalence relation.

Proof. Let $x_j \in \alpha(i_j)$, j = 1.2.3 with $x_1 \sim x_2$ and $x_2 \sim x_3$.

There exist morphisms visualized by the diagram:



Such that $\alpha(s_1)x_1 = \alpha(s_2)x_2 = \alpha(t_2)x_2 = \alpha(t_3)x_3$ and

$vou_1os_2 = vou_2ot_2$.

Set $w_1 = vou_1os_1$, $w_2 = vou_1os_2 = vou_2ot_2$ and $w_3 = vou_2ot_3$.

Then $\alpha(w_1)x_1 = \alpha(w_2)x_2 = \alpha(w_3)x_3$. Hence $x_1 \sim x_3$ [6.71].

Corollary (2.6.19): Let α: I → set be a functor, with I filtrant.

(i) Let s be a finite sub set in $\varinjlim \alpha$. Then there exists i∈I such that s is contained in image of α(i) by the natural map $\alpha(i) \to \xrightarrow{\text{Lim}} \alpha$.

(ii) Let i∈I and let x and y be elements of α(i) with the same image in $\xrightarrow{\text{Lim}} \alpha$. Then there exists s: i → j such that α(s) (x) = α(s) (y) in α(j).

Proof, (i) Denote by $\alpha: U_{i\in I}\alpha(i) \to \xrightarrow{\text{Lim}} \alpha$ the quotient map. Let $s = \{x_{1.} \ldots\ldots, x_n\}$. For j = 1,..., n there exists yj∈ α (i_j) such that x; = α (y_j).

Choose k∈I such that there exist morphism s_j: α (i_j) →α(k). Then $x_j = \alpha\{(\alpha[s_j(y_j)]\}$.

(iii) For x,y ∈α (i), xRy if and only if thee exits s: i → j with α (s) (x) = α(s) (y) in α(j) [6.71.72.93].

Corollary (2.6.20): Let A be a ring and denote by the forget full functor Mod (A) → set. Then the functor for commutes with filtrant inductive limits. In other words, if I is filtrant and α: I → Mod (A) is a functor, then

$$\text{Foro}\,(\underset{i}{\xrightarrow{\text{Lim}}}\alpha(i) = \underset{i}{\xrightarrow{\text{Lim}}}\ [\text{for o } \alpha(i)]. \qquad \underset{j}{\xrightarrow{\text{Lim}}}\underset{i}{\xleftarrow{\text{Lim}}}$$

Inductive limits with values in set indexed by filtrant categories commute with finite projective limits.

More precisely: [6.71.72.93.94].

Proposition (2.6.21):

For a filtrant category I, a finite category J and functor $\alpha: IxJ^{op} \to$ set, one has $\underset{j}{\xrightarrow{\text{lim}}}\underset{i}{\xleftarrow{\text{lim}}}\alpha(i,j) \to \underset{i}{\xleftarrow{\text{lim}}}\underset{j}{\xrightarrow{\text{lim}}}\alpha(i,j)$ In other words, the functor $\underset{j}{\xrightarrow{\text{lim}}}$: Fct (I, set) → set.

Commutes with finite projective limits.

Proof. It is enough to prove that $\xrightarrow{\lim}$ commutes with kernels and with finite products.

(i) $\xrightarrow{\lim}$ Commutes with kernels. Let α, β: I → set be two functor and let f,g: $\alpha \rightrightarrows \beta$ be two morphisms of functor.Define γ as the kernel of (f,g), that is, we have exact sequences .

$$\gamma(i) \rightarrow \alpha(i) \rightrightarrows \beta(i).$$

Let Z denote the kernel of $\xrightarrow{\lim} \alpha(i) \rightrightarrows \xrightarrow[i]{\lim} \beta(i)$.

We have to prove that the natural map $\lambda: \xrightarrow[i]{\lim} \gamma(i) \rightarrow Z$ is bijective.

(i) (a) The map λ is surjective. Indeed for x∈Z, represent x by some $x_i \in \alpha(i)$. Then fi (x_i) and $g_j(x_i)$ in β (i) having the same image in $\xrightarrow{\lim} \beta$, there exists s: i → j such that β (s) $f_i(x_i) = \beta(s)\, g_i(x_i)$. Set $x_i = \alpha(s)\, x_i$. Then $f_j(x_i) = g_j(x_i)$, which means that $x_i \in \gamma(j)$. Clearly, $\lambda(x_j) = x$,

(i) (b) the map λ is injective. Indeed, let $x, y \in \xrightarrow{\lim} \gamma$ with λ (x) = λ (y). We map represent x and y by elements x_i for some i∈I. Since x_i and y_i have the same image in $\xrightarrow{\lim} \alpha$, there exists i → j such that they have the same image in α(j). Therefore their image in γ (j) will be same.

(ii) $\xrightarrow{\lim}$ Commutes with finite products. The proof is similar to the preceding one. [6].

Corollary (2.6.22): Let A be a ring and let I be a filtrant category. Then the functor Mod (A)' → Mod (A) commutes with finite projective limits.[6.71.72].

Co final functor:

Let Ψ: J → I be a functor. If there are no risks of confusion, we still denote by Ψ the associated functor $\Psi: J^{op} \rightarrow I^{op}$. For two functors

α: I → C and β: J^{op}→ C, we have natural morphism;

(2.26) $\underleftarrow{\text{Lim}}\ (\beta o \Psi) \leftarrow \underleftarrow{\text{Lim}}\ \beta$,

(2.27) $\underrightarrow{\text{Lim}}\ (\alpha o \Psi) \leftarrow \underrightarrow{\text{Lim}}\ \alpha$,

This follows immediately of eq (2 ,15) and eq (2 , 13).

Definition (2.6.11):[6. 71. 72] Assume that Ψ is fully faith full and I is filtrate. One say that Ψ is cofinal if for any i ∈I there exists j ∈J and amorphism S: I ⟶Ψ(i).

Proposition (2. 6. 23): Let Ψ: J → I be fully faithful functor. Assume that I is filtrant and Ψ is co final. Then

(i) For any category c and any functor β: I^{op} → c, the morphism (2.26) is an isomorphism.
(ii) For any category c and any functor α: I → c, the morphism (2.27) is an isomorphism.

Proof. Let us prove (ii), the other proof being similar. By the hypothesis, for each i∈I we get a morphism $\alpha(i) \rightarrow \underset{i \in j}{\overleftarrow{\lim}} [\ \alpha o \Psi\ (j)]$ from which onededuce.

a morphism $\underset{i \in j}{\overrightarrow{\lim}} \alpha(i) \rightarrow \underset{i \in j}{\overleftarrow{\lim}}\ [\ \alpha o \Psi\ (j)]$.

One checks easily that his morphism is inverse to the morphism in (2.24) [6.71.72].

Example (2.6.24): Let X be a topological space, x∈X and denote by I_x the set of open neighborhoods of x in X. we endow I_x with the order: U ≤ V if V ⊂U. Given U and V in I_x, and setting W = U∩V, we have U ≤ W and V ≤ W.

Therefore, $I_x$is filtrant.

Denote by C^o (U) the c-vector space of complex valued continuous functions on U. The restriction maps $C^o(U) \rightarrow C^o(V)$, $V \subset U$ define an inductive system of c-vector spaces indexed by I_x. One sets

$$(2.28)\quad C^0_{X,x} = \underset{U\in i}{\xrightarrow{\lim}} C^o(U)$$

An element Ψ of $C^0_{X,x}$ is called a germ of continuous function at 0. Such a germ is an equivalence class $(U,\Psi U)$ / ~ with U a neighborhood of x, Ψua continuous functions on U and $(U,\Psi U) \sim o$ if there exists neighborhood V of x with $V \subset U$ such that the restriction of Ψu to V is the zero function. Hence, a germ of function is zero at x if this function is identically zero in neighborhood of x. [6. 71. 72].

Passage to the limit in sheaf co homology.

Proposition (2.6.25): Let c and ć be a belian categories. We assume that every object of c is isomorphic to a sub-object of an injective, and that ć satisfies Axiom AB, which in particular makes it possible to take inductive limits in ć.

Let $(F_i)_{i\in I}$ be an inductive system of covariant additive functor from c to ć. Let $F = \xrightarrow{\text{Lim}} F_i$be the inductive limit functor of the F_i, defined by $F(A) = \xrightarrow{\text{Lim}} F_i(A)$ for every $A \in C$. The homomorphism $F_i \rightarrow F$ define natural transformation of δ-functors $(R^P F_i) \rightarrow R^P F$ from which we derive a natural transformation of δ-functors

$$\xrightarrow{\text{Lim}} R^P F_i(A) \rightarrow R^P F(A)$$

(The co boundary homorphisms for the sequence of functor $\xrightarrow{\text{Lim}}$ $R^P F_i$ are defined as the inductive limit of the co boundary homorphisms relative to the $R^P F_i$). The natural transformations are equivalence.

To see this, it suffices to take an injective resolution C = C (A) of A. Then the left hand side of is $\xrightarrow{\text{Lim}} H^P [F_i\ C(A)]$ and the right side is $H^P [\xrightarrow{\text{Lim}} F_i\ C(A)]$. They are thus isomorphic since the functor $\xrightarrow{\text{Lim}}$ on the category of inductive systems on I with values in ć is exact and, in particular, commutes with taking homology of complexes. [11].

Corollary (2.6.27): Let X be a topological space and Y be subspace of X admitting a basis of para compact neighborhoods (it suffices, for example, that x be Metris able or locally compact para compact). Then for every abelian sheaf F over X, we have

$$H^P (Y, F) = \xrightarrow{\text{Lim}} H^P (U, F) =$$

The limit taken over the decreasing directed set of open neighborhoods $U \subseteq Y$.

In fact, this follows from the assumption that $H^o (Y,F) = \xrightarrow{\text{Lim}} H^o (U,F)$. The derived functor of $F \rightarrow H^o (U,F)$ are the $H^p (U,F)$ so that corollary is special case of the proposition. We should note that we also have $H^o (Y,F) = \xrightarrow{\text{Lim}} H^o (U,F)$ and therefore corollary follows if Y is closed and is contained in a single Para compact neighborhood. We also find a simple counter-example (with p=o) for the case in which no hypothesis of Para compactness is made.

By way of completeness, we indicate the following result without proof, a special case of general results on projective systems. Let X be a locally compact space. We consider the increasing directed set of the relatively compact open sub aspics U of X. Then for every abelian sheaf F over X, the restriction

homomorphism $H^p (X,F) \rightarrow \xrightarrow{\text{Lim}} H^p (U,F)$ define canonical homomorphisms (which are obviously natural transformations of δ-functors);

$$H^p (X,F) \rightarrow \xleftarrow{\text{Lim}} H^p (U,F)$$

Which are obviously bijective for p=o [11].

Chapter Three

Additive Categories

Many results or constructions in the category Mod (A) of Modules over a ring A have their counterparts in other contexts, such as finitely generated A-Modules, or graded Modules over a graded ring, or sheaves of A-Modules, etc. Hence, it is natural to look for a common language which avoids repeating the same arguments. This is the language of additive and abelian categories. In this chapter, we give the main properties of additive categories.

Section (3.1) Additive categories:

An additive category is category c for which is given, for any pair A, B of objects of c an abelian group law in Hom (A,B) such that the composition of morphisms is a bilinear operation. We suppose also

that the sum and the product of any two objects A, B of c exist. It is sufficient, moreover, to assume the existence of the sum or the product of A and B exists; the existence of the other can be easily deduced and, in addition, $A \otimes B$ is canonically isomorphic to $A\times B$.

(Supposing, for example, that $A\times B$ exists, we consider the morphism $A \rightarrow A\times B$ and $B \rightarrow A\times B$ whose components (i_A,o), respectively, (o,i_B), we check that we obtain thereby a representation of $A\times B$ as a direct sum of A and B). Finally, we assume the existence of an object A such that $i_A = o$; we call it a zero object ofc. It comes to the same thing to say that Hom (A,A) is reduced to o, or that for any $B\in C$, Hom (A,B) [or Hom (B,A)] is reduced to o. |f A and A' are zero object, there exists an unique isomorphism of A to A' (that is, the antique zero element of Hom [A,A')].

The dual category of an additive category is still additive.

Now let C be an additive category and u: $A \rightarrow B$ amorphism in C. For u to be injective (respectively, surjective) it is necessary and sufficient that there not exist a non-zero morphism whose left, respectively, right, composite with u is o. We call a generalized kernel of u any monomorphism i: A' → A such that morphism from c → A which are right zero divisors of u are exactly the ones that factor through c → A' $\xrightarrow{I}$ A. Such a monomorphism is defined up to equivalence, so among the generalized kernels of u, if any there is exactly one that is a sub object of A. We call it the kernel of u and denote it by Keru. Dually we define the co-kernel of u (which is a quotient object of B if it exists), denoted Coker u. We cal image (respectively, co-image) of the morphism u the kernel of its co-kernel (respectively, the co-kernel of its kernel) if it exists.

It is thus a sub object of B (a quotient object of A).

We denote them as lm u and coim u. If u has an image and a coim age, there exists a unique morphism ū: coim → lm u such that u is the composite A → coim u $\xrightarrow{\bar{u}}$ lm u → B, the extreme morphism being the canonical ones.

A functor F from one additive category c to another additive category ć is called an additive functor if for morphism u, v: A → B in c, w have that F (u+v) = F (u) + F (v). The composite of additive functors is additive. If F is an additive functor, F transforms a finite direct sum of object A_i, into the direct sum of F (A_i).

Definition (3.1.1) [6. 71. 72. 93. 94]: A category c is additive if it satisfies condition (i)-(v) below:

(i) For any X, Y∈C, Hom_c (x,y) ∈Ab,
(ii) The composition law o is bilinear,
(iii) Three exists a zero object in c,
(iv) The category c admits finite co products,
(v) The category c admits finite products.

Note that Hom_c (X,Y) ≠ Ø since it is a group and for all X∈C, Hom_c (X,0) = Hom_c(0,X) = 0.

(The morphism o should not be confused with the object o.)

Notation (3.1.1) [6. 71. 72. 93. 94] If x and y are two objects of c, one denotes by $X \otimes Y$ (instead of XUY) their co product, and calls it their direct sum. One denotes as usual by XxY their product. This change of notations is motivated by the fact that if A is a ring, the forget full functor Mod (A) → set does not commute with co products.

Lemma (3.1.2) [6.71.72]: Let c be a category satisfying conditions (i) – (iii) in definition (3.1.1) consider the condition (vi) for any two objects x and y in c, three exists $Z \in C$ and morphisms $x_1: X \to Z$, x_2: $Y \to Z$, p_1: $\mathbf{Z} \to X$ andp_2: $\mathbf{Z} \to Y$ satisfying.

(3.1) $p_1 o\ i_1 = id_x$, $p_1 o\ i = o$

(3.2) $p_2 o\ i_2 = id_y$, $p_2 o\ i_1 = o$

(3.3) $i_1 o\ p_1\ i_2 o\ p_2 = id_z$

Then the condition (iv), (v) and (vi) are equivalent and the objects $X \otimes Y$ X, Y and Z are naturally isomorphic.

Proof:

(a) Let us assume condition (iv). The identity of x and the zero morphism y → x define the morphism p_1: $X \otimes Y \to X$ satisfying eq (3.1). We construct similarly the morphism p_2: $Y \otimes Y \to Y$ satisfying eq (3.2). To check eq (3.3), we use the fact that if f: $X \otimes Y \to X \otimes Y$ satisfies $f\ oi_1 = i_1$and $f\ oi_2 = i_2$ then $f = id_{x\ y}$.

(b) Let us assume condition (vi). Let $W \in C$ and consider morphismsf: → X → W and g: Y → W. Set ⊗h: = f o p_1 go p_2. Then
h: Z → W satisfies h oi_1 = f and h oi_2 = g and such an h is unique. Hence Z X ⊗Y.

(c) We have proved that conditions (iv) and (vi) are equivalent and moreover that if they are satisfied, then $Z \simeq X \otimes Y$, Replacing c with C^{op}, we get that these condition are equivalent to (v) and $Z \simeq$ Xx Y.

Example (3.1.3) [6. 71. 72. 93. 94]

(i) If A is a ring, Mod (A) and Mod^f (A) are additive categories.
(ii) Bankh, the category of C–Bankh spaces and linear continuous maps is additive.
(iii) If c is additive, then C^{op} is additive.
(iv) Let I be category, if c is additive, the category c^I of functors form I to c, is additive.
(v) If c and ć are additive, then cxć are additive.
Let F: c → ć be a functor of additive categories. One says that F is additive if for X, Y∈C, Hom_c (x,y) → Hom_c [F (x), F (y)] is a morphism of groups.

Proposition (3.1.4)[6. 71. 72]: Let F: c → ć be a functor of additive categories. Then F additive if and only if it commutes with direct sum, that is, for x and y in c;

$$F(o) \simeq o$$

$$F(x \otimes y) \simeq F(x)\, F \otimes (y)$$

Unless otherwise specified, functors between additive categories will be assumed to be additive. [6. 71.72].

Example (3. 1. 5) [6. 71. 72] Consider the category Δ and for n > 0, denote be S_i^n:[0, n] [0, n -1] (0 ≤ i ≤ n-1).

The surjective order – preserving map which takes the same value ati and i+1 in other words S_i^n (k) $\begin{cases} k & for\ k \leq i, \\ k-1 & for\ k > i, \end{cases}$

Generalization: Let k be a commutative ring. One defines the notion of ak0additive category by assuming that for x and y in c, $Hom_c(x,y)$ is a k-module and composition is k-bilinear.

Section (3.2) Complexes in additive categories:

Definition (3. 2. 2 [6. 71. 72]: (i) A differential object ($x^\cdot$, $d^\cdot_x$) in c is a sequence of objects x^k and morphism d^k (k∈Z):

$$(3.4)\ \cdots\cdots \to X^{k-1} \xrightarrow{d^{k-1}} X^{k} \xrightarrow{d^{k}} X^{k-1} \to \cdots\cdots$$

(ii) A complex is a differential object $(x^{\cdot}, d^{\cdot}_{x})$ such that

$d^{k} \circ d^{k-1} = o$ for all $k \in Z$.

A morphism of differential objects $f^{\cdot}: x^{\cdot} \to y^{\cdot}$ is visualized by a commutative diagram:

$$\begin{array}{ccccccc}
\cdots\cdots \to & X^{n} & \xrightarrow{d_x^n} & X^{n+1} & \longrightarrow & \cdots\cdots \\
& \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} & & \\
\cdots\cdots \to & y^{n} & \xrightarrow{d_y^n} & X^{n+1} & \longrightarrow & \cdots\cdots
\end{array}$$

One defines naturally the direct sum of two differential objects. Hence, we get a new additive category, the category Diff(c) of differential objects in c. One denotes by c(C) the full additive sub category of Diff (c) consisting of complex.

From now, we shall concentrate our study on the category c (C). A complex is bounded (resp., bounded below, bounded above).

If $X^{n} = o$ for $|n| >> o$ (resp., $n << o$, $n >> o$). One denotes by c* (c) (* = b, +, –) the full additive sub category of C^{ub} (c) consisting of bounded complexes (resp., bounded below, bounded above). We also use the notation C (C) = c (C) (ub for "unbounded").

One considers c as a full sub category of $C^{b}(C)$ by indentifying an object XC^{ub} C with the complex $x^{\cdot}$ "concentrated in degree 0".

$$X^{\cdot} := \ldots\ldots \to o \to x \to o \to \ldots\ldots$$

Where x stands in degree 0.

Definition (3. 2. 3): Shift functor

Let c be an additive category, let $X \in C$ (c) and let $p \in Z$. One defines the shifted complex x [p] by:

$$\begin{cases} (x [p])^n = X^{n+p} \\ d^n_{x[p]} = (-1) p\, d^{n+p}_x \end{cases}$$

If f: x → y is a morphism in c (c) one defines

f [p]: x [p] → y [p] by (f [p])^n = f^{n+p}

The shift functor [I]: x → x [I] is an auto-orphism (i.e an invertible functor) of c (c) [6].

Definition (3. 2. 4) [6]: Mapping cone

Let f: x → y be a morphism in c (c). The mapping cone of f, denoted Mc (f), is the object of c (c) defined by:

$M_c (f)^k = (X [I])^k \otimes y^k$

$$d^k_{Mc\,(f)} = \begin{pmatrix} d^k_{x\,[I]} & o \\ f^{k+1} & d^k_y \end{pmatrix}$$

of course, before to state this definition, one should check that

$d^{k+1}_x \; o \; d^k_{Mc\,(f)} = o$. Indeed:

$$\begin{pmatrix} -d^{k+2}_x & o\ -d \\ f^{k+1} & d^{k+1}_y \end{pmatrix} o \begin{pmatrix} o^{k+2}_x & \\ f^{k+1} & d^k_y \end{pmatrix} = o$$

Notice that although $M_c (f)^k = (X [I])^k \otimes y^k$, M_c (f) is not isomorphis to X [I] ⊗ y in c (c) unless f is the zero morphism.

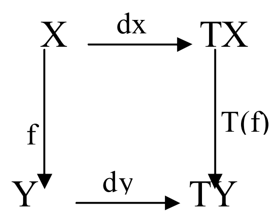
There are natural morphisms of complexes.

(3.5) α (f): y → M_c (f), β (f): M_c (f) →

X (I), and β (f) = o α (f) = o.

If F: c → Ć is an additive functor, then G [M_c (f) ≃ M_c F (f)].

Example (3.2.6) [6] A category with translation (A , T) is a category A together with equivalence T:A $\longrightarrow$ A. A differential object (X, d_x)in a category with translation (A,T) is an object X ∈ A together with morphism d_x : X $\longrightarrow$T (x). A morphism f: (x, d_x) $\longrightarrow$(y, d_y) of differential objects a commutative diagram

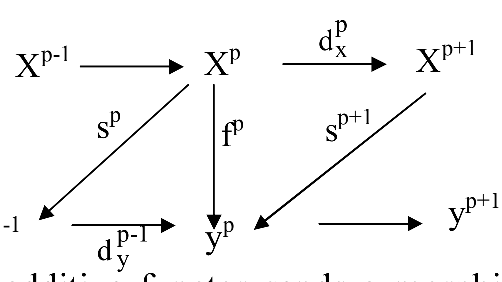


One denotes by A_d category consisting of differential objects and morphism of such objects. If A is additive, one says that a deferential objects (x, d_x) in (A, T) is a complex if the composition X $\xrightarrow{dx}$ T (x) $\xrightarrow{T(dx)}$ T^2 (x) is Zero. One denotes by A_c the null sub category of A_d consisting of complexes.

Definition (3.2.5) Homotopy [6] Let c be an additive category.

(i) A morphism f: x → y in (c) is homotopic to zero if for all p there exists a morphism s^p: $x^p \rightarrow y^{p-1}$
such that: $f^p = s^{p+1} \circ d_x^p + d_y^{p-1} \circ s^p$.
two morphism f, g: x → y are homotopic if f – g is homotopic to zero.

(ii) An object x in c (c) is homotopic to o if id_x is homotopic to zero. A morphism homotopic to zero is visualized by diagram (which is not commutative).



Note that an additive functor sends a morphism homotopic to zero to a morphism homotopic to zero.

Example (3.2.7): The complex o → X' → X' ⊗ X" → X" → o is homotoic to zero.

Example (3.2.8) [6.71.72]: We shall construct a new category by deciding that a morphism in c (c) homotopic to zero is isomorphic to the zero morphism. Set:

Ht (x, y) = { f: x → y; f is homotopicto o }.

If f: x → y and g: y→ z are two morphism in c (c) and if f or g is homotpic to zero, then go f is homotopic to zero. This allows us to state:

Definition (3.2.6): The homotopy category k (c) id defined by:

ob [k (c)] = ob [c (c)]

$Hom_{k(c)}$ (X, Y) = $Hom_{c(c)}$ (X, Y) / Ht (X, Y).

In other words, a morphism homotopic to zero in c (c) becomes the zero morphism in k (c) and homotopy equivalence becomes an isomorphism.

One defines similarly K* (c), (* = ub, b, +, -). They are clearly additive categories endowed with an autorphism, the shift functor [1]: x → x [1].

Sextion (3.3) Double complexes:

Definition (3.3.7) [6. 93. 94]: Let c be as above an additive category. A double complex ($X^{\cdot\cdot}$ d_x) in c is the data of

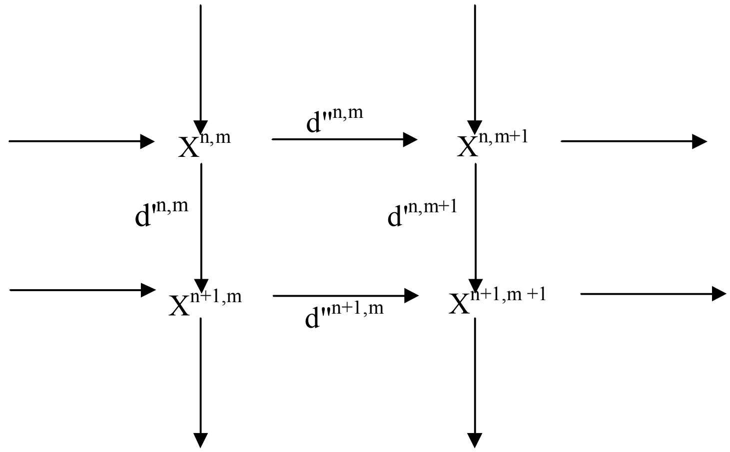
$$\{ x^{n,m}, d'^{n,m}_{x}, d''^{n,m}_{x} ; (n,m) \in Z x Z \}$$

Where $x^{n,m} \in C$ and the "differentials" $d'^{n,m}_{x} : x^{n,m} \to$ $x^{n+1.m}$, d : $d''^{n,m} : x^{n,m} \to x^{n+1.m}$ satisfy:

eq (3. 6) $d^{2}_{x} = d''^{2}_{x} = o$, d' o d" = d" o d'.

One can represent a double complex by a commutative diagram

(3.7)



One defines naturally the notion of a morphism of double complexes, and one obtains the additive category c_2 (2) of double complexes.

There are two functors, $F_{1.}$ F_{11} : c^2 (c) → c [c (c)] which associate to a double complex x the complex whose objects are the rows (resp., the cohumns) of x. There two functors are clearly isomorphisms of categories.

Now consider the finiteness conditions:

(3.8) for all p∈Z. { (m,n) ∈ZxZ; $X^{n,m} \neq$ o,m+n = p }

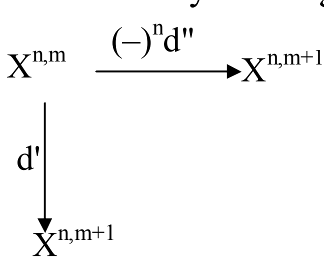
Is finite and denote by c_f^2 (c) the full sub category of c^2 (c) consisting of objects x satisfying eq (3.8).

To such an x one associates its "total complex" tot (x) by setting:

$$\mathrm{tot}\,(x)^p = \otimes_{m+n=p}\ x^{n,m}$$

$$d^p_{\mathrm{tot}(x)}|\ x^{n,m} = d'^{n,m} + (-1)^n\, d''^{n,m}$$

This is visualized by the diagram:



Proposition (3.3.9) The differential object $\{ \text{tot}(x)^p, d^{p}_{\text{tot}(x)} \}_{p \in Z}$ is a complex

(i.e., $d^{P+1}_{\text{tot}(x)} \text{ o } d^{p}_{\text{tot}(x)} = o$) and tot : c^2_f (c) → c (c) is fucntor of additive categories.

Proof. For (n,m) ∈ZxZ, one has

$$\text{dod } (x^{n,m}) = d'' \text{ o } d'' (x^{n,m}) + d' \text{ o } d' (x^{n,m})$$

$$+ (-)^n d'' \text{ o } d' (x^{n,m}) + (-)^{n+1} d' \text{ o } d' (x^{n,m}) = 0$$

We check that tot is an additive functor.

Example (3.3.10): Let $f^{\cdot}: x^{\cdot} \rightarrow y^{\cdot}$ be morphism in c (c). Consider the double complex $Z^{\cdot\cdot}$ such that $Z^{-1.\cdot} = X^{\cdot}, Z^{o,\cdot} = Y^{\cdot}, Z^{i,\cdot}$

= o for i ≠ 1. o, with differentials $f^j: Z^{-1.j} \rightarrow Z^{o,j}$.

Then eq (3.9) tot ($Z^{\cdot\cdot}$) $M_c(f^{\cdot})$. [6. 71. 72. 93. 94]

Bifunctor: Let c,c' and c'' be additive categories and let → c'' be an additive bifunctor (i.e, F (.,.) is additive with respect to each argument).

It defines an additive bifunctor c^2 (F): c (c) x c (c') → c^2 (c''). In other words, if X∈c (c) x'∈c)c') are complexes, then c^2 (F) (x, x') is a double complex.

Example (3.3.11): Consider the bifunctor **..**⊗: Mod (A^{op}) x Mod (A)→ Mod (Z). We shall simply write ⊗instead of c^2 (⊗). Hence, for x∈C^- [Mod (A^{op})] and y∈C^- [Mod (A^{op})], one has

$$(x \otimes y)^{n,m} = x^n \otimes y^m$$

$$d'^{n,m} = d^n_x \otimes y^m, d''^{n,m} = x^n \otimes d^m_y. \quad [6]$$

The complex $\text{Hom}^{\cdot}$

Consider the bifunctor $\text{Hom}_c : c^{op} \text{ x } c \rightarrow \text{Mod (z)}$.

We shall write $Hom_c^{\cdot\cdot}$ instead of c^2 (Hom_c), if x and y are two objects of c (c), one has

$Hom_c^{\cdot\cdot} (X,Y)^{n,m} = Hom_c (X^{-m}, y^n)$,

$d'^{n,m} = Hom_c (X^{-m}, d_y^n), d''^{n,m} = Hom_c [(-)^n d_x^{-n-1}, y^m]$

Note that $Hom_c^{\cdot\cdot}$ (X,Y) is a double ocmp;lex in the category Mod (z) and should not be confused with the group $Hom_{c(c)}$ (X,Y).

Let $x \in C^-$ (c) and $y \in C^+$ (c), one sets

(3.10) $Hom_c^{\cdot} (X,Y) = tot [Hom_c^{\cdot\cdot} (X,Y)]$

Hence, $Hom_c(X,Y)^n = \otimes_k Hom_c (X^k, Y^{n+k})$ and

$d^n : Hom_c (X,Y)^n \rightarrow Hom_c (X,Y)^{n+1}$

is defined as follows. To $f = \{f^k\}_k \in \otimes_{k \in Z} Hom_c(X^k,Y^{n+k})$

one associates $\otimes d^n f = \{g^k\}_k \in_{k \in Z} Hom_c(X^k,Y^{n+k})$,

with $g^k = d'^{n+k,-k} f^k + (-)^{k+n+1} d''^{k+n+1.-k-1} f^{k+1}$

In other words the components of d f in $Hom_c (X,Y)^{n+1}$ will be

eq (3.11) $(d^n f)^k = d_y^{k+n} \circ f + (-)^{n+1} f^{k+1} \circ d_x^k$.

Proposition (3.3.12): Let c be an additive category and let x, y∈ c (c) thereare isomorphism

$Z^o [Hom_c^{\cdot} (X,Y)] = kerd^o \simeq Hom_{c(c)} (X,Y)$,

$B^o [Hom_c^{\cdot} (X,Y)] = | md^{-1} \simeq Ht (X,Y)$,

$H^o [Hom_c^{\cdot} (X,Y)] = kerd^o / (| md^{-1}) \simeq Hom_{k(c)} (X,Y)$.

Proof:

(i) Let us calculate Z^o [$Hom_c^{\cdot}$ (X,Y)]. By eq (3.11), the component of d^o $\{f^k\}_k$ in Hom_c (X^k, Y^{k+1}) will be zero if and only if d^k_x o$f^k = f^{k+1}$ o d^k_y, that is , if the family $\{f^k\}_k$ defines a morphism of complexes.

(ii) Let us calculate B^o [$Hom_c^{\cdot}$ (X,Y)]. An element $f^k \in$ Hom_c (X^k,Y^k) will be in the image of d^{-1} if it is in the sum of the image of Hom_c (X^k,Y^{k-1}) by d^{k-1}_y and the image of Hom_c (x^{k+1}, y^k) by d^k_x. Hence, if it can be written as $f^k = d^{k-1}_y$ o s^k + s^{k-1} o d^k_x . [6. 71. 72].

Section (3.4) Simplicial constructions:

We shall define the simplical category and use it to construct complexes and homotopies in additive categories.

Definition (3.4.8) [6. 71. 72]:

(a) The simplical category, denoted by Δ, is the category whose objects are the finite totally ordered sets and the morphism are the order-preserving maps.

(b) We denote by Δ_{inj} the sub category of Δ such that ob (Δ_{inj}) = ob (Δ), the morphisms being the injective order-preserving maps. For integers n,m denote by [n,m] the totally ordered set { k∈Z; n ≤ m }.

Proposition (3.4.13) [6]:

(i) The natural functor $\Delta \rightarrow set^f$ is faithful,

(ii) The full sub category of Δ consisting of objects$\{[o,n]\}n \geq -1$ is equivalent to Δ

(iii) Δ admits an initial object, namely Ø, and a terminal object, namely {o}.

The proof is obvious. Let us denote by

d : [o,n] → [o,n–1] (o ≤ i ≤ n+1)

The injective order-preserving map which does not take the value i. In other words

$$d_i^n(k) = \begin{cases} k & \text{for } k < i, \\ k+1 & \text{for } k \geq i, \end{cases}$$

one checks immediately that

(3.12) $d_j^{n+1} \circ d_i^n = d_i^{n+1} \circ d_{j-1}^n$ for $o \leq i \leq j \leq n+2$

Indeed, both morphisms are the unique injective order-preserving map which does not take the values i and j.

The category Δ_{inj} is visualized by:

(3.13) $\emptyset - d_0^{-1} \rightarrow [o] \quad - d_0^0 \rightarrow \quad [o,1] \quad - d_1^1 \rightarrow \quad [0,1.2]$

$- d_1^0 \rightarrow \qquad - d_1^0 \rightarrow$

$- d_2^1 \rightarrow$

Let c be an additive category and F: $\Delta_{inj} \rightarrow$ c a functor, we set for $n \in Z$;

$$F^n = \begin{cases} F[(o,n)] & \text{for } n \geq -1. \\ o & \text{other wise} \end{cases}$$

$$d_F^n : F^n \rightarrow F^{n+1}, \ d_f^n = \sum_{i=0}^{n+1} (-)^i F(d_i^n)$$

Consider the differential object

(3.14) $F^{\bullet} := \ldots o \rightarrow F^{-1} \xrightarrow{d_F^{-1}} F^o \xrightarrow{d_F^o} F^1 \rightarrow \ldots \rightarrow F^n \xrightarrow{d_F^n} \ldots$

Theorem (3.4.14) [6. 71. 72]:

(i) The differential object $F^{\bullet}$ is a complex.

(ii) Assume that thee exist morphism $S_f^n : F^n \rightarrow F^{-1}$ $(n \geq o)$

Satisfying;
$$\begin{cases} S_F^{n+1} \circ F(d_o^n) = 1\, d_{Fn} & \text{for } n \geq -1 \\ S_F^{n+1} \circ F(d_{i+1}^n) = F(d_i^{n-1}) \circ S_F^n & \text{for } i > o, n \geq 0. \end{cases}$$

Then $F^{\bullet}$ is homotopic to zero.

Proof (i) By eq (3.12) we have

$$d_F^{n+1} \circ d_F^{n} = \sum_{i=0}^{n+2} \sum_{i=0}^{n+1} (-)^{i+j} F(d_j^{n+1} \circ d_i^{n})$$

$$= \sum_{0\le j\le i\le n+1} (-)^{i+j} F(d_j^{n+1} \circ d_i^{n}) + \sum_{0\le i<j\le n+2} (-)^{i+j} F(d_j^{n+1} \circ d_i^{n})$$

$$= \sum_{0\le j\le i\le n+1} (-)^{i+j} F(d_j^{n+1} \circ d_i^{n}) + \sum_{0\le i\le j\le n+2} (-)^{i+j} F(d_j^{n+1} \circ d_{i-1}^{n})$$

$$= o$$

Hence, we have used

$$\sum_{0\le i\le j\le n+1} (-)^{i+j} F(d_{n+1_i} \circ d_{j-1}^{n}) = \sum_{0\le i\le j\le n+1} (-)^{i+j+1} F(d_i^{n+1} \circ d_j^{n})$$

$$= \sum_{0\le j\le i\le n+1} (-)^{i+j+1} F(d_j^{n+1} \circ d\ )_i^{n}$$

(ii) We have

$$s_P^{n+1} \circ d_F^{n} + d_F^{n-1} \circ s^n$$

$$= \sum_{i=o}^{n+1} (-1)^i\ s_F^{n+1} \circ F(d_{n_i}) + \sum_{i=o}^{n} (-1)^i F(d_i^{n-1} \circ s_F^{n})$$

$$= s_F^{n+1} \circ F(d_o^n) + \sum_{i=o}^{n} (-1)^{i+1} s_F^{n+1} \circ F(d_{i+1}^n) + \sum_{i=o}^{n} (-1)^i F(d_i^{n-1} \circ s\ )_F^n$$

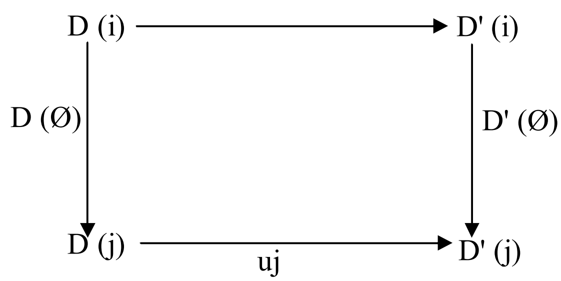
$$= id_{Fn} + \sum_{i=o}^{n} (-1)^{i+1} F(d_i^{n-1} \circ s_F^n) + \sum_{i=o}^{n} (-1)^i F(d_i^{n-1} \circ s_F^n)$$

$$= id_{Fn}.$$

Example (3.4.15) Torus T^2. We can view T^2 as quotient of a rectangle, this makes the drowning of triangles easier. There is a simple cw-triangulation where one divides the rectangle by a diagonal into two triangles. It gives a fast calculation of homology. One can gate into ninerectangle and each of this into twotriangles. Then H_0 and H_2 compiled from the dimension of H_i can be compiled from the invariance of Euler character – is tic under taken homology.

Section (3.5) Categories of diagrams:

A diagram scheme is a triple (I, Φ, d) made up of two sets I and Φ and a function d from Φ to IxI. The elements of I are retraces, the elements of Φ are arrows of the diagram and if Ø is an arrow of the diagram d (Ø) is called its direction, characterized as the source and target of the arrow (these are therefore vertices of the scheme). A composite arrow with source I and target j is, by definition, an non-empty finite sequence of arrows of the diagram, the source of the first being I, the target of each being the source of the next and the target of the last one being j. If c is a category, we call diagram in c from the scheme s a function D which associates to each i∈I and an object D (i) ∈C and to any arrow Ø∈Φ with source I and target j, a morphism D (Ø): D (i) → D (j). The class of such diagram will be denoted c^s; it will be considered a category, taking as morphism from D to D' a family of morphism u_i: D (j) → D' (i) such that for any arrow Ø with source i and target j the following diagram commutes:



Morphisms of diagrams compose in the obvious way, and it is trivial to verify the category axioms. If D is a diagram on the schemes s, then for any composite arrow Ø = ($Ø_{1.}$ ……., $Ø_k$) in s, wedefine D(Ø)=D($Ø_k$) ……D($Ø_1$); itis a morphismfrom D(i)→ D(j) if I and j are, respectively, the source and target ofØ. We call D a commutative diagram if we have D(Ø) = D(Ø') wheneverØ are two composite arrows with the same source and same target. More generally, if R is a set consisting of pairs (Ø,Ø') of composite arrows having the same source and target, and of composite arrows whose source equals its target, we consider the subcategory $c^{s,R}$ of c^s consisting of diagrams satisfying the commutatively condition D(Ø) for (Ø, Ø') ∈R and D(Ø) is the identity morphism of D(i) if Ø∈R has I as its source and target.

We have consider still other types of commutation for diagrams, whose nature varies according to the category in question. What follows seems t cover the most important cases. For any (i, j) ∈IxI we take a set R_{ij} of formal linear combinations with integer coefficients of composite arrow with values in an additive category c, then for any L∈R_{ij}, we can define the morphsim D (L): D(i) → D (i), by replacing, in the expression of L. a composite arrow Ø and e_i by the identity element of D (i). If we denote by R the union of the R_{ij}, we will say that D is R-commutative if all the D (L), L∈R, are o. we call diagram scheme with commutatively conditions a pair (S,R) ∑ consisting of a diagram scheme s and a set R as above. For any additive category c, we can they consider the sub category $c^{\in}$ of c^{s} consisting of the r-commutative diagram.

Proposition(3.5.16)[11] Let ∑ be a diagram scheme with commutatively conditions and c an additive category. Then the category c^{ε} is an additive category and if c has infinite direct (respectively, infinite direct sums), so does c^{ε}. Moreover, if c satisfies any one of the axioms.

Moreover, if D,D' ∈ C^{ε}, and if U is a morphism from D to D', then its Kernel (respectively, co kernel, image co image).

Is the diagram formed b the Kernels (respectively,….. of the components u_i, the morphism in this diagram (corresponding to the arrows of the scheme) being obtained from those of D (respectively, those of D',…..) in the usual way by restriction (respectively, passage to the quotient). We interpret analogously the direct sum of the direct product of a family of diagrams. Sub objects D' of the diagram D are identified as families (D' (j) of sub objects of d(i) such that or any arrow Ø with source i and target j we have D (Ø): D' (i) →D' (j); then D'(Ø) is defined as the morphism D'(i) →D'(j) defined by D (Ø). The quotient objects of D are defined dually.

If s is a diagram scheme, we call the dual scheme and denote it by s^{o}, the scheme with the same retraces and the same sets of arrows as s, but with the source and target o the arrows of s interchanged. If, moreover, we give a set R of commutatively conditions for s, we will keep the same set for s^o. Using this convention, for an additive category c, the dual category of $c^{\in}$ can be identified as $(c^o)^{\varepsilon o}$.

Let c and c' be two additive categories and ∈ be a diagram scheme with commutatively condition. For any functor F from c to ć, we define in the obvious way the functor F^{ε} from c^{ε} to $c^{''\varepsilon}$, called the canonical extension of F to the diagram. F^{ε}behaves formally like a functor with respect to the argument F, in particular, a natural transformation F→ F'^{ε}. Finally we note that for a composite functor, we have $(GF)^{\varepsilon} = F^{\varepsilon}G^{\varepsilon}$, and the exactness properties of a functor are preserved by extension to a class of diagrams.

Example (3.5.17) [11] Inductive systems and projective systems. We take as a set of vertices preordered set O, with arrows being pairs (I, j) of retraces' with $i \leq j$, the source and target of (i. j) of vertices with $i \leq j$, the source and target of (i, j) being i and j, respectively. The commutability relation are (i , j) (j , k) = (i , k) and (i , i)) = e_i. The corresponding diagrams (for a give category c, not necessarily additive) are known as inductive systems over I with values in c. If we change I to the opposite preordered set, or change c to c^o we get projective system over I with values in c. An important case is the one in which I is the lattice of open sets of a topological space x, ordered by containment: we then obtain the notions of pre-sheaf over x with values in c.

Section (3.6) Combinatorial topology of simplicial complexes:

Some to polemical spaces in combinatorial terms. This will then be used to calculate their invariants purely algebraically using the combinatories of the space rather than the space itself.

A simplical complex is a set v together with a family k of finite non-empty subsets of v such that with any element A∈K, family k also contains all subsets of A. [9. 73. 74.75. 40. 41].

Lemma (3.6.18):

(a) Any simplicial triangulation T defines a simplicial complex k(T).

(b) To any simplicial complex k we can associate a topological space [k] called its realization. It comes with a triangulation T such that k (T) is naturally identified with k.

Proof:

Procedure (a) has been described above. In (b) we start by associating to each finite set A∈K a topological simplex σ_A with vertices A (i.e., with vertices parameterized by A) this gives a topological space $\tilde{k} \stackrel{def}{=} \cup_{A\in K} \sigma_A$. Then the topological space |k| is obtained as a quotient k/ ~ of k by the equivalence relation ~ on x given by x∈ σ_A and y ∈ σ_B are equivalent if (i) x lies in the facet σ_A, A∩B ⊆ σ_B (iii) the coordinates of x and y with respect to the set of vertices A∩B are the same (i.e., x and y are identified by the canonical identification of topological simplices σ_A, A∩B and σ_B, A∩B given by the obvious identification of the sets of vertices of these two simplices).

Notice that the canonical map Π: k → |k| is injective on each simplex $\sigma_A \subseteq \tilde{k}$ gives a homomorphism Π A: $\sigma_A \to \Pi(\sigma_A)$ so, one can identify the image withσ_A and then σ_A's cover |k| and one check that they form a triangulation T of K. [9. 73.75.77. 79. 42].

Theorem (3.6.19) If we start with a triangulated topological space (X,T) then the realization |k(T)| of the corresponding simplicial complex K (T) is canonically Homomorphism to X.

Proof:

It is easy to construct a continuous map Π: | k → x for k = k (T). Since | k | has a quotient topology from $\tilde{k}$ such map is the same as a continuous map $\tilde{\Pi}$: $\tilde{k}$ → x such that x ~ y⟹$\tilde{\Pi}$ (x) = $\tilde{\Pi}$ (y). Now, for any simplex $\alpha \in T$ I will denote by $\gamma\alpha$ its set of vertices. Then a simplices $\sigma\gamma_\alpha \tilde{K}$ and $\alpha \subseteq$ x can be canonically identified since the sets $\gamma\alpha$ for some $\alpha \in T$ and then $\sigma_A = \sigma_{\gamma\alpha}$ $\underset{\longrightarrow}{\alpha \tilde{=}}$ $\subseteq$ x.

Since $\tilde{\Pi}$ is continuous on each σ_A it is continuous on the disjoint union $\tilde{k}$. [9,73.74.75 – 79].

Lemma (3.6.20): The above formula for ∂_i gives a well defined k-map

∂: c_i (X,T;K) $\rightarrow$ c_{i-1} (X,T;K)

Proof:

(i) First one checks that the formula only depends on the orientation. For instance for two orderings xyz and zxy which give the same orientation one has $\partial\,^{\partial}_{zxy} = \partial_{xy}$ - $\partial_{yz} + \partial_{zx} = \partial_{yz} - \partial_{xz} + \partial_{xy}$ and$\partial\,^{\partial}_{zxy}$ = coincide.

Now we have defined a map from the basis of $\tilde{}$ c_i to $c_{i-1.}$ i,e, a k–linear map $c_i \rightarrow c_{i-1.}$

(ii) Next one needs to check hat the map descends to $c_i \rightarrow c_{i-1.}$ i.e, that opposite orientations produce opposite results. For instance for two orderings xyz and yxz which give opposite orientations one has $\partial\,^{\partial}_{zxy} = \partial_{xz}$ - $\partial_{yz} + = \partial_{yx}$ which is opposite of the $\partial\,^{\partial}_{zxy} = \partial_{yz} - \partial_{xz} + \partial_{xy}$

The two requirements together say that for any permutation T of o,……..,i onehas$\partial\,^{\partial}_{y,0\,.....\,vi} = \varepsilon_T - \sigma_{vo....vi}$ where ε_r is the sign of the permutation T. This statement it suffices to check when T is one of the transpositions T_p which exchange p-1 and p, $1 \leq p \leq I$, [9,73 – 79,40,41].

Remark (3.6.21) [9.73.79]The above formula for ∂ is for the complex associated to a triangulation T, if one uses an oriented triangulation Z^ε = (T,o) then one can adjust the formula so then one needs on extra orientation of simplices in T. The boundary operatory operator $\partial_i : c_i \rightarrow c_{i-1}$ sends an oriented i–simplex y $\in \Sigma$ to the sum of its faces, with certain orientation and a certain sign. For a give face z if the orientation from ∂_I y agrees with orientation on z from Σ we do not need any adjustments, otherwise we change the orientation from ∂_I y to the one from Σ and change the sign.

Corollary (3.6.22) [9.73. 79]: C_* (X,T;K) is a complex (of chains).

Proof. Homology groups of a topological space. We have seen that any triangulation T of x associates to a topological space x the homology groups.

$$H_i(X,T;K) \overset{def}{=} H_i[(C_*(X,T;K), \partial)]$$

However, by the next theorem these groups are really invariants of x itself so we call them the homology groups of x and denote them by H_i (X,K).

Theorem (3.6.23) [9.73. 79] The homology groups H_i (X,T;K) do not depend on the choice of a triangulation T, in the sense that for any two triangulations of x there is a canonical isomorphism

$$ØT'', T' : H_i(X,T';K) \xrightarrow{\cong} H_i(X,T'';K).$$

Proof. We say that a triangulation s is a refinement of a triangulation T if for each α ∈ T the subset $S_\alpha = \{ \sigma \in S; \subseteq\sigma \quad \alpha \}$ is triangulation of α. Now the theorem follows from the following lemma.

Lemma (3.6.24) [9.73. 79]:

(a) For a refinement S of T there is a canonical isomorphism $H_i(X,T;K) \xrightarrow{\cong} H_i$ (X,S;K) obtained by sending $\alpha \in T_i$ with orientation u to $\Sigma_{\alpha \in S\alpha \cap Si}$ (σ,0/σ) where % is the orientation o restricted to σ.

(b) Any two triangulation T',T'' of x have a common refinement T.

Example (3.6.25) [91 73. 79.40.41] S^3 is the unit sphere $S \subseteq R^3$ which we can think of as C^2.

Then $S = \{x \in R^1; (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 1\}$ can be written as

$$S = \{ z \in R^2; | z^1 |^2 + | z^2 |^2 = 1 \}.$$

This point of view makes it obvious that the group $T \simeq s^1$ of unit complex numbers acts on s by $z (z^{1.} z^2) = (zz^{1.} zz^2)$.

This is a free action (i.e; there are no stabilizers), and the quotient is homomrhic to s^2. The quotient map $s \to s^2$ is called Hopf map. This is one basic example of a nontrivial vibration; all fibers are homorphic (to s^1) but the map is still quite nontrivial. We will revisit the Hopf map once we acquire the machinery of spectral sequences.

However let us consider the quotients S/μ_n where $\mu_n \subseteq T$ is the group of all n^{th} roots of unity in c. Then H_* $(S/\mu_n; R)$ is naturally identified with H_* $(S;R)$ and the same is true for homology with coefficients in Z/mz as long as m is prime to n.

However when m is not prime to n then H_* $(S/\mu_n; Z/mZ)$ is more complicated then H_* $(S;R)$. All such complications (for all m's) are already stored in H_* $(S/\mu_n; Z)$.

One can check the above statement using simplicial triangulations; however it will be much easier to do it with the machinery of sheaves. It provides as systematic use of maps in calculating homology.

Example (3. 6. 26) Triangulations of spheres. To describe Triangulation of S^- we choose an orientation of S^- and n distinct points A, … A_n that go in the direction of the orientation. The Triangulations is given by 0 – simplices $T_0 = \{A_{1.} \ldots, A_n\}$ And I – simplices $T_1 = \{\sigma A_1 A_2 \ldots, \sigma A_n A_1\}$ (I denote by σAB or gust AB the sengment from A to B) $If_n = 1$ this is not a simplice complex since A_1A_1 is not really a I – simplex y or detention its circle hence not homomorphic to σ1. N =2 still does not give simplicail complex since the intersection

$\sigma A_1A_2 \cap \sigma A_1A_2$ consist of two points so it is not a simplicail. For $n \geq 3$ we do not get a simplicail complex. The associated simplicail complex has vertices $\gamma = \{A_{1.} \ldots, A_n\}$and $K = \{A_{1.} \ldots, A_n\{A_1A_2\}, \ldots\{A_1A_2\}\}$. For any finite set γ with n elements $=\{A\subset \gamma; A\subseteq \emptyset\}$ is asimplicail complex. Its realization $|K|$is the simplex or of dimension $|\gamma|$.

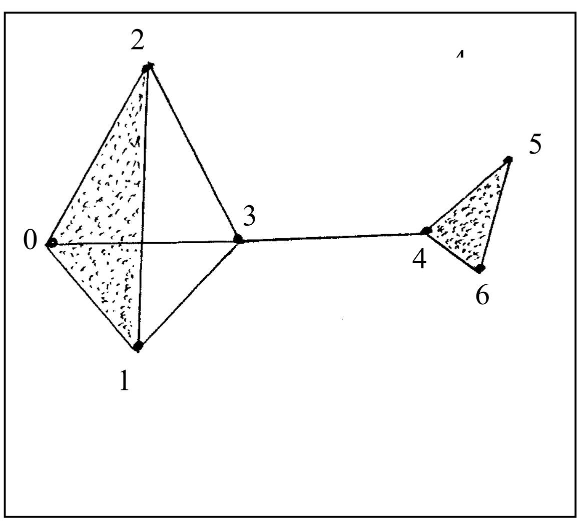
However, if remove the largest simplex $L = \{A \subseteq \gamma;\ \gamma \neq A \neq \emptyset\}$ the realization is the boundary of $\sigma\gamma$, ie, a sphere of dimension $|\gamma|_{-1}$.

Section (3.7) Simplicial complexes:

Definition (3. 7. 8) [9] A simplicial complex k is a pair k = (v,s) where:

- The component v is a totally ordered[2] set, the set of vertices of k.
- The component s is a set of non-empty finite rates of v, the simplice of k, satisfying the properties;
  For every $\upsilon \in V$, the singleton $(\upsilon) \in S$.
  For every $\sigma \in V$, then $\emptyset \neq \sigma \subset \sigma$ simplies $\sigma \in V$.

For example the small simplicial complex drawn here



The butterfly simplicial complex (y von sire's terminology

is mathematically defined as the object B = (v,s) with

$$V = (0,1.2.3.4.5.6)$$

$$S = \left\{ \begin{array}{l} (0), (1), (2), (3), (4), (5), (6). \\ (0,1), (0,2), (0,3), (1.2), (1.3), (2.3), (3.4), (4.5), (5.6), (6). \\ (0,1.2), (4.5.6). \end{array} \right\}$$

In other words, the second component, the simplex list, gives the list of all vertex combinations which are (abstractly) spanned by a simplex,. The vertex set v could be for example ordered as the integers are. Note also, because the vertex set is ordered, the list of vertices of a simplex is also ordered, which allows us to use a sequence notation (….) and not a subset notation {…..} for a simplex and also for the total vertex list.

A simplicial complex can be infinite. For example if v = N and $s = \{ (n) \}_{n \in N} \cup \{ (o,n) \}\ n \geq 1$m the simplicial complex so obtained could be understood as an infinite bunch of seginents. Standard algebraic topology proves that most "sensible" homogony types can be modeled as simplical complexes, often infinite. We will see the notion of simplicial set, roughly similar but more sophisticated, is also much more powerful to teach this goal.

Definition (3. 7. 9) [9]: For example the set of simplices s_o (k) is the set of singletonss$_o$ (k) $= \{ (v) \}_{v \in V}$. The set of 2-simplices of the butterfly B is { (0,1.2), (4.5.6)}; in the same case, the set of 1-simplices has ten elements.

Definition (3. 7. 10): Let k = (v, s) be a simplicial complex. Then the chain-complex c_* (k) canonically associated with k is defined as follows. The chain group c_n (k) is the free module generated by s_n (k). Let (u_o ……., u_n) be an n-simplex, that is, a generator of s_n (k). The boundary of this generator is then defined as;

d_n [(v_o,…., v_n)] = ($v_{1.}$ $v_{2.}$…., v_n)– (v_0, $v_{2.}$ $v_{3.}$ ….. v_n) + ….+

$(-1)^n$(v_0,$v_{1.}$….v_{n-1}) and this definition is linearly extended to c_n (k).

A variant of this definition is important.

Definition (3. 7. 11) [9]: Let k = (v, s) be a simplicial complex. Let $n \geq 1$ and $o \leq i \leq n$ be two integers n and i. Then the face operator ∂_i^n is the linear map ∂_i^n(k); c_n (k) → c_{n-1} (k) defined by;

$$\partial \, [\, (v_0, \ldots, v_n) \,] = (v_0, \ldots\ldots, v_{i-1.}\ v_{i+1.} \ldots\ldots v_n);$$

the i–th vertex of the simplex is removed, so that an (n–1) – simplex is obtained.

Application (3.7.27): A computing a homology group amounts to computing the relevant boundary matrices, and to determine a kernel, and image and the equation first one by the second one. For example, if we want to compute the homology group H_i (B), the 1-dimensional homology group of or butter f we have to describe the kernel of d_1.

$$\text{Ker } d_1 = R\Big((0, 1) + (1.\ 2) - (0, 2)\Big)$$

$$\Theta\ R\Big((0, 1) + (1.\ 3) - (0, 3)\Big)$$

$$\Theta\ R\Big((0, 2) + (2.\ 3) - (0, 3)\Big)$$

$$\Theta\ R\Big((0, 5) + (5.\ 6) - (4.\ 6)\Big)$$

and the image of d_2.

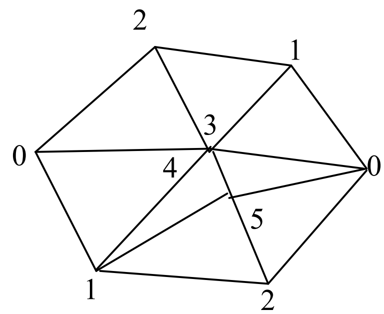
$$\text{Im}d_2 = R\Big((0, 1) + (1.\ 2) - (0, 2)\Big)$$

$$\Theta\ R\Big((4, 5) + (5.\ 6) - (4.\ 6)\Big)$$

Note un particular the limping cycle (1. 2) + (2. 3) – (1. 3) is the alternate sum of the first there ones is the discretion of ker d_1. So that the homology group H_1 (B) is isomorphic to R^2 with (0, 1) +(1. 3) -

(0, 3) and (0, 2) + (2. 3) as possible represent ants of genre lore, but adding to such a represent ants and arbitrary boundary gives another represent ants of the same homology class.

Let us examine for example the case of the real projective plane P^2 R. It can be proved the minimal triangulation of P^2 R as asimplicial complex is described by this figure:



This simplical complex has six vertices, fifteen edges and ten triangles. The 1 – skeleton is a complete group with six vertices, any to vertices are connected by an edge 4. Computing by hand the homology groups of this simplical complex is a little lengthy. The Kanzo program obtains the result.

Chapter Four

Abelian Categories

In this chapter we are dealing with an abelian category. This chapter also develops chapter three and gives the relations between additive categories and abelian categories c, we shall assume that c is a full abelian sub-category of a category Mod (A) for some ring A. This makes the proofs much easier and more over there exists a famous theorem (due to Freyd & Mitchell) that assets that this is in fact always the case (up to equivalence of categories).

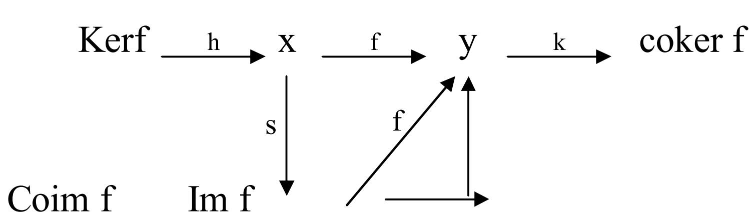
For us to be able to do any kind of useful homological algebra, we need to work with structures that are richer than just plain categories. We want to be able to talk about products, kernels and pull-backs (none of which necessarily exist in an arbitrary category) and to do more interesting things with exact sequences. We know that we can do all these things in the category of R-modules but we want to work with something a little more general than what. This compromise between abstractness and usefulness motivates the following definition:

Section (4.1) Abelian categories [6.71.72.93.94]: Let c be an additive category which admits kernels and co kernels. Let f: x → y be a morphism in c. one defines.

Coim f = Coker h, where h: ker f → x

Im f = kerk, where k: y → coker f

Consider the diagram:



Since f oh = o, f fa$\acute{f}$ctors uniquely through f, and k of factors through k o $\acute{f}$. Since k of = Ko$\acute{f}$os = o and s is an epimorphism, we get that k 0$\acute{f}$ = 0. Hence $\acute{f}$ factors through ker k = Imf, we have thus constructed a canonical morphism.

(4.1) CoImf $\xrightarrow{u}$Imf

Definition (4.1.1) [73.79.40. 42] An additive category u is a category with zero object in which any two objects have a product and in which the sets of morphism us (A,B) and abelian groups such that the composition.

u (A,B) x u (B,c) → u (A, c) is bilinear

Definition (4.1.2) [9.73 . 79]: If u and B are additive categories, then a functor

F: u → B is called additive if, for every A, B∈U,

F: u (A, B) → B (FA, FB) is a homomorphism.

Equivalently, F preserves direct sums (of two objects).

Definition (4.1.3) [9]: An abelian category is an additive category in which:

1- Every morphism has a kernel and a co kernel.
2- Every monomorphism is the kernel of its co kernel, and every epimorphism is the co kernel of its kernel.
3- Every morphism f can be written as f = moe, where m is a monomorphism and e is an epimorphism.

Example (4.1.1) [9.73. **79]:** The category of abelian groups is the archetypal example of an abelian category, as is the category of left (or right) modules over a ring T. The category of free abelian groups is additive but not abelian.

Now we are ready to give a more general definition of a short exact sequence.

Definition (4.1.4) [9. 73 . 79]: A short exact sequence in an abelian category is a sequence. $\xrightarrow{u} . \xrightarrow{\varepsilon}$, in which u is the kernel of ∈ and ∈ is the co kernel of μ. In particular, this means that ε o μ = o.

A sequence $\ldots \xrightarrow{\emptyset n} A \xrightarrow{\emptyset n+1} \ldots$ in an abelian category is exact at A if when we factor $\emptyset_n = \mu_n \varepsilon_n$ with μ_n monomorphism $\in_n$ epimorphic, then the dequence, $\xrightarrow{\mu n}, \xrightarrow{\varepsilon n+1}$ is short exact in the sense described above. Note again that the condition $\emptyset_{n+1}$ o $\emptyset_n$ = o necessarily holds.

It is also worth noting that the concepts of projective and injective objects can be applied to any abelian category and not just to the category of R–modules.

Examples (4.1.2) [6.71.72]:

(i) For a ring A and a morphism f in Mod (A), eq(4.1) is an isomorphism.

(ii) The category Ban admits kernels and co kernels. If f: X→Y is a morphism of Banach spaces, define ker = $f^{-1}(o)$ and coker f = Y Imf where Imf denotes the closure of the space Imf. Its well known that there exist continuous linear maps f: X → Y which are injective, with dense and non uosed image. For such an f, ker f = coker f = o although f is not an isomorphism. Thus coim f $\simeq$ X and Imf $\simeq$ Y. Hence, the morphism eq(4.1) is not an isomorphism.

(iii) Let A be a ring. I an ideal which is not finitely generated and let M = A/I. Then the natural morphism A → M in Mod f (A) has no kernel.

Definition (4.1.5) [6]: Let c be an additive category, one says that c is abelian if:

(i) Any f: X → Y admits a kernel and co kernel.

(ii) For any morphism f in c, the natural morphism coIm f → Imf is an isomorphism.

Exemple (4.1.3) [6. 71. 73. 93. 94]:

(i) If A is a ring, Mod (A) is an abelian category. if A is no etherian, then Mod^f (A) is abelian.

(ii) The category Ban admits kernels and co kernels but is not abelian (see example (4.1.2) (ii).

(iii) If c is abelian, then C^{0p} is abelian

(iv) If c is abelian, then the categories of complexes c^* (c) (* = ub, b, +, –) are abelian.

For example, if f: X→Y is a morphism in c (c), the complex z defined by z =ker(f^n:X^n→Y^n), with differential induced by those of x, will be a kernel for f, and similarly for co kernel f.

(v) Let I be category. Then if c is abelian, the category ć of functor from I to c, is abelian. If F, G: I → c are two functors and Ø: F → G is a morphism of functors, the functor ker Ø is given by ker Ø (x) = ker [F (x) → G (x)] and similarly with Coker Ø. Then the natural morphism coim Ø→ Im Ø is an isomorphism.

The following results are easily checked.

- An abelian category admits finite projective limits and finite inductive limits.
- In an abelian category, a morphism f is a monomorphism (resp., an epimorphism) if and only if ker f ≃ o (resp., Coker f ≃ o). If f is both a monomerphism nd an epimorphism. Then it is an isomorphism. Unless otherwise specified, we assume unit the end of this chapter c is abelian.

Consider a complex $x' \xrightarrow{f} x \xrightarrow{g} x''$ (hence, go f = o).

It defines a morphism coim f → ker g. hence, c being abelian, a morphism Imf → ker g.

Definition (4.1.6) [6]:

(i) One says that a complex $x' \overset{f}{\rightarrow} x \overset{g}{\rightarrow} x''$ is exact if Imf $\longrightarrow$ ker g.

(ii) More generally, a sequence of morphism $x \overset{Pdp}{\longrightarrow} \ldots \rightarrow x^n$ with d^i o d^{i+1} = of for alli ∈ [p, n–1] is exact if Im di $\tilde{\rightarrow}$ker d^{i+1} for all i ∈ [p, n–1].

(iii) A short exact sequence is an exact sequence o → x' → x → x'' → o.

Any morphism f: x → y may be decomposed into short exact sequences;

o → Ker f → x → coim f → o,

o → Imf → y → Coker f → o,

with coim f $\simeq$Imf.

Proposition (4.1.4) [6.71.72]: Let

(4.2) $0 \rightarrow x' \overset{f}{\rightarrow} x \overset{g}{\rightarrow} x \overset{n}{\rightarrow} 0$

Be a short exact sequence in c. Then the conditions (a) to (e) are equivalent.

(a) There exists h: $x^n \longrightarrow x$ such that goh = id x^n.
(b) Thee exists k: x → x' such that k of – id x.
(c) There exists Ψ = (k,g) and Ψ = (h^f) such that $x \overset{\psi}{\rightarrow} x' \otimes x^n$ and $x' \otimes x^n \overset{\psi}{\rightarrow} x$ are isomorphisms inverse to each other.
(d) The complex eq(4.2) is homotopic to o.
(e) The complex eq(4.2) is isomorphic to the complex o → x' → x' ⊗ x'' → x'' → o

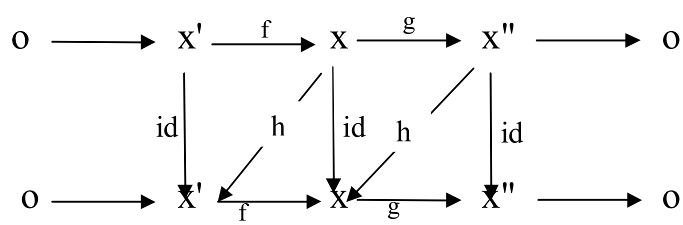
proof (a) ⟹ (c), Since g = go hog, we get go (id x → hog) = 0, which implies that id x – hog factors through kerg, that is, through x'. Hence, there exists k: x → x' such that idx – hog – fok.

(b) $\Longrightarrow$ (c) follows by reversing the arrows.

(c) $\Longrightarrow$ (a). Since gof = 0, we find g = go hog, that is (go h – idx") og = 0. Since g is an epimorphism, this implies goh – idx" = 0.

(c) $\Longrightarrow$ (b) follows by reversing the arrows.

(d) Bt definition, the complex eq(4.2) is homotopic to zero if nd only if there exists a diagram.



Such that idx' – kof, idx" – goh and idx = hog + fok. (e) is obvious by (c).

Definition (4.1.7) [6.71.72]: In the above situation, one says that the exact sequence splits.

Note that an additive functor of abelian categories sends split exact sequences in to spit exact sequences.

If A is a field, all exact sequences split, but this is not the case in general. For example, the exact sequence of Z-modules.

$$0 \to z \xrightarrow{2} z \to z/2Z \to 0$$

doesn't split.

Section (4.2) Exact functors:

Definition (4.2.8): Let F: c → c' be a functor of abelian categories. One says that:

(i) F is left exact if it commutes with finite projective limits.
(ii) F is right exact if it commutes with finite inductive limits.
(iii) F is exact if it is both left and right exact.

Lemma (4.2.5) [6.71.72]: Consider an additive functor F: c → c'

(a) The conditions below are equivalent.

- (i) F is left exact,
- (ii) F commutes with Kernels, that is, for any orphism f: X → Y, F [Ker (f)] $\tilde{\to}$Ker [F (f)],
- (iii) For any exact sequence o→ x' → x → x" in c, the sequence o → F (x') → F (x) → F (x") is exact in c'.

(b) The conditions below are equivalents

- (i) F is exact,
- (ii) For any exact sequence x' →x → x" in c, the sequence o → F (x') → F (x) → F (x") → o is exact in c'.

There is a similar result to (a) for right exact functors.

Proof:

Since F is additive, it commutes with terminal objects and products of two objects. Hence F is left exact if and only if it commutes with Kernels.

Example (4.2.6) [6.71.72.93.94]: Let A b a ring and let N be a right A-module. Since the functor N ⊗ A$^{\bullet}$ admits a right adjoint, it is right exact. Let us show that the functors Hom_A (•,•) and N ⊗ A$^{\bullet}$ are not exact in general. In the sequel, we choose A=k [x], with k a field, and we consider the exact sequence of a–modules;

$$(4.3)\quad o \to A \xrightarrow{x} A \to A/A_x \to 0.$$

Where x means multiplication by x.

(i) Apply the fucntor Hom_A (•,A) to the exact sequence eq(4.3) we get the sequence;

$$0 \to Hom_A\,(A/A_x) \xrightarrow{x} A \to 0.$$

Which is not exact since x, is not surjective. On the other hand, since x, is injective and Hom_A (.,A) is left exact, we find that $Hom_A(A/A_x,A/A_x)\rightarrow 0$. Cines Hom_A $(A/A_x,A/A_x)=0$ and $Hom_A(A/A_x,A/A_x)\neq 0$,this sequence not exact.

(ii) Apply $.\otimes A/A_x$ to the exact sequence eq(4.3). We get the sequence

$o \rightarrow A/A_x \xrightarrow{x} A/A_x \rightarrow A/xA \otimes_A A/A_x \rightarrow 0$

Multiplication by x is o on A/A_x. Hence this sequence is the same as; $o \rightarrow A/A_x \xrightarrow{0} A/A_x \otimes_A A/A_x \rightarrow o$ which shows that $A/A_x \otimes_A A/A_x \sim A/A_x$ and moreover that this sequence is not exact.

(iii) Notice that the functor Hom_A (.,A) being additive, it sends split exact sequences to split exact sequences. This shows that eq(4.3) does not split.

Example (4.2.7) [6]: We shall show that the functor $\overset{lim}{\longleftarrow}; Mod(k) \overset{top}{\longrightarrow} Mod\ (k)$ is not right exact in general.

Consider as above the k-algebra A: = k [x] over a field k.

Denote by I = A.x the ideal generated by x. Notice that $A/I^{n+1} \sim k[x]^{\leq n}$, where $k[x]^{\leq n}$, denotes the k–vector space consisting of polynomials of degree $\leq$ n.

For $p \leq n$ denote by U_{pn}; $A/I^n \rightarrow A/I^P$ the natural epimorphosis. They define a projective system of A–modules. One checks easily that

$$\underset{n}{\overset{lim}{\longleftarrow}} A/I^n \simeq k\,[\,(x)\,].$$

The ring of formal series with coefficients in k, on the other hand, for $p \leq n$ the monarchisms $I^n \rightarrow I^P$ define a projective system of A– modules and one has

$$\underset{n}{\overset{lim}{\longleftarrow}} /I^n \simeq o$$

Now consider the projective system of exact sequences of A–modules o → I^n→A → A/I^n→ o

By taking the projective limit of these exact sequences one gets the sequence o → o k [x] → k [(x)] → o which is no more exact, neither in the category Mod (A) nor in the category Mod (k).

Proposition (4.2.8) [6.71.72]: Let A^t_nbe a ring and let $o \rightarrow \{M\} \overset{fn}{\rightarrow}$ $\{M_n\} \overset{gn}{\rightarrow}$ {M"$_n$}→o be an exact sequence of projective systems of A–modules indexed by N. Assume that for each n, the map M'$_{n+1}$→ M'$_n$ is surjective. Then the sequence

$$o \rightarrow \underset{n}{\overset{\lim}{\longleftarrow}} M'_n \xrightarrow{f} \underset{n}{\overset{\lim}{\longleftarrow}} M_n \xrightarrow{g} \underset{n}{\overset{\lim}{\longleftarrow}} M''_n \rightarrow o \text{ is exact.}$$

Proof:

Let us denote for short by up the morphisms M_p→ M_{p-1} which define the projective system {M_p}, and similarly for u'$_p$, u"$_p$.

Let $\{x''_p\}_p \in \underset{n}{\overset{\lim}{\longleftarrow}} M''_n$.

Hence x"$_p$∈ M"$_p$, and ű$_p$, (x"$_p$) = $_{p-1}$.

We shall first show that u$_n$: g_n^{-1} ($\acute{x}_n$) →g_{n-1}^{-1} (x"$_{n-1}$) is surjective. Letx$_{n-1}$∈g_{n-1}^{-1} (x"$_{n-1}$). Take x$_n$∈g_{n-1}^{-1} (x"$_n$).

Then g$_{n-1}$ [x$_n$ (x$_n$) – x$_{n-1}$] = o. Hence x$_n$ (x$_n$) – x$_{n-1}$ = f$_{n-1}$ (x'$_{n-1}$). By the hypothesis f$_{n-1}$ (x'$_{n-1}$) = f$_{n-1}$ [u'$_n$ (x'$_n$)] for some x'$_n$ and thus u$_n$ (x$_n$– f$_n$ (x'$_n$) = x$_{n-1}$.

Then we can choose x$_n$∈g_n^{-1} (x"$_n$) inductively such that u'$_n$ (x$_n$) = x$_{n-1}$

Section (4.3) Injective and projective objects:
Definition (4.3.9) [6.71.72]:

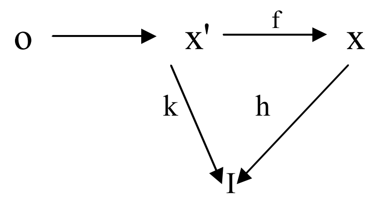
(i) An object I of c is injective if the functor Hom$_c$ (•,I) is exact.

(ii) One says that c has enough injective if for any x ∈ C there exists a monomorphism x → f with I injective.

(iii) An object p is projective in c if it is in jective in c^{op}, if the functor Hom_c (P,.) is exact.

(iv) One says that c has enough projective if for any $x \in C$ there exists an epimorphism $p \to x$ with p projective.

Proposition (4.3.9): The object $I \in C$ is injective if and only if, for any x, y∈C and any diagram in which the row is exact.



The dotted arrow may be completed, making the solid diagram commutative.

Proof. (i) Assume that I is injective and let x'' denote the co-kernel of the morphism $x' \to x$. Applying Hom_c (.,I) to the sequence $o \to x' \to x''$, one gets the exact sequence; $0 \to x' \to x \to x''$

$$Hom_c(X'',I) \to Hom_c(X,I) \xrightarrow{of} Hom_c(X',I) \to o$$

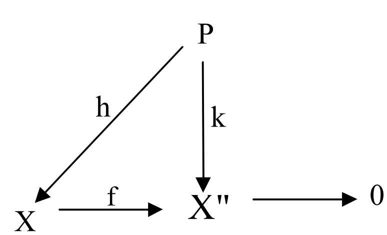
Thus there exists $h: x \to I$ such that $hof = k$,

(ii) Conversely, consider an exact sequence

$o \to x' \xrightarrow{f} x \xrightarrow{g} x'' \to o$. Then the sequence $o \to$

$Hom_c(X'',I) \xrightarrow{oh} Hom_c(X',I) \xrightarrow{of} Hom_c(X',I) \to 0$ is exact by the hypothesis.

By reversing the arrows, we get that p is projective if and only if for any solid diagram in which the row is exact;



The dotted arrow may be completed, making the diagram commutative. [6].

Lemma (4.3.10) [6.71.72]: Let $o \longrightarrow x' \xrightarrow{f} x \xrightarrow{g} x'' \longrightarrow o$ be an exact sequence in c, and assume that x' is injective. Then the sequence splits.

Proof. Applying the preceding result with k = idx', we find

h: x→ x' such that kof = idx'. Then apply.

Proposition (4.1.11): It follows that if F: c → c' is an additive functor of abelian categories, and the hypotheses of the lemma are satisfied, then the sequence o → F (x') → F (x) F (x") → o splits and in particular is exact.

Lemma (4.3.12) [6]: Let x', x" belong to c. Then x' ⊗ x" is injective if and only if x' and x" are injective.

Proof. It is enough to remark that for two additive functors of abelian categories F and G, x → F (x) ⊗ G (x) is exact if and only if F and G are exact. Apply lemmas (4.3.10) and (4.3.11), we get;

Proposition (4.3.13): Let o → x' → x→ x"→o be an exact sequence in c and assume x' and x are injectives. Then x" is injective.

Example (4.3.14) [6.71.72]:

(i) Let A be a ring. An A–modules M free if it is isomorphic to a direct sum of copies of A, that is, $M \simeq A^{(I)}$. It follows from proposition (4.2.8).

(ii) That free modules are projective.

Let M∈ Mod (A). For m∈M, denote by A_m a copy of A and denote by $Im \in A_m$ the unit. Define the linear map.

$$\Psi \otimes_{m \in M} A_m \rightarrow M$$

By setting Ψ (I_m) = m and extending by linearity. This map is clearly surjective. Since the left A–modules $\otimes_{m\in M}$ $A_m \rightarrow$ M is free, it is projective.

(ii) If k is a field, then any object of Mod (k) is both injective and projective.

(iii) Let A be a k–algebra and let M∈ Mod (A^{op}). One says that M is flat if the functor M $\otimes_A \bullet$: Mod (A) → Mod (k) is exact. Clearly, projective modules are flat.

Definition (4.3.10) [1.25. 35]: Let ψ and g be abelian categories and let F: ψ→g be a covariant functor. We say that F is;

- Left exact if o → A → B → c is exact implies that o → F (A) → F (B) → F (c) is exact.
- Right exact if A → B → c → o is exact implies that F (A) → F (B) → F (c) → o is exact.
- Exact if o → A → B → c → o is exact implies that o → F (A) → F (B) → F (c) → o is exact.

If F: Ψ→ G be is contra variant functor, we say that D is left exact if for every exact sequence A → B → c → o, the sequence o → D (c) → D (B) → D (A) is exact.

Similar contra variant definitions held for right exact and exact functors.

Example (4.3.15) [1.25. 35]: The functor Hom (–,B) is left exact contra variant. In other words, if A' $\overset{u}{\rightarrow}$ A $\overset{\varepsilon}{\rightarrow}$ A" → o is exact, then the induced

$$o \rightarrow \text{Hom}(A'',B) \overset{\varepsilon*}{\rightarrow} \text{Hom}(A,B) \overset{u*}{\rightarrow} \text{Hom}(A',B), \text{ is exact.}$$

Proof. First we show injectivity of ε*. Let g: A" → B, and suppose ε* (g) = g ε = o.

By surjectivity of ∈ this implies that g is the zero map, giving us injectivity of ε^*.

Secondly we show that $Im\varepsilon^* \subset Ker\ \mu^*$. A map in | m ε * is of the form g∈ for some g. Clearly g ∈ μ is the zero map, since ε μ already is.

Finally, we show that Ker $\mu^* \subset$ | m ε^*. Suppose h: A → B is Ker μ^*, so h μ is the zero map. This means that Ker h⊃|m μ = Ker ε, since ∈ is surjective, this means we can find a unique map o: A" → B such that Ø∈ = h, but then h = ε* (Ø) ∈ | mε*. Note also that the functor Hom (A,–) is an example of a left exact covariant functor.

Example (4.3.16) [1.25 . 35]: Consider the exact sequence $o \rightarrow Z \xrightarrow{\Psi} Z \xrightarrow{\emptyset} Z_3 \rightarrow o$, where Ø is multiplication by three, and Ψ is reduction module three. Apply the functor Hom $(-,Z_3)$

$$o \rightarrow Hom\ (Z_3, Z_3) \xrightarrow{\Psi^*} Hom\ (Z,Z_3) \xrightarrow{\emptyset^*} Hom\ (Z,Z_3)$$

However, if we recall the definition of we see that for β: z → Z, Ø* (β) = β 0 Ø, which is the zero map as β is homomorphism. This Ø* is not surjective, and the sequence above is not exact.

So Hom (–, B) is a contra variant functor which is left exact and not exact.

From this example a natural question a rises; how can we make a left (or right) exact functor in to an exact fucntor? Another way of phrasing this question is the following; given a short exact sequence.

$$o \rightarrow A \rightarrow B \rightarrow c \rightarrow o$$

and a left exact functor F: Ψ→ Gm how can we extend the exact sequence

$$o \rightarrow F\ (A) \rightarrow F\ (B) \rightarrow F\ (c)$$

to the right to from a long exact sequence? The theory of derived fucntors will provide an answer to this question, provided that Ψ is a 'nice' enough category. What we will do is find ourselves a sequence of functors R' F: Ψ→ G and continue the above sequence like this.

$$o \to F(A) \to F(B) \to F(c) \to R'F(A) \to R'F(B) \to R'F(c) \to R^2F(A) \to \ldots\ldots \quad (1)$$

Then F will be exact if f R'F = o, and we will have a measure of 'how exact' F is by the n for which R^n F = o.

Lemma (4.3. 17) [1. 25 . 35]: If p is projective then Ext' (P,B) = o for any R–module B.

Note: A similar proof shows that Ext' (A,I) = o for all R–module A when I is injective.

Proof. We want to show that Hom (p,–) is an exact functor, then from the definition of a derived functor it follows that Ext' (p,–) is zero (and actually that Ext' (p,–) is zero (and actually that Ext' (p,–) = o for all I ≥ 1). To this end, let $o \to A \xrightarrow{u} B \xrightarrow{\varepsilon} c \to o$ be short exact sequence and consider the sequence.

$$\text{Hom}(P,A) \xrightarrow{u^*} \text{Hom}(P,B) \xrightarrow{\varepsilon^*} \text{Hom}(P,C)$$

This means that if we have a projective presentation of A,i.e. a short exact sequence of R–modules $S \xrightarrow{u} P \xrightarrow{\varepsilon^*} A$ with p projective, then by applying, Hom (–, B) we get ourselves an exact sequence.

$$o \longrightarrow \text{Hom}(A,B) \xrightarrow{\varepsilon^*} \text{Hom}(P,B) \xrightarrow{u} \text{Hom}(S,B)$$

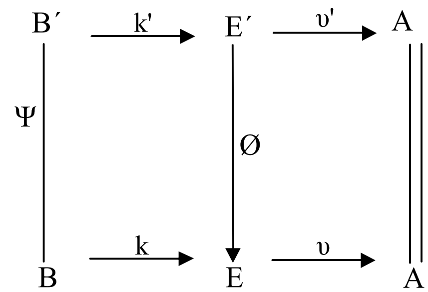
$$\longrightarrow \text{Ext}^*(A,B) \longrightarrow o$$

It then follows that we can think about Ext^- (A,B) as the co-kernel of μ*, in the traditional sense of the word, i.e. $\text{Ext}^-A,B) \simeq$ Hom (S,B) / Im μ*. Then Ext' is composed of equivalence classes, with Ψ∈ [Ø] if and only if Ψ = Ø + α o μ*. some α: p →B.

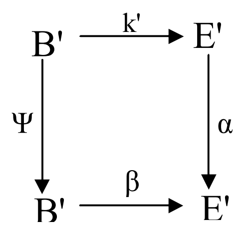
[Recall that u* (α) =α μ.].

Note: Ext' can also be computed using an injective presentation.

Lemma (4.3.18) [1.25. 35}: Let the following be a commutative diagram with exact rows.

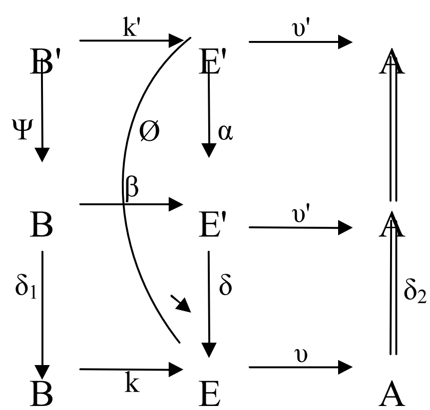


Then the left-hand square si a push-out diagram proof. Let



Be another push-out diagram. We deduce that if k' is a homomorphism then so B. It also tells us that α induces an isomorphism Coker β' → Coker k'. This means that there is a injective map μ: p → A such that B' $\xrightarrow{\beta}$ p $\xrightarrow{\mu}$ A is an extension.

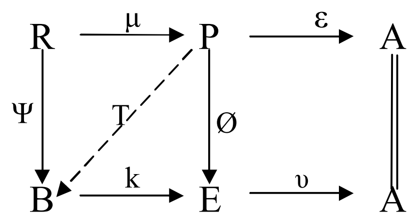
But in our original diagram we had another candidate for a push-out, so by the universal property of push-outs there must exist a map δ: p → E with Ø = δα and k = δβ.



All this information gives us the above diagram, and by the last assertions made concerning the commutatively of δ we see that δ_1 and δ_2 are identity maps and thus isomorphism. An application of the Lemma, then allows us to conclude that δ is an isomorphism and we are done.

Theorem (4.3.19) [1.25.35]: For R–modules A.B there is an isomorphism between the E (A,B) and Ext., (A,B).

Proof. Let $R \xrightarrow{\mu} p \xrightarrow{\varepsilon} A$ be a projective presentation of A and let $B \xrightarrow{k} E \xrightarrow{\upsilon} A$ be an element of E (A,B),



The map Ø exists since p is projective.

This map then induces a map Ψ which makes the diagram commute, and Ψ in turn defines an equivalence class [Ψ] ∈ Ext' (A,B). We need to show that this is well defined, so suppose Ø_1 and Ø_2 are two maps inducing $\Psi_{1.2} : R \rightarrow B$. Then $\text{Ø}_1 - \text{Ø}_2 = KT$ for some T: P → B. This, together with commutatively of the diagram tells us:

$$\Rightarrow \text{Ø}_i \text{ o } \mu = K \text{ o } \Psi_i$$

$$\Rightarrow (\text{Ø}_1 - \text{Ø}_2) \text{ o } \mu = k \text{ o } (\Psi_1 - \Psi_2)$$

$$\Rightarrow T o \mu \; \Psi_1 - \Psi_2 \text{ since k is a monomorphism}$$

$$\Rightarrow (\Psi_1) = (\Psi_2)$$

It is clear that if we had taken a different representative of the same element of E (A,B) then it would have induced the same element in Ext' (A,B), so we have a well-defined map η: E (A,B) → Ext' (A,B).

Conversely, let Ψ: R → B a representative of an element in Ext' (A,B), and take the push-out of Ψ and μ. This gives us a similar diagram to before;

$$\begin{array}{ccccc} R & \xrightarrow{\mu} & P & \xrightarrow{\varepsilon} & A \\ {\scriptstyle\Psi}\downarrow & & \downarrow{\scriptstyle Ø} & & \| \\ B & \xrightarrow{k} & E & \xrightarrow{\upsilon} & A \end{array}$$

We deduce that k is a homomorphism and υ is the co kernel of k, so that ok = o. So the bottom sequence is exact, and is thus an extension. As before we need to show that this extension is well defined, i.e., it does not depend on the particular representative Ψ. However, this follows without too much effort from the definition of representatives of Ext' and from Lemma (4.4.17).

We thus obtain a well-defined map ζ : Ext' (A,B) → E (A,B). Applying Lemma (4.4.17) one more time also tells us that ζ are inverses to each other, which finishes the proof.

Remark (4.3.20) [1. 25. 35]: In fat it turns out that the isomorphism is canonical, as the maps η and ζ are independent of the projective presentation of A chosen in the first line of the proof. Furthermore, the isomorphism is natural in both A and B.

Remark (4.3.21) [1.35. 35]: This equivalence of notions is very useful because it means that Ext' is defined in a general abelian category, even if that category has no projective or injective.

However, in practice Ext' is calculated using projective and injective resolutions, and we will see some more examples of this.

Section (4.4) Complexes in abelian categories [6.71.72]:

co homology

Recall that the categories c* (c) are abelian for * = +, –, b.

Let X∈C (c). One defines the following objects of c;

$Z^n (X) := \text{Ker } d_X^n$

$B^n (X) := \text{Im } d_X^{n-1}$

$H^n (X) := Z^n (X)/B^n (X)$ [$:= \text{Coker } (B^n (X) \to Z^n (X)$]

One calls $H^n (X)$ the n-th co homology object of x, if f: $x \to y$ is amorphism in c (c), then it induces morphisms $Z^n (X)\to Z^n (y)$ and $B^n (X) \to B^n (y)$, thus amorphism $H^n (f): H^n (X) \to H^n (y)$. Clearly, $H^n (X\otimes Y) \simeq H^n(X). \otimes H^n (y)$. Hence we have obtained an additive functor: $H^n (\bullet) : c (c) \to c$

Notice that $H^n (X) = H^o (X [n])$.

There are exact sequences

$X^{n-1} \xrightarrow{d^{n-1}} \ker d_x^n \to H^n (X) \to o, \quad o \to H^n (X) \to \text{Coker } d_x^{n-1} \xrightarrow{d^n} X^{n+1}$

The next result is easily checked.

Lemma (4.4.21). The sequences below are exact; [6].

$$(4.4) \quad o \to H^n (X) \to \text{Coker } (d_x^{n-1}) \xrightarrow{d_x^n} \text{Ker } d_x^{n+1} \to H^{n+1} (X) \to o.$$

One defines the truncation functors;

$$(4.5) \qquad T^{\leq n}, \tilde{T}^{\leq n}: c (c) \to c' (c)$$

$$(4.6) \qquad T^{\geq n}, \tilde{T}^{\geq n}: c (c) \to c^+ (c)$$

As follows. Let $x: = \ldots\ldots \to X^{n-1}\to X^n\to X^{n+1}\to \ldots\ldots$

One sets;

$T^{\leq n}(X) := \ldots\ldots \to X^{n-2} \to X^{n-1} \to \ker d_x^n \to o \to \ldots\ldots$

$T^{\leq n}(X) := \ldots\ldots \to X^{n-1} \to X^n \to \operatorname{Im} d_x^n \to o \to \ldots\ldots$

$T^{\geq n}(X) := \to o \to \operatorname{coker} d_x^{n-1} \to X^{n+1} \to X^{n+2} \ldots\ldots$

$\tilde{T}^{\geq n}(X) := \to o \to \operatorname{Im} d_x^{n-1} \to X^n \to X^{n+1} \ldots\ldots$

There is a chain of morphism in c (c);

$T^{\leq n} X \to \tilde{T}^{\leq n} X \to X \to \tilde{T}^{\geq n} X \to T^{\geq n} X,$

And thee are exact sequences in c (c);

$$(4,7)\quad \begin{cases} o \to \tilde{T}^{\leq n-1} X \to T^{\leq n} X \to H^n(X)[-n] \to o, \\ o \to N^n(X)[-n] \to T^{\geq n} X \to \tilde{T}^{\geq n-1}(X) \to o, \\ o \to T^{\leq n} X \to X \to T^{\geq n-1}(X) \to o, \\ o \to \tilde{T}^{\leq n-1} X \to X \to T^{\geq n}(X) \to o. \end{cases}$$

$$(4.8)\quad (T^{\leq n} X) \xrightarrow{\sim} H^j(T^{\leq n} X) \simeq \begin{cases} H^j(x) & j \geq n, \\ o & j < n, \end{cases}$$

$$(T^{\leq n} X) \xrightarrow{\sim} H^j(T^{\geq n} X) \simeq \begin{cases} H^j(x) & j \leq n, \\ 0 & j < n, \end{cases}$$

The verification is straight forward.

Lemma (4.4.22) [6.71.72]: Let c be an abelian category and let f: x → y be a morphism in c (c) homotopic to zero. Then $H^n(f): H^n(X) \to H^n(y)$ is the morphism.

Proof. Let $f^n = S^{n+1} \circ d_x^n + d_y^{n-1} \circ s^n$. Then $d_x^n = o$ on $\ker d_x^n$ and $d_y^{n-1} \circ s^n = o$ on $\ker d_y^n / \operatorname{Im} d_y^{n-1}$. Hence $H^n(f)$;

$\ker d_x^n / \operatorname{Im} d_x^{n-1} \to \ker d_x^{n+1} / \operatorname{Im} d_y^{n-1}$ is the zero morphism.

In view of lemma (4.4.22), the functor H^o: c (c) → c extends as a functor.

H^o: k (c) → c

One shall be a ware that the additive category k (c) is not abelian in general.

Definition (4.4.11) [6.71.72.93.94]: One says that a morphism f: x → y in c (c) is a quasi-isomorphism (a qis, for short) if H^k (f) is an isomorphism for all k∈Z. In such a case, one says that x and y are quasi-isomorphic.

In particular, x∈C (c) is qis to o if and only if the complex x is exact.

Remark (4.4.23) [6.71.72]: By lemma (4.4.22), a complex homotopic to o is qis to o, but the converse is false. One shall be aware that the property for a complex of being homotopic to o is preserved when applying an additive functor, contrarily to the property of being qis to o.

Remark(4.4.24) [6.71.72]: Consider a bounded complex $x^{\cdot}$ and denote by y* the complex gives by $y^j = H^j(x^{\cdot})$, $d^j_y \equiv o$, one has;

(4.9) $y^{\cdot} = \otimes_i H^i(x^{\cdot})[-i]$

The complexes $x^{\cdot}$ and $y^{\cdot}$ have the same co homology objects. In other words, $H(y^{\cdot}) \simeq H^j(x^{\cdot})$. However, in general these isomorphisms are neither induced by a morphism from $x^{\cdot} \to y^{\cdot}$, nor by a morphism from $y^{\cdot} \to x^{\cdot}$, and the two complexes $x^{\cdot}$ and $y^{\cdot}$ are not quasi-isomorphic.

Long exact sequence

Lemma (4.4.25) (The "five lemma") [6.71.72] Consider a commutative diagram; and assume that the

$$
\begin{array}{ccccccc}
X^0 & \xrightarrow{\alpha_1} & X^1 & \xrightarrow{\alpha_1} & X^2 & \xrightarrow{\alpha_2} & X^3 \\
\downarrow f^0 & & \downarrow f^1 & & \downarrow f^2 & & \downarrow f^3 \\
Y^0 & \xrightarrow[\beta_0]{} & Y^1 & \xrightarrow[\beta_1]{} & Y^2 & \xrightarrow[\beta_2]{} & Y^3
\end{array}
$$

Rows are exact sequence.

(i) If f^0 is an epimorphism and $f^{1.}$ f^3 are monomorphisms, then f^2 is a monomorphism.

(ii) If f^3 is a monomorphism and f^0, f^2 are epimorphisms, then f^1 is an epimorphism.

According to convention, we shall assume that c is a full abelian subcategory of mod (A) for some ring A. hence we may choose dements in the objects of c.

Proof (i) Let $x_2 \in X_2$ and assume that f^2 $(x_2) = 0$. Then f^3 0 α_2 $(x_2) = 0$ and f^3 being a monomorphism, this implies α_2 $(x_2) = 0$. Since the first row is exact, there exists $x_1 \in X_1$ such that α_1 $(x_1) = x_2$. Set $y_1 = f^1$ (x_1). Since β_1 o f^1 $(x_1) = 0$ and the second row is exact. There exists $y_0 \in Y^0$ such that β_0 $(y_0) = f^1$ (x_1). Since f^0 is an epimorphism, there exists $x_0 \in X^0$ such that $y_0 = f^0$ (x_0). Since f^1 0 α_0 $(x_0) =$ f^1 (x_1) and f^1 is monomorphism, α_0 $(x_0) = x$. Therefore, $x_2 = \alpha_1$ $(x_1) =$ 0.

(iii) Is nothing but (i) in C^{op}.

Lemma(4.4.26)(The snake lemma)[6.71.72] Consider the commutative diagram in c below with exact rows;

$$
\begin{array}{ccccccccc}
 & & X' & \xrightarrow{f} & X & \xrightarrow{g} & X'' & \xrightarrow{\upsilon} & o \\
 & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\
o & \longrightarrow & y^1 & \xrightarrow{f'} & y & \xrightarrow{g'} & y'' & &
\end{array}
$$

Then it gives rise to an exact sequence;

Ker $\alpha \rightarrow$ Ker $\beta \rightarrow$ Ker $\gamma \xrightarrow{\delta}$ co ker $\alpha \rightarrow$ co ker $\beta \rightarrow$ co ker γ The proof is similar to that of lemma (4.4.25)

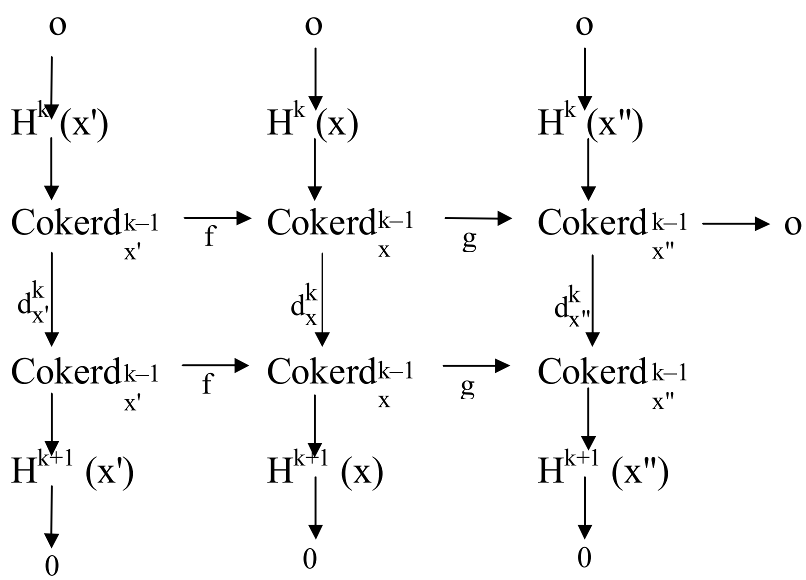
Theorem (4.4.25) [6.71.72]: Let $o \to x' \xrightarrow{f} x \xrightarrow{g} x'' \to o$ be an exact sequence in c (c).

(i) For each $k \in Z$, the sequence $H^k(x') \to H^k(x) \to H^k(x^n)$ is exact.

(ii) For each $k \in Z$, there exists $\delta^k : H^k(x'') \to H^{k+1}(x')$ making the long sequence.

(4.10) $\ldots . \to H^k(x) \to H^k(x'') \xrightarrow{\delta^k} H^{k+1}(x') \to H^{k+1}(x') \to$ exact. Moreover, one can construct δ^k functorial with respect to short exact sequence of c (c).

Proof. Consider the commutative diagrams;



The columns are exact by lemma (4.4.21) and the rows are exact by the hypothesis. Hence. The result follows from lemma 94.4.22).

Corollary (4.4.26) [6.71.72]: Consider a morphism $f: x \to y$ in c (c) and recall the Mc (f) denotes the mapping cone of f. There is along exact sequence;

(4.11) $\ldots \to H^{k+1}[Mc(f)] \to H^k(x) \xrightarrow{f} H^k(y) \to H^k[Mc(f)] \to$

(4.12) $o \to y \to Mc(f) \to x[1] \to o$

Clearly this complex is exact. Indeed, in degree, it gives the split exact sequence $o \to y^n \to y^n$ $x^{n+1} \to x^{n+1} \to o$. Applying theorem (4.4.27), we find a long exact sequence.

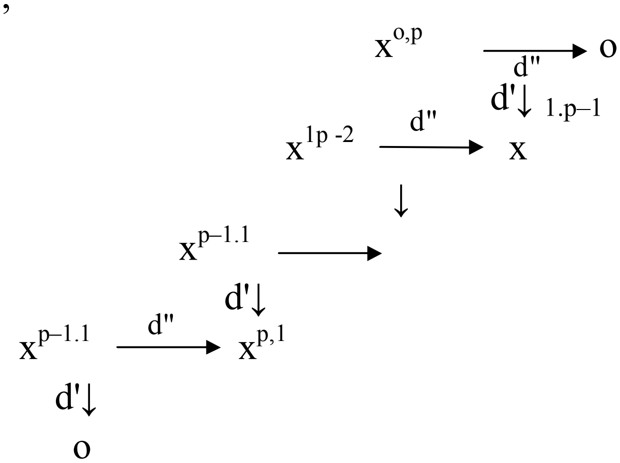
(4.13) $\ldots \to H^{k-1}[Mc(f)] \to H^{k-1}[x(1)] \xrightarrow{\delta^{k-1}} H^k(y) \to H^k[Mc(f)] \to \ldots\ldots$

It remains to heck that, up to a sign, the morphism δ^{k-1}; $(x) \to H^k(x) \to H^k(y)$ is $H^k(f)$. We shall not give the proof here.

Theorem (4.4.27) [6.71.72.93.94]: Let $x^{\cdot\cdot}$ be a double complex. Assume that all rows $x^{j,\cdot}$ and columns $x^{j,\cdot}$ are o for $j < o$ and are exact for $j > o$, Then $H^p(x^{o,\cdot}) \simeq H(x^{\cdot,o})$ for all p.

Proof. We shall only describe the first isomorphism $H^p(x^{o,\cdot}) \simeq H^p(x^{\cdot,o})$ in the case where c = Mod (A), by the so-called "weil procedure". Let $x^{p,o} \in X^{p,o}$, with d' $x^{p,o} = o$ which represents $y \in H^p(x^{\cdot,o})$. Define $x^{p,o}$ = d'' $x^{p,o}$. The d' $x^{p,o}$ = o, and the first column being exact, there exists $x^{p-1.1} \in X^{p-1.1}$ with d' $x^{p-1.1} = x^{p,1}$. One can iterate this procedure until getting $x^{o,p} \in X^{o,p}$.

Since d'' $x^{o,p}$ = o, and d' is injective on $x^{o,p}$ for $p > o$ by the hypothesis, we get d'' $x^{o,p}$= o. The class of $x^{o,p}$in $H^p(x^{o,\cdot})$ will be the image of y by the wail procedure. Of course, one has to check that his image close not depend of the various choices we have made, and that it induces as isomorphism. This can be visualized by diagram;



Section (4.5) Resolutions [6.71.72]: The aim of this subsection is to illustrate and motivate the constructions which will appear further. In this subsection. We work in the category Mod (A) for a k0algebra A. recall that the category Mod (A) admits enough projective.

Suppose one is interested in studying a system of linear equations.

(4.14) $\sum_{j=1}^{No} p_{ij}\, u_j = u_i,$ $(i = 1. \ldots\ldots., N_1)$

Where the p_{ij} is belong to the ring A and u_j, u_i belong to some left A–modules. Using matrix notations, one can write equation (4.14).

eq(4.15) Pou $= \gamma$

Where P_o is the matrix (p_{ij}) with N_1 rows and N_o columns, defining the A–linear map $P_o.: S^{No} \rightarrow S^{N1}$.

Now consider the right A–linear map

eq(4.16) $\cdot P_o : A^{N1} \rightarrow A^{No}$.

Where $\cdot$ P_o operates on the right and the elements of A^{No} and A^{N1} are written as rows. Let $(e_1 \ldots\ldots\ldots., e_{No})$ and $(f_1 \ldots\ldots\ldots., f_{N1})$ denote the canonical basis of A^{No} and $A^{N1.}$ respectively. One gets;

(4.17) $f_i \cdot P_o = \sum_{j=1}^{No} P_{ij}\, e_j$ $(i = 1. \ldots\ldots., N_1)$

Hence Im P_o is generated by the element $\sum_{j=1}^{No} p_{ij}\, e_j$ for $i = 1. N_1$.

Denote by M the quotient module $A^{No}/A^{N1} \cdot P_o$ and by : Ψ $A^{No} \rightarrow$ M the natural A–linear map.

Let $(u_1 \ldots\ldots., u_{no})$ denote the images by Ψ of $(e_1 \ldots\ldots., e_{no})$ and relation $\sum_{j=1}^{No} p_{ij}\, u_j = o$ for $i = 1. \ldots\ldots., N_1$. By construction, we have an exact sequence of left A–modules;

(4.18) $A^{N1} \xrightarrow{.Po} A^{No} \xrightarrow{\Psi} M \to o.$

Applying the left exact functor Hom_A (.,S) to this sequence, we find the exact sequence of k–modules;

eq(4.19) $o \to Hom_A(M,S) \to S^{No} \xrightarrow{p_o} S^{N1}$

(where P_o. operates on the left). Hence, the k–modules of solutions of the homogeneous equation associated to eq(4.14) is described by Hom_A (M,S).

Assume now that A is left Noetherian, that is, any sub module of a free A–modules of finite rank is of finite type. In this case, arguing as in the proof of proposition (4.5.31), we construct an exact sequence.

$\ldots\ldots \to A^{N2} \xrightarrow{P_1} A^{N1} \xrightarrow{P_o} A^{No} \xrightarrow{\Psi} M \to o.$

In other words, we have a projective resolution $L^\bullet \to M$ of M by finite free left A–modules

$$L^\bullet : \ldots.. \to L^n \to L^{n-1} \to \ldots.. \to L^o \to o$$

Applying the left exact functor Hom_A (., S) to $L^\bullet$, e find the complex of A–modules;

(4.20) $o \to S^{No} \xrightarrow{P_o} S^{N1} \xrightarrow{P_1} S^{N1}$

Then $\begin{cases} H^o(Hom_A(L^\bullet, S) \simeq \ker P_o, \\ H^1(Hom_A(L^\bullet, S) \simeq \ker(P_1)/\mathrm{Im}(P_o) \end{cases}$

Hence, a necessary condition is sufficient if $H^1(Hom_A\, 4^\bullet, S) \simeq 0$. As we shall see the co-homology groups $H^j(Hom_A(L^\bullet, S)$ do not depend, up to isomorphism's, of the choice of the projective resolution $L^\bullet$ of $M^\bullet$

Definition (4.5.12) [6.71.72]: Let J be a full additive sub category of c. We say that J is cogenerating if for all x in c, there exist y ∈ J and monomorphism x→ y.

If J is cogenerating in C^{op}, one says that J is generating.

Notations (4. 5. 28) [6.71.72]: Consider an exact sequence in c, o → x → J^o→ …. → J^n→ …. and denote by $J^{\bullet}$ the complex o → J^o→ …. → J^n→ …., we shall say for short that o → x → $J^{\bullet}$ is resolution of x. If the $J^{k,s}$ belong to J, we shall say that this is a J, we shall say that this is a J–resolution of x. When J denotes the category of injective objects one says this is an injective resolution.

Proposition (4. 5. 29) [6.71.72]: Let c be an abelian category and, let J be cogenerating full additive subcategory. Then, for any X∈ C, there exists an exact sequence.

(4.21) $o \to x \to J^o \to \dots \to J^n \to \dots$

With $J^n \in J$ for all $n \geq o$

Proof we proceed by induction Assume to have constructed;

$$o \to x \to J^o \to \dots \to J^n \to \dots$$

For n = o this is the hypothesis. Set B^n = coker ($J^{n-1} \to J^n$) (with $J^{-1} = x$). Then $J^{n-1} \to J^n \to B^n \to o$ is exact. Embed B^n in an object of J; $o \to B^n \to J^{n-1}$. Then $J^{n-1} \to J^n \to J^{n-1}$ is exact, and the induction proceeds. Then sequence.

(4.22) $J^{\bullet} = o \to J^o \to \dots \to J^n \to \dots$

Is called a right J-resolution of X. If J is the category of injective objects in c, one says that $J^{\bullet}$ is an injective resolution. Note that, identifying x and $J^{\bullet}$ to objects of c^+ (c),

(4.23) $x \to J^{\bullet}$ is apis

Of course, there is a similar result for left resolution. If for any x∈C there is an exact sequence y → x → o with y∈J, then one can construct a left J- resolution of x, that is, aqis $y^{\bullet} \to x$, here the $y^{n,s}$ belong to J. If J is the category of projuective objects of c, one says that $J^{\bullet}$ is a projective resolution.

Proposition (4.5.31) is a particular case of a result that we state without proof.

Proposition (4.5.30) [6.71.72]: Assume J is cogenerating. Then for any $x^{\bullet} \in c^{+}$ (c), thee exists $y^{\bullet} \in C^{+}$ (J) and a quasi–isomorphism$x^{\bullet} \to y^{\bullet}$

Injective resolutions

In this section. c denotes an abelian category and I_c its full additive subcategory consisting of injective objects. We shall assume.

eq(4.24) The abelian category c admits enough injective.

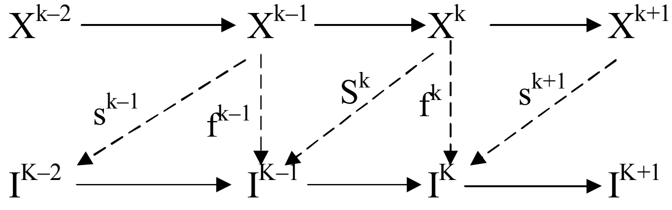
In other words, the category I_c is cogenerating.

Proposition (4. 5. 31) [6.71.72]:

(i) Let $f^{\bullet}: x^{\bullet} \to I^{\bullet}$ be a morphism in c^{+} (c). Assume $I^{\bullet}$ belong to c^{+} (I_c) and assume $X^{\bullet}$ is exact. Then $f^{\bullet}$ is homotopic to o.

(ii) Let $I^{\bullet} \in c^{+}$ (c) and assume $I^{\bullet}$ is exact. Then $I^{\bullet}$ is homotopic to o.

Proof, (i) consider the diagram;



We shall construct by induction morphism s^k satisfying;

$$f^k = s^{k+1} d_x^k + o_x^{k-1} s^k.$$

For $j << o$, $s^j = o$. Assume we have constructed the s^j for $j \leq k$. Define $g^k = f^k - d_x^{k-1} \circ s^k$. One has

$$g^k \circ d_x^{k-1} = f^k \circ d_x^{k-1} - d_x^{k-1} \circ s^k \circ d_x^{k-1}$$

$$= f^k \circ d_x^{k-1} - d_I^{k-1} \circ f^{k-1} + d_I^{k-1} \circ d_I^{k-1} \circ s^{k-1} = o$$

Hence, g^k facrizes through x^k / Imd_x^{k-1}. Since the complex $x^\bullet$ is exact, the sequence $o \to x^k / Imd_x^{k-1} \to x^{K+1}$ is exact. Consider.

$$\begin{array}{ccccc} o \to & x^k / Imd_X^{k-1} & \longrightarrow & x^{k+1} & \\ & g^k \searrow & & \swarrow s^{k-1} & \\ & & I^k & & \end{array}$$

the dotted arrow may be completed by proposition (4.3.12)

(ii) Apply the result of (i) with $X^\bullet = I^\bullet$ and $f = idx$

Proposition (4. 5. 32) [6.71.72]:

(i) Let: $X \to Y$ be a morphism in c, let $o \to x \to x^\bullet$ be a resolution of x and let $o \to y \to J^\bullet$ be acomlex with the $J^{k,s}$ injective. Then there exists morphism $f : X^\bullet \to J^\bullet$ making the diagram below commutative:

(4.25)

$$\begin{array}{ccccc} o & \longrightarrow & x & \longrightarrow & x^\bullet \\ & & f \downarrow & & \downarrow f^\bullet \\ o & \longrightarrow & x & \longrightarrow & j^\bullet \end{array}$$

(ii) The morphism $f^{\cdot}$ in $c^{\cdot}$ (c) constructed in (i) is unique up to homotopy.

Proof.

(i) Let us denote by d_x (resp. dy) the differential of the complex $x^{\cdot}$ (resp. $J^{\cdot}$), by d_y^{-1} resp. d_x^{-1}) the morphism x $\to x^o$ (resp. $y \to J^o$) and set $f^{-1} = f$.
We shall construct the $f^{n,s}$ by induction. Morphism f constructed $f^o, \ldots . f^n$. Let $g^n = d_y^n$ o
$F^n : x^n \to J^{n+1}$.
The morphism g^n factories through h^n: x^n / $Imd_x^{n-1} \to J^{n+1}$
Since $x^{\cdot}$ is exact, the sequence o $\to x^n$ / $Imd_x^{n+1} \to x$ is exact. Since J^{n+1} is injective, h^n extends as $f^{n+1} : x^{n+1} \to J^{n+1}$.

(ii) We may assume f = o and we have to prove that in this case $f^{\cdot}$ is homotopic to zeno, since the sequence o $\to x \to x^{\cdot}$ is exact, this follows from proposition (4.5.33) (i), replacing the exact sequence o $\to y \to J^{\cdot}$ by the complex o $\to 0 \to J^{\cdot}$.

Section (4.6) Derived functors [6.71.72]:

Let c be an abelian category satisfying (4.24). Recall that I_c denotes the full additive sub category of consisting of injective objects in c. we look at the additive category k (I_c) as a full ad dive sub category of the abelian category k (c).

Theorem (4. 6. 33) [6.71.72]: Assuming eq(4.24), there exists a functor λ: c → k (I_c) and for each x∈C, a qis x → λ (x). functorially in x ∈ C.

Proof. (i) Let x ∈ C and let $I_x^{\cdot}$ ∈ c^+ (I_c) be an injective resolution of x. the image of $I_x^{\cdot}$ in k^+ (c) is unique up to unique isomorphism, by proposition (4.5.34). Indeed consider two injective resolution $I_x^{\cdot}$ and $J_x^{\cdot}$ of x. By proposition (4.5.34) applied to id_x, there exists a morphism $f_x^{\cdot}$ $I_x^{\cdot} \rightarrow J_x^{\cdot}$ making thediagram eq(4.25) commutative and this morphism is unique up to homotopy, hence is unique in $k^+(c)$.Similarly, there exists a unique morphism $g^{\cdot}: J \rightarrow I$ in $k^+(c)$. Hence, $f^{\cdot}$ and $g^{\cdot}$ are isomorphism inverse one to each other.

(ii) Let f: $x \rightarrow y$ be a morphism in c, let $\dot{I}_x$ and $\dot{I}_y$be injective resolutions of x and y respectively, and let $f^{\cdot}$: $\dot{I}_x \rightarrow \dot{I}_y$ be a morphism of complexes such as in proposition (4.5.34). Then the image of $f^{\cdot}$ in $Hom_{k+(Ic)}$ ($\dot{I}_x$, $\dot{I}_y$) does not depend on the choice of $I^{\cdot}$ by proposition (4.5.34). In particular, we get that if g: $y \rightarrow z$ is another morphism in c and $\dot{I}_z$ is an injective resolution of z, then $g^{\cdot}$ o $f^{\cdot}$ = $(gof)^{\cdot}$ as morphisms in k^+ (I_c).

Let F: $c \rightarrow c$ be a left exact functor of abelian categories and recall that c satisfies eq(4.24). Consider the functors.

$$c \xrightarrow{\lambda} k^+(I_c) \xrightarrow{f} k^+(c') \xrightarrow{H^m} c'$$

Definition (4.6.13) [6.71.72]:

(4.26) $R^n F = H^n \text{ o } F \text{ o } \lambda$

And calls R^nF the n-th right derived functor of F. By its definition the receipt to construct R^nF (x) is as follows;

- Choose an injective resolution $I^{\cdot}$ of x, rthat is, construct an exact sequence $o \rightarrow x \rightarrow I_x^{\cdot}$ with $I_x^{\cdot} \in C^+$ (I_c).
- Apply F to this resolution.
- Take the n-th cohomology.

In other words, $R^n F(x) \simeq H^n [F(\dot{I}_x)]$. Note that:

- R^nF is an additive functor from c to ć.
- $R^nF(x) \simeq o$ for $n < o$ since $\dot{I}_x = o$ f_x f j < o,

- R^oF (x) $\simeq$F (x) since F being left exact, it commutes with kernels,
- R^nF (x) $\simeq$o for n ≠ o if F is exact,
- R^nF (x) $\simeq$o for n ≠ o if F is injective, by the construction of R^nF (x)

Definitions (4.6.14) [6.71.72]: An object x of c such that R^kF (x) ~ o for all k > o is called F-acyclic.

Hence, injective objects are Facyclic for all left exact functors F.

Theorem (4.6.34) [6.71.72]: Let $o \to x' \xrightarrow{f} x \xrightarrow{g} x'' \to o$ be an exact sequence in c. Then there exists a long exact sequence

$$o \to F(x')\ F(x) \to \ldots \to R^k F(x') \to R^k F(x) \to R^k F(x'') \to \ldots$$

Sketch of the proof. One constructs an exact sequence of complexes $o \to x'^{\bullet} \to x^{\bullet} \to x'' \to o$ whose objects are injective and this sequence si quasi-isomorphic to the sequence $o \to x' \xrightarrow{f} x \xrightarrow{g} x'' \to o$ in c (c). $0 \to F(x^{\bullet\prime}) \to F(x^{\bullet}) \to F(x'^{\prime\bullet}) \to 0$

Since the objects x" are injective, we get a short exact sequence in c (c');

Then one applies theorem (4.4.27).

Definition (4. 6. 15): Let J be a full additive subcategory of c. one says that J is F–injective if:

(i) J is cogenerating.
(ii) For any exact sequence o → x' → x → x" → o in with x' ∈J, x ∈ J, then x" ∈ J.
(iii) For any exact sequence o → x' → x → x" → o in c with x' ∈J, the sequence o → F (x') → F (x) → F (x") → o is exact.

By considering C^{op}, one obtains the notion of an F projective subcategory, F being right exact.

Lemma (4. 6. 35) [6.71.72]: Assume J is F–injective and let $x^{\bullet} \in C^{+}$ (J) be a complex qis to zero.

Proof:

We decompose $x^{\bullet}$ into short exact sequences (assuming that this complex starts at step o for convenience):

$$o \to x^{o} \to x^{1} \to z^{1} \to o$$

$$o \to z^{1} \to x^{2} \to z^{2} \to o$$

$$o \to z^{n-1} \to x^{n} \to z^{n} \to o$$

by induction we find that all the $z^{j,s}$ belong to J, hence all the sequence,

$$o \to F\,(z^{n-1}) \to F\,(x^{n}) \to (z^{n}) \to o$$

are exact. Hence the sequence

$$o \to F\,(x^{o}) \to F\,(x^{1}) \to \ldots\ldots..$$

is exact.

Theorem (4.6.36) [6.71.72.93.94]: Assume J is F–injective and contains the category I_c of injective objects. Let $x \in C$ and let $o \to x \to y^{\bullet}$ be a resolution o x with $y^{\bullet}x \in C^{+}$ (J). Then for each n, there is an isomorphism $R^{n}\,F\,(x) \simeq H^{n}\,[F\,(y^{\bullet})]$.

In other words, in order to calculate the derived functors R^{n} F (x), it is enough to replace x with a right J–resolution.

Proof:

Consider a right J–resolutiony$^{\cdot}$ of x and an injective resolution $I^{\bullet}$ of x. by the result o proposition (4.5.34), the identity morphism $x \to x$

will extend to a morphism of complexes $f^{\bullet}: y^{\bullet} \rightarrow I^{\bullet}$ making the diagram below commutative;

$$\begin{array}{ccccc} o & \rightarrow & x & \rightarrow & y^{\bullet} \\ & & \big\downarrow{\scriptstyle id} & & \big\downarrow{\scriptstyle f^{\bullet}} \\ o & \rightarrow & x & \rightarrow & \dot{I} \end{array}$$

Define the complex $k^{\bullet} = Mc\ (f^{\bullet})$, the mapping cone of $f^{\bullet}$. By the hypothesis $k^{\bullet}$ belongs to c^{+} (J) and this complex is qis to zero by corollary (4.4.28). By lemma (4.6.37). F ($k^{\bullet}$) is qis to zero.

On the other-hand, F [Mc (f)] is isomorphic to Mc [F (f)], the mapping we find a long exact sequence.

$$\ldots. \rightarrow H^{n}\,[F\,(J^{\bullet})] \rightarrow H^{n}\,[F\,(I^{\bullet})] \rightarrow H^{n}\,[F\,(k^{\bullet})] \rightarrow \ldots$$

Since F ($k^{\bullet}$) is qis to zero, the result follows.

Theorem (4.6.37) [6.71.73]: Left F: → c' and G: c' → c'' be left exact functors of abelian categories and assume that c and c' have enough injective.

(i) Assume that G is exact. Then R^{j} (GoF) $\simeq$ Go R^{j} F.

(ii) Assume that F is exact. There is a natural morphism R^{j} (GoF) → (R^{j} G) of.

(iii) Let x ∈ C and assume that R^{j} F (x) $\simeq$ of or j > o and that F sends the injective objects of c to G-a cyclic objects of c'. Then R^{j} (Go F) $\simeq$ (R^{j}G) of.

Proof:

For x∈C, let o → x → I be an injective resolution of x. Then R^{j} (G o F) (x) $\simeq$ H^{j} [G o F ($\dot{I}_{x}$)].

(i) If G is exact. H^{j} [GoF ($\dot{I}_{x}$)] is isomorphic to G [H^{j}(F ($\dot{I}_{x}$)].

(ii) Consider an injective resolution o → F (x) →$j_{F(x)}$ F (x). By the result of proposition (4.5.34), there exists a morphism F($\dot{I}_x$)→G ($j_{F(x)}$). Applying G we get a morphism of complexes; (Go F) (j). → G ($j_{F(x)}$). Since H^j [GoF ($\dot{I}_x$)] ≃ R^j (Go F) (x) and H^j [$G_{F(x)}$)] ≃ R^j G [F (x), we get the result.

(iii) Denote by I the full additive sub category of c' consisting of G-acyclic objects (see Example letter (4.6.40). By the hypothesis, F($\dot{I}_x$) is qis to F(x) and belongs to c^+ (I_x). Hence R^j G [F (x)] ≃ H^j [G (F ($\dot{I}_x$)] by theorem (4.6.38).

Example (4.6.38)[6.71.72.93.94]: Let F: c→ć be a left exact functor and assume that c admits enough injective.

(i) The category I_c of injective objects of c is f–injective.

(ii) Denote by I_F the full sub category of c consisting of F–acyclic objects. Then I_F contains, I_c, hence is cogenerating. It easily follows from theorem (4.6.36) that conditions (ii) and (iii) of Definition (4.6.15) are satisfied,. Hence I_F is F–injective.

Derived bi functor:

Let F: c x c' → c'' be a left exact additive bi functor of abelian categories. Assume that c and c' admit enough injective. For x∈C and y∈C', one can thus construct [R'F (x,.)] (y) and [R'F (.,y)] (x).

Theorem (4.6.39) [6.71.72]: Assume that for each injective object I∈C the functor F (I,.): c'→c'' is exact and for each injective object I'∈C' the functor G (.,I'): c'→c''. Then, for j ∈ Z. X∈C and y ∈ C', there is an isomorphism, functiorial in x and y: [R^j F(x,.)] (y) ≃ [R^j F(.,y)] (x).

Proof:

Let o → x → $\dot{I}_x$ and o → y → $\dot{I}_x$ be injective resolution o x and y, respectively consider the double complex;

$$
\begin{array}{ccccccccc}
 & & o & & o & & o & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
o & \longrightarrow & o & \longrightarrow & F(I_x^0, y) & \longrightarrow & F(I_x^1, y) & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
o & \longrightarrow & F(x, I_y^0) & \longrightarrow & F(I_x^0, I_y^0) & \longrightarrow & F(I_x^0, I_y^0) & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
o & \longrightarrow & F(x, I_x^1) & \longrightarrow & F(I_x^0, I_y^1) & \longrightarrow & F(I_x^1, I_y^1) & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}
$$

The co homology of the first row (resp., column) calculates $R^k F$ (., y) (x) [res., R^k F (x, .) (y)], since the other rows and columns are exact by the hypotheses, the result follows from theorem (4.4.29). Assume that c has enough injectives and enough projective.

Then one can define the j-th derived functor of Hom_c (x,.) and the j-th derived functor of Hom_c (.,y). By theorem (4.6.41) there exists an isomorphism.

$$R^j \, Hom_c \, (x,.) \, (y) \simeq R^j \, Hom_c \, (.,y) \, (x)$$

Functorial with respect to x and y. Hence, if c has enough injectives or enough projective, we can denote by the same symbol the derived fucntor either of the functor Hom_c (x,.) or of the functor Hom_c (., y).

A similar remark applies to the bifunctor$\otimes_A$: Mod () x Mod $(A^{op})\rightarrow$ Mod (k).

Definition (4.6.16) [6.71.72.93.94):

(i) If c has enough injectives or enough projective, one denotes by Ext]c_J^A (.,.) the j-th right derived functor of

Hom_c.

(ii) For a ring A, one denotes by Tor (.,.) the left derived functor of $\cdot \otimes_A \cdot$

Hence, the derived functors of Hom_c are calculated as follows.

Let x, y ∈ C. If c has enough injectives one chooses an injective resolution $I^{\cdot}_y$ of y and we get

(4.27) $Ext^j_c(x,y) \simeq H^j[Hom_c(x, I^{\cdot}_y)]$.

If c has enough projectives, one chooses a projective resolution $p^{\cdot}_x$ of x and we get.

(4.28) $Ext'_c(x,y) \simeq H^j[Hom_c(p^{\cdot}_x, y)]$.

If c admits both enough injectives and projectives, one can choose to use either eq(4.27) or eq(4.28). When dealing with the category Mod (A), projective resolutions are in general much easier to construct.

Similarly, the derived functors of $\otimes_A$ are calculated as follows. Let N∈ Mod (A^{op}) and M∈ Mod (A) One constructs a projective resolution $\dot{p}_N$ of N or a projective resolution $\dot{P}_M$ of M. Then.

$Tor^A_j(N,M) \simeq H^{-j}(\dot{P}_N \otimes_A M) \simeq H^{-j}(N \otimes_A \dot{P}_M)$.

In fact, it is enough to take flat resolution instead of projective ones.

Section (4.7) Koszul complexes [6.71.72]:

In this section, we do not work in abstract abelian categories but in the category Mod (A), for a non necessarily commutative k-algebra A. If afinite free k-Module of rank n, one denotes by $\Lambda^j L$ the k-Module consisting of j-multilinear alternate forms on the dual space L* and calls it the j-th exterior power of L. (Recall that L*= Hom_A (L,K). Note that $\Lambda^1 L \simeq L$ and $\Lambda^n L \simeq K$. one sets $\Lambda^0 L = K$.

If $(e_{1.} \ldots\ldots., e_n)$ is a basis of L and $I = \{ i_1 < \ldots\ldots. < i_j \} \subset$

$\{ i, \ldots\ldots\ldots., n \}$, one sets.

For a subset $I_c \{ 1. \ldots\ldots\ldots., n \}$, one denotes by | I | its cardinal. Recall that;

Λ^j L is free with basis $\{ \Lambda e_{i1} \ldots \Lambda e_{i1}: i_j < i_1 < i_{2.} < \ldots.. < i_j \leq n\}$.

If $i_{1.}$., i_m belong to the set (1. …, n), oendefines $\Lambda e_{i1} \ldots. \Lambda e_{im}$

by reducing to the case where $i_1 < \ldots.. < i_j$, using the convention $e_i \Lambda\ e_j = - e_j \Lambda\ e_i$

let M be an A–modules and let $\Psi = (\Psi_{1.} \ldots., \Psi_n)$ be an endomorphism of M over A which commute with one another;

$\{ \Psi_i, \Psi_j \} = o,\ 1 \leq i, j \leq n$

(Recall the notation [a,b]: = a b – ba). Set $M (j) = M \otimes \Lambda^j k^n$.

Hence $M^{(o)} = M$ and $M^{(n)} \simeq M. \simeq M$. Denote by $(e_{1.} \ldots., e_n)$ the canonical basis of k. Hence, any element of $M^{(j)}$ may be written uniquely as a sum.

$$m = \sum_{[1] = j} m_1 \otimes e_1.$$

One defines $d \in Hom_A (M^{(j)}, M^{(j+1)})$ by;

$$d(m \otimes e_1 = \sum_{j = 1} \psi^i\ (m_1) \Lambda e_1.$$

And extending d by linearity. Using the commutatively of the $\Psi_{i's}$ one checks easily that do d – o. Hence we get a complex, called a Koszul complex and denoted $K^{\bullet} (M,\Psi)$;

$$o \rightarrow M^{(o)} \xrightarrow{d} \ldots\ldots \rightarrow M^{(n)} \rightarrow o$$

When n = 1. the chomology of this complex gives the Kernel and co-kernel of Ψ_1. More generally

$H^o [K^{\cdot} (M,\Psi)] \simeq \text{Ker } \Psi_1 \cap \ldots\ldots \cap \text{Ker } \Psi_n,$

$H^n [K^{\cdot} (M,\Psi)] \simeq M /[\Psi_1 (M)+ \ldots.. + \Psi_n (M)]$

Set $\Psi' = \{\Psi_{1.} \ldots\ldots.. \Psi_{n-1} \}$ and denote by d' the differential in $K^{\cdot}$ (M,Ψ'). Then Ψ defines a morphism

(4. 29) $\Psi n : K^{\cdot} (M,\Psi') \rightarrow K^{\cdot}(M,\Psi')$

Lemma (4.7. 40) [6.71.72]: The complex $K^{\cdot}(M,\Psi')$ [1] is isomorphic to the mapping cone $o\text{–}\Psi_n$ ~

Proof. Consider the diagram

$$\begin{array}{ccc} M_c (\Psi_n)^p & \xrightarrow{d_m^p} & M_c (\Psi_n)^{p+1} \\ \downarrow \lambda^p & & \downarrow \lambda^{b-\lambda} \\ K^{p+1} (M,\Psi) & \xrightarrow{d_k^{p+1}} & K^{p+1} (M,\Psi) \end{array}$$

Given explicitly by;

$$\begin{array}{ccc} \rightarrow(M \otimes \Lambda^{p+1}k^{n-1}) \otimes \Lambda \ (M \otimes \Lambda^p\ k^{n-1}) & \xrightarrow[\begin{smallmatrix} -\acute{d} & 0 \\ -\psi n & \acute{d} \end{smallmatrix}]{} & (M \otimes \Lambda^{p+1}k^{n+1}) \otimes \rightarrow \\ \rightarrow(M \otimes \Lambda^{p+1}k^{n-1}) & & \\ \downarrow \text{id}\otimes\ (\text{id}\otimes e_n\Lambda\ \ ) & & \downarrow \text{id}\otimes(\text{id} \otimes e_n\Lambda) \\ M \otimes \Lambda^{p+1}k^n & \xrightarrow{-d} & M \otimes \Lambda^{p+2}\ k^n \end{array}$$

Then

$d_P (a \otimes e_j + b \otimes e_k) = - d' (a\otimes e_j) + [d' (b \otimes e_k) - \Psi_n^M (a)\otimes e_j)$

$\lambda^p (a \otimes e_j + b \otimes e_k) = - d'\ a\otimes e_j + b \otimes e_n\ \Lambda e_k.$

(i) The vertical arrows are isomorphisms. Indeed, let us treat the first one. It is described by

$$(4.30)\quad \sum_j a_j \otimes e_j + \sum_k b_k \otimes e_k \rightarrow \sum_j a_j \otimes e_j + \sum_k b_k \otimes e_n \wedge e_k$$

With $|J| = p+1$ and $|k| = p$. Any element of $M \otimes \Lambda^{P+1} k^n$ may uniquely be written as in the right hand side of (4.30)

(i) The diagram commutes. Indeed.

$$\lambda^{P+1} \circ d_m^p (a \otimes e_j + b \otimes e_k) = -d'(a \otimes e_j) + e_n \wedge d'(b \otimes e_k) + \Psi_n(a)$$

$$\otimes e_n \wedge e_j = d'(a \otimes e_j) - d'(b \otimes e_n \wedge e_k) - \Psi_n(a) \otimes e_n \wedge e_j,$$

$$d_k^{p+1} \circ \lambda^P (a \otimes e_j + b \otimes e_k) = -d(a \otimes e_j) + b \otimes e_n \wedge e_k$$

$$= d'(a \otimes e_j) - \Psi_n(a) \otimes e_n \wedge e_j - d'(b \otimes e_n \wedge e_k)$$

Theorem (4.7.41): There exists a long exact sequence.

$$(4.31)\ \ldots \rightarrow H^j[k^{\cdot}(M,\Psi)] \xrightarrow{\Psi n} H^j[k^{\cdot}(M,\Psi)] \rightarrow$$

$$H^{j+1}[k^{\cdot}(M,\Psi)] \rightarrow \ldots\ldots\ldots$$

Proof. apply lemma (4.7.42) and the long exact sequence eq(4.11).

Definition (4.7.17) [6.71.72]:

(i) If for each j, $1 \leq j \leq n$, Ψ^j is injective as an endomorphism of $M/[\Psi^1(M) + \ldots.. + \Psi_{j-1}(M)]$, one says $(\Psi_1 \ldots..,\Psi_n)$ is a regular sequence.

(ii) If for each j, $1 \leq j \leq n$, Ψ_i $(\Psi_1 \ldots..,\Psi_n)$ is co regular sequence.

Corollary (4.7.42) [6.71.72.93.94]:

(i) Assume $(\Psi_1\ldots..,\Psi_n)$ is a regular sequence. Then

$H^j[K^{\cdot}(M,\Psi)] \simeq o$ for $j \neq n$,

(ii) Assume $(\Psi_1\ldots..,\Psi_n)$ is a co regular sequence. Then

$H^j\,[K^{\bullet}\,(M,\Psi)] \simeq o$ for $j \neq o$

Proof:

assume for example that $(\Psi_1\ldots..,\Psi_n)$ is a regular sequence, and let us a rouge by induction on n. The co homology of $K^{\bullet}\,(M,\Psi')$ is thus concentrated in degree n–1 and is isomorphic to $M/[\Psi_1\,(M) + \ldots\ldots.. + \Psi_{n-1}\,(M)]$.

By the hypothesis, Ψ_n is injective on this group, and corollary (4.7.44) follows. q.e.d.

Second proof. let us give a direct proof of the corollary in case n=2 for co regular sequences. Hence we consider the complex;

$$o \rightarrow M \xrightarrow{d} M\times M \xrightarrow{d} M \rightarrow o$$

Where $d\,(x) = [\Psi_1\,(x), \Psi_2\,(x)]$, $d\,(y,z) = \Psi_2\,(y) - \Psi 1\,(z)$ and we assume Ψ_1 is surjective on M, Ψ_2 is surjective on Ker Ψ_1.

Let $(y,z) \in M\times M$ with $\Psi_2(y)=\Psi_1\,(z)$. We look for $x\in M$ solution of $\Psi_1\,(x) = y$. Then $\Psi_2\,(x) = Z$. First choose $x' \in M$ with $\Psi_1\,(x') = y$. Then $\Psi_2\,(x)\;\Psi_1\;o\;\Psi_2\,(x')$. Thus $\Psi_1\,[z - \Psi_2\,(x')] = p$ and there exists $t \in M$ with $\Psi_1\,(t) = o$, $\Psi_2\,(t) = z - \Psi_2\,(x')$. Hence $y = \Psi_1\,(t+x')$ and $x = t + x'$ is a solution to our problem. q.e.d.

Example (4.7.43) [6.71.72]: Let k be a field of characteristic o and let $A = k\,\{x_{1.}\,\ldots\ldots, x_n\}$.

(i) Denote by x_i, the multiplication by x_i in A. We get the complex;

$$o \rightarrow A^{(0)} \xrightarrow{d} \ldots\ldots.. \xrightarrow{d} A^{(n)} \rightarrow o$$

Where;

$$d\,(\sum_{1} a_1 \otimes e_1) = \sum_{j=1}^{n} \sum_{1} x_j, a_{1.} \otimes e_j, \wedge\, e_1.$$

The sequence ($x_1, \ldots\ldots, x_n$) is a regular sequence in A. considered as an A–module. Hence the Koszul complex is exact except in degree n where its co homology is isomorphic to k.

(ii) Denote by δ_i the partial derivation with respect to x_i. This is a K–linear map on the k–rector space A. Hence we get a Koszul complex.

$o \to A^{(0)} \xrightarrow{d} \ldots\ldots\ldots \xrightarrow{d} A^{(n)} \to o$

Where;

$$d\left(\sum_{1} a_1 \otimes e_1\right) = \sum_{i=1}^{n} \sum_{1} \partial_j, (a_1), \otimes e_j, \wedge e_1.$$

The sequence ($\partial_1, \ldots\ldots, \partial_n.$) is a co regular sequence, and the above complex is exact except in degree o where its co homology is isomorphic to k. writing dx_j instead of e_j, we recognize the "de Rham complex".

Example (4.7.44) [6.71.72]: Let k be a field and let A = k [x,y], M = K $\simeq$ A/x A + yA and let as calculate the k–modules Ex (M,A), since injective resolutions

are not easy to calculate, itis much simpler to calculate a free (hence, projective) resolution of M. since (x,y) is a regular sequence of endomorphism of A (viewed as an A–modules), Mis quais-isomorphic to the complex:

$$M^{\bullet}: o \to A \xrightarrow{u} A \xrightarrow{o} A \to o$$

Where u (a) = ($y_a - x_a$), u (b, c) = x b + y c and the module A on the right stands in degree o. Therefore, Ext^j_A (M,N) is the j-th co homology object of the complex Hom_A ($M^{\bullet}$,N), that is

$$o \to N \xrightarrow{\acute{v}} N^2 \xrightarrow{\acute{u}} N \to o$$

Where v' = Hom (V,N), u' = Hom (u,N_ and the modul N on theleft stands in degree o. Since v' (n) = (x_n , y_n) and u' (m,I) = ym – xI, we find again a Koszul complex, choosing N = A, its chomology is concentrated in degree 2. Hence. Ext^j_A (M,A) $\simeq$ o for j $\neq$ 2 and $\simeq$k for j = 2.

Example (4.7.45) [6.71.72]: Let w=w(k) be the Wey algebra introduced in Example (1.2.5) and denote by. ∂_j. The (∂_1......,∂_n) is a regular sequence on w (considered as an w-module) and we get the Koszul complex:

$$o \rightarrow w^{(o)} \xrightarrow{\partial} \ldots\ldots. \rightarrow w^{(n)} \rightarrow o$$

Where

$$\partial \left(\sum_{1} a_1 \otimes e_1 \right) = \sum_{i=1}^{n} \sum_{1} a_{1.}\, \partial j \otimes e_j, \wedge e_1.$$

This complex is exact except degree n where its co homology is isomorphic to k [n].

Remark (4.7.46) [6.71.72]: One may also encounter co Koszul complexes. For I = ($i_{1.}$......,i_k) introduce.

$$e_j \mid e_j = \begin{cases} o & \text{if } j \in \{ i_{1.}\ldots\ldots, i_k \} \\ (-1)^{i+1} e_{1i} := (-1)^{i+1} e_{i1} \wedge \ \wedge e_{i1} \wedge \ldots \wedge e_{ik} & \text{if} \end{cases}$$

$$\left. e_{i1} = e_i \right\}$$

Where $e_i, \wedge \ldots \wedge \overset{\wedge}{e_{i1}} \wedge \ldots \wedge e_{ik}$ means that e_{i1} should be C mittled in $e_{i1} \wedge \ldots \wedge e_{ik}$. Denote ∂.by

$$\partial (m \otimes e_I) = \sum_{i=1}^{n} \Psi_i (m) \mid e_i.$$

Hence again one checks easily that ∂ o ∂ = o, and we get the complex.

$$K_{\bullet} (M,\Psi):\ o \rightarrow M^{(n)} \xrightarrow{\partial} \ldots. \rightarrow \ldots.. M^{(O)} \rightarrow O,$$

This complex is in fact isomorphic to a Koszul complex x, consider the isomorphism

$$* : \wedge^j \ k^n \xrightarrow{\sim} \wedge^{u-j} k^n$$

Which associates ε_1 m ⊗ e_1 to m ⊗ e_1. where I^n = (1……,n) \ I and ε_1 is the signature of the permutation which send (1……,n) to I U I (any i ∈ I is smaller than any j∈ I).

Then, up to a sign * in ter changes d and ∂.

De Rham Complexes [6.71.72]: Let E be v real rector space of dimension n and let u be an open subset of E. Denote as usual by C^∞(u) the c–algebra of C–valued function on u of class C^∞. Recall that Ω^1(u) denotes the C^∞(u)–module of C–functions on u with values in $E^* \otimes_R C \simeq Hom_R$ (E,c).

Hence $\Omega^1(u) \simeq E^* \otimes_R C^\infty(u)$

For p∈N, one sets

$$\Omega^P (u) := \Lambda^{p1} \Omega(u)$$

$$\simeq (\Lambda^p E^*) \otimes_R C^\infty(u).$$

(The first exterior product is taken over the commutative ring c∞ (u) and the second one over R). Hence, Ω^0 (u) = C^∞(u), Ω^P(u) =o for p>n and Ω^n(u) is free of rank 1 over C^∞(u). The differential is c–linear map.

$$d : C^\infty(u) \rightarrow \Omega^P(u)$$

The differential extends by multi linearity as a C–linear map d: $\Omega^P(u) \rightarrow \Omega^{P-1}(u)$ satisfying

(4,32)

$$\begin{cases} d2 = o, \\ d (w_1 \wedge w_2) = d w_1 \wedge w_2 + (-)^p w_1 \wedge w_2 \text{ for any} \\ w_1 \in \Omega^P(u) \rightarrow \Omega^{P-1}(u) \text{ satisfying} \end{cases}$$

(4.33) $$D_R (u): = o \rightarrow \Omega^0(u) \xrightarrow{d} \ldots \rightarrow \Omega^n(u) \rightarrow o$$

Let us choose a basis (e_1 ……., e_n) of E and denote by x_i the function which, to $x = \sum_{i=1}^{n} x_i \cdot e_i \in E$, associates its i-th coordinate x_i. then (dx_1 ……., dx_n) is the dual basis on E^* and the differential of a function Ψ is given by

$$d\Psi = \sum_{i=1}^{n} \partial_i \Psi \, dx_i$$

Where $\partial_i \Psi = \partial\Psi/\partial_{xi}$. By its construction, the Koszul complex of (∂_1 ……., ∂_n) acting on $C^\infty(u)$ is nothing, but the De Rham complex.

$$K^{\bullet} [\overset{\infty}{C} (u), (\partial_1 \ldots\ldots, \partial_n)] = D_R (u).$$

Note that H^O (D_R (u) is the space of locally constant functions on u, and therefore is isomorphic to $C^{\neq cc}$ (u) where ≠ cc (u) denotes the cardinal of the set of connected components of u. Using sheaf theory, one proves that all co homology groups H^j [D_R (u)] are topological invariants of u.

Definition (4.7.18) [9.73. 79.40. 41.42]: The Koszul complex KSZ (m) of the R–module M is a chain complex of R–module constructed as follows. The chain group in degree n > o is Kszn (m) = M ⊗∧ V and the differential d: Ksz_{n-1} (m) $\xrightarrow{Kszn-1}$ (m) is defined by the formula:

$$d(\alpha dx_{i1} \ldots\ldots dx_{in}) = \alpha^{xi} dx_{i2} \ldots\ldots dx_{in}$$
$$- \alpha x_{i2} dx_{i1} . dx_{i3} \ldots\ldots dx_{in}$$
$$+ \ldots\ldots\ldots$$
$$+ (-1)^{n-1} \alpha x_{in} dx_{i1} \ldots\ldots dx_{in-1}$$

Observe we write simply $\alpha dx_{i4} . dx_{i5}$ instead o f

$\alpha \otimes (dx_2 \wedge dx_4 \wedge dx_5)$ if $\alpha \in M$.

The definiton can be generalized to an arbitrary collection o elements (∂_1 ……., ∂_n); of dx_i ($1 \leq i \leq p$) is then α_i. Of R instead of "variables" ($x_{1.}$ …, x_n); the differential are usual sign game shaws the Kozul complex actually is a chain complex.

Furthermore this will be also a consequence of arecursive construction given soon.

Section(4.8) Derived catogory of K - modules

Definiton (4.8.19):[9.73. 79] Making the definiton of the derived rersion of duality.

L d (m) def d ($P^{\bullet}$) for any free resoluton

$P^{\bullet}$ o f M;

Completely correct, depends on resolvign two problems:

(1) Existence of a free resoluton $P^{\bullet}$ of M.

(2) Independence of choice ofa free resoluton $P^{\bullet}$.

The first one has already been delat with. For htesecond one recall that a resoluton is a quasi-isomorphism $P^{\bullet} \rightarrow M \neq$. Our problem would disappear if this quasiisoorphism were an isomorphism since we would be repalcing $M \neq$ with an isomorphic object. So our problem will be resolved if we can find a setting in which al quasi-isomorphisms in $C^{\bullet}$ [m (k)] become isomorphism. Such setting exists, the so called derived category of K–modules D [m (k)].

The passage from $C^{\bullet}$ [m (k)] to D [m (k)] requires inverting all quasi-isomorphisms in $C^{\bullet}$ [m (k)]. This can be done either by (i) universal abstract construction of inverting morphisms in a category, or

(ii) Using some convenient sub cateory of m (k). We will eventually do both since both dieas are useful in applications.

For the approach (i) we wil firstrecal the solution ofen analogous problem in rings rather then categories.

Derived category of Modules and complexes of free Moudles. According to the above defintion(4.8.18). D[m (k)] is a very abstract construction it will turn outthatthere is a simple description of D [m (k)] in terms of homotopy in the cateogry of complexes over the sub cateogry of free modules. (This is the aproach (ii) above).

Do we really weant the derived category? The historical origin of the idea is as we have introduced it: it is a good setting for doing calculations with complexes. However, thederived category D(A) of category A (say A= m(k) above), may be more "real" than the simple category A we started with. One indication is thatthere are pairs of very different cateogry A and B such htat their derived categories D(A) and D(B) are canonically equivalent. For instance A and B could be the categories of graded modules for the symmetric algebras s (V) and the exterior algerbra

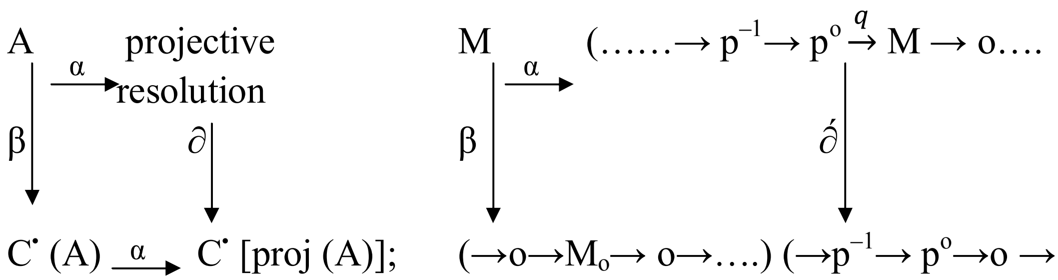
$^{\bullet}$ $V^{*}\wedge$for dual vector speces V and V^{*}.

This turns out to be important, but there are more exciting examples: the relation between linear differential equations and their solutons, minor symmetry.

Bounded category of complexes. [9.73 . 79]:

We say that a complex $C^{\bullet}$ is bounded from above if $C^{\bullet n}$ = o, n >> o. The categories of such complexes is denoted $C^{\bullet}$ (A) (meaning that the complexes are allawed to streth in the negative direction) simlarly one has C^{+} (A) and D^{+} (A). We say that a complex $C^{\bullet}$ is bounded (or finite) if C^{+n} = o for all but a finited many n∈Z, this gives C^{b} (A) and D^{b} (A) [9.73. 79]:

Definition (4. 8. 20): [9.73. 79] Improving objects m∈A. Let A = m (k) and m∈A. We improve m by replacing it with a cmplex $P^{\bullet}$ of free (or say, projective) modules. This can be schematically described as



Notice that rertical arrows a natural constructions (i.e., fucntion), while horizontal arrows require some choices.

The compositon of α and ∂ is a description of m in terms of complexes of projective modules. the other route α' oβ indicates a more formal formulation of the same idea – we first view modules as complexes via β and then α' means describing complexes in A in terms of quasi-isomorphic complexes in proj (A).

Any (additive) functor D: A → B extends to complexes. Let A = m (k) and B = m (k') be categories of modules over two rings, and let D be a way to construct from a module for k a module for k', i.e., a functor D: A → B it extends to a functor from

A–complexes to B–complexes $D^{\bullet}: C^{\bullet}(A) \rightarrow C^{\bullet}(B)$, that assings to each A–complex

$A^{\bullet} = (\ldots \rightarrow A^{-1} \xrightarrow{d^{-1}} A^{o} \xrightarrow{d^{o}} A^{1} \xrightarrow{d^{1}} \ldots)$ a B–complex

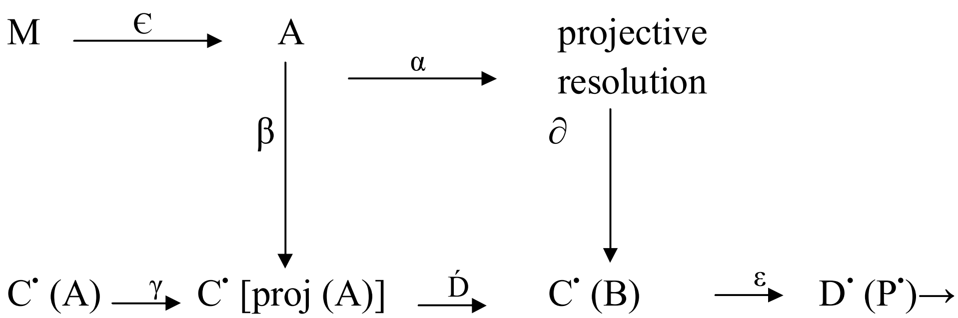
$D^{\bullet}(A^{\bullet}) = [\ldots \rightarrow D(A^{-1}) \xrightarrow{D(d-1)} D(A^{o}) \xrightarrow{D(do)} D(A^{1}) \rightarrow$

$\xrightarrow{D(d1)} \ldots\ldots\ldots]$

[As we know, if D is contravariant–for instance if D is a functor it preserves compositon of morphisms, hence $D(d^{n})$ o $D(d^{n-1}) = D(o) = o$. Asking that D is additive i.e. D (A' ⊗A") = D (A') ⊗D (A"), A', A" ∈A, is needed for the laststep: D (o) = o

Left derived version LD of D. step D. [9.73. 79].

It realy means that we do not apply D directly to M bbut to its improved version $p^{\bullet}$:



(left and right drived categories).

Example (4. 8. 47): In order to say LD is realy an improvement of D, we need to know that H^0 [LD(M)], I > 0. This is joing to be true presisely if D has property called right exactnec (oluality Dis right ecxact). There are important functors which are notright exact put have a"dual" proprty of gift ezactness, they will require a"dual" strategy; aright resolution of M:,

$...M \to I^0 \to I^1 \to ....$

by injective Modules. Will back to that.

Appliction(4. 8. 48): The commutative ring K=C [x_1....., x_n]is algebra of functions on the n – dimensional of fine sbace A^n<u>def</u>C^n. Natural examples of K-modules have geometric meaing. We say that affine algabraic variety is asubset Y of sune A^n which is given by polynomial conditions: Y = {z = (z_1. ...., z_n) $\in c^n$;f_1(z) = ...=f_c (z) = 0}. The set Iy of functions that vanish on y is an ideal in K (i. e., a K-submodule of the K – module K). we difine the ring O(Y) of polynomial functions on y as the all rotrictions f/Yof polynomials f ∈ K to y. So O (y) = K/ I_y is alos a module for K = O (A^n). we will cosider the k module O (y) where y is origin A_n. Then Iy = $\sum x_i$. K and there fore O(y) = K/ $\sum x_i$. K is isomorphic to C as ring (C valued functions on apoint). However its more intersting as aK-module n= 1. Here C [x] and Iy = x c[x], so we hve a resolution ... → 0 → C [x] $\overset{x}{\to}$ C [x]$\overset{g}{\to}$ O(y) →0 → ... and the conputation of the dual of O(y) is same as in the cese of x_n.One finds that D [O(y)] ≅ O(Y) [¯1] n=2. Then $O(A^2)$ = C [x-y] and O(Y) = C [x-y] / <x,y>= C [x-y] / (xc [x, y] + y c[x,y] = K/(xk+ yk).

The kerel of the cvering p0 = K $\overset{q}{\rightarrow}$ O(y) is xk + yk. We can coverit turn with p0 = K ⊗ K $\overset{\alpha}{\rightarrow}$xk + yk, α (f, g) = x α + y β. This covering still contains surplus: ker (α) = {(-yh, xh); h ∈ K}. However this is afree module so next covering p-2 = K $\overset{\beta}{\rightarrow}$ kre (α) ⊆ $P^{1.}$ β (h) (-yh, xh). This gives aresolution

…→ 0 →c[x,y] $\overset{\beta}{\rightarrow}$ c[x,y] ⊗ c[x,y] $\overset{\alpha}{\rightarrow}$c[x,y] $\overset{q}{\rightarrow}$O(Y) → 0 →…

As acomplex this resolution is $P^{\cdot}$[… → 0 → c[x,y] $\xrightarrow{\beta=(-xy)}$ c[x,y] ⊗c[x,y] $\xrightarrow{\alpha=(x,y)}$c[x,y] → 0 →….]

Chapter Five

Categories, Localization and chain complexes

In this chapter we construct the derived category of an abelian category c and the right derived functor RF of a left exact functor F: c → c' of abelian categories.

We shall be a ware that in general, the derived category D+(c) of all-category c is no more all-category.

Consider category C and a family's of morphism in C, the aim of localization is fine anew category C_s and a functors Q: C→Cs which sends the morphism belonging toss is morphismes in Cs (Q, Cs) being universal for such a property. In this chapter we shall the construct the localization of a category. When S satisfies satiable condition and the localization of functors, the study shall be a ware that in general the localization of au – category C in more au – category.

Section (5.1) The homotopy category k (c). [6.71.72]:

Let c be an additive category. R call that the homotopy category k(c) is defined by identifying to zero the morphism in C(c) homotopic to zero.

Also recall that if f: x → y is a morphism in c (c) m one defines its mapping cone Mc(f), an object of c (c), and there is a natural triangle.

$$(5.1) \quad y \xrightarrow{\alpha(f)} Mc\,(f) \xrightarrow{\beta(f)} X\,[1] \xrightarrow{f[1]} y\,[1].$$

Such a triangle is called a mapping cone triangle. Clearly, a triangle in c (c) gives rise to a triangle in the homotopy category k(c).

Definition (5.1.1): A distinguished triangle (d.t. for short) in k (c) is a triangle isomorphic in k (c) to a mapping cone triangle.

Theorem (5.1.1). [6.71.72]: The category k (c) endowed with the shift functor [1] and the family of d.t. is a triangulated category.

We shall not give the proof of this fundamental result here.

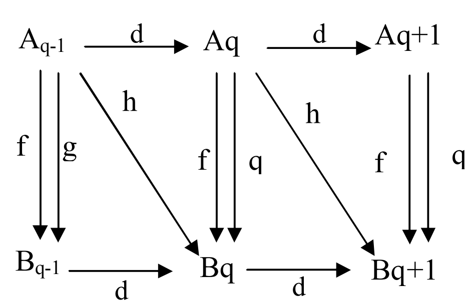
Notation (5.1.2). [6.71.72]: For short, we shall sometimes write $x \to y \to z \xrightarrow{+1}$ instead of $x \to y \to z \to x[1]$ to denote a d.t. in k (c).

Definition (5.1.2). [6.73.72]: Let $A_* = \{A_q, d_q\}$ and $B_* = \{B_q, d_q\}$ be two chain complexes, A homotopy operator $h: A_* \to B_*$ is a collection $h = \{h_q : A_q, \to B_{q+1}\}_q$ of linear maps. In other words, it is a linear map $h: A_* \to B_{*+1}$ of degree +1. this degree being implicitly implied by the index '* + 1' of B_{*+1}.

In particular, no compatibility condition is required with the respective differentials of A_* and B_*. In the interesting cases, the homotopy operator is rather "seriously non-compatible" with these differentials.

Definition (5.1.3)[6.73. 79.40,41.42]: Let f, g: $A_* \to B_*$ be two chain complex morphism. A homotop operator h: $A_* \to B_{*+1}$ is a homotopy between f and g if the relation g-f – dh + h d is satisfied.

The next diagram shows there is a unique way to understand this relation when you start from A_q and arrive at B_q



Proposition (5.1.3) [9.73. 79]: If two chain-complex morphisms f, q: $A_* \to B_*$ are homotopic, then the induced maps f, q: $H_* (A_*) \to H_*$

(B_*) are equal.

Proof. Let h be a homotopy between f and q. if z is a q-cycle representing the homotopy class h ∈ H_q (A_*), then the relation qz – fz = dhz + hdz is satisfied; butz is a cycle and hdz = o, so that qz – fz = dhz, which expresses the cycles fz and qz representing the homology classes fh and qh are homologous, their difference is a boundary; and therefore fh - qh.

Definition (5.1.4). [9.73. 79]: A homology equivalence between two chain – complexes A_* and B_* is a pair (f,q) of chain-complex morphism f: $A_* \to B_*$ and q: $A_* \to B_*$ such that qf is homotopic to id_{A*} and fq is homotopic to id_{B*}.

The terminology is not well stabilized, many authors use rather chain equivalence, or homotopy equivalence. We feel more simple and clear our terminology. We can also say that q: $A_* \to B_*$ is a homology equivalence if there exists a homological inverse q:$B_* \to A_*$ such that the pair (f,q) satisfies the above definition.

Proposition (5.1.4): If f: $A_* \to B_*$ is a homology equivalence, then the induced maps { f_q: H_q (A_*) → H_q (B_*) $\}_q$ are isomorphism.

Proof. The maps 1f and fq are respectively homotopic to id_{A*} and id_{B*}, so that the induced maps qf: H_q (A_*) → H_q (A_*) and fq: H_q (B_*) → H_q (B_*) are equal to the corresponding identities.

Definition(5.1.5)[9.73.79]: The standard n-simplex Δ"of dimension n is the simlicial complex [n̲, P_* (n̲)] where n̲ is the set of integers n̲ = (o, ……, n) from o to n and P_* (n̲) is the set of non-empty subsets of n̲.

Theorem (5.1.5)[9.73. 79]: The homology groups of the standard simplex Δ^n are null except H_o (Δ^n) = R, the ground ring.

Proof:

The result is obvious when n = o. Otherwise we can consider two simplicial morphisms f: $\Delta^o \to \Delta^n$ and q: $\Delta^n \to \Delta^q$ where f (o) = o and q(i) = o for every.

The composition qf is the identity, the composition fq is not, but the induced map fq: $C_* (\Delta^n) \rightarrow C_* (\Delta^n)$ is homotopic to the identity. The needed homotopy operator h: $C_* (\Delta^n) \rightarrow C_{*+1} (\Delta^n)$ is defined as follows; let = $\sigma = (i_o, \ldots\ldots\ldots, i_k)$ a k-simplex generator of $c_k (\Delta^n)$, that is, an ordered sequence of k+1 integers $i_o < \ldots\ldots < i_k$ of $\underline{n}$. if $i_o > o$, we decide $h(\sigma) = (o, i_o, \ldots\ldots\ldots, i_k)$; if on the contrary $i_o = o$, then we decide $h(\sigma) = o$. An interesting but elementary computation then shows $dh + hd = id_{c*} (\Delta^n) - fq$. So that the map fq: $H_* (\Delta^n) \rightarrow H_* (\Delta^n)$ is simply equal to the identity and f: $H_* (\Delta^n) \rightarrow H_* (\Delta^n)$ is an isomorphism.

Example (5. 1.6)[6. 71. 72]: Let W be the Wey algebra in one variable over a field K; W = K [x, ∂] with the relation [x, ∂] = -1. let Q = W/ W. ∂, Ω = W/ ∂. W and let us calculate $\Omega \otimes_w^l$ Q, we have an exact sequence.

$$0 \rightarrow w \xrightarrow{\partial} w \rightarrow \Omega \rightarrow 0$$

Hence Ω qis to the complex

$$0 \rightarrow w^{-1} \xrightarrow{\partial} w \rightarrow 0$$

Where $Q^{-1} = Q^0 = Q$ and Q^0 is in degree 0. Since ə: Q → Q is surjective and has K as kernel, w oblain $\Omega \otimes_w^l Q \cong K [1]$.

Section (5.2) Derived categories [6.71.72]:

From now on, c will denote an abelian category. Recall that if f: x → y is a morphism in c (c), one says that f is aquasi-isomorphism (aqis, for short) if $H^k (f): H^k (x) \rightarrow H^k (y)$ is an isomorphism for all k. one extends this definition to morphism in k (c).

If one embeds f into ad.t $x \xrightarrow{f} y \rightarrow z \xrightarrow{+1}$, then f is aqis iff $H^k (z) \simeq o$ for all KZ, that is, if z is qi to o.

Proposition(5.2.7)[6.71.72]: Let c be an abelian category. The functor H^o: k (c) → c is a co- homological functor.

Proof:

M_c (f) $\xrightarrow{\beta(f)}$ x [1] $\xrightarrow{+1}$. Since the sequence in c (c):

o → y → M_c (f) → x [1] → o. Is exact, if it follows from, that the sequence.

H^k (y) → H^k [M_c(f)] → H^{k+1} (x)

Is exact. Therefore, H^k (y) → H^k (z) → H^{k+1} (x) is exact.

Corollary (5.2.8)[6.71.72]: Let o $\xrightarrow{f}$ x $\xrightarrow{q}$ y → z → o be an exact sequence in c (c) and define Ψ M_c (f) → z as $\Psi^n = (o, g^n)$. Then Ψ is a qis.

Proof. Consider the exact sequence in c (c);

o → M (id_*) $\xrightarrow{\upsilon}$ M_c (f) $\xrightarrow{\Psi}$ z → o

Where υ^n: $(x^n \otimes x^n) \to x^{n+1} \otimes y^n$ is defined by:

$\upsilon^n = \begin{bmatrix} Id_x^{n+1} & o \\ o & f^n \end{bmatrix}$. Since H^k [M_c(idx)] $\simeq$ o for all k, we get the result.

We shall localize k (c) with respect to the family of objects qis to zero

(see definition (5.5.1).

N(c) = { x ∈ K(c), H^k (x) $\simeq$ o for all k }.

One also defines $N^*(c) = N(c) \cap K^*(c)$ for $^* = b, +, -$.

Clearly, N^*(c) is a null system in K^*(c),

Definition(5.2.6) [6.71.72]: One defines the derived categories D^*(c) as $K^*(c)/N^*(c)$,

where * = ub,b, + , – . one denotes by Q the localization

functor $K^*(c) \to D^*(c)$. By theorem(4.5.23), these are triangulated categories. Hence, a quasi-isomorphism in K(c) becomes an isomorphism in D(c). Recall the truncation functors in eq(4.5) and eq(4.5). These functors send a complex homotopic to zero to a complex homotopic to zero, hence are well defined on $K^{+}(c)$. Moreover, they send aqis to aqis to aqis. Hence the functors below are well defined:

$H^{j}(.): D(c) \to C,$

$T^{\leq n}, \tilde{T}^{\leq n}: D(c) \to D^{-}(c),$

$T^{\geq n}, \tilde{T}^{\geq n}: D(c) \to D^{+}(c),$

Note that there are isomorphisms of functor $T^{\leq n} \simeq \tilde{T}^{\leq n}$ and $T^{\geq n}$, $\tilde{T}^{\geq n}$. Moreover, $H^{j}(.)$ is a cohomological functor on $D^{*}(c)$. In fact, if $x \in N(c)$, then $H^{j}(x) \simeq o$ in c, and if f: x → y is aqis in K(c), then $T^{\leq n}(f)$ and $T^{\geq n}(f)$ are qis.

In particular, if $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{+1}$ is ad.t in D(c), we get along exact sequence:

(5.2) $\ldots \to H^{k}(x) \to H^{k}(y) \to H^{k}(z) \to H^{k+1}(x) \to \ldots$

Let $x \in K(c)$, with $H^{j}(x) = o$ for $j > n$. Then the morphism

$T^{\leq} x \to x$ in K(c) is aqis, hence an isomorphism in D(c).

It follows from proposition (4.5.25) that $D^{+}(c)$ is equivalent to the full sub category of D(c) consisting of objects x satisfying $H^{j}(x) \simeq o$ for $j \ll o$, and similarly for $D^{-}(c)$, $D^{b}(c)$. Moreover. C is equivalent to the full sub category of D(c) consisting of objects x satisfying $D^{j}(x) \simeq o$ for $j \neq o$.

Definition (5.2.17)[6.71.72]: Let x,y be objects of c. one sets.

$$\mathrm{Ext}^{k}_{c}(x,y) = \mathrm{Hom}_{D(c)}(x,y[k])$$

We shall see in Theorem (6.5.26) below that if c has enough injectives, this definition is compatible Definition.

Notation (5.2.9)[6.71.72]: Let Ab a ring, We shall write for short

$D^*(A)$ instead of $D^*[\text{mod}(A)]$, for $^* = \emptyset, b, +, -$.

Remark (5.2.10). [6.71.72]:

(i) Let $x \in K(c)$, and let $Q(x)$ denote its image in $K(c)$. one can prove that;
$Q(x) \simeq o \leftrightarrow x$ is qis to o in $K(c)$.

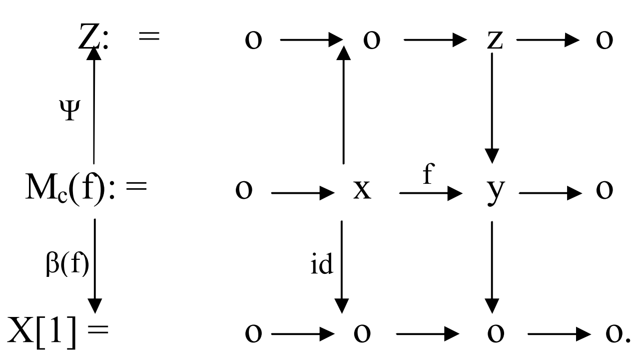
(ii) Let $f: x \to y$ be a morphism in $C(c)$. Then $f \simeq o$ in $D(c)$ iff there exists x' and a qis $g: x' \to x$ such that fog is homotopic to o, or else iff there exists y' and a qis $h: y \to y'$ such that hof is homotopic to o.

Remark (5.2.11)[6.71.72]: Consider the morphism $\upsilon: z \to x\ [1]$ in $D(c)$. If x,y,z belong to c (i.e are concentrated in degree o), the morphism $H^k(\upsilon)$:

$H^k(z) \to H^{kH}(x)$ is o for all $k \in Z$. However, υ is not the zero morphism in $D(c)$ in general (this happens if the short exact sequence splits). In fact, let us apply the cohomological functor $Hom_c(w_i)$ to the d.t above. It gives rise to the long exact sequence:

$\ldots \to Hom_c(w,y) \to Hom_c(w,z) \qquad Hom_c[w,s(1)]$

Where $\upsilon = (w\upsilon)$. Since $Hom_c(w,y) \to Hom_c(w,z)$ is not an epimorphism in general, υ is not zero. Therefore υ is not zero in general. The morphism υ may be described as follows.



Proposition (5.2.12)[6.71.72]: Let $x \in D(c)$.

(i) There are d.t. in $D(c)$.

(5.3) $T^{\leq n} x \longrightarrow x \longrightarrow T^{\leq n+1} x \xrightarrow{+1}$

(5.4) $T^{\leq n-1} x \longrightarrow T^{\leq n} \xrightarrow{f} H^n(x)\,[-n] \xrightarrow{+1}$

$$(5.5)\quad H^n(x)[-n] \longrightarrow T^{\geq n} \longrightarrow T^{\geq n+1} \xrightarrow{+1}$$

(ii) Moreover, $H^n(x)[-n] \simeq T^{\leq n} T^{\geq n}(x) \simeq T^{\geq n} T^{\leq n} x$.

Corollary(5.2.13)[6.71.72]: Let c be an abelian category and assume that for any x, y ∈ X, $Ext^k(x,y)$ = of for $k \geq 2$. Let $x \in D^b$ (c). Then:

$$X \simeq \otimes_j H^j(x)[-j].$$

Proof:

Call "amplitude of x" the smallest integer k such that $H^j(x) = o$ for j not belonging to some interval of length k.

If k = o, this means that there exists some I with $H^j(x) = o$ for $j \neq i$, hence $x \simeq H^j(x)[-i]$. Now we argue by induction on the amplitude. Consider the d.t. (6.4).

$$T^{\leq n+1} x \to T^{\leq n} x \to H^n(x)[-n] \xrightarrow{+1}$$

and assume $T^{\leq n+1} x \simeq_{j<n} H^j(x)[-j]$. By the result, it enough to show that $Hom_{D^b(c)}(H^n(x)[-n], H^j(x)[-j=1] = o$ for $j < n$.

Since $n+1 - \leq 2$. the result follows.

Example (5.2.14). [6.71.72]:

(i) If a ring A is a principal ideal domain (such as a field, or z, or k [x] for k a field), then the category Mod (A) satisfies the hypotheses of corollary (6.2.12).

(ii) See Example (6.5.29) to see an object which does split.

Example (5. 2. 15) Assume C has enough injective. Then R Hom_c: $\acute{D}(c)^{0p} \times D^+(c) \to D^+(Ab)$ exacts and my be calculated as follows.

Let $x \in \acute{D}(c)$, $y \in D^+(c)$. There exists aqis in K^+ (c), $y \to I$, the I^j s being injective. Then R $Hom_c(X, Y) \cong Hom^{\cdot}_c(X, I)$ if C has enough projective, and $P \to X$ is aqis in K^- (c), the $P^{j,s}$ being projective, one aloes has;

R Hom_c (X, Y) ≅ $\text{Hom}^{\cdot}_c$ (p, y).

These isomorphism hold D^+ (Ab)

Section(5.3) Resolutions:

Definition(5.3.8)[9.73.79]: Let M be an R–module. A free R–resolution of M, in short a resolution of M, is a chain complex Rs` (M) null negative degrees, made of free R–module, every differential is an R–morphism, every homology group H_n [Rs_2 (M)] $\simeq$ M is given.

Note the isomorphism is a component of the date defining the resolution; strictly speaking to resolution is the pair Rs_1 (M), ∈). You can also consider the isomorphism ∈ : Rsl_o (M) → M. If you "add" RsL_1 (m): = M and this augmentation, you obtain the exact sequence:

$$o \leftarrow M \xrightarrow{\overline{\varepsilon}} Rsl_o\,(M) \leftarrow Rsl_1\,(M) \leftarrow \ldots$$

Definition (5.3.9)[9.73. 79]: Let M be an R–module. An effective resolution Rsl (M) is a resolution with a *R,e.e) –reduction p = (f,g,h): Rsl (M) → M_* where the small chain–complex M_* is made from M concentrated in degree o.

The prefix (R,e,e) for our reduction means we require f is an R–morphism, but g and h in general are only e–morphims.

Definition (5.3.10) [9.73.79]: The definition of the Kosul complex is extended as follows. We denote by Ksz^q (M) the sub–chain–complex Ksz^q_z (M) =

M $\otimes_e{}^k\Lambda_2$ of Ksz (M). The only difference between Ksz^q (M) and Ksz (M) is that in the first case a dx_i with i ≤ q is excluded.

Theorem (5.3.16)[9.73. 79]: Ksz (R) is an effective free R–resolution of the R–module e. It is the particular cxse q = o of the next theorem to be proved by decreasing induction.

Theorem (5.3.17)[9.73. 79]: Ksz^q (R) is an effective free R–resolution of the R–module R_2. Note strictly speaking such a statement is improper. When we claim some object is effective, we mean some collection of algorithms, more or less difficult to be constructed, will allow us to justify the qualifier.

Proof. The theorem is obvious for q = m: the chain–complex

o ← R ← o concentrated in degree o is a resolution of R.

let us assume the theorem is proved for q and let us prove it for q-1. A resolution P_q = (fq, gq, hq): Ksz^q (R) → R_q is available. Our simple example above is easily adapted to prove:

Lemma (5.3.18) [9.73. 79]: The chain complex

$$o \leftarrow R_q \xrightarrow{x^{xq}} R_q \leftarrow o$$

Is an effective free resolution of R_{q-1}. It's a sophisticated and precise way to express the map XX_q as injective and its cokerel is R_{q1}. The relevant reduction is made of the projection $f_{2-1.q}$ which is an R–morphism, the injection $f_{q-1.q}$ which an f_{q-1} –morphism only, and the homotopy operator $h_o(\alpha)=[\alpha - \alpha(x_q = o)/x_q]$ which is an R_{q-1} morphism.

Proof:

Thanks to the reduction P, the object Ksz^q (R) is "above'. The morphism Xx_q is trivially lifted into a chian-complex morphism: Xx_q: Ksz^q (R)←Ksz^q (R); the source and the target of this morphism are reduced though P_q over R_q and we can apply the cone Reduction theorem combining with the other reduction already available, we obtain:

$$\text{Cone } (Ksz^q (R) \xrightarrow{X^{xq}} Ksz^q (R) \rightarrow \text{cone } (R_{q,*} \xrightarrow{X^{xq}} R_{q,*}) \rightarrow R_{2-1}$$

Lemma (5.3.19)[6.71.72]: Let J be an additive subcategory of c, and assume that J is cogenerating. Let $X^{\bullet} \in C^{+}$ (c).

Then there exists Let $Y^{\bullet} \in K^{+}(J)$ and aqis $x^{\bullet} \rightarrow y^{\bullet}$

Proof:

The proof is of the same kind of those.

We set $N^{+}(J) := N$ (c) $\cap$ $K^{+}(J)$. It is clear that $N^{+}(J)$ is a null system in $K^{+}(J)$.

Proposition (5.3.20)[6.71.72]: Assume J is cogenerating in c. Then the natural functor θ: $K^{+}(J)/N^{+}(J) \rightarrow D^{+}(c)$ is an equivalence of categories.

Proof:

Apply Lemma (3.6.18) and proposition (4.5.32).

Let us apply the preceding proposition to the category Ic of injective objects of c.

Corollary (5.3.21)[6.71.72]: Assume that c admits enough injective. Then $K^{+}(Ic) \rightarrow D^{+}(c)$ is an equivalence of categories.

Proof. Recall that if $x^{\bullet} \in C^{+}(Ic)$ is qis to o, then $x^{\bullet}$is homotopic to o.

Remark (5.3.22): Assume that c admit enough injective. then $D^{+}(c)$ is a u-category.

Example(5. 3.23): Let Ab airing. The functor $\cdot \otimes_{A}^{L} \cdot : \acute{D}(\text{mod } (A^{0p})) \times \acute{D}$ (mod (A)) $\rightarrow \acute{D}$ (Ab) is well defined.

$N \otimes_{A}^{L} M \simeq S (N \otimes_{A} P)$

$\simeq (\otimes_{A} M)$

Where P (resp. q) is a complex of projective A – modules qis to M (resp. N). In the preceding situation, one has

To r_{-K}^{A}(N, M) = H^{K} (N $\otimes_{A}^{L}$ M). The following result relies the derived functor of Hom_c and Hom_p (c)

Section (5.4) Derived functors[6.71.72]:

In this section, c and ć will denote abelian categories.

Let F: c →ć be a left exact functor. It defines naturally a functor.

$$K^{+} F : K^{+} (c) \rightarrow K^{+}(ć)$$

For short, one often writes F instead of K^{+} F. Applying the results of chapter 5. we shall construct (under suitable hypotheses) the right localization of F. [6].

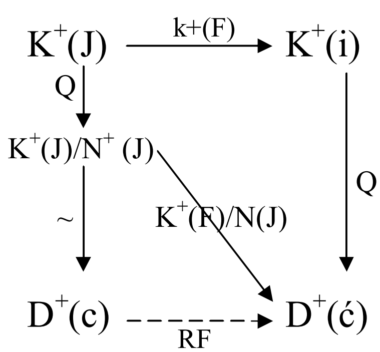
Definition (5.4.11): If the functor K^{+}(F): K^{+}(c) →D^{+}(ć) admit a right localization (with respect to the qis in K^{+}(c), one says that F admits a right derived functor and one denotes by RF: → D^{+}(c) → D^{+}(ć) the right localization of F.

Theorem (5.4.24)[6.71.72]: Let F: c→ć be a left exact functor of abelian categories, and let Jcc be a full additive sub category. Assume that J is F-injective. Then F admits a right derived functor RF: D^{+}(c)→ D^{+}(ć).

Proof:

This follows immediately from Lemma (3.6.20) and proposition (4.5.32) applied of K^{+}(f): K^{+}(c)→ D^{+}(ć).

It is visualized by the diagram



Since ob ($K^+(J)/N^+(J)$ = ob ($K^+(J)$, we get that for $x \in K^+(c)$, if there is aqis $x \to y$ with $y \in K^+(J)$, then RF (x) $\simeq$ F (y) in $D^+(ć)$. Note that if c admits enough injectives, them (5.6).

$$(5.6) \qquad R^kF = H^k \text{ o } RF.$$

Recall that the derived functor RF is triangulated, and does not depend on the category J. Hence, if $x' \to x \to x'' \xrightarrow{+1}$ is a d.t in $D^+(c)$, the RF (x') $\to$ RF (x) $\to$ RF (x'') $\xrightarrow{+1}$ is a d.t in $D^+(c)$. (Recall that an exact sequence $o \to x' \to x \to x'' \to o$ in c gives rise to a d.t in D (c). Applying the cohomological functor H^o, we get the long exact sequence in ć.

$$\ldots \to R^k F(x') \to R^k F(x) \to R^k F(x'') \to R^{k+1} F(x') \to$$

By considering the category C^{op}, one defines the notion of left derived functor of a right exact functor F.

We shall study the derived functor of a composition.

Let F: $c \to ć$ and GL $ć \to c''$ be left exact functor of abelian categories. Then Go F: $c \to c$ is left exact. Using the universal property of the localization, one shows that if G,G and GoF are right derivable, then there exists a natural morphism of functors.

$$(5.7) \qquad R(GoF) \to RGoRF$$

Proposition (5. 4. 25)[6. 71.72]: Assume that there exist full additive sub- categories $J \subset c$ and

$J' \subset c'$ such that J is F-injective, J' is G-injective and $G(J) \subset J'$.
Then J is (G o G)-injective and the morphism in (6.7) is an isomorphism: R (G o F) $\simeq$ R G o R F.

Proof:

The fact that J is (GoF) injective follows immediately from the definition. Let $x \in K^+(c)$ and let $y \in K^+(J)$ with aqi $x \to y$. Then RF(x) is represented by the complex F(y) which belongs to $K^+(J)$. Hence

RG (RG(x) is represented by G [F(y)] = (GoF) (y), and this last complex also represents R (GoF) (y) since y ∈ J and J is GoF injective. note that in general F does not send injective objects of c to injective objects of c', and that is why we had to introduce the notion of "F-injective" category.

Section (5.5) Bi functors [6.71.72]:

Now consider three abelian categories c, c', c" and an additive bi functor:

$$F: c \times c' \rightarrow c".$$

E shall assume that F is left exact with respect to each of its arguments.

Let x∈ $K^+(c)$, x' ∈ $K^+(c')$ and assume x(or x') is homotopic to o. Then one checks easily that tot [F(x,x')] is homotopic to zero. Hence one can naturally define.

$K^+(F)$: $K^+(c)$ x $K^+(c')$ →$K^+(c")$ by setting:

$K^+(F)$ (x,x') = tot [F(x,x')]

If there is not risk of confusition, we shall sometimes write F instead of K+F.

Definition (5.5.12)[6.71.72]: One says (J.H') is F-injective if:

(i) For all x ∈ J, J' is F (x,.)-injective
(ii) For all x' ∈ J, J' is F (.x')-injective

Lemma (5. 5. 26)[6. 71. 72]: Let x∈$K^+(J)$,x' ∈ K+(J). If x or x' is qis to o, then F(x,x') is qis to zero.

Proof. The double complex F(x,y) will satisfy the hypothesis of theorem (4.6.41).

Using Lemma (5. 3. 19) and proposition (5. 5. 26) one gets that F admit a right derived functor.

$$RF: D^+(c) \times D^+(c') \to D^+(c'')$$

Example (5.5.27). [6.71.72]: Assume c has enough injective. then R Hom_c: $D^-(c)^{op} \times D^+(c) \to D^+(Ab)$.

Exists and may be calculated as follows. Let $x \in D^-(c)$, $y \in D^+(c)$. There exists a qis in $K^+(c)$, $y \to I$, the $I^{S,s}$ being injective. Then $RHom_c(x,y) \simeq Hom_c^\bullet(X,I)$. If c has enough projectives, and $p \to x$ is a qis $K^-(c)$, the p^{j}'s being projective, one also has:

$$RHom_c(x,y) \simeq Hom_c^\bullet(p,y).$$

These isomorphisms hold in $D^+(Ab)$.

Example (5.5.28). [6.71.72]: Let A be a ring. The functor

$\otimes_A^{L\,\bullet\bullet}$: $D^-[Mod(A^{op})]\, D^-[Mod(A)] \to D^-(Ab)$ is well defined

$$N \otimes_A^L M \simeq S(N \otimes_A P)$$

$$\simeq S(Q \otimes_A M)$$

Where P (resp. Q) is a complex of projective A–modules qis to M (resp. N).

In the preceding situation, one has

$$Tor \otimes_{-K}^L (N,M) = H^k(M \otimes_A^L M).$$

The following result relies the derived functor of Hom_c and$Hom_{D(c)}$.

Theorem (5.5.29)[6.71.72]: Let c be an abelian category with enough injective. Then for $x \in D^-(c)$ and $y \in D^+(c)$,

$$H^o R\, Hom_c(x,y) \simeq Hom_{D(c)}(x,y),$$

Proof:

There exists $I_y \in D^+(I)$ and a qis $y \to I_y$.

Then we have the isomorphism:

$$\mathrm{Hom}_{D(c)}(x,y[k]) \simeq \mathrm{Hom}_{K(c)}(x,I_y[k])$$

$$\simeq H^o om(\mathrm{Hom}^{\bullet}_{c}(x,I_y[k])$$

$$\simeq R^k\mathrm{Hom}(\mathrm{Hom}_c(x,y)$$

Where the second isomorphism follows from theorem (5.5.29) implies the isomorphism

$$\mathrm{Ext}^k_c(x,y) \simeq H^kR\,\mathrm{Hom}_c(x,y).$$

Example (5.5.30)[6.71.72]: Let w be the Wey algebra in one variable over a field k:w = k[x,δ] with the relation [x,δ] = –1.

Let Q = w/w.δ, Ωw/δ.w and let us calculate $\Omega \otimes^L_W Q$. we have an exact sequence:

$$o \to w \xrightarrow{\delta} w \to\to \Omega o$$

hence Ω is qis to the complex

$$o \to W^{-1} \xrightarrow{\delta} W^o \to o$$

Where $W^{-1} = W^o = W$ and W^o is in degree o.

Then $\Omega \otimes^L_W Q$ is qis to the complex

$$o \to Q^{-1} \xrightarrow{\delta} Q^o \to o$$

Where $Q^{-1} = Q^o = Q$ and Q^o is in degree o. Since δ: Q → Q is surjective and has k as kernel, we obtain:

$$\Omega \otimes^L_W Q \;\; k\,[I].$$

Example (5.5.31)[6.71.72]: Ket k be a field and let A= k[x_1….., x_n]. This is a commutative noetherian ring and it is known (Hilbert) that any finitely generatedA–module M admits a finite free presentation of length at most n, i.e. M is qis to a complex:

$$L: = o \to L^{-n} \to \ldots \xrightarrow{Po} L^o \to o$$

Where the $L^{j,s}$ are free of finite rank. Consider the functor

$$Hom_A (\bullet,A): Mod (A) \rightarrow Mod (A).$$

It is contra variant and left exact. Since free A–modules are projective, we find that $RHom_A$ (M,A) is isomorphic in D^b (A) to the complex

$$L^*: = o \leftarrow L^{-n*} \leftarrow \ldots \xleftarrow{Po} L^{o*} \leftarrow o$$

Where $L^{S*} = Hom_A (L^j,A)$. Set for short$^* = RHom_A (\bullet,A)$ using eq(6.7), we find a natural morphism of functors.

$$id \rightarrow ^{**}.$$

Applying $RHom_A (\bullet,A)$ to the object $RHom_A$ (M,A) we find:

$$RHom_A [RHom_A (M,A), A] \simeq RHom_A (L^*,A)$$

$$\simeq L$$

$$\simeq M.$$

In other words, we have proved the isomorphism in $D^b(A)$: $M \simeq M^{**}$.

Assume now n I, i.e. A = K [x] and consider the natural morphism in Mod (A): f: A → A/Ax. Applying the functor * = $RHom_A (\bullet,A)$, we get the morphism in D^b (A):

$$f^*: RHom_A (A/Ax, A) \rightarrow A.$$

Remember that $RHom_A (A/Ax, A) \simeq A/x A [1-]$. Hence $H^j (f^*)$

$= o$ for all $j \in Z$, although $f^* \neq o$ since $f^{**} = f$.

Let us give an example of an object of a derived category which is not isomorphic to the direct sum of its co homology objects (hence, a situation in which corollary (5. 2. 13) does not apply).

Example (5.5.32)[6.71.72]: Let k be a field and let $A=k[x_{1.}x_{2.}]$. Define the A–modules $M' =A/(Ax_1+Ax_2)$, $M=A/(Ax_1+Ax_1x_2)$ and $M'' =A/Ax_1$. There is an exact sequence.

(5.8) $0 \to M' \to M \to M'' \to o$

And this exact sequence does not split since x_1 kills M' and M'' but not M. Gor N an A–module set $N^* = R\,Hom_A\,(N, A)$, an object of D^b (A) (see example (5. 2. 29). We have $M'^* \simeq H^2(M'^*)\,[-2]$, and $N''^* \cong \acute{H}\,(\acute{m}^*)\,[-1]$and the functor* = $RHom_A\,(\bullet,A)$ applied to the exact sequence (6.8) gives rise to the long exact sequence

$$o \to H'\,(M''^*) \to H'\,(M^*) \to o \to o \to H^2\,(M^*) \to$$

$$H^2\,(M'^*) \to o$$

Hence $H'\,(M^*)\,[-1] \simeq H'\,(M''^*)\,[-1] \simeq M''^*$ and $H^2\,(M^*)\,[-2] \simeq H^2\,(M'^*)\,[-2] \simeq M'^*$. Assume for a while $M^* \simeq \oplus_j H^j\,(M^*)[-j]$. This implies $M^* \otimes M'^*$hence (by Applying again the functor*), $M' \simeq M''$, which is a contradiction.

Localization [6.71.72]: Consider a category c and or family s of morphisms in c. The aim of localization is not find a new category C_s and a functor Q: C→ C_s which send the morphisms belonging to s to isomorphisms in C_s,(Q,C_s) being "universal" for such a property.

In this chapter, we shall construct the localization of a category when s satisfies suitable conditions and the localization of functors. We shall be aware that in general, the localization of all-category c is no more all-category.

Localization of categories appears in particular in the construction of derived categories.

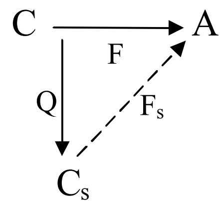
Sectiob (5.6) Localization of categories:

Let C be a category and let S be a family of morphisms in C.

Definition(5.6.1)[6]:A Localization of C be S is the data of category $C_s$and afunctor Q: C → C_s satisfying;

(a) For all s ∈ S, Q (S) is an isomorphism.

(b) For any functor $F: C \to A$ such that $F(S)$ is an isomorphism for all $s \in S$, there exists a functor $Fs: C_s \to A$ and an isomorphism $F \simeq F_s \circ Q$,



(c) If G_1 and G_2 are two objects of Fct (C_s, A), then the natural map.

(5.9) $\mathrm{Hom}_{Fct}(C_s,A)(G_1.G_2) \to \mathrm{Hom}_{Fct}(C,A)(G_1 o Q, G_2 o Q,)$ is bijective.

Note that (c) means that the functor o Q: Fct $(C_s,A) \to$ Fct (C,A) is fully faithful. This implies that F_s in (b) is unique up to unique isomorphism.

Proposition(5. 6. 33):

(i) If C_s exists, it is unique up to equivalence of categories.

(ii) If C_s exists, then, denoting by C^{op} the image of S in C^{op} by the functor op, $(C^{op})_{Sop}$ exists and there is an equivalence of categories:

$$(C_s)^{op} \simeq (C^{op})_{Sop}$$

Proof:

(i) is obvious.

(ii) Assume C_s exists. Set $(C^{op})_{Sop}: = (C_s)^{op}$ and define

$Q^{op}\ C^{op} \to (C^{op})_{Sop}$ by Q^{op} = op o Qo op. Then properties (a), (b) and (c) of Definition (5. 1. 1) are clearly satisfied.

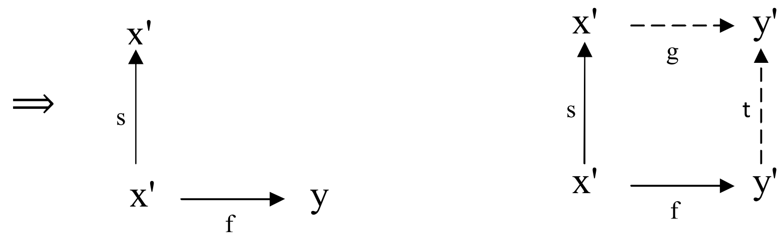
Definition (5. 6. 2): One says that S is a right multiplicative system if it satisfies the axioms S_1–S_4 below.

S_1 For all $x \in C$, $id_x \in S$.

S_2 For all f ∈ S, g ∈ S, if g of exists then go f ∈ S.

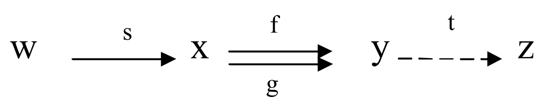
S_3 Given two morphisms, f: x → y and s: x → x'

With tes and gos = t of. This can be visualized by the diagram:



Let f,g: x → y be two parallel morphisms. If there exists s ∈ S:w → x such that fos = gos then there exists t ∈ S: y → z such that t of = tog.

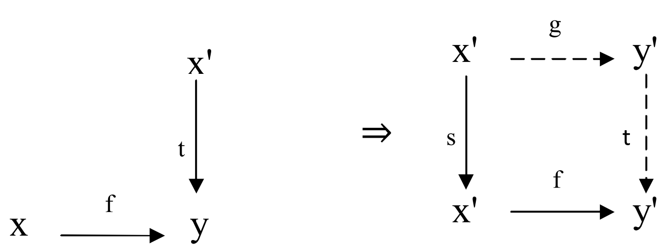
This can be visualized by the diagram:



notice that these axioms are quite natural if one wants to invert the elements of s. In other words, if the element of s would be invertible, then these axioms would clearly be satisfied.

Remark (5. 6. 34) [6. 71. 72]: Axioms S_1– S_2 asserts that S is the family of morphisms of sub category S of C with ob (S) = ob (c).

Remark (5. 6. 35): One defines the notion of a let multiplicative systems by reversing the arrows. This means that the conditionsS_3 is replaced by; given two morphisms, f: x → y and t: y' → y, with t ∈ S, there exist s: x' → x and g: x' → y' with s ∈ S and tog = fos. This can be visualized by the diagram;



and S_4 is replaced by: if there exists t∈S: y→z such that tof= top then there exists $s \in S: w \to x$ such that $fos = gos$. Then is visualized by the diagram

$$w \xrightarrow{s} x \overset{f}{\underset{g}{\rightrightarrows}} y \dashrightarrow^{t} z$$

In this literature, one often calls a multiplicative system a system which is both right and left multiplicative.

Many multiplicative systems that we shall encounter satisfy a useful property that we introduce now. [6].

Definition(5. 6. 3)[6]: Assume that S satisfies the axioms $S_1 - S_2$ and let $x \in C$.

One defines the categories S_x and S^x as follows.

$$ob\,(S^x) = \{\, s: x \to s';\ s \in S \,\}$$

$$Hom_{Sx}\,((s: x \to x'), (s: x \to x'')) = \{\, h: x' \to x'';\ hos = s' \,\}$$

$$ob\,(SX) = \{\, s: x' \to x;\ s \in S \,\}$$

$$Hom_{Sx}\,((s: x' \to x), (s': x'' \to x)) = \{\, h: x' \to x'';\ s'oh = s \,\}$$

Proposition (5. 6. 36)[6.71.72]: Assume that S is a right (resp. left) multiplicative system. Then the category S^x (resp. S_x) is filtrate.

Proof:

By reversing the arrows, both results are equivalent. We treat the case of S^x.

(a) Let $s: x \to x'$ and $s': x \to x''$ belong to S. By S, thee exists $t: x' \to x''$ and $t': x'' \to x''$ such that $t'os' = tos$, and $t \in S$. Hence, $tos \in S$ by S_2 and $(x \to x'')$ belongs to S^x.

(b) Let $s: x \to x'$ and $s': x \to x''$ belong to s, and consider two morphism f,gL $x' \to x''$, with $fos = gos = s'$. By S_4 there exists $t: x'' \to w$, $t \in S$ such that $tof = tog$.
Hence $tos': x \to w$ belongs to S^x.

One defines the functors,

$\alpha\, x: S^{x} \rightarrow c$ $\quad$ $(s: x \rightarrow x') \rightarrow x'$.

$\beta_x: S_x^{op} \rightarrow c$ $\quad$ $(s: x \rightarrow x') \rightarrow x'$.

Wwe shall concentrate on right multiplicative system

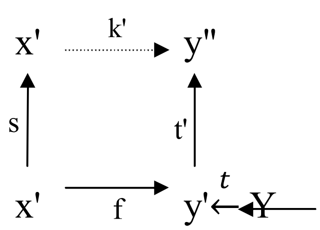
definition (5.6.4)[6.71.72]: let S be a right multiplicative system, and let x, y ∈ ob (c). we set.

$$\mathrm{Hom}_{c_s^r}(x,y) = \varinjlim_{(y\ y') \in S^r} \mathrm{Hom}_c(x,y).$$

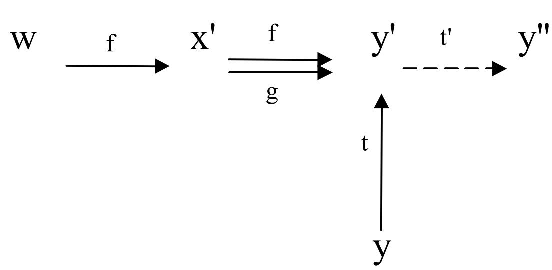
Lemma(5. 6. 37)[6]: Assume that S is a right multiplicative system. Let y ∈ C and let s: x → x' S.

Then S induces an isomorphism

$$\mathrm{Hom}_{c_s^r}(x,y) = \xrightarrow[os]{\sim} \mathrm{Hom}_{c_s^r}(x,y).$$



(iii) The map os is injective. this follows from S, as visualized by the diagram in which s, t, t' ∈ S;



Using lemma (5. 9. 36), we define the composition

(5.10) $\mathrm{Hom}_{c_s^r}(x,y) = x\, \mathrm{Hom}_{c_s^r}(y,z)\, \mathrm{Hom}_{c_s^r}(y,z) = \rightarrow \mathrm{Hom}_{c_s^r}(x,z)$ as

$$\varinjlim_{y \rightarrow y'} \mathrm{Hom}_c(x,y) = x \varinjlim_{z \rightarrow z'} \mathrm{Hom}_c(y,z')$$

$$\simeq \varinjlim_{y\to y'} (\mathrm{Hom}_c(x,y') \times \varinjlim_{z\to z'} \mathrm{Hom}_c(y,z'))$$

$$\xleftarrow{\sim} \varinjlim_{y\to y'} (\mathrm{Hom}_c(x,y') \times \varinjlim_{z\to z'} \mathrm{Hom}_c(y',z'))$$

$$\longrightarrow \varinjlim_{y\to y'} \varinjlim_{z\to z'} \mathrm{Hom}_c(x,z')$$

$$\sim \varinjlim_{z'\to z'} \mathrm{Hom}_c(x,z')$$

Lemma (5. 6. 38)[71.72]: The composition eq(5. 2) is associative.

Hence we get a category c whose objects are those of c and morphism are given by definition (5.1.4).

Let us denote by Qs: $c \to c^r_s$ the natural functor associated with

$$\mathrm{Hom}_c(x,y) \to \varinjlim_{(y\to y')\in s^r} \mathrm{Hom}_c(x,y').$$

If there is no risk of confusion, we denote this functor simply by Q.

Lemma(5. 6. 39)[6]: If s: x→y belongs to s, then Q (s) is invertible.

Proof:

For any $Z \in C^r_s$, the map $\mathrm{Hom}_c(y,z) \to \mathrm{Hom}_c(z,z)$ is bijective by lemma (5.6.6).

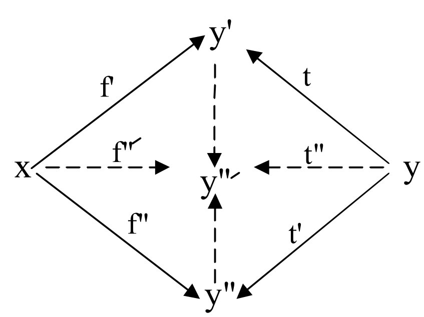
A morphism f: x → y in C^r_s is thus given by an equivalence class of triplets (y', t, 7') with t: y → y', t ∈ S and f': x → y', that is

$$x \xrightarrow{f'} y' \overset{t}{\leftrightarrows} y',$$

The equivalence relation being defined as follows:

(y',t,f') ~ (y'',t',f'') if there exists (y''',t'',f''')

(t,t',t'' ∈ S) and commutative diagram:

(5.11) 

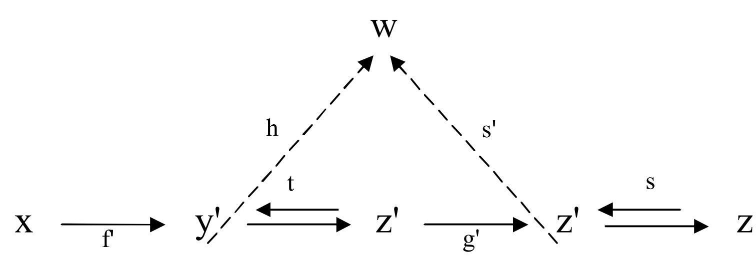
Note that the morphism (y',t,f') in C_s^r is $Q(t)^{-1}$ o Q (f'), that is,

(5.12) $f = Q(t)^{-1} \circ Q(f')$.

For two parallel arrows f, g: x $\rightrightarrows$ y in c we have the equivalence.

(5. 13) Q (f) = Q (g) ∈ C_s ↔ there exist s: y → y', s ∈ S with s o f = s o g.

The composition of two morphism (y', t, f'): x → y and (z', s, f'); y → z is defined by the diagram below in which t, s, s' ∈ S:



Theorem(5. 6.40)[6]: Assume that S is a right multiplicative system. The category C_S^r and the functor Q define a localization of C by S.

(i) For a morphism f; x → y, Q (f) is an isomorphism in C_s^r if and only if there exist g: y → z and h: z → w such that gof ∈ S and hog ∈ S.
Notation. From now on, we shall write Cs instead of C_s^r .
This is justified by Theorem (5. 1. 15).

Remark (5. 6. 41)

(i) In the above construction, we have used the property of S of being a right multiplicative system. If S is a left multiplicative system, one sets.
$\mathrm{Hom}_{C's}(x,y) = \varinjlim_{(x'\to x) \in S_x} \mathrm{Hom}_c(x',y)$.

By proposition (5. 1. 3) (i), the two constructions give equivalent categories.

(ii) If S is both a right and left multiplicative system.

$$\mathrm{Hom}_{C's}(x,y) \simeq \varinjlim_{(x'\to x) \in S_x, (y\to y') \in S^y} \mathrm{Hom}_c(x',y').$$

Remark (5. 6. 42) [6]: In general, c_s is no more all-category. However, if one assumes that for any $x \in C$ the category S_s is small (or more generally, co finally small, which means that there exists a small category confinal to it), then c_s is all-category, and there is a similar result with the S_x 's.

Example (5. 6. 43): Let C (resp.C □) be a category and S(resp. S □) a right multicity system in C (resp.C □). One checks immediately that S × S □ is a right multicity system in the category C ×C □ and (C ×C □ $)_{S \times S□}$ is equivalent to C_S× C $□_{S □}$. Since abifunctor is afunctor on the product C ×C □, we my apply the preceding results to the case of bifunctor. In the sequel, we shall write $F_{SS □}$ instead of $F_{S \times S □}$.

Proposition(5. 6. 44)[6. 71. 72]: Let c be a category. I a full sub category, s a right multiplicative system in c, I the family of morphisms in I which belong to s.

(i) Assume that I is a right multiplicative system in I. Then $I_I \to C_s$ is well-defined.

(ii) Assume that for every f: y → x. f ∈ S, y ∈ I, there exists g: x→w, w∈I, with gof∈S. Then I is a right multiplicative system and $I_I \to C_s$ is fully faithful.

Proof (i) is obvious.

(iii) We check that I is a right multiplicative system.

For x∈I, I^x is full subcategory of S^x whose objects are the morphisms s: x→y with y∈I. By proposition (5.1.4) and the hypothesis, the functor $I^x \to S^x$ is co final, and the result follows from Definition (5.1.4).

Corollary(5.6.45)[6.71.72]: Let c be a category, I a full sub category, S a right multiplicative system in c, I the family of morphisms in I which belong to s.

Assume that for any $x \in C$ there exists $s: x \rightarrow w$ with $w \in I$ and $s \in S$. Then I is a right multiplicative system and I_T is equivalent to c_s.

Proof.

The natural functor $I_T \rightarrow c_s$ is full faithful by proposition (5. 2. 12) and is essentially surjective by the assumption.

Example (5. 6. 46): The localization of a category A with respect to a class of morphism $S \subseteq$ mod (A) is the (universal) functor, i. e, morphism of category, $A \overset{i}{\rightarrow} A_S$ such that the images of all morphism is S are isomorphism in A_S(i. e, have inverses in A_S). Again, localization exists can be described under some condition.

Section (5.7) Localization of functors:

Let c be a category, S a right multiplicative system in c and F: $c \rightarrow A$ a functor, In general, F does not send morphisms in S to isomorphism in A. In other words, F does not factorize through c_s. It is however possible in some cases to define a localization of F as follows.

Definition (5. 7. 5)[6. 71.72]: A right localization of F (if it exists) is a functor F_s $c_s \rightarrow A$ and a morphism of functor T: $F \rightarrow F_s \circ Q$ such that for any functor G: $c_s \rightarrow A$ the map.

(5.14) $Hom_{Fct\ (Cs,A)}(F_s, G) \rightarrow Hom_{Fct\ (C,A)}(F, G \circ Q)$ is bijective. (This map is obtained as the compositon $Hom_{Fct\ (C,A)}(F_{S,} G) \rightarrow Hom_{Fct(c,\ A)}(F_S \circ Q, G \circ Q) \rightarrow Hom_{Fct(c,\ A)}(F, G \circ Q)$.

We shall say that F is right localizable if it admits a right localization.

One defines similarly the left localization. Since we mainly consider right localization, we shall sometimes omit the word "right" as far there is no risk of con fusition.

If (T,Fs) exists it is unique up to unique isomorphism's. Indeed, Fs is a representative of functor.

$G \rightarrow Hom_{Fct\,(C,A)}(F, G \circ Q)$.

(This last functor is defined on the category Fct (c_s,A) with values in set).

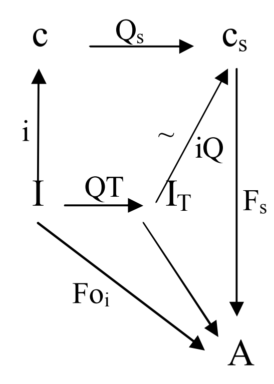
Proposition (5. 7. 47)[6]: Let c be a category, I a full sub category, S a right multiplicative system in c, I the family of morphisms in I which belong to S. Let F: c → A be functor. Assume that

(i) For any x∈C there exists s: x→w with w∈ I and s∈ S.
(ii) For anuy t ∈T, F (t) is an isomorphism.

Then F is right localizable.

Proof. We shall apply corollary (5. 2. 13).

Denote by i: → c the natural functor. By the hypothesis, the localization F_T of Fo_i exists,. Consider the diagram:



Denote by $i_{\frac{-1}{Q}}$ a qusi-inverse of i Q and set Fs: = $F_{To\frac{i-1}{Q}}$.

Let us show that F_s is the localization of F. Let

G: C_s→ A be a functor. We have the chain of morphism:

$\xrightarrow{\lambda}$

$Hom_{Fct\,(C,A)}(F, G \circ Q_s) \qquad Hom_{Fct\,(I,A)}(Fol, G \circ Qsol)$

$$\simeq Hom_{Fct\,(I,A)}(F_{TO}Q_T, G_{ocQoQT})$$

$$\simeq Hom_{Fct\,(I_T,A)}(F_T G_{OCQ})$$

$$\simeq Hom_{Fct\,(Cs,A)}(F_{TO}\, L_Q^{-1}, G)$$

$$\simeq Hom_{Fct\,(Cs,A)}(F_S, G)$$

We shall not prove here that λ is an isomorphism. The first isomomorphism above (after λ) follows from the fact that QT is a localization functor [see Definition eq(5.1.1) (c)]. The other isomorphisms are obvious.

Remark(5. 7. 48)[6. 71. 72]: Let c (resp. c') be a category and S (resp. S') a right multiplicative system in c (resp. c'). One checks immediately that S X S' is a right multiplicative system in the category c X C' and (cxc'$_s$) six' is equivalent to $c_s$xc'$_s$. Since a bifunctor is a functor on the product cxc', we may apply the preceding results to the case of bifunctors. In the sequel we shall write $F_{s's}$ instead of $F_{sxs'}$.

Section (5.8) Triangulated categories [6.71.72]:

Definition (5. 8. 6): Let D be an additive category endowed with an auto morphism T (i.e., an invertible functor T: D → D). A triangle in D is a sequence of morphisms:

(5.15) $\quad x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{\lambda} T(x).$

A morphism of triangles is a commutative diagram:

$$\begin{array}{ccccccc} x & \xrightarrow{f} & y & \xrightarrow{g} & z & \xrightarrow{\lambda} & T(x) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle T_\gamma} \\ x' & \xrightarrow{f'} & y' & \xrightarrow{g'} & z' & \xrightarrow{\lambda'} & T(x') \end{array}$$

Example (5. 8. 49). [6]: The triangle $x \xrightarrow{f} y \xrightarrow{-g} z \xrightarrow{-h} T(x)$
$\xrightarrow{f} \quad \xrightarrow{-g}$

–h

is isomorphic to the triangle eq(6.1), but the triangle x y z $\longrightarrow$ T(x) is not isomorphic to the triangle eq(6.1) in general.

Definition (5. 8. 7). [6]: A triangulated category is an additive category D endowed with an auto morphism T and a family of triangles called distinguished triangles (d.t. for short),m this family satisfying axioms TRO – TR5 below.

TR_0 A triangle isomorphic to a d.t. is a d.t.

TR_1 The triangle $x \xrightarrow{id_x} x \to o \to T(x)$ is a d.t.

TR_2 For all f: $x \to y$ there exists a d. t. $x \xrightarrow{f} y \to z \to T(x)$.

TR_3 A triangle $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} T(x)$ is a d.t. if and only if $y \xrightarrow{g} z \xrightarrow{h} T(x) \xrightarrow{-T(x)} T(y)$ is a d.t.

TR_4 Given two d.t. $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} T(x)$ and $x' \xrightarrow{f'} y' \xrightarrow{g'} z' \xrightarrow{h} T(x')$ and morphisms α: x x' and

β: y → y' with f' o α = βof, there exists a morphism

γ: z → z' giving rise to a morphism of d.

$$\begin{array}{ccccccc} x & \xrightarrow{f} & y & \xrightarrow{g} & z & \xrightarrow{h} & T(x) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle T(\alpha)} \\ x' & \xrightarrow{f'} & y' & \xrightarrow{g'} & z' & \xrightarrow{h'} & T(x') \end{array}$$

\TR5 (octahedrad axiom) Given three d.t.

$$\begin{array}{ccccccc} x' & \xrightarrow{f} & y' & \xrightarrow{h} & z' & \longrightarrow & T(x), \\ y & \xrightarrow{g} & z & \xrightarrow{k} & x' & \longrightarrow & T(x), \\ x & \xrightarrow{gof} & z & \xrightarrow{l} & y' & \longrightarrow & T(x'), \end{array}$$

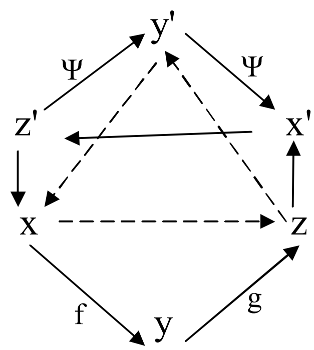
there exists a distinguished triangle $z' \xrightarrow{\Psi} y' \xrightarrow{\Psi} x' \longrightarrow T(z')$ making the diagram below commutative:

(5,16)
$$\begin{array}{ccccccc} x & \xrightarrow{f} & y & \xrightarrow{h} & z' & \longrightarrow & T(x) \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle id} \\ x & \xrightarrow{gof} & z & \xrightarrow{l} & y' & \longrightarrow & T(x) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle T(f)} \\ & \xrightarrow{g} & & \xrightarrow{k} & & \longrightarrow & \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle id} & & \downarrow \\ & \xrightarrow{\Psi} & & \xrightarrow{\Psi} & & & \end{array}$$

y ⟶ z ⟶ x' T(y)T(h)

z' y' x' T(z')

diagram eq(5.8) is often called the octahedron diagram. Indeed, it can be written using the vertexes of an octahedron.



Remark (5. 8. 50)[6]: The morphism γ in T_{R4} is not unique and this is the origin of many troubles.

Remark (5. 8. 51): The category D^{op} endowed with the image by the contra variant functor op: $D \rightarrow D^{op}$ of family of the d.t. in d, is a triangulated category.

Definition (5. 8. 8). [6]:

(i) A triangulated functor of triangulated categories F: (D,T) → (D',T') is an additive functor which satisfies FoT $\simeq$ T'of and which sends distinguished triangles to distinguished triangles.

(ii) A triangulated sub category D' of D is a subcategory D' of D which is triangulated and such that the functor D' → D is triangulated.

(iii) Let (D,T) be a triangulated category, c an abelian category, F: D → c an additive functor. One says that F is a co homological functor if for any d.t. x → y → z → T(x) in D, the sequence F(x) → F(y) F(z) is exact in c.

Remark(5. 8. 52): By $TR_{3.}$ a co homological functor gives rise to along exact sequence:

(5.17) ….→ F(x) → F(y) → F(z) → F [T(x)] →…..

Proposition (5. 8. 53) [6]:

(i) If $x \xrightarrow{f} y \xrightarrow{g} z \rightarrow T(x)$ is a d.t. then gof = o.

(ii) For any w∈ D, the functor Hom_D (w,.) and Hom_D (.,w) are co homological.

Note that (ii) means that if Ψ: w → y (resp.Ψ: y → w) satisfies goΨ = o (resp. Ψof = o), then Ψ factorizes through f (resp. through g).

proof:

(i) Appling T_{R_1} and T_{R_4} we get a commutative diagram:

$$\begin{array}{ccccccc} x & \xrightarrow{id} & x & \longrightarrow & o & \longrightarrow & T(x) \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle id} \\ x & \xrightarrow{f} & y & \xrightarrow{g} & z & \longrightarrow & T(x) \end{array}$$

Then gof factorizes through o.

(ii) Let x → y → z → T(x) be a d.t. and Let w ∈ D. We want to show that

$$Hom\,(w,x) \xrightarrow{f} Hom\,(w,y) \xrightarrow{go} Hom\,(w,z)$$

is exists, i.e.,: for all Ψ: w → y such that goΨ = o, there exists Ψ : w → x such that Ψ = f o Ψ. This means that the dotted arrow below may be completed, and this follows form the axioms T_{R_4} and T_{R_3}.

$$\begin{array}{ccccccc} w & \xrightarrow{id} & w & \longrightarrow & o & \longrightarrow & T(w) \\ \downarrow & & \downarrow{\scriptstyle \Psi} & & \downarrow & & \downarrow \\ x & \xrightarrow{f} & y & \xrightarrow{g} & z & \longrightarrow & T(x) \end{array}$$

The proof for Hom (.,w) is similar.

Proposition (5. 8. 54). [6]: Consider a morphism of d.t.:

$$\begin{array}{ccccccc} x & \xrightarrow{f} & y & \xrightarrow{g} & z & \xrightarrow{h} & T(x) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle T(\alpha)} \\ x' & \xrightarrow{f'} & y' & \xrightarrow{g'} & z' & \xrightarrow{h'} & T(x') \end{array}$$

If α and β are isomorphism, then so is γ.

Proof:

Apply Hom (w,.) to this diagram and write $\tilde{x}$ instead of Hom (w,x), α instead of Hom (w,α), etc. we get the commutative diagram;

$$\begin{array}{ccccccc} \tilde{x} & \longrightarrow & \tilde{y} & \longrightarrow & \tilde{z} & \longrightarrow & \tilde{T}(x) \\ \downarrow{\scriptstyle\tilde{\alpha}} & & \downarrow{\scriptstyle\tilde{\beta}} & & \downarrow{\scriptstyle\tilde{\gamma}} & & \downarrow{\scriptstyle T(\alpha)} \\ \tilde{x}' & \xrightarrow{\tilde{f}'} & \tilde{y}' & \xrightarrow{\tilde{g}'} & \tilde{z}' & \xrightarrow{\tilde{h}'} & \tilde{T}(x') \end{array}$$

The rows are exact in view of the preceding proposition, and

$\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{T}$ (α), $\tilde{T}$ (β) are isomorphism's. Therefore $\tilde{\gamma}$ =

Hom (w,γ): Hom (w,z) $\rightarrow$ Hom (w,z') is an isomorphism.

This implies that γ is an isomorphism by the Yoneda lemma.

Corollary (5. 8. 55)[6]: Let D' be a full triangulated category of D.

(i) Consider a triangle $x \xrightarrow{f} y \rightarrow z \rightarrow T(x)$ in D' and assume that this triangle is distinguished in D. Then it is distinguished in D'.

(ii) Consider a d.t. $x \rightarrow y \rightarrow z \rightarrow T(x)$ in D, with x and y in D'.

Then there exists z' $\in$ D' and an isomorphism z $\simeq$ z'.

Proof (i) there exists a d.t. $x \xrightarrow{f} y \rightarrow z' \rightarrow T(x)$ in D'.

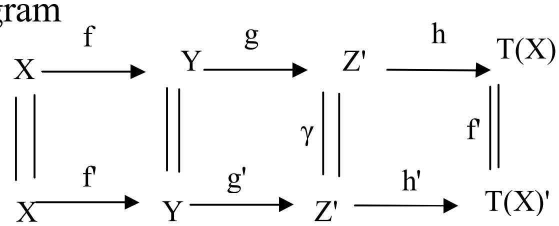
Then z' is isomorphic to z by TR_4 and proposition (5.4. 25)

(ii) Apply TR_2 to the morphism $x \rightarrow y$ in D'.

Remark (5. 8. 56)[6]: The proof of proposition (5. 4. 25) does not make use of axiom T_{R_5}. and this proposition implies that T_{R_5} is equivalent to the axiom: $T_{R'_5}$.: given f: x → y and g; y → z, there exists a commutative diagram eq(5.8) such that all rows are d. t.

By proposition (5. 4. 25), one gets that the object z given in T_{R_2} is unique up to isomorphism. However, this isomorphism is not unique, and this is the source of many difficulties (e.g., gluing problems in sheaf theory).

Example(5. 8. 57) Let D be a triangulated category and consider a commutate diagram



Assume that T (f) o h□ = o and the first row is ad. T. we prove that the second row is also ad. t. Under one of the hypotheses;

(i) For any p ∈D, the sequence below is exact
Hom_D (P, X) →Hom_D(P, Y)→ Hom_D (P, Z□ □) → Hom_D(P, T(X)).

(ii) For any P ∈ D, the sequence below is exact.
Hom_D (T(Y), P) →Hom_D (T(X), P) → Hom_D(Z □, P) → Hom_D(Y, P).

Section (5. 9) Localization of triangulated categories.

Definition (5. 9. 9): Let D be a triangulated category and let N ⊂ob (D).

One says that N is a null system if it satisfies:

N_1 o ∈ B,

N_2 x ∈ N, if and only if T(x) ∈ N,

N_3 if $x \rightarrow y \rightarrow z \rightarrow T(x)$ is a d.t. in D and $x, y \in N$ then $z \in N$.

To a null system one associates a multiplicative system as follows. Define: $s = \{f: x \rightarrow y$, there exists a d. t.

$x \rightarrow y \rightarrow z \rightarrow T(x)$ with $z \in N\}$.

Theorem (5. 9. 58). [6]:

(i) S is a right and left multiplicative system.

(ii) Denote as usual by Ds the localization of D by S and by Q the localization functor. Then Ds is an additive category endowed with an auto orphism (the image of T. still denoted by T).

(iii) Define a d.t. in Ds as being isomorphic to the image by Q of a d. t. in D. Then Ds is a triangulated category.

(iv) If $x \in N$ then $Q(x) \simeq o$.

(v) Left $F: D \rightarrow D'$ be a functor of triangulated categories such that $F(x) \simeq o$ for any $x \in N$. Then F factors uniquely through Q. The proof is tedious and will not be given here.

Notation (5. 9. 59): We will write D/N instead of Ds.

Now consider a full triangulated sub category I of D. We shall write $N \cap I$ instead of $N \cap ob(I)$. This is clearly a null system in I.

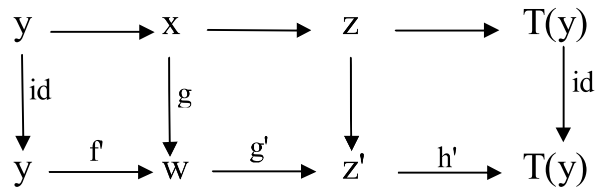
Proposition (5. 9. 60): Let D be a triangulated category, N a null system, I a full triangulated category of D. Assume condition (i) or (ii) below (i) any morphism $y \rightarrow z$ with $y \in I$ and $z \in N$, factorizes as $y \rightarrow z' \rightarrow z$ with $z' \in N \cap I$.

Then $I/(N \cap I) \rightarrow D/N$ is fully faithful.

Proof:

We shall apply proposition (5. 2. 12). We may assume (ii), the case (i) being deduced by considering D^{op}. Let $f: y \rightarrow x$ is a morphism in s with $y \in I$. We shall show that there exists $g: x \rightarrow w$

with w ∈ I and fof ∈ S. The morphism f is embedded in a d. t. y → x → z → T (x), with z ∈ N. By the hypothesis, the morphism z → T(y) factorizes through an object z' ∈ N∩I. We may embed z' → T(y) into a d.t. and obtain a commutative diagram of d.t.



By TR_4. the dotted arrow g may be completed, and z' belonging to N, this implies that gof ∈ S.

Proposition(5. 9. 61). [6]: Led D be a triangulated category, N a null system, I a full triangulated sub category of D, and assume conditions (i) or (ii) below:

(i) For any x ∈ D, there exists a d. t. x → y → z → T(x) with z ∈ N and y ∈ I.

(ii) For any x ∈ D, there exists a d.t. x → y → z → T(x) with z ∈ N and y ∈ I.

Then I/N∩I → D/N is an equivalence of categories.

Apply corollary (5. 9. 62). Let F: D → D' be functor of triangulated categories, N a null system in D, one defines the localization of F similar as in the usual case, replacing all categories and functors by triangulated ones. Applying proposition (5. 3.20) we get;

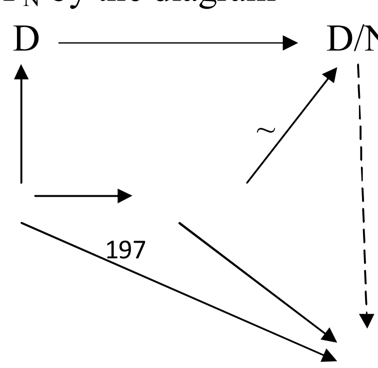
Proposition (5. 9. 63)[6]: Let D be a triangulated category, N a null system, I a full triangulated category of D. Let F: D → D' be triangulated functor, and assume.

(i) For any x ∈ D, there exists a d.t. x → y → z → T(x)with z ∈ N and y ∈ I,

(ii) For any y ∈ N∩I, F(y) $\simeq$ o.

Then F is right localizable.

One can define F_N by the diagram



I I/N∩I

D'

If one replace condition (i) in proposition (5. 4. 25) by the condition (i)' for any $x \in D$, there exists a d. t. $x \to y \to z \to T(x)$ with $z \in N$ and $y \in I$,

One gets that F is left localizable.

Finally, let us consider triangulated bifunctors, i.e. bifunctors which are additive and triangulated with respect to each of their arguments.

Proposition (5. 9. 64): Let D,N.I and D', N', I' be as in proposition (5.5. 27).

Let F: DXD' → D'' be triangulated bifunctor, Assume:

(i) For any $x \in D$, there exists A d.t. $x \to y \to z \to T(x)$ with $z \in N$ and $y \in I$.
(ii) For any $x' \in D'$, there exists d.t. $x \to y \to z \to T(x)$ with $z' \in N'$ and $y' \in I$.
(iii) For any $y \in I$ and $y' \in I' \cap N'$, $F(y, y') \simeq o$,
(iv) For any $y \in I \cap N$, and $y' \in I'$ $F(y, y') \simeq o$

Then F is right localizable.

One denotes by FNN' its localization.

Of coruse, there exists a similar result for left localizable functors by reversing the arrows in the hypotheses (i) and (ii) above.

Example(5. 9. 65): Let D be a triangulated category and let $X_1 \to Y_1 \to Z_1 \to T(X_1)$ and $X_2 \to Y_2 \to Z_2 \to T(X_2)$ be to d. t. show that

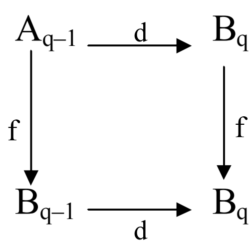
$X_1 \otimes X_2 \rightarrow Y_1 \otimes Y_2 \rightarrow Z_1 \otimes Z_2 \rightarrow T(X_1) \otimes T(X_2)$ be to d. t. show that $X_1 \otimes X_2 \rightarrow Y_1 \otimes Y_2 \rightarrow Z_1 \otimes Z_2 \rightarrow T(X_1) \otimes T(X_2)$ is ad. t.

In particular, $X \rightarrow X \otimes Y \rightarrow Y \overset{o}{\rightarrow} T(X)$ is ad. t.

(Hint; consider ad. t, $X_1 \otimes X_2 \rightarrow Y_1 \otimes Y_2 \rightarrow H \rightarrow T(X_1) \otimes T(X_2)$ and reconstruct the morphism $H \rightarrow Z_1 \otimes Z_2$. Then apply the result of example (5. 8. 57).

Section(5. 10) Effective Chain complexes:

Definition (5. 10. 10)[9. 73. 79]: Let $A_* = \{A_q, d_q\}_2$ and $B_* = \{B_q, d_q\}_2$ be two chain complexes. A chain complex morphism $f: A_* \rightarrow B_*$ is a collection of linear morphisms $f = \{f_q, : A_q \rightarrow B_q\}_q$ satisfying the differential condition: for every q, the relation $f_{q-1}\, d_q = f_q\, d_q$, or more simply $df = fd$:



is satisfied.

More and more frequently, we will not indicate the indices of morphisms, clearly implied by context. Also we use the same notation for a morphism and some other morphisms directly deduced from the first one.

If $f: A_* \rightarrow B_*$ is a chain-complex morphism, many other maps are naturally induced; most often they aredenoted bythe same symbol, f in this case. Because of the differential condition, the image of a cycle is a cycle and we have induced maps

$f: Z_q(A_*) \rightarrow Z_q(B_*)$, the same for the boundaries $f: B_q(A_*) \rightarrow B_q(B_*)$, and for homolog classes and homology groups $f: H_*(A_*) \rightarrow H_*(B_*)$.

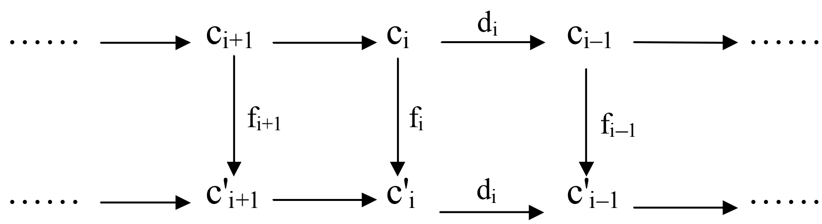
Definition (5. 10. 11) [7]: A chain complex is a collection of $\{c_i\}_{i \in Z}$ of R–modules and maps $\{d_i: c_i \rightarrow c_{i-1}\}$ called differentials such that

d_{i+1} d_i

d_{i-1} odi = o. similarly, a cochain complex is a collection of $\{c_i\}_{i \in Z}$ of R–modules and maps $\{d^i: c^i \to c^{i+1}\}$ such that d^{i+1} od^i od^i = o

...... c_{i+1} c_i x_{i-1} ......

Definition (5. 10. 12) [7]: Given two chain complexes c =(c_i,d) and ć = (c'_i,d'), chain map between them is a collection of maps f =\{f_i; $c_i \to c'_i$\} such that d'ofi =f_{i-1} od, i.e., the following diagram commutes.



Given a ring R, the collection of chain complexes of R–modules and chain maps between them forms a category, which we shall denote ch (R). Let c be a chain complex. Let z_i = Ker d_i be the cycles of c_i and B_i = in d_{i+1} be the boundaries ofc_i. Since d^2 = o, we have that, for each i, $B_i \subset Z_i$ call the quotient by H_i (c) = Z_i / B_i, the ith homology of c. similarly, for a co chain complex, we define the ith co homology H^i (c).

Definition(5. 10. 13)[7]: Two chain maps f, g: c → c' are chain homotopic, written f ~ g, if there exist c_i: $c_i \to c'_{i+1}$ such that f = g + d's + sd. The terminology comes from topology, where two maps which are homotopic at the level of topological spaces induce maps on corresponding chain complexes which are chain homotopic.

Proposition (5. 10. 66): If f, g: c →c' and f ~ g, then $f_* = g_*$.

Proof. It suffices to show that if f = d's + sd then f_* = o.

First note that d'f = fd = d'sd, and sof is actually a chain map. Let [x] ∈ Zn / Bn. Then f_* ([x]) = [d's (x) + sd (x)]=[d' sx) + s (o)] = [o].

There are certain kinds of chain complexes and chain maps which, due to their usefulness, have names. A map is f : c →c' a quasi-isomorphism if f_* is an isomorphism, and in this event, c and c' are said to be a homotopy equivalence and c and c' are homotopy equivalent.

If f and g are inverse chain homotopy equivalences, then f_* and g_* are inverses, and thus f and g are quasi = isomorphism. Not all quasi-isomorphims are chain homotopy equivalence. If idc ~ o, then c is said to be contractible. If c is c not ratable, then at the level of homology the identity map and the zero map are the same, and thus all homology groups are zero. This is not a necessary condition for the homology groups to vanish.

Remark(5.10. 67)[7]: The only difference between a chain complex and co chain complex is whether the maps go up in degree (are of degree1) or go down in degree (are of digree–1). Every chain complex is canonically a co chain complex by setting $c^i = c_{i-1}$ and $d=d_{i-1}$

Remark (5. 10. 68) [7]: While we have assumed complexes to be infinite in both directions, if a complex begins or ends with an infinite number of zeros, we can suppress these zeros and discuss finite or bounded complexes. Additionally, if c_i = o for all sufficiently large or sufficiently small values of I, then we say that the complex is bounded above or bounded below.

To ease notation, the sub scripts and superscripts on differentials will be suppressed. For example, the condition that one has chain complex becomes d^2 = o.

Remark (5. 10. 69) [7]: As we shall see later, there is a nice way to associate a chain complex to a space with a given triangulation. While two different triangulations of a space usually give rise to different chain complexes, the homology of these chain complexes will be isomorphic. This observation, one of the first applications of homology, crated a powerful family of algebraic in variants for a topological space. In general, most homology theories follow a similar pattern. Given an object (e.g., a topological space, a module, a pair of modules, a graph, a cow, a herd of chattel, etc.), we have a way to generate a chain complex, unfortunately the chain complex is not what we want: either it is too unwieldy to work with, there is not a canonical way to create it, similar objects will have dissimilar chain complex or something else will go wrong. However, when we pass to homology, our problems go away and we get an easy to compute algebraic in variant of our object from which we can easily read useful information.

Given a map f: c → c' between two chain complexes, f maps cycles and boundaries to boundaries, and thus f induces a map f_*: H(c) → H (c'). It often happens that two different chain aps induce the same maps on homology. The following is a useful sufficient condition for this to occur.

proposition (5. 10. 70) [7]: Given a short exact sequence $o \rightarrow A \xrightarrow{f} B \rightarrow c \rightarrow o$ of chain complexes, there are maps δ, natural in the sense of natural transfor-mations such that

$$\ldots \longrightarrow H_i(A) \xrightarrow{f_*} H_i(B) \xrightarrow{\delta_*} H_i(c) \xrightarrow{\delta} H_{i-1}(A) \xrightarrow{f_*} H_{i-1}(B) \xrightarrow{g_*} H_{i-1}(c) \longrightarrow \ldots$$

The following are all examples of complexes.

Examples (5. 10. 71) [7. 95]:

1. The complex … $o \rightarrow z \xrightarrow{o} z \rightarrow o \rightarrow$ … has two non zero homology groups, both isomorphic to z. In general, if all the maps in a complex are zero, then $H_i(c) \simeq c_i$.

2. The complex … $o \rightarrow z \xrightarrow{1} z \rightarrow o \rightarrow$ … is exact. In fact, it is contractible.
3. The complex … $o \rightarrow z \xrightarrow{2} z \rightarrow o \rightarrow$ … has $H_o(c) \simeq z / 2z$ and $H_1(c) \simeq o$.
4. The complex … $o \rightarrow z \longrightarrow z/2z \rightarrow o \rightarrow$ … has $H_o(c) \simeq o$ and $H_1(c) \simeq z$.

Examples (5. 10. 72) [7. 95]: Assume that one has a surface x with a triangulation T, namely a collection of (oriented) vertices, edges, and faces such that every point not on an edge is in the interior of a face, every face is bounded by three edges, and no vertex is in the interior of an edge. We can associate a chain complex c^T to this triangulation by denoting C_i^T to be the free abelian group on the i–1–cells of the triangulation and defining the differential on a generator of c_i^T to be an alternating sum of the i – cells on its boundary. Given a refinement T' ⊃ T, there is a natural inclusion map o: $c^T \to c^{T'}$ which is a quasi-isomorphism. Given two triangulations T' and T'', we can consider a common refinement T, and since c^T is quasi-isomorphic to both $c^{T'}$ and $c^{T''}$, we see that $H_n(c^{T'}) \simeq H_n(c^{T''})$ for every n, and thus $H_*(c^T)$ depends only on x.

This is the beginning of simplified homology, which is an important tool in the proof of the classification of surface.

Section (5. 11) Locally effective chain complexes:

Definition(5. 11. 73)[9. 73. 79]: A reduction p: $\hat{c}_* \Longrightarrow c_*$ is a diagram:

$$P = \boxed{h \circlearrowleft \hat{C}_* \underset{f}{\overset{g}{\rightleftarrows}} C_*}$$

Where:

1. $\hat{C}_*$ and c_* are chain-complexes.
2. f and g are chain-complex morphisms.
3. h is a homotopy operator (degree+1).
4. These relations are satisfied:
   (a) $fg = id_{c_*}$.
   (b) $gf + dh + hd = id_{\hat{c}_*}$.
   (c) $Fh = hg = hh = o$.

A reduction is a particular homology equivalence between a big chain complex $\hat{c}_*$ and a small one c_*. This point is deleted in the next proposition (5. 11. 66) be a reduction. This reduction is equivalent to a decomposition let be = $\hat{C}_* \Rightarrow C_* = A_* \otimes B_* \otimes \acute{C}_*$:

1. $\hat{C}_* \supset \acute{C}_* = 1mg$ is a sub complex of $\acute{C}_*$.
2. $A_* \otimes B_* = \mathrm{Ker}\, f$ is a sub complex of $\hat{c}_*$.
3. $\hat{C}_* \supset A_*$ Ker f ∩ Ker h is not in general a sub complex of $\hat{c}_*$.
4. $\hat{C}_* \supset B_* = \mathrm{Ker}\, f \cap \mathrm{Ker}\, h$ is a sub complex of $\hat{c}_*$ with null differentials.
5. The chain-complex morphism f and g inverse isomorphism's between $\acute{c}_*$ and c_*.
6. The arrows d and h are module isomorphism's of respective degrees–1 and +1 between A_* and B_*.

Theorem(5.11.73)[9]: Let $p=(f, g, h): \hat{c}_* \to c_*$ be a reduction where the chain complexes and $\hat{c}_* \to c_*$ are locally effective. If the homological problem is solved on the small chain-complex c_*, then the reduction p induces a solution of the homological problem for the big chain-complex $\hat{c}_*$.

Proof:

Let us examine the criteria of Definition (5.7.13).

1. Let $c \in \hat{C}_*$; the chain-complex $\hat{c}_*$ is locally effective and the "local" calculation dc can be achieved, which allows you to determine whether the chain c satisfies ds = o or not, where c is a cycle or not.
2. The known relation $id c_* = fg$ and $id c_* = gf + dh + hd$ imply f and g are inverse homology equivalences.
The homology groups H_n $(\hat{c}_*)$ and H_n (c_*) are canonically isomorphic. Let σ_* be the algorithms provided by the solution of the homological problem for c_* and let us call σ_* the algorithms to be constructed for $\hat{c}_*$. We can choose in particular $\sigma_{2.n} = \sigma_{2.n}$, the last equality being a genuine one.
3. The chain morphism f induces an isomorphism between $H_n(\hat{c}_*)$ and H_n (c_*). This allows us to choose $\sigma_{3.n}$ (z):
$= \sigma_{3.n}$ [f(z)].
4. In the same way, choose $\sigma_{4.n}$ (h): = f [$\sigma_{4.n}$ (h)].
5. Finally, if $z \in \hat{C}_n$ is a cycle known homologous to zero, a

boundary pre image is $\sigma_{5.n}$ (z): = h (z) + g [$\sigma_{5.n}$ (f (z)]. In fact: d (hz + g ($\sigma_{5.n}$ (f (z)] = dhz + gd $\sigma_{5.n}$ [f (z)]= dhz + gfz = z – hdz = z, for g is a chain complex morphism, $\sigma_{5.n}$ finds boundary pre images, and z is a cycle.

Corollary (5. 11. 74)[9]: If p = (f, g, h): $\hat{c}_* \to c_*$ is a reduction where $\hat{c}_*$ is locally effective and c_* is effective, then this reduction produces a solution of the homological problem for $\hat{c}_*$.

Proof:

The small chain-complex c_* is effective and a solution of the homological problem for c_* therefore is elementary.

Application(5.11.75): We want to concretely illustrate how reduction between locally effective and effective change complexes, allow a user to obtain and use the corresponding solution at homological problem. We considered the polynomial a RQ [t, x, z) and this ring the ideal I = $<$ T^5 – x, t^3y - $x^{2.}$ t^2y^2-x z -$y^{2.}$ t^2 – y - tx^2-, x^3 - t$y^{2.}$ y^3-x^2 z, xy$>$

It happens the homology of the Kesul complex K(R / I) effects deep properties of the ideal, I. the Koszul Complex is a Q -vector space of finite dimension, yet an algorithm can compute it is effective homology.

Keno constructs the ideal as a list of generator, each generator being a combination (cmbn) of monomials, each monomials being a list exponents. For example,(3 0 1 0) codes t^3y.

(Setf ideal).

(cmobn 0 1. (5 0 0 0)-1. (0 1 0 0))

(cmobn 0 1. (3 0 1 0)-1. (0 2 0 0))

[… 6 lines deleted …]

(cmohn 0 1. (0 1 1 0)-1. (1 0 0 1))

(cmobn)

<1 * (5 0 0 0)<1 *(0 1 0 0)>

[… other lines deleted….]

The display is simp0ly the list of generators, only the first one is given here.

The Kosul complex Ks(R/I) is then constructed.

Appendix

Symbol	**Terminologies (the meaning)**
Λ	Rank
$\oplus$	Direct sum
$\otimes$	Tensor product
H^o	Zero CoHomology Groub
$\ldots\longrightarrow F_n\longrightarrow F_{n-1}\longrightarrow F_0\longrightarrow m\longrightarrow 0\longrightarrow\ldots..$	Long exact sequence
$\longrightarrow X\longrightarrow X'\longrightarrow 0$	Short exact sequence
C^{op}	Object
$H_n\ (X)$	The nth homology group (X)
$H^n(X)$	The nth cohomology group (X)

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